博士論文

論文題目 Special Lagrangian submanifolds and mean curvature flows

(特殊ラグランジュ部分多様体と平均曲率流について)

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Preface

In this thesis we study special Lagrangian submanifolds, mean curvature flows (especially, Lagrangian mean curvature flows) and some related topics. This thesis consists of self-contained four parts, Part I, Part II, Part III and Part IV. In this preface we outline each of parts.

Recently, the study of special Lagrangian submanifolds have acquired an important role in Mirror Symmetry. For example, they are key words in the Strominger-Yau-Zaslow Conjecture [48] which explains Mirror Symmetry of 3-dimensional Calabi-Yau manifolds. Historically, special Lagrangian submanifolds in Calabi-Yau manifolds are defined in the paper of Harvey and Lawson [22] as calibrated submanifolds. As a general property of calibrated submanifolds, a special Lagrangian submanifold is a minimal submanifold. In general, constructing explicit examples of special Lagrangian submanifolds is difficult, since these conditions are locally written by nonlinear elliptic PDE. However some examples are constructed in the case that the ambient Calabi-Yau manifold has symmetries, especially in $\mathbb{C}^m$.

In Part I (cf. [50]), we construct some examples of special Lagrangian submanifolds and Lagrangian self-similar solutions in almost Calabi–Yau cones over toric Sasaki manifolds. The cone of a toric Sasaki manifold is a kind of generalization of $\mathbb{C}^m$, since it has $\mathbb{R}^+$-action and T^m-action, where T^m is a real m-dimensional torus. As a corollary of this part, for any integer $g \geq 1$, we can construct a real 6-dimensional toric almost Calabi–Yau cone M_g and a 3-dimensional special Lagrangian submanifold $F^1_g : L^1_g \to M_g$ which is diffeomorphic to $\Sigma_g \times \mathbb{R}$, where Σ_g is a closed surface of genus g. This is a generalization of the construction of special Lagrangian submanifold in $\mathbb{C}^m$ by Harvey-Lawson[22] and Joyce [24]. Furthermore, in this part, for any integer $g \geq 1$, we construct a compact Lagrangian self-shrinker $F^2_g : L^2_g \to M_g$ which is diffeomorphic to $\Sigma_g \times S^1$. The meaning and importance of Lagrangian self-shrinkers are mentioned in other paragraphs below. This construction of Lagrangian self-shrinkers is a generalization of one of Joyce, Lee and Tsui [26] in $\mathbb{C}^m$.

Although the chief aim of Part I is constructing explicit examples of special Lagrangian submanifolds, there is an abstract way to get special Lagrangian submanifolds. It is the Lagrangian mean curvature flow. A Lagrangian mean curvature flow is one of potential approaches to find a special Lagrangian submanifold in a given Calabi-Yau manifold as the following meaning. If there exists a long time solution of a Lagrangian mean curvature flow $\{ L_t \,;\, t \in [0, \infty) \}$ starting from a given Lagrangian submanifold L_0 and the flow converges to some smooth manifold L_∞, then it is a minimal Lagrangian submanifold, that is, a special Lagrangian submanifold. Here we used a well-known magical fact that if an initial submanifold L_0 is a Lagrangian submanifold then its deformation L_t along the mean curvature flow is also a Lagrangian submanifold if the ambient space is a Calabi-Yau manifold, that is, the Lagrangian condition is preserved under the mean curvature flow. Indeed, the method of Lagrangian mean curvature flow has more deep background related to Mirror Symmetry proposed by Thomas and Yau [49]. Roughly speaking, they introduce a stability condition on Lagrangian submanifolds and conjecture that the Lagrangian mean curvature flow $\{ L_t \,;\, t \in [0, T) \}$ starting from a stable Lagrangian submanifold exists for all time, that is, $T = \infty$, and converges to a special Lagrangian submanifold in its Hamiltonian deformation class. This conjecture is called Thomas-Yau conjecture. Recently, Joyce [25] has updated the Thomas-Yau conjectures to achieve more plausible statement. In [25], he discussed the possibility that the Lagrangian mean curvature flow develops singularities many times even if an initial Lagrangian submanifold is stable and mentioned the necessity of surgeries of Lagrangian mean curvature flows. Thus it is meaningful to construct examples of Lagrangian mean curvature flows with singularities to understand the motion of Lagrangian mean curvature flows and to develop this program.

In Part II (cf. [51]), we construct explicit examples of special or weighted Hamiltonian stationary Lagrangian submanifolds in toric almost Calabi–Yau manifolds and construct solutions of generalized Lagrangian mean curvature flows with singularities and topological changes starting from these examples. These examples can be considered as some kind of generalization of examples of Lee and Wang [29] in $\mathbb{C}^m$ to toric almost Calabi–Yau manifolds. In general, the topology of a toric almost Calabi–Yau manifold is not simple and there are many fixed points of the torus action. Hence we can get examples of special or weighted Hamiltonian stationary Lagrangian submanifolds with various topologies. Furthermore, its generalized Lagrangian mean curvature flow develops singularities many times, though examples of Lee and Wang in $\mathbb{C}^m$ develops a singularity once. Note that, in this part, we use notions of weighted Hamiltonian stationary and generalized Lagrangian mean curvature flow. These notions are modifications of the ordinary notions of Hamiltonian stationary and Lagrangian mean curvature flow defined in Calabi–Yau manifolds to almost Calabi–Yau manifolds. See Section 13 for precise definitions.

As mentioned above, it is important to study singularities of Lagrangian mean curvature flows. In the study of mean curvature flows, there is a well-known result of Huisken [23]. He studied asymptotic behavior of a mean curvature flow in $\mathbb{R}^m$ when it develops a singularity of special type I, and proved that its rescaled flow converges to a self-shrinker in $\mathbb{R}^m$. Here a self-shrinker is an immersion $F : L \to \mathbb{R}^m$ from some manifold L which satisfies $H(F)_x = -\frac{1}{2}\overrightarrow{x}^{\perp}$ for all points $x \in F(L)$. Hence a self-shrinker is considered as a local model of a singularity of a mean curvature flow.

In Part III (cf. [52]), we try to generalize the result of Huisken in $\mathbb{R}^m$ to in a more general Riemannian manifold, to study singularities of a Lagrangian mean curvature flow in a Calabi-Yau manifold. As a result of such an attempt, we have generalized the result of Huisken in $\mathbb{R}^m$ for a Ricci-mean curvature flow moving along a Ricci flow constructed from a gradient shrinking Ricci soliton, although it is not a Calabi-Yau manifold. Here a Ricci-mean curvature flow is a coupled parabolic PDE system of a Ricci flow and a mean curvature flow, that is, we consider a Ricci flow (N, g_t) with $\frac{\partial}{\partial t} g_t = -2\mathrm{Ric}(g_t)$ and a mean curvature flow $F_t : L \to (N, g_t)$ with $\frac{\partial}{\partial t} F_t = H(F_t)$, where the mean curvature vector field $H(F_t)$ is calculated by g_t at each time t. In Part III, the Ricci flow we consider is the one generated by a gradient shrinking Ricci soliton (N, g) with potential function f. Then, under the special type I assumption, we prove that the rescaled flow converges to a self-shrinker in (N, g, f). Here a self-shrinker in a gradient shrinking Ricci soliton $(N, g, f)$is an immersion $F : L \to N$ from some manifold L which satisfies $H(F) = -\frac{1}{2}\nabla f^{\perp}$. This is a generalization of the notion of self-shrinker in $\mathbb{R}^m$ to in a gradient shrinking Ricci soliton (N, g, f), given by Lott [32].

There are many results about self-shrinkers (more generally, self-similar solutions) in $\mathbb{R}^m$. By a generalization of the notion of a self-similar solution in $\mathbb{R}^m$ to in a gradient shrinking Ricci soliton (N, g, f), we can discuss which results about self-similar solutions in $\mathbb{R}^m$ also hold in a gradient shrinking Ricci soliton (N, g, f). As an example of such results, it is proved that a result due to Smoczyk partially holds in a gradient shrinking Kähler-Ricci soliton. More precisely, in the proof of Theorem 2.3.5 in [44], Smoczyk proved that every compact Lagrangian self-similar solution with exact mean curvature form is a minimal submanifold in $\mathbb{C}^n$, and as a generalization of this statement, we can prove that every compact Lagrangian self-similar solution with exact mean curvature form is a minimal submanifold in a gradient shrinking Kähler-Ricci soliton.

In Part IV (cf. [53]), we give further two results which are already established when (N, g, f) is $\mathbb{R}^m$. The first result is an analog of Theorem 4.3 of Futaki, Li and Li [15] under the Lagrangian assumption. It gives a lower diameter bound of a compact Lagrangian self-shrinker (with under some assumptions) in a complex m-dimensional gradient shrinking Kähler-Ricci soliton (N, g, f) by using an estimate of the first eigenvalue of the weighted Laplacian. The second result is an analog of Proposition 5.3 of Cao and Li [6] under the Lagrangian assumption. To be more precise, we prove that if $R(g) > 2m$ there exists no compact Lagrangian self-shrinker in N, if $R(g) < 2m$ there exists no compact Lagrangian self-expander in N and if $R(g) = 2m$ every compact Lagrangian self-similar solution in N is a minimal submanifold, where $R(g)$ is the scalar curvature of (N, g).

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この博士論文を作成するにあたり, 私は多くの方々にお世話になりました. その感謝の気持ちをより正確に自分の言葉で表すため, 以下の謝辞を日本語で書くことをお許しください. まず初めに, 上の英語での謝辞でも述べましたが, 私の修士課程及び博士課程での指導教員であった二木昭人氏 (以下, 二木先生と呼ばせて頂きます) にもう一度, 感謝の言葉を述べたいと思います. 二木先生には私が東京工業大学の学部３年生の卒業研究のセミナーの時からお世話になりました. 卒業研究では二木先生との１対１でのセミナーで, リーマン面の基本的な本を読み, そのセミナーが自分にとって大変楽しいものであったことを今でも忘れません. その後, 二木先生は, 東京工業大学での修士課程, そして東京大学へ移動した後の博士課程でも, 一貫して私の指導教員であり, 非常に熱心に数学の指導をしてくださいました. 私の数学的に基本的な内容に関するセミナー発表からも, 何か論文の種になるものはないかと常に一緒に考えてくださり, 私の向学心を絶えず刺激してくださりました. そうしたセミナーでの二木先生からの思いがけない指摘から研究が前進したことも何度もあり, 心から感謝しております. 純粋に数学的な内容のみならず, 例えば講演発表に関することや, 論文の書き方に関する私の質問にも, 二木先生はいつでも真摯に答えてくださりました. 研究者として, 修士そして博士と成長するにあたって, 様々な面で何度となく助けていただいたように思います. 本当にありがとうございます.

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この他, 私の博士課程の３年間に私を数学的に, または非数学的に助けてくれた人の名前を挙げれば数限りなく, 一人の名前を挙げれば, 全ての人の名前も挙げなければならないと思うため, これより先で, 特筆して誰かの名前を挙げることは控えさせて頂きますが, 例えば東京工業大学で私と何度となく会話をした友人や先輩, 後輩, そして東京学大での友人や先輩, 後輩, また, 研究集会に私を講演者として呼んで下さった各大学の先生方や, その研究集会で私の研究について有益な指摘や応援をしてくださった全ての方々に感謝いたします. また, 大学での関係を離れ, 私との個人的な付き合いのあった全ての方々にも感謝致します. そうした方々との数学的, 精神的な交流があってこそ, このような博士論文を書くことができたと思っております.

最後に, 私の家族に感謝の言葉を述べてこの謝辞を締めくくろうと思います. 私の両親は, 私が大学で数学を勉強することを様々な面で絶えずサポートしてくれました. 例えば, 私は大学生になってから一人暮らしになりましたが, その間に金銭的に助けてくれたこともありましたし, 数学の内容は分からなくても, お前はすごいことをやっているから自信を持て, と精神的に鼓舞してくれたこともありました. そうした, 何も心配せずに自由に数学だけ勉強していれば良い, という大変恵まれた環境を私に用意し続けてくれた両親に, 言葉では言い尽くせないほどの感謝をいたします. 博士課程の３年間の数学の研究は山あり谷ありで, 楽しいことも大変なこともありましたが, その間の両親の絶え間なく暖かい見守りがあってこそ, 私は幾つかの難所を乗り越え, 前進することができたように思います. この私に健康な体と, 自由に数学をしてよいという環境を揃えてくれた両親に最後にもう一度感謝致します. 本当にありがとうございます.

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Contents

Part I

Special Lagrangians and Lagrangian self-similar solutions in cones over toric Sasaki manifolds

Abstract. We construct some examples of special Lagrangian submanifolds and Lagrangian self-similar solutions in almost Calabi–Yau cones over toric Sasaki manifolds. For example, for any integer $g \geq 1$, we can construct a real 6-dimensional Calabi–Yau cone M_g and a 3-dimensional special Lagrangian submanifold $F_g^1 : L_g^1 \to M_g$ which is diffeomorphic to $\Sigma_g \times \mathbb{R}$ and a compact Lagrangian self-shrinker $F_g^2 : L_g^2 \to M_g$ which is diffeomorphic to $\Sigma_g \times S^1$, where Σ_g is a closed surface of genus g.

1 Introduction

Special Lagrangian submanifolds are defined in almost Calabi-Yau manifolds. Recently special Lagrangian submanifolds have acquired an important role in Mirror Symmetry. For example, they are key words in the Strominger–Yau–Zaslow Conjecture [48] which explains Mirror Symmetry of 3-dimensional Calabi–Yau manifolds. Furthermore Thomas and Yau [49] introduced a stability condition for graded Lagrangians and conjectured that a stable Lagrangian converges to a special Lagrangian submanifold by the mean curvature flow.

In this conjecture, the mean curvature flow is also one of important key words. Simply stated, mean curvature flows are gradient flows of volume functionals of manifolds. In a precise sense, it is a flow of a manifold in a Riemannian manifold moving along its mean curvature vector field. Let (M, g) be a Riemannian manifold, N a manifold and $F : N \times [0, T) \to M$ a smooth family of immersions, then F is called a mean curvature flow if it satisfies

$$\frac{\partial F}{\partial t}(p, t) = H_t(p) \qquad \text{for all } (p, t) \in N \times [0, T)$$

where H_t is the mean curvature vector field of the immersion $F_t := F(\cdot, t) : N \to M$. If the ambient is $\mathbb{R}^m$, there is an important class of solutions called *self-similar solution.* An immersion of a manifold $F : N \to \mathbb{R}^m$ is called a self-similar solution if it satisfies

$$H = \lambda F^{\perp}$$

where $\lambda \in \mathbb{R}$ is a constant and $F^{\perp}$ is the normal part of the position vector F. Huisken [23] has studied mean curvature flows in $\mathbb{R}^m$ and proved that if the mean curvature flow in $\mathbb{R}^m$ has the type I singularity, then there exists a smoothly convergent subsequence of the rescaling such that its limit becomes a self-similar solution. In this sense, a self-similar solution can be thought of as an asymptotical model of a mean curvature flow which develops a type I singularity at the time when it blowups.

In this Part, we construct Lagrangian self-similar solutions in *cone manifolds.* To define self-similar solutions in cone manifolds, we use the generalization of position vectors in $\mathbb{R}^m$ to cone manifolds defined by Futaki, Hattori and the author in [14].

Here we introduce some notations over cone manifolds. First, for a Riemannian manifold (S, g), we say that $(C(S), \overline{g})$ is a cone over (S, g), if $C(S) \cong S \times \mathbb{R}^+$ and $\overline{g} = r^2 g + dr^2$ where r is the standard coordinate of $\mathbb{R}^+$. We denote two projections by $\pi : C(S) \to S$ and $r : C(S) \to \mathbb{R}^+$. On the cone $C(S)$, there is a natural $\mathbb{R}^+$-action defined below. This action can be considered as an expansion or shrinking on the cone.

Definition 1.1. We define the $\mathbb{R}^+$-action on $C(S)$ by

$$\rho \cdot p_0 = (s_0, \rho r_0) \quad \in C(S) \cong S \times \mathbb{R}^+$$

for all $\rho \in \mathbb{R}^+$ and $p_0 = (s_0, r_0) \in C(S)$.

Definition 1.2. For a point $p_0 = (s_0, r_0) \in S \times \mathbb{R}^+ \cong C(S)$, we define the position vector $\overrightarrow{p_0}$ by

$$\overrightarrow{p_0} = r_0 \frac{\partial}{\partial r}\bigg|_{r=r_0} \in T_{p_0}C(S).$$

Furthermore, for a map $F : N \to C(S)$ from a manifold N, we define the position vector $\overrightarrow{F}$ of F by $\overrightarrow{F}(x) := \overrightarrow{F(x)}$ at $x \in N$. Note that $\overrightarrow{F}$ is a section of $F^*(TC(S))$ over N.

Clearly $\overrightarrow{p_0}$ coincides with the derivative of the curve $c(\rho) := \rho \cdot p_0$ in $C(S)$ at $\rho = 1$, that is,

$$\overrightarrow{p_0} = \frac{d}{d\rho}\bigg|_{\rho=1} (\rho \cdot p_0).$$

Using this generalization of the position vector, we can define self-similar solutions in cone manifolds.

Definition 1.3. Let N be a manifold. An immersion $F : N \to C(S)$ is called a self-similar solution if

$$H = \lambda \overrightarrow{F}^{\perp}$$

where $\lambda \in \mathbb{R}$ is a constant. It is called a self-shrinker if $\lambda < 0$ and self-expander if $\lambda > 0$.

Here $\perp$ is the orthogonal projection map from $F^*(TC(S))$ to $T^{\perp}N$ which is an orthogonal complement of $F_*(TN)$. Furthermore if a self-similar solution in a Kähler manifold is a Lagrangian submanifold, then we call it a Lagrangian self-similar solution.

The typical results in $\mathbb{R}^n$ studied by Huisken [23] are extended to the mean curvature flow in a cone manifold by Futaki, Hattori and the author in [14]. For example, it is proved in [14] that if a mean curvature flow in a cone manifold has the type I_c singularity, then there exists a smoothly convergent subsequence of the rescaling such that its limit becomes a self-similar solution. Type I_c singularity is a certain kind of singularity similar to type I singularity, and for more details refer to [14].

In this part, we present a method of constructing special Lagrangian submanifolds and Lagrangian self-similar solutions in toric Calabi–Yau cones. First we construct Lagrangian submanifolds in toric Kähler cone in Theorem 3.4. Next, if the canonical line bundle of the toric Kähler cone is trivial, that is, it is a toric almost Calabi–Yau cone, then we construct special Lagrangian submanifolds in Theorem 6.1 and Theorem 6.2, and Lagrangian self-similar solutions in Theorem 7.1. These constructions are considered to be a kind of extension of special Lagrangian submanifolds in $\mathbb{C}^m$ by Harvey and Lawson [22] and Lagrangian self-similar solutions in $\mathbb{C}^m$ by Joyce, Lee and Tsui in [26], see Remark 6.3 and Remark 7.2. As an application of these theorems, we concretely construct some examples.

Example 1.4 (cf. Example 8.4)**.** For any integer $g \geq 1$, we construct a real 6-dimensional Calabi–Yau cone M_g and a 3-dimensional special Lagrangian submanifold $F_g^1 : L_g^1 \to M_g$ which is diffeomorphic to $\Sigma_g \times \mathbb{R}$ and a compact Lagrangian self-similar solution (self-shrinker) $F_g^2 : L_g^2 \to M_g$ which is diffeomorphic to $\Sigma_g \times S^1$ concretely, where Σ_g is a closed surface of genus g.

This part is organized as follows. In Section 2, we introduce some basic definitions and propositions in toric Sasaki manifolds. In Section 3, we construct Lagrangian submanifolds in cones over toric Sasaki manifolds. In Section 4, we explain some details about almost Calabi–Yau manifolds, Lagrangian angles, special Lagrangian submanifolds and generalized mean curvature vectors. In Section 5, we compute the Lagrangian angles of Lagrangians constructed in Section 3 when the ambient is a toric almost Calabi–Yau cone. Section 6 is devoted to the proofs of Theorem 6.1 and 6.2. Section 7 is devoted to the proofs of Theorem 7.1. In Section 8, for an application of our theorems, we construct some concrete examples in toric Calabi–Yau 3-folds.

2 Toric Sasaki manifold

In this section we introduce some definitions and propositions in toric Sasaki manifolds. Proofs of the results in this section are summarized in the papers of Boyer and Galicki [3] and Martelli, Sparks and Yau [36]. First of all, we define Sasaki manifolds.

Definition 2.1. Let (S, g) be a Riemannian manifold and ∇ the Levi-Civita connection of the Riemannian metric g. Then (S, g) is said to be a Sasaki manifold if and only if it satisfies one of the following two equivalent conditions.

(2.1.a) There exists a Killing vector field ξ of unit length on S so that the tensor field Φ of type $(1, 1)$, defined by $\Phi(X) = \nabla_X \xi$, satisfies

$$(\nabla_X \Phi)(Y) = g(\xi, Y)X - g(X, Y)\xi.$$

(2.1.b) There exists a complex structure J on $C(S)$ compatible with $\overline{g}$ so that $(C(S), \overline{g}, J)$ becomes a Kähler manifold.

We call the quadruple (ξ, η, Φ, g) on S the Sasaki structure. S is often identified with the submanifold $\{r = 1\} = S \times \{1\} \subset C(S)$. By the definition, the dimension of S is odd and denoted by $2m - 1$. Hence the complex dimension of $C(S)$ is m. Note that $C(S)$ does not contain the apex.

The equivalence of (2.1.a) and (2.1.b) can be seen as follows. If (S, g) satisfies the condition (2.1.a), we can define a complex structure J on $C(C)$ as

$$JY = \Phi(Y) - \eta(Y) r \frac{\partial}{\partial r} \quad \text{and} \quad J r \frac{\partial}{\partial r} = \xi.$$

for all $Y \in \Gamma(TS)$ and $r(\partial/\partial r) \in \Gamma(T\mathbb{R}^+)$, where η is a 1-form on S defined by $\eta(Y) = g(\xi, Y)$. Conversely, if (S, g) satisfies condition (2.1.b), we have a Killing vector field ξ defined as $\xi = J\frac{\partial}{\partial r}$.

We can extend ξ and η also on the cone $C(S)$ by putting

$$\xi = J r \frac{\partial}{\partial r}, \qquad \eta(Y) = \frac{1}{r^2} \overline{g}(\xi, Y)$$

where Y is any smooth vector field on $C(S)$. Of course η on $C(S)$ is the pull-back of η on S by the projection $\pi : C(S) \to S$. Furthermore the 1-form η is expressed on $C(S)$ as

$$\eta = 2d^c \log r \tag{1}$$

where $d^c = \frac{i}{2}(\bar{\partial} - \partial)$. From (1), the Kähler form ω of the cone $(C(S), \overline{g})$ is expressed as

$$\omega = \frac{1}{2} d(r^2 \eta) = \frac{1}{2} dd^c r^2 = \frac{i}{2} \partial\bar{\partial} r^2. \tag{2}$$

Remember that we have defined $\mathbb{R}^+$-action on $C(S)$ in Definition 1.1. By (2), it is clear that $\rho^*\omega = \rho^2\omega$, where we denote the transition map with respect to $\rho \in \mathbb{R}^+$ by the same symbol $\rho : C(S) \to C(S)$; $\rho(p) = \rho \cdot p$. Next, we introduce the notion of toric Sasaki manifolds.

Definition 2.2. A Sasaki manifold with Sasaki structure (S, ξ, η, Φ, g) of dimension $2m - 1$ is a *toric* Sasaki manifold if and only if it satisfies one of the following two equivalent conditions.

(2.2.a) There is an effective action of m-dimensional torus T^m on S preserving the Sasaki structure.

(2.2.b) There is an effective holomorphic action of m-dimensional torus T^m on $C(S)$ preserving $\overline{g}$. Furthermore two projections $\pi : C(S) \to S$ and $r : C(S) \to \mathbb{R}^+$ satisfy $\pi(\tau \cdot p) = \tau \cdot \pi(p)$ and $r(\tau \cdot p) = r(p)$ for all $\tau \in T^m$ and $p \in C(S)$.

It is clear that $\mathbb{R}^+$-action and T^m-action is commutative. The most typical example of the toric Sasaki manifold is the sphere S^{2m-1}, because $C(S) = \mathbb{C}^m \setminus \{0\}$ is toric Kähler.

The equivalence of (2.2.a) and (2.2.b) can be seen as follows. If a Sasaki manifold (S, g) satisfies the condition (2.2.a), let $\tau \in T^m$ act on $C(S)$ as

$$\tau \cdot p_0 = (\tau \cdot s_0, r_0)$$

for all $p_0 = (s_0, r_0) \in C(S)$. Then this action on $C(S)$ satisfies the condition (2.2.b). Conversely, if a Sasaki manifold (S, g) satisfies the condition (2.2.b), then the restriction of T^m-action to S satisfies the condition (2.2.a).

Let $\mathfrak{g} \cong \mathbb{R}^m$ be the Lie algebra of T^m and $\mathfrak{g}^*$ be the dual vector space. We identify the vector field on $C(S)$ generated by $v \in \mathfrak{g}$ and v itself. That is, for $p \in C(S)$ we write

$$v(p) = \left.\frac{d}{dt}\right|_{t=0} \exp(tv) \cdot p.$$

A toric Sasaki manifold and its cone have a moment map $\mu : C(S) \to \mathfrak{g}^*$ with respect to the Kähler form $\omega = \frac{1}{2}d(r^2\eta)$. It is given by

$$\langle \mu(p), v\rangle = \frac{1}{2}r^2(p)\eta(v(p)), \tag{3}$$

for all $p \in C(S)$ and $v \in \mathfrak{g}$ and it satisfy

$$d\langle \mu, v\rangle = -\omega(v, \cdot).$$

On the other hand, since $C(S)$ is a toric variety, there exists a fan Σ of $C(S)$ and the complex structure on $C(S)$ is determined by Σ. Moreover there exists an m-dimensional complex torus $T^m_{\mathbb{C}} (\cong (\mathbb{C}^\times)^m)$ contains T^m as a compact subgroup, and $T^m_{\mathbb{C}}$ acts on $C(S)$ as a bi-holomorphic automorphism and has an open dense $T^m_{\mathbb{C}}$-orbit. Hence, over $C(S)$, there exists an intrinsic anti-holomorphic involution $\sigma : C(S) \to C(S)$ determined by Σ, that is, $\sigma^2 = id$ and $\sigma_* J = -J\sigma_*$. This involution satisfies

$$\sigma(w \cdot p) = \overline{w} \cdot \sigma(p), \tag{4}$$

where $w \in T^m_{\mathbb{C}}$ and $p \in C(S)$. We denote the set of fixed points of σ by

$$C(S)^\sigma = \{\, p \in C(S) \mid \sigma(p) = p \,\}.$$

Then it is a real m-dimensional submanifold of $C(S)$, and we call it a real form of $C(S)$. Now we consider some properties of σ and $C(S)^\sigma$.

Proposition 2.3. *The involution $\sigma : C(S) \to C(S)$ is anti-symplectic. Thus it is also isometry.*

Proof. Let U_0 be an open dense $T^m_{\mathbb{C}}$-orbit. For $(w^1, \ldots, w^m) \in U_0 \cong T^m_{\mathbb{C}} \cong (\mathbb{C}^\times)^m$, we take a logarithmic holomorphic coordinates $(z^1, \ldots, z^m)$ defined by $e^{z^k} = w^k$. Since ω is T^m-invariant and the action of T^m is Hamiltonian, there exists a function $F(x) \in C^\infty(\mathbb{R}^m)$ with the property

$$\omega = \frac{i}{2}\sum_{k,\ell=1}^{m} \frac{\partial^2 F}{\partial x^k \partial x^\ell} dz^k \wedge d\overline{z}^\ell \quad \text{on } U_0,$$

where $z^k = x^k + iy^k$. (See Guillemin [17].) On U_0, the involution σ coincides with the standard complex conjugate $\sigma(z) = \overline{z}$, where $\overline{z} = (\overline{z}^1, \ldots, \overline{z}^m)$. Note that F is independent of the coordinates $(y^k)_{k=1}^m$. Thus we have $\sigma^*\omega = -\omega$ on U_0. Since U_0 is open and dense in $C(S)$, thus we have $\sigma^*\omega = -\omega$ on $C(S)$. Second statement follows immediately by combining the property that σ is anti-holomorphic. □

Here we have some remarks.

Remark 2.4. Take a point p in real form $C(S)^\sigma$ and two vectors X, Y in $T_pC(S)^\sigma$. Since $\sigma_*X = X$ and $\sigma_*Y = Y$, we have

$$\omega(X, Y) = \omega(\sigma_*X, \sigma_*Y) = -\omega(X, Y)$$

by Proposition 2.3, hence $\omega = 0$ on $C(S)^\sigma$. This means that the real form $C(S)^\sigma$ is a Lagrangian submanifold in $C(S)$. Moreover if we apply the condition (4) for p and $\tau \in T^m$, we have $\sigma(\tau \cdot p) = \tau^{-1} \cdot p$, hence for all $v \in \mathfrak{g}$ we have $\sigma_*v(p) = -v(p)$. This means that $v(p)$ is orthogonal to $T_pC(S)^\sigma$ with respect to $\overline{g}$.

In general we do not know for p in $C(S)^\sigma$ whether its position vector $\overrightarrow{p}$ is tangent to $C(S)^\sigma$. However if we assume the Reeb field ξ is generated by an element in $\mathfrak{g}$, then it is ensured. For such a toric Sasaki manifold, we identify the Reeb vector field ξ and an element in $\mathfrak{g}$ that generates ξ.

Proposition 2.5. *Let (S, ξ, η, Φ, g) be a toric Sasaki manifold. If the Reeb field ξ is generated by an element in $\mathfrak{g}$, then for all p in $C(S)^\sigma$ its position vector $\overrightarrow{p}$ is tangent to $C(S)^\sigma$.*

Proof. Remember Remark 2.4. Since $C(S)^\sigma$ is a Lagrangian submanifold, we have orthogonal decomposition

$$T_pC(S) = T_pC(S)^\sigma \oplus J(T_pC(S)^\sigma),$$

with respect to $\overline{g}$. Now ξ is in $\mathfrak{g}$, hence $\xi(p)$ is orthogonal to $T_pC(S)^\sigma$, that is, $\xi(p)$ is in $J(T_pC(S)^\sigma)$. On the other hand, $\xi(p) = J(r\frac{\partial}{\partial r})|_{\text{at } p} = J(\overrightarrow{p})$. Thus we have $\overrightarrow{p} \in T_pC(S)^\sigma$. □

In this Part we always assume that the Reeb field ξ of toric Sasaki manifold is generated by an element in $\mathfrak{g}$. By Proposition 2.5, it follows that $C(S)^\sigma$ is also a cone manifold. If we write $S^\sigma = \{ p \in S \mid \sigma(p) = p \}$, then $C(S)^\sigma = C(S^\sigma)$.

In the last of this section, we remark some facts that is well known in the toric contact geometry and the algebraic toric geometry. Let $C(S)$ be the cone of a toric Sasaki manifold S with dimension $2m-1$ and with the Reeb field ξ. Let $\mathbb{Z}_\mathfrak{g} \cong \mathbb{Z}^m$ be the integral lattice of $\mathfrak{g}$, that is the kernel of the exponential map $\exp : \mathfrak{g} \to T^m$. Let Σ be a fan of $C(S)$ and $\Lambda = \{\lambda_1, \ldots, \lambda_d\} \subset \mathbb{Z}_\mathfrak{g}$ be the primitive generators of the 1-dimensional cones of Σ. Let $\Delta = \mu(C(S))$ be a moment image of $C(S)$ and let Δ_0^* be a (open) dual cone of Δ defined by

$$\Delta_0^* := \{ x \in \mathfrak{g} \mid \langle y, x \rangle > 0 \text{ for all } y \in \Delta \}.$$

Remark 2.6. In fact, Δ is a *good* rational polyhedral cone defined below and the Reeb field ξ is an element of Δ_0^*.

The second statement in Remark 2.6 is clear since for all p in $C(S)$ we have

$$\langle \mu(p), \xi \rangle = \frac{1}{2} r^2(p) \eta(\xi(p)) = \frac{1}{2} r^2(p) > 0.$$

Definition 2.7 (Good cone, cf. [30]). First we say that a subset $\Delta \subset \mathfrak{g}^*$ is a *rational polyhedral cone* if there exists a finite set of primitive vectors $\Lambda = \{\lambda_1, \ldots, \lambda_d\} \subset \mathbb{Z}_\mathfrak{g}$ such that

$$\Delta = \{ y \in \mathfrak{g}^* \mid \langle y, \lambda \rangle \geq 0 \text{ for } \lambda \in \Lambda \} - \{0\}.$$

We assume that the set Λ is minimal, that is, we can not express Δ by any subset $\Lambda' \subset \Lambda$, $\Lambda' \neq \Lambda$. Furthermore we say that Δ is *strongly convex* if $\Delta \cup \{0\}$ does not contain any straight lines of the form $\ell = \{ p + vt \mid t \in \mathbb{R} \}$ for some p and v in $\mathfrak{g}^*$. Under these assumptions a strongly convex rational polyhedral cone Δ with non-empty interior is *good* if the following condition holds. If a subset $\Lambda' \subset \Lambda$ satisfies

$$\{ y \in \Delta \mid \langle y, \lambda \rangle = 0 \text{ for } \lambda \in \Lambda' \} \neq \emptyset,$$

then Λ' is linearly independent over $\mathbb{Z}$ and

$$\left\{ \sum_{\lambda \in \Lambda'} a_\lambda \lambda \;\middle|\; a_\lambda \in \mathbb{R} \right\} \cap \mathbb{Z}_\mathfrak{g} = \left\{ \sum_{\lambda \in \Lambda'} m_\lambda \lambda \;\middle|\; m_\lambda \in \mathbb{Z} \right\}. \tag{5}$$

By the standard algebraic toric geometry theory, we know that the canonical line bundle $K_{C(S)}$ of $C(S)$ is trivial or not. That is the following remark.

Remark 2.8. The canonical line bundle $K_{C(S)}$ of $C(S)$ is trivial if and only if there exists an element $\gamma \in (\mathbb{Z}_\mathfrak{g})^* \cong \mathbb{Z}^m$ such that

$$\langle \gamma, \lambda \rangle = 1$$

for all $\lambda \in \Lambda$. In fact, by using this element $\gamma = (\gamma_1, \ldots, \gamma_m)$, we can construct canonical non-vanishing holomorphic $(m, 0)$-form on $C(S)$ by purely algebraic toric geometry way, and we denote it by Ω_γ. On the open dense $T^m_\mathbb{C}$-orbit $U_0 \cong (\mathbb{C}^\times)^m$, we can express Ω_γ by the logarithmic holomorphic coordinates $(z^k)_{k=1}^m$ by

$$\Omega_\gamma = \exp(\gamma_1 z^1 + \cdots + \gamma_m z^m) dz^1 \wedge \cdots \wedge dz^m.$$

3 Construction of Lagrangian submanifolds

Let (S, g) be a toric Sasaki manifold with $\dim_\mathbb{R} S = 2m - 1$ and $(C(S), \overline{g})$ be the toric Kähler cone. In this section we construct the explicit examples of Lagrangian submanifolds in $C(S)$. Let $\mu : C(S) \to \mathfrak{g}^*$ be a moment map and $\Delta = \mu(C(S))$ be the moment image of $C(S)$. As explained in Section 2, there exists a finite set of primitive vectors $\Lambda = \{\lambda_1, \ldots, \lambda_d\} \subset \mathbb{Z}_\mathfrak{g}$ such that

$$\Delta = \{ y \in \mathfrak{g}^* \mid \langle y, \lambda \rangle \geq 0 \text{ for } \lambda \in \Lambda \} - \{0\}.$$

To construct Lagrangian submanifolds, first of all, take $\zeta \in \mathfrak{g}$ and $c \in \mathbb{R}$, and we denote the hyperplane $\{ y \in \mathfrak{g}^* \mid \langle y, \zeta \rangle = c \}$ by $H_{\zeta,c}$. We assume that

$$\mathrm{Int}\Delta \cap H_{\zeta,c} \neq \emptyset \quad \text{and} \tag{6}$$

$$\zeta \notin \mathfrak{z}_y \quad \text{for any } y \in \Delta \cap H_{\zeta,c}, \tag{7}$$

where we define $\mathfrak{z}_y$ for $y \in \Delta$ by

$$\mathfrak{z}_y = \mathrm{Span}_{\mathbb{R}}\{\, \lambda_i \mid \langle y, \lambda_i \rangle = 0 \}.$$

For example, if $y \in \mathrm{Int}\Delta$ then $\mathfrak{z}_y = \{0\}$. We denote the intersection of Δ and $H_{\zeta,c}$ by

$$\Delta_{\zeta,c} = \Delta \cap H_{\zeta,c}.$$

First assumption (6) means that $\Delta_{\zeta,c}$ is codimension one in Δ. Second assumption (7) means that if $p \in C(S)$ is in $\mu^{-1}(\Delta_{\zeta,c})$ then $\zeta(p) \neq 0$, where we identify $\zeta \in \mathfrak{g}$ and the vector field on $C(S)$ generated by $\zeta \in \mathfrak{g}$.

Let $\sigma : C(S) \to C(S)$ be the involution explained in Section 2 and $C(S)^\sigma$ be the real form. Let $\mu^\sigma : C(S)^\sigma \to \Delta$ be the restriction of μ on the real form. In fact, μ^σ is a 2^m-fold ramified covering of Δ. We define a subset of $C(S)^\sigma$ as the pull-back of $\Delta_{\zeta,c}$ by μ^σ by

$$\begin{aligned} C(S)^\sigma_{\zeta,c} &= (\mu^\sigma)^{-1}(\Delta_{\zeta,c}) \\ &= \{\, p \in C(S)^\sigma \mid \langle \mu(p), \zeta \rangle = c \,\}. \end{aligned}$$

By the assumptions (6) and (7), in fact $C(S)^\sigma_{\zeta,c}$ is a real $(m-1)$-dimensional submanifold in the real form $C(S)^\sigma$. Since μ^σ is a 2^m-fold covering of Δ, $C(S)^\sigma_{\zeta,c}$ is a 2^m-fold covering of $\Delta_{\zeta,c}$.

Remark 3.1. If ζ and c do not satisfy the assumptions (6) and (7), then $C(S)^\sigma_{\zeta,c}$ may become a singular submanifold.

To construct a Lagrangian submanifold, we move $C(S)^\sigma_{\zeta,c}$ by a one parameter action of $\mathbb{R}^+$ and torus T^m. Take an open interval $I \subset \mathbb{R}$. Let $f : I \to \mathbb{R}$ and $\rho : I \to \mathbb{R}^+$ be two functions on I, and τ_0 be an element of torus T^m. We assume that $\dot{f}$ is non-vanishing on I. We denote the 1-parameter orbit $\{\exp(f(t)\zeta) \cdot \tau_0\}_{t \in I}$ in torus by $\{\tau(t)\}_{t\in I}$. We define a real m-dimensional manifold by

$$L_{\zeta,c} = C(S)^\sigma_{\zeta,c} \times I.$$

Definition 3.2. We define a map $F : L_{\zeta,c} \to C(S)$ by

$$F(p,t) := \rho(t) \cdot \tau(t) \cdot p$$

for $(p,t) \in C(S)^\sigma_{\zeta,c} \times I = L_{\zeta,c}$.

Remark 3.3. If $\rho(t) \cdot \tau(t)$ is defined on $I = \mathbb{R}$ and periodic, then we can reduce I to S^1 and take $L_{\zeta,c}$ as $C(S)^\sigma_{\zeta,c} \times S^1$.

Theorem 3.4. *$F : L_{\zeta,c} \to C(S)$ is a Lagrangian submanifold in $C(S)$.*

Proof. Fix $x_0 = (p_0, t_0) \in L_{\zeta,c}$. For any $X \in T_{p_0} C(S)^\sigma_{\zeta,c}$, we have

$$F_* X = (\rho(t_0) \cdot \tau(t_0))_* X \tag{8}$$

and for $\partial/\partial t \in T_{t_0} I$ we have

$$F_* \frac{\partial}{\partial t} = (\rho(t_0) \cdot \tau(t_0))_* \left(\frac{\dot\rho(t_0)}{\rho(t_0)} \overrightarrow{p_0} + \dot f(t_0) \zeta(p_0) \right). \tag{9}$$

By the assumption, $\dot f(t_0)\zeta(p_0) \neq 0$ and it is orthogonal to all tangent vectors on $C(S)^\sigma$, it follows that F is an immersion. Next, it is clear that

$$\begin{aligned} &\omega(F_* X, F_* Y) = \rho^2(t_0)\omega(X, Y) = 0, \\ &\omega(F_* \partial/\partial t, F_* \partial/\partial t) = 0 \quad \textit{and} \\ &\omega(F_* \partial/\partial t, F_* X) = \rho^2(t_0) \dot f(t_0) \omega(\zeta(p_0), X). \end{aligned}$$

As mentioned in Remark 2.4, if two vectors X and Y are tangent to the real form then $\omega(X,Y) = 0$ and note that position vector $\overrightarrow{p_0}$ is tangent to the real form. Finally, in fact $\omega(\zeta(p_0), X) = 0$ since

$$\omega(\zeta(p_0), X) = X(\langle \mu, \zeta \rangle)$$

and by definition of $C(S)^\sigma_{\zeta,c}$ the function $\langle \mu, \zeta \rangle$ is a constant c on $C(S)^\sigma_{\zeta,c}$. Thus we have $F^*\omega = 0$ and F is a Lagrangian immersion. □

4 almost Calabi–Yau manifold

In this section, we recall the details about almost Calabi–Yau manifolds, special Lagrangian submanifolds and so on.

Definition 4.1. Let (M,ω) be a Kähler manifold with complex dimension m. If the canonical line bundle K_M is trivial, we can take a non-vanishing holomorphic $(m,0)$-form Ω on M. Then we call a triple (M,ω,Ω) an almost Calabi–Yau manifold. Furthermore if the function $\psi : M \to \mathbb{R}$ defined below is identically constant, we call it a Calabi–Yau manifold.

On an almost Calabi–Yau manifold (M,ω,Ω), we define a function ψ by

$$e^{2m\psi}\frac{\omega^m}{m!} = (-1)^{\frac{m(m-1)}{2}}\left(\frac{i}{2}\right)^m \Omega\wedge\bar{\Omega}.$$

In this section, we always assume that (M,ω,Ω) is an almost Calabi–Yau manifold with complex dimension m. Next, we define the Lagrangian angle of a Lagrangian submanifold.

Definition 4.2. Let $F : L \to M$ be a Lagrangian submanifold. The Lagrangian angle of F is the map $\theta_F : L \to \mathbb{R}/\pi\mathbb{Z}$ defined by

$$F^*(\Omega) = e^{i\theta_F + mF^*(\psi)}\mathrm{d}V_{F^*(g)},$$

where g is the Riemannian metric on M with respect to ω.

Note that we do not assume that L is oriented. Thus $dV_{F^*(g)}$ has ambiguity of the sign. Since $F : L \to M$ is a Lagrangian submanifold, θ_F is well defined. For details, see for example Harvey and Lawson [22, III.1] or Behrndt [2].

Remark 4.3. Note that $F^*\Omega$ is a non-vanishing *complex-valued* m-form on L. Hence on each local coordinates $(U, x^1, \ldots, x^m)$ we can express $F^*\Omega$ as

$$F^*\Omega = h(x)dx^1 \wedge \cdots \wedge dx^m.$$

Here h is a non-vanishing complex-valued function on U. Then the Lagrangian angle θ_F is exactly $\arg h$ the argument of h.

Now we can define special Lagrangian submanifolds.

Definition 4.4. Take a constant $\theta \in \mathbb{R}$. We say that $F : L \to M$ is a special Lagrangian submanifold with phase $e^{i\theta}$ if the Lagrangian angle θ_F is identically constant θ. This condition is equivalent to that

$$F^*(\mathrm{Im}(e^{-i\theta}\Omega)) = F^*(\cos\theta\,\mathrm{Im}\,\Omega - \sin\theta\,\mathrm{Re}\,\Omega) = 0.$$

If $F : L \to M$ is a special Lagrangian submanifold with phase $e^{i\theta}$, then there is a unique orientation on L in which $F^*(\mathrm{Re}(e^{-i\theta}\Omega)) = F^*(\cos\theta\,\mathrm{Re}\,\Omega + \sin\theta\,\mathrm{Im}\,\Omega)$ is positive.

Historically Harvey and Lawson [22] have defined special Lagrangian submanifolds by calibrations. Of course we can define special Lagrangian submanifolds in almost Calabi–Yau manifolds by calibrations as follows. Let g be a Riemannian metric with respect to ω. Here we define a new Riemannian metric $\tilde{g}$ on M by conformally rescaling by $\tilde{g} = e^{2\psi}g$. Then the m-form $\mathrm{Re}(e^{-i\theta}\Omega)$ becomes a calibration on the Riemannian manifold $(M,\tilde{g})$ and the definition of special Lagrangian submanifolds in (M,ω,Ω) is restated as a calibrated submanifold in the Riemannian manifold $(M,\tilde{g})$ with respect to $\mathrm{Re}(e^{-i\theta}\Omega)$.

