博士論文

論文題目　Mathematical analysis for diffusion equations with generalized fractional time derivatives

（一般化された非整数階時間微分項を持つ拡散方程式に対する数学解析について）

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Contents

Preface

Within the last few decades, the anomalous diffusion processes which cannot be adequately modeled by the classical diffusion equation have been observed and confirmed in several different application areas in biology, geological sciences, medicine, etc.. One of the possibilities to interpret the anomalous phenomenon in some anomalous diffusion processes on the macro level is to use the time-fractional diffusion equations. There have been numerous important works involving these kinds of time-fractional diffusion equations from physical or mathematical aspects. Here we do not intend any lists of references and we refer only to [36], [37] and [56] and the references therein. In Chapter 1–3, we continue the research activities of [36], [37] and [56], and investigate the forward and inverse problems for diffusion equation of the single- or multi-term fractional derivatives.

However, by integrating the fractional derivatives over the order of the derivative within a given range (say, $[0,1]$), we arrive at the distributed order derivatives

$$\mathbb{D}_t^{(\mu)}\varphi(t) = \int_0^1 \partial_t^\alpha \varphi(t)\mu(\alpha)\mathrm{d}\alpha,$$

where μ is a non-negative continuous function on $[0,1]$, and ∂_t^α is the Caputo fractional derivative of order α:

$$\partial_t^\alpha \varphi(t) = \begin{cases} \varphi(t), & \alpha = 0, \\ \dfrac{1}{\Gamma(1-\alpha)}\displaystyle\int_0^t \frac{\varphi'(\tau)}{(t-\tau)^\alpha}\mathrm{d}\tau, & 0<\alpha<1, \\ \varphi'(t), & \alpha = 1. \end{cases}$$

The distributed order derivative was introduced for the first time in [9] and soon attracted attention of physicists who recognized that these equations can serve as models for a so-called ultra slow diffusion processes which have been found in polymer physics, kinetics of particles moving in the quenched random force fields etc..

Let us mention that the single or multi-term time-fractional derivatives can be formally obtained from the distributed order derivative by setting the weight function in form of a finite linear combination of the Dirac δ-functions with the positive weight coefficients. In Chapter 4, forward and inverse problems for distributed order fractional diffusion equation

$$\begin{cases} \mathbb{D}_t^{(\mu)}u = -\mathcal{A}u + B(x)\cdot\nabla u + c(x)u + F & \text{in } \Omega\times(0,T), \\ u(x,0) = a, & \text{in } \Omega, \\ u(x,t) = 0, & \text{on } \partial\Omega\times(0,T) \end{cases} \tag{0.1}$$

are considered. Here Ω is assumed to be a bounded domain in $\mathbb{R}^d$ with sufficiently smooth boundary $\partial\Omega$. The operator $\mathcal{A}$ denotes a symmetric elliptic operator (e.g., $\mathcal{A} = -\Delta$).

Chapter 1

In this chapter, we consider the initial-boundary value problem for the single-term time-fractional diffusion equation. On the basis of unique continuation for parabolic equations, we establish the following unique continuation principle, which is slightly weaker than that for the parabolic case.

Theorem 0.1 *Let* $0 < \alpha < 1$, $F = 0$ *and* $(B, c) \in (L^\infty(\Omega))^{d+1}$ *in* (1.7). *Furthermore, we suppose that* $u \in C([0,T]; L^2(\Omega)) \cap C((0,T]; H^{2\gamma}(\Omega) \cap H_0^1(\Omega))$ $(\gamma \in (\frac{1}{2}, 1))$ *satisfies* (1.7). *Let* $\omega \subset \Omega$ *be an arbitrarily chosen subdomain.*

Then

$$u = 0 \text{ in } \omega \times (0,T) \text{ implies } u = 0 \text{ in } Q$$

Next, we consider an inverse source problem for (0.1) under the assumption that the inhomogeneous term F is in form of separation of variables.

Problem 0.1 *Let the subdomain* $\omega \subset \Omega$ *and* $T > 0$ *be any given. Assume that the initial value* $a = 0$ *and the source term* $F(x,t) = \rho(t)f(x)$ *in* (0.1) *where* ρ *is given, and let* u *satisfy* (0.1). *Determine* $f(x)$ *by the interior observation*

$$u|_{\omega \times (0,T)}.$$

As an application of the weak unique continuation, a uniqueness for determining the source term by interior measurement is proved.

Theorem 0.2 *Let* $a = 0$, $f \in L^2(\Omega)$ *and* $\rho \in C^1[0,T]$ *with* $\rho(0) \neq 0$ *in the initial-boundary value problem* (0.1), *and suppose that* ω *be an arbitrary open subset of* Ω. *Then* $u = 0$ *in* $\omega \times (0,T)$ *implies* $f = 0$ *in* Ω.

Chapter 2

As a natural extension, we consider the case of $\mu = \sum_{j=1}^{\ell} q_j(x,t)\delta(\cdot - \alpha_j)$, $0 < \alpha_\ell < \cdots < \alpha_1 < 1$ which derives the multi-term time-fractional diffusion equation. This equation is expected to improve the modeling accuracy in depicting the anomalous diffusion due to its potential feasibility.

Firstly, we apply the eigenfunction expansion and Fredholm principle for compact operator to prove unique existence as well as regularity of solution.

Theorem 0.3 *Let* $0 < \alpha_\ell < \cdots < \alpha_1 < 1$ *and* $T > 0$ *be fixed constants. Assuming that* $q_1 = 1$, $q_j \in W^{2,\infty}(\Omega)$ $(j = 2, \cdots, \ell)$, *and* $(B, c) \in (L^\infty(\Omega))^{d+1}$. *Then for any fixed constant* $\gamma \in [\frac{1}{2}, 1)$, *the initial-boundary value problem* (0.1) *with* $F = 0$ *and* $a \in L^2(\Omega)$ *admits a unique mild solution* $u \in C([0,T]; L^2(\Omega)) \cap C((0,T] : H^{2\gamma}(\Omega) \cap H_0^1(\Omega))$ *such that*

$$\|u(t)\|_{H^{2\gamma}(\Omega)} \leq Ct^{-\alpha_1\gamma}e^{CT}\|a\|_{L^2(\Omega)}, \quad t \in (0,T].$$

Moreover $u : (0,T] \to H^{2\gamma}(\Omega)$ *can be analytically extended to the sector* $\{z \neq 0; |\arg z| < \frac{\pi}{2}\}$.

Theorem 0.4 *Let* $0 < \alpha_\ell < \cdots < \alpha_1 < 1$ *and* $T > 0$ *be given. Assuming that* $(B, c) \in (L^\infty(\Omega))^{d+1}$, $q_j \in L^\infty(\Omega)(j = 2, \cdots, \ell)$. *Let* $F \in L^2(0,T; L^2(\Omega))$, $a = 0$, *then the initial-boundary value problem* (0.1) *admits a unique weak solution* $u \in L^2(0,T; H^2(\Omega) \cap H_0^1(\Omega)) \cap H^{\alpha_1}(0,T; L^2(\Omega))$. *Moreover the following estimate holds:*

$$\|u\|_{H^{\alpha_1}(0,T;L^2(\Omega))} + \|u\|_{L^2(0,T;H^2(\Omega))} \leq C_T\|F\|_{L^2(0,T;L^2(\Omega))}.$$

Secondly, we investigated some further properties of the solution to our initial-boundary value problem. In the case when all the coefficients q_j are positive, $B = 0$ and $F = 0$, according to the argument in the proof of Watson's lemma, we derived that the decay rate of the solution is determined by the lowest order α_ℓ, that is,

Theorem 0.5 *Let $\alpha_j \in (0,1)$ be constants such that $\alpha_\ell < \cdots < \alpha_1$, and $\{q_j\}_{j=1}^{\ell}$ be in $W^{2,\infty}(\Omega)$ with $q_j > 0$ in $\overline{\Omega}$. We further assume that $c \in L^\infty(\Omega)$ such that $c \le 0$ in Ω. Then there exists $v(t) \in H_0^1(\Omega) \cap H^2(\Omega)$, the unique solution of the initial-boundary value problem*

$$\begin{cases} q_\ell(x)\partial_t^{\alpha_\ell} v(x,t) = -\mathcal{A}v(x,t) + c(x)v(x,t), & x \in \Omega,\ t > 0, \\ v(x,0) = a(x), & x \in \Omega, \\ v(x,t) = 0, & x \in \partial\Omega,\ t > 0, \end{cases}$$

where has the same asymptotic behavior as u, in the sense that

$$\|u(\cdot\,,t) - v(\cdot\,,t)\|_{H^2(\Omega)} = O(t^{-\min\{2\alpha_\ell,\alpha_{\ell-1}\}})\|a\|_{L^2(\Omega)} \text{ as } t \to \infty.$$

It is a remarkable property of fractional diffusion equation since the classical advection equation admits non-zero solutions decaying exponentially. This is one description of the slower diffusion, compared to the classical one.

Finally, when we consider (0.1) as model equation for describing e.g., anomalous diffusion in inhomogeneous media, the orders α_j of fractional derivatives should be determined by the inhomogeneity of the media, but it is not clear which physical law can correspond the inhomogeneity to the orders α_j. Thus one reasonable way for estimating α_j is an inverse problem of determining $\alpha_1, ..., \alpha_\ell$ in order to match available data such as using the interior observation or by using the method of Dirichlet-to-Neumann map. We firstly investigate inverse problem of identifying fractional orders from pointwise observation. Secondly, we consider an inverse boundary value problem for diffusion equation (0.1). We prove that the Dirichlet-to-Neumann map uniquely determines the number of fractional time-derivative terms, the orders of the derivatives and spatially varying coefficients.

Chapter 3

In the case of $\mu = \delta(\cdot - 1) + \sum_{j=1}^{\ell} q_j(x,t)\delta(\cdot - \alpha_j)$, $0 < \alpha_\ell < \cdots < \alpha_1 < \frac{1}{2}$, letting $d \in C^2(\overline{\Omega})$ and $|\nabla d| \ne 0$ on $\overline{\Omega}$ and setting $\psi = \zeta(x) - \beta t^{2-2\alpha_1}$ with $\beta > 0$. We first discuss the derivation of a Carleman estimate for $L_0 = \partial_t - \sum_{i,j=1}^{d} a_{ij}(x,t)\partial_i\partial_j$ with the new weight function $\varphi := e^{\lambda\psi}$. Namely

Theorem 0.6 *Let $\Sigma_0 = \overline{\Omega} \times \{0\}$ and $D \subset Q$ be bounded domain whose boundary ∂D is composed of a finite number of smooth surfaces. Then there exists a constant $\lambda_0 > 0$ such that for arbitrary $\lambda \ge \lambda_0$, we can choose a constant $s_0(\lambda) > 0$ satisfying: there exists a constant $C = C(s_0, \lambda_0)$ such that*

$$\begin{aligned} &\int_D \left\{ \frac{1}{s\varphi}|\partial_t u|^2 + s\lambda^2\varphi|\nabla u|^2 + s^3\lambda^4\varphi^3 u^2 \right\} e^{2s\varphi} dxdt \\ \le & C\int_D |L_0 u|^2 e^{2s\varphi} dxdt + e^{C(\lambda)s}\int_{\partial D}(|\nabla u|^2 + |u|^2)dSdt + e^{C(\lambda)s}\int_{\partial D\setminus\Sigma_0}|\partial_t u|^2 dSdt \end{aligned}$$

for all $s > s_0$ and all $u \in H^{2,1}(D)$.

On the basis of the above Carleman estimates for parabolic equations with the weight function ψ, we prove a Hölder stability for the generalized fractional advection dispersion equations by considering the fractional derivative as perturbation for the first order time-derivative.

Theorem 0.7 *Let $\Gamma \subset \partial\Omega$ be an arbitrary non-empty sub-boundary of $\partial\Omega$. For any $\varepsilon > 0$ and an arbitrary bounded domain Ω_0 such that $\overline{\Omega_0} \subset \Omega \cup \Gamma$, $\partial\Omega_0 \cap \partial\Omega \subsetneq \Gamma$ is a non-empty open subset of $\partial\Omega$, there exist constants $C > 0$ and $\theta \in (0,1)$ such that*

$$\|u\|_{H^{1,1}(\Omega_0\times(0,\varepsilon))} \le C\|u\|_{H^{1,1}(Q)}^{1-\theta}\widetilde{F}^{\theta},$$

where $\widetilde{F} := \|u(\cdot,0)\|_{L^2(\Omega)} + \|F\|_{L^2(Q)} + \|u\|_{H^1(\Gamma\times(0,T))} + \|\partial_{\nu_A}u\|_{L^2(\Gamma\times(0,T))}$.

Chapter 4

In Chapter 4, we investigate the well-posedness and some important properties of the solutions to initial-boundary value problems for time-fractional diffusion equations of distributed orders.

First, by exploiting eigenfunction expansion and carrying out the inversion Laplace transforms on several integral loops, various estimates are established.

Theorem 0.8 *For any fixed $T > 0$. Let $F = 0$, $a \in L^2(\Omega)$, $B = 0$, $c(\le 0) \in C(\overline{\Omega})$ and $\mu \in C[0,1]$ be non-negative and not vanish in $[0,1]$. Then the initial-boundary value problem (0.1) admits a unique solution $u(\cdot,t) \in H_0^1(\Omega) \cap H^2(\Omega)$ such that*

$$\|\partial_t^m u(\cdot,t)\|_{H^2(\Omega)} \le C\|a\|_{L^2(\Omega)} \frac{M^m m!}{t^{m+1}\log(2T/t)}$$

holds true for $t \in (0,T]$ and $m = 0,1,\cdots$. Moreover, $u(\cdot,t)$ is real analytic in $t \in (0,T]$ and can be analytically extended to $(0,\infty)$.

Theorem 0.9 *Let $a = 0$, $F \in L^2(0,T;L^2(\Omega))$, $B = 0$, and $c(\le 0) \in C(\overline{\Omega})$. We assume the weight function $\mu \in C[0,1]$ is nonnegative, and does not vanish in $[0,1]$. Then the solution $u \in L^2(0,T;H^2(\Omega) \cap H_0^1(\Omega))$ and admits*

$$\|u\|_{L^2(0,T;H^2(\Omega))} \le C\|F\|_{L^2(0,T;L^2(\Omega))}.$$

Based on the above theorems, we further verify the Lipschitz continuous dependency of the solution to (0.1) with respect to μ and the diffusion coefficient in $\mathcal{A}$, which is fundamental for the optimization approach to the related coefficient inverse problem.

Second, by a Laplace transform argument, under some suitable assumptions on the weight function μ, it turns out that the solutions decay logarithmically as $t \to \infty$.

Theorem 0.10 *Let $\mu \in C[0,1]$ be a non-negative function and not vanish in $[0,1]$. We further assume that $a \in L^2(\Omega)$, $F = 0$, $B = 0$ and $c(\le 0) \in C(\overline{\Omega})$.*

Then

$$\|u(\,\cdot\,,t)\|_{H^2(\Omega)} \le C\|a\|_{L^2(\Omega)}(\log t)^{-1}$$

for the solution u to the initial-boundary value problem (0.1) for sufficiently large $t > 0$.

Moreover, if the weight function $\mu(\alpha)$ admits the representation $\mu(\alpha) = \mu(0) + o(\alpha^\delta)$, $\mu(0) > 0$, with some $\delta > 0$ as $\alpha \to 0$, then the asymptotic formula

$$\|u(\cdot\,,t) - \tfrac{\mu(0)}{\log t}(\mathcal{A} - c)^{-1}a\|_{H^2(\Omega)} = o((\log t)^{-1})\|a\|_{L^2(\Omega)}, \quad t \to \infty$$

holds true. The last formula holds uniformly dependently on Ω, the spatial dimension d, the initial condition a, the coefficients a_{ij} of the spatial differential operator of the equation (0.1), and the exponent δ of the asymptotic expansion of the weight function μ.

Finally, we consider an inverse problem of the determination of the weight function for (0.1). For the statement of our main problem, we introduce an admissible set of unknown weight function,

$$\mathcal{U} = \{\mu \in C[0,1]; \mu \ge 0, \not\equiv 0\}.$$

Problem 0.2 *Assume $F = 0$ in (0.1). Let $x_0 \in \Omega$ be fixed and let $I \subset (0,T)$ be a non-empty open interval. Let u, v be the solutions to the initial-boundary value problems (0.1) with respect to μ_1, $\mu_2 \in \mathcal{U}$ separately. We will investigate whether $u = v$ in $\{x_0\} \times I$ can derive $\mu_1 = \mu_2$.*

As an application of the analyticity, we give a uniqueness result for the above inverse problem on the determination of the weight function μ.

Theorem 0.11 *Let $\mu_1, \mu_2 \in \mathcal{U}$. Assume that $F = 0$, $B = 0$, $c(\le 0) \in C(\overline{\Omega})$. We further assume that $a \ge 0$ in Ω, $a \not\equiv 0$ and $a \in H^{2\gamma}(\Omega)$ with $\gamma > \max\{\frac{d}{2} + \delta - 1, 0\}$, $\delta > 0$ can be sufficiently small. Then $\mu_1 = \mu_2$ provided*

$$u(x_0,t) = v(x_0,t),\ x_0 \in \Omega,\ t \in I.$$

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Chapter 1

Single-term time-fractional diffusion equations

In this chapter, we investigate the unique continuation principle for single-term time-fractional diffusion equations. On the basis of unique continuation for parabolic equations, a weak unique continuation principle for the time-fractional diffusion equations is proved. As an application of the weak unique continuation principle, the uniqueness for an inverse problem of the determination of the spatial component of the source term in the time-fractional diffusion equation by interior measurement is proved.
Keywords: fractional diffusion equation, analyticity, weak unique continuation, inverse source problem

1.1 Introduction and main results

The classical diffusion equations with integer-order derivative have played important roles in modelling contaminants diffusion processes. However, in recent two decades, more and more experimental data in some diffusion processes in highly heterogeneous media, show that the classical model may be inadequate in order to interpret experimental data. For example, Adams and Gelhar [2] points out that field data in a saturated zone of a highly heterogeneous aquifer indicate a long-tailed profile in the spatial distribution of densities as the time passes, which is difficult to be interpreted by the classical diffusion equation.

For better model, a diffusion equation where the first-order time derivative is replaced by a derivative of fractional order $\alpha \in (0,1)$ has been proposed, that is

$$\partial_t^\alpha u = -\mathcal{A}u + B(x)\cdot\nabla u + c(x)u + F(x,t), \quad (x,t)\in\Omega\times(0,T), \tag{1.1}$$

where $T>0$, Ω is an open bounded domain in $\mathbb{R}^d$ with a smooth boundary (for example, of C^2 class), $\alpha\in(0,1)$, and ∂_t^α denotes the Caputo derivative with respect to t:

$$\partial_t^\alpha g(t) := J^{1-\alpha}\left(\frac{\mathrm{d}g}{\mathrm{d}t}\right)(t),$$

where J^β $(\beta>0)$ is the Riemann-Liouville fractional integral operator defined by

$$J^\beta g(t) = \frac{1}{\Gamma(\beta)}\int_0^t (t-\tau)^{\beta-1}g(\tau)\mathrm{d}\tau.$$

The operator $\mathcal{A}$ is uniformly elliptic operator (e.g., $\mathcal{A}=-\Delta$).

This modified model is presented as a useful approach for the description of transport dynamics in complex system that are governed by anomalous diffusion and non-exponential relaxation patterns, and attracted great attention from different areas. For numerical calculation, see Beson et al [5], Diethelm and Luchko [14], Meerschaert et al [48] and the references therein. For theoretical aspects, see Gorenflo et al [17], Luchko [33], Luchko and Gorenflo [36], Sakamoto and Yamamoto [60], Xu et al [64], etc.. For the inverse source problems, see Luchko et al [39], Sakamoto and Yamamoto [61], and Xu and Zhang [67]. For other kind of inverse problems for fractional diffusion equations, see, e.g., Cheng et al [11], Liu and Yamamoto [42], Yamamoto and Zhang [68], and the references therein.

However, to the best of the authors' knowledge, there are very few works on forward and inverse problems for non-symmetric time-fractional diffusion equations in spite of the physical and practical importance.

In this chapter, we are interested in the unique continuation principle for fractional diffusion equations. As one of the remarkable characterizations of parabolic equations, the classical unique continuation asserts that any solution of a parabolic equation that is defined on a domain D must vanish in all of D if it vanishes on an open subset in D (See, e.g., [65]), which can be further applied to deal with inverse source problems. The main focuses of this chapter is to construct the unique continuation principle for the differential equation in (1.7), namely,

Theorem 1.1 (Weak unique continuation) *Let $0 < \alpha < 1$, $F = 0$ and $(B, c) \in (L^\infty(\Omega))^{d+1}$ in* (1.1). *Furthermore, we suppose that $u \in C([0,T]; L^2(\Omega)) \cap C((0,T]; H^{2\gamma}(\Omega) \cap H_0^1(\Omega))$ $(\gamma \in (\frac{1}{2}, 1))$ satisfies* (1.7). *Let $\omega \subset \Omega$ be an arbitrarily chosen subdomain.*
Then

$$u = 0 \text{ in } \omega \times (0,T) \text{ implies } u = 0 \text{ in } \Omega \times (0,T).$$

Then we are devoted to studying the unique determination of the spatial component of the inhomogeneous term by the extra data on the solution u to the initial-boundary value problem

$$\begin{cases} u(x,0) = a, & x \in \Omega, \\ u(x,t) = 0, & (x,t) \in \partial\Omega \times (0,T). \end{cases} \tag{1.2}$$

for (1.1):

Problem 1.1 *Let the subdomain $\omega \subset \Omega$ and $T > 0$ be any given. Assume that the initial value $a = 0$ and the source term $F(x,t) = \rho(t)f(x)$ in* (1.1) *where ρ is given, and let u satisfy* (1.1) *respectively. Determine $f(x)$ by the interior observation*

$$u|_{\omega \times (0,T)}.$$

From the weak unique continuation, we have

Theorem 1.2 (Uniqueness) *Let $a = 0$, $f \in L^2(\Omega)$ and $\rho \in C^1[0,T]$ with $\rho(0) \neq 0$ in the initial-boundary value problem* (1.2) *for* (1.1), *and suppose that ω be an arbitrary open subset of Ω. Then $u = 0$ in $\omega \times (0,T)$ implies $f = 0$ in Ω.*

We only obtain the uniqueness result for the determination of the source term from interior observation and cannot show the stability for this kind of inverse problems for the fractional diffusion equations since integration by parts fails which is the key point of the Carleman estimate. (See, e.g., [65])

The remainder of this chapter is organized as follows. A proof of the unique continuation for the equation in (1.1) is given in Section 1.2 on the basis of unique continuation for parabolic equations and elliptic equations. In Section 1.3, based on Duhamel's principle and the established unique continuation, we give the proof uniqueness result for inverse source problem (1.1) with interior measurement. Finally, concluding remarks are given in Section 1.4.

1.2 Proof of Theorem 1.1

In this section, we prove the weak unique continuation for the differential equation in (1.1) by the unique continuation for the parabolic equations. The key point is that we obtain a relation between parabolic equations and time-fractional diffusion equations by the Laplace transform. Sakamoto and Yamamoto [60] proved this lemma provided $B \equiv 0$. However, their method mainly based on the eigenfunction expansion argument which very depends on the symmetric of the system, so their methods cannot work for the non-symmetric case.

Before giving the proof, we first fix some general settings and notations. Let $L^2(\Omega)$ be a usual L^2- space with the inner product $(\cdot,\cdot)$ and $H_0^1(\Omega)$, $H^\gamma(\Omega)$ $(\gamma > 0)$ denote the Sobolev spaces (see, e.g., Adams [1]). The elliptic operator $\mathcal{A}$ is defined for $\varphi \in H_0^1(\Omega) \cap H^2(\Omega)$ defined as

$$\mathcal{A}\varphi(x) = -\sum_{i,j=1}^{d} \partial_{x_i}\left(a_{ij}(x)\partial_{x_j}\varphi(x)\right), \tag{1.3}$$

where $a_{ij} = a_{ji} \in C^1(\overline{\Omega})$, $1 \le i,j \le d$. Moreover there exists a constant $\sigma > 0$ such that

$$\sigma \sum_{i=1}^{d} \xi_i^2 \le \sum_{i,j=1}^{d} a_{ij}(x)\xi_i\xi_j, \quad x \in \overline{\Omega},\ \xi \in \mathbb{R}^d.$$

We recall a useful lemma,

Lemma 1.1 (Theorem 4.1 in [43]) *Under the assumptions in Theorem 1.1, let $\frac{1}{2} < \gamma < 1$ be given constant. Then there exists a unique mild solution $u \in C((0,T]; H^{2\gamma}(\Omega) \cap H_0^1(\Omega)) \cap C([0,T]; L^2(\Omega))$ to the initial-boundary value problem* (1.2) *for* (1.1). *Moreover, the solution $u : (0,T) \to H^{2\gamma}(\Omega)$ is analytic and can be analytically extended to $(0,\infty)$, and there exists a constant $C = C(d,\alpha,\mathcal{A},B,c,\gamma) > 0$ such that*

$$\|u(\cdot,t)\|_{H^{2\gamma}(\Omega)} \le Ce^{CT}t^{-\alpha\gamma}\|f\|_{L^2(\Omega)}, \quad 0 < t < T. \tag{1.4}$$

Remark 1.1 We note that in the case of $B \in W^{1,\infty}(\Omega)$ and $c \le 0$, we can moreover see that the initial-boundary value problem (1.7) admits a unique solution $u \in C([0,T]; L^2(\Omega)) \cap C((0,T]; H_0^1(\Omega) \cap H^2(\Omega)))$ and u can be analytically extended to $(0,\infty)$.

Proof of Theorem 1.1. By our assumptions and Lemma 1.1, we can analytically extend the function $u(x,t)$ from $(0,T]$ to $(0,\infty)$, by the same notation we denote the extension. We consider the following initial-boundary value problem

$$\begin{cases} \partial_t^\alpha u = -\mathcal{A}u + B(x)\cdot\nabla u + c(x)u & \text{in } \Omega\times(0,\infty), \\ u = a & \text{in } \Omega\times\{0\}, \\ u = 0 & \text{on } \partial\Omega\times(0,\infty). \end{cases} \tag{1.5}$$

Now we apply the Laplace transform $\mathcal{L}$ (we also use the notation $\hat{\cdot}$) on both sides of the equation in (1.5), and use the formula

$$\mathcal{L}(\partial_t^\alpha\varphi)(s) = s^\alpha\mathcal{L}\varphi(s) - s^{\alpha-1}\varphi(0+)$$

to derive the transformed algebraic equation

$$\begin{cases} s^\alpha\widehat{u}(x;s) - s^{\alpha-1}a = -\mathcal{A}\widehat{u}(x;s) + B(x)\cdot\nabla\widehat{u}(x;s) + c(x)\widehat{u}(x;s), & x \in \Omega, \\ \widehat{u}(x;s) = 0, & x \in \partial\Omega, \end{cases}$$

where $s > M_1$ with $M_1 > 0$ being sufficiently large. Multiplying both sides of the equations on the above by $s^{1-\alpha}$, and denoting $\widehat{v}(x;s) := s^{1-\alpha}\widehat{u}(x;s)$, we then obtain the following elliptic type equations with parameter s

$$\begin{cases} s^\alpha\widehat{v}(x;s) - a = -\mathcal{A}\widehat{v}(x;s) + B(x)\cdot\nabla\widehat{v}(x;s) + c(x)\widehat{v}(x;s), & x \in \Omega, \\ \widehat{v}(x;s) = 0, & x \in \partial\Omega, \end{cases} \tag{1.6}$$

for $s > M_1$. On the other hand, let us turn to consider an initial-boundary value problem for parabolic type equation defined as follows

$$\begin{cases} \partial_t w = -\mathcal{A}w + B(x)\cdot\nabla w + c(x)w & \text{in } \Omega\times(0,\infty),\\ w = a & \text{in } \Omega\times\{0\},\\ w = 0 & \text{on } \partial\Omega\times(0,\infty). \end{cases}$$

Again by applying Laplace transforms on both sides of the equations in the above initial-boundary value problem, it follows that

$$\begin{cases} \eta\widehat{w}(x;\eta) - a = -\mathcal{A}\widehat{w}(x;\eta) + B(x)\cdot\nabla\widehat{w}(x;\eta) + c(x)\widehat{w}(x;\eta), & x\in\Omega,\\ \widehat{w}(x;\eta) = 0, & x\in\partial\Omega, \end{cases}$$

where $\eta > M_2$ and $M_2 > 0$ is a sufficiently large constant. After the change of variable $\eta = s^\alpha$, we find

$$\begin{cases} s^\alpha\widehat{w}(x;s^\alpha) - a = -\mathcal{A}\widehat{w}(x;s^\alpha) + B(x)\cdot\nabla\widehat{w}(x;s^\alpha) + c(x)\widehat{w}(x;s^\alpha), & x\in\Omega,\\ \widehat{w}(x;s^\alpha) = 0, & x\in\partial\Omega \end{cases}$$

for $s^\alpha > M_2$. Now noting (1.6), from the uniqueness result for boundary value problems for the elliptic type equations, we then obtain

$$\widehat{w}(x;s^\alpha) = \widehat{v}(x;s) = s^{1-\alpha}\widehat{u}(x;s), \quad (x;s)\in\Omega\times\{s>M\},$$

where $M := \max\{M_2^{\frac{1}{\alpha}}, M_1\}$, hence that

$$\widehat{w}(x;\eta) = 0, \quad (x;\eta)\in\omega\times\{\eta > M^\alpha\}$$

in view of Lemma 1.1, by the uniqueness of Laplace transform, finally that

$$w(x,t) = 0, \quad (x,t)\in\omega\times(0,\infty).$$

We conclude from the unique continuation principle of parabolic equations (see, e.g., [65]) that

$$w(x,t) = 0, \quad (x,t)\in\Omega\times(0,\infty),$$

which implies $a = w(\cdot,0) = 0$ in Ω, finally that

$$u(x,t) = 0, \quad (x,t)\in\Omega\times(0,\infty).$$

This completes the proof of the first part of Theorem 1.1. □

1.3 Proof of Theorem 1.2

This section is devoted to the proof of Theorem 1.2. The argument is mainly based on the weak unique continuation proved in Section 1.2 and Duhamel's principle

Lemma 1.2 (Duhamel's principle) *Let $F(x,t) = \rho(t)f(x)$ in (1.1). Assuming u is the weak solution of initial-boundary value problem (1.2) for (1.1). Then*

$$u(x,t) = \int_0^t \theta(t-s)v(x,s)\mathrm{d}s \ \textit{in}\ \Omega\times(0,T),$$

where $J^{1-\alpha}\theta(t) = \rho(t)$, and v such that

$$\begin{cases} \partial_t^\alpha v = -\mathcal{A}v + B(x)\cdot\nabla v + c(x)v & \textit{in } \Omega\times(0,T),\\ v(\cdot,0) = f & \textit{in } \Omega\\ v = 0 & \textit{on } \partial\Omega\times(0,T). \end{cases} \tag{1.7}$$

The proof can be found in Lemma 4.1 in [40].

We are ready to give the proof of Theorem 1.2.

Proof of Theorem 1.2. Let u satisfy the problem (1.1) and the interior observation $u|_{\omega\times(0,T)} = 0$. From Lemma 1.2, we have

$$\Theta(t) := \int_0^t \theta(t-s)v(x,s)\mathrm{d}s = 0, \quad (x,t) \in \omega \times (0,T),$$

where v is the solution to the problem (1.7). Noting that Fubini's theorem implies

$$J_{0+}^{1-\alpha}\Theta(t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t (t-\tau)^{-\alpha} \int_0^\tau \theta(\tau-s)v(x,s)\mathrm{d}s\mathrm{d}\tau$$
$$= \int_0^t \left(\frac{1}{\Gamma(1-\alpha)} \int_s^t (t-\tau)^{-\alpha}\theta(\tau-s)\mathrm{d}\tau \right) v(x,s)\mathrm{d}s = \int_0^t (J_{0+}^{1-\alpha}\theta)(t-s)v(x,s)\mathrm{d}s,$$

which implies

$$\int_0^t \rho(t-s)v(x,s)\mathrm{d}s = 0, \quad (x,t) \in \omega \times (0,T).$$

Consequently taking derivative with respect to t we find

$$\rho(0)v(x,t) + \int_0^t \rho'(t-s)v(x,s)\mathrm{d}s = 0, \ (x,t) \in \omega \times (0,T).$$

Then $\rho(0) \neq 0$ leads to

$$\|v(\cdot,t)\|_{L^2(\omega)} \le \frac{1}{|\rho(0)|} \int_0^t \|\rho\|_{C^1[0,T]} \|v(\cdot,s)\|_{L^2(\omega)}\mathrm{d}s, \ t \in (0,T).$$

By Gronwall's inequality, we obtain $v|_{\omega\times(0,T)} = 0$. Hence according to Theorem 1.1, we derive $v = 0$ in Q, implying $f = v(\cdot,0) = 0$. This completes the proof of theorem. □

1.4 Conclusions and open problems

As an application of the analyticity, we proved the weak type unique continuation principle by noting the relation between time-fractional diffusion equations and parabolic equations. Letting $\alpha = 1$ in Theorem 1.1, we have that the uniqueness holds without $v|_{\partial\Omega} = 0$ which is the unique continuation (e.g., [65]). However, for $\alpha \in (0,1)$, we do not know whether the uniqueness holds without $v|_{\partial\Omega} = 0$.

In Theorem 1.2, we only proved the uniqueness result for the inverse source problem. It is known that such kind of inverse problem to the parabolic or hyperbolic equation admits the well-posedness in Hadamard sense by Carleman estimate or Multiplier method. But such kinds of methods for the parabolic or hyperbolic equation are failed in the case of fractional diffusion equation since there is no classical integration by part for the fractional derivatives. For a general fractional diffusion equation, the stability remains open.

Chapter 2

Multi-term time fractional diffusion equations

In this chapter, we discuss the initial-boundary value problems for diffusion equations with multi-term time-fractional derivatives. In the case of homogeneous equations, by means of the Mittag-Leffler function and the eigenfunction expansion, we reduce the problem to an integral equation for a solution to show the unique existence and the analyticity of solution for the equation. Different to the homogeneous case, we extend the Caputo derivative in Sobolev space and regard the lower order terms as perturbation to prove the unique existence of solutions to nonhomogeneous equations. Especially, in the case of homogeneous equation where all the coefficients of the time-fractional derivatives are positive, by the Laplace transform argument, it turns out that the decay rate of the solution for long time is dominated by the lowest order of the time-fractional derivatives. Finally, several inverse problems of the determination of the orders are investigated by mainly using the analyticity of the solution to the differential equation.

Keywords: multi-term time-fractional diffusion equation, uniqueness and existence, analyticity, asymptotic behavior, Laplace transform, Fredholm alternative, Determination of fractional orders

2.1 Introduction

Let Ω be an open bounded domain in $\mathbb{R}^d$ with a smooth boundary (for example, of C^∞ class), and for an arbitrarily fixed $T > 0$, we set $Q := \Omega \times (0, T)$, $\Sigma := \partial\Omega \times (0, T)$. We deal with the following initial-boundary value problem for the multi-term time-fractional diffusion equation

$$\begin{cases} \sum_{j=1}^{\ell} q_j(x,t)\partial_t^{\alpha_j} u = -\mathcal{A}u + B(x,t)\cdot\nabla u + b(x,t)u + F(x,t), & (x,t) \in Q, \\ u(x,0) = a(x), & (x,t) \in \Omega, \\ u(x,t) = 0, & (x,t) \in \Sigma. \end{cases} \tag{2.1}$$

where α_j $(j = 1, \cdots, \ell)$ are positive constants such that $1 \le \alpha_\ell < \cdots < \alpha_1 < 1$, $q_j \in L^2(\Omega \times (0,T))$, $(j = 1, \cdots, \ell)$, and we assume $q_1 = 1$ without loss of generality. $-\mathcal{A}$ is a symmetric uniformly elliptic operator defined by (1.3) in Chapter 1.

In the case of $\ell = 1$, equation (2.1) is reduced to its single-term counterpart (1.1). As we mentioned in Chapter 1, the fractional diffusion equation is presented as a useful model for the anomalous diffusion in heterogeneous medium. As a natural extension for the single-term time-fractional diffusion equation, the multi-term time-fractional diffusion equation modifies the model simulating the advection and attracted great attention from different aspect due to its potential feasibility. In [13], a solution to an initial-boundary value problem is formally represented by Fourier series. However no proofs for the convergence of the series and for the uniqueness of the solution. A proof of the convergence of the series defining the solution of the multi-term time-fractional diffusion equation with positive constant coefficients can be found in the paper [37]. [28] discusses the case where the spatial dimension is one, the coefficients are constants and the spatial fractional derivative is considered, and establishes the formula of the solution. The paper [34] proves the unique existence of the classical solution, the maximum principle and related properties in the case where the coefficients of the time derivatives are positive and independent on x, and the arguments are based on the Fourier method, that is, the separation of the variables. These papers mainly discuss the case where the spatial differential operators is a symmetric elliptic operator and the coefficients of time-fractional derivatives are positive constants.

In this chapter, we are concerned with the forward and inverse problems for the initial-boundary value problem (2.1). For the forward problem, firstly, the continuous dependency of the solution to (2.1) with respect to initial value and source term is proved by using fixed point method and the Fredholm alternative under some suitable assumption on the coefficients. Moreover, in the case when all of the coefficients of the fractional derivatives are positive, the use of Laplace transform and analyticity yields that for homogeneous equation, the asymptotic behavior of the solution is dominated by the lowest fractional orders, which can be regarded as a generalization of the asymptotic behavior result in Li et al [37] where they dealt with the case of positive-constant coefficients. Next on basis of these established results for the forward problem, we can deduce the Lipschitz stability of the solution to (2.17) with respect to α_j, q_j $(j = 1, \cdots, \ell)$ and diffusion coefficients. As a direct corollary, we can establish an existence result for the optimization approach to the simultaneous reconstruction of various coefficients.

Finally, as an application of the analyticity, we investigate two kinds of inverse problems of identifying fractional orders and other quantities in diffusion equations with multiple time-fractional derivatives, and establish the uniqueness of our inverse problems.

The rest of this chapter is organized as follows: In Section 2.2.1 for the homogeneous equations, the unique existence and the analyticity of the solution to the problem (2.1) are proved under some suitable assumptions on the coefficients. In Section 2.2.2, for the non-homogeneous equations with initial value $a = 0$, the Fredholm alternative is applied to show unique existence as well as regularity of the solution. The long-time asymptotic behavior are given in Section 2.3. Finally, in Section 2.4.1, the uniqueness in determining orders α_j by measured data at one endpoint is proved. In Section 2.4.2, by applying the Laplace transforms of the solutions to (2.1) and reducing our inverse problem to the inverse boundary value problem for elliptic equations, we complete the proof of Theorem 2.6.

2.2 Wellposedness

In this section, we consider the initial-boundary value problem (2.1). Our overall plan is first to define and then construct an appropriate weak solution u of (2.1), and later to investigate the asymptotic, analytic and other properties of the solution u.

Recently, more and more publications related to fractional diffusion equations, show that there are lots of big different properties between the homogeneous equations and nonhomogeneous equations, see, e.g., [16], [41], [66] and the references therein, which show that there is a big gap between these two cases. Therefore, it should be suitable to define the weak solution separately. To this end, we start with some general settings and notations. First we define an operator A by

$$A\psi = -\mathcal{A}\psi, \quad \psi \in \mathcal{D}(A) := H^2(\Omega) \cap H_0^1(\Omega).$$

Let $\{\lambda_k, \varphi_k\}_{k=1}^\infty$ be an eigensystem of the elliptic operator A. That is, $0 < \lambda_1 < \lambda_2 \le \cdots$, $\lim_{k\to\infty} \lambda_k = \infty$, $A\varphi_k = \lambda_k\varphi_k$ and $\varphi_k \in \mathcal{D}(A)$. Then the fractional power A^γ is defined for $\gamma \in \mathbb{R}$ (e.g., [50]) by

$$A^\gamma\psi = \sum_{n=1}^\infty \lambda_n^\gamma (\psi, \varphi_n)\varphi_n,$$

where

$$\psi \in \mathcal{D}(A^\gamma) := \left\{ \psi \in L^2(\Omega) : \sum_{n=1}^{\infty} \lambda_n^{2\gamma} |(\psi, \varphi_n)|^2 < \infty \right\}$$

and that $\mathcal{D}(A^\gamma)$ is a Hilbert space with the norm

$$\|\psi\|_{\mathcal{D}(A^\gamma)} = \left(\sum_{n=1}^{\infty} \lambda_n^{2\gamma} |(\psi, \varphi_n)|^2 \right)^{\frac{1}{2}}.$$

Moreover we define the Mittag-Leffler function by

$$E_{\alpha,\beta}(z) := \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + \beta)}, \quad z \in \mathbb{C}, \tag{2.2}$$

where $\alpha > 0$ and $\beta \in \mathbb{R}$ are arbitrary constants. The above formula and the classical asymptotics

$$\Gamma(\eta) = e^{-\eta} \eta^{\eta - \frac{1}{2}} (2\pi)^{\frac{1}{2}} \left(1 + O\left(\frac{1}{\eta} \right) \right) \qquad \text{as } \eta \to +\infty \tag{2.3}$$

(e.g., Abramowitz and Stegun [3], p.257) imply that the radius of convergence is ∞ and so $E_{\alpha,\beta}(z)$ is an entire function of $z \in \mathbb{C}$. Furthermore, the following useful lemma holds:

Lemma 2.1 (i) *Let $0 < \alpha < 2$ and $\beta \in \mathbb{R}$ be arbitrary. We suppose that μ is such that $\frac{\pi}{2}\alpha < \mu < \min\{\pi, \pi\alpha\}$. Then there exists a constant $C = C(\alpha, \beta, \mu) > 0$ such that*

$$|E_{\alpha,\beta}(z)| \le \frac{C}{1 + |z|}, \quad \mu \le |\arg z| \le \pi.$$

(ii) *For $\lambda > 0$, $\alpha > 0$ and positive integer $n \in \mathbb{N}$, we have*

$$\frac{d^n}{dt^n} E_{\alpha,1}(-\lambda t^\alpha) = -\lambda t^{\alpha - n} E_{\alpha, \alpha - n + 1}(-\lambda t^\alpha), \quad t > 0.$$

Moreover, $E_{\alpha,1}(-\lambda t^\alpha)$ with $0 < \alpha < 1$ is completely monotonic, that is, $(-1)^n \frac{\mathrm{d}^n}{\mathrm{d}t^n} E_{\alpha,1}(-\lambda t^\alpha) \ge 0$ for all $t > 0$ and $n = 0, 1, \cdots$.

The proof can be found in Gorenflo and Mainardi [18], on p. 35 in Podlubny [52] and Lemma 3.2 in Sakamoto and Yamamoto [60].

2.2.1 Homogeneous equation

Throughout this section, we set $F = 0$. We assume that B, c are independent of t and that

$$(B, c) \in (L^\infty(\Omega))^{d+1}.$$

Next we give the definition of the solution to the initial-boundary value problem (2.1). To this end, we formally show an integral equation which is equivalent to (2.1), which is only composed of $u, \nabla u$ without the time derivative of the solution.

We define an operator $S(z) : L^2(\Omega) \to L^2(\Omega)$ for $z \in \{z \neq 0; |\arg z| < \frac{\pi}{2}\}$ by

$$S(z)a := \sum_{n=1}^{\infty} (a, \varphi_n) E_{\alpha,1}(-\lambda_n z^\alpha) \varphi_n, \quad a \in L^2(\Omega). \tag{2.4}$$

In view of (ii) in Lemma 2.1, the term-wise differentiations are possible and give

$$S^{(j)}(z)a := -\sum_{n=1}^{\infty} \lambda_n (a, \varphi_n) z^{\alpha_1 - j} E_{\alpha_1, \alpha_1 - j + 1}(-\lambda_n z^{\alpha_1}) \varphi_n, \quad j = 1, 2$$

for $a \in L^2(\Omega)$. Moreover, From the definition of (2.4) and the property of the Mittag-Leffler function, $S^{(j)}(z)$ $(j = 0, 1, 2)$ are analytic in the sector Ω_θ and by Theorem 1.6 in [52] (p.35), we can prove that there exists a constant $C > 0$ which is independent of z such that

$$\|A^{\gamma-1}S^{(j)}(z)\|_{L^2(\Omega)\to L^2(\Omega)} \le C|z|^{\alpha_1-j-\alpha_1\gamma}, \quad j = 0, 1, 2 \tag{2.5}$$

for $z \in \Omega_\theta$ and $0 \le \gamma \le 1$, where $\|\cdot\|_{L^2(\Omega)\to L^2(\Omega)}$ denotes the operator norm from $L^2(\Omega)$ to $L^2(\Omega)$.

