

博士論文

論文題目 Volume renormalization for the Blaschke metric
on strictly convex domains
(強凸領域上のブラシュケ計量の体積繰り込みについて)

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VOLUME RENORMALIZATION FOR THE BLASCHKE METRIC ON STRICTLY CONVEX DOMAINS

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ABSTRACT. We consider the volume expansion of the Blaschke metric, which is a projectively invariant metric on a strictly convex domain in a locally flat projective manifold. The coefficient L of the logarithmic term is shown to be a global projective invariant of the boundary, and we express it as the integral of affine invariants. We also formulate the geometry of the boundary as conformal Codazzi structure and define the GJMS operators and Q -curvature. We show that L agrees with the integral of the Q -curvature, which is a global conformal invariant, and derive the variation formulas for the deformation of the domain.

1. INTRODUCTION

Renormalization of the volume of a complete metric has been an important topic in conformal and CR geometries. In conformal geometry, the complete metric we consider is the Poincaré-Einstein metric ([G2], [HS]), which generalizes the hyperbolic metric on the unit ball. The volume renormalization of Poincaré-Einstein metric was motivated by theoretical physics and it has been intensively studied also in mathematical viewpoints ([Al], [An], [GMS]). A Riemannian metric g_+ on the interior X of a compact manifold with boundary is called a Poincaré-Einstein metric if it satisfies Einstein equation and $x^2 g_+$ extends to a metric on $\bar{X}$, where x is a defining function of X which is positive inside. The conformal class of $x^2 g_+|_{T\partial X}$ is independent of the choice of x and defines a conformal structure $[h]$ on $M = \partial X$, which is called the conformal infinity of g_+ . The volume of X with respect to g_+ is infinite, but one can renormalize it by expanding the volume of subdomains; A representative metric $h \in [h]$ determines a special choice of a defining function x with $x^2 g_+|_{TM} = h$ and the volume of each sublevel set of x has the following expansion:

$$(1.1) \quad \text{Vol}(\{x < \epsilon\}) = \sum_{j=0}^{\lceil n/2 \rceil - 1} c_{-n+2j} \epsilon^{-n+2j} + \begin{cases} V + o(1) & (n : \text{odd}) \\ L \log \frac{1}{\epsilon} + V + o(1) & (n : \text{even}), \end{cases}$$

where $n = \dim M$. The constant term V , called the *renormalized volume*, is independent of the choice of $h \in [h]$ when n is odd. When n is even V depends on the choice, but instead the coefficient L of the logarithmic term becomes an invariant of g_+ . Moreover, it is shown in [GZ], [FG1] that L agrees with the integral of the Q -curvature over M , which is a global conformal invariant.

The complex counterpart of this setting is the Cheng-Yau metric on a strictly pseudoconvex domain with the CR boundary in a complex manifold. The Cheng-Yau metric is a complete Kähler-Einstein metric constructed from the solution to the complex Monge-Ampère equation ([CY2]). In this case, there are two ways of choosing a defining function for the volume renormalization. Seshadri [Se] uses a defining function which satisfies a normalization condition analogous to that in conformal case and shows that the coefficient of the logarithmic term agrees with the total CR Q -curvature, though it vanishes for a natural class of contact forms, namely the class of pseudo-Einstein contact forms. On the other hand, the volume expansion with respect to the approximate solution to the Monge-Ampère equation

is considered in [HMM]. The expansion has no logarithmic term but the constant term is shown to agree, up to a topological term, with the total Q -prime curvature, which is a global CR invariant defined by Case–Yang [CaY] and Hirachi [H] for pseudo-Einstein contact forms.

Conformal geometry and CR geometry are both examples of so called *parabolic geometries*. A parabolic geometry is a Cartan geometry modeled on the homogeneous space G/P , where G is a semi-simple Lie group and P is a parabolic subgroup, and can be treated in a unified way by Cartan connections, tractor calculus, and BGG machinery ([ČG1], [ČS], [ČSS]).

In this article, we consider the volume renormalization in the setting of another important example of such geometries: *projective differential geometry*. Recall that two torsion-free affine connections are said to be projectively equivalent if they have the same geodesics up to parametrization. A differentiable manifold N equipped with a projective equivalence class $[\nabla]$ (a projective structure) is called a projective manifold. The flat model is the projective space $\mathbb{RP}^{n+1}$ with the canonical projective structure and we say $(N, [\nabla])$ is locally flat if it is locally projectively equivalent to the flat model. We view a locally flat projective structure as a real analogue of complex structure, and consider a strictly convex domain Ω in a projective manifold instead of a strictly pseudocovex domain. Along the analogy to the complex case, we define the real Monge–Ampère equation for a projective density $\rho \in \mathcal{E}(2)$ as

$$(1.2) \quad \det D_I D_J \rho = -1, \quad \Omega = \{\rho < 0\}$$

with the D -operator on the tractor bundle. Then, with the (approximate) solution to (1.2), we define a projectively invariant metric g on Ω by

$$g_{ij} = \frac{\nabla_i \nabla_j \rho}{-2\rho} + \frac{\nabla_i \rho \nabla_j \rho}{4\rho^2} - P_{ij},$$

where P_{ij} is the projective Schouten tensor of $\nabla \in [\nabla]$. We call g the *Blaschke metric* by following Sasaki [Sa1]. The Blaschke metric is a generalization of the Klein (projective) model hyperbolic metric while the Poincaré–Einstein metric is a generalization of the Poincaré model hyperbolic metric. Fixing a projective scale $\tau \in \mathcal{E}(1)$, we obtain the same volume expansion as in (1.1) with $x = (-2\tau^{-2}\rho)^{1/2}$, and V (resp. L) is independent of the choice of τ when n is odd (resp. even), where n is the dimension of $M = \partial\Omega$ (Theorem 3.4). We then relate the invariant L to the geometry of the boundary. If we fix a scale τ , the boundary M is endowed with several affine tensors such as the affine metric $h_{\alpha\beta}$, the affine shape operator S_α^β , and the Fubini–Pick form $A_{\alpha\beta\gamma}$. Under a rescaling $\hat{\tau} = e^{-\Upsilon}\tau$ with $\Upsilon \in C^\infty(N)$, $h_{\alpha\beta}$ and $A_{\alpha\beta\gamma}$ transform conformally and the transformation formula of S_α^β involves the 1-jet of Υ along M . Given a 1-jet of a projective scale along the boundary, we introduce a normalized scale via the tractor parallel transport so that L is written in terms of the integral of affine invariants (Theorem 4.10).

Although the shape operator S_α^β is an extrinsic curvature of M , there is an intrinsic definition of the geometric structure on a strictly convex hypersurface in a locally flat projective manifold: A *conformal Codazzi structure* is a conformal structure $[h]$ (or a Möbius structure when $n = 2$) together with a trace-free symmetric tensor $A_{\alpha\beta\gamma} \in \mathcal{E}_{(\alpha\beta\gamma)_0}[2]$ which satisfies certain integrability conditions called the *Gauss–Codazzi equations* (see Definition 5.6). This is a special case of the framework of *parabolic subgeometries* proposed by Burstall–Calderbank [BC]. They develop the general theory of parabolic geometry on a submanifold of a generalized flag manifold, and derive the Gauss–Codazzi equations in terms of Lie algebra homology and the BGG operators. In this paper, we derive an explicit form of the equations in a rather direct way based on the comparison of the ambient projective

tractor bundle and the conformal tractor bundle over the boundary; These two bundles are canonically isomorphic and the projective tractor connection induces a connection on the conformal tractor bundle over M , whose flatness corresponds to the Gauss–Codazzi equations. The form of the equations depends on whether $n = 2, 3$ or ≥ 4 . When $n \geq 4$, the equations are

$$\begin{aligned} \text{(Gauss)} \quad & 2 \operatorname{tf}(A_{\nu\gamma[\alpha} A_{\beta]\mu}{}^{\nu}) + W_{\alpha\beta\gamma\mu}^h = 0, \\ \text{(Codazzi)} \quad & \nabla_{[\alpha}^h A_{\beta]\gamma\mu} - \frac{1}{n} \left(h_{\mu[\alpha} (\delta A)_{\beta]\gamma} + h_{\gamma[\alpha} (\delta A)_{\beta]\mu} \right) = 0, \end{aligned}$$

where $W_{\alpha\beta\gamma\mu}^h$ is the Weyl tensor and $(\delta A)_{\alpha\beta} = \nabla_{\gamma}^h A_{\alpha\beta}{}^{\gamma}$. See Proposition 5.3 for the Gauss–Codazzi equations in $n = 2, 3$. The differential operator applied to $A_{\alpha\beta\gamma}$ in the Codazzi equation is known as the second BGG operator associated with the representation $S_0^2(\mathbb{R}^{n+2})^*$ of $SO(n+1, 1)$. Conversely, given a tensor $A_{\alpha\beta\gamma}$ on a conformal manifold with the Gauss–Codazzi equations, one can define a local immersion to the projective space such that the induced conformal Codazzi structure agrees with the original one (projective Bonnet theorem). Thus it contains complete information on the local geometric structure on a hypersurface in a projective manifold.

We define the ambient metric for a conformal Codazzi structure by the projective tractor $\tilde{g}_{IJ} = D_I D_J \boldsymbol{\rho}$ with the approximate solution $\boldsymbol{\rho}$ to the Monge–Ampère equation. It is identified with a homogeneous Lorentz metric on the projective density bundle and we can construct the GJMS operators and the Q -curvature with this metric as in conformal and CR geometries ([GJMS], [FH]). The GJMS operator P_m is a conformally invariant linear differential operator whose symbol agrees with that of the power of the Laplacian Δ_h^m , and is different from usual conformal GJMS operator. In particular we can define $P_{n/2+2}$ for even n , which does not exist for general conformal manifolds ([GH]). The conformal Codazzi Q -curvature integrates to a global conformal invariant and we show that it coincides with L . Recall, however, that the expression of L involves $S_{\alpha}{}^{\beta}$, which is not intrinsic to the conformal Codazzi structure. To resolve this issue, we introduce a normalization on the 1-jet of the projective scale by imposing the harmonicity $\tilde{\Delta}\tau = O(\boldsymbol{\rho})$ with respect the ambient metric. Any conformal scale on the boundary extends to such a scale and we call it a *harmonic scale*. In a harmonic scale, the shape operator is expressed with the Fubini–Pick form and we obtain a representation of L as a conformal invariant on a conformal Codazzi manifold (Theorem 5.11). Explicit formulas are given for $n = 2, 4$ (see (4.22), (5.29)).

We show that the first variation of L for a deformation of a strictly convex domain is given by the obstruction density $\mathcal{O} \in \mathcal{E}[-n-2]$, which is a real analogue of Graham’s obstruction in CR geometry (Theorem 7.1). As a corollary it is shown that a strictly convex surface in $\mathbb{R}^3$ is a critical point of L if and only if it is projectively equivalent to the sphere (Corollary 7.2). We also show that the second variation for a deformation parametrized by a density on the boundary is described by the GJMS operator $P_{n/2+2}$ (Theorem 7.6). For the spheres, the operator agrees with the ordinary GJMS operator, and it turns out that its kernel corresponds to the deformation via a family of projective linear transformations, which gives a trivial deformation of the conformal Codazzi manifold.

This paper is organized as follows. In §2, we review basic notions in projective differential geometry including projective tractor calculus. In §3, we define the Blaschke metric via the solution to the real Monge–Ampère equation and consider the volume expansion to obtain invariants of the domain. In §4, we express the logarithmic coefficient L as the integral of affine invariants over the boundary. We

then derive the explicit form of the Gauss–Codazzi equations in §5 and define the conformal Codazzi structures. We also introduce harmonic scales in order to represent L as a conformal invariant. In §6, we construct GJMS operators and Q -curvature for conformal Codazzi manifolds and show that the total Q -curvature agrees with L . Finally, in §7 we compute the first and second variation formulas of L for the deformation of a strictly convex domain.

Notations. We adopt Einstein’s summation convention and assume that

- uppercase Latin indices $I, J, K, \dots$ run from 0 to $n + 1$;
- lowercase Latin indices $i, j, k, \dots$ run from 1 to $n + 1$;
- lowercase Greek indices $\alpha, \beta, \gamma, \dots$ run from 1 to n .

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2. PROJECTIVE STRUCTURE AND TRACTOR BUNDLE

2.1. Projective structures. Let N be an oriented C^∞ -manifold of dimension $n + 1 \geq 3$. We use abstract index notation to denote tensors and tensor bundles over N . For example, $\mathcal{E}^i$ and $\mathcal{E}_i$ stand for TN and T^*N respectively, and sections of the bundles are denoted by X^i or T_i . The symmetrization and skew symmetrization of a tensor $T_{i_1 i_2 \dots i_k}$ are denoted by $T_{(i_1 i_2 \dots i_k)}$ and $T_{[i_1 i_2 \dots i_k]}$ respectively. The corresponding tensor bundles are denoted by $\mathcal{E}_{(i_1 i_2 \dots i_k)}$ or $\mathcal{E}_{[i_1 i_2 \dots i_k]}$, and we also use the same symbols to denote the space of sections of the bundles. Since N is oriented we can define, for each $w \in \mathbb{R}$, an oriented real line bundle $\mathcal{E}(w) = (\wedge^{n+1} T^*N)^{-(w/(n+2))}$, which we call the *projective density bundle of weight w* . A connection ∇ on TN induces a connection on $\mathcal{E}(w)$, which we also denote by ∇ . A positive section of $\mathcal{E}(1)$ is called a *projective scale*. We put (w) to a bundle to indicate the tensor product with $\mathcal{E}(w)$, as in $\mathcal{E}_{ij}(w)$.

Let ∇ and ∇' be torsion-free affine connections on TN . We say ∇ and ∇' are *projectively equivalent* when there exists a 1-form p_i such that

$$\nabla'_i X^j = \nabla_i X^j + p_i X^j + p_l X^l \delta_i^j,$$

where δ_i^j is the Kronecker delta. We write this relation as $\nabla' = \nabla + p$. It is known that two affine connections are projectively equivalent if and only if they have the same unparametrized geodesic paths; see, e.g., [ČS] for a proof. An equivalence class $[\nabla]$ is called a *projective structure* on N . The curvature and Ricci tensors of $\nabla \in [\nabla]$ are defined by

$$(\nabla_i \nabla_j - \nabla_j \nabla_i) X^k = R_{ij}{}^k{}_l X^l, \quad \text{Ric}_{jl} = R_{ij}{}^i{}_l.$$

Note that the Ricci tensor is not symmetric in general. To deal with this matter, we introduce a subclass of $[\nabla]$, which consists of Ricci symmetric connections.

Lemma 2.1. *For each choice of a projective scale $\tau \in \mathcal{E}(1)$, there exists a unique $\nabla \in [\nabla]$ such that $\nabla \tau = 0$. Moreover the Ricci tensor of ∇ is symmetric.*

Proof. Let $\overset{\circ}{\nabla}$ be an arbitrary representative connection. Take a local unimodular frame $\{e_i\}$ with respect to the volume $\tau^{-(n+2)}$ and let $\overset{\circ}{\omega}_i{}^j$ be the connection forms of $\overset{\circ}{\nabla}$. Then for $\nabla = \overset{\circ}{\nabla} + p \in [\nabla]$, we have

$$\nabla \tau = ((n+2)^{-1} \overset{\circ}{\omega}_l{}^l + p) \otimes \tau.$$

Thus there is a unique p for which $\nabla \tau = 0$. The Bianchi identity $R_{ij}{}^k{}_k = 2 \text{Ric}_{[ji]}$ implies that ∇ is Ricci symmetric. $\square$

From now on, we will restrict ourselves to representative connections given as in the above lemma. If we change a projective scale as $\hat{\tau} = e^{-\Upsilon}\tau$, $\Upsilon \in C^\infty(N)$, the connection transforms by $\hat{\nabla} = \nabla + d\Upsilon$. We define the *projective Schouten tensor* $P_{ij} \in \mathcal{E}_{(ij)}$ and the *projective Weyl tensor* $C_{ij}{}^k{}_l \in \mathcal{E}_{[ij]}{}^k{}_l$ by

$$P_{ij} = \frac{1}{n} \text{Ric}_{ij}, \quad C_{ij}{}^k{}_l = R_{ij}{}^k{}_l - 2\delta_{[i}{}^k P_{j]l}.$$

For a change of the scale $\hat{\tau} = e^{-\Upsilon}\tau$, the projective Weyl tensor is invariant and the projective Schouten tensor satisfies

$$\hat{P}_{ij} = P_{ij} - \nabla_i \Upsilon_j + \Upsilon_i \Upsilon_j,$$

where $\Upsilon_i = d\Upsilon$. A projective structure is said to be *locally flat* if $C_{ij}{}^k{}_l = 0$. This is equivalent to the condition that on a neighborhood of any point in N there exists a projective scale such that the corresponding connection in the projective class is flat. We call such a projective scale an *affine scale*. From the Bianchi identity $\nabla_{[i} R_{jk]}{}^l{}_m = 0$ it follows that $\nabla_k C_{ij}{}^k{}_l = 2(n-1)\nabla_{[i} P_{j]l}$, so if $[\nabla]$ is locally flat $\nabla_{[i} P_{j]l}$ also vanishes.

2.2. Projective tractor bundle and tractor connection. We review the construction of the tractor bundles and tractor connections for projective manifolds by following [BEG]. We refer to [ČG1] for the general construction of tractor bundles for parabolic geometries.

Let $\pi : \tilde{N} \rightarrow N$ be the $\mathbb{R}_+$ -bundle of positive elements in $\mathcal{E}(-1)$. Let $\mathbb{R}_+$ act on $T\tilde{N}$ by $s \cdot u := s^{-1}(\delta_s)_* u$, where δ_s is the dilation on $\tilde{N}$. Then $\tilde{\mathcal{E}}^I := T\tilde{N}/\mathbb{R}_+$ becomes a rank $(n+2)$ vector bundle over N , and is called the *projective tractor bundle*. We call an element $\nu^I \in \tilde{\mathcal{E}}^I$ a *tractor*. We often identify a section of $\tilde{\mathcal{E}}^I$ with a vector field on $\tilde{N}$ homogeneous of degree -1 . Since the inclusion $\pi^* \mathcal{E}(-1) \subset T\tilde{N}$ is $\mathbb{R}_+$ -invariant, $\tilde{\mathcal{E}}^I$ contains $\mathcal{E}(-1)$ as a line subbundle, which is the kernel of the well-defined projection $\tilde{\mathcal{E}}^I \rightarrow \mathcal{E}^i(-1)$. Thus we have an exact sequence

$$(2.1) \quad 0 \longrightarrow \mathcal{E}(-1) \longrightarrow \tilde{\mathcal{E}}^I \longrightarrow \mathcal{E}^i(-1) \longrightarrow 0.$$

If we choose a projective scale $\tau \in \mathcal{E}(1)$ then $\tilde{N}$ is trivialized by τ^{-1} and accordingly the sequence (2.1) splits:

$$\tilde{\mathcal{E}}^I \cong \begin{array}{c} \mathcal{E}^i(-1) \\ \oplus \\ \mathcal{E}(-1) \end{array} \xrightarrow{\tau}.$$

So a tractor is represented as $\nu^I = {}^t(\nu^i, \lambda)$ with $\nu^i \in \mathcal{E}^i(-1)$ and $\lambda \in \mathcal{E}(-1)$. Let $\nu^I = {}^t(\hat{\nu}^i, \hat{\lambda})$ be the expression in terms of another scale $\hat{\tau} = e^{-\Upsilon}\tau$. Then two expressions are related as

$$(2.2) \quad \begin{pmatrix} \hat{\nu}^i \\ \hat{\lambda} \end{pmatrix} = \begin{pmatrix} \nu^i \\ \lambda - \Upsilon_k \nu^k \end{pmatrix}.$$

The inclusion $\mathcal{E}(-1) \hookrightarrow \tilde{\mathcal{E}}^I$ is described as $\lambda \mapsto \lambda T^I$, where $T^I := {}^t(0, 1) \in \tilde{\mathcal{E}}^I(1)$ is the *Euler field*.

The *tractor connection* is a projectively invariant linear connection on $\tilde{\mathcal{E}}^I$ defined by

$$(2.3) \quad \nabla_i \begin{pmatrix} \nu^j \\ \lambda \end{pmatrix} = \begin{pmatrix} \nabla_i \nu^j + \lambda \delta_i{}^j \\ \nabla_i \lambda - P_{ik} \nu^k \end{pmatrix}$$

in a chosen projective scale. We can also consider the dual or tensor products of the projective tractor bundles, and we also call them “tractor bundles”. The

transformation laws and the induced tractor connections are computed from (2.2) and (2.3). For the dual bundle $\tilde{\mathcal{E}}_I \cong \mathcal{E}(1) \oplus \mathcal{E}_i(1)$, we have

$$\begin{pmatrix} \hat{\sigma} \\ \hat{\mu}_i \end{pmatrix} = \begin{pmatrix} \sigma \\ \mu_i - \Upsilon_i \sigma \end{pmatrix}, \quad \nabla_i \begin{pmatrix} \sigma \\ \mu_j \end{pmatrix} = \begin{pmatrix} \nabla_i \sigma - \mu_i \\ \nabla_i \mu_j + P_{ij} \sigma \end{pmatrix}.$$

We define the curvature of the tractor connection by

$$(\nabla_i \nabla_j - \nabla_j \nabla_i) \nu^K = \Omega_{ij}{}^K{}_L \nu^L.$$

Then we have

$$\Omega_{ij}{}^K{}_L = \begin{pmatrix} C_{ij}{}^k{}_l & 0 \\ -2\nabla_{[i} P_{j]l} & 0 \end{pmatrix}.$$

Thus the projective tractor connection is flat if and only if the projective structure $[\nabla]$ is locally flat.

Let $\tilde{\mathcal{E}}_*(w)$ denote a weighted tractor bundle with arbitrary tractor indices. The *projective tractor D-operator* $D_I : \tilde{\mathcal{E}}_*(w) \rightarrow \tilde{\mathcal{E}}_*(w-1)$ is defined by

$$D_I f_* = \begin{pmatrix} w f_* \\ \nabla_i f_* \end{pmatrix} \in \begin{pmatrix} \tilde{\mathcal{E}}_*(w) \\ \tilde{\mathcal{E}}_{i*}(w) \end{pmatrix} \xrightarrow{\tau} \tilde{\mathcal{E}}_{I*}(w-1),$$

where in the second slot, we use the connection induced by the tractor connection and the flat connection on $\mathcal{E}(w)$ relative to a chosen scale $\tau \in \mathcal{E}(1)$. It is immediate to check that D_I is a projectively invariant differential operator. A calculation shows that the commutator of the D -operator satisfies $T^I D_{[I} D_{J]} f_* = 0$ and it projects to $\nabla_{[i} \nabla_{j]} f_*$. Therefore $D_{[I} D_{J]} f_* = 0$ for locally flat projective structures.

We can give a characterization of the affine scale via projectively invariant differential operators:

Proposition 2.2. *Assume that $[\nabla]$ is locally flat and let $\tau \in \mathcal{E}(1)$ be a projective scale. Then the following are equivalent:*

- (i) *The scale τ is affine;*
- (ii) *It holds that $D_I D_J \tau = 0$;*
- (iii) *It holds that $\nabla_i \nabla_j \tau + P_{ij} \tau = 0$ for any $\nabla \in [\nabla]$.*

Proof. The equivalence of (ii) and (iii) follows from the equation

$$D_I D_J \tau = \begin{pmatrix} 0 & 0 \\ 0 & \nabla_i \nabla_j \tau + P_{ij} \tau \end{pmatrix}.$$

If we compute in the scale τ , we have $\nabla_i \nabla_j \tau + P_{ij} \tau = P_{ij}^\tau \tau$, where P_{ij}^τ is the projective Schouten tensor of the corresponding representative connection $\nabla \in [\nabla]$. Thus $\nabla_i \nabla_j \tau + P_{ij} \tau = 0$ if and only if $P_{ij}^\tau = 0$, which is equivalent to the flatness of ∇ by $C_{ij}{}^k{}_l = 0$. $\square$

3. MONGE-AMPÈRE EQUATION AND BLASCHKE METRIC

3.1. Monge–Ampère equation on strictly convex domains. Let $(N, [\nabla])$ be an oriented $(n+1)$ -dimensional projective manifold. Let Ω be a relatively compact domain in N with smooth boundary $\partial\Omega = M$. We say M is *strictly convex* if the Hessian $(\nabla_i \nabla_j \rho)|_{TM}$ is positive definite for a connection $\nabla \in [\nabla]$ and a defining function ρ . (Here and after, the defining functions or defining densities are assumed to be *negative* inside the domain.) Note that the condition is independent of the choices of ∇ and ρ .