Here we introduce the generalized mean curvature vector field. The generalized mean curvature vector field was introduced by Behrndt in [1, §3] and later generalized by Smoczyk and Wang in [46].

Definition 4.5. The generalized mean curvature vector field H^g of $F : L \to M$ is a normal vector field defined by

$$H^g = H - m(\nabla\psi)^\perp.$$

Here H is the ordinary mean curvature vector field of $F : L \to M$, ∇ is the gradient with respect to g, and $\perp$ is the projection from TM to $T^\perp L$ it is the g-orthogonal complement of $F_*(TL)$.

Note that if ψ is constant or equivalently (M,ω,Ω) is Ricci-flat, then $H^g \equiv H$. As well known, if the ambient space is a Calabi–Yau manifold, then the Lagrangian angle θ_F of a Lagrangian submanifold $F : L \to M$ and its mean curvature vector field H satisfy the equation

$$H = J\nabla\theta_F.$$

More precisely, $H = JF_*(\nabla_{F^*g}\theta_F)$ where ∇_{F^*g} is the (F^*g)-gradient on L, however we write it as above for short. On the other hand, if the ambient space is an almost Calabi–Yau manifold, the above equation does not hold in general. However if we take H^g instead of H, the above equation holds. This is proved by Behrndt [1, Prop. 4].

Proposition 4.6 (cf. [1, Prop. 4])**.** *Let $F : L \to M$ be a Lagrangian submanifold in an almost Calabi–Yau manifold. Then the generalized mean curvature vector field satisfies $H^g = J\nabla\theta_F$.*

It is clear that if L is connected, then L is a special Lagrangian submanifold if and only if $H^g \equiv 0$. For more motivation to introduce the generalized mean curvature vector field and some properties, refer the paper of Behrndt [2].

5 Lagrangian angle

Let $(C(S), \overline{g})$ be the toric Kähler cone over a $(2m-1)$-dimensional toric Sasaki manifold (S, g). In this section we assume that the canonical line bundle $K_{C(S)}$ is trivial. As mentioned in Remark 2.8, this assumption is equivalent to that there exists an element $\gamma \in (\mathbb{Z}_{\mathfrak{g}})^* \cong \mathbb{Z}^m$ such that

$$\langle \gamma, \lambda \rangle = 1$$

for all $\lambda \in \Lambda$. Then we can take a non-vanishing holomorphic $(m, 0)$-form Ω_γ which is expressed as

$$\Omega_\gamma = \exp(\gamma_1 z^1 + \cdots + \gamma_m z^m) dz^1 \wedge \cdots \wedge dz^m$$

on the open dense $T^m_{\mathbb{C}}$-orbit $U_0 \cong (\mathbb{C}^\times)^m$ by the logarithmic holomorphic coordinates $(z^k)_{k=1}^m$. Thus we have a toric almost Calabi–Yau cone manifold $(C(S), \omega, \Omega_\gamma)$.

Remember that in Section 3 we took the data $c \in \mathbb{R}$, $\zeta \in \mathfrak{g}$, $I \subset \mathbb{R}$, $f : I \to \mathbb{R}$, $\rho : I \to \mathbb{R}^+$ and $\tau_0 \in T^m$, and we denoted $\tau(t) = \exp(f(t)\zeta) \cdot \tau_0$. We have defined a submanifold

$$C(S)^\sigma_{\zeta,c} = \{\, p \in C(S)^\sigma \mid \langle \mu(p), \zeta \rangle = c \,\},$$

an m-dimensional manifold

$$L_{\zeta,c} = C(S)^\sigma_{\zeta,c} \times I$$

and a map $F : L_{\zeta,c} \to C(S)$ by

$$F(p, t) = \rho(t) \cdot \tau(t) \cdot p.$$

Then by Theorem 3.4, $F : L_{\zeta,c} \to C(S)$ is a Lagrangian submanifold.

In this section, we want to compute $F^*\Omega_\gamma$ and the Lagrangian angle θ_F. Let $U_0 \cong (\mathbb{C}^\times)^m$ be an open dense $T^m_{\mathbb{C}}$-orbit and $(z^k)_{k=1}^m$ be the logarithmic holomorphic coordinates on U_0. Then $C(S)^\sigma \cap U_0 = \{\, (x^1, \ldots, x^m) \in \mathbb{R}^m \,\}$ and

$$C(S)^\sigma_{\zeta,c} \cap U_0 = \{\, (x^1, \ldots, x^m) \mid \langle \mu(x), \zeta \rangle = c \,\}.$$

We have only to compute $F^*\Omega_\gamma$ on this open dense subset. If we denote $\tau_0 = (e^{i\nu^1}, \ldots e^{i\nu^m}) \in T^m$ then we have the following lemma.

Lemma 5.1. *The Lagrangian angle of $F : L_{\zeta,c} \to (C(S), \omega, \Omega_\gamma)$ is given by*

$$\begin{aligned} \theta_F(x,t) =& f(t) \sum_{k=1}^m \gamma_k \zeta^k + \sum_{k=1}^m \gamma_k \nu^k \\ &+ \arg\left(\sum_{k=1}^m \left(\left(\frac{\dot\rho(t)}{\rho(t)} \xi^k + i \dot f(t) \zeta^k \right) \frac{\partial \langle \mu(x), \zeta \rangle}{\partial x^k} \right) \right) \mod \pi, \end{aligned} \tag{10}$$

where $\xi = (\xi^1, \ldots, \xi^m)$ is the Reeb field on $C(S)$.

Proof. Let $\tilde L = C(S)^\sigma \times I$ and $\iota : L_{\zeta,c} \to \tilde L$ be an inclusion map. If we define $\tilde F : \tilde L \to C(S)$ by

$$\tilde F(p, t) = \rho(t) \cdot \tau(t) \cdot p,$$

then $F = \tilde F \circ \iota$ and $F^*\Omega_\gamma = \iota^*(\tilde F^*\Omega_\gamma)$. For $\tau = (e^{i\theta^1}, \ldots, e^{i\theta^m}) \in T^m$, the transition map $\tau : U_0 \to U_0$ is expressed by

$$\tau \cdot (z^1, \ldots z^m) = (z^1 + i\theta^1, \ldots, z^m + i\theta^m).$$

Since $J(r\frac{\partial}{\partial r}) = \xi$ and

$$\xi = \xi^1 \frac{\partial}{\partial y^1} + \cdots + \xi^m \frac{\partial}{\partial y^m},$$

we have

$$r\frac{\partial}{\partial r} = \xi^1 \frac{\partial}{\partial x^1} + \cdots + \xi^m \frac{\partial}{\partial x^m}.$$

Hence for $\rho \in \mathbb{R}^+$ the transition map $\rho : U_0 \to U_0$ is expressed by

$$\rho \cdot (z^1, \ldots z^m) = (z^1 + \xi^1 \log \rho, \ldots, z^m + \xi^m \log \rho).$$

Then we have

$$(\tilde{F}^* z^k)(x^1, \ldots, x^m, t) = x^k + \xi^k \log \rho(t) + i(f(t)\zeta^k + \nu^k).$$

Since

$$\Omega_\gamma = \exp(\gamma_1 z^1 + \cdots + \gamma_m z^m) dz^1 \wedge \cdots \wedge dz^m$$

on U_0 we have

$$\tilde{F}^* \Omega_\gamma = \exp(h_1(x,t) + i h_2(x,t)) d(\tilde{F}^* z^1) \wedge \cdots \wedge d(\tilde{F}^* z^m),$$

where we put

$$\begin{aligned} h_1(x,t) &= \sum_{k=1}^m \gamma_k x^k + \log \rho(t) \sum_{k=1}^m \gamma_k \xi^k, \\ h_2(x,t) &= f(t) \sum_{k=1}^m \gamma_k \zeta^k + \sum_{k=1}^m \gamma_k \nu^k \quad \textit{and} \\ d(\tilde{F}^* z^k) &= dx^k + \left(\frac{\dot{\rho}(t)}{\rho(t)} \xi^k + i \dot{f}(t) \zeta^k \right) dt. \end{aligned}$$

Fix a point $p_0 \in C(S)^\sigma_{\zeta,c} \cap U_0$. If we put $\phi(x) := \langle \mu(x), \zeta \rangle - c$, then $C(S)^\sigma_{\zeta,c}$ is locally expressed around p_0 as $\{ (x^1, \ldots, x^m) \mid \phi(x^1, \ldots, x^m) = 0 \}$. By the definition of a moment map and the non-degeneracy of Kähler form, we have $d\phi = -\omega(\zeta, \cdot) \neq 0$ at p_0. Hence there exists $k_0 \in \{1, \ldots, m\}$ such that $\frac{\partial \phi}{\partial x^{k_0}}(p_0) \neq 0$. Thus by the implicit function theorem, x^{k_0} is locally represented as $x^{k_0} = x^{k_0}(x^1, \ldots, x^{k_0-1}, x^{k_0+1}, \ldots, x^m)$. Note that since $\phi(x^1, \ldots, x^m) = 0$, we have

$$\frac{\partial \phi}{\partial x^\ell} + \frac{\partial \phi}{\partial x^{k_0}} \frac{\partial x^{k_0}}{\partial x^\ell} = 0$$

for all $\ell \neq k_0$. If we take $(x^1, \ldots, x^{k_0-1}, x^{k_0+1}, \ldots, x^m)$ as a local coordinates on $C(S)^\sigma_{\zeta,c}$, we have

$$\begin{aligned} &\iota^*(d(\tilde{F}^* z^1) \wedge \cdots \wedge d(\tilde{F}^* z^m)) \\ =& h_3(x,t) dx^1 \wedge \cdots \wedge dx^{k_0-1} \wedge dx^{k_0+1} \wedge \cdots \wedge dx^m \wedge dt, \end{aligned}$$

where

$$h_3(x,t) = (-1)^{m-k_0} \left(\frac{\partial \langle \mu(x), \zeta \rangle}{\partial x^{k_0}} \right)^{-1} \left(\sum_{\ell=1}^m \left(\frac{\dot{\rho}(t)}{\rho(t)} \xi^\ell + i \dot{f}(t) \zeta^\ell \right) \frac{\partial \langle \mu(x), \zeta \rangle}{\partial x^\ell} \right).$$

As mentioned in Remark 4.3, the Lagrangian angle θ_F is

$$\arg(h_3 \exp(h_1 + i h_2)) = h_2 + \arg(h_3).$$

One can prove that this coincides with the right hand side of the equation (10). □

6 Construction of special Lagrangian submanifolds

Let $(C(S), \omega, \Omega_\gamma)$ be a toric almost Calabi–Yau cone over a toric Sasaki manifold (S, g). In this section, we construct the special Lagrangian submanifolds in $C(S)$. Let $F : L(\zeta, c) \to C(S)$ be a Lagrangian submanifold explained in Section 3. Then we find the conditions such that F is a special Lagrangian submanifold. Remember that we denote the Reeb field ξ and write $\tau_0 = (e^{i\nu^1}, \ldots, e^{i\nu^m}) \in T^m$. Here we put

$$N := \langle \zeta, \gamma \rangle = \sum_{k=1}^m \gamma_k \zeta^k \quad \text{and} \quad \theta := \sum_{k=1}^m \gamma_k \nu^k.$$

Theorem 6.1. *Assume that the function $\rho : I \to \mathbb{R}^+$ is identically constant. Take a constant $\theta_0 \in \mathbb{R}$. Then $F : L_{\zeta,c} \to C(S)$ is a special Lagrangian submanifold with phase $e^{i\theta_0}$ if and only if*

$$N = 0 \quad \text{and} \quad \theta + \frac{\pi}{2} = \theta_0.$$

Proof. Since $\dot\rho(t) = 0$, by Lemma 5.1 we have the Lagrangian angle

$$\theta_F(p,t) = f(t)N + \theta + \frac{\pi}{2}.$$

Note that we have assumed that $f(t)$ is not constant. Thus the statement follows clearly. □

Theorem 6.2. *We assume that $\zeta = \xi$, and put $\kappa(t) := \log \rho(t)$. Take a constant $\theta_0 \in \mathbb{R}$. Then $F : L_{\zeta,c} \to C(S)$ is a special Lagrangian submanifold with phase $e^{i\theta_0}$ if and only if*

$$\operatorname{Im}(e^{i(\theta-\theta_0)}e^{N(\kappa(t)+if(t))}) = \text{const} \tag{11}$$

Proof. Since $\zeta = \xi$, by Lemma 5.1, we have the Lagrangian angle

$$\begin{aligned}\theta_F(p,t) &= f(t)N + \theta + \arg(\dot\kappa(t) + i\dot f(t)) \\ &= \arg((\dot\kappa(t) + i\dot f(t))e^{i(f(t)N+\theta)}).\end{aligned} \tag{12}$$

Note that γ is in Δ since $\langle\gamma, \lambda\rangle = 1$ for all $\lambda \in \Lambda$ and, as mentioned in Remark 2.6, the Reeb field $\xi = \zeta$ is in Δ_0^* and this means that $N = \langle\gamma, \zeta\rangle > 0$. Since the argument of a complex valued function is unchanged by a multiplication of a positive function, we can multiply the term in the argument in (12) by $Ne^{N\kappa(t)}$ and we have

$$\begin{aligned}\theta_F(p,t) &= \arg((\dot\kappa(t) + i\dot f(t))e^{i(f(t)N+\theta)}) \\ &= \arg(N(\dot\kappa(t) + i\dot f(t))e^{N\kappa(t)+i(f(t)N+\theta)}).\end{aligned}$$

If we put

$$h(t) = e^{N\kappa(t)+i(f(t)N+\theta)},$$

then it is clear that $\theta_F(p,t) = \arg(\dot h(t))$. Thus it follows that $\theta_F \equiv \theta_0$ constant if and only if

$$\operatorname{Im}(e^{i(\theta-\theta_0)}e^{N(\kappa(t)+if(t))}) = \text{const}.$$

□

Remark 6.3. If we define the curves $c_j : I \to \mathbb{C}^\times$ by

$$c_j(t) := \rho^{\xi^j}(t)e^{i(f(t)\xi^j+\nu^j)},$$

then the equality (11) in Theorem 6.2 is equivalent to the equality

$$\operatorname{Im}\big(e^{-i\theta_0}c_1^{\gamma_1}\cdots c_m^{\gamma_m}\big) = \text{const}.$$

For example in $\mathbb{C}^m$, the canonical Reeb field is $\xi = (1, \dots, 1)$ and we can take $\gamma = (1, \dots, 1)$. Then if we take $\theta_0 = 0$ and $\nu^1 = \cdots = \nu^m = 0$ for example, then $c_1(t) = \cdots = c_m(t)$, and we put $c(t) := c_1(t)$. Then the equality (11) in Theorem 6.2 becomes

$$\operatorname{Im}(c^m(t)) = \text{const},$$

and the image of $F : L_{\zeta,c} \to \mathbb{C}^m$ coincides with

$$\{\, (c(t)x^1, \dots, c(t)x^m) \in \mathbb{C}^m \mid t \in I,\ x^j \in \mathbb{R},\ (x^1)^2 + \cdots + (x^m)^2 = c \,\}.$$

Hence this is an extension of examples of special Lagrangian submanifolds mentioned in Theorem 3.5 in Section III.3.B. in the paper of Harvey and Lawson [22].

7 Construction of Lagrangian self-similar solutions

Let $(C(S), \omega, \Omega_\gamma)$ be a toric almost Calabi–Yau cone over a toric Sasaki manifold (S, g). Since $C(S)$ has both the cone structure and the almost Calabi–Yau structure, we can consider both the position vector and the generalized mean curvature vector. Then we can defined the generalized self-similar solution. Let M be a manifold and $F : M \to C(S)$ be an immersion. Then we say that F is a generalized self-similar solution if

$$H^g = \lambda \overrightarrow{F}^\perp$$

for some $\lambda \in \mathbb{R}$. In this section, we construct the Lagrangian generalized self-similar solutions in $C(S)$. Let $F : L_{\zeta,c} \to C(S)$ be a Lagrangian submanifold explained in Section 3. Remember that we denote the Reeb field ξ and write $\tau_0 = (e^{i\nu^1}, \dots, e^{i\nu^m}) \in T^m$, and in Section 6, we put

$$N = \langle \zeta, \gamma \rangle = \sum_{k=1}^{m} \gamma_k \zeta^k \quad \text{and} \quad \theta = \sum_{k=1}^{m} \gamma_k \nu^k.$$

Theorem 7.1. *Let us assume that $\zeta = \xi$, and put $c(t) := \rho(t)e^{if(t)} \in \mathbb{C}^\times$. If there exist a function $\theta : I \to \mathbb{R}/\pi\mathbb{Z}$ and a constant $A \in \mathbb{R}$, and $\theta(t)$ and $c(t)$ satisfy the differential equations*

$$\begin{cases} \dot{c}(t) = e^{i(\theta(t)-\theta)}\overline{c(t)}^{N-1} \\ \dot{\theta}(t) = A\rho(t)^N \sin(f(t)N + \theta - \theta(t)), \end{cases} \tag{13}$$

then $F : L_{\zeta,c} \to C(S)$ is a Lagrangian generalized self-similar solution with

$$2cH^g = A\overrightarrow{F}^{\perp}$$

and Lagrangian angle $\theta_F(p,t) = \theta(t)$.

Proof. First of all, we prove that the Lagrangian angle $\theta_F(p,t)$ is equal to $\theta(t)$. Since $\zeta = \xi$, by Lemma 5.1 we have the Lagrangian angle

$$\theta_F(p,t) = f(t)N + \theta + \arg(\dot{\kappa}(t) + i\dot{f}(t)),$$

where $\kappa(t) = \log \rho(t)$. Since the argument of a complex valued function is unchanged under the multiplication of a positive real valued function, by multiplying $2\rho(t)^2$ we have

$$\begin{aligned} \arg(\dot{\kappa}(t) + i\dot{f}(t)) &= \arg(2\rho(t)^2\dot{\kappa}(t) + 2i\rho(t)^2\dot{f}(t)) \\ &= \arg\left(\frac{d}{dt}(\rho(t)^2) + 2i\rho(t)^2\dot{f}(t)\right). \end{aligned}$$

Since $c(t) = \rho(t)e^{if(t)}$, we have

$$\dot{c}(t) = \dot{\rho}(t)e^{if(t)} + i\rho(t)\dot{f}(t)e^{if(t)}$$

and multiplying this equation by $2\rho(t)e^{-if(t)} (= 2\overline{c(t)})$ we have

$$2\overline{c(t)}\dot{c}(t) = \frac{d}{dt}(\rho(t)^2) + 2i\rho(t)^2\dot{f}(t). \tag{14}$$

If we use the differential equation (13) with respect to $c(t)$ then the left hand side of (14) is equal to

$$2\overline{c(t)}\dot{c}(t) = 2e^{i(\theta(t)-\theta)}\overline{c(t)}^N = 2\rho(t)^N e^{i(\theta(t)-\theta-f(t)N)}. \tag{15}$$

Thus we have

$$\arg(\dot{\kappa}(t) + i\dot{f}(t)) = \theta(t) - \theta - f(t)N.$$

Consequently we have proved that

$$\theta_F(p,t) = \theta(t).$$

We turn to the proof of $2cH^g = A\overrightarrow{F}^{\perp}$. Since ω is non-degenerate and we have the orthogonal decomposition

$$T_{F(p)}C(S) = F_*(T_pL_{\zeta,c}) \oplus J(F_*(T_pL_{\zeta,c}))$$

for all p in $L_{\zeta,c}$, we have only to prove that

$$\omega(2cH^g, F_*X) = \omega(A\overrightarrow{F}^{\perp}, F_*X)$$

for all X tangent to $L_{\zeta,c}$. Furthermore, since $\omega(A\overrightarrow{F}^{\perp}, F_*X) = \omega(A\overrightarrow{F}, F_*X)$, it is equivalent to prove that

$$\omega(2cH^g, F_*X) = \omega(A\overrightarrow{F}, F_*X).$$

Remember that $L_{\zeta,c} = C(S)^\sigma_{\zeta,c} \times I$. Fix $x_0 = (p_0, t_0)$ in $L_{\zeta,c}$, X in $T_{p_0}C(S)^\sigma_{\zeta,c}$ and $\partial/\partial t$ in $T_{t_0}I$. See the equalities (8) and (9) in the proof of Theorem 3.4, we have

$$\begin{aligned} F_*X &= (\rho(t_0) \cdot \tau(t_0))_*X \\ F_*\frac{\partial}{\partial t} &= (\rho(t_0) \cdot \tau(t_0))_*\left(\frac{\dot{\rho}(t_0)}{\rho(t_0)}\overrightarrow{p_0} + \dot{f}(t_0)\xi(p_0)\right). \end{aligned}$$

By Proposition 4.6 we have

$$H^g = JF_*(\nabla_{F^*g}\theta_F),$$

where ∇_{F^*g} is the (F^*g)-gradient on L. By the definition of the position vector, one can prove that

$$\overrightarrow{F}(x_0) = (\rho(t_0)\cdot\tau(t_0))_*(\overrightarrow{p_0})$$

at $x_0 = (p_0, t_0)$. Note that we have proved that the Lagrangian angle

$$\theta_F(p,t) = \theta(t)$$

and this function is independent of any points in $C(S)^\sigma_{\zeta,c}$. Thus if X is tangent to $C(S)^\sigma_{\zeta,c}$ at p_0, then we have

$$\begin{aligned}\omega(2cH^g, F_*X) &= 2c\,\omega(JF_*(\nabla_{F^*g}\theta_F), F_*X) = -2c(F^*g)(\nabla_{F^*g}\theta_F, X)\\ &= -2cX(\theta_F) = 0.\end{aligned}$$

Since if we substitute two vectors tangent to the real form into ω then it is zero, and $\overrightarrow{p_0}$ is tangent to the real form, for X tangent to $C(S)^\sigma_{\zeta,c}$ at p_0 we have

$$\omega(A\overrightarrow{F}, F_*X) = A\rho^2(t_0)\omega(\overrightarrow{p_0}, X) = 0.$$

Thus we have

$$\omega(2cH^g, F_*X) = 0 = \omega(A\overrightarrow{F}, F_*X)$$

for all X tangent to $C(S)^\sigma_{\zeta,c}$ at p_0. Next, for $\partial/\partial t$ tangent to I at t_0, we have

$$\begin{aligned}\omega(2cH^g, F_*\frac{\partial}{\partial t}) &= 2c\,\omega(JF_*(\nabla_{F^*g}\theta_F), F_*\frac{\partial}{\partial t}) = -2c(F^*g)(\nabla_{F^*g}\theta_F, \frac{\partial}{\partial t})\\ &= -2c\frac{\partial}{\partial t}\theta_F = -2c\dot\theta(t_0)\\ &= -2cA\rho(t_0)^N\sin(f(t_0)N + \theta - \theta(t_0)).\end{aligned}$$

In the last equality, we use the differential equation (13) with respect to $\theta(t)$. On the other hand, we have

$$\begin{aligned}\omega(A\overrightarrow{F}, F_*\frac{\partial}{\partial t}) &= A\rho^2(t_0)\dot f(t_0)\omega(\overrightarrow{p_0}, \xi(p_0)) = A\rho^2(t_0)\dot f(t_0)\overrightarrow{p_0}(\langle\mu,\xi\rangle))\\ &= A\rho^2(t_0)\dot f(t_0)\frac{d}{d\rho}\bigg|_{\rho=1}\langle\mu(\rho\cdot p_0),\xi\rangle\\ &= A\rho^2(t_0)\dot f(t_0)\frac{d}{d\rho}\bigg|_{\rho=1}\rho^2\langle\mu(p_0),\xi\rangle\\ &= 2cA\rho^2(t_0)\dot f(t_0).\end{aligned}$$

In the fourth equality, we use $\langle\mu(\rho\cdot p_0),\xi\rangle = \rho^2\langle\mu(p_0),\xi\rangle$ for a $\rho\in\mathbb{R}^+$ action and it follows by the definition of the moment map (3). In the last equality, remember that for p_0 in $C(S)^\sigma_{\zeta,c}$ (now $\zeta=\xi$ by the assumption) $\langle\mu(p_0),\zeta\rangle = c$ by the definition of $C(S)^\sigma_{\zeta,c}$. By the equality (14), we know that $2\rho^2(t_0)\dot f(t_0)$ is the imaginary part of $2\overline{c(t_0)}\dot c(t_0)$, and using the equality (15) we show that

$$2\rho^2(t_0)\dot f(t_0) = 2\rho^N(t_0)\sin(\theta(t_0) - \theta - f(t_0)N)$$

Thus we have

$$\omega(2cH^g, F_*\frac{\partial}{\partial t}) = \omega(A\overrightarrow{F}, F_*\frac{\partial}{\partial t}).$$

This means that $2cH^g = A\overrightarrow{F}^{\perp}$. □

Remark 7.2. Here we assume that all $\xi^j \neq 0$. If we define curves $c_j : I \to \mathbb{C}^*$ by

$$c_j(t) := \rho^{\xi^j}(t)e^{i(f(t)\xi^j + \nu^j)},$$

then the differential equations (13) in Theorem 7.1 are equivalent to the following differential equations.

$$\begin{cases}\frac{d}{dt}c_j^{1/\xi^j}(t) = e^{i\theta(t)}\overline{c_1^{\gamma_1}(t)\cdots c_j^{\gamma_j-1/\xi^j}(t)\cdots c_m^{\gamma_m}(t)} \quad (j=1,\dots,m)\\ \frac{d}{dt}\theta(t) = A\,\mathrm{Im}(e^{-i\theta(t)}c_1^{\gamma_1}(t)\cdots c_m^{\gamma_m}(t)).\end{cases} \tag{16}$$

For example in $\mathbb{C}^m$, the canonical Reeb field is $\xi = (1,\dots,1)$ and $\gamma = (1,\dots,1)$. Then if we take

$\theta_0 = 0$ and $\nu^1 = \cdots = \nu^m = 0$ for example, then the above equality (16) becomes

$$\begin{cases} \frac{d}{dt} c_j(t) = e^{i\theta(t)} \overline{c_1(t) \cdots c_{j-1}(t) \cdot c_{j+1}(t) \cdots c_m(t)} & (j = 1, \dots, m) \\ \frac{d}{dt}\theta(t) = A \operatorname{Im}(e^{-i\theta(t)} c_1(t) \cdots c_m(t)), \end{cases}$$

and the image of $F : L_{\zeta,c} \to \mathbb{C}^m$ coincides with

$$\{ (c_1(t)x^1, \dots, c_m(t)x^m) \in \mathbb{C}^m \mid t \in I,\, x^j \in \mathbb{R},\, (x^1)^2 + \cdots + (x^m)^2 = c \}.$$

This differential equations appear in Theorem A in the paper of Joyce, Lee and Tsui [26]. Hence this is one of extension of the paper of Joyce, Lee and Tsui in $\mathbb{C}^m$ to the toric almost Calabi–Yau cone.

8 Examples

In this section, we apply the theorems and construct some concrete examples of special Lagrangians and Lagrangian self-similar solutions. As explained in Remark 2.6 in Section 2, the moment image of a toric Kähler cone is a strongly convex good rational polyhedral cone. Conversely, we can construct a toric Kähler cone from a strongly convex good rational polyhedral cone by the Delzant construction.

Let

$$\Delta = \{ y \in \mathfrak{g}^* \mid \langle y, \lambda_i \rangle \geq 0 \text{ for } \ i = 1, \cdots, d \} - \{0\}$$

be a strongly convex good rational polyhedral cone and put the (open) dual cone

$$\Delta_0^* = \{ \xi \in \mathfrak{g} \mid \langle v, \xi \rangle > 0 \text{ for all } v \in \Delta \}.$$

Proposition 8.1. *For Δ and $\xi \in \Delta_0^*$, there exists a compact connected toric Sasaki manifold (S, g) whose moment image is equal to Δ and whose Reeb vector field is generated by ξ.*

This proposition is proved by the Delzant construction, for details see [30] and [35]. Of course the cone $(C(S), \overline{g})$ of (S, g) is a toric Kähler manifold whose moment image is equal to Δ.

As mentioned in Remark 2.8 in Section 2, the canonical line bundle $K_{C(S)}$ is trivial if and only if there exists an element γ in $(\mathbb{Z}_{\mathfrak{g}})^* \cong \mathbb{Z}^m$ such that $\langle \gamma, \lambda_j \rangle = 1$ for all $j = 1, \dots, d$, and using γ we can construct a non-vanishing holomorphic $(m, 0)$-form Ω_γ that is written by

$$\Omega_\gamma = \exp(\gamma_1 z^1 + \dots \gamma_m z^m) dz^1 \wedge \cdots \wedge dz^m \tag{17}$$

on an open dense $T^m_{\mathbb{C}}$-orbit by the logarithmic holomorphic coordinates. This condition is called the *height 1* and in fact there exists a definition of the *height* ℓ for some $\ell \in \mathbb{Z}$, for example see Cho-Futaki-Ono [10]. Here we want to introduce the results in [10].

Theorem 8.2 (cf. Theorem 1.2 in [10])**.** *Let S be a compact toric Sasaki manifold with $c_1^B > 0$ and $c_1(D) = 0$. Then by deforming the Sasaki structure varying the Reeb vector field, we obtain a Sasaki-Einstein structure.*

We do not explain the meanings of c_1^B and $c_1(D)$ in this Part, but in [10] it is proved that the condition with $c_1^B > 0$ and $c_1(D) = 0$ is equivalent to the *height* ℓ for some $\ell \in \mathbb{Z}$. Note that (S, g) is Sasaki-Einstein if and only if $(C(S), \omega)$ is Ricci flat. Thus, if we use Theorem 8.2, then we get a toric Calabi–Yau cone $(C(S), \omega, \Omega_\gamma)$ rather than *almost* Calabi–Yau . The merit of using the toric Calabi–Yau is that H^g coincides with H.

From now on, we restrict ourselves to the case of $\dim_{\mathbb{C}} C(S) = 3$. There is a useful proposition (c.f. [10]) to check whether given inward conormal vectors λ_i satisfy the goodness condition (5) of Definition 2.7.

Proposition 8.3. *Let Δ be a strongly convex rational polyhedral cone in $\mathbb{R}^3$ given by*

$$\Delta = \{ y \in \mathbb{R}^3 \mid \langle y, \lambda_i \rangle \geq 0,\ j = 1, \cdots, d \} - \{0\}$$

$$\lambda_1 = \begin{pmatrix} 1 \\ p_1 \\ q_1 \end{pmatrix}, \cdots, \lambda_d = \begin{pmatrix} 1 \\ p_d \\ q_d \end{pmatrix}.$$

Then Δ is good in the sense of Definition 2.7 if and only if either

1. *$|p_{i+1} - p_i| = 1$ or*

2. $|q_{i+1} - q_i| = 1$ *or*

3. $p_{i+1} - p_i$ *and* $q_{i+1} - q_i$ *are relatively prime non-zero integers*

for $i = 1, \cdots, d$ *where we have put* $\lambda_{d+1} = \lambda_1$*.*

Example 8.4. Take an integer $g \geq 1$. If $g = 1$, let Δ be the strongly convex rational polyhedral cone defined by

$$\Delta = \Delta_1 = \{ y \in \mathbb{R}^3 \mid \langle y, \lambda_i \rangle \geq 0, \, i = 1, 2, 3, 4 \} - \{0\}$$

with

$$\lambda_1 := \begin{pmatrix} 1 \\ -1 \\ -1 \end{pmatrix}, \; \lambda_2 := \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}, \; \lambda_3 := \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \; \lambda_4 := \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}.$$

If $g \geqq 2$ let Δ be the strongly convex rational polyhedral cone defined by

$$\Delta = \Delta_g = \{ y \in \mathbb{R}^3 \mid \langle y, \lambda_i \rangle \geq 0, \; i = 1, \ldots, g+3 \} - \{0\}$$

with

$$\lambda_1 := \begin{pmatrix} 1 \\ -1 \\ -1 \end{pmatrix}, \; \lambda_k := \begin{pmatrix} 1 \\ k-2 \\ (k-2)^2 - 1 \end{pmatrix} \; (k = 2, 3, \ldots, g+2), \; \lambda_{g+3} := \begin{pmatrix} 1 \\ -2 \\ g^2 \end{pmatrix},$$

Then by Proposition 8.3, Δ is a strongly convex *good* rational polyhedral cone. Since we can take γ as $(1, 0, 0)$ so that $\langle \gamma, \lambda_j \rangle = 1$ for $j = 1, \ldots, g+3$, this condition satisfies the *height 1* and we can use Theorem 8.2. Let $(C(S), \omega)$ be a toric Kähler manifold whose moment image is equal to Δ. The existence of it is guaranteed by Proposition 8.1. If necessary, we deform the Kähler form ω and Reeb field ξ on $C(S)$ so that $(C(S), \omega)$ is Ricci flat by Theorem 8.2. Thus we can assume that $(C(S), \omega)$ is Ricci flat. Furthermore, since we can take γ as above, the canonical line bundle $K_{C(S)}$ is trivial and we have a non-vanishing holomorphic $(3,0)$-form Ω_γ on $C(S)$. Thus we have a Calabi–Yau cone $M_g = (C(S), \omega, \Omega_\gamma)$ and denote its Reeb field by ξ.

For example, if we take

$$c := \frac{1}{2} \langle \gamma, \xi \rangle \quad \text{and} \quad \zeta := \xi,$$

then ζ and c satisfy the assumptions (6) and (7) in Section 3, which proved in Proposition 9.1 in Section 9. Then the shape of $\Delta_{\zeta,c} = \Delta \cap H_{\zeta,c}$ is a $(g+3)$-gon, which proved in Proposition 9.2 in Section 9. For example if $g = 1$ then $\Delta_{\zeta,c}$ is a quadrilateral and if $g = 2$ then $\Delta_{\zeta,c}$ is a pentagon.

Remember that μ^σ, the restriction of the moment map μ to the real form $C(S)^\sigma$, is a $2^3 (= 8)$-fold covering of Δ, and we have defined $C(S)^\sigma_{\zeta,c} = (\mu^\sigma)^{-1}(\Delta_{\zeta,c})$. Hence the topological shape of the $C(S)^\sigma_{\zeta,c}$ is a 2-dimensional surface constructed from 8-copies of $\Delta_{\zeta,c}$ that is glued with certain boundaries. In this setting, we can see that

$$C(S)^\sigma_{\zeta,c} \cong \Sigma_g,$$

where Σ_g is a closed surface of genus g. This will be explained in Proposition 9.3 in Section 9.

Special Lagrangian. First we construct special Lagrangian submanifolds using Theorem 6.2. Now $N = \langle \gamma, \zeta \rangle > 0$. For example take $\theta_0 = 0$. Then, for example, take an open interval $I = (0, \pi)$, and define $f : I \to \mathbb{R}$ and $\rho : I \to \mathbb{R}^+$ by

$$f(t) = \frac{1}{N} t \quad \text{and} \quad \rho(t) = \left(\frac{1}{\sin t} \right)^{1/N},$$

and take $\tau_0 = (e^{i\nu^1}, e^{i\nu^2}, e^{i\nu^3})$ in T^3 as $\nu^1 = \nu^2 = \nu^3 = 0$. Then $\theta = \gamma_1 \nu^1 + \gamma_2 \nu^2 + \gamma_3 \nu^3 = 0$. This setting satisfies the equality (11). Thus $F : L_{\zeta,c} \to M_g$ is a special Lagrangian submanifold and $L_{\zeta,c}$ is diffeomorphic to

$$L_{\zeta,c} \cong \Sigma_g \times \mathbb{R}.$$

Note that of course the map F and $L_{\zeta,c}$ depend on g, and in Example 1.4 we denote these by $F^1_g : L^1_g \to M_g$.

Lagrangian self-similar solution. Next we construct Lagrangian (generalized) self-similar solutions using Theorem 7.1. Now $N = \langle \gamma, \zeta \rangle > 0$. For example take

$$\theta(t) = Nt + \frac{\pi}{2} \quad \text{and} \quad A = -N.$$

Then, for example, take an interval $I = \mathbb{R}$, and define $f : I \to \mathbb{R}$ and $\rho : I \to \mathbb{R}^+$ by

$$f(t) = t \quad \text{and} \quad \rho(t) = 1,$$

and take $\tau_0 = (e^{i\nu^1}, e^{i\nu^2}, e^{i\nu^3})$ in T^3 as $\nu^1 = \nu^2 = \nu^3 = 0$. Then $\theta = \gamma_1\nu^1 + \gamma_2\nu^2 + \gamma_3\nu^3 = 0$. This setting satisfies the differential equations (13). Thus $F : L_{\zeta,c} \to C(S)$ is a Lagrangian self-similar solution (self-shrinker). Furthermore as mentioned in Remark 3.3, we can reduce I to S^1, hence we have a *compact* Lagrangian self-shrinker $F : L_{\zeta,c} \to M_g$ with

$$H^g = -\overrightarrow{F}^{\perp}$$

which is diffeomorphic to

$$L_{\zeta,c} \cong \Sigma_g \times S^1.$$

Note that of course the map F and $L_{\zeta,c}$ depend on g, and in Example 1.4 we denote these by $F_g^2 : L_g^2 \to M_g$.

Remark 8.5. In $M_g(= C(S))$ constructed above, it is clear that the real form $C(S)^\sigma$ itself is one of the most typical examples of special Lagrangian submanifold in $C(S)$, and it is a cone. Hence $C(S)^\sigma$ is also diffeomorphic to $\Sigma_g \times \mathbb{R}$. However the above example $F_g^1 : L_g^1 \to M_g$ is different from the real form itself, especially it dose not have a cone shape.

9 Appendix

In this Section, we give some proofs for the statements mentioned in Example 8.4 in Section 8.

Proposition 9.1. *ζ and c in Example 8.4 satisfy the assumptions (6) and (7) in Section 3.*

Proof. First, it is clear that $\frac{1}{2}\gamma$ is in $\operatorname{Int}\Delta$ and it is also in $H_{\zeta,c}$. This proves that ζ and c satisfy the assumption (6). Next we prove that ζ and c satisfy the assumption (7) by the proof of contradiction. Assume that there exists y in $\Delta \cap H_{\zeta,c}$ such that ζ is in $\mathfrak{z}_y$. Here remember that

$$\mathfrak{z}_y = \operatorname{Span}_{\mathbb{R}}\{\, \lambda_j \mid \langle y, \lambda_j\rangle = 0 \,\}.$$

Since y is in Δ and, as mentioned in Remark 2.6, the Reeb field ξ is in Δ_0^*, this means that $\langle y, \zeta\rangle = \langle y, \xi\rangle > 0$. On the other hand, the pairing of y and all elements in $\mathfrak{z}_y$ is zero. This is in contradiction to that ζ is in $\mathfrak{z}_y$. Thus we have proved that ζ and c satisfy the assumption (7). □

Proposition 9.2. *The shape of $\Delta_{\zeta,c} = \Delta \cap H_{\zeta,c}$ in Example 8.4 is a $(g+3)$-gon.*

Proof. First, we denote the facet of Δ defined by λ_j by

$$F_j = \{\, y \in \Delta \mid \langle y, \lambda_j\rangle = 0 \,\}$$

for $j = 1, \ldots, g+3$. Next, take an element y in F_j and put $\kappa := \frac{c}{\langle y,\zeta\rangle}$. Since $\frac{1}{2}\gamma$ and y are in Δ and $\zeta = \xi$ is in Δ_0^*, it follows that $c = \frac{1}{2}\langle \gamma, \xi\rangle > 0$, $\langle y, \zeta\rangle > 0$ and $\kappa > 0$. Then κy is in F_j and $H_{\zeta,c}$. This means that the hyperplane $H_{\zeta,c}$ intersects all facets of Δ. Thus we have proved that $\Delta_{\zeta,c}$ is a $(g+3)$-gon. □

Proposition 9.3. *Under the setting in Example 8.4,*

$$C(S)^\sigma_{\zeta,c} \cong \Sigma_g,$$

where Σ_g is a closed surface of genus g.

Proof. There exists an open dense $T^3_{\mathbb{C}}$-orbit on $C(S)$. We identify $T^3_{\mathbb{C}}$ with $(\mathbb{C}^\times)^3$. It is clear that the real form of $(\mathbb{C}^\times)^3$ is $(\mathbb{R}^\times)^3$ and it has 8 connected components $\mathbb{R}^3(\kappa_1, \kappa_2, \kappa_3)$, where κ_i are $+1$ or -1 and we define

$$\mathbb{R}^3(\kappa_1, \kappa_2, \kappa_3) = \{\, (x_1, x_2, x_3) \in \mathbb{R}^3 \mid \kappa_1 x_1 > 0, \kappa_2 x_2 > 0, \kappa_3 x_3 > 0 \,\}.$$

There is a standard diffeomorphism from each $\mathbb{R}^3(\kappa_1, \kappa_2, \kappa_3)$ to $\mathbb{R}^3$ defined by

$$-\log|\cdot| : \mathbb{R}^3(\kappa_1, \kappa_2, \kappa_3) \to \mathbb{R}^3,$$

that is , (x_1, x_2, x_3) maps to $(-\log|x_1|, -\log|x_2|, -\log|x_3|)$. In the algebraic toric geometry, there is a concept of manifolds with corner associated with toric varieties. From this view point, we can consider that $\mathbb{R}^3$ is rescaled and embedded into Δ, that is a manifold with corner. This means that the infinity toward the direction of λ_j in $\mathbb{R}^3$ corresponds to the facet F_j of Δ defined by λ_j. For more general treatment, see Oda [42]. In this sense, we identify $\mathbb{R}^3$ and Int Δ, and we identify the infinity toward the direction of λ_j in $\mathbb{R}^3$ and the facet F_j of Δ defined by λ_j. For each inward conormal $\lambda_j = (\lambda_j^1, \lambda_j^2, \lambda_j^3)$ of Δ, then consider a curve $c_j(t)$ in $\mathbb{R}^3 \cong \mathrm{Int}\,\Delta$ defined by

$$c_j(t) = t\lambda_j = (\lambda_j^1 t, \lambda_j^2 t, \lambda_j^3 t).$$

Then the pull back of $c_j(t)$ to $\mathbb{R}^3(\kappa_1, \kappa_2, \kappa_3)$ by $-\log|\cdot|$ is

$$\tilde{c}_j(t) = (\kappa_1 e^{-\lambda_j^1 t}, \kappa_2 e^{-\lambda_j^2 t}, \kappa_3 e^{-\lambda_j^3 t})$$

and if we put $s = e^{-t} > 0$ then this curve $\tilde{c}_j(t)$ in $\mathbb{R}^3(\kappa_1, \kappa_2, \kappa_3)$ is written by

$$\tilde{c}_j(s) = (\kappa_1 s^{\lambda_j^1}, \kappa_2 s^{\lambda_j^2}, \kappa_3 s^{\lambda_j^3}).$$

If this curve tends to the facet F_j, then it is equivalent to $t \to +\infty$ and also $s \to +0$. If we allow to take $s = 0$, then the point $\tilde{c}_j(0)$ can be considered as in the facet F_j and furthermore if we allow to take $s < 0$, then the curve $\tilde{c}_j(s)$ is in

$$\mathbb{R}^3((-1)^{\lambda_j^1}\kappa_1, (-1)^{\lambda_j^2}\kappa_2, (-1)^{\lambda_j^3}\kappa_3).$$

This means that if we prepare 8 copies of Δ and give the labels formally to each Δ as

$$\begin{array}{llll} \Delta(+1,+1,+1), & \Delta(+1,+1,-1), & \Delta(+1,-1,+1), & \Delta(+1,-1,-1), \\ \Delta(-1,+1,+1), & \Delta(-1,+1,-1), & \Delta(-1,-1,+1), & \Delta(-1,-1,-1), \end{array} \tag{18}$$

then $\Delta(\kappa_1, \kappa_2, \kappa_3)$ and $\Delta((-1)^{\lambda_j^1}\kappa_1, (-1)^{\lambda_j^2}\kappa_2, (-1)^{\lambda_j^3}\kappa_3)$ are glued together along the facet F_j defined by λ_j.

In the above observation, we consider the gluing relation of 8 copies of Δ however, the glueing relation of $\Delta_{\zeta,c}$ is the same as Δ. That is, if we prepare 8 copies of $\Delta_{\zeta,c}$ and give the labels formally to each $\Delta_{\zeta,c}$ as same as (18), then $\Delta_{\zeta,c}(\kappa_1, \kappa_2, \kappa_3)$ and $\Delta_{\zeta,c}((-1)^{\lambda_j^1}\kappa_1, (-1)^{\lambda_j^2}\kappa_2, (-1)^{\lambda_j^3}\kappa_3)$ are glued together along the edge $E_j = F_j \cap \Delta_{\zeta,c}$ defined by λ_j. This is the topological shape of $C(S)^\sigma_{\zeta,c}$.