In fact, by using $\Gamma(z+1) = z\Gamma(z)$ for Re $z > 0$, the termwise differentiations yield

$$\frac{d^j}{dz^j}E_{\alpha_1,1}(-\lambda z^{\alpha_1}) = -\lambda z^{\alpha_1-j}E_{\alpha_1,\alpha_1-j+1}(-\lambda z^{\alpha_1}), \quad z \in \Omega_\theta,\ \lambda > 0,\ j = 1, 2.$$

Therefore we have

$$A^{\gamma-1}S^{(j)}(z)a = -\sum_{n=1}^{\infty}\lambda_n^\gamma(a,\varphi_n)z^{\alpha_1-j}E_{\alpha_1,\alpha_1-j+1}(-\lambda_n z^{\alpha_1})\varphi_n,$$

and so

$$\begin{aligned}\|A^{\gamma-1}S^{(j)}(z)a\|^2_{L^2(\Omega)} &\le |z|^{2(\alpha_1-j)}\sum_{n=1}^{\infty}\lambda_n^{2\gamma}|(a,\varphi_n)|^2|E_{\alpha_1,\alpha_1-j+1}(-\lambda_n z^{\alpha_1})|^2\\ \le& C|z|^{2(\alpha_1-j)}\sum_{n=1}^{\infty}\lambda_n^{2\gamma}|(a,\varphi_n)|^2\left(\frac{1}{1+|\lambda_n||z^{\alpha_1}|}\right)^2.\end{aligned}$$

Here we used

$$|E_{\alpha_1,\alpha_1-j+1}(-\lambda_n z^{\alpha_1})| \le \frac{C}{1+|\lambda_n||z^{\alpha_1}|}$$

if $0 < \arg z^{\alpha_1} < \frac{\pi}{2}\alpha_1$ (e.g., Theorem 1.6 (p.35) in [52]). Therefore

$$\begin{aligned}\|A^{\gamma-1}S^{(j)}(z)a\|^2_{L^2(\Omega)} &\le C|z|^{2(\alpha_1-j-\gamma\alpha_1)}\sum_{n=1}^{\infty}|(a,\varphi_n)_{L^2(\Omega)}|^2\left(\frac{(|\lambda_n||z|^{\alpha_1})^\gamma}{1+|\lambda_n||z^{\alpha_1}|}\right)^2\\ \le& C|z|^{2(\alpha_1-j-\gamma\alpha_1)}\max_{\eta\ge 0}\left(\frac{\eta^\gamma}{1+\eta}\right)^2\sum_{n=1}^{\infty}|(a,\varphi_n)|^2,\end{aligned}$$

which proves (2.5).

By regarding the term $B(x)\cdot\nabla u(x,t) + c(x)u(x,t)$ as a source term, from Theorem 2.2 in [60], we find an implicit form of the solution u to the problem (1.1), say, the following integral equation of u,

$$u(t) = S(t)a - \int_0^t A^{-1}S'(t-s)(B\cdot\nabla u(s) + cu(s))\mathrm{d}s + \sum_{j=2}^{\ell}\int_0^t A^{-1}S'(t-s)q_j\partial_t^{\alpha_j}u(s)\mathrm{d}s.$$

We consider the last term

$$\sum_{j=2}^{\ell}\int_0^t A^{-1}S'(t-s)q_j\partial_t^{\alpha_j}u(s)\mathrm{d}s.$$

Noting by the definition of Caputo fractional derivative, we have

$$\int_0^t A^{-1}S'(t-s)\Big(q_j\partial_t^{\alpha_j}u(s)\Big)\mathrm{d}s = \int_0^t A^{-1}S'(t-s)\frac{1}{\Gamma(1-\alpha_j)}\left(\int_0^s(s-r)^{-\alpha_j}q_ju'(r)\mathrm{d}r\right)\mathrm{d}s,$$

where we denote $u'(t) := \frac{\mathrm{d}u}{\mathrm{d}t}(t)$, by Fubini's theorem we exchange the orders of integrals and change the variable $s \to \xi$ by $\xi := \frac{s-r}{t-r}$ to obtain

$$\int_0^t A^{-1}S'(t-s)q_j\partial_t^{\alpha_j}u(s)\mathrm{d}s = \int_0^t\left(\int_r^t A^{-1}S'(t-s)\frac{(s-r)^{-\alpha_j}}{\Gamma(1-\alpha_j)}\mathrm{d}s\right)q_ju'(r)\mathrm{d}r$$

$$= \int_0^t \left(\int_0^1 A^{-1} S'\big((1-\xi)(t-r)\big) \frac{\xi^{-\alpha_j}}{\Gamma(1-\alpha_j)} \mathrm{d}\xi \right) (t-r)^{1-\alpha_j} q_j u'(r) \mathrm{d}r =: \frac{I(t)}{\Gamma(1-\alpha_j)}.$$

Since the integrands have singularities at $\xi = 0, 1$ and $r = t$, we should understand that

$$\begin{aligned} I(t) &= \lim_{\epsilon_1,\epsilon_2,\epsilon_3 \downarrow 0} \int_0^{t-\epsilon_3} \Big(\int_{\epsilon_2}^{1-\epsilon_1} A^{-1} S'\big((1-\xi)(t-r)\big) \xi^{-\alpha_j} \mathrm{d}\xi \Big) (t-r)^{1-\alpha_j} q_j u'(r) \mathrm{d}r \\ &=: \lim_{\epsilon_1,\epsilon_2,\epsilon_3 \downarrow 0} I_{\epsilon_1,\epsilon_2,\epsilon_3}(t) \end{aligned}$$

Integration by parts yields

$$\begin{aligned} I_{\epsilon_1,\epsilon_2,\epsilon_3}(t) &= \left(\int_{\epsilon_2}^{1-\epsilon_1} A^{-1} S'\big((1-\xi)(t-r)\big) \xi^{-\alpha_j} \mathrm{d}\xi \right) q_j u(r) (t-r)^{1-\alpha_j} \Big|_{r=0}^{r=t-\epsilon_3} \\ &\quad + \int_0^{t-\epsilon_3} \left(\int_{\epsilon_2}^{1-\epsilon_1} A^{-1} S''\big((1-\xi)(t-r)\big) (1-\xi) \xi^{-\alpha_j} \mathrm{d}\xi \right) (t-r)^{1-\alpha_j} q_j u(r) \mathrm{d}r \\ &\quad + \int_0^{t-\epsilon_3} \left(\int_{\epsilon_2}^{1-\epsilon_1} A^{-1} S'\big((1-\xi)(t-r)\big) \xi^{-\alpha_j} \mathrm{d}\xi \right) (1-\alpha_j)(t-r)^{-\alpha_j} q_j u(r) \mathrm{d}r \\ &=: \sum_{k=1}^{3} I^{(k)}_{\epsilon_1,\epsilon_2,\epsilon_3}(t). \end{aligned}$$

We evaluate each of the above three terms separately. First for $I^{(1)}_{\epsilon_1,\epsilon_2,\epsilon_3}(t)$, from (2.5) and $\alpha_1 > \alpha_j$ for $j = 2, ..., \ell$, it follows that

$$\left\| \int_{\epsilon_2}^{1-\epsilon_1} A^{-1} S'\big((1-\xi)(t-r)\big) \xi^{-\alpha_j} \mathrm{d}\xi \right\|_{L^2(\Omega) \to L^2(\Omega)} \le C \int_{\epsilon_2}^{1-\epsilon_1} \big((1-\xi)(t-r)\big)^{\alpha_1 - 1} \xi^{-\alpha_j} \mathrm{d}\xi.$$

Moreover from the property of the Beta function that

$$\int_0^1 (1-\xi)^{\alpha-1} \xi^{\beta-1} \mathrm{d}\xi = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)} < \infty, \ \alpha, \beta > 0, \tag{2.6}$$

by $\alpha_1 > \alpha_j$ for $j = 2, ..., \ell$, for $r = t - \epsilon_3$ we have

$$\left\| \left(\int_{\epsilon_2}^{1-\epsilon_1} A^{-1} S'((1-\xi)(t-r)) \xi^{-\alpha_j} \mathrm{d}\xi \right) q_j u(r) (t-r)^{1-\alpha_j} \right\|_{L^2(\Omega)} \le C \epsilon_3^{\alpha_1 - \alpha_j} \|u\|_{L^\infty(0,T;L^2(\Omega))} \to 0$$

as $\epsilon_3 \to 0$. Hence by $u(0) = a$, we see that

$$\lim_{\epsilon_1,\epsilon_2,\epsilon_3 \downarrow 0} I^{(1)}_{\epsilon_1,\epsilon_2,\epsilon_3}(t) = t^{1-\alpha_j} \int_0^1 A^{-1} S'((1-\xi)t) \xi^{-\alpha_j} q_j a \mathrm{d}\xi.$$

Next we estimate $I^{(2)}_{\epsilon_1,\epsilon_2,\epsilon_3}(t)$, again by using (2.5) and $\alpha_1 > \alpha_j$ for $j = 2, ..., \ell$, it follows that

$$\|I^{(2)}_{\epsilon_1,\epsilon_2,\epsilon_3}(t)\|_{L^2(\Omega)} \le \int_0^{t-\epsilon_3} \left(\int_{\epsilon_2}^{1-\epsilon_1} (1-\xi)^{\alpha_1-1} \xi^{-\alpha_j} \mathrm{d}\xi \right) (t-r)^{\alpha_1-\alpha_j-1} \mathrm{d}r \|u\|_{L^\infty(0,T;L^2(\Omega))},$$

the integrand is integrable in $0 < \xi < 1$ and $0 < r < t$ in view of (2.6) and we take the limit as $\epsilon_1, \epsilon_2, \epsilon_3 \downarrow 0$ to derive

$$\lim_{\epsilon_1,\epsilon_2,\epsilon_3 \downarrow 0} I^{(2)}_{\epsilon_1,\epsilon_2,\epsilon_3}(t) =: \int_0^t \left(\int_0^1 A^{-1} S''\big((1-\xi)(t-r)\big) (1-\xi) \xi^{-\alpha_j} \mathrm{d}\xi \right) (t-r)^{1-\alpha_j} q_j u(r) \mathrm{d}r$$

Finally, for $I^{(3)}_{\epsilon_1,\epsilon_2,\epsilon_3}(t)$ we argue similarly to obtain

$$\lim_{\epsilon_1,\epsilon_2,\epsilon_3 \downarrow 0} I^{(3)}_{\epsilon_1,\epsilon_2,\epsilon_3}(t) =: (1-\alpha_j) \int_0^t \left(\int_0^1 A^{-1} S'((1-\xi)(t-r)) \xi^{-\alpha_j} \mathrm{d}\xi \right) (t-r)^{-\alpha_j} q_j u(r) \mathrm{d}r.$$

Thus

$$\begin{aligned}I(t) =& t^{1-\alpha_j}\int_0^1 A^{-1}S'((1-\xi)t)\xi^{-\alpha_j}q_j a\mathrm{d}\xi\\&+\int_0^t\left(\int_0^1 A^{-1}S''((1-\xi)(t-r))\xi^{-\alpha_j}\mathrm{d}\xi\right)(t-r)^{1-\alpha_j}q_j u(r)\mathrm{d}r\\&+(1-\alpha_j)\int_0^t\left(\int_0^1 A^{-1}S'((1-\xi)(t-r))\xi^{-\alpha_j}\mathrm{d}\xi\right)(t-r)^{-\alpha_j}q_j u(r)\mathrm{d}r.\end{aligned}$$

Consequently we have

$$\begin{aligned}u(t) =& S(t)a+\sum_{j=2}^{\ell}\frac{1}{\Gamma(1-\alpha_j)}\int_0^t A^{-1}S'(t-r)r^{-\alpha_j}q_j a\mathrm{d}r\\&-\int_0^t A^{-1}S'(t-r)(B\cdot\nabla u(r)+cu(r))\mathrm{d}r\\&+\sum_{j=2}^{\ell}\frac{1}{\Gamma(1-\alpha_j)}\int_0^t\int_0^1 A^{-1}S''\big((1-s)(t-r)\big)(1-s)(t-r)^{1-\alpha_j}s^{-\alpha_j}q_j u(r)\mathrm{d}s\mathrm{d}r\\&+\sum_{j=2}^{\ell}\frac{1-\alpha_j}{\Gamma(1-\alpha_j)}\int_0^t\int_0^1 A^{-1}S'\big((1-s)(t-r)\big)(t-r)^{-\alpha_j}s^{-\alpha_j}q_j u(r)\mathrm{d}s\mathrm{d}r =: \sum_{j=1}^{5}I_j.\end{aligned}\tag{2.7}$$

Setting $F=0$ in initial-boundary value problem (2.1), since the mild solution to (2.1) satisfies the integral equation (2.7), after the change of variables, we find

$$\begin{aligned}u(t) =& S(t)a-\sum_{j=2}^{\ell}\frac{t^{1-\alpha_j}}{\Gamma(1-\alpha_j)}\int_0^1 A^{-1}S'(rt)(1-r)^{-\alpha_j}q_j a\mathrm{d}r - t\int_0^1 A^{-1}S'(rt)\widetilde{u}\big((1-r)t\big)\mathrm{d}r\\&+\sum_{j=2}^{\ell}\frac{(1-\alpha_j)t^{1-\alpha_j}}{\Gamma(1-\alpha_j)}\int_0^1\int_0^1 A^{-1}S'\big((1-s)rt\big)r^{-\alpha_j}s^{-\alpha_j}q_j u\big((1-r)t\big)\mathrm{d}s\mathrm{d}r\\&+\sum_{j=2}^{\ell}\frac{t^{2-\alpha_j}}{\Gamma(1-\alpha_j)}\int_0^1\int_0^1 A^{-1}S''\big((1-s)rt\big)(1-s)r^{1-\alpha_j}s^{-\alpha_j}q_j u\big((1-r)t\big)\mathrm{d}s\mathrm{d}r,\end{aligned}\tag{2.8}$$

where $\widetilde{u} := B\cdot\nabla u + cu$.

Based on the above integral equation for u, we have the following definition of the weak solution to the problem (2.1).

Definition 2.1 (Weak solution) *Let $a\in L^2(\Omega)$. We call a function u a solution to* (2.1) *if $u\in C((0,T];H^2(\Omega)\cap H_0^1(\Omega))\cap C([0,T];L^2(\Omega))$ and satisfies* (2.8) *and*

$$\lim_{t\to 0}\|u(\cdot,t)-a\|_{L^2(\Omega)}=0.$$

We now investigate the properties of the solution to initial-boundary value problem (2.1) with $F=0$. We extend the variable t in (2.8) from $(0,T)$ to the sector $\{z\neq 0; |\arg z|<\frac{\pi}{2}\}$, and setting $u_0=0$, we define $u_{n+1}(z)$ $(n=0,1,\cdots)$ as follows:

$$\begin{aligned}u_{n+1}(z) =& S(z)a-\sum_{j=2}^{\ell}\frac{z^{1-\alpha_j}}{\Gamma(1-\alpha_j)}\int_0^1 A^{-1}S'(rz)(1-r)^{-\alpha_j}q_j a\mathrm{d}r\\&-z\int_0^1 A^{-1}S'(rz)\widetilde{u}_n\big((1-r)z\big)\mathrm{d}r\\&+\sum_{j=2}^{\ell}\frac{(1-\alpha_j)z^{1-\alpha_j}}{\Gamma(1-\alpha_j)}\int_0^1\int_0^1 A^{-1}S'\big((1-s)rz\big)r^{-\alpha_j}s^{-\alpha_j}q_j u_n\big((1-r)z\big)\mathrm{d}s\mathrm{d}r\end{aligned}$$

$$+\sum_{j=2}^{\ell}\frac{z^{2-\alpha_j}}{\Gamma(1-\alpha_j)}\int_0^1\int_0^1 A^{-1}S''\big((1-s)rz\big)(1-s)r^{1-\alpha_j}s^{-\alpha_j}q_ju_n\big((1-r)z\big)\mathrm{d}s\mathrm{d}r. \quad (2.9)$$

Here $\widetilde{u}_n(z) := B\cdot\nabla u_n(z) + cu_n(z)$. We conclude from the definition (2.4) of $S(z)$ and the properties of Mittag-Leffler function that $u_n(z)$ defined in (2.9) uniformly converges to the solution to the initial-boundary value problem (2.1) with $F=0$ as $n\to\infty$ for any compact subset of the section $\{z\neq 0; |\arg z|<\frac{\pi}{2}\}$. Namely the following theorem holds.

Theorem 2.1 *Let $0<\alpha_\ell<\cdots<\alpha_1<1$ and $T>0$ be fixed constants. Assuming that $q_j\in W^{2,\infty}(\Omega)$ $(j=2,\cdots,\ell)$, $(B,c)\in(L^\infty(\Omega))^{d+1}$. Then for any fixed constant $\gamma\in[\frac{1}{2},1)$, the initial-boundary value problem (2.1) with $F=0$ and $a\in L^2(\Omega)$ admits a unique mild solution $u\in C([0,T];L^2(\Omega))\cap C((0,T]:H^{2\gamma}(\Omega)\cap H_0^1(\Omega))$ such that*

$$\|u(t)\|_{H^{2\gamma}(\Omega)}\le Ct^{-\alpha_1\gamma}e^{CT}\|a\|_{L^2(\Omega)},\quad t\in(0,T].$$

Moreover $u:(0,T]\to H^{2\gamma}(\Omega)$ can be analytically extended to the sector $\{z\neq 0; |\arg z|<\frac{\pi}{2}\}$.

Here and henceforth in this section, $C>0$ denotes the constants which are independent of T, n, a and u, but may depend on γ, $\{\alpha_i\}_{i=1}^{\ell}$, d, B, c, Ω, $\{q_j\}_{j=2}^{\ell}$ and the coefficients of $\mathcal{A}$.

Remark 2.1 In [8] a similar fractional diffusion equation is discussed for $F=0$ and $B=0$ and a similar regularity is proved. However [8] assumes an extra condition $\alpha_1+\alpha_\ell>1$, and our main result needs not such an assumption.

Proof. For any $n\in\mathbb{N}$, taking the operator A^γ on both sides of (2.9), from (2.5) for the $z\in\Omega_{\theta,T}:=\{z\neq 0; |\arg z|<\theta, |z|\le T\}$ with $\theta\in(0,\frac{\pi}{2})$, we claim that the following estimate holds:

$$\|u_{n+1}(z)-u_n(z)\|_{D(A^\gamma)}\le M_1M^n\left(\sum_{j=1}^{\ell}J^{\beta_j}\right)^n(g)(|z|)\|a\|_{L^2(\Omega)},\quad n\in\mathbb{N}, \quad (2.10)$$

where $g(t):=t^{-\alpha_1\gamma}$, $\beta_1:=\alpha_1-\alpha_1\gamma$, $\beta_j:=\alpha_1-\alpha_j$, $j=2,\cdots,\ell$, the constant M is independent of T, $t>0$, $z\in\Omega_{\theta,T}$, but may dependent on γ, d, Ω, θ, p, $p_1,...,p_\ell$, $\alpha_1,...,\alpha_\ell$, and by J^α we denote the Riemann-Liouville fractional integral

$$(J^\alpha f)(t):=\frac{1}{\Gamma(\alpha)}\int_0^t(t-\tau)^{\alpha-1}f(\tau)\mathrm{d}\tau,\quad \alpha>0,$$

and denote $J^0f(t)=f(t)$. We now proceed by induction on n to prove (2.10). Firstly, for $n=0$, using the estimate (2.5), noting $u_0=0$, then for $z\in\Omega_{\theta,T}$, we have

$$\begin{aligned}\|u_1(z)\|_{D(A^\gamma)}\le&\sum_{j=2}^{\ell}\left\|\frac{-z^{1-\alpha_j}}{\Gamma(1-\alpha_j)}\int_0^1 A^{\gamma-1}S'(rz)(1-r)^{-\alpha_j}q_ja\mathrm{d}r\right\|_{L^2(\Omega)}+\|A^\gamma S(z)a\|_{L^2(\Omega)}\\ \le&C\sum_{j=2}^{\ell}|z|^{\alpha_1-\alpha_j-\alpha_1\gamma}\int_0^1 r^{\alpha_1-1-\alpha_1\gamma}(1-r)^{-\alpha_j}\mathrm{d}r\|a\|_{L^2(\Omega)}+|z|^{-\alpha_1\gamma}\|a\|_{L^2(\Omega)}.\end{aligned}$$

Since $\gamma\in[\frac{1}{2},1)$, and noting that $|z|^{\alpha_1-\alpha_j-\alpha_1\gamma}\le T^{\alpha_1-\alpha_j}|z|^{-\alpha_1\gamma}$, we see that

$$\|u_1(z)-u_0(z)\|_{D(A^\gamma)}\le C(\sum_{j=2}^{\ell}T^{\alpha_1-\alpha_j}+1)|z|^{-\alpha_1\gamma}\|a\|_{L^2(\Omega)}=:M_1|z|^{-\alpha_1\gamma}\|a\|_{L^2(\Omega)}.$$

Next, for any $n\in\mathbb{N}$, in view of the inequalities $\|B\cdot\nabla v\|_{L^2(\Omega)}\le C\|v\|_{D(A^{\frac{1}{2}})}\le C\|v\|_{D(A^\gamma)}$ for $v\in D(A^\gamma)$ and $\gamma\in[\frac{1}{2},1)$, we derive

$$\|A^{-1}S'(rz)\big(\widetilde{u}_{n+1}-\widetilde{u}_n\big)(z)\|_{D(A^\gamma)}\le C\|A^{\gamma-1}S'(rz)\|_{L^2(\Omega)\to L^2(\Omega)}\|(u_n-u_{n-1})(z)\|_{D(A^\gamma)}.$$

Combining the above inequalities with (2.5) for $z \in \Omega_{\theta,T}$, we can prove that

$$\begin{aligned}&\|u_{n+1}(z)-u_n(z)\|_{D(A^\gamma)}\\ \le& C|z|^{\beta_1}\int_0^1 r^{\beta_1-1}\|u_n((1-r)z)-u_{n-1}((1-r)z)\|_{D(A^{\frac12})}\mathrm{d}r\\ &+C\sum_{j=2}^{\ell}|z|^{\beta_j}\left(\int_0^1(1-s)^{\beta_1-1}s^{-\alpha_j}\mathrm{d}s\right)\int_0^1 r^{\beta_j-1}\|u_n((1-r)z)-u_{n-1}((1-r)z)\|_{D(A^\gamma)}\mathrm{d}r,\end{aligned}$$

where $\beta_1 := \alpha_1-\alpha_1\gamma$, $\beta_j := \alpha_1-\alpha_j$, $j=2,\cdots,\ell$. Noting that, for $0<\alpha_j<\alpha_1<1$, $j=2,\cdots,\ell$ and (2.6), we have

$$\|u_{n+1}(z)-u_n(z)\|_{D(A^\gamma)}\le C\sum_{j=1}^{\ell}|z|^{\beta_j}\int_0^1 r^{\beta_j-1}\|u_n((1-r)z)-u_{n-1}((1-r)z)\|_{D(A^\gamma)}\mathrm{d}r.$$

Consequently, by inductive assumption, we can prove

$$\|u_{n+1}(z)-u_n(z)\|_{D(A^\gamma)}\le C\sum_{j=1}^{\ell}|z|^{\beta_j}\int_0^1 r^{\beta_j-1}M_1M^{n-1}\left(\sum_{i=1}^{\ell}J^{\beta_i}\right)^{n-1}(g)((1-r)|z|)\mathrm{d}r.$$

After making the change of variable $r\to(1-r)|z|$ and from the definition of the Riemann-Liouville fractional integral, we see that

$$\begin{aligned}&\|u_{n+1}(z)-u_n(z)\|_{D(A^\gamma)}\le CM_1M^{n-1}\sum_{j=1}^{\ell}\int_0^{|z|}(|z|-r)^{\beta_j-1}\left(\sum_{i=1}^{\ell}J^{\beta_i}\right)^{n-1}(g)(r)\mathrm{d}r\\ =&CM_1M^{n-1}\sum_{j=1}^{\ell}\Gamma(\beta_j)J^{\beta_j}\left(\left(\sum_{i=1}^{\ell}J^{\beta_i}\right)^{n-1}(g)\right)(|z|)\\ \le&CM_1M^{n-1}\max_{1\le j\le\ell}\{\Gamma(\beta_j)\}\sum_{j=1}^{\ell}J^{\beta_j}\left(\left(\sum_{i=1}^{\ell}J^{\beta_i}\right)^{n-1}(g)\right)(|z|)=M_1M^n\left(\sum_{i=1}^{\ell}J^{\beta_i}\right)^{n}(g)(|z|),\end{aligned}$$

where we set $M := C\max_{1\le j\le\ell}\{\Gamma(\beta_j)\}$. Therefore by indcution, (2.10) holds true. Moreover, noting the semigroup property

$$J^\alpha J^\beta = J^{\alpha+\beta},\quad \alpha\ge 0,\ \beta\ge 0,$$

and the effect of the operator J^α on the power functions

$$J^\alpha t^\beta=\frac{\Gamma(\beta+1)}{\Gamma(\beta+1+\alpha)}t^{\alpha+\beta},\quad \alpha\ge0,\ \beta>-1,\ t>0,$$

we derive

$$\begin{aligned}&\|u_{n+1}(z)-u_n(z)\|_{D(A^\gamma)}\le M_1M^n\sum_{k_1+\cdots+k_\ell=n}\binom{n}{k_1}\cdots\binom{n}{k_\ell}J^{\beta_1k_1+\cdots+\beta_\ell k_\ell}(g)(|z|)\\ =&M_1M^n\sum_{k_1+\cdots+k_\ell=n}\binom{n}{k_1}\cdots\binom{n}{k_\ell}\frac{\Gamma(1-\alpha_1\gamma)|z|^{\beta_1k_1+\cdots+\beta_\ell k_\ell-\alpha_1\gamma}}{\Gamma(\beta_1k_1+\cdots+\beta_\ell k_\ell-\alpha_1\gamma+1)},\quad z\in\Omega_{\theta,T}.\end{aligned}\tag{2.11}$$

Here we noticed for any subset K compacted in $\Omega_{\theta,T}$ that

$$\sum_{n=0}^{\infty}M^n\sum_{k_1+\cdots+k_\ell=n}\binom{n}{k_1}\cdots\binom{n}{k_\ell}\frac{\Gamma(1-\alpha_1\gamma)|z|^{\beta_1k_1+\cdots+\beta_\ell k_\ell-\alpha_1\gamma}}{\Gamma(\beta_1k_1+\cdots+\beta_\ell k_\ell-\alpha_1\gamma+1)}$$

converges uniformly in K. In fact, the asymptotic behavior (2.3) yields

$$\Gamma(\beta_1 k_1 + \cdots + \beta_\ell k_\ell - \alpha_1\gamma + 1) \ge C\Gamma(\underline{\beta}(k_1 + \cdots + k_\ell) - \alpha_1\gamma + 1) = C\Gamma(\underline{\beta}n - \alpha_1\gamma + 1),$$

and noting that $\sum_{k_1+\cdots+k_\ell=n} \binom{n}{k_1}\cdots\binom{n}{k_\ell} = \ell^n$, it follows for $z \in \Omega_{\theta,T}$ that

$$\sum_{n=0}^{\infty} \sum_{k_1+\cdots+k_\ell=n} \binom{n}{k_1}\cdots\binom{n}{k_\ell} \frac{M^n |z|^{\beta_1 k_1+\cdots+\beta_\ell k_\ell - \alpha_1\gamma}}{\Gamma(\beta_1 k_1 + \cdots + \beta_\ell k_\ell - \alpha_1\gamma + 1)} \le C\sum_{n=0}^{\infty} \ell^n \frac{M^n T^{\overline{\beta}n} |z|^{-\alpha_1\gamma}}{\Gamma(\underline{\beta}n - \alpha_1\gamma + 1)},$$

where $\overline{\beta} := \max_{1\le j\le \ell}\{\beta_j\}$, $\underline{\beta} := \min_{1\le j\le\ell}\{\beta_j\}$. Again using the asymptotic behavior (2.3), we find

$$\frac{\ell^{n+1} M^{n+1} T^{\overline{\beta}(n+1)}}{\Gamma(\underline{\beta}(n+1) - \alpha_1\gamma + 1)} \Big/ \frac{\ell^n M^n T^{\overline{\beta}n}}{\Gamma(\underline{\beta}n - \alpha_1\gamma + 1)} \longrightarrow 0 \quad \text{as } n \to \infty,$$

so that

$$\sum_{n=0}^{\infty} M^n \sum_{k_1+\cdots+k_\ell=n} \binom{n}{k_1}\cdots\binom{n}{k_\ell} \frac{|z|^{\beta_1 k_1+\cdots+\beta_\ell k_\ell - \alpha_1\gamma}}{\Gamma(\beta_1 k_1 + \cdots + \beta_\ell k_\ell - \alpha_1\gamma + 1)} < \infty.$$

Hence the majorant test implies $\sum_{n=1}^{\infty} \|u_{n+1}(z) - u_n(z)\|_{D(A^\gamma)}$ is convergent uniformly in any compact subset of $\Omega_{\theta,T}$. Therefore there exists $u_*(z) \in L^2(\Omega)$ such that $\|u_n(z) - u_*(z)\|_{D(A^\gamma)}$ tends to 0 as $n \to \infty$ uniformly in any compact subset of $\Omega_{\theta,T}$. We thus assert that $u = u_*|_{\Omega\times(0,T]}$ is the unique solution to the integral equation (2.7).

Furthermore, we can see from (2.10) that $\|A^\gamma u_*(t)\|_{L^2(\Omega)} = O(\mathrm{e}^{Ct})$, as $t \to \infty$. Indeed, for any $T \ge 1$ and $0 < t \le T$, we have

$$\begin{aligned}
&\|A^\gamma u_*(t)\|_{L^2(\Omega)} \le \sum_{n=0}^{\infty} \|A^\gamma u_{n+1}(t) - A^\gamma u_n(t)\|_{L^2(\Omega)} \\
\le& M_1 \sum_{n=0}^{\infty} M^n \sum_{k_1+\cdots+k_\ell=n} \binom{n}{k_1}\cdots\binom{n}{k_\ell} \frac{\Gamma(1-\alpha_1\gamma) T^{\beta_1 k_1+\cdots+\beta_\ell k_\ell - \alpha_1\gamma}}{\Gamma(\beta_1 k_1 + \cdots + \beta_\ell k_\ell - \alpha_1\gamma + 1)} \\
=&: M_1\Gamma(1-\alpha_1\gamma)H(T).
\end{aligned}$$

The estimate of $H(t)$ as $t \to \infty$ follows from the fact that the Laplace transform

$$\begin{aligned}
\mathcal{L}H(s) :=& \int_0^\infty \sum_{n=0}^{\infty} \sum_{k_1+\cdots+k_\ell=n} \binom{n}{k_1}\cdots\binom{n}{k_\ell} \frac{M^n t^{\beta_1 k_1+\cdots+\beta_\ell k_\ell - \alpha_1\gamma}}{\Gamma(\beta_1 k_1 + \cdots + \beta_\ell k_\ell - \alpha_1\gamma + 1)} \mathrm{e}^{-st}\mathrm{d}t \\
=& \frac{s^{\alpha_1\gamma - 1}}{1 - M\sum_{j=1}^{\ell} s^{-\beta_j}},
\end{aligned}$$

where $\mathrm{Re}\, s > M_2$ and $M_2 > 0$ is a sufficiently large constant, has only finite simple poles in the main sheet of Riemann surface cutting off the negative axis. We denote the poles as $\{s_1, \cdots, s_\ell\}$. Moreover, we can see that $s_i \in \mathbb{R}$ and $s_i > 0$, $i = 1, \cdots, m$. Indeed, for $s := r\mathrm{e}^{\mathrm{i}\theta}$ with $\theta \in [-\pi, \pi]$ such that $1 - M\sum_{j=1}^{\ell} s^{-\beta_j} = 0$, that is,

$$\sum_{j=1}^{\ell} r^{-\beta_j}\cos\beta_j\theta - \mathrm{i}\sum_{j=1}^{\ell} r^{-\beta_j}\sin\beta_j\theta = \frac{1}{M},$$

which implies $\sum_{j=1}^{\ell} r^{-\beta_j}\sin\beta_j\theta = 0$, and noting that $\sin\beta_j\theta$ $(j = 1, \cdots, \ell)$ have the same signals, hence $\theta = 0$. Now by Fourier-Mellin formula (e.g., [55]), we have

$$H(t) = \frac{1}{2\pi\mathrm{i}} \int_{M_2 - \mathrm{i}\infty}^{M_2 + \mathrm{i}\infty} \mathcal{L}H(s)\mathrm{e}^{st}\mathrm{d}s.$$

Now we choose a small constant $0 < \gamma < \min\{s_1, \cdots, s_\ell\}$ such that $1 - M\sum_{j=1}^{\ell} s^{-\beta_j} \neq 0$ for $s \neq s_1, \cdots, s_\ell$, $\operatorname{Re} s \geq \gamma$, and then by Residue Theorem (e.g., [53]), for $t > 0$ we see that

$$H(t) = \sum_{j=1}^{m} a_j \mathrm{e}^{s_j t} + \frac{1}{2\pi\mathrm{i}} \int_{\gamma-\mathrm{i}\infty}^{\gamma+\mathrm{i}\infty} \mathcal{L}H(s)\mathrm{e}^{st}\mathrm{d}s,$$

where $a_j := \lim_{s\to s_j}(s - s_j)\mathcal{L}H(s)$, and the shift in the line of integration is justified by the fact $e^{st}\mathcal{L}H(s) \to 0$ as $\operatorname{Im} s \to \infty$ with $\operatorname{Re} s$ bounded.

Integration by parts shows

$$\int_{\gamma-\mathrm{i}\infty}^{\gamma+\mathrm{i}\infty} \mathcal{L}H(s)\mathrm{e}^{st}\mathrm{d}s = \frac{s^{\alpha_1\gamma-1}}{1 - M\sum_{j=1}^{\ell} s^{-\beta_j}} \frac{\mathrm{e}^{st}}{t}\Bigg|_{s=\gamma-\mathrm{i}\infty}^{s=\gamma+\mathrm{i}\infty}$$
$$- \int_{\gamma-\mathrm{i}\infty}^{\gamma+\mathrm{i}\infty} \frac{\mathrm{e}^{st}}{t} \frac{(\alpha_1\gamma-1)s^{\alpha_1\gamma-2}(1 - M\sum_{j=1}^{\ell} s^{-\beta_j}) - s^{\alpha_1\gamma-1}M\sum_{j=1}^{\ell}\beta_j s^{-\beta_j-1}}{(1 - M\sum_{j=1}^{\ell} s^{-\beta_j})^2}\mathrm{d}s.$$

Therefore

$$\left|H(t) - \sum_{j=1}^{m} a_j \mathrm{e}^{s_j t}\right| = O\left(\frac{1}{t}\mathrm{e}^{\gamma t}\right), \quad t \to \infty.$$

See e.g., [19] as for a similar argument. Consequently, we derive that

$$H(t) = O(\mathrm{e}^{Ct}), \quad t \to \infty,$$

which implies

$$\|u_*(t)\|_{D(A^\gamma)} \leq Ct^{-\alpha_1\gamma}\mathrm{e}^{CT}\|a\|_{L^2(\Omega)}, \quad t \in (0, T]. \tag{2.12}$$

Let us turn to show the analyticity of the solution with respect to z. For this, by induction, we first prove that $u_n : \Omega_{\theta,T} \to D(A^\gamma)$ is analytic for $n = 0, 1, \cdots$. By $u_0 \equiv 0$, it is obvious for $n = 0$. We assume that $u_n : \Omega_{\theta,T} \to D(A^\gamma)$ is analytic in z. We estimate the integrands in (2.9). The use of (2.5) implies

$$\begin{aligned}
&\|A^{-1}S'(rz)(1-r)^{-\alpha_j}q_j a\|_{D(A^\gamma)}\\
\leq &C(rz)^{\alpha_1-\alpha_1\gamma-1}(1-r)^{-\alpha_j}\|a\|_{L^2(\Omega)}\\
\leq &C|z|^{\alpha_1-\alpha_1\gamma-1}r^{\alpha_1-\alpha_1\gamma-1}(1-r)^{-\alpha_j}\|a\|_{L^2(\Omega)},
\end{aligned}$$

$$\begin{aligned}
&\|A^{-1}S'(rz)(B\cdot\nabla u_n((1-r)z) + cu_n((1-r)z))\|_{D(A^\gamma)}\\
\leq &C(rz)^{\alpha_1-\alpha_1\gamma-1}\|u_n((1-r)z)\|_{D(A^{\frac12})}\\
\leq &C|z|^{\alpha_1-\alpha_1\gamma-1}r^{\alpha_1-\alpha_1\gamma-1}\|u_n((1-r)z)\|_{D(A^{\frac12})},
\end{aligned}$$

$$\begin{aligned}
&\|A^{-1}S'((1-s)rz)r^{-\alpha_j}s^{-\alpha_j}q_j u_n((1-r)z)\|_{D(A^\gamma)}\\
\leq &C((1-s)rz)^{\alpha_1-1}r^{-\alpha_j}s^{-\alpha_j}\|u_n((1-r)z)\|_{D(A^\gamma)}\\
\leq &C|z|^{\alpha_1-1}r^{\alpha_1-\alpha_j-1}(1-s)^{-\alpha_j}s^{-\alpha_j}\|u_n((1-r)z)\|_{D(A^\gamma)}
\end{aligned}$$

and

$$\begin{aligned}
&\|A^{-1}S''((1-s)rz)(1-s)r^{1-\alpha_j}s^{-\alpha_j}q_j u_n((1-r)z)\|_{D(A^\gamma)}\\
\leq &C((1-s)rz)^{\alpha_1-2}(1-s)r^{1-\alpha_j}s^{-\alpha_j}\|u_n((1-r)z)\|_{D(A^\gamma)}\\
\leq &C|z|^{\alpha_1-2}r^{\alpha_1-\alpha_j-1}s^{-\alpha_j}(1-s)^{\alpha_1-1}\|u_n((1-r)z)\|_{D(A^\gamma)}.
\end{aligned}$$

Here in view of (2.3) and (2.11), it follows that

$$\|u_n(z)\|_{D(A^\gamma)} \leq C_n|z|^{-\alpha_1\gamma}\|a\|_{L^2(\Omega)}, \quad z \in \Omega_{\theta,T},$$

hence that the $D(A^\gamma)$-norm of the integrands in (2.9) are integrable in $r, s \in (0,1)$. Therefore $u_{n+1}((1-r)z) : \Omega_{\theta,T} \to D(A^\gamma)$ is analytic, we see that also $u_{n+1} : \Omega_{\theta,T} \to D(A^\gamma)$ is analytic. Thus by induction $u_n : \Omega_{\theta,T} \to D(A^\gamma)$ is analytic for all $n \in \mathbb{N}$. Since we have proved that $\sum_{n=1}^\infty \|u_{n+1}(z) - u_n(z)\|_{D(A^\gamma)}$ is convergent uniformly in any compact subset of $\Omega_{\theta,T}$, therefore $u_* : \Omega_{\theta,T} \to D(A^\gamma)$ is analytic. Moreover, since T and θ are arbitrarily chosen, we deduce u_* is analytic in the sector $\{s \in \mathbb{C}; s \neq 0, |\arg s| < \frac{\pi}{2}\}$.

Finally, we see that $u(\cdot,t)$ $(t \in (0,T])$ is just the solution to (2.1) and such that

$$\|u(\cdot,t)\|_{H^{2\gamma}(\Omega)} \le Ct^{-\alpha_1\gamma} e^{CT} \|a\|_{L^2(\Omega)}, \quad \gamma \in [\frac{1}{2}, 1),\ t \in (0,T]$$

in view of (2.12). This completes the proof of the theorem. □

Remark 2.2 If we furthermore assume that

$$B \in \{W^{1,\infty}(\Omega)\}^d \text{ and } c \in W^{1,\infty}(\Omega),$$

we point out that the solution $u(t)$ to the initial-boundary value problem (2.1) can achieve more regularity on time and space, that is, $u \in C((0,T]; H_0^1(\Omega) \cap H^2(\Omega))$ and

$$\|u(t)\|_{H^2(\Omega)} \le Ct^{-\alpha_1} e^{CT} \|a\|_{L^2(\Omega)}, \quad 0 < t \le T.$$

Theorem 2.1 shows that the spatial regularity can be as close as possible to, but cannot achieve the maximal regularity, say, $H^2(\Omega)$-regularity, providing the continuity of the solution with respect to time $t \in (0,T]$. However, next theorem demonstrates for a.e. $t \in (0,T]$, the solution $u(t)$ can achieve the maximal spatial regularity, that is, $u(t) \in H^2(\Omega)$ for almost all $t \in (0,T)$.

Theorem 2.2 ($H^2(\Omega)$–regularity) *Let $0 < \alpha_\ell < \cdots < \alpha_1 < 1$ and $T > 0$ be given. Assuming that $a \in L^2(\Omega)$, $F = 0$, $q_j \in W^{2,\infty}(\Omega)$ $(j = 2, \cdots, \ell)$, $(B, c) \in (L^\infty(\Omega))^{d+1}$. Then the solution u to the initial-boundary value problem* (2.1) *belongs to $L^p(0,T; H_0^1(\Omega) \cap H^2(\Omega))$ with $1 \le p < \min\{2, \frac{1}{\alpha_1}\}$. Moreover the following estimate*

$$\|u\|_{L^p(0,T;H^2(\Omega))} \le C_T \|a\|_{L^2(\Omega)}$$

holds true.

Remark 2.3 This is a very different property compared with parabolic equations whose solutions can not be in $L^p(0,T; H^2(\Omega))$ for any $p \ge 1$ providing the initial value is in $L^2(\Omega)$.

Proof. Now let us take the operator A on the both sides of (2.7), we evaluate each of the five terms separately. Estimate of $I_1(t)$. We conclude from (2.5) that

$$\|I_1(t)\|_{D(A)} \le Ct^{-\alpha_1} \|a\|_{L^2(\Omega)}, \quad t \in (0,T].$$

In order to estimate $\|I_2(t)\|_{D(A)}$, we break up the integral in I_2 into two integrals as follows

$$\begin{aligned} AI_2(t) &= \sum_{j=2}^{\ell} \int_0^{\frac{t}{2}} S'(r)(t-r)^{-\alpha_j} \frac{q_j a}{\Gamma(1-\alpha_j)} \mathrm{d}r + \sum_{j=2}^{\ell} \int_{\frac{t}{2}}^{t} S'(r)(t-r)^{-\alpha_j} \frac{q_j a}{\Gamma(1-\alpha_j)} \mathrm{d}r \\ &=: AI_{21}(t) + AI_{22}(t). \end{aligned}$$

For $AI_{21}(t)$. Integrating by parts derives

$$AI_{21}(t) = \sum_{j=2}^{\ell} \frac{q_j}{\Gamma(1-\alpha_j)} S(r)(t-r)^{-\alpha_i} a \Big|_{r=0}^{r=\frac{t}{2}} - \sum_{j=2}^{\ell} \alpha_j \int_0^{\frac{t}{2}} S(r)(t-r)^{-\alpha_j - 1} \frac{q_j a}{\Gamma(1-\alpha_j)} \mathrm{d}r.$$

Since $\|S(t)\|_{L^2(\Omega)\to L^2(\Omega)}$ is uniformly bounded for $t \in [0,T]$, it is easily seen that

$$\|I_{21}(t)\|_{D(A)} \le C\sum_{j=2}^{\ell} t^{-\alpha_j}\|a\|_{L^2(\Omega)}, \quad t\in(0,T].$$

For $AI_{22}(t)$, from (2.5), it follows

$$\|I_{22}(t)\|_{D(A)} \le C\sum_{j=2}^{\ell}\int_{\frac{t}{2}}^{t} r^{-1}(t-r)^{-\alpha_j}\mathrm{d}r\|a\|_{L^2(\Omega)} \le C\sum_{j=2}^{\ell} t^{-\alpha_j}\|a\|_{L^2(\Omega)}.$$

For $I_3(t)$, $0<t\le T$, from (2.5), recalling the definition of $S(t)$ in (2.4), we derive

$$\|I_3(t)\|_{D(A)}^2 = \left\|\int_0^t S'(t-r)(B\cdot\nabla u(r)+cu(r))\mathrm{d}r\right\|_{L^2(\Omega)}^2$$
$$\le \sum_{n=1}^{\infty}\left|\int_0^t \lambda_n(t-r)^{\alpha_1-1}E_{\alpha_1,\alpha_1}(-\lambda_n(t-r)^{\alpha_1})(B\cdot\nabla u(r)+cu(r),\varphi_n)\mathrm{d}r\right|^2.$$

Thus the Young inequality implies

$$\int_0^T \|I_3(t)\|_{D(A)}^2\mathrm{d}t \le \sum_{n=1}^{\infty}\Big(\int_0^T \lambda_n r^{\alpha_1-1}\left|E_{\alpha_1,\alpha_1}(-\lambda_n r^{\alpha_1})\right|\mathrm{d}r\Big)^2\int_0^T |(B\cdot\nabla u(r)+cu(r),\varphi_n)|^2\mathrm{d}r.$$

Moreover, the use of Lemma 2.1 derives that

$$\int_0^T \left|\lambda_n r^{\alpha_1-1}E_{\alpha_1,\alpha_1}(-\lambda_n r^{\alpha_1})\right|\mathrm{d}r = \int_0^T \lambda_n r^{\alpha_1-1}E_{\alpha_1,\alpha_1}(-\lambda_n r^{\alpha_1})\mathrm{d}r = 1-E_{\alpha_1,1}(-\lambda_n T^{\alpha_1}) \le C_T,$$

thereby obtaining the inequalities

$$\int_0^T \|I_3(t)\|_{D(A)}^2\mathrm{d}t \le C_T\int_0^T\sum_{n=1}^{\infty}|(B\cdot\nabla u(r)+cu(r),\varphi_n)|^2\mathrm{d}r \le C_T\int_0^T\|u(t)\|_{H^1(\Omega)}^2\mathrm{d}t$$
$$\le C_T\int_0^T (t^{-\frac{1}{2}\alpha_1}\|a\|_{L^2(\Omega)})^2\mathrm{d}t \le C_T\|a\|_{L^2(\Omega)}^2.$$

Here in the third inequality we used the estimate (2.12). For $I_4(t)$, from (2.5), select $\epsilon>0$ small enough, and similar to the argument used in Theorem 2.1, it follows that

$$\|I_4(t)\|_{D(A)} \le C\sum_{j=2}^{\ell}\left(\int_0^1 (1-s)^{\alpha_1-\alpha_1\epsilon-1}s^{-\alpha_j}\mathrm{d}s\right)\int_0^t (t-r)^{\alpha_1(1-\varepsilon)-1-\alpha_j}\|A^{1-\varepsilon}u(r)\|_{L^2(\Omega)}\mathrm{d}r.$$

Again the use of (2.12) leads to

$$\|I_4(t)\|_{D(A)} \le C_T\sum_{j=2}^{\ell} t^{-\alpha_j}\|a\|_{L^2(\Omega)}, \quad 0<t\le T.$$

For $I_5(t)$ we argument similarly to obtain

$$\|I_5(t)\|_{D(A)} \le C_T\sum_{i=2}^{\ell} t^{-\alpha_i}\|a\|_{L^2(\Omega)}.$$

Finally, we proved that for any $t\in(0,T]$, the solution u satisfies

$$\left(\int_0^T \|u(t)\|_{H^2(\Omega)}^p\mathrm{d}t\right)^{\frac{1}{p}} \le C_T\|a\|_{L^2(\Omega)}, \quad 1\le p<\min\{2,\tfrac{1}{\alpha_1}\}.$$

This completes the proof of the theorem. □

2.2.2 Nonhomogeneous equation

In this section, we are concerned to initial-boundary value problem (2.1) providing $a = 0$ and $F \in L^2(Q)$ ($Q := \Omega \times (0,T)$). Different to the initial-boundary value problem for homogeneous equation discussed in Section 2.2.1 where the unique solution is shown to be continuous with respect to $t \in (0,T]$, the time-regularity cannot make sense pointwisely any more in view of Theorem 1.1 in Gorenflo et al [16] where the maximal time-regularity of the solution is shown to be $H^{\alpha_1}(0,T;L^2(\Omega))$. Therefore we have to revise the definition of the solution used in Definition 2.1 as follows:

Definition 2.2 (Weak solution) *Let $a = 0$ and $F \in L^2(Q)$. We call u a weak solution to the initial-boundary value problem* (2.1) *if $u \in L^2(0,T;H^2(\Omega) \cap H_0^1(\Omega))$ satisfies*

$$J^{-\alpha_1} u \in L^2(0,T;L^2(\Omega))$$

and

$$\partial_t^{\alpha_1} u + \sum_{j=2}^{\ell} q_j \partial_t^{\alpha_j} u = -\mathcal{A}u + B \cdot \nabla u + cu + F \text{ in } L^2(0,T;L^2(\Omega)).$$

Here we should understand the Caputo derivative $\partial_t^\alpha (0 < \alpha < 1)$ as a unique extension of the operator $\partial_t^\alpha : C^\infty(0,T) \to L^2(0,T)$ on $H^\alpha(0,T)$ according to Gorenflo et al [16].