In order to formulate the real Monge–Ampère equation on Ω , we will define the determinant of an element in $\tilde{\mathcal{E}}_{ij}(w)$ or $\tilde{\mathcal{E}}_{IJ}(w)$. Take a projective scale $\tau \in \mathcal{E}(1)$. First we define

$$\det : \tilde{\mathcal{E}}_{ij}(w) \longrightarrow \mathcal{E}((n+1)w - 2n - 4)$$

by

$$\det \phi_{ij} := \tau^{(n+1)w-2n-4} \det_{\tau}(\tau^{-w} \phi_{ij}),$$

where in the right-hand side we take the determinant with respect to the volume $\tau^{-(n+2)}$. It can be shown that $\det \phi_{ij}$ is independent of τ . Next we define

$$\det : \tilde{\mathcal{E}}_{IJ}(w) \longrightarrow \mathcal{E}((n+2)w)$$

by

$$\det \begin{pmatrix} \sigma & \nu_j \\ \mu_i & \lambda_{ij} \end{pmatrix} := \tau^{(n+2)w} \det_{\tau} \left[\tau^{-w-2} \begin{pmatrix} \sigma & \nu_j \\ \mu_i & \lambda_{ij} \end{pmatrix} \right].$$

Note that each component in the matrix has weight $w+2$. One can show that the determinant is invariant under the change $\hat{\tau} = e^{-\Upsilon} \tau$ by using the transformation law

$$\begin{pmatrix} \hat{\sigma} & \hat{\nu}_j \\ \hat{\mu}_i & \hat{\lambda}_{ij} \end{pmatrix} = \begin{pmatrix} \sigma & \nu_j + \Upsilon_j \sigma \\ \mu_i + \Upsilon_i \sigma & \lambda_{ij} + \Upsilon_j \mu_i + \Upsilon_i \nu_j + \Upsilon_i \Upsilon_j \sigma \end{pmatrix}.$$

Now the real Monge–Ampère equation for a density $\rho \in \mathcal{E}(2)$ is defined by

$$(3.1) \quad \begin{cases} \mathcal{J}[\rho] = -1, \\ \Omega = \{\rho < 0\}, \end{cases}$$

where

$$\mathcal{J}[\rho] = \det(D_I D_J \rho) = \det \begin{pmatrix} 2\rho & \nabla_j \rho \\ \nabla_i \rho & \nabla_i \nabla_j \rho + 2\rho P_{ij} \end{pmatrix}.$$

For a defining density $\rho \in \mathcal{E}(2)$, we call $\tilde{g}_{IJ} = D_I D_J \rho$ a *pre-ambient metric*. By strict convexity of M , $\tilde{g}_{IJ}$ defines a Lorentz metric on $\tilde{N}$ near M which is homogeneous of degree 2. Then the Monge–Ampère equation is equivalent to $\det \tilde{g}_{IJ} = -1$. Note, however, that the equation does not imply that $\tilde{g}_{IJ}$ is Ricci flat, unlike in the Kähler case.

The real Monge–Ampère equation can be rewritten as an equation for a density $\mathbf{u} \in \mathcal{E}(1)|_{\Omega}$ which is continuous up to M but may not be a defining density. Setting $\mathbf{u} = -(-2\rho)^{1/2}$, we have

$$\det(D_I D_J \rho) = -(-\mathbf{u})^{n+3} \det(\nabla_i \nabla_j \mathbf{u} + P_{ij} \mathbf{u}).$$

Thus (3.1) is equivalent to

$$(3.2) \quad \begin{cases} (-\mathbf{u})^{n+3} \det(\nabla_i \nabla_j \mathbf{u} + P_{ij} \mathbf{u}) = 1, \\ \mathbf{u}|_M = 0. \end{cases}$$

The differential operator $\nabla_i \nabla_j + P_{ij}$ on $\mathcal{E}(1)$ is the projecting part of $D_I D_J$ so it is projectively invariant. It is also known as the first BGG operator associated with the representation of $SL(n+2, \mathbb{R})$ on $(\mathbb{R}^{n+2})^*$. The generalization of real Monge–Ampère equation via the first BGG operator is also considered by Fox in [Fo].

With a solution ρ or $\mathbf{u}$ to the Monge–Ampère equation, we define a projectively invariant metric on Ω by

$$(3.3) \quad \begin{aligned} g_{ij} &= \frac{\nabla_i \nabla_j \mathbf{u}}{-\mathbf{u}} - P_{ij} \\ &= \frac{\nabla_i \nabla_j \rho}{-2\rho} + \frac{\nabla_i \rho \nabla_j \rho}{4\rho^2} - P_{ij} \in \mathcal{E}_{(ij)}. \end{aligned}$$

We call g_{ij} the *Blaschke metric* by following [Sal]. In the case where $N = \mathbb{R}^{n+1}$ and $[\nabla]$ is the canonical flat projective structure, unique existence of the solution to (3.2) has been proved by Cheng–Yau:

Theorem 3.1 ([CY1]). *Let Ω be a bounded strictly convex domain in $\mathbb{R}^{n+1}$. Then there exists a unique function $u \in C^\infty(\Omega) \cap C(\overline{\Omega})$ such that $(-u)^{n+3} \det(\partial^2 u / \partial x^i \partial x^j) = 1$ and $u|_{\partial\Omega} = 0$.*

For $\Omega = \mathbb{B}^{n+1}$, the solution is given by $u(x) = -(1 - |x|^2)^{1/2}$ and the Blaschke metric agrees with the Klein model hyperbolic metric.

3.2. Fefferman's approximate solution. We do not have the existence theorem for the Monge–Ampère equation on convex domains in general projective manifolds, but we can construct an asymptotic solution to (3.1) by the method of Fefferman [Fe] (see also [Sa2] for the real case).

Lemma 3.2. *Let $\rho \in \mathcal{E}(2)$ be a defining density of Ω .*

(i) *For any positive function $f \in \mathcal{E}$, it holds that*

$$(3.4) \quad \mathcal{J}[f\rho] = f^{n+2} \mathcal{J}[\rho] + O(\rho).$$

(ii) *Let $m \geq 2$ and suppose that ρ satisfies $\mathcal{J}[\rho] = -1 + K\rho^{m-1}$ with $K \in \mathcal{E}(2-2m)$. Then for any density $\phi \in \mathcal{E}(2-2m)$, it holds that*

$$(3.5) \quad \mathcal{J}[\rho + \phi\rho^m] = -1 + (K - m(n+4-2m)\phi)\rho^{m-1} + O(\rho^m).$$

Moreover, if $m \geq 3$ we also have

$$(3.6) \quad \mathcal{J}[\rho + \phi\rho^m] = -1 + (K - m(n+4-2m)\phi)\rho^{m-1} + \rho^m \tilde{\Delta}\phi + O(\rho^{m+1}),$$

where $\tilde{\Delta} = -\tilde{g}^{IJ} \tilde{\nabla}_I \tilde{\nabla}_J$ is the Laplacian of $\tilde{g}_{IJ} = D_I D_J \rho$.

Proof. (i) Using row and column transformation, we have

$$\begin{aligned} \mathcal{J}[f\rho] &= \det \begin{pmatrix} 0 & f\nabla_j \rho \\ f\nabla_i \rho & f\nabla_i \nabla_j \rho + 2\nabla_{(i} f \nabla_{j)} \rho \end{pmatrix} + O(\rho) \\ &= \det \begin{pmatrix} 0 & f\nabla_j \rho \\ f\nabla_i \rho & f\nabla_i \nabla_j \rho \end{pmatrix} + O(\rho) \\ &= f^{n+2} \mathcal{J}[\rho] + O(\rho). \end{aligned}$$

Thus we obtain (3.4).

(ii) We note that $\tilde{\Delta}\phi = -\tilde{g}^{IJ} D_I D_J \phi + O(\rho)$ holds, since the Christoffel symbols $\tilde{\Gamma}_{IJ}^K$ satisfy $\tilde{g}^{IJ} \tilde{\Gamma}_{IJ}^K = O(\rho)$ by the Monge–Ampère equation. In the following calculation, we compute modulo $O(\rho^{m+1})$ for $m \geq 3$ and modulo $O(\rho^2)$ for $m = 2$. First, we divide each entry in the matrix $D_I D_J (\rho + \phi\rho^m)$ by $1 + m\phi\rho^{m-1}$ and perform some row and column transformations to obtain

$$\begin{aligned} \mathcal{J}[\rho + \phi\rho^m] &\equiv (1 + m\phi\rho^{m-1})^{n+2} \\ &\cdot \det \begin{pmatrix} 2\rho + 2(1-m)\phi\rho^m & (1-2m(m-1)\phi\rho^{m-1})\nabla_j \rho \\ & + (1-2m)\rho^m \nabla_j \phi \\ \nabla_i \rho + (1-2m)\rho^m \nabla_i \phi & \nabla_i \nabla_j \rho + 2\rho P_{ij} \\ & + \rho^m (\nabla_i \nabla_j \phi + 2(1-m)\phi P_{ij}) \end{pmatrix}. \end{aligned}$$

Then, dividing the first row by $1 - 2m(m-1)\phi\boldsymbol{\rho}^{m-1}$, we have

$$\begin{aligned}
\mathcal{J}[\boldsymbol{\rho} + \phi\boldsymbol{\rho}^m] &\equiv (1 + m(n+2)\phi\boldsymbol{\rho}^{m-1})(1 - 2m(m-1)\phi\boldsymbol{\rho}^{m-1}) \\
&\quad \cdot \det \begin{pmatrix} 2\boldsymbol{\rho} + 2(1-m)(1-2m)\phi\boldsymbol{\rho}^m & \nabla_j \boldsymbol{\rho} + (1-2m)\boldsymbol{\rho}^m \nabla_j \phi \\ \nabla_i \boldsymbol{\rho} + (1-2m)\boldsymbol{\rho}^m \nabla_i \phi & \nabla_i \nabla_j \boldsymbol{\rho} + 2\boldsymbol{\rho} P_{ij} \\ & + \boldsymbol{\rho}^m (\nabla_i \nabla_j \phi + 2(1-m)\phi P_{ij}) \end{pmatrix} \\
&\equiv (1 + m(n+4-2m)\phi\boldsymbol{\rho}^{m-1}) \det(\tilde{g}_{IJ} + \boldsymbol{\rho}^m D_I D_J \phi) \\
&\equiv (1 + m(n+4-2m)\phi\boldsymbol{\rho}^{m-1}) \mathcal{J}[\boldsymbol{\rho}](1 - \boldsymbol{\rho}^m \tilde{\Delta} \phi) \\
&\equiv -1 + (K - m(n+4-2m)\phi)\boldsymbol{\rho}^{m-1} + \boldsymbol{\rho}^m \tilde{\Delta} \phi.
\end{aligned}$$

Thus we obtain (3.5) and (3.6). $\square$

Proposition 3.3. *There exists a defining density $\boldsymbol{\rho} \in \mathcal{E}(2)$ such that*

$$(3.7) \quad \mathcal{J}[\boldsymbol{\rho}] = \begin{cases} -1 + O(\boldsymbol{\rho}^\infty) & \text{if } n \text{ is odd,} \\ -1 + \mathcal{O}\boldsymbol{\rho}^{n/2+1} & \text{if } n \text{ is even,} \end{cases}$$

where $\mathcal{O} \in \mathcal{E}(-n-2)$. Moreover, such a density is unique modulo $O(\boldsymbol{\rho}^\infty)$ for n odd, and unique modulo $O(\boldsymbol{\rho}^{n/2+2})$ for n even.

Proof. Take an arbitrary defining density $\boldsymbol{\rho} \in \mathcal{E}(2)$. Since M is strictly convex, we have $\mathcal{J}[\boldsymbol{\rho}] < 0$ near M . Using (3.4) with $f = (-\mathcal{J}[\boldsymbol{\rho}])^{-(1/n+2)}$, we obtain a defining density $\boldsymbol{\rho}'$ such that $\mathcal{J}[\boldsymbol{\rho}'] = -1 + O(\boldsymbol{\rho}')$. Then we inductively improve the approximate solution by using (3.5) up to $m = n/2 + 1$ when n is even and for all m when n is odd. We use Borel's lemma to obtain the infinite order approximate solution for n odd. The uniqueness up to the stated order also follows from (3.5). $\square$

We call a defining density which satisfies (3.7) a *Fefferman defining density*. By Lemma 3.2, we can show that $\mathcal{O}|_M$ is independent of the choice of a Fefferman defining density and it vanishes if and only if there exists $\boldsymbol{\rho}$ such that $\mathcal{J}[\boldsymbol{\rho}] = -1 + O(\boldsymbol{\rho}^\infty)$. We call $\mathcal{O}$ the *obstruction density*.

A pre-ambient metric $\tilde{g}_{IJ} = D_I D_J \boldsymbol{\rho}$ is called the *ambient metric* if $\boldsymbol{\rho}$ is a Fefferman defining density.

3.3. Volume expansion of the Blaschke metric. Let $\boldsymbol{\rho} \in \mathcal{E}(2)$ be a Fefferman defining density and set $\mathbf{u} = -(-2\boldsymbol{\rho})^{1/2} \in \mathcal{E}(1)$. We define the Blaschke metric g_{ij} by (3.3) near the boundary and extend it to a Riemannian metric on Ω . Then the volume expansion of g_{ij} is given by the following theorem:

Theorem 3.4. *Let $\tau \in \mathcal{E}(1)$ be a projective scale and put $u := \tau^{-1}\mathbf{u} \in \mathcal{E}$. Then we have the expansion*

$$(3.8) \quad \text{Vol}(\{u < -\epsilon\}) = \sum_{j=0}^{\lceil n/2 \rceil - 1} c_{-n+2j} \epsilon^{-n+2j} + \begin{cases} V + o(1) & (n : \text{odd}) \\ L \log \frac{1}{\epsilon} + V + o(1) & (n : \text{even}). \end{cases}$$

The constant term V is independent of the choice of τ when n is odd while L is independent of the choice of τ when n is even.

We call the constant term V the *renormalized volume* of g .

Proof of Theorem 3.4. The approximate Monge–Ampère equation (3.7) is equivalent to

$$(-u)^{n+3} \det_\tau (\nabla_i \nabla_j u + P_{ij} u) = 1 + O(u^m),$$

where $m = n + 2$ for n even and $m = \infty$ for n odd. Thus we have

$$\det_\tau g_{ij} = (-u)^{-2n-4} (1 + O(u^m)).$$

Near the boundary we identify Ω with the product $M \times (-\epsilon_0, 0)$, where the second component is given by $\rho = \tau^{-2}\boldsymbol{\rho}$. Take a volume form vol_M on M , and write $\tau^{-(n+2)} = v d\rho \wedge vol_M$ with $v \in C^\infty(M \times (-\epsilon_0, 0])$. Then since v is even in u , we have

$$\begin{aligned} vol_g &= (\det \tau g_{ij})^{1/2} \tau^{-(n+2)} \\ &= (-u)^{-n-2} (1 + O(u^m)) v d\rho \wedge vol_M \\ &= \left(\sum_{j=0}^{[n/2]} a_j u^{-n-1+2j} + O(1) \right) du \wedge vol_M. \end{aligned}$$

Integrating both sides, we obtain the desired volume expansion.

Let $\hat{\tau} = e^{-\Upsilon} \tau$ be another projective scale, and set $\hat{\rho} = \hat{\tau}^{-2}\boldsymbol{\rho}$, $\hat{u} = \hat{\tau}^{-1}\mathbf{u}$. Near the boundary, u and $\hat{u}$ are related as $u = b(x, \hat{\rho})\hat{u}$ with some $b \in C^\infty(M \times (-\hat{\epsilon}_0, 0])$. Then the difference of the volumes of sublevel sets is

$$\begin{aligned} \text{Vol}(\{\hat{u} < -\epsilon\}) - \text{Vol}(\{u < -\epsilon\}) \\ = \int_{-\epsilon}^{-b(x, -\frac{1}{2}\epsilon^2)\epsilon} \int_M \left(\sum_{j=0}^{[n/2]} a_j u^{-n-1+2j} + O(1) \right) du \wedge vol_M. \end{aligned}$$

When n is odd, we can see that this integral is odd in ϵ so it does not contain the constant term. Therefore, V is invariant under rescaling of τ . When n is even, the difference does not contain the logarithmic term, so L is invariant. $\square$

4. RELATION TO GEOMETRY ON THE BOUNDARY

4.1. Geometry of hypersurfaces in projective manifolds. Let $(N, [\nabla])$ be an oriented projective manifold of dimension $n+1$, and let $\Omega \subset N$ be a relatively compact domain with strictly convex boundary M . We assume that the projective structure $[\nabla]$ is locally flat. We review the geometric structure induced on the hypersurface M ; see, e.g., [NS] for the detail.

We fix a projective scale $\tau \in \mathcal{E}(1)$ and a defining function ρ . Take a vector field $\xi \in \Gamma(M, TN)$ with $d\rho(\xi) > 0$. Then according to the decomposition $TN|_M = TM \oplus \mathbb{R}\xi$, we have

$$\begin{aligned} \nabla_X Y &= \nabla_X^\xi Y - h(X, Y)\xi, \\ \nabla_X \xi &= S(X) + \eta(X)\xi \end{aligned}$$

for $X, Y \in \Gamma(TM)$. Since ∇ is torsion-free, ∇^ξ defines a torsion-free affine connection on TM and h becomes a symmetric 2-tensor, called the *affine second fundamental form*. Moreover, strict convexity of M implies that h is positive definite, so it is called the *affine metric*. We denote the Levi-Civita connection of h and its curvature by ∇^h and R^h respectively. The endomorphism S is called the *affine shape operator*. These quantities depend on the choice of the transverse field ξ . However we can take ξ canonically by imposing some normalization conditions:

Lemma 4.1. *There exists a unique $\xi \in \Gamma(M, TN)$ with $d\rho(\xi) > 0$ such that $(\xi \lrcorner \tau^{-(n+2)})|_{TM} = vol_h$ and $\eta = 0$.*

Proof. We take an arbitrary $\xi \in \Gamma(M, TN)$ with $d\rho(\xi) > 0$. Then any other $\xi' \in \Gamma(M, TN)$ such that $d\rho(\xi') > 0$ can be written in the form $\xi' = \phi\xi + Z$ with $\phi > 0$ and $Z \in \Gamma(TM)$. A calculation shows that the corresponding tensor fields are given by $h' = \phi^{-1}h$ and $\eta'(X) = \eta(X) + \phi^{-1}X\phi - \phi^{-1}h(X, Z)$. Thus we have $vol_{h'} = \phi^{-n/2}vol_h$. On the other hand we have $(\xi' \lrcorner \tau^{-(n+2)})|_{TM} = \phi(\xi \lrcorner \tau^{-(n+2)})|_{TM}$, so there is a unique ϕ such that $(\xi' \lrcorner \tau^{-(n+2)})|_{TM} = vol_{h'}$. Then since h is non-degenerate, the condition $\eta' = 0$ uniquely determines Z . $\square$

The transverse vector field ξ in Lemma 4.1 is called the *affine normal field* associated with the projective scale τ . The transformation laws under rescaling of τ are given by the following

Proposition 4.2. *Let $\hat{\tau} = e^{-\Upsilon} \tau$ be another projective scale. Then it holds that*

$$\begin{aligned}\hat{\xi} &= e^{-2\Upsilon}(\xi - \text{grad}_h \Upsilon), \\ \hat{h} &= e^{2\Upsilon} h, \\ \hat{\nabla}_X^{\hat{\xi}} Y &= \nabla_X^{\xi} Y + d\Upsilon(X)Y + d\Upsilon(Y)X - h(X, Y) \text{grad}_h \Upsilon, \\ \hat{S}(X) &= e^{-2\Upsilon}(S(X) + (\xi \Upsilon - |d\Upsilon|_h^2)X + d\Upsilon(X) \text{grad}_h \Upsilon - \nabla_X^{\xi} \text{grad}_h \Upsilon),\end{aligned}$$

where $\text{grad}_h \Upsilon$ denotes the gradient vector field of $\Upsilon|_M$ with respect to the metric h .

Since the affine metric transforms conformally under a change of τ , the conformal structure $[h]$ is well-defined. Recall that the conformal density bundle on M is defined by $\mathcal{E}[1] = (\wedge^n TM)^{-1/n}$. Using affine metric we can define a canonical isomorphism $\mathcal{E}(1)|_M \rightarrow \mathcal{E}[1]$ by $\tau|_M \mapsto (\text{vol}_h)^{-1/n}$. Thus we identify $\mathcal{E}(w)|_M$ with $\mathcal{E}[w]$ for each w . The conformally invariant weighted tensor $\mathbf{h}_{\alpha\beta} = \tau^2|_M h_{\alpha\beta} \in \mathcal{E}_{(\alpha\beta)}[2]$ is called the *conformal metric*. We raise and lower the indices with $\mathbf{h}_{\alpha\beta}$ and its inverse $\mathbf{h}^{\alpha\beta}$. We put conformal weight -2 on the affine shape operator and write $\mathbf{S}_{\alpha}{}^{\beta} = \tau|_M^{-2} S_{\alpha}{}^{\beta} \in \mathcal{E}_{\alpha}{}^{\beta}[-2]$. Note that $\mathbf{S}_{\alpha}{}^{\beta}$ depends on 1-jet of the scale along the boundary.