Then one can check that $C(S)^\sigma_{\zeta,c} \cong \Sigma_g$ by the straight forward observations glueing 8 copies of $\Delta_{\zeta,c}$ as above relations. In Figure 1 and Figure 2, we draw the image of the way of gluing in the case $g = 1$ and $g = 2$ respectively. In these figures, we write $\Delta_{\zeta,c}(\kappa_1, \kappa_2, \kappa_3)$ by $(\kappa_1, \kappa_2, \kappa_3)$ for short and the edge E_j by j for short, and glue same labels together. Note that in Figure 2 we write a pentagon as a quadrilateral by joining edge 4 and edge 5 flatly to write a picture easily. $\square$

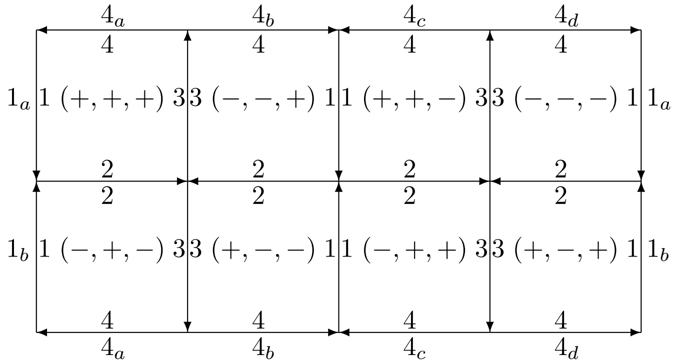

Figure 1. $g = 1$

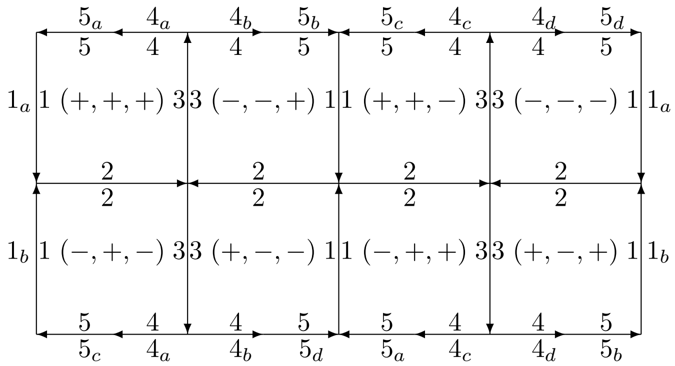

Figure 2. $g = 2$

Part II

Weighted Hamiltonian stationary Lagrangian submanifolds and generalized Lagrangian mean curvature flows in toric almost Calabi–Yau manifolds

Abstract. In this Part, we generalize examples of Lagrangian mean curvature flows constructed by Lee and Wang in $\mathbb{C}^m$ to toric almost Calabi–Yau manifolds. To be more precise, we construct examples of weighted Hamiltonian stationary Lagrangian submanifolds in toric almost Calabi–Yau manifolds and solutions of generalized Lagrangian mean curvature flows starting from these examples. We allow these flows to have some singularities and topological changes.

10 Introduction

Recently, study of Lagrangian submanifolds acquire much importance in association with Mirror Symmetry. There are several classes of Lagrangian submanifolds. For example, special Lagrangian submanifolds are defined in Calabi–Yau manifolds by Harvey and Lawson [22] and they have an important role in the Strominger–Yau–Zaslow conjecture [48]. A class of Hamiltonian stationary Lagrangian submanifolds is also defined in Calabi–Yau manifolds, especially a special Lagrangian submanifold is a Hamiltonian stationary Lagrangian submanifold. In general, constructing explicit examples of special or Hamiltonian stationary Lagrangian submanifolds is difficult since these conditions are locally written by nonlinear PDE. However some examples are constructed in the case that the ambient Calabi–Yau manifold has symmetries, especially in $\mathbb{C}^m$.

First, we introduce some previously known examples of special or Hamiltonian stationary Lagrangian submanifolds and Lagrangian mean curvature flows. One of examples of special Lagrangian submanifolds in $\mathbb{C}^m$ constructed by Harvey and Lawson [22, III.3.A] is defined by

$$M_c := \{ (z_1, \dots, z_m) \in \mathbb{C}^m \mid \mathrm{Im}(z_1 \cdots z_m) = c_1,\ |z_1|^2 - |z_j|^2 = c_j\ (j = 2, \dots, m)\},$$

where $c = (c_1, \dots, c_m) \in \mathbb{R}^m$. Note that the phase of M_c is i^m. We remark that if $c_1 = 0$ and $z_j = x_j e^{i\theta_j}$ for $x_j \in \mathbb{R}$ and $\theta_j \in \mathbb{R}$, then M_c is written by

$$\{ \exp(\theta_2\zeta_2 + \cdots + \theta_m\zeta_m) \cdot x \in \mathbb{C}^m \mid x \in \mathbb{R}^m,\ \theta_j \in \mathbb{R},\ \langle \mu(x), \zeta_j \rangle = \frac{c_j}{2}\ (j = 2, \dots, m)\},$$

where $\zeta_j := (1, 0, \dots, 0, -1, 0, \dots, 0) = e_1 - e_j \in \mathbb{R}^m$, $\mu(x) := \frac{1}{2}(x_1^2, \dots, x_m^2)$ and $\exp(v) \cdot x = (x_1 e^{2\pi i v_1}, \dots, x_m e^{2\pi i v_m})$ for $v = (v_1, \dots, v_m) \in \mathbb{R}^m$. This is a T^{m-1}-invariant special Lagrangian submanifold in $\mathbb{C}^m$.

Next examples of special Lagrangian submanifolds in $\mathbb{C}^m$ are constructed by Joyce [24, Example 9.4]. He considered a family of T^1-invariant Lagrangian submanifolds denoted by

$$N_c^{a_1,\dots,a_m} := \{ (x_1 e^{2\pi i a_1 \theta}, \dots, x_m e^{2\pi i a_m \theta}) \in \mathbb{C}^m \mid \theta \in \mathbb{R},\ a_1 x_1^2 + \cdots + a_m x_m^2 = c\},$$

where $a = (a_1, \dots, a_m) \in \mathbb{R}^m$ and $c \in \mathbb{R}$, and he proved that $N_c^{a_1,\dots,a_m}$ is a special Lagrangian submanifold if and only if

$$a_1 + \cdots + a_m = 0. \tag{19}$$

He constructed these examples by using a moment map of T^1-action on $\mathbb{C}^m$. Of course, in the same way as M_c, the Lagrangian submanifold $N_c^{a_1,\dots,a_m}$ can be written by

$$\{ \exp(\theta a) \cdot x \mid x \in \mathbb{R}^m,\ \theta \in \mathbb{R},\ \langle \mu(x), a \rangle = \frac{c}{2} \}.$$

These two examples suggest that a torus action, a real structure and a moment map are useful to construct special Lagrangian submanifolds. From this view point, the author [50] generalized Joyce's example $N_c^{a_1,\dots,a_m}$ in $\mathbb{C}^m$ to in an m-dimensional toric almost Calabi–Yau cone manifold. To be more precisely, the author constructed examples of special Lagrangian submanifolds of the form

$$\{ \exp(t\zeta) \cdot p \mid p \in M^\sigma,\, t \in \mathbb{R},\, \langle \mu(p), \zeta \rangle = c \}$$

in a toric almost Calabi–Yau cone manifold $(M, \omega, g, J, \Omega_\gamma)$, where M^σ is the real form of M, μ is a moment map of T^m-action on M, ζ is a vector in $\mathbb{R}^m$ satisfying a special condition and c is a constant. This is a T^1-invariant special Lagrangian submanifold in a toric almost Calabi–Yau cone manifold $(M, \omega, g, J, \Omega_\gamma)$.

This type of constructions is also effective to construct examples of Hamiltonian stationary Lagrangian submanifolds and its mean curvature flows. Actually, Lee and Wang [29] proved that V_t defined by

$$\left\{ (x_1 e^{2\pi i \zeta_1 s}, \ldots, x_m e^{2\pi i \zeta_m s}) \in \mathbb{C}^m \;\middle|\; \begin{array}{l} 0 \le s \le 1, \\ \displaystyle\sum_{j=1}^m \zeta_j x_j^2 = -4\pi t \sum_{j=1}^m \zeta_j,\, x = (x_1, \ldots, x_m) \in \mathbb{R}^m \end{array} \right\}$$

is Hamiltonian stationary Lagrangian submanifolds for all $\zeta \in \mathbb{R}^m$ and $c \in \mathbb{R}$. Furthermore, they proved that this family $\{V_t\}_{t\in\mathbb{R}}$ is a solution of mean curvature flow and it has a singularity when $t = 0$. To be more precise, they proved that it is a solution of Brakke flow. Here Brakke flow proposed by Brakke [4] is a weak formulation of a mean curvature flow with singularities .

A mean curvature flow is one of potential approaches to find a special Lagrangian submanifold in a given Calabi–Yau manifold as the following meaning. If there exists a long time solution of a mean curvature flow starting from a given Lagrangian submanifold and the flow converges to a smooth manifold, then it is a minimal Lagrangian submanifold, that is, a special Lagrangian submanifold. Indeed, this method has more deep background related to Mirror Symmetry proposed by Thomas and Yau [49]. Roughly speaking, they introduce a stability condition on Lagrangian submanifolds and conjecture that the Lagrangian mean curvature flow starting from a stable Lagrangian submanifold exists for all time and converges to a special Lagrangian submanifold in its Hamiltonian deformation class. This conjecture is called Thomas–Yau conjecture. Recently, Joyce [25] has updated the Thomas–Yau conjectures to achieve more plausible statement. In [25], he discusses the possibility that the Lagrangian mean curvature flow develops singularities many times even if an initial Lagrangian submanifold is stable and mentions the necessity of surgeries of Lagrangian mean curvature flows. Thus it is meaningful to construct examples of Lagrangian mean curvature flows with singularities to understand the motion of Lagrangian mean curvature flows and to develop this program.

In this Part, we construct explicit examples of special or weighted Hamiltonian stationary Lagrangian submanifolds in toric almost Calabi–Yau manifolds and construct solutions of generalized Lagrangian mean curvature flows with singularities and topological changes starting from these examples. These examples can be considered as some kind of generalization of examples of Lee and Wang [29] in $\mathbb{C}^m$ to toric almost Calabi–Yau manifolds. When the ambient space is a general toric almost Calabi–Yau manifold, its topology is not simple and there are many fixed points of torus action. Hence we can get examples of special or weighted Hamiltonian stationary Lagrangian submanifolds with various topologies. Furthermore, its generalized Lagrangian mean curvature flow develops singularities many times though examples of Lee and Wang in $\mathbb{C}^m$ develops a singularity once.

Note that, in this Part, we use notions of *weighted* Hamiltonian stationary and *generalized* Lagrangian mean curvature flow. These notions are modifications of the ordinary notions of Hamiltonian stationary and Lagrangian mean curvature flow defined in Calabi–Yau manifolds to almost Calabi–Yau manifolds. See Section 13 for precise definitions.

Here we give a short description of the main results of this Part. Let $(M, \omega, g, J, \Omega_\gamma)$ be a real $2m$-dimensional toric almost Calabi–Yau manifold with torus T^m action. To be more precise, that is a toric Kähler manifold with a nonvanishing holomorphic $(m, 0)$-form Ω_γ. We see in Section 13 that Ω_γ is constructed by a vector γ in $\mathbb{Z}^m$ which is canonically determined by the toric structure of (M, J). Note that we do not assume that (M, ω, g, J) is Ricci-flat. Since (M, ω, g, J) is a toric Kähler manifold, there exist a moment map $\mu : M \to \Delta$ with a moment polytope Δ and an anti-holomorphic and anti-symplectic involution $\sigma : M \to M$, see Section 11 for more precise settings. We denote the fixed point set of σ by M^σ and call it the real form of M. This is a real m-dimensional submanifold in M. Fix an integer n with $0 \le n \le m$. Take a set of n vectors $\zeta = \{ \zeta_1, \ldots, \zeta_n \} \subset \mathbb{Z}^m$ and a set of

n constants $c = \{ c_1, \dots, c_n \} \subset \mathbb{R}$ and consider the set

$$M^\sigma_{\zeta,c} := \{ p \in M^\sigma \mid \langle \mu(p), \zeta_i \rangle = c_i,\ i = 1, \dots, n \}.$$

We assume that $M^\sigma_{\zeta,c}$ is a real $(m-n)$-dimensional submanifold in M^σ and $T_\zeta := V_\zeta/(V_\zeta \cap \mathbb{Z}^m)$ is isomorphic to a subtorus T^n in T^m, where $V_\zeta := \mathrm{Span}_\mathbb{R}\{\zeta_1, \dots, \zeta_n\}$. Then we put a real m-dimensional manifold as

$$L_{\zeta,c} := M^\sigma_{\zeta,c} \times T_\zeta \tag{20}$$

and define a map $F_{\zeta,c} : L_{\zeta,c} \to M$ by

$$F_{\zeta,c}(p, [v]) := \exp v \cdot p.$$

Then the main theorems in this Part are the following.

Theorem 10.1. *$F_{\zeta,c} : L_{\zeta,c} \to M$ is a T^n-invariant weighted Hamiltonian stationary Lagrangian submanifold for all ζ and c, and its Lagrangian angle $\theta_{\zeta,c} : L_{\zeta,c} \to \mathbb{R}/\pi\mathbb{Z}$ is given by $\theta_{\zeta,c}(p, [v]) = 2\pi\langle \gamma, v \rangle + \frac{\pi}{2} n\,(\mathrm{mod}.\,\pi)$. Thus $F_{\zeta,c} : L_{\zeta,c} \to M$ is a special Lagrangian submanifold if and only if $\langle \gamma, \zeta_i \rangle = 0$ for all $i = 1, \dots, n$.*

Theorem 10.2. *The family of the images of $\{F_{\zeta,c(t)} : L_{\zeta,c(t)} \to M\}_{0 \le t \le T}$ is a solution of generalized Lagrangian mean curvature flow with singularities and topological changes with initial condition $F_{\zeta,c}$, where $c(t) := \{ c_1(t), \dots, c_n(t) \}$ and each $c_j(t)$ is given by $c_j(t) := c_j - 2\pi t \langle \gamma, \zeta_j \rangle$. Here T is the first time that $M^\sigma_{\zeta,c(t)}$ becomes empty set.*

Theorem 10.1 is a summary of Theorem 13.2, Corollary 13.3 and Theorem 13.5. Theorem 10.2 is a part of Theorem 14.2.

The definitions of Lagrangian angle and weighted Hamiltonian stationary are given in Section 13. The meaning of weighted Hamiltonian stationary is explained in Section 16. The notion of generalized Lagrangian mean curvature flow with singularities and topological changes is defined in Section 14. Roughly speaking, this flow is parametrized by a smooth flow except for some m-dimensional Hausdorff measure zero sets. In Example 15.1 of Section 15, we see that our construction is a kind of generalization of the example of Lee and Wang [29]. In Example 15.2, we give a concrete example of generalized Lagrangian mean curvature flow with singularities and topological changes in $K_{\mathbb{P}^2}$, the total space of the canonical bundle over $\mathbb{P}^2$.

We note that the example M_c of Harvey and Lawson is in the case when $n = m-1$, and $N_c^{a_1,\dots,a_m}$ of Joyce, V_t of Lee and Wang and the previous work of the author in [50] are in the case when $n = 1$. After finishing my work, the author learned from H. Konno that the Mironov and Panov [38] constructed examples of T^n-invariant Hamiltonian stationary Lagrangian submanifolds in m-dimensional toric varieties for $0 \le n \le m$. First, Mironov [37] constructed T^n-invariant Hamiltonian stationary or minimal Lagrangian submanifolds in $\mathbb{C}^m$ and $\mathbb{CP}^m$. These examples can be written as the form (20) in $\mathbb{C}^m$. In [38], they used a Kähler quotient of $\mathbb{C}^m$ to construct new examples in toric varieties. We remark that our method is different from theirs in the point that we use the real form and a moment map rather than Kähler quotient, and furthermore we study the motion of generalized Lagrangian mean curvature flows starting from these examples.

11 Toric Kähler manifold

In this section, we fix our notations of toric Kähler geometry and introduce an anti-holomorphic involution and its properties. Let $T^m \cong (S^1)^m$ be an m-dimensional real torus and (M, ω, g, J) be a toric Kähler manifold with complex dimension m. Then T^m acts on M effectively and the Kähler form ω is invariant under the action. Let $\mu : M \to \mathfrak{g}^*$ be a moment map and $\Delta := \mu(M)$ be a moment polytope, where $\mathfrak{g}$ is a Lie algebra of T^m and $\mathfrak{g}^*$ is its dual. Since (M, J) is a toric variety, there is a complex torus $T^m_\mathbb{C} \cong (\mathbb{C}^\times)^m$ which is a complexification of T^m and $T^m_\mathbb{C}$ acts on (M, J) as biholomorphic automorphisms. Then M has an open dense $T^m_\mathbb{C}$-orbit and we denote the fan of (M, J) by Σ. Let $\Sigma(1) := \{ \rho \in \Sigma \mid \dim \rho = 1 \}$ be the set of 1-dimensional cones in Σ. We assume that $\Sigma(1)$ is a finite set and write $\Sigma(1) = \{\rho_1, \dots, \rho_d\}$. Let λ_i be the primitive element in $\mathbb{Z}^m$ that generates ρ_i for $i = 1, \dots, d$, that is, $\rho_i = \mathbb{R}^+ \lambda_i$. Note that, in general, Δ is not a closed subset in $\mathfrak{g}^*$. For example, if we consider a toric Kähler manifold constructed by removing all fixed points of torus action from some toric Kähler manifold, then its moment polytope has a shape that all vertices are removed from the original polytope and this is not a closed subset.

We assume that there exist κ_i in $\mathbb{R}$ for $i = 1, \ldots, d$ so that the closure of Δ is given by

$$\overline{\Delta} = \bigcap_{i=1}^{d} H^+_{\lambda_i,\kappa_i}.$$

Here for a nonzero vector λ in $\mathfrak{g}$ and κ in $\mathbb{R}$, we define the affine hyperplane $H_{\lambda,\kappa}$ and closed half-space $H^+_{\lambda,\kappa}$ by

$$H_{\lambda,\kappa} := \{\, y \in \mathfrak{g}^* \mid \langle y, \lambda\rangle = \kappa \,\} \quad \text{and} \quad H^+_{\lambda,\kappa} := \{\, y \in \mathfrak{g}^* \mid \langle y, \lambda\rangle \geq \kappa \,\}.$$

A subset $F \subset \overline{\Delta}$ is called a face of $\overline{\Delta}$ if and only if there exist a vector v in $\mathfrak{g}$ and a constant c such that

$$\overline{\Delta} \subset H^+_{v,c} \quad \text{and} \quad F = \overline{\Delta} \cap H_{v,c}.$$

We denote the set of all faces of $\overline{\Delta}$ by $\mathcal{F}$. Then we assume that there exists a subset $\mathcal{G}$ of $\mathcal{F}$ such that Δ is of the form

$$\overline{\Delta} - \bigcup_{F \in \mathcal{G}} F.$$

For a point y in Δ, we define $\mathfrak{z}_y$ a subspace of $\mathfrak{g}$ by

$$\mathfrak{z}_y := \mathrm{Span}_{\mathbb{R}}\{\, \lambda_i \mid y \in H_{\lambda_i,\kappa_i} \,\}.$$

For example, if y is in the interior of Δ then $\mathfrak{z}_y$ is $\{0\}$. For a point p in M, if we denote the stabilizer at p by $Z_p = \{\, t \in T^m \mid t \cdot p = p \,\}$, then the Lie algebra of Z_p coincides with $\mathfrak{z}_{\mu(p)}$. Thus, if $\mu(p)$ is in the interior of Δ then torus action is free at p, and if μ maps p to a vertex of Δ then p is a fixed point.

Since (M, J) is a toric variety, there exists the intrinsic anti-holomorphic involution $\sigma : M \to M$ determined by the fan Σ, that is, $\sigma^2 = id$ and $\sigma_* J = -J\sigma_*$, where J is the complex structure on M. This involution satisfies $\sigma(u \cdot p) = \overline{u} \cdot \sigma(p)$, where $u \in T^m_{\mathbb{C}}$ acts on p. Let $M^\sigma := \{\, p \in M \mid \sigma(p) = p \,\}$ be the set of fixed points of σ, that is a submanifold of M with real dimension m, we call it the real form of M.

Proposition 11.1. *The involution $\sigma : M \to M$ is anti-symplectic, and consequently σ is isometry.*

Proof. Let U be an open dense $T^m_{\mathbb{C}}$-orbit. For $(w^1, \ldots, w^m) \in U \cong (\mathbb{C}^\times)^m$, we take the logarithmic holomorphic coordinates $(z^1, \ldots, z^m)$ with $e^{z^i} = w^i$. Since ω is T^m-invariant and the action of T^m is Hamiltonian, there exists a function $F \in C^\infty(\mathbb{R}^m)$ with the property

$$\omega = \frac{\sqrt{-1}}{2} \sum_{i,j=1}^{m} \frac{\partial^2 F}{\partial x^i \partial x^j} dz^i \wedge d\overline{z}^j \quad \text{on } U, \tag{21}$$

where $z^i = x^i + \sqrt{-1} y^i$. (See Theorem 3.3 in Appendix 2 of [18].) On U, the involution σ coincides with the standard complex conjugate $\sigma(z) = \overline{z}$, where $\overline{z} = (\overline{z}^1, \ldots, \overline{z}^m)$. Since ω is T^m-invariant, note that F is independent of the coordinates $(y^1, \ldots, y^m)$. Thus we have $\sigma^*\omega = -\omega$ on U. Since U is open and dense in M, thus we have $\sigma^*\omega = -\omega$ on M. □

12 Lagrangian submanifold

In this section, we construct our examples of Lagrangian submanifold. First of all, let n be an integer with $0 \leq n \leq m$. Next, take a set of n vectors $\zeta = \{\zeta_i\}_{i=1}^n \subset \mathfrak{g}$ and a set of n constants $c = \{c_i\}_{i=1}^n \subset \mathbb{R}$. If $n = 0$, we take no vectors and no constants. We assume that $\{\zeta_i\}_{i=1}^n$ is linearly independent. Then the intersection of n affine hyperplanes H_{ζ_i,c_i} defines a $(m-n)$-dimensional affine plane. We assume that this affine plane intersects in the interior of Δ, and we define $\Delta_{\zeta,c}$ a subset of Δ by

$$\begin{aligned} \Delta_{\zeta,c} &:= \Delta \cap \left(\bigcap_{i=1}^{n} H_{\zeta_i,c_i} \right) \\ &= \{\, y \in \Delta \mid \langle y, \zeta_i \rangle = c_i,\ (i = 1, \ldots, n) \,\}. \end{aligned}$$

Definition 12.1. Let $V_\zeta := \mathrm{Span}_{\mathbb{R}}\{\zeta_1, \dots, \zeta_n\} \subset \mathfrak{g}$. We call a point y in Δ a ζ-singular point if and only if $V_\zeta \cap \mathfrak{z}_y \neq \{0\}$, and if $V_\zeta \cap \mathfrak{z}_y = \{0\}$ we call y a ζ-regular point. We denote the set of all ζ-singular points and all ζ-regular points in Δ by $\Delta_{\zeta sing}$ and $\Delta_{\zeta reg}$ respectively. Note that $\Delta_{\zeta reg}$ is open dense in Δ.

For a point p in M, a vector v in $\mathfrak{g}$ generates a tangent vector at p denoted by

$$v_p = \left.\frac{d}{dt}\right|_{t=0} \exp(tv) \cdot p.$$

This map $\mathfrak{g} \to T_pM$ is a homomorphism. Then it is clear that y is a ζ-regular point if and only if the restricted homomorphism $V_\zeta \to T_pM$ is injective for a p in $\mu^{-1}(y)$. For example, vertices of Δ are always ζ-singular points and interior points are always ζ-regular points.

Definition 12.2. We call a point p in M^σ a ζ-singular point if and only if $\mu(p)$ is a ζ-singular point, and if not, we call p a ζ-regular point. We denote the set of all ζ-singular points and all ζ-regular points in M^σ by $M^\sigma_{\zeta sing}$ and $M^\sigma_{\zeta reg}$ respectively. Note that $M^\sigma_{\zeta reg}$ is open dense in M^σ.

Definition 12.3. We denote the restriction of the moment map on the real form by $\mu^\sigma : M^\sigma \to \mathbb{R}^m$. We define a subset of M^σ as the pull-back of $\Delta_{\zeta,c}$ by μ^σ by

$$\begin{aligned} M^\sigma_{\zeta,c} &:= (\mu^\sigma)^{-1}(\Delta_{\zeta,c}) \\ &= \{\, p \in M^\sigma \mid \langle \mu(p), \zeta_i \rangle = c_i,\ i = 1, \dots, n \,\}. \end{aligned}$$

Proposition 12.4. *If $\Delta_{\zeta,c}$ is contained in $\Delta_{\zeta reg}$, then $M^\sigma_{\zeta,c}$ is a smooth submanifold of M^σ with* $\dim_{\mathbb{R}} M^\sigma_{\zeta,c} = m - n$.

Proof. We define n functions f_i $(i = 1, \dots, n)$ on M^σ by

$$f_i(p) := \langle \mu(p), \zeta_i \rangle - c_i.$$

Then $M^\sigma_{\zeta,c} = \{\, p \in M^\sigma \mid f_i(p) = 0,\ i = 1, \dots, n \,\}$. By a property of the moment map, for all p in $M^\sigma_{\zeta,c}$, we have

$$df_i(p) = d\langle \mu, \zeta_i \rangle(p) = -\omega(\zeta_{i,p}, \cdot).$$

Since every point in $\Delta_{\zeta,c}$ is ζ-regular, the restricted homomorphism $V_\zeta \to T_pM$ is injective for all p in $M^\sigma_{\zeta,c}$. Thus $\{df_i\}_{i=1}^n$ are linearly independent 1-forms on $M^\sigma_{\zeta,c}$. This means that $M^\sigma_{\zeta,c}$ is a smooth submanifold of M^σ by the implicit function theorem. ☐

In this section, we assume that $\Delta_{\zeta,c}$ is contained in $\Delta_{\zeta reg}$. Then $M^\sigma_{\zeta,c}$ is a smooth submanifold of M^σ. Let $\exp : \mathfrak{g} \to T^m$ be the exponential map. Let $\mathbb{Z}_{\mathfrak{g}} (\cong \mathbb{Z}^m)$ be a integral lattice of $\mathfrak{g}$, that is a kernel of $\exp : \mathfrak{g} \to T^m$ and $\mathfrak{g}/\mathbb{Z}_{\mathfrak{g}} \cong T^m$. Let $\frac{1}{2}\mathbb{Z}_{\mathfrak{g}}$ be the set of all elements y in $\mathfrak{g}$ such that $2y$ is in $\mathbb{Z}_{\mathfrak{g}}$. Then $\frac{1}{2}\mathbb{Z}_{\mathfrak{g}}/\mathbb{Z}_{\mathfrak{g}} \cong \{1, -1\}^m$ is a subgroup of T^m considered as all elements t in T^m such that $t^2 = e$ identity element. Let $V_\zeta = \mathrm{Span}_{\mathbb{R}}\{\zeta_1, \dots, \zeta_n\} \subset \mathfrak{g}$. Now we construct a manifold $L_{\zeta,c}$ with real dimension m.

(I) Generic case. For a generic case, let U be an open small ball in V_ζ centered at 0 such that U and $\frac{1}{2}\mathbb{Z}_{\mathfrak{g}}$ intersect only at 0. Then we define an m-dimensional manifold $L_{\zeta,c}$ and a map $F_{\zeta,c} : L_{\zeta,c} \to M$ by

$$L_{\zeta,c} = M^\sigma_{\zeta,c} \times U \quad \textit{and} \quad F_{\zeta,c}(p, v) := \exp(v) \cdot p,$$

for p in $M^\sigma_{\zeta,c}$ and v in U. Then $F_{\zeta,c}$ is injective and its image is

$$L'_{\zeta,c} := \{\, \exp(v) \cdot p \mid v \in U,\ p \in M^\sigma, \langle \mu(p), \zeta_j \rangle = c_j,\ j = 1, \dots, n \,\}. \tag{22}$$

(II) Unimodular case. If the set of vectors $\zeta = \{\zeta_i\}_{i=1}^n$ satisfies the following unimodular condition then we can take $L_{\zeta,c}$ as explained below.

Definition 12.5. We say that ζ satisfies the unimodular condition if there exists a set of n vectors $v = \{v_j\}_{j=1}^n$ in $V_\zeta \cap \mathbb{Z}_{\mathfrak{g}}$ such that v is a base of V_ζ and v is a generator of $V_\zeta \cap \mathbb{Z}_{\mathfrak{g}}$ over $\mathbb{Z}$.

If ζ satisfies the unimodular condition, we replace U in the case (I) by $T_\zeta := V_\zeta/(V_\zeta \cap \mathbb{Z}_\mathfrak{g})$ and we define an m-dimensional manifold $L_{\zeta,c}$ and a map $F_{\zeta,c} : L_{\zeta,c} \to M$ by

$$L_{\zeta,c} = M^\sigma_{\zeta,c} \times T_\zeta \quad \textit{and} \quad F_{\zeta,c}(p,[v]) := \exp(v) \cdot p,$$

for p in $M^\sigma_{\zeta,c}$ and $[v]$ in $T_\zeta = V_\zeta/(V_\zeta \cap \mathbb{Z}_\mathfrak{g})$, this map is well defined. Since $T_\zeta \cong T^n$ which is a subtorus of T^m, the product manifold $L_{\zeta,c}$ is diffeomorphic to $M^\sigma_{\zeta,c} \times T^n$. We denote the subgroup $(V_\zeta \cap \frac{1}{2}\mathbb{Z}_\mathfrak{g})/(V_\zeta \cap \mathbb{Z}_\mathfrak{g})$ of T_ζ by K_ζ. Then, of course, K_ζ acts on T_ζ freely and K_ζ also acts on $M^\sigma_{\zeta,c}$ as

$$[k] \cdot p := \exp(k) \cdot p$$

for $[k]$ in K_ζ and p in $M^\sigma_{\zeta,c}$. Thus K_ζ acts on $L_{\zeta,c} = M^\sigma_{\zeta,c} \times T_\zeta$ as a diagonal action and this action is free. Hence we have an m-dimensional manifold $\tilde{L}_{\zeta,c}$ by

$$\tilde{L}_{\zeta,c} := (M^\sigma_{\zeta,c} \times T_\zeta)/K_\zeta.$$

In this case (II), $F_{\zeta,c} : L_{\zeta,c} \to M$ is not injective and one can show that $F_{\zeta,c}(p_1,[v_1]) = F_{\zeta,c}(p_2,[v_2])$ if and only if there exists a $[k]$ in K_ζ such that $[k] \cdot (p_1,[v_1]) = (p_2,[v_2])$. Thus the image of $F_{\zeta,c}$ written by

$$L'_{\zeta,c} := \{ \exp(v) \cdot p \mid v \in V_\zeta,\, p \in M^\sigma,\, \langle \mu(p), \zeta_j \rangle = c_j,\, j = 1,\dots,n \} \tag{23}$$

is diffeomorphic to $\tilde{L}_{\zeta,c}$. Note that $\tilde{L}_{\zeta,c}$ is a T^n-bundle over a smooth $(m-n)$-dimensional manifold $M^\sigma_{\zeta,c}/K_\zeta$.

Remark 12.6. Here we explain the meaning of $L_{\zeta,c}$ and the number n, that is the number of vectors in ζ. In an m-dimensional toric Kähler manifold M, there are two typical Lagrangian submanifolds, one is the real form M^σ and the other is a torus fiber T^m, and these two Lagrangians M^σ and T^m intersect transverse and orthogonal just like $\mathbb{R}^m$ and $i\mathbb{R}^m$ in $\mathbb{C}^m$. First, if we take $n = 0$ then we take no vectors ζ and no constants c. Then $L_{\zeta,c}$ becomes the real form M^σ, hence $L_{\zeta,c}$ has no torus factors. On the other hand, if n is full, that is, $n = m$, then $M^\sigma_{\zeta,c} = \{pt\}$, thus $L_{\zeta,c}$ is diffeomorphic to a torus fiber T^m. Hence, roughly speaking, $L_{\zeta,c}$ is a hybrid (or interpolation) of the real form M^σ and a torus fiber T^m, and n is the dimension of torus factors in $L_{\zeta,c}$.

From now, we consider both cases (I) and (II) above.

Theorem 12.7. *$F_{\zeta,c} : L_{\zeta,c} \to M$ is a Lagrangian immersion.*

Proof. In this proof, we write $F_{\zeta,c}$ by F for short. Since the case (II) is locally diffeomorphic to the cace (I), it is clear that we only have to prove in the case (I). First we prove that F is an immersion map. Fix a point $x = (p,v)$ in $L_{\zeta,c} = M^\sigma_{\zeta,c} \times U$. Then we have a decomposition

$$T_x L_{\zeta,c} = T_p M^\sigma_{\zeta,c} \oplus T_v U,$$

and note that $T_v U \cong V_\zeta$ since U is an open ball in a vector space V_ζ. Take tangent vectors X, X_1, X_2 in $T_p M^\sigma_{\zeta,c}$. We have

$$F_* X = t_{v*} X,$$

where we put $t_v := \exp(v)$ for short, and we identify an element t_v in T^m with a left transition map $t_v : M \to M$. Take vectors Y, Y_1, Y_2 in $T_v U \cong V_\zeta$. We have

$$F_* Y = t_{v*} Y_p.$$

Here Y_p is the tangent vector at p generated by $Y \in V_\zeta \subset \mathfrak{g}$. Since g is torus-invariant, that is, $t_v^* g = g$, we have

$$g(F_* X, F_* Y) = g(t_{v*} X, t_{v*} Y_p) = (t_v^* g)(X, Y_p) = g(X, Y_p). \tag{24}$$

Note that $\sigma_* X = X$ since X is tangent to the real form, and $\sigma_* Y_p = -Y_p$ since the direction of the curve of the exponential map generated by Y is reversed by σ because of the reration $\sigma(u \cdot p) = u^{-1} \cdot p$ for all u in T^m. Since σ is isometry, that is $\sigma^* g = g$, by Proposition 11.1, we have

$$g(X, Y_p) = (\sigma^* g)(X, Y_p) = g(\sigma_* X, \sigma_* Y_p) = -g(X, Y_p),$$

and this means that $g(X, Y_p) = 0$ and also $g(F_* X, F_* Y) = 0$ by (24). Thus $F_*(T_p M^\sigma_{\zeta,c})$ and $F_*(T_v U)$ are orthogonal to each other. It is clear that F_* restricted on $T_p M^\sigma_{\zeta,c}$ is injective and F_* restricted on $T_v U$ is also injective. Thus F_* is injective on $T_x L_{\zeta,c}$ and F is an immersion map.

Next we prove that F is a Lagrangian, that is, $F^* \omega = 0$. It is easy to see $(F^* \omega)(X_1, X_2) = 0$ and $(F^* \omega)(Y_1, Y_2) = 0$ since the real form and a torus fiber are typical Lagrangians. We can also prove that $(F^*\omega)(X,Y)=0$ easily. Since ω is torus-invariant and $\omega(\cdot,Y_p)=d\langle\mu,Y\rangle$, we have

$$(F^*\omega)(X,Y)=\omega(F_*X,F_*Y)=\omega(X,Y_p)=X(\langle\mu,Y\rangle).$$

Since Y is in $T_vU\cong V_\zeta=\mathrm{Span}_{\mathbb{R}}\{\zeta_1,\dots,\zeta_n\}$, we can write Y as $Y=a^1\zeta_1+\cdots+a^n\zeta_n$ for some coefficients $a^k\in\mathbb{R}$, and we have

$$\langle\mu,Y\rangle=a^1\langle\mu,\zeta_1\rangle+\cdots+a^n\langle\mu,\zeta_n\rangle.$$

By the definition of $M^\sigma_{\zeta,c}$, this function $\langle\mu,Y\rangle$ is a constant

$$a^1c_1+\cdots+a^nc_n$$

on $M^\sigma_{\zeta,c}$, and now X is a tangent vector on $M^\sigma_{\zeta,c}$, thus it is clear that

$$X(\langle\mu,Y\rangle)=0.$$

Hence we have $F^*\omega=0$. □

13 Lagrangian angle

In above sections, the ambient space (M,ω,g,J) is a toric Kähler manifold. From this section, we assume that the canonical line bundle K_M of (M,J) is trivial. This condition is equivalent to that there exists a vector γ in $\mathbb{Z}^*_g$ such that $\langle\gamma,\lambda_i\rangle=1$ for all $i=1,\dots,d$, where λ_i is a primitive generator of a 1-dimensional cone of fan Σ of M, see Section 11. In fact, if such a vector $\gamma=(\gamma_1,\dots,\gamma_m)$ exists, a holomorphic $(m,0)$-form

$$\Omega_\gamma:=e^{\gamma_1z^1+\cdots+\gamma_mz^m}dz^1\wedge\cdots\wedge dz^m \tag{25}$$

written by logarithmic holomorphic coordinates on an open dense $(\mathbb{C}^*)^m$-orbit can be extend over M as a nowhere vanishing holomorphic $(m,0)$-form. We call this $(M,\omega,g,J,\Omega_\gamma)$ a toric almost Calabi–Yau manifold.

In general, an m-dimensional Kähler manifold (M,ω,g,J) with nowhere vanishing holomorphic $(m,0)$-form Ω is called an almost Calabi–Yau manifold, and for a Lagrangian immersion $F:L\to M$ we can define the Lagrangian angle $\theta_F:L\to\mathbb{R}/\pi\mathbb{Z}$ as follows. For a point x in L, take a local chart $(U,(x^1,\dots,x^m))$ around x, then $F^*\Omega$ is a $\mathbb{C}^*$-valued m-form on U, so there exists a $\mathbb{C}^*$-valued function h_U on U such that

$$F^*\Omega=h_U(x^1,\dots,x^m)dx^1\wedge\cdots\wedge dx^m$$

on U, and we define the Lagrangian angle $\theta_F:L\to\mathbb{R}/\pi\mathbb{Z}$ by

$$\theta_F(x):=\arg(h_U(x))\mod\pi.$$

This definition is independent of the choice of local charts. It is clear that if L is oriented we can lift θ_F to a $\mathbb{R}/2\pi\mathbb{Z}$-valued function $\theta_F:L\to\mathbb{R}/2\pi\mathbb{Z}$. If we can lift θ_F to a $\mathbb{R}$-valued function $\theta_F:L\to\mathbb{R}$ then $F:L\to M$ is called Maslov zero, and furthermore if θ_F is constant θ_0 then $F:L\to M$ is called a special Lagrangian submanifold with phase $e^{i\theta_0}$. Note that the definition of special Lagrangian condition depends on the choice of holomorphic volume form Ω.

In [1], Behrndt introduced the notion of the generalized mean curvature vector field K for a Lagrangian immersion $F:L\to M$ in an almost Calabi–Yau manifold. The generalized mean curvature vector field K is defined by

$$K:=H-m\nabla\psi^\perp, \tag{26}$$

where H is the mean curvature vector field of the immersion $F:L\to(M,g)$, ψ is a function on M defined by the following equation;

$$e^{2m\psi}\frac{\omega^m}{m!}=(-1)^{\frac{m(m-1)}{2}}\left(\frac{i}{2}\right)^m\Omega\wedge\overline{\Omega}, \tag{27}$$

and $\nabla\psi^\perp$ is the normal part of the gradient of ψ. By the definition of K, if M is a Calabi–Yau manifold, that is, $\psi\equiv0$, then the generalized mean curvature vector field K coincides with the mean curvature vector field H. In Proposition 4.8 in [2], Behrndt proved the relation between K and θ_F which is written by

$$K=J\nabla\theta_F. \tag{28}$$

Thus $K\equiv0$ is equivalent to that L is a special Lagrangian submanifold.

Furthermore, in this Part, we introduce the notion of *weighted Hamiltonian stationary* for a Lagrangian immersion $F : L \to M$ into an almost Calabi–Yau manifold $(M, \omega, g, J, \Omega)$ with ψ defined by (27).

Definition 13.1. Let θ_F be the Lagrangian angle of $F : L \to M$. If $\Delta_f \theta_F = 0$ then we call $F : L \to M$ a weighted Hamiltonian stationary Lagrangian submanifold.

Here f is a function on L defined by $f := -mF^*\psi$ and Δ_f is the weighted Laplacian on Riemannian manifold (L, F^*g). In general, for a Riemannian manifold (N, h) with a function f, the weighted Laplacian with respect to f is defined by $\Delta_f u := \Delta u + \langle \nabla u, \nabla f\rangle$. Thus, if M is a Calabi–Yau manifold, that is, $\psi = 0$, then the notion of weighted Hamiltonian stationary is equivalent to the Hamiltonian stationary condition, namely $\Delta\theta_F = 0$. For the meaning of the weighted Hamiltonian stationary condition, See Section 16. Note that Δ_f is the standard Laplace operator on L with respect to a Riemannian metric $F^*(e^{2\psi}g)$.

In this section, we compute the Lagrangian angle of our example $F_{\zeta,c} : L_{\zeta,c} \to M$ constructed in Section 12, and show some properties of $F_{\zeta,c} : L_{\zeta,c} \to M$.

Let $(M, \omega, g, J, \Omega_\gamma)$ be an m-dimensional toric almost Calabi–Yau manifold and $F_{\zeta,c} : L_{\zeta,c} \to M$ be a Lagrangian immersion constructed by $\zeta = \{\zeta_1, \dots, \zeta_n\} \subset \mathfrak{g}$ and $c = \{c_1, \dots, c_n\} \subset \mathbb{R}$, explained in Section 12.

Theorem 13.2. *The Lagrangian angle θ of $F_{\zeta,c} : L_{\zeta,c} \to M$ is given by*

$$\theta(x) = 2\pi\langle\gamma, v\rangle + \frac{\pi}{2}n \mod \pi$$

for $x = (p, v)$ in $L_{\zeta,c} = M^\sigma_{\zeta,c} \times U$ in the case (I) *and for $x = (p, [v])$ in $L_{\zeta,c} = M^\sigma_{\zeta,c} \times T_\zeta$ in the case* (II).

Proof. In this proof, we write $F_{\zeta,c}$ by F for short. It is clear that we only have to prove in the case (I). Let M^σ be a real form of M and $\mathfrak{g}$ be a Lie algebra of T^m. We define a map $\tilde{F} : M^\sigma \times \mathfrak{g} \to M$ by

$$\tilde{F}(p, v) := \exp(v) \cdot p.$$

Remember that $L_{\zeta,c} = M^\sigma_{\zeta,c} \times U$, and $M^\sigma_{\zeta,c}$ is an $(m-n)$-dimensional submanifold in M^σ and U is an n-dimensional submanifold in $\mathfrak{g}$. Thus we have the inclusion map $L_{\zeta,c}$ into $M^\sigma \times \mathfrak{g}$ by

$$\iota = (\iota_1, \iota_2) : L_{\zeta,c} = M^\sigma_{\zeta,c} \times U \hookrightarrow M^\sigma \times \mathfrak{g}.$$

Then the map $F : L_{\zeta,c} \to M$ coincides with $\tilde{F} \circ \iota$ by the definition of F, so we compute $\iota^*(\tilde{F}^*\Omega_\gamma)$ to compute $F^*\Omega_\gamma$. It is enough to prove this theorem on an open dense $(\mathbb{C}^*)^m$-orbit, so we take a logarithmic holomorphic coordinates $(z^1, \dots, z^m)$, then $(x^1, \dots, x^m)$ define local coordinates on the real form M^σ, where $z^j = x^j + iy^j$. Let $(t^1, \dots, t^m)$ be coordinates of $\mathfrak{g} \cong \mathbb{R}^m$, then we have a local expression of a map $\tilde{F} : M^\sigma \times \mathfrak{g} \to M$ by

$$\tilde{F}(x^1, \dots, x^m, t^1, \dots, t^m) = (x^1 + 2\pi i t^1, \dots, x^m + 2\pi i t^m).$$

Since $\Omega_\gamma = e^{\gamma_1 z^1 + \cdots + \gamma_m z^m} dz^1 \wedge \cdots \wedge dz^m$, we have

$$\tilde{F}^*\Omega_\gamma = e^{(\gamma_1 x^1 + \cdots + \gamma_m x^m) + 2\pi i(\gamma_1 t^1 + \cdots + \gamma_m t^m)}(dx^1 + 2\pi i dt^1) \wedge \cdots \wedge (dx^m + 2\pi i dt^m).$$

Since $L_{\zeta,c} = M^\sigma_{\zeta,c} \times U$, and $M^\sigma_{\zeta,c}$ is an $(m-n)$-dimensional submanifold in M^σ and U is an n-dimensional submanifold in $\mathfrak{g}$, in the expansion of $(dx^1 + 2\pi i dt^1) \wedge \cdots \wedge (dx^m + 2\pi i dt^m)$, differential forms such as

$$(2\pi i)^n dx^I \wedge dt^J$$

with $\sharp I = m - n$ and $\sharp J = n$ do not vanish after pull-back by ι, and other forms vanish, where I and J are multi-indices. Thus the argument of $F^*\Omega_\gamma = \iota^*(\tilde{F}^*\Omega_\gamma)$ at (p, v) is the argument of

$$(2\pi i)^n e^{\langle\gamma, p\rangle + 2\pi i\langle\gamma, v\rangle},$$

that is, $2\pi\langle\gamma, v\rangle + \frac{\pi}{2}n \mod \pi$. □

Then the following corollary is clear.