Moreover, we note that $J^{-\alpha_1} u \in L^2(0,T;L^2(\Omega))$ implies $u(x,\cdot) \in \mathcal{R}(J^{\alpha_1})$ for almost all $x \in \Omega$. Hence if $\frac{1}{2} < \alpha_1 < 1$, then by Theorem 1.1 (i) in [16], we have the initial condition $u(x,0) = 0$ is valid automatically. However, for $0 < \alpha \le \frac{1}{2}$, the initial condition in $u(x,0) = 0$ is not meaningful at all. We should understand the behavior of $u(x,\cdot) \in \mathcal{R}(J^{\alpha_1})$ near $t = 0$ in the case of $0 < \alpha_1 \le \frac{1}{2}$ in the following sense

$$\lim_{t\to 0} J^{1-\alpha_1} u(x,t) = \lim_{t\to 0} J^{1-\alpha_1} J^{\alpha_1} \phi(x,t) = \lim_{t\to 0} J\phi(x,t) = \lim_{t\to 0} \int_0^t \phi(x,s)\mathrm{d}s = 0,$$

where $\phi = J^{-\alpha_1} u$. For more detailed interpretation of the above definition, see e.g., [16].

We have

Theorem 2.3 *Let $0 < \alpha_\ell < \cdots < \alpha_1 < 1$ and $T > 0$ be given. Assuming that $B \in (L^\infty(Q))^d$, $q_j \in L^\infty(Q) (j = 2, \cdots, \ell)$, $b \in L^\infty(Q)$. Let $F \in L^2(0,T;L^2(\Omega))$, $a = 0$, then the initial-boundary value problem* (2.1) *admits a unique weak solution $u \in L^2(0,T;H^2(\Omega) \cap H_0^1(\Omega)) \cap H^{\alpha_1}(0,T;L^2(\Omega))$. Moreover the following estimate holds:*

$$\|u\|_{H^{\alpha_1}(0,T;L^2(\Omega))} + \|u\|_{L^2(0,T;H^2(\Omega))} \le C_T \|F\|_{L^2(0,T;L^2(\Omega))}.$$

Proof. Similar to the argument used in Section 2.2.1, we have

$$u(t) = \sum_{j=2}^{\ell} \int_0^t A^{-1} S'(t-s) q_j \partial_t^{\alpha_j} u(s)\mathrm{d}s - \int_0^t A^{-1} S'(t-s)(B \cdot \nabla u(s) + cu(s) + F(\cdot,s))\mathrm{d}s. \quad (2.13)$$

We set

$$Lu = -\int_0^t A^{-1} S'(t-s)(B \cdot \nabla u(s) + cu(s))\mathrm{d}s,$$

$$(K_j u)(t) = \int_0^t A^{-1} S'(t-s) q_j(s) \partial_t^{\alpha_j} u(s)\mathrm{d}s, \quad 2 \le j \le \ell$$

and $K = \sum_{j=2}^{\ell} K_j$. Then our equation (2.13) is rewritten as

$$u = (K + L)u + g \quad \text{in } L^2(Q).$$

Here we set $g(t) = -\int_0^t A^{-1} S'(t-s) F(\cdot,s)\mathrm{d}s$, and

$$X_\eta = H^\alpha(0,\eta;L^2(\Omega)) \cap L^2(0,\eta;H^2(\Omega))$$

for $\eta \in (0,T)$ with the norm

$$\|v\|_{X_\eta} := \|v\|_{H^\alpha(0,\eta;L^2(\Omega))} + \|v\|_{L^2(0,\eta;H^2(\Omega))}.$$

From Sakamoto and Yamamoto [60], we have

$$\|g\|_{X_\eta} \le C\|F\|_{L^2(0,\eta;L^2(\Omega))}.$$

We next prove that $K+L : X_\eta \to X_\eta$ is compact. In fact, by Theorem 4.2 in [16], we can directly prove

$$\|Lu\|_{X_\eta} \le C(\|B\cdot\nabla u\|_{L^2(0,\eta;L^2(\Omega))} + \|cu\|_{L^2(0,\eta;L^2(\Omega))}) \le C\|u\|_{L^2(0,\eta;H^1(\Omega))}. \tag{2.14}$$

Then the embedding $X_\eta \to L^2(0,\eta;H^1(\Omega))$ is compact, and so the operator $L : X_\eta \to X_\eta$ is compact.

We see

$$\|K_j u\|_{X_\eta} \le C\|u\|_{H^{\alpha_j}(0,\eta;L^2(\Omega))} \le C\|u\|_{H^{\alpha_2}(0,\eta;L^2(\Omega))}, \quad j = 2,\cdots,\ell, \tag{2.15}$$

where the constant $C>0$ is independent of $\eta \in (0,T)$ (see p.434 in Sakamoto and Yamamoto [60]). By $\alpha_1 > \alpha_j$ for $j = 2,\cdots,\ell$, the embedding $X_\eta \to H^{\alpha_2}(0,\eta;L^2(\Omega))$ is compact (e.g., Temam [63], Chap. III §2, or we can prove directly similarly to Chapter 5 of Baumeister [4]). Hence $K : X_\eta \to X_\eta$ is a composition of compact operators. Hence $K+L : X_\eta \to X_\eta$ is a compact operator.

Next we are to verify that 1 is not an eigenvalue of $K+L$, that is, $(K+L)u = u$ implies $u = 0$. First we prove

$$\|\partial_t^\beta u\|_{L^2(0,\eta;L^2(\Omega))} \le C\eta^{\alpha_1-\beta}\|\partial_t^{\alpha_1} u\|_{L^2(0,\eta;L^2(\Omega))} \tag{2.16}$$

for $0 \le \beta < \alpha_1$ and $u \in \mathcal{R}(J^{\alpha_1})$. Indeed, since J^γ with $\gamma \in \mathbb{R}$ is defined by the fractional power, we have

$$J^{-\beta}u = J^{\alpha_1-\beta}J^{-\alpha_1}u, \quad u \in \mathcal{R}(J^{\alpha_1})$$

(e.g., Pazy [50], Theorem 6.8 (p.72)). Therefore

$$\|\partial_t^\beta u(x,\cdot)\|_{L^2(0,\eta)} = \|J^{-\beta}u(x,\cdot)\|_{L^2(0,\eta)} = \|J^{\alpha_1-\beta}J^{-\alpha_1}u(x,\cdot)\|_{L^2(0,\eta)}.$$

On the other hand, by the Young inequality, for $w \in H^{\alpha_1}(0,\eta)$, we have

$$\begin{aligned}\|J^{\alpha_1-\beta}w\|_{L^2(0,\eta)} &= \frac{1}{\Gamma(\alpha_1-\beta)}\left\|\int_0^t (t-s)^{(\alpha_1-\beta)-1}w(s)\mathrm{d}s\right\|_{L^2(0,\eta)}\\ \le&\frac{1}{\Gamma(\alpha_1-\beta)}\int_0^\eta s^{(\alpha_1-\beta)-1}\mathrm{d}s\left(\int_0^\eta |w(s)|^2\mathrm{d}s\right)^{\frac{1}{2}} = \frac{1}{\Gamma(\alpha_1-\beta+1)}\eta^{\alpha_1-\beta}\|w\|_{L^2(0,\eta)}.\end{aligned}$$

Hence

$$\|\partial_t^\beta u(x,\cdot)\|_{L^2(0,\eta)} \le C\eta^{\alpha_1-\beta}\|J^{-\alpha_1}u(x,\cdot)\|_{L^2(0,\eta)} = C\eta^{\alpha_1-\beta}\|\partial_t^{\alpha_1}u(x,\cdot)\|_{L^2(0,\eta)}.$$

Thus, by taking the norm in $L^2(\Omega)$, the proof of (2.16) is completed.

Now we estimate $K_j u$ for $j = 2,\cdots,\ell$. The inequalities (2.15) and (2.16) yield

$$\begin{aligned}&\|K_j u\|_{X_\eta} \le C\|\partial_t^{\alpha_j}u\|_{L^2(0,\eta;L^2(\Omega))}\\ \le&C\eta^{\alpha_1-\alpha_j}\|\partial_t^{\alpha_1}u\|_{L^2(0,\eta;L^2(\Omega))} \le C\eta^{\alpha_1-\alpha_2}\|u\|_{H^{\alpha_1}(0,\eta;L^2(\Omega))}, \quad j = 2,\cdots,\ell.\end{aligned}$$

From the estimate (2.14), by the interpolation inequality, for any $\epsilon > 0$, there exists a constant $C_\epsilon > 0$ such that

$$\|u\|_{L^2(0,\eta;H^1(\Omega))} \le \epsilon\|u\|_{L^2(0,\eta;H^2(\Omega))} + C_\epsilon\|u\|_{L^2(0,\eta;L^2(\Omega)))}.$$

Therefore by (2.16) with $\beta = 0$, we have

$$\begin{aligned}\|u\|_{X_\eta} \le& \epsilon\|u\|_{X_\eta} + C_\epsilon\|u\|_{L^2(0,\eta;L^2(\Omega)))}\\ \le& \epsilon\|u\|_{X_\eta} + C_\epsilon\eta^{\alpha_1}\|u\|_{H^{\alpha_1}(0,\eta;L^2(\Omega)))} \le (\epsilon + C_\epsilon\eta^{\alpha_1})\|u\|_{X_\eta}.\end{aligned}$$

Hence

$$\|Ku + Lu\|_{X_\eta} \le (C\eta^{\alpha_1-\alpha_2} + \epsilon + C_\epsilon\eta^{\alpha_1})\|u\|_{X_\eta}.$$

We fix $0 < \epsilon < 1$ arbitrarily. Then we choose $\eta > 0$ sufficiently small, so that

$$C\eta^{\alpha_1-\alpha_2} + \epsilon + C_\epsilon\eta^{\alpha_1} < 1.$$

Hence, since $u = (K + L)u$ in X_η, we have $u(x,t) = 0$ for $x \in \Omega$ and $0 < t < \eta$.

Next we continue this argument over η. By $u = 0$ for $0 < t < \eta$, we have

$$\partial_t^{\alpha_j} u(t) = \frac{1}{\Gamma(1-\alpha_j)}\int_\eta^t (t-s)^{-\alpha_j}\frac{\partial u}{\partial s}(s)\mathrm{d}s, \quad t > \eta.$$

We set $\widetilde{u}(t) = u(t+\eta)$. Therefore

$$\begin{aligned}\partial_t^{\alpha_j}\widetilde{u}(t-\eta) =& \frac{1}{\Gamma(1-\alpha_j)}\int_\eta^t (t-s)^{-\alpha_j}\frac{\partial \widetilde{u}}{\partial s}(s-\eta)\mathrm{d}s\\ =& \frac{1}{\Gamma(1-\alpha_j)}\int_0^{t-\eta} (t-\eta-\xi)^{-\alpha_j}\frac{\partial \widetilde{u}}{\partial \xi}(\xi)\mathrm{d}\xi, \quad t > \eta,\end{aligned}$$

that is,

$$\partial_t^{\alpha_j}\widetilde{u}(t) = \frac{1}{\Gamma(1-\alpha_j)}\int_0^t (t-\xi)^{-\alpha_j}\frac{\partial \widetilde{u}}{\partial \xi}(\xi)\mathrm{d}\xi, \quad t > 0.$$

Therefore $\widetilde{u} = (K + L)\widetilde{u}$ for $t > 0$. The same argument yields $\widetilde{u} = 0$ for $0 < t < \eta$. Hence $u(t+\eta) = 0$ for $0 < t < \eta$, that is, $u(t) = 0$ for $0 < t < 2\eta$. Repeating the above argument, we see that $u(t) = 0$ for $0 < t < T$.

Consequently, by the Fredholm alternative, we complete the proof of Theorem 2.3. □

2.3 Long-time asymptotic behavior

For a fixed positive integer ℓ, let $\alpha_j \in (0,1)$ be constants such that $\alpha_\ell < \cdots < \alpha_1$. Setting $B = 0$ in (2.1), we consider the following initial-boundary value problem

$$\begin{cases}\displaystyle\sum_{j=1}^{\ell} q_j(x)\partial_t^{\alpha_j} u(x,t) = -\mathcal{A}u(x,t) + c(x)u(x,t), & x \in \Omega,\ t > 0,\\ u(x,0) = a(x), & x \in \Omega,\\ u(x,t) = 0, & x \in \partial\Omega,\ t > 0,\end{cases} \tag{2.17}$$

Based on the results in Section 2.2.1 and Remark 2.2, we know that the mild solution u to (2.1) uniquely exists in $H_0^1(\Omega) \cap H^2(\Omega)$ for any $t \in (0,T]$ and admits

$$\|u(\cdot,t)\|_{H^2(\Omega)} \le Ct^{-\alpha_1}e^{CT}\|a\|_{L^2(\Omega)}, \quad t \in (0,T],\ \gamma \in [\frac{1}{2},1). \tag{2.18}$$

Thus the asymptotic behavior near 0 is only related to the largest order of the fractional derivatives. As for the long-time asymptotic behavior, for $\ell = 1$, Sakamoto and Yamamoto [60] asserts that the solution decays in polynomial $t^{-\alpha_1}$ as $t \to \infty$, which is a typical property of fractional diffusion equations in contrast to the exponential decay in the classical diffusion equations. In Li et al. [37], the initial-boundary value problem for the multi-term time-fractional diffusion equation with positive-constant coefficients was investigated. The Laplace transform in time was applied to show that the decay rate is indeed $t^{-\alpha_\ell}$ at best as $t \to \infty$, where α_ℓ is the lowest order of the time-fractional derivatives.

Here in this section we are devoted to the long-time asymptotic behavior of the solution to the initial-boundary value problem (2.1), and attempt to establish results parallel to that for the case of positive-constant coefficients. Namely, the following theorem holds.

Theorem 2.4 *Let $\alpha_j \in (0,1)$ be constants such that $\alpha_\ell < \cdots < \alpha_1$, and $\{q_j\}_{j=1}^{\ell}$ be in $W^{2,\infty}(\Omega)$ with $q_j > 0$ in $\overline{\Omega}$. We further assume that $c \in W^{1,\infty}(\Omega)$ such that $b \le 0$ in Ω. Then there exists $v \in H_0^1(\Omega) \cap H^2(\Omega)$, the unique solution of the initial-boundary value problem*

$$\begin{cases} q_\ell(x)\partial_t^{\alpha_\ell} v(x,t) = -\mathcal{A}v(x,t) + c(x)v(x,t), & x \in \Omega,\ t > 0,\\ v(x,0) = a(x), & x \in \Omega,\\ v(x,t) = 0, & x \in \partial\Omega,\ t > 0, \end{cases} \tag{2.19}$$

where has the same asymptotic behavior as u, in the sense that

$$\|u(\cdot,t) - v(\cdot,t)\|_{H^2(\Omega)} = O(t^{-\min\{2\alpha_\ell,\alpha_{\ell-1}\}})\|a\|_{L^2(\Omega)} \text{ as } t \to \infty,$$

where $(\mathcal{A}-c)^{-1}(q_\ell a)$ denotes the unique solution of $(\mathcal{A}-c)w = q_\ell a$ for $w \in H^2(\Omega) \cap H_0^1(\Omega)$, and the constant $C > 0$ is independent of t, a and u, but may depend on d, Ω, $\{\alpha_j\}_{j=1}^{\ell}$, $\{q_j\}_{j=1}^{\ell}$, c and $\{a_{ij}\}$.

Corollary 2.1 *Under the same assumptions in Theorem 2.4, then u admits the following estimate*

$$\left\| u(\cdot,t) - \frac{(\mathcal{A}-b)^{-1}(q_\ell a)t^{-\alpha_\ell}}{\Gamma(1-\alpha_\ell)} \right\|_{H^2(\Omega)} \le C\|a\|_{L^2(\Omega)} t^{-\min\{2\alpha_\ell,\alpha_{\ell-1}\}} \quad \text{for sufficiently large } t > 0.$$

Moreover, suppose that $\|u(\cdot,t)\|_{H^2(\Omega)} = o(t^{-\alpha_\ell})$ as $t \to \infty$, then $u(x,t) = 0$ for all $x \in \Omega$ and $t > 0$.

The above theorem shows that the solution u to the initial-boundary value problem (2.1) is approximated by the solution v to the initial-boundary value problem (2.19). Moreover, from Corollary 2.1, we can see that the decay rate of u is $t^{-\alpha_\ell}$ at best. The assumption $b \le 0$ and $q_j > 0$ in Ω are necessary for proving that the Laplace transform $\widehat{u}(x,s)$ of the solution u to our problem (2.1) has no poles in the main sheet of Riemann surface cutting off the negative axis, which is essential for the proof of Theorem 2.4. In the case of negative coefficients $\{q_j\}$, a counterexample can be found in [37].

For the statement of our main results, we set

$$S_\theta := \{s \in \mathbb{C}; s \ne 0,\ |\arg s| < \theta\}, \quad \tfrac{\pi}{2} < \theta < \min\{\tfrac{\pi}{2\alpha_1}, \pi\}.$$

From (2.18), we can apply the Laplace transform $\widehat{\cdot}$ on both sides on the equation in (2.1), and use the formula

$$\widehat{\partial_t^\alpha f}(s) = s^\alpha \widehat{f}(s) - s^{\alpha-1}f(0+),$$

to derive the transformed algebraic equation

$$(\mathcal{A} - b(x))\widehat{u}(x;s) + Q(x;s)\widehat{u}(x;s) = s^{-1}Q(x;s)a(x), \quad \Omega \times \{\operatorname{Re} s > M\},$$

where we set $Q(x;s) := \sum_{j=1}^{\ell} q_j(x)s^{\alpha_j}$. We check at once that $\widehat{u} : \{\operatorname{Re} s > M\} \to H^2(\Omega)$ is analytic, which is clear from the property of Laplace transform. Moreover, we claim that $\widehat{u}(x,s)$ ($\operatorname{Re} s > M$) can be analytically extended to the sector S_θ. Namely, the following lemma holds.

Lemma 2.2 *Under the assumptions in Theorem 2.1, the Laplace transform $\widehat{u}$ of the unique mild solution u to the initial-boundary value problem* (2.1) *can be analytically extended to the sector S_θ. Moreover, there exists a constant C only depending on d, Ω, θ, b, $\{\alpha_j\}$, $\{q_j\}$, $\{a_{ij}\}$ such that*

$$\|\mathcal{L}u(\cdot;s)\|_{H^1(\Omega)} \le C\sum_{j=1}^{\ell} r^{\alpha_j-1}\|a\|_{L^2(\Omega)}, \quad \forall s = re^{i\rho} \in S_\theta. \tag{2.20}$$

Proof. Firstly from Theorem 2.1, we see that the solution u to the initial-boundary value problem 2.1 can be analytically extended to the sector $\{s \in \mathbb{C}; |\arg s| < \frac{\pi}{2}\}$. Therefore by an argument similar to the proof of Theorem 0.1 in [51], we can prove that the Laplace transform $\widehat{u} : \{\operatorname{Re} s > M\} \to H^2(\Omega)$ can be analytically extended to the sector S_θ.

Now let us turn to give an estimate for the Laplace transform $\widehat{u} : S_\theta \to H^2(\Omega)$. For this, we define a bilinear operator $B[\Phi, \Psi; s] : H_0^1(\Omega) \times H_0^1(\Omega) \to \mathbb{C}$ by

$$B[\Phi, \Psi; s] := \int_\Omega \big((\mathcal{A} - b)\Phi(x)\big)\overline{\Psi(x)} + Q(x; s)\Phi(x)\overline{\Psi(x)}\mathrm{d}x, \quad s \in S_\theta,$$

where $\overline{\Psi}$ denotes conjugate of Ψ. Integration by parts yields

$$B[\Phi, \Psi; s] = \int_\Omega \sum_{i,j=1}^d a_{ij}\partial_i\Phi(x)\overline{\partial_j\Psi(x)} + (Q(x; s) - b(x))\Phi(x)\overline{\Psi(x)}\mathrm{d}x.$$

Taking $\Phi = \Psi$ implies

$$B[\Phi, \Phi; s] = \int_\Omega \sum_{i,j=1}^d a_{ij}\partial_i\Phi(x)\overline{\partial_j\Phi(x)} + (Q(x; s) - b(x))|\Phi(x)|^2\mathrm{d}x,$$

hence that

$$\operatorname{Re}(B[\Phi, \Phi; s]) = \int_\Omega \sum_{i,j=1}^d a_{ij}(\operatorname{Re}\partial_i\Phi \operatorname{Re}\partial_j\Phi + \operatorname{Im}\partial_i\Phi \operatorname{Im}\partial_j\Phi) + (\operatorname{Re} Q - b)|\Phi|^2\mathrm{d}x.$$

From $\operatorname{Re} Q(x; s) = \sum_{j=1}^\ell q_j(x)r^{\alpha_j}\cos\alpha_j\rho > 0$ in view of $s = r\mathrm{e}^{\mathrm{i}\rho} \in S_\theta$, and $b \le 0$, it follows that

$$\operatorname{Re}(B[\Phi, \Phi; s]) \ge \int_\Omega \sum_{i,j=1}^d a_{ij}\big(\operatorname{Re}\partial_i\Phi(x)\operatorname{Re}\partial_j\Phi(x) + \operatorname{Im}\partial_i\Phi(x)\operatorname{Im}\partial_j\Phi(x)\big)\mathrm{d}x.$$

The ellipticity of $\{a_{ij}\}$ and the use of Poincaré's inequality imply

$$\operatorname{Re}(B[\Phi, \Phi; s]) \ge C\|\operatorname{Re}\nabla\Phi\|^2_{L^2(\Omega)} + C\|\operatorname{Im}\nabla\Phi\|^2_{L^2(\Omega)} \ge C\|\Phi\|^2_{H^1(\Omega)}.$$

Consequently

$$\begin{aligned} C\|\widehat{u}(\cdot\,; s)\|^2_{H^1(\Omega)} \le& |B[\widehat{u}(\cdot\,; s), \widehat{u}(\cdot\,; s)\,; s] = \big|(s^{-1}Q(\cdot; s))a, \widehat{u}(\cdot\,; s))\big| \\ \le& C\sum_{j=1}^\ell r^{\alpha_j-1}\|a\|_{L^2(\Omega)}\|\widehat{u}(\cdot\,; s)\|_{H^1(\Omega)} \end{aligned}$$

in view of the Hölder inequality, finally that

$$\|\widehat{u}(\cdot\,; s)\|_{H^1(\Omega)} \le C\sum_{j=1}^\ell r^{\alpha_j-1}\|a\|_{L^2(\Omega)}, \text{ for } s \in S_\theta.$$

The proof of Lemma 2.2 is completed. □

Proof of Theorem 2.4. By Fourier-Mellin formula (e.g., [55]), we have

$$u(x, t) = \frac{1}{2\pi\mathrm{i}}\int_{M-\mathrm{i}\infty}^{M+\mathrm{i}\infty} \widehat{u}(x; s)\mathrm{e}^{st}\mathrm{d}s.$$

From Lemma 2.2, we see that the Laplace transform $\widehat{u}(x; s)$ of the solution to the initial-boundary value problem (2.1) is analytic in the sector S_θ. Therefore by Residue Theorem (e.g.,

[53]), for $t > 0$ we see that the inverse Laplace transform of $\widehat{u}$ can be represented by an integral on the contour $\gamma(\epsilon,\theta)$ defined as $\{s \in \mathbb{C}; \arg s = \theta,\ |s| \ge \epsilon\} \cup \{s \in \mathbb{C}; |\arg s| \le \theta,\ |s| = \epsilon\}$, that is

$$u(x,t) = \frac{1}{2\pi \mathrm{i}} \int_{\gamma(\epsilon,\theta)} \widehat{u}(x;s) \mathrm{e}^{st} \mathrm{d}s,$$

where in fact the shift in the line of integration is justified by the estimate (2.20). Moreover, again from the estimate (2.20), we can let ϵ tend to 0, then we have

$$u(x,t) = \frac{1}{2\pi \mathrm{i}} \int_{\gamma(0,\theta)} \widehat{u}(x;s) \mathrm{e}^{st} \mathrm{d}s.$$

We repeat the above argument to derive that $\widehat{v}(\cdot\,;s) \in H_0^1(\Omega)$, where v solves the problem (2.19) and

$$\|\widehat{v}(\cdot\,;s)\|_{H^1(\Omega)} \le C|s|^{\alpha_\ell - 1} \|a\|_{L^2(\Omega)}, \text{ for } s \in \gamma(0,\theta), \tag{2.21}$$

hence $v(t) = \frac{1}{2\pi \mathrm{i}} \int_{\gamma(0,\theta)} \widehat{v}(x;s) \mathrm{e}^{st} \mathrm{d}s$. Thus

$$\|u(\cdot\,,t) - v(\cdot\,,t)\|_{H^2(\Omega)} \le C \int_{\gamma(0,\theta)} \|\widehat{u}(\cdot\,;s) - \widehat{v}(\cdot\,;s)\|_{H^2(\Omega)} |\mathrm{e}^{st} \mathrm{d}s| \tag{2.22}$$

Noting that $\widehat{u} - \widehat{v}$ satisfies the following problem

$$\begin{cases} (\mathcal{A} - b)(\widehat{u} - \widehat{v}) + q_\ell(x) s^{\alpha_\ell} (\widehat{u} - \widehat{v}) + \sum_{j=1}^{\ell-1} q_j(x) s^{\alpha_j} \widehat{u} = \sum_{j=1}^{\ell-1} q_j(x) s^{\alpha_j - 1} a(x), & x \in \Omega,\ s \in S_\theta \\ \widehat{u}(x;s) - \widehat{v}(x;s) = 0, & x \in \partial\Omega,\ s \in S_\theta. \end{cases}$$

Then again using the boundary regularity estimates in elliptic equation combining (2.20) we deduce that

$$\begin{aligned} &\|\widehat{u}(\cdot\,;s) - \widehat{v}(\cdot\,;s)\|_{H^2(\Omega)} \\ \le& C r^{\alpha_\ell} \|\widehat{u}(\cdot\,;s) - \widehat{v}(\cdot\,;s)\|_{L^2(\Omega)} + C \left(\sum_{i=1}^{\ell} \sum_{j=1}^{\ell-1} r^{\alpha_i + \alpha_j - 1} + \sum_{j=1}^{\ell-1} r^{\alpha_j - 1} \right) \|a\|_{L^2(\Omega)}. \end{aligned} \tag{2.23}$$

Now for $0 < \delta_0 < 1$ small enough such that $C\delta_0^{\alpha_\ell} \le \frac{1}{2}$, we break up the integral in (2.22) into two parts

$$\begin{aligned} \|u(\cdot\,,t) - v(\cdot\,,t)\|_{H^2(\Omega)} \le& C \left(\int_0^{\delta_0} + \int_{\delta_0}^{\infty} \right) \|\widehat{u}(\cdot\,; r\mathrm{e}^{\mathrm{i}\theta}) - \widehat{v}(\cdot\,; r\mathrm{e}^{\mathrm{i}\theta})\|_{H^2(\Omega)} \mathrm{e}^{rt\cos\theta} \mathrm{d}r \\ =&: I_1(t;\delta_0) + I_2(t;\delta_0). \end{aligned}$$

For $I_1(t;\delta_0)$ $(t > 0)$, from Poincaré's inequality, we conclude from (2.23) that

$$\|\widehat{u}(\cdot\,;s) - \widehat{v}(\cdot\,;s)\|_{H^2(\Omega)} \le 2C \Big(\sum_{i=1}^{\ell} \sum_{j=1}^{\ell-1} r^{\alpha_i + \alpha_j - 1} + \sum_{j=1}^{\ell-1} r^{\alpha_j - 1} \Big) \|a\|_{L^2(\Omega)}, \quad |s| \le \delta_0,$$

which implies

$$I_1(t;\delta_0) \le \int_0^{\delta_0} \|\widehat{u}(\cdot\,; r\mathrm{e}^{\mathrm{i}\theta}) - \widehat{v}(\cdot\,; r\mathrm{e}^{\mathrm{i}\theta})\|_{H^2(\Omega)} \mathrm{e}^{rt\cos\theta} \mathrm{d}r \le C \Big(\sum_{i=1}^{\ell} \sum_{j=1}^{\ell-1} t^{-\alpha_i - \alpha_j} + \sum_{j=1}^{\ell-1} t^{-\alpha_j} \Big) \|a\|_{L^2(\Omega)}.$$

For $I_2(t;\delta_0)$ $(t > 0)$, the use of (2.23) yields

$$\|\widehat{u}(\cdot\,;s) - \widehat{v}(\cdot\,;s)\|_{H^2(\Omega)}$$

$$\leq Cr^{\alpha_\ell}\left(\|\widehat{u}(\cdot\,;s)\|_{L^2(\Omega)}+\|\widehat{v}(\cdot\,;s)\|_{L^2(\Omega)}\right)+C\left(\sum_{i=1}^{\ell}\sum_{j=1}^{\ell-1}r^{\alpha_i+\alpha_j-1}+\sum_{j=1}^{\ell-1}r^{\alpha_j-1}\right)\|a\|_{L^2(\Omega)},$$

where $|s|\geq\delta_0$, hence combining (2.20) with (2.21) gives

$$I_2(t;\delta_0)\leq C\left(\sum_{j=1}^{\ell}t^{-\alpha_\ell-\alpha_j}+t^{-2\alpha_\ell}+\sum_{i=1}^{\ell}\sum_{j=1}^{\ell-1}t^{-\alpha_i-\alpha_j}+\sum_{j=1}^{\ell-1}t^{-\alpha_j}\right)\|a\|_{L^2(\Omega)}.$$

Substituting the estimates for $I_1(t;\delta_0)$ and $I_2(t;\delta_0)$ into (2.22), we can assert that

$$\|u(\cdot\,,t)-v(\cdot\,,t)\|_{H^2(\Omega)}\leq Ct^{-\alpha}\|a\|_{L^2(\Omega)},\ t>0 \text{ large enough.}$$

where $\alpha:=\min\{2\alpha_\ell,\alpha_{\ell-1}\}$. This completes the proof of Theorem 2.4. □

Proof of Corollary 2.1. In order to prove the asymptotic behavior of u, we denote $u_\ell=\frac{(\mathcal{A}-b)^{-1}(q_\ell a)t^{-\alpha_\ell}}{\Gamma(1-\alpha_\ell)}$ and notice that the Laplace transform $\widehat{u}_\ell$ of u_ℓ is $(\mathcal{A}-b)^{-1}(q_\ell a)s^{\alpha_\ell-1}$ and satisfies $\mathcal{A}\widehat{u}_\ell-b\widehat{u}_\ell=q_\ell s^{\alpha_\ell-1}a$ and $\widehat{u}_\ell(\cdot\,;s)\in H_0^1(\Omega)$ for $s\in S_\theta$. Thus $\widehat{v}-\widehat{u}_\ell$ satisfies

$$\begin{cases}(\mathcal{A}-b)(\widehat{v}(x;s)-\widehat{u}_\ell(x;s))=-q_\ell(x)s^{\alpha_\ell}\widehat{v}(x;s), & x\in\Omega,\ s\in S_\theta,\\ \widehat{v}(x;s)-\widehat{u}_\ell(x;s)=0, & x\in\partial\Omega,\ s\in S_\theta,\end{cases}$$

Therefore, the regularity estimate for elliptic equations and (2.21) combined yield

$$\|\widehat{v}(\cdot\,;s)-\widehat{u}_\ell(\cdot\,;s)\|_{H^2(\Omega)}\leq Cr^{2\alpha_\ell-1}\|a\|_{L^2(\Omega)}$$

An argument similar to the proof in Theorem 2.4 implies

$$\|v(\cdot\,,t)-u_\ell(\cdot\,,t)\|_{H^2(\Omega)}\leq Ct^{-2\alpha_\ell}\|a\|_{L^2(\Omega)},$$

hence

$$\|u(\cdot\,,t)-u_\ell(\cdot\,,t)\|_{H^2(\Omega)}\leq Ct^{-\min\{2\alpha_\ell,\alpha_{\ell-1}\}}\|a\|_{L^2(\Omega)},\ \text{for } t>0 \text{ large enough,}$$

which completes the proof of Corollary 2.1. □

2.4 Inverse problems of determination of fractional orders

Nowadays pollution of the environment has become a global problem. For the accurate prediction of the diffusion of the pollution, the investigation of the behavior of the solution to the initial-boundary value problem becomes critical. In view of results in Section 2.3, it turns out that the fractional orders are very important for the prediction of the asymptotic behavior of the solution to the initial-boundary value problem (2.17) is only dominated by the lowest fractional order as $t\to\infty$. As $t\to0$, the decay rate of the solutions is dominated by the highest order $t^{-\alpha_1}$. In this section, we focus on inverse problems of determination of fractional orders in the diffusion model, which are important for experimentally evaluating the anomaly of the diffusion in heterogeneous media.

When we consider (2.1) as model equation for describing e.g., anomalous diffusion in inhomogeneous media, the orders α_j of fractional derivatives should be determined by the inhomogeneity of the media, but it is not clear which physical law can correspond the inhomogeneity to the orders α_j. Thus one reasonable way for estimating α_j is an inverse problem of determining $\alpha_1,...,\alpha_n$ in order to match available data such as $u(x_0,t)$, $0<t<T$ at a monitoring point $x_0\in\Omega$.

In case of all the coefficients of the fractional derivatives are positive constants, by means of eigenfunction expansion, the unique determination of fractional orders is proved by using one-point interior measurement. In its space-dependent counterpart, the analyticity of the solution and Laplace transform are applied to show the Dirichlet-to-Neumann map can simultaneously identify the number and orders of derivatives and coefficients. For the uniqueness for the above two inverse problems, see, e.g., [44] and [45] in the list of major publications.

In this section, we investigate two kinds of inverse problems of identifying fractional orders and other quantities in diffusion equations with multiple time-fractional derivatives, and establish the uniqueness of our inverse problems.

2.4.1 Inverse problem with $L^2(\Omega)$ initial value

In this section, we consider a bounded domain $\Omega \subset \mathbb{R}^d$ with smooth boundary $\partial\Omega$. Let $T > 0$ be fixed arbitrarily. Consider the following initial value - boundary value problem

$$\begin{cases} \sum_{j=1}^{n} q_j \partial_t^{\alpha_j} u + \mathcal{A}u = 0 & \text{in } Q, \\ u(\cdot, 0) = a & \text{in } \Omega, \\ u(x,t) = 0 & \text{on } \Sigma, \end{cases} \tag{2.24}$$

where α_j and $q_j > 0$, $j = 1, \cdots, n$, are constants such that

$$0 < \alpha_1 < \cdots < \alpha_n < 1, \tag{2.25}$$

$\mathcal{A}$ is the elliptic operator which is defined by (1.3). We define the operator A in $D(A) = H^2(\Omega) \cap H_0^1(\Omega)$ as follows:

$$A\psi = \mathcal{A}\psi, \quad \psi \in D(A).$$

We recall the eigensystem $\{\lambda_k, \varphi_k\}_{k=1}^{\infty}$ of the elliptic operator A: $0 < \lambda_1 < \lambda_2 < \cdots$, $\lim_{k\to\infty} \lambda_k = \infty$, and $A\varphi_k = \lambda_k \phi_k$, $\{\varphi_k\}_{k=1}^{\infty} \subset D(A)$ forms an orthogonal basis of $L^2(\Omega)$.

Henceforth $(\cdot, \cdot)$ denotes the scalar product in $L^2(\Omega)$. Moreover we can define a fractional power A^γ of A with $\gamma > 0$ (e.g., Tanabe [62]).

We discuss

Problem 2.1 *Let $x_0 \in \Omega$ be fixed and let $I \subset (0, T)$ be a non-empty open interval. Determine the number n of fractional orders α_j, fractional orders $\{\alpha_j\}_{j=1}^n$ of the time derivatives, and constant coefficients $\{q_j\}_{j=1}^n$ of the fractional derivatives from interior measurement $u(x_0, t)$, $t \in I$.*

Theorem 2.5 (Uniqueness) *Assuming that $a \geq 0$ in Ω, $a \not\equiv 0$ and $a \in D(A^\gamma)$ with $\gamma > \max\{\frac{d}{2} + \delta - 1, 0\}$, $\delta > 0$ can be sufficiently small. Let u be the weak solution to (2.24), and let v be the weak solution to (2.26) with the same initial and boundary conditions as (2.24):*

$$\begin{cases} \sum_{j=1}^{m} r_j \partial_t^{\beta_j} v + \mathcal{A}v = 0 & \text{in } Q, \\ v(\cdot, 0) = a & \text{in } \Omega, \\ v = 0 & \text{on } \Sigma, \end{cases} \tag{2.26}$$

where $r_i > 0$, $i = 1, \cdots, m$ are constants, and

$$0 < \beta_1 < \cdots < \beta_\ell < 1. \tag{2.27}$$

Then for any fixed $x_0 \in \Omega$, $u(x_0, t) = v(x_0, t)$, $t \in I$, implies $m = n$, $\alpha_i = \beta_i$, $q_i = r_i$, $i = 1, \cdots, n$.

Proof. We know that

$$u(\cdot, t) = \sum_{j=1}^{\infty} (1 - \lambda_j t^{\alpha_n} E^{(j)}_{\boldsymbol{q}, \boldsymbol{\alpha}', 1+\alpha_n}(t))(a, \phi_j)\phi_j, \tag{2.28}$$

$$v(\cdot,t)=\sum_{j=1}^{\infty}(1-\lambda_j t^{\beta_\ell}E^{(j)}_{\boldsymbol{r},\boldsymbol{\beta}',1+\beta_\ell}(t))(a,\phi_j)\phi_j \quad \text{in } L^2(\Omega) \tag{2.29}$$

for each $t\in[0,T]$ (e.g., Theorem 2.4 in [37]). The Sobolev embedding inequality yields that $\|\phi_j\|_{C(\overline{\Omega})}\le C\|A^{\frac{d}{4}+\varepsilon}\phi_j\|_{L^2(\Omega)}$ with sufficiently small $\varepsilon>0$, and we have $C_0 j^{\frac{2}{d}}\le\lambda_j\le C_1 j^{\frac{2}{d}}$ (see, e.g., [10]). Therefore, fixing $t_0>0$ arbitrarily, similarly to the proof of Theorem 2.5, for $t\in[t_0,T]$, we obtain

$$\sum_{j=1}^{\infty}|(1-\lambda_j t^{\alpha_n}E^{(j)}_{\boldsymbol{q},\boldsymbol{\alpha}',1+\alpha_n}(t))|\|(a,\phi_j)\phi_j\|_{C(\overline{\Omega})}\le C\sum_{j=1}^{\infty}\sum_{i=1}^{n-1}\frac{t^{\alpha_n-\alpha_i}}{1+\lambda_j t^{\alpha_n}}\|(A^\gamma a,\phi_j)A^{-\gamma}\phi_j\|_{C(\overline{\Omega})}$$

$$\le C\sum_{j=1}^{\infty}|(A^\gamma a,\phi_j)|\sum_{i=1}^{n-1}\frac{t^{\alpha_n-\alpha_i}}{1+\lambda_j t^{\alpha_n}}\lambda_j^{\frac{d}{4}+\varepsilon-\gamma}\le C\sum_{j=1}^{\infty}|(A^\gamma a,\phi_j)|\lambda_j^{\frac{d}{4}+\varepsilon-\gamma-1}$$

$$\le C\left(\sum_{j=1}^{\infty}|(A^\gamma a,\phi_j)|^2\right)^{\frac{1}{2}}\left(\sum_{j=1}^{\infty}\lambda_j^{\frac{d}{2}+2\varepsilon-2\gamma-2}\right)^{\frac{1}{2}}.$$

By $\lambda_j\sim j^{\frac{2}{d}}$ as $j\to\infty$ (e.g., [10]) and $\gamma>\frac{d}{2}-1$, we see that $\sum_{j=1}^{\infty}\lambda_j^{\frac{d}{2}+2\varepsilon-2\gamma-2}<\infty$. Hence

$$\sum_{j=1}^{\infty}|(1-\lambda_j t^{\alpha_n}E^{(j)}_{\boldsymbol{q},\boldsymbol{\alpha}',1+\alpha_n}(t))|\|(a,\phi_j)\phi_j\|_{C(\overline{\Omega})}<\infty,\quad t_0\le t\le T. \tag{2.30}$$

Therefore, we see that the series on the right-hand side of (2.28) and (2.29) are convergent uniformly in $x\in\overline{\Omega}$ and $t\in[t_0,T]$. Moreover, since the solutions u and v can be analytically extended to $t>0$ in view of the analyticity of the multinomial Mittag-Leffler function (e.g., [43]), we have $u(x_0,t)=v(x_0,t)$ for $t>0$. Consequently by the Laplace transform, we obtain

$$\sum_{j=1}^{\infty}\rho_j\frac{\sum_{i=1}^{n}q_i\eta^{\alpha_i-1}}{\sum_{i=1}^{n}q_i\eta^{\alpha_i}+\lambda_j}=\sum_{j=1}^{\infty}\rho_j\frac{\sum_{i=1}^{m}r_i\eta^{\beta_i-1}}{\sum_{i=1}^{m}r_i\eta^{\beta_i}+\lambda_j},\quad \eta>0,$$

where $\rho_j=(a,\phi_j)\phi_j(x_0)$. Moreover, noting $\gamma>\frac{d}{2}-1$, similarly to (2.30), we have $\sum_{j=1}^{\infty}|\rho_j|<\infty$. Therefore

$$\sum_{j=1}^{\infty}\frac{\lambda_j\rho_j}{\sum_{i=1}^{n}q_i\eta^{\alpha_i}+\lambda_j}=\sum_{j=1}^{\infty}\frac{\lambda_j\rho_j}{\sum_{i=1}^{m}r_i\eta^{\beta_i}+\lambda_j},\quad \eta\in\mathbb{R}\text{ with }|\eta|\text{ small enough}, \tag{2.31}$$

where the series on both sides are uniformly convergent for $|\eta|$ small enough. On the other hand, we set

$$p_k=(-1)^k\sum_{j=1}^{\infty}\frac{\rho_j}{\lambda_j^k}.$$

Then

$$0<(-1)^k p_k<\infty,\quad k\in\mathbb{N}.$$

In fact, since $\sum_{j=1}^{\infty}|\rho_j|<\infty$, and $\lambda_j>0$, $\lim\lambda_j=\infty$, we see that $p_k<\infty$. By the assumption of a, we have $p_1=-\sum_{j=1}^{\infty}\lambda_j^{-1}(a,\phi_j)\phi_j(x_0)=-(A^{-1}a)(x_0)$. Setting $b=-A^{-1}a$, we have $Ab=-a$ and $b|_{\partial\Omega}=0$. By the strong maximum principle for $Au=-\sum_{i,j=1}^{d}\partial_j(a_{ij}\partial_i u)-cu$ with $c\le 0$ and $a\ge 0$, we have $b<0$ in Ω. Hence $p_1<0$. Similarly, we can prove $(-1)^k p_k>0$ for $k=2,3,\cdots$.