From the transformation laws of the induced connection ∇^{ξ} and the Levi-Civita connection ∇^h , it follows that their difference $A_{\alpha\beta}{}^{\gamma} = \nabla_{\alpha}^h - \nabla_{\alpha}^{\xi}$, called the *Fubini-Pick form*, is a conformally invariant tensor. This tensor is a fundamental invariant of M , and it is a classical theorem that the Fubini-Pick form vanishes identically if and only if M is locally equivalent to the sphere (see [NS] for a proof):

Theorem 4.3 (Blaschke-Pick-Berwald). *Let M be a strictly convex hypersurface in a locally flat projective manifold N of dimension $n+1$. Then the Fubini-Pick form vanishes identically if and only if for each point $p \in M$ there exists a neighborhood $U \subset N$, an open subset $V \subset \mathbb{R}^{n+1}$, and a projective equivalent map $\varphi : U \rightarrow V$ such that $\varphi(U \cap M) = V \cap S^n$.*

As a consequence of the flatness of the projective structure $[\nabla]$, $A_{\alpha\beta}{}^{\gamma}$ and $\mathbf{S}_{\alpha\beta}$ have the following symmetries:

Proposition 4.4. *The Fubini-Pick form $A_{\alpha\beta}{}^{\gamma}$ and the affine shape operator $\mathbf{S}_{\alpha\beta}$ satisfy*

$$A_{\alpha\beta}{}^{\gamma} = A_{(\alpha\beta)}{}^{\gamma}, \quad A_{\alpha\gamma}{}^{\gamma} = 0, \quad \mathbf{S}_{[\alpha\beta]} = 0.$$

Proof. Take a local frame $\{e_{\alpha}\}$ for TM . Then $\{e_{\infty} = \xi, e_{\alpha}\}$ forms a local frame for $TN|_M$. Let $\{\theta^{\infty}, \theta^{\alpha}\}$ be the dual frame. Since ξ is affine normal, the connection forms of ∇ satisfy

$$\omega_{\alpha}{}^{\beta}|_{TM} = \omega_{\alpha}{}^{\beta}, \quad \omega_{\infty}{}^{\beta}|_{TM} = S_{\gamma}{}^{\beta} \theta^{\gamma}, \quad \omega_{\alpha}{}^{\infty}|_{TM} = -h_{\alpha\gamma} \theta^{\gamma}, \quad \omega_{\infty}{}^{\infty}|_{TM} = 0,$$

where $\omega_{\alpha}{}^{\beta}$ is the connection forms of ∇^{ξ} . Thus the curvature forms satisfy

$$\Omega_{\alpha}{}^{\infty}|_{TM} = -\nabla_{\mu}^{\xi} h_{\gamma\alpha} \theta^{\mu} \wedge \theta^{\gamma}, \quad \Omega_{\infty}{}^{\infty}|_{TM} = -h_{\mu\gamma} S_{\alpha}{}^{\mu} \theta^{\gamma} \wedge \theta^{\alpha}.$$

On the other hand, from $C_{ij}{}^k{}_l = 0$ we have $R_{ij}{}^k{}_l = 2\delta_{[i}{}^k P_{j]l}$ so that $\Omega_{\alpha}{}^{\infty}|_{TM} = \Omega_{\infty}{}^{\infty}|_{TM} = 0$. Therefore we obtain $\nabla_{[\mu}^{\xi} h_{\gamma]\alpha} = 0$ and $\mathbf{S}_{[\alpha\gamma]} = 0$. It follows that $A_{\alpha\beta}{}^{\gamma} = (1/2)h^{\gamma\mu} \nabla_{\alpha}^{\xi} h_{\beta\mu}$ and $A_{\alpha\beta}{}^{\gamma} \in \mathcal{E}_{(\alpha\beta)}{}^{\gamma}[2]$. To prove that $A_{\alpha\beta}{}^{\gamma}$ is trace-free, it

suffices to show $\nabla^\xi \text{vol}_h = 0$. For any $X, X_1, \dots, X_n \in \Gamma(TM)$, we have

$$\begin{aligned}
0 &= (\nabla_X \tau^{-(n+2)})(\xi, X_1, \dots, X_n) \\
&= X \cdot \tau^{-(n+2)}(\xi, X_1, \dots, X_n) - \tau^{-(n+2)}(\nabla_X \xi, X_1, \dots, X_n) \\
&\quad - \sum_{i=1}^n \tau^{-(n+2)}(\xi, X_1, \dots, \nabla_X X_i, \dots, X_n) \\
&= X \cdot \text{vol}_h(X_1, \dots, X_n) - \sum_{i=1}^n \text{vol}_h(X_1, \dots, \nabla_X^\xi X_i, \dots, X_n) \\
&= (\nabla_X^\xi \text{vol}_h)(X_1, \dots, X_n).
\end{aligned}$$

Thus $A_{\alpha\beta\gamma}$ is trace-free. $\square$

4.2. Volume expansion and affine invariants. By an *affine invariant* of the boundary, we mean a linear combination of complete contractions of $h_{\alpha\beta}$, $R_{\alpha\beta\gamma\mu}^h$, $A_{\alpha\beta\gamma}$, $S_{\alpha\beta}$ and their covariant derivatives with respect to ∇^h in a chosen projective scale. Recall that affine invariants depend only on 1-jet of the projective scale along the boundary. In this subsection we prove that, given a 1-jet of the projective scale along the boundary, there exists an appropriate extension of the scale such that the coefficients c_j and L in (3.8) can be written as the integrals of some affine invariants over M .

Let $\rho \in \mathcal{E}(2)$ be a Fefferman defining density and g_{ij} be the associated Blaschke metric on Ω . Until the normalization of the projective scale is involved, we work in an arbitrary projective scale $\tau \in \mathcal{E}(1)$. Set $\rho = \tau^{-2}\rho$ and let $\nabla \in [\nabla]$ be the representative connection such that $\nabla\tau = 0$. By strict convexity of M , there exists a unique $\xi \in \Gamma(TN)$ near M which satisfies

$$(4.1) \quad \xi\rho = 1, \quad \xi^i X^j \nabla_i \nabla_j \rho = 0 \quad \text{if} \quad X\rho = 0.$$

We call $r := \xi^i \xi^j \nabla_i \nabla_j \rho$ the *transverse curvature*. If we take a local frame $\{e_\alpha\}$ for the subbundle $\text{Ker } d\rho \subset TN$, the set $\{e_\infty = \xi, e_\alpha\}$ forms a local frame for TN . Let $\{\theta^\infty = d\rho, \theta^\alpha\}$ be the dual frame. Then we have

$$(4.2) \quad \nabla d\rho = h_{\alpha\beta} \theta^\alpha \otimes \theta^\beta + r d\rho \otimes d\rho,$$

where $h_{\alpha\beta}$ is the affine metric on each level set of ρ . The ambient metric $\tilde{g}_{IJ}$ satisfies

$$(4.3) \quad \tilde{g}_{IJ}|_M = \begin{pmatrix} 0 & 1 & \\ 1 & \mathbf{r}|_M & \\ & & \mathbf{h}_{\alpha\beta} \end{pmatrix}, \quad \tilde{g}^{IJ}|_M = \begin{pmatrix} -\mathbf{r}|_M & 1 & \\ 1 & 0 & \\ & & \mathbf{h}^{\alpha\beta} \end{pmatrix},$$

where $\mathbf{r} := \tau^{-2}r$

Lemma 4.5. *The vector field $\xi|_M$ agrees with the affine normal field.*

Proof. First, using the characterization (4.1) we have

$$\begin{aligned}
0 &= X^i \xi^j \nabla_i \nabla_j \rho \\
&= X^i \nabla_i (\xi^j \nabla_j \rho) - X^i \nabla_i \xi^j \nabla_j \rho \\
&= -d\rho(\nabla_X \xi)
\end{aligned}$$

for $X \in \Gamma(TM)$. This implies $\eta = 0$. Next, if we take $\{\xi, e_\alpha\}$ so that the dual frame $\{d\rho, \theta^\alpha\}$ satisfies $\tau^{-(n+2)} = d\rho \wedge \theta^1 \wedge \dots \wedge \theta^n$, we have

$$-1 = \mathcal{J}[\rho]|_M = \det \begin{pmatrix} 0 & 1 & \\ 1 & \mathbf{r} & \\ & & \mathbf{h}_{\alpha\beta} \end{pmatrix} = -\det h_{\alpha\beta}.$$

Thus, at the boundary it holds that

$$vol_h = (\det h_{\alpha\beta})^{1/2} \theta^1 \wedge \cdots \wedge \theta^n = \theta^1 \wedge \cdots \wedge \theta^n = \xi \lrcorner \tau^{-(n+2)}.$$

Therefore $\xi|_M$ is the affine normal field. $\square$

Let $F : M \times (-\epsilon_0, 0] \rightarrow U \subset \overline{\Omega}$ be the diffeomorphism defined by the flow generated by ξ . Since $\xi\rho = 1$ and $\rho|_M = 0$, it holds that the second component of F^{-1} equals ρ and $F_*\partial/\partial\rho = \xi$. We regard differential forms on M as those on $M \times (-\epsilon_0, 0]$ by pulling them back with the first projection. Take a local frame $\{e_\alpha\}$ for $\text{Ker } d\rho$ such that the frame $\{e_\infty = \xi, e_\alpha\}$ is unimodular with respect to $\tau^{-(n+2)}$, and let $\{\theta^\infty = d\rho, \theta^\alpha\}$ be the dual frame. Then we can write as

$$\begin{aligned} F^* \tau^{-(n+2)} &= F^*(d\rho \wedge \theta^1 \wedge \cdots \wedge \theta^n) \\ &= v(x, \rho) d\rho \wedge \theta^1|_{TM} \wedge \cdots \wedge \theta^n|_{TM} \\ &= v(x, \rho) d\rho \wedge vol_h, \end{aligned}$$

for a function v with $v(x, 0) = 1$. In view of the proof of Theorem 3.4, the coefficients c_j and L are the integrals of the Taylor coefficients in ρ of $v(x, \rho)$ at the boundary. In order to obtain the expansion of v , we will find a differential equation which v satisfies. Taking exterior derivatives of both sides of

$$F^*(\theta^1 \wedge \cdots \wedge \theta^n) = v(x, \rho) vol_h,$$

we have

$$(4.4) \quad F^* d(\theta^1 \wedge \cdots \wedge \theta^n) = \frac{\partial v}{\partial \rho} d\rho \wedge vol_h.$$

We write the connection forms of ∇ as $\omega_j^i = \Gamma_{kj}^i \theta^k$. Since ∇ is torsion-free, it holds that $d\theta^i = \theta^j \wedge \omega_j^i$, and thus we compute as

$$\begin{aligned} d(\theta^1 \wedge \cdots \wedge \theta^n) &= -\omega_\gamma^\gamma \wedge \theta^1 \wedge \cdots \wedge \theta^n + \Gamma_{\gamma\infty}^\gamma d\rho \wedge \theta^1 \wedge \cdots \wedge \theta^n \\ &= (\Gamma_{\gamma\infty}^\gamma - \Gamma_{\infty\gamma}^\gamma) d\rho \wedge \theta^1 \wedge \cdots \wedge \theta^n. \end{aligned}$$

From $\nabla(d\rho \wedge \theta^1 \wedge \cdots \wedge \theta^n) = 0$ it follows that $\Gamma_{\infty k}^k = 0$, so we have $\Gamma_{\infty\gamma}^\gamma = -\Gamma_{\infty\infty}^\infty = r$ by (4.2). Therefore we obtain

$$(4.5) \quad F^* d(\theta^1 \wedge \cdots \wedge \theta^n) = F^*(\text{tr} S - r) v d\rho \wedge vol_h,$$

where S is the affine shape operator on each level set of ρ . From (4.4) and (4.5), we obtain the following differential equation for v :

$$(4.6) \quad \begin{cases} \xi v = (\text{tr} S - r)v, \\ v|_M = 1. \end{cases}$$

Here, we have regarded v as a function on U . Thus the expansion of v is reduced to those of r and $\text{tr} S$. We will work in a special local frame:

Lemma 4.6. *Let $\{e_\alpha\}$ be a local frame for TM which is unimodular with respect to h . If one extends $\{e_\alpha\}$ by the parallel transport relative to ∇ along the integral curves of ξ , then it becomes a local frame for $\text{Ker } d\rho \subset TN$.*

Proof. We set $f = d\rho(e_\alpha)$ for fixed α . By $\nabla_\xi e_\alpha = 0$ and (4.2), we have $\xi f = (\nabla_\xi d\rho)(e_\alpha) = rf$. Thus f is the solution to the ODE $\xi f = rf$, $f|_M = 0$. It follows that $f = 0$ and so $e_\alpha \in \text{Ker } d\rho$. $\square$

We call $\{e_\infty = \xi, e_\alpha\}$ an *adapted frame* for TN when $\{e_\alpha\}$ is given as above. In an adapted frame, the connection forms of ∇ are given by

$$(4.7) \quad \begin{aligned} \omega_\alpha^\beta &= \omega_\alpha^\xi \theta^\beta, & \omega_\infty^\alpha &= S_\gamma^\alpha \theta^\gamma + C^\alpha d\rho, \\ \omega_\beta^\infty &= -h_{\beta\gamma} \theta^\gamma, & \omega_\infty^\infty &= -rd\rho, \end{aligned}$$

where C^α is a function, and $\omega^\xi_{\alpha^\beta}$ restricts to the connection form of the induced connection on each level set and satisfies $\omega^\xi_{\alpha^\beta}(\xi) = 0$. In the sequel, we will derive equations for the computation of the expansions of S_{α^β} , $h_{\alpha\beta}$, C^α , r , and P_{ij} .

First, the structure equation $\Omega_i^j = d\omega_i^j - \omega_i^k \wedge \omega_k^j$ gives

$$\begin{aligned}\Omega_\infty^\beta &= (\xi S_{\alpha^\beta} + S_{\alpha^\gamma} S_{\gamma^\beta} + r S_{\alpha^\beta} - \nabla_\alpha^\xi C^\beta) d\rho \wedge \theta^\alpha + \nabla_\alpha^\xi S_{\gamma^\beta} \theta^\alpha \wedge \theta^\gamma, \\ \Omega_\alpha^\infty &= (\xi h_{\alpha\beta} - r h_{\alpha\beta} + h_{\alpha\gamma} S_{\beta^\gamma}) \theta^\beta \wedge d\rho - \nabla_\beta^\xi h_{\alpha\gamma} \theta^\beta \wedge \theta^\gamma, \\ \Omega_\infty^\infty &= (C^\gamma h_{\gamma\alpha} + r_\alpha) d\rho \wedge \theta^\alpha + S_{\gamma^\mu} h_{\mu\beta} \theta^\gamma \wedge \theta^\beta, \\ \Omega_\alpha^\beta &\equiv \left(\frac{1}{2} R_{\gamma\mu}^{\xi\beta}{}_\alpha + h_{\alpha\gamma} S_{\mu^\beta} \right) \theta^\gamma \wedge \theta^\mu \mod d\rho,\end{aligned}$$

where $R_{\gamma\mu}^{\xi\beta}{}_\alpha$ is the curvature tensor of ∇^ξ . Comparing the coefficients of both sides of the above equations and using $C_{ij}^k{}_l = R_{ij}^k{}_l - 2\delta_{[i}^k P_{j]l} = 0$, we have

$$(4.8) \quad \xi S_{\alpha^\beta} + S_{\alpha^\gamma} S_{\gamma^\beta} + r S_{\alpha^\beta} - \nabla_\alpha^\xi C^\beta + \delta_\alpha^\beta P_{\infty\infty} = 0,$$

$$(4.9) \quad \nabla_{[\alpha}^\xi S_{\gamma]}^\beta - \delta_{[\alpha}^\beta P_{\gamma]\infty} = 0,$$

$$(4.10) \quad \xi h_{\alpha\beta} - r h_{\alpha\beta} + h_{\alpha\gamma} S_{\beta^\gamma} + P_{\alpha\beta} = 0,$$

$$(4.11) \quad C^\gamma h_{\gamma\alpha} + r_\alpha - P_{\alpha\infty} = 0,$$

$$(4.12) \quad R_{\gamma\mu}^{\xi\beta}{}_\alpha + 2h_{\alpha[\gamma} S_{\mu]}^\beta - 2\delta_{[\gamma}^\beta P_{\mu]\alpha} = 0.$$

The contractions of (4.9) and (4.12) yield

$$(4.13) \quad (n-1)P_{\alpha\infty} + \nabla_\alpha^\xi(\text{tr}S) - \nabla_\gamma^\xi S_{\alpha^\gamma} = 0,$$

$$(4.14) \quad (n-1)P_{\alpha\beta} - \text{Ric}_{\alpha\beta}^\xi - S_{\alpha\beta} + (\text{tr}S)h_{\alpha\beta} = 0,$$

where $\text{Ric}_{\alpha\beta}^\xi$ is the Ricci tensor of ∇^ξ .

Next, we take the exterior derivatives of both sides of

$$\Omega_\alpha^\beta = d\omega_\alpha^\beta - \omega_\alpha^\gamma \wedge \omega_\gamma^\beta - \omega_\alpha^\infty \wedge \omega_\infty^\beta$$

and compare the coefficients of $d\rho \wedge \theta^\gamma \wedge \theta^\mu$ to obtain

$$\delta_{[\gamma}^\beta \xi P_{\mu]\alpha} + \delta_{[\gamma}^\beta S_{\mu]}^\nu P_{\nu\alpha} - \delta_{[\gamma}^\beta \nabla_{\mu]}^\xi P_{\alpha\infty} + h_{\alpha[\gamma} \delta_{\mu]}^\beta P_{\infty\infty} = 0.$$

Taking the trace of the equation, we have

$$(4.15) \quad \xi P_{\alpha\beta} + S_{\alpha^\gamma} P_{\gamma\beta} - \nabla_\alpha^\xi P_{\beta\infty} - P_{\infty\infty} h_{\alpha\beta} = 0.$$

We shall derive an equation for r from the Monge–Ampère equation. If we write

$$\tau^{-(n+2)} = e^\varphi d\rho \wedge \theta^1 \wedge \cdots \wedge \theta^n$$

with a function φ , we have

$$0 = \nabla_\xi \tau^{-(n+2)} = (\xi\varphi - \Gamma_{\infty j}^j) \tau^{-(n+2)} = (\xi\varphi + r) \tau^{-(n+2)}$$

and thus

$$\xi\varphi = -r.$$

Let ψ_i^j be the connection forms of the Levi–Civita connection of g_{ij} . Then, since

$$d \log \det_\tau(g_{ij}) = 2\psi_k^k - 2d\varphi,$$

we have

$$(4.16) \quad \xi \log \det_\tau(g_{ij}) = 2\psi_k^k(\xi) + 2r.$$

In an adapted frame, the Blaschke metric is given by

$$g = \frac{\tilde{h}_{\alpha\beta}}{-2\rho} \theta^\alpha \cdot \theta^\beta - 2P_{\alpha\infty} \theta^\alpha \cdot d\rho + \frac{1 - 2r\rho - 4\rho^2 P_{\infty\infty}}{4\rho^2} d\rho^2,$$

where $\tilde{h}_{\alpha\beta} = h_{\alpha\beta} + 2\rho P_{\alpha\beta}$. Therefore, setting $\tilde{h}^{\alpha\beta} = (\tilde{h}_{\alpha\beta})^{-1}$, we have

$$\begin{aligned} 2\psi_k^k(\xi) &= \xi \log \det \begin{pmatrix} (1 - 2r\rho - 4\rho^2 P_{\infty\infty})/4\rho^2 & -P_{\beta\infty} \\ -P_{\alpha\infty} & \tilde{h}_{\alpha\beta}/(-2\rho) \end{pmatrix} \\ &= \xi \log \det \begin{pmatrix} (1 - 2r\rho - 4\rho^2 P_{\infty\infty})/4\rho^2 + 2\rho\tilde{h}^{\gamma\mu}P_{\gamma\infty}P_{\mu\infty} & -P_{\beta\infty} \\ 0 & \tilde{h}_{\alpha\beta}/(-2\rho) \end{pmatrix} \\ &= -(n+2)\rho^{-1} + \xi \log \det(\tilde{h}_{\alpha\beta}) \\ &\quad + \xi \log(1 - 2r\rho - 4\rho^2 P_{\infty\infty} + 8\rho^3 \tilde{h}^{\gamma\mu}P_{\gamma\infty}P_{\mu\infty}). \end{aligned}$$

By (4.10) we also have

$$\begin{aligned} \xi \log \det(\tilde{h}_{\alpha\beta}) &= h^{\alpha\beta} \xi h_{\alpha\beta} + \xi \log \det(\delta_\alpha^\beta + 2\rho h^{\beta\gamma}P_{\gamma\alpha}) \\ &= nr - \text{tr}S - h^{\alpha\beta}P_{\alpha\beta} + \xi \log \det(\delta_\alpha^\beta + 2\rho h^{\beta\gamma}P_{\gamma\alpha}). \end{aligned}$$

Consequently the right-hand side of (4.16) is written as

$$\begin{aligned} 2\psi_k^k(\xi) + 2r &= -(n+2)\rho^{-1} + \xi \log(1 - 2r\rho - 4\rho^2 P_{\infty\infty} + 8\rho^3 \tilde{h}^{\gamma\mu}P_{\gamma\infty}P_{\mu\infty}) \\ &\quad + \xi \log \det(\delta_\alpha^\beta + 2\rho h^{\beta\gamma}P_{\gamma\alpha}) + (n+2)r - \text{tr}S - h^{\alpha\beta}P_{\alpha\beta}. \end{aligned}$$

On the other hand, by the approximate Monge–Ampère equation (3.7), we have

$$(-2\rho)^{n+2} \det_\tau(g_{ij}) = 1 - \underline{\mathcal{Q}}\rho^{n/2+1},$$

where $\underline{\mathcal{Q}} = \tau^{n+2}\mathcal{Q}$. (We will write the formulas only in the case where n is even; when n is odd, terms which come from $\underline{\mathcal{Q}}\rho^{n/2+1}$ are replaced by $O(\rho^\infty)$.) It follows that the left-hand side of (4.16) satisfies

$$(n+2)\rho^{-1} + \xi \log \det_\tau(g_{ij}) = -\left(\frac{n}{2} + 1\right)\underline{\mathcal{Q}}\rho^{n/2} + O(\rho^{n/2+1}).$$

As a result we obtain

$$\begin{aligned} (4.17) \quad (n+2)r + \xi \log(1 - 2r\rho - 4\rho^2 P_{\infty\infty} + 8\rho^3 \tilde{h}^{\gamma\mu}P_{\gamma\infty}P_{\mu\infty}) \\ + \xi \log \det(\delta_\alpha^\beta + 2\rho h^{\beta\gamma}P_{\gamma\alpha}) - \text{tr}S - h^{\alpha\beta}P_{\alpha\beta} \\ = -\left(\frac{n}{2} + 1\right)\underline{\mathcal{Q}}\rho^{n/2} + O(\rho^{n/2+1}). \end{aligned}$$

Finally we consider the expansion of $P_{\infty\infty}$, which involves a normalization of the projective scale. We set

$$\mathcal{S} = \left\{ \tau_I = \begin{pmatrix} \tau \\ \mu_i \end{pmatrix} \in \Gamma(M, \tilde{\mathcal{E}}_I) \mid \tau > 0, X^i(\nabla_i \tau - \mu_i) = 0 \text{ for } X^i \in TM \right\}.$$

Note that the condition $X^i(\nabla_i \tau - \mu_i) = 0$ is independent of the choice of $\nabla \in [\nabla]$ since it is the projecting part of $X^i \nabla_i \tau_I$. The set $\mathcal{S}$ can be viewed as the space of 1-jets of projective scales at the boundary M , so a choice of an element of $\mathcal{S}$ determines affine invariants on M . For each $\tau_I \in \mathcal{S}$, we construct a projective scale near M as follows: Take the affine normal field ξ along the boundary, which is determined by τ_I . Then we extend τ_I by the parallel transport with respect to the tractor connection along the geodesic paths of $[\nabla]$ with initial velocity ξ . The top slot of the extension of τ_I is positive near M so it defines a projective scale whose 1-jet is given by τ_I . We call a projective scale constructed in this way a *normalized scale*. A global affine scale τ is a normalized scale since $D_I \tau$ is a parallel tractor by Proposition 2.2.

Lemma 4.7. *Let $\tau \in \mathcal{E}(1)$ be a normalized scale and $\nabla \in [\nabla]$ the corresponding representative connection. Let c be the geodesic of ∇ whose initial velocity is given by the affine normal field ξ . Then the velocity vector field V^i of c satisfies*

$$P_{ij}V^iV^j = 0,$$

where P_{ij} is the projective Schouten tensor of ∇ .