Corollary 13.3. *$F_{\zeta,c} : L_{\zeta,c} \to M$ is a special Lagrangian submanifold if and only if $\langle\gamma, \zeta_i\rangle = 0$ for all $i = 1, \dots, n$.*

Remark 13.4. It is clear that the real form M^σ, that is the case of $n = 0$, is always a special Lagrangian submanifold, and every torus fiber, that is the case of $n = m$, is not a special Lagrangian submanifold with respect to this holomorphic volume form Ω_γ. If $M = \mathbb{C}^m$, we take $\gamma = (1, \ldots, 1)$, see also Example 15.1. Then the special Lagrangian condition (19) by Joyce introduced in Section 10 coincides with the condition $\langle \gamma, a \rangle = 0$ in Corollary 13.3.

Theorem 13.5. *$F_{\zeta,c} : L_{\zeta,c} \to M$ is weighted Hamiltonian stationary.*

Proof. In this proof, we write $F_{\zeta,c}$ by F for short. We only have to prove that $\Delta_f \theta = 0$ in the case (I) that $L_{\zeta,c} = M^\sigma_{\zeta,c} \times U$. As noted above, Δ_f is the standard Laplace operator on L with respect to a Riemannian metric $F^*(e^{2\psi} g)$. Since g is invariant under the torus action and it is easily seen that ψ is also torus invariant by the equation (25) and (27), so the metric $e^{2\psi} g$ is also a torus invariant metric on M. Since $F : L_{\zeta,c} \to M$ is given by $F(p, v) := \exp(v) \cdot p$ and $e^{2\psi} g$ is a torus invariant metric on M, the metric $F^*(e^{2\psi} g)$ on L is independent of the U-factor of $L_{\zeta,c}$. Furthermore, in the proof of Theorem 12.7 we prove that $F_*(TM^\sigma_{\zeta,c})$ and $F_*(TU)$ are orthogonal, thus $F^*(e^{2\psi} g)$ is a product metric over $M^\sigma_{\zeta,c}$ and U locally. By Theorem 13.2, the Lagrangian angle is given by $\theta(p, v) = 2\pi \langle \gamma, v \rangle + \frac{\pi}{2} n$, it is independent of $M^\sigma_{\zeta,c}$-factor of $L_{\zeta,c}$ and affine on U-factor. Then one can easily prove that $\Delta_f \theta = 0$. $\square$

14 Mean curvature flow

In this section, we consider generalized Lagrangian mean curvature flows. In general, a generalized Lagrangian mean curvature flow is defined in an almost Calabi–Yau manifold $(M, \omega, g, J, \Omega)$. Let $F_0 : L \to M$ be a Lagrangian immersion, then a one parameter family of Lagrangian submanifolds $F : L \times I \to M$ is called a solution of a generalized Lagrangian mean curvature flow with initial condition F_0, if it moves along its generalized Lagrangian mean curvature vector field K defined in (26), that is,

$$\left(\frac{\partial F}{\partial t} \right)^\perp = K_t \quad \textit{and} \quad F(\cdot, 0) = F_0, \tag{29}$$

where K_t is the generalized Lagrangian mean curvature vector field of immersion $F_t : L \to M$ defined by $F_t(p) := F(p, t)$. Of course, if M is a Calabi–Yau manifold then a generalized Lagrangian mean curvature flow is an ordinary Lagrangian mean curvature flow. It is clear that $K = 0$ on a special Lagrangian submanifold by the equation (28), thus a special Lagrangian submanifold is a stationary solution of a generalized Lagrangian mean curvature flow. In general, a generalized Lagrangian mean curvature flow develops some singularities in a finite time, so here we define a notion of a generalized Lagrangian mean curvature flow with some singularities and topological changes.

Definition 14.1. Let $(M, \omega, g, J, \Omega)$ be a real $2m$-dimensional almost Calabi–Yau manifold and $\{L_t\}_{t \in I}$ be a one parameter family of subsets in M. Then we call $\{L_t\}_{t \in I}$ a solution of a generalized Lagrangian mean curvature flow with singularities and topological changes if there exists a real m-dimensional manifold L and a solution of a generalized Lagrangian mean curvature flow $F : L \times I \to M$ such that $F_t : L \to M$ is an embedding into L_t and m-dimensional Hausdorff measure of $L_t \setminus F_t(L)$ is zero, i.e.

$$F_t(L) \subset L_t \quad \textit{and} \quad \mathcal{H}^m(L_t \setminus F_t(L)) = 0. \tag{30}$$

It means that $\{L_t\}_{t \in I}$ is almost parametrized by a smooth solution of a generalized Lagrangian mean curvature flow.

The purpose of this section is to observe how our concrete examples $F_{\zeta,c} : L_{\zeta,c} \to M$ move along the generalized Lagrangian mean curvature flow. Let $(M, \omega, g, J, \Omega_\gamma)$ be a toric almost Calabi–Yau manifold and $F_{\zeta,c} : L_{\zeta,c} \to M$ be a Lagrangian submanifold constructed in Section 12 by data $\zeta = \{\zeta_1, \ldots, \zeta_n\} \subset \mathfrak{g}$ and $c = \{c_1, \ldots, c_n\} \subset \mathbb{R}$. Let

$$c_i(t) := c_i - 2\pi \langle \gamma, \zeta_i \rangle t$$

for $t \in \mathbb{R}$ and we denote $c(t) := \{c_1(t), \ldots, c_n(t)\}$. We define an open interval I by

$$I := \left\{ t \in \mathbb{R} \,\middle|\, \mathrm{Int}\Delta \cap \left(\bigcap_{i=1}^n H_{\zeta_i, c_i(t)} \right) \neq \emptyset \right\},$$

by the assumption of ζ and c we have $0 \in I$.

Theorem 14.2. *A one parameter family of subsets $\{L'_{\zeta,c(t)}\}_{t\in I}$ defined by* (22) *in the case* (I) *or by* (23) *in the case* (II) *is a solution of a generalized Lagrangian mean curvature flow with singularities and topological changes.*

Proof. It is sufficient to prove this theorem in the case (I). First we define

$$\Delta''_{\zeta,c(t)} := \mathrm{Int}\Delta \cap \left(\bigcap_{i=1}^{n} H_{\zeta_i,c_i(t)}\right).$$

Remember that $\Delta_{\zeta,c(t)}$ is defined by

$$\Delta_{\zeta,c(t)} := \Delta \cap \left(\bigcap_{i=1}^{n} H_{\zeta_i,c_i(t)}\right).$$

Since $\Delta_{\zeta,c(t)} \setminus \Delta''_{\zeta,c(t)}$ is contained in $\partial\Delta_{\zeta,c(t)}$, it is clear that $(m-n)$-dimensional Hausdorff measure of $\Delta_{\zeta,c(t)} \setminus \Delta''_{\zeta,c(t)}$ is zero. Since each $\Delta''_{\zeta,c(t)}$ is an $(m-n)$-dimensional connected convex affine open subset in $\mathbb{R}^m$, all $\Delta''_{\zeta,c(t)}$ are diffeomorphic to each other.

Next we define

$$M''^{\sigma}_{\zeta,c(t)} := (\mu^{\sigma})^{-1}(\Delta''_{\zeta,c(t)}) \quad \text{and} \quad L''_{\zeta,c(t)} := M''^{\sigma}_{\zeta,c(t)} \times U.$$

Then $M''^{\sigma}_{\zeta,c(t)}$ is an $(m-n)$-dimensional open dense submanifold in M, and $L''_{\zeta,c(t)}$ is an m-dimensional open dense submanifold in $L_{\zeta,c(t)}$. As same as $\Delta''_{\zeta,c(t)}$, all $M''^{\sigma}_{\zeta,c(t)}$ are diffeomorphic to each other, and $(m-n)$-dimensional Hausdorff measure of $M^{\sigma}_{\zeta,c(t)} \setminus M''^{\sigma}_{\zeta,c(t)}$ is zero, and m-dimensional Hausdorff measure of $L_{\zeta,c(t)} \setminus L''_{\zeta,c(t)}$ is also zero. Thus we can take a one parameter family of diffeomorphisms

$$G_t : M''^{\sigma}_{\zeta,c} \to M''^{\sigma}_{\zeta,c(t)},$$

for all $t \in I$, and G_t induces a one parameter family of diffeomorphisms

$$\tilde{G}_t : L''_{\zeta,c} \to L''_{\zeta,c(t)}$$

by $\tilde{G}_t(p,v) := (G_t(p),v)$. Then we have a one parameter family of maps $F : L''_{\zeta,c} \times I \to M$ by

$$F_t(p,v) := F_{\zeta,c(t)} \circ \tilde{G}_t(p,v) = \exp(v)\cdot G_t(p).$$

It is clear that

$$F_t(L''_{\zeta,c}) = F_{\zeta,c(t)}(\tilde{G}_t(L''_{\zeta,c})) = F_{\zeta,c(t)}(L''_{\zeta,c(t)}) \subset L'_{\zeta,c(t)},$$

where remember that

$$L'_{\zeta,c(t)} = \{\, \exp(v)\cdot p \mid v \in U,\, p \in M^{\sigma},\, \langle \mu(p), \zeta_j\rangle = c_j(t),\, j = 1,\ldots,n \,\}.$$

Since torus action is free on $M''^{\sigma}_{\zeta,c(t)}$, one can easily prove that F_t is embedding for all t, and m-dimensional Hausdorff measure of $L'_{\zeta,c(t)} \setminus F_t(L''_{\zeta,c})$ is zero.

Hence the remainder we have to prove is to prove that $F : L''_{\zeta,c}\times I \to M$ is a solution of a generalized Lagrangian mean curvature flow. Since both K_t and the normal part of $\partial F/\partial t$ are sections of normal bundle and $F_t : L''_{\zeta,c} \to M$ is a Lagrangian submanifold, it is enough to prove

$$\omega(\frac{\partial F}{\partial t}, F_{t*}Z) = \omega(K_t, F_{t*}Z) \tag{31}$$

for all tangent vectors Z on $L''_{\zeta,c}$ to prove the equation (29). Fix a point $x = (p,v)$ in $L''_{\zeta,c} = M''^{\sigma}_{\zeta,c}\times U$. Since we have a decomposition

$$T_x L''_{\zeta,c} = T_p M''^{\sigma}_{\zeta,c} \oplus T_v U$$

and note that $T_vU \cong V_\zeta$, a tangent vector Z is written by $Z = X + Y$ for some tangent vectors X in $T_pM''^{\sigma}_{\zeta,c}$ and Y in V_ζ. For X and Y, we have

$$F_{t*}X = \exp(v)_*(G_{t*}X) \quad \text{and} \quad F_{t*}Y = \exp(v)_*(Y_{G_t(p)}).$$

For X, we have

$$\omega(\frac{\partial F}{\partial t}, F_{t*}X) = \omega(\exp(v)_*(\frac{\partial G}{\partial t}), \exp(v)_*(G_{t*}X)) = \omega(\frac{\partial G}{\partial t}, G_{t*}X) = 0.$$

The second equality follows from the torus invariance of ω, and the third equality follows from that both $\partial G/\partial t$ and $G_{t*}X$ are tangent to real form and it is a Lagrangian. If we use the equation (28), we have

$$\omega(K_t, F_{t*}X) = \omega(J\nabla\theta_{F_t}, F_{t*}X) = -g(\nabla\theta_{F_t}, F_{t*}X) = -X\theta_{F_t} = 0,$$

since $\theta_{F_t}(p,v) = 2\pi\langle\gamma, v\rangle + \frac{\pi}{2}n$ by Theorem 13.2 and it is independent of $M''^{\sigma}_{\zeta,c}$ part. Thus the equation (31) holds for X. Next, for Y, we have

$$\begin{aligned}
\omega(\frac{\partial F}{\partial t}, F_{t*}Y) &= \omega(\frac{\partial G}{\partial t}, Y_{G_t(p)}) = \frac{\partial G}{\partial t}\langle\mu, Y\rangle = \frac{\partial}{\partial t}\langle\mu\circ G_t, Y\rangle \\
&= \frac{\partial}{\partial t}\langle\mu\circ G_t, a^1\zeta_1 + \cdots + a^n\zeta_n\rangle \\
&= \frac{\partial}{\partial t}(a^1c_1(t) + \cdots + a^nc_n(t)) \\
&= -2\pi\langle\gamma, Y\rangle.
\end{aligned}$$

The second equality follows from the assumption of the moment map μ. In the fourth equality we put $Y = a^1\zeta_1 + \cdots + a^n\zeta_n$ for some coefficients a^i and the fifth equality follows from the definition of $M''^{\sigma}_{\zeta,c(t)}$. In the last equality, remember that $c_i(t)$ is defined by $c_i(t) := c_i - 2\pi\langle\gamma,\zeta_i\rangle t$. If we use the equation (28), we have

$$\omega(K_t, F_{t*}Y) = \omega(J\nabla\theta_{F_t}, F_{t*}Y) = -g(\nabla\theta_{F_t}, F_{t*}Y) = -Y\theta_{F_t} = -2\pi\langle\gamma, Y\rangle.$$

Thus the equation (31) holds for Y and it is proved that $F : L''_{\zeta,c}\times I \to M$ is a solution of a generalized Lagrangian mean curvature flow. □

15 Examples

In this section, we give some examples of our main theorems. First we explain that if the ambient space M is $\mathbb{C}^m$ then our examples coincide with those constructed by Lee and Wang in [29].

Example 15.1. Let $(\mathbb{C}^m, \omega, g, J, \Omega)$ be a standard complex plane with a holomorphic volume form $\Omega = dw_1\wedge\cdots\wedge dw_m$ by the standard coordinates w. If we write $w_i = e^{z_i}$ where $w_i \neq 0$, then Ω is written by $\Omega = e^{z_1+\cdots+z_m}dz_1\wedge\cdots\wedge dz_m$. Hence we can take γ as $\gamma = (1,\ldots,1)$. A moment map is given by $\mu(w) = \frac{1}{2}(|w_1|^2,\ldots,|w_m|^2)$ and a moment polytope is given by

$$\Delta = \{\, y\in\mathbb{R}^m \mid \langle y,\lambda_i\rangle \geq 0,\, i = 1,\ldots,m \,\},$$

where $\lambda_i := e_i$, the i-th standard base, and then we have $\langle\gamma,\lambda_i\rangle = 1$ for all i. The real form of $\mathbb{C}^m$ is $\mathbb{R}^m$ and note that $\mathbb{R}^m$ can be constructed by gluing from 2^m-copies of Δ. Take one vector $\zeta = (\zeta_1,\ldots,\zeta_m)\in\mathbb{R}^m$ satisfying $\langle\gamma,\zeta\rangle > 0$ and $c = 0$. Since

$$c(t) = c - 2\pi\langle\gamma,\zeta\rangle t = -2\pi t\langle\gamma,\zeta\rangle = -2\pi t\sum_{j=1}^m \zeta_j$$

and $\Delta_{\zeta,c(t)} = \{\, y\in\Delta \mid \langle y,\zeta\rangle = c(t)\,\}$, we have

$$\begin{aligned}
M^{\sigma}_{\zeta,c(t)} &= (\mu|_{\mathbb{R}^m})^{-1}(\Delta_{\zeta,c(t)}) \\
&= \left\{\, x\in\mathbb{R}^m \;\middle|\; \sum_{j=1}^m \zeta_j x_j^2 = -4\pi t\sum_{j=1}^m\zeta_j \,\right\},
\end{aligned}$$

and $L'_{\zeta,c(t)}$, the image of $F_{\zeta,c(t)} : L_{\zeta,c}\to\mathbb{C}^m$, is given by

$$\begin{aligned}
L'_{\zeta,c(t)} = \Bigg\{\, (x_1e^{2\pi i\zeta_1 s},\ldots,x_me^{2\pi i\zeta_m s})\in\mathbb{C}^m \;\Bigg|\; & 0\leq s\leq 1, \\
& \sum_{j=1}^m\zeta_jx_j^2 = -4\pi t\sum_{j=1}^m\zeta_j,\; x = (x_1,\ldots,x_m)\in\mathbb{R}^m \,\Bigg\}.
\end{aligned}$$

This $L'_{\zeta,c(t)}$ coincides with V_t in Theorem 1.1 in [29], and Lee and Wang proved that V_t is Hamiltonian stationary and $\{V_t\}_{t\in\mathbb{R}}$ forms an eternal solution for Brakke flow. Hence our theorems can be considered as a kind of generalization of example of Lee and Wang to toric almost Calabi–Yau manifolds.

Example 15.2. Let $M = K_{\mathbb{P}^2}$ be the total space of the canonical line bundle of $\mathbb{P}^2$. Then a moment polytope is given by $\Delta = \{ y \in \mathbb{R}^3 \mid \langle y, \lambda_i \rangle \geq \kappa_i,\ i = 1, \dots, 4 \}$, where

$$\lambda_1 = (0,0,1),\ \lambda_2 = (1,0,1),\ \lambda_3 = (0,1,1),\ \lambda_4 = (-1,-1,1)$$

and $\kappa_1 = \kappa_2 = \kappa_3 = 0$, $\kappa_4 = -1$. Of course, M is a toric almost Calabi–Yau manifold since we can take $\gamma = (0,0,1)$ so that $\langle \gamma, \lambda_i \rangle = 1$ for all i. For example, take $n = 1$, and take one vector and one constant as

$$\zeta = (3,1,5) \quad \text{and} \quad c = 5.$$

Then $\Delta_{\zeta,c(t)}$ is written by

$$\Delta_{\zeta,c(t)} = \{ y \in \Delta \mid \langle y, \zeta \rangle = 5 - 10\pi t \},$$

since $c(t) = c - 2\pi\langle \gamma, \zeta \rangle t$ and $t \geq 0$. We write each facet of Δ by $F_i := \{ y \in \Delta \mid \langle y, \lambda_i \rangle = \kappa_i \}$ for $i = 1, 2, 3, 4$.

By simple calculation, one can easily see the following.

- On $0 \leq t < \frac{1}{5\pi}$, $\Delta_{\zeta,c(t)}$ intersects with F_2, F_3 and F_4, so $\Delta_{\zeta,c(t)}$ is a triangle.
- At $t = \frac{1}{5\pi}$, $\Delta_{\zeta,c(t)}$ across $(1,0,0)$, a vertex of Δ, and a topological change happens.
- On $\frac{1}{5\pi} < t < \frac{2}{5\pi}$, $\Delta_{\zeta,c(t)}$ intersects with F_1, F_2, F_3 and F_4, so $\Delta_{\zeta,c(t)}$ is a square.
- At $t = \frac{2}{5\pi}$, $\Delta_{\zeta,c(t)}$ across $(0,1,0)$, a vertex of Δ, and a topological change happens.
- On $\frac{2}{5\pi} < t < \frac{1}{2\pi}$, $\Delta_{\zeta,c(t)}$ intersects with F_1, F_2 and F_3, so $\Delta_{\zeta,c(t)}$ is a triangle.
- At $t = \frac{1}{2\pi}$, $\Delta_{\zeta,c(t)}$ is one point $\{(0,0,0)\}$, this means that $\Delta_{\zeta,c(t)}$ vanishes.

Hence a solution $\{L'_{\zeta,c(t)}\}_{t\in I}$ of a generalized Lagrangian mean curvature flow with singularities and topological changes exists for $t \in I = [0, \frac{1}{2\pi})$. It forms singularities and topological changes when $t = \frac{1}{5\pi}$ and $t = \frac{2}{5\pi}$, and vanishes when $t = \frac{1}{2\pi}$.

One can see the topology of $L_{\zeta,c(t)} = M^{\sigma}_{\zeta,c(t)} \times S^1$ (since now $T_\zeta \cong S^1$) by the same argument as explained in the proof of Proposition A.3 in [50]. In fact the topology of $M^{\sigma}_{\zeta,c(t)}$ is S^2 when $0 \leq t < \frac{1}{5\pi}$, is T^2 when $\frac{1}{5\pi} < t < \frac{2}{5\pi}$, is S^2 when $\frac{2}{5\pi} < t < \frac{1}{2\pi}$.

16 Appendix

In Section 13, we introduce the notion of the weighted Hamiltonian stationary. In this appendix, we explain the meaning of it. Let $(M, \omega, g, J, \Omega)$ be a $2m$-dimensional almost Calabi–Yau manifold with the function ψ defined by (27) and $F : L \to M$ be a Lagrangian immersion with the Lagrangian angle θ_F. Then we say that $F : L \to M$ is a weighted Hamiltonian stationary if $\Delta_f \theta_F = 0$. Here f is a function on L defined by $f := -mF^*\psi$ and Δ_f is the weighted Laplacian on Riemannian manifold (L, F^*g) defined by $\Delta_f u := \Delta u + \langle \nabla u, \nabla f \rangle$, where Δ is the standard Laplacian on L with respect to a metric F^*g.

Let $\tilde{g} := e^{2\psi} g$ be a conformal rescaling of g on M, then we get a new Riemannian manifold $(M, \tilde{g})$. For an immersion $F : L \to M$, we define a weighted volume functional Vol_ψ by

$$\mathrm{Vol}_\psi(F) := \int_L dV_{F^*\tilde{g}},$$

where $dV_{F^*\tilde{g}}$ is the volume form on L with respect to a metric $F^*\tilde{g}$. Note that the relation between $dV_{F^*\tilde{g}}$ and dV_{F^*g} is given by

$$dV_{F^*\tilde{g}} = e^{mF^*\psi} dV_{F^*g} = e^{-f} dV_{F^*g}.$$

Then we consider a symplectic manifold (M, ω) with the weighted volume functional Vol_ψ. The following proposition is the meaning of the weighted Hamiltonian stationary.

Proposition 16.1. *A Lagrangian immersion $F : L \to M$ is weighted Hamiltonian stationary if and only if F is a critical point of the weighted volume functional Vol_ψ along Hamiltonian deformations with respect to ω.*

Proof. Let $\{F_t : L \to M\}_t$ be a Hamiltonian deformation of F with Hamiltonian functions $\{h_t : L \to \mathbb{R}\}_t$, that is, $F_0 = F$ and

$$\omega(\frac{\partial F}{\partial t}, \cdot\,) = -dh_t. \tag{32}$$

If L is non-compact, we assume that each h_t has a compact support. Then the first variation of Vol_ψ at F along $\{F_t : L \to M\}_t$ is derived by the first variation formula as

$$\begin{aligned}
\frac{d}{dt}\bigg|_{t=0} \mathrm{Vol}_\psi(F_t) &= \frac{d}{dt}\bigg|_{t=0} \int_L e^{mF_t^*\psi} dV_{F_t^*g} \\
&= -\int_L g(e^{mF^*\psi}H - me^{mF^*\psi}\nabla\psi^\perp, \frac{\partial F}{\partial t}\bigg|_{t=0})dV_{F^*g} \\
&= -\int_L g(H - m\nabla\psi^\perp, \frac{\partial F}{\partial t}\bigg|_{t=0})e^{-f}dV_{F^*g}.
\end{aligned}$$

Next we remember the definition of the generalized mean curvature vector filed K, see (26), and use the equation (28), then we have

$$\begin{aligned}
-\int_L g(H - m\nabla\psi^\perp, \frac{\partial F}{\partial t}\bigg|_{t=0})e^{-f}dV_{F^*g} &= -\int_L g(K, \frac{\partial F}{\partial t}\bigg|_{t=0})e^{-f}dV_{F^*g} \\
&= -\int_L g(J\nabla\theta_F, \frac{\partial F}{\partial t}\bigg|_{t=0})e^{-f}dV_{F^*g}.
\end{aligned}$$

Since the equation (32) is equivalent to $\frac{\partial F}{\partial t} = J\nabla h_t$, we have

$$\begin{aligned}
-\int_L g(J\nabla\theta_F, \frac{\partial F}{\partial t}\bigg|_{t=0})e^{-f}dV_{F^*g} &= -\int_L g(J\nabla\theta_F, J\nabla h_0)e^{-f}dV_{F^*g} \\
&= -\int_L \langle d\theta_F, dh_0\rangle_{F^*g} e^{-f}dV_{F^*g} \\
&= -\int_L (\Delta_f\theta_F)h_0 e^{-f}dV_{F^*g} \\
&= -\int_L (\Delta_f\theta_F)h_0 dV_{F^*\tilde{g}}.
\end{aligned}$$

In the third equality, we use the another definition of $\Delta_f u = \delta_f(du)$, where δ_f is the formal adjoint of d with respect to a weighted measure $e^{-f}dV_{F^*g}$. One can easily show that $\delta_f(du) = \Delta u + \langle\nabla u, \nabla f\rangle_{F^*g}$. Now we can take any h_0, thus it is clear that the first variation of Vol_ψ at F along all Hamiltonian deformations is zero if and only if $\Delta_f\theta_F = 0$. $\square$

Part III

Ricci-mean curvature flows in gradient shrinking Ricci solitons

Abstract. Huisken [23] studied asymptotic behavior of a mean curvature flow in a Euclidean space when it develops a singularity of type I, and proved that its rescaled flow converges to a self-shrinker in the Euclidean space. In this Part, we generalize this result for a Ricci-mean curvature flow moving along a Ricci flow constructed from a gradient shrinking Ricci soliton.

17 Introduction

Let M and N be manifolds with dimension m and n respectively, satisfying $m \leq n$. Let $g = (\, g_t \,;\, t \in [0, T_1)\,)$ be a smooth 1-parameter family of Riemannian metrics on N and $F\colon M \times [0, T_2) \to N$ be a smooth 1-parameter family of immersions with $T_2 \leq T_1$, that is, $F_t\colon M \to N$ defined by $F_t(\,\cdot\,) := F(\cdot, t)$ is an immersion map. We say that the pair of g and F is a solution of the *Ricci-mean curvature flow* if it satisfies the following coupled equation of the Ricci flow and the mean curvature flow:

$$\frac{\partial g_t}{\partial t} = -2\mathrm{Ric}(g_t) \tag{33a}$$

$$\frac{\partial F_t}{\partial t} = H(F_t), \tag{33b}$$

where $H(F_t)$ denotes the mean curvature vector field of $F_t\colon M \to N$ computed by the ambient Riemannian metric g_t at the time t. Note that this coupling is partial, that is, the Ricci flow equation (33a) does not depend on F. It is clear that a Ricci-mean curvature flow is a mean curvature flow when the ambient Riemannian manifold (N, g_0) is Ricci flat (especially $(\mathbb{R}^n, g_{\mathrm{st}})$).

Huisken [23] studied asymptotic behavior of a mean curvature flow in a Euclidean space when it develops a singularity of type I, and proved that its rescaled flow converges to a self-shrinker in the Euclidean space. In this Part, we generalize this result to a Ricci-mean curvature flow moving along a Ricci flow constructed from a gradient shrinking Ricci soliton. Before stating our main results, we review the definition of self-similar solutions in $\mathbb{R}^n$ and the results due to Huisken [23].

On $\mathbb{R}^n$, we naturally identify a point $x = (x^1, \ldots, x^n) \in \mathbb{R}^n$ with a tangent vector $\overrightarrow{x} \in T_x\mathbb{R}^n$ by

$$\overrightarrow{x} := x^1 \frac{\partial}{\partial x^1} + \cdots + x^n \frac{\partial}{\partial x^n}.$$

For an immersion map $F\colon M \to \mathbb{R}^n$, we have a section $\overrightarrow{F} \in \Gamma(M, F^*(T\mathbb{R}^n))$ defined by $\overrightarrow{F}(p) := \overrightarrow{F(p)}$ for all $p \in M$. Then $F\colon M \to \mathbb{R}^n$ is called a self-similar solution if it satisfies

$$H(F) = \frac{\lambda}{2} \overrightarrow{F}^{\perp} \tag{34}$$

for some constant $\lambda \in \mathbb{R}$, where $\perp$ denotes the projection onto the normal bundle of M. A self-similar solution is called a self-expander, steady or self-shrinker when $\lambda > 0$, $\lambda = 0$ or $\lambda < 0$ respectively.

Let M be an m-dimensional compact manifold and $F\colon M \times [0, T) \to \mathbb{R}^n$ be a mean curvature flow with the maximal time $T < \infty$, that is, we can not extend the flow over the time T. Further assume that F satisfies the following two conditions (A1) and (B1).

(A1) The norm of the second fundamental form of F_t (denoted by $A(F_t)$) satisfies

$$\limsup_{t \to T} \bigl(\sqrt{T - t}\, \max_M |A(F_t)|\bigr) < \infty.$$

(B1) There exists a point p_0 in M such that $F_t(p_0) \to \mathrm{O} \in \mathbb{R}^n$ as $t \to T$.

If a mean curvature flow satisfies (A1), then we say that it develops a singularity of type I, and for the remaining case we say that it develops a singularity of type II. The condition (B1) guarantees that there exists at least one point in M such that its image of the rescaled flow remains in a bounded region in $\mathbb{R}^n$, thus the limiting submanifold is nonempty. In [23], it is also assumed that $|A(F_t)|(p_0) \to \infty$

as $t \to T$ for p_0 given in (B1). However, this assumption is not necessary to prove Theorem 17.1 introduced below.

For each $t \in (-\infty, T)$, let $\Phi_t \colon \mathbb{R}^n \to \mathbb{R}^n$ be a diffeomorphism of $\mathbb{R}^n$ defined by

$$\Phi_t(x) := \frac{1}{\sqrt{T-t}} x.$$

Define the rescaled flow $\tilde{F} \colon M \times [-\log T, \infty) \to \mathbb{R}^n$ by

$$\tilde{F}_s := \Phi_t \circ F_t \quad \text{with} \quad s = -\log(T-t).$$

Then it satisfies the *normalized mean curvature flow* equation:

$$\frac{\partial \tilde{F}_s}{\partial s} = H(\tilde{F}_s) + \frac{1}{2} \overrightarrow{\tilde{F}_s}.$$

Huisken proved the following (cf. Proposition 3.4 and Theorem 3.5 in [23]).

Theorem 17.1. *Under the assumptions (A1) and (B1), for each sequence $s_j \to \infty$, there exists a subsequence s_{j_k} such that the sequence of immersed submanifolds $\tilde{M}_{s_{j_k}} := \tilde{F}_{s_{j_k}}(M)$ converges smoothly to an immersed nonempty limiting submanifold $\tilde{M}_\infty \subset \mathbb{R}^n$, and $\tilde{M}_\infty$ is a self-shrinker with $\lambda = -1$ in (34).*

By this theorem, a self-shrinker can be considered as a local model of a singularity of type I for a mean curvature flow in $\mathbb{R}^n$.

On the other hand, there is also the notion of type I singularity for a Ricci flow $g = (\, g_t \,;\, t \in [0, T)\,)$ on a manifold N. Assume that $T < \infty$ is the maximal time. We say that g forms a singularity of type I if

$$\limsup_{t \to T} \big((T-t) \sup_N |\mathrm{Rm}(g_t)| \big) < \infty,$$

where $\mathrm{Rm}(g_t)$ denotes the Riemannian curvature tensor of g_t. In the Ricci flow case, a gradient shrinking Ricci soliton can be considered as a local model of a singularity of type I (cf. [13, 41, 43]). Actually, from a gradient shrinking Ricci soliton, we can construct a Ricci flow which develops a singularity of type I by the action of diffeomorphisms and scaling. In this Part, we consider a Ricci-mean curvature flow along this Ricci flow, and assume that the mean curvature flow and the Ricci flow develop singularities at the same time. Then we prove the convergence of the rescaled flow to a self-shrinker in the gradient shrinking Ricci soliton under the type I assumption (more precisely, under the assumption (A2) when N is compact, and (A2) and (B2) when N is non-compact). The precise settings and main results are the following.

Let $(N, \tilde{g}, \tilde{f})$ be an n-dimensional complete gradient shrinking Ricci soliton with

$$\mathrm{Ric}(\tilde{g}) + \mathrm{Hess}\, \tilde{f} - \frac{1}{2} \tilde{g} = 0. \tag{35}$$

As Hamilton's proof of Theorem 20.1 in [20], one can easily see that $R(\tilde{g}) + |\nabla \tilde{f}|^2 - \tilde{f}$ is a constant, where $R(\tilde{g})$ denotes the scalar curvature of $\tilde{g}$. Hence by adding some constant to $\tilde{f}$ if necessary, we may assume that the potential function $\tilde{f}$ also satisfies

$$R(\tilde{g}) + |\nabla \tilde{f}|^2 - \tilde{f} = 0. \tag{36}$$

For an immersion $F \colon M \to N$, we get a section $(\nabla \tilde{f}) \circ F \in \Gamma(M, F^*(TN))$, and we usually omit the symbol $\circ F$, for short.

Definition 17.2. If an immersion map $F \colon M \to N$ satisfies

$$H(F) = \lambda \nabla \tilde{f}^{\perp} \tag{37}$$

for some constant $\lambda \in \mathbb{R}$, we call it a self-similar solution. A self-similar solution is called a self-expander, steady or self-shrinker when $\lambda > 0$, $\lambda = 0$ or $\lambda < 0$ respectively.

Definition 17.3. If a 1-parameter family of immersions $\tilde{F} \colon M \times [0, S) \to N$ satisfies

$$\frac{\partial \tilde{F}_s}{\partial s} = H(\tilde{F}_s) + \nabla \tilde{f}, \tag{38}$$

we call it a normalized mean curvature flow.

Fix a positive time $0 < T < \infty$. Let $\{\Phi_t \colon N \to N\}_{t\in(-\infty,T)}$ be the 1-parameter family of diffeomorphisms with $\Phi_0 = \mathrm{id}_N$ generated by the time-dependent vector field $V_t := \frac{1}{T-t}\nabla\tilde{f}$. For $t \in (-\infty, T)$, define

$$g_t := (T-t)\Phi_t^*\tilde{g} \quad \text{and} \quad f_t := \Phi_t^*\tilde{f}.$$

Then g_t satisfies the Ricci flow equation (33a). Assume that $F \colon M \times [0,T) \to N$ is a solution of Ricci-mean curvature flow (33b) along this Ricci flow $g = (\, g_t \,;\, t \in [0,T)\,)$. We consider the following two conditions (A2) and (B2).

(A2) The norm of the second fundamental form of F_t (denoted by $A(F_t)$) satisfies

$$\limsup_{t\to T}\big(\sqrt{T-t}\,\max_M |A(F_t)|\big) < \infty.$$

(B2) There exists a point $p_0 \in M$ such that when $t \to T$

$$\ell_{F_t(p_0),t} \to f \quad \text{pointwise on } N \times [0,T),$$

where $f \colon N \times [0,T) \to \mathbb{R}$ is a function on $N \times [0,T)$ defined above and $\ell_{*,\bullet} \colon N \times [0,\bullet) \to \mathbb{R}$ is the reduced distance based at $(*,\bullet)$.

Remark 17.4. The condition (A2) corresponds to (A1). In (A2), note that $A(F_t)$ and its norm $|A(F_t)|$ are computed by the ambient metric g_t at each time t. In this Part, we do not assume that

$$\limsup_{t\to T}\sup_M |A(F_t)| = \infty. \tag{39}$$

If $F \colon M\times[0,T) \to N$ satisfies (39) and (A2), we say that F develops a singularity of type I. Hence (A2) is slightly weaker than the condition that F develops a singularity of type I. Especially, non-singular case (that is, $\limsup_{t\to T}\sup_M |A(F_t)| < \infty$) is contained in (A2).

Remark 17.5. The condition (B2) corresponds to (B1). In (B2), $\ell_{F_t(p_0),t}$ is the reduced distance for the Ricci flow g based at $(F_t(p_0), t)$ introduced by Perelman. Here we explain this briefly. Let (N, g_t) be a Ricci flow on $[0,T)$. For any curve $\gamma \colon [t_1,t_2] \to N$ with $0 \le t_1 < t_2 < T$, we define the $\mathcal{L}$-length of γ by

$$\mathcal{L}(\gamma) := \int_{t_1}^{t_2} \sqrt{t_2 - t}\big(R(g_t) + |\dot{\gamma}|^2\big)dt,$$

where $|\dot{\gamma}|$ is the norm of $\dot{\gamma}(t)$ measured by g_t. For a fixed point (p_2,t_2) in the space-time $N \times (0,T)$, we get the reduced distance

$$\ell_{p_2,t_2} \colon N \times [0,t_2) \to \mathbb{R}$$

based at (p_2,t_2) defined by

$$\ell_{p_2,t_2}(p_1,t_1) := \frac{1}{2\sqrt{t_2-t_1}}\inf_\gamma \mathcal{L}(\gamma),$$

where the infimum is taken over all curves $\gamma \colon [t_1,t_2] \to N$ with $\gamma(t_1) = p_1$ and $\gamma(t_2) = p_2$. In Remark 17.10, we see that (B1) and (B2) are equivalent when $(N,\tilde{g},\tilde{f})$ is the Gaussian soliton $(\mathbb{R}^n, g_{\mathrm{st}}, \frac{1}{4}|x|^2)$.

If $(N,\tilde{g},\tilde{f})$ is compact (resp. non-compact), we assume that F satisfies (A2) (resp. (A2) and (B2)). As in the Euclidean case, we consider the rescaled flow $\tilde{F} \colon M \times [-\log T, \infty) \to N$ defined by

$$\tilde{F}_s := \Phi_t \circ F_t \quad \text{with} \quad s = -\log(T-t), \tag{40}$$

and we can see that $\tilde{F}$ becomes a normalized mean curvature flow in $(N,\tilde{g},\tilde{f})$ (cf. Proposition 20.4). Then the main results in this Part are the following.

Theorem 17.6. *Assume that $(N,\tilde{g},\tilde{f})$ is compact. Let $F \colon M\times[0,T) \to N$ be a Ricci-mean curvature flow along the Ricci flow (N,g_t) defined by $g_t := (T-t)\Phi_t^*\tilde{g}$. Assume that M is compact and F satisfies (A2). Let $\tilde{F} \colon M \times [-\log T, \infty) \to N$ be defined by (40). Then, for any sequence $s_1 < s_2 < \cdots < s_j < \cdots \to \infty$ and points $\{x_j\}_{j=1}^\infty$ in M, there exist sub-sequences s_{j_k} and x_{j_k} such that the family of immersion maps $\tilde{F}_{s_{j_k}} \colon M \to N$ from pointed manifolds (M, x_{j_k}) converges to an immersion map $\tilde{F}_\infty \colon M_\infty \to N$ from some pointed manifold (M_∞, x_∞). Furthermore, M_∞ is a complete Riemannian manifold with metric $\tilde{F}_\infty^*\tilde{g}$ and $\tilde{F}_\infty$ is a self-shrinker in $(N,\tilde{g},\tilde{f})$ with $\lambda = -1$, that is, $\tilde{F}_\infty$ satisfies*

$$H(\tilde{F}_\infty) = -\nabla\tilde{f}^\perp.$$

Theorem 17.7. *Assume that $(N, \tilde{g}, \tilde{f})$ is non-compact and satisfies the assumption in Remark 17.8. Under the same setting in Theorem 17.6, assume that M is compact and F satisfies (A2) and (B2). Then, for any sequence of times s_j, the same statement as Theorem 17.6 holds, where we fix $x_j := p_0$ for all j.*

Remark 17.8. For a complete non-compact Riemannian manifold $(N, \tilde{g})$, we assume that there is an isometrically embedding $\Theta\colon N \to \mathbb{R}^L$ into some higher dimensional Euclidean space with

$$|\nabla^p A(\Theta)| \leq \tilde{D}_p < \infty$$

for some constants $\tilde{D}_p > 0$ for all $p \geq 0$. Under this assumption, one can see that $(N, \tilde{g})$ must have the bounded geometry by Theorem 23.5 and Gauss equation (71) (and its iterated derivatives).

Remark 17.9. The notion of the convergence of immersions from pointed manifolds is defined in Section 23 (cf. Definition 23.7). Roughly speaking, it is the immersion version of the Cheeger–Gromov convergence of pointed Riemannian manifolds.

Remark 17.10. We see that Theorem 17.7 implies Theorem 17.1 in $\mathbb{R}^n$. Consider $\mathbb{R}^n$ as the Gaussian soliton with potential function $\tilde{f}(x) := \frac{1}{4}|x|^2$. Since $\overrightarrow{x} = 2\nabla\tilde{f}(x)$, Definition 17.2 coincides with (34) in $\mathbb{R}^n$. It is trivial that $(\mathbb{R}^n, g_{\mathrm{st}})$ satisfies the assumption in Remark 17.8. We take $T = 1$ for simplicity. Then we have

$$\Phi_t(x) = \frac{1}{\sqrt{T-t}}x, \quad g_t \equiv g_{\mathrm{st}}, \quad f(x,t) = \frac{|x|^2}{4(T-t)}.$$

Since g_t is the trivial Ricci flow, the condition (A1) and (A2) coincides. Furthermore, one can easily see that in this trivial Ricci flow Perelman's reduced distance bases at $(*, \bullet)$ is given by

$$\ell_{*,\bullet}(x,t) := \frac{|x - *|^2}{4(\bullet - t)}.$$

Hence it is clear

$$\ell_{F_t(p_0),t} \to f \quad \text{pointwise on } \mathbb{R}^n \times [0, T)$$

when $F_t(p_0) \to \mathrm{O}$ as $t \to T$, that is, the condition (B1) implies (B2). Conversely, under the assumption (B2) we can see that $F_t(p_0) \to \mathrm{O}$ as $t \to T$ since

$$\frac{1}{4t}|F_t(p_0)|^2 = \ell_{F_t(p_0),t}(\mathrm{O}, 0) \to f(\mathrm{O}, 0) = 0$$

as $t \to T(< \infty)$. Hence (B1) and (B2) are equivalent in $\mathbb{R}^n$, and Theorem 17.7 implies Theorem 17.1.

Example 17.11. Here we consider compact examples of self-similar solutions embedded in compact gradient shrinking Ricci solitons. Let $(N, \tilde{g}, \tilde{f})$ be a compact gradient shrinking Ricci soliton. Then N itself and a critical point P (0-dimensional submanifold) of $\tilde{f}$ are trivially compact self-similar solutions, since $H = 0$ and $\nabla\tilde{f}^{\perp} = 0$. The next examples are given in *Kähler-Ricci solitons*. Let $(N, \tilde{g}, \tilde{f})$ be a compact gradient shrinking Kähler Ricci soliton. Let $M \subset N$ be a compact *complex* submanifold such that the gradient $\nabla\tilde{f}$ is tangent to M. Then M is a compact self-similar solution, since $H = 0$ (by a well-known fact that a complex submanifold in a Kähler manifold is minimal) and $\nabla\tilde{f}^{\perp} = 0$ on M. Actually, Cao [5] and Koiso [27] (for notations and assumptions, see [28]) constructed examples of compact gradient shrinking Kähler Ricci solitons. By their construction, each soliton is the total space of some complex $\mathbb{P}^1$-fibration and the gradient of the potential function is tangent to every $\mathbb{P}^1$-fiber. Hence each $\mathbb{P}^1$-fiber is a compact self-similar solution with real dimension 2.

Finally, we give some comments for Lagrangian self-similar solutions. For a Lagrangian immersion $F\colon L \to N$ in a Kähler manifold N with a Kähler form ω, a 1-form ω_H on L defined by $\omega_H(X) := \omega(H(F), F_*X)$ is called the mean curvature form. In Theorem 2.3.5 in [44], Smoczyk proved that there exists no compact Lagrangian self-similar solution with exact mean curvature form in $\mathbb{C}^n$. In his proof, it is proved that a compact Lagrangian self-similar solution with exact mean curvature form is a minimal submanifold in $\mathbb{C}^n$. However there exists no compact minimal submanifold in $\mathbb{C}^n$. Hence the assertion holds. As an analog of this theorem, we have the following theorem and its proof is given at the end of Section 20.