We consider the asymptotic expansion of (2.31) near $\eta=0$. Since $\lambda_j>0$ for $j\in\mathbb{N}$, we have $\left|\frac{\sum_{i=1}^{n}q_i\eta^{\alpha_i}}{\lambda_j}\right|<1$, $\left|\frac{\sum_{i=1}^{m}r_i\eta^{\beta_i}}{\lambda_j}\right|<1$ for small η and all $j\in\mathbb{N}$. Consequently

$$\sum_{k=1}^{\infty}p_k\left(\sum_{i=1}^{n}q_i\eta^{\alpha_i}\right)^k=\sum_{k=1}^{\infty}p_k\left(\sum_{i=1}^{m}r_i\eta^{\beta_i}\right)^k,\text{uniformly converges for small }|\eta|. \tag{2.32}$$

Firstly, we prove $m = n$. Otherwise, we can assume $m > n$. Now we proceed by induction to prove that $\alpha_i = \beta_i$, $q_i = r_i$, $i = 1, \cdots, n$. First we prove $\alpha_1 = \beta_1$, $q_1 = r_1$. From (2.32), we see that

$$p_1 q_1 \eta^{\alpha_1} + p_1 \sum_{i=2}^{n} q_i \eta^{\alpha_i} + \sum_{k=2}^{\infty} p_k \left(\sum_{i=1}^{n} q_i \eta^{\alpha_i} \right)^k = p_1 r_1 \eta^{\beta_1} + p_1 \sum_{i=2}^{m} r_i \eta^{\beta_i} + \sum_{k=2}^{\infty} p_k \left(\sum_{i=1}^{m} r_i \eta^{\beta_i} \right)^k.$$

We see that $\alpha_1 = \beta_1$ from $p_1 < 0$, $q_1 > 0$ and $r_1 > 0$. If not, we can assume that $\alpha_1 > \beta_1$. Dividing both sides of the above equality by η^{β_1}, we obtain

$$\begin{aligned} & p_1 q_1 \eta^{\alpha_1 - \beta_1} + p_1 \sum_{i=2}^{n} q_i \eta^{\alpha_i - \beta_1} + \sum_{k=2}^{\infty} p_k \left(\sum_{i=1}^{n} q_i \eta^{\alpha_i} \right)^k \eta^{-\beta_1} \\ = & p_1 r_1 + p_1 \sum_{i=2}^{m} r_i \eta^{\beta_i - \beta_1} + \sum_{k=2}^{\infty} p_k \left(\sum_{i=1}^{m} r_i \eta^{\beta_i} \right)^k \eta^{-\beta_1}. \end{aligned} \tag{2.33}$$

Now letting $\eta \to 0$, from $\alpha_1 > \beta_1$, (2.25) and (2.27), we derive that the left-hand side of (2.33) tends to 0, but the right-hand side tends to $p_1 r_1 \neq 0$, which is a contradiction. Hence $\alpha_1 \leq \beta_1$. By a similar argument, we have $\alpha_1 \geq \beta_1$. Therefore $\alpha_1 = \beta_1$ and $q_1 = r_1$.

Suppose for $j \in \mathbb{N}$, $1 \leq j \leq n-1$ that $\alpha_i = \beta_i$, $q_i = r_i$, for $i = 1, \cdots, j$, that is

$$\sum_{k=1}^{\infty} p_k \left(\sum_{i=1}^{j} q_i \eta^{\alpha_i} + \sum_{i=j+1}^{n} q_i \eta^{\alpha_i} \right)^k = \sum_{j=1}^{\infty} p_k \left(\sum_{i=1}^{j} q_i \eta^{\alpha_i} + \sum_{i=j+1}^{m} r_i \eta^{\beta_i} \right)^k, \tag{2.34}$$

uniformly converges for small $|\eta|$. We show that (2.34) holds also for $j + 1$.

By S_1 and S_2 we denote the sets of the orders ℓ of the terms of η^ℓ of each side of (2.34) respectively. For the case

$$\alpha_{j+1} > \beta_{j+1} \quad \text{and} \quad \beta_{j+1} \notin \left\{ \sum_{i=1}^{j} k_i \alpha_i; \quad k_i \in \mathbb{N} \right\}, \tag{2.35}$$

from (2.25) and (2.27), it follows that

$$\beta_{j+1} \notin \left\{ \sum_{i=1}^{j} k_i \alpha_i + \sum_{i=j+1}^{n} k_i \alpha_i; \quad k_i \in \mathbb{N} \right\}.$$

In fact, if not, then there exist $k_i^0 \in \mathbb{N}$ for $i = 1, \cdots, n$ such that

$$\beta_{j+1} = \sum_{i=1}^{j} k_i^0 \alpha_i + \sum_{i=j+1}^{n} k_i^0 \alpha_i.$$

Then (2.25), (2.27) and (2.35) show that $\beta_{j+1} < \alpha_{j+1} < \cdots < \alpha_n$. Hence $k_i^0 = 0$ for $i = j+1, \cdots, n$. This means $\beta_{j+1} \in \left\{ \sum_{i=1}^{j} k_i \alpha_i; \quad k_i \in \mathbb{N} \right\}$, which is a contradiction. Moreover we can find $\beta_{j+1} \notin S_1$ in view of

$$S_1 \subset \left\{ \sum_{i=1}^{j} k_i \alpha_i + \sum_{i=j+1}^{n} k_i \alpha_i; \quad k_i \in \mathbb{N} \right\},$$

which is a contraction since (2.35) yields that $\beta_{j+1} \in S_2$. Indeed the coefficient of $\eta^{\beta_{j+1}}$ on the right-hand side of (2.34) is $p_1 r_{j+1} \neq 0$.

For the case

$$\alpha_{j+1} > \beta_{j+1} \quad \text{and} \quad \beta_{j+1} \in \left\{ \sum_{i=1}^{j} k_i \alpha_i; \quad k_i \in \mathbb{N} \right\}, \tag{2.36}$$

we now proceed to show that the coefficients of $\eta^{\beta_{j+1}}$ on both sides of (2.34) are different. Indeed, again using assumptions (2.25) and (2.27), we find that the coefficient of $\eta^{\beta_{j+1}}$ on the left-hand side of (2.34) is composed only of the coefficients of η^{α_i}, $i = 1, \cdots, j$, that is

$$\sum_{k_1\alpha_1 \cdots + k_j\alpha_j = \beta_{j+1}} p_{k_1+\cdots+k_j} q_1^{k_1} \cdots q_j^{k_j}.$$

Similarly, we see that the coefficient of $\eta^{\beta_{j+1}}$ on the right-hand side of (2.34) is

$$p_1 r_{j+1} + \sum_{k_1\alpha_1 \cdots + k_j\alpha_j = \beta_{j+1}} p_{k_1+\cdots+k_j} q_1^{k_1} \cdots q_j^{k_j}.$$

This is a contradiction since $p_1 < 0$ and $r_{j+1} > 0$. Consequently, $\alpha_{j+1} \le \beta_{j+1}$. In the same manner, we can see $\alpha_{j+1} \ge \beta_{j+1}$. Therefore

$$\sum_{k=1}^{\infty} p_k \left(\sum_{i=1}^{j+1} q_i \eta^{\alpha_i} + \sum_{i=j+2}^{n} q_i \eta^{\alpha_i} \right)^k = \sum_{k=1}^{\infty} p_k \left(\sum_{i=1}^{j+1} q_i \eta^{\alpha_i} + \sum_{i=j+2}^{m} r_i \eta^{\beta_i} \right)^k.$$

By induction, we can derive $\alpha_i = \beta_i$ and $q_i = r_i$ for $i = 1, \cdots, n$, that is

$$\sum_{k=1}^{\infty} p_k \left(\sum_{i=1}^{n} q_i \eta^{\alpha_i} \right)^k = \sum_{k=1}^{\infty} p_k \left(\sum_{i=1}^{n} q_i \eta^{\alpha_i} + \sum_{i=n+1}^{m} r_i \eta^{\beta_i} \right)^k. \tag{2.37}$$

Consequently

$$\beta_{n+1} \in \left\{ \sum_{i=1}^{n} k_i \alpha_i; \quad k_i \in \mathbb{N} \right\}.$$

This is impossible. In fact, we find that the coefficient of $\eta^{\beta_{n+1}}$ on the left-hand side of (2.37) is

$$\sum_{k_1\alpha_1 \cdots + k_n\alpha_n = \beta_{n+1}} p_{k_1+\cdots+k_n} q_1^{k_1} \cdots q_n^{k_n}.$$

and the coefficient of $\eta^{\beta_{n+1}}$ on the right-hand side of (2.37) is

$$p_1 r_{n+1} + \sum_{k_1\alpha_1 \cdots + k_n\alpha_n = \beta_{n+1}} p_{k_1+\cdots+k_n} q_1^{k_1} \cdots q_n^{k_n},$$

which is a contradiction in view of $r_{n+1} > 0$. Therefore, we see that $m > n$ is impossible. Hence $m \le n$. Similarly, we can prove that $m \ge n$. Finally we obtain $m = n$ and repeat the above argument to obtain $\alpha_i = \beta_i$, $q_i = r_i$, $i = 1, \cdots, n$. □

2.4.2 Inverse problem with many measurement

We consider

$$\begin{cases} \sum_{j=1}^{\ell} p_j(x) \partial_t^{\alpha_j} u = \Delta u + p(x) u, & (x,t) \in Q, \\ u(x,t) = 0, & x \in \Omega, \\ u(x,t) = \lambda(t) g(x), & (x,t) \in \Sigma. \end{cases} \tag{2.38}$$

Henceforth, Let ν be the outward unit normal vector to $\partial\Omega$. We denote $\frac{\partial u}{\partial \nu} = \nabla u \cdot \nu$. For $\ell \in \mathbb{N}$, we set $\boldsymbol{\alpha} = (\alpha_1, \cdots, \alpha_\ell) \in (0,1)^\ell$ where $\alpha_\ell < \alpha_{\ell-1} < \cdots < \alpha_1$. We note that also ℓ is unknown parameter in the inverse problem.

We state

Problem 2.2 *Let $\lambda \not\equiv 0$ be fixed. For $g \in H^{\frac{3}{2}}(\partial\Omega)$, we define the Dirichlet-to-Neumann map by*

$$\Lambda(\ell, \boldsymbol{\alpha}, p_j, p)g := \frac{\partial u}{\partial \nu}\Big|_{\partial\Sigma} \in L^2(0,T; H^{\frac{1}{2}}(\partial\Omega)).$$

Then we discuss whether $(\ell, \boldsymbol{\alpha}, p_j, p)$ is uniquely determined by the Dirichlet-to-Neumann map $\Lambda(\ell, \boldsymbol{\alpha}, p_j, p) : H^{\frac{3}{2}}(\partial\Omega) \longrightarrow L^2(0,T; H^{\frac{1}{2}}(\partial\Omega))$.

Our inverse problem is based on the Dirichlet-to-Neumann map, and for elliptic equations, there have been numerous important works. Here we do not intend any lists of references and we refer only to Imanuvilov and Yamamoto [25], Isakov [26], Sylvester and Uhlmann [59] and the references therein.

For the statement of our main results, we introduce some notations. As an admissible set of unknown fractional orders including numbers and coefficients, we set

$$\mathcal{U} = \{(\ell, \boldsymbol{\alpha}, p_j, p) \in \mathbb{N} \times (0,1)^\ell \times C^\infty(\overline{\Omega})^{\ell+1};\ p_1 > 0,\ p_j \gneqq 0,\ 2 \le j \le \ell,\ p \le 0 \text{ on } \overline{\Omega}\}.$$

where $\boldsymbol{\alpha} := (\alpha_1, \cdots, \alpha_\ell)$ such that $\alpha_\ell < \alpha_{\ell-1} < \cdots < \alpha_1$. For $\theta \in \left(0, \frac{\pi}{2}\right)$ and $T > 0$, we further set

$$\Omega_\theta := \{z \in \mathbb{C};\ z \ne 0,\ |\arg z| < \theta\}, \quad \Omega_{\theta,T} := \{z \in \Omega_\theta;\ |z| < T\}.$$

We are ready to state our main result.

Theorem 2.6 (Uniqueness) *Let $(\ell, \boldsymbol{\alpha}, p_j, p), (m, \boldsymbol{\beta}, q_j, q) \in \mathcal{U}$. Assume that for some $\theta \in (0, \frac{\pi}{2})$ the function $\lambda \not\equiv 0$ can be analytically extended to Ω_θ with $\lambda(0) = 0$ and $\lambda'(0) = 0$ and there exists a constant $C_0 > 0$ such that $|\lambda^{(k)}(t)| \le C_0 e^{C_0 t}$, $t > 0$, $0 \le k \le 2$. Then $\ell = m$, $\boldsymbol{\alpha} = \boldsymbol{\beta}$, $p_j = q_j$, $1 \le j \le \ell$ and $p = q$ provided*

$$\Lambda(\ell, \boldsymbol{\alpha}, p_j, p)g = \Lambda(m, \boldsymbol{\beta}, q_j, q)g, \quad \forall g \in H^{\frac{3}{2}}(\partial\Omega). \tag{2.39}$$

The assumption $p_j \ge 0$, $j = 2, \cdots, \ell$, and $p \le 0$ on $\overline{\Omega}$ is sufficient for proving that $|u(x,t)| = O(e^{C_1 t})$ as $t \to \infty$ with some constant $C_1 > 0$. Such an estimate is sufficient for taking the Laplace transforms of u, which is a key element of the proof of Theorem 2.6. In this section, we do not discuss the inverse problem without the condition $p_j \ge 0$ and $p \le 0$.

In $\mathcal{U}$, we can relax the regularity of $p, p_1, ..., p_\ell$ but we do not discuss here. Moreover, in the two dimensional case $d = 2$, thanks to Imanuvilov and Yamamoto [24], we can prove a sharp uniqueness result where Dirichlet inputs and Neumann outputs can be restricted on an arbitrary subboundary and the required regularity of unknown coefficients is relaxed.

Corollary 2.2 *Let $\Omega \subset \mathbb{R}^2$ be a bounded domain with smooth boundary $\partial\Omega$ and $\Gamma \subset \partial\Omega$ be an arbitrarily given subboundary and let $\gamma > 2$ be arbitrarily fixed. We assume the λ satisfies the same conditions as in Theorem 2.6. We set*

$$\widehat{\mathcal{U}} = \{(\ell, \boldsymbol{\alpha}, p_j, p) \in \mathbb{N} \times (0,1)^\ell \times (W^{2,\infty}(\Omega))^\ell \times W^{1,\gamma}(\Omega);\ p_1 > 0,\ p_j \gneqq 0,\ 2 \le j \le \ell,\ p \le 0 \text{ on } \overline{\Omega}\}.$$

If $(\ell, \boldsymbol{\alpha}, p_j, p), (m, \boldsymbol{\beta}, q_j, q) \in \widehat{\mathcal{U}}$ satisfy

$$\Lambda(\ell, \boldsymbol{\alpha}, p_j, p)g = \Lambda(m, \boldsymbol{\beta}, q_j, q)g \text{ on } \Gamma$$

for all $g \in H^{\frac{3}{2}}(\partial\Omega)$ with $\operatorname{supp} g \subset \Gamma$, then $\ell = m$, $\boldsymbol{\alpha} = \boldsymbol{\beta}$, $p_j = q_j$, $1 \le j \le \ell$ and $p = q$.

As for inverse problems for single-term time-fractional diffusion equations, recently researches are extended and we can refer e.g., to see e.g., Cheng et al [11], Hatano et al [21], Li et al [46]. The references are rapidly increasing and we do not intend to create a complete list. The Gel'fand-Levitan theory which plays an essential role in [11] and [46], but the asymptotic behavior of the solution used in [21] cannot be applied to show the uniqueness of for inverse problems for multi-term time-fractional diffusion equations.

As for inverse problems for multi-term time-fractional diffusion equations, to the best knowledge of the authors, there are no published results except for Li and Yamamoto [44].

Before we give a proof of Theorem 2.6, we first establish the analyticity of the solution u to the initial-boundary value problem (2.1), which are necessary for the proof of Theorem 2.6.

We start from the following observation. By the Sobolev extension theorem, the assumption $g \in H^{\frac{3}{2}}(\partial\Omega)$ allows us to choose $\widetilde{g} \in H^2(\Omega)$ such that $\widetilde{g}|_{\partial\Omega} = g$. Now introducing the new unknown function $\widetilde{u}(x,t) = u(x,t) - \lambda(t)\widetilde{g}(x)$, we can rewrite (2.1) as

$$\begin{cases} \partial_t^{\alpha_1}\widetilde{u} + \sum_{j=2}^{\ell} \widetilde{p}_j \partial_t^{\alpha_j}\widetilde{u} = \operatorname{div}\left(\frac{1}{p_1}\nabla\widetilde{u}\right) + B\cdot\nabla\widetilde{u} + b\widetilde{u} + F & \text{in } Q, \\ \widetilde{u} = 0 & \text{on } \Omega\times\{0\}, \\ \widetilde{u} = 0 & \text{on } \Sigma, \end{cases} \tag{2.40}$$

where $\widetilde{p}_j(x) := \frac{p_j(x)}{p_1(x)}$, $j = 2,\cdots,\ell$, $B(x) := -\nabla(\frac{1}{p_1(x)})$, $b(x) := \frac{p(x)}{p_1(x)}$ and

$$F(x,t) := \frac{1}{p_1(x)}\left(\lambda(t)\Delta\widetilde{g}(x) + \lambda(t)p(x)\widetilde{g}(x) - \sum_{j=1}^{\ell}(\partial_t^{\alpha_j}\lambda)(t)p_j(x)\widetilde{g}(x)\right). \tag{2.41}$$

By $p_j, p \in C^\infty(\overline{\Omega})$ for $j = 1, ..., \ell$, we see that $\widetilde{p}_j$, B, $b \in C^\infty(\overline{\Omega})$ for $j = 2, ..., \ell$.

Later in the proof of Theorem 2.7, we can see that $F \in W^{1,\infty}(0,T;L^2(\Omega))$ under the assumptions in Theorem 2.6. Thus here for convenience we only give the definition of the weak solution to (2.40) in the case of $F \in W^{1,\infty}(0,T;L^2(\Omega))$.

Definition 2.3 (Weak solution) *Let $F \in W^{1,\infty}(0,T;L^2(\Omega))$, we call $\widetilde{u}$ a weak solution to the initial-boundary value problem* (2.40) *if $\widetilde{u} \in C^1((0,T];L^2(\Omega)) \cap C([0,T];H^2(\Omega)\cap H_0^1(\Omega))$, and satisfies the differential equation in* (2.40) *and $\lim_{t\to 0+}\|\widetilde{u}(\cdot,t)\|_{L^2(\Omega)} = 0$.*

For the weak solution with the right-hand side in $L^2(Q)$, see Gorenflo et al [16].

Based on the above definition of a weak solution to the initial-boundary value problem (2.40) which is equivalent to our original problem (2.1), we see that $u(x,t) = \widetilde{u}(x,t) + \lambda(t)\widetilde{g}(x)$ is a weak solution to the initial-boundary value problem (2.1) under the assumptions of Theorem 2.6.

For easily proceeding the estimates for the solution defined in the above definition, similar to the argument used in Section 2.2.1, we show an equivalent integral equation of the solution, which is only composed of $\widetilde{u}, \nabla\widetilde{u}$ without the time derivative of the solution. To this end, we start from fixing some general settings and notations. When no ambiguity is possible, we use A to denote the following operator

$$(A\psi)(x) := -\operatorname{div}\left(\frac{1}{p_1(x)}\nabla\psi(x)\right), \quad x\in\Omega, \quad \psi \in H^2(\Omega)\cap H_0^1(\Omega),$$

and denote $\{\lambda_n, \varphi_n\}_{n=1}^\infty$ as the eigensystem of the elliptic operator A, that is, $0 < \lambda_1 < \lambda_2 \le \lambda_3 \cdots$, $A\varphi_n = \lambda_n\varphi_n$ and $\{\varphi_n\}_{n=1}^\infty \subset H^2(\Omega)\cap H_0^1(\Omega)$ forms an orthonormal basis of $L^2(\Omega)$. We note that $\lim_{n\to\infty}\lambda_n = \infty$. Then we can define the fractional power A^γ for $\gamma \in \mathbb{R}$ of the operator A (e.g., Tanabe [62]), and we see that

$$A_1^\gamma\psi = \sum_{n=1}^\infty \lambda_n^\gamma(\psi,\varphi_n)\varphi_n, \quad \psi \in D(A^\gamma) := \left\{\psi\in L^2(\Omega) : \sum_{n=1}^\infty \lambda_n^{2\gamma}|(\psi,\varphi_n)|^2 < \infty\right\}$$

and $D(A^\gamma)$ is a Hilbert space with the norm

$$\|\psi\|_{D(A^\gamma)} = \left(\sum_{n=1}^\infty \lambda_n^{2\gamma}|(\psi,\varphi_n)_{L^2(\Omega)}|^2\right)^{\frac{1}{2}}.$$

Recall the definition S in (2.4), here we use eigensystem of the operator A to define the operator $S(z): L^2(\Omega) \to L^2(\Omega)$ for $z \in \Omega_\theta$ in the same way, that is,

$$S(z)a := \sum_{n=1}^{\infty} (a, \varphi_n) E_{\alpha_1,1}(-\lambda_n z^{\alpha_1})\varphi_n, \quad 0 < \alpha_1 < 1$$

for $a \in L^2(\Omega)$, where $E_{\alpha,\beta}(z)$ is the Mittag-Leffler function defined in (2.2). It is not difficult to see that all the properties established in Section 2.2 hold true.

Henceforth we write $u(t) = u(\cdot, t)$, $F(t) = F(\cdot, t)$, etc., which mean functions in t with values in $L^2(\Omega)$.

Now by an argument similar to the calculation used in Section 2.2, by regarding the term $-\sum_{j=2}^{\ell} \widetilde{p}_j \partial_t^{\alpha_j} \widetilde{u} + B \cdot \nabla \widetilde{u} + b\widetilde{u} + F$ as new source term in (2.40), we can represent

$$\widetilde{u}(t) = -\int_0^t A^{-1}S'(t-s)(B \cdot \nabla \widetilde{u}(s) + b\widetilde{u}(s) + F(s))\mathrm{d}s + \sum_{j=2}^{\ell} \int_0^t A^{-1}S'(t-s)\widetilde{p}_j \partial_t^{\alpha_j} \widetilde{u}(s)\mathrm{d}s.$$

which is equivalent to (2.40). Based on the calculation used in the derivation of (2.7), we conclude that the solution $\widetilde{u}$ to (2.40) satisfies the following integral equation

$$\begin{aligned}\widetilde{u}(t) = &-\int_0^t A^{-1}S'(t-r)(B \cdot \nabla \widetilde{u}(r) + b\widetilde{u}(r) + F(r))\mathrm{d}r \\ &+ \sum_{j=2}^{\ell} \frac{1}{\Gamma(1-\alpha_j)} \int_0^t \left(\int_0^1 A^{-1}S''\big((1-s)(t-r)\big)(1-s)(t-r)^{1-\alpha_j} s^{-\alpha_j} \widetilde{p}_j \widetilde{u}(r)\mathrm{d}s \right) \mathrm{d}r \\ &+ \sum_{j=2}^{\ell} \frac{1-\alpha_j}{\Gamma(1-\alpha_j)} \int_0^t \left(\int_0^1 A^{-1}S'\big((1-s)(t-r)\big)(t-r)^{-\alpha_j} s^{-\alpha_j} \widetilde{p}_j \widetilde{u}(r)\mathrm{d}s \right) \mathrm{d}r. \end{aligned} \tag{2.42}$$

Based on the above assumptions and notations, we are ready to give the proof of the following theorem which is a key of the proof of Theorem 2.6.

Theorem 2.7 *Let $(\ell, \boldsymbol{\alpha}, p_j, p) \in \mathcal{U}$ and $T > 0$ be arbitrarily given. Assume that $g \in H^{\frac{3}{2}}(\partial\Omega) \cap C(\partial\Omega)$, $\lambda(0) = 0$, $\lambda'(0) = 0$, for $\theta \in (0, \frac{\pi}{2})$, the function $\lambda(t)$ can be analytically extended to Ω_θ and $\lambda \in W^{2,\infty}(\Omega_{\theta,T})$. Then there exists a unique weak solution u to the problem (2.1) such that $u \in C([0,T]; H^2(\Omega) \cap H_0^1(\Omega)) \cap C^1((0,T]; H^2(\Omega))$, and u: $(0,T] \to H^2(\Omega)$ can be analytically extended to Ω_θ.*

Moreover there exists a constant $C = C(g)$, depending also on T, Ω such that

$$\|u\|_{C(\overline{\Omega}\times[0,T])} \le C(g)\|\lambda\|_{C[0,T]}. \tag{2.43}$$

Proof. The uniqueness of u follows directly from (2.43). Thus it suffices to prove the existence of analytic u and (2.43).

First we point out that $F(x, \cdot)$ defined in (2.41) can be analytically extended to Ω_θ. In fact, it is sufficient to prove that $\partial_t^\alpha \lambda$ can be analytically extended to $\Omega_{\theta,T}$ with any $\alpha \in (0,1)$ and $T > 0$. Let $z \in \Omega_{\theta,T}$ be arbitrarily fixed. We set $\lambda'(s) := \frac{\mathrm{d}\lambda(s)}{\mathrm{d}s}$ and

$$\lambda_\alpha(z) := \frac{z^{1-\alpha}}{\Gamma(1-\alpha)} \int_0^1 (1-\eta)^{-\alpha}\lambda'(\eta z)\mathrm{d}\eta = \frac{1}{\Gamma(1-\alpha)} \int_0^z (z-s)^{-\alpha}\lambda'(s)\mathrm{d}s. \tag{2.44}$$

The last integral is considered on the segment from 0 to z in $\mathbb{C}$, and we see that $\lambda_\alpha(z)$ is well-defined for $z \in \Omega_{\theta,T}$. By the definition of $\partial_t^\alpha \lambda(t)$ for $t > 0$ we can verify that $\lambda_\alpha(t) = \partial_t^\alpha \lambda(t)$ for $t > 0$. For any small $\epsilon > 0$, we set

$$\lambda_\alpha^\epsilon(z) := \frac{z^{1-\alpha}}{\Gamma(1-\alpha)} \int_0^{1-\epsilon} (1-\eta)^{-\alpha}\lambda'(\eta z)\mathrm{d}\eta.$$

By the analyticity of λ in Ω_θ, we see that λ_α^ϵ is analytic in $\Omega_{\theta,T}$. For any $z \in \Omega_{\theta,T}$, we have

$$|\lambda_\alpha^\epsilon(z) - \lambda_\alpha(z)| \le \frac{T^{1-\alpha}}{\Gamma(1-\alpha)} \int_{1-\epsilon}^1 (1-\eta)^{-\alpha} |\lambda'(\eta z)| \,\mathrm{d}\eta \le C \sup_{s\in[0,z]} |\lambda'(s)| \int_{1-\epsilon}^1 (1-\eta)^{-\alpha} \mathrm{d}\eta.$$

Here $[0, z]$ denotes the closed segment in $\mathbb{C}$ from 0 to z. Therefore, for any fixed $T > 0$, we see that

$$\sup_{z\in\Omega_{\theta,T}} |\lambda_\alpha^\epsilon(z) - \lambda_\alpha(z)| \longrightarrow 0 \quad \text{as } \epsilon \to 0.$$

Since λ_α^ϵ is analytic in $\Omega_{\theta,T}$, we see that λ_α is analytic in $\Omega_{\theta,T}$, because λ_α is the uniform convergent limit of analytic functions λ_α^ϵ in $\Omega_{\theta,T}$. Thus we completed the proof that $F(\cdot, t)$ can be analytically extended to Ω_θ and $F(\cdot, z) : \Omega_\theta \longrightarrow D(A)$ is analytic in z, and by the same notation we denote the extension, that is

$$F(x, z) := \frac{1}{p_1(x)} \left(\lambda(z)\Delta\widetilde{g}(x) + \lambda(z)p(x)\widetilde{g}(x) - \sum_{j=1}^{\ell} \lambda_{\alpha_j}(z)\widetilde{p}_j(x)\widetilde{g}(x) \right).$$

Next we estimate $F(x, z)$ for $x \in \overline{\Omega}$ and $z \in \Omega_{\theta,T}$. Without loss of generality, we assume that $T \ge 1$. First we have

$$\|F\|_{L^\infty(\Omega_{\theta,T};L^2(\Omega))} \le C \left(\|\lambda\|_{L^\infty(\Omega_{\theta,T})} + \sum_{j=1}^{\ell} \|\lambda_{\alpha_j}\|_{L^\infty(\Omega_{\theta,T})} \right).$$

Here and henceforth $C > 0$ denotes a generic constant which is independent of T, $z \in \Omega_{\theta,T}$, but dependent on $d, \Omega, T, g, \theta, p, p_1, ..., p_\ell, \alpha_1, ..., \alpha_\ell$. By $T \ge 1$, direct calculations yield

$$|\lambda_{\alpha_j}(z)| = \frac{|z|^{1-\alpha_j}}{\Gamma(1-\alpha_j)} \left| \int_0^1 (1-\eta)^{-\alpha_j} \lambda'(\eta z) \mathrm{d}\eta \right| \le CT^{1-\alpha_j} \int_0^1 (1-\eta)^{-\alpha_j} \mathrm{d}\eta \|\lambda\|_{W^{1,\infty}(\Omega_{\theta,T})}, \tag{2.45}$$

and so

$$\|F\|_{L^\infty(\Omega_{\theta,T};L^2(\Omega))} \le CT\|\lambda\|_{W^{1,\infty}(\Omega_{\theta,T})}.$$

Moreover, by $0 < \alpha_j < 1$, $\lambda'(0) = 0$ and integration by parts yield

$$\begin{aligned}
\lambda_{\alpha_j}(z) &= \frac{z^{1-\alpha_j}}{\Gamma(1-\alpha_j)} \int_0^1 (1-\eta)^{-\alpha_j} \lambda'(\eta z) \mathrm{d}\eta \\
&= \frac{z^{1-\alpha_j}}{\Gamma(1-\alpha_j)} \left(\left[\lambda'(\eta z) \frac{(1-\eta)^{1-\alpha_j}}{1-\alpha_j} \right]_{\eta=1}^{\eta=0} + \int_0^1 \frac{(1-\eta)^{1-\alpha_j}}{1-\alpha_j} \lambda''(\eta z) z \mathrm{d}\eta \right) \\
&= \frac{z^{1-\alpha_j}}{\Gamma(1-\alpha_j)} \int_0^1 \frac{(1-\eta)^{1-\alpha_j}}{1-\alpha_j} \lambda''(\eta z) z \mathrm{d}\eta = \frac{1}{\Gamma(1-\alpha_j)} \int_0^z \frac{(z-s)^{1-\alpha_j}}{1-\alpha_j} \lambda''(s) \mathrm{d}s.
\end{aligned}$$

Therefore we can differentiate in z to have

$$\lambda'_{\alpha_j}(z) = \frac{1}{\Gamma(1-\alpha_j)} \int_0^z (z-s)^{-\alpha_j} \lambda''(s) \mathrm{d}s,$$

and again by change of the variables $s = \eta z$, similarly to (2.45), we obtain

$$\|\lambda'_{\alpha_j}\|_{L^\infty(\Omega_{\theta,T})} \le CT\|\lambda\|_{W^{2,\infty}(\Omega_{\theta,T})}.$$

Hence $\|\partial_z F\|_{L^\infty(\Omega_{\theta,T};L^2(\Omega))} \le CT\|\lambda\|_{W^{2,\infty}(\Omega_{\theta,T})}$. Consequently, noting that $\widetilde{g} \in H^2(\Omega)$, $p_1, \widetilde{p}_j \in C^\infty(\overline{\Omega})$ for $j = 2, ..., \ell$, we see

$$F : \Omega_{\theta,T} \longrightarrow L^2(\Omega) \text{ is analytic}, \quad \|F\|_{W^{1,\infty}(\Omega_{\theta,T};L^2(\Omega))} \le CT\|\lambda\|_{W^{2,\infty}(\Omega_{\theta,T})}. \tag{2.46}$$

Now let us turn to consider the uniqueness existence and the analyticity of the solution to the integral equation (2.42). For this, after the change $r \to \eta := 1 - \frac{r}{t}$ of variables in (2.42), by $r = (1-\eta)t$ we find

$$\begin{aligned}\widetilde{u}(t) = &- t\int_0^1 A^{-1}S'(rt)\Big(B\cdot\nabla\widetilde{u}\big((1-r)t\big) + b\widetilde{u}\big((1-r)t\big) + F\big((1-r)t\big)\Big)\mathrm{d}r\\ &+\sum_{j=2}^{\ell}\frac{t^{2-\alpha_j}}{\Gamma(1-\alpha_j)}\int_0^1\int_0^1 A^{-1}S''\big((1-s)rt\big)(1-s)r^{1-\alpha_j}s^{-\alpha_j}\widetilde{p}_j\widetilde{u}\big((1-r)t\big)\mathrm{d}s\mathrm{d}r\\ &+\sum_{j=2}^{\ell}\frac{(1-\alpha_j)t^{1-\alpha_j}}{\Gamma(1-\alpha_j)}\int_0^1\int_0^1 A^{-1}S'\big((1-s)rt\big)r^{-\alpha_j}s^{-\alpha_j}\widetilde{p}_j\widetilde{u}\big((1-r)t\big)\mathrm{d}s\mathrm{d}r. \end{aligned}\tag{2.47}$$

Moreover, we extend the variable t in (2.47) from $(0,T)$ to the sector $\Omega_{\theta,T}$, and setting $\widetilde{u}_0 = 0$, we inductively define $\widetilde{u}_{n+1}(z)$ $(n = 0,1,\cdots)$ as follows:

$$\begin{aligned}\widetilde{u}_{n+1}(z) = &- z\int_0^1 A^{-1}S'(rz)\Big(B\cdot\nabla\widetilde{u}_n\big((1-r)z\big) + b\widetilde{u}_n\big((1-r)z\big) + F\big((1-r)z\big)\Big)\mathrm{d}r\\ &+\sum_{j=2}^{\ell}\frac{z^{2-\alpha_j}}{\Gamma(1-\alpha_j)}\int_0^1\int_0^1 A^{-1}S''\big((1-s)rz\big)(1-s)r^{1-\alpha_j}s^{-\alpha_j}\widetilde{p}_j\widetilde{u}_n\big((1-r)z\big)\mathrm{d}s\mathrm{d}r\\ &+\sum_{j=2}^{\ell}\frac{(1-\alpha_j)z^{1-\alpha_j}}{\Gamma(1-\alpha_j)}\int_0^1\int_0^1 A^{-1}S'\big((1-s)rz\big)r^{-\alpha_j}s^{-\alpha_j}\widetilde{p}_j\widetilde{u}_n\big((1-r)z\big)\mathrm{d}s\mathrm{d}r. \end{aligned}\tag{2.48}$$

Now by induction we will prove that

$$\|\widetilde{u}_{n+1}(z) - \widetilde{u}_n(z)\|_{D(A)} \le \frac{C^n(|z|^{\alpha_0}T^{\beta_0}\Gamma(\alpha_0))^n}{\Gamma(n\alpha_0+1)}M_1, \quad n = 0,1,2,..., \ \forall z\in\Omega_{\theta,T}, \tag{2.49}$$

where $\alpha_0 = \min_{j=2,3,\cdots,\ell}\{\frac{\alpha_1}{2}, \alpha_1-\alpha_j\}$, $\beta_0 = \max_{j=2,3,\cdots,\ell}\{\frac{\alpha_1}{2},\alpha_1-\alpha_j\} - \alpha_0$ and M_1 is chosen later. First for $n=0$, integrating by parts and using (2.5), (2.46) and $\widetilde{u_0} = 0$, we see

$$\begin{aligned}&\|\widetilde{u}_1(z) - \widetilde{u}_0(z)\|_{D(A)} = \|A\widetilde{u}_1(z)\|_{L^2(\Omega)} = \left\|z\int_0^1 S'(rz)F((1-r)z)\mathrm{d}r\right\|_{L^2(\Omega)}\\ =&\left\|S(rz)F((1-r)z)\Big|_{r=0}^{r=1} - \int_0^1 S(rz)F'((1-r)z)(-z)\mathrm{d}r\right\|_{L^2(\Omega)}\\ \le&\|S(z)F(0) - F(z)\|_{L^2(\Omega)} + C\int_0^1\|F'((1-r)z)\|_{L^2(\Omega)}\mathrm{d}r\\ \le& CT\|\lambda\|_{W^{2,\infty}(\Omega_{\theta,T})} =: M_1.\end{aligned}$$

For the first equality, by the definition of the norm in $D(A)$, we note $\|v\|_{D(A)} = \|Av\|_{L^2(\Omega)}$.

Next we estimate $\|\widetilde{u}_{n+1}(z) - \widetilde{u}_n(z)\|_{D(A)}$. By (2.48) we have

$$\begin{aligned}A\widetilde{u}_{n+1}(z) - A\widetilde{u}_n(z) = &-z\int_0^1 S'(rz)\Big(B\cdot(\nabla\widetilde{u}_n - \nabla\widetilde{u}_{n-1})\big((1-r)z\big) + b(\widetilde{u}_n - \widetilde{u}_{n-1})\big((1-r)z\big)\Big)\mathrm{d}r\\ +&\sum_{j=2}^{\ell}\frac{z^{2-\alpha_j}}{\Gamma(1-\alpha_j)}\int_0^1\int_0^1 A^{-1}S''\big((1-s)rz\big)(1-s)r^{1-\alpha_j}s^{-\alpha_j}A(\widetilde{p}_j(\widetilde{u}_n - \widetilde{u}_{n-1})\big((1-r)z\big))\mathrm{d}s\mathrm{d}r\\ +&\sum_{j=2}^{\ell}\frac{(1-\alpha_j)z^{1-\alpha_j}}{\Gamma(1-\alpha_j)}\int_0^1\int_0^1 A^{-1}S'\big((1-s)rz\big)r^{-\alpha_j}s^{-\alpha_j}A(\widetilde{p}_j(\widetilde{u}_n - \widetilde{u}_{n-1})\big((1-r)z\big))\mathrm{d}s\mathrm{d}r.\end{aligned}$$

For any $n\in\mathbb{N}$, in view of (2.5), the assumption $B\in C^\infty(\overline{\Omega})$ and the inequality $\|B\cdot\nabla v\|_{D(A^{\frac{1}{2}})} \le C\|v\|_{D(A)}$ for $v\in D(A)$, we derive

$$\|A^{-1}S'(rz)B\cdot(\nabla\widetilde{u}_n - \nabla\widetilde{u}_{n-1})((1-r)z)\|_{D(A)}$$

$$
\begin{aligned}
&=\|A^{-\frac{1}{2}}S'(rz)A^{\frac{1}{2}}(B\cdot(\nabla\widetilde{u}_n-\nabla\widetilde{u}_{n-1})((1-r)z))\|_{L^2(\Omega)}\\
&\le C\|A^{-\frac{1}{2}}S'(rz)\|_{L^2(\Omega)\to L^2(\Omega)}\|B\cdot(\nabla\widetilde{u}_n-\nabla\widetilde{u}_{n-1})((1-r)z)\|_{H^1(\Omega)}\\
&\le C\|A^{-\frac{1}{2}}S'(rz)\|_{L^2(\Omega)\to L^2(\Omega)}\|(\widetilde{u}_n-\widetilde{u}_{n-1})((1-r)z)\|_{D(A)}\\
&\le C|z|^{\frac{\alpha_1}{2}-1}r^{\frac{\alpha_1}{2}-1}\|(\widetilde{u}_n-\widetilde{u}_{n-1})((1-r)z)\|_{D(A)}
\end{aligned}\tag{2.50}
$$

for $z\in\Omega_{\theta,T}$. Estimating other terms similarly, in view of (2.5), we obtain

$$
\begin{aligned}
&\|\widetilde{u}_{n+1}(z)-\widetilde{u}_n(z)\|_{D(A)}\\
\le& C|z|^{\frac{\alpha_1}{2}}\int_0^1 r^{\frac{\alpha_1}{2}-1}\|\widetilde{u}_n((1-r)z)-\widetilde{u}_{n-1}((1-r)z)\|_{D(A)}\mathrm{d}r\\
&+C\sum_{j=2}^{\ell}|z|^{\alpha_1-\alpha_j}\int_0^1(1-s)^{\alpha_1-1}s^{-\alpha_j}\mathrm{d}s\int_0^1 r^{\alpha_1-\alpha_j-1}\|\widetilde{u}_n((1-r)z)-\widetilde{u}_{n-1}((1-r)z)\|_{D(A)}\mathrm{d}r.
\end{aligned}
$$

Noting that $0<\alpha_\ell<\cdots<\alpha_1<1$ and (2.6), we have

$$
\begin{aligned}
&\|\widetilde{u}_{n+1}(z)-\widetilde{u}_n(z)\|_{D(A)}\\
\le& C\left(|z|^{\frac{\alpha_1}{2}}+\sum_{j=2}^{\ell}|z|^{\alpha_1-\alpha_j}\right)\int_0^1(r^{\frac{\alpha_1}{2}-1}+\sum_{j=2}^{\ell}r^{\alpha_1-\alpha_j-1})\|\widetilde{u}_n((1-r)z)-\widetilde{u}_{n-1}((1-r)z)\|_{D(A)}\mathrm{d}r\\
\le& C|z|^{\alpha_0}T^{\beta_0}\int_0^1 r^{\alpha_0-1}\|\widetilde{u}_n((1-r)z)-\widetilde{u}_{n-1}((1-r)z)\|_{D(A)}\mathrm{d}r,\quad z\in\Omega_{\theta,T}.
\end{aligned}
$$

By the assumption (2.49) of induction, we again use (2.6) to derive

$$
\begin{aligned}
&\|\widetilde{u}_{n+1}(z)-\widetilde{u}_n(z)\|_{D(A)}\le CM_1|z|^{\alpha_0}T^{\beta_0}\int_0^1 r^{\alpha_0-1}\frac{C^{n-1}(|z|^{\alpha_0}T^{\beta_0}(1-r)^{\alpha_0}\Gamma(\alpha_0))^{n-1}}{\Gamma((n-1)\alpha_0+1)}\mathrm{d}r\\
=&M_1\frac{(C|z|^{\alpha_0}T^{\beta_0}\Gamma(\alpha_0))^n}{\Gamma(n\alpha_0+1)}.
\end{aligned}\tag{2.51}
$$

Hence the proof of (2.49) is completed by induction. Therefore by (2.3), we have

$$
\|\widetilde{u}_n(z)\|_{D(A)}\le\sum_{k=1}^{n-1}\|\widetilde{u}_{k+1}(z)-\widetilde{u}_k(z)\|_{D(A)}\le\sum_{k=1}^{\infty}M_1\frac{(CT^{\alpha_0+\beta_0}\Gamma(\alpha_0))^n}{\Gamma(n\alpha_0+1)}.<\infty.\tag{2.52}
$$

Next by indutction, we will prove that $\widetilde{u}_n:\Omega_{\theta,T}\longrightarrow D(A)$ is analytic in z for $n=0,1,2,....$ By $\widetilde{u}_0\equiv 0$, it is obvious for $n=0$. We assume that $\widetilde{u}_n:\Omega_{\theta,T}\longrightarrow D(A)$ is analytic in z. We estimate the integrands in (2.48). Similarly to (2.50), we have

$$
\begin{aligned}
&\|A^{-1}S'(rz)(B\cdot\nabla\widetilde{u}_n((1-r)z)+b\widetilde{u}_n((1-r)z))\|_{D(A)}\\
\le& C|z|^{\frac{\alpha_1}{2}-1}r^{\frac{\alpha_1}{2}-1}\|\widetilde{u}_n((1-r)z)\|_{D(A)},
\end{aligned}
$$

$$
\begin{aligned}
&\|A^{-1}S''((1-s)rz)(1-s)r^{1-\alpha_j}s^{-\alpha_j}\widetilde{p}_j\widetilde{u}_n((1-r)z)\|_{D(A)}\\
\le& C((1-s)r|z|)^{\alpha_1-2}(1-s)r^{1-\alpha_j}s^{-\alpha_j}\|\widetilde{u}_n((1-r)z)\|_{D(A)}\\
\le& C|z|^{\alpha_1-2}(1-s)^{\alpha_1-1}r^{\alpha_1-\alpha_j-1}s^{-\alpha_j}\|\widetilde{u}_n((1-r)z)\|_{D(A)}
\end{aligned}
$$

and

$$
\begin{aligned}
&\|A^{-1}S'((1-s)rz)r^{-\alpha_j}s^{-\alpha_j}\widetilde{p}_j\widetilde{u}_n((1-r)z)\|_{D(A)}\\
\le& C((1-s)r|z|)^{\alpha_1-1}r^{-\alpha_j}s^{-\alpha_j}\|\widetilde{u}_n((1-r)z)\|_{D(A)}\\
\le& C|z|^{\alpha_1-1}(1-s)^{\alpha_1-1}r^{\alpha_1-\alpha_j-1}s^{-\alpha_j}\|\widetilde{u}_n((1-r)z)\|_{D(A)}.
\end{aligned}
$$

Hence, in view of (2.46) and (2.52), the $D(A)$-norms of the integrands in (2.48) are integrable in $r, s \in (0,1)$. Therefore, since $\widetilde{u}_n((1-r)z) : \Omega_{\theta,T} \longrightarrow D(A)$ is analytic, we see that also $\widetilde{u}_{n+1} : \Omega_{\theta,T} \longrightarrow D(A)$ is analytic. Thus by induction $\widetilde{u}_n : \Omega_{\theta,T} \longrightarrow D(A)$ is analytic for all $n \in N$.

We proceed to the completion of the proof of Theorem 2.2. Using (2.3), we see that

$$\sum_{n=0}^{\infty} \frac{(CT^{\alpha_0+\beta_0}\Gamma(\alpha_0))^n}{\Gamma(n\alpha_0+1)} < \infty.$$

Hence the majorant test implies $\sum_{n=0}^{\infty} \|\widetilde{u}_{n+1}(z) - \widetilde{u}_n(z)\|_{D(A)}$ converges uniformly in $z \in \Omega_{\theta,T}$. Therefore there exists $\widetilde{u}(z) \in L^2(\Omega)$ such that $\|A\widetilde{u}_n(z) - A\widetilde{u}(z)\|_{L^2(\Omega)}$ tends to 0 as $n \to \infty$ uniformly in $z \in \Omega_{\theta,T}$. Recalling the analyticity of $\widetilde{u}_n$ in $z \in \Omega_{\theta,T}$ for $n = 1, 2, \cdots$, we see that $A\widetilde{u}(z)$ is analytic in $\Omega_{\theta,T}$. Moreover, since T is arbitrarily chosen, we deduce $A\widetilde{u}(z)$ is analytic in the sector Ω_θ.