Proof. Let $\tau_I = (\tau, \mu_i)$ be the extension of the 1-jet of τ by the parallel transport along c . By the normalization condition on τ it holds that

$$(4.18) \quad 0 = V^i \nabla_i \begin{pmatrix} \tau \\ \mu_j \end{pmatrix} \stackrel{\tau}{=} \begin{pmatrix} -V^i \mu_i \\ V^i \nabla_i \mu_j + V^i P_{ij} \tau \end{pmatrix}.$$

Since $V^j \nabla_j V^i = 0$, the first slot of (4.18) implies

$$0 = V^j \nabla_j (V^i \mu_i) = V^j V^i \nabla_j \mu_i.$$

Then it follows from the second slot of (4.18) that

$$0 = V^i V^j \nabla_i \mu_j + V^i V^j P_{ij} \tau = V^i V^j P_{ij} \tau.$$

Thus we have $P_{ij}V^iV^j = 0$. $\square$

We write $V = V^\infty \xi + V^\alpha e_\alpha$ with an adapted frame $\{\xi, e_\alpha\}$. Then since $\nabla_V V = 0$, $P_{ij}V^iV^j = 0$ and $V|_M = \xi$, we have

$$(4.19) \quad \begin{cases} V^\infty \xi V^\infty + V^\gamma e_\gamma V^\infty - r(V^\infty)^2 - h_{\gamma\mu} V^\gamma V^\mu = 0, \\ V^\infty \xi V^\alpha + V^\gamma \nabla_\gamma^\xi V^\alpha + S_\gamma^\alpha V^\gamma V^\infty + C^\alpha (V^\infty)^2 = 0, \\ (V^\infty)^2 P_{\infty\infty} + 2P_{\gamma\infty} V^\gamma V^\infty + P_{\gamma\mu} V^\gamma V^\mu = 0, \\ V^\infty|_M = 1, \\ V^\alpha|_M = 0. \end{cases}$$

These equations enable us to compute the expansion of $P_{\infty\infty}$ together with the expansions of V^∞ and V^α .

Now we have completed the derivation of all equations that we use in the calculation of expansions of affine invariants. We will need some commutation relations in the computations. In the following lemma, we present the commutation relation for a tensor t_β^γ , but the equation can be extended in an obvious way for general type of tensors.

Lemma 4.8. *Let $\{\xi, e_\alpha\}$ be an adapted frame. For a tensor field $t_\beta^\gamma \in \Gamma(\text{Ker } d\rho \otimes (\text{Ker } d\rho)^*)$, one has*

$$\begin{aligned} \xi \nabla_\alpha^\xi t_\beta^\gamma &= \nabla_\alpha^\xi (\xi t_\beta^\gamma) - S_\alpha^\mu \nabla_\mu^\xi t_\beta^\gamma + (C^\gamma h_{\alpha\mu} - \delta_\alpha^\gamma P_{\mu\infty}) t_\beta^\mu \\ &\quad - (C^\mu h_{\alpha\beta} - \delta_\alpha^\mu P_{\beta\infty}) t_\mu^\gamma. \end{aligned}$$

Proof. We extend t_β^γ to t_j^k by setting $t_\infty^\gamma = t_\beta^\infty = t_\infty^\infty = 0$. By (4.7), we have

$$\nabla_\infty \nabla_\alpha t_\beta^\gamma = \xi \nabla_\alpha t_\beta^\gamma + C^\gamma \nabla_\alpha t_\beta^\infty = \xi \nabla_\alpha^\xi t_\beta^\gamma - C^\gamma h_{\alpha\mu} t_\beta^\mu.$$

Using $R_{ij}{}^k{}_l = 2\delta_{[i}{}^k P_{j]l}$, we also have

$$\begin{aligned} \nabla_\infty \nabla_\alpha t_\beta^\gamma &= \nabla_\alpha \nabla_\infty t_\beta^\gamma - R_{\infty\alpha}{}^\mu{}_\beta t_\mu^\gamma + R_{\infty\alpha}{}^\gamma{}_\mu t_\beta^\mu \\ &= \nabla_\alpha^\xi (\xi t_\beta^\gamma) - S_\alpha^\mu \nabla_\mu^\xi t_\beta^\gamma - h_{\alpha\beta} C^\mu t_\mu^\gamma + P_{\beta\infty} t_\alpha^\gamma - \delta_\alpha^\gamma P_{\mu\infty} t_\beta^\mu. \end{aligned}$$

Comparing the above two expressions for $\nabla_\infty \nabla_\alpha t_\beta^\gamma$, we obtain the desired equation. $\square$

Proposition 4.9. *Let $\tau \in \mathcal{E}(1)$ be a normalized projective scale and $\rho \in \mathcal{E}(2)$ a Fefferman defining density. Then in an adapted frame $\{\xi, e_\alpha\}$, the boundary values of*

$$\begin{aligned} &\xi^{k+1}S_\alpha^\beta, \quad \xi^{k+1}h_{\alpha\beta}, \quad \xi^{k+1}P_{\alpha\beta}, \quad \xi^{k+1}V^\infty, \quad \xi^{k+1}V^\alpha, \\ &\xi^k r, \quad \xi^k P_{\alpha\infty}, \quad \xi^k P_{\infty\infty}, \quad \xi^k C^\alpha \end{aligned}$$

are determined by (4.8), (4.10), (4.11), (4.13), (4.14), (4.15), (4.17), and (4.19) for $k \leq n/2 - 1$ when n is even and for all k when n is odd. Moreover they are expressed in terms of $h_{\alpha\beta}$, S_α^β , $\text{Ric}_{\alpha\beta}^h$, $A_{\alpha\beta\gamma}$ and their covariant derivatives with respect to ∇^h .

The proof is given by the induction on k ; we differentiate the equations in the ξ -direction repeatedly and use Lemma 4.8 to obtain the Taylor coefficients as tensors on the boundary. If we differentiate (4.17) k times, the coefficient of $\xi^k r$ becomes $n - 2k$, so when n is even, $\xi^k r|_M$ is determined up to $k = n/2 - 1$. When $k = n/2$, we obtain the expression of the obstruction $\underline{Q}$ in terms of tensors on the boundary.

By (4.6) and Proposition 4.9, we obtain the following theorem:

Theorem 4.10. *Let $\tau \in \mathcal{E}(1)$ be a normalized projective scale. Then the coefficients c_j and L in the volume expansion (3.8) are given by the integrals of linear combinations of complete contractions of $h_{\alpha\beta}$, $\text{Ric}_{\alpha\beta}^h$, S_α^β , $A_{\alpha\beta\gamma}$ and their covariant derivatives with respect to ∇^h .*

4.3. Explicit calculations. We calculate the expansion of v and present an explicit formula for L when $n = 2$. Setting $\rho = 0$ in (4.14), we have

$$\begin{aligned} (4.20) \quad (n-1)P_{\alpha\beta}|_M &= \text{Ric}_{\alpha\beta}^\xi + S_{\alpha\beta} - (\text{tr}S)h_{\alpha\beta} \\ &= \text{Ric}_{\alpha\beta}^h - (\delta A)_{\alpha\beta} - A_{\alpha\mu\nu}A_\beta^{\mu\nu} + S_{\alpha\beta} - (\text{tr}S)h_{\alpha\beta}, \end{aligned}$$

where $(\delta A)_{\alpha\beta} := \nabla_\gamma^h A_{\alpha\beta}^\gamma$. Then setting $\rho = 0$ in (4.17) gives

$$\begin{aligned} (4.21) \quad r|_M &= \frac{1}{n}(\text{tr}S - h^{\alpha\beta}P_{\alpha\beta}|_M) \\ &= \frac{2}{n}\text{tr}S - \frac{1}{n(n-1)}(\text{Scal}_h - |A|^2). \end{aligned}$$

By (4.6) we have

$$\xi v|_M = \text{tr}S - r|_M = \frac{1}{n(n-1)}(\text{Scal}_h - |A|^2) + \frac{n-2}{n}\text{tr}S.$$

Therefore, when $n = 2$, L agrees with a nonzero multiple of

$$(4.22) \quad 4\pi\chi(M) - \int_M |A|^2 \text{vol}_h$$

by the Gauss–Bonnet theorem.

We will give a formula for L when $n = 4$ in a suitably chosen scale in §5.5.

5. CONFORMAL CODAZZI STRUCTURE

5.1. Correspondence between projective and conformal tractor bundles.

Let $(N, [\nabla])$ be an $(n+1)$ -dimensional locally flat projective manifold and let $M \subset N$ be a strictly convex hypersurface. Then we have two bundles over M : the restriction of the projective tractor bundle $\tilde{\mathcal{E}}^I$ and the conformal tractor bundle for the affine metric, which we denote by $\mathcal{E}^I$. We will see that there exists a canonical isomorphism between these two bundles.

First we recall the conformal tractor bundle and conformal tractor connection. For each choice of a conformal scale $\kappa \in \mathcal{E}[1]$, the conformal tractor bundle is expressed as

$$\begin{array}{c} \mathcal{E}[1] \\ \oplus \\ \mathcal{E}^I \cong \mathcal{E}^\alpha[-1] \\ \oplus \\ \mathcal{E}[-1] \end{array}$$

If we change the scale as $\widehat{\kappa} = e^{-\Upsilon} \kappa$, the expression transforms as

$$(5.1) \quad \begin{pmatrix} \widehat{\sigma} \\ \widehat{\mu}^\alpha \\ \widehat{\varphi} \end{pmatrix} = \begin{pmatrix} \sigma \\ \mu^\alpha + \Upsilon^\alpha \sigma \\ \varphi - \Upsilon_\gamma \mu^\gamma - \frac{1}{2} \Upsilon_\gamma \Upsilon^\gamma \sigma \end{pmatrix}.$$

The *tractor metric* $h_{IJ} \in \mathcal{E}_{(IJ)}$ is a conformally invariant symmetric form defined by

$$h_{IJ} U^I U'^J = \mathbf{h}_{\alpha\beta} \mu^\alpha \mu'^\beta + \sigma \varphi' + \varphi \sigma',$$

where $U^I = {}^t(\sigma, \mu^\alpha, \varphi)$ and $U'^I = {}^t(\sigma', \mu'^\alpha, \varphi')$. When $n \geq 3$, the *conformal tractor connection* is defined by

$$(5.2) \quad \nabla_\alpha \begin{pmatrix} \sigma \\ \mu^\beta \\ \varphi \end{pmatrix} = \begin{pmatrix} \nabla_\alpha^h \sigma - \mu_\alpha \\ \nabla_\alpha^h \mu^\beta + \delta_\alpha^\beta \varphi + P_\alpha^{h\beta} \sigma \\ \nabla_\alpha^h \varphi - P_{\alpha\gamma}^h \mu^\gamma \end{pmatrix},$$

where

$$P_{\alpha\beta}^h = \frac{1}{n-2} \left(\text{Ric}_{\alpha\beta}^h - \frac{\text{Scal}_h}{2(n-1)} \mathbf{h}_{\alpha\beta} \right)$$

is the (*conformal*) *Schouten tensor* of the representative metric $h_{\alpha\beta} = \kappa^{-2} \mathbf{h}_{\alpha\beta}$. The tractor connection is conformally invariant and preserves the tractor metric. The curvature of the connection is given by

$$(5.3) \quad \Omega_{\alpha\beta}{}^K{}_L = \begin{pmatrix} 0 & 0 & 0 \\ 2Y_{\alpha\beta}{}^\gamma & W_{\alpha\beta}{}^{\gamma\delta} & 0 \\ 0 & -2Y_{\alpha\beta\delta} & 0 \end{pmatrix},$$

where

$$Y_{\alpha\beta\gamma} = \nabla_{[\alpha}^h P_{\beta]\gamma}^h$$

is the *Cotton-York tensor* and

$$W_{\alpha\beta\gamma\delta}^h = R_{\alpha\beta\gamma\delta}^h - 2(\mathbf{h}_{\gamma[\alpha} P_{\beta]\delta}^h - \mathbf{h}_{\delta[\alpha} P_{\beta]\gamma}^h)$$

is the *conformal Weyl tensor*. When $n = 2$, we need an additional structure on $(M, [h])$ in order to define a tractor connection. For a fixed conformal scale $\kappa \in \mathcal{E}[1]$, we set

$$(5.4) \quad \mathbb{P}_{\alpha\beta} = \mathbb{P}_{\alpha\beta}^\kappa := \text{tf} \mathbf{S}_{\alpha\beta} - \frac{1}{2} (\delta A)_{\alpha\beta} + \frac{1}{4} \text{Scal}_h \mathbf{h}_{\alpha\beta} \in \mathcal{E}_{(\alpha\beta)},$$

where $\text{tf} \mathbf{S}_{\alpha\beta}$ denotes the trace-free part of $\mathbf{S}_{\alpha\beta}$ with respect to $\mathbf{h}_{\alpha\beta}$. Then it holds that

$$\mathbb{P}_\alpha{}^\alpha = \frac{1}{2} \text{Scal}_h$$

and

$$\widehat{\mathbb{P}}_{\alpha\beta} = \mathbb{P}_{\alpha\beta} - \nabla_\alpha^h \Upsilon_\beta + \Upsilon_\alpha \Upsilon_\beta - \frac{1}{2} \Upsilon_\gamma \Upsilon^\gamma \mathbf{h}_{\alpha\beta}$$

for a rescaling $\widehat{\kappa} = e^{-\Upsilon} \kappa$. Thus $\mathbb{P}_{\alpha\beta}$ defines a *Möbius structure* on $(M, [h])$ in the sense of [C], [R]. We denote the set $\{\mathbb{P}_{\alpha\beta}^\kappa \mid 0 < \kappa \in \mathcal{E}[1]\}$ by $[\mathbb{P}]$. The tensor $\mathbb{P}_{\alpha\beta}$ satisfies $R_{\alpha\beta\gamma\delta}^h = 2\mathbf{h}_{\gamma[\alpha} \mathbb{P}_{\beta]\delta} - 2\mathbf{h}_{\delta[\alpha} \mathbb{P}_{\beta]\gamma}$, so it is a 2-dimensional analogue of the

conformal Schouten tensor. We set $P_{\alpha\beta}^h := \mathbb{P}_{\alpha\beta}$ for a Möbius surface and define the *Möbius tractor connection* by (5.2). The curvature is given by (5.3) with $W_{\alpha\beta\gamma\delta}^h = 0$.

Let us denote the inclusion map by $\iota : M \rightarrow N$. The isomorphism between $\tilde{\mathcal{E}}^I|_M$ and $\mathcal{E}^I$ is given by the following proposition:

Proposition 5.1. *Let $\tau \in \mathcal{E}(1)$ be a projective scale and set $J' = \frac{1}{2(n-1)}(\text{Scal}_h - |A|^2) \in \mathcal{E}[-2]$ and $\xi^i = \tau^{-2}\xi^i \in \mathcal{E}^i(-2)|_M$, where ξ^i is the affine normal field. Then the bundle isomorphism*

$$\Phi : \mathcal{E}^I \xrightarrow{\tau|_M} \mathcal{E}^\alpha[-1] \oplus \mathcal{E}^\alpha[-1] \ni \begin{pmatrix} \sigma \\ \mu^\alpha \\ \varphi \end{pmatrix} \mapsto \begin{pmatrix} \sigma \\ \varphi - \frac{1}{n}(\text{tr}\mathbf{S} - J')\sigma \end{pmatrix} \in \begin{matrix} \mathcal{E}^i(-1)|_M \\ \oplus \\ \mathcal{E}(-1)|_M \end{matrix} \xrightarrow{\tau} \tilde{\mathcal{E}}^I|_M$$

is independent of τ . Moreover, if $\rho \in \mathcal{E}(2)$ is a Fefferman defining density of M , it holds that $\Phi^*\tilde{g}_{IJ} = h_{IJ}$, where $\tilde{g}_{IJ} = D_I D_J \rho$ is the ambient metric.

Proof. Let $\hat{\tau} = e^{-\Upsilon}\tau$ be another projective scale. By Proposition 4.2 and (5.1), we have

$$\begin{aligned} \iota_*\hat{\mu}^\alpha + \hat{\sigma}\hat{\xi}^i &= \iota_*\mu^\alpha + \sigma\iota_*\Upsilon^\alpha + \sigma(\xi^i - \iota_*\Upsilon^\alpha) \\ &= \iota_*\mu^\alpha + \sigma\xi^i, \\ \hat{\varphi} - \frac{1}{n}(\text{tr}\hat{\mathbf{S}} - \hat{J}')\hat{\sigma} &= \varphi - \Upsilon_\gamma\mu^\gamma - \frac{1}{2}\Upsilon_\gamma\Upsilon^\gamma\sigma - \frac{1}{n}(\text{tr}\mathbf{S} - J' + n\xi^i\Upsilon_i - \frac{n}{2}\Upsilon_\gamma\Upsilon^\gamma)\sigma \\ &= \varphi - \frac{1}{n}(\text{tr}\mathbf{S} - J')\sigma - \Upsilon_i(\iota_*\mu + \sigma\xi^i). \end{aligned}$$

This agrees with the transformation formula for the projective tractor, so Φ is invariant under rescaling. Let ρ is a Fefferman defining density of M . Using (4.3) and (4.21), we have

$$\begin{aligned} (\Phi^*\tilde{g}_{IJ})U^I U'^J &= h_{\alpha\beta}\mu^\alpha\mu'^\beta + r|_M\sigma\sigma' + \left(\varphi - \frac{1}{n}(\text{tr}\mathbf{S} - J')\sigma\right)\sigma' \\ &\quad + \sigma\left(\varphi' - \frac{1}{n}(\text{tr}\mathbf{S} - J')\sigma'\right) \\ &= h_{\alpha\beta}\mu^\alpha\mu'^\beta + \varphi\sigma' + \sigma\varphi' \\ &= h_{IJ}U^I U'^J \end{aligned}$$

for $U^I = {}^t(\sigma, \mu^\alpha, \varphi)$ and $U'^I = {}^t(\sigma', \mu'^\alpha, \varphi')$. Thus $\Phi^*\tilde{g}_{IJ} = h_{IJ}$ holds. $\square$

Remark 5.2. The correspondence between the projective and conformal tractor bundles is also considered by Čap and Gover [ČG2] in the context of projective compactification. In our setting, when $\nabla \in [\nabla]$ is a representative connection corresponding to a projective scale τ and $\rho = \tau^2\rho$ is a Fefferman defining density, the connection $\nabla' = \nabla - (2\rho)^{-1}d\rho$ is projectively compact of order 2 and the isomorphism described in [ČG2] agrees with the one given above.

5.2. Comparison of connections on the conformal tractor bundle. We denote by $\nabla_\alpha^{\text{proj}}$ the restriction of the projective tractor connection to $\tilde{\mathcal{E}}^I|_M$, and set $\nabla_\alpha := \Phi^{-1} \circ \nabla_\alpha^{\text{proj}} \circ \Phi$. A calculation shows that the flat connection ∇ and the conformal (Möbius) tractor connection ∇ are related as

$$(5.5) \quad \nabla_\alpha U^J = \nabla_\alpha U^J - \tilde{A}_{\alpha K}{}^J U^K, \quad \tilde{A}_{\alpha K}{}^J = \begin{pmatrix} 0 & 0 & 0 \\ E_\alpha^\beta & A_{\alpha\gamma}{}^\beta & 0 \\ F_\alpha & G_{\alpha\gamma} & 0 \end{pmatrix},$$

where

$$\begin{aligned}
(5.6) \quad E_{\alpha\beta} &= -\text{tf} \mathbf{S}_{\alpha\beta} - \frac{1}{n} J' \mathbf{h}_{\alpha\beta} + P_{\alpha\beta}^h \in \mathcal{E}_{(\alpha\beta)}, \\
G_{\alpha\beta} &= P_{\alpha\beta} + \frac{1}{n} (\text{tr} \mathbf{S} - J') \mathbf{h}_{\alpha\beta} - P_{\alpha\beta}^h \in \mathcal{E}_{(\alpha\beta)}, \\
F_{\alpha} &= \tau^{-2} P_{\alpha\infty} + \frac{1}{n} \nabla_{\alpha}^h (\text{tr} \mathbf{S} - J') \in \mathcal{E}_{\alpha}[-2]
\end{aligned}$$

in any projective scale $\tau \in \mathcal{E}(1)$. By (4.14), it holds that $E_{\gamma}{}^{\gamma} + G_{\gamma}{}^{\gamma} = 0$.

The following proposition asserts that the flatness of the connection ∇ determines $E_{\alpha\beta}$, $G_{\alpha\beta}$, F_{α} and imposes some equations on $A_{\alpha\beta\gamma}$:

Proposition 5.3. *Let ∇ be a connection on $\mathcal{E}^I$ of the form (5.5) with some tensors $E_{\alpha\beta} \in \mathcal{E}_{(\alpha\beta)}$, $G_{\alpha\beta} \in \mathcal{E}_{(\alpha\beta)}$, $F_{\alpha} \in \mathcal{E}_{\alpha}[-2]$ and $A_{\alpha\beta\gamma} \in \mathcal{E}_{(\alpha\beta\gamma)_0}[2]$ in a fixed conformal scale. We assume that $E_{\gamma}{}^{\gamma} + G_{\gamma}{}^{\gamma} = 0$. When $n = 2$, we also assume that $\text{tf}(E_{\alpha\beta} - G_{\alpha\beta}) = 0$. Then ∇ is flat if and only if the following two conditions hold:*

(i) *The tensors $E_{\alpha\beta}$, $G_{\alpha\beta}$ and F_{α} satisfy*

$$(5.7) \quad E_{\alpha\beta} = -\frac{1}{n} (\delta A)_{\alpha\beta} + P_{\alpha\beta}^A,$$

$$(5.8) \quad G_{\alpha\beta} = -\frac{1}{n} (\delta A)_{\alpha\beta} - P_{\alpha\beta}^A,$$

$$(5.9) \quad F_{\alpha} = \frac{1}{n(n-1)} (\delta^2 A)_{\alpha} + \frac{1}{n-1} A_{\alpha}{}^{\mu\nu} (P_{\mu\nu}^h - P_{\mu\nu}^A),$$

where

$$P_{\alpha\beta}^A = \begin{cases} \frac{1}{n-2} (A_{\alpha\mu\nu} A_{\beta}{}^{\mu\nu} - \frac{|A|^2}{2(n-1)} \mathbf{h}_{\alpha\beta}) & (n \geq 3) \\ \frac{1}{4} |A|^2 \mathbf{h}_{\alpha\beta} & (n = 2). \end{cases}$$

(ii) *The tensor $A_{\alpha\beta\gamma}$ satisfies the Gauss equation*

$$(5.10) \quad 2 \text{tf}(A_{\nu\gamma[\alpha} A_{\beta]\mu}{}^{\nu}) + W_{\alpha\beta\gamma\mu}^h = 0 \quad (n \geq 4),$$

$$(5.11) \quad \nabla_{[\alpha}^h P_{\beta]\gamma}^A + \frac{1}{3} A_{\mu\gamma[\alpha} (\delta A)_{\beta]}{}^{\mu} - Y_{\alpha\beta\gamma} = 0 \quad (n = 3),$$

$$(5.12) \quad \nabla_{\alpha}^h |A|^2 + 2 A_{\alpha}{}^{\mu\nu} (\delta A)_{\mu\nu} - 8 Y_{\alpha\mu}{}^{\mu} = 0 \quad (n = 2),$$

and the Codazzi equation

$$(5.13) \quad \nabla_{[\alpha}^h A_{\beta]\gamma\mu} - \frac{1}{n} (\mathbf{h}_{\mu[\alpha} (\delta A)_{\beta]\gamma} + \mathbf{h}_{\gamma[\alpha} (\delta A)_{\beta]\mu}) = 0 \quad (n \geq 3),$$

$$(5.14) \quad \nabla_{[\alpha}^h (\delta^2 A)_{\beta]} + 4 P_{\mu[\alpha}^h (\delta A)_{\beta]}{}^{\mu} - 2 A^{\gamma\mu}{}_{[\alpha} \nabla_{\beta]}^h P_{\gamma\mu}^h = 0 \quad (n = 2).$$

