博士論文 (要約)

論文題目　Characteristic cycle and ramification of a rank 1 sheaf
(階数１の層の特性サイクルと分岐)

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Characteristic cycle and ramification of a rank 1 sheaf

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Recently, Saito ([S2]) has defined the characteristic cycle $CC(\mathcal{K})$ of a constructible complex $\mathcal{K}$ of étale sheaves on a smooth variety X over a perfect field k of characteristic $p > 0$ using the singular support $SS(\mathcal{K})$ defined by Beilinson ([B]). The characteristic cycle $CC(\mathcal{K})$ is a cycle on the cotangent bundle T^*X defined as a $\mathbb{Z}$-linear combination of irreducible components of the singular support $SS(\mathcal{K}) \subset T^*X$ and characterized by the Milnor formula ([S2]) computing the total dimension of the space of vanishing cycles as an intersection number. If k is algebraically closed and X is projective over k, the characteristic cycle $CC(\mathcal{K})$ satisfies the index formula

$$\chi(X, \mathcal{K}) = (CC(\mathcal{K}), T^*_X X)_{T^*X}$$

computing the Euler characteristic $\chi(X, \mathcal{K})$ as an intersection number with the zero-section $T^*_X X$. The index formula is a generalization of the classical Grothendieck-Ogg-Shafarevich formula ([SGA5]).

Computations of characteristic cycle and singular support has been given in some special cases in ([S2]) such as the case where the complex satisfies some good conditions on ramification along a divisor, for example, the tameness or the non-degeneration. However it is difficult to compute the characteristic cycle in general because of the complexity of the space of vanishing cycles. In the thesis, we mainly compute the characteristic cycle $CC(j_!\mathcal{F})$ and the singular support $SS(j_!\mathcal{F})$ for a rank 1 sheaf $\mathcal{F}$ on the complement U of a divisor D on X with simple normal crossings and the canonical open immersion $j \colon U \to X$ using ramification theory in the case where X is a surface. The invariants of ramification which we use are obtained explicitly by computing Witt vectors and blowing-ups at closed points of the divisor.

The thesis consists of two parts. In the first part, we compare Matsuda's ramification theory ([M]) and Abbes-Saito's ramification theory ([AS1], [S1]) in non-logarithmic case for a complete discrete valuation field. Matsuda's ramification theory is a non-logarithmic variant of Kato's ramification theory ([K1]). Kato has given a definition of logarithmic characteristic cycle using this ramification theory which will be used in the computation of $CC(j_!\mathcal{F})$ in the second part. In the first part, we prove the equivalence of Matsuda's ramification filtration $\{\mathrm{fil}'_m H^1(K, \mathbb{Q}/\mathbb{Z})\}_{m \in \mathbb{Z}_{\geq 1}}$ ([M]) and Abbes-Saito's ramification filtration $\{G^r_K\}_{r \in \mathbb{Q}_{\geq 1}}$ ([AS1]) in non-logarithmic case for a complete discrete valuation field K of characteristic $p > 0$:

Theorem 0.1. *Let $\chi \in H^1(K, \mathbb{Q}/\mathbb{Z})$. For $m \in \mathbb{Z}_{>0}$ and $r \in \mathbb{Q}_{\geq 1}$ such that $m \leq r < m + 1$, the following are equivalent:*

(i) $\chi \in \mathrm{fil}'_m H^1(K, \mathbb{Q}/\mathbb{Z})$.

(ii) $\chi(G_K^{m+}) = 0$.

(iii) $\chi(G_K^{r+}) = 0$.