Theorem 17.12. *Let (N, g, f) be a gradient shrinking Kähler Ricci soliton and $F\colon L \to N$ be a compact Lagrangian self-similar solution with exact mean curvature form. Then $F\colon L \to N$ is a minimal Lagrangian immersion.*

Relation to previous literature. Recently there has been some studies in Ricci-mean curvature flows. One of main streams of the study is to generalize results established for mean curvature flows in Kähler-Einstein manifolds to Ricci-mean curvature flows along Kähler-Ricci flows. For example, some results for Lagrangian mean curvature flows can be generalized (cf. [21, 31]). Another main stream of the study is to generalize Huisken's monotonicity formula in $\mathbb{R}^n$ to Ricci-mean curvature flows along Ricci flows. In this direction, Lott considered a mean curvature flow in a gradient Ricci soliton in Section 5 in [32], and a certain kind of monotonicity formula is obtained in gradient steady soliton case. He also gave a definition of a self-similar solution for hypersurfaces in a gradient Ricci soliton. Our definition of a self-similar solution (cf. Definition 17.2) coincides with Lott's one for hypersurfaces. In Remark 5 in [32], he pointed out the existence of an analog of a monotonicity formula in gradient shrinking soliton case. Actually, a monotonicity formula for a mean curvature flow moving in a gradient shrinking Ricci soliton was also given by Magni, Mantegazza and Tsatis (cf. Proposition 3.1 in [33]) more directly. In this Part, we reintroduce their monotonicity formula in Section 20. There is also a generalization of Huisken's work to a mean curvature flow in a Riemannian cone manifold (cf. [14]).

Organization of this paper. The rest of this Part is organized as follows. In Section 18, we prove Theorem 17.6 and 17.7, after reviewing the proof of Theorem 17.1. In this proof, we use lemmas and propositions proved in the following sections and appendices. In Section 19, we introduce some general formulas for the first variation of a certain kind of weighted volume functional. In Section 20, we study some properties of Ricci-mean curvature flows along Ricci flows constructed from gradient shrinking Ricci solitons, and introduce the monotonicity formula. Furthermore, we prove the estimates for higher derivatives of the second fundamental forms of a rescaled flow and give an analog of Stone's estimate. In Section 21, we give a general treatment of evolution equations for tensors along Ricci-mean curvature flows. In Section 22, we give an estimate which is used in the proof of Lemma 20.10. In Section 23, we give a definition of convergence of immersion maps into a Riemannian manifold and prove some propositions.

18 Proofs of main theorems

In this section, we give proofs of Theorem 17.6 and 17.7. First of all, we review the proof of Theorem 17.1. The key results to prove Theorem 17.1 are the following (i), (ii) and (iii).

(i) The monotonicity formula for the weighted volume functional (cf. Theorem 3.1 and Corollary 3.2 in [23]).

Here the weighted volume functional is defined by

$$\int_{\tilde{M}} e^{-\frac{|x|^2}{4}} d\mu_{\tilde{M}}$$

for a submanifold $\tilde{M}$ in $\mathbb{R}^n$. This result corresponds to Proposition 20.5 and 20.6. For a submanifold $\tilde{M}$ (or immersion $\tilde{F}\colon M \to N$) in a gradient shrinking Ricci soliton $(N, \tilde{g}, \tilde{f})$, we consider the weighted volume functional $\int_M e^{-\tilde{f}} d\mu(\tilde{F}^*\tilde{g})$. The monotonicity formula decides the profile of the limiting submanifold $\tilde{M}_\infty$ if it exists.

(ii) Uniform estimates for all derivatives of second fundamental forms of $\tilde{M}_{s_j}$ (cf. Proposition 2.3 in [23]).

This result corresponds to Proposition 20.9. It is proved by the parabolic maximum principle for the evolution equation of $|\tilde{\nabla}^k \tilde{A}_s|^2$ and the argument of degree (it is explained in the proof of Proposition 20.9). This result implies the sub-convergence of $\tilde{M}_{s_j}$ to some limiting submanifold $\tilde{M}_\infty$.

(iii) A uniform estimate for the second derivative of the weighted volume functional. It is proved by Stone's estimate (cf. Lemma 2.9 in [47]) and the result (ii).

In this Part we prepare an analog of Stone's estimate in Lemma 20.7, and by combining Lemma 20.7 and Proposition 20.9 we prove Proposition 20.10 which is an analog of the result (iii). This result is necessary in the following sense. In general, if we know $\frac{d}{ds}\mathcal{F}(s) \leq 0$ for some smooth non-negative function $\mathcal{F}\colon [0,\infty) \to [0,\infty)$, we can say that $\mathcal{F}$ is monotone decreasing and converges to some value as $s \to \infty$. However we can not say that $\frac{d}{ds}\mathcal{F}(s_j) \to 0$ for any sequence $s_1 < s_2 < \cdots \to \infty$. If we further know that $|\frac{d^2}{d^2s}\mathcal{F}(s)| \leq C$ uniformly, then we can say that. In our situation, $\mathcal{F}(s)$ is the weighted volume of $\tilde{M}_s$. This argument is pointed out right before Lemma 3.2.7 in [34].

Proof of Theorem 17.6. First, we prove the existence of a smooth manifold M_∞ and a smooth map $\tilde{F}_\infty\colon M_\infty \to N$. Next, we show that this $\tilde{F}_\infty$ is a self-shrinker by using the monotonicity formula (55) in Proposition 20.6.

By Proposition 20.9, for all $k = 0, 1, 2, \dots$, there exist constants $C_k > 0$ such that

$$|\tilde{\nabla}^k A(\tilde{F}_s)| \leq C_k \quad \text{on} \quad M \times [-\log T, \infty).$$

Since N is compact, by Theorem 23.9, we get a sub-sequence j_k, a pointed manifold (M_∞, x_∞) and an immersion map $\tilde{F}_\infty\colon M_\infty \to N$ with a complete Riemannian metric $F_\infty^*\tilde{g}$ on M_∞ such that $\tilde{F}_{s_{j_k}}\colon (M, x_{j_k}) \to N$ converges to $\tilde{F}_\infty\colon (M_\infty, x_\infty) \to N$ in the sense of Definition 23.7 as $k \to \infty$. We denote $\tilde{F}_{s_{j_k}}$ by $\tilde{F}_k$ for short. Then, there exist an exhaustion $\{U_k\}_{k=1}^\infty$ of M_∞ with $x_\infty \in U_k$ and a sequence of diffeomorphisms $\Psi_k\colon U_k \to V_k := \Psi_k(U_k) \subset M$ with $\Psi_k(x_\infty) = x_{j_k}$ such that $\Psi_k^*(\tilde{F}_k^*\tilde{g})$ converges in C^∞ to $\tilde{F}_\infty^*\tilde{g}$ uniformly on compact sets in M_∞, and furthermore the sequence of maps $\tilde{F}_k \circ \Psi_k\colon U_k \to N$ converges in C^∞ to $F_\infty\colon M_\infty \to N$ uniformly on compact sets in M_∞.

Let $K \subset M_\infty$ be any compact set. Then we will prove that

$$\int_K \left| H(\tilde{F}_\infty) + \nabla \tilde{f}^{\perp_{\tilde{F}_\infty}} \right|_{\tilde{g}}^2 e^{-\tilde{f}\circ\tilde{F}_\infty} d\mu(\tilde{F}_\infty^*\tilde{g}) = 0.$$

It is clear that this implies that $\tilde{F}_\infty\colon M_\infty \to N$ satisfies

$$H(\tilde{F}_\infty) = -\nabla \tilde{f}^{\perp_{\tilde{F}_\infty}}$$

on M_∞, where $\perp_{\tilde{F}_\infty}$ denotes the normal projection with respect to $\tilde{F}_\infty$. Its proof is the following. For K, there exists k_0 such that $K \subset U_k$ for all $k \geq k_0$. Since $\tilde{F}_k \circ \Psi_k\colon U_k \to N$ converges to $F_\infty\colon M_\infty \to N$ in C^∞ uniformly on K for $k \geq k_0$, we have

$$\begin{aligned} &\int_K \left| H(\tilde{F}_k \circ \Psi_k) + \nabla \tilde{f}^{\perp_{\tilde{F}_k\circ\Psi_k}} \right|_{\tilde{g}}^2 e^{-\tilde{f}\circ(\tilde{F}_k\circ\Phi_k)} d\mu((\tilde{F}_k \circ \Phi_k)^*\tilde{g}) \\ \to &\int_K \left| H(\tilde{F}_\infty) + \nabla \tilde{f}^{\perp_{\tilde{F}_\infty}} \right|_{\tilde{g}}^2 e^{-\tilde{f}\circ\tilde{F}_\infty} d\mu(\tilde{F}_\infty^*\tilde{g}) \end{aligned} \tag{41}$$

as $k \to \infty$. Since $\Psi_k\colon U_k \to V_k \subset M$ is a diffeomorphism, it is clear that

$$\begin{aligned} &\int_K \left| H(\tilde{F}_k \circ \Psi_k) + \nabla \tilde{f}^{\perp_{\tilde{F}_k\circ\Psi_k}} \right|_{\tilde{g}}^2 e^{-\tilde{f}\circ(\tilde{F}_k\circ\Psi_k)} d\mu((\tilde{F}_k \circ \Psi_k)^*\tilde{g}) \\ = &\int_{\Psi_k(K)} \left| H(\tilde{F}_k) + \nabla \tilde{f}^{\perp_{\tilde{F}_k}} \right|_{\tilde{g}}^2 e^{-\tilde{f}\circ\tilde{F}_k} d\mu(\tilde{F}_k^*\tilde{g}) \\ \leq &\int_M \left| H(\tilde{F}_k) + \nabla \tilde{f}^{\perp_{\tilde{F}_k}} \right|_{\tilde{g}}^2 e^{-\tilde{f}\circ\tilde{F}_k} d\mu(\tilde{F}_k^*\tilde{g}). \end{aligned} \tag{42}$$

By using the monotonicity formula (55) and Lemma 20.10, one can prove that

$$\int_M \left| H(\tilde{F}_k) + \nabla \tilde{f}^{\perp_{\tilde{F}_k}} \right|_{\tilde{g}}^2 e^{-\tilde{f}\circ\tilde{F}_k} d\mu(\tilde{F}_k^*\tilde{g}) \to 0 \tag{43}$$

as $k \to \infty$ by the argument of contradiction. Actually, assume that there exist a constant $\delta > 0$ and a subsequence $\{\ell\} \subset \{k\}$ with $\ell \to \infty$ such that

$$\int_M \left| H(\tilde{F}_\ell) + \nabla \tilde{f}^{\perp_{\tilde{F}_\ell}} \right|_{\tilde{g}}^2 e^{-\tilde{f}\circ\tilde{F}_\ell} d\mu(\tilde{F}_\ell^*\tilde{g}) \geq \delta.$$

Then one can easily see that

$$\int_M \left| H(\tilde{F}_s) + \nabla \tilde{f}^{\perp_{\tilde{F}_s}} \right|_{\tilde{g}}^2 e^{-\tilde{f}\circ\tilde{F}_s} d\mu(\tilde{F}_s^*\tilde{g}) \geq \frac{\delta}{2},$$

for $s \in [s_\ell, s_\ell + \frac{\delta}{2C'}]$, where we used Lemma 20.10 and C' is the constant appeared in that lemma. Hence we have that

$$\int_{-\log T}^\infty \int_M \left| H(\tilde{F}_s) + \nabla \tilde{f}^{\perp_{\tilde{F}_s}} \right|_{\tilde{g}}^2 e^{-\tilde{f}\circ\tilde{F}_s} d\mu(\tilde{F}_s^*\tilde{g}) ds = \infty.$$

On the other hand, by the monotonicity formula (55):

$$\frac{d}{ds}\int_M e^{-\tilde f\circ\tilde F}\,d\mu(\tilde F^*\tilde g)=-\int_M\left|H(\tilde F)+\nabla\tilde f^{\perp_{\tilde F}}\right|^2_{\tilde g}e^{-\tilde f\circ\tilde F}d\mu(\tilde F^*\tilde g)\le 0,$$

the weighted volume

$$\int_M e^{-\tilde f\circ\tilde F_s}d\mu(\tilde F_s^*\tilde g)$$

is monotone decreasing and non-negative, thus it converges to some value

$$\alpha:=\lim_{s\to\infty}\int_M e^{-\tilde f\circ\tilde F_s}d\mu(\tilde F_s^*\tilde g)<\infty.$$

Hence we have

$$\int_{-\log T}^{\infty}\int_M\left|H(\tilde F_s)+\nabla\tilde f^{\perp_{\tilde F_s}}\right|^2_{\tilde g}e^{-\tilde f\circ\tilde F_s}d\mu(\tilde F_s^*\tilde g)=-\alpha+\int_M e^{-\tilde f\circ\tilde F_\bullet}d\mu(\tilde F_\bullet^*\tilde g)<\infty,$$

where $\bullet=-\log T$. This is a contradiction. Thus, by combining (41)-(43), it follows that

$$\int_K\left|H(\tilde F_\infty)+\nabla\tilde f^{\perp_{\tilde F_\infty}}\right|^2_{\tilde g}e^{-\tilde f\circ\tilde F_\infty}d\mu(\tilde F_\infty^*\tilde g)=0.$$

Here we completed the proof. □

Next, we give the proof of the non-compact version of the above theorem.

Proof of Theorem 17.7. We will prove that $\tilde F_{s_j}(p_0)$ is a bounded sequence in $(N,\tilde g)$. For any t_1, t_2 with $0\le t_1<t_2<T$, we can take $\{F_t(p_0)\}_{t\in[t_1,t_2]}$ as a curve joining $F_{t_1}(p_0)$ and $F_{t_2}(p_0)$. Hence we have

$$\begin{aligned}\ell_{F_{t_2}(p_0),t_2}(F_{t_1}(p_0),t_1)\le&\frac{1}{2\sqrt{t_2-t_1}}\int_{t_1}^{t_2}\sqrt{t_2-t}\left(R(g_t)+\left|\frac{\partial F_t}{\partial t}\right|^2\right)dt\\=&\frac{1}{2\sqrt{t_2-t_1}}\int_{t_1}^{t_2}\sqrt{t_2-t}\big(R(g_t)+|H(F_t)|^2\big)dt\end{aligned}$$

By the assumption (A2), $(T-t)|H(F_t)|^2$ is bounded, and it is clear that $(T-t)R(g_t)=R(g_0)$ and it is also bounded by the assumption in Remark 17.8. Hence we have $R(g_t)+|H(F_t)|^2\le\frac{C}{T-t}$ for some $C>0$ and

$$\begin{aligned}\ell_{F_{t_2}(p_0),t_2}(F_{t_1}(p_0),t_1)\le&\frac{C}{2\sqrt{t_2-t_1}}\int_{t_1}^{t_2}\frac{\sqrt{t_2-t}}{T-t}dt\\\le&\frac{C}{2\sqrt{t_2-t_1}}\int_{t_1}^{t_2}\frac{1}{\sqrt{T-t}}dt\\\le&C\frac{\sqrt{T-t_1}}{\sqrt{t_2-t_1}}.\end{aligned}$$

By the assumption (B2), by taking the limit as $t_2\to T$, we have

$$f(F_{t_1}(p_0),t_1)\le C.$$

Since $f(F_t(p_0),t)=f_t(F_t(p_0))=\tilde f(\tilde F_s(p_0))$, the above bound means that

$$\tilde f(\tilde F_s(p_0))\le C$$

on $s\in[-\log T,\infty)$. In [7] (cf, Theorem 1.1), Cao and Zhou proved that there exist positive constants C_1 and C_2 such that

$$\frac14(r-C_1)^2\le\tilde f\le\frac14(r+C_2)^2$$

on N, where $r(q)=d_{\tilde g}(q_0,q)$ is the distance function from some fixed point q_0 in N. Hence we have

$$d_{\tilde g}(q_0,\tilde F_s(p_0))\le 2\sqrt{C}+C_1,$$

that is, $\tilde F_s(p_0)$ moves in a bounded region in N. Hence we can use Theorem 23.9 with a bounded sequence $\tilde F_{s_j}(p_0)$. Then the remainder part of the proof is completely same as the proof of the case that N is compact. □

19 Monotonicity formulas

In this section, we introduce some general formulas which are useful in the following sections and appendices. Let M and N be manifolds with dimension m and n respectively, and assume that $m \leq n$ and M is compact. We denote the space of all immersion maps from M to N by $\mathfrak{Imm}(M, N)$ and the space of all Riemannian metrics on N by $\mathfrak{Met}(N)$. Consider the following functional:

$$\begin{aligned} \mathcal{F} \colon C^\infty(M) \times \mathfrak{Imm}(M, N) \times C^\infty(N)_{>0} \times \mathfrak{Met}(N) \to \mathbb{R} \\ \mathcal{F}(u, F, \rho, g) := \int_M u\, F^*\!\rho \, d\mu(F^* g)\,. \end{aligned} \tag{44}$$

Here u is a smooth function on M and ρ is a positive smooth function on N. First of all, we remark some elementary symmetric properties associated with $\mathcal{F}$. Here we denote diffeomorphism groups of M and N by $\mathrm{Diff}(M)$ and $\mathrm{Diff}(N)$ respectively.

Remark 19.1. For $\varphi \in \mathrm{Diff}(M)$ and $\psi \in \mathrm{Diff}(N)$, we have

$$\mathcal{F}(\varphi^* u, \psi^{-1} \circ F \circ \varphi, \psi^* \rho, \psi^* g) = \mathcal{F}(u, F, \rho, g),$$

and for a positive constant $\lambda > 0$ we have

$$\mathcal{F}(\lambda^{n-m} u, F, \lambda^{-n} \rho, \lambda^2 g) = \mathcal{F}(u, F, \rho, g).$$

Let $p := (u, F, \rho, g)$ be a point in $C^\infty(M) \times \mathfrak{Imm}(M, N) \times C^\infty(N)_{>0} \times \mathfrak{Met}(N)$ and $v := (w, V, k, h)$ be a tangent vector of $C^\infty(M) \times \mathfrak{Imm}(M, N) \times C^\infty(N)_{>0} \times \mathfrak{Met}(N)$ at p. Namely, $w \in C^\infty(M)$, $V \in \Gamma(M, F^*(TN))$, $k \in C^\infty(N)$ and $h \in \mathrm{Sym}^2(N)$. Then we consider the first variation of $\mathcal{F}$ at p in the direction v, denoted by $\delta_v \mathcal{F}(p)$.

Proposition 19.2. *We have*

$$\begin{aligned} \delta_v \mathcal{F}(p) = &- \int_M u\, g(V + \nabla f^{\perp_F}, H(F) + \nabla f^{\perp_F})\, F^*\!\rho \, d\mu(F^* g) \\ &+ \int_M u\, F^* \left(\Delta_g \rho + k + \frac{1}{2} \rho \operatorname{tr} h \right) d\mu(F^* g) \\ &+ \int_M \Bigg(w - \Delta_{F^* g} u - g(V, F_* \nabla u) \\ &\qquad\qquad + u \operatorname{tr}^{\perp_F} \left(\operatorname{Hess} f - \frac{1}{2} h \right) \Bigg) F^*\!\rho \, d\mu(F^* g), \end{aligned} \tag{45}$$

where we define f by $\rho = (4\pi\tau)^{-\frac{n}{2}} e^{-f}$ for a positive function $\tau = \tau(t)$ (which depends only on t) and $H(F)$ is the mean curvature vector field of immersion F from M to a Riemannian manifold (N, g).

Remark 19.3. Here, note that there is an ambiguity of a choice of a function τ, but the gradient and Hessian of f do not depend on the choice of τ.

Notation 19.4. By $\perp_F$, we denote the normal projection with respect to the orthogonal decomposition

$$F^*(TN) = F_*(TM) \oplus T^{\perp_F} M$$

defined by the immersion F, and by $\operatorname{tr}^{\perp_F}$ we denote the normal trace, that is, for a 2-tensor η on N and a point $p \in M$, $(\operatorname{tr}^{\perp_F} \eta)(p)$ is defined by

$$(\operatorname{tr}^{\perp_F} \eta)(p) := \sum_{j=1}^{n-m} \eta(F(p))(\nu_j, \nu_j),$$

where $\{\nu_j\}_{j=1}^{n-m}$ is an orthonormal basis of $T_p^{\perp_F} M$.

Proof. Let $\{F_s : M \to N\}_{s \in (-\epsilon, \epsilon)}$ be a smooth 1-parameter family of immersions with

$$F_0 = F \quad \text{and} \quad \left.\frac{\partial F_s}{\partial s}\right|_{s=0} = V.$$

Let $u_s := u + sw$, $\rho_s := \rho + sk$ and $g_s := g + sh$. Then $p_s := (u_s, F_s, \rho_s, g_s)$ is a curve in $C^\infty(M) \times \mathfrak{Imm}(M, N) \times C^\infty(N)_{>0} \times \mathfrak{Met}(N)$ with $p_0 = p$ and $\dot{p}_0 = v$. Then the first variation of $\mathcal{F}$ at p in the direction v is calculated as

$$\delta_v \mathcal{F}(p) = \frac{d}{ds}\bigg|_{s=0} \mathcal{F}(p_s) = \frac{d}{ds}\bigg|_{s=0} \int_M u_s\, F_s^* \rho_s \, d\mu(F_s^* g_s),$$

and we have

$$\begin{aligned}
&\frac{d}{ds}\bigg|_{s=0} \int_M u_s\, F_s^* \rho_s \, d\mu(F_s^* g_s) \\
&= \int_M w\, F^*\rho \, d\mu(F^* g) + \int_M u\, g(V, \nabla\rho)\, d\mu(F^* g) + \int_M u\, F^* k \, d\mu(F^* g) \\
&\quad + \int_M u\, F^*\rho \bigg(\frac{d}{ds}\bigg|_{s=0} d\mu(F_s^* g) \bigg) + \int_M u\, F^*\rho \bigg(\frac{d}{ds}\bigg|_{s=0} d\mu(F^* g_s) \bigg).
\end{aligned} \tag{46}$$

It is well-known that the first variation of the induced measure $d\mu(F_s^* g)$ is given by

$$\frac{d}{ds}\bigg|_{s=0} d\mu(F_s^* g) = \{ \mathrm{div}_{F^* g}\, F_*^{-1}(V^{\top_F}) - g(H(F), V) \} d\mu(F^* g).$$

On the right hand side of the above equation, we decompose V as $V = V^{\top_F} + V^{\perp_F} \in F_*(TM) \oplus T^{\perp_F} M$, and we take the divergence of $F_*^{-1}(V^{\top_F})$ on a Riemannian manifold $(M, F^* g)$.

On the other hand, $F^* g_s$ is a time-dependent metric on M. Since $g_s = g + sh$, we have $F^* g_s = F^* g + s F^* h$. Thus, the derivation of $F^* g_s$ is $F^* h$ at $s = 0$. In such a situation, it is also well-known that the first variation of the induced measure $d\mu(F^* g_s)$ of a time-dependent metric on M is given by

$$\frac{d}{ds}\bigg|_{s=0} d\mu(F^* g_s) = \frac{1}{2}\, \mathrm{tr}(F^* h) d\mu(F^* g),$$

where the trace is taken with respect to a metric $F^* g$ on M. By the divergence formula on $(M, F^* g)$, we have

$$\begin{aligned}
&\int_M u\, F^*\rho \, \mathrm{div}_{F^* g}\, F_*^{-1}(V^{\top_F}) d\mu(F^* g) \\
&= -\int_M (F^* g)(F_*^{-1}(V^{\top_F}), \nabla(u\, F^*\rho)) d\mu(F^* g) \\
&= -\int_M g(V, F_* \nabla u)\, F^*\rho \, d\mu(F^* g) - \int_M u\, g(V, \nabla\rho^{\top_F})\, d\mu(F^* g).
\end{aligned}$$

Since $\nabla\rho = -\rho \nabla f$, we have

$$\begin{aligned}
&\int_M u\, g(V, \nabla\rho)\, d\mu(F^* g) + \int_M u\, F^*\rho \bigg(\frac{d}{ds}\bigg|_{s=0} d\mu(F_s^* g) \bigg) \\
&= -\int_M g(V, F_* \nabla u)\, F^*\rho \, d\mu(F^* g) - \int_M u\, g(V, H(F) + \nabla f^{\perp_F})\, F^*\rho \, d\mu(F^* g).
\end{aligned} \tag{47}$$

It is clear that

$$\mathrm{tr}(F^* h) = F^*(\mathrm{tr}\, h) - \mathrm{tr}^{\perp_F} h.$$

Hence we have

$$\begin{aligned}
&\int_M u\, F^* k \, d\mu(F^* g) + \int_M u\, F^*\rho \bigg(\frac{d}{ds}\bigg|_{s=0} d\mu(F^* g_s) \bigg) \\
&= \int_M u F^* \bigg(k + \frac{1}{2}\rho\, \mathrm{tr}\, h \bigg) d\mu(F^* g) - \int_M \frac{1}{2} u\, F^*\rho (\mathrm{tr}^{\perp_F} h) d\mu(F^* g).
\end{aligned}$$

Furthermore, one can easily see that

$$\begin{aligned}
F^*(\Delta_g \rho) &= \Delta_{F^* g}(F^*\rho) - g(H(F), \nabla\rho) + \mathrm{tr}^{\perp_F}(\mathrm{Hess}\, \rho) \\
&= \Delta_{F^* g}(F^*\rho) + F^*\rho\, g(\nabla f^{\perp_F}, H(F) + \nabla f^{\perp_F}) - F^*\rho\, \mathrm{tr}^{\perp_F}(\mathrm{Hess}\, f).
\end{aligned}$$

Hence we have

$$
\begin{aligned}
&\int_M u\, F^*k\, d\mu(F^*g) + \int_M u\, F^*\!\rho\bigg(\frac{d}{ds}\bigg|_{s=0} d\mu(F^*g_s)\bigg)\\
=&\int_M uF^*\bigg(\Delta_g\rho + k + \frac{1}{2}\rho\operatorname{tr} h\bigg)d\mu(F^*g) - \int_M \frac{1}{2}u\, F^*\!\rho(\operatorname{tr}^{\perp_F} h)d\mu(F^*g)\\
&-\int_M uF^*(\Delta_g\rho)d\mu(F^*g)\\
=&\int_M u\, F^*\bigg(\Delta_g\rho + k + \frac{1}{2}\rho\operatorname{tr} h\bigg)d\mu(F^*g)\\
&-\int_M u\, g(\nabla f^{\perp_F}, H(F) + \nabla f^{\perp_F})\, F^*\!\rho\, d\mu(F^*g)\\
&+\int_M \bigg(-\Delta_{F^*g}u + u\operatorname{tr}^{\perp_F}(\operatorname{Hess} f - \frac{1}{2}h)\bigg)\, F^*\!\rho\, d\mu(F^*g),
\end{aligned}
\tag{48}
$$

where we used

$$
\int_M u\Delta_{F^*g}(F^*\!\rho)d\mu(F^*g) = \int_M (\Delta_{F^*g}u)\, F^*\!\rho\, d\mu(F^*g).
$$

Finally, by combining equations (46)-(48), we get the formula (45). □

By using this general formula (45), we get the following monotonicity formula for Ricci-mean curvature flows.

Proposition 19.5. *Assume that the pair $g = (\, g_t\, ;\, t\in[0,T_1)\,)$ and $F\colon M\times[0,T_2)\to N$ is a solution of Ricci-mean curvature flow with $T_2\le T_1$, that is, these satisfy (33a) and (33b). Further assume that a smooth positive function $\rho\colon N\times[0,T_1)\to\mathbb{R}^+$ on N and a smooth non-negative function $u\colon M\times[0,T_2)\to\mathbb{R}$ on M satisfy the following coupled equations:*

$$
\frac{\partial\rho_t}{\partial t} = -\Delta_{g_t}\rho_t + R(g_t)\rho_t \tag{49a}
$$

$$
\frac{\partial u_t}{\partial t} = \Delta_{F_t^*g_t}u_t - u_t\operatorname{tr}^{\perp_{F_t}}\big(\operatorname{Ric}(g_t) + \operatorname{Hess} f_t\big), \tag{49b}
$$

where we define f by $\rho = (4\pi\tau)^{-\frac{n}{2}}e^{-f}$ for a positive function $\tau=\tau(t)$. Then we have, for all $t\in(0,T_2)$,

$$
\frac{d}{dt}\mathcal{F}(u_t,F_t,\rho_t,g_t) = -\int_M u_t\Big|H(F_t)+\nabla f_t^{\perp_{F_t}}\Big|^2_{g_t} F_t^*\rho_t d\mu(F_t^*g_t) \le 0. \tag{50}
$$

Proof. Since g_t is a solution of the Ricci flow (33a), $h = -2\mathrm{Ric}(g_t)$ in the equation (45) of Proposition 19.2. Furthermore, $V = H(F_t)$ in this case, and $g(V, F_{t*}\nabla u_t) = 0$ since $V(=H(F_t))$ is a normal vector and $F_{t*}\nabla u_t$ is a tangent vector. Then, the equality (50) is clear by Proposition 19.2. □

Remark 19.6. The equation (49a) is called the conjugate heat equation for the Ricci flow, and the equation (49b) is a linear heat equation with time-dependent potential $\operatorname{tr}^{\perp_{F_t}}\big(\operatorname{Ric}(g_t)+\operatorname{Hess} f_t\big)$ on M.

Proposition 19.7. *Assume that $\tau(t) = T - t$. Let $u\colon M\times[0,T)\to\mathbb{R}$ be a solution for (49b). Define $v\colon M\times[0,T)\to\mathbb{R}$ by $u = (4\pi\tau)^{\frac{n-m}{2}}v$. Then v satisfies*

$$
\frac{\partial v}{\partial t} = \Delta_{F^*g}v - v\operatorname{tr}^{\perp_F}(\operatorname{Ric} + \operatorname{Hess} f - \frac{g}{2\tau}), \tag{49b$'$}
$$

and the converse is also true.

Proof. We have

$$
\begin{aligned}
&\frac{\partial u}{\partial t} - \Delta_{F^*g}u + u\operatorname{tr}^{\perp_F}(\operatorname{Ric} + \operatorname{Hess} f)\\
=&-\frac{n-m}{2\tau}u + (4\pi\tau)^{\frac{n-m}{2}}\frac{\partial v}{\partial t} - \Delta_{F^*g}u + u\operatorname{tr}^{\perp_F}(\operatorname{Ric} + \operatorname{Hess} f)\\
=&(4\pi\tau)^{\frac{n-m}{2}}\bigg(\frac{\partial v}{\partial t} - \Delta_{F^*g}v + v\operatorname{tr}^{\perp_F}(\operatorname{Ric} + \operatorname{Hess} f - \frac{g}{2\tau})\bigg).
\end{aligned}
$$

Thus, the equivalence is clear. □

Example 19.8. If the ambient space is a Euclidean space, that is, $(N, g) = (\mathbb{R}^n, g_{\mathrm{st}})$, we can reduce Huisken's monotonicity formula from (50). Let M be an m dimensional compact manifold and $F\colon M \times [0, T) \to \mathbb{R}^n$ be a mean curvature flow. Fix a point $y_0 \in \mathbb{R}^n$. Let $\rho\colon \mathbb{R}^n \times [0, T) \to \mathbb{R}^+$ be the standard backward heat kernel on $\mathbb{R}^n$ at (y_0, T), that is, ρ is defined by

$$\rho(y, t) := \frac{1}{(4\pi(T-t))^{\frac{n}{2}}} e^{-\frac{|y-y_0|^2}{4(T-t)}}.$$

Of course, ρ satisfies the backward heat equation (49a) with $R = 0$. In this case, since f is $|y - y_0|^2/(4(T-t))$, we have

$$\mathrm{Hess}\, f = \frac{g_{\mathrm{st}}}{2(T-t)} \quad \text{and} \quad \mathrm{tr}^{\perp}(\mathrm{Hess}\, f) = \frac{n-m}{2(T-t)}.$$

Thus one can easily see that $u\colon M \times [0, T) \to \mathbb{R}$ defined by

$$u(p, t) := (4\pi(T-t))^{\frac{n-m}{2}}$$

is the non-negative solution of (49b) with initial condition $u(\cdot, 0) = (4\pi T)^{\frac{n-m}{2}}$. Hence by Theorem 19.5 we have

$$\frac{d}{dt}\mathcal{F}(u_t, F_t, \rho_t, g_{\mathrm{st}}) = -\int_M u_t \left| H(F_t) + \nabla f_t^{\perp_{F_t}} \right|^2 F_t^* \rho_t d\mu(F_t^* g_{\mathrm{st}}).$$

By definitions, we have

$$\begin{aligned}\mathcal{F}(u_t, F_t, \rho_t, g_{\mathrm{st}}) &= \int_M u_t\, F_t^* \rho_t \, d\mu(F_t^* g_{\mathrm{st}}) \\ &= \int_M \frac{1}{(4\pi(T-t))^{\frac{m}{2}}} e^{-\frac{|F_t - y_0|^2}{4(T-t)}} d\mu(F_t^* g_{\mathrm{st}})\end{aligned}$$

and

$$\nabla f_t(F_t(p)) = \frac{\overrightarrow{F_t}(p) - \overrightarrow{y_0}}{2(T-t)}$$

at $p \in M$. Then we get Huisken's monotonicity formula:

$$\begin{aligned}&\frac{d}{dt}\int_M \frac{1}{(4\pi(T-t))^{\frac{m}{2}}} e^{-\frac{|F_t - y_0|^2}{4(T-t)}} d\mu(F_t^* g_{\mathrm{st}}) \\ &= -\int_M \frac{1}{(4\pi(T-t))^{\frac{m}{2}}} e^{-\frac{|F_t - y_0|^2}{4(T-t)}} \left| H(F_t) + \frac{(\overrightarrow{F_t}(p) - \overrightarrow{y_0})^{\perp_{F_t}}}{2(T-t)} \right|^2 d\mu(F_t^* g_{\mathrm{st}}) \le 0.\end{aligned}$$

20 mean curvature flows in gradient shrinking Ricci solitons

In this section, we recall some definitions and properties of gradient shrinking Ricci solitons and self-similar solutions (cf. Definition 17.2), and prove the monotonicity formula for a Ricci-mean curvature flow along a Ricci flow constructed from a gradient shrinking Ricci soliton and also prove an analog of Stone's estimate.

Recall that if an n-dimensional Riemannian manifold $(N, \tilde{g})$ and a function $\tilde{f}$ on N satisfies the equation (35):

$$\mathrm{Ric}(\tilde{g}) + \mathrm{Hess}\, \tilde{f} - \frac{1}{2}\tilde{g} = 0,$$

it is called a gradient shrinking Ricci soliton. In this Part we assume that $(N, \tilde{g})$ is a complete Riemannian manifold. Then by the result due to Zhang [54], it follows that $\nabla \tilde{f}$ is a complete vector field on N. As Theorem 20.1 in Hamilton's paper [20], one can easily see that $R(\tilde{g}) + |\nabla \tilde{f}|^2 - \tilde{f}$ is a constant. Hence by adding some constant to $\tilde{f}$ if necessary, we can assume that the potential function $\tilde{f}$ satisfying (35) also satisfy the equation (36):

$$R(\tilde{g}) + |\nabla \tilde{f}|^2 - \tilde{f} = 0.$$

As a special case of a more general result for complete ancient solutions by Chen [8] (cf. Corollary 2.5), we can see that $(N, \tilde{g}, \tilde{f})$ must have the nonnegative scalar curvature $R(\tilde{g}) \ge 0$. Hence we have

$$0 \le |\nabla \tilde{f}|^2 \le \tilde{f} \quad \text{and} \quad 0 \le R(\tilde{g}) \le \tilde{f}.$$

Fix a positive time $T > 0$ arbitrary. Let $\{\Phi_t\colon N \to N\}_{t\in(-\infty,T)}$ be the 1-parameter family of diffeomorphisms with $\Phi_0 = \mathrm{id}_N$ generated by the time-dependent vector field $V(t) := \frac{1}{T-t}\nabla\tilde{f}$. For $t \in (-\infty, T)$, define

$$g_t := (T-t)\Phi_t^*\tilde{g}, \quad f_t := \Phi_t^*\tilde{f}, \quad \rho_t := (4\pi(T-t))^{-\frac{n}{2}}e^{-f_t}.$$

Then by the standard calculation, one can prove the following (cf. [40]).

Proposition 20.1. *g is the solution of the Ricci flow, $\frac{\partial g}{\partial t} = -2\mathrm{Ric}$, on the time interval $(-\infty, T)$ with $g_0 = T\tilde{g}$, and ρ and f satisfy the following equations:*

$$\frac{\partial \rho}{\partial t} = -\Delta_g\rho + R(g)\rho \tag{51}$$

$$\mathrm{Ric}(g) + \mathrm{Hess}\, f - \frac{g}{2(T-t)} = 0. \tag{52}$$

$$R(g) + |\nabla f|^2 - \frac{f}{T-t} = 0. \tag{53}$$

Recall that an immersion map $F\colon M \to N$ is called a self-similar solution if it satisfies the equation (37):

$$H(F) = \lambda\nabla\tilde{f}^{\perp},$$

and it is called shrinking when $\lambda < 0$, steady when $\lambda = 0$ and expanding when $\lambda > 0$. A self-similar solution corresponds to a minimal submanifold in a conformal rescaled ambient space. The precise statement is the following.

Proposition 20.2. *Let $F\colon M \to N$ be an immersion map in a gradient shrinking Ricci soliton $(N, \tilde{g}, \tilde{f})$. Then the following two conditions are equivalent.*

1. *F is a self-similar solution with coefficient λ.*
2. *F is a minimal immersion with respect to a metric $e^{2\lambda\tilde{f}/m}\tilde{g}$ on N.*

Here m is the dimension of M.

Proof. One can easily see that in general if we denote the mean curvature vector field of F in $(N, \tilde{g})$ by $H(F)$ then the mean curvature vector field in the conformal rescaling $(N, e^{2\varphi}\tilde{g})$ is given by

$$e^{-2\varphi}(H(F) - m\nabla\varphi^{\perp}).$$

Hence, by putting $\varphi := \lambda\tilde{f}/m$, the equivalence is clear. □

From a self-shrinker, we can construct a solution of Ricci-mean curvature flow canonically.

Proposition 20.3. *Let $\tilde{F}\colon M \to N$ be a self-shrinker with $\lambda = -1$. For a fixed time $T > 0$, let Φ_t and g_t be defined as above, and define $\Psi_t := \Phi_t^{-1}$. Then $F\colon M\times[0,T) \to N$ defined by $F(p,t) := \Psi_t(\tilde{F}(p))$ satisfies*

$$\left(\frac{\partial F}{\partial t}\right)^{\perp} = H(F_t),$$

in the Ricci flow (N, g_t) defined on $t \in [0, T)$, that is, F becomes a solution of the Ricci-mean curvature flow in (N, g_t) up to a time-dependent re-parametrization of M.

Proof. By differentiating the identity $\Phi_t \circ \Psi_t = \mathrm{id}_N$, we have

$$\frac{1}{T-t}\nabla\tilde{f} + \Phi_{t*}\left(\frac{\partial\Psi_t}{\partial t}\right) = 0.$$

Hence we can see that

$$\frac{\partial\Psi_t}{\partial t} = -\Psi_{t*}\left(\frac{1}{T-t}\nabla\tilde{f}\right).$$

Since $H(\tilde{F}) = -\nabla \tilde{f}^{\perp}$, more precisely $H(\tilde{F}) = -\nabla \tilde{f}^{\perp_{\tilde{F},\tilde{g}}}$ (note that the notion of the normal projection depends on an immersion map and an ambient metric), we have

$$\begin{aligned}\left(\frac{\partial F}{\partial t}\right)^{\perp_{F_t,g_t}} =&\left(-\Psi_{t*}\left(\frac{1}{T-t}\nabla \tilde{f}\right)\right)^{\perp_{F_t,g_t}}\\ =&-\Psi_{t*}\left(\frac{1}{T-t}\nabla \tilde{f}^{\perp_{\tilde{F},\tilde{g}}}\right)\\ =&\frac{1}{T-t}\Psi_{t*}(H(\tilde{F}))\\ =&H(F_t),\end{aligned}$$

where $H(\tilde{F})$ is the mean curvature vector field with respect to the metric $\tilde{g}$ and $H(F_t)$ is the one with respect to the metric g_t. $\square$

There exists a one to one correspondence between Ricci-mean curvature flows in (N, g_t) and normalized mean curvature flows (cf. Definition 17.3) in $(N, \tilde{g})$.

Proposition 20.4. *For a fixed time $T > 0$, let Φ_t and g_t be defined as above. If $F\colon M\times[0,T) \to N$ is a Ricci-mean curvature flow along the Ricci flow (N, g_t), then the rescaled flow $\tilde{F}\colon M\times[-\log T,\infty) \to N$ defined by the equation (40):*

$$\tilde{F}_s := \Phi_t \circ F_t \quad \text{with} \quad s = -\log(T-t)$$

for $s \in [-\log T, \infty)$ becomes a normalized mean curvature flow in $(N, \tilde{g})$, that is, it satisfies

$$\frac{\partial \tilde{F}}{\partial s} = H(\tilde{F}) + \nabla \tilde{f}.$$

Conversely, if $\tilde{F}\colon M \times [-\log T, \infty) \to N$ is a normalized mean curvature flow in $(N, \tilde{g})$, then the flow $F\colon M \times [0, T) \to N$ defined by (40) becomes a Ricci-mean curvature flow along the Ricci flow (N, g_t).

Proof. By differentiating $\tilde{F}$, we have

$$\frac{\partial \tilde{F}}{\partial s} = \nabla \tilde{f} + (T-t)\Phi_{t*}\left(\frac{\partial F}{\partial t}\right).$$

Furthermore, it is clear that

$$(T-t)\Phi_{t*}(H(F_t)) = H(\tilde{F}_s).$$

Hence, the correspondence between Ricci-mean curvature flows along (N, g_t) and normalized mean curvature flows in $(N, \tilde{g})$ is clear. $\square$

Here the monotonicity formula for a Ricci-mean curvature flow moving along the Ricci flow (N, g_t) is almost clear by Proposition 19.5.

Proposition 20.5. *For a fixed time $T > 0$, let g_t, f_t, and ρ_t be defined as above, and define $u_t := (4\pi(T-t))^{\frac{n-m}{2}}$. If $F\colon M \times [0, T) \to N$ is a Ricci-mean curvature flow along the Ricci flow (N, g_t) and M is compact, then we have the monotonicity formula:*

$$\frac{d}{dt}\int_M u\, F^{*}\!\rho\, d\mu(F^*g) = -\int_M u\left|H(F) + \nabla f^{\perp_F}\right|_g^2 F^{*}\!\rho\, d\mu(F^*g) \le 0. \tag{54}$$

Proof. By Proposition 20.1, we see that ρ satisfies the conjugate heat equation (49a). To see that u satisfies the equation (49b), we use the equivalent equation (49b$'$). In this case, by Proposition 20.1, the equation (49b$'$) becomes

$$\frac{\partial v}{\partial t} = \Delta_{F^*g} v,$$

the standard heat equation on M, where u and v are related by $u = (4\pi(T-t))^{\frac{n-m}{2}} v$. Then $v \equiv 1$ is a trivial solution of (49b$'$). Hence $u_t = (4\pi(T-t))^{\frac{n-m}{2}}$ becomes a solution of (49b). Thus, by Proposition 19.5, we have the above monotonicity formula (54). $\square$

By Proposition 20.5, we can deduce the following monotonicity formula of the weighted volume functional for a normalized mean curvature flow, immediately.

Proposition 20.6. *If $\tilde{F}\colon M\times[-\log T,\infty)\to N$ is a normalized mean curvature flow in $(N,\tilde{g},\tilde{f})$ and M is compact, then we have the monotonicity formula:*

$$\frac{d}{ds}\int_M e^{-\tilde{f}\circ\tilde{F}}\,d\mu(\tilde{F}^*\tilde{g})=-\int_M\Big|H(\tilde{F})+\nabla\tilde{f}^{\perp_{\tilde{F}}}\Big|^2_{\tilde{g}}e^{-\tilde{f}\circ\tilde{F}}d\mu(\tilde{F}^*\tilde{g})\leq 0. \tag{55}$$

Proof. In this proof, we follow the notations in Proposition 20.4. It is clear that $f_t\circ F_t=\tilde{f}\circ\tilde{F}_s$ and $F_t^*g_t=(T-t)\tilde{F}_s^*\tilde{g}$. Hence we have

$$\begin{aligned}u_t\,F_t^*\rho_t\,d\mu(F_t^*g_t)=&(4\pi(T-t))^{-\frac{m}{2}}e^{-f_t\circ F_t}\,d\mu(F_t^*g_t)\\=&(4\pi)^{-\frac{m}{2}}e^{-\tilde{f}\circ\tilde{F}_s}\,d\mu(\tilde{F}_s^*\tilde{g})\,.\end{aligned}$$

Since $H(\tilde{F}_s)=(T-t)\Phi_{t*}H(F_t)$ and $\nabla\tilde{f}=(T-t)\Phi_{t*}\nabla f_t$, we have

$$(T-t)\Big|H(F_t)+\nabla f_t^{\perp_{F_t}}\Big|^2_{g_t}=\Big|H(\tilde{F}_s)+\nabla\tilde{f}^{\perp_{\tilde{F}_s}}\Big|^2_{\tilde{g}}.$$

Thus, by the equality (54), one can easily see that the equality (55) holds. □

To prove the main theorems, we need the following key lemma. Its proof is an analog of the proof of Stone's estimate (cf. Lemma 2.9 in [47]). Stone considered the weight $e^{-\sqrt{\tilde{f}}}$ in the Euclidean case, where $\tilde{f}:=|x|^2/4$. However we consider the weight $e^{-\frac{\tilde{f}}{2}}$ here, since $-\frac{\tilde{f}}{2}$ is a smooth function and we can apply Proposition 19.2.

