博士論文

論文題目　Hom complexes and chromatic numbers of graphs
（グラフ Hom 複体と彩色数について）

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HOM COMPLEXES AND CHROMATIC NUMBERS OF GRAPHS

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1. Introduction

1.1. **Graph coloring problem.** A *graph* is a pair $G = (V(G), E(G))$ consisting of a set $V(G)$ together with a symmetric subset $E(G)$ of $V(G) \times V(G)$, i.e. $(x, y) \in E(G)$ implies $(y, x) \in E(G)$. Hence our graphs are undirected, may have loops, but have no multiple edges. We call a graph *simple* if it has no looped vertices. A *graph homomorphism* is a map $f : V(G) \to V(H)$ with $(f \times f)(E(G)) \subset E(H)$.

An *n-coloring of* G is a map $c : V(G) \to \langle n \rangle = \{0, 1, \cdots, n-1\}$ such that $(x, y) \in E(G)$ implies $c(x) \neq c(y)$. The chromatic number is defined to be the number

$$\chi(G) = \inf\{n \geq 0 \mid \text{There is an } n\text{-coloring of } G.\}.$$

Consider the infimum of the empty set is $+\infty$.

Let K_n be the complete graph with n-vertices. Namely, $V(K_n) = \langle n \rangle = \{0, 1, \cdots, n-1\}$ and $E(K_n) = \{(x, y) \mid x \neq y\} \subset V(G) \times V(G)$. Then an n-coloring is identified with a graph homomorphism from G to K_n. Let $f : G \to H$ be a graph homomorphism. If H has an n-coloring $c : H \to K_n$, then G has an n-coloring $c \circ f$, and hence we have $\chi(G) \leq \chi(H)$.

The neighborhood complexes were introduced by Lovász in his proof of the Kneser conjecture. He related the connectivity of the neighborhood complex of a graph G to the chromatic number of G. This is the first application of algebraic topology to the graph coloring problem. After that, several complexes have been considered by many authors in this context.

The Hom complex $\mathrm{Hom}(T, G)$ of graphs is a poset assigned to a pair of graphs T and G (The definition will be found in Section 1.3). This is functorial with respect to T and G, and hence if a group Γ acts on T then the Hom complex $\mathrm{Hom}(T, G)$ becomes a Γ-poset. We regard K_2 as a $\mathbb{Z}_2$-graph by the exchange of the two vertices. The box complex $B(G)$ is the $\mathbb{Z}_2$-poset $\mathrm{Hom}(K_2, G)$.

In this thesis, we study how far the chromatic numbers are determined by the neighborhood complexes, the box complexes, and the Hom complexes. A homotopy test graph is a graph T such that the connectivity of $\mathrm{Hom}(T, G)$ plus $\chi(T) + 1$ is a lower bound for the chromatic number of a graph G. We show that every bipartite graph is a homotopy test graph. This is conjectured by Kozlov. Next we show that the homotopy types of the Hom complexes (and hence the box complexes and the neighborhood complexes) do not determine the chromatic numbers of graphs. Moreover, we show that no homotopy invariant of $\mathrm{Hom}(T, G)$ is an upper bound for the chromatic number of G.

Hence to determine the chromatic number of G, we need to observe more rigid structures on the Hom complexes. The box complex $B(G) = \mathrm{Hom}(K_2, G)$ is a $\mathbb{Z}_2$-poset. We show that the $\mathbb{Z}_2$-poset structure of the box complex $B(G)$ determines the graph G up to isolated vertices. On the other hand, we show that there are graphs having the same box complexes, the same neighborhood complex, yet different chromatic numbers. This implies that the non-equivariant poset structure and the neighborhood complex do not determine the chromatic number. In the paper [30] of the proof of the Kneser conjecture, Lovász asked if there is a topological property which is equivalent to the k-colorability. Therefore the above example gives a negative answer to his question.

To deduce the non-existence of graph homomorphisms, we assign a graph to a the box complex $B(G)$, and consider that there is a $\mathbb{Z}_2$-map between the box complexes. Hence it is interesting to compare the category of graphs with the category of $\mathbb{Z}_2$-spaces. For example, it is important to understand that which $\mathbb{Z}_2$-homotopy class of $\mathbb{Z}_2$-continuous maps between the box complexes is induced by a graph homomorphism. In general, there is no graph homomorphism from G to H although there is a $\mathbb{Z}_2$-map between their box complexes. In fact the $\mathbb{Z}_2$-homotopy type of the box complex does not determine the chromatic number as was mentioned. However, if we consider the localization of the category of graphs by the class of graph homomorphisms which induce $\mathbb{Z}_2$-homotopy equivalences between box complexes, then the resulting category and the homotopy category of $\mathbb{Z}_2$-spaces are equivalent.

1.2. **Neighborhood complex.** For a vertex v of a graph G, the set of vertices adjacent to v is denoted by $N(v)$, and is called the *neighborhood of* v. The *neighborhood complex* $N(G)$ *of* G is the abstract simplicial complex whose vertex set is the set of non-isolated vertices of G and simplices are finite subsets of $V(G)$ contained in the neighborhood of some vertex of G. The neighborhood complex is introduced by Lovász [30], and he showed the following theorem.

Theorem 1.1 (Lovász [30]). *If the neighborhood complex $N(G)$ is n-connected, then $\chi(G) \geq n+3$.*

Lovász applied the above theorem to solve the Kneser conjecture described as follows. Let X be the family of subsets having k elements of $\langle n \rangle = \{0, \cdots, n-1\}$. Consider a partition X into m subsets $X_1, \cdots, X_m$ for some positive integer m. Kneser conjectured in 1955 that if $m \leq n-2k+1$, then there is X_i containing two elements which are disjoint as subsets in $\langle n \rangle$.

Let n, k be positive integers with $n \geq 2k$. The Kneser graph $KG_{n,k}$ is defined by

$$V(KG_{n,k}) = \{\sigma \subset \langle n \rangle \mid \#\sigma = k\},$$

and

$$E(KG_{n,k}) = \{(\sigma, \tau) \mid \sigma \cap \tau = \emptyset\}.$$

The Kneser conjecture is equivalent to $\chi(KG_{n,k}) = n-2k+2$. (It is easy to show that $\chi(KG_{n,k}) \leq n-2k+2$.) Lovász proved that $N(KG_{n,k})$ is $(n-2k)$-connected and showed that $\chi(KG_{n,k}) = n-2k+2$, using Theorem 1.1.

In the next section we shall explain how to obtain obstructions of the exitence of a coloring, using the Hom complexes.

1.3. **Hom complex.** A *multi-homomorphism from G to H* is a map $\eta : V(G) \to V(H)$ which satisfies $\eta(v) \times \eta(w) \subset E(H)$ for all $(v, w) \in E(G)$. We write $\eta \leq \eta'$ if $\eta(v) \subset \eta'(v)$ for all $v \in V(G)$. The *Hom complex* $\mathrm{Hom}(G, H)$ is the poset of multi-homomorphisms from G to H. Note that a graph homomorphism f is identified with a multi-homomorphism $v \mapsto \{f(v)\}$.

Let $f : G_1 \to G_2$ be a graph homomorphism. Then we have an order preserving map

$$\mathrm{Hom}(G_2, H) \to \mathrm{Hom}(G_1, H)$$

corresponding η to $\eta \circ f$. On the other hand, for a graph homomorphism $g : H_1 \to H_2$, define the order preserving map

$$g_* : \mathrm{Hom}(G, H_1) \to \mathrm{Hom}(G, H_2)$$

by $g_*(\eta)(x) = g(\eta(x))$. Thus we have a functor

$$\mathcal{G}^{\mathrm{op}} \times \mathcal{G} \to \mathcal{P}, (G, H) \mapsto \mathrm{Hom}(G, H),$$

where $\mathcal{G}$ is the category of graphs and $\mathcal{P}$ is the category of posets.

Suppose that a group Γ acts on a graph T from the right. Then the functorial property of the Hom complex implies that $\mathrm{Hom}(T, G)$ is a left Γ-poset, and a graph homomorphism $f : G \to H$ induces a Γ-equivariant map $f_* : \mathrm{Hom}(T, G) \to \mathrm{Hom}(T, H)$. Hence if there is no Γ-equivariant map from $\mathrm{Hom}(T, G)$ to $\mathrm{Hom}(T, H)$, then we have that there is no graph homomorphism from G to H.

Consider K_2 as a $\mathbb{Z}_2$-graph by the involution $0 \leftrightarrow 1$. We often write $B(G)$ instead of $\mathrm{Hom}(K_2, G)$, and call it the *box complex of* G.

Theorem 1.2 (Babson-Kozlov [1]). *The box complex $B(G)$ is homotopy equivalent to the neighborhood complex $N(G)$.*

Theorem 1.3 (Babson-Kozlov [1]). *The box complex $B(K_n)$ is $\mathbb{Z}_2$-homeomorphic to the $(n-2)$-sphere S^{n-2}.*

Here we consider the n-sphere S^n as a $\mathbb{Z}_2$-space by the antipodal map.

For a $\mathbb{Z}_2$-space X, the *$\mathbb{Z}_2$-index of X* is defined to be the number

$$\mathrm{ind}(X) = \inf\{n \geq -1 \mid \text{There is a } \mathbb{Z}_2\text{-map from } X \text{ to } S^n.\}.$$

Corollary 1.4 (Walker [37]). *The inequality*

$$\chi(G) \geq \mathrm{ind}(B(G)) + 2$$

holds for every graph G.

For a topological space X, define the *connectivity of X* to be the number

$$\mathrm{conn}(X) = \sup\{n \geq -1 \mid X \text{ is } n\text{-connected.}\}.$$

Since $\mathrm{ind}(X) > \mathrm{conn}(X)$, Theorem 1.1 follows from Corollary 1.4.

1.4. **Test graphs.** To generalize the case of the box complex, Kozlov introduced the notion of test graphs. A graph T is a test graph if a certain inequality between the chromatic number $\chi(G)$ and some homotopy invariant of $\mathrm{Hom}(T, G)$ holds for every graph G. There are several type of test graphs. We deal with homotopy test graphs and Stiefel-Whitney test graphs.

Definition 1.5 (Kozlov [21]). A graph T is a *homotopy test graph* if the inequality

$$\chi(G) > \mathrm{conn}(\mathrm{Hom}(T, G)) + \chi(T)$$

holds for every graph G.

An *involution of a graph* T is a graph homomorphism $\alpha : T \to T$ with $\alpha^2 = \mathrm{id}_T$. The involution α is *flipping* if there is a vertex v such that $\alpha(v)$ is adjacent to v. An involution is identified with a $\mathbb{Z}_2$-action on the graph T, and we call a $\mathbb{Z}_2$-graph T *flipping* if its $\mathbb{Z}_2$-action is flipping. It is easy to see that if T is a flipping $\mathbb{Z}_2$-graph, then $\mathrm{Hom}(T, G)$ is a free $\mathbb{Z}_2$-space (see [1] for example).

Let X be a free $\mathbb{Z}_2$-space. Let $\overline{X}$ denote the orbit space of X and suppose that the quotient map $X \to \overline{X}$ is a double covering. Let $w_1(X) \in H^1(\overline{X}; \mathbb{Z}_2)$ be the 1st Stiefel-Whitney class of the $O(1)$-bundle $X \to \overline{X}$. The *$\mathbb{Z}_2$-height of X* is defined to be the number

$$h(X) = \sup\{n \geq 0 \mid w_1(X)^n \neq 0\}.$$

Definition 1.6 (Kozlov). A flipping $\mathbb{Z}_2$-graph T is a *Stiefel-Whitney test graph* if

$$h(\mathrm{Hom}(T, K_n)) = n - \chi(T)$$

for every n with $n \geq \chi(T)$.

Remark 1.7. Dochtermann and Schultz [11] call a flipping $\mathbb{Z}_2$-graph T a "Stiefel-Whitney test graph" if the inequality

$$\chi(G) \geq h(\mathrm{Hom}(T, G)) + \chi(T) \tag{1}$$

holds for every graph G.

Suppose that T is a Stiefel-Whitney test graph in the sense of Definition 1.6. Let G be a graph and let $f : G \to K_n$ be an n-coloring of G. Then f induces a $\mathbb{Z}_2$-map

$$f_* : \mathrm{Hom}(T, G) \to \mathrm{Hom}(T, K_n).$$

This implies that $h(\mathrm{Hom}(T, G)) \leq \mathrm{Hom}(T, K_n) = n - \chi(T)$. Hence T is a "Stiefel-Whitney test graph" in the sense of Dochtermann and Schultz [11]. On the other hand, it is not known that these two notions coincide. In this thesis, we call a flipping $\mathbb{Z}_2$-graph T a *weak Stiefel-Whitney test graph* if the inequality (1) holds.

Remark 1.8. By the Gysin sequence, we have that $h(X) > \mathrm{conn}(X)$. This implies that a weak Stiefel-Whitney test graph (Remark 1.7) is a homotopy test graph.

There are several results concerning with test graphs. Theorem 1.1 and Theorem 1.2 imply that K_2 is a homotopy test graph. In general, Babson and Kozlov [1] showed that the complete graph K_n for $n \geq 2$ is a Stiefel-Whitney test graph. Here we consider the involution of K_n as the exchange of 0 and 1. Lovász conjectured that an odd cycle C_{2r+1} for a positive integer r is a homotopy test graph, and this conjecture was proved by Babson and Kozlov [2]. Babson and Kozlov conjectured that an odd cycle with reflection is a Stiefel-Whitney test graph, and this conjecture was proved by Schultz [34], and was later solved in [24] and [35]. It is clear that a graph having no edges is not a homotopy test graph. The first non-trivial example of a non-homotopy test graph is found in Hoory and Linial [17]. For further references relating to test graphs, we refer to [11].

Kozlov suggested several problems concerning with Hom complexes in [21]. The following are two of them.

Problem 1.9 (Kozlov, Conjecture 6.2.1 of [21])**.** *Does $\chi(T) = 2$ imply that T is a homotopy test graph?*

Remark 1.10. The precise statement of the conjecture is "Every bipartite graph is a homotopy test graph". Here we should consider that the term "bipartite graph" means a graph with chromatic number 2. In fact it is clear that graphs having no edges are not homotopy test graphs, as was mentioned.

Problem 1.11 (Kozlov, Section 6.1 of [21])**.** *Is there a graph T having two flipping involutions α_0 and α_1 such that (T, α_0) is a Stiefel-Whitney test graph but (T, α_1) is not?*

The purpose of Section 3 is to solve the above problems. The following theorem is the answer to Problem 1.9.

Theorem 1.12. *A graph T with $\chi(T) = 2$ is a homotopy test graph.*

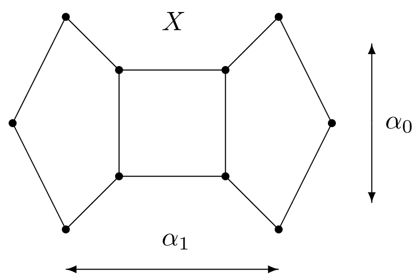

Figure 1.1

Let X be a graph illustrated in Figure 1.1. Let α_0 be the reflection in the horizontal line, and let α_1 be the reflection in the vertical line. The following theorem is the answer to Problem 1.11.

Theorem 1.13. *Let X be the graph, and let α_0 and α_1 be the involutions described in Figure 1.1. Then (X, α_0) is a Stiefel-Whitney test graph but (X, α_1) is not.*

1.5. **Homotopy types of Hom complexes.** In his pioneer paper [30], Lovász asked that there is a homotopy, or topological invariant of neighborhood complexes which is equivalent to the chromatic number. The homotopy case of this question was negatively solved by Walker [37]. He constructed graphs

G_1 and G_2 such that their box complexes are $\mathbb{Z}_2$-homotopy equivalent but their chromatic numbers are different (see Theorem 1.3 and Figure 1.2).

In Section 4, we will generalize his result to Hom complexes in the following form:

Theorem 1.14. *Let T be a finite graph and let G be a non-bipartite graph. For each integer n, there is a graph H such that $\mathrm{Hom}(T,G)$ and $\mathrm{Hom}(T,H)$ are homotopy equivalent and $\chi(H) > n$. In case T is a flipping $\mathbb{Z}_2$-graph, we can take H so that $\mathrm{Hom}(T,G)$ and $\mathrm{Hom}(T,H)$ are $\mathbb{Z}_2$-homotopy equivalent. Moreover, if G is finite or connected, then we can take H to be finite or connected, respectively.*

This theorem will be proved in Section 4.

In Section 1.3 and Section 1.4, we mentioned that some homotopy invariants of the Hom complex $\mathrm{Hom}(T,G)$ gives a lower bound for the chromatic number of G. On the other hand, Theorem 1.14 implies that there is no ($\mathbb{Z}_2$-equivariant) homotopy invariant of $\mathrm{Hom}(T,G)$ gives an upper bound for the chromatic number. However, we should say that this result is almost known by Walker [37] in the case $T = K_2$. In fact, he noticed that for every integer n, there is a finite graph G such that $\chi(G) > n$ and $B(G)$ is $\mathbb{Z}_2$-homotopy equivalent to some 1-dimensional $\mathbb{Z}_2$-complex.

1.6. **Kronecker double coverings.** Therefore to determine the chromatic number, one should observe more rigid structures of the Hom complex. The next result (Theorem 1.15) concerns with the poset structure and the $\mathbb{Z}_2$-poset structure of the box complex.

Let G and H be graphs. The *(tensor or categorical) product $G \times H$ of G and H* is the graph defined by

$$V(G \times H) = V(G) \times V(H)$$

and

$$E(G \times H) = \{((x,y),(x',y')) \mid (x,x') \in E(G), (y,y') \in E(H)\}.$$

It is easy to see that the product $G \times H$ is actually the categorical product of the category of graphs.

A graph homomorphism $p : G \to H$ is a *covering* if $p|_{N(v)} : N(v) \to N(p(v))$ is bijective for all $v \in V(G)$. The *Kronecker double covering over G* is the 2nd projection $K_2 \times G \to G$. It is easy to see that the Kronecker double covering is a covering.

The following result shows that there is a remarkable relation between box complexes (or neighborhood complexes) and Kronecker double coverings.

Theorem 1.15. *Let G and H be graphs without isolated vertices. Then the following hold.*

(1) *The Kronecker double coverings $K_2 \times G$ and $K_2 \times H$ are isomorphic if and only if their box complexes $B(G)$ and $B(H)$ are isomorphic as posets.*
(2) *The graphs G and H are isomorphic if and only if their box complexes $B(G)$ and $B(H)$ are isomorphic as $\mathbb{Z}_2$-posets.*
(3) *If the Kronecker double coverings $K_2 \times G$ and $K_2 \times H$ are isomorphic, then their neighborhood complexes are isomorphic. On the other hand, if G and H are stiff (mentioned below), then the converse holds.*

A graph G is *stiff* if $v, w \in V(G)$ and $N(v) \subset N(w)$ imply $v = w$.

Let m, n be positive integers greater than 3. In Example 5.16, we construct graphs G and H such that $K_2 \times G \cong K_2 \times H$, $\chi(G) = m$, and $\chi(H) = n$. It follows from (1) of Theorem 1.15, there are graphs G and H with isomorphic box complexes and isomorphic neighborhood complex, yet with different chromatic numbers. As was mentioned, Lovász asked if there is a topological property of $N(G)$ which is equivalent to the chromatic number of G. The above example gives a negative answer to his question.

1.7. **Simplicial methods.** The category of graphs is denoted by $\mathcal{G}$. For a non-negative integer n, define the graph Σ_n as follows: The vertex set of Σ_n is $[n] = \{0, 1, \cdots, n\}$, and the edge set of Σ_n is $V(\Sigma_n) \times V(\Sigma_n)$. Let T be a graph. Define $\mathrm{Sing}(T, G)$ to be the simiplicial set whose n-simplices are graph homomorphisms from $T \times \Sigma_n$ to G, i.e. $\mathrm{Sing}(T, G)_n = \mathcal{G}(T \times \Sigma_n, G)$. We will show the following theorem in Section 7.

Theorem 1.16. *There is a natural homotopy equivalence*

$$|\mathrm{Sing}(T, G)| \xrightarrow{\simeq} |\mathrm{Hom}(T, G)|.$$

As an application of this theorem, we show that the category $\mathcal{G}$ of graphs has a model structure which is Quillen equivalent to the category of $\mathbb{Z}_2$-spaces.

Let $\mathbf{SSet}$ be the category of simplicial sets and let $\mathbf{SSet}^{\mathbb{Z}_2}$ be the category of $\mathbb{Z}_2$-simplicial sets.

Let $\mathcal{B}(G)$ be the $\mathbb{Z}_2$-simplicial set $\mathrm{Sing}(K_2, G)$. We will call $\mathcal{B}(G)$ the *singular box complex of* G. One can show that $\mathcal{B}(G)$ is $\mathbb{Z}_2$-homotopy equivalent to the $B(G)$. Then the functor

$$\mathcal{B} : \mathcal{G} \longrightarrow \mathbf{SSet}^{\mathbb{Z}_2}$$

has a left adjoint. (In fact the usual box complex functor B is not a right adjoint functor since B does not preserve products up to homotopy.) Let $\mathcal{A} : \mathbf{SSet}^{\mathbb{Z}_2} \to \mathcal{G}$ be the left adjoint functor of $\mathcal{A}$. If K is an ordered $\mathbb{Z}_2$-simplicial complex, then it turns out that $\mathcal{A}(K)$ is isomorphic to G_K constructed in Csorba [7]. Namely, the vertex set of $\mathcal{A}(K)$ is the vertex set of K. Two vertices v, w are adjacent if and only if $\{\alpha(v), w\}$ is a simplex of K, where α is the involution of K.

A graph homomorphism $f : G \to H$ induces a $\mathbb{Z}_2$-map $f : G \to H$. So it is important to compare the category of graphs with the category of $\mathbb{Z}_2$-spaces. In general, there is not a graph homomorphism from G to H if there is a $\mathbb{Z}_2$-map from $B(G)$ to $B(H)$ (see Section 1.5). However, the following theorem asserts that the localization of the category of graphs with respect to the class of graph homomorphisms which induce $\mathbb{Z}_2$-homotopy equivalences between box complexes coincides with the homotopy category of $\mathbb{Z}_2$-spaces. Let $\mathrm{Sd} : \mathbf{SSet} \to \mathbf{SSet}$ be the barycentric subdivision functor, and let Ex be the right adjoint of Sd. Then we have the following.

Theorem 1.17. *The category $\mathcal{G}$ of graphs has a model structure described as follows:*

(1) *A graph homomorphism $f : G \to H$ is a weak equivalence if and only if $f_* : B(G) \to B(H)$ is a $\mathbb{Z}_2$-homotopy equivalence.*
(2) *A graph homomorphism $f : G \to H$ is a cofibration if there is an inclusion $i : K \hookrightarrow L$ of $\mathbb{Z}_2$-simplicial sets such that $f \cong \mathcal{A} \circ \mathrm{Sd}^3(i)$.*

Moreover, the adjoint pair

$$\mathcal{A} \circ \mathrm{Sd}^3 : \mathbf{SSet}^{\mathbb{Z}_2} \longrightarrow \mathcal{G} : \mathrm{Ex}^3 \circ \mathcal{B}$$

is a Quillen equivalence.

Recall that the inequality

$$\chi(G) \geq \mathrm{ind}(B(G)) + 2$$

holds for every graph G (see Corollary 1.4). Namely, the chromatic number of $\chi(G)$ is bounded below by the $\mathbb{Z}_2$-homotopy invariant of $B(G)$. As another application of the singular box complex, we show that the lower bound $\mathrm{ind}(B(G)) + 2$ is best possible in the following sense: Let u be a $\mathbb{Z}_2$-homotopy invariant of a $\mathbb{Z}_2$-space, which assign an integer to a $\mathbb{Z}_2$-space, and suppose that $\chi(G) \geq u(B(G))$ for every finite graph G. Then we have that $\mathrm{ind}(X) + 2 \geq u(X)$. In fact we show that for a finite $\mathbb{Z}_2$-complex X, there is a finite graph G with $B(G) \simeq_{\mathbb{Z}_2} X$ and $\chi(G) = \mathrm{ind}(X) + 2$. However, we should note that this is also deduced from Theorem 1.6 and Theorem 1.7 in Dochtermann and Schultz [11].

1.8. **Organization of the thesis.** The rest of the thesis is organized as follows. Section 3 is devoted to the proofs of Theorem 1.11 and Theorem 1.12 which answers Problem 1.9 and Problem 1.10, respectively. Section 4 is devoted to the proof of Theorem 1.13. In Section 5, we review the theory of the Kronecker double coverings, and prove Theorem 1.14. Here we construct graphs such that their Kronecker double coverings are isomorphic but their chromatic numbers are different.

In Section 2, we review definitions and facts we will need in Section 3, Section 4, and Section 5. For the reader who is not familiar with topology, we often give precise proofs.

In Section 6, we review simplicial sets and model categories. In Section 7, we introduce the singular complex mentioned in Section 1.6 and prove Theorem 1.16 and Theorem 1.17.

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2. Preliminaries

2.1. **Classical theorems in algebraic topology.** Let us recall the following three theorems which are well-known in algebraic topology.

Theorem 2.1 (Whitehead's theorem, Theorem 4.5 of [14])**.** *A continuous map $f : X \to Y$ between CW-complexes is a homotopy equivalence if and only if f induces a bijection $\pi_0(X) \to \pi_0(Y)$ and an isomorphism $\pi_n(X,x) \to \pi_n(Y, f(x))$ for every $n > 0$ and $x \in X$.*

Theorem 2.2 (Theorem 4.5 of [14])**.** *For a CW-pair (X, A), A is a deformation retract of X if and only if the inclusion $A \hookrightarrow X$ is a homotopy equivalence.*

Theorem 2.3 (Cellular approximation theorem, Theorem 4.8 of [14])**.** *A continuous map $f : X \to Y$ between CW-complexes is homotopic to a cellular map. If f is already cellular on the subcomplex $A \subset X$, we can take the homotopy to be stationary on A.*

We often use the following property of CW-complexes.

Proposition 2.4 (Proposition A.1 of [14])**.** *A compact subset of a CW-complex is contained in some finite subcomplex of it.*

Proposition 2.5 (Gluing lemma)**.** *Let $f : X \to Y$ be a continuous map between CW-complexes, let A_0 and A_1 be subcomplexes of X with $X = A_0 \cup A_1$, and let B_0 and B_1 be subcomplexes of Y with $Y = B_0$, B_1. Suppose that $f(A_i) \subset B_i$ for $i = 0, 1$. If $f|_{A_0} : A_0 \to B_0$, $f|_{A_1} : A_1 \to B_1$, and $f|_{A_0 \cap A_1} : A_0 \cap A_1 \to B_0 \cap B_1$ are homotopy equivalences, then f is a homotopy equivalence.*

Proof. It follows from the cellular approximation theorem that $f|_{A_0\cap A_1} : A_0 \cap A_1 \to B_0 \cap B_1$ is homotopic to a cellular map. The homotopy extension property implies that f is homotopic to a continuous map g such that $g|_{A_i} \simeq f|_{A_i} : A_i \to B_i$ for each $i = 0, 1$ and $g|_{A_0\cap A_1}$ is cellular. Applying the cellular approximation theorem to each of the maps $g|_{A_i} : A_i \to B_i$, we have that there is a cellular map $h : X \to Y$ such that $h|_{A_i} \simeq g|_{A_i}$ for each $i = 0, 1$ and $h|_{A_0\cap A_1} = g|_{A_0\cap A_1}$. Replacing h to f, we can assume that f is cellular.

Let M_f be the mapping cylinder of f. We want to show that the inclusion $X \hookrightarrow M_f$ is a homotopy equivalence. For $i = 0, 1$, consider the sequence

$$A_i \hookrightarrow (M_{f|_{A_0 \cap A_1}} \cup A_i) \hookrightarrow M|_{f|_{A_i}}.$$

The first inclusion and the composition are deformation retracts. Hence the second inclusion is a homotopy equivalence. Therefore $M_{f|_{A_0 \cap A_1}} \cup A_i$ is a deformation retract of $M|_{f|_{A_i}}$.

Next consider the sequence

$$X \hookrightarrow (M_{f|_{A_0 \cap A_1}} \cup X) \hookrightarrow M_f = (M_{f|_{A_0}} \cup M_{f|_{A_1}}).$$

Clearly, the first inclusion is a deformation retract. It follows from the previous paragraph that the second inclusion is a deformation retract. Hence the composition $X \hookrightarrow M_f$ is a homotopy equivalence. □

Proposition 2.5 has a vast generalization (see Proposition 6.18).

Proposition 2.6. *Let $f : X \to Y$ be a map between CW-complexes. Suppose that for every finite subcomplexes X' of X and Y' of Y with $f(X') \subset Y'$, there are subcomplexes A of X and B of Y such that $X' \subset A$, $Y' \subset B$, $f(A) \subset B$, and $f|_A : A \to B$ is a homotopy equivalence. Then f is a homotopy equivalence.*

Proof. It suffices to show that f induces a bijection $\pi_0(X) \to \pi_0(Y)$ and isomorphisms $\pi_n(X, x) \to \pi_n(Y, f(x))$ for every $x \in X$ and $n > 0$. Here we only prove that the map $f_* : \pi_n(X, x) \to \pi_n(Y, f(x))$ is injective for $n > 0$ since the other parts are similarly proved.

Let $\alpha \in \pi_n(X, x)$ and suppose that $f_*(\alpha) = 1$. Let $\varphi : S^n \to X$ be a representative of α and let $\psi : D^{n+1} \to Y$ be an extension of $f \circ \varphi$. By Proposition 2.4 and the hypothesis, there are subcomplexes A of X and B of Y such that $\varphi(S^n) \subset A$, $\psi(D^{n+1}) \subset B$, $f(A) \subset B$, and $f|_A : A \to B$ is a homotopy equivalence. Since $f \circ \varphi$ is null-homotopic in B and $f|_A$ is a homotopy equivalence, φ is null-homotopic. Hence $f_* : \pi_n(X, x) \to \pi_n(Y, f(x))$ is injective. □

Corollary 2.7. *If every finite subcomplex of X is contained in some contractible subcomplex, then X is contractible.*

Proposition 2.8 (Infinite version of the gluing lemma)**.** *Let $f : X \to Y$ be a continuous map between CW-complexes. Let S be a set, and let $\{A_i \mid i \in S\}$ and $\{B_i \mid i \in S\}$ be S-indexed families of subcomplexes of X and subcomplexes of Y, respectively. Suppose that the following conditions hold:*

(1) $\bigcup_{i \in S} A_i = X$ *and* $\bigcup_{i \in S} B_i = Y$.
(2) $f(A_i) \subset B_i$ *for every* $i \in S$.
(3) *For a non-empty finite subset* $\{i_0, \cdots, i_k\} \subset S$, *the map*

$$f|_{A_0 \cap \cdots \cap A_k} : A_0 \cap \cdots \cap A_k \to B_0 \cap \cdots \cap B_k$$

is a homotopy equivalence.

Then f is a homotopy equivalence.

Proof. First we consider the case S is finite. This follows from the induction on the cardinality of S. Set $S = \{i_0, \cdots, i_k\}$. For simplicity, we write A_j or B_j instead of A_{i_j} or B_{i_j}, respectively.