Proof. We denote the curvature of ∇_{α} and ∇_{α} by $\Omega_{\alpha\beta M}^{\nabla}{}^K$ and $\Omega_{\alpha\beta M}^{\nabla}{}^K$ respectively. Then we can compute as

$$\begin{aligned}
(1/2) \Omega_{\alpha\beta M}^{\nabla}{}^K &= -\nabla_{[\alpha} \tilde{A}_{\beta]M}{}^K - \tilde{A}_{[\alpha|M}{}^L \tilde{A}_{\beta]L}{}^K + (1/2) \Omega_{\alpha\beta M}^{\nabla}{}^K \\
&= \begin{pmatrix} 0 & 0 & 0 \\ \Omega_{\alpha\beta\infty}{}^{\gamma} & \Omega_{\alpha\beta\mu}{}^{\gamma} & 0 \\ \Omega_{\alpha\beta\infty}{}^0 & \Omega_{\alpha\beta\mu}{}^0 & 0 \end{pmatrix},
\end{aligned}$$

where

$$(5.15) \quad \Omega_{\alpha\beta\mu}{}^\gamma = -\nabla_{[\alpha}^h A_{\beta]\mu}{}^\gamma - \mathbf{h}_{\mu[\alpha} E_{\beta]}{}^\gamma - \delta_{[\alpha}{}^\gamma G_{\beta]\mu} - A_{\nu\mu[\alpha} A_{\beta]}{}^{\gamma\nu} + \frac{1}{2} W_{\alpha\beta}^h{}^\gamma{}_\mu,$$

$$(5.16) \quad \Omega_{\alpha\beta\infty}{}^\gamma = -\nabla_{[\alpha}^h E_{\beta]}{}^\gamma + P_{\nu[\alpha}^h A_{\beta]}{}^{\nu\gamma} - \delta_{[\alpha}{}^\gamma F_{\beta]} - E_{\nu[\alpha} A_{\beta]}{}^{\gamma\nu} + Y_{\alpha\beta}{}^\gamma,$$

$$(5.17) \quad \Omega_{\alpha\beta\mu}{}^0 = -\nabla_{[\alpha}^h G_{\beta]\mu} + P_{\nu[\alpha}^h A_{\beta]\mu}{}^\nu - \mathbf{h}_{\mu[\alpha} F_{\beta]} - A_{\nu\mu[\alpha} G_{\beta]}{}^\nu - Y_{\alpha\beta\mu},$$

$$(5.18) \quad \Omega_{\alpha\beta\infty}{}^0 = -\nabla_{[\alpha}^h F_{\beta]} + P_{\nu[\alpha}^h E_{\beta]}{}^\nu + P_{\nu[\alpha}^h G_{\beta]}{}^\nu - E_{\nu[\alpha} G_{\beta]}{}^\nu.$$

By taking the symmetric part of (5.15), we have

$$\Omega_{\alpha\beta(\mu\gamma)} = -\nabla_{[\alpha}^h A_{\beta]\mu\gamma} - \frac{1}{2} \mathbf{h}_{\mu[\alpha} (E_{\beta]\gamma} + G_{\beta]\gamma}) - \frac{1}{2} \mathbf{h}_{\gamma[\alpha} (E_{\beta]\mu} + G_{\beta]\mu}).$$

Using the assumption $E_\gamma{}^\gamma + G_\gamma{}^\gamma = 0$, we see that $\Omega_{\alpha\beta(\mu\gamma)} = 0$ if and only if $E_{\beta\gamma} + G_{\beta\gamma} = -(2/n)(\delta A)_{\beta\gamma}$ and $A_{\alpha\beta\gamma}$ satisfies (5.13). Note that the equation (5.13) always holds when $n = 2$ since a tensor $K_{\alpha\beta\mu\gamma} = K_{[\alpha\beta](\mu\gamma)}$ is determined by the trace $K_{\alpha\beta}{}^\alpha{}_\gamma$. Next we take the skew symmetric part of (5.15) to obtain

$$2\Omega_{\alpha\beta[\mu\gamma]} = -\mathbf{h}_{\mu[\alpha} (E_{\beta]\gamma} - G_{\beta]\gamma}) + \mathbf{h}_{\gamma[\alpha} (E_{\beta]\mu} - G_{\beta]\mu}) - 2A_{\nu\mu[\alpha} A_{\beta]\gamma}{}^\nu + W_{\alpha\beta\gamma\mu}^h.$$

It follows that when $n \geq 3$, $\Omega_{\alpha\beta[\mu\gamma]} = 0$ if and only if $E_{\beta\gamma} - G_{\beta\gamma} = 2P_{\beta\gamma}^A$ and $A_{\alpha\beta\gamma}$ satisfies (5.10), but in the case $n = 3$ the equation (5.10) becomes vacuous since a tensor with the Weyl curvature symmetry automatically vanishes. When $n = 2$, using $W_{\alpha\beta\gamma\mu}^h = 0$ and the fact that a tensor $K_{\alpha\beta\mu\gamma}$ with the Riemannian curvature symmetry is determined by its trace $K_{\alpha\beta}{}^{\alpha\beta}$, we see that $\Omega_{\alpha\beta[\mu\gamma]} = 0$ if and only if $E_\gamma{}^\gamma - G_\gamma{}^\gamma = |A|^2$. Since we have assumed that $\text{tf}(E_{\beta\gamma} - G_{\beta\gamma}) = 0$, this is equivalent to $E_{\beta\gamma} - G_{\beta\gamma} = (1/2)|A|^2 \mathbf{h}_{\beta\gamma}$.

Thus the condition $\Omega_{\alpha\beta\mu\gamma} = 0$ determines $E_{\alpha\beta}$ and $G_{\alpha\beta}$ as in (5.7) and (5.8), and imposes the equations (5.10) and (5.13).

By (5.7), (5.8), (5.16) and (5.17), we have

$$\Omega_{\gamma\beta\infty}{}^\gamma + \Omega_{\gamma\beta}{}^{\gamma 0} = \frac{1}{n}(\delta^2 A)_\beta + A_\beta{}^{\gamma\mu}(P_{\mu\gamma}^h - P_{\mu\gamma}^A) - (n-1)F_\beta,$$

$$\Omega_{\gamma\beta\infty}{}^\gamma - \Omega_{\gamma\beta}{}^{\gamma 0} = \nabla_\beta^h P_\gamma^A - \nabla_\gamma^h P_\beta^A + \frac{1}{n} A_\beta{}^{\mu\gamma}(\delta A)_{\mu\gamma} - 2Y_{\beta\gamma}{}^\gamma.$$

When $n \geq 3$, as $Y_{\beta\gamma}{}^\gamma = 0$, we see that $\Omega_{\gamma\beta\infty}{}^\gamma - \Omega_{\gamma\beta}{}^{\gamma 0} = 0$ follows from (5.13), so the condition $\Omega_{\gamma\beta\infty}{}^\gamma = \Omega_{\gamma\beta}{}^{\gamma 0} = 0$ is equivalent to the equation (5.9). On the other hand, when $n = 2$ it holds that

$$\Omega_{\gamma\beta\infty}{}^\gamma - \Omega_{\gamma\beta}{}^{\gamma 0} = \frac{1}{4} \nabla_\beta^h |A|^2 + \frac{1}{2} A_\beta{}^{\mu\gamma}(\delta A)_{\mu\gamma} - 2Y_{\beta\gamma}{}^\gamma.$$

Therefore $\Omega_{\gamma\beta\infty}{}^\gamma = \Omega_{\gamma\beta}{}^{\gamma 0} = 0$ is equivalent to (5.9) and (5.12).

Let us consider the Bianchi identity

$$\nabla_{[\nu} \Omega_{\alpha\beta]}^{\nabla}{}^K{}_M - \tilde{A}_{[\nu|L]}{}^K \Omega_{\alpha\beta]}^{\nabla}{}^L{}_M + \Omega_{[\nu\alpha|L]}^{\nabla}{}^K \tilde{A}_{\beta]}{}^L{}_M = 0.$$

When $\Omega_{\alpha\beta\mu\gamma} = 0$, the above equation for $(M, K) = (\mu, \gamma)$ and $(M, K) = (\infty, \gamma)$ gives

$$(5.19) \quad \mathbf{h}_{\mu[\nu} \Omega_{\alpha\beta]\infty}{}^\gamma + \delta_{[\nu}{}^\gamma \Omega_{\alpha\beta]\mu}{}^0 = 0,$$

$$(5.20) \quad \nabla_{[\nu}^h \Omega_{\alpha\beta]\infty}{}^\gamma + \delta_{[\nu}{}^\gamma \Omega_{\alpha\beta]\infty}{}^0 - A_\gamma \delta_{[\nu} \Omega_{\alpha\beta]\infty}{}^\delta = 0.$$

Taking the traces of (5.19) under the condition $\Omega_{\gamma\beta\infty}{}^\gamma = \Omega_{\gamma\beta}{}^{\gamma 0} = 0$, we have

$$(n-2)\Omega_{\alpha\beta\infty}{}^\gamma + \Omega_{\alpha\beta}{}^{\gamma 0} = 0, \quad \Omega_{\alpha\beta\infty}{}^\gamma + (n-2)\Omega_{\alpha\beta}{}^{\gamma 0} = 0.$$

It follows that $\Omega_{\alpha\beta\infty}{}^\gamma = \Omega_{\alpha\beta}{}^{\gamma 0} = 0$ when $n \neq 3$ and that $\Omega_{\alpha\beta\infty}{}^\gamma + \Omega_{\alpha\beta}{}^{\gamma 0} = 0$ when $n = 3$. Then, when $n \neq 3$ the equation (5.20) implies $(n-2)\Omega_{\alpha\beta\infty}{}^0 = 0$ so that ∇ is flat when $n \geq 4$. When $n = 2$, it follows from (5.18), (5.7), (5.8), and (5.9) that $\Omega_{\alpha\beta\infty}{}^0 = 0$ is equivalent to the equation (5.14). When $n = 3$, the equation

$\Omega_{\alpha\beta\infty}^\gamma = \Omega_{\alpha\beta}^{\gamma 0} = 0$ holds if and only if $\Omega_{\alpha\beta\infty}^\gamma - \Omega_{\alpha\beta}^{\gamma 0} = 0$, which is equivalent to (5.11), and if the equation holds, $\Omega_{\alpha\beta\infty}^0 = 0$ follows from (5.20). Thus we complete the proof. $\square$

Remark 5.4. When $n = 2$ there are surfaces which have the same conformal structure and the same Fubini–Pick form but have different Möbius structures, so we need the condition $\text{tf}(E_{\alpha\beta} - G_{\alpha\beta}) = 0$ to fix a Möbius structure. In the case of $\nabla_\alpha = \Phi^{-1} \circ \nabla_\alpha^{\text{proj}} \circ \Phi$, we can directly prove the equations (5.7) and (5.8) by using (4.14), (5.4) and the fact that $\text{Ric}_{\alpha\beta} = (1/2)\text{Scal}_h \mathbf{h}_{\alpha\beta}$, $A_{\alpha\mu\nu} A_\beta^{\mu\nu} = (1/2)|A|^2 \mathbf{h}_{\alpha\beta}$.

Comparing (5.6) with (5.7) and (5.8), we obtain the following identities:

$$(5.21) \quad \text{tf} \mathbf{S}_{\alpha\beta} = \frac{1}{n}(\delta A)_{\alpha\beta} + \text{tf}(P_{\alpha\beta}^h - P_{\alpha\beta}^A),$$

$$(5.22) \quad \text{tf} P_{\alpha\beta} = -\frac{1}{n}(\delta A)_{\alpha\beta} + \text{tf}(P_{\alpha\beta}^h - P_{\alpha\beta}^A).$$

Next we compare the Levi–Civita connection $\tilde{\nabla}$ of the ambient metric and the conformal tractor connection. We decompose $\tilde{A}_{\alpha J}^K$ into the symmetric and the skew symmetric parts:

$$\tilde{A}_{\alpha J}^K = \tilde{A}'_{\alpha J}{}^K + \tilde{A}''_{\alpha J}{}^K, \quad \tilde{A}'_{\alpha JK} = \tilde{A}_{\alpha(JK)}, \quad \tilde{A}''_{\alpha JK} = \tilde{A}_{\alpha[JK]}.$$

Then we have the following proposition:

Proposition 5.5. *Let ρ be a Fefferman defining density and let $\tilde{\nabla}_I$ be the Levi–Civita connection of the ambient metric $\tilde{g}_{IJ} = D_I D_J \rho$. Then the restriction $\tilde{\nabla}_\alpha$ is well-defined and satisfies*

$$\Phi^{-1} \circ \tilde{\nabla}_\alpha \circ \Phi = \nabla_\alpha - \tilde{A}''_{\alpha J}{}^K,$$

where ∇_α is the conformal (Möbius) tractor connection. In particular, $\Phi^{-1} \circ \tilde{\nabla}_\alpha \circ \Phi = \nabla_\alpha$ if and only if $A_{\alpha\beta\gamma} = 0$.

Proof. Since D_I is flat, the connection $\tilde{\nabla}_I$ satisfies $\tilde{\nabla}_I = D_I + \tilde{\Gamma}_{IJ}^K$, where $\tilde{\Gamma}_{IJK} = (1/2)D_I D_J D_K \rho \in \tilde{\mathcal{E}}_{(IJK)}(-1)$. It follows that $T^I \tilde{\nabla}_I \nu^J = 0$ for any $\nu^I \in \tilde{\mathcal{E}}^I$, so one can define $\tilde{\nabla}_i \nu^J$ invariantly and hence $\tilde{\nabla}_\alpha \nu^J$ is well-defined on $\tilde{\mathcal{E}}^I|_M$. The projecting part of $\tilde{\Gamma}_{IJK}$ is computed as

$$2\tilde{\Gamma}_{ijk} = \nabla_i \nabla_j \nabla_k \rho + 2\rho \nabla_i P_{jk} + 2P_{jk} \nabla_i \rho + P_{ij} \nabla_k \rho + P_{ik} \nabla_j \rho.$$

We fix a projective scale $\tau \in \mathcal{E}(1)$, and work in an adapted frame $\{e_\infty = \xi, e_\alpha\}$. Using (4.7), (4.21), and (5.6), we can show that the identities

$$\tilde{\Gamma}_{\alpha\beta\gamma} = A_{\alpha\beta\gamma}, \quad \tilde{\Gamma}_{\alpha\beta\infty} = \frac{1}{2}(E_{\alpha\beta} + G_{\alpha\beta}), \quad \tilde{\Gamma}_{\alpha\infty\infty} = F_\alpha$$

hold on the hypersurface M . It follows that

$$\Phi^* \tilde{\Gamma}_{\alpha J}^K = \begin{pmatrix} 0 & 0 & 0 \\ \frac{1}{2}(E_\alpha^\gamma + G_\alpha^\gamma) & A_{\alpha\beta}^\gamma & 0 \\ F_\alpha & \frac{1}{2}(E_{\alpha\beta} + G_{\alpha\beta}) & 0 \end{pmatrix} = \tilde{A}'_{\alpha J}{}^K.$$

Thus we have $\Phi^{-1} \circ \tilde{\nabla}_\alpha \circ \Phi = \nabla_\alpha + \tilde{A}'_{\alpha J}{}^K = \nabla_\alpha - \tilde{A}''_{\alpha J}{}^K$.

By (5.7) and (5.8), it holds that

$$\tilde{A}''_{\alpha JK} = \begin{pmatrix} 0 & 0 & 0 \\ P_{\alpha\gamma}^A & 0 & 0 \\ 0 & -P_{\alpha\beta}^A & 0 \end{pmatrix}.$$

Since $\text{tr} P^A = (1/2(n-1))|A|^2$, the condition $\tilde{A}''_{\alpha JK} = 0$ is equivalent to $A_{\alpha\beta\gamma} = 0$. $\square$

5.3. Conformal Codazzi manifold. As an intrinsic geometric structure of a hypersurface in a locally flat projective manifold, we define the *conformal Codazzi structure* as follows:

Definition 5.6. Let M be a C^∞ -manifold of dimension n . When $n \geq 3$, a conformal Codazzi structure is a pair $([h], A_{\alpha\beta\gamma})$, where $[h]$ is a conformal structure and $A_{\alpha\beta\gamma} \in \mathcal{E}_{(\alpha\beta\gamma)_0}[2]$ satisfies the Gauss–Codazzi equations (5.10), (5.13) for $n \geq 4$ and (5.11), (5.13) for $n = 3$. When $n = 2$, a conformal Codazzi structure is a triad $([h], [\mathbb{P}], A_{\alpha\beta\gamma})$, where $[h]$ is a conformal structure, $[\mathbb{P}]$ is a Möbius structure and $A_{\alpha\beta\gamma} \in \mathcal{E}_{(\alpha\beta\gamma)_0}[2]$ satisfies the Gauss–Codazzi equations (5.12), (5.14).

We will call $(M, [h], A_{\alpha\beta\gamma})$ a conformal Codazzi manifold, suppressing the Möbius structure $[\mathbb{P}]$ when $n = 2$. On the conformal tractor bundle over a conformal Codazzi manifold, there is a flat connection ∇ defined by (5.5) with (5.7), (5.8), and (5.9). We call the connection ∇ the *conformal Codazzi tractor connection*.

We observe that when $A_{\alpha\beta\gamma}$ satisfies the Gauss–Codazzi equations, then $A_{\alpha\beta\gamma}^* := -A_{\alpha\beta\gamma}$ also satisfies the equations. We call $([h], A_{\alpha\beta\gamma}^*)$ the *dual conformal Codazzi structure*. The corresponding conformal Codazzi tractor connection ∇^* is given by

$$\nabla_\alpha^* = \nabla_\alpha - \tilde{A}'_{\alpha J}{}^K + \tilde{A}''_{\alpha J}{}^K,$$

and called the *dual conformal Codazzi tractor connection*. It is the dual connection of ∇ with respect to the conformal tractor metric h_{IJ} in the sense that

$$\nabla_\alpha^h(h_{IJ}U^I V^J) = h_{IJ}(\nabla_\alpha U^I)V^J + h_{IJ}U^I(\nabla_\alpha^* V^J)$$

holds for $U^I, V^I \in \mathcal{E}^I$.

Recall from [BEG] that the conformal tractor connection is defined via the prolongation of the conformal-to-Einstein equation $\text{tf}(\nabla_\alpha^h \nabla_\beta^h \sigma + P_{\alpha\beta}^h \sigma) = 0$ for $\sigma \in \mathcal{E}[1]$. We will see that when $n \geq 3$ the conformal Codazzi tractor connection is also defined via the prolongation of a conformally invariant differential equation. We set

$$\mathcal{P}_{\alpha\beta} := -\frac{1}{n}(\delta A)_{\alpha\beta} + \text{tf}(P_{\alpha\beta}^h - P_{\alpha\beta}^A) \in \mathcal{E}_{(\alpha\beta)_0}.$$

When M is a hypersurface of a locally flat projective manifold, the tensor $\mathcal{P}_{\alpha\beta}$ agrees with $\text{tf}P_{\alpha\beta}$ by (5.21). We consider a conformally invariant differential equation

$$(5.23) \quad \text{tf}(\nabla_\alpha^h \nabla_\beta^h \sigma) + A_{\alpha\beta}{}^\gamma \nabla_\gamma^h \sigma + \mathcal{P}_{\alpha\beta} \sigma = 0$$

for a density $\sigma \in \mathcal{E}[1]$. Suppose that σ is a solution to (5.23) and set $\mu_\alpha = \nabla_\alpha^h \sigma$. Then there exists a density $\varphi \in \mathcal{E}[-1]$ such that

$$(5.24) \quad \nabla_\alpha^h \mu_\beta + A_{\alpha\beta}{}^\gamma \mu_\gamma + \varphi h_{\alpha\beta} + \left(-\frac{1}{n}(\delta A)_{\alpha\beta} + P_{\alpha\beta}^h - P_{\alpha\beta}^A\right) \sigma = 0.$$

Thus the 2-jet of σ at a point $x \in M$ is determined by $(\sigma_x, (\mu_\alpha)_x, \varphi_x)$. If we change the conformal scale by Υ , the triad ${}^t(\sigma, \mu^\alpha, \varphi)$ transforms as (5.1). Therefore we have the following isomorphism:

$$\mathcal{E}_I \cong \{j^2 \sigma \in J^2 \mathcal{E}[1] \mid \text{tf}(\nabla_\alpha^h \nabla_\beta^h \sigma) + A_{\alpha\beta}{}^\gamma \nabla_\gamma^h \sigma + \mathcal{P}_{\alpha\beta} \sigma = 0\},$$

where $j^2 \sigma$ denotes the 2-jet of σ and $J^2 \mathcal{E}[1]$ denotes the jet bundle. Taking the divergence of (5.24) and using (5.13), we have

$$\nabla_\alpha^h \varphi + (E_{\alpha\gamma} - P_{\alpha\gamma}^h) \mu^\gamma + F_\alpha \sigma = 0.$$

It follows that the linear connection on $\mathcal{E}_I$ defined by the prolongation of the equation (5.23) coincides with ∇ . Thus, $\sigma \in \mathcal{E}[1]$ is a solution to (5.23) if and only if there exist $\mu_\alpha \in \mathcal{E}_\alpha[1]$ and $\varphi \in \mathcal{E}[-1]$ such that

$$\nabla_\alpha \begin{pmatrix} \sigma \\ \mu_\beta \\ \varphi \end{pmatrix} = 0.$$

We shall consider the dual of this construction. We set

$$\mathcal{S}_{\alpha\beta} := \frac{1}{n}(\delta A)_{\alpha\beta} + \text{tf}(P_{\alpha\beta}^h - P_{\alpha\beta}^A) \in \mathcal{E}_{(\alpha\beta)_0}.$$

The tensor $\mathcal{S}_{\alpha\beta}$ is the dual of $\mathcal{P}_{\alpha\beta}$ and agrees with $\text{tf}\mathcal{S}_{\alpha\beta}$ by (5.21) when M is a hypersurface of a locally flat projective manifold. The dual conformal Codazzi tractor connection ∇^* is given by the prolongation of the equation dual to (5.23):

$$(5.25) \quad \text{tf}(\nabla_\alpha^h \nabla_\beta^h \sigma) - A_{\alpha\beta}{}^\gamma \nabla_\gamma^h \sigma + \mathcal{S}_{\alpha\beta} \sigma = 0.$$

A conformal scale $\sigma \in \mathcal{E}[1]$ is a solution to (5.23) or (5.25) if and only if it holds in the scale σ that $\mathcal{P}_{\alpha\beta} = 0$ or $\mathcal{S}_{\alpha\beta} = 0$ respectively. Since ∇ and ∇^* are flat, such a scale always exists locally. Thus we obtain the following proposition:

Proposition 5.7. *Let $(M, [h], A_{\alpha\beta\gamma})$ be a conformal Codazzi manifold of dimension $n \geq 3$. Then there exists a local conformal scale for which $\mathcal{P}_{\alpha\beta} = 0$ holds, and there also exists one for which $\mathcal{S}_{\alpha\beta} = 0$.*

5.4. Projective Bonnet theorem. We will prove that the conformal Codazzi structure recovers the local immersion to the projective space and hence it has complete information on the local geometric structure of a hypersurface in a locally flat projective manifold.