Next we prove (2.43). In view of $p \le 0$, $p_1 > 0$ and $p_j \ge 0$ on $\overline{\Omega}$ for $2 \le j \le \ell$, we can prove

$$u(x,t) \le \max\{0, \max_{x\in\partial\Omega, 0\le t\le T} g(x)\lambda(t)\} \quad \text{for } x \in \overline{\Omega},\ 0 \le t \le T. \tag{2.53}$$

In fact, we can repeat the proof of Theorem 2 in Luchko [34] which assumes that $p_1, ..., p_\ell$ are all constants and $p_1 > 0$, $p_j \ge 0$ for $j = 2, ..., \ell$. Therefore (2.53) holds if u is sufficiently smooth. For our solution with the boundary value $g(x)\lambda(t)$, applying an approximating argument, we see that (2.53) is valid for the solutions constructed in the theorem.

Replacing u by $-u$ and applying (2.53), we obtain

$$-u(x,t) \le \max\{0, \max_{x\in\partial\Omega, 0\le t\le T} (-g(x)\lambda(t))\},$$

that is,

$$u(x,t) \ge \min\{0, \min_{x\in\partial\Omega, 0\le t\le T} g(x)\lambda(t)\}$$

for $x \in \overline{\Omega}$ and $0 \le t \le T$. Combining (2.53), we obtain

$$|u(x,t)| \le \max_{x\in\partial\Omega, 0\le t\le T} |g(x)\lambda(t)|$$

for $x \in \overline{\Omega}$ and $0 \le t \le T$. Therefore the proof of (2.43) is completed.

Finally we show that $\widetilde{u}(z)$ is the weak solution to (2.40) when the variable z is restricted to $(0,T)$. In fact, denoting the imaginary part of $\widetilde{u}(t)$, $\forall t \in (0,T)$ as $\operatorname{Im}\widetilde{u}(t)$, we see that $v := \operatorname{Im}\widetilde{u}(t)$ is a weak solution to the following initial-boundary problem:

$$\begin{cases} \partial_t^{\alpha_1} v + \sum_{j=2}^{\ell} \widetilde{p}_j(x)\partial_t^{\alpha_j} v = \operatorname{div}\left(\frac{1}{p_1(x)}\nabla v\right) + B(x)\cdot\nabla v + b(x)v & \text{in } \Omega \times (0,T], \\ v(x,0) = 0 & \text{in } \Omega, \\ v(x,t) = 0 & \text{on } \partial\Omega \times (0,T]. \end{cases}$$

Using the uniqueness result of the above problem (e.g., Theorem 2.4 in [43]), we have $\operatorname{Im}\widetilde{u}(t) = 0$, $\forall t \in (0,T)$. Thus again by the uniqueness argument we see that $\widetilde{u}(t) = \operatorname{Re}\widetilde{u}(t)$, $\forall t \in (0,T)$ solves (2.40). Consequently, we see that $u(t) = \widetilde{u}(t) + \lambda(t)\widetilde{g}$ is the weak solution to (2.1) and is analytic from $(0,T]$ to $H^2(\Omega)$ in view of the analyticity of $\lambda(t)$. This completes the proof of the theorem. □

Now let us turn to the proof of our main theorem in this subsection. The proofs rely on relevant uniqueness for Calderón problems. For convenience, we describe uniqueness results which are applied for our proof.

Let $\Gamma_+, \Gamma_- \subset \partial\Omega$ be sub-boundaries. By

$$\Lambda(q, \Gamma_-, \Gamma_+)f = \frac{\partial u}{\partial\nu}|_{\Gamma_+}$$

we define the Dirichlet-to-Neumann map for

$$\Delta u + qu = 0 \quad \text{in} \quad \Omega, \quad u|_{\Gamma_-} = f, \quad u|_{\partial\Omega\setminus\Gamma_-} = 0.$$

The sub-boundaries Γ_- and Γ_+ can be regarded as input sub-boundary and output sub-boundary respectively.

Sylvester and Uhlmann [59] proved the uniqueness in determining $q(x)$ by the Dirichlet-to-Neumann map $\Lambda(q, \Gamma_-, \Gamma_+)$ for the case of $\Gamma_+ = \Gamma_- = \partial\Omega$ where one must change all the Dirichlet data on the whole boundary and measure all the corresponding Neumann data on the whole boundary. On the other hand, the uniqueness by $\Lambda(q, \Gamma_-, \Gamma_+)$ with limited Γ_+ and Γ_- is important and there have been several works and in particular we apply Imanuvilov and Yamamoto [24] which proved the uniqueness for the Caldrón problem with arbitrary sub-boundary $\Gamma_+ = \Gamma_-$.

The analyicity in t of the solution to (1) (Theorem 2.2) reduces the Dirichlet-to-Neumann map for (1) to the Calderón problem, so that we can apply various existing uniqueness results. Here we apply only [24] and [59] as such known results and we note that we can obtain various types of uniqueness for the inverse problem for (1) if we apply other uniqueness results for Calderón problems. As for other uniqueness results for the Calderón problem, we refer for example to Bukhgeim and Uhlmann [6], Imanuvilov and Yamamoto [25], Kenig and Salo [31], and the references therein.

The proof of Corollary 2.2 is the same as the proof of Theorem 2.6, and the only difference is that instead of the uniqueness result in Theorem 0.1 of [59], we have to use Theorem 0.1 in [24]. Thus it is sufficient to prove Theorem 2.6

Proof of Theorem 2.6. Our key idea is to reduce the inverse problem to the corresponding inverse boundary value problem for the Schrödinger equation

$$\begin{cases} \Delta v(x,s) + P_s(x)v(x,s) = 0 & \text{in } \Omega, \\ v(x,s) = g(x) & \text{on } \partial\Omega, \end{cases}$$

for all large $s > 0$. Here and henceforth we set $P_s(x) := p(x) - \sum_{j=1}^{\ell} p_j(x)s^{\alpha_j}$.

Let $u_1(g)(x,t)$ and $u_2(g)(x,t)$ be the solutions to (2.1) with $(\ell, \boldsymbol{\alpha}, p_j, p)$ and $(m, \boldsymbol{\beta}, q_j, q)$ respectively. Since $\lambda(t)$ is t-analytic in $t > 0$, Theorem 2.7 implies that $u_1(g)(x,t)$ and $u_2(g)(x,t)$ are t-analytic in $t > 0$ under the norm $H^2(\Omega)$. Therefore, since $w \mapsto \frac{\partial w}{\partial \nu}\colon H^{\frac{3}{2}}(\partial\Omega) \to H^{\frac{1}{2}}(\partial\Omega)$ is continuous, equality (2.39) implies

$$\frac{\partial u_1(g)}{\partial \nu}(x,t) = \frac{\partial u_2(g)}{\partial \nu}(x,t), \quad x \in \partial\Omega,\ 0 < t < \infty \quad \text{for } g \in H^{\frac{3}{2}}(\partial\Omega).$$

Let $(Lu)(x,s) := \int_0^\infty u(x,t)\mathrm{e}^{-st}\mathrm{d}t$ be the Laplace transform of $u(x,t)$ in t for each fixed $x \in \overline{\Omega}$. By (2.43) in Theorem 2.7 and assumption $|\lambda(t)| \le C_0 e^{C_0 t}$ for $t > 0$, we see that $|u(x,t)| \le Ce^{C_0 t}$ for $t > 0$, where $C > 0$ is a constant and is independent of $t > 0$ and $x \in \Omega$. Therefore $(Lu_k(g))(x,s)$, $k = 1,2$, exist for $s > C_1$ where $C_1 > 0$ is some constant depending only on λ. Using $u_k(g)(x,0) = 0$, by [52], we have

$$L(\partial_t^\alpha u_k(g))(x,s) = s^\alpha (Lu_k(g))(x,s), \quad s > C_1,\ x \in \Omega,\ k = 1,2.$$

Therefore $L(\partial_t^\alpha u_k)(x,s)$ exists for $s > C_1$ and $x \in \Omega$, $k = 1,2$. Since (1) holds for all $t > 0$, it follows that $L(\Delta u_k(g))(x,s)$, $k = 1,2$, exist for $s > C_1$ and $x \in \Omega$. Hence

$$\begin{cases} \Delta L(u_1(g))(x,s) + P_s(x)L(u_1(g))(x,s) = 0, & x \in \Omega,\ s > C_1, \\ L(u_1(g))(x,s) = (L\lambda)(s)g(x), & x \in \partial\Omega,\ s > C_1, \end{cases}$$

$$\begin{cases} \Delta L(u_2(g))(x,s) + Q_s(x)L(u_2(g))(x,s) = 0, & x \in \Omega,\ s > C_1, \\ L(u_2(g))(x,s) = (L\lambda)(s)g(x), & x \in \partial\Omega,\ s > C_1, \end{cases}$$

and

$$\frac{\partial L(u_1(g))}{\partial \nu}(x,s) = \frac{\partial L(u_2(g))}{\partial \nu}(x,s), \quad \forall x \in \partial\Omega, \quad \text{and } \forall s > C_1,$$

where $Q_s(x) := q(x) - \sum_{j=1}^{m} q_j(x)s^{\beta_j}$. On the other hand, we consider the following two boundary value problems

$$\begin{cases} \Delta v_1(x,s) + P_s(x)v_1(x,s) = 0, & x \in \Omega,\ s > C_1, \\ v_1(x,s) = g(x), & x \in \partial\Omega,\ s > C_1. \end{cases} \tag{2.54}$$

and

$$\begin{cases} \Delta v_2(x,s) + Q_s(x)v_2(x,s) = 0, & x \in \Omega,\ s > C_1, \\ v_2(x,s) = g(x), & x \in \partial\Omega,\ s > C_1. \end{cases} \tag{2.55}$$

Then we define the Dirichlet-to-Neumann maps corresponding to the boundary value problems (2.54) and (2.55) as $\Lambda(P_s)$ and $\Lambda(Q_s)$ respectively by

$$\Lambda(P_s)g := \frac{\partial v_1(g)}{\partial \nu}\Big|_{\partial\Omega}, \quad \Lambda(Q_s)g := \frac{\partial v_2(g)}{\partial \nu}\Big|_{\partial\Omega}.$$

Now we prove that there exists a subset $\sigma \subset (C_1, \infty)$ such that σ contains a non-empty open interval and

$$\Lambda(\ell, \boldsymbol{\alpha}, p_j, p)g = \Lambda(m, \boldsymbol{\beta}, q_j, q)g \Longrightarrow \Lambda(P_s)g = \Lambda(Q_s)g \quad \text{for all } g \in H^{\frac{3}{2}}(\partial\Omega) \text{ and } s \in \sigma. \tag{2.56}$$

In fact, $(L\lambda)(z)$ is analytic in Re $z > C_1$ and $\{s;\ (L\lambda)(s) = 0,\ s > C_1\}$ has no accumulation points except for ∞. Therefore $\sigma := (C_1, \infty) \setminus \{s;\ (L\lambda)(s) = 0,\ s > C_1\}$ contains a non-empty open interval. Then we can set $\widetilde{v}_j(g)(x,s) = \frac{L(u_j(g))(x,s)}{(L\lambda)(s)}$ for $j = 1, 2$ and $s \in \sigma$. It is not very difficult to see that $\widetilde{v}_1(g)$ and $\widetilde{v}_2(g)$ are the solutions to (2.54) and (2.55) respectively. From the uniqueness of the boundary value problem, we see that $\widetilde{v}_j(g) = v_j(g)$, $j = 1, 2$ for $s \in \sigma$.

Here by the density of $H^{\frac{3}{2}}(\partial\Omega)$ in $H^{\frac{1}{2}}(\partial\Omega)$ and the continuity of $\Lambda(P_s) : H^{\frac{1}{2}}(\partial\Omega) \longrightarrow H^{-\frac{1}{2}}(\partial\Omega)$, we see that (2.56) holds for all $g \in H^{\frac{1}{2}}(\partial\Omega)$.

Therefore from the uniqueness in determining a potential by Dirichlet-to-Neumann map (e.g., in [59]), we see that $P_s(x) = Q_s(x)$ for all $x \in \Omega$ and $s \in \sigma$. Since σ contains a non-empty open interval, we obtain $\ell = m$, $\boldsymbol{\alpha} = \boldsymbol{\beta}$, $p_j = q_j$, $1 \le j \le \ell$ and $p = q$. Thus the proof of Theorem 1.1 is completed. □

2.5 Conclusions and open problems

In this paper, we mainly dealt with the forward and inverse problems to the initial-boundary value problem for the multi-term time-fractional diffusion equations.

For the forward problem, by means of Fourier methods and Fredholm theory for the compact operator, we show the unique existence of the solutions to the initial-boundary value problems, and we succeed in determining the solutions by initial values and the source terms. Indeed, in the case of the non-homogeneous equation, we considered the initial-boundary value problem provided all the coefficients are spatial and temporal dependence, whereas for the homogeneous counterpart, due to our methods, the assumption that all the coefficients are independent of the time is necessary. Furthermore, by a Laplace transform argument, it turns out that the decay rate of the solution for long time is dominated by the lowest order of the time-fractional derivatives, which generalized that for the case of positive-constant coefficients discussed in [37].

For the inverse problems, first, in case of all the coefficients of the fractional derivatives are positive constants, by means of eigenfunction expansion, Theorem 2.5 shows the uniqueness in determining fractional orders in the d–dimensional diffusion equation with $L^2(\Omega)$-initial function by using one interior point measurement. In its space-dependent counterpart, as an application of the analyticity of the solution and Laplace transform, Theorem 2.6 gives an affirmative answer for the uniqueness in determining the number of fractional time-derivative terms, the orders of the derivatives and spatially varying coefficients under the assumption that the boundary input is in form of separation of variables. For the general boundary input with time t being fixed, it is still expected to get the uniqueness of the same inverse problem and remains open.

Chapter 3

Fractional advection-diffusion equations

In this chapter, we continue to consider a multi-term time-fractional diffusion equation, but here the multi-term derivatives including the first order derivative ∂_t, that is, fractional advection diffusion equations (FADE). Some properties of FADE are expected to be partly inherited by parabolic equations, which enable us apply Carleman estimates for parabolic equations to prove a Carleman estimate for the generalized fractional-in-time advection dispersion equations by considering the fractional derivative as perturbation for the first order time-derivative. The point is a special choice of the time factor of the weight function. As an application of the Carleman estimate, we show a Hölder dependency of solutions with respect to initial values, Cauchy data and source terms.

Keywords: fractional advection dispersion equation, Carleman estimate, Conditional stability

3.1 Introduction and main results

The advection dispersion equation (ADE) based on Fick's law has been widely used to solve a range of problems in analysing mass transport. Recently numerous field experiments for the solute transport in highly heterogeneous media demonstrate that solute concentration profiles exhibited anomalous non-Fickian growth rates, skewness, sharp leading edges and so-called "long tails"(See e.g., Benson et al [7], Hatano and Hatano [20], and Levy and Berkowitz [35]), which are poorly characterized by the conventional mass transport equations. To sufficiently predict these effects, the non-Fickian diffusion model has been proposed to mass transport model, say, fractional-in-time advection-dispersion equation (FADE):

$$\partial_t u(x,t) + \partial_t^\alpha u(x,t) = \Delta u(x,t), \quad (x,t) \in \mathbb{R}^d \times (0,\infty). \tag{3.1}$$

See, e.g., [22], [23], [57] and the references therein for the FADEs. In Hornung and Showalter [22], diffusion models in fractured media are described, say, a generalized form of equation (3.1), and a general elementary proof for well-posedness with the additional appropriate initial and boundary conditions are given. the macro advection-dispersion experiment (MADE) site mobile tritium mass decline is consistent with a fractional time derivative of order $\alpha = 0.33$, while Haggerty et al [23] stream tracer test is well modeled by a fractional time derivative of order $\alpha = 0.28$ In Schumer et al [57], in the case of 1-dimension, the solution to (3.1) is gained by performing an integral transform on the solution of any boundary value problem for transport in the absence of an immobile phase.

In this chapter, assuming $0 < \alpha_\ell < \cdots < \alpha_1 < \frac{1}{2}$, we consider a generalized FADE

$$(Lu)(x,t) \equiv \partial_t u + \sum_{j=1}^{\ell} q_j(x,t)\partial_t^{\alpha_j} u - \sum_{i,j=1}^{d} a_{i,j}(x,t)\partial_i\partial_j u - \sum_{i=1}^{d} b_i(x,t)\partial_i u - c(x,t)u = f, \tag{3.2}$$

where (x,t) in $\mathbb{R}^d \times (0,\infty)$. The equation here is very different from [22], [23] and [57]. In our case, fully taking into account the impact of space, time and time delay, the generalized FADE can be regarded as more feasible equation than the advection diffusion equations in modeling the diffusion in the heterogeneous media. All the previous methods used in Chapter 1 and Chapter 2, e.g., eigenfunction expansion, Laplace transform, fail, which forces us to adopt a completely different approach to the forward and inverse problems for FADEs.

We investigate the continuous dependency on initial values, boundary values and source terms for the equation (3.2) by the Carleman estimate for parabolic equations. To the best knowledge of the authors, the stability results of the equation (3.2) were not yet established. To this end, we start from fixing some general settings and notations. Let $T>0$ be fixed constant and $\Omega \subset \mathbb{R}^d$ is a bounded domain, $d \geq 1$, with sufficiently smooth boundary $\partial\Omega$, for example, of C^2-class. We set $Q := \Omega \times (0,T)$. Assume that $a_{ij} = a_{ji} \in C^1(\overline{Q})$, $1 \leq i,j \leq d$, satisfies that

$$\sigma \sum_{j=1}^{d} \xi_j^2 \leq \sum_{j,k=1}^{d} a_{jk}(x,t)\xi_j\xi_k, \quad (x,t) \in \overline{Q},\ \xi \in \mathbb{R}^d,$$

where $\sigma > 0$ is a constant independent of x,t,ξ. We set $\partial_{\nu_A} u = \sum_{i,j=1}^{d} a_{ij}\nu_i\partial_j u$ where $(\nu_1,\cdots,\nu_d)$ denotes the unit outwards normal vector to the boundary $\partial\Omega \times (0,T)$. Let $L^2(\Omega)$ and $H^{k,\ell}(Q)$ $(k \geq 0, \ell \geq 0)$ denote Sobolev spaces (See, e.g., Adams [1] and Yamamoto [65]).

Theorem 3.1 *Let $\Gamma \subset \partial\Omega$ be an arbitrary non-empty sub-boundary of $\partial\Omega$. For any $\varepsilon > 0$ and an arbitrary bounded domain Ω_0 such that $\overline{\Omega_0} \subset \Omega \cup \Gamma$, $\partial\Omega_0 \cap \partial\Omega \subsetneq \Gamma$ is a non-empty open subset of $\partial\Omega$, there exist constants $C>0$ and $\theta \in (0,1)$ such that*

$$\|u\|_{H^{1,1}(\Omega_0\times(0,\varepsilon))} \leq C\|u\|_{H^{1,1}(Q)}^{1-\theta} F^\theta, \tag{3.3}$$

where $F := \|u(\cdot,0)\|_{L^2(\Omega)} + \|f\|_{L^2(Q)} + \|u\|_{H^1(\Gamma\times(0,T))} + \|\partial_{\nu_A} u\|_{L^2(\Gamma\times(0,T))}$.

Remark 3.1 Different to the results in Yamamoto [65], here due to the choice of the weight function in the derivation of the Carleman estimate in Section 3.2, we can only prove that the continuous dependency of the solution with respect to initial values, boundary values and source terms in the case of $\alpha \in (0, \frac{1}{2})$.

The rest of this chapter is organized in three sections. In section 3.2, we derive a Carleman type estimate for parabolic equations with a new weight function. In Section 3.3, by regarding the fractional-order terms as non-homogeneous term and applying the Carleman estimate for the parabolic equations in Section 3.2, we prove the continuous dependency on initial values, Cauchy data and source terms. Finally, concluding remarks are given in Section 3.4.

3.2 Carleman estimate for parabolic equations

In this section, letting $\zeta \in C^2(\overline{\Omega})$ and $|\nabla\zeta| \neq 0$ on $\overline{\Omega}$ and setting $\psi = \zeta(x) - \beta t^{2-2\alpha_1}$ with $\beta > 0$, we discuss the derivation of a Carleman estimate for $L_0 = \partial_t - \sum_{i,j=1}^{d} a_{ij}(x,t)\partial_i\partial_j$ with the new weight function $\varphi := \mathrm{e}^{\lambda\psi}$. Namely

Lemma 3.1 *Let $\Sigma_0 = \overline{\Omega} \times \{0\}$ and $D \subset Q$ be bounded domain whose boundary ∂D is composed of a finite number of smooth surfaces. Then there exists a constant $\lambda_0 > 0$ such that for arbitrary $\lambda \geq \lambda_0$, we can choose a constant $s_0(\lambda) > 0$ satisfying: there exists a constant $C = C(s_0,\lambda_0)$ such that*

$$\begin{aligned}&\int_D \left\{ \frac{1}{s\varphi}|\partial_t u|^2 + s\lambda^2\varphi|\nabla u|^2 + s^3\lambda^4\varphi^3 u^2 \right\} \mathrm{e}^{2s\varphi} \mathrm{d}x\mathrm{d}t \\ \leq &C \int_D |L_0 u|^2 \mathrm{e}^{2s\varphi} \mathrm{d}x\mathrm{d}t + \mathrm{e}^{C(\lambda)s} \int_{\partial D} (|\nabla u|^2 + |u|^2) \mathrm{d}S\mathrm{d}t + \mathrm{e}^{C(\lambda)s} \int_{\partial D \setminus \Sigma_0} |\partial_t u|^2 \mathrm{d}S\mathrm{d}t\end{aligned} \tag{3.4}$$

for all $s > s_0$ and all $u \in H^{2,1}(D)$.

Proof. We follow the arguments on pp.9-19 of the survey paper [65] to prove the estimate (3.4). We use the same notations, where we must modify locally because our choice of the time dependency of ψ is different.

We set

$$\sigma(x,t) = \sum_{i,j=1}^{d} a_{ij}(x,t)(\partial_i\zeta)\partial_j\zeta, \quad (x,t) \in \overline{Q}$$

and

$$w(x,t) = \mathrm{e}^{s\varphi(x,t)}u(x,t)$$

and

$$\begin{aligned} Pw = \mathrm{e}^{s\varphi}L_0(\mathrm{e}^{-s\varphi}w) =&\partial_t w - \sum_{i,j=1}^{d} a_{ij}\partial_j\partial_j w + 2s\lambda\varphi \sum_{i,j=1}^{d} a_{ij}(\partial_i\zeta)\partial_j w \\ &- s^2\lambda^2\varphi^2\sigma w + s\lambda^2\varphi\sigma w + s\lambda\varphi w \sum_{i,j=1}^{d} a_{ij}\partial_i\partial_j\zeta - s\lambda\varphi w(\partial_t\psi). \end{aligned}$$

Similar to the argument used in [65], we decompose P into the parts P_1 and P_2, where P_1 is composed of second-order and zero-order terms in x, and P_2 is composed of first-order terms in t and first-order terms in x, say

$$Pw = P_1w + P_2w,$$

where

$$P_1w = -\sum_{i,j=1}^{d} a_{ij}\partial_j\partial_j w - s^2\lambda^2\varphi^2\sigma w + \left(s\lambda^2\varphi\sigma + s\lambda\varphi \sum_{i,j=1}^{d} a_{ij}\partial_i\partial_j\zeta - s\lambda\varphi w(\partial_t\psi) \right) w,$$

and

$$P_2w = \partial_t w + 2s\lambda\varphi \sum_{i,j=1}^{d} a_{ij}(\partial_i\zeta)\partial_j w.$$

We first estimate $\int_D |P_2w|^2 + 2(P_1w)(P_2w)\mathrm{d}x\mathrm{d}t$ from below. By $\|\mathrm{e}^{s\varphi}L_0u\|^2_{L^2(D)} = \|P_1w + P_2w\|^2_{L^2(D)}$, we have

$$2\int_D (P_1w)(P_2w)\mathrm{d}x\mathrm{d}t + \|P_2w\|^2_{L^2(D)} \le \int_D |L_0u|^2\mathrm{e}^{2s\varphi}\mathrm{d}x\mathrm{d}t.$$

We estimate

$$\begin{aligned} \int_D (P_1w)(P_2w)\mathrm{d}x\mathrm{d}t =& -\sum_{i,j=1}^{d}\int_D a_{ij}(\partial_i\partial_j w)\partial_t w\mathrm{d}x\mathrm{d}t \\ &- \sum_{i,j=1}^{d}\int_D a_{ij}(\partial_i\partial_j w)2s\lambda\varphi \sum_{k,l=1}^{d} a_{k,l}(\partial_k\zeta)\partial_l w\mathrm{d}x\mathrm{d}t \\ &- \int_D s^2\lambda^2\varphi^2\sigma w\partial_t w\mathrm{d}x\mathrm{d}t - \int_D 2s^3\lambda^3\varphi^3\sigma w \sum_{i,j=1}^{d} a_{ij}(\partial_i\zeta)\partial_j w\mathrm{d}x\mathrm{d}t \\ &+ \int_D (A_1w)\partial_t w\mathrm{d}x\mathrm{d}t + \int_D (A_1w)2s\varphi \sum_{i,j=1}^{d} a_{ij}(\partial_i\zeta)\partial_j w\mathrm{d}x\mathrm{d}t \\ =:& \sum_{k=1}^{6} J_k, \end{aligned}$$

where $A_1 w = s\lambda^2\varphi\sigma w + s\lambda\varphi w\sum_{i,j=1}^d a_{ij}\partial_i\partial_j\zeta - s\lambda\varphi w\partial_t\psi =: s\lambda^2\varphi a_1(x,t;\lambda,s)w$. Now, applying the integration by parts and $a_{ij}=a_{ji}$, we estimate J_k, $k=1,\cdots,6$ separately.

$$J_1 = -\sum_{i,j=1}^d \int_D a_{ij}(\partial_i\partial_j w)\partial_t w \mathrm{d}x\mathrm{d}t$$
$$= \sum_{i,j=1}^d \int_D (\partial_i a_{ij})(\partial_j w)\partial_t w\mathrm{d}x\mathrm{d}t + \sum_{i,j=1}^d \int_D a_{ij}(\partial_j w)\partial_i\partial_t w\mathrm{d}x\mathrm{d}t - \sum_{i,j=1}^d \int_{\partial D\setminus\Sigma_0} a_{ij}(\partial_j w)\nu_i\partial_t w\mathrm{d}S\mathrm{d}t.$$

Here and henceforth $\nu := (\nu_1,\cdots,\nu_d,\nu_{d+1})$ denotes the unit normal exterior with respect to the boundary ∂D of D. In particular, ν_{d+1} is the component in the time direction. Therefore integration by parts yields

$$\begin{aligned}J_1 =& \sum_{i,j=1}^d \int_D (\partial_i a_{ij})(\partial_j w)\partial_t w\mathrm{d}x\mathrm{d}t + \frac{1}{2}\sum_{i,j=1}^d \int_D a_{ij}\partial_t\big((\partial_i w)\partial_j w\big)\mathrm{d}x\mathrm{d}t \\ &- \sum_{i,j=1}^d \int_{\partial D\setminus\Sigma_0} a_{ij}(\partial_j w)\nu_i\partial_t w\mathrm{d}S\mathrm{d}t \\ =& \sum_{i,j=1}^d \int_D (\partial_i a_{ij})(\partial_j w)\partial_t w\mathrm{d}x\mathrm{d}t - \frac{1}{2}\sum_{i,j=1}^d \int_D (\partial_t a_{ij})(\partial_i w)\partial_j w\mathrm{d}x\mathrm{d}t \\ &+ \frac{1}{2}\sum_{i,j=1}^d \int_{\partial D} a_{ij}(\partial_i w)(\partial_j w)\nu_{n+1}\mathrm{d}S\mathrm{d}t - \sum_{i,j=1}^d \int_{\partial D\setminus\Sigma_0} a_{ij}(\partial_j w)\nu_i\partial_t w\mathrm{d}S\mathrm{d}t.\end{aligned}$$

Thus

$$\begin{aligned}|J_1| \le& C\int_D |\nabla w||\partial_t w|\mathrm{d}x\mathrm{d}t + C\int_D |\nabla w|^2\mathrm{d}x\mathrm{d}t + C\int_{\partial D}|\nabla w|^2\mathrm{d}S\mathrm{d}t + C\int_{\partial D\setminus\Sigma_0}|\nabla w||\partial_t w|\mathrm{d}S\mathrm{d}t \\ \le& C\int_D |\nabla w||\partial_t w|\mathrm{d}x\mathrm{d}t + C\int_D |\nabla w|^2\mathrm{d}x\mathrm{d}t + C\int_{\partial D}|\nabla w|^2\mathrm{d}S\mathrm{d}t + C\int_{\partial D\setminus\Sigma_0}|\partial_t w|^2\mathrm{d}S\mathrm{d}t.\end{aligned}$$

Since the Cauchy-Schwarz inequality implies that

$$|\nabla w||\partial_t w| = s^{\frac{1}{2}}\lambda^{\frac{1}{2}}\varphi^{\frac{1}{2}}|\nabla w| s^{-\frac{1}{2}}\lambda^{-\frac{1}{2}}\varphi^{-\frac{1}{2}}|\partial_t w| \le \frac{1}{2}s\lambda\varphi|\nabla w|^2 + \frac{1}{2}\frac{1}{s\lambda\varphi}|\partial_t w|^2,$$

we have

$$\begin{aligned}|J_1| \le& C\int_D \frac{1}{s\lambda\varphi}|\partial_t w|^2\mathrm{d}x\mathrm{d}t + C\int_D s\lambda\varphi|\nabla w|^2\mathrm{d}x\mathrm{d}t \\ &+ C\int_{\partial D}|\nabla w|^2\mathrm{d}S\mathrm{d}t + C\int_{\partial D\setminus\Sigma_0}|\partial_t w|^2\mathrm{d}S\mathrm{d}t. \end{aligned} \tag{3.5}$$

Next similar to the argument on pp. 12-13 in [65], we have

$$\begin{aligned}J_2 =& -\sum_{i,j=1}^d\sum_{k,l=1}^d \int_D 2s\lambda\varphi a_{ij}a_{kl}(\partial_k\zeta)(\partial_l w)\partial_i\partial_j w\mathrm{d}x\mathrm{d}t \\ =& 2s\lambda\int_D \sum_{i,j=1}^d\sum_{k,l=1}^d \lambda(\partial_i\zeta)\varphi a_{ij}a_{kl}(\partial_k\zeta)(\partial_l w)\partial_j w\mathrm{d}x\mathrm{d}t \\ &+ 2s\lambda\int_D \sum_{i,j=1}^d\sum_{k,l=1}^d \varphi\partial_i(a_{ij}a_{kl}\partial_k\zeta)(\partial_l w)\partial_j w\mathrm{d}x\mathrm{d}t\end{aligned}$$

$$+ 2s\lambda \int_D \sum_{i,j=1}^{d} \sum_{k,l=1}^{d} \varphi a_{ij} a_{kl} (\partial_k \zeta)(\partial_i \partial_l w) \partial_j w \mathrm{d}x\mathrm{d}t$$
$$- 2s\lambda \int_{\partial D} \sum_{i,j=1}^{d} \sum_{k,l=1}^{d} \varphi a_{ij} a_{kl} (\partial_k \zeta)(\partial_l \zeta)(\partial_l w)(\partial_j w) \nu_i \mathrm{d}S\mathrm{d}t.$$

We have

$$(\text{first term}) = 2s\lambda^2 \int_D \varphi \left| \sum_{i,j=1}^{d} a_{ij} (\partial_i \zeta) \partial_j w \right|^2 \mathrm{d}x\mathrm{d}t \geq 0$$

and

$$\begin{aligned}(\text{third term}) =& s\lambda \int_D \varphi \sum_{i,j=1}^{d} \sum_{k,l=1}^{d} a_{ij} a_{kl} (\partial_k \zeta) \partial_l \big((\partial_i w)(\partial_j w) \big) \mathrm{d}x\mathrm{d}t \\ =& s\lambda \int_{\partial D} \varphi \sum_{i,j=1}^{d} \sum_{k,l=1}^{d} a_{ij} a_{kl} (\partial_k \zeta)(\partial_i w)(\partial_j w) \nu_l \mathrm{d}S\mathrm{d}t \\ &- s\lambda^2 \int_D \varphi \sum_{i,j=1}^{d} \sigma a_{ij} (\partial_i w) \partial_j w \mathrm{d}x\mathrm{d}t \\ &- s\lambda \int_D \varphi \sum_{i,j=1}^{d} \sum_{k,l=1}^{d} \partial_l \big(a_{ij} a_{kl} (\partial_k \zeta) \big) (\partial_i w) \partial_j w \mathrm{d}x\mathrm{d}t,\end{aligned}$$

which imply

$$J_2 \geq -\int_D s\lambda^2 \varphi \sigma \sum_{i,j=1}^{d} a_{ij} (\partial_i w) \partial_j w \mathrm{d}x\mathrm{d}t - C \int_D s\lambda \varphi |\nabla w|^2 \mathrm{d}x\mathrm{d}t - C \int_{\partial D} s\lambda \varphi |\nabla w|^2 \mathrm{d}S\mathrm{d}t. \quad (3.6)$$

$$\begin{aligned}|J_3| =& \left| -\frac{1}{2} \int_D s^2 \lambda^2 \varphi^2 \sigma \partial_t (w^2) \mathrm{d}x\mathrm{d}t \right| \\ =& \left| \int_D s^2 \lambda^3 \varphi^2 \beta (2\alpha_1 - 2) t^{1-2\alpha_1} \sigma w^2 \mathrm{d}x\mathrm{d}t + \frac{1}{2} \int_D s^2 \lambda^2 \varphi^2 (\partial_t \sigma) w^2 \mathrm{d}x\mathrm{d}t - \frac{1}{2} \int_{\partial D} s^2 \lambda^2 \varphi^2 \sigma w^2 \nu_{d+1} \mathrm{d}S\mathrm{d}t \right| \\ \leq& C \int_D s^2 \lambda^3 \varphi^2 w^2 \mathrm{d}x\mathrm{d}t + C \int_{\partial D} s^2 \lambda^2 \varphi^2 w^2 \mathrm{d}S\mathrm{d}t. \end{aligned} \quad (3.7)$$

$$\begin{aligned}J_4 =& -\int_D s^3 \lambda^3 \varphi^3 \sum_{i,j=1}^{d} \sigma a_{ij} (\partial_i \zeta) \partial_j (w^2) \mathrm{d}x\mathrm{d}t \\ =& 3 \int_D s^3 \lambda^4 \varphi^3 \sigma^2 w^2 \mathrm{d}x\mathrm{d}t + \int_D s^3 \lambda^3 \varphi^3 \sum_{i,j=1}^{d} \partial_j (\sigma a_{ij} \partial_i \zeta) w \mathrm{d}x\mathrm{d}t \\ &- \int_{\partial D} s^3 \lambda^3 \varphi^3 \sum_{i,j=1}^{d} \sigma a_{ij} (\partial_i \zeta) w^2 \nu_j \mathrm{d}S\mathrm{d}t \\ \geq& 3 \int_D s^3 \lambda^4 \varphi^3 \sigma^2 w^2 \mathrm{d}x\mathrm{d}t - C \int_D s^3 \lambda^3 \varphi^3 w^2 \mathrm{d}x\mathrm{d}t - C \int_{\partial D} s^3 \lambda^3 \varphi^3 w^2 \mathrm{d}S\mathrm{d}t. \end{aligned} \quad (3.8)$$

From

$$J_6 = \int_D a_1 s^2 \lambda^3 \varphi^2 \sum_{i,j=1}^{d} a_{ij} (\partial_i \zeta) \partial_j (w^2) \mathrm{d}x\mathrm{d}t$$

$$= \int_{\partial D} a_1 s^2\lambda^3\varphi^2 \sum_{i,j=1}^{d} a_{ij}(\partial_i\zeta)w^2\nu_j \mathrm{d}S\mathrm{d}t - 2\int_D s^2\lambda^4\varphi^2 a_1\sigma w^2 \mathrm{d}x\mathrm{d}t$$
$$- 2\int_D s^2\lambda^3\varphi^2 \sum_{i,j=1}^{d} \partial_j(a_1 a_{ij}(\partial_i\zeta))w^2 \mathrm{d}x\mathrm{d}t,$$

we have

$$|J_6| \le C\int_D s^2\lambda^4\varphi^2 w^2 \mathrm{d}x\mathrm{d}t + C\int_{\partial D} s^2\lambda^3\varphi^2 w^2 \mathrm{d}S\mathrm{d}t. \tag{3.9}$$

We estimate J_5 by a different way. First we note that

$$A_1 w = s\lambda^2\varphi\sigma w + s\lambda\varphi\sum_{i,j=1}^{d} a_{ij}(\partial_i\partial_j\zeta)w + s\lambda\varphi(2-2\alpha_1)\beta t^{1-2\alpha_1}w$$

and

$$(A_1 w)\partial_t w = \frac{1}{2}s\lambda^2\varphi\sigma\partial_t(w^2) + O(s\lambda\varphi)w\partial_t w.$$

Therefore

$$J_5 = \int_D O(s\lambda\varphi)w\partial_t w \mathrm{d}x\mathrm{d}t + (1-\alpha_1)\beta\int_D t^{1-2\alpha_1}s\lambda^3\varphi\sigma w^2 \mathrm{d}x\mathrm{d}t + \frac{1}{2}\int_D s\lambda^2\varphi(\partial_t\sigma)w^2 \mathrm{d}x\mathrm{d}t$$
$$+ \frac{1}{2}\int_{\partial D} s\lambda^2\varphi\sigma w^2\nu_{d+1} \mathrm{d}S\mathrm{d}t.$$

Therefore

$$|J_5| \le C\int_{\partial D} s\lambda^2\varphi w^2 \mathrm{d}S\mathrm{d}t + C\int_D s\lambda^3\varphi w^2 \mathrm{d}x\mathrm{d}t + C\int_D s^{\frac{3}{2}}\lambda^{\frac{3}{2}}\varphi^{\frac{3}{2}}|w|s^{-\frac{1}{2}}\lambda^{-\frac{1}{2}}\varphi^{-\frac{1}{2}}|\partial_t w| \mathrm{d}x\mathrm{d}t$$
$$\le C\int_D s^3\lambda^3\varphi^3 w^2 \mathrm{d}x\mathrm{d}t + C\int_D \frac{1}{s\lambda\varphi}|\partial_t w|^2 \mathrm{d}x\mathrm{d}t + C\int_{\partial D} s\lambda^2\varphi w^2 \mathrm{d}S\mathrm{d}t. \tag{3.10}$$

Hence

$$\int_D (P_1 w)(P_2 w)\mathrm{d}x\mathrm{d}t \ge \int_D 3s^3\lambda^4\varphi^3\sigma^2 w^2 \mathrm{d}x\mathrm{d}t - \int_D s\lambda^2\varphi\sigma\sum_{i,j=1}^{d} a_{ij}(\partial_i w)\partial_j w \mathrm{d}x\mathrm{d}t$$
$$- C\int_D (s^3\lambda^3\varphi^3 + s^2\lambda^4\varphi^2)w^2 \mathrm{d}x\mathrm{d}t - C\int_D \frac{1}{s\lambda\varphi}|\partial_t w|^2 \mathrm{d}x\mathrm{d}t - C\int_D s\lambda\varphi|\nabla w|^2 \mathrm{d}x\mathrm{d}t$$
$$- C\int_{\partial D}(s\lambda\varphi|\nabla w|^2 + s^3\lambda^3\varphi^3 w^2)\mathrm{d}S\mathrm{d}t - C\int_{\partial D\setminus\Sigma_0}|\partial_t w|^2 \mathrm{d}S\mathrm{d}t.$$

By the definition of P_2, we have

$$\epsilon\int_D \frac{1}{s\varphi}|\partial_t w|^2 \mathrm{d}x\mathrm{d}t = \epsilon\int_D \frac{1}{s\varphi}\left|P_2 w - 2s\lambda\varphi\sum_{i,j=1}^{d} a_{ij}(\partial_i\zeta)\partial_j w\right|^2 \mathrm{d}x\mathrm{d}t$$
$$\le C\int_D |P_2 w|^2 \mathrm{d}x\mathrm{d}t + C\epsilon\int_D s\lambda^2\varphi|\nabla w|^2 \mathrm{d}x\mathrm{d}t.$$

Hence

$$3\int_D s^3\lambda^4\varphi^3\sigma^2 w^2 \mathrm{d}x\mathrm{d}t - \int_D s\lambda^2\varphi\sigma\sum_{i,j=1}^{d} a_{ij}(\partial_i w)\partial_j w \mathrm{d}x\mathrm{d}t + \left(\epsilon - \frac{C}{\lambda}\right)\int_D \frac{1}{s\varphi}|\partial_t w|^2 \mathrm{d}x\mathrm{d}t$$
$$\le C\int_D |L_0 u|^2 e^{2s\varphi}\mathrm{d}x\mathrm{d}t + C\int_D (s\lambda\varphi + \epsilon s\lambda^2\varphi)|\nabla w|^2 \mathrm{d}x\mathrm{d}t + C\int_D (s^3\lambda^3\varphi^3 + s^2\lambda^4\varphi^2)w^2 \mathrm{d}x\mathrm{d}t$$

$$+ C\int_{\partial D}(s\lambda\varphi|\nabla w|^2 + s^3\lambda^3\varphi^3 w^2)\mathrm{d}S\mathrm{d}t + C\int_{\partial D\setminus\Sigma_0}|\partial_t w|^2\mathrm{d}S\mathrm{d}t. \tag{3.11}$$

The first and second terms on the left-hand side have different signs and so we need another estimate. Thus we will execute another estimation for

$$\int_D s\lambda^2\varphi\sigma\sum_{i,j=1}^d a_{ij}(\partial_i w)\partial_j w\mathrm{d}x\mathrm{d}t$$

by means of

$$\int_D (P_1 w + P_2 w)\times s\lambda^2\varphi\sigma w\mathrm{d}x\mathrm{d}t.$$

That is, multiplying

$$\mathrm{e}^{s\varphi}L_0 u = \partial_t w + 2s\lambda\varphi\sum_{i,j=1}^d a_{ij}(\partial_i\zeta)\partial_j w - \sum_{i,j=1}^d a_{ij}\partial_j\partial_j w - s^2\lambda^2\varphi^2\sigma w + A_1 w$$

with $s\lambda^2\varphi\sigma w$, we obtain

$$\begin{aligned}
&\int_D s\lambda^2\varphi\sigma w\mathrm{e}^{s\varphi}L_0 u\mathrm{d}x\mathrm{d}t\\
=&\frac{1}{2}\int_D s\lambda^2\varphi\sigma\partial_t(w^2)\mathrm{d}x\mathrm{d}t + \int_D s^2\lambda^3\varphi^2\sigma\sum_{i,j=1}^d a_{ij}(\partial_i\zeta)\partial_j(w^2)\mathrm{d}x\mathrm{d}t - \int_D s\lambda^2\varphi\sigma\sum_{i,j=1}^d a_{ij}w\partial_j\partial_j w\mathrm{d}x\mathrm{d}t\\
&-\int_D s^3\lambda^4\varphi^3\sigma^2 w^2\mathrm{d}x\mathrm{d}t + \int_D s\lambda^2\varphi\sigma w A_1 w\mathrm{d}x\mathrm{d}t =: \sum_{k=1}^5 I_k.
\end{aligned}$$

First, in terms of the integration by parts, noting that $|\sigma|\le C$, $|\partial_t\sigma|\le C$, $\partial_i\varphi = \lambda\varphi\partial_i\zeta$ and $\partial_t\varphi = 2(\alpha_1-1)\lambda\beta\varphi t^{1-2\alpha_1}$, we have

$$\begin{aligned}
|I_1| =&\frac{1}{2}\left|\int_{\partial D} s\lambda^2\varphi\sigma w^2\nu_{d+1}\mathrm{d}S\mathrm{d}t - \int_D s\lambda^2(\partial_t\varphi)\sigma w^2\mathrm{d}x\mathrm{d}t - \int_D s\lambda^2\varphi(\partial_t\sigma)w^2\mathrm{d}x\mathrm{d}t\right|\\
\le& C\int_D s\lambda^3\varphi w^2\mathrm{d}x\mathrm{d}t + C\int_{\partial D} s\lambda^2\varphi w^2\mathrm{d}S\mathrm{d}t.
\end{aligned}\tag{3.12}$$

$$\begin{aligned}
|I_2| =&\left| -2\int_D s^2\lambda^4\varphi^2\sigma^2 w^2\mathrm{d}x\mathrm{d}t - \int_D s^2\lambda^3\varphi^2\sigma\sum_{i,j=1}^d \partial_j(a_{ij}(\partial_i\zeta))w^2\mathrm{d}x\mathrm{d}t\right.\\
&\left.+\int_{\partial D} s^2\lambda^3\varphi^2\sigma\sum_{i,j=1}^d a_{ij}(\partial_i\zeta)w^2\nu_j\mathrm{d}S\mathrm{d}t\right|\\
\le& C\int_D s^2\lambda^4\varphi^2 w^2\mathrm{d}x\mathrm{d}t + C\int_{\partial D} s^2\lambda^3\varphi^2 w^2\mathrm{d}S\mathrm{d}t.
\end{aligned}\tag{3.13}$$

$$\begin{aligned}
I_3 =& \int_D s\lambda^2\varphi\sigma\sum_{i,j=1}^d a_{ij}(\partial_i w)\partial_j w\mathrm{d}x\mathrm{d}t + \int_D s\lambda^2\sum_{i,j=1}^d \partial_i(\varphi\sigma a_{ij})w\partial_j w\mathrm{d}x\mathrm{d}t\\
&-\int_{\partial D} s\lambda^2\varphi\sigma\sum_{i,j=1}^d a_{ij}w(\partial_j w)\nu_i\mathrm{d}S\mathrm{d}t\\
\ge&\int_D s\lambda^2\varphi\sigma\sum_{i,j=1}^d a_{ij}(\partial_i w)\partial_j w\mathrm{d}x\mathrm{d}t - C\int_D s\lambda^3\varphi|\nabla w||w|\mathrm{d}x\mathrm{d}t - C\int_{\partial D} s\lambda^2\varphi|\nabla w||w|\mathrm{d}S\mathrm{d}t.
\end{aligned}$$