The case $k = 0$ is obvious. By the induction hypothesis, we have that

$$f|_{A_0 \cup \cdots \cup A_{k-1}} : A_0 \cup \cdots \cup A_{k-1} \to B_0 \cup \cdots \cup B_{k-1}$$

and

$$f|_{(A_0 \cup \cdots \cup A_{k-1}) \cap A_k} : (A_0 \cup \cdots \cup A_{k-1}) \cap A_k \to (B_0 \cup \cdots \cup B_{k-1}) \cap B_k$$

are homotopy equivalences since $(A_0 \cup \cdots \cup A_{k-1}) \cap A_k = (A_0 \cap A_k) \cup \cdots \cup (A_{k-1} \cap A_k)$ and $(B_0 \cup \cdots \cup B_{k-1}) \cap B_k = (B_0 \cap B_k) \cup \cdots \cup (B_{k-1} \cap B_k)$. It follows from Proposition 2.5 that $f|_{A_1 \cup \cdots \cup A_k} : A_1 \cup \cdots \cup A_k \to B_1 \cup \cdots \cup B_k$ is a homotopy equivalence. This completes the proof of the case S is finite.

If S is infinite, the proposition follows from Proposition 2.4 and Proposition 2.6. □

Let Γ be a group and let $f, g : X \to Y$ be Γ-maps. A homotopy $(h_t)_{t \in I}$ is a Γ-*homotopy* if h_t is Γ-equivariant for every $t \in I$. We call f and g Γ-*homotopic* if there is a Γ-homotopy from f to g, and write $f \simeq_\Gamma g$. A Γ-map $f : X \to Y$ is a Γ-*homotopy equivalence* if there is a Γ-map $g : Y \to X$ with $g \circ f \simeq_\Gamma \mathrm{id}_X$ and $f \circ g \simeq_\Gamma \mathrm{id}_Y$.

A Γ-action on a CW-complex X is *free* if $\gamma e \cap e = \emptyset$ for every cell e of X and $\gamma \in \Gamma \setminus \{1\}$.

Proposition 2.9. *Let $f : X \to Y$ be a Γ-map between free Γ-CW-complexes. Then f is a Γ-homotopy equivalence if and only if f is a homotopy equivalence.*

Proof. If Γ is finite, then this follows from Chapter II of Bredon [6]. However, we will use this theorem in the case that Γ is infinite. Although his proof works well if the action is free, we give an alternative proof for the reader's convenience.

We write $\overline{X}$ and $\overline{Y}$ to indicate the orbit spaces of X and Y, respectively. We show that the map $\overline{f} : \overline{X} \to \overline{Y}$ induced by f is a homotopy equivalence. By the cellular approximation theorem, $\overline{f}$ is homotopic to a cellular map. By the homotopy lifting property of covering spaces, we have that f is Γ-homotopic to a cellular map. So we can assume that f is cellular.

Let M_f be the mapping cylinder of f. Then Γ acts freely on M_f and the orbit space is identified with the mapping cylinder $M_{\overline{f}}$ of $\overline{f}$. Let $\varphi : (D^n, S^{n-1}) \to (M_{\overline{f}}, \overline{X})$ be a map of pairs. Since D^n is contractible, there is a lift $\tilde{\varphi} : (D^n, S^{n-1}) \to (M_f, X)$. Since f is a homotopy equivalence, $\tilde{\varphi}$ is homotopic rel S^{n-1} to a map with image contained in X. Hence φ is homotopic rel S^{n-1} to a map with image contained in $\overline{X}$. This implies that $\pi_n(M_{\overline{f}}, \overline{X})$ is trivial and hence $\overline{f}$ is a homotopy equivalence.

Thus $\overline{X}$ is a deformation retract of $M_{\overline{f}}$. By the homotopy lifting property of covering spaces, this deformation retract lifts a deformation retract $(h_t : M_f \to M_f)$ of M_f to X. It is clear that h_t is a Γ-equivariant for every t. $\square$

2.2. **Abstract simplicial complex.** In this section, we review some definitions relating to the abstract simplicial complex. We review the strong homotopy theory of simplicial complexes introduced by Barmak and Minian [5].

An *abstract simplicial complex* is a pair (V, Δ) consisting of a set V equipped with a family of finite subsets of V, and we require the following conditions:

(1) $v \in V$ implies $\{v\} \in \Delta$.
(2) $\tau \in \Delta$ and $\sigma \subset \tau$ imply $\sigma \in \Delta$.

We often write Δ to indicate the simplicial complex (V, Δ). In this terminology, the vertex set of Δ is denoted by $V(\Delta)$.

A *simplicial map* is a map $f : V(\Delta_0) \to V(\Delta_1)$ such that $\sigma \in \Delta_1$ implies $f(\sigma) \in \Delta_2$. Two simplicial maps f and g are *contiguous* if $\sigma \in \Delta_0$ implies $f(\sigma) \cup g(\sigma) \in \Delta_1$. Let $\simeq_s$ be the equivalence relation generated by the contiguity. Two simplicial maps f and g from Δ_0 to Δ_1 are *strongly homotopic* if $f \simeq_s g$.

As was the case of Hom complexes of graphs, we shall consider the following construction. A *simplicial multi-map from Δ_0 to Δ_1* is a map $\eta : V(\Delta_0) \to 2^V(\Delta_1) \setminus \{\emptyset\}$ such that $\sigma \in \Delta_0$ implies

$$\bigcup_{v \in \sigma} \eta(v) \in \Delta_2.$$

For simplicial multi-maps η and η', we write $\eta \leq \eta'$ if $\eta(v) \subset \eta'(v)$ for every $v \in V(\Delta_0)$. The poset of simplicial multi-maps is denoted by $\mathrm{Map}(\Delta_0, \Delta_1)$. Note that a simplicial map is identified with a minimal point of $\mathrm{Map}(\Delta_0, \Delta_1)$.

Lemma 2.10. *Let f, g be simplicial maps from Δ_0 to Δ_1. Then the following are equivalent.*

(1) *f and g are contiguous.*
(2) *The map $V(\Delta_0) \to 2^{V(\Delta_1)} \setminus \{\emptyset\}$, $v \mapsto \{f(v), g(v)\}$ is a simplicial multi-map.*
(3) *There is an element $\eta \in \mathrm{Hom}(\Delta_0, \Delta_1)$ with $f \leq \eta$ and $g \leq \eta$.*

This lemma implies that f and g are strongly homotopic if and only if they belong to the same connected component of $\mathrm{Map}(\Delta_0, \Delta_1)$.

Let K, L, M be simplicial complexes. For $\eta \in \mathrm{Map}(K, L)$ and $\tau \in \mathrm{Map}(L, M)$, define the composition $\tau * \eta \in \mathrm{Map}(K, M)$ by

$$(\tau * \eta)(x) = \bigcup_{y \in \eta(x)} \tau(y).$$

The composition map is an order-preserving map $\mathrm{Map}(L, M) \times \mathrm{Map}(K, L) \to \mathrm{Map}(K, M)$. If $f : K \to L$ and $g : L \to M$ are simplicial maps, then $g * f$ coincides with the composition $g \circ f$ of maps.

Let $f_i : K \to L$ $(i = 0, 1)$ and $g_i : L \to M$ $(i = 0, 1)$ be simplicial maps. If $f_0 \simeq_s g_0$ and $f_1 \simeq_s g_1$, then $g_0 \circ f_0 \simeq_s g_1 \circ f_1$ since the composition map $*$ is an order-preserving map.

Definition 2.11 (Barmak-Minian [5])**.** Let K be a simplicial complex. A vertex x of K is a *cone point* if there is another vertex y of K such that $\sigma \in K$ and $x \in \sigma$ imply $\sigma \cup \{y\} \in K$.

Let K be a simplicial complex and let S be a subset of $V(K)$. The maximal subcomplex of K whose vertex set is $V(K) \setminus S$ is denoted by $K \setminus S$. If $S = \{x\}$ we write $K \setminus x$ instead of $K \setminus \{x\}$.

Lemma 2.12 (Barmak-Minian [5])**.** *Let K be a simplicial complex and let x be a cone point of K. Then the inclusion $K \setminus x \hookrightarrow K$ is a strong homotopy equivalence.*

Proof. Define the simplicial map $f : K \to K$ by the correspondence

$$f(v) = \begin{cases} y & (v = x) \\ v & (v \neq x). \end{cases}$$

Then the pair of f and id_K satisfies the condition (2) of Lemma 2.10. □

Let K be a simplicial complex. Let $\mathbb{R}^{(V(K))}$ denote the free $\mathbb{R}$-module generated by $V(K)$. Consider that the topology of $\mathbb{R}^{(V(K))}$ is induced by the finitely generated $\mathbb{R}$-submodules. For a vertex v, the element of $\mathbb{R}^{(V(K))}$ associated to v is denoted by e_v. For a simplex $\sigma \in K$, set

$$\Delta_\sigma = \{\sum_{v \in \sigma} a_v e_v \mid a_v \geq 0 \ (v \in \sigma), \ \sum_{v \in \sigma} a_v = 1\}.$$

The *geometric realization of* K is the union

$$|K| = \bigcup_{\sigma \in K} \Delta_\sigma.$$

A simplicial map $f : K \to L$ induces a continuous map $|f| : |K| \to |L|$.

The geometric realization functor allows us to assign topological terms to simplicial complexes. For example, a simplicial map f is a homotopy equivalence if the continuous map $|f|$ induced by f is a homotopy equivalence.

Let K and L be simplicial complexes. The *join* $K * L$ is the simplicial complex whose vertex set is the disjoint union $V(K) \sqcup V(L)$, and a subset σ of $V(K) \sqcup V(L)$ is a simplex of $K * L$ if and only if $\sigma \cap V(K) \in K$ and $\sigma \in V(L)$.

Let K be a simplicial complex and let σ be a simplex of K. The *star of* σ is the subcomplex $\{\tau \in K \mid \sigma \cup \tau \in K\}$.

2.3. **Posets.** A partially ordered set is called a *poset*, for short. A *chain of a poset* P is a subset c of P such that for every pair of elements in c is comparable. The *order complex* $\Delta(P)$ *of* P is the abstract simplicial complex whose simplices are finite chains of P. The *classifying space of* P is the geometric realization of the order complex, and is denoted by $|P|$. As is the case of simplicial complexes, we assign topological terms to posets by the classifying space functor.

Let P, Q be posets. The *product* $P \times Q$ *of posets* is defined as follows: The underlying set of $P \times Q$ is the direct product of the underlying sets of P and Q, and $(x, y) \leq (x', y')$ if and only if $x \leq x'$ and $y \leq y'$.

Proposition 2.13. *There is a natural isomorphism* $|P| \times |Q| \cong |P \times Q|$.

Proof. See Theorem 10.21 of Kozlov [22]. $\square$

Let $f : P \to Q$ be an order-preserving map. If c is a finite chain of P, then $f(c)$ is a chain. Hence f induces a continuous map $|f| : |P| \to |Q|$.

Corollary 2.14. *Let* $f, g : P \to Q$ *be order-preserving maps and suppose that* $f(x) \leq g(x)$ *for every* $x \in P$. *Then* $|f| \simeq |g|$.

Proof. Consider $[1] = \{0, 1\}$ as a poset ordered by the usual ordering. Define the order preserving map

$$F : P \times [1] \to Q$$

to be the map $F(x, 0) = f(x)$ and $F(x, 1) = g(x)$. Then $|F| : |P| \times |[1]| \cong |P| \times [0, 1] \to |Q|$ is a homotopy from $|f|$ to $|g|$. $\square$

Let P, Q be posets. For a pair of order-preserving maps $f, g : P \to Q$, we write $f \leq g$ if $f(x) \leq g(x)$ for every $x \in P$. The poset of order-preserving maps from P to Q is denoted by $\mathrm{Poset}(P, Q)$. We call two order-preserving maps $f, g : P \to Q$ *strongly homotopic* if they belong to the same connected component of $\mathrm{Poset}(P, Q)$, and in this case we write $f \simeq_s g$. An order-preserving map $f : P \to Q$ is a *strong equivalence* if there is an order-preserving map $g : Q \to P$ with $g \circ f \simeq_s \mathrm{id}_P$ and $f \circ g \simeq_s \mathrm{id}_Q$.

In fact, the term "strongly homotopic" is not standard. However, this notion was known in terms of finite space theory (see Barmak [4]). A finite space is a topological space whose underlying set is finite. Recall that the category of finite posets is isomorphic to the category of finite T_0-spaces. Then the order-preserving maps f and g are strongly homotopic if and only if the continuous maps associated to f and g are homotopic.

Lemma 2.15. *Let* P*,* Q*, and* R *be posets. Then there is a natural isomorphism*

$$\mathrm{Poset}(P \times Q, R) \xrightarrow{\cong} \mathrm{Poset}(P, \mathrm{Poset}(Q, R)).$$

Let $f_i : P \to Q$ $(i = 0, 1)$ and $g_i : Q \to R$ $(i = 0, 1)$ be order-preserving maps. Suppose that $f_0 \simeq_s f_1$ and $g_0 \simeq_s g_1$. Then we have $g_0 \circ f_0 \simeq_s g_1 \circ f_1$ since the composition

$$\mathrm{Poset}(Q, R) \times \mathrm{Poset}(P, Q) \to \mathrm{Poset}(P, R)$$

is an order-preserving map.

The face poset FK of an abstract simplicial complex K is the set of non-empty simplices ordered by inclusion.

Lemma 2.16. *Let* f *and* g *be simplicial maps from* K *to* L*. If* f *and* g *are strongly homotopic, then* Ff *and* Fg *are strongly homotopic.*

Proof. Define the map $F : \mathrm{Map}(K, L) \to \mathrm{Poset}(FK, FL)$ by

$$F\eta(\sigma) = \bigcup_{x \in \sigma} \eta(x).$$

The map F is a well-defined order-preserving map. Hence if the simplicial maps f and g belong to the same connected component of $\mathrm{Map}(K, L)$, then Ff and Fg belong to the same connected component of $\mathrm{Poset}(FK, FL)$. $\square$

An *ascending closure operator of* P is an order-preserving map $c : P \to P$ such that $c \geq \mathrm{id}_P$ and $c^2 = c$. A *descending closure operator of* P is an order-preserving map $c : P \to P$ such that $c \leq \mathrm{id}_P$ and $c^2 = c$. A *closure operator of* P is an order-preserving map $c : P \to P$ such that c is either an ascending or descending closure operator.

Lemma 2.17. *Let c be a closure operator of P. Then the inclusion $c(P) \hookrightarrow P$ is a strong homotopy equivalence.*

Proof. Suppose that c is an ascending closure operator. Let $c' : P \to c(P)$ be the order-preserving map defined by the correspondence $x \mapsto c(x)$. Let $i : c(P) \hookrightarrow P$ be an inclusion. Then $c'i = \mathrm{id}_P$ and $ic' \geq \mathrm{id}_P$. Hence the inclusion $c(P) \hookrightarrow P$ is a strong homotopy equivalence. The case that c is descending is similarly proved. □

Let P be a poset. An element x is a *lower beat point* if $P_{<x}$ has the maximum. An element x is an *upper beat point* if $P_{>x}$ has the minimum. An element x is a *beat point* if x is either a lower beat point or an upper beat point.

Suppose that x_0 is a lower beat point and let y_0 be the maximum of $P_{<x}$. Define the map $c : P \to P$ by

$$c(x) = \begin{cases} x & (x \neq x_0) \\ y_0 & (x = x_0). \end{cases}$$

Then c is order-preserving, $c^2 = c$, and $c \leq \mathrm{id}_P$. Therefore the inclusion $P \setminus x = c(P) \hookrightarrow P$ is a strong homotopy equivalence.

It is clear that if x is a beat point of P, then x is a cone point of $\Delta(P)$.

Lemma 2.18. *Let P and Q be finite posets. Let f and g be order-preserving maps from P to Q. If f and g are strongly homotopic, then $\Delta(f)$ and $\Delta(g)$ are strongly homotopic as simplicial maps from P to Q.*

Proof. We can assume that $f \leq g$. Let $\{x_1, \cdots, x_k\}$ be a linear order of P such that $x_i \leq x_j$ implies $i \leq j$. For $j = 0, 1, \cdots, k$, define the map $\varphi_j : P \to Q$ by

$$\varphi_j(x_i) = \begin{cases} f(x_i) & (i \leq j) \\ g(x_i) & (i > j). \end{cases}$$

Then $\Delta(\varphi_j),\ \Delta(\varphi_{j-1}) : \Delta(P) \to \Delta(Q)$ are contiguous for $j = 1, \cdots, k$. Since $\varphi_0 = g$ and $\varphi_k = f$, we have that f and g are strongly homotopic. □

Let Q be a poset. A subposet P of Q is an *induced subposet* if for every $x, y \in P$, $x \leq y$ in Q implies $x \leq y$ in P. Whenever we regard a subset A of Q as a poset, we consider that A is an induced subposet of Q unless otherwise stated.

Theorem 2.19 (Quillen's theorem A)**.** *Let $f : P \to Q$ be an order-preserving map. Suppose that for every $y \in Q$, the subposet $f^{-1}(Q_{\leq x})$ of P is contractible. Then f is a homotopy equivalence.*

Proof. The following is essentially the same as the proof of Barmak [3]. A little modification allows us to generalize his proof to the case that Q is infinite.

We first consider the case that P is an induced subposet of Q, f is an inclusion, and $Q \setminus P$ is finite. Let $\{x_1, \cdots, x_n\}$ be a linear order on $Q \setminus P$ such that $x_i \leq x_j$ implies $i \leq j$. For $i = 0, 1, \cdots, n$, define P_i ($i = 0, \cdots, n$) to be the induced subposet of Q consisting of the elements of P and $x_{i+1}, \cdots, x_n$. Then we have a sequence

$$P = P_n \subset P_{n-1} \subset \cdots \subset P_0 = Q.$$

Note that $\Delta(P_i) \cap \mathrm{st}_{\Delta(P_{i-1})(x_i)} = \Delta(Q_{>x}) * \Delta(P_{i,<x_i})$. Since $\Delta(P_{i,<x_i}) = \Delta(P \cap Q_{\leq x_i})$ is contractible, we have that $\Delta(P_i) \cap \mathrm{st}_{\Delta(P_{i-1}(x_i))}$ is contractible. Hence $\Delta(P_i)$ is a deformation retract of $\Delta(P_{i-1}) = \Delta(P_i) \cup \mathrm{st}_{\Delta(P_{i-1})}(x_i)$. Thus P is a deformation retract of Q.

Next we consider the case that P is an induced subposet of Q and f is an inclusion. By Whitehead's theorem (Theorem 2.1), it suffices to show that every map $(D^n, S^{n-1}) \to (|Q|, |P|)$ is a homotopic rel S^{n-1} to a map with image contained in $|P|$. Let $\varphi : (D^n, S^{n-1}) \to (|Q|, |P|)$ be a map. Then there is an induced subposet Q' of P such that $\varphi(D^n) \subset |Q'|$ and $Q' \setminus P$ is finite. If follows from the previous paragraph that $|P|$ is a deformation retract of $|Q'|$. Hence φ is homotopic rel S^{n-1} to a map with image contained in $|P|$.

Finally, we consider the general case. Define the mapping space construction B_f as follows. The underlying set of B_f is the disjoint union of P and Q. Moreover, P and Q are induced subposets of B_f. For $x \in P$ and $y \in Q$, define $x \leq y$ in B_f if $f(x) \leq y$. No other ordering is defined.

Define the order-preserving map $c : B_f \to Q$ by

$$c(x) = \begin{cases} f(x) & (x \in P) \\ x & (x \in Q). \end{cases}$$

Let $j : Q \hookrightarrow B_f$ be the inclusion. Then jc is an ascending closure operator and $cj = \mathrm{id}_Q$. Therefore c induces a homotopy equivalence $|B_f| \to |Q|$.

Let $i : P \hookrightarrow B_f$ be the inclusion. It follows from the third paragraph of this proof that i is a homotopy equivalence. Hence $f = ci$ is a homotopy equivalence. This completes the proof. □

2.4. **$\times$-homotopy theory.** In this section we shall review the $\times$-homotopy theory of graphs established by Dochtermann [8]. The definition of the Hom complex is found in Section 1.3.

Two graph homomorphisms $f, g : G \to H$ are *$\times$-homotopic* if they belong to the same connected component of $\mathrm{Hom}(G, H)$, and we write $f \simeq_\times g$. A graph homomorphism $f : G \to H$ is a *$\times$-homotopy equivalence* if there is a graph homomorphism $g : H \to G$ with $g \circ f \simeq_\times \mathrm{id}_G$ and $f \circ g \simeq_\times \mathrm{id}_H$.

Let G_0, G_1, G_2 be graphs and let $\eta \in \mathrm{Hom}(G_0, G_1)$ and $\eta' \in \mathrm{Hom}(G_1, G_2)$. Then the *composition* $\eta' * \eta$ is the multi-homomorphism from G_0 to G_2 defined by the correspondence

$$V(G_0) \to 2^{V(G_2)} \setminus \{\emptyset\},\ v \mapsto \bigcup_{w \in \eta(v)} \eta'(w).$$

The composition gives an order-preserving map

$$\mathrm{Hom}(G_1, G_2) \times \mathrm{Hom}(G_0, G_1) \to \mathrm{Hom}(G_0, G_2),\ (\eta', \eta) \mapsto \eta' * \eta.$$

Lemma 2.20. *Let $f, g : G \to H$ be graph homomorphisms. If $f \simeq_\times g$, then $f_* \simeq_s g_* : \mathrm{Hom}(T, G) \to \mathrm{Hom}(T, H)$ and $f^* \simeq_s g^* : \mathrm{Hom}(H, X) \to \mathrm{Hom}(G, X)$.*

Proof. The following proof is due to the author. We only show $f_* \simeq_s g_*$ since the other is similarly proved. Consider the composition map

$$* : \mathrm{Hom}(G, H) \times \mathrm{Hom}(T, G) \to \mathrm{Hom}(T, H).$$

By Lemma 2.15, we have an order-preserving map $\Phi : \mathrm{Hom}(G, H) \to \mathrm{Poset}(\mathrm{Hom}(T, G), \mathrm{Hom}(T, H))$. Then we have $\Phi(f) = f_*$ and $\Phi(g) = g_*$. Since f and g belong to the same connected component of $\mathrm{Hom}(G, H)$, f_* and g_* are strongly homotopic. □

In particular, if f is a $\times$-homotopy equivalence, then both $f_* : \mathrm{Hom}(T, G) \to \mathrm{Hom}(T, H)$ and $f^* : \mathrm{Hom}(H, X) \to \mathrm{Hom}(G, X)$ are strong homotopy equivalences.

For a non-negative integer n, define I_n to be the graph whose vertex set is $[n] = \{0, 1, \cdots, n\}$ and $x, y \in [n]$ are adjacent in I_n if and only if $|x - y| \leq 1$. Let f and g be graph homomorphisms. A *$\times$-homotopy from f to g* is a graph homomorphism

$$h : G \times I_n \to H$$

for a non-negative integer n such that $h(x,0) = f(x)$ and $h(x,n) = g(x)$ for all $x \in V(G)$. Here the product $G \times I_n$ is the categorical product in $\mathcal{G}$. Namely, for graphs G, H, the product $G \times H$ is defined by

$$V(G \times H) = V(G) \times V(H)$$

and

$$E(G \times H) = \{((x,y),(x',y')) \mid (x,x') \in E(G) \text{ and } (y,y') \in E(H)\}.$$

Lemma 2.21. *Let $f, g : G \to H$ be graph homomorphisms. Then f and g are $\times$-homotopic if and only if there is a $\times$-homotopy from f to g.*

This lemma is deduced from the following lemma.

Lemma 2.22. *Let $f, g : G \to H$ be graph homomorphisms. Then the following assertions are equivalent.*

(1) *There is a multi-homomorphism $\eta \in \mathrm{Hom}(G,H)$ with $f \le \eta$ and $g \le \eta$.*
(2) *The map $V(G) \to 2^{V(H)} \setminus \{\emptyset\}$, $x \mapsto \{f(x), g(x)\}$ is a multi-homomorphism.*
(3) *The map $h : V(G \times I_1) \to V(H)$, $(x,0) \mapsto f(x)$, $(x,1) \mapsto g(x)$ is a graph homomorphism from $G \times I_1$ to H.*

Proof. It is clear that (2) implies (1).

Suppose that (1) holds. We show that the condition (3) holds. Let $\eta \in \mathrm{Hom}(G,H)$ with $f \le \eta$ and $g \le \eta$. Let $((x,i),(y,j)) \in E(G \times I_1)$. Then we have $(x,y) \in E(G)$. Since

$$(h(x,i), h(y,j)) \in \{f(x), g(x)\} \times \{f(y), g(y)\} \subset \eta(x) \times \eta(y) \subset E(H).$$

This implies that h is a graph homomorphism. Thus the condition (3) holds.

Suppose that (3) holds. We show that the condition (2) holds. Let $(x,y) \in E(G)$. We want to show that $\{f(x), g(x)\} \times \{f(y), g(y)\} = \{(f(x), g(x)), (f(x), g(y)), (f(y), g(x)), (f(y), g(y))\} \subset E(H)$. Since f and g are graph homomorphisms, we have $(f(x), f(y)), (g(x), g(y)) \in E(H)$. Note that $f(x) = h(x,0)$, $g(y) = h(y,1)$, and $(x,0)$ and $(y,1)$ are adjacent in $G \times I_1$. Therefore $(f(x), g(y)) \in E(H)$. Similarly, we can show that $(f(y), g(x)) \in E(H)$. Thus the condition (1) holds. □

For a subset S of $V(G)$, we write $G \setminus S$ to indicate the maximal subgraph of G whose vertex set is $V(G) \setminus S$. If $S = \{v\}$, then we write $G \setminus v$ instead of $G \setminus \{v\}$.

Definition 2.23. A vertex v of G is *dismantlable* if there is $w \in V(G)$ such that $v \ne w$ and $N(v) \subset N(w)$.

Lemma 2.24. *Suppose that v is a dismantlable vertex of a graph G. Then the inclusion $i : G \setminus v \hookrightarrow G$ is a $\times$-homotopy equivalence.*

Proof. Since v is dismantlable, there is a vertex $w \in V(G)$ such that $v \ne w$ and $N(v) \subset N(w)$. Define the graph homomorphism $r : G \to G \setminus v$ by

$$r(x) = \begin{cases} x & (x \ne v) \\ w & (x = v). \end{cases}$$

Then the pair of graph homomorphisms id_G and ir satisfies the condition (2) of Lemma 2.10. Since $ri = \mathrm{id}_{G \setminus v}$, we have that the inclusion $i : G \setminus v \hookrightarrow G$ is a $\times$-homotopy equivalence. □

3. Test graphs

The purpose of this section is to prove the following theorems. The definitions of homotopy test graphs and Stiefel-Whitney test graphs are found in Section 1.4.

Theorem 3.1 (M. [26]). *Every graph T with $\chi(T) = 2$ is a homotopy test graph.*

Let X be the graph and let α_0, α_1 be involutions described in Figure 1.1.

Theorem 3.2 (M. [26]). *The flipping $\mathbb{Z}_2$-graph (X, α_0) is a Stiefel-Whitney test graph but (X, α_1) is not.*

As was mentioned in Section 1.3, these theorems give answers to some problems suggested by Kozlov.

We now prove Theorem 3.1. Suppose $\chi(T) = 2$. Then K_2 is a retract of T. Since K_2 is a homotopy test graph (see Section 1.4), it suffices to prove the following lemma.

Lemma 3.3. *Suppose that a graph S is a retract of T. If S is a homotopy test graph, then T is a homotopy test graph.*

Proof. Since there are graph homomorphisms between S and T from each to the other, we have $\chi(T) = \chi(S)$.

Suppose that $\mathrm{Hom}(T, G)$ is n-connected. Since $\mathrm{Hom}(S, G)$ is a retract of $\mathrm{Hom}(T, G)$, we have that $\mathrm{Hom}(S, G)$ is n-connected. Since S is a homotopy test graph, we have the inequality

$$\begin{aligned} \chi(G) &> \mathrm{conn}(\mathrm{Hom}(S, G)) + \chi(S) \\ &\geq n + \chi(T). \end{aligned}$$

It follows that T is a homotopy test graph. $\square$

Next we show Theorem 3.2. To prove that (X, α_0) is a Stiefel-Whitney test graph, we need the following.

Theorem 3.4 (Schultz [34]). *For a positive integer r, the odd cycle C_{2r+1} with reflection is a Stiefel-Whitney test graph.*

Proposition 3.5 (Kozlov, Proposition 6.1.5 of [21]). *Let A, B, and C be flipping $\mathbb{Z}_2$-graphs satisfying the following conditions:*

(a) *A and C are Stiefel-Whitney test graphs.*
(b) $\chi(A) = \chi(C)$
(c) *There are $\mathbb{Z}_2$-equivariant graph homomorphisms $f : A \to B$ and $g : B \to C$.*

Then B is a Stiefel-Whitney test graphs.

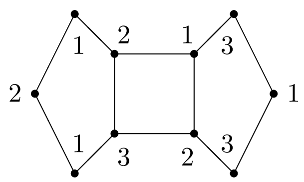
Combining Theorem 3.4 and Proposition 3.5, we have that the $\mathbb{Z}_2$-graph (T, α_0) is a Stiefel-Whitney test graph, since there are $\mathbb{Z}_2$-equivariant graph homomorphisms $C_5 \to X$ and $X \to C_5$.

Next we prove that (X, α_1) is not a Stiefel-Whitney test graph. We first note the following lemma.

Lemma 3.6. *Let φ and ψ be graph homomorphisms from G to H. If there is a non-looped vertex v of G such that $\varphi(x) = \psi(x)$ for $x \neq v$, then φ and ψ belong to the same connected component of* $\mathrm{Hom}(G, H)$.

Proof. This is deduced from Lemma 2.10. $\square$

From now on, we consider $\mathrm{Hom}(X, K_3)$ as a $\mathbb{Z}_2$-poset by the involution induced by α_1. It suffices to show that $w_1(\mathrm{Hom}(X, K_3) \neq 0$. Let f be a graph homomorphism depicted by Figure 3.1. The front and the back of each arrow in Figure 3.2 satisfy the hypothesis of Lemma 3.6. Therefore f and $f \circ \alpha_1$ belong to the same connected component of $\mathrm{Hom}(X, K_3)$. This implies that there is a $\mathbb{Z}_2$-map from S^1 to $\mathrm{Hom}(X, K_3)$, and hence $w_1(\mathrm{Hom}(X, K_3)) \neq 0$. This completes the proof of Theorem 3.2.



The graph homomorphism f

Figure 3.1.

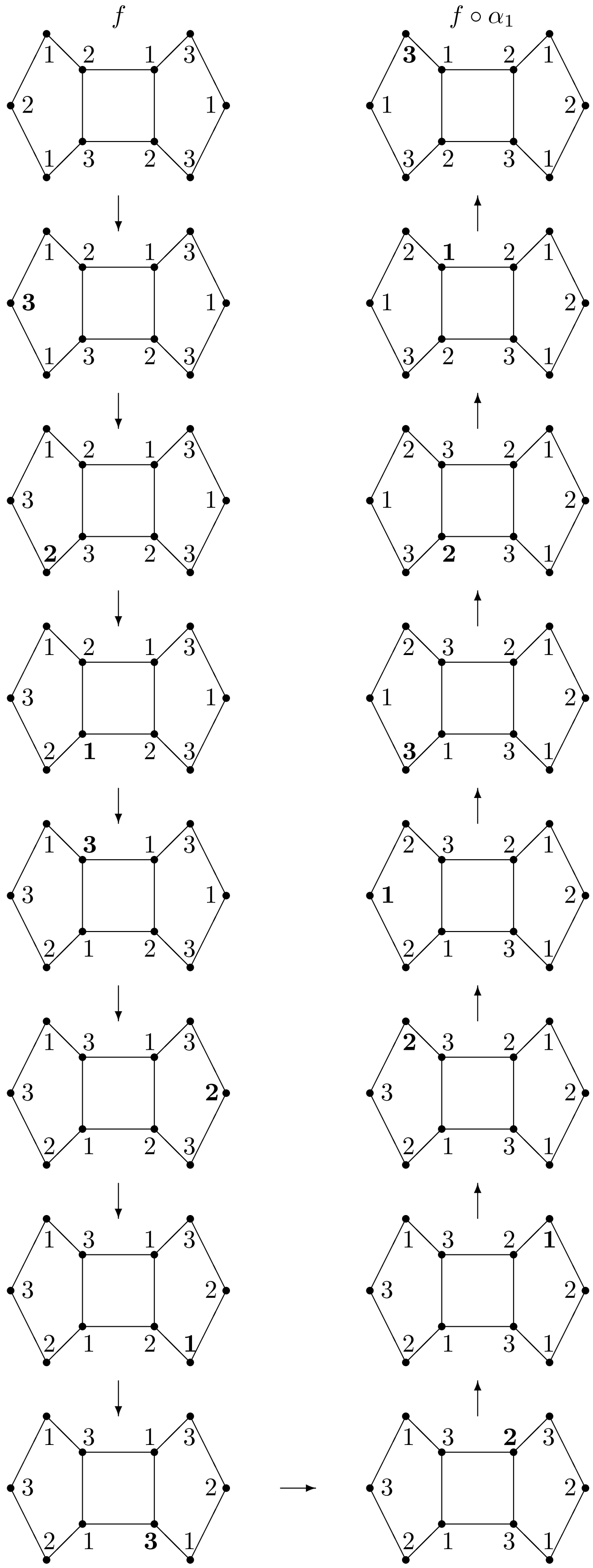


Figure 3.2

4. Chromatic numbers and homotopy types of Hom complexes

The purpose of this section is to prove the following theorem.