Lemma 20.7. *Assume that $(N,\tilde{g})$ has bounded geometry. If $\tilde{F}\colon M\times[-\log T,\infty)\to N$ is a normalized mean curvature flow in $(N,\tilde{g},\tilde{f})$ and M is compact, then there exists a constant $C>0$ such that*

$$\int_M e^{-\frac{\tilde{f}}{2}\circ\tilde{F}}\,d\mu(\tilde{F}^*\tilde{g})\leq C. \tag{56}$$

uniformly on $[-\log T,\infty)$.

Proof. In this proof, we follow the notations in Proposition 20.4. As the proof of Proposition 20.6, we have

$$\int_M e^{-\frac{\tilde{f}}{2}\circ\tilde{F}_s}\,d\mu(\tilde{F}_s^*\tilde{g})=(4\pi)^{\frac{m}{2}}\int_M u_t\,F_t^*\bar{\rho}_t\,d\mu(F_t^*g_t),$$

where

$$\bar{\rho}_t:=\frac{1}{(4\pi(T-t))^{\frac{n}{2}}}e^{-\frac{f_t}{2}}\quad\text{and}\quad u_t:=(4\pi(T-t))^{\frac{n-m}{2}}.$$

By Proposition 19.2, we have

$$\begin{aligned}&\frac{d}{dt}\int_M u_t\,F_t^*\bar{\rho}_t\,d\mu(F_t^*g_t)\\=&-\int_M u_t\Big|H(F_t)+\frac{1}{2}\nabla f_t^{\perp_{F_t}}\Big|^2_{g_t}F_t^*\bar{\rho}_t\,d\mu(F_t^*g_t)\\&+\int_M u_tF_t^*\left(\frac{\partial\bar{\rho}_t}{\partial t}+\Delta_{g_t}\bar{\rho}_t-R(g_t)\bar{\rho}_t\right)d\mu(F_t^*g_t)\\&+\int_M\left(\frac{\partial u_t}{\partial t}-\Delta_{F_t^*g_t}u_t+u_t\,\mathrm{tr}^{\perp_{F_t}}\left(\frac{1}{2}\operatorname{Hess}f_t+\operatorname{Ric}(g_t)\right)\right)F_t^*\bar{\rho}_t\,d\mu(F_t^*g_t)\,.\end{aligned}$$

By using $\frac{\partial f}{\partial t}=|\nabla f|^2$ and $|\nabla f|^2=\frac{f}{T-t}-R(g)$, we have

$$\frac{\partial\bar{\rho}}{\partial t}=\bar{\rho}\bigg(\frac{n}{2(T-t)}-\frac{f}{2(T-t)}+\frac{1}{2}R(g)\bigg).$$

By using $\Delta_g f=-R(g)+\frac{n}{2(T-t)}$ and $|\nabla f|^2=\frac{f}{T-t}-R(g)$, we have

$$\Delta_g\bar{\rho}=\bar{\rho}\bigg(\frac{f}{4(T-t)}+\frac{1}{4}R(g)-\frac{n}{4(T-t)}\bigg).$$

Hence we have

$$\frac{\partial\bar{\rho}}{\partial t}+\Delta_g\bar{\rho}-R(g)\bar{\rho}=\bar{\rho}\bigg(\frac{n}{4(T-t)}-\frac{f}{4(T-t)}-\frac{1}{4}R(g)\bigg)\leq\frac{\bar{\rho}}{4(T-t)}(n-f).$$

Furthermore, since u satisfies

$$\frac{\partial u_t}{\partial t} - \Delta_{F_t^* g_t} u_t + u_t \operatorname{tr}^{\perp_{F_t}} (\operatorname{Hess} f_t + \operatorname{Ric}(g_t)) = 0,$$

we have

$$\frac{\partial u_t}{\partial t} - \Delta_{F_t^* g_t} u_t + u_t \operatorname{tr}^{\perp_{F_t}} \left(\frac{1}{2} \operatorname{Hess} f_t + \operatorname{Ric}(g_t)\right) = -\frac{1}{2} u_t \operatorname{tr}^{\perp_{F_t}} \operatorname{Hess} f_t.$$

By using $\operatorname{Hess} f_t = \frac{1}{2(T-t)} g_t - \operatorname{Ric}(g_t)$, we have

$$-\frac{1}{2} u_t \operatorname{tr}^{\perp_{F_t}} \operatorname{Hess} f_t = u_t \left(-\frac{n-m}{4(T-t)} + \frac{1}{2} \operatorname{tr}^{\perp_{F_t}} \operatorname{Ric}(g_t)\right).$$

It is clear that

$$\operatorname{tr}^{\perp_{F_t}} \operatorname{Ric}(g_t) \leq (n-m) |\operatorname{Ric}(g_t)|_{g_t} = (n-m) \frac{|\operatorname{Ric}(\tilde{g})|_{\tilde{g}}}{T-t} \leq \frac{C''}{T-t},$$

where $C'' := (n-m) \max_N |\operatorname{Ric}(\tilde{g})|_{\tilde{g}}$ is a bounded constant since $(N, \tilde{g})$ has bounded geometry. Hence we have

$$\frac{d}{dt} \int_M u_t \, F_t^* \bar{\rho}_t \, d\mu(F_t^* g_t) < \frac{1}{4(T-t)} \int_M \Big(C_0 - f_t \circ F_t\Big) u_t \, F_t^* \bar{\rho}_t \, d\mu(F_t^* g_t),$$

where $C_0 := m + 4C'' + 1$. Since $s = -\log(T-t)$, we have

$$\frac{d}{ds} \int_M e^{-\frac{\tilde{f}}{2} \circ \tilde{F}_s} \, d\mu(\tilde{F}_s^* \tilde{g}) = (4\pi)^{\frac{m}{2}} (T-t) \frac{d}{dt} \int_M u_t \, F_t^* \bar{\rho}_t \, d\mu(F_t^* g_t).$$

Hence we have

$$\frac{d}{ds} \int_M e^{-\frac{\tilde{f}}{2} \circ \tilde{F}_s} \, d\mu(\tilde{F}_s^* \tilde{g}) < \frac{1}{4} \int_M \Big(C_0 - \tilde{f} \circ \tilde{F}_s\Big) e^{-\frac{\tilde{f}}{2} \circ \tilde{F}_s} \, d\mu(\tilde{F}_s^* \tilde{g}).$$

Here we divide M into time-dependent three pieces as follows:

$$\begin{aligned} M_{1,s} &:= \tilde{F}_s^{-1}(\{\tilde{f} \leq C_0\}), \\ M_{2,s} &:= \tilde{F}_s^{-1}(\{C_0 < \tilde{f} \leq 2C_0\}), \\ M_{3,s} &:= \tilde{F}_s^{-1}(\{2C_0 < \tilde{f}\}). \end{aligned}$$

On each component, we have

$$\begin{aligned} &\int_{M_{1,s}} \Big(C_0 - \tilde{f} \circ \tilde{F}_s\Big) e^{-\frac{\tilde{f}}{2} \circ \tilde{F}_s} \, d\mu(\tilde{F}_s^* \tilde{g}) \leq C_0 \int_{M_{1,s}} e^{-\frac{\tilde{f}}{2} \circ \tilde{F}_s} \, d\mu(\tilde{F}_s^* \tilde{g}), \\ &\int_{M_{2,s}} \Big(C_0 - \tilde{f} \circ \tilde{F}_s\Big) e^{-\frac{\tilde{f}}{2} \circ \tilde{F}_s} \, d\mu(\tilde{F}_s^* \tilde{g}) \leq 0 \\ &\int_{M_{3,s}} \Big(C_0 - \tilde{f} \circ \tilde{F}_s\Big) e^{-\frac{\tilde{f}}{2} \circ \tilde{F}_s} \, d\mu(\tilde{F}_s^* \tilde{g}) \leq -C_0 \int_{M_{3,s}} e^{-\frac{\tilde{f}}{2} \circ \tilde{F}_s} \, d\mu(\tilde{F}_s^* \tilde{g}). \end{aligned}$$

Thus we have

$$\begin{aligned} &\frac{d}{ds} \int_M e^{-\frac{\tilde{f}}{2} \circ \tilde{F}_s} \, d\mu(\tilde{F}_s^* \tilde{g}) \\ <&\frac{C_0}{4} \left(\int_{M_{1,s}} e^{-\frac{\tilde{f}}{2} \circ \tilde{F}_s} \, d\mu(\tilde{F}_s^* \tilde{g}) - \int_{M_{3,s}} e^{-\frac{\tilde{f}}{2} \circ \tilde{F}_s} \, d\mu(\tilde{F}_s^* \tilde{g})\right). \end{aligned} \tag{57}$$

On the other hand, by the monotonicity formula (cf. Proposition 20.6), we have

$$\int_M e^{-\tilde{f} \circ \tilde{F}_s} \, d\mu(\tilde{F}_s^* \tilde{g}) \leq C',$$

where C' is the value of the left hand side at the initial time $s = -\log T$. We further define a region in M by

$$M_{4,s} := \tilde{F}_s^{-1}(\{\tilde{f} \leq 2C_0\}) = M_{1,s} \cup M_{2,s}.$$

Since $e^{-\frac{\tilde{f}}{2}} = e^{\frac{\tilde{f}}{2}} e^{-\tilde{f}} \leq e^{\frac{C_0}{2}} e^{-\tilde{f}}$ on $M_{1,s}$, we have

$$\int_{M_{1,s}} e^{-\frac{\tilde{f}}{2} \circ \tilde{F}_s} \, d\mu(\tilde{F}_s^* \tilde{g}) \leq e^{\frac{C_0}{2}} \int_M e^{-\tilde{f} \circ \tilde{F}_s} \, d\mu(\tilde{F}_s^* \tilde{g}) \leq e^{\frac{C_0}{2}} C' =: C_1.$$

As on $M_{1,s}$, we have

$$\int_{M_{4,s}} e^{-\frac{\tilde{f}}{2}\circ \tilde{F}_s}\, d\mu(\tilde{F}_s^*\tilde{g}) \leq e^{C_0}C' =: C_2. \tag{58}$$

Hence, by the inequality (57), we see that for each $s \in [-\log T, \infty)$ we must have either

$$\frac{d}{ds}\int_M e^{-\frac{\tilde{f}}{2}\circ \tilde{F}_s}\, d\mu(\tilde{F}_s^*\tilde{g}) < 0 \quad \text{or} \quad \int_{M_{3,s}} e^{-\frac{\tilde{f}}{2}\circ \tilde{F}_s}\, d\mu(\tilde{F}_s^*\tilde{g}) \leq C_1.$$

Since $M = M_{3,s} \cup M_{4,s}$ and we have the bound (58), we see that for each $s \in [-\log T, \infty)$ we must have either

$$\frac{d}{ds}\int_M e^{-\frac{\tilde{f}}{2}\circ \tilde{F}_s}\, d\mu(\tilde{F}_s^*\tilde{g}) < 0 \quad \text{or} \quad \int_{M} e^{-\frac{\tilde{f}}{2}\circ \tilde{F}_s}\, d\mu(\tilde{F}_s^*\tilde{g}) \leq C_1 + C_2.$$

This condition implies that

$$\int_{M} e^{-\frac{\tilde{f}}{2}\circ \tilde{F}_s}\, d\mu(\tilde{F}_s^*\tilde{g}) \leq \max\{C_1 + C_2, C_3\} =: C,$$

where C_3 is the value of the left hand side at the initial time $s = -\log T$. □

Remark 20.8. In Section 17, we consider the condition (A2):

$$\limsup_{t\to T}(\sqrt{T-t}\sup_M |A(F_t)|_{g_t}) < \infty,$$

for a Ricci-mean curvature flow $F\colon M \times [0,T) \to N$ along the Ricci flow g_t. Note that if M is compact then this condition is equivalent to that there exists a constant $C_0 > 0$ such that

$$\max_M |A(F_t)|_{g_t} \leq \frac{C_0}{\sqrt{T-t}} \quad \text{on} \quad [0,T).$$

Proposition 20.9. *Let $(N, \tilde{g}, \tilde{f})$ be a gradient shrinking Ricci soliton with bounded geometry. For a fixed time $T > 0$, let Φ_t and g_t be defined as above, and let $F\colon M \times [0,T) \to N$ be a Ricci-mean curvature flow along the Ricci flow (N, g_t). Assume that M is compact and F satisfies the condition (A2). Let $\tilde{F}$ be the normalized mean curvature flow defined by (40). Then, for all $k = 0, 1, 2, \dots,$ there exist constants $C_k > 0$ such that*

$$|\tilde{\nabla}^k A(\tilde{F}_s)|_{\tilde{g}} \leq C_k \quad \text{on} \quad M \times [-\log T, \infty),$$

where $\tilde{\nabla}$ is the connection defined by the Levi–Civita connection on $(N, \tilde{g})$ and the one on $(M, \tilde{F}_s^*\tilde{g})$.

Proof. First of all, by the definitions of $g_t = (T-t)\Phi_t^*\tilde{g}$ and $\tilde{F}_s = \Phi_t \circ F_t$, one can easily see that

$$\tilde{\nabla}^k A(\tilde{F}_s) = \Phi_{t*}\nabla^k A(F_t),\ |\tilde{\nabla}^k A(\tilde{F}_s)|_{\tilde{g}} = (T-t)^{\frac{1}{2}+\frac{1}{2}k}|\nabla^k A(F_t)|_{g_t}, \tag{59}$$

$$|\tilde{\nabla}^k \mathrm{Rm}(\tilde{g})|_{\tilde{g}} = (T-t)^{1+\frac{1}{2}k}|\nabla^k \mathrm{Rm}(g_t)|_{g_t} \tag{60}$$

for all $k = 0, 1, 2, \dots,$ where $\tilde{\nabla}$ is the connection defined by the Levi–Civita connection on $(N, \tilde{g})$ and the one on $(M, \tilde{F}_s^*\tilde{g})$, and ∇ is the connection defined by the Levi–Civita connection on (N, g_t) and the one on $(M, F_t^* g_t)$. In this sense, as Huisken done in [23], we can consider the degree of $\nabla^k A(F_t)$ is $\frac{1}{2} + \frac{1}{2}k$ and the degree of $\nabla^k \mathrm{Rm}(g_t)$ is $1 + \frac{1}{2}k$. We will write $A(F_t)$ and $A(\tilde{F}_s)$ by A and $\tilde{A}$ respectively, and also write $\mathrm{Rm}(g_t)$ and $\mathrm{Rm}(\tilde{g})$ by Rm and $\widetilde{\mathrm{Rm}}$ respectively, for short. To use the argument of degree more rigorously, we define a set $V_{a,b}$ and a vector space $\mathcal{V}_{a,b}$ as follows. First, we recall the notion of $*$-product here. For tensors T_1 and T_2, we write $T_1 * T_2$ to mean a tensor formed by a sum of terms each one of them obtained by contracting some indices of the pair T_1 and T_2 by using g, F^*g and these inverses, and there is a property that

$$|T_1 * T_2| \leq C|T_1||T_2|, \tag{61}$$

where $C > 0$ is a constant which depends only on the algebraic structure of $T_1 * T_2$. Then, for $a, b \in \mathbb{N}$, we define a set $V_{a,b}$ as the set of all (time-dependent) tensors T on M which can be expressed as

$$T = (\nabla^{k_1}\mathrm{Rm} * \cdots * \nabla^{k_I}\mathrm{Rm}) * (\nabla^{\ell_1} A * \cdots * \nabla^{\ell_J} A) * (\overset{p}{*} DF)$$

with $I, J, p, k_1, \dots, k_I, \ell_1, \dots, \ell_J \in \mathbb{N}$ satisfying

$$\sum_{i=1}^{I}\left(1 + \frac{1}{2}k_i\right) + \sum_{j=1}^{J}\left(\frac{1}{2} + \frac{1}{2}\ell_j\right) = a \quad \text{and} \quad \sum_{j=1}^{J}\ell_j \leq b,$$

and we define a vector space $\mathcal{V}_{a,b}$ as the set of all tensors T on M which can be expressed as

$$T = a_1 T_1 + \cdots + a_r T_r$$

for some $r \in \mathbb{N}$, $a_1 \dots a_r \in \mathbb{R}$ and $T_1, \dots, T_r \in V_{a,b}$.

For the case $k = 0$, as noted in Remark 20.8, there exists a constant $C_0 > 0$ such that

$$|A| \leq \frac{C_0}{\sqrt{T-t}} \quad \text{on} \quad M \times [0, T),$$

since F satisfies the condition (A2). Hence we have

$$|\tilde{A}| = \sqrt{T-t}|A| \leq C_0.$$

For the case $k \geq 0$, we work by induction on $k \in \mathbb{N}$. The case $k = 0$ has already proved above. For a fixed $k \geq 1$, assume that there exist positive constants $C_0, C_1, \dots, C_{k-1}$ such that

$$|\tilde{\nabla}^i \tilde{A}| \leq C_i \quad \text{on} \quad M \times [-\log T, \infty)$$

for $i = 0, 1, \dots, k-1$. We consider the evolution equation of $|\tilde{\nabla}^k \tilde{A}|^2$, and finally we will prove the bound of $|\tilde{\nabla}^k \tilde{A}|^2$ by the parabolic maximum principle. Since $|\tilde{\nabla}^k \tilde{A}|^2 = (T-t)^{k+1}|\nabla^k A|^2$ and $\frac{\partial}{\partial s} = (T-t)\frac{\partial}{\partial t}$, we have that

$$\begin{aligned}
\frac{\partial}{\partial s}|\tilde{\nabla}^k \tilde{A}|^2 &= -(k+1)|\tilde{\nabla}^k \tilde{A}|^2 + (T-t)^{k+2}\frac{\partial}{\partial t}|\nabla^k A|^2 \\
&\leq (T-t)^{k+2}\frac{\partial}{\partial t}|\nabla^k A|^2.
\end{aligned}$$

By Proposition 21.19, there exist tensors $\mathcal{E}[k] \in \mathcal{V}_{\frac{3}{2}+\frac{1}{2}k,k}$, $\mathcal{C}[k] \in \mathcal{V}_{\frac{3}{2}+\frac{1}{2}k,k+1}$ and $\mathcal{G}[k] \in \mathcal{V}_{\frac{1}{2}+\frac{1}{2}k,k-1}$ such that

$$\frac{\partial}{\partial t}|\nabla^k A|^2 = \Delta|\nabla^k A|^2 - 2|\nabla^{k+1}A|^2 + \mathcal{E}[k] * \nabla^k A + \mathcal{C}[k] * \mathcal{G}[k],$$

where Δ is the Laplacian on $(M, F_t^* g_t)$. Let $\tilde{\Delta}$ be the Laplacian on $(M, \tilde{F}_s^* \tilde{g})$, then we have $(T-t)\Delta = \tilde{\Delta}$. Hence we have

$$(T-t)^{k+2}(\Delta|\nabla^k A|^2 - 2|\nabla^{k+1}A|^2) = \tilde{\Delta}|\tilde{\nabla}^k \tilde{A}|^2 - 2|\tilde{\nabla}^{k+1}\tilde{A}|^2.$$

Since $\mathcal{G}[k] \in \mathcal{V}_{\frac{1}{2}+\frac{1}{2}k,k-1}$, there exist $r \in \mathbb{N}$, $a_1 \dots a_r \in \mathbb{R}$ and

$$\mathcal{G}[k]_1, \dots, \mathcal{G}[k]_r \in V_{\frac{1}{2}+\frac{1}{2}k,k-1}$$

such that

$$\mathcal{G}[k] = a_1\mathcal{G}[k]_1 + \cdots + a_r\mathcal{G}[k]_r.$$

Hence we have

$$|\mathcal{G}[k]| \leq |a_1||\mathcal{G}[k]_1| + \cdots + |a_r||\mathcal{G}[k]_r|.$$

By the definition of $V_{\frac{1}{2}+\frac{1}{2}k,k-1}$, each $\mathcal{G}[k]_\bullet$ can be expressed as

$$(\nabla^{k_1}\mathrm{Rm} * \cdots * \nabla^{k_I}\mathrm{Rm}) * (\nabla^{\ell_1}A * \cdots * \nabla^{\ell_J}A) * (\overset{p}{*} DF)$$

with some $I, J, p, k_1, \dots, k_I, \ell_1, \dots, \ell_J \in \mathbb{N}$ satisfying

$$\sum_{i=1}^{I}\left(1 + \frac{1}{2}k_i\right) + \sum_{j=1}^{J}\left(\frac{1}{2} + \frac{1}{2}\ell_j\right) = \frac{1}{2} + \frac{1}{2}k \quad \text{and} \quad \sum_{j=1}^{J}\ell_j \leq k-1.$$

Hence, by using (59), (60), and (61), we have

$$\begin{aligned}
&(T-t)^{\frac{1}{2}+\frac{1}{2}k}|\mathcal{G}[k]_\bullet| \\
\leq &C(T-t)^{\frac{1}{2}+\frac{1}{2}k}|\nabla^{k_1}\mathrm{Rm}| \cdots |\nabla^{k_I}\mathrm{Rm}||\nabla^{\ell_1}A| \cdots |\nabla^{\ell_J}A||DF|^p \\
= &C(\sqrt{m})^p|\tilde{\nabla}^{k_1}\widetilde{\mathrm{Rm}}| \cdots |\tilde{\nabla}^{k_I}\widetilde{\mathrm{Rm}}||\tilde{\nabla}^{\ell_1}\tilde{A}| \cdots |\tilde{\nabla}^{\ell_J}\tilde{A}|
\end{aligned}$$

for some constant $C > 0$. Here note that $|DF| = \sqrt{m}$. Since $(N, \tilde{g})$ has bounded geometry, each $|\tilde{\nabla}^{k_i}\widetilde{\mathrm{Rm}}|$ is bounded. Furthermore, since $\ell_j \leq k-1$, each $|\tilde{\nabla}^{\ell_j}\tilde{A}|$ is bounded by the assumption of induction. Hence there exists a constant $C' > 0$ such that

$$(T-t)^{\frac{1}{2}+\frac{1}{2}k}|\mathcal{G}[k]| \leq C'.$$

Since $\mathcal{E}[k] \in \mathcal{V}_{\frac{3}{2}+\frac{1}{2}k,k}$, there exist $r' \in \mathbb{N}$, $b_1 \dots b_{r'} \in \mathbb{R}$ and

$$\mathcal{E}[k]_1, \dots, \mathcal{E}[k]_{r'} \in V_{\frac{3}{2}+\frac{1}{2}k,k}$$

such that

$$\mathcal{E}[k] = b_1\mathcal{E}[k]_1 + \cdots + b_{r'}\mathcal{E}[k]_{r'}.$$

Hence we have

$$|\mathcal{E}[k]| \le |b_1||\mathcal{E}[k]_1| + \cdots + |b_{r'}||\mathcal{E}[k]_{r'}|.$$

By the definition of $V_{\frac{3}{2}+\frac{1}{2}k,k}$, each $\mathcal{E}[k]_\bullet$ can be expressed as

$$(\nabla^{k_1}\mathrm{Rm} * \cdots * \nabla^{k_I}\mathrm{Rm}) * (\nabla^{\ell_1}A * \cdots * \nabla^{\ell_J}A) * (\overset{p}{*}DF)$$

with some $I, J, p, k_1, \ldots, k_I, \ell_1, \ldots, \ell_J \in \mathbb{N}$ satisfying

$$\sum_{i=1}^{I}\left(1+\frac{1}{2}k_i\right) + \sum_{j=1}^{J}\left(\frac{1}{2}+\frac{1}{2}\ell_j\right) = \frac{3}{2}+\frac{1}{2}k \quad \text{and} \quad \sum_{j=1}^{J}\ell_j \le k.$$

If $\max\{\ell_1, \ldots, \ell_J\} \le k-1$, we can prove that $(T-t)^{\frac{3}{2}+\frac{1}{2}k}|\mathcal{E}[k]_\bullet|$ is bounded by the same argument as the case of $\mathcal{G}[k]_\bullet$. If $\max\{\ell_1, \ldots, \ell_J\} = k$, one can easily see that the possible forms of $\mathcal{E}[k]_\bullet$ are

$$A * A * \nabla^k A * (\overset{p}{*}DF) \quad \text{and} \quad \mathrm{Rm} * \nabla^k A * (\overset{p}{*}DF).$$

In both cases, we can see by the same argument as the case of $\mathcal{G}[k]_\bullet$ that there exists a constant $\tilde{C} > 0$ such that $(T-t)^{\frac{3}{2}+\frac{1}{2}k}|\mathcal{E}[k]_\bullet| \le \tilde{C}|\tilde{\nabla}^k\tilde{A}|$. Hence we can see that there exists a constant $C'' > 0$ such that

$$(T-t)^{\frac{3}{2}+\frac{1}{2}k}|\mathcal{E}[k]| \le C''(1 + |\tilde{\nabla}^k\tilde{A}|).$$

Since $\mathcal{C}[k] \in \mathcal{V}_{\frac{3}{2}+\frac{1}{2}k,k+1}$, there exist $r'' \in \mathbb{N}$, $c_1 \ldots c_{r''} \in \mathbb{R}$ and

$$\mathcal{C}[k]_1, \ldots, \mathcal{C}[k]_{r''} \in V_{\frac{3}{2}+\frac{1}{2}k,k+1}$$

such that

$$\mathcal{C}[k] = c_1\mathcal{C}[k]_1 + \cdots + c_{r''}\mathcal{C}[k]_{r''}.$$

Hence we have

$$|\mathcal{C}[k]| \le |c_1||\mathcal{C}[k]_1| + \cdots + |c_{r''}||\mathcal{C}[k]_{r''}|.$$

By the definition of $V_{\frac{3}{2}+\frac{1}{2}k,k+1}$, each $\mathcal{C}[k]_\bullet$ can be expressed as

$$(\nabla^{k_1}\mathrm{Rm} * \cdots * \nabla^{k_I}\mathrm{Rm}) * (\nabla^{\ell_1}A * \cdots * \nabla^{\ell_J}A) * (\overset{p}{*}DF)$$

with some $I, J, p, k_1, \ldots, k_I, \ell_1, \ldots, \ell_J \in \mathbb{N}$ satisfying

$$\sum_{i=1}^{I}\left(1+\frac{1}{2}k_i\right) + \sum_{j=1}^{J}\left(\frac{1}{2}+\frac{1}{2}\ell_j\right) = \frac{3}{2}+\frac{1}{2}k \quad \text{and} \quad \sum_{j=1}^{J}\ell_j \le k+1.$$

If $\max\{\ell_1, \ldots, \ell_J\} \le k-1$, we can prove that $(T-t)^{\frac{3}{2}+\frac{1}{2}k}|\mathcal{C}[k]_\bullet|$ is bounded by the same argument as the case of $\mathcal{G}[k]_\bullet$. If $\max\{\ell_1, \ldots, \ell_J\} = k$, one can easily see that the possible forms of $\mathcal{C}[k]_\bullet$ are

$$A * A * \nabla^k A * (\overset{p}{*}DF) \quad \text{and} \quad \mathrm{Rm} * \nabla^k A * (\overset{p}{*}DF),$$

and we have $(T-t)^{\frac{3}{2}+\frac{1}{2}k}|\mathcal{C}[k]_\bullet| \le \tilde{C}|\tilde{\nabla}^k\tilde{A}|$ as the case of $\mathcal{E}[k]_\bullet$. If $\max\{\ell_1, \ldots, \ell_J\} = k+1$, one can easily see that the possible form of $\mathcal{C}[k]_\bullet$ is

$$\nabla^{k+1}A * (\overset{p}{*}DF),$$

and we have $(T-t)^{\frac{3}{2}+\frac{1}{2}k}|\mathcal{C}[k]_\bullet| \le \tilde{C}'|\tilde{\nabla}^{k+1}\tilde{A}|$ for some constant $\tilde{C}' > 0$. Hence we can see that there exists a constant $C''' > 0$ such that

$$(T-t)^{\frac{3}{2}+\frac{1}{2}k}|\mathcal{C}[k]| \le C'''(1 + |\tilde{\nabla}^k A| + |\tilde{\nabla}^{k+1}\tilde{A}|).$$

Hence we have

$$\begin{aligned}\frac{\partial}{\partial s}|\tilde{\nabla}^k\tilde{A}|^2 \le& (T-t)^{k+2}\frac{\partial}{\partial t}|\nabla^k A|^2\\ \le& \tilde{\Delta}|\tilde{\nabla}^k\tilde{A}|^2 - 2|\tilde{\nabla}^{k+1}\tilde{A}|^2 + C''(1+|\tilde{\nabla}^k\tilde{A}|)|\tilde{\nabla}^k\tilde{A}|\\ &+ C'C'''(1+|\tilde{\nabla}^k\tilde{A}| + |\tilde{\nabla}^{k+1}\tilde{A}|).\end{aligned}$$

Since $-|\tilde{\nabla}^{k+1}\tilde{A}|^2 + C'C'''|\tilde{\nabla}^{k+1}\tilde{A}| \le \frac{(C'C''')^2}{4}$, we have

$$\begin{aligned}\frac{\partial}{\partial s}|\tilde{\nabla}^k\tilde{A}|^2 \le& \tilde{\Delta}|\tilde{\nabla}^k\tilde{A}|^2 - |\tilde{\nabla}^{k+1}\tilde{A}|^2 \\ &+ C''|\tilde{\nabla}^k\tilde{A}|^2 + (C'' + C'C''')|\tilde{\nabla}^k\tilde{A}| + C'C''' + \frac{(C'C''')^2}{4}.\end{aligned}$$

By putting $\bar{C}_k := C'' + (C'' + C'C''') + C'C''' + \frac{(C'C''')^2}{4}$, we have

$$\frac{\partial}{\partial s}|\tilde{\nabla}^k\tilde{A}|^2 \le \tilde{\Delta}|\tilde{\nabla}^k\tilde{A}|^2 - |\tilde{\nabla}^{k+1}\tilde{A}|^2 + \bar{C}_k(1 + |\tilde{\nabla}^k\tilde{A}|^2). \tag{62}$$

Hence immediately we have

$$\frac{\partial}{\partial s}|\tilde{\nabla}^k\tilde{A}|^2 \le \tilde{\Delta}|\tilde{\nabla}^k\tilde{A}|^2 + \bar{C}_k(1 + |\tilde{\nabla}^k\tilde{A}|^2). \tag{63}$$

Note that the inequality (62) also holds for $k-1$, that is, we have

$$\frac{\partial}{\partial s}|\tilde{\nabla}^{k-1}\tilde{A}|^2 \le \tilde{\Delta}|\tilde{\nabla}^{k-1}\tilde{A}|^2 - |\tilde{\nabla}^k\tilde{A}|^2 + \bar{C}_{k-1}(1 + |\tilde{\nabla}^{k-1}\tilde{A}|^2), \tag{64}$$

for some constant $\bar{C}_{k-1} > 0$. Hence by combining the inequality (63) and (64), we have

$$\begin{aligned}\frac{\partial}{\partial s}(|\tilde{\nabla}^k\tilde{A}|^2 + 2\bar{C}_k|\tilde{\nabla}^{k-1}\tilde{A}|^2) \le& \tilde{\Delta}(|\tilde{\nabla}^k\tilde{A}|^2 + 2\bar{C}_k|\tilde{\nabla}^{k-1}\tilde{A}|^2) \\ &+ \bar{C}_k - \bar{C}_k|\tilde{\nabla}^k\tilde{A}|^2 \\ &+ 2\bar{C}_k\bar{C}_{k-1}(1 + |\tilde{\nabla}^{k-1}\tilde{A}|^2).\end{aligned} \tag{65}$$

Since we have

$$\begin{aligned}&\bar{C}_k - \bar{C}_k|\tilde{\nabla}^k\tilde{A}|^2 + 2\bar{C}_k\bar{C}_{k-1}(1 + |\tilde{\nabla}^{k-1}\tilde{A}|^2) \\ =& -\bar{C}_k(|\tilde{\nabla}^k\tilde{A}|^2 + 2\bar{C}_k|\tilde{\nabla}^{k-1}\tilde{A}|^2) \\ &+ \bar{C}_k(1 + 2\bar{C}_{k-1} + 2(\bar{C}_k + \bar{C}_{k-1})|\tilde{\nabla}^{k-1}\tilde{A}|^2)\end{aligned}$$

and $|\tilde{\nabla}^{k-1}\tilde{A}|^2$ is bounded by the assumption of induction, one can easily see that there exists a constant $\bar{\bar{C}}_k > 0$ such that

$$\begin{aligned}\frac{\partial}{\partial s}(|\tilde{\nabla}^k\tilde{A}|^2 + 2\bar{C}_k|\tilde{\nabla}^{k-1}\tilde{A}|^2 - \bar{\bar{C}}_k) \le& \tilde{\Delta}(|\tilde{\nabla}^k\tilde{A}|^2 + 2\bar{C}_k|\tilde{\nabla}^{k-1}\tilde{A}|^2 - \bar{\bar{C}}_k) \\ &- \bar{C}_k(|\tilde{\nabla}^k\tilde{A}|^2 + 2\bar{C}_k|\tilde{\nabla}^{k-1}\tilde{A}|^2 - \bar{\bar{C}}_k).\end{aligned}$$

Thus, by putting $\mu := e^{\bar{C}_k s}(|\tilde{\nabla}^k\tilde{A}|^2 + 2\bar{C}_k|\tilde{\nabla}^{k-1}\tilde{A}|^2 - \bar{\bar{C}}_k)$, we have

$$\frac{\partial}{\partial s}\mu \le \tilde{\Delta}\mu.$$

Since M is compact, μ is bounded at initial time $s = -\log T$. Then, by the parabolic maximum principle, it follows that μ is also bounded on $M \times [-\log T, \infty)$, that is, there exists a constant $\tilde{C}_k > 0$ such that $\mu \le \tilde{C}_k$ on $M \times [-\log T, \infty)$. Hence we have

$$|\tilde{\nabla}^k\tilde{A}|^2 \le e^{-\bar{C}_k s}\tilde{C}_k - 2\bar{C}_k|\tilde{\nabla}^{k-1}\tilde{A}|^2 + \bar{\bar{C}}_k \le T^{\bar{C}_k}\tilde{C}_k + \bar{\bar{C}}_k.$$

Thus, by putting $C_k := T^{\bar{C}_k}\tilde{C}_k + \bar{\bar{C}}_k$, we have

$$|\tilde{\nabla}^k\tilde{A}| \le C_k.$$

Hence the induction argument can be proceeded, and we completed the proof. □

Combining Lemma 20.7 and Proposition 20.9, we can deduce the following uniform bound of the second derivative of the weighted volume.

Lemma 20.10. *Let $(N, \tilde{g}, \tilde{f})$ be a gradient shrinking Ricci soliton with bounded geometry. For a fixed time $T > 0$, let Φ_t and g_t be defined as above, and let $F\colon M \times [0, T) \to N$ be a Ricci-mean curvature flow along the Ricci flow (N, g_t). Assume that M is compact and F satisfies the condition (A2). Let $\tilde{F}$ be the normalized mean curvature flow defined by (40). Then there exists a constant $C' > 0$ such that*

$$\left|\frac{d^2}{ds^2}\int_M e^{-\tilde{f}\circ\tilde{F}_s}\, d\mu(\tilde{F}_s^*\tilde{g})\right| = \left|\frac{d}{ds}\int_M \left|H(\tilde{F}_s) + \nabla\tilde{f}^{\perp_{\tilde{F}_s}}\right|_{\tilde{g}}^2 e^{-\tilde{f}\circ\tilde{F}_s} d\mu(\tilde{F}_s^*\tilde{g})\right| \le C'$$

uniformly on $[-\log T, \infty)$.

Proof. As the proof of Proposition 20.6, we have

$$\begin{aligned}&\int_M \left|H(\tilde F_s)+\nabla \tilde f^{\perp_{\tilde F_s}}\right|^2_{\tilde g} e^{-\tilde f\circ \tilde F_s} d\mu(\tilde F_s^*\tilde g)\\ =&(4\pi)^{\frac{m}{2}}(T-t)\int_M u_t\left|H(F_t)+\nabla f_t^{\perp_{F_t}}\right|^2_{g_t} F_t^*\rho_t\, d\mu(F_t^* g_t),\end{aligned}$$

where $u:=(4\pi(T-t))^{\frac{n-m}{2}}$. Since $\frac{d}{ds}=(T-t)\frac{d}{dt}$, we have

$$\begin{aligned}&\frac{d}{ds}\int_M \left|H(\tilde F_s)+\nabla \tilde f^{\perp_{\tilde F_s}}\right|^2_{\tilde g} e^{-\tilde f\circ \tilde F_s} d\mu(\tilde F_s^*\tilde g)\\ =&-(4\pi)^{\frac{m}{2}}(T-t)\int_M u_t\left|H(F_t)+\nabla f_t^{\perp_{F_t}}\right|^2_{g_t} F_t^*\rho_t\, d\mu(F_t^* g_t)\\ &+(4\pi)^{\frac{m}{2}}(T-t)^2\frac{d}{dt}\int_M u_t\left|H(F_t)+\nabla f_t^{\perp_{F_t}}\right|^2_{g_t} F_t^*\rho_t\, d\mu(F_t^* g_t)\\ =&-\int_M \left|H(\tilde F_s)+\nabla \tilde f^{\perp_{\tilde F_s}}\right|^2_{\tilde g} e^{-\tilde f\circ \tilde F_s} d\mu(\tilde F_s^*\tilde g)\\ &+(4\pi)^{\frac{m}{2}}(T-t)^2\frac{d}{dt}\int_M u_t\left|H(F_t)+\nabla f_t^{\perp_{F_t}}\right|^2_{g_t} F_t^*\rho_t\, d\mu(F_t^* g_t).\end{aligned}\tag{66}$$

First, we consider the term

$$-\int_M \left|H(\tilde F_s)+\nabla \tilde f^{\perp_{\tilde F_s}}\right|^2_{\tilde g} e^{-\tilde f\circ \tilde F_s} d\mu(\tilde F_s^*\tilde g).$$

Since $|H(\tilde F_s)|\le \sqrt{m}|A(\tilde F_s)|$ and we know that $|A(\tilde F_s)|\le C_0$ by Proposition 20.9, we can see that

$$\begin{aligned}\left|H(\tilde F_s)+\nabla \tilde f^{\perp_{\tilde F_s}}\right|^2 \le&|H(\tilde F_s)|^2+2|H(\tilde F_s)||\nabla\tilde f|+|\nabla \tilde f|^2\\ \le&C''(1+|\nabla\tilde f|^2)\\ \le&C''(1+\tilde f\circ\tilde F_s)\end{aligned}$$

for some constant $C''>0$, where we used $0\le|\nabla\tilde f|^2\le\tilde f$. Hence we have

$$\begin{aligned}&\left|-\int_M \left|H(\tilde F_s)+\nabla \tilde f^{\perp_{\tilde F_s}}\right|^2_{\tilde g} e^{-\tilde f\circ \tilde F_s} d\mu(\tilde F_s^*\tilde g)\right|\\ \le&C''\int_M(1+\tilde f\circ\tilde F_s)e^{-\tilde f\circ \tilde F_s} d\mu(\tilde F_s^*\tilde g).\end{aligned}\tag{67}$$

Next we consider the term

$$(4\pi)^{\frac{m}{2}}(T-t)^2\frac{d}{dt}\int_M u_t\left|H(F_t)+\nabla f_t^{\perp_{F_t}}\right|^2_{g_t} F_t^*\rho_t\, d\mu(F_t^* g_t).$$

By Proposition 19.2, we have

$$\begin{aligned}&\frac{d}{dt}\int_M u_t\left|H(F_t)+\nabla f_t^{\perp_{F_t}}\right|^2_{g_t} F_t^*\rho_t\, d\mu(F_t^* g_t)\\ =&-\int_M u_t\left|H(F_t)+\nabla f_t^{\perp_{F_t}}\right|^4_{g_t} F_t^*\rho_t\, d\mu(F_t^* g)+\int_M L\bar u_t\, F_t^*\rho_t\, d\mu(F_t^* g_t),\end{aligned}\tag{68}$$

where we put

$$\begin{aligned}\bar u_t:=&u_t\left|H(F_t)+\nabla f_t^{\perp_{F_t}}\right|^2_{g_t},\\ L\bar u_t:=&\frac{\partial}{\partial t}\bar u_t-\Delta_{F_t^*g_t}\bar u_t+\bar u_t\operatorname{tr}^\perp(\operatorname{Ric}(g_t)+\operatorname{Hess} f_t)\\ =&u_t\left(\frac{\partial}{\partial t}-\Delta_{F_t^*g_t}\right)\left|H(F_t)+\nabla f_t^{\perp_{F_t}}\right|^2_{g_t}.\end{aligned}$$

First, as the above argument, we can see that

$$\begin{aligned}&\left|-(4\pi)^{\frac{m}{2}}(T-t)^2\int_M u_t\left|H(F_t)+\nabla f_t^{\perp_{F_t}}\right|_{g_t}^4 F_t^*\rho_t\, d\mu(F_t^*g_t)\right|\\ \leq &C'''\int_M(1+\tilde f^2\circ\tilde F_s)e^{-\tilde f\circ\tilde F_s}d\mu(\tilde F_s^*\tilde g)\end{aligned} \tag{69}$$

for some constant $C'''>0$. Next we consider

$$\left(\frac{\partial}{\partial t}-\Delta_{F_t^*g_t}\right)\left|H(F_t)+\nabla f_t^{\perp_{F_t}}\right|_{g_t}^2.$$

In fact, by the long computation (cf. Lemma 22.1), it follows that there exists a constant $C''''>0$ such that

$$(T-t)^2\left|\left(\frac{\partial}{\partial t}-\Delta_{F_t^*g_t}\right)\left|H(F_t)+\nabla f_t^{\perp_{F_t}}\right|_{g_t}^2\right|\leq C''''(1+\tilde f\circ\tilde F_s). \tag{70}$$

By combining (66)-(70), it follows that there exists a constant $\bar C>0$ such that

$$\begin{aligned}&\left|\frac{d}{ds}\int_M\left|H(\tilde F_s)+\nabla\tilde f^{\perp_{\tilde F_s}}\right|_{\tilde g}^2 e^{-\tilde f\circ\tilde F_s}d\mu(\tilde F_s^*\tilde g)\right|\\ \leq&\bar C\int_M(1+\tilde f^2\circ\tilde F_s)e^{-\tilde f\circ\tilde F_s}d\mu(\tilde F_s^*\tilde g).\end{aligned}$$

Note that $(1+\tilde f^2)e^{-\tilde f}=(1+\tilde f^2)e^{-\frac{\tilde f}{2}}e^{-\frac{\tilde f}{2}}$ and $(1+\tilde f^2)e^{-\frac{\tilde f}{2}}$ is a bounded function on N, that is, $(1+\tilde f^2)e^{-\frac{\tilde f}{2}}\leq\bar C'$ for some constant $\bar C'$. Thus we have

$$\begin{aligned}&\left|\frac{d}{ds}\int_M\left|H(\tilde F_s)+\nabla\tilde f^{\perp_{\tilde F_s}}\right|_{\tilde g}^2 e^{-\tilde f\circ\tilde F_s}d\mu(\tilde F_s^*\tilde g)\right|\\ \leq&\bar C\int_M(1+\tilde f^2\circ\tilde F_s)e^{-\tilde f\circ\tilde F_s}d\mu(\tilde F_s^*\tilde g)\\ \leq&\bar C\bar C'\int_M e^{-\frac{\tilde f}{2}\circ\tilde F_s}d\mu(\tilde F_s^*\tilde g)\leq\bar C\bar C'C=:C',\end{aligned}$$

where C is the constant appeared in (56) of Lemma 20.7. □

Finally, here we give the proof of Theorem 17.12.