By

$$s\lambda^3\varphi|\nabla w||w| = (s\lambda^2\varphi|w|)(\lambda|\nabla w|) \le \frac{1}{2}s^2\lambda^4\varphi^2 w^2 + \frac{1}{2}\lambda^2|\nabla w|^2,$$

and

$$s\lambda^2\varphi|\nabla w||w| = (s\lambda^{\frac{3}{2}}\varphi|w|)(\lambda^{\frac{1}{2}}|\nabla w|) \le \frac{1}{2}s^2\lambda^3\varphi^2 w^2 + \frac{1}{2}\lambda|\nabla w|^2,$$

we have

$$\begin{aligned} I_3 \ge &\int_D s\lambda^2\varphi\sigma\sum_{i,j=1}^d a_{ij}(\partial_i w)\partial_j w \mathrm{d}x\mathrm{d}t - C\int_D s^2\lambda^4\varphi^2 w^2 \mathrm{d}x\mathrm{d}t - C\int_D \lambda^2|\nabla w|^2\mathrm{d}x\mathrm{d}t \\ &- C\int_{\partial D}\lambda|\nabla w|^2\mathrm{d}S\mathrm{d}t - C\int_{\partial D} s^2\lambda^3\varphi^2 w^2\mathrm{d}S\mathrm{d}t. \end{aligned} \tag{3.14}$$

From the definition of A_1, we see that

$$|I_5| = \left|\int_D s\lambda^2\varphi\sigma w A_1 w\mathrm{d}x\mathrm{d}t\right| \le C\int_D (s^2\lambda^4\varphi^2 + s^2\lambda^3\varphi^2)w^2\mathrm{d}x\mathrm{d}t \le C\int_D s^2\lambda^4\varphi^2 w^2\mathrm{d}x\mathrm{d}t. \tag{3.15}$$

Therefore

$$\begin{aligned} &\int_D s\lambda^2\varphi\sigma\sum_{i,j=1}^d a_{ij}(\partial_i w)\partial_j w\mathrm{d}x\mathrm{d}t - \int_D s^3\lambda^4\varphi^3\sigma^2 w^2\mathrm{d}x\mathrm{d}t \\ \le &\int_D |s\lambda^2\varphi\sigma w\mathrm{e}^{s\varphi}L_0 u|\mathrm{d}x\mathrm{d}t + C\int_D \lambda^2|\nabla w|^2\mathrm{d}x\mathrm{d}t + C\int_D s^2\lambda^4\varphi^2 w^2\mathrm{d}x\mathrm{d}t \\ &+ \int_{\partial D}(s\lambda\varphi|\nabla w|^2 + s^3\lambda^3\varphi^2 w^2)\mathrm{d}S\mathrm{d}t. \end{aligned}$$

Furthermore, from

$$|s\lambda^2\varphi\sigma w\mathrm{e}^{s\varphi}L_0 u| \le \frac{1}{2}|L_0 u|^2\mathrm{e}^{2s\varphi} + \frac{1}{2}s^2\lambda^4\varphi^2\sigma^2 w^2,$$

we have

$$\begin{aligned} &\int_D s\lambda^2\varphi\sigma^2\sum_{i,j=1}^d a_{ij}(\partial_i w)\partial_j w\mathrm{d}x\mathrm{d}t - \int_D s^3\lambda^4\varphi^3\sigma^2 w^2\mathrm{d}x\mathrm{d}t \\ \le &\int_D |L_0 u|^2\mathrm{e}^{2s\varphi}\mathrm{d}x\mathrm{d}t + C\int_D \lambda^2|\nabla w|^2\mathrm{d}x\mathrm{d}t + C\int_D s^2\lambda^4\varphi^2 w^2\mathrm{d}x\mathrm{d}t \\ &+ \int_{\partial D}(s\lambda\varphi|\nabla w|^2 + s^3\lambda^3\varphi^2 w^2)\mathrm{d}S\mathrm{d}t. \end{aligned} \tag{3.16}$$

Finally, we consider $(3.16)\times 2 + (3.11)$. By the ellipticity of a_{ij} and $\sigma_0 := \inf_{(x,t)\in Q}\sigma(x,t) > 0$, we obtain

$$\begin{aligned} &\int_D s^3\lambda^4\varphi^3\sigma_0^2 w^2 dxdt + \int_D (\sigma_0^2 - C\epsilon)s\lambda^2\varphi|\nabla w|^2\mathrm{d}x\mathrm{d}t + \left(\epsilon - \frac{C}{\lambda}\right)\int_D \frac{1}{s\varphi}|\partial_t w|^2\mathrm{d}x\mathrm{d}t \\ \le &C\int_D |L_0 u|^2 e^{2s\varphi}\mathrm{d}x\mathrm{d}t + C\int_D (s^3\lambda^3\varphi^3 + s^2\lambda^4\varphi^2)w^2\mathrm{d}x\mathrm{d}t + C\int_D (s\lambda\varphi + \lambda^2)|\nabla w|^2\mathrm{d}x\mathrm{d}t \\ &+ \int_{\partial D}(s\lambda\varphi|\nabla w|^2 + s^3\lambda^3\varphi^2 w^2)\mathrm{d}S\mathrm{d}t + \int_{\partial D\setminus\Sigma_0}|\partial_t w|^2\mathrm{d}S\mathrm{d}t. \end{aligned}$$

Thus choosing $\epsilon > 0$ small, and choosing λ and then s large, we can absorb terms suitably to obtain

$$\int_D \left\{\frac{1}{s\varphi}|\partial_t w|^2 + s\lambda^2\varphi|\nabla w|^2 + s^3\lambda^4\varphi^3 w^2\right\}\mathrm{d}x\mathrm{d}t$$

$$\leq C\int_D |L_0u|^2 e^{2s\varphi}\mathrm{d}x\mathrm{d}t + \int_{\partial D}(s\lambda\varphi|\nabla w|^2 + s^3\lambda^3\varphi^2 w^2)\mathrm{d}S\mathrm{d}t + \int_{\partial D\setminus\Sigma_0}|\partial_t w|^2\mathrm{d}S\mathrm{d}t.$$

Noting $w = ue^{s\varphi}$, we have

$$\int_D\left\{\frac{1}{s\varphi}|\partial_t u|^2 + s\lambda^2\varphi|\nabla u|^2 + s^3\lambda^4\varphi^3u^2\right\}e^{2s\varphi}\mathrm{d}x\mathrm{d}t$$
$$\leq C\int_D |L_0u|^2e^{2s\varphi}\mathrm{d}x\mathrm{d}t + e^{C(\lambda)s}\int_{\partial D}(|\nabla u|^2 + u^2)\mathrm{d}S\mathrm{d}t + e^{C(\lambda)s}\int_{\partial D\setminus\Sigma_0}|\partial_t u|^2\mathrm{d}S\mathrm{d}t.$$

□

3.3 Proof of the main theorem

The key idea for proving Theorem 3.1 is to regard the fractional derivatives as the perturbation for the first order time-derivative and use the Carleman estimate for the parabolic equation derived in Section 3.2. The first problem which we have to overcome is to evaluate the fractional derivative by the first order time-derivative under some suitable norm. Namely, the following lemma holds.

Lemma 3.2 *Let $T > 0$ and $0 < \alpha \leq \alpha_1 < \frac{1}{2}$ be given constants, then for any sub-domain D of $Q := \Omega \times (0, T)$, the following inequality*

$$\int_D |\partial_t^\alpha u|^2 e^{2s\varphi}\mathrm{d}x\mathrm{d}t \leq C\int_D \frac{1}{s\lambda\varphi}|\partial_t u|^2 e^{2s\varphi}\mathrm{d}x\mathrm{d}t \tag{3.17}$$

holds true for all $u \in H^{2,1}(Q)$, where $\varphi = e^{\lambda\psi}$ with $\psi(x,t) = \zeta(x) - \beta t^{2-2\alpha_1}$.

Proof. From the definition of the Caputo derivative, it follows that

$$\int_D |\partial_t^\alpha u|^2 e^{2s\varphi}\mathrm{d}x\mathrm{d}t = \int_D\left|\frac{1}{\Gamma(1-\alpha)}\int_0^t (t-r)^{-\alpha}\partial_r u(x,r)\mathrm{d}r\right|^2 e^{2s\varphi}\mathrm{d}x\mathrm{d}t.$$

We note that $\varphi(x,t) \geq c_0$, where $c_0 > 0$ is a constant. We have

$$\partial_t\psi = \beta(2\alpha_1 - 2)t^{1-2\alpha_1}, \quad \partial_t\varphi = \lambda(\partial_t\psi)\varphi = (2\alpha_1 - 2)\beta\lambda\varphi t^{1-2\alpha_1}.$$

Hence

$$t^{1-2\alpha_1}e^{2s\varphi} = -\frac{1}{4\beta s\lambda\varphi(1-\alpha_1)}\partial_t(e^{2s\varphi}).$$

By the Cauchy-Schwarz inequality and (3), we have

$$\int_0^T\left|\int_0^t (t-r)^{-\alpha}\partial_r u(r)\mathrm{d}r\right|^2 e^{2s\varphi(x,t)}\mathrm{d}t \leq \int_0^T\left(\int_0^t (t-r)^{-2\alpha}\mathrm{d}r\right)\left(\int_0^t |\partial_r u(r)|^2\mathrm{d}r\right)e^{2s\varphi(x,t)}\mathrm{d}t$$
$$=\frac{1}{1-2\alpha}\int_0^T t^{1-2\alpha}\left(\int_0^t |\partial_r u(r)|^2\mathrm{d}r\right)e^{2s\varphi(x,t)}\mathrm{d}t$$

Moreover, since $0 < \alpha < \alpha_1$, we see that

$$\int_0^T\left|\int_0^t (t-r)^{-\alpha}\partial_r u(r)\mathrm{d}r\right|^2 e^{2s\varphi(x,t)}\mathrm{d}t \leq \frac{T^{2(\alpha_1-\alpha)}}{1-2\alpha}\int_0^T t^{1-2\alpha_1}\left(\int_0^t |\partial_r u(r)|^2\mathrm{d}r\right)e^{2s\varphi(x,t)}\mathrm{d}t$$
$$=\frac{T^{2(\alpha_1-\alpha)}}{(1-2\alpha)(1-\alpha_1)}\int_0^T \frac{-1}{4\beta s\lambda\varphi}\partial_t(e^{2s\varphi(x,t)})\left(\int_0^t |\partial_r u(r)|^2\mathrm{d}r\right)\mathrm{d}t.$$

Integration by parts yields

$$\int_0^T t^{1-2\alpha_1}\left(\int_0^t |\partial_r u(r)|^2 dr\right)e^{2s\varphi(x,t)}\mathrm{d}t$$

$$
\begin{aligned}
=&\frac{1}{1-\alpha_1}\left(\frac{-1}{4\beta s\lambda\varphi}\int_0^t|\partial_r u|^2\mathrm{d}r\right)\mathrm{e}^{2s\varphi(x,t)}\Big|_{t=0}^{t=T}\\
&+\frac{1}{1-\alpha_1}\int_0^T\frac{1}{4\beta s\lambda\varphi}|\partial_t u|^2\mathrm{e}^{2s\varphi}\mathrm{d}t+\int_0^T\frac{t^{1-2\alpha_1}}{2s\varphi}\left(\int_0^t|\partial_r u(r)|^2\mathrm{d}r\right)\mathrm{e}^{2s\varphi}\mathrm{d}t\\
\le&\frac{1}{1-\alpha_1}\int_0^T\frac{1}{4\beta s\lambda\varphi}|\partial_t u|^2\mathrm{e}^{2s\varphi}\mathrm{d}t+\int_0^T\frac{t^{1-2\alpha_1}}{2s\varphi}\left(\int_0^t|\partial_r u(r)^2dr\right)\mathrm{e}^{2s\varphi}\mathrm{d}t.
\end{aligned}
$$

The last term on the right-hand side can be absorbed into the left-hand side by choosing $s>0$ large and we have

$$
\int_0^T t^{1-2\alpha_1}\left(\int_0^t|\partial_r u(r)|^2\mathrm{d}r\right)\mathrm{e}^{2s\varphi(x,t)}\mathrm{d}t\le C\int_0^T\frac{1}{s\lambda\varphi}|\partial_t u(t)|^2\mathrm{e}^{2s\varphi}\mathrm{d}t.
$$

Thus

$$
\int_Q|\partial_t^\alpha u|^2\mathrm{e}^{2s\varphi}\mathrm{d}x\mathrm{d}t\le C\int_Q\frac{1}{s\lambda\varphi}|\partial_t u|^2\mathrm{e}^{2s\varphi}\mathrm{d}x\mathrm{d}t. \tag{3.18}
$$

Firstly, in the case of $u\in H^{2,1}(Q)$ satisfying $\operatorname{supp}u\subset D$, (3.18) implies that the estimate (3.17) holds true. Next, in view of the fact that $C_c^\infty(D)$ is dense in $H^{2,1}(D)$, by an approximation argument, we can show that the estimate (3.17) is valid for any $u\in H^{2,1}(Q)$. □

Before giving the proof of Theorem 3.1, we introduce some notations.

For arbitrary given domain Ω_0 such that $\overline{\Omega_0}\subset\Omega$, similar to Theorem 5.1 in [65], we will choose a suitable weight function $\psi(x,t):=\zeta(x)-\beta t^{2-2\alpha_1}$. For this, we first choose a bounded domain Ω_1 with smooth boundary such that

$$
\Omega\subsetneqq\Omega_1,\quad\overline{\Gamma}=\overline{\partial\Omega\cap\Omega_1},\quad\partial\Omega\setminus\Gamma\subset\partial\Omega_1.
$$

We then apply Lemma 4.1 in [65] to obtain $d\in C^2(\overline{\Omega_1})$ satisfying

$$
\zeta(x)>0,\ x\in\Omega_1,\quad\zeta(x)=0,\ x\in\partial\Omega_1,\quad|\nabla\zeta(x)|>0,\ x\in\overline{\Omega}.
$$

Then we can choose $\beta>0$ and $\varepsilon>0$ such that

$$
\beta\varepsilon^{2-2\alpha_1}<\|\zeta\|_{C(\overline{\Omega_1})}<2^{2-2\alpha_1}\beta\varepsilon^{2-2\alpha_1}. \tag{3.19}
$$

Moreover, since $\overline{\Omega_0}\subset\Omega_1$, we can choose a sufficiently large $N>1$ such that

$$
\Omega_0\subset\overline{\Omega}\cap\{x\in\Omega_1;\ \zeta(x)>\frac{4}{N}\|\zeta\|_{C(\overline{\Omega_1})}\}. \tag{3.20}
$$

We set $\mu_k=\exp\{\lambda(\frac{k}{N}\|\zeta\|_{C(\overline{\Omega_1})}-\frac{\beta\varepsilon^{2-2\alpha_1}}{N})\}$, $k=1,2,3,4$. Then we can verify from (3.19) and (3.20) that

$$
\Omega_0\times(0,\frac{\varepsilon}{M})\subset D_3\subset D_1\subset\overline{\Omega}\times(0,2\varepsilon), \tag{3.21}
$$

where $M:=N^{\frac{1}{2-2\alpha_1}}$, $D_j:=\{(x,t);\ x\in\overline{\Omega},\ t>0,\ \varphi(x,t)>\mu_j\}$, $j=1,3$, and

$$
\partial D_1\subset\Sigma_0\cup\Sigma_1\cup\Sigma_2, \tag{3.22}
$$

where $\Sigma_0=\{(x,0);\ x\in\overline{\Omega}\}$, $\Sigma_1\subset\Gamma\times(0,T)$ and $\Sigma_2=\{(x,t);\ x\in\Omega,\ t>0,\varphi(x,t)=\mu_1\}$.

Now we are ready to give the proof of our main theorem.

Proof of Theorem 3.1. We start from the Cauchy problem

$$
\begin{cases}
u(x,t)=g_0(x,t) & \text{on }\Gamma\times(0,T],\\
\partial_{\nu_A}u(x,t)=g_1(x,t) & \text{on }\Gamma\times(0,T]
\end{cases}
$$

for the equation (3.2).

Henceforth $C > 0$ denotes generic constants depending on λ, but independent of s and the choice of g_0, g_1, u. For it, we need a cut-off function because we have no data $\partial_{\nu_A} u$ on $\partial D \setminus \Gamma \times (0, T)$. Let $\chi \in C^\infty(\mathbb{R}^{n+1})$ such that $0 \le \chi \le 1$ and

$$\chi(x,t) = \begin{cases} 1, & \varphi(x,t) > \mu_3, \\ 0, & \varphi(x,t) < \mu_2. \end{cases} \tag{3.23}$$

Setting $v := \chi u$, $\widetilde{L} := L - \sum_{j=1}^{\ell} q_j \partial_t^{\alpha_j}$, and then using Leibniz's formula for the differential of the product we have

$$\widetilde{L} v = \chi L u - \chi \sum_{j=1}^{\ell} q_j \partial_t^{\alpha_j} u + A_1 u = \chi f - \chi \sum_{j=1}^{\ell} q_j \partial_t^{\alpha_j} u + A_1 u. \tag{3.24}$$

Here the last term $A_1 u$ involves only the linear combination of $(\partial_t \chi) u$, $(\partial_i \partial_j \chi) u$, $(\partial_i \chi)(\partial_j u)$ and $(\partial_i \chi) u$, $i, j = 1, \cdots, n$.

By (3.22) and (3.23), we see that $v = |\nabla v| = 0$ on Σ_2. Hence using the Carleman estimate in Lemma 3.1, from $D_3 \subset D_1$ by an argument similar to Theorem 3.2 in [65] in D_1 to (3.24), we find

$$\begin{aligned} &\int_{D_3} \left\{ \frac{1}{s} |\partial_t v|^2 + s\lambda^2 \varphi |\nabla v|^2 + s^3 \lambda^4 \varphi^3 v^2 \right\} \mathrm{e}^{2s\varphi} \mathrm{d}x\mathrm{d}t \\ \le& \int_Q f^2 \mathrm{e}^{2s\varphi} \mathrm{d}x\mathrm{d}t + C \int_{D_1} \sum_{j=1}^{\ell} |\partial_t^{\alpha_j} u|^2 \mathrm{e}^{2s\varphi} \mathrm{d}x\mathrm{d}t + C \int_{D_1} |A_1 u|^2 \mathrm{e}^{2s\varphi} \mathrm{d}x\mathrm{d}t \\ &+ \mathrm{e}^{C(\lambda)s} \int_{\Sigma_0 \cup (\Gamma \times (0,T))} (|\nabla v|^2 + v^2) \mathrm{d}S\mathrm{d}t + \mathrm{e}^{C(\lambda)s} \int_{\Gamma \times (0,T)} |\partial_t v|^2 \mathrm{d}S\mathrm{d}t. \end{aligned} \tag{3.25}$$

for all $s \ge s_0$ and $\lambda \ge \lambda_0$.

By (3.23), $A_1 u$ does not vanish only if $\mu_2 \le \varphi(x,t) \le \mu_3$ and so

$$\int_{D_1} |A_1 u|^2 \mathrm{e}^{2s\varphi} \mathrm{d}x\mathrm{d}t \le C \mathrm{e}^{2s\mu_3} \|u\|^2_{H^{1,0}(Q)}.$$

Moreover, from (3.21) and Lemma 3.2, by taking λ large enough, we conclude that the term $\int_{D_3} \sum_{j=1}^{\ell} |\partial_t^{\alpha_j} u|^2 \mathrm{e}^{2s\varphi} \mathrm{d}x\mathrm{d}t$ can be absorbed by the left-hand side of (3.25), which implies

$$\begin{aligned} &\int_{D_3} \left\{ \frac{1}{s} |\partial_t u|^2 + s\lambda^2 \varphi |\nabla u|^2 + s^3 \lambda^4 \varphi^3 u^2 \right\} \mathrm{e}^{2s\varphi} \mathrm{d}x\mathrm{d}t \\ \le& C \mathrm{e}^{C(\lambda)s} \|f\|_{L^2(Q)} + C \mathrm{e}^{2s\mu_3} \|u\|^2_{H^{1,0}(Q)} + C \sum_{j=1}^{\ell} \int_{D_1 \setminus D_3} |\partial_t^{\alpha_j} u|^2 \mathrm{e}^{2s\varphi} \mathrm{d}x\mathrm{d}t \\ &+ \mathrm{e}^{C(\lambda)s} \int_{\Sigma_0 \cup (\Gamma \times (0,T))} (|\nabla v|^2 + v^2) \mathrm{d}S\mathrm{d}t + \mathrm{e}^{C(\lambda)s} \int_{\Gamma \times (0,T)} |\partial_t v|^2 \mathrm{d}S\mathrm{d}t. \end{aligned}$$

By (3.20), we can directly verify that $\varphi(x,t) \le \mu_3$ in $D_1 \setminus D_3$, and if $(x,t) \in \Omega_0 \times (0, \frac{\varepsilon}{N})$, then $\varphi(x,t) > \mu_4$. Then combined with (3.21) and (3.22), again from Lemma 3.2, we have

$$\begin{aligned} &\mathrm{e}^{2s\mu_4} \int_0^{\frac{\varepsilon}{N}} \int_{\Omega_0} \left\{ \frac{1}{s} |\partial_t u|^2 + s |\nabla u|^2 + s^3 u^2 \right\} \mathrm{d}x\mathrm{d}t \\ \le& C \mathrm{e}^{C(\lambda)s} \|f\|_{L^2(Q)} + C \mathrm{e}^{2s\mu_3} \|u\|^2_{H^{1,0}(Q)} + C \mathrm{e}^{2s\mu_3} \int_{D_1 \setminus D_3} |\partial_t u|^2 \mathrm{d}x\mathrm{d}t \\ &+ \mathrm{e}^{Cs} \int_\Omega (|\nabla v(x,0)|^2 + v(x,0)^2) \mathrm{d}x + \mathrm{e}^{Cs} \int_{\Gamma \times (0,T)} (|\partial_t v|^2 + |\nabla v|^2 + v^2) \mathrm{d}S\mathrm{d}t. \end{aligned}$$

for $s \geq s_0$. Then dividing both sides by $e^{2s\mu_4}$, since

$$se^{-2s(\mu_4-\mu_3)} \leq Ce^{-s(\mu_4-\mu_3)} \leq Ce^{-Cs},$$

by replacing $\frac{\varepsilon}{N}$ by ε and C by Ce^{Cs_0}, we have

$$\|u\|^2_{H^{1,1}(\Omega_0\times(0,\varepsilon))} \leq Ce^{-(\mu_4-\mu_3)s}\|u\|^2_{H^{1,1}(Q)} + Ce^{Cs}F^2 \tag{3.26}$$

for all $s > 0$ and $\varepsilon > 0$ and $u \in H^{2,1}(Q)$.

First, if $F = 0$, letting $s \to \infty$, we conclude that $u = 0$ in $\Omega_0 \times (0, \varepsilon)$, so that the conclusion of Theorem 3.1 holds true. Next let $F \neq 0$. First let $F \geq \|u\|_{H^{1,1}(Q)}$. Then (3.26) implies

$$\|u\|_{H^{1,1}(\Omega_0\times(0,\varepsilon))} \leq Ce^{Cs}F, \quad s > 0,$$

which already proves the theorem. Second let $F \leq \|u\|_{H^{1,1}(Q)}$, we choose $s > 0$ minimizing the right-hand side of (3.26), that is

$$e^{-s(\mu_4-\mu_3)}\|u\|^2_{H^{1,1}(Q)} = e^{Cs}F^2.$$

By $F \neq 0$, we can choose

$$s = \frac{2}{C + \mu_4 - \mu_3} \log \frac{\|u\|_{H^{1,1}(Q)}}{F} > 0.$$

Then (3.26) gives

$$\|u\|_{H^{1,1}(\Omega_0\times(0,\varepsilon))} \leq 2C\|u\|^{\frac{C}{C+\mu_4-\mu_3}}_{H^{1,1}(Q)} F^{\frac{\mu_4-\mu_3}{C+\mu_4-\mu_3}}.$$

The proof of Theorem (3.1) is completed.

□

3.4 Conclusions and open problems

We firstly established a Carleman estimate for the parabolic equation with a new weight function. Theorem 3.1 was then proved by regarding the fractional order terms as perturbation and the use of the Carleman estimate for the parabolic equations. All the above arguments are valid only for the case of order $\alpha_1 \in (0, \frac{1}{2})$. Moreover, due to our choice of the weight function, we do not know whether the estimate is valid without the initial value.

On the other hand, that regarding the fractional order terms as perturbation and the choice of the new weight function are not suitable for the study of the inverse problems. As is well known, for dealing with the inverse problems, the Carleman type estimate derived by $\psi := \zeta(x) - \beta(t - t_0)^2$ ($t_0 \in (0, T)$) should be better according to the series of theories in [65]. However, the weight function used in [65] cannot work for our FADEs. The inverse problems for the equation (3.2) remain open.

Chapter 4

Distributed order time-fractional diffusion equations

This chapter is mainly devoted to the investigation of some important properties of solutions to initial-boundary value problems for time-fractional diffusion equations of distributed orders in bounded multi-dimensional domains. Using eigenfunction expansion and carrying out the inversion Laplace transforms on several integral loops, we prove the uniqueness and continuous dependency of the solutions on initial values and source terms as well as the analyticity. Furthermore, under some suitable assumptions on the weight function μ, by a Laplace transform argument, it turns out that the solutions decay logarithmically as $t \to \infty$. As $t \to 0$, the decay rate of the solutions is dominated by the term $(t \log 1/t)^{-1}$. Finally, as an application of the analyticity, we give a uniqueness result for a related inverse problem on the determination of the weight function μ in the distributed derivative from one interior point observation.

Keywords: distributed order time-fractional diffusion equation, initial-boundary value problem, analyticity, Laplace transform, eigenfunction expansion, asymptotic behavior, inverse problem

4.1 Introduction

Within the last few decades, some recent publications by physicists (see [12], [49], [58] and the references therein) are devoted to some ultraslow diffusion where the mean square displacement is of logarithmic growth. The usual way to model such processes is to employ the so called diffusion equation of distributed orders with a continuous weight function μ in $[0, 1]$

$$\begin{cases} \mathbb{D}_t^{(\mu)} u(x,t) = -\mathcal{A}u(x,t) + F(x,t), & (x,t) \in Q, \\ u(x,0) = a(x), & x \in \Omega, \\ u(x,t) = 0, & (x,t) \in \Sigma, \end{cases} \tag{4.1}$$

where $T > 0$ is a fixed constant, and Ω is an open bounded domain in $\mathbb{R}^d$ with a smooth boundary $\partial\Omega$, which is defined e.g., by some C^2 functional relations. $a \in L^2(\Omega)$ and $\mathcal{A}$ is a symmetric uniformly elliptic operator defined in Chapter 1.

From the mathematical viewpoint, the forward problems for such equations were investigated e.g. in [29], [30], [33] [47] and the references therein. In [29, 30], the fundamental solutions to the Cauchy problems for both the ordinary and the partial fractional differential equations with distributed order derivatives with continuously differential weight functions have been derived and investigated in detail. [33] applied an appropriate maximum principle to show the uniqueness results for the initial-boundary value problems for the diffusion equation of distributed orders. The existence of the solution is illustrated by constructing a formal solution using the Fourier method of variables separation, but no proofs for the convergence of the series (i.e. the obtained formal solutions are in fact solutions) were given in [33]. [47] provided explicit strong solutions and stochastic analogues for distributed-order time-fractional diffusion equations, with Dirichlet boundary conditions provided that the weight function in the definition of distributed order derivative is in $C^1(0,1)$.

In this chapter, we are concerned with the well-posedness and the asymptotic behavior of the solution to the initial-boundary value problem (4.1), and we attempt to establish results parallel to that for the multi-term case. On the basis of eigenfunction expansion, by exploiting the inversion Laplace transforms on several integral loops, we give estimates for the solution, which imply the analyticity and continuous dependency of the solutions on initial values and source terms. The employed technique follows the lines of the paper [33], where solutions of the corresponding initial-boundary value problems for the diffusion equations of distributed orders were formally represented in the form of the Fourier series via the Laplace and inversion Laplace transform. Furthermore, for the asymptotic behavior of the solution, we employ the Laplace transform argument to show that the solutions decay logarithmically as $t \to \infty$. As $t \to 0$, the decay rate of the solutions is dominated by the term $(t \log 1/t)^{-1}$ under some suitable assumptions on the weight function μ. Thus the asymptotic behavior of the solutions to the initial-boundary value problem for the time-fractional diffusion equation of distributed orders is shown to be different compared from equations with a finite number of time fractional derivatives which is discussed in Chapter 2. Finally, as an application of the analyticity, we give a uniqueness result for a related inverse problem on the determination of the weight function μ in the distributed derivative from one interior point observation.

The rest of this chapter is organized as follows. The proof of the uniqueness existence and the analyticity of solution to initial-boundary value problem for distributed order time-fractional diffusion equation are given in Section 4.2.1. In Section 4.3, the Strichartz estimates are established. The proofs of the long-time asymptotic formulas are given in Section 4.4. Finally, the last section is devoted to the proof of the uniqueness of the inverse problem of determining the weight function in the distributed derivative.

4.2 Wellposedness

In this section, we first introduce a definition of the weak solution to the initial-boundary value problem for the distributed order time fractional diffusion equations by means of Laplace transform, and then consider the unique existence of the weak solution.

Before we start with the investigation of uniqueness and existence of the solution to the initial-boundary value problem for (4.1), we point out that it is difficult to follow the way used in Definition 2.2 in Chapter 2 to introduce the definition of weak solutions of (4.1) since it is not very easy to find the inverse operator of $\mathbb{D}_t^{(\mu)}$ and a suitable Sobolev space $H^\gamma(0,T;L^2(\Omega))$, $\gamma > 0$ to give the definition. In other words, distributed order fractional diffusion equation (4.1) is completely different from the single-term or multi-term time-fractional diffusion equation. To overcome this gap we follow Definition 1.1 in [32] to give the following definition of weak solutions to the initial-boundary value problem (4.1) by means of Laplace transform. Let $\chi_{(0,T)}$ be the characteristic function of $(0,T)$.

Definition 4.1 (Weak solution) *Let $a \in L^2(\Omega)$, and $F \in L^1(0,T;L^2(\Omega))$. We say that initial-boundary value problem (4.1) admits a weak solution u if there exists $v \in L^\infty_{loc}(\mathbb{R}^+;L^2(\Omega))$ such that:*
1) $v|_Q = u$ and $\inf\{s > 0 : \mathrm{e}^{-st}v \in L^1(\mathbb{R}^+;L^2(\Omega))\} = s_0 < \infty$,
2) for all $s > s_0$ the Laplace transform $\widehat{v}(s) := \mathcal{L}v(s) := \int_0^\infty \mathrm{e}^{-st}v(t,\cdot)\mathrm{d}t$ of v solves

$$\begin{cases} (\mathcal{A} + sw(s))\widehat{v}(s) = \widehat{F}(s) & \text{in } \Omega, \\ \widehat{v}(s) = 0 & \text{on } \partial\Omega, \end{cases}$$

where $w(s) := \int_0^1 s^{\alpha-1}\mu(\alpha)\mathrm{d}\alpha$ and $\widehat{F}(s) := \mathcal{L}(F\chi_{(0,T)})(s) = \int_0^T \mathrm{e}^{-st}F(t,\cdot)\mathrm{d}t$.

We recall the eigensystem $\{\lambda_n, \varphi_n\}$ of the operator $A := \mathcal{A}$, that is, $\{\varphi_n\}_{n=1}^{\infty}$ satisfy $A\varphi_n = \lambda_n\varphi_n$ and $\varphi_n \in D(A)$, and we assume the solution u to (4.1) can be formally represented in the form

$$u(\,\cdot\,,t) = \sum_{n=1}^{\infty} u_n(t)\varphi_n, \quad t > 0,$$

and substituting this representation into (4.1), by orthogonality of $\{\varphi_n\}$, we obtain the following equations for the coefficients u_n:

$$\begin{cases} \mathbb{D}_t^{(\mu)} u_n(t) + \lambda_n u_n(t) = F_n := (F, \varphi_n), \quad t > 0, \\ u_n(0) = a_n := (a, \varphi_n), \end{cases} \qquad n = 1, 2, \cdots. \tag{4.2}$$

Application of the Laplace transform

$$(\mathcal{L}g(t))(s) = \widehat{g}(s) := \int_0^{\infty} g(t)\,\mathrm{e}^{-st}\,\mathrm{d}t$$

to (4.2) along with the known formula

$$(\mathcal{L}\partial_t^{\alpha} g(t))(s) = s^{\alpha}\widehat{g}(s) - s^{\alpha-1}g(0+)$$

leads to an algebraic equation

$$\widehat{u}_n(s)\int_0^1 s^{\alpha}\mu(\alpha)\mathrm{d}\alpha + \lambda_n\widehat{u}_n(s) = a_n\int_0^1 s^{\alpha-1}\mu(\alpha)\mathrm{d}\alpha + \widehat{F_n}, \quad \operatorname{Re} s > s_0$$

for the unknown function $\widehat{u}_n$. This equation can be directly solved:

$$\widehat{u}_n(s) = \frac{w(s)}{sw(s) + \lambda_n}a_n + \frac{1}{sw(s) + \lambda_n}\widehat{F_n}, \quad \operatorname{Re} s > s_0,$$

where the auxiliary function w is determined by the weight function μ as follows:

$$w(s) = \int_0^1 s^{\alpha-1}\mu(\alpha)\mathrm{d}\alpha = \int_0^1 \mathrm{e}^{(\alpha-1)\log s}\mu(\alpha)\mathrm{d}\alpha.$$

By Fourier-Mellin formula (see e.g., [55]), we have

$$u_n(t) = \frac{1}{2\pi\mathrm{i}}\int_{s_0-\mathrm{i}\infty}^{s_0+\mathrm{i}\infty} \widehat{u}_n(s)\mathrm{e}^{st}\mathrm{d}s, \quad n = 1, 2, \cdots,$$

which yields the formal representation of the solution to the initial-boundary value problem (4.1). Moreover, taking the principal value of the logarithmic function on the complex plane cut along the negative part of the real axis, we can extend $w(s)$ to an analytic function on the complex plane with the cut. For any $\lambda_n > 0$, the function $sw(s) + \lambda_n$ has no zeros on the main sheet of the Riemann surface for the logarithmic function. Indeed, we can prove

Lemma 4.1 *Let $\mu \in C[0,1]$ be a non-negative function satisfying the condition $\mu \not\equiv 0$. Then the estimates*

$$|w(s)| \le \frac{C(|s|-1)}{|s|\log|s|},$$

and

$$|sw(s) + \lambda| \ge C > 0$$

hold true for any $s \in \mathbb{C} \setminus \{0\}$, where $C > 0$ is a constant independent of λ and s.

Proof. By the definition of $w(s)$, we have

$$|w(s)| \le \|\mu\|_{C[0,1]} \int_0^1 \mathrm{e}^{(\alpha-1)\log|s|}\mathrm{d}\alpha = \|\mu\|_{C[0,1]} \frac{|s|-1}{|s|\log|s|}.$$

Since $\theta \in (\frac{\pi}{2}, \pi)$ and $\mu \not\equiv 0$, we obtain

$$|sw(s)+\lambda| \ge |\operatorname{Im} sw(s)| = \int_0^1 |s|^\alpha \sin(\alpha\theta)\mu(\alpha)\mathrm{d}\alpha \ge \int_0^1 \sin(\alpha\theta)\mu(\alpha)\mathrm{d}\alpha > 0,$$

which completes the proof of the lemma. □

Therefore, $\widehat{u}_n(s)$ is an analytic function on the main sheet of the Riemann surface of the complex plane cut along the negative real semi-axis.

On the basis of the above argument and the estimates in Lemma 4.1, from the residue theorem (see e.g., [53]), for $t > 0$ we deduce that the inverse Laplace transform of $\widehat{u}_n$ can be represented by

$$u_n(t) = \frac{1}{2\pi \mathrm{i}} \int_{\gamma(\varepsilon,\theta)} \widehat{u}_n(s)\mathrm{e}^{st}\mathrm{d}s, \tag{4.3}$$

here $\gamma(\varepsilon,\theta)$ ($\theta \in (\frac{\pi}{2}, \pi)$ and $\varepsilon > 0$) denotes a contour in $\mathbb{C}$ consisting of the following three parts:

1. $\gamma_-(\varepsilon,\theta)$: $\arg s = -\theta$, $|s| \ge \varepsilon$;
2. $\gamma_c(\varepsilon,\theta)$: $-\theta \le \arg s \le \theta$, $|s| = \varepsilon$;
3. $\gamma_+(\varepsilon,\theta)$: $\arg s = \theta$, $|s| \ge \varepsilon$.

Now we introduce the operators $S_1(t), S_2(t) : L^2(\Omega) \to L^2(\Omega)$ for $t > 0$ which are defined as follows:

$$S_1(t)\psi := \sum_{n=1}^\infty \frac{1}{2\pi \mathrm{i}} \int_{\gamma(\varepsilon,\theta)} \frac{w(s)}{sw(s)+\lambda_n} \mathrm{e}^{st}\mathrm{d}s\, (\psi, \varphi_n)\varphi_n \tag{4.4}$$

and

$$S_2(t)\psi := \sum_{n=1}^\infty \frac{1}{2\pi \mathrm{i}} \int_{\gamma(\varepsilon,\theta)} \frac{1}{sw(s)+\lambda_n} \mathrm{e}^{st}\mathrm{d}s\, (\psi, \varphi_n)\varphi_n. \tag{4.5}$$

Then the formal representation of the solution to the problem (4.1) is given by

$$u(t) = S_1(t)a + \int_0^t S_2(t-\tau)F(\tau)\mathrm{d}\tau, \quad t \in (0,T). \tag{4.6}$$

In the following two subsections, we are to show that the above formal representation (4.6) of the solution is indeed the weak solution to the initial-boundary value problem (4.1) and give the estimate.

4.2.1 Homogeneous equation

In this section, we are mainly devoted to the investigation of the properties, e.g., wellposedness and analyticity of the solutions to the initial-boundary value problems (4.1) provided the source term F vanishes.

Theorem 4.1 *For any fixed $T > 0$. Let $F = 0$, $a \in L^2(\Omega)$ and $\mu \in C[0,1]$ be non-negative and not vanish in $[0,1]$. Then the initial-boundary value problem (4.1) admits a unique solution $u(\cdot,t) \in H_0^1(\Omega) \cap H^2(\Omega)$ such that*

$$\|\partial_t^m u(\cdot,t)\|_{H^2(\Omega)} \le C\|a\|_{L^2(\Omega)} \frac{M^m m!}{t^{m+1}\log(2T/t)}$$

holds true for $t \in (0,T]$ and $m = 0, 1, \cdots$. Moreover, $u(\cdot,t)$ is real analytic in $t \in (0,T]$ and can be analytically extended to $(0,\infty)$.

Here and henceforth, $C > 0$ denotes generic constant which is independent of t, λ_n, a, u and may change line by line.

Before giving a proof of Theorem 4.1, some important auxiliary results are formulated and proved. We first derive estimates of the auxiliary function $w(s)$ which are needed for the proofs of our main results.

Lemma 4.2 *Let $s \in \gamma(\varepsilon, \theta)$ and suppose $\mu(\geq 0) \in C[0,1]$ does not identically zero. Then*

$$|sw(s) + \lambda_n| \geq C\lambda_n$$

is valid for sufficiently large n, where $C > 0$ is a constant which is independent of s, n, λ_n.

Proof. We first introduce two useful parameters K_n and K_n' defined as follows

$$\int_0^1 (K_n)^\alpha \mu(\alpha)\mathrm{d}\alpha = \delta^{-1}\lambda_n, \quad \int_0^1 (K_n')^\alpha \mu(\alpha)\mathrm{d}\alpha = \delta\lambda_n,$$

where $\delta > 0$ is small enough. In the case of $|s| = |re^{\mathrm{i}\theta}| \geq K_n$, we see that

$$|sw(s) + \lambda_n| \geq |\operatorname{Re} sw(s) + \lambda_n| = \int_0^{\frac{\pi}{2\theta}} |s|^\alpha \cos(\theta\alpha)\mu(\alpha)\mathrm{d}\alpha + \lambda_n \geq \lambda_n$$

if $\mu(\alpha) = 0$ for any $\alpha \in [\frac{\pi}{2\theta}, 1]$. On the other hand, if there exists $\delta_1 \in (\frac{\pi}{2\theta}, 1)$ such that $\mu(\delta_1) > 0$, it follows that

$$|sw(s) + \lambda_n| \geq |\operatorname{Re} sw(s) + \lambda_n| \geq \int_{\delta_1}^1 r^\alpha |\cos(\theta\alpha)|\mu(\alpha)\mathrm{d}\alpha - \lambda_n \geq |\cos\theta\delta_1| \int_{\delta_1}^1 r^\alpha \mu(\alpha)\mathrm{d}\alpha - \lambda_n.$$

In view of the fact that

$$\int_0^{\delta_1} r^\alpha \mu(\alpha)\mathrm{d}\alpha \leq \frac{1}{2}\int_{\delta_1}^1 r^\alpha \mu(\alpha)\mathrm{d}\alpha \tag{4.7}$$

for sufficiently large $r > 0$, hence that

$$|sw(s) + \lambda_n| \geq \frac{|\cos(\theta\delta_1)|}{2}\int_0^1 r^\alpha \mu(\alpha)\mathrm{d}\alpha - \lambda_n \geq \frac{|\cos(\theta\delta_1)|}{2\delta}\lambda_n - \lambda_n \geq C\lambda_n, \quad r \geq K_n, \ n \geq N,$$

where $N \in \mathbb{N}$ is sufficiently large, and $\delta > 0$ is small enough. In the case of $|s| \in [K_n', K_n]$, similar to the above argument, we see that

$$|sw(s) + \lambda_n| \geq C\lambda_n$$

if $\mu(\alpha) = 0$ for any $\alpha \in [\frac{\pi}{2\theta}, 1]$. On the other hand, if there exists $\delta_1 \in (\frac{\pi}{2\theta}, 1)$ such that $\mu(\delta_1) > 0$, we conclude from (4.7) that

$$|sw(s) + \lambda_n| \geq |\operatorname{Im} sw(s)| \geq \int_{\delta_1}^1 |s|^\alpha \sin(\theta\alpha)\mu(\alpha)\mathrm{d}\alpha \geq \sin\theta \int_{\delta_1}^1 r^\alpha \mu(\alpha)\mathrm{d}\alpha \geq \frac{\sin\theta}{2}\int_0^1 r^\alpha \mu(\alpha)\mathrm{d}\alpha$$

for sufficiently large $r > 0$. Thus noting the definition of K_n', we see that for $|s| \in [K_n', K_n]$, the estimates

$$|sw(s) + \lambda_n| \geq \frac{\sin\theta}{2}\int_0^1 (K_n')^\alpha \mu(\alpha)\mathrm{d}\alpha = \frac{\delta}{2}\lambda_n \sin\theta, \quad n \geq N$$

holds true for sufficiently large $N \in \mathbb{N}$. In the case of $|s| \leq K_n'$, we find

$$|sw(s) + \lambda_n| \geq \lambda_n - |sw(s)| \geq \lambda_n - \int_0^1 r^\alpha \mu(\alpha)\mathrm{d}\alpha \geq \lambda_n - \int_0^1 (K_n')^\alpha \mu(\alpha)\mathrm{d}\alpha \geq (1-\delta)\lambda_n$$

for sufficiently small $\delta > 0$. Combining all the estimates above, we finally obtain

$$|sw(s) + \lambda_n| \geq C\lambda_n, \quad n \geq N,$$

where $s \in \gamma(\varepsilon, \theta)$ and $N \in \mathbb{N}$ is sufficiently large. □

Lemma 4.3 *Under the assumptions in Theorem 4.1. Then the function $u_n(t)$ defined by* (4.3) *admits the following estimates*

$$\left|\frac{\mathrm{d}^m u_n(t)}{\mathrm{d}t^m}\right| \le \frac{C|a_n|}{\lambda_n}\frac{M^m}{t^{m+1}\log(2T/t)}, \quad m=0,1,\cdots \tag{4.8}$$

for $t\in(0,T]$ and $n=1,2,\cdots$. Moreover, $|u_n(t)|\le C|a_n|$ holds true for $t\in[0,T]$.