Theorem 4.1 (M. [29]). *Let T be a finite graph and let G be a non-bipartite graph. For each integer n, there is a graph H such that* $\mathrm{Hom}(T, G)$ *and* $\mathrm{Hom}(T, H)$ *are homotopy equivalent and $\chi(H) > n$. In case T is a flipping $\mathbb{Z}_2$-graph, we can take H so that* $\mathrm{Hom}(T, G)$ *and* $\mathrm{Hom}(T, H)$ *are $\mathbb{Z}_2$-homotopy equivalent. Moreover, if G is finite or connected, then we can take H to be finite or connected, respectively.*

To explain the outline of the proof, we review Walker's remark (Section 12 of [37]). The girth $g(G)$ of G is the minimal length of cycles embedded into G. Walker showed that if the girth of G is greater than 4, then the box complex of G is $\mathbb{Z}_2$-homotopy equivalent to a 1-dimensional free $\mathbb{Z}_2$-CW-complex. On the other hand, the following famous theorem by Erdős asserts that there is a graph whose girth and chromatic number are both quite large:

Theorem 4.2 (Erdős [12]). *Let n and m be positive integers. There is a finite graph G such that $\chi(G) > n$ and $g(G) > m$.*

Since the $\mathbb{Z}_2$-index of a 1-dimensional free $\mathbb{Z}_2$-CW-complex is smaller than or equal to 1, Walker showed that the difference of the both sides of the inequality

$$\chi(G) \geq \mathrm{ind}(B(G)) + 2$$

can be arbitrarily bad (see Section 1.3).

Now we write the outline of the proof of Theorem 4.1 in the case $T = K_2$. Let n be a positive integer. It follows from Theorem 4.2 that there is a graph X such that $\chi(X) > n$ and $g(X) > 4$. In Section 4.2, we show that there are a graph Y and graph homomorphisms $f : Y \to X$ and $g : Y \to G$ such that f induces a $\mathbb{Z}_2$-homotopy equivalence $\mathrm{Hom}(K_2, Y) \to \mathrm{Hom}(K_2, X)$. Let k be an integer greater than 2. The graph H is constructed by attaching the ends Y of the "cylinder" $Y \times I_k$ to G and X, respectively. Since X is a subgraph of H, we have $\chi(H) \geq \chi(X) > n$.

The reader who is familiar with algebraic topology may notice that this construction is similar to the homotopy pushout of spaces. In fact it turns out that $\mathrm{Hom}(K_2, H)$ is the homotopy pushout of $f_* : \mathrm{Hom}(K_2, Y) \to \mathrm{Hom}(K_2, X)$ and $g_* : \mathrm{Hom}(K_2, Y) \to \mathrm{Hom}(K_2, G)$. Since f_* is a $\mathbb{Z}_2$-homotopy equivalence, we have that $\mathrm{Hom}(K_2, G) \to \mathrm{Hom}(K_2, H)$ is a $\mathbb{Z}_2$-homotopy equivalence. This is the outline of the proof in the case $T = K_2$.

4.1. Deformations of box complexes.

Let G be a finite graph. Recall that the box complex $B(G)$ is isomorphic to the face poset of some regular CW-complexes (see [1]). In fact, let $\mathbb{R}^{V(G)}$ be the free $\mathbb{R}$-module generated by $V(G)$ and let $\Delta^{V(G)}$ be the standard simplex, i.e.,

$$\Delta^{V(G)} = \{ \sum_{v \in V(G)} a_v e_v \mid a_v \geq 0, \sum_{v \in V(G)} a_v = 1 \}.$$

Note that there is a 1-1-correspondence between subsets of $V(G)$ and subsimplices of $\Delta^{V(G)}$. For a subset σ of $V(G)$, we write Δ_σ to indicate the subsimplex associated to σ. Then the box complex $B(G)$ is isomorphic to the face poset of the CW-complex

$$X = \bigcup_{(\sigma,\tau) \in V(G)} \Delta_\sigma \times \Delta_\tau.$$

In this section we show that some deformations of graphs do not change the $\mathbb{Z}_2$-homotopy type of the box complex $B(G)$. For a positive integer k, define the graph L_k by $V(L_k) = \{0, 1, \cdots, k\}$ and $E(L_k) = \{(a, b) \mid |a - b| = 1\}$.

For a pair of vertices x, y of G, the subset $\{(x, y), (y, x)\}$ of $V(G) \times V(G)$ is denoted by $\langle x, y \rangle$.

Lemma 4.3. *Let G be a finite graph and let $e = \langle x, w\rangle$ be an edge of G. Suppose that either x or w is non-looped and there is a unique graph homomorphism f from L_3 to $G \setminus e$ which takes 0 to x and 3 to w, respectively. Then the inclusion $B(G \setminus e) \hookrightarrow B(G)$ is a homotopy equivalence.*

Proof. Since $B(G)$ is a face poset of a certain regular CW-complex, one can apply the discrete Morse Theory (see Section 11 of [22]). Throughout this proof, we identify $B(G)$ with the poset

$$B(G) = \{(\sigma, \tau) \mid \sigma \text{ and } \tau \text{ are non-empty subsets of } V(G) \text{ with } \sigma \times \tau \subset E(G).\}$$

ordered by $(\sigma, \tau) \leq (\sigma', \tau') \Leftrightarrow \sigma \subset \sigma'$ and $\tau \subset \tau'$. We want to construct a partial matching of $B(G)$ whose set of critical points coincides with $B(G \setminus e)$. Here we consider a partial matching as a pair (M, φ) consisting of a subset $M \subset B(G)$ and an injection $\varphi : M \to P \setminus M$ such that $\varphi(x)$ covers x for every $x \in M$.

Let $f : L_3 \to G \setminus e$ be the unique homomorphism joining x to w, and put $y = f(1)$ and $z = f(2)$. We can assume that x is not looped. Set

$$A = \{(\sigma, \tau) \in B(G) \mid x \in \sigma \text{ and } w \in \tau.\}$$

and

$$B = \{(\sigma, \tau) \in B(G) \mid w \in \sigma \text{ and } x \in \tau.\}.$$

Then we have that $B(G) \setminus B(G \setminus e) = A \cup B$, and $A \cap B = \emptyset$ since x is not looped.

Let $(\sigma, \tau) \in A$. We show that either $(\sigma \cup \{z\}, \tau)$ or $(\sigma, \tau \cup \{y\})$ belongs to $B(G)$. Suppose that neither of these belongs to $B(G)$. Then $(\sigma \cup \{z\}, \tau) \notin B(G)$ implies that there is an element $y' \in \tau$ with $(z, y') \notin E(G)$. The condition that y' is not adjacent to z implies $y' \neq y, w$. Since $y' \in \tau$ and $x \in \sigma$, y' is adjacent to x. Similarly, the assumption $(\sigma, \tau \cup \{y\}) \notin B(G)$ implies that there is $z' \in \sigma$ such that $z' \neq x, z$ and $z' \sim w$. Since $y' \in \sigma$ and $z' \in \tau$, we have that y' and z' are adjacent in G. This implies that the map $f' : V(L_3) \to V(G)$ defined by the correspondence

$$f'(0) = x,\ f'(1) = y',\ f'(2) = z',\ f'(3) = w$$

is a homomorphism from L_3 to $G \setminus e$. This contradicts the uniqueness of f.

We now construct a partial matching on A. Set

$$M_1 = \{(\sigma, \tau) \in A \mid z \notin \sigma \text{ and } (\sigma \cup \{z\}, \tau) \in B(G).\},$$

and

$$M_2 = \{(\sigma, \tau) \in A \mid (\sigma \cup \{z\}, \tau) \notin B(G) \text{ and } y \notin \tau.\}.$$

Define the matching $(M_1 \cup M_2, \varphi)$ on A by

$$\varphi(\sigma, \tau) = \begin{cases} (\sigma \cup \{z\}, \tau) & ((\sigma, \tau) \in M_1) \\ (\sigma, \tau \cup \{y\}) & ((\sigma, \tau) \in M_2). \end{cases}$$

It follows from the previous paragraph that this matching has no critical points. Since the verification of the acyclicity is easy, we omit the details.

Let T denote the involution of $B(G)$. Note that $T(A) = B$. Then we have an acyclic partial matching (M, ψ) on $A \cup B$ defined by

$$M = (M_1 \cup M_2) \cup T(M_1 \cup M_2),$$

and

$$\psi(\sigma, \tau) = \begin{cases} \varphi(\sigma, \tau) & ((\sigma, \tau) \in A) \\ T\varphi T(\sigma, \tau) & ((\sigma, \tau) \in B). \end{cases}$$

Applying Theorem 11.13 of [22], we have that the inclusion $B(G \setminus e) \hookrightarrow B(G)$ is a homotopy equivalence. □

Remark 4.4. In fact the above acyclic matching is $\mathbb{Z}_2$-equivariant in the sense of [36]. Therefore we have that the box complex $B(G \setminus e) \hookrightarrow B(G)$ is a $\mathbb{Z}_2$-simple homotopy equivalence (Definition 4.8 and Proposition 4.9 of [36]).

Let G be a graph and let $e = \langle v, w \rangle$ be an edge of G. We define the graph G_e as follows. The vertex set of G_e is $V(G) \coprod \{0,1\}$ and the edge set is defined by

$$E(G_e) = (E(G) \setminus e) \cup \langle 0, 1 \rangle \cup \langle 0, v \rangle \cup \langle 1, w \rangle.$$

Then we have a natural homomorphism $r_e : G_e \to G$ defined by the correspondence $r_e(x) = x$ for $x \in V(G)$, $r_e(0) = w$, and $r_e(1) = v$. In general r_e does not induce a homotopy equivalence between their box complexes. For example, consider C_6 and C_4 ($B(C_6) \simeq S^1 \sqcup S^1$ and $B(C_4) \simeq S^0$).

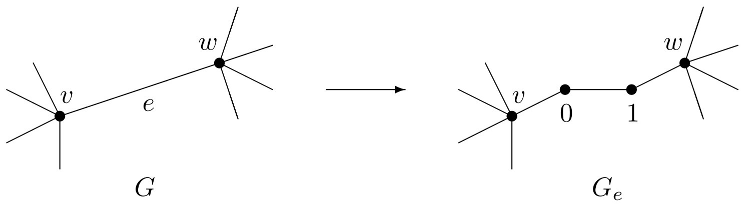


Figure 4.1.

Lemma 4.5. *Let G be a graph and let $e = \langle v, w \rangle$ be an edge of G. If there is no graph homomorphism $L_3 \to G \setminus e$, then the map $r_e : G_e \to G$ induces a homotopy equivalence $B(G_e) \to B(G)$.*

Proof. Let H be the graph defined by $V(H) = V(G_e)$ and $E(H) = E(G_e) \cup e$ (see Figure 4.2). Let $s_e : H \to G$ be the extension of r_e. Since the deletion of a dismantlable vertex does not change the $\mathbb{Z}_2$-homotopy type of the box complex, we have that the inclusion $i : G \to H$ induces a homotopy equivalence $B(G) \to B(H)$. Since $s_e i = \mathrm{id}_G$, s_e induces a homotopy equivalence $B(H) \to B(G)$. It follows from Lemma 4.3 that the inclusion $G_e \hookrightarrow H$ induces a $\mathbb{Z}_2$-homotopy equivalence $B(G_e) \to B(H)$. Hence the composition $B(G_e) \to B(H) \to B(G)$ is a $\mathbb{Z}_2$-homotopy equivalence. $\square$

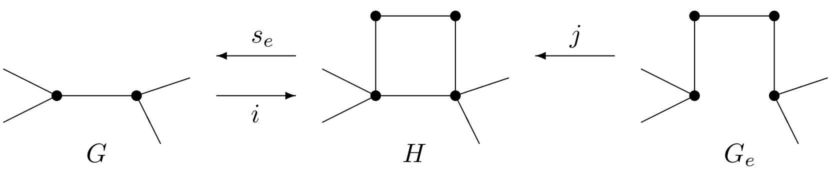


Figure 4.2.

Note that if the girth $g(G)$ of G is greater than 4, then the hypothesis of Lemma 4.5 always holds.

Example 4.6 (Walker [37])**.** Let G_1 and G_2 be graphs illustrated in Figure 4.3. Clearly they have different chromatic numbers. On the other hand, applying Lemma 4.5, we have that $B(G_1)$ and $B(G_2)$ are $\mathbb{Z}_2$-homotopy equivalent.

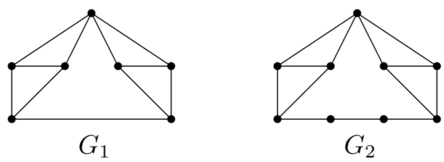


Figure 4.3.

4.2. **Trees.** Let T be a connected bipartite graph having at least one edge, namely, $\chi(T) = 2$. The purpose of this section is to prove Proposition 4.10. This implies that if the girth of G and H are sufficiently large, then a graph homomorphism $f : G \to H$ which induces a homotopy equivalence $f_* : \mathrm{Hom}(K_2, G) \to \mathrm{Hom}(K_2, H)$ also induces a homotopy equivalence $f_* : \mathrm{Hom}(T, G) \to \mathrm{Hom}(T, H)$.

Recall that a tree means a connected graph having no embedded cycles. Throughout this thesis, we assume that a tree has at least one edge.

Let G and H be graphs. A multi-homomorphism $\eta \in \mathrm{Hom}(G, H)$ is *locally finite* if $\eta(v)$ is finite for every $v \in V(G)$. The induced subposet of $\mathrm{Hom}(G, H)$ consisting of locally finite multi-homomorphisms is denoted by $\mathrm{Hom}_f(G, H)$.

Lemma 4.7. *The inclusion* $\mathrm{Hom}_f(G, H) \hookrightarrow \mathrm{Hom}(G, H)$ *is a homotopy equivalence.*

This is deduced from Theorem 2.19 and the following lemma.

Lemma 4.8. *Let P be a poset. If every finite subset of P has an upper bound, then P is contractible.*

Proof. This is deduced from Corollary 2.7. □

Proposition 4.9. *Let X be a tree, let T be a finite connected graph with $\chi(T) = 2$, and let $\iota : K_2 \to T$ be a graph homomorphism. Then* $\iota^* : \mathrm{Hom}(T, X) \to \mathrm{Hom}(K_2, X) \simeq S^0$ *is a homotopy equivalence.*

Proof. We first consider the case that X is finite. Note that a tree X has a dismantlable vertex if X has at least three vertices. Since the deletion of a dismantlable vertex does not change the homotopy type of Hom complexes (see Section 2.4), we have that $\mathrm{Hom}(T, X) \simeq \mathrm{Hom}(T, K_2) = S^0$.

Suppose that X is infinite. Let $u : X \to K_2$ be a graph homomorphism. Let $\mathcal{F}$ be the family of finite subtrees of X containing $u(K_2)$. Then we have

$$\bigcup_{X' \in \mathcal{F}} \mathrm{Hom}_f(K_2, X') = \mathrm{Hom}_f(K_2, X)$$

and

$$\bigcup_{X' \in \mathcal{F}} \mathrm{Hom}_f(T, X') = \mathrm{Hom}_f(T, X).$$

Proposition 2.6 implies that $u_* : \mathrm{Hom}(K_2, X) \to \mathrm{Hom}(K_2, K_2) = S^0$ and $u_* : \mathrm{Hom}(T, X) \to \mathrm{Hom}(T, K_2) = S^0$ are homotopy equivalences.

Then we have the commutative diagram

$$\begin{array}{ccc} \mathrm{Hom}_f(T, X) & \xrightarrow{u_*} & \mathrm{Hom}_f(T, K_2) = S^0 \\ {\scriptstyle \iota^*}\downarrow & & \downarrow{\scriptstyle \iota^*} \\ \mathrm{Hom}_f(K_2, X) & \xrightarrow{u_*} & \mathrm{Hom}_f(K_2, K_2) = S^0. \end{array}$$

By the previous paragraph, the horizontal arrows are homotopy equivalences. It is clear that the right vertical arrow is a homeomorphism. Hence the left vertical arrow is a homotopy equivalence. □

Proposition 4.10. *Let T be a finite connected graph with $\chi(T) = 2$ and let $\iota : K_2 \to T$ be a graph homomorphism. There is a positive integer n such that for every graph X with $g(X) > n$, the map*

$$\iota^* : \mathrm{Hom}(T, X) \to \mathrm{Hom}(K_2, X)$$

is a homotopy equivalence.

Proof. Let $X_1, \cdots, X_k$ be the connected components of T. Then we have

$$\mathrm{Hom}(T, X) \cong \mathrm{Hom}(T, X_1) \sqcup \cdots \sqcup \mathrm{Hom}(T, X_k),$$

and $g(x_i) \geq g(X)$ for $i = 1, \cdots, k$. Hence we can assume that X is connected. Let Δ be the diameter of T. Let n be an integer greater than $2\Delta + 4$. Let X be a connected graph whose girth is greater than

n. Let $p : \tilde{X} \to X$ be the universal covering of X. Since the girth of X is greater than 4, p induces a covering map $\mathrm{Hom}(T, \tilde{X}) \to \mathrm{Hom}(T, X)$. Similarly, $\mathrm{Hom}(K_2, \tilde{X}) \to \mathrm{Hom}(K_2, X)$ is a covering map.

Note that $\mathrm{Hom}(T, \tilde{X}) \simeq S^0$, and the complex $\mathrm{Hom}(T, X)$ has at least two connected components. We claim that $\mathrm{Hom}(T, \tilde{X}) \to \mathrm{Hom}(T, X)$ is surjective and hence $\mathrm{Hom}(T, \tilde{X})/\pi_1(X) \cong \mathrm{Hom}(T, X)$. To see this, it suffices to show that for each graph homomorphism $f : T \to X$, there is a graph homomorphism $\tilde{f} : T \to \tilde{X}$ with $p \circ \tilde{f} = f$. Let $f : T \to X$ be a graph homomorphism. Then the image of f is contained in a certain subgraph Y of X whose diameter is smaller than or equal to Δ.

Suppose that Y is not a tree. Since Y is a subgraph of X, we have that $g(Y) \geq g(X) \geq 2\Delta + 4$. Let $\varphi : C_k \to Y$ be an embedding of a cycle with $k = g(Y)$. The distance between $\varphi(0)$ and $\varphi([k/2])$ in Y is equal to $[k/2]$. Let $\Delta(Y)$ denote the diameter of Y. Then we have the inequality

$$\Delta \geq \Delta(Y) \geq \left[\frac{k}{2}\right] \geq \left[\frac{2\Delta + 4}{2}\right] = \Delta + 2.$$

This is a contradiction.

Therefore Y is a tree and the inclusion $Y \to X$ has a lift $Y \to \tilde{X}$. Hence $f : T \to X$ has a lift. By the same way, we can show that $p_* : \mathrm{Hom}(K_2, \tilde{X}) \to \mathrm{Hom}(K_2, X)$ is a surjective covering map.

Consider the commutative diagram

$$\begin{array}{ccc} \mathrm{Hom}(T, \tilde{X}) & \xrightarrow{p_*} & \mathrm{Hom}(T, X) \\ {\scriptstyle \tilde{\iota}^*}\downarrow & & \downarrow{\scriptstyle \iota^*} \\ \mathrm{Hom}(K_2, \tilde{X}) & \xrightarrow{p_*} & \mathrm{Hom}(K_2, X). \end{array}$$

The left vertical arrow $\tilde{\iota}^*$ of the above diagram is a homotopy equivalence (see Proposition 4.7) and is $\pi_1(X)$-equivariant. Therefore it follows from Proposition 2.9 that $\iota^* : \mathrm{Hom}(T, X) \to \mathrm{Hom}(K_2, X)$ is a homotopy equivalence. □

4.3. **Proof of the main theorem.** Let G be a graph. Recall that the *odd girth of* G to be the number

$$g_o(G) = \inf\{2r + 1 \mid r \text{ is a non-negative integer and there is a graph homomrophism from } C_{2r+1} \text{ to } G.\}.$$

It is easy to see that if $g_o(G) > g_o(H)$, then there is no graph homomorphism from G to H.

Let $\mathcal{F}$ be a (not necessarily small) family of finite connected graphs and suppose that there is a positive integer n which satisfies the following conditions:

- The diameter of a graph belonging to $\mathcal{F}$ is smaller than n.
- If $T \in \mathcal{F}$ is not bipartite, then the odd girth of T is smaller than n.

We call a family $\mathcal{F}$ *uniformly small* if such an integer n exists. Note that if $\mathcal{F}$ is a finite family of finite graphs, then $\mathcal{F}$ is uniformly small. Let T be a finite graph and let $\{T_1, \cdots, T_k\}$ be the set of connected components of T. Then we have

$$\mathrm{Hom}(T, G) \cong \mathrm{Hom}(T_1, G) \times \cdots \times \mathrm{Hom}(T_k, G).$$

Hence the non-equivariant part of Theorem 4.1 is deduced from the following theorem:

Theorem 4.11. *Let $\mathcal{F}$ be a uniformly small family of graphs and let G be a non-bipartite graph. Then for every positive integer m, there is an inclusion $f : G \hookrightarrow H$ such that f induces a homotopy equivalence $\mathrm{Hom}(T, G) \to \mathrm{Hom}(T, H)$ for every $T \in \mathcal{F}$ but $\chi(H) \geq m$. Moreover, if G is finite and connected, then one can take H to be finite and connected.*

Proof. Let n be an integer satisfying the following conditions:

- For a bipartite graph T contained in $\mathcal{F}$, we have $n > 2\Delta(T) + 4$. Here $\Delta(T)$ denotes the diameter of T.
- n is greater than the odd girth of a non-bipartite graph belonging to $\mathcal{F}$.

By Theorem 4.2, there is a finite graph X whose girth is greater than n and $\chi(X) \geq m$. We can assume that X is connected. Let G be a non-bipartite graph, and let k be the odd girth of G. Define the graph Y to be the subdivision of X so that each edge of X is subdivided into L_k (see the beginning of Section 3 for the definition of L_k). It follows from Lemma 4.5 that there is a map $f : Y \to X$ which induces a $\mathbb{Z}_2$-homotopy equivalence between their box complexes. Clearly there is a homomorphism from Y to the k-cycle C_k, and hence there is a homomorphism g from Y to G.

For a pair of integers a, b with $a \leq b$, define the graph $I_{[a,b]}$ by $V(I_{[a,b]}) = \{x \in \mathbb{Z} \mid a \leq x \leq b\}$ and $E(I_{[a,b]}) = \{(x, y) \mid |x - y| \leq 1\}$. Define the graph H as the colimit of the diagram

$$X \xleftarrow{f} Y \xrightarrow{\iota_0} Y \times I_{[0,n+1]} \xleftarrow{\iota_{n+1}} Y \xrightarrow{g} G$$

in the category of graphs. Precisely speaking, the vertex set of H is the quotient of $V(X) \sqcup V(Y \times I_{[0,n+1]}) \sqcup V(G)$ with respect to the equivalence relation $\sim$ generated by the relations $f(y) \sim \iota_0(y)$ and $\iota_{n+1}(y) \sim g(y)$. Let π denote the quotient map $V(X) \sqcup V(Y \times I_{n+1}) \sqcup V(G) \to V(H)$. Then two vertices α and β of H are adjacent if there is $(x, y) \in E(X) \sqcup E(Y \times I_{[0,n+1]}) \sqcup E(G)$ with $\pi(x) = \alpha$ and $\pi(y) = \beta$. Since X is a subgraph of H, we have $\chi(H) \geq \chi(X) \geq m$. We want to show that $\mathrm{Hom}(T, G) \to \mathrm{Hom}(T, H)$ is a homotopy equivalence for every $T \in \mathcal{F}$.

Let H_1 be the colimit (pushout) of the diagram

$$X \xleftarrow{f} Y \xrightarrow{\iota_0} Y \times I_{[0,n]},$$

and let H_2 be the colimit of the diagram

$$Y \times I_{[1,n+1]} \xleftarrow{\iota_{n+1}} Y \xrightarrow{g} G.$$

Here we write $\iota_k : Y \to Y \times I_k$ to indicate the injection $Y \to Y \times I_k$, $x \mapsto (x, k)$. Note that these are subgraphs of H and the inclusions $X \hookrightarrow H_1$ and $G \hookrightarrow H_2$ are $\times$-homotopy equivalences (see Section 2.4).

Let T be a graph contained in $\mathcal{F}$. Since the diameter of T is smaller than n, a multi-homomorphism from T to G factors through either H_1 or H_2. Therefore we have

$$\mathrm{Hom}(T, H_1) \cup \mathrm{Hom}(T, H_2) = \mathrm{Hom}(T, H)$$

and

$$\mathrm{Hom}(T, H_1) \cap \mathrm{Hom}(T, H_2) = \mathrm{Hom}(T, H_1 \cap H_2) = \mathrm{Hom}(T, Y \times I_{[1,n]}).$$

Recall that we want to prove that the composition of the sequence

$$\mathrm{Hom}(T, G) \hookrightarrow \mathrm{Hom}(T, H_2) \hookrightarrow \mathrm{Hom}(T, H) \tag{2}$$

is a homotopy equivalence. To show this, it is enough to show that the inclusion $i : \mathrm{Hom}(T, H_1 \cap H_2) \hookrightarrow \mathrm{Hom}(T, H_1)$ is a homotopy equivalence. In fact if i is a homotopy equivalence, then $\mathrm{Hom}(T, H_1 \cap H_1)$ is a deformation retract of $\mathrm{Hom}(T, H_1)$ (see Theorem 2.2). Since $\mathrm{Hom}(T, H_1) \cap \mathrm{Hom}(T, H_2) = \mathrm{Hom}(T, H_1 \cap H_2)$, we have that $\mathrm{Hom}(T, H_2)$ is a deformation retract of $\mathrm{Hom}(T, H) = \mathrm{Hom}(T, H_1) \cup \mathrm{Hom}(T, H_2)$. Since the inclusion $G \hookrightarrow H_2$ is a $\times$-homotopy equivalence, we have that the composition of the sequence (2) is a homotopy equivalence.

Note that the graph homomorphism $f : Y \to X$ is the composition of the sequence

$$Y \xrightarrow{\iota_n} Y \times I_{[1,n]} = H_1 \cap H_2 \longrightarrow H_1 \xrightarrow{r'} X.$$

Here $r : H_1 \to X$ is the natural retraction of the inclusion $X \hookrightarrow H_1$. Since ι_n and r are $\times$-homotopy equivalences, the inclusion $\mathrm{Hom}(T, H_1 \cap H_2) \to \mathrm{Hom}(T, H_1)$ is a homotopy equivalence if and only if $f_* : \mathrm{Hom}(T, Y) \to \mathrm{Hom}(T, X)$ is a homotopy equivalence. Thus it suffices to prove that $f_* : \mathrm{Hom}(T, Y) \to \mathrm{Hom}(T, X)$ is a homotopy equivalence.

Suppose that T is not bipartite. Then there is no graph homomorphism from T to X since the girth of X is greater than the odd girth of T. Since there is a graph homomorphism r from Y to X, there is no graph homomorphism from T to Y. This implies that $\mathrm{Hom}(T, Y)$ and $\mathrm{Hom}(T, X)$ are empty and hence $f_* : \mathrm{Hom}(T, Y) \to \mathrm{Hom}(T, X)$ is a homotopy equivalence.

Suppose that T is bipartite. If T has no edges, then both of $\mathrm{Hom}(T,Y)$ and $\mathrm{Hom}(T,X)$ are contractible. Hence $f_* : \mathrm{Hom}(T,Y) \to \mathrm{Hom}(T,X)$ is a homotopy equivalence. Thus we assume that T has at least one edge. Let $\iota : K_2 \to T$ be a graph homomorphism. Proposition 4.2 implies that the maps $\iota^* : \mathrm{Hom}(T,X) \to \mathrm{Hom}(K_2,X)$ and $\iota^* : \mathrm{Hom}(T,Y) \to \mathrm{Hom}(K_2,Y)$ induced by ι are homotopy equivalences. It follows from the diagram

$$\begin{array}{ccc} \mathrm{Hom}(T,Y) & \xrightarrow{\simeq} & \mathrm{Hom}(K_2,Y) \\ {\scriptstyle f_*}\downarrow & & \downarrow{\scriptstyle \simeq} \\ \mathrm{Hom}(T,X) & \xrightarrow{\simeq} & \mathrm{Hom}(K_2,X) \end{array}$$

that $f_* : \mathrm{Hom}(T,Y) \to \mathrm{Hom}(T,X)$ is a homotopy equivalence. $\square$

The equivariant part of Theorem 4.1 is deduced from the following.

Corollary 4.12. *Let $\mathcal{F}$ be a uniformly small family of flipping $\mathbb{Z}_2$-graphs. Then for every graph G with $\chi(G) \geq 3$ and for every positive integer m, there is an inclusion $f : G \hookrightarrow H$ such that f induces a $\mathbb{Z}_2$-homotopy equivalence* $\mathrm{Hom}(T,G) \to \mathrm{Hom}(T,H)$ *for every $T \in \mathcal{F}$ but $\chi(H) \geq m$. Moreover, if G is finite and connected), then one can take H to be finite and connected.*

Proof. If G has a looped vertex, then put $H = G$. If G has no looped vertex, then the graph H constructed in the proof of Theorem 4.11 has the desired properties. In fact $\mathrm{Hom}(T,G)$ and $\mathrm{Hom}(T,H)$ are free $\mathbb{Z}_2$-complexes and $i_* : \mathrm{Hom}(T,G) \to \mathrm{Hom}(T,H)$ is a homotopy equivalence. By Proposition 2.9), we have that i_* is a $\mathbb{Z}_2$-homotopy equivalence. $\square$

5. Kronecker double coverings and box complexes of graphs

The purpose of this section is to prove the following theorem.

Theorem 5.1 (M. [29])**.** *Let G and H be graphs without isolated vertices. Then the following hold.*

1. *The Kronecker double coverings $K_2 \times G$ and $K_2 \times H$ are isomorphic if and only if their box complexes $B(G)$ and $B(H)$ are isomorphic as posets.*
2. *The graphs G and H are isomorphic if and only if their box complexes $B(G)$ and $B(H)$ are isomorphic as $\mathbb{Z}_2$-posets.*
3. *If the Kronecker double coverings $K_2 \times G$ and $K_2 \times H$ are isomorphic, then their neighborhood complexes are isomorphic. On the other hand, if G and H are stiff (mentioned below), then the converse holds.*

5.1. **Kronecker double covering.** In this section we shall review the theory of the Kronecker double coverings. Most of the results mentioned here are essentially known (see [19]). We formulate the theory in terms of "2-colored graphs" mentioned below. For the sake of our treatment, we have a simple description provided as Theorem 3.1.