Proof of Theorem 17.12. We denote the Kähler form and the complex structure on (N,g,f) by ω and J respectively. Since $F\colon L\to N$ is a self-similar solution, F satisfies

$$H(F)=\lambda\nabla f^{\perp}$$

for some constant $\lambda\in\mathbb{R}$. Then, by the definition of the mean curvature form ω_H, for a tangent vector X on L, we have

$$\omega_H(X)=\omega(H(F),F_*X)=\lambda\omega(\nabla f^{\perp},F_*X)=\lambda\omega(\nabla f,F_*X),$$

where we used the Lagrangian condition in the last equality. Since the mean curvature form is exact, there exists a smooth function θ on L such that $\omega_H=d\theta$. Let $\{e_i\}_{i=1}^n$ be an orthonormal local frame on L with respect to the metric F^*g. Since ω and J are parallel, we have

$$\begin{aligned}\Delta\theta=&\nabla_{e_i}\omega_H(e_i)-\omega_H(\nabla_{e_i}e_i)\\ =&\lambda\nabla_{e_i}\omega(\nabla f,F_*e_i)-\omega_H(\nabla_{e_i}e_i)\\ =&-\lambda\operatorname{Hess}f(F_*e_i,JF_*e_i)+\lambda\omega(\nabla f,\nabla_{F_*e_i}F_*e_i)-\lambda\omega(\nabla f,F_*(\nabla_{e_i}e_i))\\ =&-\lambda\operatorname{Hess}f(F_*e_i,JF_*e_i)+\lambda\omega(\nabla f,H(F)).\end{aligned}$$

Since the ambient is a gradient shrinking Kähler Ricci soliton, we have

$$\operatorname{Hess}f(F_*e_i,JF_*e_i)=-\operatorname{Ric}(F_*e_i,JF_*e_i)+\frac{1}{2}g(F_*e_i,JF_*e_i)=0.$$

Furthermore, we have

$$\omega(\nabla f,H(F))=\omega(\nabla f^{\top},H(F))=\omega(F_*\nabla(F^*f),H(F))=-(F^*g)(\nabla(F^*f),\nabla\theta).$$

Hence θ satisfies the following linear elliptic equation:

$$\Delta\theta + \lambda(F^*g)(\nabla(F^*f), \nabla\theta) = 0.$$

Since L is compact, by the maximum principle, we obtain that θ is a constant, and this implies that $H(F) = 0$. □

21 Evolution equations

In this Section, we give a general treatment of evolution equations for tensors with Ricci-mean curvature flows along Ricci flows. Note that, in this Section, we do not assume that g_t is the Ricci flow constructed by a gradient shrinking Ricci soliton.

Let M and N be manifolds with dimension m and n respectively, and assume that $m \leq n$. Let $g = (\, g_t\, ;\, t \in [0, T_1)\,)$ be a solution of Ricci flow (33a) and $F\colon M \times [0, T_2) \to N$ be a solution of Ricci-mean curvature flow (33b) with $T_2 \leq T_1$. Here we introduce the notion of the covariant time derivative ∇_t as in [45]. Assume that, for each $t \in [0, T_2)$, $T(t)$ is a smooth section of

$$E_t := (\overset{A}{\otimes} F_t^*(TN)) \otimes (\overset{B}{\otimes} F_t^*(T^*N)) \otimes (\overset{C}{\otimes} TM) \otimes (\overset{D}{\otimes} T^*M)$$

over M, and its correspondence $t \mapsto T(t)$ is smooth. Then for each $t \in [0, T_2)$ we define $(\nabla_t T)(t)$ as follows, and it is also a smooth section of E_t. Denote T by local coordinates $(y^\alpha)_{\alpha=1}^n$ on N and $(x^i)_{i=1}^m$ on M as

$$T^{\alpha_1 \ldots \alpha_A}{}_{\beta_1 \ldots \beta_B}{}^{i_1 \ldots i_C}{}_{j_1 \ldots j_D}.$$

This is the coefficient of

$$\frac{\partial}{\partial y^{\alpha_1}} \otimes \cdots \otimes \frac{\partial}{\partial y^{\alpha_A}} \otimes dy^{\beta_1} \otimes \cdots \otimes dy^{\beta_B} \otimes \frac{\partial}{\partial x^{i_1}} \otimes \cdots \otimes \frac{\partial}{\partial x^{i_C}} \otimes dx^{j_1} \otimes \cdots \otimes dx^{j_D}$$

of T. Then the coefficients of $(\nabla_t T)(t)$ is defined by

$$\begin{aligned}
(\nabla_t T)^{\alpha_1 \ldots \alpha_A}{}_{\beta_1 \ldots \beta_B}{}^{i_1 \ldots i_C}{}_{j_1 \ldots j_D} :=& \frac{\partial}{\partial t} T^{\alpha_1 \ldots \alpha_A}{}_{\beta_1 \ldots \beta_B}{}^{i_1 \ldots i_C}{}_{j_1 \ldots j_D} \\
&+ \sum_{p=1}^{A} \Gamma^{\alpha_p}_{\gamma\delta} H^\gamma T^{\alpha_1 \ldots \delta \ldots \alpha_A}{}_{\beta_1 \ldots \beta_B}{}^{i_1 \ldots i_C}{}_{j_1 \ldots j_D} \\
&- \sum_{p=1}^{B} \Gamma^{\delta}_{\gamma\beta_p} H^\gamma T^{\alpha_1 \ldots \alpha_A}{}_{\beta_1 \ldots \delta \ldots \beta_B}{}^{i_1 \ldots i_C}{}_{j_1 \ldots j_D},
\end{aligned}$$

where $\Gamma^\alpha_{\beta\gamma}$ is the Christoffel symbol of the Levi–Civita connection of g_t on N for each time t. Then one can easily check that this definition does not depend on the choice of local coordinates and defines a global smooth section of E_t over M.

Remark 21.1. One can easily check that ∇_t satisfies Leibniz rule for tensor contractions. For example, for tensors S^α_{ij}, $T^\beta_{k\ell}$, $U_{\alpha\beta}$, V^{ik}, $W^{j\ell}$, we have

$$\begin{aligned}
\nabla_t(S^\alpha_{ij} T^\beta_{k\ell} U_{\alpha\beta} V^{ik} W^{j\ell}) =& \nabla_t S^\alpha_{ij} T^\beta_{k\ell} U_{\alpha\beta} V^{ik} W^{j\ell} + S^\alpha_{ij} \nabla_t T^\beta_{k\ell} U_{\alpha\beta} V^{ik} W^{j\ell} \\
&+ S^\alpha_{ij} T^\beta_{k\ell} \nabla_t U_{\alpha\beta} V^{ik} W^{j\ell} \\
&+ S^\alpha_{ij} T^\beta_{k\ell} U_{\alpha\beta} \nabla_t V^{ik} W^{j\ell} + S^\alpha_{ij} T^\beta_{k\ell} U_{\alpha\beta} V^{ik} \nabla_t W^{j\ell}.
\end{aligned}$$

Note that in this Part we define

$$\begin{aligned}
&\mathrm{Rm}(X, Y)Z := (\nabla_X \nabla_Y - \nabla_Y \nabla_X - \nabla_{[X,Y]})Z \\
&R_{\alpha\beta\gamma\delta} := g\left(\frac{\partial}{\partial y^\alpha}, \mathrm{Rm}\left(\frac{\partial}{\partial y^\gamma}, \frac{\partial}{\partial y^\delta}\right)\frac{\partial}{\partial y^\beta}\right), \\
&R_{\alpha\gamma} := \mathrm{Ric}_{\alpha\gamma} := g^{\beta\delta} R_{\alpha\beta\gamma\delta},
\end{aligned}$$

and we define

$$F_i^\alpha = F_i^\alpha(t) := \frac{\partial F_t^\alpha}{\partial x^i},$$

that is a coefficient of the tensor $DF_t (= F_{t*}) \in \Gamma(M, F_t^*(TN) \otimes T^*M)$. By the straightforward computation with the definition of ∇_t, we get the following formulas, Lemma 21.2, 21.3, and 21.4.

Lemma 21.2. *We have*

$$\begin{aligned}&\nabla_t\nabla_j T^{\alpha}_{i_1\dots i_k} - \nabla_j\nabla_t T^{\alpha}_{i_1\dots i_k}\\ =&R^{\alpha}{}_{\gamma\delta\beta}H^{\delta}F^{\beta}_j T^{\gamma}_{i_1\dots i_k} + \frac{\partial}{\partial t}\Gamma^{\alpha}_{\beta\gamma}F^{\beta}_j T^{\gamma}_{i_1\dots i_k} - \sum_{p=1}^{k}\frac{\partial}{\partial t}\Gamma^{\ell}_{ji_p}T^{\alpha}_{i_1\dots\ell\dots i_k},\end{aligned}$$

where Γ^i_{jk} is the Christoffel symbol of the Levi–Civita connection of $F_t^* g_t$ on M for each time t.

Lemma 21.3. *By the restriction, we consider g_t, more precisely $g_t \circ F_t$, as a section of $F_t^*(T^*N)\otimes F_t^*(T^*N)$ over M. Then we have*

$$\nabla_t g_{\alpha\beta} = -2R_{\alpha\beta}.$$

Lemma 21.4. *We have*

$$\nabla_t F^{\alpha}_i = \nabla_i H^{\alpha}.$$

Combining above lemmas, we have the following.

Lemma 21.5. *Put $g_{ij} = (F_t^* g_t)_{ij} = g_{\alpha\beta}F^{\alpha}_i F^{\beta}_j$. Then we have*

$$\frac{\partial}{\partial t}g_{ij} = \nabla_t g_{ij} = -2((F^*\mathrm{Ric})_{ij} + g(H, A_{ij})).$$

Proof. By the definition of ∇_t, the first equality $\frac{\partial}{\partial t}g_{ij} = \nabla_t g_{ij}$ is clear. By the remark that ∇_t satisfies Leibniz rule for tensor contractions and by Lemma 21.3 and 21.4, we have

$$\begin{aligned}\nabla_t g_{ij} &= \nabla_t(g_{\alpha\beta}F^{\alpha}_i F^{\beta}_j)\\ &= \nabla_t g_{\alpha\beta}F^{\alpha}_i F^{\beta}_j + g_{\alpha\beta}\nabla_t F^{\alpha}_i F^{\beta}_j + g_{\alpha\beta}F^{\alpha}_i\nabla_t F^{\beta}_j\\ &= -2R_{\alpha\beta}F^{\alpha}_i F^{\beta}_j + g_{\alpha\beta}\nabla_i H^{\alpha}F^{\beta}_j + g_{\alpha\beta}F^{\alpha}_i\nabla_j H^{\beta}.\end{aligned}$$

Since H is a normal vector field, we have $g_{\alpha\beta}H^{\alpha}F^{\beta}_i = 0$. By differentiating both sides by ∇_j, we have

$$0 = g_{\alpha\beta}\nabla_j H^{\alpha}F^{\beta}_i + g_{\alpha\beta}H^{\alpha}A^{\beta}_{ji}.$$

Here we used $A^{\beta}_{ji} = \nabla_j F^{\beta}_i$. Since A_{ij} is symmetric, we have

$$g_{\alpha\beta}\nabla_i H^{\alpha}F^{\beta}_j + g_{\alpha\beta}F^{\alpha}_i\nabla_j H^{\beta} = -2g_{\alpha\beta}H^{\alpha}A^{\beta}_{ij}.$$

Here we completed the proof. □

By using $\frac{\partial}{\partial t}g_{\alpha\beta} = -2R_{\alpha\beta}$ and the Koszul formula, one can deduce the following formula immediately.

Lemma 21.6. *We have*

$$\frac{\partial}{\partial t}\Gamma^{\gamma}_{\alpha\beta} = -g^{\gamma\delta}(\nabla_{\alpha}R_{\delta\beta} + \nabla_{\beta}R_{\alpha\delta} - \nabla_{\delta}R_{\alpha\beta}).$$

As an analog of Lemma 21.6, we can prove the following.

Lemma 21.7. *We have*

$$\frac{\partial}{\partial t}\Gamma^{k}_{ij} = -g^{k\ell}(\nabla_i T_{\ell j} + \nabla_j T_{i\ell} - \nabla_{\ell}T_{ij}),$$

where we put $T_{ij} := (F^*\mathrm{Ric})_{ij} + g(H, A_{ij})$.

Here we introduce the notion of $*$-product following Hamilton [19].

Notation 21.8. For tensors S and T, we write $S * T$ to mean a tensor formed by a sum of terms each one of them obtained by contracting some indices of the pair S and T by using g and F^*g and these inverse. There is a property of $*$-product that

$$|S * T| \leq C|S||T|,$$

where $C > 0$ is a constant which depends only on the algebraic structure of $S * T$.

Definition 21.9. For $a, b \in \mathbb{N}$, we define a set $V_{a,b}$ as the set of all (time-dependent) tensors T on M which can be expressed as

$$T = (\nabla^{k_1}\mathrm{Rm} * \cdots * \nabla^{k_I}\mathrm{Rm}) * (\nabla^{\ell_1}A * \cdots * \nabla^{\ell_J}A) * (\overset{p}{*} DF)$$

with $I, J, p, k_1, \ldots, k_I, \ell_1, \ldots, \ell_J \in \mathbb{N}$ satisfying

$$\sum_{i=1}^{I}\left(1 + \frac{1}{2}k_i\right) + \sum_{j=1}^{J}\left(\frac{1}{2} + \frac{1}{2}\ell_j\right) = a \quad \text{and} \quad \sum_{j=1}^{J}\ell_j \le b,$$

and we define a vector space $\mathcal{V}_{a,b}$ as the set of all tensors T on M which can be expressed as

$$T = a_1T_1 + \cdots + a_rT_r$$

for some $r \in \mathbb{N}$, $a_1 \ldots a_r \in \mathbb{R}$ and $T_1, \ldots, T_r \in V_{a,b}$.

Since $\nabla DF = A$, the following is clear.

Proposition 21.10. *Assume that $T_1 \in \mathcal{V}_{a_1,b_1}$, $T_2 \in \mathcal{V}_{a_2,b_2}$ and $T_3 \in \mathcal{V}_{a_3,b_3}$. Then we have*

$$T_1 * T_2 \in \mathcal{V}_{a_1+a_2,b_1+b_2} \quad \text{and} \quad \nabla T_3 \in \mathcal{V}_{a_3+\frac{1}{2},b_3+1},$$

whenever $T_1 * T_2$ makes sense.

Combining Lemma 21.2, 21.6, 21.7, and Proposition 21.10, the following is clear.

Lemma 21.11. *For a time dependent tensor $T = (T^{\alpha}_{i_1\ldots i_k}) \in \mathcal{V}_{a,b}$, we have*

$$\nabla_t\nabla_jT^{\alpha}_{i_1\ldots i_k} - \nabla_j\nabla_tT^{\alpha}_{i_1\ldots i_k} \in \mathcal{V}_{a+\frac{3}{2},b+1}.$$

Lemma 21.12. *For a tensor $T = (T^{\alpha}_{i_1\ldots i_k}) \in \mathcal{V}_{a,b}$, we have*

$$\nabla_j\Delta T^{\alpha}_{i_1\ldots i_k} - \Delta\nabla_jT^{\alpha}_{i_1\ldots i_k} \in \mathcal{V}_{a+\frac{3}{2},b+1}.$$

Proof. First of all, we have

$$\begin{aligned}\nabla_j\Delta T^{\alpha}_{i_1\ldots i_k} =& \nabla_j\nabla_p\nabla^pT^{\alpha}_{i_1\ldots i_k}\\ =& \nabla_p\nabla_j(\nabla^pT^{\alpha}_{i_1\ldots i_k}) + R^{\alpha}{}_{\beta\gamma\delta}F^{\gamma}_jF^{\delta}_p\nabla^pT^{\beta}_{i_1\ldots i_k}\\ &+ R^p{}_{\ell jp}\nabla^{\ell}T^{\alpha}_{i_1\ldots i_k} - \sum_{s=1}^{k}R^{\ell}{}_{i_sjp}\nabla^pT^{\alpha}_{i_1\ldots\ell\ldots i_k},\end{aligned}$$

where $R^i{}_{jk\ell}$ is the Riemannian curvature tensor of $F_t^*g_t$ on M. Then, by the Gauss equation:

$$R_{ki\ell j} = R_{\epsilon\beta\gamma\delta}F^{\epsilon}_kF^{\beta}_iF^{\gamma}_{\ell}F^{\delta}_j - A^{\beta}_{kj}A_{\beta i\ell} + A^{\beta}_{k\ell}A_{\beta ij} \in \mathcal{V}_{1,0}, \tag{71}$$

we can see that

$$\nabla_j\Delta T^{\alpha}_{i_1\ldots i_k} - \nabla_p\nabla_j\nabla^pT^{\alpha}_{i_1\ldots i_k} \in \mathcal{V}_{a+\frac{3}{2},b+1}.$$

As above computations, we have

$$\nabla_j\nabla^pT^{\alpha}_{i_1\ldots i_k} - \nabla^p\nabla_jT^{\alpha}_{i_1\ldots i_k} \in \mathcal{V}_{a+1,b}.$$

Hence, by differentiating by ∇_p, we have

$$\nabla_p\nabla_j\nabla^pT^{\alpha}_{i_1\ldots i_k} - \nabla_p\nabla^p\nabla_jT^{\alpha}_{i_1\ldots i_k} \in \mathcal{V}_{a+\frac{3}{2},b+1}.$$

Note that $\nabla_p\nabla^p\nabla_jT^{\alpha}_{i_1\ldots i_k} = \Delta\nabla_jT^{\alpha}_{i_1\ldots i_k}$. Here we completed the proof. □

Lemma 21.13. *For a time dependent tensor $T = (T^{\alpha}_{i_1\ldots i_k}) \in \mathcal{V}_{a,b}$ there exists a tensor $\mathcal{D} = \mathcal{D}(T) \in \mathcal{V}_{a+1,b}$ such that*

$$\nabla_t|T|^2 = 2\langle\nabla_tT, T\rangle + \mathcal{D} * T.$$

Proof. We have

$$\begin{aligned}\nabla_t|T|^2 =& \nabla_t(g_{\alpha\beta}g^{i_1j_1}\ldots g^{i_kj_k}T^{\alpha}_{i_1\ldots i_k}T^{\beta}_{j_1\ldots j_k})\\ =& 2\langle\nabla_tT, T\rangle + \nabla_tg * T * T + \nabla_t((F^*g)^{-1}) * T * T\\ =& 2\langle\nabla_tT, T\rangle + (\nabla_tg * T + \nabla_t((F^*g)^{-1}) * T) * T.\end{aligned}$$

By Lemma 21.3 and 21.5, we have

$$\nabla_tg, \nabla_t((F^*g)^{-1}) \in \mathcal{V}_{1,0}.$$

Thus the statement is clear. □

Lemma 21.14. *For $k \geq 1$, by definitions, it is clear that*
$$F_p^\alpha \nabla_{i_1} \dots \nabla_{i_k} A_\alpha \in \mathcal{V}_{\frac{1}{2}+\frac{1}{2}k,k}.$$
Actually, it is true that
$$F_p^\alpha \nabla_{i_1} \dots \nabla_{i_k} A_\alpha \in \mathcal{V}_{\frac{1}{2}+\frac{1}{2}k,k-1}.$$
For $k = 0$, it is clear that
$$F_p^\alpha A_\alpha = 0$$
since A is a normal bundle valued 2-tensor.

Proof. By differentiating the equation $F_p^\alpha A_\alpha = 0$, we have
$$F_p^\alpha \nabla_{i_1} A_\alpha = -A_{i_1 p}^\alpha A_\alpha \in \mathcal{V}_{1,0}.$$
Hence the statement is true for $k = 1$. Assume that for $k-1$ the statement is true. Then, for k, the statement is also true since we have
$$F_p^\alpha \nabla_{i_1} \dots \nabla_{i_k} A_\alpha = \nabla_{i_1}(F_p^\alpha \nabla_{i_2} \dots \nabla_{i_k} A_\alpha) - A_{i_1 p}^\alpha \nabla_{i_2} \dots \nabla_{i_k} A_\alpha.$$
We completed the proof. □

Lemma 21.15. *There exist tensors $\mathcal{B} = (\mathcal{B}_{ij}^\alpha) \in \mathcal{V}_{\frac{3}{2},0}$ and $\mathcal{C} = (\mathcal{C}_{ij}^p) \in \mathcal{V}_{\frac{3}{2},1}$ such that*
$$\nabla_t A_{ij}^\alpha = \Delta A_{ij}^\alpha + \mathcal{B}_{ij}^\alpha + \mathcal{C}_{ij}^p F_p^\alpha.$$

Proof. By identies $A_{ij}^\alpha = \nabla_i F_j^\alpha$ and $\nabla_t F_j^\alpha = \nabla_j H^\alpha$ and Lemma 21.2, we have
$$\begin{aligned}
\nabla_t A_{ij}^\alpha =& \nabla_t \nabla_i F_j^\alpha \\
=& \nabla_i \nabla_t F_j^\alpha + R^\alpha{}_{\gamma\delta\beta} H^\delta F_i^\beta F_j^\gamma + \frac{\partial}{\partial t}\Gamma_{\beta\gamma}^\alpha F_i^\beta F_j^\gamma - \frac{\partial}{\partial t}\Gamma_{ij}^\ell F_\ell^\alpha \\
=& \nabla_i \nabla_j H^\alpha + R^\alpha{}_{\gamma\delta\beta} H^\delta F_i^\beta F_j^\gamma + \frac{\partial}{\partial t}\Gamma_{\beta\gamma}^\alpha F_i^\beta F_j^\gamma - \frac{\partial}{\partial t}\Gamma_{ij}^\ell F_\ell^\alpha.
\end{aligned}$$
Furthermore, by using Simons' identity:
$$\begin{aligned}
\nabla_k \nabla_\ell H^\alpha =& \Delta A_{k\ell}^\alpha + (\nabla_\epsilon R^\alpha{}_{\beta\gamma\delta} + \nabla_\gamma R^\alpha{}_{\delta\beta\epsilon}) F_i^\epsilon F_\ell^\beta F_k^\gamma F^{\delta i} \\
&+ R^\alpha{}_{\beta\gamma\delta}(2A_{ik}^\beta F_\ell^\gamma F^{\delta i} + 2A_{i\ell}^\beta F_k^\gamma F^{\delta i} + H^\delta F_\ell^\beta F_k^\gamma + A_{\ell k}^\gamma F_i^\beta F^{\delta i}) \\
&- (\nabla_k R^p{}_\ell + \nabla_\ell R^p{}_k - \nabla^p R_{k\ell}) F_p^\alpha \\
&+ 2R_k{}^i{}_\ell{}^j A_{ij}^\alpha - R^p{}_k A_{p\ell}^\alpha - R^p{}_\ell A_{pk}^\alpha,
\end{aligned}$$
we have
$$\begin{aligned}
\nabla_t A_{ij}^\alpha =& \Delta A_{ij}^\alpha \\
&+ (\nabla_\epsilon R^\alpha{}_{\beta\gamma\delta} + \nabla_\gamma R^\alpha{}_{\delta\beta\epsilon}) F_k^\epsilon F_j^\beta F_i^\gamma F^{\delta k} \\
&+ R^\alpha{}_{\beta\gamma\delta}(2A_{ki}^\beta F_j^\gamma F^{\delta k} + 2A_{kj}^\beta F_i^\gamma F^{\delta k} + H^\delta F_j^\beta F_i^\gamma + A_{ji}^\gamma F_k^\beta F^{\delta k}) \\
&- (\nabla_i R^p{}_j + \nabla_j R^p{}_i - \nabla^p R_{ij}) F_p^\alpha \\
&+ 2R_i{}^k{}_j{}^\ell A_{k\ell}^\alpha - R^p{}_i A_{pj}^\alpha - R^p{}_j A_{pi}^\alpha \\
&+ R^\alpha{}_{\gamma\delta\beta} H^\delta F_i^\beta F_j^\gamma + \frac{\partial}{\partial t}\Gamma_{\beta\gamma}^\alpha F_i^\beta F_j^\gamma - \frac{\partial}{\partial t}\Gamma_{ij}^p F_p^\alpha.
\end{aligned}$$
By putting
$$\begin{aligned}
\mathcal{B}_{ij}^\alpha :=& (\nabla_\epsilon R^\alpha{}_{\beta\gamma\delta} + \nabla_\gamma R^\alpha{}_{\delta\beta\epsilon}) F_k^\epsilon F_j^\beta F_i^\gamma F^{\delta k} \\
&+ R^\alpha{}_{\beta\gamma\delta}(2A_{ki}^\beta F_j^\gamma F^{\delta k} + 2A_{kj}^\beta F_i^\gamma F^{\delta k} + H^\delta F_j^\beta F_i^\gamma + A_{ji}^\gamma F_k^\beta F^{\delta k}) \\
&+ 2R_i{}^k{}_j{}^\ell A_{k\ell}^\alpha - R^p{}_i A_{pj}^\alpha - R^p{}_j A_{pi}^\alpha \\
&+ R^\alpha{}_{\gamma\delta\beta} H^\delta F_i^\beta F_j^\gamma + \frac{\partial}{\partial t}\Gamma_{\beta\gamma}^\alpha F_i^\beta F_j^\gamma \\
\mathcal{C}_{ij}^p :=& -(\nabla_i R^p{}_j + \nabla_j R^p{}_i - \nabla^p R_{ij}) - \frac{\partial}{\partial t}\Gamma_{ij}^p,
\end{aligned}$$
We have
$$\nabla_t A_{ij}^\alpha = \Delta A_{ij}^\alpha + \mathcal{B}_{ij}^\alpha + \mathcal{C}_{ij}^p F_p^\alpha.$$

Furthermore, by using Lemma 21.6, 21.7 and Gauss equation (71), one can easily see that

$$\mathcal{B} = (\mathcal{B}_{ij}^{\alpha}) \in \mathcal{V}_{\frac{3}{2},0} \quad \text{and} \quad \mathcal{C} = (\mathcal{C}_{ij}^{p}) \in \mathcal{V}_{\frac{3}{2},1}.$$

Here we completed the proof. □

Proposition 21.16. *There exists a tensor $\mathcal{E} \in \mathcal{V}_{\frac{3}{2},0}$ such that*

$$\frac{\partial}{\partial t}|A|^2 = \Delta|A|^2 - 2|\nabla A|^2 + \mathcal{E} * A.$$

Proof. By Lemma 21.13, there exists a tensor $\mathcal{D} = \mathcal{D}(A) \in \mathcal{V}_{\frac{3}{2},0}$ such that

$$\frac{\partial}{\partial t}|A|^2 = \nabla_t|A|^2 = 2\langle\nabla_t A, A\rangle + \mathcal{D} * A.$$

By Lemma 21.15, we have

$$\begin{aligned}2\langle\nabla_t A, A\rangle =& 2(\Delta A_{ij}^{\alpha} + \mathcal{B}_{ij}^{\alpha} + \mathcal{C}_{ij}^{p} F_p^{\alpha}) A_{\alpha}^{ij} \\ =& \Delta|A|^2 - 2|\nabla A|^2 + \mathcal{B} * A.\end{aligned}$$

Here we used $F_p^{\alpha} A_{\alpha}^{ij} = 0$. Hence, by putting $\mathcal{E} := \mathcal{D} + \mathcal{B}$, we have

$$\frac{\partial}{\partial t}|A|^2 = \Delta|A|^2 - 2|\nabla A|^2 + \mathcal{E} * A,$$

and, we have that

$$\mathcal{E} \in \mathcal{V}_{\frac{3}{2},0}.$$

Here we completed the proof. □

Proposition 21.17. *We have*

$$\frac{\partial}{\partial t}|A|^2 \leq \Delta|A|^2 - 2|\nabla A|^2 + C_1|A|^4 + C_2|\mathrm{Rm}||A|^2 + C_3|\nabla\mathrm{Rm}||A|,$$

for positive constants C_1, C_2, C_3 which depend only on the dimension of M and N, where Rm *is the Riemannian curvature tensor of (N, g_t).*

Proof. By Proposition 21.16, we know that

$$\frac{\partial}{\partial t}|A|^2 = \Delta|A|^2 - 2|\nabla A|^2 + \mathcal{E} * A.$$

Since $\mathcal{E} \in \mathcal{V}_{\frac{3}{2},0}$, the tensor $\mathcal{E}$ can be constructed by

$$A * A * A * (*^i DF), \ \mathrm{Rm} * A * (*^j DF), \ \nabla\mathrm{Rm} * (*^k DF).$$

Note that we can not decide i, j, k from the information that $\mathcal{E} \in \mathcal{V}_{\frac{3}{2},0}$. However this is not a matter when we consider the norm of tensors since the norm of DF is a constant $\sqrt{m}$. Hence we see that there exist positive constants C_1, C_2, C_3 which depend only on the dimensions of M and N such that

$$|\mathcal{E} * A| \leq C_1|A|^4 + C_2|\mathrm{Rm}||A|^2 + C_3|\nabla\mathrm{Rm}||A|.$$

Here we completed the proof. □

Proposition 21.18. *For all $k \geq 0$ there exist tensors $\mathcal{B}[k] \in \mathcal{V}_{\frac{3}{2}+\frac{1}{2}k,k}$ and $\mathcal{C}[k] \in \mathcal{V}_{\frac{3}{2}+\frac{1}{2}k,k+1}$ such that*

$$\nabla_t \nabla_{\ell_1} \ldots \nabla_{\ell_k} A_{ij}^{\alpha} = \Delta \nabla_{\ell_1} \ldots \nabla_{\ell_k} A_{ij}^{\alpha} + \mathcal{B}[k] + \mathcal{C}[k]^p F_p^{\alpha}.$$

Proof. We work by induction on $k \in \mathbb{N}$. For the case $k = 0$, the statement is true by Lemma 21.15. Assume that for $k-1$ the statement is true. Since

$$\nabla_{\ell_2} \ldots \nabla_{\ell_k} A_{ij}^{\alpha} \in \mathcal{V}_{\frac{3}{2}+\frac{1}{2}(k-1),k-1},$$

by defining $\mathcal{D}$ as

$$\nabla_t \nabla_{\ell_1} \nabla_{\ell_2} \ldots \nabla_{\ell_k} A_{ij}^{\alpha} = \nabla_{\ell_1} \nabla_t \nabla_{\ell_2} \ldots \nabla_{\ell_k} A_{ij}^{\alpha} + \mathcal{D},$$

we have, by Lemma 21.11, that

$$\mathcal{D} \in \mathcal{V}_{\frac{3}{2}+\frac{1}{2}k,k}.$$

By the assumption of the induction for $k-1$, we have

$$\begin{aligned}&\nabla_{\ell_1} \nabla_t \nabla_{\ell_2} \ldots \nabla_{\ell_k} A_{ij}^{\alpha} \\ =& \nabla_{\ell_1}(\Delta \nabla_{\ell_2} \ldots \nabla_{\ell_k} A_{ij}^{\alpha} + \mathcal{B}[k-1] + \mathcal{C}[k-1]^p F_p^{\alpha}) \\ =& \nabla_{\ell_1} \Delta \nabla_{\ell_2} \ldots \nabla_{\ell_k} A_{ij}^{\alpha} + \nabla\mathcal{B}[k-1] + \nabla\mathcal{C}[k-1]^p F_p^{\alpha} + \mathcal{C}[k-1] * A.\end{aligned}$$

Furthermore by Lemma 21.12, by defining $\mathcal{D}'$ as

$$\nabla_{\ell_1}\Delta\nabla_{\ell_2}\dots\nabla_{\ell_k}A^{\alpha}_{ij} = \Delta\nabla_{\ell_1}\nabla_{\ell_2}\dots\nabla_{\ell_k}A^{\alpha}_{ij} + \mathcal{D}',$$

we have that

$$\mathcal{D}' \in \mathcal{V}_{\frac{3}{2}+\frac{1}{2}k,k}.$$

Hence, by putting

$$\begin{aligned}\mathcal{B}[k] :=& \nabla\mathcal{B}[k-1] + \mathcal{C}[k-1]*A + \mathcal{D} + \mathcal{D}'.\\ \mathcal{C}[k] :=& \nabla\mathcal{C}[k-1],\end{aligned}$$

we have

$$\nabla_t\nabla_{\ell_1}\dots\nabla_{\ell_k}A^{\alpha}_{ij} = \Delta\nabla_{\ell_1}\dots\nabla_{\ell_k}A^{\alpha}_{ij} + \mathcal{B}[k] + \mathcal{C}[k]^p F^{\alpha}_p.$$

and

$$\mathcal{B}[k] \in \mathcal{V}_{\frac{3}{2}+\frac{1}{2}k,k} \quad \text{and} \quad \mathcal{C}[k] \in \mathcal{V}_{\frac{3}{2}+\frac{1}{2}k,k+1}.$$

Here we completed the proof. $\square$

Proposition 21.19. *For all $k \geq 0$ there exist tensors $\mathcal{E}[k] \in \mathcal{V}_{\frac{3}{2}+\frac{1}{2}k,k}$, $\mathcal{C}[k] \in \mathcal{V}_{\frac{3}{2}+\frac{1}{2}k,k+1}$ and $\mathcal{G}[k] \in \mathcal{V}_{\frac{1}{2}+\frac{1}{2}k,k-1}$ such that*

$$\frac{\partial}{\partial t}|\nabla^k A|^2 = \Delta|\nabla^k A|^2 - 2|\nabla^{k+1}A|^2 + \mathcal{E}[k]*\nabla^k A + \mathcal{C}[k]*\mathcal{G}[k].$$

Proof. Put $T^{\alpha}_{\ell_1\dots\ell_k ij} := \nabla_{\ell_1}\dots\nabla_{\ell_k}A^{\alpha}_{ij}$. Since $T \in \mathcal{V}_{\frac{1}{2}+\frac{1}{2}k,k}$, by Lemma 21.13 there exists a tensor $\mathcal{D}[k] = \mathcal{D}(T) \in \mathcal{V}_{\frac{3}{2}+\frac{1}{2}k,k}$ such that

$$\frac{\partial}{\partial t}|T|^2 = \nabla_t|T|^2 = 2\langle\nabla_t T, T\rangle + \mathcal{D}[k]*T.$$

By Proposition 21.18, there exist tensors $\mathcal{B}[k] \in \mathcal{V}_{\frac{3}{2}+\frac{1}{2}k,k}$ and $\mathcal{C}[k] \in \mathcal{V}_{\frac{3}{2}+\frac{1}{2}k,k+1}$ such that

$$\nabla_t T = \Delta T + \mathcal{B}[k] + \mathcal{C}[k]^p F^{\alpha}_p.$$

Hence we have

$$\begin{aligned}2\langle\nabla_t T, T\rangle =& 2\langle\Delta T, T\rangle + \mathcal{B}[k]*T + \mathcal{C}[k]^p F^{\alpha}_p T_{\alpha}\\ =& \Delta|T|^2 - 2|\nabla T|^2 + \mathcal{B}[k]*T + \mathcal{C}[k]^p F^{\alpha}_p T_{\alpha}.\end{aligned}$$

By Lemma 21.14, we have

$$\mathcal{G}[k] := F^{\alpha}_p T_{\alpha} \in \mathcal{V}_{\frac{1}{2}+\frac{1}{2}k,k-1}.$$

Hence, by putting $\mathcal{E}[k] := \mathcal{D}[k] + \mathcal{B}[k] \in \mathcal{V}_{\frac{3}{2}+\frac{1}{2}k,k}$, we have

$$\frac{\partial}{\partial t}|T|^2 = \Delta|T|^2 - 2|\nabla T|^2 + \mathcal{E}[k]*T + \mathcal{C}[k]*\mathcal{G}[k].$$

Here we completed the proof. $\square$

22 An estimate in the proof of Lemma 20.10

In this Section, we give a proof for the following estimate which is used in the proof of Lemma 20.10. It is just a straightforward long computation.

Lemma 22.1. *In the situation of Lemma 20.10, there exists a constant $C'''' > 0$ such that*

$$(T-t)^2\left|\left(\frac{\partial}{\partial t} - \Delta_{F_t^* g_t}\right)\left|H(F_t) + \nabla f_t{}^{\perp_{F_t}}\right|^2_{g_t}\right| \leq C''''(1 + \tilde{f}\circ\tilde{F}_s). \tag{72}$$

Proof. First of all, we define $W_{c,d}$ and $\mathcal{W}_{c,d}$ as analogs of $V_{a,b}$ and $\mathcal{V}_{a,b}$. For $c, d \in \mathbb{N}$, we define a set $W_{c,d}$ as the set of all (time-dependent) tensors T on M which can be expressed as

$$T = \frac{1}{(T-t)^q}(\overset{r}{*}\nabla f)*(\nabla^{k_1}\mathrm{Rm}*\dots*\nabla^{k_I}\mathrm{Rm})*(\nabla^{\ell_1}A*\dots*\nabla^{\ell_J}A)*(\overset{p}{*}DF)$$

with $q, r, I, J, p, k_1, \dots, k_I, \ell_1, \dots, \ell_J \in \mathbb{N}$ satisfying

$$q + \frac{1}{2}r + \sum_{i=1}^{I}\left(1 + \frac{1}{2}k_i\right) + \sum_{j=1}^{J}\left(\frac{1}{2} + \frac{1}{2}\ell_j\right) = c \quad \text{and} \quad r \leq d,$$

and we define a vector space $\mathcal{W}_{c,d}$ as the set of all tensors T on M which can be expressed as

$$T = a_1 T_1 + \cdots + a_v T_v$$

for some $v \in \mathbb{N}$, $a_1 \dots a_v \in \mathbb{R}$ and $T_1, \dots, T_v \in W_{c,d}$. By the definition, it is clear that $\mathcal{V}_{a,b} \subset \mathcal{W}_{a,0}$, and if $T_1 \in \mathcal{W}_{c_1,d_1}$ and $T_2 \in \mathcal{W}_{c_2,d_2}$ then $T_1 * T_2 \in \mathcal{W}_{c_1+c_2,d_1+d_2}$.

Note that we consider $\nabla^\alpha f$ as a tensor field over M by pulling it back by F_t. However we sometimes omit the symbol $\circ F_t$. Then we have

$$\nabla_i \nabla^\alpha f = F_i^\beta \nabla_\beta \nabla^\alpha f = -\mathrm{Ric}_\beta{}^\alpha F_i^\beta + \frac{1}{2(T-t)} F_i^\alpha \in \mathcal{W}_{1,0}, \tag{73}$$

where we used $\mathrm{Ric}_{\alpha\beta} + \nabla_\alpha \nabla_\beta f = \frac{1}{2(T-t)} g_{\alpha\beta}$. Hence we can see that if $T \in \mathcal{W}_{c,d}$ then $\nabla T \in \mathcal{W}_{c+\frac{1}{2},d}$.

To prove this lemma, we use the identity

$$\left| H(F_t) + \nabla f_t^{\perp_{F_t}} \right|_{g_t}^2 = |H(F_t)|_{g_t}^2 + 2 g_t(H(F_t), \nabla f_t) + |\nabla f_t|_{g_t}^2 - |\nabla f_t{}^{\top_{F_t}}|_{g_t}^2.$$

By Lemma 21.5 and 21.15, we have

$$\begin{aligned}
\nabla_t H^\alpha =& \nabla_t (g^{ij} A_{ij}^\alpha) \\
=& 2(\mathrm{Ric}_{\beta\gamma} F^{\beta i} F^{\gamma j} + H^\beta A_\beta^{ij}) A_{ij}^\alpha + g^{ij}(\Delta A_{ij}^\alpha + \mathcal{B}_{ij}^\alpha + \mathcal{C}_{ij}^p F_p^\alpha) \\
=& \Delta H^\alpha + \bar{\mathcal{B}}^\alpha + g^{ij} \mathcal{C}_{ij}^p F_p^\alpha,
\end{aligned}$$

where we put $\bar{\mathcal{B}}^\alpha := 2(\mathrm{Ric}_{\beta\gamma} F^{\beta i} F^{\gamma j} + H^\beta A_\beta^{ij}) A_{ij}^\alpha + g^{ij} \mathcal{B}_{ij}^\alpha \in \mathcal{V}_{\frac{3}{2},0}$. Since $H \in \mathcal{V}_{\frac{1}{2},0}$, by Lemma 21.13 there exists a tensor $\mathcal{D} = \mathcal{D}(H) \in \mathcal{V}_{\frac{3}{2},0}$ such that

$$\begin{aligned}
\frac{\partial}{\partial t} |H|^2 =& 2\langle \nabla_t H, H \rangle + \mathcal{D}^\alpha H_\alpha \\
=& 2\langle \Delta H, H \rangle + 2\bar{\mathcal{B}}^\alpha H_\alpha + \mathcal{D}^\alpha H_\alpha \\
=& \Delta |H|^2 - 2|\nabla H|^2 + 2\bar{\mathcal{B}}^\alpha H_\alpha + \mathcal{D}^\alpha H_\alpha,
\end{aligned}$$

where we used $F_p^\alpha H_\alpha = 0$. Thus we have

$$\left(\frac{\partial}{\partial t} - \Delta_{F^* g} \right) |H|^2 = -2|\nabla H|^2 + 2\bar{\mathcal{B}}^\alpha H_\alpha + \mathcal{D}^\alpha H_\alpha,$$

and it is clear that

$$\left(\frac{\partial}{\partial t} - \Delta_{F^* g} \right) |H|^2 \in \mathcal{W}_{2,0}. \tag{74}$$

Next, we consider $\nabla_t \nabla^\alpha f$. Then, by the definition of ∇_t, we have

$$\begin{aligned}
\nabla_t \nabla^\alpha f =& \frac{\partial}{\partial t} \nabla^\alpha f + H^\beta \nabla_\beta \nabla^\alpha f \\
=& \frac{\partial}{\partial t} \nabla^\alpha f - \mathrm{Ric}^\alpha{}_\beta H^\beta + \frac{1}{2(T-t)} H^\alpha,
\end{aligned}$$

where we used $\mathrm{Ric}_{\alpha\beta} + \nabla_\alpha \nabla_\beta f = \frac{1}{2(T-t)} g_{\alpha\beta}$. Furthermore, by using $\frac{\partial}{\partial t} g^{\alpha\beta} = 2\mathrm{Ric}^{\alpha\beta}$ and $\frac{\partial}{\partial t} f = |\nabla f|^2$, one can easily see that

$$\frac{\partial}{\partial t} \nabla^\alpha f = \frac{1}{T-t} \nabla^\alpha f.$$

Thus we have

$$\nabla_t \nabla^\alpha f = \frac{1}{T-t} \nabla^\alpha f - \mathrm{Ric}^\alpha{}_\beta H^\beta + \frac{1}{2(T-t)} H^\alpha \in \mathcal{W}_{\frac{3}{2},1}.$$

Hence we can see that

$$\nabla_t (g_{\alpha\beta} H^\alpha \nabla^\beta f) = \nabla_t g_{\alpha\beta} H^\alpha \nabla^\beta f + g_{\alpha\beta} \nabla_t H^\alpha \nabla^\beta f + g_{\alpha\beta} H^\alpha \nabla_t \nabla^\beta f \in \mathcal{W}_{2,1}$$

and

$$\Delta (g_{\alpha\beta} H^\alpha \nabla^\beta f) \in \mathcal{W}_{2,1}$$

Thus we have

$$\left(\frac{\partial}{\partial t} - \Delta_{F^* g} \right) g(H(F), \nabla f) \in \mathcal{W}_{2,1}. \tag{75}$$

Next, one can easily see that

$$\nabla_t|\nabla f|^2 = \nabla_t g_{\alpha\beta}\nabla^\alpha f \nabla^\beta f + 2\nabla_t \nabla^\alpha f \nabla_\alpha f \in \mathcal{W}_{2,2}$$

and

$$\Delta|\nabla f|^2 \in \mathcal{W}_{2,2}.$$

Hence we have

$$\left(\frac{\partial}{\partial t} - \Delta_{F^*g}\right)|\nabla f|^2 \in \mathcal{W}_{2,2}. \tag{76}$$

Finally, one can easily see that

$$\nabla_t|\nabla f_t{}^{\top_{F_t}}|^2_{g_t} = \nabla_t((F^*g)^{k\ell} g_{\alpha\beta} g_{\gamma\delta} F_k^\beta F_\ell^\delta \nabla^\alpha f \nabla^\gamma f) \in \mathcal{W}_{2,2}$$

and

$$\Delta|\nabla f_t{}^{\top_{F_t}}|^2_{g_t} \in \mathcal{W}_{2,2}.$$

Hence we have

$$\left(\frac{\partial}{\partial t} - \Delta_{F^*g}\right)|\nabla f_t{}^{\top_{F_t}}|^2_{g_t} \in \mathcal{W}_{2,2}. \tag{77}$$

Hence, by (74)–(77), we have

$$\left(\frac{\partial}{\partial t} - \Delta_{F^*g}\right)\left|H(F) + \nabla f^{\perp_F}\right|^2_g \in \mathcal{W}_{2,2}.$$

By the definition of $\mathcal{W}_{2,2}$, there exist $v \in \mathbb{N}$, $a_1 \dots a_v \in \mathbb{R}$ and $T_1, \dots, T_v \in W_{2,2}$ such that

$$\left(\frac{\partial}{\partial t} - \Delta_{F^*g}\right)\left|H(F) + \nabla f^{\perp_F}\right|^2_g = a_1 T_1 + \cdots + a_v T_v.$$

Hence we have

$$\left|\left(\frac{\partial}{\partial t} - \Delta_{F^*g}\right)\left|H(F) + \nabla f^{\perp_F}\right|^2_g\right| \leq |a_1||T_1| + \cdots + |a_v||T_v|.$$

By the definition of $W_{2,2}$, each $T_\bullet$ can be expressed as

$$T_\bullet = \frac{1}{(T-t)^q}(\overset{r}{*}\nabla f) * (\nabla^{k_1}\mathrm{Rm} * \cdots * \nabla^{k_I}\mathrm{Rm}) * (\nabla^{\ell_1}A * \cdots * \nabla^{\ell_J}A) * (\overset{p}{*} DF)$$

with some $I, J, q, r, p, k_1, \dots, k_I, \ell_1, \dots, \ell_J \in \mathbb{N}$ satisfying

$$q + \frac{1}{2}r + \sum_{i=1}^{I}\left(1 + \frac{1}{2}k_i\right) + \sum_{j=1}^{J}\left(\frac{1}{2} + \frac{1}{2}\ell_j\right) = 2 \quad \text{and} \quad r \leq 2,$$

Here note that by Proposition 20.9 and the equation (59) it follows that $(T-t)^{\frac{1}{2}+\frac{1}{2}\ell}|\nabla^\ell A|$ is bounded for all $\ell \geq 0$. Furthermore by the equation (60) it is clear that $(T-t)^{1+\frac{1}{2}k}|\nabla^k \mathrm{Rm}|$ is bounded for all $k \geq 0$. Hence, for $T_\bullet$ above, we have

$$(T-t)^2|T_\bullet| \leq C(T-t)^{\frac{1}{2}r}|\nabla f|^r$$

for some $C > 0$. Furthermore, we have

$$|\nabla f|_g = \frac{1}{\sqrt{T-t}}|\nabla \tilde{f}|_{\tilde{g}} \leq \frac{1}{\sqrt{T-t}}\sqrt{\tilde{f}}.$$

Thus we have

$$(T-t)^2|T_\bullet| \leq C\tilde{f}^{\frac{1}{2}r}.$$

Now each $T_\bullet$ is in $\mathcal{W}_{2,2}$, so $r \leq 2$. For $r = 0, 1, 2$, it is clear that $\tilde{f}^{\frac{1}{2}r} \leq 1 + \tilde{f}$. Thus we have proved that there exists a constant $C'''' > 0$ such that

$$(T-t)^2\left|\left(\frac{\partial}{\partial t} - \Delta_{F^*g}\right)\left|H(F) + \nabla f^{\perp_F}\right|^2_g\right| \leq C''''(1 + \tilde{f}).$$

□

23 convergence of submanifolds

In this Section, we give a definition of the convergence of immersion maps into a Riemannian manifolds and prove some propositions.