Proof. Recalling the representation (4.3), for $m=0,1,\cdots$, we see that

$$\frac{\mathrm{d}^m u_n(t)}{\mathrm{d}t^m} = \frac{a_n}{2\pi\mathrm{i}}\int_{\gamma(\varepsilon,\theta)}\frac{w(s)s^m}{sw(s)+\lambda_n}\mathrm{e}^{st}\mathrm{d}s.$$

We first evaluate the right-hand side of the above equation for the case where $n>N$ with sufficiently large N. Letting $\varepsilon>1$ in the definition of the contour $\gamma(\varepsilon,\theta)$, by Lemma 4.2, a direct calculation yields

$$\begin{aligned}\left|\frac{\mathrm{d}^m u_n(t)}{\mathrm{d}t^m}\right| &\le C|a_n|\int_\varepsilon^\infty \frac{|w(s)s^m\mathrm{e}^{st}|}{\lambda_n}\Big|_{s=r\mathrm{e}^{\mathrm{i}\theta}}\mathrm{d}r + C|a_n|\int_{-\theta}^{\theta}\frac{(\varepsilon-1)\varepsilon^m}{\lambda_n\varepsilon\log\varepsilon}\varepsilon\mathrm{d}\rho\\ &=: I_{m,n,1}(t)+I_{m,n,2}(t), \quad n\ge N,\end{aligned}$$

where $t\in(0,T]$, $m=0,1,\cdots$ and N is sufficiently large. We start with the estimate for $I_{m,n,2}$. Noting that $\frac{(\varepsilon-1)}{\log\varepsilon}\le C$ for fixed $\varepsilon>1$, then the following estimate

$$I_{m,n,2}(t)\le\frac{C|a_n|}{\lambda_n}\varepsilon^m$$

holds true for $t\in(0,T]$, $m=0,1,\cdots$ and $n>N$. From the definition of $w(s)$, a direct calculation implies

$$I_{n,1}(t)\le\frac{C|a_n|}{\lambda_n}\int_\varepsilon^\infty\int_0^1 r^{m+\alpha-1}\mu(\alpha)\mathrm{d}\alpha\mathrm{e}^{-rt|\cos\theta|}\mathrm{d}r.$$

From the Fubini theorem and noting that the Laplace transform of $r^{m+\alpha-1}$ is $\frac{\Gamma(m+\alpha)}{s^{m+\alpha}}$, it follows that

$$I_{m,n,1}(t)\le\frac{C|a_n|}{\lambda_n}\int_0^1\int_\varepsilon^\infty r^{m+\alpha-1}\mathrm{e}^{-rt|\cos\theta|}\mathrm{d}r\mu(\alpha)\mathrm{d}\alpha\le\frac{C|a_n|}{\lambda_n}\int_0^1\frac{\Gamma(m+\alpha)}{(t|\cos\theta|)^{m+\alpha}}\mathrm{d}\alpha.$$

Therefore

$$I_{m,n,1}(t)\le\frac{C|a_n|m!}{|\cos\theta|^{m+1}\lambda_n t^m}\int_0^1 t^{-\alpha}\mathrm{d}\alpha=\frac{C|a_n|m!}{|\cos\theta|^{m+1}\lambda_n}\frac{1-\frac{1}{t}}{\log t}\le\frac{C|a_n|}{\lambda_n}\frac{M^m m!}{t^{m+1}\log(2T/t)}, \quad m=0,1,\cdots,$$

where $t\in(0,T]$ and $N\in\mathbb{N}$ is sufficiently large. Consequently, we have

$$\left|\frac{\mathrm{d}^m u_n(t)}{\mathrm{d}t^m}\right|\le\frac{C|a_n|}{\lambda_n}\frac{M^m m!}{t^{m+1}\log(2T/t)}, \quad t\in(0,T],\ n\ge N,\ m=0,1,\cdots.$$

In the case of $n\le N$, similarly to the above argument for $n>N$, we see that the inequality

$$|sw(s)+\lambda_n|\ge|\operatorname{Im} sw(s)|\ge\int_{\delta_2}^1|s|^\alpha\sin(\theta\alpha)\mu(\alpha)\mathrm{d}\alpha\ge\varepsilon^{\delta_2}\int_{\delta_2}^1\sin(\theta\alpha)\mu(\alpha)\mathrm{d}\alpha\ge C>0$$

are valid for $|s|\ge\varepsilon$, where $\delta_2\in(0,1)$ such that $\mu(\delta_2)\ne0$, which implies

$$\left|\frac{\mathrm{d}^m u_n(t)}{\mathrm{d}t^m}\right|\le C|a_n|\int_\varepsilon^\infty|w(s)s^m\mathrm{e}^{st}|_{s=r\mathrm{e}^{\mathrm{i}\theta}}\mathrm{d}r+C|a_n|\int_{-\theta}^{\theta}\frac{(\varepsilon-1)\varepsilon^m}{\varepsilon\log\varepsilon}\varepsilon\mathrm{d}\rho.$$

Again using the above argument, we see that

$$\left|\frac{\mathrm{d}^m u_n(t)}{\mathrm{d}t^m}\right| \le C|a_n|\frac{M^m m!}{t^{m+1}\log(2T/t)}, \quad m = 0, 1, \cdots .$$

holds true for $t \in (0, T]$ and $n < N$, which completes the proof of (4.8).

Now let us turn to the estimate $|u_n(t)| \le C|a_n|$ for $t \in [0, T]$ and $n = 1, 2, \cdots$. Recalling the representation (4.3), by the arguments similar to (2.19) in [29], we let $\theta \to \pi$ and take $\varepsilon = K_n$ to derive that

$$\begin{aligned} u_n(t) &= \frac{a_n\lambda_n}{2\pi\mathrm{i}}\int_{K_n}^{\infty} r^{-1}\mathrm{e}^{-rt}\Phi(r)\mathrm{d}r + \frac{a_n}{2\pi\mathrm{i}}\int_{\gamma(K_n,\pi)\cap\{|s|=K_n\}} \frac{w(s)}{sw(s)+\lambda_n}\mathrm{e}^{st}\mathrm{d}s \\ &=: I_{n,1}(t) + I_{n,2}(t), \end{aligned}$$

where $t \in [0, T]$, $K_n > 0$ $(n = 1, 2, \cdots)$ are defined in Lemma 4.2, and

$$\Phi(r) := \frac{\int_0^1 r^\alpha \sin(\pi\alpha)\mu(\alpha)\mathrm{d}\alpha}{(\int_0^1 r^\alpha \cos(\pi\alpha)\mu(\alpha)\mathrm{d}\alpha + \lambda_n)^2 + (\int_0^1 r^\alpha \sin(\pi\alpha)\mu(\alpha)\mathrm{d}\alpha)^2}.$$

Based on the notation of K_n, using the proof in Lemma 4.2, we see that the following estimate

$$\left|\int_0^1 r^\alpha \cos(\pi\alpha)\mu(\alpha)\mathrm{d}\alpha + \lambda_n\right| \ge C\lambda_n \tag{4.9}$$

is valid for $r \ge K_n$ and $n \ge N$ with $N \in \mathbb{N}$ being large enough. Now we define R_n as follows

$$\int_0^1 R_n^\alpha \sin(\pi\alpha)\mu(\alpha)\mathrm{d}\alpha = \delta\lambda_n,\ \delta > 0 \text{ small enough.}$$

Without loss of generalization, we assume $K_n \le R_n$ and break the interval $[K_n, \infty)$ into $[K_n, R_n]$ and $[R_n, \infty)$ and have

$$|I_{n,1}(t)| \le C|a_n|\lambda_n \int_{K_n}^{R_n} r^{-1}\Phi(r)\mathrm{d}r + C|a_n|\lambda_n \int_{R_n}^{\infty} r^{-1}\Phi(r)\mathrm{d}r =: Q_{n,1}(t) + Q_{n,2}(t),$$

where $t \in [0, T]$. From the estimate (4.9) and the Fubini theorem, the following estimates

$$\begin{aligned} Q_{n,1}(t) &\le C|a_n|\lambda_n \int_{K_n}^{R_n} r^{-1}\frac{\int_0^1 r^\alpha \sin(\pi\alpha)\mu(\alpha)\mathrm{d}\alpha}{\lambda_n^2}\mathrm{d}r \\ &= \frac{C|a_n|}{\lambda_n}\int_0^1 \mu(\alpha)\int_{K_n}^{R_n} r^{\alpha-1}\mathrm{d}r \sin(\pi\alpha)\mathrm{d}\alpha \end{aligned}$$

are valid for $t \in [0, T]$, hence that

$$Q_{n,1}(t) \le \frac{C|a_n|}{\lambda_n}\int_0^1 R_n^\alpha \mu(\alpha)\sin(\pi\alpha)\mathrm{d}\alpha \le C|a_n|$$

for $t \in [0, T]$ and $n \ge N$ in view of the definition of R_n. For $Q_{n,2}(t)$, $t \in [0, T]$. Noting the fact

$$\left|\int_0^1 r^\alpha \cos(\pi\alpha)\mu(\alpha)\mathrm{d}\alpha + \lambda_n\right| \ge 0$$

implies

$$\Phi(r) \le \frac{1}{\int_0^1 r^\alpha \sin(\pi\alpha)\mu(\alpha)\mathrm{d}\alpha}, \tag{4.10}$$

hence that

$$Q_{n,2}(t) \le C|a_n|\lambda_n \int_{R_n}^{\infty} \frac{1}{\int_0^1 r^{\alpha+1}\sin(\pi\alpha)\mu(\alpha)\mathrm{d}\alpha}\mathrm{d}r$$
$$\le \frac{C|a_n|\lambda_n}{\int_0^1 R_n^{\alpha}\sin(\pi\alpha)\mu(\alpha)\mathrm{d}\alpha} = \frac{C}{\delta}|a_n|$$

in view of inequality

$$\int_{R_n}^{\infty} \frac{1}{\int_0^1 r^{\alpha+1}\sin(\pi\alpha)\mu(\alpha)\mathrm{d}\alpha}\mathrm{d}r \le \frac{C}{\int_0^1 R_n^{\alpha}\sin(\pi\alpha)\mu(\alpha)\mathrm{d}\alpha}, \quad n \ge N, \tag{4.11}$$

which can be derived from the L'Hospital theorem. Let us trun to evaluate $I_{n,2}(t)$. Noting that the integrand in $I_{n,2}(t)$ is analytic on the complex plane cutting off the negative real axis, then by the Cauchy theorem, the contour $\gamma(K_n,\pi)\cap\{|s|=K_n\}$ can be shifted to the contour $\tilde{\gamma}$ consisting of the following three parts

1. γ_1: $s\in\gamma(K_n,\pi)\cap\{|s|=K_n\}$, $\frac{3\pi}{4}<|\arg s|\le\pi$,
2. γ_2: $\varepsilon<|s|<K_n$, $|\arg s|=\frac{3\pi}{4}$,
3. γ_3: $|s|=\varepsilon$, $|\arg s|<\frac{3\pi}{4}$,

where $\varepsilon>0$ is sufficiently small, and hence that

$$|I_{n,2}(t)| \le C|a_n|\sum_{j=1}^{3}\int_{\gamma_j}\frac{|w(s)|}{|sw(s)+\lambda_n|}|\mathrm{e}^{st}\mathrm{d}s| =: \sum_{j=1}^{3}P_{n,j}(t), \quad t\in[0,T].$$

Let $\theta=\frac{3\pi}{4}$ in Lemma 4.2, we see that

$$P_{n,1}(t) \le C|a_n|\int_{\frac{3\pi}{4}}^{\pi}\frac{\int_0^1 K_n^{\alpha}\mu(\alpha)\mathrm{d}\alpha}{\lambda_n}d\rho \le C|a_n|, \quad t\in[0,T]$$

in view of the definition of K_n. Next we give estimates for $P_{n,3}(t)$, $t\in[0,T]$. For this, since $\varepsilon>0$ is small enough, we see that

$$P_{n,3}(t) \le C|a_n|\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\frac{\varepsilon-1}{\lambda_n\varepsilon\log\varepsilon}\varepsilon\mathrm{e}^{\varepsilon t\cos\rho}d\rho \le \frac{C|a_n|}{\lambda_n}.$$

For $P_{n,2}(t)$. Because of the increasing of the function $\int_0^1 r^{\alpha}\mu(\alpha)\mathrm{d}\alpha$, we see that $K_n>K_n'$ and break the integral in $P_{n,2}$ into two parts as follows

$$P_{n,2}(t) \le C|a_n|\int_{\varepsilon}^{K_n'}\frac{|w(s)||\mathrm{e}^{st}|}{|sw(s)+\lambda_n|}\Big|_{s=r\mathrm{e}^{\frac{3\pi}{4}\mathrm{i}}}\mathrm{d}r + C|a_n|\int_{K_n'}^{K_n}\frac{|w(s)||\mathrm{e}^{st}|}{|sw(s)+\lambda_n|}\Big|_{s=r\mathrm{e}^{\frac{3\pi}{4}\mathrm{i}}}\mathrm{d}r$$
$$=: U_{n,1}(t)+U_{n,2}(t)$$

for $t\in[0,T]$. Again using Lemma 4.2, we conclude that

$$\frac{|w(s)|}{|sw(s)+\lambda_n|} \le \frac{\int_0^1 r^{\alpha-1}\mu(\alpha)\mathrm{d}\alpha}{C\lambda_n}, \quad s=r\mathrm{e}^{\frac{3\pi}{4}\mathrm{i}},\ \varepsilon\le r\le K_n',$$

hence that

$$U_{n,1}(t) \le \frac{C|a_n|}{\lambda_n}\int_{\varepsilon}^{K_n'}\int_0^1 r^{\alpha-1}\mu(\alpha)\mathrm{d}\alpha\mathrm{d}r \le \frac{C|a_n|}{\lambda_n}\int_0^1 (K_n')^{\alpha}\mu(\alpha)\mathrm{d}\alpha \le C|a_n|, \quad t\in[0,T],$$

where we used the Fubini theorem and the definition of K_n'. For $U_{n,2}$, first we claim that $\frac{K_n}{K_n'} \le C$ for any $n \in \mathbb{N}$. In fact, if not, we can choose a subsequence of K_{n_k} and K_{n_k}' such that

$$\frac{K_{n_k}}{K_{n_k}'} \ge k, \quad k \in \mathbb{N}.$$

But from the definition of K_n and K_n', we see that

$$\frac{\int_0^1 (K_{n_k})^\alpha \mu(\alpha)\mathrm{d}\alpha}{\int_0^1 (K_{n_k}')^\alpha \mu(\alpha)\mathrm{d}\alpha} = \delta^{-2}.$$

Since $\mu \in C[0,1]$ does not identically equal zero, we can choose $\delta_1 \in (0,1)$ such that $\mu(\delta_1) > 0$, therefore

$$\delta^{-2} = \frac{\int_0^1 (K_{n_k}/K_{n_k}')^\alpha (K_{n_k}')^\alpha \mu(\alpha)\mathrm{d}\alpha}{\int_0^1 (K_{n_k}')^\alpha \mu(\alpha)\mathrm{d}\alpha} \ge \frac{\int_{\delta_1}^1 (K_{n_k}/K_{n_k}')^\alpha (K_{n_k}')^\alpha \mu(\alpha)\mathrm{d}\alpha}{\int_0^1 (K_{n_k}')^\alpha \mu(\alpha)\mathrm{d}\alpha} \ge \frac{k^{\delta_1}\int_{\delta_1}^1 (K_{n_k}')^\alpha \mu(\alpha)\mathrm{d}\alpha}{\int_0^1 (K_{n_k}')^\alpha \mu(\alpha)\mathrm{d}\alpha}.$$

By noting that

$$\int_0^{\delta_1} (K_{n_k}')^\alpha \mu(\alpha)\mathrm{d}\alpha \le \frac{1}{2}\int_{\delta_1}^1 (K_{n_k}')^\alpha \mu(\alpha)\mathrm{d}\alpha,$$

for k large enough, we finally conclude that $\delta^{-2} \ge Ck^{\delta_1}$, which is a contradiction when k is sufficiently large. Combined the above argument, we see that the following inequalities

$$U_{n,2}(t) \le C|a_n| \int_{K_n'}^{K_n} \frac{\int_0^1 r^{\alpha-1}\mu(\alpha)\mathrm{d}\alpha}{\int_0^1 r^\alpha \sin(\frac{3\pi}{4}\alpha)\mu(\alpha)\mathrm{d}\alpha} \le C|a_n| \int_{K_n'}^{K_n} r^{-1}\mathrm{d}r \le C|a_n| \log\frac{K_n}{K_n'} \le C|a_n|$$

are valid for $t \in [0,T]$ and $n \ge N$ with sufficiently large $N \in \mathbb{N}$. Finally we obtain

$$|u_n(t)| \le C|a_n|, \ t \in [0,T], \, n \ge N \text{ with sufficiently large } N \in \mathbb{N}.$$

In the case of $n \le N$, we have

$$u_n(t) = \frac{a_n\lambda_n}{2\pi \mathrm{i}} \int_\varepsilon^\infty r^{-1}\mathrm{e}^{-rt}\Phi(r)\mathrm{d}r + \frac{a_n}{2\pi\mathrm{i}} \int_{\gamma(\varepsilon,\pi)\cap\{|s|=\varepsilon\}} \frac{w(s)}{sw(s)+\lambda_n}\mathrm{e}^{st}\mathrm{d}s.$$

In view of estimate (4.10) for Φ, it follows that

$$|u_n(t)| \le C|a_n| \int_\varepsilon^\infty \frac{1}{\int_0^1 r^{\alpha+1}\sin(\pi\alpha)\mu(\alpha)\mathrm{d}\alpha}\mathrm{d}r + C|a_n| \int_{-\pi}^{\pi} \frac{\varepsilon - 1}{\log\varepsilon}\mathrm{d}\rho \le C|a_n|. \tag{4.12}$$

Here we used the inequality similar to (4.11). Then combining the above estimates, we finally complete the proof of the lemma. □

One important point we would like to mention here is that we in fact proved that the formal solution $u(\cdot, t)$ belongs to the Sobolev space $H_0^1(\Omega) \cap H^2(\Omega)$ for any $t \in (0,T]$ and satisfies the differential equation in our initial-boundary value problem (4.1). What we then need to do is to prove $u(\cdot, t)$ tends to a as $t \to 0$ in $L^2(\Omega)$. Indeed the following lemma holds.

Lemma 4.4 *Under the same assumptions in Theorem 4.1. Then u tends to a as $t \to 0$ in the sense of $L^2(\Omega)$.*

Proof. We first show that for each $n \in \mathbb{N}$, $u_n(t)$ tends to a_n as $t \to \infty$. By noting that the Laplace transform of 1 is s^{-1}, we first have the representation

$$u_n(t) - a_n = \frac{a_n}{2\pi\mathrm{i}} \int_{M-\mathrm{i}\infty}^{M+\mathrm{i}\infty} \left(\frac{w(s)}{sw(s)+\lambda_n} - \frac{1}{s}\right)\mathrm{e}^{st}\mathrm{d}s \tag{4.13}$$

and so

$$u_n(t) - a_n = \frac{-a_n\lambda_n}{\pi} \operatorname{Im} \int_0^\infty \frac{e^{st}}{s(sw(s)+\lambda_n)}\Big|_{s=M+ir} i dr.$$

We set $M = \frac{1}{t}$ and proceed with estimations of the above integral:

$$\begin{aligned} |u_n(t) - a_n| \le & \frac{|a_n|\lambda_n}{\pi} \left| \int_0^\infty \frac{e^{(ir+\frac{1}{t})t}}{(ir+\frac{1}{t})(sw(s)+\lambda_n)}\Big|_{s=\frac{1}{t}+ir} dr \right| \\ \le & \int_0^\infty \frac{C|a_n|\lambda_n}{(r^2+\frac{1}{t^2})^{\frac{1}{2}}|sw(s)+\lambda_n|}\Big|_{s=\frac{1}{t}+ir} dr. \end{aligned}$$

Since $0 \le \arg s < \frac{\pi}{2}$ for $s = \frac{1}{t} + ir$, and noting that $\mu \in C[0,1]$ does not identically zero, there exists a constant δ_0 and δ_1 such that $0 < \delta_1 < \delta_0 < 1$ and $\mu(\alpha) > 0$ for $[\delta_1, \delta_0]$. We obtain

$$|sw(s) + \lambda_n| \ge \int_0^1 |s|^\alpha \cos(\alpha \arg s)\mu(\alpha) d\alpha \ge C\frac{|s|^{\delta_0} - |s|^{\delta_1}}{\log|s|}.$$

Applying this to the last estimate for $|u_n(t) - a_n|$, we have

$$|u_n(t) - a_n| \le C\lambda_n|a_n| \int_0^\infty \frac{\log(r^2+\frac{1}{t^2})}{(r^2+\frac{1}{t^2})^{\frac{1}{2}}\left((r^2+\frac{1}{t^2})^{\frac{\delta_0}{2}} - (r^2+\frac{1}{t^2})^{\frac{\delta_1}{2}}\right)} dr$$

for small enough $t > 0$. Moreover, since

$$\frac{\log(r^2+\frac{1}{t^2})}{(r^2+\frac{1}{t^2})^{\frac{1+\delta_0}{2}}\left((r^2+\frac{1}{t^2})^{\frac{1-\delta_0}{2}} - (r^2+\frac{1}{t^2})^{\frac{\delta_1-\delta_0}{2}}\right)} \le \frac{C}{(r^2+1)^{\frac{1+\delta_0}{2}}(r^2+1)^{\frac{1-\delta_0}{4}}}$$

for small enough $t > 0$, the Lebesgue dominated convergence theorem can be applied for the estimate $|u_n(t) - a_n|$. Thus

$$|u_n(t) - a_n| \to 0, \text{ as } t \to 0, \ n = 1, 2, \ldots.$$

Combining the estimate in Lemma 4.3, we see that $|u_n(t) - a_n| \le C|a_n|$, and then again using the Lebesgue theorem, we find

$$\lim_{t\to 0} \|u(t) - a\|^2_{L^2(\Omega)} = \lim_{t\to 0} \sum_{n=1}^\infty |u_n(t) - a_n|^2 = \sum_{n=1}^\infty \lim_{t\to 0} |u_n(t) - a_n|^2 = 0$$

which completes the proof of Lemma 4.4. □

Proof of Theorem 4.1. Now Lemma 4.3 and Lemma 4.4 together show that $u(\cdot, t)$ defined by (4.3) belongs to $H_0^1(\Omega) \cap H^2(\Omega)$ and solves the initial-boundary value problem (4.1).

It remains to prove the analyticity of the solution to the problem (4.1). We conclude from Lemma 4.3 that the following two statements

1. $u(\cdot, t) \in C^\infty((0,\infty); L^2(\Omega))$;
2. For any interval $[\delta, T]$, there exists positive constant $M = M(\mu, \delta, T)$ such that

$$\left\| \frac{\partial^m u(\cdot, t)}{\partial t^m} \right\|_{H^2(\Omega)} \le CM^m m! \|a\|_{L^2(\Omega)}, \quad \forall t \in [\delta, T], \quad m = 0, 1, \cdots$$

hold true for any $m = 0, 1, \cdots$. Therefore $u(\cdot, t)$ is real analytic in $t \in (0, T]$ in $H^2(\Omega)$, e.g., pp. 65-66 in John [27]. Moreover, since $\delta > 0$ and $T > 0$ can be arbitrarily chosen, we thus conclude that $u(\cdot, t)$ can be analytically extended to $t \in (0, \infty)$, which completes the proof of Theorem 4.1. □

4.2.2 Nonhomogeneous equation

Throughout this section, we assume $a = 0$ and $F \in L^2(Q)$ in (4.1), and then establish the wellposedness for our initial-boundary value problem.

We have

Theorem 4.2 (Wellposedness) *Let $a = 0$, and $F \in L^2(Q)$. We assume the weight function $\mu \in C[0,1]$ is nonnegative, and there exist $\alpha_0 \in (0,1)$ and small $\delta > 0$ such that*

$$\mu > 0 \text{ in } [\alpha_0 - \delta, \alpha_0]. \tag{4.14}$$

Then the problem (4.1) admits a unique solution $u \in L^2(0,T;H^2(\Omega))$ satisfying

$$\|u\|_{L^2(0,T;H^2(\Omega))} \le C\|F\|_{L^2(Q)}.$$

Recalling the positivity of the multinomial Mittag-Leffler function (see, e.g., Lemma 6.4 in [37]) which plays an essential role for obtaining the $H^2(\Omega)$-regularity for the solution provided in the case of $F \in L^2(Q)$, we are expected to get the sharp regularity by establishing the paralysed property of

$$G_n^{(\mu)}(t) := \frac{1}{2\pi \mathrm{i}} \int_{\gamma(\varepsilon,\theta)} \frac{1}{sw(s) + \lambda_n} \mathrm{e}^{st} \mathrm{d}s, \quad t \in (0,T) \tag{4.15}$$

for each $n \in \mathbb{N}$, so that similar arguments are feasible for distributed order time-fractional diffusion equations.

Lemma 4.5 *Assume the weight function $\mu \in C[0,1]$ is nonnegative. Then*

$$G_n^{(\mu)}(t) < 0, \quad \text{for } t \in (0,T) \text{ and } n \in \mathbb{N}.$$

Proof. We first observe that the integral in J_n is the sum of the integral over $\gamma_c(\varepsilon,\theta)$ and the functions

$$G_{n,\pm}^{(\mu)}(t) = \frac{1}{2\pi \mathrm{i}} \int_{\gamma_\pm(\varepsilon,\theta)} \frac{1}{sw(s) + \lambda_n} \mathrm{e}^{st} \mathrm{d}s.$$

We have

$$G_{n,+}^{(\mu)}(t) + G_{n,-}^{(\mu)}(t) = \frac{1}{\pi} \operatorname{Im} \left\{ \mathrm{e}^{\mathrm{i}\theta} \int_\varepsilon^\infty \frac{\mathrm{e}^{rt\mathrm{e}^{\mathrm{i}\theta}}}{r\mathrm{e}^{\mathrm{i}\theta} w(r\mathrm{e}^{\mathrm{i}\theta}) + \lambda_n} \mathrm{d}r \right\}.$$

We have to consider the expression

$$\int_\varepsilon^\infty \left\{ \operatorname{Im}\left(\mathrm{e}^{\mathrm{i}\theta}\mathrm{e}^{rt\mathrm{e}^{\mathrm{i}\theta}}\right) \operatorname{Re}\left(\frac{1}{r\mathrm{e}^{\mathrm{i}\theta} w(r\mathrm{e}^{\mathrm{i}\theta}) + \lambda_n}\right) + \operatorname{Re}\left(\mathrm{e}^{\mathrm{i}\theta}\mathrm{e}^{rt\mathrm{e}^{\mathrm{i}\theta}}\right) \operatorname{Im}\left(\frac{1}{r\mathrm{e}^{\mathrm{i}\theta} w(r\mathrm{e}^{\mathrm{i}\theta}) + \lambda_n}\right) \right\} \mathrm{d}r.$$

We have

$$\operatorname{Im}\left(\mathrm{e}^{\mathrm{i}\theta}\mathrm{e}^{rt\mathrm{e}^{\mathrm{i}\theta}}\right) = \mathrm{e}^{rt\cos\theta} \sin(\theta + rt\sin\theta).$$

It is easy to see that the above expression tends to zero as $\theta \to \pi$, and noting that

$$\operatorname{Im}\left(\frac{1}{r\mathrm{e}^{\mathrm{i}\pi} w(r\mathrm{e}^{\mathrm{i}\pi}) + \lambda_n}\right) = \frac{\int_0^1 r^\alpha \sin(\pi\alpha)\mu(\alpha)\mathrm{d}\alpha}{\left[\int_0^1 r^\alpha \cos(\pi\alpha)\mu(\alpha)\mathrm{d}\alpha + \lambda_n\right]^2 + \left[\int_0^1 r^\alpha \sin(\pi\alpha)\mu(\alpha)\mathrm{d}\alpha\right]^2} =: \Phi_n(r),$$

so that

$$G_n^{(\mu)}(t) = \frac{1}{2\pi \mathrm{i}} \int_{\gamma_c(\varepsilon,\pi)} \frac{1}{sw(s) + \lambda_n} \mathrm{e}^{st} \mathrm{d}s - \frac{1}{\pi} \int_\varepsilon^\infty \Phi_n(r)\mathrm{e}^{-rt} \mathrm{d}r.$$

By an argument similar to Lemma 3.1 in [38], we have that the first term on the right hand side of the above equality tends to 0 as $\varepsilon \to 0$, hence that

$$G_n^{(\mu)}(t) = -\frac{1}{\pi} \int_0^\infty \Phi_n(r)\mathrm{e}^{-rt} \mathrm{d}r < 0$$

in view of the notation of $\Phi_n(r)$. □

Proof of Theorem 4.2. Now let us take the operator A on the both sides of (4.6), we derive

$$\|u\|_{D(A)}^2 = \left\| \int_0^t AS_2(t-\tau)F(\tau)\mathrm{d}\tau \right\|_{L^2(\Omega)}^2 \le \sum_{n=1}^{\infty} \left| \int_0^t \lambda_n G_n^{(\mu)}(t-\tau)(F(\tau),\varphi_n)\mathrm{d}\tau \right|^2.$$

Thus the Young inequality implies

$$\int_0^T \|u(t)\|_{D(A)}^2 \mathrm{d}t \le \sum_{n=1}^{\infty} \left(\int_0^T \lambda_n |G_n^{(\mu)}(t)|\mathrm{d}t \right)^2 \int_0^T |(F(t),\varphi_n)|^2 \,\mathrm{d}t.$$

Moreover, the use of Lemma 4.5 and Fubini's theorem derive that

$$\int_0^T \lambda_n |G_n^{(\mu)}(t)|\mathrm{d}t = \frac{\lambda_n}{\pi} \int_0^\infty \Phi_n(r)\mathrm{d}r \int_0^T \mathrm{e}^{-rt}\mathrm{d}t \le \frac{C\lambda_n}{\pi} \int_0^\infty \frac{\Phi_n(r)}{r}\mathrm{d}r \le C,$$

here in the inequality, we used an argument similar to the proof of Lemma 4.3, thereby obtaining the estimates

$$\int_0^T \|u(t)\|_{D(A)}^2 \mathrm{d}t \le C \sum_{n=1}^{\infty} \int_0^T (F(t),\varphi_n)^2 \mathrm{d}t = C\|F\|_{L^2(Q)}^2,$$

that is

$$\|u\|_{L^2(0,T;H^2(\Omega))} \le C\|F\|_{L^2(Q)}^2,$$

which completes the proof of the theorem. □

4.3 Strichartz estimate

Throughout this section, we assume $F \in L^1(0,T;L^2(\Omega))$ and $a \in D(A^\gamma)$, $\gamma \in (0,1]$, in (4.1), and establish the Strichartz estimate (see, e.g., [54]) which should be regarded as the starting point for some further research concerning the theory of nonlinear fractional diffusion equations.

By an argument similar to the proof for Theorem 1.3 in [32], we have the following Strichartz estimate for the solution u to the initial-boundary value problem (4.1).

Theorem 4.3 (Strichartz estimates) *Let $a \in D(A^\gamma)$, $\gamma \in (0,1]$ and μ satisfy (4.14). Assume that $1 \le q \le \infty$ satisfies*

$$\begin{cases} q = \infty, & \dfrac{d}{4} < \gamma < 1, \\ 2 < q < \infty, & \gamma = \dfrac{d}{4}, \\ q = \dfrac{2d}{d-4\gamma}, & 0 < \gamma < \dfrac{d}{4}, \end{cases} \tag{4.16}$$

and we set $F \in L^1(0,T;L^2(\Omega))$. Then for any $1 \le p < \frac{1}{1-\alpha_0(1-\gamma)}$, the weak solution u to (4.1) admits

$$\|u\|_{L^p(0,T;L^q(\Omega))} \le C(\|a\|_{H^{2\gamma}(\Omega)} + \|F\|_{L^1(0,T;L^2(\Omega))}).$$

Before proceeding to the proof of the Strichartz estimate of the solution to the initial-boundary value problem (4.1) stated in Theorem 4.3, we introduce some key lemmas for showing the theorem.

Lemma 4.6 *Let $\varepsilon > 0$ and $\gamma \in [0,1]$ be arbitrarily fixed constants. We assume the weight function μ fulfils the condition (4.14). Then for $s = r\mathrm{e}^{\mathrm{i}\theta}$ satisfying $r > \varepsilon$ and $\theta \in (\frac{\pi}{2},\pi)$, there exists a positive constant $C = C(\alpha_0,\gamma,\mu,\varepsilon,\theta)$ such that*

$$\frac{\lambda_n^\gamma |\operatorname{Im}(sw(s))|^{1-\gamma}}{|sw(s)+\lambda_n|} \le C$$

Proof. Indeed, in the case of $|\,\mathrm{Re}\,(sw(s))| \le \frac{1}{2}\lambda_n$, we see that $(\,\mathrm{Re}\,(sw(s)) + \lambda_n)^2 \ge \frac{1}{4}\lambda_n^2$, therefore

$$\frac{\lambda_n^\gamma |\,\mathrm{Im}\,(sw(s))|^{1-\gamma}}{|sw(s)+\lambda_n|} \le \frac{\lambda_n^\gamma (\,\mathrm{Im}\,(sw(s)))^{1-\gamma}}{\sqrt{\frac{1}{4}\lambda_n^2 + (\,\mathrm{Im}\,(sw(s)))^2}} \le C.$$

On the other hand, for $|\,\mathrm{Re}\,(sw(s))| \ge \frac{1}{2}\lambda_n$, by recalling the definition of $w(s)$, from the inequality $|\cos(\alpha\theta)| \le C\sin(\alpha\theta)$ for $\alpha \in [\alpha_0, 1]$ and $\theta \in (\frac{\pi}{2}, \pi)$, we can conclude the following estimates

$$|\,\mathrm{Re}\,(sw(s))| = \int_0^1 |s|^\alpha |\cos(\alpha\theta)|\mu(\alpha)\mathrm{d}\alpha \le C\int_0^1 |s|^\alpha \sin(\alpha\theta)\mu(\alpha)\mathrm{d}\alpha = C\,\mathrm{Im}\,(sw(s)) \quad (4.17)$$

are valid for $s = r\mathrm{e}^{\mathrm{i}\theta}$ satisfying $r > \varepsilon$ and $\theta \in (\frac{\pi}{2}\pi)$, hence we have

$$\frac{\lambda_n^\gamma |\,\mathrm{Im}\,(sw(s))|^{1-\gamma}}{|sw(s)+\lambda_n|} \le \frac{\lambda_n^\gamma (\,\mathrm{Im}\,(sw(s)))^{1-\gamma}}{\sqrt{(\,\mathrm{Im}\,(sw(s)))^2}} \le 2^\gamma \frac{|\,\mathrm{Re}\,(sw(s))|^\gamma |\,\mathrm{Im}\,(sw(s))|^{1-\gamma}}{\mathrm{Im}\,(sw(s))} \le C.$$

This completes the proof of the lemma. □

Lemma 4.7 *Assume the weight function $\mu \in C[0,1]$ is nonnegative and satisfies (4.14). Then there exist $C = C(\mu,\gamma,\tau,T) > 0$ such that*

$$\|A^\tau S_1(t)\psi\|_{L^2(\Omega)} \le C\|\psi\|_{H^{2\gamma}(\Omega)} t^{\gamma-\tau}, \quad t \in (0,T],\ \gamma \in (0,1],\ \tau \ge \gamma. \quad (4.18)$$

Proof. Recalling the notations (4.3) and (4.4), it is sufficient to give an estimate for $E_n^{(\mu)}(t)$ defined by

$$E_n^{(\mu)}(t) := \frac{1}{2\pi\mathrm{i}} \int_{\gamma(\varepsilon,\theta)} \frac{w(s)\mathrm{e}^{st}}{sw(s)+\lambda_n} \mathrm{e}^{st}\mathrm{d}s$$

for $t \in (0,T)$, $n \in \mathbb{N}$. For this, we have

$$|E_n^{(\mu)}(t)| \le C\left(\int_{\gamma_c(\varepsilon,\theta)} + \int_{\gamma_\pm(\varepsilon,\theta)}\right) \frac{|w(s)\mathrm{e}^{st}|}{|sw(s)+\lambda_n|}\Big|_{s=r\mathrm{e}^{\mathrm{i}\theta}} \mathrm{d}r =: E_{n,c}(t) + E_{n,\pm}(t),$$

where $t \in (0,T]$ and $n \in \mathbb{N}$.

Letting $\varepsilon > 0$ in the definition of the contour $\gamma(\varepsilon,\theta)$ small enough, we have $E_{n,c}(t) \le \frac{C}{\lambda_n}$ is valid for $t \in (0,T]$ and $n \in \mathbb{N}$. Next, for $E_{n,\pm}(t)$, from the estimate (4.17), a direct calculation implies

$$E_{n,\pm}(t) \le C\int_\varepsilon^\infty \frac{\mathrm{Im}\,(sw(s))}{|sw(s)+\lambda_n|} \mathrm{e}^{-rt|\cos\theta|} \frac{\mathrm{d}r}{r}.$$

We break up the integral into $(\varepsilon, \varepsilon_0\lambda_n)$ and $(\varepsilon_0\lambda_n, \infty)$ to derive

$$E_{n,\pm}(t) \le C\left(\int_\varepsilon^{\varepsilon_0\lambda_n} + \int_{\varepsilon_0\lambda_n}^\infty\right) \frac{\mathrm{Im}\,(sw(s))}{|sw(s)+\lambda_n|} \mathrm{e}^{-rt|\cos\theta|} \frac{\mathrm{d}r}{r} =: E_{n,\pm}^{(1)}(t) + E_{n,\pm}^{(2)}(t),$$

where $\varepsilon_0 > 0$ is small enough, and estimate them separately. For $E_{n,\pm}^{(1)}(t)$, since $|s| \le \varepsilon_0\lambda_n$ yields $|sw(s)+\lambda_n| \ge C\lambda_n$, and noting that $|\,\mathrm{Im}\,(sw(s))| \le Cr$, we have

$$E_{n,\pm}^{(1)}(t) \le \frac{C}{\lambda_n}\int_\varepsilon^{\varepsilon_0\lambda_n} \mathrm{e}^{-rt|\cos\theta|}\mathrm{d}r = \frac{C}{\lambda_n}\int_\varepsilon^{\varepsilon_0\lambda_n} \frac{1}{r^{\tau-\gamma}t^{\tau-\gamma}}\mathrm{d}r \le \frac{C\lambda_n^\gamma}{\lambda_n^\tau} t^{\gamma-\tau},$$

where we choose $\tau \ge \gamma$. Moreover, in view of Lemma 4.6, it follows that

$$E_{n,\pm}^{(2)}(t) \le C\int_{\varepsilon_0\lambda_n}^\infty r^{-1}\mathrm{e}^{-rt|\cos\theta|}\mathrm{d}r \le C\int_{\varepsilon_0\lambda_n}^\infty r^{-1}\frac{1}{r^{\tau-\gamma}t^{\tau-\gamma}}\mathrm{d}r \le \frac{C\lambda_n^\gamma}{\lambda_n^\tau} t^{\gamma-\tau},$$

where $\tau \geq \gamma$. Consequently, we have

$$|E_n^{(\mu)}(t)| \leq \frac{C\lambda_n^\gamma}{\lambda_n^\tau} t^{\gamma-\tau}, \quad t \in (0,T),\ \tau \geq \gamma,\ \gamma \in (0,1].$$

Thus for $\psi \in D(A^\gamma)$, we find

$$\|A^\tau S_1(t)\psi\|_{L^2(\Omega)}^2 = \sum_{n=1}^\infty \lambda_n^{2\tau} |E_n^{(\mu)}(t)^2| (\psi, \varphi_n)^2 \leq Ct^{2(\gamma-\tau)} \sum_{n=1}^\infty \lambda_n^{2\gamma} (\psi, \varphi_n)^2,$$

that is

$$\|A^\tau S_1(t)\psi\|_{L^2(\Omega)} \leq C\|\psi\|_{D(A^\gamma)} t^{\gamma-\tau}, \quad t \in (0,T),\ \tau \geq \gamma.$$

This completes the proof of the lemma.

□

Now let us turn to the evaluation of the operator $S_2(t)$.

Lemma 4.8 *Under the same assumptions in Lemma 4.7. Then there exist* $C = C(\mu, \kappa, \kappa_0, T) > 0$ *such that*

$$\|A^\kappa S_2(t)\phi\|_{L^2(\Omega)} \leq C\|\phi\|_{L^2(\Omega)} t^{-\kappa_0}, \quad 0 < t < T,$$

where $\kappa \in [0,1)$, *and* $\kappa_0 \in (1 - \alpha_0(1-\kappa), 1)$.

Proof. For $0 < t \leq T$, we recall (4.15), and have

$$G_n^{(\mu)}(t) = \frac{1}{2\pi \mathrm{i}} \left(\int_{\gamma_c(\varepsilon,\theta)} + \int_{\gamma_\pm(\varepsilon,\theta)} \right) \frac{\mathrm{e}^{st}}{sw(s) + \lambda_n} \mathrm{d}s =: G_{n,1}(t) + G_{n,1}(t).$$

We set $\varepsilon > 0$ small enough, and then by an argument used in Lemma 4.7, we obtain that $|G_{n,1}(t)| \leq \frac{C}{\lambda_n}$, $0 < t \leq T$, $n \in \mathbb{N}$. Now we consider the integral $G_{n,2}(t)$ with $0 < t \leq T$. Again, breaking the integral in $G_{n,2}$ into the following two parts $(\varepsilon, \varepsilon_0\lambda_n)$ and $(\varepsilon_0\lambda_n, \infty)$, and employing the similar argument used in Lemma 4.7 give

$$\begin{aligned} G_{n,2}(t) &\leq C \int_\varepsilon^{\varepsilon_0\lambda_n} \frac{1}{\lambda_n} \mathrm{e}^{-rt|\cos\theta|} \mathrm{d}r + C \int_{\varepsilon_0\lambda_n}^\infty \frac{1}{|sw(s) + \lambda_n|} \mathrm{e}^{-rt|\cos\theta|} \mathrm{d}r \\ &\leq \frac{C}{\lambda_n^\kappa t^\kappa} + \frac{C}{\lambda_n^\kappa} \int_{\varepsilon_0\lambda_n}^\infty \frac{\lambda_n^\kappa (\operatorname{Im}(sw(s)))^{1-\kappa}}{|sw(s) + \lambda_n|} \frac{\mathrm{e}^{-rt|\cos\theta|}}{(\operatorname{Im}(sw(s)))^{1-\kappa}} \mathrm{d}r \end{aligned}$$

where $\varepsilon,\ \varepsilon_0 > 0$ are small enough. We further note that the assumption (4.14) yields

$$\operatorname{Im}(sw(s)) = \int_0^1 r^\alpha \sin(\theta\alpha)\mu(\alpha)\mathrm{d}\alpha \geq C \int_{\alpha_0-\delta}^{\alpha_0} r^\alpha \mathrm{d}\alpha = \frac{Cr^{\alpha_0}}{\log r}, \quad r > \varepsilon_0\lambda_n.$$

Then we can choose $\kappa_0 \in (0,1)$ such that $\kappa_0 > 1 - \alpha_0(1-\kappa)$, hence that

$$|G_{n,2}(t)| \leq \frac{C}{\lambda_n^\kappa t^\kappa} + \frac{C}{\lambda_n^\kappa} \int_{\varepsilon_0\lambda_n}^\infty \frac{1}{r^{\alpha_0(1-\kappa)}} \frac{1}{r^{\kappa_0} t^{\kappa_0}} \mathrm{d}r \leq \frac{C}{\lambda_n^\kappa t^{\kappa_0}}, \quad 0 < t < T.$$

Combining all the above estimates for $G_{n,1}$ and $G_{n,2}$, it follows that

$$|G_n^{(\mu)}(t)| \leq \frac{C}{\lambda_n^\kappa t^{\kappa_0}}, \quad n \in \mathbb{N},\ 0 < t < T,$$

where $1 - \alpha_0(1-\kappa) < \kappa_0 < 1$.

Finally, for $\phi \in L^2(\Omega)$, we obtain

$$\|A^\kappa S_2(t)\phi\|_{L^2(\Omega)}^2 = \sum_{n=1}^\infty \lambda_n^{2\kappa} |G_n^{(\mu)}(t)^2| (\phi, \varphi_n)^2 \leq Ct^{-2\kappa_0} \sum_{n=1}^\infty (\phi, \varphi_n)^2,$$

that is

$$\|A^\kappa S_2(t)\phi\|_{L^2(\Omega)} \leq C\|\phi\|_{L^2(\Omega)} t^{-\kappa_0}, \quad t \in (0,T),$$

where $1 - \alpha_0(1-\kappa) < \kappa_0 < 1$. The proof of the lemma is complete. □

Now we are ready to give the proof of Theorem 4.3

Proof of Theorem 4.3. Let $a \in D(A^\gamma)$, $\gamma \in (0,1]$, and $F \in L^1(0,T;L^2(\Omega))$, taking the operator A^κ, $\gamma \le \kappa < 1$, on both sides of (4.6), we conclude from Lemma 4.7 and Lemma 4.8 that

$$\|A^\kappa S_1(t)a\|_{L^2(\Omega)} \le \|A^\gamma a\|_{L^2(\Omega)} t^{\gamma-\kappa}$$

and

$$\|A^\kappa S_2(t-\tau)F(\tau)\|_{L^2(\Omega)} \le (t-\tau)^{-\kappa_0}\|F(\tau)\|_{L^2(\Omega)}, \quad 0<t<T,$$

where $\kappa \ge \gamma$ and $1-\alpha_0(1-\kappa) < \kappa_0 < 1$.

Consequently, by letting $\kappa = \gamma$, and for $p \in [1, \frac{1}{\kappa_0})$, from Young's inequality, we have

$$\left(\int_0^T \|u(t)\|^p_{D(A^\gamma)}\mathrm{d}t\right)^{\frac{1}{p}} \le CT^{1/p}\|A^\gamma a\|_{L^2(\Omega)} + C\left(\int_0^T t^{1-\kappa_0 p}\mathrm{d}t\right)^{\frac{1}{p}} \int_0^T \|F(t)\|_{L^2(\Omega)}\mathrm{d}t,$$

that is

$$\|u\|_{L^p(0,T;H^{2\gamma}(\Omega))} \le CT^{1/p}\|a\|_{H^{2\gamma}(\Omega)} + CT^{1/p-\kappa_0}\|F\|_{L^1(0,T;L^2(\Omega))}.$$

By the Sobolev embedding theorem, for all $0<t<T$, we have

$$\|u(t)\|_{L^q(\Omega)} \le C\|u(t)\|_{H^{2\gamma}(\Omega)},$$

where $1 \le q \le \infty$ satisfies (4.16). Applying this estimate, we obtain

$$\|u\|_{L^p(0,T;L^q(\Omega))} \le CT^{1/p}\|a\|_{H^{2\gamma}(\Omega)} + CT^{1/p-\kappa_0}\|F\|_{L^1(0,T;L^2(\Omega))}$$

where κ_0 such that $1-\alpha_0(1-\gamma) < \kappa_0 < 1$ which completes the proof of the theorem. □

Remark 4.1 If $a=0$ and $F \in L^1(0,T;L^2(\Omega))$ does not vanish, then Lemma 4.8 asserts the solution $u \in L^p(0,T;D(A^\gamma))$, where γ can be arbitrarily close to 1 but is never touch 1, except for the special case that F is in $L^2(Q)$. The technical reason is that only in case of $F \in L^2(Q)$ one can take advantage of the property in Lemma 4.5 for getting the $H^2(\Omega)$– regularity.