A graph homomorphism $p : X \to G$ is a *covering* if $p|_{N(v)} : N(v) \to N(p(v))$ is bijective for every $v \in V(G)$. The *Kronecker double covering over G* is the second projection $K_2 \times G \to G$. It is clear that the Kronecker double covering is actually a covering.

If a graph G is bipartite, then its Kronecker double covering is the direct sum $G \sqcup G$. If G is not bipartite and is connected, then its Kronecker double covering is connected. These facts are easily proved.

A *2-colored graph* is a pair (X, ε) consisting of a graph X equipped with a graph homomorphism $\varepsilon : X \to K_2$. Let (X, ε_X) and (Y, ε_Y) be 2-colored graphs. A *2-colored homomorphism from (X, ε_X) to (Y, ε_Y)* is a graph homomorphism $f : X \to Y$ which satisfies $\varepsilon_Y \circ f = \varepsilon_X$. We often abbreviate (X, ε) to X.

The category of graphs whose morphisms are graph homomorphisms is denoted by $\mathcal{G}$. If we use the language of category theory [20], a 2-colored graph is a graph over K_2, and a 2-colored graph homomorphism is a morphism between graphs over K_2. So we write $\mathcal{G}_{/K_2}$ to indicate the category of 2-colored graphs.

Let X be a 2-colored graph. An *odd involution τ of X* is a graph homomorphism $\tau : X \to X$ such that $\tau^2 = \mathrm{id}_X$ and $\varepsilon_X \circ \tau(x) \neq \varepsilon_X(x)$ for every $x \in X$. Note that this involution is not a morphism in $\mathcal{G}_{/K_2}$ if X is not empty.

Consider a triple (X, ε, τ) consisting of a 2-colored graph (X, ε) with an odd involution τ of X. A 2-colored homomorphism $f : X \to Y$ between such triples $(X, \varepsilon_X, \tau_X)$ and $(Y, \varepsilon_X, \tau_Y)$ is *equivariant* if $\tau_Y \circ f = f \circ \tau_X$. Let $\mathcal{G}^{odd}_{/K_2}$ denote the category whose objects are 2-colored graphs with odd involutions, and whose morphisms are equivariant 2-colored homomorphisms.

Note that for a graph G, the Kronecker double covering $K_2 \times G$ is naturally 2-colored by the first projection $K_2 \times G \to K_2$and the involution $(1, x) \leftrightarrow (2, x)$ is an odd involution. Therefore the Kronecker double covering gives a functor $K_2 \times (-) : \mathcal{G} \to \mathcal{G}^{odd}_{/K_2}$. On the other hand, the correspondence $(X, \tau) \mapsto X/\tau$ gives a functor from $\mathcal{G}^{odd}_{/K_2} \to \mathcal{G}$. Here X/τ denotes the quotient graph defined by

$$V(X/\tau) = \{\{x, \tau(x)\} \mid x \in V(X)\},$$

$$E(X/\tau) = \{(\alpha, \beta) \mid \alpha, \beta \in V(X/\tau), (\alpha \times \beta) \cap E(X) \neq \emptyset\}.$$

In terms of these notions, we can formulate the following theorem. The proof is straightforword and is omitted.

Theorem 5.2. *The above two functors give categorical equivalences between $\mathcal{G}$ and $\mathcal{G}^{odd}_{/K_2}$.*

In particular, there is a natural isomorphism $X \cong K_2 \times (X/\tau)$ for an object (X, ε, τ) of $\mathcal{G}^{odd}_{/K_2}$.

Next we consider the case of bipartite graphs. Let $\mathcal{G}_{K_2}$ be the full subcategory of $\mathcal{G}$ consisting of bipartite graphs. An involution $\tau : X \to X$ of a bipartite graph X is *odd* if for every $x \in V(X)$, there is no path with even length joining x to $\tau(x)$. If X is 2-colored and τ is an odd involution in the sense of 2-colored graphs, then τ is an odd involution in the sense of bipartite graphs.

Define the category $\mathcal{G}^{odd}_{K_2}$ whose object consists of a pair (X, τ) of a bipartite graph X equipped with an odd involution τ, and whose morphism is an equivariant graph homomorphism. Clearly the Kronecker double covering gives a functor $\mathcal{G} \to \mathcal{G}^{odd}_{K_2}$, and the quotient $(X, \tau) \mapsto X/\tau$ gives a functor $\mathcal{G}^{odd}_{K_2} \to \mathcal{G}$. However, these functors are *not* categorical equivalences. Indeed, for an object (X, τ) of $\mathcal{G}^{odd}_{K_2}$, the involution $\tau : X \to X$ is a morphism of $\mathcal{G}_{K_2}$ which induces the identity $X/\tau \to X/\tau$.

However, one can easily show that if (X, τ) is an object of $\mathcal{G}^{odd}_{K_2}$, then there is a 2-coloring $\varepsilon : X \to K_2$ such that the involution τ is an odd involution of the 2-colored graph (X, ε). In other words the forgetful functor $\mathcal{G}^{odd}_{/K_2} \to \mathcal{G}^{odd}_{K_2}$ is essentially surjective. Hence for an object (X, τ) of $\mathcal{G}^{odd}_{K_2}$, there is an isomorphism $X \cong K_2 \times (X/\tau)$, but this isomorphism is not natural.

Note that $K_2 \times G \cong K_2 \times H$ does not imply $G \cong H$. In fact Imrich and Pisanski [19] show that the Desargues graph is not only the Kronecker double covering over the Peterson graph but is also the Kronecker double covering over the graph which is not isomorphic to the Peterson graph. In Example 4.13 we will construct graphs having different chromatic numbers but having the same Kronecker double coverings.

We conclude this section with the following proposition. This may be classically known to experts, but I could not find it.

Proposition 5.3. *Suppose that G is connected but is not bipartite. Let $p : X \to G$ be a double covering. If X is bipartite, then X is the Kronecker double covering over G.*

Proof. For a vertex x of X, let $\tau(x)$ be a vertex of X with $\tau(x) \neq x$ and $p(\tau(x)) = p(x)$. Then $\tau : V(X) \to V(X)$ is an involution of X and $X/\tau \cong G$.

We now show that X is connected. Since X is bipartite and G is not bipartite, X is not isomorphic to the direct sum $G \sqcup G$ of two copies of G. Hence there is $x \in V(X)$ such that x and $\tau(x)$ belong to the same connected component of X. Let y be a vertex of X. Since G is connected, there is a path φ of G joining $p(y)$ to $p(x)$. Consider the lift ψ of φ whose initial point is y. Then the terminal point of ψ is either x or $\tau(x)$. Since x and $\tau(x)$ belong to the same connected component of X, y and x belong to the same connected component of X. Therefore X is connected.

Suppose that τ is not an odd involution. Since X is connected, for every vertex x of X, there is no path joining x to $\tau(x)$ with odd length. This implies that there is no graph homomorphism from an odd cycle to G. This contradicts the assumption that G is not bipartite.

Hence τ is an odd involution and $X \cong K_2 \times (X/\tau) \cong K_2 \times G$. This completes the proof. □

5.2. **Complexes of bipartite graphs.** The purpose of this section is to prove Theorem 5.1. We also construct graphs G and H such that their Kronecker double coverings are isomorphic but their chromatic numbers are different (Example 5.16).

The category of posets is denoted by $\mathcal{P}$, and the category of $\mathbb{Z}_2$-posets is denoted by $\mathcal{P}^{\mathbb{Z}_2}$. Recall that $\mathcal{G}_{K_2}$ denotes the full subcategory of $\mathcal{G}$ consisting of bipartite graphs. We start with the construction of the functor $B_0 : \mathcal{G}_{K_2} \to \mathcal{P}$ which makes the following diagram commute:

$$\begin{array}{ccc} \mathcal{G} & \xrightarrow{B} & \mathcal{P}^{\mathbb{Z}_2} \\ {\scriptstyle K_2\times(-)}\downarrow & & \downarrow \\ \mathcal{G}_{K_2} & \xrightarrow{B_0} & \mathcal{P} \end{array}$$

Here the right vertical arrow is the forgetful functor.

Definition 5.4. Let X be a bipartite graph. We define the poset $B_0(X)$ of X as follows. The underlying set of $B_0(X)$ consists of an unordered pair $\{\sigma, \tau\}$ of subsets of $V(X)$ such that $\sigma\times\tau \subset E(X)$. For elements α, β of $B_0(X)$, we write $\alpha \leq \beta$ if for each element σ of α, there is an element τ of β which contains α. One can easily show that this relation "$\leq$" is actually an ordering, by noting that bipartite graphs have no looped vertices.

Let (X, ε) be a 2-colored graph. Then the poset $B_0(X)$ is identified with the induced subposet of the box complex $B(X)$ consisting of a pair $(\sigma, \tau) \in B(X)$ with $\sigma \subset \varepsilon^{-1}(1)$ and $\tau \subset \varepsilon^{-1}(2)$.

Proposition 5.5. *For a graph G, there is a natural isomorphism $B(G) \cong B_0(K_2 \times G)$ as $\mathbb{Z}_2$-posets. Here we consider $B_0(K_2 \times G)$ as a $\mathbb{Z}_2$-poset by the involution of $K_2 \times G$.*

Proof. Consider the correspondence

$$\Phi : B(G) \to B_0(K_2 \times G), (\sigma, \tau) \mapsto \{\{1\} \times \sigma, \{2\} \times \tau\}.$$

Clearly Φ is a $\mathbb{Z}_2$-equivariant poset map. So it suffices to prove that Φ is an isomorphism as posets. Now we regard $B_0(K_2 \times G)$ as the induced subposet of the box complex $B(K_2 \times G)$ by the first projection $K_2 \times G \to K_2$ (see the previous paragraph of this proposition). Then Φ is rewritten as the map $(\sigma, \tau) \mapsto (\{1\} \times \sigma, \{2\} \times \tau)$. It is easy to see that the map

$$\Psi : B_0(K_2 \times G) \to B(G), (\sigma, \tau) \mapsto (p_1(\sigma), p_1(\tau))$$

is the inverse of Φ. □

Corollary 5.6. *Let G and H be graphs. If $K_2 \times G \cong K_2 \times H$ as graphs, then $B(G) \cong B(H)$ as posets.*

Next we consider the converse of Corollary 5.6.

Proposition 5.7. *Let X and Y be bipartite graphs without isolated vertices. Then $B_0(X) \cong B_0(Y)$ implies $X \cong Y$.*

Proof. Let $f : B_0(X) \to B_0(Y)$ be an isomorphism of posets. Let $x \in V(X)$. We assert that $f(\{\{x\}, N(x)\})$ is written as $\{\{y\}, N(y)\}$ for some $y \in V(H)$. To prove this we need some preparation.

Let a be an element of a poset P. We say that a has a finite height if there is a non-negative integer n such that there is an integer n such that $P_{\leq a}$ does not have chains with length greater than n. Consider the following condition concerning an element x of a poset P.

($*$) For an element y of $P_{\leq x}$ with finite height, the poset $P_{\leq y}$ is isomorphic to the face poset of a finite dimensional simplex.

Clearly the element $\{\{x\}, N(x)\} \in B_0(X)$ satisfies the condition ($*$). On the other hand, the following two conditions concerning with an element α of $B_0(X)$ are equivalent:

(1) The element α is not minimal. Moreover, α is maximal among the points of $B_0(X)$ satisfying the condition ($*$).
(2) There is a vertex x such that $\alpha = \{\{x\}, N(x)\}$ and the degree of x is greater than 1.

Clearly the vertex x in the condition (2) is unique.

Now we are ready to construct the graph homomorphism $g : X \to Y$. Let $x \in V(X)$. If $\#N(x) > 1$, then we define $g(x)$ by $f(\{\{x\}, N(x)) = \{\{g(x)\}, N(g(x))\}$. Next suppose $N(x) = \{x'\}$. If $\#N(x') > 1$, then define $g(x)$ by $f(\{\{x\}, \{x'\}\}) = \{\{g(x)\}, \{g(x')\}\}$. (Note that we have already defined $g(x')$.) Suppose that $N(x') = \{x\}$, namely, x and x' form a connected component isomorphic to K_2. This is equivalent that $\{\{x\}, \{x'\}\}$ is an isolated point of $B_0(X)$. Hence $f(\{\{x\}, \{x'\}\})$ is also an isolated point of $B_0(Y)$. Define $g(x), g(x')$ simultaneously so that $\{\{g(x)\}, \{g(x')\}\} = f(\{\{x\}, \{x'\}\})$.

Now we show that g is actually a graph homomorphism. Let $(x, x') \in E(X)$. It is clear that $(g(x), g(x')) \in E(Y)$ if $\#N(x) = 1$ or $\#N(x') = 1$. Suppose $\#N(x) > 1$ and $\#N(x') > 1$. Then

$$\{\{x\}, \{x'\}\} \in B_0(X)_{\leq \{\{x\}, N(x)\}} \cap B_0(X)_{\leq \{\{x'\}, N(x')\}} \neq \emptyset,$$

and hence

$$B_0(Y)_{\leq \{\{g(x)\}, N(g(x))\}} \cap B_0(Y)_{\leq \{\{g(x')\}, N(g(x'))\}} \neq \emptyset.$$

This implies that $(g(x), g(x')) \in E(Y)$.

Construct the graph homomorphism $h : Y \to X$ from $f^{-1} : B_0(Y) \to B_0(X)$ in a similar way. Indeed gh and hg may not be the identities. (Recall that the homomorphism g is not uniquely determined on the connected components isomorphic to K_2.) However, they become the identities after flipping some of connected components isomorphic to K_2. Hence g is an isomorphism. □

Combining Proposition 5.5 and Proposition 5.7, we have the following.

Corollary 5.8. *Let G and H be graphs having no isolated vertices. If $B(G) \cong B(H)$ as posets, then $K_2 \times G \cong K_2 \times H$ as graphs.*

Next we show that the $\mathbb{Z}_2$-poset structure of $B(G)$ determines the graph G.

Proposition 5.9. *Let G and H be graphs having no isolated vertices. If $B(G) \cong B(H)$ as $\mathbb{Z}_2$-posets, then $G \cong H$ as graphs.*

Proof. Let X and Y be bipartite graphs without isolated vertices, and let τ_X and τ_Y be odd involutions of X and Y respectively. Suppose that there is a $\mathbb{Z}_2$-poset isomorphism $f : B_0(X) \to B_0(Y)$. From the discussion in Section 3, it is enough to show that there is a $\mathbb{Z}_2$-equivariant isomorphism $X \to Y$.

Let $g : X \to Y$ be the homomorphism constructed in the proof of Proposition 5.7. Let $x \in V(X)$ with $\# N(x) > 1$. Then

$$\begin{aligned} \{\{g(\tau_X x)\}, N(g(\tau_X x))\} &= f(\{\{\tau_X x\}, N(\tau_X x)\}) \\ &= f(\tau_X \{\{x\}, N(x)\}) \\ &= \tau_X f(\{\{x\}, N(x)\}) \\ &= \{\{\tau_X g(x)\}, N(\tau_X g(x))\} \end{aligned}$$

implies $g(\tau_X x) = \tau_X g(x)$. Next suppose $\# N(x) = \{x'\}$ and $\# N(x') > 1$. Then we have

$$\{\{g(\tau_X x)\}, \{g(\tau_X x')\}\} = f(\{\{\tau_X x\}, \{\tau_X x'\}\}) = \tau_X f(\{\{x\}, \{x'\}\}) = \tau_X \{\{g(x)\}, \{g(x')\}\}.$$

Since we have already proved $g(\tau_X x') = \tau_X g(x')$, we have $\tau_X g(x) = g(\tau_X x)$.

Suppose that $N(x) = \{x'\}$ and $N(x') = \{x\}$. If $\{\{x\}, \{x'\}\}$ is a fixed point of $B_0(X)$, then $\tau_X(x) = x'$ and $\tau_X(x') = x$. Since $f(\{\{x\}, \{x'\}\}) = \{\{g(x)\}, \{g(x')\}\}$ is also a fixed point, we have that $\tau_X g(x) = g(x') = g(\tau_X x)$ and $\tau_X g(x') = g(x) = g(\tau_X x')$. Suppose that $\{\{x\}, \{x'\}\}$ is not a fixed point of $B_0(X)$. Set $y = \tau_X(x)$ and $y' = \tau_X(x')$. If $\tau_X g(x) \neq g(\tau_X x)$, then we replace $g \circ \varphi$ to g where $\varphi : X \to X$ is the graph isomorphism exchange x and x' and fixing the other elements. Then we have $g(\tau_X v) = \tau_X g(v)$ for $v = x, x', y, y'$. After these modifications, we have a $\mathbb{Z}_2$-equivariant isomorphism $g : X \to Y$. (If X is infinite one needs transfinite induction.) $\square$

Next we discuss the case of neighborhood complexes. Let X be a 2-colored graph with a 2-coloring $\varepsilon : X \to K_2$. Define $N_i(X)$ $(i = 1, 2)$ to be the induced subcomplex of the neighborhood complex $N(X)$ of X whose vertex set is $\varepsilon^{-1}(i) \cap V(N(X))$. In general, $N_1(X)$ and $N_2(X)$ are not isomorphic, but we will see that $N_1(X)$ and $N_2(X)$ are homotopy equivalent in Lemma 5.10.

Let $B_0^f(X)$ be the induced subposet of $B_0(G)$ consisting of an element $\alpha \in B_0(X)$ such that each element $\sigma \in \alpha$ is a finite set. Then the inclusion $B_0^f(X) \hookrightarrow B(X)$ is a homotopy equivalence. This is shown in the same way as Lemma 4.7, and is the details is omitted.

Lemma 5.10. *Let X be a 2-colored graph. Then $B_0(X)$, $N_1(X)$, and $N_2(X)$ are homotopy equivalent.*

Proof. It suffices to show that $B_0^f(X)$, $N_1(X)$, and $N_2(X)$ are homotopy equivalent. Consider the map

$$p_1 : B_0^f(X) \to FN_1(X),\ (\sigma, \tau) \mapsto \sigma.$$

Let $\sigma_0 \in FN_1(X)$. By Quillen's lemma A (Theorem 2.19), it suffices to show that $p_1^{-1}(FN_1(X))_{\geq \sigma_0}$ is contractible. Define the order-preserving map

$$c : p_1^{-1}(FN_1(X)_{\geq \sigma_0}) \to p_1^{-1}(FN_1(X)_{\geq \sigma_0})$$

by the correspondence $(\sigma, \tau) \mapsto (\sigma_0, \tau)$. Then c is a descending closure operator. Therefore

$$c(FN_1(X)_{\geq \sigma_0}) = \{(\sigma_0, \tau) \mid \sigma_0 \times \tau \subset E(X)\}$$

is a deformation retract of $FN_1(X)_{\geq \sigma_0}$. It follows from Lemma 4.8 that $c(FN_1(X)_{\geq \sigma_0})$ is contractible. Hence we have shown that $B_0^f(X) \simeq N_1(X)$. The proof of $B_0^f(X) \simeq N_2(X)$ is similar. $\square$

On the other hand the following lemma holds. The proof is obvious and is omitted.

Lemma 5.11. *Suppose that the 2-colored graph X admits an odd involution. Then $N_1(X)$ and $N_2(X)$ are isomorphic. Hence for a graph G, we have that $N_1(K_2 \times G) \cong N_2(K_2 \times G) \cong N(G)$.*

Proposition 5.12. *Let G and H be graphs. If $K_2 \times G \cong K_2 \times H$ as graphs, then $N(G) \cong N(H)$.*

Proof. By Lemma 4.8, we have

$$\begin{aligned} N(G) \sqcup N(G) &\cong N_1(K_2 \times G) \sqcup N_2(K_2 \times G) \cong N(K_2 \times G) \\ &\cong N(K_2 \times H) \cong N_1(K_2 \times H) \sqcup N_2(K_2 \times H) \cong N(H) \sqcup N(H). \end{aligned}$$

This implies $N(G) \cong N(H)$. $\square$

On the other hand, the converse does not hold in general. Consider the path P_4 with 4-vertices and the 4-cycle C_4. Their neighborhood complexes are two copies of a closed interval, but clearly $K_2 \times P_4 \cong P_4 \sqcup P_4$ and $K_2 \times C_4 \cong C_4 \sqcup C_4$ are not isomorphic. However, we will show that if $N(G) \cong N(H)$, then their Kronecker double coverings are $\times$-homotopy equivalent in the sense of [8].

Now we review the $\times$-homotopy theory established by Dochtermann, following his paper [8]. Define the graph I_n ($n \geq 0$) by $V(I_n) = \{0, 1, \cdots, n\}$ and $E(I_n) = \{(x,y) \mid |x-y| \leq 1\}$. Two graph homomorphisms $f, g : G \to H$ are *$\times$-homotopic* if there are $n \geq 0$ and a graph homomorphism $K : G \times I_n \to H$ which satisfies $K(x, 0) = f(x)$ and $K(x,n) = g(x)$ for every $x \in V(G)$. In this case we write $f \simeq_\times g$. A graph homomorphism $f : G \to H$ is a *$\times$-homotopy equivalence* if there is a graph homomorphism $h : H \to G$ such that $hf \simeq_\times \mathrm{id}_G$ and $fh \simeq_\times \mathrm{id}_H$.

If $f : G \to H$ is a $\times$-homotopy equivalence, then for each graph T the poset map $\mathrm{Hom}(T, G) \to \mathrm{Hom}(T, H)$ induced by f is a homotopy equivalence. Since our box complex is isomorphic to $\mathrm{Hom}(K_2, G)$, a $\times$-homotopy equivalence gives rise to a homotopy equivalence between the box complexes or the neighborhood complexes. A graph homomorphism $f : G \to H$ between stiff graphs (see Section 2 for the definition) is a $\times$-homotopy equivalence if and only if f is an isomorphism.

We are now ready to formulate the precise statement of the converse of Proposition 5.12.

Proposition 5.13. *Let G and H be non-empty locally finite graphs. If $N(G) \cong N(H)$ as simplicial complexes, then $K_2 \times G \simeq_\times K_2 \times H$.*

Proof. We can assume that G and H have no isolated vertices. Let $f : N(G) \to N(H)$ be an isomorphism. Let $g : V(G) \to V(H)$ and $h : V(H) \to V(G)$ be maps which satisfy

$$f(N(x)) \subset N(g(x)),\ f^{-1}(N(y)) \subset N(h(x))\ (x \in V(G), y \in V(H)).$$

Note that such maps exist since G and H are locally finite. Define the graph homomorphisms $F : K_2 \times G \to K_2 \times H$ and $F' : K_2 \times H \to K_2 \times G$ by

$$F(1, x) = (1, f(x)), F(2, x) = (2, g(x)),$$
$$F'(1, y) = (1, f^{-1}(y)), F'(2, y) = (2, h(y)).$$

Then we have that $F'F(1,x) = (1,x)$ and $F'F(2,x) = (2, hg(x))$. Since $N(x) = f^{-1}f(N(x)) \subset f^{-1}(N(g(x))) \subset N(hg(x))$, we have that $N(v) \subset N(F'F(v))$ for every $v \in V(K_2 \times G)$. Applying Lemma 5.14 mentioned below, we have that $F'F \simeq_\times \mathrm{id}$. Similarly we can prove that $FF' \simeq_\times \mathrm{id}$. $\square$

Lemma 5.14. *Let $f, g : G \to H$ be graph homomorphisms which satisfy $N(f(x)) \subset N(g(x))$ for all $x \in V(G)$. Then $f \simeq_\times g$.*

Proof. One can show that the map $H : V(G \times I_1) \to V(H \times I_1)$, $(x, 0) \mapsto f(x)$, $(x, 1) \mapsto g(x)$ is a graph homomorphism. $\square$

Note that a graph G is stiff if and only if $K_2 \times G$ is stiff. Since the maps F and F' constructed in the proof of Proposition 5.13 preserve the 2-colorings of their Kronecker double coverings, we have the following.

Corollary 5.15. *Let G and H be locally finite stiff graphs having no isolated vertices. If $N(G) \cong N(H)$ as simplicial complexes, then $K_2 \times G \cong K_2 \times H$ as 2-colored graphs.*

Proof of Theorem 5.1. The assertion (1) is deduced from Corollary 5.6 and Corollary 5.8. The assertion (2) is deduced from Proposition 5.9. The assertion (3) is deduced from Proposition 5.12 and Corollary 5.15. $\square$

Let m, n be integers greater than 3. We conclude this section with the construction of connected graphs G and H such that $K_2 \times G \cong K_2 \times H$, $\chi(G) = m$, and $\chi(H)$. It follows from (1) of Theorem 5.1

that their box complexes are isomorphic as posets, their neighborhood complexes are isomorphic, but they have different chromatic numbers.

Example 5.16. Set $X_1 = X_2 = K_2 \times K_n$ and $Y_1 = Y_2 = K_2 \times K_m$. Define the bipartite graph Z by identifying the following vertices of $X_1 \sqcup X_2 \sqcup Y_1 \sqcup Y_2$.

- The vertex $(1,1)$ of X_1 and the vertex $(1,1)$ of Y_1.
- The vertex $(2,1)$ of X_1 and the vertex $(1,1)$ of Y_2.
- The vertex $(1,1)$ of X_2 and the vertex $(2,1)$ of Y_1.
- The vertex $(2,1)$ of X_1 and the vertex $(2,1)$ of Y_2.

The graphs X_1, X_2, Y_1, and Y_2 are considered as subgraphs of Z.

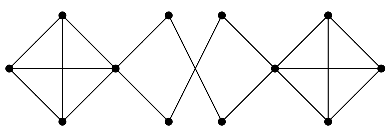
Next we define the odd involutions τ_1, τ_2 of Z as follows:

- τ_1 maps X_i to X_i for each $i = 1, 2$, and $\tau_1|_{X_i} : X_i \to X_i$ is the natural odd involution of $X_i = K_2 \times K_n$. On the other hand, τ_1 exchanges Y_1 and Y_2. The maps $\tau_1|_{Y_1} : Y_1 \to Y_2$ and $\tau_2|_{Y_2} : Y_2 \to Y_1$ are the identity of $K_2 \times K_m$.
- τ_2 maps Y_i to Y_i for each $i = 1, 2$, and $\tau_2|_{Y_i} : Y_i \to Y_i$ is the natural odd involution of $Y_i = K_2 \times K_m$. On the other hand, τ_2 exchanges X_1 and X_2. The maps $\tau_2|_{X_1} : X_1 \to X_2$ and $\tau_2|_{X_2} : X_2 \to X_1$ are the identity of $K_2 \times K_n$.

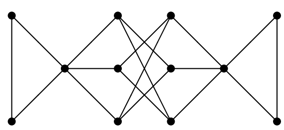
Set $G = Z/\tau_1$ and $H = Z/\tau_2$. Since τ_1 and τ_2 are odd involutions, we have that $K_2 \times G \cong Z$ and $K_2 \times H \cong Z$ (see the sentence after Theorem 3.1). To complete the proof, we need to show $\chi(G) = n$ and $\chi(H) = m$. We only prove $\chi(G) = n$ since the other is similarly proved. However, it is enough to note that G is obtained by identifying the following vertices of $X_1' \sqcup X_2' \sqcup (K_2 \times K_m)$. Here X_1' and X_2' are the copies K_n.

- The vertex of $1 \in X_1'$ and the vertex of $(1,1)$ of $K_2 \times K_m$.
- The vertex of $1 \in X_2'$ and the vertex of $(2,1)$ of $K_2 \times K_m$.

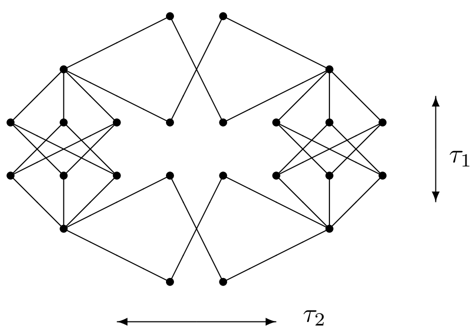
Figure 5.1 describes the graphs G, H, Z in the case $n = 4$ and $m = 3$. In this figure the involution τ_1 is the reflection in the horizontal line, and the involution τ_2 is the reflection in the vertical line.



The graph G.



The graph H.



The graph Z.

Figure 5.1.

5.3. **Comparison with another box complex.** In this short section we shall discuss the case of another box complex $B'(G)$ discussed in Matoušek and Ziegler [31]. Namely, we consider that theorems similar to Theorem 5.1 holds for $B'(G)$.

We recall the definition of $B'(G)$. The $\mathbb{Z}_2$-subcomplex $B'(G)$ of $N(G) * N(G)$ by

$$B'(G) = \{\sigma \uplus \tau \mid \sigma, \tau \in N(G), \sigma \cap \tau = \emptyset\}.$$

Here we write $\sigma \uplus \tau$ to indicate the subset $\sigma \times \{0\} \sqcup \tau \times \{1\}$ of $V(N(G) * N(G)) = V(N(G)) \times \{0, 1\}$. The involution is given by $(x, 0) \leftrightarrow (x, 1)$. There is a $\mathbb{Z}_2$-equivariant inclusion $B(G) \hookrightarrow FB'(G)$, and Živaljević showed that this is a $\mathbb{Z}_2$-homotopy equivalence [38].

As we constructed $B_0(X)$ in the case of $B(X)$, we can define the complex $B'_0(X)$ for a bipartite graph X as follows. Let X be a bipartite graph. Fix a 2-coloring $\varepsilon : X \to K_2$. We define $B'_0(X)$ to be the subcomplex

$$B'_0(X) = \{\sigma \uplus \tau \mid \sigma \in N_1(X), \tau \in N_2(X), \sigma \times \tau \subset E(G)\}$$

of $N_1(X) * N_2(X)$. Clearly this definition does not depend on the choice of 2-colorings. Moreover, it is easy to see that the following holds.

Theorem 5.17. *There is a $\mathbb{Z}_2$-poset isomorphism $B'_0(K_2 \times G) = B'(G)$ for a graph G.*

However, Proposition 5.7 for B'_0 fails. Let X and Y be bipartite graphs described in Figure 5.2. Then we have that $B'_0(X)$ and $B'_0(Y)$ are the 4-dimensional simplex. I do not know that whether the assertion similar to Theorem 5.1 for $B'(X)$ holds or not.

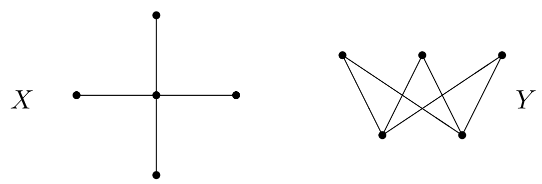


Figure 5.2.

6. Simplicial sets and model categories

In this section, we review the definitions and facts related to simplicial sets and model categories. For a concrete introduction to the subjects, we refer to [13], [16], and [18].

6.1. **Simplicial sets.** For a non-negative integer n, let $[n]$ be the totally ordered set $\{0, 1, \cdots, n\}$. Let Δ be the small category described as follows: the objects are the ordered sets $[n]$ for $n \geq 0$, and the morphisms are order-preserving maps. Let **Set** denote the category of small sets. A *simplicial set* is a functor from Δ^{op} to **Set**. A *simplicial map* is a natural transformation between two simplicial sets. The category of simplicial sets is denoted by **SSet**.

For a simplicial set K, we write K_n instead of $K[n]$. An *n-simplex of K* is an element of K.