Let (N, g) be an n-dimensional Riemannian manifold and E be a real vector bundle over N with a metric h. Take a compatible connection ∇ over E, that is, for all smooth sections $e, f \in \Gamma(N, E)$ and a vector field $X \in \mathfrak{X}(N)$ we have

$$X(h(e, f)) = h(\nabla_X e, f) + h(e, \nabla_X f).$$

Definition 23.1. Let $p \in \mathbb{N}$. Let $K \subset N$ be a compact set and $\Omega \subset N$ be an open set satisfying $K \subset \Omega$. Let $\{\xi_k\}_{k=1}^{\infty}$ be a sequence of sections of E defined on Ω and ξ_∞ be a section of E defined on Ω. We say that ξ_k converges in C^p to ξ_∞ uniformly on K if for every $\epsilon > 0$ there exists $k_0 = k_0(\epsilon)$ such that for $k \geq k_0$,

$$\sup_{0 \leq \alpha \leq p} \sup_{x \in K} |\nabla^\alpha(\xi_k - \xi_\infty)|_{g^\alpha \otimes h} < \epsilon.$$

Furthermore, we say ξ_k converges in C^∞ to ξ_∞ uniformly on K if ξ_k converges in C^p to ξ_∞ uniformly on K for every $p \in \mathbb{N}$.

Let $\{U_k\}_{k=1}^{\infty}$ be a sequence of open sets in N. We call $\{U_k\}_{k=1}^{\infty}$ an exhaustion of N if $\overline{U}_k$ is compact and $\overline{U}_k \subset U_{k+1}$ for all k, and $\cup_{k=1}^{\infty} U_k = N$.

Definition 23.2. Let $\{U_k\}_{k=1}^{\infty}$ be an exhaustion of N. Let $\{\xi_k\}_{k=1}^{\infty}$ be a sequence of locally defined sections of E such that each ξ_k is defined on U_k. Let ξ_∞ be a section of E defined on N. We say that ξ_k converges in C^∞ to ξ_∞ uniformly on compact sets in N if for any compact set $K \subset N$ there exists $k_0 = k_0(K)$ such that $K \subset U_k$ for all $k \geq k_0$ and the sequence $\{\xi_k|_{U_{k_0}}\}_{k=k_0}^{\infty}$ converges in C^∞ to $\xi_\infty|_{U_{k_0}}$ uniformly on K.

Definition 23.3. A sequence $\{(N_k, g_k, x_k)\}_{k=1}^{\infty}$ of complete pointed Riemannian manifolds converges to a complete pointed Riemannian manifold $(N_\infty, g_\infty, x_\infty)$ if there exists

(1) an exhaustion $\{U_k\}_{k=1}^{\infty}$ of N_∞ with $x_\infty \in U_k$ and

(2) a sequence of diffeomorphisms $\Psi_k \colon U_k \to V_k \subset N_k$ with $\Psi_k(x_\infty) = x_k$

such that $\Psi_k^* g_k$ converges in C^∞ to g_∞ uniformly on compact sets in N_∞.

This notion of convergence is often referred as (smooth) Cheeger–Gromov convergence, C^∞-convergence or geometric convergence. A basic fact of Cheeger–Gromov convergence is the following. For the proof, see [39].

Theorem 23.4. *Let $\{(N_k, g_k, x_k)\}_{k=1}^{\infty}$ be a sequence of n-dimensional complete pointed Riemannian manifolds. Suppose that*

(1) for each integer $p \geq 0$, there exists a constant $0 < C_p < \infty$ such that

$$|\nabla^p \mathrm{Rm}(g_k)|_{g_k} \leq C_p \quad \text{for all } k \geq 1$$

(2) there exists a constant $0 < \eta < \infty$ such that

$$\mathrm{inj}(x_k, g_k) \geq \eta \quad \text{for all } k \geq 1$$

where $\mathrm{Rm}(g_k)$ is the Riemannian curvature tensor of (N_k, g_k) and $\mathrm{inj}(x_k, g_k)$ is the injectivity radius at x_k with respect to g_k. Then, there exist a complete pointed Riemannian manifold $(N_\infty, g_\infty, x_\infty)$ and a subsequence $\{k_\ell\}_{\ell=1}^{\infty}$ such that the subsequence $\{(N_{k_\ell}, g_{k_\ell}, x_{k_\ell})\}_{\ell=1}^{\infty}$ converges to $(N_\infty, g_\infty, x_\infty)$.

To prove the convergence of submanifolds in a Riemannian manifold, we need the following estimate for the injectivity radius of a submanifold. This estimate is proved by combining Klingenberg's lemma and Hessian comparison theorem of the square of the distance function (cf. Theorem 2.1 in [9]).

Theorem 23.5. *Let (N, g) be an n-dimensional complete Riemannian manifold with*

$$|\mathrm{Rm}(g)| \leq C \quad \text{and} \quad \mathrm{inj}(N, g) \geq \eta$$

for some constants $C, \eta > 0$. Let M be a compact manifold and $F \colon M \to N$ be an immersion map with

$$|A(F)| \leq D$$

with some constant $D > 0$. Then there exists a constant $\delta = \delta(C, \eta, D, n) > 0$ such that

$$\mathrm{inj}(M, F^* g) \geq \delta.$$

The following remark partially overlaps with Remark 17.8.

Remark 23.6. In the remainder of this Section, for a complete Riemannian manifold (N, g), we assume that we have an isometrically embedding $\Theta\colon N \to (\mathbb{R}^L, g_{\mathrm{st}})$ into some higher dimensional Euclidean space with

$$|\nabla^p A(\Theta)| \le \tilde{D}_p < \infty$$

for all $p \ge 0$. Under this assumption, one can see that (N, g) must have the bounded geometry by Theorem 23.5 and Gauss equation (71) (and its iterated derivatives), and note that all compact Riemannian manifolds always satisfy this condition. For a map $F\colon U \to N$ from an open set U in some Riemannian manifold (M, h), by composing Θ, we have a map $\Theta \circ F\colon U \to \mathbb{R}^L$, and furthermore we consider $\Theta \circ F$ as a section of the trivial $\mathbb{R}^L$-bundle over U with a fiber metric g_{st}. We write the standard flat connection of the trivial $\mathbb{R}^L$ bundle by $\bar{\nabla}$. Then $\bar{\nabla}(\Theta \circ F)$ is a section of $T^*M \otimes \mathbb{R}^L$ over U. The Levi–Civita connection on TM and the connection $\bar{\nabla}$ on $\mathbb{R}^L$ induce the connection on $T^*M \otimes \mathbb{R}^L$, and we use the same symbol $\bar{\nabla}$ to denote this connection.

The following is the definition of the convergence of (pointed) immersions. It is the immersion map version of the Cheeger–Gromov convergence.

Definition 23.7. Let (N, g) be a complete n-dimensional Riemannian manifold satisfying the assumption in Remark 23.6 (= Remark 17.8). Assume that for each $k \ge 1$ we have an m-dimensional pointed manifold (M_k, x_k) and an immersion map $F_k\colon M_k \to N$. Then we say that a sequence of immersion maps $\{F_k\colon M_k \to N\}_{k=1}^{\infty}$ converges to an immersion map $F_\infty\colon M_\infty \to N$ from an m-dimensional pointed manifold (M_∞, x_∞) if there exist

(1) an exhaustion $\{U_k\}_{k=1}^{\infty}$ of M_∞ with $x_\infty \in U_k$ and

(2) a sequence of diffeomorphisms $\Psi_k\colon U_k \to V_k \subset M_k$ with $\Psi_k(x_\infty) = x_k$ such that the sequence of maps $F_k \circ \Psi_k\colon U_k \to N$ converges in C^∞ to $F_\infty\colon M_\infty \to N$ uniformly on compact sets in M_∞.

It is clear that if $\{F_k\colon (M_k, x_k) \to N\}_{k=1}^{\infty}$ converges to $F_\infty\colon (M_\infty, x_\infty) \to N$ then $\{(M_k, F_k^* g, x_k)\}_{k=1}^{\infty}$ converges to $(M_\infty, F_\infty^* g, x_\infty)$ in the sense of Cheeger–Gromov convergence. To prove Theorem 23.9, we need the following Lemma. For the proof, see Corollary 4.6 in [11].

Lemma 23.8. *Let M be a manifold, K be a compact set in M and U be an open set in M with $K \subset U$. Assume that we have Riemannian metrics g and $\hat{g}$ on U, and these two satisfy*

$$|\nabla^\ell (g - \hat{g})|_g \le \epsilon_\ell \quad \text{on} \quad K$$

for some constants ϵ_ℓ for all $\ell \ge 0$, where ∇ is the Levi–Civita connection with respect to g. Let $E \to U$ be a vector bundle over U with a fiber metric h and a compatible connection $\bar{\nabla}$, and T be a section of E over U which satisfies

$$|\hat{\nabla}^\ell T|_{\hat{g} \otimes h} \le \hat{C}_\ell \quad \text{on} \quad K$$

for some constants $\hat{C}_\ell$ for all $\ell \ge 0$, where $\hat{\nabla}$ is the connection induced by the Levi–Civita connection with respect to $\hat{g}$ and the connection $\bar{\nabla}$. Then for each $\ell \ge 0$ there exists a constant C_ℓ which depends only on $\{\epsilon_p\}_{p=0}^{\ell}$ and $\{\hat{C}_p\}_{p=0}^{\ell}$ such that

$$|\nabla^\ell T|_{g \otimes h} \le C_\ell \quad \text{on} \quad K,$$

where ∇ is the connection induced by the Levi–Civita connection with respect to g and the connection $\bar{\nabla}$.

Theorem 23.9. *Let (N, g) be a complete n-dimensional Riemannian manifold satisfying the assumption in Remark 23.6 (= Remark 17.8). Let $\{(M_k, x_k)\}_{k=1}^{\infty}$ be a sequence of compact pointed m-dimensional manifolds and $\{F_k\colon M_k \to N\}_{k=1}^{\infty}$ be a sequence of immersions with*

$$|\nabla^p A(F_k)| \le D_p < \infty$$

for all $p \ge 0$. In the case that (N, g) is non-compact, we further assume that $\{F_k(x_k)\}_{k=1}^{\infty}$ is a bounded sequence in N. Then, there exist a pointed manifold (M_∞, x_∞), an immersion $F_\infty\colon M_\infty \to N$ and a subsequence $\{k_\ell\}_{\ell=1}^{\infty}$ such that $\{F_{k_\ell}\colon M_{k_\ell} \to N\}_{\ell=1}^{\infty}$ converges to $F_\infty\colon M_\infty \to N$ and $(M_\infty, F_\infty^* g)$ is a complete Riemannian manifold.

Proof. First of all, we prove that the sequence $\{(M_k, F_k^* g, x_k)\}_{k=1}^{\infty}$ sub-converges to some complete pointed Riemannian manifold $(M_\infty, h_\infty, x_\infty)$. By Remark 23.6 (= Remark 17.8), (N, g) has bounded geometry, that is,

$$|\nabla^p \mathrm{Rm}(g)| \le C_p < \infty \quad \text{and} \quad \mathrm{inj}(N, g) \ge \eta > 0$$

for some positive constants C_p and η. Then, by Theorem 23.5, there exists a constant $\delta = \delta(C_0, \eta, D_0, n) > 0$ such that

$$\mathrm{inj}(M_k, F_k^* g) \ge \delta > 0.$$

We denote the Riemannian curvature tensor of $(M_k, F_k^* g)$ by $\mathrm{Rm}(F_k^* g)$. Then, by Gauss equation (71) and its iterated derivatives, we can see that there exist constants $\tilde{C}_p > 0$ such that

$$|\nabla^p \mathrm{Rm}(F_k^* g)| \le \tilde{C}_p < \infty,$$

for all $p \ge 0$, where each $\tilde{C}_p$ does not depend on k. Then, by Theorem 23.4, $\{(M_k, F_k^* g, x_k)\}_{k=1}^{\infty}$ sub-converges to some complete pointed Riemannian manifold $(M_\infty, h_\infty, x_\infty)$. Note that, in the following in this proof, we continue to use the letter k for indices of subsequences. Since $(M_k, F_k^* g, x_k)$ converge to $(M_\infty, h_\infty, x_\infty)$, there exist an exhaustion U_k of M_∞ with $x_\infty \in U_k$ and a sequence of diffeomorphisms $\Psi_k \colon U_k \to \Psi_k(U_k) \subset M_k$ with $\Psi(x_\infty) = x_k$.

Next, we prove that the sequence of smooth maps $F_k \circ \Psi_k \colon U_k \to N$ sub-converge to some smooth map $F_\infty \colon M_\infty \to N$ uniformly on compact sets in M_∞. We denote $\Theta \circ F_k \circ \Psi_k \colon U_k \to \mathbb{R}^L$ by $\bar{F}_k$ for short. We will use the standard diagonal argument to construct a map $F_\infty \colon M_\infty \to N$. Take a sequence of radii $R_1 < R_2 < \cdots \to \infty$, and consider balls $B_i := B_{h_\infty}(x_\infty, R_i) \subset M_\infty$.

First of all, we work on B_1. Since U_k is an exhaustion, there exists k_1 such that $\overline{B_1} \subset U_k$ for all $k \ge k_1$. Hence we have a sequence of C^∞-maps $\bar{F}_k = \Theta \circ F_k \circ \Psi_k \colon (U_k \supset) B_1 \to \mathbb{R}^L$ restricted on B_1 for all $k \ge k_1$.

<u>(0): C^0-estimate.</u> First, we derive a C^0-bound for $\bar{F}_k$. If N is compact, then the image $\Theta(N)$ is a compact set in $\mathbb{R}^L$ and contained in some ball

$$B_{g_{\mathrm{st}}}(0, \hat{C}_0) = \{\, y \in \mathbb{R}^L \mid |y|_{g_{\mathrm{st}}} < \hat{C}_0 \,\}$$

with radius $\hat{C}_0$. Since each image $\bar{F}_k(B_1)$ is contained in $\Theta(N)$, we have

$$|\bar{F}_k|_{g_{\mathrm{st}}} \le \hat{C}_0 \quad \text{on} \quad \overline{B_1}.$$

It is clear that the constant $\hat{C}_0$ does not depend on k. If N is non-compact, we need some additional argument to get a C^0-bound. Since $|\bar{F}_k^* g_{\mathrm{st}} - h_\infty|_{h_\infty} = |\Psi_k^*(F_k^* g) - h_\infty|_{h_\infty} \to 0$ uniformly on $\overline{B_1}$, for a given $\epsilon > 0$ there exists $k_1'(\ge k_1)$ such that on $\overline{B_1}$

$$|\bar{F}_k^* g_{\mathrm{st}} - h_\infty|_{h_\infty} < \epsilon \quad \text{for} \quad k \ge k_1',$$

and this implies that

$$|\bar{F}_k(x_\infty) - \bar{F}_k(x)|_{g_{\mathrm{st}}} \le \sqrt{1+\epsilon}\, d_{h_\infty}(x_\infty, x) \le \sqrt{1+\epsilon} R_1$$

for all $x \in B_1$ and $k \ge k_1'$. Furthermore, by the assumption for the non-compact case, $\{F_k(x_k)\}_{k=1}^{\infty}$ is a bounded sequence in N. Hence $\bar{F}_k(x_\infty) = (\Theta \circ F_k)(x_k)$ is also a bounded sequence in $\mathbb{R}^L$, that is, there exists a constant $\hat{C}_0'$ such that $|\bar{F}_k(x_\infty)|_{g_{\mathrm{st}}} \le \hat{C}_0'$. Hence we have

$$|\bar{F}_k|_{g_{\mathrm{st}}} \le \hat{C}_0' + \sqrt{1+\epsilon} R_1 =: \hat{C}_0$$

for $k \ge k_1'$. It is clear that $\hat{C}_0$ does not depend on k. Hence we get a C^0-bound.

<u>(1): C^1-estimate.</u> Next, we consider a C^1-bound for $\bar{F}_k$. One can easily see that $\nabla_{g_{\mathrm{st}}} \bar{F}_k = D\bar{F}_k$. Since $\bar{F}_k \colon (B_1, \bar{F}_k^* g_{\mathrm{st}}) \to (\mathbb{R}^L, g_{\mathrm{st}})$ is an isometric immersion, we have a C^1-bound

$$|\nabla_{g_{\mathrm{st}}} \bar{F}_k|_{\bar{F}_k^* g_{\mathrm{st}} \otimes g_{\mathrm{st}}} = |D\bar{F}_k|_{\bar{F}_k^* g_{\mathrm{st}} \otimes g_{\mathrm{st}}} = \sqrt{m} =: \hat{C}_1.$$

<u>(2): C^2-estimate.</u> Next, we derive a C^2-bound for $\bar{F}_k$. Let $\hat{\nabla}$ be the connection on $(\otimes^p T^* M) \otimes \mathbb{R}^L$ $(p \ge 0)$ over B_1 induced by the metric $\bar{F}_k^* g_{\mathrm{st}}$ and g_{st}. Note that $\hat{\nabla} = \nabla_{g_{\mathrm{st}}}$ for $p = 0$. Since $\hat{\nabla} \bar{F}_k = D\bar{F}_k$, we have

$$\hat{\nabla}^2 \bar{F}_k = A(\bar{F}_k),$$

the second fundamental form of the isometric immersion $\bar{F}_k = \Theta \circ F_k \circ \Psi_k \colon (B_1, \bar{F}_k^* g_{\mathrm{st}}) \to (\mathbb{R}^L, g_{\mathrm{st}})$.

Hence, by using the composition rule for the second fundamental forms of immersions, we have

$$\begin{aligned}\hat{\nabla}^2 \bar{F}_k(X,Y) =& A(\bar{F}_k)(X,Y)\\ =& A(\Theta)((F_k \circ \Psi_k)_* X, (F_k \circ \Psi_k)_* Y) + \Theta_*(A(F_k)(\Psi_{k*}X, \Psi_{k*}Y))\end{aligned}$$

for any tangent vectors X and Y on M. By using the notion of $*$-product, this identity is written as

$$\hat{\nabla}^2 \bar{F}_k = A(\Theta) * (\overset{2}{*} D(F_k \circ \Psi_k)) + A(F_k) * D\Theta * (\overset{2}{*} D\Psi_k). \tag{78}$$

Since $|D(F_k \circ \Psi_k)|_{\bar{F}_k^* g_{\mathrm{st}} \otimes g} = |D\Psi_k|_{\bar{F}_k^* g_{\mathrm{st}} \otimes F_k^* g} = \sqrt{m}$ and $|D\Theta|_{g \otimes g_{\mathrm{st}}} = \sqrt{n}$, we have

$$|\hat{\nabla}^2 \bar{F}_k|_{\bar{F}_k^* g_{\mathrm{st}} \otimes g_{\mathrm{st}}} \le \hat{C}_2' |A(\Theta)|_{g \otimes g_{\mathrm{st}}} + \hat{C}_2'' |A(F_k)|_{F_k^* g \otimes g}$$

for some constants $\hat{C}_2'$ and $\hat{C}_2''$ which do not depend on k. Furthermore, by the assumptions, we have $|A(\Theta)|_{g \otimes g_{\mathrm{st}}} \le \tilde{D}_0$ and $|A(F_k)|_{F_k^* g \otimes g} \le D_0$. Hence we have a C^2-bound

$$|\hat{\nabla}^2 \bar{F}_k|_{\bar{F}_k^* g_{\mathrm{st}} \otimes g_{\mathrm{st}}} \le \hat{C}_2' \tilde{D}_0 + \hat{C}_2'' D_0 =: \hat{C}_2.$$

It is clear that $\hat{C}_2$ does not depend on k.

<u>(p): C^p-estimate.</u> By differentiating (78), we can get a C^p-bound. We only observe a C^3-bound. Note that for any tangent vectors X and Y on M we have

$$(\nabla_{\bar{F}_k^* g_{\mathrm{st}} \otimes g} D(F_k \circ \Psi_k))(X,Y) = A(F_k \circ \Psi_k)(X,Y) = A(F_k)(\Psi_{k*}X, \Psi_{k*}Y).$$

By using the notion of $*$-product, this identity is written as

$$\nabla_{\bar{F}_k^* g_{\mathrm{st}} \otimes g} D(F_k \circ \Psi_k) = A(F_k) * (\overset{2}{*} D\Psi_k).$$

Furthermore, note that $\nabla_{g \otimes g_{\mathrm{st}}} D\Theta = A(\Theta)$ and $\nabla_{\bar{F}_k^* g_{\mathrm{st}} \otimes F_k^* g} D\Psi_k = 0$. Hence we have

$$\begin{aligned}\hat{\nabla}^3 \bar{F}_k =& \nabla_{g \otimes g_{\mathrm{st}}} A(\Theta) * (\overset{2}{*} D(F_k \circ \Psi_k))\\ &+ 2A(\Theta) * D(F_k \circ \Psi_k) * A(F_k) * (\overset{2}{*} D\Psi_k)\\ &+ \nabla_{F_k^* g \otimes g} A(F_k) * D\Theta * (\overset{2}{*} D\Psi_k)\\ &+ A(F_k) * A(\Theta) * (\overset{2}{*} D\Psi_k).\end{aligned}$$

By the assumptions, norms of all tensors appeared in the above inequality is bounded. Hence we have a C^3-bound

$$|\hat{\nabla}^3 \bar{F}_k|_{\bar{F}_k^* g_{\mathrm{st}} \otimes g_{\mathrm{st}}} \le \hat{C}_3$$

for some constant $\hat{C}_3$ which does not depend on k. For higher derivatives, one can prove that there exists a constant $\hat{C}_p > 0$ which does not depend on k such that

$$|\hat{\nabla}^p \bar{F}_k|_{\bar{F}_k^* g_{\mathrm{st}} \otimes g_{\mathrm{st}}} \le \hat{C}_p,$$

by induction.

On the above argument, we have proved that there exist constants $\hat{C}_p$ ($p \ge 0$) which do not depend on k such that $|\hat{\nabla}^p \bar{F}_k|_{\bar{F}_k^* g_{\mathrm{st}} \otimes g_{\mathrm{st}}} \le \hat{C}_p$. Hence by Lemma 23.8 we can prove that there exist constants C_p ($p \ge 0$) which do not depend on k such that

$$|\nabla^p \bar{F}_k|_{h_\infty \otimes g_{\mathrm{st}}} \le C_p.$$

Hence, by The Arzelà–Ascoli Theorem, there exists a smooth map $\bar{F}_{1,\infty} \colon \overline{B}_1 \to \mathbb{R}^L$ and $\bar{F}_k$ sub-converges to $\bar{F}_{1,\infty}$ in C^∞ on $\overline{B}_1$. Since all images $\bar{F}_k(\overline{B}_1)$ are contained in $\Theta(N)$, the image $\bar{F}_{1,\infty}(\overline{B}_1)$ is also contained in $\Theta(N)$. Furthermore $\bar{F}_{1,\infty} \colon \overline{B}_1 \to \mathbb{R}^L$ has the property that

$$\bar{F}_{1,\infty}^* g_{\mathrm{st}} = h_\infty,$$

since $|\bar{F}_{1,\infty}^* g_{\mathrm{st}} - h_\infty|_{h_\infty} \le |\bar{F}_{1,\infty}^* g_{\mathrm{st}} - \bar{F}_k^* g_{\mathrm{st}}|_{h_\infty} + |\bar{F}_k^* g_{\mathrm{st}} - h_\infty|_{h_\infty}$ and the right hand side converges to 0 as $k \to \infty$ on $\overline{B}_1$. Thus, especially, $\bar{F}_{1,\infty} \colon \overline{B}_1 \to \mathbb{R}^L$ is an immersion map.

Next, for the subsequence of $\bar{F}_k$ which converges to $\bar{F}_{1,\infty}$, we work on B_2. Then all the above argument also work on B_2 and we can show that there exists a smooth immersion map $\bar{F}_{2,\infty} \colon \overline{B}_2 \to \Theta(N) \subset \mathbb{R}^L$ with $\bar{F}_{2,\infty}^* g_{\mathrm{st}} = h_\infty$ and $\bar{F}_{2,\infty} = \bar{F}_{1,\infty}$ on $\overline{B}_1$ and $\bar{F}_k$ sub-converges to $\bar{F}_{2,\infty}$ in C^∞ on $\overline{B}_2$. By iterating this construction and the diagonal argument, finally we get a smooth immersion map $\bar{F}_\infty \colon M_\infty \to \Theta(N) \subset \mathbb{R}^L$ with $\bar{F}_\infty^* g_{\mathrm{st}} = h_\infty$ and $\bar{F}_k$ sub-converges to $\bar{F}_\infty$ uniformly on compact sets in M_∞ in C^∞, and the map defined by $F_\infty := \Theta^{-1} \circ \bar{F}_\infty \colon M_\infty \to N$ is the requiring one satisfying the properties in the statement. $\square$

Part IV

Lagrangian self-similar solutions in gradient shrinking Kähler-Ricci solitons

abstract In this Part, we give a lower bound estimate for the diameter of a Lagrangian self-shrinker in a gradient shrinking Kähler-Ricci soliton as an analog of a result of A. Futaki, H. Li and X.-D. Li [15] for a self-shrinker in a Euclidean space. We also prove an analog of a result of H.-D. Cao and H. Li [6] about the non-existence of compact self-expanders in a Euclidean space.

24 Introduction

A gradient shrinking Kähler-Ricci soliton is a Kähler manifold (N, ω, g, J) with a smooth function $f : N \to \mathbb{R}$ satisfying

$$\mathrm{Ric}(g) + \mathrm{Hess}_g f = g. \tag{79}$$

By the equation (79), it follows that the $(2,0)$-part of $\mathrm{Hess}\, f$ is zero. Hence it is clear that the $(1,0)$-part of ∇f is a holomorphic vector field on N. By a simple calculation, it is proved that the gradient of $R(g) + |\nabla f|^2 - 2f$ is zero, and we put a constant C_0 by

$$C_0 := R(g) + |\nabla f|^2 - 2f, \tag{80}$$

where $R(g)$ is the scalar curvature of (N, g). It is proved that $R(g) \geq 0$ for a complete gradient shrinking Ricci soliton by an application of Corollary 2.5 in [8].

For an immersion $F\colon L \to N$, we get a section $(\nabla f) \circ F \in \Gamma(L, F^*(TN))$, and we usually omit the symbol $\circ F$, for short.

Definition 24.1. An immersion map $F\colon L \to N$ is called a self-similar solution if it satisfies

$$H = \lambda \nabla f^{\perp} \tag{81}$$

for some constant $\lambda \in \mathbb{R}$, where H is the mean curvature vector field of F and $\perp$ denotes the projection onto the normal bundle of L. It is called a self-shrinker, a steady soliton or a self-expander when $\lambda < 0$, $\lambda = 0$ or $\lambda > 0$, respectively.

For example, a function $f(z^1, \ldots, z^m) := \frac{1}{2}(|z^1|^2 + \cdots + |z^m|^2)$ on $\mathbb{C}^m$ with the standard Kähler structure satisfies the identity (79), and it satisfies $\nabla f(x) = x$ under a natural identification of points and tangent vectors for all points $x \in \mathbb{C}^m \cong \mathbb{R}^{2m}$. Hence the equation (81) coincides with $H_x = \lambda x^{\perp}$ for all points $x \in F(L) \subset \mathbb{C}^m \cong \mathbb{R}^{2m}$, and Definition 24.1 can be considered as a generalization of a self-similar solution in a Euclidean space to in a gradient shrinking Ricci soliton.

There are many results about self-similar solutions in a Euclidean space. By a generalization of the notion of a self-similar solution in a Euclidean space to in a gradient shrinking Ricci soliton as in Definition 24.1, we can discuss which results about self-similar solutions in a Euclidean space also hold in a gradient shrinking Ricci soliton. As an example of such results, it is proved that a part of a result due to Smoczyk also holds in a gradient shrinking Kähler-Ricci soliton. More precisely, in the proof of Theorem 2.3.5 in [44], Smoczyk proved that every compact Lagrangian self-similar solution with exact mean curvature form is a minimal submanifold in $\mathbb{C}^n$, and as a generalization of this statement, it is proved in [53] that every compact Lagrangian self-similar solution with exact mean curvature form is a minimal submanifold in a gradient shrinking Kähler-Ricci soliton.

In this Part, we give further two results which are already established when (N, g) is a Euclidean space. The first result is an analog of Theorem 4.3 of [15] under the Lagrangian assumption.

Theorem 24.2. *Let (N, ω, g, J) be a $2m$-dimensional gradient shrinking Kähler-Ricci soliton with potential function $f\colon N \to \mathbb{R}$ satisfying the equation (79). Let $F\colon L \to N$ be a compact Lagrangian self-shrinker with*

$$H = -\frac{1}{2}\nabla f^{\perp}.$$

Assume that $F(L)$ is not contained in $\{ f = m - \frac{C_0}{2} \}$, where C_0 is a constant defined by (80). Then we have

$$\mathrm{diam}(L, F^*g) \geq \frac{\pi}{\sqrt{\frac{3}{4} + \frac{m}{2}(K_0 + A_0^2)}},$$

for constants $K_0, A_0 \geq 0$ satisfying $|K_N| \leq K_0$ and $|A| \leq A_0$, where K_N is the sectional curvature of (N, g) and A is the second fundamental form of F.

The second result is an analog of Proposition 5.3 of [6] under the Lagrangian assumption.

Theorem 24.3. *Let (N, ω, g, J) be a $2m$-dimensional gradient shrinking Kähler-Ricci soliton with potential function $f\colon N \to \mathbb{R}$ satisfying the equation (79). Then we have the following.*

- *If $R(g) > 2m$, there exists no compact Lagrangian self-shrinker in N.*
- *If $R(g) < 2m$, there exists no compact Lagrangian self-expander in N.*
- *If $R(g) = 2m$, every compact Lagrangian self-similar solution in N is a minimal submanifold.*

The rest of this Part is organized as follows. In Section 25, we give some characterization of self-similar solutions in gradient shrinking Ricci solitons. In Section 26, we give a proof of Theorem 24.2 and 24.3.

25 Characterization of self-similar solutions

In this section, we give some characterization of self-similar solutions in gradient shrinking Ricci solitons and review a result in [53].

The first characterization is given by the variation of the weighted volume as follows. Let (N, g, f) be a n-dimensional gradient shrinking Ricci soliton with potential function f satisfying (79). For an m-dimensional compact manifold L and a constant $\lambda \in \mathbb{R}$, we define the weighted volume functional $\mathcal{F}_\lambda$ by

$$\mathcal{F}_\lambda(F) := \int_L e^{\lambda f} d\mu(F^*g)$$

for each immersion $F\colon L \to N$, where $d\mu(F^*g)$ is the induced measure on L with respect to the metric F^*g.

Proposition 25.1. *Let $F\colon L \to N$ be an immersion and $\lambda \in \mathbb{R}$ be a constant. Then the following three conditions are equivalent.*

1. *F is a self-similar solution with $H = \lambda \nabla f^\perp$.*
2. *F is a minimal immersion with respect to a metric $e^{2\lambda f/m} g$ on N.*
3. *F is a critical point of $\mathcal{F}_\lambda$.*

The equivalence of (1) and (2) is proved in [53], and the equivalence of (2) and (3) can be easily proved by the equality

$$\int_L e^{\lambda f} d\mu(F^*g) = \int_L d\mu(F^*(e^{2\lambda f/m} g)).$$

The equivalence of (1) and (3) can be considered as a generalization of Proposition 3.6 in [12].

The second characterization is given by the asymptotic behavior of a Ricci-mean curvature flow, the coupled equation of the Ricci flow and the mean curvature flow. Let (N, ω, g, J) be a compact $2m$-dimensional complete gradient shrinking Kähler-Ricci soliton with a potential function $f\colon N \to \mathbb{R}$ satisfying (79). Fix a time $T > 0$. Then for $t \in [0, T)$ we define $g_t := (T - t)\Phi_t^* g$, where $\{\Phi_t\colon N \to N\}_{t \in (-\infty, T)}$ is the 1-parameter family of holomorphic automorphisms of (N, J) with $\Phi_0 = \mathrm{id}_N$ generated by the time dependent vector field $\frac{1}{2(T-t)} \nabla f$. Then g_t is a solution of Kähler-Ricci flow, that is, the associated Kähler form $\omega_t(\cdot, \cdot) := g_t(J\cdot, \cdot)$ satisfies

$$\frac{\partial}{\partial t} \omega_t = -\rho(\omega_t),$$

where $\rho(\cdot,\cdot) := \mathrm{Ric}(J\cdot,\cdot)$ is the Ricci form of ω_t. Here we review the main result in [53]. Let L be an m-dimensional compact manifold and $F\colon L\times[0,T)\to N$ be a solution of Ricci-mean curvature flow along $g_t = (T-t)\Phi_t^* g$, that is, F satisfies

$$\frac{\partial}{\partial t}F_t = H(F_t),$$

where $H(F_t)$ is the mean curvature vector field of $F_t(\cdot) := F(\cdot,t)$ calculated by g_t at each time t. Assume that the initial immersion $F_0\colon L\to N$ is a Lagrangian immersion for the initial Kähler form ω. Then, it follows that $F_t\colon L\to N$ is also a Lagrangian immersion with respect to ω_t for all $t\in[0,T)$ (c.f. [31]). That is, the Lagrangian condition is preserved under a Ricci-mean curvature flow along a Kähler-Ricci flow. We further assume that F develops a singularity of type I, that is, the norm of the second fundamental form of F_t (denoted by $A(F_t)$) satisfies

$$\limsup_{t\to T}\bigl(\sqrt{T-t}\,\max_L|A(F_t)|\bigr) < \infty.$$

Then, in [53], it is proved that for any sequence $t_j\to T$ and any point $p_0\in L$ the family of pointed immersions $\tilde F_j\colon (L,p_0)\to N$ defined by $\tilde F_j := \Phi_{t_j}\circ F_{t_j}$ subconverges to a pointed immersion $\tilde F_\infty\colon (L_\infty,p_\infty)\to N$ satisfying

$$H(\tilde F_\infty) = -\frac{1}{2}\nabla f^\perp.$$

This result can be considered as a generalization of Huisken's result in [23] for a mean curvature flow in a Euclidean space.

Since each $\tilde F_j$ is a Lagrangian immersion in (N,ω) and the Lagrangian condition ($\tilde F_j^*\omega = 0$) is a closed condition, it follows that $\tilde F_\infty\colon L_\infty\to N$ is a Lagrangian immersion, that is, the Lagrangian self-shrinker. Hence a Lagrangian self-shrinker is an asymptotic model of a Lagrangian mean curvature flow with a type I singularity along a Kähler-Ricci flow constructed from a gradient shrinking Kähler-Ricci soliton.

Remark 25.2. Actually, the same statement also holds under some additional assumptions even though N is non-compact and complete, see [53] for detail. The differences of factor 2 or 1/2 between coefficients appeared in some formula in this Part and those in [53] arise from the difference of factor 2 between the Ricci flow equation $\partial_t g_t = -2\mathrm{Ric}(g_t)$ and the Kähler-Ricci flow equation $\partial_t\omega_t = -\rho(\omega_t)$.

26 Proofs of Theorem 24.2 and 24.3

First, we give a proof of Theorem 24.2. The proof is an analog of the proof of Theorem 4.3 of [15]. As the first step, we prove that the weighted Laplacian $\underline{\Delta}_\phi$ defined below has an eigenvalue 1. In the second step, we use Theorem 1.1 in [15] giving an estimate for the first eigenvalue of the weighted Laplacian.

Proof of Theorem 24.2. Put a smooth function on L by $\phi := \frac{1}{2}f\circ F$ and consider the weighted Laplacian

$$\mathcal{L} := \underline{\Delta}_\phi := \underline{\Delta} - \overline{\nabla}\phi\cdot\overline{\nabla}$$

which acts on $C^\infty(L)$, where $\underline{\Delta}$ and $\overline{\nabla}$ is the Laplacian and the gradient on (L,F^*g). Then we have

$$\begin{aligned}\mathcal{L}\phi =& \frac{1}{2}\underline{\Delta}(f\circ F) - \frac{1}{4}(F^*g)(\overline{\nabla}(f\circ F),\overline{\nabla}(f\circ F))\\ =& \frac{1}{2}\underline{\Delta}(f\circ F) - \frac{1}{4}|\nabla f^\top|^2\\ =& \frac{1}{2}\left(\mathrm{tr}^\top\mathrm{Hess}_g f + g(\nabla f,H)\right) - \frac{1}{4}|\nabla f^\top|^2,\end{aligned}$$

where $\mathrm{tr}^\top$ is the tangential trace, that is, $\mathrm{tr}^\top B := \mathrm{tr}_{F^*g}(F^*B)$ for a 2-tensor B on N. Since $H = -\frac{1}{2}\nabla f^\perp$, we have

$$\frac{1}{2}g(\nabla f,H) - \frac{1}{4}|\nabla f^\top|^2 = -\frac{1}{4}|\nabla f^\perp|^2 - \frac{1}{4}|\nabla f^\top|^2 = -\frac{1}{4}|\nabla f|^2.$$

Since (N, g) is a gradient shrinking Ricci soliton satisfying (79), we have

$$\mathrm{tr}^{\top}\mathrm{Hess}_g f = \mathrm{tr}^{\top}(g - \mathrm{Ric}(g)) = m - \mathrm{tr}^{\top}\mathrm{Ric}(g).$$

Furthermore, since $F : L \to N$ is a Lagrangian immersion and (N, g, J) is a Kähler manifold, we have

$$\begin{aligned}\mathrm{tr}^{\top}\mathrm{Ric}(g) =& \sum_{i=1}^{m} \mathrm{Ric}(g)(F_*e_i, F_*e_i)\\ =& \frac{1}{2}\sum_{i=1}^{m} \mathrm{Ric}(g)(F_*e_i, F_*e_i) + \frac{1}{2}\sum_{i=1}^{m} \mathrm{Ric}(g)(JF_*e_i, JF_*e_i) = \frac{1}{2}R(g),\end{aligned}$$

for an orthonormal basis $e_1, \dots, e_m$ on (L, F^*g). Hence we have

$$\mathcal{L}\phi = \frac{m}{2} - \frac{1}{4}(R(g) + |\nabla f|^2) = \frac{m}{2} - \frac{1}{4}(C_0 + 2f) = \left(\frac{m}{2} - \frac{C_0}{4}\right) - \phi.$$

Since $\mathcal{L}(\mathrm{const}) = 0$, we have

$$\mathcal{L}\left(\left(\frac{m}{2} - \frac{C_0}{4}\right) - \phi\right) = -\left(\left(\frac{m}{2} - \frac{C_0}{4}\right) - \phi\right).$$

By the assumption, $\frac{m}{2} - \frac{C_0}{4} - \phi \neq 0$. Hence we have proved that 1 is an eigenvalue of the weighted Laplacian $\mathcal{L} = \Delta_{\phi}$.

Next, we prove that

$$\mathrm{Ric}(F^*g) + \mathrm{Hess}_{F^*g}\phi \geq \kappa F^*g, \tag{82}$$

for a Riemannian manifold (L, F^*g), where

$$\kappa := \frac{1}{2} - m(K_0 + A_0^2).$$

Let X be a tangent vector on L and $e_1, \dots, e_m$ be an orthonormal basis on (L, F^*g). Then, by the Gauss equation, we have

$$\begin{aligned}\mathrm{Ric}(F^*g)(X, X) =& \sum_{i=1}^{m} \mathrm{Rm}(F^*g)(X, e_i, X, e_i)\\ =& \sum_{i=1}^{m} \mathrm{Rm}(g)(F_*X, F_*e_i, F_*X, F_*e_i)\\ & - \sum_{i=1}^{m} |A(X, e_i)|^2 + g(A(X, X), H).\end{aligned}$$

Furthermore, we have

$$\begin{aligned}\mathrm{Hess}_{F^*g}\phi(X, X) =& \frac{1}{2}\mathrm{Hess}_{F^*g}(f \circ F)(X, X)\\ =& \frac{1}{2}\Big(\mathrm{Hess}_g f(F_*X, F_*X) + g(A(X, X), \nabla f)\Big)\\ =& \frac{1}{2}|X|^2 - \frac{1}{2}\mathrm{Ric}(g)(F_*X, F_*X) - g(A(X, X), H),\end{aligned}$$

where we used $\mathrm{Hess}_g f = g - \mathrm{Ric}$ and $\nabla f^{\perp} = -2H$. Since

$$\begin{aligned}&\sum_{i=1}^{m} \mathrm{Rm}(g)(F_*X, F_*e_i, F_*X, F_*e_i) - \frac{1}{2}\mathrm{Ric}(g)(F_*X, F_*X)\\ =& \frac{1}{2}\sum_{i=1}^{m} \mathrm{Rm}(g)(F_*X, F_*e_i, F_*X, F_*e_i) - \frac{1}{2}\sum_{i=1}^{m} \mathrm{Rm}(g)(F_*X, JF_*e_i, F_*X, JF_*e_i)\\ \geq& -\frac{1}{2}mK_0|X|^2 - \frac{1}{2}mK_0|X|^2 = -mK_0|X|^2\end{aligned}$$

and

$$-\sum_{i=1}^{m} |A(X, e_i)|^2 \geq -mA_0|X|^2,$$

we have

$$\mathrm{Ric}(F^*g)(X,X) + \mathrm{Hess}_{F^*g}\phi(X,X) \geq \left(\frac{1}{2} - m(K_0 + A_0^2)\right)|X|^2.$$

Hence the inequality (82) holds.

Thus, by Theorem 1.1 in [15], we have

$$1 \geq \sup_{s\in(0,1)} \left\{4s(1-s)\frac{\pi^2}{d^2} + s\kappa\right\}, \tag{83}$$

where $d := \mathrm{diam}(L, F_*g)$. Choosing $s = \frac{1}{2}$ in (83), we have

$$d \geq \frac{\pi}{\sqrt{1-\frac{1}{2}\kappa}} = \frac{\pi}{\sqrt{\frac{3}{4} + \frac{m}{2}(K_0 + A_0^2)}}.$$

Hence the proof is completed. □

Next, we prove Theorem 24.3. The proof is an analog of the proof of Proposition 5.3 of [6].

Proof of Theorem 24.3. Let $F\colon L^m \to N^{2m}$ be a compact Lagrangian self-similar solution with $H = \lambda\nabla f^\perp$ for some constant $\lambda \in \mathbb{R}$. Using computations in the proof of Theorem 24.2, we have

$$\begin{aligned}\underline{\Delta}(f\circ F) &= \mathrm{tr}^\top \mathrm{Hess}_g f + g(\nabla f, H)\\ &= m - \frac{1}{2}R(g) + \frac{1}{\lambda}|H|^2\end{aligned}$$

if $\lambda \neq 0$, where $\underline{\Delta}$ is the Laplacian on (L, F^*g). Hence we have

$$0 = \int_L \underline{\Delta}(f\circ F)d\mu(F^*g) = \int_L \left(m - \frac{1}{2}R(g) + \frac{1}{\lambda}|H|^2\right)d\mu(F^*g),$$

and the theorem holds immediately. □

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