Theorem 5.8. *Let $(M, [h], A_{\alpha\beta\gamma})$ be a conformal Codazzi manifold of dimension $n \geq 2$. Then for any point $p_0 \in M$ there exists a neighborhood $U \subset M$ and an immersion $f : U \rightarrow \mathbb{RP}^{n+1}$ such that the induced conformal Codazzi structure agrees with $([h], A_{\alpha\beta\gamma})$.*

Proof. We denote by $\mathcal{T}$ the conformal tractor bundle over M . Let $\Psi \in \wedge^{n+2} \mathcal{T}^*$ be the volume form of the tractor metric. Then since $\tilde{A}_{\alpha J}{}^J = 0$, the conformal Codazzi tractor connection ∇ preserves Ψ . It follows from the flatness of ∇ that there exist a neighborhood U of any point $p_0 \in M$ and a frame $\{e_I\}$ for $\mathcal{T}|_U$ which is parallel with respect to ∇ and unimodular with respect to Ψ . We write $\phi : \mathcal{T}|_U \rightarrow U \times \mathbb{R}^{n+2}$ for the trivialization via the frame $\{e_I\}$. Then we have $\phi(\nabla_X Y) = X\phi(Y)$. Since $\mathcal{T}$ contains $\mathcal{E}[-1]$ as a line subbundle, we can define the map $f : U \rightarrow \mathbb{RP}^{n+1}$ by $f(p) := [\phi(\mathcal{E}[-1]_p)]$. Let $\tilde{\mathcal{T}}$ and $\tilde{\mathcal{E}}(-1)$ be respectively the projective tractor bundle over $\mathbb{RP}^{n+1}$ and the projective density bundle of weight -1 over $\mathbb{RP}^{n+1}$. Then $\tilde{\mathcal{T}}$ is the trivial bundle and $\tilde{\mathcal{E}}(-1)$ coincides with the tautological line bundle. Moreover the conformal Codazzi tractor connection on $\tilde{\mathcal{T}}$, which we denote by $\tilde{\nabla}$, is the canonical flat connection on $\mathbb{RP}^{n+1} \times \mathbb{R}^{n+2}$. By the definition of f it holds that

$$f^* \tilde{\mathcal{T}} = \mathcal{T}|_U, \quad f^* \tilde{\nabla} = \nabla, \quad f^* \tilde{\mathcal{E}}(-1) = \mathcal{E}[-1].$$

Let $(x^0, \dots, x^{n+1})$ be the homogeneous coordinates on $\mathbb{RP}^{n+1}$. As we may assume that $f(U) \subset \{x^0 \neq 0\}$, we identify f with a map to the affine space $\mathbb{R}^{n+1} \cong \{x^0 = 1\}$. The hyperplane $\{x^0 = 1\}$ in $\mathbb{R}^{n+2}$ defines a section of $\tilde{\mathcal{E}}(-1)$ and hence defines a projective scale $\tilde{\tau} \in \tilde{\mathcal{E}}(1)$. We decompose $\mathcal{T}|_U$ by the conformal scale $\tau := f^* \tilde{\tau}$ as

$$\mathcal{T}|_U = \mathcal{E} \oplus TU \oplus \mathcal{E}.$$

Setting $T := (0, 0, 1) \in \Gamma(\mathcal{T}|_U)$, we have $f = \phi(T)$ and hence

$$f_* X = X\phi(T) = \phi(\nabla_X T) = \phi(X)$$

for any $X \in \Gamma(TU)$. Therefore, $f_* = \phi : TU \rightarrow \mathbb{R}^{n+1}$ and f is an immersion.

Let us consider the conformal Codazzi structure induced on the hypersurface $f(U) \subset \mathbb{R}^{n+1}$. We set $Y := (1, 0, 0) \in \mathcal{T}|_U$. Let $\pi : \tilde{\mathcal{T}}|_{\mathbb{R}^{n+1}} \rightarrow T\mathbb{R}^{n+1}$ be the projection along $\phi(T)$ and set $\xi := \pi(\phi(Y)) \in \Gamma(f(U), T\mathbb{R}^{n+1})$. We will see that ξ is identical to the affine normal field on $f(U)$ for the projective scale $\tilde{\tau}$. First, let $\hat{h}$ be the affine second fundamental form on $f(U)$ with respect to ξ . We note

that the representative connection $\nabla^{\dot{\tau}}$ for the scale $\dot{\tau}$ agrees with the canonical flat connection on $\mathbb{R}^{n+1}$. For $X, Z \in \Gamma(TU)$, we have

$$\nabla_{f_*X}^{\dot{\tau}} f_*Z = \overset{\circ}{\nabla}_{f_*X} f_*Z = f_*\nabla_X Z = \phi(\nabla_X Z),$$

where $\nabla_X Z$ is of the form

$$(5.26) \quad \nabla_X Z = \begin{pmatrix} -h_{\alpha\beta} X^\alpha Z^\beta \\ X^\beta (\nabla_\beta^h Z^\alpha - A_{\beta\gamma}{}^\alpha Z^\gamma) \\ * \end{pmatrix}$$

with $h_{\alpha\beta} := \tau^{-2} h_{\alpha\beta}$. Since the ξ -component of $\phi(\nabla_X Z)$ equals the Y -component of $\nabla_X Z$, it follows that $f^* \dot{h} = h$. We set $\dot{\Psi} := dx^0 \wedge \cdots \wedge dx^{n+1} \in \wedge^{n+2} \dot{\mathcal{T}}^*$ so that $\phi^* \dot{\Psi} = \Psi$ holds. Noting that $\dot{\tau}^{-(n+2)} = dx^1 \wedge \cdots \wedge dx^{n+1}$, $\phi(T) = \partial/\partial x^0 + x^i \partial/\partial x^i$, and $\phi(Y) \equiv \xi \bmod \phi(T)$, we have

$$\begin{aligned} f^*(\xi \lrcorner \dot{\tau}^{-(n+2)})|_{Tf(U)} &= \phi^*(\phi(Y) \lrcorner \phi(T) \lrcorner \dot{\Psi})|_{Tf(U)} \\ &= (Y \lrcorner T \lrcorner \Psi)|_{TU} \\ &= \text{vol}_h \\ &= f^* \text{vol}_{\dot{h}}. \end{aligned}$$

Thus $\xi \lrcorner \dot{\tau}^{-(n+2)}|_{Tf(U)} = \text{vol}_{\dot{h}}$ holds. Next, for any $X \in TU$ we have

$$\begin{aligned} \nabla_{f_*X}^{\dot{\tau}} \xi &\equiv \pi(\overset{\circ}{\nabla}_{\phi(X)} \phi(Y)) \\ &= \pi(\phi(\nabla_X Y)) \bmod Tf(U), \end{aligned}$$

where $\nabla_X Y$ is of the form

$$(5.27) \quad \nabla_X Y = \begin{pmatrix} 0 \\ X^\beta (P_\beta^h{}^\alpha + \frac{1}{n}(\delta A)_\beta{}^\alpha - P_\beta^A{}^\alpha) \\ * \end{pmatrix}.$$

The ξ -component of $\nabla_{f_*X}^{\dot{\tau}} \xi$ equals the Y -component of $\nabla_X Y$, so it vanishes. Thus we have proved that ξ is the affine normal field.

We already proved that $f^* \dot{h} = h$. By equation (5.26), we can see that the induced connection on U is given by $\nabla_\alpha^h - A_{\alpha\beta}{}^\gamma$. Therefore, the Fubini–Pick form is equal to $A_{\alpha\beta\gamma}$. When $n = 2$, it follows from equation (5.27) that the affine shape operator on U is given by $S_{\alpha\beta} = \mathbb{P}_{\alpha\beta} + (1/2)(\delta A)_{\alpha\beta} - (1/4)|A|^2 h_{\alpha\beta}$, and hence the induced Möbius structure is $\text{tf} S_{\alpha\beta} - (1/2)(\delta A)_{\alpha\beta} + (1/4)\text{Scal}_h h_{\alpha\beta} = \mathbb{P}_{\alpha\beta}$. Thus we complete the proof. $\square$

5.5. Harmonic scales. Let M be the boundary of a strictly convex domain in a locally flat projective manifold N of dimension $n + 1$. Then the coefficient L of the logarithmic term in the volume expansion of the Blaschke metric gives a projective invariant of M when n is even. We will show that in fact it is a conformal invariant of the conformal Codazzi manifold $(M, [h], A_{\alpha\beta\gamma})$. Recall that $\text{tr} \mathbf{S}$ depends on the 1-jet of the projective scale along the boundary, and cannot be expressed in terms of the conformal Cosazzi structure in general. Thus we need to introduce a normalization condition on the 1-jet of a projective scale. Let $\tilde{\Delta} = -\tilde{g}^{IJ} \tilde{\nabla}_I \tilde{\nabla}_J$ be the Laplacian of the ambient metric $\tilde{g}_{IJ} = D_I D_J \rho$.

Definition 5.9. A projective scale $\tau \in \mathcal{E}(1)$ is called a harmonic scale if it satisfies $\tilde{\Delta} \tau = O(\rho)$.

Note that the definition does not depend on the choice of ρ . The following proposition shows that the harmonicity normalizes the transverse derivative of the scale:

Proposition 5.10. (i) For any conformal scale $\kappa \in \mathcal{E}[1]$, there exists a harmonic scale $\tau \in \mathcal{E}(1)$, unique modulo $O(\rho^2)$, such that $\tau|_M = \kappa$.

(ii) When $\tau \in \mathcal{E}(1)$ is a harmonic scale, $\hat{\tau} = e^{-\Upsilon} \tau$ is harmonic if and only if

$$\xi \Upsilon = \frac{1}{n} (\Delta_h \Upsilon + |d\Upsilon|_h^2),$$

where ξ is the (weighted) affine normal field for τ and $\Delta_h = -\mathbf{h}^{\alpha\beta} \nabla_\alpha^h \nabla_\beta^h$.

(iii) A projective scale $\tau \in \mathcal{E}(1)$ is harmonic if and only if

$$(5.28) \quad \text{tr} \mathbf{S} = \frac{1}{n-1} (\text{Scal}_h - |A|^2)$$

holds.

Proof. (i) Let $\hat{\tau}$ be an arbitrary extension of κ and set $\tau = \hat{\tau} + \rho\phi$ for a density $\phi \in \mathcal{E}(-1)$. Then using $\tilde{\Delta}\rho = -(n+2)$ and $\tilde{g}^{IJ} D_J \rho = T^I$, we have

$$\tilde{\Delta}\tau = \tilde{\Delta}\hat{\tau} - n\phi + O(\rho).$$

Thus there is a ϕ unique modulo $O(\rho)$ such that $\tilde{\Delta}\tau = O(\rho)$.

(ii) By the Monge–Ampère equation $\det \tilde{g}_{IJ} = -1 + O(\rho^{n/2+1})$, the Christoffel symbols of $\tilde{\nabla}$ satisfy $\tilde{g}^{IJ} \tilde{\Gamma}_{IJ}^K = O(\rho^{n/2})$. Thus we have

$$\begin{aligned} \tilde{\Delta}\hat{\tau} &= -\tilde{g}^{IJ} D_I D_J \hat{\tau} + O(\rho) \\ &= e^{-\Upsilon} \tilde{g}^{IJ} (\tau D_I D_J \Upsilon - \tau D_I \Upsilon D_J \Upsilon + 2D_I \Upsilon D_J \tau) + O(\rho). \end{aligned}$$

In an adapted frame $\{e_\infty = \xi, e_\alpha\}$ for the scale τ , it holds at the boundary that

$$\begin{aligned} D_I \tau &= \begin{pmatrix} \tau \\ 0 \\ 0 \end{pmatrix}, \quad D_J \Upsilon = \begin{pmatrix} 0 \\ \xi \Upsilon \\ \nabla_\beta^h \Upsilon \end{pmatrix}, \\ D_I D_J \Upsilon &= \begin{pmatrix} 0 & -\xi \Upsilon & -\nabla_\beta^h \Upsilon \\ -\xi \Upsilon & * & * \\ -\nabla_\alpha^h \Upsilon & * & \nabla_\alpha^h \nabla_\beta^h \Upsilon + A_{\alpha\beta}{}^\gamma \nabla_\gamma^h \Upsilon + (\xi \Upsilon) \mathbf{h}_{\alpha\beta} \end{pmatrix}. \end{aligned}$$

Therefore using (4.3) we have

$$\tilde{\Delta}\hat{\tau} = e^{-\Upsilon} \tau (n \xi \Upsilon - \Delta_h \Upsilon - |d\Upsilon|_h^2) + O(\rho),$$

which proves the assertion.

(iii) In an adapted frame for the scale τ , it holds at the boundary that

$$D_I D_J \tau = \tau \begin{pmatrix} 0 & 0 & 0 \\ 0 & P_{\infty\infty} & P_{\beta\infty} \\ 0 & P_{\alpha\infty} & P_{\alpha\beta} \end{pmatrix}.$$

Thus, by (4.14), we have

$$\tilde{\Delta}\tau|_M = -\tau \mathbf{h}^{\alpha\beta} P_{\alpha\beta} = \tau \left(\text{tr} \mathbf{S} - \frac{1}{n-1} (\text{Scal}_h - |A|^2) \right),$$

from which (iii) follows. $\square$

By the above proposition, a conformal scale on M determines the 1-jet of a harmonic scale and we can extend it to a normalized projective scale. Thus, it follows from Proposition 4.10 that L gives a conformal invariant of M :

Theorem 5.11. Let M be the boundary of a strictly convex domain in a locally flat projective manifold, and let $\kappa \in \mathcal{E}[1]$ be a conformal scale on M . Then the coefficient of the logarithmic term in the volume expansion of the Blaschke metric can be written in the form

$$L = \int_M F \text{vol}_h,$$

where F is a linear combination of complete contractions of $h_{\alpha\beta}$, $\text{Ric}_{\alpha\beta}^h$, $A_{\alpha\beta\gamma}$ and their covariant derivatives with respect to ∇^h . Moreover the integral is independent of the choice of κ .

We shall compute the conformal invariant L for $n = 4$. Let $\tau \in \mathcal{E}(1)$ be a harmonic normalized scale. By (4.6), L is given by the integral of

$$\xi^2 v|_M = (\xi \text{tr} S - \xi r + (\text{tr} S - r)^2)|_M$$

over M . Since τ is harmonic, we have

$$r|_M = \frac{1}{n} \text{tr} S = \frac{1}{n(n-1)} (\text{Scal}_h - |A|^2).$$

By differentiating (4.17) and using (4.10) and (4.15), we have

$$\xi r|_M = \frac{1}{2} (|P|^2 - |S|^2) + (\text{div}),$$

where (div) denotes divergence terms. The equation (4.8) gives

$$\xi \text{tr} S|_M = -|S|^2 - \frac{1}{4} (\text{tr} S)^2 + (\text{div}).$$

Then it follows from (5.21) and (5.22) that

$$\begin{aligned} \xi^2 v|_M &= -\frac{1}{2} (|\text{tf} S|^2 + |\text{tf} P|^2) + \frac{3}{16} (\text{tr} S)^2 + (\text{div}) \\ &= \frac{1}{8} \left(|W^h|^2 - 2|\text{Ric}^h|^2 + \frac{2}{3} \text{Scal}_h^2 \right) - \frac{1}{8} |W^h|^2 - \frac{1}{4} |A^2|^2 + \frac{1}{12} |A|^4 \\ &\quad - \frac{1}{16} |\delta A|^2 + \frac{1}{6} |A|^2 \text{Scal}_h + \frac{1}{2} \langle \text{Ric}^h, A^2 \rangle + (\text{div}), \end{aligned}$$

where the tensor A^2 is defined by $(A^2)_{\alpha\beta} = A_{\alpha\mu\nu} A_{\beta}^{\mu\nu}$. Thus by the Gauss–Bonnet theorem, we obtain

$$\begin{aligned} (5.29) \quad L &= 4\pi^2 \chi(M) - \int_M \left(\frac{1}{8} |W^h|^2 + \frac{1}{4} |A^2|^2 - \frac{1}{12} |A|^4 \right) \text{vol}_h \\ &\quad + \int_M \left(\frac{1}{6} |A|^2 \text{Scal}_h + \frac{1}{2} \langle \text{Ric}^h, A^2 \rangle - \frac{1}{16} |\delta A|^2 \right) \text{vol}_h. \end{aligned}$$

6. GJMS OPERATORS AND Q -CURVATURE

In this section we construct conformally invariant powers of the Laplacian and the Q -curvature in conformal Codazzi geometry by applying the method in [GJMS] and [FH].

6.1. GJMS construction. Let M be the boundary of a strictly convex domain Ω in an $(n+1)$ -dimensional locally flat projective manifold N . Let $\tilde{\nabla}_I$ be the Levi–Civita connection of the ambient metric $\tilde{g}_{IJ} = D_I D_J \rho$, where ρ is a Fefferman defining density of Ω . Since D_I is flat, we have $\tilde{\nabla}_I = D_I + \tilde{\Gamma}_{IJ}^K$ with

$$\tilde{\Gamma}_{IJK} = \frac{1}{2} D_I D_J D_K \rho \in \tilde{\mathcal{E}}_{(IJK)}(-1).$$

When n is even the Monge–Ampère equation $\det \tilde{g} = -1 + \mathcal{O}(\rho^{n/2+1})$ implies

$$(6.1) \quad \tilde{\Gamma}_{IK}^K = -\frac{1}{2} \left(\frac{n}{2} + 1 \right) \mathcal{O}(\rho^{n/2}) D_I \rho - \frac{1}{2} \rho^{n/2+1} D_I \mathcal{O} + \mathcal{O}(\rho^{n/2+2}),$$

while the right-hand side is replaced by $\mathcal{O}(\rho^\infty)$ when n is odd. Since $T^I \tilde{\Gamma}_{IJK} = 0$, we have $\tilde{\Delta} \rho = -(n+2)$.

As in conformal and CR cases, $\tilde{\Delta}$ satisfies the following commutation relations:

Lemma 6.1. For a density $\tilde{f} \in \mathcal{E}(w)$ and a positive integer m , we have

$$(6.2) \quad \begin{aligned} [\tilde{\Delta}, \rho^m] \tilde{f} &= -m(2w + 2m + n) \rho^{m-1} \tilde{f}, \\ [\tilde{\Delta}^m, \rho] \tilde{f} &= -m(2w - 2m + n + 4) \tilde{\Delta}^{m-1} \tilde{f}. \end{aligned}$$

By using the above lemma, we can construct conformally invariant differential operators as follows: Let $f \in \mathcal{E}[w]$ be a conformal density on M . We extend f to a density $\tilde{f} \in \mathcal{E}(w)$ and apply $\tilde{\Delta}^m$. By (6.2), we have

$$\tilde{\Delta}^m(\tilde{f} + \phi \rho) = \tilde{\Delta}^m \tilde{f} - m(2w - 2m + n) \tilde{\Delta}^{m-1} \phi + \rho \tilde{\Delta}^m \phi$$

for any density $\phi \in \mathcal{E}(w - 2)$. Therefore $\tilde{\Delta}^m \tilde{f}|_M$ is independent of the choice of an extension when $w = m - n/2$, and defines a differential operator

$$P_m : \mathcal{E}[m - n/2] \longrightarrow \mathcal{E}[-m - n/2],$$

which we call the (*conformal Codazzi*) *GJMS operator*.

The following proposition is proved in the same way as in [GJMS] and [FH]:

Proposition 6.2. Let $f \in \mathcal{E}[m - n/2]$.

(i) There exists an extension $\tilde{f} \in \mathcal{E}(m - n/2)$ of f such that

$$\tilde{\Delta} \tilde{f} = O(\rho^{m-1}).$$

Moreover, such an extension is unique modulo $O(\rho^m)$ and satisfies $c_m \rho^{1-m} \tilde{\Delta} \tilde{f}|_M = P_m f$ with $c_m = 2^{m-1} \{(m-1)!\}^2$.

(ii) Let $\tau \in \mathcal{E}(1)$ be a projective scale and set $\rho = \tau^{-2} \rho$. Then there exist $A \in \mathcal{E}(m - n/2)$ and $B \in \mathcal{E}(-m - n/2)$ such that $A|_M = f$ and

$$\tilde{\Delta}(A + B \rho^m \log |\rho|) = O(\rho^\infty).$$

Moreover, A and B are unique modulo $O(\rho^m)$ and $O(\rho^\infty)$ respectively, and it holds that $c'_m B|_M = P_m f$ with $c'_m = -2^m m \{(m-1)!\}^2$.

Next we examine the dependence on the choice of a Fefferman defining density. When n is odd, the Fefferman defining density is unique to infinite order, so P_m does not depend on the choice of ρ . On the other hand, we must restrict the range of m when n is even:

Proposition 6.3. Let n be even. Then the GJMS operator P_m is independent of the choice of a Fefferman defining density ρ if $1 \leq m \leq n/2$.

Proof. Let ρ' be another Fefferman defining density and $\tilde{g}'_{IJ}$ the associated ambient metric. Since $\rho' = \rho + O(\rho^{n/2+2})$, the Laplacian of $\tilde{g}'_{IJ}$ satisfies $\tilde{\Delta}' = -\tilde{g}'^{IJ} D_I D_J + O(\rho^{n/2}) = \tilde{\Delta} + O(\rho^{n/2})$. Let $f \in \mathcal{E}[m - n/2]$. We extend f to $\tilde{f} \in \mathcal{E}(m - n/2)$ so that

$$\tilde{\Delta} \tilde{f} = \psi \rho^{m-1}, \quad \psi|_M = P_m f$$

holds. Then $\tilde{f}$ satisfies $\tilde{\Delta}' \tilde{f} = (\psi + O(\rho)) \rho^{m-1}$ if $m \leq n/2$, so we have $P'_m = P_m$. $\square$

By following the CR case ([HMM]), we will introduce a further normalization on ρ so that $P_{n/2+2}$ is well-defined for n even.

Lemma 6.4. Let n be even. Then there exists a defining density ρ of Ω such that

$$\mathcal{J}[\rho] = -1 + O(\rho^{n/2+1}), \quad \tilde{\Delta} \rho = O(\rho),$$

where $\tilde{\Delta}$ is the Laplacian of $\tilde{g}_{IJ} = D_I D_J \rho$. Moreover, a defining density ρ' satisfies the same equations if and only if there exists $\phi \in \mathcal{E}(-n - 2)$ such that $\rho' = \rho + \phi \rho^{n/2+2}$ and $\tilde{\Delta} \phi = O(\rho)$.

Proof. Take a defining density ρ such that $\mathcal{J}[\rho] = -1 + \mathcal{O}\rho^{n/2+1}$. Then any Fefferman defining density is written in the form $\rho' = \rho + \phi\rho^{n/2+2}$ with a density $\phi \in \mathcal{E}(-n-2)$. By (3.6) it holds that

$$\mathcal{J}[\rho'] = -1 + (\mathcal{O} + \rho\tilde{\Delta}\phi)\rho^{n/2+1} + O(\rho^{n/2+3})$$

so the obstruction density of ρ' satisfies $\mathcal{O}' = \mathcal{O} + \rho\tilde{\Delta}\phi + O(\rho^2)$. Using the commutation relation (6.2), we have

$$(6.3) \quad \tilde{\Delta}'\mathcal{O}' = \tilde{\Delta}\mathcal{O} + (n+6)\tilde{\Delta}\phi + O(\rho).$$

Thus, setting $\phi = -(n+6)^{-2}\rho\tilde{\Delta}\mathcal{O}$, we obtain ρ' such that $\tilde{\Delta}'\mathcal{O}' = O(\rho)$. The second statement of the lemma also follows from (6.3). $\square$

A defining density given as in the above lemma is called a *strict Fefferman defining density*.