Let $\mathcal{C}$ be a category. A *cosimplicial object of $\mathcal{C}$* is a functor from Δ to $\mathcal{C}$, and a *simplicial object of $\mathcal{C}$* is a functor from Δ^{op} to $\mathcal{C}$. Suppose that $\mathcal{C}$ admits all small colimits, and let $A^\bullet : \Delta \to \mathcal{C}$ be a cosimplicial object of $\mathcal{C}$. Then we have the functor $\mathcal{C}(A^\bullet, -) : \mathcal{C} \to \mathbf{SSet}$ defined by

$$\mathcal{C}(A^\bullet, X)_n = \mathcal{C}(A^n, X).$$

Proposition 6.1. *If the category $\mathcal{C}$ admits all small colimits, then the functor $\mathcal{C}(A^\bullet, -)$ is a right adjoint functor.*

In fact the left Kan extension of $A_\bullet : \Delta \to \mathcal{C}$ along the Yoneda functor $\Delta \to \mathbf{SSet}$ is the left adjoint of $\mathcal{C}(A^\bullet, -)$.

The above proposition is easily generalized to the following form. Let $\mathcal{J}$ be a small category and let $\mathcal{C}$ be a category which admits all small colimits. We write $\mathcal{C}_{\mathcal{J}}$ to indicate the functor category from $\mathcal{J}^{\mathrm{op}}$ to $\mathcal{C}$. Let $A^\bullet : \mathcal{J} \to \mathcal{C}$ be a functor. Define the functor $\mathcal{C}(A^\bullet, -) : \mathcal{C} \to \mathbf{Set}_{\mathcal{J}}$ in the same way. Then this functor is a right adjoint functor, and its left adjoint is the left Kan extension of $A^\bullet : \Delta \to \mathcal{C}$ along the Yoneda functor $\Delta \to \mathbf{Set}_{\mathcal{J}}$. This generalization will be used in the proof of Lemma 7.9.

Example 6.2. We write **Top** to indicate the category of k-spaces. For a non-negative integer n, we let

$$\Delta^n = \{(t_0, \cdots, t_n) \in \mathbb{R}^{n+1} \mid t_i \geq 0, \ \sum_{i=0}^{n} t_i = 1\}.$$

Then $\Delta^\bullet : \Delta \to \mathbf{Top}$ is a cosimplicial object of **Top**. The functor

$$\mathrm{Sing} := \mathbf{Top}(\Delta^\bullet, -) : \mathbf{Top} \to \mathbf{SSet}$$

is called the *singular functor*. The *geometric realization functor*

$$| - | : \mathbf{SSet} \to \mathbf{Top}, \ K \mapsto |K|$$

is the left adjoint of the singular functor.

Let K be a simplicial set. We describe the precise construction of the geometric realization $|K|$ of K. Assign the standard n-simplex $\Delta_\sigma = \Delta^n$ to each n-simplex σ of K. Consider the following equivalence relation $\sim$ on the disjoint union of all Δ_σ: Let $f : [m] \to [n]$ be an order-preserving map. Then f induces a continuous map f_* from $\Delta_{f^*\sigma} = \Delta^m$ to $\Delta_\sigma = \Delta^n$. Then we set $x \sim f_*(x)$ for all $x \in \Delta_{f^*\sigma}$. Then $\sim$ is the equivalence relation generated by these identification. Then $|K|$ is the quotient space

$$|K| = \coprod_{n \geq 0, \sigma \in K_n} \Delta_\sigma / \sim .$$

It is known that the geometric realization functor preserves equalizers and finite products (see Section 3 of [18]).

6.2. **Model categories.** Let $\mathcal{C}$ be a category and let

$$\begin{array}{ccc} A & \xrightarrow{f} & X \\ {\scriptstyle i}\downarrow & & \downarrow{\scriptstyle p} \\ B & \xrightarrow{g} & Y \end{array}$$

be a commutative square in $\mathcal{C}$. A *lift of the square* is a morphism r from B to X which commutes the diagram, i.e. $pr = g$ and $ri = f$.

Let i, p be morphisms in the category $\mathcal{C}$. The map i *has the left lifting property with respect to* p, or p *has the right lifting property with respect to* i if and only if every commutative square such as

$$\begin{array}{ccc} A & \longrightarrow & X \\ {\scriptstyle i}\downarrow & & \downarrow{\scriptstyle p} \\ B & \longrightarrow & Y \end{array}$$

has a lift. In other words, the natural map

$$(i^*, p_*) : \mathcal{C}(B, X) \to \mathcal{C}(A, X) \times_{\mathcal{C}(A,Y)} \mathcal{C}(B, Y)$$

is surjective.

Recall that an *adjoint pair from* $\mathcal{C}$ *to* $\mathcal{D}$ is a pair of functors $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{D} \to \mathcal{C}$ such that there is a natural isomorphism

$$\mathcal{D}(FA, X) \cong \mathcal{C}(A, UX)$$

as functors from $\mathcal{C}^{\mathrm{op}} \times \mathcal{D}$ to **Set**. A natural isomorphism φ between the above functors $\mathcal{C}^{\mathrm{op}} \times \mathcal{D} \to \mathbf{Set}$ is called an *adjunction*.

Lemma 6.3. *Let $(F, U, \varphi) : \mathcal{C} \to \mathcal{D}$ be an adjunction. Consider a diagram*

$$\begin{array}{ccc} FA & \xrightarrow{f} & X \\ {\scriptstyle Fi}\big\downarrow & & \big\downarrow{\scriptstyle p} \\ FB & \xrightarrow{g} & Y \end{array}$$

in $\mathcal{D}$. This diagram is commutative if and only if the diagram

$$\begin{array}{ccc} A & \xrightarrow{\varphi(f)} & UX \\ {\scriptstyle f}\big\downarrow & & \big\downarrow{\scriptstyle Up} \\ B & \xrightarrow{\varphi(g)} & UY \end{array}$$

is commutative in $\mathcal{C}$.

Proof. Consider the commutative diagram

$$\begin{array}{ccccc} \mathcal{D}(FA, X) & \xrightarrow{p_*} & \mathcal{D}(FA, Y) & \xleftarrow{Fi^*} & \mathcal{D}(FB, Y) \\ {\scriptstyle \varphi}\big\downarrow & & {\scriptstyle \varphi}\big\downarrow & & \big\downarrow{\scriptstyle \varphi} \\ \mathcal{C}(A, UX) & \xrightarrow{Up_*} & \mathcal{C}(A, UY) & \xleftarrow{i^*} & \mathcal{C}(B, UY). \end{array}$$

The upper square is commutative, namely, $g \circ Fi = p \circ f$ if and only if $f \in \mathcal{D}(FA, X)$ and $g \in \mathcal{D}(FB, Y)$ are taken to the same element of $\mathcal{D}(FA, Y)$. Similarly, the lower square is commutative, namely, $\varphi(g) \circ i = Up \circ \varphi(f)$ if and only if $\varphi(f) \in \mathcal{C}(A, UX)$ and $\varphi(g) \in \mathcal{C}(B, UY)$ are taken to the same element of $\mathcal{C}(A, UY)$. □

Lemma 6.4. *Let $F : \mathcal{C} \to \mathcal{D} : U$ be an adjoint pair. Let i be a morphism in $\mathcal{C}$ and let p be a morphism in $\mathcal{D}$. Then Fi has the left lifting property with respect to p if and only if i has the left lifting property with respect to Up.*

Proof. Consider the following commutative diagram

$$\begin{array}{ccc} \mathcal{D}(FB, X) & \xrightarrow{(Fi^*, p_*)} & \mathcal{D}(FA, X) \times_{\mathcal{D}(FA,Y)} \mathcal{D}(FB, Y) \\ {\scriptstyle \cong}\big\downarrow & & \big\downarrow{\scriptstyle \cong} \\ \mathcal{C}(B, GX) & \xrightarrow{(i^*, Gp_*)} & \mathcal{C}(A, UX) \times_{\mathcal{C}(A,UY)} \mathcal{C}(B, UY). \end{array}$$

The upper horizontal arrow is surjective if and only if the lower horizontal arrow is surjective. □

Let S be a (not necessarily small) family of morphisms in $\mathcal{C}$. A map f in $\mathcal{C}$ is *S-injective* if f has the right lifting property with respect to every map belonging to S. The family of S-injective maps is denoted by *S-inj*. A morphism f in $\mathcal{C}$ is *S-projective* if f has the right lifting property with respect to every map belonging to S. The family of S-projective maps is denoted by *S-proj*. We write *S-cof* instead of (S-*inj*)-*proj*. A map belonging to *S-cof* is called an *S-cofibration.*

It is easy to show that both *S-inj* and *S-proj* are closed under compositions, and contain all isomorphisms. *S-proj* is closed under pushouts. Dually, *S-inj* is closed under pullbacks.

Lemma 6.5. *S-inj = (S-cof)-inj*

Proof. Since $S \subset$ *S-cof*, we have *S-inj* $\supset$ (*S-cof*)-*inj*. On the other hand, we have *S-inj* $\subset$ ((*S-inj*)-*proj*)-*inj* = (*S-cof*)-*inj*. □

Let P be a poset. We regard the poset P as a small category in the usual way: The object set is the underlying set of P, and there is a unique morphism from x to y if and only if $x \leq y$.

Let $\mathcal{C}$ be a category and let $\mathcal{J}$ be a small category. The functor category from $\mathcal{J}$ to $\mathcal{C}$ is denoted by $\mathcal{C}^{\mathcal{J}}$. A functor from $[n]$ to a category $\mathcal{C}$ is identified with a composable sequence $(f_n, \cdots, f_1)$ of morphisms in $\mathcal{C}$, i.e. the domain of f_i coincide with the codomain of f_{i-1} for $i = 2, \cdots, n$. In particular, a functor from $[1]$ to $\mathcal{C}$ is identified with a morphism in $\mathcal{C}$, and a morphism between objects in $\mathcal{C}^{[1]}$ is identified with a commutative square. For a pair of morphisms f, g in $\mathcal{C}$, we call f a *retract of* g if f is a retract of g in $\mathcal{C}^{[1]}$. A family S of morphisms in $\mathcal{C}$ is *closed under retracts* if a map f in $\mathcal{C}$ which is a retract of a map g belonging to S belongs to S. For example, S-*inj* and S-*proj* are closed under retracts.

Define $\partial^i : [n] \to [n+1]$ $(i = 0, 1, \cdots, n)$ by the correspondence

$$\partial^i(j) = \begin{cases} j & (j < i) \\ j+1 & (j \geq i). \end{cases}$$

Example 6.6. Recall that an object of $\mathcal{C}^{[2]}$ is identified with a pair (g, f) of morphisms in $\mathcal{C}$ such that the composition $g \circ f$ is defined. Let $d_i : \mathcal{C}^{[2]} \to \mathcal{C}^{[1]}$ denote the dual of $\partial^i : [1] \to [2]$. Then

$$d_0 : \mathcal{C}^{[2]} \to \mathcal{C}^{[1]}, (g, f) \mapsto g,$$
$$d_2 : \mathcal{C}^{[2]} \to \mathcal{C}^{[1]}, (g, f) \mapsto f,$$

and

$$d_1 : \mathcal{C}^{[1]} \to \mathcal{C}^{[1]}, (g, f) \mapsto g \circ f.$$

Definition 6.7. A *functorial factorization of* $\mathcal{C}$ is a functor $\Phi : \mathcal{C}^{[1]} \to \mathcal{C}^{[2]}$ such that $d_1 \circ \Phi$ is the identity functor of $\mathcal{C}^{[1]}$.

Set $\beta = d_0 \circ \Phi$ and $\alpha = d_2 \circ \Phi$. Then the functorial factorization Φ is determined by the pair of functors $\beta, \alpha : \mathcal{C}^{[1]} \to \mathcal{C}^{[1]}$. So we write (β, α) to indicate the functorial factorization Φ.

We are now ready to define the model category, following [16].

Definition 6.8. Let $\mathcal{M}$ be a category which admits all small colimits and limits. A *model structure on the category* $\mathcal{M}$ is a triple $(\mathcal{W}, \mathcal{C}, \mathcal{F})$ of (not necessarily) families of morphisms in $\mathcal{M}$ which satisfies the following conditions:

(1) The families $\mathcal{W}$, $\mathcal{C}$, and $\mathcal{F}$ are closed under retracts.
(2) Let g and f be morphisms in $\mathcal{M}$ such that $g \circ f$ is defined. If two of f, g, and $g \circ f$ belong to $\mathcal{W}$, then so does the third. This axiom is called the two out of three axiom.
(3) A map belonging to $\mathcal{C}$ has the left lifting property with respect to every map belonging to $\mathcal{W} \cap \mathcal{F}$. A map belonging to $\mathcal{C} \cap \mathcal{W}$ has the left lifting property with respect to every map belonging to $\mathcal{F}$.
(4) There are functorial factorizations (β, α) and (δ, γ) which satisfies the following. For every morphism f in $\mathcal{M}$, $\beta(f)$ belongs to $\mathcal{W} \cap \mathcal{F}$, $\alpha(f)$ belongs to $\mathcal{C}$, $\delta(f)$ belongs to $\mathcal{F}$, and $\gamma(f)$ belongs to $\mathcal{C} \cap \mathcal{W}$.

A *model category* is a category equipped with a model structure on it. A map belonging to $\mathcal{W}$ ($\mathcal{C}$, $\mathcal{F}$, $\mathcal{W} \cap \mathcal{C}$, or $\mathcal{W} \cap \mathcal{F}$) is called a *weak equivalence* (*cofibration*, *fibration*, *trivial cofibration*, or *trivial fibration*, respectively).

Lemma 6.9 (Proposition 7.2.3 of [16]). *Let* $\mathcal{M}$ *be a model category and let* $(\mathcal{W}, \mathcal{C}, \mathcal{F})$ *be the model structure. Then we have* $\mathcal{C} = (\mathcal{W} \cap \mathcal{F})$*-proj,* $\mathcal{W} \cap \mathcal{C} = \mathcal{F}$*-proj,* $\mathcal{F} = (\mathcal{W} \cap \mathcal{C})$*-inj, and* $\mathcal{W} \cap \mathcal{F} = \mathcal{C}$*-inj.*

In particular, every isomorphism in a model category $\mathcal{M}$ is a weak equivalence, a cofibration, and a fibration. Moreover, a composition of cofibrations (or fibrations) is again a cofibration (or fibration, respectively).

An initial object is denoted by $\emptyset$ and a terminal object is denoted by $*$. An object A of $\mathcal{M}$ is *cofibrant* if $\emptyset \to A$ is a cofibration, and A is *fibrant* if $A \to *$ is a fibration. Clearly this definition does not depend on the choices of an initial object and a terminal object.

Let $\mathcal{M}$ and $\mathcal{N}$ be model categories. A *left Quillen functor from $\mathcal{M}$ to $\mathcal{N}$* is a left adjoint functor $F : \mathcal{M} \to \mathcal{N}$ which preserves cofibrations and trivial cofibrations. A *right Quillen functor from $\mathcal{N}$ to $\mathcal{M}$* is a right adjoint functor $U : \mathcal{N} \to \mathcal{M}$ which preserves fibrations and trivial fibrations. Let $(F, U, \varphi) : \mathcal{M} \to \mathcal{N}$ be an adjunction. Then F is a left Quillen functor if and only if U is a right Quillen functor (Proposition 8.5.7 of [16]).

Since a left adjoint functor preserves initial objects, a left Quillen functor preserves cofibrant objects. A Quillen functor preserves weak equivalences between cofibrant objects (Proposition 8.5.7 of [16]). Dually, a right Quillen functor preserves fibrant objects, and weak equivalences between fibrant objects.

A Quillen adjoint pair $F : \mathcal{M} \to \mathcal{N} : U$ is a *Quillen equivalence* if for every cofibrant object A of $\mathcal{M}$ and every fibrant object X of $\mathcal{N}$, a morphism $f : FA \to X$ is a weak equivalence in $\mathcal{N}$ if and only if the adjoint $A \to UX$ of f is a weak equivalence in $\mathcal{M}$.

Next we recall the transfinite composition. From now on, we assume that the category $\mathcal{C}$ admits all small colimits. A totally ordered set λ is an ordinal if every non-empty subset of λ has a minimal element. We write 0 to indicate the minimal element of an ordinal. Let λ be an ordinal. A *λ-sequence in $\mathcal{C}$* is a colimit preserving functor from λ to $\mathcal{C}$. The *composition of the λ-sequence* $X_\bullet : \lambda \to \mathcal{C}$ is the natural map

$$X_0 \to \mathrm{colim}\,(X_\bullet)$$

For a set X, recall that the cardinality $|X|$ of X is the minimal ordinal κ such that there is a bijvection from κ to X. A *cardinal* is an ordinal κ with $|\kappa| \cong \kappa$. Let κ be a cardinal. An ordinal λ is *κ-filtered* if every cofinal subset A of λ has the cardinality greater than or equal to κ. In other words, a subset A of λ has an upper bound if $|A| < \kappa$.

Let $\mathcal{C}$ be a category and let $\mathcal{D}$ be a subcategory. Let κ be a small cardinal. An object A of $\mathcal{C}$ is *κ-small relative to $\mathcal{D}$* if and only if for every κ-filtered ordinal λ and every λ-sequence $X_\bullet : \lambda \to \mathcal{C}$ such that the map

$$X_\alpha \to X_{\alpha+1}$$

belongs to $\mathcal{D}$ for every $\alpha < \lambda$, the natural map

$$\mathrm{colim}_{\alpha<\lambda}\mathcal{C}(A, X_\alpha) \to \mathcal{C}(A, \mathrm{colim}(X_\bullet))$$

is bijective. An object A is *small* if it is κ-small relative to $\mathcal{D}$ for some cardinal κ. If $\mathcal{C} = \mathcal{D}$, a small object relative to $\mathcal{D}$ is simply called a small object of $\mathcal{C}$.

Example 6.10. A graph G is small if $V(G)$ is a small set. This implies that $E(G)$ is also small. Let $\mathcal{G}$ be the category of (small) graphs. Then every graph is a small object in $\mathcal{G}$.

Proof. Let G be a small graph. Let κ be a small infinite cardinal greater than $\max\{|V(G)|, |E(G)|\}$. Let λ be a κ-filtered ordinal and let $X_\bullet : \lambda \to \mathcal{G}$ be a λ-sequence. We shall write X_λ to indicate the colimit of $X_\bullet$. For $\alpha, \beta < \lambda$ with $\alpha \le \beta$, the map $X_\alpha \to X_\beta$ is denoted by $\iota_{\beta\alpha}$, and the natural map $X_\alpha \to X_\lambda$ is denoted by ι_α. We want to show that the natural map

$$\Phi : \mathrm{colim}_{\alpha<\lambda}\mathcal{G}(G, X_\alpha) \to \mathcal{G}(G, \mathrm{colim}(X_\bullet))$$

is bijective.

Let $\alpha < \lambda$, and let f and g be graph homomorphisms from G to X_α with $\iota_\alpha \circ f = \iota_\alpha g$. Let $v \in V(G)$. Since $\iota_\alpha(f(v)) = \iota_\alpha(g(v))$, there is $\alpha(v) < \lambda$ such that $\alpha \le \alpha(v)$ and $\iota_{\alpha(v)\alpha}(f(v)) = \iota_{\alpha(v)\alpha}(g(v))$. Since λ is κ-filtered and $|V(G)| < \kappa$, there is an upper bound $\beta < \lambda$ for $\{\alpha(v) \mid v \in V(G)\}$. Then we have that

$$\iota_{\beta\alpha}(f(v)) = \iota_{\beta\alpha}(g(v))$$

for every $v \in V(G)$. This implies that Φ is injective.

Next we show that Φ is surjective. Let $f : G \to \mathrm{colim}(X_\bullet)$ be a graph homomorphism. For each element v of $V(G)$, there are $\alpha(v) < \lambda$ and $v'_\alpha \in X_{\alpha(v)}$ such that $\iota_{\alpha(v)}(v'_\alpha) = f(v)$. Let α be an upper bound for the set $\{\alpha(v) \mid v \in V(G)\}$, and let $v_\alpha = \iota_{\alpha,\alpha(v)}(v'_\alpha)$.

For each $(v, w) \in E(G)$, there is $\beta(v, w) \geq \alpha$ such that $(\iota_{\beta(v,w),\alpha}(v_\alpha), \iota_{\beta(v,w),\alpha}(w_\alpha)) \in E(X_{\beta(v,w)})$. Let β be an upper bound for the set $\{\beta(v, w) \mid (v, w) \in E(G)\}$. Then $(v, w) \in E(G)$ implies $(v_\beta, w_\beta) \in E(X_\beta)$. Namely, the correspondence $V(G) \to V(X_\beta)$, $v \mapsto v_\beta$ is a graph homomorphism $f_\beta : G \to X_\beta$. Since

$$\iota_\beta \circ f_\beta(v) = \iota_\beta(v_\beta) = \iota_\alpha(v_\alpha) = \iota_{\alpha'}(v'_\alpha) = f(v),$$

we have $\iota_\beta \circ f_\beta = f$. This implies that the map Φ is surjective. This completes the proof. □

Definition 6.11. Let $\mathcal{C}$ be a category and let I be a small family of morphisms in $\mathcal{C}$. We say that I *permits the small object argument* if I is a small set and the domains of elements of I are small relative to *I-cell*.

Let I be a family of morphisms in $\mathcal{C}$. A map f in $\mathcal{C}$ is an I-cell complex if f is isomorphic to the composition of some sequence $X_\bullet \lambda \to \mathcal{C}$ from an ordinal λ such that

$$X_\alpha \to X_{\alpha+1}$$

is the pushout of an element of I. The class of I-cell complexes is denoted by *I-cell*. The class *I-cof* of I-cofibrations is closed under pushouts and transfinite compositions. Hence we have *I-cell* $\subset$ *I-cof*.

Definition 6.12. A model category $\mathcal{M}$ is *cofibrantly generated* if there are small families I and J which satisfy the following conditions:

(1) I permits the small object argument, and the class of cofibrations coincide with *I-cof*.
(2) J permits the small object argument, and the class of trivial cofibrations coincide with *J-cof*.

We call I a set of *generating cofibrations* and call J a set of *generating trivial cofibrations.*

Proposition 6.13 (The small object argument, Proposition 10.5.16 of [16])**.** *Let $\mathcal{C}$ be a category which admits all small colimits. Let I be a family of morphisms in $\mathcal{C}$ which permits the small object argument. Then there is a functorial factorization $(\beta, \alpha) : \mathcal{C}^{[1]} \to \mathcal{C}^{[2]}$ such that $\beta(f) \in$ I-inj and $\alpha(f) \in$ I-cell for every morphism f in $\mathcal{C}$.*

Theorem 6.14 (Kan, Theorem 11.3.1 of [16])**.** *Let $\mathcal{M}$ be a category which admits all small colimits and limits. Let $\mathcal{W}$ be a family of morphisms in $\mathcal{M}$ which is closed under retracts and satisfies the two out of three axiom, and let I and J be small families of morphisms in $\mathcal{M}$. Suppose that the following conditions hold:*

(1) *Both I and J permit the small object argument.*
(2) *J-cof $\subset \mathcal{W} \cap$ (I-cof)*
(3) *I-inj $\subset \mathcal{W} \cap$ (J-inj)*
(4) *Either J-cof $= \mathcal{W} \cap$ (I-cof) or I-inj $= \mathcal{W} \cap$ (J-inj) holds.*

Then $\mathcal{M}$ has a cofibrantly generated model structure in which $\mathcal{W}$ is the family of weak equivalences, I is a set of generating cofibrations, and J is a set of generating trivial cofibrations.

Theorem 6.15 (Kan, Theorem 11.3.2 of [16])**.** *Let $\mathcal{M}$ be a cofibrantly generated model category with generating cofibration I and generating trivial cofibrations J, and let $F : \mathcal{M} \to \mathcal{N} : U$ be an adjoint pair. Suppose that $\mathcal{N}$ admits all small colimits and limits. Let FI denote the set $\{Ff \mid f \in I\}$ and FJ denote the set $\{Ff \mid f \in J\}$. Suppose the following conditions hold.*

(1) *Both FI and FJ permit the small object argument.*
(2) *U takes an (FJ)-cell complex to a weak equivalence in $\mathcal{M}$.*

Then $\mathcal{N}$ has a cofibrantly generated model structure with generating cofibrations I and generating trivial cofibrations J. A map f in $\mathcal{N}$ is a weak equivalence if and only if Uf is a weak equivalence in $\mathcal{M}$. Moreover, the adjoint pair (F, U) is a Quillen adjoint pair with respect to this model structure.

Example 6.16 (Chapter 2 of [18]). Let **Top** be the category of k-spaces. We call a continuous map $f : X \to Y$ a *weak homotopy equivalence* if f induces a bijection $\pi_0(X) \to \pi_0(Y)$ and a group isomorphism $f_* : \pi_n(X, x) \to \pi_n(Y, f(x))$ for every $n > 0$ and $x \in X$. Set

$$S^n = \{(x_0, \cdots, x_n) \mid x_0^2 + \cdots + x_n^2 = 1\},$$
$$D^n = \{(x_1, \cdots, x_n) \mid x_1^2 + \cdots + x_n^2 \leq 1\}.$$

Define the map $i_0 : D^n \hookrightarrow D^n \times [0, 1]$ by the correspondence $x \mapsto (x, 0)$. Set

$$I = \{S^{n-1} \hookrightarrow D^n \mid n \geq 0\},$$
$$J = \{i_0 : D^n \hookrightarrow D^n \times [0, 1] \mid n \geq 0\}.$$

Then **Top** has the cofibrantly generated model structure with generating cofibrations I and generating trivial cofibrations J. The class of weak equivalences coincides with the class of weak homotopy equivalences. Note that a continuous map f is a fibration if and only if f is a Serre fibration. If (X, A) be a pair of CW-complexes, then the inclusion $A \hookrightarrow X$ is a cofibration.

Example 6.17 (Chapter 3 of [18]). Let **SSet** be the category of simplicial sets. For a non-negative integer n, we write $\Delta[n] \in$ **SSet** to indicate the Yoneda functor $[m] \mapsto \Delta([m], [n])$. Define the subcomplexes $\partial\Delta[n]$ and $\Lambda_r[n]$ $(0 \leq r \leq n)$ of $\Delta[n]$ as follows:

$$\partial\Delta[n]_m = \{f : [m] \to [n] \mid f \text{ is order-preserving and } \mathrm{Im}(f) \neq [n].\},$$
$$\Lambda_r[n]_m = \{f : [m] \to [n] \mid f \text{ is order-preserving and } \mathrm{Im}(f) \cup \{r\} \neq [n].\}.$$

Set

$$I = \{\partial\Delta[n] \hookrightarrow \Delta[n] \mid n \geq 0\},$$
$$J = \{\Lambda_r[n] \hookrightarrow \Delta[n] \mid n \geq 1, 0 \leq r \leq n\}.$$

The category **SSet** of simiplicial sets has the cofibrantly generated structure with generating cofibrations I and generating trivial cofibrations J. A simplicial map f is a weak equivalence if and only if ithe continuous map $|f| : |K| \to |L|$ induced by f is a homotopy equivalence. A simplicial map $i : K \hookrightarrow L$ is a cofibration if and only if i is an inclusion. A fibration of **SSet** is called a *Kan fibration*, and a fibrant object is called a *Kan complex*.

The geometric realization functor

$$|-| : \mathbf{SSet} \to \mathbf{Top}$$

is a Quillen equivalence. Moreover, it is known that the geometric realization functor preserves fibrations (see Chapter 3 of [18]).

The following lemma is a generalization of the gluing lemma (Proposition 2.5).

Proposition 6.18 (Proposition 15.10.10 of [16]). *Let $\mathcal{C}$ be a model catgory and let*

$$\begin{tikzcd} A \arrow[d, "f_A"'] & B \arrow[l, "i"'] \arrow[r] \arrow[d, "f_B"'] & C \arrow[d, "f_C"] \\ A' & B' \arrow[l, "i'"'] \arrow[r] & C' \end{tikzcd}$$

be a commutative diagram in $\mathcal{C}$. Suppose that all objects appearing in the above diagram are cofibrant, and i and i' are cofibrations. Then the natural map

$$A \cup_B C \to A' \cup_{B'} C'$$

is a weak equivalence.

Proposition 6.19. *Let $\mathcal{C}$ be a model category. Let κ be an ordinal, let $X_\bullet, Y_\bullet : \kappa \to \mathcal{C}$ be κ-sequences, and let $u : X_\bullet \to Y_\bullet$ be a natural transformation. Moreover, suppose the following conditions.*

(1) *For every $\alpha < \kappa$, $u_\alpha : X_\alpha \to Y_\alpha$ is a weak equivalence.*

(2) X_0 *is cofibrant and* $X_\alpha \hookrightarrow X_{\alpha+1}$ *is a cofibration for every* $\alpha < \kappa$.
(3) Y_0 *is cofibrant and* $Y_\alpha \hookrightarrow Y_{\alpha+1}$ *is a cofibration for every* $\alpha < \kappa$.

Then the colimit $u_\infty : X_\infty \to Y_\infty$ *of* u *is a weak equivalence.*

Proof. The proof is almost the same as the proof of Proposition 15.10.11 of [16]. □

We will use the following criterion to prove the Quillen equivalence.

Proposition 6.20 (Corollary 1.3.16 of [18]). *A Quillen adjoint pair* $F : \mathcal{M} \to \mathcal{N} : U$ *is a Quillen equivalence if and only if both of the following conditions hold:*

(1) *Let* $f : X \to Y$ *be a map between fibrant objects in* $\mathcal{N}$*. If* Uf *is a weak equivalence, then* f *is a weak equivalence.*
(2) *For a cofibrant object* A *of* $\mathcal{M}$*, the composition*

$$A \to UFA \to URFA$$

is a weak equivalence.

6.3. **Model structure on $\mathbf{SSet}^{\mathbb{Z}_2}$.** The category of $\mathbb{Z}_2$-simplicial sets is denoted by $\mathbf{SSet}^{\mathbb{Z}_2}$. For a $\mathbb{Z}_2$-simplicial set K, the subcomplex of fixed points of K is denoted by $K^{\mathbb{Z}_2}$. Note that $K^{\mathbb{Z}_2}$ is the equalizer of the identity of K and the involution of K. We call a $\mathbb{Z}_2$-simplicial map $f : K \to L$ a $\mathbb{Z}_2$*-weak equivalence* if both $f : K \to L$ and $f^{\mathbb{Z}_2}$ are weak equivalences in $\mathbf{SSet}$.

If we regard a simplicial set K as a $\mathbb{Z}_2$-simplicial set, we consider the $\mathbb{Z}_2$-action on K is trivial unless otherwise stated. Let $i : \mathbf{SSet} \to \mathbf{SSet}^{\mathbb{Z}_2}$ be the inclusion functor. Then i is the left adjoint of $(-)^{\mathbb{Z}_2} : \mathbf{SSet}^{\mathbb{Z}_2} \to \mathbf{SSet}$. We often abbreviate $i(K)$ to K.

Regard $\mathbb{Z}_2$ as a $\mathbb{Z}_2$-simplicial set consisting of two vertices $\Delta[0] \sqcup \Delta[0]$ whose involution is the exchange of the vertices. Consider the composition

$$\mathbf{SSet} \xrightarrow{\ i\ } \mathbf{SSet}^{\mathbb{Z}_2} \xrightarrow{\mathbb{Z}_2 \times (-)} \mathbf{SSet}^{\mathbb{Z}_2},$$

which is also denoted by $\mathbb{Z}_2 \times (-)$. Then $\mathbb{Z}_2 \times (-) : \mathbf{SSet} \to \mathbf{SSet}^{\mathbb{Z}_2}$ is the left adjoint functor of the forgetful functor $\mathbf{SSet}^{\mathbb{Z}_2} \to \mathbf{SSet}$.