4.4 Long-time asymptotic behavior

In this section, we mainly discuss the long-time asymptotic behavior of solutions to the initial-boundary value problem (4.1) under the assumption that the source term $F=0$, and compare our results with those obtained for the initial-boundary value problems for the multi-term time-fractional diffusion equations (see [8], [34], [37] and [43]).

Henceforth C denotes a positive constant that is independent of the time variable t, the initial condition a, and the solution u, but may depend on Ω, d, a_{ij} and μ.

Theorem 4.4 *Let $\mu(\ge 0) \in C[0,1]$ not vanish in $[0,1]$. We further assume $a \in L^2(\Omega)$, and $F=0$.*

Then

$$\|u(\,\cdot\,,t)\|_{H^2(\Omega)} \le C\|a\|_{L^2(\Omega)}(\log t)^{-1}$$

for the solution u to the initial-boundary value problem (4.1) for sufficiently large $t>0$.

Moreover, if the weight function $\mu(\alpha)$ admits the representation $\mu(\alpha) = \mu(0) + o(\alpha^\delta)$, $\mu(0) > 0$, with some $\delta > 0$ as $\alpha \to 0$, then the asymptotic formula

$$\|u(\,\cdot\,,t) - \tfrac{\mu(0)}{\log t}\mathcal{A}^{-1}a\|_{H^2(\Omega)} = o((\log t)^{-1})\|a\|_{L^2(\Omega)}, \quad t \to \infty$$

holds true. The last formula holds uniformly dependently on Ω, the spatial dimension d, the initial condition a, the coefficients a_{ij} of the spatial differential operator of the equation (4.1), and the exponent δ of the asymptotic expansion of the weight function μ.

In particular, the second part of Theorem 4.4 means that the decay rate $(\log t)^{-1}$ of a solution u to the problem under consideration is the best possible one. More precisely, we have the following statement:

Corollary 4.1 *Let the weight function μ admit $\mu(\alpha) = \mu(0) + o(\alpha^\delta)$ with some $\delta > 0$ as $\alpha \to 0$. Moreover suppose that*

$$\|u(\,\cdot\,,t)\|_{H^2(\Omega)} = o((\log t)^{-1}), \ \ \textit{as } t \to \infty.$$

Then $u(x,t) = 0$ for all $x \in \Omega$ and $t > 0$.

Let us recall that the multi-term time-fractional diffusion equation

$$\sum_{j=1}^{\ell} q_j \partial_t^{\alpha_j} u + \mathcal{A}u = 0, \quad (x,t) \in \Omega \times (0,\infty)$$

can be formally obtained from the equation (4.1) by setting the weight function $\mu(\alpha) = \sum_{j=1}^{\ell} q_j \delta(\alpha - \alpha_j)$, $0 < \alpha_\ell < \cdots \alpha_1 < 1$, $0 < q_j$, $j = 1, \ldots, \ell$. In Chapter 2, the asymptotic behavior of solutions to initial-boundary value problems for the multi-term fractional diffusion equations was investigated. In particular, it was shown there that the decay rate of solution is $t^{-\alpha_\ell}$ as $t \to \infty$, α_ℓ being the smallest exponent of the multi-term fractional derivative. Thus a multi-term fractional diffusion equation cannot simulate a very slow logarithmic decay rate of its solutions. In other words, distributed order fractional diffusion equation (4.1) is completely different from the case of a singular measure μ of the type $\mu(\alpha) = \sum_{j=1}^{\ell} q_j \delta(\alpha - \alpha_j)$, which is a multi-term fractional diffusion equation.

The proof of Theorem 4.4 will be complete if we show the asymptotic bahavior of the solution u_n to the Cauchy problem (4.2). In fact, (4.2) is a Cauchy problem for an ordinary fractional differential equation of distributed orders. For a fixed value of λ_n, this problem was investigated in [29] and [30]. In this part of the section, we follow the analysis method employed in [30] for all λ_n. Our target is to describe long-time asymptotic behavior of the solution $u_n(t)$ to the Cauchy problem (4.2). To reach this aim, we need a more detailed analysis compared to [29] and [30].

We start from

Lemma 4.9 *Let $\mu \in C[0,1]$ be a non-negative function satisfying the conditions $\mu(0) > 0$ and*

$$\mu(\alpha) = \mu(0) + o(\alpha^\delta), \quad \alpha \to 0, \tag{4.19}$$

where δ is a positive constant. Then the estimate

$$\left| \frac{w(s)}{sw(s) + \lambda_n} - \frac{1}{\lambda_n} \frac{\mu(0)(s-1)}{s \log s} \right| \le \frac{C}{\lambda_n |s| (\log 1/|s|)^{1+\delta}}, \quad 0 < |s| < \varepsilon_0 \tag{4.20}$$

holds true, where $C > 0$ is a constant independent of λ_n and s and $\varepsilon_0 > 0$ is sufficiently small.

Proof. First we have

$$\left| \frac{w(s)}{sw(s) + \lambda_n} - \frac{\mu(0)(s-1)}{\lambda_n s \log s} \right| \le \left| \frac{w(s)}{sw(s) + \lambda_n} - \frac{w(s)}{\lambda_n} \right| + \left| \frac{w(s)}{\lambda_n} - \frac{\mu(0)(s-1)}{\lambda_n s \log s} \right|$$
$$=: Q_{n,1}(s) + Q_{n,2}(s).$$

Since ε_0 is sufficiently small and $\lambda_n \ge \lambda_1 > 0$, the function $Q_{n,1}(s)$ can be estimated by Lemma 4.2:

$$Q_{n,1}(s) = \frac{|sw^2(s)|}{\lambda_n |sw(s) + \lambda_n|} \le C \frac{\frac{1}{|s|}\left(\frac{|s|-1}{\log |s|}\right)^2}{\lambda_n^2} \le \frac{C}{\lambda_n |s| (\log 1/|s|)^2}, \quad 0 < |s| < \varepsilon_0.$$

As to $Q_{n,2}(s)$, we employ the integral $\int_0^1 s^{\alpha-1} \mathrm{d}\alpha = \frac{s-1}{s \log s}$ and the asymptotic formula (4.19) to obtain

$$Q_{n,2}(s) = \frac{1}{\lambda_n} \left| \int_0^1 s^{\alpha-1} \mu(\alpha) \mathrm{d}\alpha - \int_0^1 s^{\alpha-1} \mu(0) \mathrm{d}\alpha \right| \le \frac{C}{\lambda_n} \int_0^1 |s|^{\alpha-1} \alpha^\delta \mathrm{d}\alpha$$

and then the inequality

$$Q_{n,2}(s) \leq \frac{C}{\lambda_n |s|} \int_0^1 \mathrm{e}^{-\alpha \log(1/|s|)} \alpha^\delta \mathrm{d}\alpha, \quad 0 < |s| < \varepsilon_0.$$

After the change of variables $\alpha \log(1/|s|) \to \alpha$ in the last integral, we have

$$Q_{n,2}(s) \leq \frac{C}{\lambda_n |s| (\log 1/|s|)^{\delta+1}} \int_0^\infty \mathrm{e}^{-\alpha} \alpha^\delta \mathrm{d}\alpha \leq \frac{C}{\lambda_n |s| (\log 1/|s|)^{\delta+1}}.$$

We thus obtain the estimate

$$\left| \frac{w(s)}{sw(s) + \lambda_n} - \frac{1}{\lambda_n} \frac{\mu(0)(s-1)}{s \log s} \right| \leq \frac{C}{\lambda_n} \frac{1}{|s|(\log 1/|s|)^{1+\delta}}, \quad 0 < |s| < \varepsilon_0,$$

which completes the proof of the lemma. □

Now we are ready to prove Theorem 4.4.

Proof of Theorem 4.4. For $n \geq N$ with sufficiently large N, the representation (4.3) and Lemma 4.2 yield

$$|u_n(t)| \leq \frac{C|a_n|}{\lambda_n} \int_\varepsilon^\infty |w(r\mathrm{e}^{\mathrm{i}\theta})| \mathrm{e}^{rt\cos\theta} \mathrm{d}r + \frac{C|a_n|}{\lambda_n} \int_{-\theta}^{\theta} \frac{\varepsilon - 1}{\log \varepsilon} \mathrm{e}^{\varepsilon t \cos\rho} \mathrm{d}\rho =: I_{n,1}(t) + I_{n,2}(t).$$

Taking $\varepsilon = \frac{1}{t}$, we can assert that

$$I_{n,2}(t) \leq \frac{C|a_n|}{\lambda_n} \int_{-\theta}^{\theta} \frac{1}{\log \frac{1}{\varepsilon}} \mathrm{e}^{\varepsilon t \cos\rho} d\rho = \frac{C|a_n|}{\lambda_n} \int_{-\theta}^{\theta} \frac{1}{\log t} \mathrm{e}^{\cos\rho} \mathrm{d}\rho \leq \frac{C|a_n|}{\lambda_n \log t}$$

for sufficiently large t. In order to estimate $I_{n,1}(t)$, we proceed as follows:

$$I_{n,1}(t) = \frac{C|a_n|}{\lambda_n} \int_{\frac{1}{t}}^\infty |w(r\mathrm{e}^{\mathrm{i}\theta})| \mathrm{e}^{rt\cos\theta} \mathrm{d}r \leq \frac{C|a_n|}{\lambda_n} \int_0^\infty \int_0^1 r^{\alpha-1} \mathrm{d}\alpha \mathrm{e}^{rt\cos\theta} \mathrm{d}r$$

for sufficiently large t. From the Fubini's lemma, and noting that

$$\int_0^\infty r^{\alpha-1} \mathrm{e}^{-rs} \mathrm{d}r = \frac{\Gamma(\alpha)}{s^\alpha},$$

we arrive at the desired estimate

$$I_{n,1}(t) \leq \frac{C|a_n|}{\lambda_n} \int_0^1 \frac{\Gamma(\alpha)}{(t|\cos\theta|)^\alpha} \mathrm{d}\alpha \leq \frac{C|a_n|}{\lambda_n} \frac{1 - 1/t}{\log t} \leq \frac{C|a_n|}{\lambda_n \log t}, \quad n \geq N$$

for sufficiently large N and sufficiently large t. If $n < N$, then we note from Lemma 4.1 that $|sw(s) + \lambda_n| > C > 0$, and estimate the integral (4.3) as follows:

$$|u_n(t)| \leq C|a_n| \int_{1/t}^\infty \int_0^1 r^{\alpha-1} \mathrm{d}\alpha \mathrm{e}^{rt\cos\theta} \mathrm{d}r + C|a_n| \int_{-\theta}^{\theta} \frac{1/t - 1}{\log 1/t} \mathrm{e}^{\cos\rho} \mathrm{d}\rho$$

for sufficiently large t. Similarly to the case $n \geq N$ we see that the inequality $|u_n(t)| \leq \frac{C|a_n|}{\log t}$ holds true if t is large enough and $n < N$.

Collecting all the above estimates, we obtain

$$|u_n(t)| \leq \frac{C|a_n|}{\lambda_n \log t}$$

for sufficiently large t and $n = 1, 2, \ldots$. It follows from the representation (4.3) that

$$\|u(\cdot,t)\|^2_{H^2(\Omega)} \le C\sum_{n=1}^{\infty}\lambda_n^2 u_n^2 \le \sum_{n=1}^{\infty}\lambda_n^2\frac{Ca_n^2}{\lambda_n^2(\log t)^2} \le \frac{C\|a\|^2_{L^2(\Omega)}}{(\log t)^2},$$

that is, $\|u(\cdot,t)\|_{H^2(\Omega)} \le C\|a\|_{L^2(\Omega)}(\log t)^{-1}$ for sufficiently large t, which proves the first part of Theorem 4.4.

Now we assume that $\mu(\alpha) = \mu(0) + o(\alpha^\delta),\ \delta > 0$ as $\alpha \to 0$ and rewrite the representation (4.3) as follows:

$$\begin{aligned} u_n(t) =& \frac{a_n}{2\pi \mathrm{i}}\int_{\gamma(\varepsilon,\theta)}\Big(\frac{w(s)}{sw(s)+\lambda_n} - \frac{\mu(0)(s-1)}{\lambda_n s\log s}\Big)\mathrm{e}^{st}\mathrm{d}s \\ &+ \frac{a_n}{2\pi \mathrm{i}}\int_{\gamma(\varepsilon,\theta)}\frac{\mu(0)(s-1)}{\lambda_n s\log s}\mathrm{e}^{st}\mathrm{d}s =: R_{n,1}(t) + R_{n,2}(t). \end{aligned}$$

In order to estimate $R_{n,1}(t)$, for sufficiently large t, we set $\varepsilon := \frac{1}{t} < \varepsilon_0$ in the definition of the contour $\gamma(\varepsilon,\theta)$ and employ inequality (4.20) to obtain

$$\begin{aligned} |R_{n,1}(t)| \le& \int_{\gamma(\varepsilon,\theta)\cap\{|s|<\varepsilon_0\}}\frac{C|a_n|}{\lambda_n|s|(\log 1/|s|)^{\delta+1}}|\mathrm{e}^{st}\mathrm{d}s| \\ &+ C\int_{\gamma(\varepsilon,\theta)\cap\{|s|\ge\varepsilon_0\}}\left(\frac{|w(s)a_n|}{|sw(s)+\lambda_n|} + \frac{\mu(0)|a_n||s-1|}{\lambda_n|s||\log s|}\right)|\mathrm{e}^{st}\mathrm{d}s| \\ \le& \int_{\frac{1}{t}}^{\varepsilon_0}\frac{C|a_n|}{\lambda_n r(\log 1/r)^{\delta+1}}\mathrm{e}^{rt\cos\theta}\mathrm{d}r + \int_{-\theta}^{\theta}\frac{C|a_n|}{\lambda_n\frac{1}{t}(\log t)^{\delta+1}}\mathrm{e}^{\cos\rho}\tfrac{1}{t}d\rho \\ &+ C|a_n|\int_{\varepsilon_0}^{\infty}\frac{\int_0^1 r^{\alpha-1}\mu(\alpha)\mathrm{d}\alpha + \mu(0)\int_0^1 r^{\alpha-1}\mathrm{d}\alpha}{\lambda_n}\mathrm{e}^{rt\cos\theta}\mathrm{d}r =: \sum_{j=1}^{3}H_{n,j}(t). \end{aligned}$$

Here in the last inequality, we used the Lemma 4.2 and facts that $\int_0^1 s^{\alpha-1}\mathrm{d}\alpha = \frac{s-1}{s\log s}$ and $|w(s)| \le \int_0^1|s|^{\alpha-1}\mu(\alpha)\mathrm{d}\alpha$. We start with the integral $H_{n,1}(t)$. Noting that $r(\log 1/r)^{\delta+1} \ge \frac{1}{t}(\log t)^{\delta+1}$ for $\frac{1}{t} \le r \le \varepsilon_0$, where t is sufficiently large and $\varepsilon_0 > 0$ is small enough, we arrive at the estimate

$$H_{n,1}(t) \le \int_{\frac{1}{t}}^{\varepsilon_0}\frac{C|a_n|}{\lambda_n\frac{1}{t}(\log t)^{\delta+1}}\mathrm{e}^{rt\cos\theta}\mathrm{d}r \le \frac{C|a_n|}{\lambda_n(\log t)^{\delta+1}}$$

Now we consider the integral $H_{n,2}(t)$. A direct calculation yields

$$H_{n,2}(t) \le \frac{C|a_n|}{\lambda_n(\log t)^{\delta+1}}, \quad n \ge N.$$

Now let us consider the integral $H_{n,3}(t)$. In view of the inequality $\mathrm{e}^{rt\cos\theta} \le \frac{C}{|rt\cos\theta|^2}$ for $r > \varepsilon_0$, we have

$$|H_{n,3}(t)| \le \frac{C|a_n|}{\lambda_n}\int_{\varepsilon_0}^{\infty}\int_0^1 r^{\alpha-1}\mathrm{d}\alpha\frac{C}{r^2t^2}\mathrm{d}r \le \frac{C|a_n|}{\lambda_n t^2}, \quad n > N.$$

for sufficiently large t and N. Now we collect the estimates for the integrals $H_{n,j}(t)$, $j = 1, 2, 3$ and thus obtain the inequality

$$R_{n,1} \le \frac{C|a_n|}{\lambda_n(\log t)^{\delta+1}}, \quad n > N$$

for sufficiently large t and N.

In order to estimate $R_{n,2}(t)$, we shall show that $R_{n,2}(t)$ is the Laplace transform of a positive function, which implies that the function $R_{n,2}(t)$ is completely monotone. Once we have established this property of $R_{n,2}(t)$, the Karamata-Feller Tauberian theorem (see e.g., Chapter XIII in [15]) along with the asymptotic formula

$$\frac{s-1}{s\log s} = \frac{1}{s\log 1/s} - \frac{1}{\log 1/s} = \frac{1}{s\log 1/s} + \frac{o(1)}{s\log 1/s}, \quad s \to 0,$$

leads to the asymptotic formula

$$R_{n,2}(t) = \frac{\mu(0)a_n}{\lambda_n \log t} + \frac{o(1)a_n}{\lambda_n \log t}, \quad t \to \infty,$$

Consequently, we arrive at the desired representation

$$u_n(t) = \frac{\mu(0)a_n}{\lambda_n \log t} + \frac{a_n}{\lambda_n \log t}o(1), \quad t \to \infty.$$

Now it remains to show that the function $R_{n,2}(t)$ is indeed the Laplace transform of a positive function. First we note that $R_{n,2}(t)$ is independent of θ and ε because the function $\frac{s-1}{s\log s}$ is analytic on the main sheet of the Riemann surface for the logarithmic function. Now the function $R_{n,2}(t)$ is represented in the form

$$\begin{aligned} R_{n,2}(t) =& \frac{a_n\mu(0)}{2\pi\lambda_n \mathrm{i}} \int_{\gamma_c(\varepsilon,\theta)} \frac{s-1}{s\log s} \mathrm{e}^{st}\mathrm{d}s \\ &+ \frac{a_n\mu(0)}{2\pi\lambda_n \mathrm{i}} \int_{\gamma_\pm(\varepsilon,\theta)} \frac{s-1}{s\log s} \mathrm{e}^{st}\mathrm{d}s =: V_{n,1}(t) + V_{n,2}(t). \end{aligned}$$

For the function $V_{n,1}(t)$, the arguments similar to the estimation of $I_{n,2}(t)$ in the first part of our proof lead to the relation

$$|V_{n,1}(t)| \le \frac{C|a_n|}{\lambda_n} \frac{\mathrm{e}^{\varepsilon t}}{\log 1/\varepsilon} \to 0, \quad \varepsilon \to 0. \tag{4.21}$$

As to the function $V_{n,2}(t)$, we first represent it in the form

$$\begin{aligned} V_{n,2}(t) =& \frac{a_n\mu(0)}{\pi\lambda_n} \operatorname{Im} \int_\varepsilon^\infty \frac{r\mathrm{e}^{\mathrm{i}\theta} - 1}{r\mathrm{e}^{\mathrm{i}\theta}\log(r\mathrm{e}^{\mathrm{i}\theta})} \mathrm{e}^{r\mathrm{e}^{\mathrm{i}\theta}t}\mathrm{e}^{\mathrm{i}\theta}\mathrm{d}r \\ =& \frac{a_n\mu(0)}{\pi\lambda_n} \operatorname{Im} \int_\varepsilon^\infty \frac{1}{r}\frac{r\mathrm{e}^{\mathrm{i}\theta} - 1}{(\log r) + \mathrm{i}\theta} \mathrm{e}^{r\mathrm{e}^{\mathrm{i}\theta}t}\mathrm{d}r. \end{aligned}$$

Because the last integral is independent of the parameter θ, we consider its limit as $\theta \to \pi$. We thus get the representation

$$\begin{aligned} V_{n,2}(t) =& \frac{a_n\mu(0)}{\pi\lambda_n} \operatorname{Im} \int_\varepsilon^\infty \frac{-r-1}{(\log r) + \mathrm{i}\pi} \frac{\mathrm{e}^{-rt}}{r}\mathrm{d}r = \frac{a_n\mu(0)}{\lambda_n} \int_\varepsilon^\infty \frac{r+1}{(\log r)^2 + \pi^2} \frac{\mathrm{e}^{-rt}}{r}\mathrm{d}r \\ =& \frac{a_n\mu(0)}{\lambda_n} \int_\varepsilon^1 \frac{r+1}{(\log r)^2 + \pi^2} \frac{\mathrm{e}^{-rt}}{r}\mathrm{d}r + \frac{a_n\mu(0)}{\lambda_n} \int_1^\infty \frac{r+1}{(\log r)^2 + \pi^2} \frac{\mathrm{e}^{-rt}}{r}\mathrm{d}r. \end{aligned}$$

Employing the variables substitution $\log r \to r$, we see that

$$\int_0^1 \frac{1}{r}\frac{r+1}{(\log r)^2 + \pi^2} \mathrm{e}^{-rt}\mathrm{d}r = \int_\infty^0 \frac{\mathrm{e}^r + 1}{r^2 + \pi^2} \mathrm{e}^{-\mathrm{e}^r t}\mathrm{d}r$$

is finite for any $t > 0$ and the following limit value exists:

$$\lim_{\varepsilon\to 0}\lim_{\theta\to\pi} V_{n,2}(t) = \frac{a_n\mu(0)}{\lambda_n} \int_0^\infty \frac{1}{r}\frac{r+1}{(\log r)^2 + \pi^2} \mathrm{e}^{-rt}\mathrm{d}r.$$

The last formula along with the limit value (4.21) leads to

$$R_{n,2}(t) = \frac{a_n \mu(0)}{\lambda_n} \int_0^\infty \frac{r+1}{r} \frac{1}{(\log r)^2 + \pi^2} \mathrm{e}^{-rt} \mathrm{d}r,$$

which proves that the function $R_{n,2}(t)$ is indeed completely monotonic, which is our desired conclusion.

Summarizing now the above estimates for the coefficients $u_n(t)$ of the Fourier series (4.6), we finally obtain the asymptotic formula

$$\begin{aligned} u(\,\cdot\,,t) =& \frac{\mu(0)}{\log t} \sum_{n=1}^\infty \frac{a_n}{\lambda_n} \varphi_n + \frac{o(1)}{\log t} \sum_{n=1}^\infty \frac{a_n}{\lambda_n} \varphi_n \\ =& \frac{\mu(0)}{\log t} A^{-1} a + \frac{o(1)}{\log t} A^{-1} a, \quad t \to \infty, \end{aligned}$$

which implies

$$\|u(\,\cdot\,,t) - \tfrac{\mu(0)}{\log t} A^{-1} a\|_{H^2(\Omega)} = o((\log t)^{-1}) \|a\|_{L^2(\Omega)}, \quad t \to \infty.$$

This finishes the proof of Theorem 4.4. □

Proof of Corollary 4.1. The relation

$$\|(\log t) u(\cdot,t) - \mu(0) A^{-1} a\|_{H^2(\Omega)} = o(1) \|a\|_{L^2(\Omega)} \to 0, \quad t \to \infty$$

easily follows from the second part of Theorem 4.4. By the assumption of the corollary, $\|(\log t) u(\cdot,t)\|_{H^2(\Omega)} \to 0$ as $t \to \infty$. Hence the relation $\mu(0) A^{-1} a = 0$ is valid. Since $\mu(0) \neq 0$, the equation $A^{-1} a = 0$ is satisfied. By the conditions we posed on the operator A, the above equation possesses only one trivial solution $a = 0$. In this case, the initial-boundary value problem (4.1) has only the trivial solution, too, e.g., $u(x,t) = 0$ for $x \in \Omega$ and $t > 0$. This completes the proof of Corollary 4.1. □

4.5 Lipschitz stability with respect to distributed orders and coefficients

Based on the above theorem, in this section, we follow the framework of the argument used in [37] to further verify the Lipschitz continuous dependency of the solution to (2.17) with respect to μ and the diffusion coefficient in $\mathcal{A}$, which is fundamental for the optimization approach to the related coefficient inverse problem.

More precisely, we evaluate the difference between the solutions u and $\widetilde{u}$ to

$$\begin{cases} \mathbb{D}_t^{(\mu)} u = L_D u & \text{in } Q, \\ u|_{t=0} = a & \text{in } \Omega, \\ u = 0 & \text{on } \Sigma \end{cases} \tag{4.22}$$

and

$$\begin{cases} \mathbb{D}_t^{(\widetilde{\mu})} \widetilde{u} = L_{\widetilde{D}} \widetilde{u} & \text{in } Q, \\ \widetilde{u}|_{t=0} = a & \text{in } \Omega, \\ \widetilde{u} = 0 & \text{on } \Sigma \end{cases} \tag{4.23}$$

respectively, where $L_D u(x,t) := \operatorname{div}(D(x) \nabla u(x,t))$ and D denotes the diffusion coefficient. To this end, we restrict the coefficients in the admissible sets

$$\begin{aligned} \mathcal{W}_{\alpha_0} &:= \{\mu \in C[0,1];\ \mu \ge 0, \mu(\alpha_0) > 0\}, \\ \mathcal{U} &:= \{D \in C^1(\overline{\Omega});\ D \ge \delta \text{ in } \overline{\Omega},\ \|D\|_{C^1(\overline{\Omega})} \le M\}. \end{aligned} \tag{4.24}$$

Lemma 4.10 *Under the assumptions in Theorem 4.1. We further assume $a \in H^{2\gamma}(\Omega)$ with $(0 < \gamma \le 1)$ and μ satisfies (4.14). Then the solution $u(t)$ to the initial-boundary value problem (4.1) with $F = 0$ admits the following estimate*

$$\|\partial_t u(t)\|_{L^2(\Omega)} \le C\|a\|_{H^{2\gamma}(\Omega)} t^{-\gamma_0}, \quad t \in (0, T], \tag{4.25}$$

where $\gamma_0 > 1 - \alpha_0\gamma$.

Proof. We will first show

$$\left|\frac{\mathrm{d}E_n^{(\mu)}(t)}{\mathrm{d}t}\right| \le C\lambda_n^\gamma t^{-\gamma_0}, \quad \gamma \in (0, 1)$$

for $t \in (0, T]$ and $n = 1, 2, \cdots$.

For this, in view of the Cauchy theorem and notation (4.15), we see that

$$\frac{\mathrm{d}E_n^{(\mu)}(t)}{\mathrm{d}t} = \frac{1}{2\pi\mathrm{i}}\int_{\gamma(\varepsilon,\theta)} \frac{sw(s)}{sw(s) + \lambda_n}\mathrm{e}^{st}\mathrm{d}s = \frac{-1}{2\pi\mathrm{i}}\int_{\gamma(\varepsilon,\theta)} \frac{\lambda_n}{sw(s) + \lambda_n}\mathrm{e}^{st}\mathrm{d}s = -\lambda_n G_n^{(\mu)}.$$

Now taking $\kappa = 1 - \gamma$ in the estimate in Lemma 4.8, it follows that

$$\lambda_n |G_n^{(\mu)}(t)| \le \frac{C\lambda_n^\gamma}{t^{\gamma_0}}, \quad 1 - \alpha_0\gamma < \gamma_0 < 1.$$

Consequently, we have

$$\left|\frac{\mathrm{d}E_n^{(\mu)}(t)}{\mathrm{d}t}\right| \le \frac{C\lambda_n^\gamma}{t^{\gamma_0}}, \quad t \in (0, T],\ \gamma \in (0, 1],\ n \in \mathbb{N},$$

where $1 - \alpha_0\gamma < \gamma_0 < 1$. Now combining the above estimates, from (4.3) and (4.6), we find

$$\|\partial_t u(t)\|_{L^2(\Omega)} = \sum_{n=1}^{\infty}(a, \varphi_n)^2 \left|\frac{\mathrm{d}E_n^{(\mu)}(t)}{\mathrm{d}t}\right|^2 \le \sum_{n=1}^{\infty}\lambda_n^{2\gamma}(a, \varphi_n)^2 t^{-2\gamma_0},$$

which derives

$$\|\partial_t u(t)\|_{L^2(\Omega)} \le C\|a\|_{H^{2\gamma}(\Omega)} t^{-\gamma_0}, \quad 0 < \gamma \le 1,\ 1 - \alpha_0\gamma < \gamma_0 < 1,$$

which completes the proof of Lemma 4.10. □

Under these settings, we can show the following result on Lipschitz stability.

Theorem 4.5 *Fix $\gamma \in (0, 1]$ and $\kappa \in (0, 1)$. Let u and $\widetilde{u}$ be the solutions to (4.22) and (4.23) respectively, where*

$$a \in D(A^\gamma), \quad \mu \in \mathcal{W}_{\alpha_0}, \quad D, \widetilde{D} \in \mathcal{U}$$

and $\mathcal{W}_{\alpha_0}$, $\mathcal{U}$ are defined in (4.24). Then for any $\kappa_0 \in (1 - \alpha_0(1 - \kappa), 1)$ and $1 \le p < \frac{1}{\kappa_0}$, there exists a constant $C > 0$ depending only on a, T, κ, κ_0, $\mathcal{W}_{\alpha_0}$ and $\mathcal{U}$ such that

$$\|u - \widetilde{u}\|_{L^p(0,T;\mathcal{D}(A^\kappa))} \le C\left(\|\mu - \widetilde{\mu}\|_{C[0,1]} + \|D - \widetilde{D}\|_{C^1(\overline{\Omega})}\right). \tag{4.26}$$

The above theorem extends a similar result in [37] for the multi-term case. It is also fundamental for the optimization method for an inverse problem of determining $\mu, D(x)$ by extra data of the solution.

As a direct application of Lemma 4.8 and Lemma 4.10, it is straightforward to show the Lipschitz stability of the solution with respect to various coefficients.

Proof of Theorem 4.5. Let $\gamma, \kappa \in (0,1]$, $a \in \mathcal{D}(A^\gamma)$ and $C > 0$ be a general constant which depends only on γ, κ, α_0, a, $\mathcal{W}_{\alpha_0}$ and $\mathcal{U}$. First, by taking the difference of systems (4.23) and (4.22), it turns out that the system for $v := u - \widetilde{u}$ reads

$$\begin{cases} \mathbb{D}_t^{(\mu)} v = L_D v + F & \text{in } Q, \\ v|_{t=0} = 0 & \text{in } \Omega \\ v = 0 & \text{on } \Sigma, \end{cases}$$

where

$$F := \mathbb{D}_t^{(\widetilde{\mu}-\mu)} \widetilde{u} + L_{D-\widetilde{D}} \widetilde{u}.$$

Therefore, it suffices to dominate F by the difference of coefficients.

To this end, first it is readily seen from Lemma 4.7 and Lemma 4.10 immediately yields

$$\|u\|_{L^1(0,T;H^2(\Omega))} \le CT^\gamma, \quad \|\partial_t u\|_{L^1(0,T;L^2(\Omega))} \le CT^{\alpha_0\gamma}, \tag{4.27}$$

where $\mu(\alpha_0) > 0$, and $\alpha_0 \in (0,1)$. Therefore, together with $D, \widetilde{D} \in C^1(\overline{\Omega})$, we see

$$\|L_{\widetilde{D}-D} \widetilde{u}\|_{L^1(0,T;H^2(\Omega))} \le C\|D - \widetilde{D}\|_{C^1(\overline{\Omega})} \|\widetilde{u}\|_{L^1(0,T;H^2(\Omega))}.$$

On the other hand, a direct application of the definition of the distributed order derivative, we obtain

$$|\mathbb{D}_t^{(\widetilde{\mu}-\mu)} \widetilde{u}(t)| \le \|\widetilde{\mu} - \mu\|_{C[0,1]} \int_0^1 |\partial_t^\alpha \widetilde{u}(t)| \mathrm{d}\alpha \le \|\widetilde{\mu} - \mu\|_{C[0,1]} \int_0^1 \frac{1}{\Gamma(1-\alpha)} \int_0^t \frac{|\partial_r \widetilde{u}(r)|}{(t-r)^\alpha} \mathrm{d}r \mathrm{d}\alpha.$$

Therefore the combination of the above estimate and Young's inequality immediately yields that

$$\begin{aligned} \|\mathbb{D}_t^{(\widetilde{\mu}-\mu)} \widetilde{u}\|_{L^1(0,T;L^2(\Omega))} &\le C\|\widetilde{\mu} - \mu\|_{C[0,1]} \int_0^1 \left(\int_0^T \frac{t^{-\alpha}}{\Gamma(1-\alpha)} \mathrm{d}t \right) \left(\int_0^T |\partial_t \widetilde{u}(t)| \mathrm{d}t \right) \mathrm{d}\alpha \\ \le & C\|\widetilde{\mu} - \mu\|_{C[0,1]} \int_0^1 \frac{T^{1-\alpha}}{(1-\alpha)\Gamma(1-\alpha)} \mathrm{d}\alpha = C\|\widetilde{\mu} - \mu\|_{C[0,1]} \int_0^1 \frac{T^{\alpha_0\gamma} T^{1-\alpha}}{\Gamma(2-\alpha)} \mathrm{d}\alpha, \end{aligned}$$

here $\alpha_0 \in (0,1)$ such that $\mu(\alpha_0) > 0$, and in the last equality, we used the property of the Gamma function $\Gamma(1+\alpha) = \alpha\Gamma(\alpha)$, $\alpha > 0$. Now from the analyticity of the Gamma, it follows that

$$\|\mathbb{D}_t^{(\widetilde{\mu}-\mu)} \widetilde{u}\|_{L^1(0,T;L^2(\Omega))} \le \frac{CT^{\alpha_0\gamma}}{|\log T|} \|\widetilde{\mu} - \mu\|_{C[0,1]}.$$

Therefore, we see $F \in L^1(0,T;L^2(\Omega))$ from (4.27), and that

$$\|F\|_{L^1(0,T;L^2(\Omega))} \le \frac{CT^{\alpha_0\gamma}}{\log T} \|\mu - \widetilde{\mu}\|_{C[0,1]} + CT^\gamma \|D - \widetilde{D}\|_{C^1(\overline{\Omega})}.$$

Consequently, it is straightforward to employ Lemma 4.8 to obtain

$$\begin{aligned} \|u - \widetilde{u}\|_{L^p(0,T;\mathcal{D}(\mathcal{A}^\kappa))} \le & CT^{1/p-\kappa_0} \|F\|_{L^1(0,T;L^2(\Omega))} \\ \le & \frac{CT^{1/p+\alpha_0\gamma-\kappa_0}}{|\log T|} \|\mu - \widetilde{\mu}\|_{C[0,1]} + CT^{1/p+\gamma-\kappa_0} \|D - \widetilde{D}\|_{C^1(\overline{\Omega})}, \end{aligned}$$

where $\alpha_0 \in (0,1)$ satisfies $\mu(\alpha_0) > 0$, $\gamma \in (0,1]$, $1 \le p < \frac{1}{\kappa_0}$, and $\kappa \in [0,1)$ such that $1 - \alpha_0(1-\kappa) < \kappa_0 < 1$. $\square$

4.6 Inverse problem

As we mentioned above, the distributed order time-fractional diffusion equations can be regarded as a kind of generalization for the multi-term case, it is natural to expect the parallel generalization for the inverse problems,

For the statement of our main problem, we introduce an admissible set of unknown weight function. We set

$$\mathcal{U} = \{\mu \in C[0,1]; \mu \geq 0, \not\equiv 0\}.$$

Problem 4.1 *Assume $F = 0$ in (4.1). Let $x_0 \in \Omega$ be fixed and let $I \subset (0,T)$ be a non-empty open interval. Let u, v be the solutions to the initial-boundary value problems (4.1) with respect to μ_1, $\mu_2 \in \mathcal{U}$ separately. We will investigate whether $u = v$ in $\{x_0\} \times I$ can derive $\mu_1 = \mu_2$.*

This is our inverse problem, and in this section we discuss the uniqueness as the fundamental theoretical topic for the inverse problem and attempt to establish results parallel to that for the multi-term case.

We have

Theorem 4.6 (Uniqueness) *Let $\mu_1, \mu_2 \in \mathcal{U}$. Assume that $a \geq 0$ in Ω, $a \not\equiv 0$ and $a \in D(A^\gamma)$ with $\gamma > \max\{\frac{d}{2} + \delta - 1, 0\}$, $\delta > 0$ can be sufficiently small. Then $\mu_1 = \mu_2$ provided*

$$u(x_0, t) = v(x_0, t),\ x_0 \in \Omega,\ t \in I.$$

Proof. From the arguments in Section 4.2.1, we know that

$$u(\cdot, t) = \frac{1}{2\pi \mathrm{i}} \sum_{n=1}^{\infty} \int_{M-\mathrm{i}\infty}^{M+\mathrm{i}\infty} \frac{w_1(s)}{sw_1(s) + \lambda_n} \mathrm{e}^{st} \mathrm{d}s (a, \varphi_n)\varphi_n, \tag{4.28}$$

$$v(\cdot, t) = \frac{1}{2\pi \mathrm{i}} \sum_{n=1}^{\infty} \int_{M-\mathrm{i}\infty}^{M+\mathrm{i}\infty} \frac{w_2(s)}{sw_2(s) + \lambda_n} \mathrm{e}^{st} \mathrm{d}s (a, \varphi_n)\varphi_n \quad \text{in } L^2(\Omega) \tag{4.29}$$

for each $t \in [0,T]$. The Sobolev embedding inequality yields that $\|\varphi_n\|_{C(\overline{\Omega})} \leq C\|A^{\frac{d}{4}+\varepsilon}\varphi_n\|_{L^2(\Omega)}$ with sufficiently small $\varepsilon > 0$, and we have $C_0 n^{\frac{2}{d}} \leq \lambda_n \leq C_1 n^{\frac{2}{d}}$ (see, e.g., [10]). Therefore, fixing $t_0 > 0$ arbitrarily, from the estimate in Lemma 4.3, for $t \in [t_0, T]$, we obtain

$$\begin{aligned}
&\frac{1}{2\pi} \sum_{n=1}^{\infty} \left| \int_{M-\mathrm{i}\infty}^{M+\mathrm{i}\infty} \frac{w_j(s)}{sw_j(s) + \lambda_n} \mathrm{e}^{st} \mathrm{d}s \right| \|(a, \varphi_n)\varphi_n\|_{C(\overline{\Omega})} \leq C \sum_{n=1}^{\infty} \frac{1}{\lambda_n t} \|(A^\gamma a, \varphi_n) A^{-\gamma}\varphi_n\|_{C(\overline{\Omega})} \\
\leq & C \sum_{n=1}^{\infty} |(A^\gamma a, \varphi_n)| \frac{1}{\lambda_n t_0} \lambda_n^{\frac{d}{4}+\varepsilon-\gamma} \leq C \sum_{n=1}^{\infty} |(A^\gamma a, \varphi_n)| \lambda_n^{\frac{d}{4}+\varepsilon-\gamma-1} \\
\leq & C \left(\sum_{n=1}^{\infty} |(A^\gamma a, \varphi_n)|^2 \right)^{\frac{1}{2}} \left(\sum_{n=1}^{\infty} \lambda_n^{\frac{d}{2}+2\varepsilon-2\gamma-2} \right)^{\frac{1}{2}}, \quad j = 1, 2.
\end{aligned}$$

By $\lambda_n \sim n^{\frac{2}{d}}$ as $n \to \infty$ (e.g., [10]) and $\gamma > \frac{d}{2} - 1$, we see that $\sum_{n=1}^{\infty} \lambda_n^{\frac{d}{2}+2\varepsilon-2\gamma-2} < \infty$. Hence

$$\frac{1}{2\pi} \sum_{n=1}^{\infty} \left| \int_{M-\mathrm{i}\infty}^{M+\mathrm{i}\infty} \frac{w_j(s)}{sw_j(s) + \lambda_n} \mathrm{e}^{st} \mathrm{d}s \right| \|(a, \varphi_n)\varphi_n\|_{C(\overline{\Omega})} < \infty, \quad t_0 \leq t \leq T,\ j = 1, 2. \tag{4.30}$$

Therefore, we see that the series on the right-hand side of (4.28) and (4.29) are convergent uniformly in $x \in \overline{\Omega}$ and $t \in [t_0, T]$. Moreover, since the solutions u and v can be analytically extended to $t > 0$ in view of Theorem 4.1, we have $u(x_0, t) = v(x_0, t)$ for $t > 0$. Consequently by the Laplace transform, we obtain

$$\sum_{n=1}^{\infty} \rho_n \frac{w_1(s)}{sw_1(s) + \lambda_n} = \sum_{n=1}^{\infty} \rho_n \frac{w_2(s)}{sw_2(s) + \lambda_n}, \quad s > 0,$$

where $\rho_n = (a, \varphi_n)\varphi_n(x_0)$. Moreover, noting $\gamma > \frac{d}{2} - 1$, similarly to (4.30), we have $\sum_{n=1}^{\infty} |\rho_n| < \infty$. Therefore

$$\sum_{n=1}^{\infty} \frac{\lambda_n \rho_n}{sw_1(s) + \lambda_n} = \sum_{n=1}^{\infty} \frac{\lambda_n \rho_n}{sw_2(s) + \lambda_n}, \quad s \in \mathbb{R} \text{ with } |s| \text{ small enough},$$

where the series on both sides are uniformly convergent for $|s|$ small enough. Consequently, we find

$$(sw_1(s) - sw_2(s)) \sum_{n=1}^{\infty} \frac{\rho_n \lambda_n}{(sw_1(s) + \lambda_n)(sw_2(s) + \lambda_n)} = 0. \tag{4.31}$$

Since $\sum_{n=1}^{\infty} |\rho_n| < \infty$, and $\lambda_n > 0$, $\lim \lambda_n = \infty$, we see that $\sum_{n=1}^{\infty} \frac{\rho_n}{\lambda_n} < \infty$. Moreover, by the assumption of a, we have $\sum_{n=1}^{\infty} \lambda_n^{-1}(a, \varphi_n)\varphi_n(x_0) = -(A^{-1}a)(x_0)$. Setting $b = -A^{-1}a$, we have $Ab = -a$ and $b|_{\partial\Omega} = 0$. By the strong maximum principle for $Au = -\sum_{i,j=1}^{d} \partial_j(a_{ij}\partial_i u)$ with $a \geq 0$, we have $b < 0$ in Ω. Hence

$$\sum_{n=1}^{\infty} \lambda_n^{-1}(a, \varphi_n)\varphi_n(x_0) > 0.$$

By noting that $\sum_{n=1}^{\infty} \frac{\rho_n}{\lambda_n} < \infty$, it is not difficult to show that the function

$$\sum_{n=1}^{\infty} \frac{\rho_n \lambda_n}{(sw_1(s) + \lambda_n)(sw_2(s) + \lambda_n)}$$

is continuous for s in $(0, \epsilon)$ with sufficiently small $\epsilon > 0$, it follows that

$$\sum_{n=1}^{\infty} \frac{\rho_n \lambda_n}{(sw_1(s) + \lambda_n)(sw_2(s) + \lambda_n)} > 0$$

is valid for $s \in (0, \varepsilon)$. Therefore (4.31) implies

$$sw_1(s) - sw_2(s) = 0, \quad s \in (0, \varepsilon),$$

The use of the analyticity of the functions sw_1 and sw_2 gives

$$\int_0^1 (\mu_1(\alpha) - \mu_2(\alpha))s^{\alpha} \mathrm{d}\alpha = 0, \quad s > 0,$$

that is

$$\int_0^1 (\mu_1(\alpha) - \mu_2(\alpha))\mathrm{e}^{\alpha \log s} \mathrm{d}\alpha = 0, \quad s > 0.$$

After the change of the variable, we find

$$\int_0^{\infty} \chi_{(0,1)}(\mu_1(\alpha) - \mu_2(\alpha))\mathrm{e}^{-\alpha s} \mathrm{d}\alpha = 0, \quad s > 0,$$

which yields

$$\mu_1(\alpha) = \mu_2(\alpha), \quad \alpha \in [0, 1]$$

in view of the uniqueness of the Laplace transform. □

4.7 Conclusions and open problems

We summarize this chapter by providing several concluding remarks. Concerning the initial-boundary value problem (4.1), we mainly dealt with forward and inverse problems for the initial-boundary value problems for the fractional diffusion equations of distributed orders on the bounded multi-dimensional domains.

For the forward problem, on the basis of the representation of solutions and a careful analysis of several contour integral representation of inversion Laplace transform, we succeed in dominating the solutions by the initial value a and the source term F (See Theorems 4.1-4.2 and Theorem 4.3). By using the above estimate, we establish the Lipschitz stability of the solution with respect to μ and the diffusion coefficient. Furthermore, we succeed in showing the long-time asymptotic behavior on the basis of the representation of solutions and the Laplace transform. These properties are very essential for analysis of the suitable numerical methods for this type of problems and for dealing with the inverse problems for the fractional diffusion equations of distributed orders.

For the inverse problem, as a direct application of the analyticity of the solution to the initial-boundary value problem (4.1), the uniqueness of the determination of the weight function μ is established. We compare the conclusions in Theorem 2.5 with those of multi-term cases obtained in Chapter 2. It turns out that Theorem 4.6 is a parallel extension of its multi-term counterpart.

As to the open problems related to the initial-boundary value problems for the fractional diffusion equations of distributed orders, let us mention the following ones: In the proofs of our results, we needed the non-negativity of the weight function $\mu(\alpha)$, $\alpha \in [0,1]$ that are necessary for ensuring that the Laplace transform $\widehat{u}(x,s)$ of the solution u has no poles in the main sheet of the Riemann surface of the logarithmic function. It would be interesting to investigate what happens with the asymptotic properties of the solutions if one or both of these assumptions are not valid. Another interesting direction of research would be to investigate the initial-boundary value problems for the fractional diffusion equations of distributed orders with the non-continuous weight functions. As we established in this chapter, the characteristic behavior of the solutions is very different in the case of the singular weight functions in the form $\mu(\alpha) = \delta(\alpha - \alpha_0)$ and the continuous functions. The case of the non-continuous but also non-singular weight functions will probably lead to yet another type of asymptotic behavior of solutions to the problem under consideration.

Finally, the question if the estimate for the short-time asymptotic of the solution that was presented in Theorem 4.5 is sharp or not still remains open and should be investigated.

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