Proposition 6.5. *When n is even, the operators $P_{n/2+2}$ is independent of the choice of a strict Fefferman defining density.*

Proof. Let ρ and ρ' be strict Fefferman defining densities. By Lemma 6.4, we can write as $\rho' = \rho + \phi\rho^{n/2+2}$ with $\phi \in \mathcal{E}(-n-2)$ satisfying $\tilde{\Delta}\phi = O(\rho)$. The corresponding obstruction densities satisfy $\mathcal{O}|_M = \mathcal{O}'|_M$ and $\tilde{\Delta}(\mathcal{O} - \mathcal{O}') = O(\rho)$, so it holds that $\mathcal{O} - \mathcal{O}' = O(\rho^2)$. Thus, from (6.1) we have

$$(6.4) \quad \tilde{g}^{IJ}\tilde{\Gamma}_{IJ}^K - \tilde{g}'^{IJ}\tilde{\Gamma}_{IJ}^K = O(\rho^{n/2+2}).$$

The associated ambient metrics satisfy

$$(6.5) \quad \begin{aligned} \tilde{g}^{IJ} &= \tilde{g}'^{IJ} - \left(\frac{n}{2} + 2\right)\left(\frac{n}{2} + 1\right)\phi T^I T^J \rho^{n/2} \\ &\quad - \left(\frac{n}{2} + 2\right)(T^I D^J \phi + T^J D^I \phi + \phi \tilde{g}^{IJ})\rho^{n/2+1} + O(\rho^{n/2+2}). \end{aligned}$$

Let $f \in \mathcal{E}[2]$. By Proposition 6.2 (i), there is an extension $\tilde{f}$ of f such that $\tilde{\Delta}\tilde{f} = \psi\rho^{n/2+1}$ and $c_{n/2+2}\psi|_M = P_{n/2+2}f$. Then by (6.4) and (6.5), we have

$$\begin{aligned} \tilde{\Delta}'\tilde{f} &= \tilde{\Delta}\tilde{f} + \left(\frac{n}{2} + 2\right)\left(\frac{n}{2} + 1\right)\phi T^I T^J D_I D_J \tilde{f} \rho^{n/2} \\ &\quad + \left(\frac{n}{2} + 2\right)(2T^I D^J \phi D_I D_J \tilde{f} - \phi \tilde{\Delta}\tilde{f})\rho^{n/2+1} + O(\rho^{n/2+2}) \\ &= \psi\rho^{n/2+1} + \left(\frac{n}{2} + 2\right)(n+2)\phi\tilde{f}\rho^{n/2} + (n+4)D^I \phi D_I \tilde{f}\rho^{n/2+1} \\ &\quad + O(\rho^{n/2+2}). \end{aligned}$$

Thus, setting $\tilde{f}' = \tilde{f} + (n/2 + 2)\phi\tilde{f}\rho^{n/2+1}$, we obtain

$$\begin{aligned} \tilde{\Delta}'\tilde{f}' &= \tilde{\Delta}\tilde{f}' - \left(\frac{n}{2} + 2\right)(n+2)\phi\tilde{f}\rho^{n/2} + \left(\frac{n}{2} + 2\right)\tilde{\Delta}(\phi\tilde{f})\rho^{n/2+1} \\ &= \psi\rho^{n/2+1} + O(\rho^{n/2+2}), \end{aligned}$$

which implies $P'_{n/2+2}f = P_{n/2+2}f$. $\square$

It is proved in [G1] and [GH] that on a general even dimensional conformal manifold there is no conformally invariant linear differential operator whose principal part agrees with that of Δ_h^m for $m \geq n/2 + 1$. In our case, an additional structure on M , namely the Fubini–Pick form with the Gauss–Codazzi equations, enables us to define an over critical GJMS operator.

As an example, we will compute a formula for $P_1 : \mathcal{E}[1 - n/2] \longrightarrow \mathcal{E}[-1 - n/2]$. We fix a projective scale $\tau \in \mathcal{E}(1)$, and take an adapted frame $\{e_\infty = \xi, e_\alpha\}$. For an arbitrary extension $\tilde{f} \in \mathcal{E}(1 - n/2)$ of $f \in \mathcal{E}[1 - n/2]$, we have

$$D_I D_J \tilde{f} = \begin{pmatrix} n/2(n/2 - 1)\tilde{f} & -n/2\nabla_j \tilde{f} \\ -n/2\nabla_i \tilde{f} & \nabla_i \nabla_j \tilde{f} + (1 - n/2)P_{ij}\tilde{f} \end{pmatrix}.$$

Thus, using (4.3), (4.7), (4.14) and (4.21), we obtain

$$\begin{aligned} P_1 f &= -\tilde{g}^{IJ} D_I D_J \tilde{f}|_M \\ &= -\mathbf{h}^{\alpha\beta} \left(\nabla_\alpha \nabla_\beta \tilde{f} + \left(1 - \frac{n}{2}\right) P_{\alpha\beta} \tilde{f} \right) + n \nabla_\infty \tilde{f} + \frac{n}{2} \left(\frac{n}{2} - 1 \right) \mathbf{r} \tilde{f} \\ &= \Delta_h f + \frac{n-2}{4(n-1)} (\text{Scal}_h - |A|^2) f. \end{aligned}$$

6.2. Self-adjointness. We will prove that the GJMS operators are self-adjoint. First we relate the ambient Laplacian $\tilde{\Delta}$ to the Laplacian Δ_g of the Blaschke metric.

Lemma 6.6. (i) *Let $u \in C^\infty(\Omega)$ and $w \in \mathbb{R}$. We assume $du = O(\rho^{-1})$ in local coordinates around each boundary point. If $w \neq 0$, we further assume that $u = O(1)$. Then when n is even we have*

$$(6.6) \quad 2(-\rho)^{1-w/2} \tilde{\Delta}((-\rho)^{w/2} u) = (\Delta_g + w(n+w))u + O(\rho^{n/2+1}).$$

When n is odd, the error term is replaced by $O(\rho^\infty)$.

(ii) *Let $\tilde{f} \in \mathcal{E}(2)$. Then when n is even we have*

$$(6.7) \quad 2\tilde{\Delta}\tilde{f} = (\Delta_g + 2(n+2))((-\rho)^{-1}\tilde{f}) + O(\rho^{n/2+2}).$$

Proof. (i) We consider the case of even n . We set $\tau' := (-\rho)^{1/2}$. Then τ' defines a projective scale inside Ω . We denote by ∇' the associated connection, which is singular along the boundary. Since $\nabla'\rho = 0$, the Blaschke metric is given by $g_{ij} = -P'_{ij}$, where P'_{ij} is the projective Schouten tensor of ∇' . We set $\mathbf{u} = (-\rho)^{w/2}u$. Then in the scale τ' we have

$$\tilde{g}_{IJ} = \begin{pmatrix} 2\rho & 0 \\ 0 & -2\rho g_{ij} \end{pmatrix}, \quad D_I D_J \mathbf{u} = \begin{pmatrix} w(w-1)\mathbf{u} & (w-1)\nabla'_j \mathbf{u} \\ (w-1)\nabla'_i \mathbf{u} & \nabla'_i \nabla'_j \mathbf{u} - w g_{ij} \mathbf{u} \end{pmatrix}.$$

Thus, using (6.1) and the assumption on u , we compute as

$$2(-\rho)^{1-w/2} \tilde{\Delta} \mathbf{u} = -g^{ij} \nabla'_i \nabla'_j \mathbf{u} + w(n+w)\mathbf{u} + O(\rho^{n/2+1}).$$

We shall compare $-g^{ij} \nabla'_i \nabla'_j \mathbf{u}$ with $\Delta_g \mathbf{u}$. Let $B_{ij}{}^k \in \mathcal{E}_{(ij)}{}^k$ be the difference tensor defined by $\nabla^g - \nabla'$. The flatness of $[\nabla]$ implies $\nabla'_i g_{jk} = -(1/2(n-1))\nabla'_l C_{ij}{}^l{}_k = 0$, so it holds that $B_{ijk} = 1/2\nabla'_i g_{jk} \in \mathcal{E}_{(ijk)}$, where the index is lowered by g_{ij} . Hence we have

$$g^{ij} B_{ij}{}^k = g^{kl} B_{li}{}^i = g^{kl} \text{vol}_g^{-1} \nabla'_l \text{vol}_g.$$

The Monge–Ampère equation gives

$$\text{vol}_g = (-2\rho)^{-(n/2+1)} \left(1 - \frac{1}{2} \mathcal{O} \rho^{n/2+1} + O(\rho^{n/2+3}) \right).$$

We take a projective scale τ on N and set $\rho = \tau^{-2} \rho$. Then the associated connection ∇ satisfies $\nabla = \nabla' + d\rho/2\rho$, so we have

$$\begin{aligned} g^{kl} \text{vol}_g^{-1} \nabla'_l \text{vol}_g &= g^{kl} \text{vol}_g^{-1} \nabla_l \text{vol}_g + \left(\frac{n}{2} + 1 \right) g^{kl} (\rho^{-1} \nabla_l \rho) \text{vol}_g \\ &= g^{kl} \nabla_l \log \left(1 - \frac{1}{2} \mathcal{O} \rho^{n/2+1} + O(\rho^{n/2+3}) \right). \end{aligned}$$

By the definition of the Blaschke metric, we have $g^{kl} = O(\rho)$. We also have $g^{kl} \nabla_l \rho = O(\rho^2)$ as follows: In the projective scale τ' , the top slot of $\tilde{g}^{KL} D_L \rho$

equals $(-2\rho)^{-1}g^{kl}\nabla_l\rho$. On the other hand, $\tilde{g}^{KL}D_L\rho = \tau^{-2}T^K + O(\rho)$. Thus we have $g^{kl}\nabla_l\rho = O(\rho^2)$. It follows from these estimates that

$$(6.8) \quad \begin{aligned} g^{ij}B_{ij}{}^k &= -\frac{1}{2}g^{kl}\nabla_l(\mathcal{O}\rho^{n/2+1}) + O(\rho^{n/2+4}) \\ &= O(\rho^{n/2+2}). \end{aligned}$$

Thus, by the assumption that $du = O(\rho^{-1})$, we obtain $-g^{ij}\nabla'_i\nabla'_j u = -g^{ij}\nabla'_i\nabla'_j g u - g^{ij}B_{ij}{}^k\nabla_k u = \Delta_g u + O(\rho^{n/2+1})$, which yields (6.6).

(ii) As in the proof of (i), we compute as

$$(6.9) \quad \begin{aligned} \tilde{\Delta}\tilde{f} &= -\tilde{g}^{IJ}D_ID_J\tilde{f} + \tilde{g}^{IJ}\tilde{\Gamma}_{IJ}{}^K D_K\tilde{f} \\ &= (n+2)(-\rho)^{-1}\tilde{f} - \frac{1}{2}g^{ij}\nabla'_i\nabla'_j((-\rho)^{-1}\tilde{f}) - \left(\frac{n}{2}+1\right)\mathcal{O}\tilde{f}\rho^{n/2} \\ &\quad - \frac{1}{2}\tilde{g}^{IJ}D_ID_J\tilde{f}\rho^{n/2+1} + O(\rho^{n/2+2}). \end{aligned}$$

By the first equality in (6.8), we have

$$-g^{ij}\nabla'_i\nabla'_j((-\rho)^{-1}\tilde{f}) = \Delta_g((-\rho)^{-1}\tilde{f}) - \frac{1}{2}g^{ij}\nabla'_i\mathcal{O}\nabla'_j\tilde{f}\rho^{n/2} + O(\rho^{n/2+2}).$$

Also, calculating in the projective scale τ' , we have

$$-\tilde{g}^{IJ}D_ID_J\tilde{f}\rho^{n/2+1} = (n+2)\mathcal{O}\tilde{f}\rho^{n/2} + \frac{1}{2}g^{ij}\nabla'_i\mathcal{O}\nabla'_j\tilde{f}\rho^{n/2}.$$

Substituting these equations to (6.9), we obtain (6.7). $\square$

Proposition 6.7. (i) *The GJMS operator P_m is self-adjoint for $1 \leq m \leq n/2$ when n is even and for all m when n is odd.*

(ii) *Let n be even and let ρ be a strict Fefferman defining density. Then $P_{n/2+2}$ is self-adjoint.*

Proof. (i) We fix a projective scale $\tau \in \mathcal{E}(1)$ and take an adapted frame $\{e_\infty = \xi, e_\alpha\}$ with the dual $\{d\rho, \theta^\alpha\}$. We set

$$\begin{aligned} \tilde{h}_{\alpha\beta} &= h_{\alpha\beta} + 2\rho P_{\alpha\beta}, \quad \tilde{\theta}^\alpha = \theta^\alpha + 2\rho\tilde{h}^{\alpha\gamma}P_{\gamma\infty}d\rho, \\ \tilde{r} &= r + 2\rho P_{\infty\infty} - 4\rho^2\tilde{h}^{\gamma\mu}P_{\gamma\infty}P_{\mu\infty}, \end{aligned}$$

where $\tilde{h}^{\alpha\beta}$ is the inverse of $\tilde{h}_{\alpha\beta}$. Then the Blaschke metric is represented as

$$g = \frac{\tilde{h}_{\alpha\beta}}{-2\rho} \tilde{\theta}^\alpha \cdot \tilde{\theta}^\beta + \frac{1-2\tilde{r}\rho}{4\rho^2} d\rho^2.$$

Thus the outward unit normal field on each level set of ρ is given by

$$(6.10) \quad \begin{aligned} \nu &= \frac{-2\rho}{(1-2\tilde{r}\rho)^{1/2}}(\xi - 2\rho\tilde{h}^{\alpha\gamma}P_{\gamma\infty}e_\alpha) \\ &\equiv -2\rho(1+O(\rho))\xi \pmod{\text{Ker } d\rho}. \end{aligned}$$

We identify a neighborhood of M in $\bar{\Omega}$ with the product $M \times (-\epsilon_0, 0]$ as in Section 4.2. Since

$$(6.11) \quad vol_g = \frac{1+O(\rho)}{(-2\rho)^{n/2+1}} d\rho \wedge vol_h,$$

we have

$$(6.12) \quad (\nu \lrcorner vol_g)|_{TM_\epsilon} = \frac{1+O(\epsilon)}{(2\epsilon)^{n/2}} vol_h,$$

where $M_\epsilon = \{\rho = -\epsilon\}$.

Let $f_1, f_2 \in \mathcal{E}[m - n/2]$. We take $A_j \in \mathcal{E}(m - n/2)$ and $B_j \in \mathcal{E}(-m - n/2)$ for $j = 1, 2$ as in Proposition 6.2 (ii), and set

$$u_j = (-\rho)^{1/2(n/2-m)}(A_j + B_j \rho^m \log |\rho|).$$

Since u_j satisfies the assumption in Lemma 6.6 (i), we have

$$\begin{aligned} (\Delta_g + (m - n/2)(m + n/2))u_j &= 2(-\rho)^{1-1/2(m-n/2)}\tilde{\Delta}(A_j + B_j \rho^m \log |\rho|) \\ &\quad + O(\rho^{n/2+1}) \\ &= O(\rho^{n/2+1}) \end{aligned}$$

for even n . When n is odd, the last term is replaced by $O(\rho^\infty)$. We shall compute

$$(6.13) \quad \text{lp} \int_{\{\rho < -\epsilon\}} (\langle du_1, du_2 \rangle_g + (m - n/2)(m + n/2)u_1 u_2) \text{vol}_g,$$

where ‘lp’ stands for the coefficient of $\log(1/\epsilon)$ in the expansion of the integral. By Green’s formula, this is equal to

$$\text{lp} \int_{\{\rho < -\epsilon\}} u_1 (\Delta_g + (m - n/2)(m + n/2))u_2 \text{vol}_g + \text{lp} \int_{M_\epsilon} (u_1 \cdot \nu u_2) \nu \lrcorner \text{vol}_g.$$

The first term equals 0 since the integrand is $O(1)$ for $1 \leq m \leq n/2$ when n is even and for all m when n is odd. It follows from (6.10) and (6.12) that the second term is given by

$$2^{-n/2}(-1)^m c'_m{}^{-1} \int_M ((n/2 + m)f_1 P_m f_2 + (n/2 - m)f_2 P_m f_1).$$

This integral must be symmetric in f_1 and f_2 since (6.13) is symmetric in u_1 and u_2 . Thus we have

$$\int_M (f_1 P_m f_2 - f_2 P_m f_1) = 0,$$

so P_m is self-adjoint.

(ii) Let $f_1, f_2 \in \mathcal{E}[2]$ and take $\tilde{f}_1, \tilde{f}_2 \in \mathcal{E}(2)$ as in Proposition 6.2 (i), and set $u_j = (-\rho)^{-1}\tilde{f}_j$ ($j = 1, 2$). By Lemma 6.6 (ii), we have

$$\begin{aligned} (\Delta_g + 2(n + 2))u_j &= 2\tilde{\Delta}((-\rho)u_j) + O(\rho^{n/2+2}) \\ &= 2\psi_j \rho^{n/2+1} + O(\rho^{n/2+2}). \end{aligned}$$

Hence,

$$\begin{aligned} \langle du_1, du_2 \rangle + 2(n + 2)u_1 u_2 &= u_1 (\Delta_g u_2 + 2(n + 2)u_2) + (\text{div}) \\ &= -2\tilde{f}_1 \psi_2 \rho^{n/2} + O(\rho^{n/2+1}) + (\text{div}). \end{aligned}$$

As in the proof of (i), we see the log term of the integration of this function. Since the divergence term does not contribute to it, we have

$$\begin{aligned} &\text{lp} \int_{\{\rho < -\epsilon\}} (\langle du_1, du_2 \rangle + 2(n + 2)u_1 u_2) \text{vol}_g \\ &= \text{lp} \int_{\{-\epsilon_0 < \rho < -\epsilon\}} (-2\tilde{f}_1 \psi_2 \rho^{n/2} + O(\rho^{n/2+1})) \frac{1 + O(\rho)}{(-2\rho)^{n/2+1}} d\rho \wedge \text{vol}_h \\ &= c \int_M f_1 P_{n/2+2} f_2, \end{aligned}$$

with a nonzero constant c . It follows from the symmetry in u_1 and u_2 that $P_{n/2+2}$ is self-adjoint. $\square$

6.3. Conformal Codazzi Q -curvature. Let n be even. We take a conformal scale $\sigma \in \mathcal{E}[1]$ on M and extend it arbitrarily to a projective scale $\tau \in \mathcal{E}(1)$ on N . We define the (conformal Codazzi) Q -curvature by

$$Q = -\tilde{\Delta}^{n/2} \log t \big|_M \in \mathcal{E}[-n],$$

where t is the fiber coordinate of the $\mathbb{R}_+$ -bundle $\tilde{N}$ associated with the trivialization by τ^{-1} . Note that the definition is independent of the choice of τ ; If we take a different extension, then $\log t$ changes by a function $f \in \mathcal{E}(0)$ with $f = O(\rho)$, so Q is invariant.

Let $\hat{\sigma} = e^{-\Upsilon} \sigma$ be another conformal scale. We extend $\hat{\sigma}$ to $\hat{\tau} = e^{-\tilde{\Upsilon}} \tau$ with an extension $\tilde{\Upsilon}$ of Υ . Then the Q -curvature changes as

$$\begin{aligned} \hat{Q} &= -\tilde{\Delta}^{n/2} (\log t - \tilde{\Upsilon}) \big|_M \\ &= Q + P_{n/2} \Upsilon. \end{aligned}$$

Since $P_{n/2}$ is self-adjoint and satisfies $P_{n/2} 1 = 0$, the total Q -curvature

$$\overline{Q} = \int_M Q$$

is a conformal invariant.

As in the usual conformal case ([FH]), the Q -curvature is related to the harmonic extension of $\log t$:

Lemma 6.8. *Let $\tau \in \mathcal{E}(1)$ be a projective scale and set $\rho = \tau^{-2} \rho$, where ρ is a Fefferman defining density. Then there exist $A \in \mathcal{E}(0)$ and $B \in \mathcal{E}(-n)$ such that $A|_M = 0$ and*

$$(6.14) \quad \tilde{\Delta}(\log t + A + B \rho^{n/2} \log |\rho|) = O(\rho^\infty).$$

Such A and B are unique modulo $O(\rho^{n/2})$ and $O(\rho^\infty)$ respectively and it holds that $2^{n/2-1} n \{(n/2-1)!\}^2 B|_M = Q$.

With this lemma, we show the following proposition:

Proposition 6.9. *There exists a nonzero universal constant a_n such that*

$$L = a_n \overline{Q},$$

where L is the coefficient of the logarithmic term in the volume expansion of the Blaschke metric g .

Proof. We fix a projective scale and take A and B as in the previous lemma. It follows from Lemma 6.6 (i) that

$$\begin{aligned} \Delta_g \log |\rho| &= -2\rho \tilde{\Delta} \log |\rho| + O(\rho^{n/2+1}) \\ &= -2\rho \tilde{\Delta} (\log |\rho| - 2 \log t) + O(\rho^{n/2+1}) \\ &= 2n + 4\rho \tilde{\Delta} \log t + O(\rho^{n/2+1}). \end{aligned}$$

Then, multiplying both sides of (6.14) by -2ρ and using Lemma 6.6 (i) again, we have

$$\Delta_g (1/2 \log |\rho| - A - B \rho^{n/2} \log |\rho|) = n + O(\rho^{n/2+1}).$$

Hence by (6.10), (6.11) and (6.12), we obtain

$$\begin{aligned} L &= \text{lp} \int_{\{\rho < -\epsilon^2/2\}} \text{vol}_g \\ &= \text{lp} \frac{2}{n} \int_{M_\epsilon} \nu \cdot \left(\frac{1}{2} \log |\rho| - A - B \rho^{n/2} \log |\rho| \right) \nu \lrcorner \text{vol}_g \\ &= a_n \overline{Q} \end{aligned}$$

with a nonzero constant a_n . □

7. VARIATION FORMULAS

7.1. The first variation formula. Let N be a locally flat projective manifold of odd dimension $n + 1$, and let Ω be a relatively compact domain in N with strictly convex boundary M . We consider a smooth family of strictly convex domains $\{\Omega_t\}$ with $\Omega_0 = \Omega$ and compute $\dot{L} = (d/dt)|_{t=0} L_t$, where L_t is the conformal invariant on $\partial\Omega_t$ given by the log term in the volume expansion of the Blaschke metric. We take Fefferman defining densities ρ_t of Ω_t which is smooth in t and denote the associated obstruction density and the Blaschke metric by $\mathcal{O}_t$ and g_{ij}^t respectively. We will omit the subscript or superscript t when $t = 0$.