In Example 6.17, recall that we set

$$I = \{\partial\Delta[n] \hookrightarrow \Delta[n] \mid n \geq 0\}$$

and

$$J = \{\Lambda_r[n] \hookrightarrow \Delta[n] \mid n \geq 1,\ 0 \leq r \leq n\}.$$

Set $I' = (\mathbb{Z}_2 \times I) \cup i(I)$ and $J' = (\mathbb{Z}_2 \times J) \cup i(J)$. The purpose of this section is to prove the following proposition.

Proposition 6.21. *The category* $\mathbf{SSet}^{\mathbb{Z}_2}$ *of* $\mathbb{Z}_2$*-simplicial sets has the cofibrantly generated model structure with generating cofibrations* I' *and generating trivial cofibrations* J'*. Moreover, the classes of weak equivalences, cofibrations, and fibrations are described as follows:*

(1) *The class of weak equivalences is the class of* $\mathbb{Z}_2$*-weak equivalences.*
(2) *The class of cofibrations is the class of inclusions.*
(3) *A* $\mathbb{Z}_2$*-simplicial map* $p : X \to Y$ *is a fibration in* $\mathbf{SSet}^{\mathbb{Z}_2}$ *if and only if both* $Up : X \to Y$ *and* $p^{\mathbb{Z}_2} : X^{\mathbb{Z}_2} \to Y^{\mathbb{Z}_2}$ *are Kan fibrations.*

Proposition 6.21 seems well-known to the experts (see Appendix of [15]). In fact the proof is straightforward. However, I could not find the proof of Proposition 6.21. So we write the proof here to make the thesis self-contained.

Lemma 6.22. *A* $\mathbb{Z}_2$*-simplicial map* i *is an* I'*-cofibration if and only if* i *is an inclusion.*

Proof. The proof is almost the same as the case of usual simplicial sets (see Section 3 of [18]). □

Lemma 6.23. *Let p be a $\mathbb{Z}_2$-simplicial map. Then p is J'-injective if and only if both Up and $p^{\mathbb{Z}_2}$ are Kan fibrations.*

Proof. Suppose that p is J'-injective. Consider a commutative square

$$\begin{tikzcd} \Lambda_r[n] \arrow[r] \arrow[d, "f"'] & UX \arrow[d, "Up"] \\ \Delta[n] \arrow[r] & UY \end{tikzcd} \tag{3}$$

in **SSet**. Then the associated diagram

$$\begin{tikzcd} \mathbb{Z}_2 \times \Lambda_r[n] \arrow[r] \arrow[d] & X \arrow[d] \\ \mathbb{Z}_2 \times \Delta[n] \arrow[r] & Y \end{tikzcd}$$

in $\mathbf{SSet}^{\mathbb{Z}_2}$ commutes (Lemma 6.3) and has a lift. Hence the diagram (3) has a lift (Lemma 6.4). This implies that Up is a Kan fibration. On the other hand, consider a commutative square

$$\begin{tikzcd} \Lambda_r[n] \arrow[r] \arrow[d] & X^{\mathbb{Z}_2} \arrow[d, "p^{\mathbb{Z}_2}"] \\ \Delta[n] \arrow[r] & Y^{\mathbb{Z}_2} \end{tikzcd} \tag{4}$$

in **SSet**. Then the associated diagram

$$\begin{tikzcd} i(\Lambda_r[n]) \arrow[r] \arrow[d] & X \arrow[d] \\ i(\Delta[n]) \arrow[r] & Y \end{tikzcd}$$

in $\mathbf{SSet}^{\mathbb{Z}_2}$ commutes (Lemma 6.3) and has a lift. Hence we have that the diagram (4) has a lift (Lemma 6.4). This implies that $p^{\mathbb{Z}_2}$ is a Kan fibration.

On the other hand, suppose that both Up and $p^{\mathbb{Z}_2}$ are Kan fibrations. Consider a commutative diagram such as

$$\begin{tikzcd} \mathbb{Z}_2 \times \Lambda_r[n] \arrow[r] \arrow[d] & X \arrow[d] \\ \mathbb{Z}_2 \times \Delta[n] \arrow[r] & Y \end{tikzcd} \tag{5}$$

in $\mathbf{SSet}^{\mathbb{Z}_2}$. By Lemma 6.4, this has a lift if and only if

$$\begin{tikzcd} \Lambda_r[n] \arrow[r] \arrow[d] & X \arrow[d, "Up"] \\ \Delta[n] \arrow[r] & Y \end{tikzcd}$$

in **SSet**. Since Up has a lift, we have that the diagram (5) has a lift. In a similar way, we can show that a commutative diagram such as

$$\begin{tikzcd} \Lambda_r[n] \arrow[r] \arrow[d] & X \arrow[d] \\ \Delta[n] \arrow[r] & Y \end{tikzcd}$$

has a lift. □

The following lemma is similarly proved.

Lemma 6.24. *Let $p : X \to Y$ be a $\mathbb{Z}_2$-simplicial map. Then p is an I'-injective map if and only if both Up and $p^{\mathbb{Z}_2}$ are I-injective in* **SSet**.

We write $\mathcal{W}'$ to indicate the class of $\mathbb{Z}_2$-weak equivalences in $\mathbf{SSet}^{\mathbb{Z}_2}$.

Corollary 6.25. *I'-inj $= \mathcal{W}' \cap (J'$-inj$)$*

Lemma 6.26. *The fixed point functor $(-)^{\mathbb{Z}_2} : \mathbf{SSet}^{\mathbb{Z}_2} \to \mathbf{SSet}$ commutes with the transfinite composition. Namely, for a λ-sequence $X_\bullet : \lambda \to \mathbf{SSet}^{\mathbb{Z}_2}$, the natural map*

$$\Phi : \mathrm{colim}_{\alpha<\lambda}(X_\alpha^{\mathbb{Z}_2}) \to (\mathrm{colim}(X_\bullet))^{\mathbb{Z}_2}$$

is an isomorphism.

Proof. The colimit of $X_\bullet$ is denoted by X_λ, for simplicity. The generator of $\mathbb{Z}_2$ is denoted by τ. For $\alpha, \beta < \lambda$ with $\alpha \le \beta$, the natural map $X_\alpha \to X_\beta$ is denoted by $\iota_{\beta\alpha}$. We write ι_α to indicate the natural map $X_\alpha \to X_\lambda$.

Let $\alpha < \lambda$ and let $\sigma_0, \sigma_1 \in X_\alpha^{\mathbb{Z}_2}$ with $\iota_\alpha(\sigma_0) = \iota_\alpha(\sigma_1)$. Then there is $\alpha_1 \ge \alpha$ such that $\iota_{\alpha_1\alpha}(\sigma_0) = \iota_{\alpha_1\alpha}(\sigma_1)$. This implies that Φ_n is injective.

Next let σ be an n-simplex of $X_\lambda^{\mathbb{Z}_2}$. Since $\sigma \in (X_\lambda)_n$, there is $\alpha < \lambda$ and $\sigma' \in X_\alpha$ such that $\iota_\alpha(\sigma') = \sigma$. Since $\iota_\alpha(\sigma') = \iota_\alpha(\tau\sigma')$, there is $\beta \ge \alpha$ such that $\iota_{\beta\alpha}(\sigma') = \iota_{\beta\alpha}(\tau\sigma') = \tau\iota_{\beta\alpha}(\sigma')$. Therefore $\sigma'' = \iota_{\beta\alpha}(\sigma')$ is contained in $X_\beta^{\mathbb{Z}_2}$ and $\iota_\beta(\sigma'') = \sigma$. This implies that Φ_n is surjective. □

Lemma 6.27. *The class $\mathcal{W}'$ of weak equivalences in $\mathbf{SSet}^{\mathbb{Z}_2}$ is closed under the transfinite compositions.*

Proof. This is deduced from Lemma 6.26 and the fact that the class of weak equivalences in **SSet** is closed under transfinite compositions. □

Lemma 6.28. *J'-cof $\subset \mathcal{W}' \cap (I'$-cof$)$*

Proof. Recall that we have already shown that J'-*cof* $\subset$ I'-*cof*. If $f : K \to L$ is a pushout of an element of J', then $|f| : |K| \to |L|$ is a $\mathbb{Z}_2$-homotopy equivalence. Hence Lemma 6.27 implies that J-cell complexes are $\mathbb{Z}_2$-weak equivalences. Since a J'-cofibration is a retract of a J'-cell complex, we have J'-*cof* $\subset \mathcal{W}'$. □

This completes the proof of Proposition 6.21

We close this section with the Quillen equivalence between $\mathbf{SSet}^{\mathbb{Z}_2}$ and the category $\mathbf{Top}^{\mathbb{Z}_2}$ of $\mathbb{Z}_2$-k-spaces. It is similarly shown that $\mathbf{Top}^{\mathbb{Z}_2}$ has the model structure described as follows:

(1) A $\mathbb{Z}_2$-continuous map $f : X \to Y$ is a weak equivalence if and only if f and $f^{\mathbb{Z}_2} : X^{\mathbb{Z}_2} \to Y^{\mathbb{Z}_2}$ are weak homotopy equivalences.
(2) A $\mathbb{Z}_2$-continuous map $p : X \to Y$ is a fibration if and only if f and $f^{\mathbb{Z}_2} : X^{\mathbb{Z}_2} \to Y^{\mathbb{Z}_2}$ are Serre fibrations.

Moreover, if (X, A) is a $\mathbb{Z}_2$-CW-pair, then the incluion $A \hookrightarrow X$ is a cofibration in $\mathbf{Top}^{\mathbb{Z}_2}$.

Lemma 6.29. *Let $\mathcal{J}$ be a small category and let $F : \mathcal{C} \to \mathcal{D} : U$ be an adjunction between categories. Then $F_* : \mathcal{C}^{\mathcal{J}} \to \mathcal{D}^{\mathcal{J}} : U_*$ is an adjoint pair.*

Proof. Let φ be the adjunction

$$\varphi_{A,X} : \mathcal{D}(FA, X) \xrightarrow{\cong} \mathcal{C}(A, UX),\ f \mapsto \varphi(f).$$

Let $A^\bullet : \mathcal{J} \to \mathcal{C}$ and $X^\bullet : \mathcal{J} \to \mathcal{D}$ be functors. Let $f^\bullet : F_*A^\bullet \to X^\bullet$ be a natural transformation. Define the natural transformation $\varphi(f^\bullet)$ by $j \mapsto \varphi(f^j)$ for an object j of $\mathcal{J}$.

We want to show that this is actually an natural transformation. Let $\alpha : j \to j'$ be a morphism in $\mathcal{J}$. Since the diagram

$$\begin{array}{ccc} FA^j & \xrightarrow{f^j} & X^j \\ {\scriptstyle FA^\alpha}\downarrow & & \downarrow{\scriptstyle X^\alpha} \\ FA^{j'} & \xrightarrow{f^{j'}} & X^{j'} \end{array}$$

commutes, the diagram

$$\begin{array}{ccc} A^j & \xrightarrow{\varphi(f^j)} & UX^j \\ {\scriptstyle A^\alpha}\downarrow & & \downarrow{\scriptstyle UX^\alpha} \\ A^{j'} & \xrightarrow{\varphi(f^{j'})} & UX^{j'} \end{array}$$

commutes. This implies that $\varphi(f^\bullet)$ is a natural transformation. □

Proposition 6.30. *The adjoint pair*

$$|-| : \mathbf{SSet}^{\mathbb{Z}_2} \to \mathbf{Top}^{\mathbb{Z}_2} : \mathrm{Sing}$$

is a Quillen equivalence.

Proof. Let K be a $\mathbb{Z}_2$-simplicial set, let X be a $\mathbb{Z}_2$-space, and let $f : |K| \to X$ be a $\mathbb{Z}_2$-continuous map. Let $\varphi(f) : K \to \mathrm{Sing}(X)$ be the adjoint. We show that the adjoint of $f^{\mathbb{Z}_2} : |K^{\mathbb{Z}_2}| \to X^{\mathbb{Z}_2}$ is

$$\varphi(f)^{\mathbb{Z}_2} : K^{\mathbb{Z}_2} \to \mathrm{Sing}(X^{\mathbb{Z}_2}).$$

To see this, consider the commutative diagram

$$\begin{array}{ccc} |K^{\mathbb{Z}_2}| & \xrightarrow{f^{\mathbb{Z}_2}} & X^{\mathbb{Z}_2} \\ \downarrow & & \downarrow \\ |K| & \xrightarrow{f} & X. \end{array}$$

Then the diagram

$$\begin{array}{ccc} K^{\mathbb{Z}_2} & \xrightarrow{\varphi(f^{\mathbb{Z}_2})} & \mathrm{Sing}(X^{\mathbb{Z}_2}) \\ \downarrow & & \downarrow \\ K & \xrightarrow{\varphi(f)} & \mathrm{Sing}(X) \end{array}$$

By taking the fixed point subcomplex, we have that $\varphi(f^{\mathbb{Z}_2}) = \varphi(f)^{\mathbb{Z}_2}$.

Suppose that $f : |K| \to X$ is a $\mathbb{Z}_2$-weak equivalence. Then $f = Uf$ and $f^{\mathbb{Z}_2}$ are weak equivalences in **SSet**. Since $|-| : \mathbf{SSet} \to \mathbf{Top} : \mathrm{Sing}$ is a Quillen equivalence, we have that $\varphi(f)$ and $\varphi(f^{\mathbb{Z}_2}) \cong \varphi(f)^{\mathbb{Z}_2}$ are $\mathbb{Z}_2$-weak equivalences. This implies that $\varphi(f) : K \to \mathrm{Sing}(X)$ is a $\mathbb{Z}_2$-weak equivalence. Similarly, we can show that if $\varphi(f)$ is a $\mathbb{Z}_2$-weak equivalence, then f is a $\mathbb{Z}_2$-weak equivalence. By definition, this shows that $|-|$ is a Quillen equivalence. □

Let X be a $\mathbb{Z}_2$-CW-complex and let Y be a $\mathbb{Z}_2$-space. Then the $\mathbb{Z}_2$-maps $f, g : X \to Y$ are left homotopic in $\mathbf{Top}^{\mathbb{Z}_2}$ if and only if they are $\mathbb{Z}_2$-homotopic. Since every object in $\mathbf{Top}^{\mathbb{Z}_2}$ is fibrant, we have the following.

Proposition 6.31 (Bredon [6])**.** *A $\mathbb{Z}_2$-map $f : X \to Y$ is a $\mathbb{Z}_2$-homotopy equivalence if and only if $f : X \to Y$ and $f^{\mathbb{Z}_2} : X^{\mathbb{Z}_2} \to Y^{\mathbb{Z}_2}$ are $\mathbb{Z}_2$-homotopy equivalences.*

Of course, Proposition 6.31 is directly proved in an obvious way.

6.4. **Barycentric subdivision.** Consider the set $2^{[n]} \setminus \{\emptyset\}$ of non-empty subsets of $[n]$ as a poset ordered by inclusion. Define the simplicial set $\mathrm{Sd}(\Delta[n])$ to be the nerve of $2^{[n]} \setminus \{\emptyset\}$. Then we have a functor

$$\Delta \to \mathbf{SSet}, [n] \mapsto \mathrm{Sd}(\Delta[n]).$$

The barycentric subdivision functor $\mathrm{Sd} : \mathbf{SSet} \to \mathbf{SSet}$ is the left Kan extension of this functor along the Yoneda functor $\Delta^{\mathrm{op}} \to \mathbf{SSet}$. The right adjoint of Sd is denoted by Ex.

Note that the poset map

$$2^{[n]} \setminus \{\emptyset\} \to [n], \sigma \mapsto \max(\sigma)$$

induces a simplicial map $\mathrm{Sd}(\Delta[n]) \to \Delta[n]$, and induces a natural weak equivalence $u : \mathrm{Sd} \to \mathrm{id}$, i.e. $u_K : \mathrm{Sd}(K) \to K$ is a weak equivalence for every simplicial set K. Let $h : K \to \mathrm{Ex}(K)$ be the adjoint of $u : \mathrm{Sd}(K) \to K$.

Lemma 6.32 ([13])**.** *For a simplicial set K, the simplicial map $h : K \to \mathrm{Ex}(K)$ is a weak equivalence.*

Let $\mathrm{Ex}^\infty(K)$ be the colimit of the sequence

$$K \xrightarrow{\ h\ } \mathrm{Ex}(K) \xrightarrow{\ h\ } \mathrm{Ex}^2(K) \longrightarrow \cdots .$$

Since the class of weak equivalences in **SSet** is closed under transfinite compositions, we have that $K \to \mathrm{Ex}^\infty(K)$ is a weak equivalence.

Proposition 6.33 ([13])**.** *The simplicial set $\mathrm{Ex}^\infty(K)$ is a Kan complex for every $K \in \mathbf{SSet}$.*

Next we consider the barycentric subdivision of $\mathbf{SSet}^{\mathbb{Z}_2}$.

We start with the following general argument: Let $F : \mathcal{C} \to \mathcal{D} : G$ be an adjoint pair and let $\mathcal{J}$ be a small category. Then F induces a functor $F_* : \mathcal{C}^{\mathcal{J}} \to \mathcal{D}^{J}$ and G induces a functor $G_* : \mathcal{D}^{\mathcal{J}} \to \mathcal{C}^{\mathcal{J}}$. Moreover, the pair $F_* : \mathcal{C}^{\mathcal{J}} \to \mathcal{D}^{\mathcal{J}} : G_*$ is an adjoint pair.

Consider the group $\mathbb{Z}_2$ as a small category in the usual way. Namely, the object set of $\mathbb{Z}_2$ is the one point subset $\{*\}$, and the morphisms set from $*$ to $*$ is $\mathbb{Z}_2$. The composition law is defined by the group multiplication of $\mathbb{Z}_2$. Then the $\mathbb{Z}_2$-simplicial set is identified with a functor from $\mathbb{Z}_2$ to **SSet**, and a $\mathbb{Z}_2$-equivariant simpilcial map coincides with a natural transformation. Hence the category $\mathbf{SSet}^{\mathbb{Z}_2}$ of $\mathbb{Z}_2$-simplicial sets is identified with the functor category from $\mathbb{Z}_2$ to **SSet** as the notation indicates. Hence the adjoint pair

$$\mathrm{Sd} : \mathbf{SSet} \to \mathbf{SSet} : \mathrm{Ex}$$

induces an adjoint pair

$$\mathrm{Sd} : \mathbf{SSet}^{\mathbb{Z}_2} \to \mathbf{SSet}^{\mathbb{Z}_2} : \mathrm{Ex}.$$

Consider the adjoint $h : K \to \mathrm{Ex}(K)$ of $u : \mathrm{Sd}(K) \to K$.

Lemma 6.34. *For a $\mathbb{Z}_2$-simplicial set K, the map $\mathrm{Sd}(K^{\mathbb{Z}_2}) \to \mathrm{Sd}(K)^{\mathbb{Z}_2}$ is an isomorphism.*

Proof. See the following commutative diagram

$$\begin{array}{ccc} |\mathrm{Sd}(K^{\mathbb{Z}_2})| & \xrightarrow{\ \cong\ } & |K^{\mathbb{Z}_2}| \\ \big\downarrow & & \big\downarrow{\scriptstyle\cong} \\ |\mathrm{Sd}(K)^{\mathbb{Z}_2}| & \xrightarrow{\ \cong\ } & |K|^{\mathbb{Z}_2}. \end{array}$$

The horizontal arrows are natural homeomorphisms. The left vertical arrow is a homeomorphism since the geometric realization preserves equalizers. Hence the left vertical arrow is a homeomorphism. It is easy to see that a simplicial map $f : K \to L$ which induces a homeomorphism from $|K|$ to $|L|$ is an isomorphism. $\square$

Corollary 6.35. *Let K be a $\mathbb{Z}_2$-simplicial set. Then the natural map map $u : \mathrm{Sd}(K) \to K$ is a $\mathbb{Z}_2$-homotopy equivalence.*

Proof. It suffices to see that $u^{\mathbb{Z}_2} : \mathrm{Sd}(K)^{\mathbb{Z}_2} \to K^{\mathbb{Z}_2}$ is a homotopy equivalence. Consider the diagram

$$\begin{tikzcd}
\mathrm{Sd}(K^{\mathbb{Z}_2}) \arrow[r, "\simeq"] \arrow[d, "\cong"'] & K^{\mathbb{Z}_2} \arrow[d, equal] \\
\mathrm{Sd}(K)^{\mathbb{Z}_2} \arrow[r] & K^{\mathbb{Z}_2}
\end{tikzcd}$$

The left vertical arrow is an isomorphism by Lemma 6.34. Hence the map $u^{\mathbb{Z}_2} : \mathrm{Sd}(K)^{\mathbb{Z}_2} \to K^{\mathbb{Z}_2}$ is a homotopy equivalence. $\square$

Lemma 6.36. *For a $\mathbb{Z}_2$-simplicial set K, the map $h : K \to \mathrm{Ex}(K)$ is a weak equivalence in $\mathbf{SSet}^{\mathbb{Z}_2}$.*

Proof. For a simplicial set K, we write $h(K)$ to indicate the natural weak equivalence $K \to \mathrm{Ex}(K)$. We want to show that $h(K) : K \to \mathrm{Ex}(K)$ is a $\mathbb{Z}_2$-weak equivalence. It suffices to show that $h(K)^{\mathbb{Z}_2} : K^{\mathbb{Z}_2} \to \mathrm{Ex}(K)^{\mathbb{Z}_2} = \mathrm{Ex}(K^{\mathbb{Z}_2})$ is a weak equivalence. Consider the diagram

$$\begin{tikzcd}
K^{\mathbb{Z}_2} \arrow[r, "h(K^{\mathbb{Z}_2})"] \arrow[d, "i"'] & \mathrm{Ex}(K^{\mathbb{Z}_2}) \arrow[d, "\mathrm{Ex}(i)"] \\
K \arrow[r, "h(K)"] & \mathrm{Ex}(K).
\end{tikzcd}$$

in $\mathbf{SSet}^{\mathbb{Z}_2}$. Here we consider that $K^{\mathbb{Z}_2}$ and $\mathrm{Ex}(K^{\mathbb{Z}_2})$ are $\mathbb{Z}_2$-simplicial set by the trivial $\mathbb{Z}_2$-actions. By taking the fixed point subcomplexes, we have the diagram

$$\begin{tikzcd}
K^{\mathbb{Z}_2} \arrow[r, "h(K^{\mathbb{Z}_2})"] \arrow[d, "\cong"'] & \mathrm{Ex}(K^{\mathbb{Z}_2}) \arrow[d, "\cong"] \\
K^{\mathbb{Z}_2} \arrow[r, "h(K)^{\mathbb{Z}_2}"] & \mathrm{Ex}(K)^{\mathbb{Z}_2}.
\end{tikzcd}$$

Since $h(K^{\mathbb{Z}_2})$ is a weak equivalence, we have that $h(K)^{\mathbb{Z}_2}$ is a weak equivalence. $\square$

Since the class of the $\mathbb{Z}_2$-weak equivalences of $\mathbf{SSet}^{\mathbb{Z}_2}$ is closed under transfinite compositions, we have that the map $K \to \mathrm{Ex}^\infty(K)$ is a $\mathbb{Z}_2$-weak equivalence.

Lemma 6.37. *For a $\mathbb{Z}_2$-simplicial set K, the $\mathbb{Z}_2$-simplicial object $\mathrm{Ex}^\infty(K)$ is fibrant in $\mathbf{SSet}^{\mathbb{Z}_2}$.*

Proof. By Proposition 6.33, the simiplicial set $\mathrm{Ex}^\infty(K)$ is a Kan complex. Since Ex preserves equalizers, we have that $\mathrm{Ex}(K^{\mathbb{Z}_2}) \cong \mathrm{Ex}(K)^{\mathbb{Z}_2}$. Since the functor $(-)^{\mathbb{Z}_2} : \mathbf{SSet}^{\mathbb{Z}_2} \to \mathbf{SSet}$ preserves transfinite compositions we have

$$\mathrm{Ex}^\infty(K)^{\mathbb{Z}_2} \cong \mathrm{colim}_{n\to\infty}(\mathrm{Ex}^n(K)^{\mathbb{Z}_2}) \cong \mathrm{colim}_{n\to\infty}(\mathrm{Ex}^n(K^{\mathbb{Z}_2})) \cong \mathrm{Ex}^\infty(K^{\mathbb{Z}_2}).$$

By Proposition 6.33, the simplicial set $\mathrm{Ex}^\infty(K)^{\mathbb{Z}_2}$ is a Kan complex. This completes the proof. $\square$

7. Simplicial methods

7.1. **Singular complex.** In this section we shall introduce a simplicial set associated to a pair of graphs, which will be called a singular complex. The main result showed that singular complexes and Hom complexes are homotopy equivalent. The results mentioned here are found in [27].

Let Σ_n be the graph defined by $V(\Sigma_n) = [n]$ and $E(\Sigma_n) = V(\Sigma_n) \times V(\Sigma_n)$. We often write $\mathbf{1}$ instead of Σ_0. Note that $\mathbf{1}$ is the terminal object of $\mathcal{G}$ and hence $\mathbf{1} \times G \cong G \times \mathbf{1} \cong G$.

Let T, G be graphs. The singular complex $\mathrm{Sing}(T, G)$ is the simplicial set

$$\mathrm{Sing}(T, G)_n = \mathcal{G}(T \times \Sigma_n, G)$$

with obvious face and degeneracy maps. The purpose of this section is to prove the following theorem.

Theorem 7.1 (M. [27]). *There is a natural homotopy equivalence*

$$\Phi : |\mathrm{Sing}(T,G)| \xrightarrow{\simeq} |\mathrm{Hom}(T,G)|.$$

Remark 7.2. Let T be a graph and let $\mathcal{P}$ be the category of poset. Then the functor

$$\mathcal{G} \to \mathcal{P},\ G \mapsto \mathrm{Hom}(T,G)$$

is not a right adjoint functor since it does not preserve products. On the other hand, the functor

$$\mathcal{G} \to \mathbf{SSet}, G \mapsto \mathrm{Sing}(T,G)$$

becomes a right adjoint functor. To see this, apply Proposition 6.1 to the cosimplicial object $\Delta \to \mathcal{G}$, $[n] \mapsto T \times \Sigma_n$.

Let G, H be graphs. The exponential graph H^G is defined by

$$V(H^G) = \{f \mid f \text{ is a map from } V(G) \text{ to } V(H).\},$$

$$E(H^G) = \{(f,g) \mid (f \times g)(E(G)) \subset E(H)\}$$

The following proposition is a well-known fact (see [8] for example).

Proposition 7.3. *There is a natural isomorphism*

$$\mathcal{G}(T \times G, H) \cong \mathcal{G}(T, H^G).$$

Corollary 7.4. *There is a natural isomorphism*

$$\mathrm{Sing}(T \times G, H) \cong \mathrm{Sing}(T, H^G).$$

Proof. For each non-negative integer n, we have an isomorphism

$$\begin{aligned} \mathrm{Sing}(T \times G, H)_n &= \mathcal{G}(\Sigma_n \times T \times G, H) \\ &\cong \mathcal{G}(\Sigma_n \times T, H^G) \\ &= \mathrm{Sing}(T, H^G)_n \end{aligned}$$

The naturality of the isomorphism of Proposition 7.3 implies that the sequence $(\Phi_n)_n : \mathrm{Sing}(T \times G, H) \to \mathrm{Sing}(T, H^G)$ is an isomorphism of simplicial sets. □

Proposition 7.5 (Dochtermann [8]). *There is a natural homoropy equivalence*

$$\mathrm{Hom}(T, H^G) \xrightarrow{\simeq} \mathrm{Hom}(T \times G, H)$$

Proof. The following proof is the same as the original proof of [8]. Here we give the proof for the reader's convenience.

Define the order-preserving map

$$\Phi : \mathrm{Hom}(T \times G, H) \to \mathrm{Hom}(T, H^G)$$

by the correspondence

$$\Phi(\eta)(x) = \{f : V(G) \to V(H) \mid v \in V(G) \Rightarrow f(v) \in \eta(x,v)\}.$$

for $\eta \in \mathrm{Hom}(T \times G, H)$ and $x \in V(T)$. On the other hand, define the order-preserving map

$$\Psi : \mathrm{Hom}(T, H^G) \to \mathrm{Hom}(T \times G, H)$$

by the correspondence

$$\Psi(\eta)(x,v) = \{f(v) \mid f \in \eta(x)\}$$

for $\eta \in \mathrm{Hom}(T, H^G)$ and $x \in V(T)$. We want to show that $\Psi \circ \Phi = \mathrm{id}$ and $\Phi \circ \Psi \geq \mathrm{id}$. For $\eta \in \mathrm{Hom}(T \times G, H)$, $x \in V(T)$, and $v \in V(G)$, we have that

$$\begin{aligned}\Psi \circ \Phi(\eta)(x,v) &= \{f(v) \mid f \in \Phi(\eta)(x)\} \\ &= \{f(v) \mid f : V(G) \to V(H) \text{ is a map such that } v \in V(G) \text{ implies } f(v) \in \eta(x,v).\} \\ &= \eta(x,v).\end{aligned}$$

Hence we have $\Psi \circ \Phi = \mathrm{id}$. On the other hand, for $\eta \in \mathrm{Hom}(T, H^G)$ and for $x \in V(T)$, we have

$$\begin{aligned}\Phi \circ \Psi(\eta)(x) &= \{f : V(G) \to V(H) \mid v \in V(G) \Rightarrow f(v) \in \Psi(\eta)(x,v)\} \\ &= \{f : V(G) \to V(H) \mid v \in V(G) \Rightarrow f(v) \in \{g(v) \mid g \in \eta(x)\}\} \\ &\geq \{f : V(G) \to V(H) \mid f \in \eta(x)\} \\ &= \eta(x).\end{aligned}$$

Hence we have $\Phi \circ \Psi \geq \mathrm{id}$. □

Recall that a multi-homomorphism $\eta \in \mathrm{Hom}(G, H)$ is locally finite if $\eta(v)$ is finite for all $v \in V(G)$, and the induced subposet of $\mathrm{Hom}(G, H)$ consisting of locally finite multi-homomorphisms is denoted by $\mathrm{Hom}_f(G, H)$. The inclusion $\mathrm{Hom}_f(G, H) \hookrightarrow \mathrm{Hom}(G, H)$ is a homotopy equivalence (see Lemma 4.7).

A *clique of a graph* G is a subset $\sigma \subset V(G)$ with $\sigma \times \sigma \subset E(G)$. The *clique complex* $C(G)$ *of* G is the abstract simplicial complex whose simplices are cliques of G. Note that a finite clique of G is identified with a locally finite multi-homomorphism from $\mathbf{1}$ to G. Moreover, the face poset of $C(G)$ is isomorphic to $\mathrm{Hom}_f(\mathbf{1}, G)$. Hence there is a natural homeomorphism

$$|C(G)| \xrightarrow{\cong} |\mathrm{Hom}(\mathbf{1}, G)|.$$

Hence we have homotopy equivalences

$$|\mathrm{Sing}(G, H)| \xrightarrow{\cong} |\mathrm{Sing}(\mathbf{1}, H^G)|$$

and

$$|C(G)| \xrightarrow{\cong} |\mathrm{Hom}_f(\mathbf{1}, H^G)| \xrightarrow{\simeq} |\mathrm{Hom}(\mathbf{1}, H^G)| \xrightarrow{\simeq} |\mathrm{Hom}(G, H)|.$$

Hence it suffices to prove the following proposition, which will be proved in Section 5.4.

Proposition 7.6. *There is a natural homotopy equivalence*

$$|\mathrm{Sing}(\mathbf{1}, G)| \longrightarrow |C(G)|.$$

In the rest of this section, we use the notation $\mathrm{Sing}(X)$ instead of $\mathrm{Sing}(\mathbf{1}, X)$.