Theorem 0.1 has been proved by Abbes and Saito ([AS2]) in the case where $m > 2$. We prove Theorem 0.1 without any assumption. Our proof of Theorem 0.1 follows the method of the proof by Abbes and Saito. For a character $\chi \in H^1(K, \mathbb{Q}/\mathbb{Z})$, Matsuda ([M]) has defined the non-logarithmic variant of the refined Swan conductor $\mathrm{rsw}(\chi)$ ([K1]) of $\chi \in H^1(K, \mathbb{Q}/\mathbb{Z})$. We call it the characteristic form and denote it by $\mathrm{char}(\chi)$. Matsuda's definition of characteristic form has been given except the case where $p = 2$ and $\chi \in \mathrm{fil}'_2 H^1(K, \mathbb{Q}/\mathbb{Z})$. We define the characteristic form $\mathrm{char}(\chi)$ of χ without any assumption and prove that this characteristic form $\mathrm{char}(\chi)$ is the same as Saito's characteristic form ([S1]). These results give a compatibility of the algebraic approach by Matsuda and the geometric approach by Abbes and Saito.

In the second part of the thesis, we prove two theorems. The first is on a sufficient condition for the equality of characteristic cycles of two complexes and the second is on the computation of characteristic cycle $CC(j_!\mathcal{F})$ for a rank 1 sheaf $\mathcal{F}$ on the complement U of D in a smooth surface X over k. In the first theorem, we prove that two constructible complexes have the same characteristic cycles if they have the same wild ramification along boundaries:

Theorem 0.2. *Let $\mathcal{K}$ and $\mathcal{K}'$ be constructible complexes of $\mathbb{F}_\ell$-modules on X. We assume that there exists a decomposition $X = \coprod_{i \in I} X_i$ of X into finitely many locally closed subsets $\{X_i\}_{i \in I}$ such that $\mathcal{H}^q(\mathcal{K})|_{X_i}$ and $\mathcal{H}^q(\mathcal{K}')|_{X_i}$ are smooth for every $i \in I$ and q. We further assume that for every $i \in I$ and q there exists a normal proper connected scheme $\bar{X}_i$ over k containing X_i as an open subscheme such that $\mathcal{H}^q(\mathcal{K})|_{X_i}$ and $\mathcal{H}^q(\mathcal{K}')|_{X_i}$ have locally the same wild ramification along $\bar{X}_i \setminus X_i$ for every q. Then we have*

$$CC(\mathcal{K}) = CC(\mathcal{K}').$$

By the index formula, Theorem 0.2 is a generalization of Deligne's equality of Euler characteristics in [I]. For the proof of Theorem 0.2, we use Deligne's equation of Euler characteristics and Deligne's method ([D]) twisting sheaves by an Artin-Schreier sheaf ramified deeply.

For the computation of the characteristic cycle $CC(j_!\mathcal{F})$ for a rank 1 sheaf $\mathcal{F}$ on U in the case where X is a surface, we use Kato's logarithmic characteristic cycle $\mathrm{Char}^{\log}(X, U, \mathcal{F})$ defined in [K2]. Kato's characteristic cycle $\mathrm{Char}^{\log}(X, U, \mathcal{F})$ is defined as a cycle on the logarithmic cotangent bundle $T^*X(\log D)$ of X with logarithmic poles along D. We construct a canonical lifting $\mathrm{Char}^K(X, U, \mathcal{F})$ on the cotangent bundle T^*X of Kato's logarithmic characteristic cycle $\mathrm{Char}^{\log}(X, U, \mathcal{F})$ using the ramification theory discussed in the first part. The canonical lifting $\mathrm{Char}^K(X, U, \mathcal{F})$ is of the form

$$\mathrm{Char}^K(X, U, \mathcal{F}) = [T^*_X X] + \sum_{i \in I} \mathrm{dt}'(\chi|_{K_i})[L'_i] + \sum_{x \in |D|} t_x [T^*_x X],$$

where $dt'(\chi)$ and t_x are invariants of ramification of $\mathcal{F}$ along D. We prove the equality of the canonical lifting and the characteristic cycle:

Theorem 0.3. *We have*

$$CC(j_!\mathcal{F}) = \mathrm{Char}^K(X, U, \mathcal{F}). \tag{0.1}$$

Theorem 0.3 gives a computation of the characteristic cycle $CC(j_!\mathcal{F})$ in terms of invariants of ramification. Since $\mathrm{Char}^K(X, U, \mathcal{F})$ is a canonical lifting of Kato's logarithmic characteristic cycle $\mathrm{Char}^{\log}(X, U, \mathcal{F})$ and the index formula for $CC(j_!\mathcal{F})$ generalizes Deligne-Laumon's formula for the Euler characteristic for surfaces ([L]), Theorem 0.3 gives the compatibility of the index formula for Kato's logarithmic characteristic cycle $\mathrm{Char}^{\log}(X, U, \mathcal{F})$ and Deligne-Laumon's formula. A partial result of this compatibility has been obtained by Matsuda ([M]). Since the characteristic cycle $CC(j_!\mathcal{F})$ is defined for a smooth variety X of general dimension over k, Theorem 0.3 gives a ground for an expectation of computation of $CC(j_!\mathcal{F})$ in terms of invariants of ramification in the higher dimensional case.

In the proof of Theorem 0.3, the equivalence Theorem 0.1 of Matsuda's and Abbes-Saito's filtrations gives the equality (0.1) of the canonical lifting and the characteristic cycle except at fibers $T^*_x X$ at finitely many closed point x of D. The equality Theorem 0.2 of characteristic cycles of smooth sheaves having the same wild ramification and the construction of $\mathrm{Char}^K(X, U, \mathcal{F})$ reduces the proof to the case where $\mathcal{F}$ is locally defined by an Artin-Schreier-Witt equation. We conclude the proof of the equality (0.1) using the index formulas for $\mathrm{Char}^K(X, U, \mathcal{F})$ and $CC(j_!\mathcal{F})$ and a homotopy invariance. The equality (0.1) further gives a computation of the singular support $SS(j_!\mathcal{F})$.

We describe the construction of the thesis. The first part of the two parts is on ramification theories and the second part is on characteristic cycles.

Section 1 is the introduction. In Section 2, we recall Kato's and Matsuda's ramification theories ([K1], [K2], [M]) and compute graded quotients of ramification filtrations. In Section 3, we give a definition of characteristic form including Matsuda's definition without any assumption. We briefly recall Abbes-Saito's ramification theory ([S1]) in Section 4 and we prove the equivalence Theorem 0.1 of Matsuda's ramification filtration and Abbes-Saito's non-logarithmic ramification filtration in Section 5. We prove the equality of characteristic form defined in Section 3 and Saito's characteristic form in Section 5. In Section 6, we recall the notions of cleanliness and non-degeneration and prove some properties. This is used in the proof of the equality Theorem 0.3 of characteristic cycle and the canonical lifting of Kato's characteristic cycle for a rank 1 sheaf locally defined by an Artin-Schreier-Witt equation.

In Section 7, we briefly recall the definition of characteristic cycle ([S2]) and prove some properties. We prove the equality Theorem 0.2 of characteristic cycles of the sheaves locally having the same wild ramification in this section. In Section 8, we introduce the notion of log transversality. The log transversality is a logarithmic variant of transversality which has much to do with the definitions of singular support and characteristic cycle. In Section 9, we briefly recall Kato's logarithmic characteristic cycle ([K2]) and construct a canonical lifting of Kato's logarithmic characteristic cycle of a rank 1 sheaf on a smooth surface. Further we construct a canonical lifting of Kato's characteristic cycle of a clean rank 1 sheaf on a smooth variety of higher dimension in a special case and compute a micro-support of the sheaf in Section 9. In Section 10, we prove a homotopy invariance of characteristic cycles of rank 1 sheaves on a surface by constructing a homotopy connecting the sheaves. The equality

Theorem 0.3 of the canonical lifting and characteristic cycle of a rank 1 sheaf on a surface is proved in Section 11. We also give some immediate corollaries in Section 11. At the end of Section 11, we prove the equality of canonical lifting and characteristic cycle for a rank 1 sheaf on a smooth variety of higher dimension in a special case.

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