The logarithmic term in the volume expansion can be described in terms of Riesz renormalization (see, e.g., [Al] for the details): We fix a projective scale $\tau \in \mathcal{E}(1)$ and set

$$\zeta_t(z) = \int_{\Omega_t} (-\rho_t)^z \text{vol}_{g^t},$$

where $\rho_t = \tau^{-2} \rho_t$. Then $\zeta_t(z)$ is holomorphic in z if $\text{Re } z$ is sufficiently large and extends to a meromorphic function on $\mathbb{C}$. Moreover, it has at most simple pole at $z = 0$ and satisfies

$$\text{Res}_{z=0} \zeta_t(z) = \text{lp} \int_{\{\rho_t < -\epsilon\}} \text{vol}_{g^t} = \frac{1}{2} L_t.$$

Hence we have

$$\begin{aligned} \frac{1}{2} \dot{L} &= \text{Res}_{z=0} \dot{\zeta}_0(z) \\ &= \text{Res}_{z=0} \int_{\Omega} \frac{d}{dt} \Big|_{t=0} (-\rho_t)^z \text{vol}_{g^t} \\ &\quad + \text{Res}_{z=0} \lim_{t \rightarrow 0} \frac{1}{t} \left(\int_{\Omega_t} (-\rho_t)^z \text{vol}_{g^t} - \int_{\Omega} (-\rho_t)^z \text{vol}_{g^t} \right). \end{aligned}$$

The limit in the second term in the last expression equals 0 when $\text{Re } z$ is large enough since $(-\rho_t)^z (\chi_{\Omega_t} - \chi_{\Omega}) \text{vol}_{g^t}$ is uniformly convergent to 0 in a neighborhood of $\partial\Omega$, where χ_{Ω_t} and χ_{Ω} are the characteristic functions. The first term is computed as

$$\begin{aligned} \text{Res}_{z=0} \int_{\Omega} \frac{d}{dt} \Big|_{t=0} (-\rho_t)^z \text{vol}_{g^t} &= \text{Res}_{z=0} z \int_{\Omega} (-\rho)^z \frac{\dot{\rho}}{\rho} \text{vol}_g + \text{Res}_{z=0} \int_{\Omega} (-\rho)^z \frac{d}{dt} \Big|_{t=0} \text{vol}_{g^t} \\ &= \text{Res}_{z=0} \int_{\Omega} (-\rho)^z \frac{d}{dt} \Big|_{t=0} \text{vol}_{g^t}. \end{aligned}$$

Here we have used the fact that the integral

$$\int_{\Omega} (-\rho)^z \frac{\dot{\rho}}{\rho} \text{vol}_g$$

has at most simple pole at $z = 0$. Thus we obtain the following expression for $\dot{L}$:

$$\dot{L} = 2 \text{lp} \int_{\{\rho < -\epsilon\}} \frac{d}{dt} \Big|_{t=0} \text{vol}_{g^t}.$$

The approximate Monge–Ampère equation for ρ_t gives

$$\det g_{ij}^t = (-2\rho_t)^{-(n+2)} (1 - \mathcal{O}_t \rho_t^{n/2+1}),$$

so we have

$$\begin{aligned} \frac{d}{dt} \Big|_{t=0} \text{vol}_{g^t} &= \frac{d}{dt} \Big|_{t=0} (\det g_{ij}^t)^{1/2} \\ &= -\left(\frac{n}{2} + 1\right) \left(\frac{\dot{\rho}}{\rho} + \frac{1}{2} \mathcal{O}_t \dot{\rho} \rho^{n/2} + O(\rho^{n/2+1}) \right) \text{vol}_g. \end{aligned}$$

We shall compute $\dot{\rho}/\rho$ by expressing $\tilde{g}^{IJ}D_ID_J\dot{\rho}$ in two ways. First we let $\nabla' \in [\nabla]$ be the representative connection with respect to the singular projective scale $\tau' = (-\rho)^{1/2}$, as in the proof of Lemma 6.6 (i). Then near the boundary it holds that

$$\tilde{g}^{IJ}D_ID_J\dot{\rho} = (n+2)\frac{\dot{\rho}}{\rho} - \frac{1}{2}g^{ij}\nabla'_i\nabla'_j\left(\frac{\dot{\rho}}{\rho}\right).$$

Using the first equality of (6.8) and the estimates $g^{kl}\nabla_l\rho = O(\rho^2)$, $g^{kl}\nabla_k\rho\nabla_l\rho = 4\rho^2(1 + O(\rho))$, we have

$$\begin{aligned} g^{ij}\nabla'_i\nabla'_j\left(\frac{\dot{\rho}}{\rho}\right) + \Delta_g\left(\frac{\dot{\rho}}{\rho}\right) &= g^{ij}B_{ij}{}^k\nabla_k\left(\frac{\dot{\rho}}{\rho}\right) \\ &= -\frac{1}{2}\left(\frac{n}{2} + 1\right)\mathcal{O}\rho^{n/2}g^{kl}\nabla_l\rho\left(\frac{\nabla_k\dot{\rho}}{\rho} - \frac{\dot{\rho}\nabla_k\rho}{\rho^2}\right) + O(\rho^{n/2+1}) \\ &= (n+2)\mathcal{O}\dot{\rho}\rho^{n/2} + O(\rho^{n/2+1}). \end{aligned}$$

Hence

$$(7.1) \quad \tilde{g}^{IJ}D_ID_J\dot{\rho} = (n+2)\frac{\dot{\rho}}{\rho} - \left(\frac{n}{2} + 1\right)\mathcal{O}\dot{\rho}\rho^{n/2} + \frac{1}{2}\Delta_g\left(\frac{\dot{\rho}}{\rho}\right) + O(\rho^{n/2+1}).$$

On the other hand, by differentiating the Monge–Ampère equation $\det(D_ID_J\rho_t) = -1 + \mathcal{O}_t\rho_t^{n/2+1}$, we have

$$(7.2) \quad \begin{aligned} \tilde{g}^{IJ}D_ID_J\dot{\rho} &= (-1 + O(\rho^{n/2+1}))^{-1}\left(\left(\frac{n}{2} + 1\right)\mathcal{O}\dot{\rho}\rho^{n/2} + O(\rho^{n/2+1})\right) \\ &= -\left(\frac{n}{2} + 1\right)\mathcal{O}\dot{\rho}\rho^{n/2} + O(\rho^{n/2+1}). \end{aligned}$$

Comparing (7.1) and (7.2) gives

$$\frac{\dot{\rho}}{\rho} = -\frac{1}{2(n+2)}\Delta_g\left(\frac{\dot{\rho}}{\rho}\right) + O(\rho^{n/2+1}),$$

which implies that the integral of $\dot{\rho}/\rho$ does not yield a logarithmic term. Consequently the variation of L is computed as

$$\begin{aligned} \dot{L} &= -\left(\frac{n}{2} + 1\right)\text{lp} \int_{\{\rho < -\epsilon\}} \mathcal{O}\dot{\rho}\rho^{n/2}\text{vol}_g \\ &= -\left(\frac{n}{2} + 1\right)\text{lp} \int_{\{-\epsilon_0 < \rho < -\epsilon\}} \mathcal{O}\dot{\rho}\rho^{n/2} \frac{1 + O(\rho)}{(-2\rho)^{n/2+1}} d\rho \wedge \text{vol}_h \\ &= k_n \int_M \mathcal{O}\dot{\rho}, \end{aligned}$$

where $k_n = (n/2 + 1)(-2)^{-n/2-1}$.

Thus we have proved the following

Theorem 7.1. *Let $\{\Omega_t\}$ be a smooth family of strictly convex domains in a locally flat projective manifold N of odd dimension $n + 1$. Let ρ_t be a Fefferman defining density of Ω_t . Then the total conformal Codazzi Q -curvatures L_t on $\partial\Omega_t$ satisfy*

$$\left.\frac{d}{dt}\right|_{t=0} L_t = k_n \int_M \mathcal{O}\dot{\rho},$$

where $\mathcal{O}$ is the obstruction density on $M = \partial\Omega_0$ and k_n is a nonzero universal constant.

Consequently, the critical point of L is given by hypersurfaces whose obstructions vanish. As an example, let us compute the obstruction density for a strictly convex surface M in $\mathbb{R}^3$. We take the canonical affine scale $\tau = (dx^1 \wedge dx^2 \wedge dx^3)^{-1/4}$, for which $P_{ij} = 0$. Then we differentiate (4.17) and use (4.8) to obtain $2\mathcal{Q}|_M = -|S|^2 - r|_M \text{tr} S + 4r^2|_M + (\text{div})$. Since τ is harmonic, $2r|_M = \text{tr} S = \text{Scal} - |A|^2$ by

(4.21) and (5.28). Noting that $\text{tf Ric}_{\alpha\beta}^h = \text{tf} A_{\alpha\mu\nu} A_{\beta}^{\mu\nu} = 0$ in dimension 2, we also have $\text{tf} S_{\alpha\beta} = (\delta A)_{\alpha\beta}$ from (4.20). Therefore the obstruction is given by

$$\mathcal{O}|_M = -\frac{1}{2}|\delta A|^2 + (\text{div}).$$

Integrating this over M , we see that $\mathcal{O} \equiv 0$ if and only if $\delta A \equiv 0$, or equivalently $\text{tf} S \equiv 0$. A hypersurface with $\text{tf} S \equiv 0$ in affine scales is known as an *affine sphere*. It is a classical theorem of Blaschke ($n = 2$) and Deicke ($n \geq 3$) that a strictly convex affine sphere in $\mathbb{R}^{n+1}$ is projectively equivalent to the sphere (see [NS] for a proof). Thus we obtain the following corollary:

Corollary 7.2. *A strictly convex surface in $\mathbb{R}^3$ is a critical point of L if and only if it is projectively equivalent to S^2 .*

7.2. The second variation formula. We will derive a second variation formula of L for the deformation of Ω parametrized by a density on the boundary M . Let ρ be a strict Fefferman defining density of Ω . We fix a conformal density $f \in \mathcal{E}[2]$ and extend it to a projective density $\tilde{f} \in \mathcal{E}(2)$ with $\tilde{\Delta}\tilde{f} = O(\rho)$, where $\tilde{\Delta}$ is the Laplacian of the ambient metric $\tilde{g}_{IJ} = D_I D_J \rho$. We set

$$X^I = \tilde{g}^{IJ} D_J \tilde{f} \in \mathcal{E}^I(1).$$

Recall that the tractor field X^I can be identified with a vector field X on $\tilde{N}$, the $\mathbb{R}_+$ -bundle of positive elements in $\mathcal{E}(-1)$. Since X is homogeneous of degree 0, it projects to a vector field $\underline{X}$ on N . Let Fl_t and $\underline{\text{Fl}}_t$ be the flows generated by X and $\underline{X}$ respectively. We consider the family of strictly convex domains $\{\Omega_t\}$ defined by $\Omega_t = \underline{\text{Fl}}_t(\Omega)$.

Lemma 7.3. *There exist Fefferman defining densities ρ_t of Ω_t which satisfy $\rho_0 = \rho$ and $\dot{\rho} := (d/dt)|_{t=0} \rho_t = -2\tilde{f} + O(\rho)$.*

Proof. We put $\rho'_t = \text{Fl}_{-t}^* \rho$. Then it holds that $\rho'_0 = \rho$ and $\dot{\rho}' = -X^I D_I \rho = -2\tilde{f}$. Since $\tilde{\Delta}\tilde{f} = O(\rho)$, we have

$$\frac{d}{dt}\Big|_{t=0} \mathcal{J}[\rho'_t] = \mathcal{J}[\rho] \tilde{g}^{IJ} D_I D_J \dot{\rho}' = O(\rho),$$

which implies $\mathcal{J}[\rho'_t]|_M = -1 + O(t^2)$. Starting with ρ'_t , we construct a Fefferman defining density ρ''_t by the algorithm described in Section 3.2. Then ρ''_t is written as

$$\rho''_t = (-\mathcal{J}[\rho'_t])^{-(1/n+2)} \rho'_t + \phi_t(\rho'_t)^2$$

with some density ϕ_t , so it holds that $\dot{\rho}'' = -2\tilde{f} + O(\rho)$. Since ρ and ρ'_0 are both Fefferman defining densities, we can write as $\rho'_0 = \rho + \psi \rho^{n/2+2}$. Thus, setting $\rho_t = \rho''_t - \text{Fl}_{-t}^*(\psi \rho^{n/2+2})$, we obtain a family of Fefferman defining densities with $\rho_0 = \rho$ and $\dot{\rho} = -2\tilde{f} + O(\rho)$. $\square$

First we present a variation formula of the obstruction density.

Proposition 7.4. *Let ρ_t be the Fefferman defining density of Ω_t given by the previous lemma. Then the obstruction densities $\mathcal{O}_t$ of ρ_t satisfy*

$$(7.3) \quad \frac{d}{dt}\Big|_{t=0} \text{Fl}_t^* \mathcal{O}_t|_M = b_n P_{n/2+2} f,$$

where b_n is a nonzero universal constant and $P_{n/2+2}$ is the GJMS operator.

Proof. Differentiating both sides of $\mathcal{J}[\rho_t] = -1 + \mathcal{O}_t \rho_t^{n/2+1}$ at $t = 0$, we have

$$\tilde{g}^{IJ} D_I D_J \dot{\rho} = -\left(\frac{n}{2} + 1\right) \mathcal{O} \dot{\rho} \rho^{n/2} - \dot{\mathcal{O}} \rho^{n/2+1} + O(\rho^{n/2+2}).$$

It follows from this equation and (6.1) that

$$\begin{aligned}\tilde{\Delta}\dot{\rho} &= -\tilde{g}^{IJ}D_ID_J\dot{\rho} + \tilde{g}^{IJ}\tilde{\Gamma}_{IJ}^K D_K\dot{\rho} \\ &= \left(\dot{\mathcal{O}} - \frac{1}{2}\tilde{g}^{IJ}D_I\mathcal{O}D_J\dot{\rho}\right)\rho^{n/2+1} + O(\rho^{n/2+2}).\end{aligned}$$

Then, since $\dot{\rho}$ and $\tilde{f}$ satisfy $\dot{\rho} + 2\tilde{f} = O(\rho)$ and $\tilde{\Delta}(\dot{\rho} + 2\tilde{f}) = O(\rho)$, it holds that $\dot{\rho} = -2\tilde{f} + O(\rho^2)$. Thus we have

$$\tilde{\Delta}\dot{\rho} = (\dot{\mathcal{O}} + X^I D_I \mathcal{O})\rho^{n/2+1} + O(\rho^{n/2+2}).$$

Applying $\tilde{\Delta}^{n/2+1}$ to both sides yields

$$b_n P_{n/2+2} f = (\dot{\mathcal{O}} + X^I D_I \mathcal{O})|_M = \frac{d}{dt}\Big|_{t=0} \text{Fl}_t^* \mathcal{O}_t|_M,$$

with a nonzero constant b_n . $\square$

Next, we prepare a lemma on the derivatives of a density $\psi \in \mathcal{E}(-n-2)$. Since ψ is identified with a homogeneous function on $\tilde{N}$, one can consider the derivative $X\psi \in \mathcal{E}(-n-2)$. On the other hand ψ can also be regarded as a form on N via the isomorphism $\mathcal{E}(-n-2) \cong \wedge^{n+1} T^*N$, so one can consider the Lie derivative $\mathcal{L}_X \psi$. The following lemma asserts that the two derivatives coincide at M .

Lemma 7.5. *For a projective density $\psi \in \mathcal{E}(-n-2)$, we have*

$$\mathcal{L}_X \psi = X\psi$$

*at M under the identification $\mathcal{E}(-n-2) \cong \wedge^{n+1} T^*N$.*

Proof. We take a projective scale $\tau \in \mathcal{E}(1)$ and let $t \in \mathbb{R}_+$ be the fiber coordinate of $\tilde{N}$ associated with the trivialization by τ^{-1} . It can be seen that the determinant of a tractor in $\mathcal{E}_{IJ}$ defined in Section 3.1 is identical to the one with respect to the volume form

$$\Psi = t^{n+1} dt \wedge \tau^{-(n+2)}$$

on $\tilde{N}$. Here we regard $\tau^{-(n+2)}$ as a form on $\tilde{N}$ by pulling it back by the projection map. By the Monge–Ampère equation, the volume form of $\tilde{g}_{IJ}$ satisfies

$$\text{vol}_{\tilde{g}} = (1 + O(\rho^{n/2+1}))\Psi.$$

Then it follows from $\mathcal{L}_X \text{vol}_{\tilde{g}} = -\tilde{\Delta}\tilde{f} \text{vol}_{\tilde{g}} = O(\rho)$ that $\mathcal{L}_X \Psi = O(\rho)$. Since X is homogeneous of degree 0, there is a function a on N such that $X = at(\partial/\partial t) + \underline{X}$. Using $Xt = at$ and $\mathcal{L}_X dt = d(at)$, we have

$$\mathcal{L}_X \Psi = t^{n+1} dt \wedge (\mathcal{L}_X \tau^{-(n+2)} + (n+2)a\tau^{-(n+2)}),$$

and thus

$$\mathcal{L}_X \tau^{-(n+2)}|_M = -(n+2)a\tau^{-(n+2)}|_M.$$

If we set $\underline{\psi} = \tau^{n+2}\psi$, the density ψ is identified with the function $t^{-(n+2)}\underline{\psi}$. Then it holds at M that

$$\mathcal{L}_X \psi = (\underline{X}\underline{\psi})\tau^{-(n+2)} + \underline{\psi}\mathcal{L}_X \tau^{-(n+2)} = (\underline{X}\underline{\psi} - (n+2)a\underline{\psi})\tau^{-(n+2)},$$

and that

$$X\psi = t^{-(n+2)}(\underline{X}\underline{\psi} - (n+2)t^{-1}(Xt)\underline{\psi}) = t^{-(n+2)}(\underline{X}\underline{\psi} - (n+2)a\underline{\psi}).$$

Thus $\mathcal{L}_X \psi|_M$ and $X\psi|_M$ coincide under the identification $\mathcal{E}(-n-2) \cong \wedge^{n+1} T^*N$. $\square$

Now we are ready to show the following

Theorem 7.6. *Let ρ be a strict Fefferman defining density of a strictly convex domain Ω in a locally flat projective manifold N of odd dimension $n + 1$. If $\{\Omega_t\}$ is the deformation of Ω defined from $f \in \mathcal{E}[2]$, the total conformal Codazzi Q -curvatures L_t on $\partial\Omega_t$ satisfy*

$$(7.4) \quad \left. \frac{d^2}{dt^2} \right|_{t=0} L_t = k'_n \int_M f P_{n/2+2} f,$$

where k'_n is a nonzero universal constant and $P_{n/2+2}$ is the GJMS operator.

Proof. We fix a projective scale $\tau \in \mathcal{E}(1)$ and take Fefferman defining densities $\rho_t = \tau^2 \rho_t$ with the obstruction densities $\mathcal{O}_t = \tau^{-(n+2)} \underline{\mathcal{O}}_t$, as in Lemma 7.3. Recall from the proof of Proposition 7.4 that the derivative $\dot{\rho}$ satisfies $\dot{\rho} = -2\tilde{f} + O(\rho^2)$.

By Theorem 7.1, we have

$$\dot{L}_t = \int_{\partial\Omega_t} \mathcal{O}_t \dot{\rho}_t = \int_M \underline{\text{Fl}}_t^* (\mathcal{O}_t \dot{\rho}_t \text{vol}_{h^t}),$$

where h^t is the affine metric on $\partial\Omega_t$ with respect to τ . Thus the second derivative of L_t is decomposed as $\ddot{L} = \text{(I)} + \text{(II)}$ with

$$\text{(I)} = \int_M \left. \frac{d}{dt} \right|_{t=0} \underline{\text{Fl}}_t^* (\mathcal{O}_t \dot{\rho}_t) \text{vol}_h, \quad \text{(II)} = \int_M \mathcal{O} \dot{\rho} \left. \frac{d}{dt} \right|_{t=0} \underline{\text{Fl}}_t^* \text{vol}_{h^t}.$$

First we compute (I). Using Proposition 7.4, we have

$$\begin{aligned} \left. \frac{d}{dt} \right|_{t=0} \underline{\text{Fl}}_t^* (\mathcal{O}_t \dot{\rho}_t)|_M &= \left. \frac{d}{dt} \right|_{t=0} \underline{\text{Fl}}_t^* (\tau^n \mathcal{O}_t \dot{\rho}_t)|_M \\ &= \mathcal{O} \dot{\rho} (X \tau^n)|_M - 2b_n \tau^n f P_{n/2+2} f + \tau^n \mathcal{O} \left. \frac{d}{dt} \right|_{t=0} \underline{\text{Fl}}_t^* \dot{\rho}_t|_M. \end{aligned}$$

Differentiating $\underline{\text{Fl}}_t^* \rho_t = O(\rho)$ gives $\underline{\text{Fl}}_t^* \dot{\rho}_t + \underline{\text{Fl}}_t^* (X^I D_I \rho_t) = O(\rho)$, so it holds that

$$\begin{aligned} \left. \frac{d}{dt} \right|_{t=0} \underline{\text{Fl}}_t^* \dot{\rho}_t &= -X^I D_I \dot{\rho} - X^J D_J (X^I D_I \rho) + O(\rho) \\ &= O(\rho). \end{aligned}$$

Thus we obtain

$$(7.5) \quad \text{(I)} = k'_n \int_M f P_{n/2+2} f + \int_M \mathcal{O} \dot{\rho} (X \tau^n)$$

with a nonzero constant k'_n .

Next we compute (II). For each t , we take a vector field ξ_t on N determined by the conditions $\xi_t \rho_t = 1$ and $\xi_t^i Z^j \nabla_i \nabla_j \rho_t = 0$ for $Z \in \text{Ker } d\rho_t$, where $\nabla \in [\nabla]$ is the representative connection associated with τ . Since ρ_t solves the Monge–Ampère equation at $\partial\Omega_t$, it follows from Lemma 4.5 that $\xi_t|_{\partial\Omega_t}$ is affine normal. Thus, using Lemma 7.5, we compute as

$$\begin{aligned} \left. \frac{d}{dt} \right|_{t=0} \underline{\text{Fl}}_t^* \text{vol}_{h^t} &= \left. \frac{d}{dt} \right|_{t=0} \underline{\text{Fl}}_t^* (\xi_t \lrcorner \tau^{-(n+2)}) \\ &= (\dot{\xi} + \mathcal{L}_{\underline{X}} \xi) \lrcorner \tau^{-(n+2)} + \xi \lrcorner \mathcal{L}_{\underline{X}} \tau^{-(n+2)} \\ &= d\rho(\dot{\xi} + \mathcal{L}_{\underline{X}} \xi) \text{vol}_h - (n+2) \tau^{-1} X \tau \text{vol}_h. \end{aligned}$$

Differentiating the equation $\underline{\text{Fl}}_t^* (\xi_t \rho_t) = 1$ and using $\dot{\rho} = -2\tilde{f} + O(\rho^2)$ and $X \rho = 2\tilde{f}$, we have

$$\begin{aligned} d\rho(\dot{\xi} + \mathcal{L}_{\underline{X}} \xi) &= -\xi(\dot{\rho} + \underline{X} \rho) \\ &= -\xi(\tau^{-2} \dot{\rho} + X(\tau^{-2} \rho)) \\ &= 2\tau^{-1} X \tau + O(\rho), \end{aligned}$$

which yields

$$(7.6) \quad (\text{II}) = -n \int_M \mathcal{O} \dot{\rho} \tau^{-1} X \tau \text{vol}_h = - \int_M \mathcal{O} \dot{\rho} (X \tau^n).$$

Combining (7.5) and (7.6), we obtain the formula (7.4). $\square$

As an example, we consider the case of the sphere $M = S^n$. Let $(\xi^0, \dots, \xi^{n+1})$ be the homogeneous coordinates of $\mathbb{RP}^{n+1}$. The projective density bundle $\mathcal{E}(-1)$ of $\mathbb{RP}^{n+1}$ is identified with the tautological line bundle, and the D -operator is given by $\partial/\partial \xi^I$. The defining density $\rho = (1/2)(-(\xi^0)^2 + (\xi^1)^2 + \dots + (\xi^{n+1})^2)$ of M is the exact solution to the Monge–Ampère equation, so the ambient metric is given by $\tilde{g} = -(d\xi^0)^2 + (d\xi^1)^2 + \dots + (d\xi^{n+1})^2$. Thus $\tilde{g}$ agrees with the Fefferman–Graham ambient metric in conformal geometry. In the standard scale on S^n , the GJMS operator is given explicitly by

$$P_{n/2+2} = \prod_{j=1}^{n/2+2} \left(\Delta_h + \left(\frac{n}{2} + j - 1 \right) \left(\frac{n}{2} - j \right) \right),$$

where h is the standard metric on S^n ; see [FG2] for the derivation of this formula.

Let E_k be the eigenspace of Δ_h with eigenvalue $k(k+n-1)$ for $k \geq 0$, so that we have the decomposition $L^2(S^n) = \sum_{k \geq 0} E_k$. It follows from the above formula that $\text{Ker } P_{n/2+2} = E_0 \oplus E_1 \oplus E_2$ and $P_{n/2+2}$ is positive definite on $(\text{Ker } P_{n/2+2})^\perp$.

The space $E_0 \oplus E_1 \oplus E_2$ has the following geometric interpretation: Since eigenfunctions of Δ_h are given by the restriction of homogeneous harmonic polynomials on $\mathbb{R}^{n+1}$, it holds that

$$E_0 \oplus E_1 \oplus E_2 = \{F(\xi) = (a_{IJ} \xi^I \xi^J)|_{S^n} \mid a_{IJ} = a_{JI} \in \mathbb{R}\}.$$

In fact, noting that we may assume that $\sum_{i=1}^{n+1} a_{ii} = 0$ by adding a multiple of ρ , we have

$$F = a_{00} + 2a_{0i}x^i + a_{ij}x^i x^j,$$

where $x^i = \xi^i/\xi^0$ are the affine coordinates, and each term in the right-hand side belongs to E_0, E_1, E_2 respectively. The gradient vector field of F is given by

$$X = -2a_{0J} \xi^J \frac{\partial}{\partial \xi^0} + 2 \sum_{i=1}^{n+1} a_{iJ} \xi^J \frac{\partial}{\partial \xi^i},$$

so the flow generated by X is of the form $\text{Fl}_t(\xi) = (I + tA)\xi$ with some matrix A . Therefore, the flow Fl_t on $\mathbb{RP}^{n+1}$ is a projective linear transformation, which does not change the conformal Codazzi structure on S^n .

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