Lemma 7.7. *If X is a non-empty and $E(X) = V(X) \times V(X)$, then $|\mathrm{Sing}(X)|$ is contractible.*

Proof. Note that $|\mathrm{Sing}(\mathbf{1})|$ is the one point space. This implies that if a graph homomorphism $f : G \to H$ is constant, then the map $|f_*| : |\mathrm{Sing}(G)| \to |\mathrm{Sing}(H)|$ is also constant.

First we show that $\mathrm{Sing}(X)$ is connected. Let a and b be 0-simplices of $\mathrm{Sing}(X)$, namely, graph homomorphisms from Σ_0 to X. Define the graph homomorphism $u : \Sigma_1 \to X$ by the correspondences $u(0) = a$ and $u(1) = b$. Then $d_0 u = b$ and $d_1 u = a$. Hence $\mathrm{Sing}(X)$ is connected.

Let $x_0 \in V(X)$. Define the graph homomorphism $r : X \times \Sigma_1$ by

$$r(x,i) = \begin{cases} x & (i = 0) \\ x_0 & (i = 1). \end{cases}$$

Since $\mathrm{Sing}(X \times \Sigma_1) \cong \mathrm{Sing}(X) \times \mathrm{Sing}(\Sigma_1)$ and $\mathrm{Sing}(\Sigma_1)$ is connected, we have that the identity of $|\mathrm{Sing}(X)|$ is homotopic to the constant map. Therefore $|\mathrm{Sing}(X)|$ is contractible. □

We are now ready to prove Proposition 7.6.

Let X be a graph and let σ be a clique of X. Define the map $\Phi_\sigma : \Delta_\sigma \to |C(G)|$ by the correspondence.

$$\Phi(a_0 e_0 + \cdots + a_n e_n) = a_0 e_{\sigma(0)} + \cdots + a_n e_{\sigma(n)}.$$

If $f : [m] \to [n]$ is an order-preserving map, then the diagram

$$\begin{tikzcd} \Delta_{f^*\sigma} \arrow[r, "f_*"] \arrow[d, "\Phi_{f^*\sigma}"'] & \Delta_\sigma \arrow[d, "\Phi_\sigma"] \\ {|C(G)|} \arrow[r, equal] & {|C(G)|} \end{tikzcd}$$

is commutative. Hence the maps Φ_σ inudces a continuous map $\Phi_G : |\mathrm{Sing}(G)| \to |C(G)|$. It is easy to see that Φ_G is natural.

Let $\sigma_0, \cdots, \sigma_n$ be finite cliques of G. We identify σ_i with the clique subgraph of G whose vertex set is σ_i. By Proposition 2.8, it suffices to show that the restriction

$$\Phi_{\sigma_0,\cdots,\sigma_n} |\mathrm{Sing}(\sigma_0)| \cap \cdots \cap |\mathrm{Sing}(\sigma_n)| \to |C(\sigma_0)| \cap \cdots \cap |C(\sigma_n)|$$

of Φ_G is a homotopy equivalence. Note that

$$|\mathrm{Sing}(\sigma_0)| \cap \cdots \cap |\mathrm{Sing}(\sigma_n)| = |\mathrm{Sing}(\sigma_0 \cap \cdots \cap \sigma_n)|$$

and

$$|C(\sigma_0)| \cap \cdots \cap |C(\sigma_n)| = |C(\sigma_0 \cap \cdots \cap \sigma_n)|.$$

Hence if $\sigma_0 \cap \cdots \cap \sigma_n = \emptyset$, then the source and range of $\Phi_{\sigma_0,\cdots,\sigma_n}$ are empty and hence $\Phi_{\sigma_0,\cdots,\sigma_n}$ is a homotopy equivalence. If $\sigma_0 \cap \cdots \cap \sigma_n \neq \emptyset$, then Lemma 7.7 implies that the source and range of $\Phi_{\sigma_0,\cdots,\sigma_n}$ are contractible and hence $\Phi_{\sigma_0,\cdots,\sigma_n}$ is a homotopy equivalence. This completes the proof of Proposition 7.6, and the proof of Theorem 7.1.

7.2. **Singular box complex.** Let G be a graph. The *singular box complex* $\mathcal{B}(G)$ *of* G is the $\mathbb{Z}_2$-simplicial set $\mathrm{Sing}(K_2, G)$. The $\mathbb{Z}_2$-action is induced by the flipping $\mathbb{Z}_2$-action on K_2.

Lemma 7.8. *The map $\Phi_{K_2} : |\mathcal{B}(G)| \to |B(G)|$ in Theorem 7.1 is a $\mathbb{Z}_2$-homotopy equivalence.*

Proof. The naturality of the map Φ in Theorem 7.1 implies that the map Φ is $\mathbb{Z}_2$-equivariant. By Proposition 6.31, it suffices to show that both Φ_{K_2} and $\Phi_{K_2}^{\mathbb{Z}_2} : |\mathcal{B}(G)|^{\mathbb{Z}_2} \to |B(G)|^{\mathbb{Z}_2}$ are homotopy equivalence. Theorem 7.1 asserts that Φ_{K_2} is a homotopy equivalence. On the other hand, consider the commutative diagram.

$$\begin{tikzcd} {|\mathrm{Sing}(\mathbf{1}, G)|} \arrow[r] \arrow[d, "\Phi_{\mathbf{1}}"'] & {|\mathrm{Sing}(K_2, G)|} \arrow[d, "\Phi_{K_2}"] \\ {|\mathrm{Hom}(\mathbf{1}, G)|} \arrow[r] & {|\mathrm{Hom}(K_2, G)|} \end{tikzcd}$$

whose horizontal arrows are induced by the graph homomorphism $K_2 \to \mathbf{1}$. The commutativity of the diagram follows from the naturality of Φ. It is easy to show that the map $\mathrm{Sing}(\mathbf{1}, G) \to \mathrm{Sing}(K_2, G)$ is an inclusion of simplicial sets and its image coincide with the fixed point subcomplex $\mathrm{Sing}(K_2, G)^{\mathbb{Z}_2}$. Similarly, the map $\mathrm{Hom}(\mathbf{1}, G) \to \mathrm{Hom}(K_2, G)$ is an inclusion and the image coincide with the fixed point subposet $\mathrm{Hom}(K_2, G)^{\mathbb{Z}_2}$. Since the geometric realization functor of simplicial sets and the classifying space functor of posets preserve equalizers, we have $|\mathrm{Sing}(K_2, G)|^{\mathbb{Z}_2} = |\mathrm{Sing}(K_2, G)^{\mathbb{Z}_2}|$ and $|\mathrm{Hom}(K_2, G)|^{\mathbb{Z}_2} = |\mathrm{Hom}(K_2, G)^{\mathbb{Z}_2}|$. Since $\Phi_{\mathbf{1}} : |\mathrm{Sing}(\mathbf{1}, G)| \to |\mathrm{Hom}(\mathbf{1}, G)|$ is a homotopy equivalence, we have that $\Phi_{K_2}^{\mathbb{Z}_2} : |\mathrm{Sing}(K_2, G)|^{\mathbb{Z}_2} \to |\mathrm{Hom}(K_2, G)|^{\mathbb{Z}_2}$ is a homotopy equivalence. □

Lemma 7.9. *The singular box complex functor $\mathcal{B} : \mathcal{G} \to \mathbf{SSet}^{\mathbb{Z}_2}$ has a left adjoint.*

Proof. Regard $\mathbb{Z}_2$ as a small category in a usual way. A $\mathbb{Z}_2$-simplicial set with a functor from $\mathbb{Z}_2$ to $\mathbf{SSet}$. Since a simplicial set is a functor from Δ^{op} to the category $\mathbf{Set}$ of small sets, a $\mathbb{Z}_2$-simplicial set is identified with a functor from $\Delta^{\mathrm{op}} \times \mathbb{Z}_2 = (\Delta \times \mathbb{Z}_2)^{\mathrm{op}}$ to the category $\mathbf{Set}$ of small sets. Define the functor $K_2 \times \Sigma_\bullet : \Delta \times \mathbb{Z}_2 \to \mathcal{G}$ by $[n] \mapsto K_2 \times \Sigma^n$. Since $\mathcal{B}(G) = \mathrm{Sing}(K_2, G) = \mathcal{G}(K_2 \times \Sigma_n, G)$, the functor $\mathcal{B}$ has the left adjoint (see the paragraph after Proposition 6.1). $\square$

We write $\mathcal{A}$ to indicate the singular box complex functor $\mathcal{B} : \mathcal{G} \to \mathbf{SSet}^{\mathbb{Z}_2}$. We shall describe the precise construction of $\mathcal{A}$. Let K be a $\mathbb{Z}_2$-simplicial set. The vertex set of $\mathcal{A}(K)$ is the set K_0 of 0-simplices of K. Let α denote the involution of K. Two vertices $x, y \in V(\mathcal{A}(K)) = K_0$ are adjacent if there is a 1-simplex connecting x with $\alpha(y)$.

If K is a simplicial complex, the same construction was obtained by Csorba in [7]. Precisely speaking, for an abstract simplicial complex K, he defines the graph G_K as follows: The vertex set of G_K is the vertex set of K. Two vertices x and y are adjacent if and only if $\{x, \alpha(y)\}$ is a 1-simplex of K. The $\mathbb{Z}_2$-action α on a simplicial complex K is *free* if $\alpha(\sigma) \cap \sigma = \emptyset$ for every $\sigma \in K$. Csorba showed the following:

Theorem 7.10 (Csorba [7]). *For a free $\mathbb{Z}_2$-simplicial complex K, there is a natural $\mathbb{Z}_2$-homotopy equivalence*

$$|B(G_{\mathrm{Sd}(K)})| \xrightarrow{\simeq} |K|.$$

As a corollary, he showed that for every free $\mathbb{Z}_2$-simplicial complex K, there is a graph G with $|B(G)| \simeq |K|$.

We conclude this section with the following remark. Recall that the inequality

$$\chi(G) \geq \mathrm{ind}(B(G)) + 2$$

holds for every graph G. We show that this is the maximal lower bound given by the $\mathbb{Z}_2$-homotopy type of the box complex. For the precise statement, see Corollary 7.12.

Proposition 7.11. *Let X be a finite $\mathbb{Z}_2$-CW-complex. Then there is a finite graph G with $B(G) \simeq_{\mathbb{Z}_2} X$ and $\chi(G) = \mathrm{ind}(B(G)) + 2$.*

Proof. Let K be a finite ordered $\mathbb{Z}_2$-simplicial complex with $|K| \simeq_{\mathbb{Z}_2} X$. We regard K as a $\mathbb{Z}_2$-simplicial set in the usual way. Set $n = \mathrm{ind}(X)$. Since $|\mathcal{B}(K_{n+2})| \simeq_{\mathbb{Z}_2} S^n$, there is a $\mathbb{Z}_2$-continuous map $|K| \to |\mathcal{B}(G)|$. Since $\mathbf{SSet}^{\mathbb{Z}_2}$ and $\mathbf{Top}^{\mathbb{Z}_2}$ are Quillen equivalent and $K \to \mathrm{Ex}^\infty(K)$ is a fibrant replacement in $\mathbf{SSet}^{\mathbb{Z}_2}$, we have that there is a $\mathbb{Z}_2$-simplicial map f from K to $\mathrm{Ex}^\infty(\mathcal{B}(K_{n+2}))$. Since K is finite, f factors through $\mathrm{Ex}^m(\mathcal{B}(K_{n+1}))$ for some $m \geq 1$. Since Ex is the right adjoint of the barycentric subdivision, there is a $\mathbb{Z}_2$-map from $\mathrm{Sd}^m(K)$ to $\mathcal{B}(K_{n+2})$. Therefore we have that there is a graph homomorphism

$$\mathcal{A}(\mathrm{Sd}^m(K)) \to K_{n+2}.$$

It follows from Theorem 7.10 that $|B \circ \mathcal{A} \circ \mathrm{Sd}^m(K)|$ and $|K|$ are $\mathbb{Z}_2$-homotopy equivalent. This completes the proof. $\square$

Corollary 7.12. *Let u be a $\mathbb{Z}_2$-homotopy invariant which assigns a $\mathbb{Z}_2$-space to an integer. Suppose that $\chi(B(G)) \geq u(G)$ for every finite graph G. Then we have that $\mathrm{ind}(X) + 2 \geq u(X)$ for every finite $\mathbb{Z}_2$-CW-complex X.*

We note that this corollary is obtained by combining Theorem 1.6 and Theorem 1.7 in Dochtermann and Schultz [11].

7.3. **Strong homotopy deformation retract.** In this section, we consider the deformation retract of finite simplicial complexes and finite posets in the sense of strong homotopy theory. We also consider the deformation retract of graphs in the sense of $\times$-homotopy theory. This construction is necessary to describe the cofibrations in the category of graphs.

Let $f : L \to L'$ be a simplicial map between simplicial complexes. Let K be a subcomplex of L. The *image $f(K)$ of f* is the subcomplex $\{f(\sigma) \mid \sigma \in K\}$ in L'.

A subcomplex K of a simplicial complex L is a *strong deformation retract* if there is a finite sequence $f_0, f_1, \cdots, f_k$ of simplicial maps from L to L which satisfies the following properties:

(1) $f_0 = \mathrm{id}_L$ and $f_k(L) \subset K$.
(2) $f_i(x) = x$ for every $x \in V(K)$ and $i = 0, 1, \cdots, k$.
(3) f_i and f_{i-1} are contiguous for $i = 1, \cdots, k$.

A typical example of strong deformation retracts of simplicial complexes is the deletion $K \setminus v$ of a cone point v (see Section 2.2). On the other hand, the following holds.

Proposition 7.13. *Let L be a finite simplicial complex, and let K be a subcomplex of L. Then K is a strong deformation retract of Q if and only if there is a linear order $\{x_1, \cdots, x_k\}$ on $V(L) \setminus V(K)$ such that x_i is a cone point of the simplicial complex $L \setminus \{x_1, \cdots, x_{i-1}\}$.*

Proof. The proof of the case $K = \emptyset$ is found in [5], and the general case is similarly proved. However, we write the proof for the reader's convenience.

The proof is obtained by the induction on the cardinality of $V(L) \setminus V(K)$. The case $\#(V(L) \setminus V(K)) = 0$ is trivial. Suppose $\#(V(L) \setminus V(K)) = n > 0$. Since K is a strong deformation retract of L and $K \neq L$, there is a simplicial map $f : L \to L$ such that $f|_K = \mathrm{id}_K$, $f \neq \mathrm{id}_L$, and f and id_L are contiguous. Let $x \in V(L)$ with $f(x) \neq x$. Note that $f|_K = \mathrm{id}_K$ implies $x \notin V(K)$.

Let σ be a simplex of L which contains x. Since id_L and f are contiguous, we have that $\sigma \cup f(\sigma)$ is a simplex of L. Therefore $\sigma \cup \{f(x)\}$ is a simplex of L. Thus x is a cone point of L. □

A subposet P of a poset Q is a *strong deformation retract* if there is a finite sequence $f_0, f_1, \cdots, f_k$ of order-preserving maps from Q to Q which satisfies the following properties:

(1) $f_0 = \mathrm{id}_Q$, and $f_k(Q)$ is contained in P.
(2) $f_i(x) = x$ for every $x \in P$ and $i = 0, 1, \cdots, k$.
(3) f_i and f_{i-1} are comparable in the poset $\mathrm{Poset}(Q, Q)$ (see Section 2 for the definition for the definition of $\mathrm{Poset}(Q, Q)$).

A typical example of strong deformation retracts of posets is the deletion $P \setminus x$ of a beat point x of P. On the other hand, the following holds.

Proposition 7.14. *Let P be a subposet of a finite poset Q. Then P is a strong deformation retract of Q if and only if there is a linear order $\{x_1, \cdots, x_k\}$ on $Q \setminus P$ such that x_i is a beat point of the poset $Q \setminus \{x_1, \cdots, x_{i-1}\}$ for $i = 1, \cdots, k$.*

Proof. The proof of the case $P = \emptyset$ is found in [4], and the general case is similarly proved. However, we write the proof for the reader's convenience.

The proof is obtained by the induction of the cardinality of $Q \setminus P$. The case $\#(Q \setminus P) = 0$ is obvious. Suppose that $\#(Q \setminus P) = n > 0$. It suffices to show that there is a beat point x of Q not contained in P. Since P is a strong deformation retract of Q and $P \neq Q$, there is an order-preserving map $f : Q \to Q$ such that $f > \mathrm{id}_Q$ or $f < \mathrm{id}_Q$. Suppose $f > \mathrm{id}_Q$. Let x_0 be a maximal element of the set $\{x \in Q \mid f(x) \neq x\}$. Since $f|_P = \mathrm{id}_P$, we have $x_0 \notin P$. We want to show that x_0 is an upper beat point. Let $y \in Q$ with $y > x_0$. The maximality of x_0 implies $f(y) = y$. Hence we have $y = f(y) \geq f(x_0) > x_0$. Therefore $f(x_0)$ is the minimum of $P_{>x_0}$. This completes the proof in the case $f > \mathrm{id}_Q$. If $f < \mathrm{id}_Q$, let x_1 be the minimal element of $\{x \in Q \mid f(x) \neq x\}$. In a similar way, we can show that x_1 is a lower beat point of Q. □

Proposition 7.15. *Let K be a subcomplex of L. If K is a strong deformation retract of L, then the face poset FK of K is a strong deformation retract of the face poset FL of L.*

Proof. Let $A(L, K)$ be the induced subposet of $\mathrm{Map}(L, L)$ (see Section 2) consisting of simplicial multi-maps η such that $\eta(x) = \{x\}$ for every $x \in V(K)$. Let $A_0(L, K)$ be the connected component of $A(L, K)$ containing the identity map id_L. Clearly, K is a strong deformation retract of L if and only if $A_0(L, K)$ contains a simplicial map $f : L \to L$ with $f(L) \subset K$.

Let P be an induced subposet of Q. Let $A(Q, P)$ be the induced subposet of $\mathrm{Poset}(Q, Q)$ consisting of order-preserving maps $f : Q \to Q$ with $f(x) = x$ for every $x \in P$. Let $A_0(Q, P)$ be the connected component of $A(Q, P)$ containing the identity map id_Q. Then P is a strong deformation retract of Q if and only if $A_0(L, K)$ contains an order-preserving map $f : Q \to Q$ with $f(Q) \subset P$.

The natural order-preserving map $F : \mathrm{Map}(L, L) \to \mathrm{Poset}(FL, FL)$ (see Section 2.3) maps $A(L, K)$ to $A(FL, FK)$. Suppose that there is $f \in A_0(L, K)$ with $f(L) \subset K$. Then $Ff(FL) \subset FK$ and $Ff \in A_0(FL, FK)$. This completes the proof. □

Proposition 7.16. *Let P be a subposet of a finite poset Q. If P is a strong deformation retract of Q, then $\Delta(P)$ is a strong deformation retract of $\Delta(Q)$.*

Proof. If x is a beat point of Q, then x is a cone point of $\Delta(P)$. Moreover, $\Delta(P \setminus x) = \Delta(P) \setminus x$. Hence the proposition follows from Proposition 7.14. □

Recall that the barycentric subdivision of a simplicial complex K is denoted by $\mathrm{Sd}(K)$. Since $\mathrm{Sd}(K) = \Delta(FK)$, we have the following.

Corollary 7.17. *Let L be a finite simplicial complex and let K be a subcomplex of L. If K is a strong deformation retract of L, then $\mathrm{Sd}(K)$ is a strong deformation retract of $\mathrm{Sd}(L)$.*

Next we consider the $\times$-homotopy deformation retract. An induced subgraph G of a graph H is a *$\times$-homotopy deformation retract* if there is a $\times$-homotopy $K : H \times I_k \to H$ such that $K(x, i) = x$ for every $x \in V(G)$ and $K(y, k) \in V(G)$ for every $y \in V(H)$.

A folding is a typical example of the $\times$-homotopy deformation retract.

Proposition 7.18. *Let G be a subgraph of a finite graph H. Then G is a $\times$-homotopy deformation retract of H if and only if there is a linear order $\{x_1, \cdots, x_n\}$ on $V(H) \setminus V(G)$ such that x_i is a dismantlable vertex of $H \setminus \{x_1, \cdots, x_{i-1}\}$.*

Proof. The proof of the case $G = \emptyset$ is found in [8], and the general case is similarly proved. However, we write the proof for the reader's convenience.

The proof is obtained by the induction on the cardinality of $V(H) \setminus V(G)$. The case $\#(V(H) \setminus V(G)) = 0$ is obvious. Suppose $\#(V(H) \setminus V(G)) = n > 0$. It suffices to show that there is a dismantlable vertex of H not contained in $V(G)$. Since G is a $\times$-deformation retract of H and $G \neq H$, there is $\eta \in \mathrm{Hom}(H, H)$ such that $\eta(v) = \{v\}$ for every $v \in V(K)$, $\eta > \mathrm{id}_H$ (see Lemma 2.22). Let $x \in V(G)$ with $\eta(x) \neq \{x\}$. By the assumption on η, we have that $x \notin V(G)$. Let $y \in \eta(x) \setminus \{x\}$. If $(x, x') \in E(H)$, then we have

$$(y, x') \in \eta(x) \times \eta(x') \subset E(H).$$

Therefore x is a dismantlable vertex. □

Let $\mathcal{A}_T$ be the left adjoint of the functor

$$\mathrm{Sing}(T, -) : \mathcal{G} \to \mathbf{SSet},$$

see Remark 7.2.

Let K be a simplicial complex. There is an ordering of $V(K)$ which satisfies $\Delta(V(K)) = K$. We write $N(V(K))$ to indicate the nerve of the poset $V(K)$. We write $\mathcal{A}_T(K)$ instead $\mathcal{A}_T(N(V(K)))$. The graph $\mathcal{A}_T(K)$ is independent of the choice of the ordering of $V(K)$.

Proposition 7.19. *Let K be a subcomplex of a finite simplicial complex L and let T be a finite graph. If K is a strong deformation retract of L, then $\mathcal{A}_T(K)$ is a $\times$-deformation retract of $\mathcal{A}_T(L)$.*

Proof. Consider the case $T = \mathbf{1}$. Then $\mathcal{A}_{\mathbf{1}}(K)$ is the graph whose vertex set is $V(K)$ and $x, y \in V(K)$ is adjacent in $\mathcal{A}_{\mathbf{1}}(K)$ if and only if $\{x, y\}$ is a simplex of K.

We show that if x is a cone point of K, then x is a dismantlable vertex of $\mathcal{A}_{\mathbf{1}}(K)$. In fact, suppose that x is a cone point. There is $y \in V(K) \setminus \{x\}$ such that $\sigma \in K$ and $x \in \sigma$ imply $\sigma \cup \{y\} \in K$. Let x' be a vertex of K and suppose that x and x' are adjacent in $\mathcal{A}_{\mathbf{1}}(K)$. Since $\{x, x'\}$ is a simplex of K, we have that $\{x, x', y\}$ is a simplex of K. Hence $\{x', y\}$ is a simplex of K. Thus we have that x' and y are adjacent in $\mathcal{A}_{\mathbf{1}}(K)$. This implies that x is dismantlable vertex of $\mathcal{A}_{\mathbf{1}}(K)$.

Thus it follows from Proposition 7.18 that $\mathcal{A}_{\mathbf{1}}(K)$ is a $\times$-deformation retract of $\mathcal{A}_{\mathbf{1}}(L)$. The general case follows from $\mathcal{A}_T(K) = T \times \mathcal{A}_{\mathbf{1}}(K)$. □

Recall that $\mathcal{A}$ denotes the left adjoint functor of the singular box complex $\mathcal{B} : \mathcal{G} \to \mathbf{SSet}^{\mathbb{Z}_2}$. For the definition of $\mathcal{B}$, see Section 7.2.

Corollary 7.20. *Let L be a finite simplicial complex and let K be a subcomplex of K. Suppose that K is a strong deformation retract of L. Then the following hold.*

1. *Regard K and L as $\mathbb{Z}_2$-simplicial complexes by the trivial $\mathbb{Z}_2$-actions. Then we have that $\mathcal{A}(K)$ is a $\times$-deformation retract of $\mathcal{A}(L)$.*
2. *$\mathcal{A}(\mathbb{Z}_2 \times K)$ is a $\times$-homotopy deformation retract of $\mathcal{A}(\mathbb{Z}_2 \times L)$.*

Proof. Let K be a simplicial complex. Consider K as the $\mathbb{Z}_2$-simplicial complex whose $\mathbb{Z}_2$-action is trivial. In this case we have $\mathcal{A}(K) \cong \mathcal{A}_{\mathbf{1}}(K)$. (Consider the construction of $\mathcal{A}$ and $\mathcal{A}_{\mathbf{1}}$.) Therefore (1) follows from Proposition 7.19 in the case $T = \mathbf{1}$. On the other hand, since $\mathcal{A}(\mathbb{Z}_2 \times K) \cong \mathcal{A}_{K_2}(K)$, (2) follows from Proposition 7.19 in the case $T = K_2$. □

7.4. **r-NDR.** Let $f : G \to H$ be a graph homomorphism. Consider the pushout diagram such as

$$\begin{array}{ccc} G & \xrightarrow{f} & H \\ \big\downarrow & & \big\downarrow \\ X & \longrightarrow & Y. \end{array}$$

This induces a commutative diagram

$$\begin{array}{ccc} B(G) & \xrightarrow{f_*} & B(H) \\ \big\downarrow & & \big\downarrow \\ B(X) & \longrightarrow & B(Y). \end{array}$$

In general, this square is not homotopy cocartesian even if the map f is an inclusion. However, if f is a 2-NDR mentioned below, then the diagram is homotopy cocartesian.

Let K be a subcomplex of a simplicial complex L. Set

$$V(\mathcal{N}(K)) = \{v \in V(L) \mid \text{There is } w \in V(K) \text{ with } \{v, w\} \in L.\}.$$

Define $\mathcal{N}(K)$ of H to be the subcomplex induced by $V(\mathcal{N}(K))$, that is, $\sigma \subset V(\mathcal{N}(K))$ is a simplex of $\mathcal{N}(K)$ if and only if $\sigma \in L$. The *r-neighborhood $\mathcal{N}_r(K)$ of K* is defined inductively by

$$\mathcal{N}_1(K) = \mathcal{N}(K),\ \mathcal{N}_{r+1}(K) = \mathcal{N}(\mathcal{N}_r(K)).$$

Definition 7.21. An inclusion $K \hookrightarrow L$ is an *r-NDR* if there is a subcomplex A containing $\mathcal{N}_r(K)$ such that K is a strong deformation retract of A.

A simplicial map $f : K \to L$ is an *r-NDR* if f is an injection as a set map from $V(K)$ to $V(L)$ and the inclusion $f(K) \hookrightarrow L$ is an r-NDR.

Recall that the barycentric subdivision of a simplicial complex K is denoted by $\mathrm{Sd}(K)$. Let Δ^n denote the abstract simplicial complex $([n], 2^{[n]})$.

Proposition 7.22. *Let K be a subcomplex of Δ^n. Then the inclusion $\mathrm{Sd}^2(K) \hookrightarrow \mathrm{Sd}^2(\Delta^n)$ is a 1-NDR.*

Proof. Note that $\mathrm{Sd}(\Delta^n)$ is the simplicial complex whose simplices are chains of $2^{[n]} \setminus \{\emptyset\}$. Hence $F\mathrm{Sd}(\Delta^n)$ is the poset of non-empty chains of $2^{[n]} \setminus \{\emptyset\}$. Set

$$X = \{\sigma \subset [n] \mid \sigma \text{ is not a simplex of } K.\}.$$

Then we have that

$$F\mathrm{Sd}(K) = \{\alpha \in F\mathrm{Sd}(\Delta^n) \mid \alpha \cap X = \emptyset\}.$$

Define P to be the induced subposet

$$P = \{\alpha \in F\mathrm{Sd}(\Delta^n) \mid \alpha \not\subset X\}$$

of $F\mathrm{Sd}(\Delta^n)$. Then $\Delta(P)$ is the 1-neighborhood of $\mathrm{Sd}^2(K)$ in $\mathrm{Sd}^2(\Delta^n)$. By Proposition 7.16, it suffices to show that $F\mathrm{Sd}(K)$ is a strong deformation retract of P.

For $i = 2, 3, \cdots, n+1$, set

$$B_i = \{c \in P \mid \#c = i, c \cap X \neq \emptyset\}.$$

Define the induced subposet P_i of P inductively by $P_0 = P$, $P_i = P_{i-1} \setminus B_{i+1}$ for $i = 1, \cdots, n$. It is easy to see that B_{i+2} is a set of lower beat points of P_i. Hence we have that P_{i+1} is a strong deformation retract of P_i. Therefore we have

$$P_n = P \setminus (B_2 \cup \cdots \cup B_{n+1}) = \{c \in P \mid c \cap X = \emptyset\} = F\mathrm{Sd}(K)$$

is a strong deformation retract of P. This completes the proof. □

Lemma 7.23. *If $f : K \to L$ is an r-NDR, then $\mathrm{Sd}(f) : \mathrm{Sd}(K) \to \mathrm{Sd}(L)$ is a $(2r)$-NDR.*

Combining Proposition 7.22 and Lemma 7.23, we have the following.

Corollary 7.24. *Let K be a subcomplex of Δ^n. For a positive integer r, the inclusion $\mathrm{Sd}^{r+1}(K) \hookrightarrow \mathrm{Sd}^{r+1}(\Delta^n)$ is an r-NDR.*

Let G be a subgraph of H. Define the subgraph $\mathcal{N}(G)$ of H as follows. A vertex of $\mathcal{N}(G)$ is a vertex of H adjacent to some vertex of G. Two vertices v and w of $\mathcal{N}(G)$ are adjacent if and only if either v or w belongs to $V(G)$. Then $\mathcal{N}_r(G)$ is defined inductively by

$$\mathcal{N}_1(G) = \mathcal{N}(G),\ \mathcal{N}_{r+1}(G) = \mathcal{N}(\mathcal{N}_r(G)).$$

Definition 7.25. Let G be a subgraph of a graph H. Then the inclusion $G \hookrightarrow H$ is an *r-NDR* if there is a subgraph A of H containing $\mathcal{N}_r(G)$ such that G is a $\times$-deformation retract of A.

A graph homomorphism $f : G \to H$ is an *r-NDR* if it is injective as a set map between vertex sets, and the inclusion $f(G) \hookrightarrow H$ is an r-NDR.

The following proposition is the main result in this section.

Proposition 7.26. *Consider a pushout square*

$$\begin{array}{ccc} G & \xrightarrow{f} & H \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle j} \\ X & \xrightarrow{i} & Y, \end{array}$$

in the category of graphs. Suppose that f is a 2-NDR. Then the square

$$\begin{array}{ccc} |B(G)| & \longrightarrow & |B(H)| \\ \downarrow & & \downarrow \\ |B(X)| & \longrightarrow & |B(Y)| \end{array}$$

is homotopy cocartesian in the category of $\mathbb{Z}_2$-spaces. In other words, the natural map

$$|B(H)| \cup_{|B(G)|} |B(X)| \to |B(Y)|$$

is a $\mathbb{Z}_2$-homotopy equivalence.

Proof. Throughout the proof, we assume that the notation $B(G)$ indicates the classifying space of the box complex of G, for simplicity.

Without loss of generality, we can assume that G is a subgraph of H and f is an inclusion. Let A be a subgraph of H such that G is a $\times$-homotopy deformation retract of A. Let A_0 denote the subgraph $X \cup_G A$ in H. Let $H \setminus G$ denote the maximal subgraph of G whose vertex set is $V(H) \setminus V(G)$. We also define $A \setminus G$ and $Y \setminus X$ in a similar way. Note that every multi-homomorphism from K_2 to Y factors through either A_0 or $Y \setminus X \cong H \setminus G$. Hence we have

$$B(Y) = B(A_0) \cup B(H \setminus G).$$

Define Y' to be the colimit of the diagram

$$X \xleftarrow{\ g\ } G \xrightarrow{\ \iota_0\ } G \times I_1 \xleftarrow{\ \iota_1\ } G \longrightarrow H.$$

Here $\iota_k : G \to G \times I_1$ $(k = 0, 1)$ is the injection $v \mapsto (v, k)$. Let A' be the subgraph $X \cup_G (G \times I_1) \cup_G A$ in Y'. Then X is a $\times$-deformation retract of A'_0. Attaching the graph homomorphisms

$$i : X \to Y,\ j : H \to Y,$$

and the composition of $G \times I_1 \to G \to Y$, we have a graph homomorphism $F : Y' \to Y$.

We show that $F_* : B(Y') \to B(Y)$ is a $\mathbb{Z}_2$-homotopy equivalence. Note that

$$B(Y') = B(A'_0) \cup B(H \setminus G)$$

and

$$B(Y) = B(A_0) \cup B(H \setminus G).$$

The graph homomorphism F induces isomorphisms $B(H \setminus G) \to B(H \setminus G)$ and $B(A'_0) \cap B(H \setminus G) = B(A \setminus G) \to B(A_0) \cap B(H \setminus G) = B(A \setminus G)$. The commutative diagram

$$\begin{array}{ccc} B(A'_0) & \xrightarrow{F|_{A'_0}} & B(A_0) \\ \simeq\uparrow & & \uparrow\simeq \\ B(X) & = & B(X) \end{array}$$

shows that $F|_{A'_0 *} : B(A'_0) \to B(A_0)$ is a $\mathbb{Z}_2$-homotopy equivalence. It follows from the gluing lemma for $\mathbb{Z}_2$-spaces (see Proposition 6.18 and Proposition 6.21) that $B(Y') \to B(Y)$ is a $\mathbb{Z}_2$-homotopy equivalence.

Next let X' denote the subgraph $X \cup_G (G \times I_1)$ of Y'. The commutative diagram

$$\begin{array}{ccc} B(G) & \xrightarrow{f_*} & B(H) \\ \iota_1\downarrow & & \downarrow \\ B(X') & \longrightarrow & B(Y') \end{array}$$

induces a map $B(X') \cup_{B(G)} B(H) \to B(Y')$. Next we show that this map is a $\mathbb{Z}_2$-homotopy equivalence. Note that $B(Y') = B(A') \cup B(H)$ and $B(A') \cap B(H) = B(A)$. By the gluing lemma for $\mathbb{Z}_2$-spaces, it suffices to show that all of the vertical arrows of the diagram

$$\begin{tikzcd} B(X') \arrow[d] & B(G) \arrow[l] \arrow[r] \arrow[d] & B(H) \arrow[d, equal] \\ B(A') & B(A) \arrow[l] \arrow[r] & B(H) \end{tikzcd}$$

are $\mathbb{Z}_2$-homotopy equivalences. However, this is clear since G (or X') is a $\times$-deformation retract of A (or A', respectively). Hence $B(X') \cup_{B(G)} B(H) \to B(Y')$ is a $\mathbb{Z}_2$-homotopy equivalence. Then $F : Y' \to Y$ defines the commutative diagram

$$\begin{tikzcd} B(X') \cup_{B(G)} B(H) \arrow[r] \arrow[d] & B(Y') \arrow[d, "F_*"] \\ B(X) \cup_{B(G)} B(H) \arrow[r] & B(Y) \end{tikzcd}$$

We have already proved that the right vertical arrow and the upper horizontal arrow are $\mathbb{Z}_2$-homotopy equivalences. It follows from the gluing lemma for $\mathbb{Z}_2$-spaces that the left vertical arrow is a $\mathbb{Z}_2$-homotopy equivalence. Therefore the lower horizontal arrow is a $\mathbb{Z}_2$-homotopy equivalence. This completes the proof. $\square$

In a similar way, we can show the following proposition. Since we will not need it, we omit the proof.

Proposition 7.27. *Let T be a connected graph and suppose that the diameter is smaller than r. Consider a pushout square*

$$\begin{tikzcd} G \arrow[r, "f"] \arrow[d, "g"'] & H \arrow[d, "j"] \\ X \arrow[r, "i"] & Y \end{tikzcd}$$

such that f is an r-NDR. Then the square

$$\begin{tikzcd} {|\mathrm{Hom}(T,G)|} \arrow[r] \arrow[d] & {|\mathrm{Hom}(T,H)|} \arrow[d] \\ {|\mathrm{Hom}(T,X)|} \arrow[r] & {|\mathrm{Hom}(T,Y)|} \end{tikzcd}$$

is homotopy cocartesian. In other words, the natural map

$$|\mathrm{Hom}(T,H)| \cup_{|\mathrm{Hom}(T,G)|} |\mathrm{Hom}(T,X)| \to |\mathrm{Hom}(T,Y)|$$

is a homotopy equivalence.

7.5. **Model structure.** First we recall the notation of Section 7.3. We set

$$I = \{\partial\Delta[n] \hookrightarrow \Delta[n] \mid n \geq 0\},$$

$$\mathbb{Z}_2 \times I = \{\mathbb{Z}_2 \times \partial\Delta[n] \hookrightarrow \mathbb{Z}_2 \times \Delta[n] \mid n \geq 0\},$$

$$J = \{\Lambda_r[n] \hookrightarrow \Delta[n] \mid n \geq 1,\ 0 \leq r \leq n\},$$

$$\mathbb{Z}_2 \times J = \{\mathbb{Z}_2 \times \Lambda_r[n] \hookrightarrow \mathbb{Z}_2 \times \Delta[n] \mid n \geq 1,\ 0 \leq r \leq n\},$$

$I' = I \cup (\mathbb{Z}_2 \times I)$, and $J' = J \cup (\mathbb{Z}_2 \times J)$. The purpose of this section is to prove the following theorem.

Theorem 7.28. *The category $\mathcal{G}$ of graphs has a cofibrantly generated model structure with generating cofibrations $\mathcal{A} \circ \mathrm{Sd}^3(I')$ and generating trivial cofibrations $\mathcal{A} \circ \mathrm{Sd}^3(J')$. A graph homomorphism $f : G \to H$ is a weak equivalence of this model structure if and only if the $\mathbb{Z}_2$-map $f_* : B(G) \to B(H)$ is a $\mathbb{Z}_2$-homotopy equivalence for every f. Moreover, the adjoint pair*

$$\mathcal{A} \circ \mathrm{Sd}^3 : \mathbf{SSet}^{\mathbb{Z}_2} \to \mathcal{G} : \mathrm{Ex}^3 \circ \mathcal{B}$$

is a Quillen equivalence.

We first show that the graph $\mathcal{G}$ has a cofibrantly generated model structure with generating cofibrations $\mathcal{A} \circ \mathrm{Sd}^3(I')$ and generating trivial cofibrations $\mathcal{A} \circ \mathrm{Sd}^3(J')$.

Let $f : G \to H$ be a graph homomorphism. Then $f_* : B(G) \to B(H)$ is a $\mathbb{Z}_2$-homotoopy equivalence if and only if the map

$$\mathrm{Ex}^3 \circ \mathcal{B}(f) : \mathrm{Ex}^3 \circ \mathcal{B}(G) \to \mathrm{Ex}^3 \circ \mathcal{B}(H)$$

is a $\mathbb{Z}_2$-homotopy equivalence. This follows from Lemma 6.36 and Lemma 7.8.

Hence it suffices to check the hypothesis of Theorem 6.15. Since every object of $\mathcal{G}$ is small, both $\mathcal{A} \circ \mathrm{Sd}^3(I')$ and $\mathcal{A} \circ \mathrm{Sd}^3(J')$ permit the small object argument. Therefore we want to show that if $f : G \to H$ is a J'-cell complex, then $\mathrm{Ex}^3 \circ \mathcal{B}(f)$ is a weak equivalence. By Lemma 6.36, it suffices to show that $f_* : \mathcal{B}(G) \to \mathcal{B}(H)$ is a $\mathbb{Z}_2$-homotopy equivalence.

Proposition 7.29. *Let $f : G \to H$ be a graph homomorphism. If f is a pushout of an element of $\mathcal{A} \circ \mathrm{Sd}^3(J)$, then $\mathcal{B}(f)$ is a $\mathbb{Z}_2$-weak equivalence.*

Recall $\mathcal{B}(G)_n = \mathcal{G}(K_2 \times \Sigma_n, G)$. Hence if $f : G \to H$ is an inclusion, then $f_* : \mathcal{B}(G) \to \mathcal{B}(H)$ is also an inclusion. Recall that a $\mathcal{A} \circ \mathrm{Sd}^3(J')$-cell complex is a transfinite composition of pushtouts of elements of $\mathcal{A} \circ \mathrm{Sd}^3(J')$. Therefore Proposition 7.29 implies that if $f : G \to H$ is an $\mathcal{A} \circ \mathrm{Sd}^3(J)$-cell complex, then $f_* : \mathcal{B}(G) \to \mathcal{B}(H)$ is a $\mathbb{Z}_2$-homotopy equivalence. Thus it suffices to show Proposition 7.29.

Recall that Δ^n denotes the simplicial complex $([n], 2^{[n]})$. Define the subcomplex Λ^n_r of Δ^n whose simplex is a subset σ of $[n]$ such that $\sigma \cup \{r\} \neq [n]$. Suppose $n \geq 1$. Let $*$ be the simplicial complex consisting of one vertex $*$. Then we have the map $i_r : * \to \Lambda^n_r$, $* \mapsto n$. We also denote by i_r the inclusion $* \to \Delta^n$, $* \mapsto r$. By the following lemma, we have that the inclusions $i_r : * \hookrightarrow \Delta^n$ and $i_r : * \hookrightarrow \Lambda^n_r$ are strong deformation retracts.

Lemma 7.30. *Let K be a finite simplicial complex. Let CK be the join of $*$ and K. Then the inclusion $* \hookrightarrow CK$ is a strong deformation retract.*

Proof. This is proved by the induction on the cardinality of $V(K)$. The case $V(K) = \emptyset$ is obvious. Suppose that $\#V(K) > 0$ and let $x \in V(K)$. Let σ be a simplex of CK which contains x. If σ contains $*$, then clearly $\sigma \cup \{*\} \in CK$. If σ does not contain $*$, then $\sigma \cup \{*\}$ is a simplex of CK by the definition of join. Hence x is a cone point of CK and hence $(CK) \setminus x$ is a strong deformation retract of CK. By the induction hypothesis, we have that $*$ is a strong deformation retract of $(CK) \setminus x = C(K \setminus x)$. □

Corollary 7.31. *For every $n \geq 1$ and $0 \leq r \leq n$, both the maps $\mathrm{Sd}^3(i_r) : * \to \mathrm{Sd}^3(\Lambda^n_r)$ and $\mathrm{Sd}^3(i_r) : * \hookrightarrow \mathrm{Sd}^3(\Delta^n)$ are strong deformation retracts.*

By Corollary 7.20, we have the following.

Corollary 7.32. *Let n be a positive integer and let r be an integer with $0 \leq r \leq n$. Then the following hold.*

(1) *The inclusions*

$$\mathcal{A} \circ \mathrm{Sd}^3(i_r) : \mathcal{A} \circ \mathrm{Sd}^3(*) \hookrightarrow \mathcal{A} \circ \mathrm{Sd}^3(\Delta^n)$$

and

$$\mathcal{A} \circ \mathrm{Sd}^3(\mathbb{Z}_2 \times i_r) : \mathcal{A} \circ \mathrm{Sd}^3(\mathbb{Z}_2) \hookrightarrow \mathcal{A} \circ \mathrm{Sd}^3(\mathbb{Z}_2 \times \Delta^n)$$

are $\times$-homotopy equivalences.

(2) *The inclusions*

$$\mathcal{A} \circ \mathrm{Sd}^3(i_r) : \mathcal{A} \circ \mathrm{Sd}^3(*) \hookrightarrow \mathcal{A} \circ \mathrm{Sd}^3(\Lambda^n_r)$$

and

$$\mathcal{A} \circ \mathrm{Sd}^3(\mathbb{Z}_2 \times i_r) : \mathcal{A} \circ \mathrm{Sd}^3(\mathbb{Z}_2) \hookrightarrow \mathcal{A} \circ \mathrm{Sd}^3(\mathbb{Z}_2 \times \Lambda^n_r)$$

are $\times$*-homotopy equivalences.*

Lemma 7.33. *Let n be a positive integer and let r be an integer with $0 \le r \le n$. Then the inclusions*

$$j_r : \mathcal{A} \circ \mathrm{Sd}^3(\Lambda^n_r) \hookrightarrow \mathcal{A} \circ \mathrm{Sd}^3(\Delta^n)$$

and

$$K_2 \times j_r : \mathcal{A} \circ \mathrm{Sd}^3(\mathbb{Z}_2 \times \Lambda^n_r) \hookrightarrow \mathcal{A} \circ \mathrm{Sd}^3(\mathbb{Z}_2 \times \Delta^n)$$

are weak equivalences in $\mathcal{G}$.

Proof. Consider the sequence

$$\mathbf{1} = \mathcal{A} \circ \mathrm{Sd}^3(*) \longrightarrow \mathcal{A} \circ \mathrm{Sd}^3(\Lambda^n_r) \xrightarrow{j_r} \mathcal{A} \circ \mathrm{Sd}^3(\Delta^n).$$

By Corollary 7.32, the first arrow and the composition of the sequence are $\times$-homotopy equivalences, and are weak equivalences. By the two out of three axiom, we have that j_r is a weak equivalence of $\mathcal{G}$. $\square$

We are now ready to prove Proposition 7.29. Let n be a positive integer and let r be an integer with $0 \le r \le n$. We first note that the graph homomorphism $\mathcal{A} \circ \mathrm{Sd}^3(\Lambda_r[n]) \to \mathcal{A} \circ \mathrm{Sd}^3(\Delta[n])$ is isomorphic to the graph homomorphism $\mathcal{A} \circ \mathrm{Sd}^3(\Lambda^n_r) \hookrightarrow \mathcal{A} \circ \mathrm{Sd}^3(\Delta^n)$. Consider the pushout diagram

$$\begin{tikzcd} \mathcal{A} \circ \mathrm{Sd}^3(\Lambda^n_r) \arrow[r] \arrow[d] & \mathcal{A} \circ \mathrm{Sd}^3(\Delta^n) \arrow[d] \\ G \arrow[r] & H \end{tikzcd}$$

in $\mathcal{G}$. By Corollary 7.24, we have that the inclusion $\mathrm{Sd}^3(\Lambda^r_n) \hookrightarrow \mathrm{Sd}^3(\Delta^n)$ is a 2-NDR. Hence the upper horizontal arrow of the above diagram is a 2-NDR. It follows from Proposition 7.26 that the natural map

$$B(G) \cup_{B(\mathcal{A} \circ \mathrm{Sd}^3(\Lambda^n_r))} B(\mathcal{A} \circ \mathrm{Sd}^3(\Delta^n)) \to B(H)$$

is a $\mathbb{Z}_2$-homotopy equivalence. Note that the map $B(\mathcal{A} \circ \mathrm{Sd}^3(\Lambda^n_r)) \to B(\mathcal{A} \circ \mathrm{Sd}^3(\Delta^n))$ is an inclusion of $\mathbb{Z}_2$-CW-complexes, which is a $\mathbb{Z}_2$-homotopy equivalence. It follows from the gluing lemma for $\mathbb{Z}_2$-spaces (see Proposition 6.18 and Proposition 6.21) that the inclusion

$$B(G) \to B(G) \cup_{B(\mathcal{A} \circ \mathrm{Sd}^3(\Lambda^n_r))} B(\mathcal{A} \circ \mathrm{Sd}^3(\Delta^n))$$

is a $\mathbb{Z}_2$-homotopy equivalence. Hence the inclusion $B(G) \hookrightarrow B(H)$ is a $\mathbb{Z}_2$-homotopy equivalence. This completes the proof of Proposition 7.29.

Next we show that the adjoint pair $\mathcal{A} \circ \mathrm{Sd}^3 : \mathbf{SSet}^{\mathbb{Z}_2} \to \mathcal{G} : \mathrm{Ex}^3 \circ \mathcal{B}$ is a Quillen equivalence. We want to show that for every $\mathbb{Z}_2$-simplicial set K, the map

$$K \to \mathrm{Ex}^3 \circ \mathcal{B} \circ \mathcal{A} \circ \mathrm{Sd}^3(K)$$

is a $\mathbb{Z}_2$-homotopy equivalence. Hence it suffices to show the following.

Proposition 7.34. *Let K be a $\mathbb{Z}_2$-simplicial set. The counit map*

$$\mathrm{Sd}^3(K) \to \mathcal{B} \circ \mathcal{A}(\mathrm{Sd}^3(K))$$

is a $\mathbb{Z}_2$-homotopy equivalence.

Lemma 7.35. $\mathrm{Sd}^3(\partial\Delta^n) \to \mathcal{B} \circ \mathcal{A}(\mathrm{Sd}^3(\partial\Delta^n))$ *and* $\mathrm{Sd}^3(\mathbb{Z}_2 \times \Delta^n) \to \mathcal{B} \circ \mathcal{A}(\mathrm{Sd}^3(\mathbb{Z}_2 \times \Delta^n))$ *are $\mathbb{Z}_2$-homotopy equivalences.*

Proof. Since the latter is simialarly proved, we only prove the former.

We prove this by the induction on n. The case $n = 0$ is clear. Suppose $n > 0$. Then we have $\partial\Delta^n = \Delta^{n-1} \cup \Lambda^n_n$ and $\Delta^{n-1} \cap \Lambda^n_n = \partial\Delta^{n-1}$. Thus we have the pushout diagram

$$\begin{tikzcd} \mathcal{A}(\mathrm{Sd}^3(\partial\Delta^{n-1})) \arrow[r] \arrow[d] & \mathcal{A}(\mathrm{Sd}^3(\Lambda^n_n)) \arrow[d] \\ \mathcal{A}(\mathrm{Sd}^3(\Delta^{n-1})) \arrow[r] & \mathcal{A}(\mathrm{Sd}^3(\partial\Delta^n)). \end{tikzcd}$$

Since the upper horizontal arrow is a 2-NDR, we have that the map

$$\mathcal{B} \circ \mathcal{A} \circ \mathrm{Sd}^3(\Lambda^n_n) \cup_{\mathcal{B}\circ\mathcal{A}\circ\mathrm{Sd}^3(\partial\Delta^{n-1})} \mathcal{B} \circ \mathcal{A} \circ \mathrm{Sd}^3(\Delta^{n-1}) \to \mathcal{B} \circ \mathcal{A} \circ \mathrm{Sd}^3(\partial\Delta^n)$$

is a $\mathbb{Z}_2$-homotopy equivalence. Next consider the diagram

$$\begin{tikzcd} \mathrm{Sd}^3(\Delta^{n-1}) \arrow[d] & \mathrm{Sd}^3(\partial\Delta^{n-1}) \arrow[l] \arrow[r] \arrow[d] & \mathrm{Sd}^3(\Lambda^n_n) \arrow[d] \\ \mathcal{B} \circ \mathcal{A} \circ \mathrm{Sd}^3(\Delta^{n-1}) & \mathcal{B} \circ \mathcal{A} \circ \mathrm{Sd}^3(\partial\Delta^{n-1}) \arrow[l] \arrow[r] & \mathcal{B} \circ \mathcal{A} \circ \mathrm{Sd}^3(\Lambda^n_n). \end{tikzcd}$$

By the gluing lemma for $\mathbb{Z}_2$-spaces, we have that the map

$$\mathrm{Sd}^3(\partial\Delta^n) \to \mathcal{B} \circ \mathcal{A} \circ \mathrm{Sd}^3(\Lambda^n_n) \cup_{\mathcal{B}\circ\mathcal{A}\circ\mathrm{Sd}^3(\partial\Delta^{n-1})} \mathcal{B} \circ \mathcal{A} \circ \mathrm{Sd}^3(\Delta^{n-1})$$

is a $\mathbb{Z}_2$-homotopy equivalence. Hence we have that the map

$$\mathrm{Sd}^3(\partial\Delta^n) \to \mathcal{B} \circ \mathcal{A} \circ \mathrm{Sd}^3(\partial\Delta^n)$$

is a $\mathbb{Z}_2$-homotopy equivalence. □

Lemma 7.36. *Let K be a $\mathbb{Z}_2$-simplicial set and let $K \hookrightarrow L$ be a pushout of an element of I' (see the beginning of this section). Suppose that the counit map*

$$\mathrm{Sd}^3(K) \to \mathcal{B} \circ \mathcal{A} \circ \mathrm{Sd}^3(K)$$

is a $\mathbb{Z}_2$-homotopy equivalence, then the counit map

$$\mathrm{Sd}^3(L) \to \mathcal{B} \circ \mathcal{A} \circ \mathrm{Sd}^3(L)$$

is a $\mathbb{Z}_2$-homotopy equivalence.

Proof. Suppose that the inclusion $K \hookrightarrow L$ is a pushout of the inclusion $\partial\Delta[n] \hookrightarrow \Delta[n]$. Then we have a pushout diagram

$$\begin{tikzcd} \mathcal{A} \circ \mathrm{Sd}^3(\partial\Delta[n]) \arrow[r] \arrow[d] & \mathcal{A} \circ \mathrm{Sd}^3(\Delta[n]) \arrow[d] \\ \mathcal{A} \circ \mathrm{Sd}^3(K) \arrow[r] & \mathcal{A} \circ \mathrm{Sd}^3(L). \end{tikzcd}$$

Since the upper horizontal arrow is a 2-NDR, we have that the map

$$\mathcal{B} \circ \mathcal{A} \circ \mathrm{Sd}^3(K) \cup_{\mathcal{B}\circ\mathcal{A}\circ\mathrm{Sd}^3(\partial\Delta[n])} \mathcal{B} \circ \mathcal{A} \circ \mathrm{Sd}^3(\Delta[n]) \to \mathcal{B} \circ \mathcal{A} \circ \mathrm{Sd}^3(L)$$

is a $\mathbb{Z}_2$-homotopy equivalence. Consider the commutative diagram

$$\begin{tikzcd} \mathrm{Sd}^3(K) \arrow[d] & \mathrm{Sd}^3(\partial\Delta[n]) \arrow[l] \arrow[r] \arrow[d] & \mathrm{Sd}^3(\Delta[n]) \arrow[d] \\ \mathcal{B} \circ \mathcal{A} \circ \mathrm{Sd}^3(K) & \mathcal{B} \circ \mathcal{A} \circ \mathrm{Sd}^3(\partial\Delta[n]) \arrow[l] \arrow[r] & \mathcal{B} \circ \mathcal{A} \circ \mathrm{Sd}^3(\Delta[n]). \end{tikzcd}$$

By the gluing lemma for $\mathbb{Z}_2$-spaces, we have that the map

$$\mathrm{Sd}^3(L) \to \mathcal{B} \circ \mathcal{A} \circ \mathrm{Sd}^3(K) \cup_{\mathcal{B}\circ\mathcal{A}\circ\mathrm{Sd}^3(\partial\Delta[n])} \mathcal{B} \circ \mathcal{A} \circ \mathrm{Sd}^3(\Delta[n])$$

is a $\mathbb{Z}_2$-homotopy equivalence. Hence we have that the map

$$\mathrm{Sd}^3(L) \to \mathcal{B} \circ \mathcal{A} \circ \mathrm{Sd}^3(L)$$

is a $\mathbb{Z}_2$-homotopy equivalence.

The case that the inclusion $K \hookrightarrow L$ is a pushout of the inclusion $\mathbb{Z}_2 \times \partial\Delta[n] \hookrightarrow \mathbb{Z}_2 \times \Delta[n]$ is similarly proved. □

We are now ready to prove Proposition 7.34.

Let K be a $\mathbb{Z}_2$-simplicial set. Let λ be an ordinal and let $X_\bullet : \lambda \to \mathbf{SSet}^{\mathbb{Z}_2}$ be a λ-sequence such that $X_\alpha \to X_{\alpha+1}$ is a pushout of an element of I' for every $\alpha < \lambda$ and $\mathrm{colim}(X_\bullet) \cong K$. By Lemma 7.35, we have that the map $\mathrm{Sd}^3(X_0) \to \mathcal{B} \circ \mathcal{A} \circ \mathrm{Sd}^3(X_0)$ is a $\mathbb{Z}_2$-homotopy equivalence. Let $\alpha < \lambda$ and suppose that for every β with $\beta < \alpha$, the map

$$\mathrm{Sd}^3(X_\beta) \to \mathcal{B} \circ \mathcal{A} \circ \mathrm{Sd}^3(X_\beta)$$

is a $\mathbb{Z}_2$-homotopy equivalence.

Suppose that $\alpha - 1$ does not exist. Since the singular box complex functor $\mathcal{B}$ commutes with the transfinite composition, we have that

$$\mathrm{colim}_{\beta<\alpha} \mathcal{B} \circ \mathcal{A} \circ \mathrm{Sd}^3(X_\beta) \cong \mathcal{B} \circ \mathcal{A} \circ \mathrm{Sd}^3(X_\alpha).$$

Hence Proposition 6.19 implies that $\mathrm{Sd}^3(X_\alpha) \to \mathcal{B} \circ \mathcal{A} \circ \mathrm{Sd}^3(X_\alpha)$ is a $\mathbb{Z}_2$-homotopy equivalence.

Next suppose that $\alpha - 1$ exists. It follows from Lemma 7.35 that $\mathrm{Sd}^3(X_\alpha) \to \mathcal{B} \circ \mathcal{A} \circ \mathrm{Sd}^3(X_\alpha)$ is a $\mathbb{Z}_2$-homotopy equivalence.

By the transfinite induction, we have that the map

$$\mathrm{Sd}^3(K) \to \mathcal{B} \circ \mathcal{A} \circ \mathrm{Sd}^3(K)$$

is a $\mathbb{Z}_2$-homotopy equivalence. This completes the proof of Proposition 7.34 and hence the proof of Theorem 7.28.

References

[1] E. Babson, D. N. Kozlov, *Complexes of graph homomorphisms*, Israel J. Math. **152** 285-316 (2006).

[2] E. Babson, D. N. Kozlov, *Proof of the Lovász conjecture*, Ann. of Math. **165** 965-1007 (2007).

[3] J. A. Barmak, *On Quillen's theorem A for posets*, J. Combin. Ser. A **118** 2445-2453 (2011).

[4] J. A. Barmak, *Algebraic topology of finite spaces and applications*, Lecture Notes in Mathematics Vol. 2032. Springer (2011).

[5] J. A. Barmak, *Strong homotopy types, nerves and collapses*, Discrete and Computational Geometry **47** 301-328 (2012).

[6] G. Bredon, *Equivariant cohomology theories*, Springer Lecture Notes in Mathematics, no. 34 (1967).

[7] P. Csorba, *Homotopy types of box complexes*, Combinatorica, **27** (6) 669-682 (2007).

[8] A. Dochtermann, *Hom complexes and homotopy theory in the category of graphs*, European J. Combin. **30** (2) 490-509, (2009).

[9] A. Dochtermann, *Homotopy groups of Hom complexes of graphs*, J. Combin. Ser. A **116** (1) 180-194 (2009).

[10] A. Dochtermann, *The universality of Hom complexes of graphs*, Combinatorica **29** (4) 433-448 (2009)

[11] A. Dochtermann, C. Schultz, *Topology of Hom complexes and test graphs for bounding chromatic number*, Israel J. Math. **187** (1) 371-417 (2012).

[12] P. Erdős, *Graph theory and probability*, Canad. J. Math. **11** 34-38 (1959) .

[13] P. G. Goerss, J. F. Jardine, *Simplicial homotopy theory*, Progress in Math., vol. 174, Birkhäuser Verlag, Basel, 1999.

[14] A. Hatcher, *Algebraic topology*, Cambridge University Press (2001).

[15] M. A. Hill, M. J. Hopkins, D. Ravenel, *On the non-existence of elements of Kervaire invariant one*, arXiv 0908.3724.

[16] P. S. Hirschhorn, *Model categories and their localizations*, Mathematical Surveys and Monographs 99, American Mathematical Society Providence R. I., 1999

[17] A. Hoory, N. Linial, *A counter example to the conjecture of Lovász on the χ-coloring complex*, J. Combin. Ser. B **95**(2) 346-349 (2005).

[18] M. Hovey, *Model categories*, Mathematical Surveys and monographs 63, American Mathematical Society Providence R. I., 2003

[19] W. Imrich, T. Pisanski, *Multiple Kronecker covering graphs*, European J. Combin. **29** 1116-1122 (2008)

[20] M. Kashiwara, P. Schapira, *Categories and sheaves*, Grundlehren der mathematischen Wissenschaften, A Series of Comprehensive Studies in Mathematics, 332, Springer (2005)

[21] D. N. Kozlov, *Chromatic numbers, morphism complexes, and Stiefel-Whitney characteristic classes*, In geometric combinatorics, volume 13 of IAS/Park City Math. Ser., pages 249-315. Amer. Math. Soc., Providence, RI, (2007)

[22] D. N. Kozlov, *Combinatorial algebraic topology*, Algorithms and Computation in Mathematics vol. 21, Springer (2008).

[23] D. N. Kozlov, *A simple proof of folds on both sides in complexes of graph homomorphisms*, Proc. Amer. Math. Soc. **134** no. 5, 1265-1270 (2006).

[24] D. N. Kozlov, *Cobounding odd cycle colorings*, Electronic Research Announcements of AMS, vol. 12, 53-55 (2006).

[25] J. P. May, *Simplicial objects in algebraic topology*, Van Nostrand Math. Studies, vol. 11, D. Van Nostrand, 1967.

[26] T. Matsushita, *Answers to some problems about graph coloring test graphs*, European J. Combin. Vol. 45, 59-64 April 2015. doi:10.1016/j.ejc.2014.10.006

[27] T. Matsushita, *Morphism complexes of sets with relations*, to appear in Osaka J. Math. (2016).

[28] T. Matsushita, *Box complex and Kronecker double covering*, arXiv:1404.1549

[29] T. Matsushita, *Non-existence of homotopical upper bounds for the chromatic numbers*, arXiv:1509.03855

[30] L. Lovász, *Kneser's conjecture, chromatic number, and homotopy*, J. Combin. Ser. A **25** (3) 319-324 (1978).

[31] J. Matoušek and G. M. Ziegler, *Topological lower bounds for the chromatic number: a hierarchy*, Jahresbericht der DMV **106** 71-90 (2004).

[32] D. Quillen, *Homotopical algebra* Lecture Notes on Mathematics 43 Springer Verlag, Berlin and New York, 1967

[33] D. Quillen, *Higher Algebraic K-theory I*, In "Algebraic K-theory I: Higher K-Theories," Lecture Notes in Mathematics 341, Springer Berlin 1973.

[34] C. Schultz, *A short proof of $w_1(\mathrm{Hom}(C_{2r+1,K_{n+2}}))^n = 0$ for all n and a graph colouring theorem by Babson and Kozlov*, Israel J. Math., **170** 125-134 (2009).

[35] C. Schultz, *Graph colorings, spaces of edges, and spaces of circuits*, Adv. Math. **221** (6) 1733-1756 (2009).

[36] T. Thansri, *Simple Σ_r-homotopy types Hom complexes and box complexes assigned to r-graphs*, Kyushu J. Math. **66** (2) 493-508

[37] J. W. Walker *From graphs to ortholattices and equivariant maps*, J. Combin. Theory Ser. B **35**, 171-192 (1983).

[38] R. T. Živaljević, *WI-posets, graph complexes and Z_2-equivalences*, J. Combin. Ser. A, **111** (2), 204-223 (2005).