Excluded Minors of Matroid Representability and Orientability, and Fast Parameterized Algorithms for Ising Partition Function

マトロイド表現可能性・向き付け可能性の禁止マイナー，および，グラフ上イジング分配関数の高速計算アルゴリズムに関する研究

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ABSTRACT

In this thesis, we deal with two problems fundamental in discrete mathematics and computer science: the representability of matroids and computation of Ising partition function.

Being originally introduced by H. Whitney in 1935 in an attempt to axiomize linearly independent relations in vector spaces, matroids have become known to be so fundamental underlying structures in discrete mathematics and combinatorial optimization. A wide variety of objects in graphs, geometries and combinatorial problems are described in terms of matroids. While this universality of matroids provides us a unified way to deal with seemingly-different objects and problems, matroids are in a certain sense a roughly-abstracted structure. A given matroid often has no corresponding representation by graphs and geometries over vector spaces. Hence, the problem of characterizing the representability of matroids, i.e. to pin down what kind of matroids have actual representations especially by linearly independent relations of vectors in vector spaces over fields, is intrinsically important.

On the other hand, Ising model was introduced by E. Ising in 1925 as a mathematical model on graphs to describe ferromagnetism in statistical physics, and later became known to bridge between several different areas such as statistical physics, quantum computation, discrete mathematics, combinatorial optimization and so on.

Matroid theory and computation of Ising partition function are known to be connected via Tutte polynomial, or Tutte-Whitney polynomial, which is the well-known matroid-invariant polynomial; computing Ising partition function of Ising model on graphs can be regarded as evaluation of Tutte polynomial of the special class of matroids called graphic matroids, i.e. the class of matroids which has the representation by forests and cycles of graphs.

In this thesis, toward the application of matroid representation theory into the computation of Ising partition function in the future, we investigate the representability of matroids and the exact computation of Ising partition function.

In the first part, we deal with the excluded minors' characterization for the two types of classes of matroids: the class of orientable matroids and the classes of matroid representable over infinite fields.

First, we investigate the excluded minors for the class of orientable matroids. Orientable matroids can be regarded as a generalization of the representability over the ordered field. We mainly show that there exist infinitely many excluded minors of rank 3 with n elements for $n \geq 7$ by constructing two new infinite families YM^1_{3n} and F_{3n-2} of excluded minors of rank 3 for the class of orientable matroids. Then the representabiliy of these two new families are investigated; YM^1_{3n} is not representable over fields of characteristic 2, and F_{3n-2} is not representable over fields of characteristic other than 2.

Second, the excluded minors' characterization for the union and intersection of the class of orientable matroids and the class of matroids representable over field is investigated. Regarding the union, we show that there exist infinitely many excluded minors of rank 3 for the union of the class of orientable matroids and the classes of matroids representable over fields. To prove this, we explicitly construct the family F^+_{3n-2} of excluded minors of rank 3 by modifying F_{3n-2}. This result implies that, even restricting the rank of the union to at most fixed constant, characterization for the union by a finite list of excluded minors is still impossible. Regarding the intersection, we show that there exist infinitely many **orientable** excluded minors of rank 3 for the intersection of the class of orientable matroids and the classes of matroid representable over fields of characteristic 0. This result implies the following two statements: (i) characterization for the intersection by a finite list of excluded minors is impossible, even restricting the rank of the union to at most fixed constant, and (ii) even within the class of orientable matroids, the characterization for the representability over fields of characteristic 0 by a finite list of excluded minors is impossible.

Third, we investigate the excluded minors' characterization for the class of matroids representable over the rational field $\mathbb{Q}$. We show that there exist infinitely many $\mathbb{Q}[x]$-**representable** excluded minors of rank 3 for $\mathbb{Q}$-representable matroids, where x is an element algebraic over $\mathbb{Q}$, the degree of whose minimal polynomial is two. This result implies that, even within the class of $\mathbb{Q}[x]$-representable matroids and restricting the rank of the class of $\mathbb{Q}$-representable matroids to at most fixed constant, characterization for the class of $\mathbb{Q}$-representable matroids by a finite list of excluded minors is impossible.

In the second part, we develop two fast parameterized exact algorithms to compute Ising partition function of the most general case of Ising model on graphs. Our algorithms exploit the dynamic programming on graph decompositions. The fast algorithm exploits the branch decomposition of graphs. Given an optimal branch decomposition of a graph G with width $bw(G)$, our algorithm runs in $O^*(2^{\frac{\omega}{2}bw(G)})$-time where ω is a matrix multiplication constant which is known to be less than 2.37287. The second algorithm exploits the rank decomposition of graphs. For a graph G, given an optimal rank decomposition of the subdivided graph $\tilde{G}$ of G with width $rw(\tilde{G})$, our algorithm runs in $O^*(4^{rw(\tilde{G})})$-time. Applied into square grids on square grids, this algorithm computes the Ising partition function of the Ising model on the $\sqrt{n} \times \sqrt{n}$ square grid in $O(n2^{\sqrt{n}})$-time. Relating to the second algorithm, we also develop the alorithm using the binary decision diagram (BDD), and analyze its time complexity with linear rank width. This algorithm is obtained via classically simulating the quantum algorithm to compute Ising partition function by BDD.

論文要旨

本論文では，離散数学や計算機科学において，本質的な意味を持ちうる二つの問題，マトロイドの表現可能性とイジング分配関数の計算を扱う．

マトロイドは 1935 年に H. Whitney により，ベクトルの一次独立性を公理化する試みの中で導入された概念である．後にマトロイドは，一次独立性に限らず，離散数学や組合せ最適化における種々の対象の疎構造となっていることが明らかになった．マトロイドは，グラフや幾何の一見異なる様々な対象を統一的に扱える普遍的な枠組みである一方で，その普遍性ゆえに，グラフや幾何における具体的な対応物によって表現できないようなマトロイドが多く存在する．それ故，どのようなマトロイドが具体的な表現（特にベクトル空間の一次独立関係における表現）を持つかを特徴づける研究は，本質的な重要性を持ち，長く研究されてきた．

他方，イジング模型は 1925 年に E. Ising により，強磁性を統計力学的に記述するモデルとして導入された．その後，イジング模型は，統計力学だけでなく，量子計算，離散数学，組合せ最適化といった幅広い分野の橋渡しをするモデルであることが知られるようになった．マトロイド理論とイジング模型における分配関数の計算は，Tutte 多項式（または Tutte-Whitney 多項式）というグラフ・マトロイド上において定義される，ある多項式を通じて結びつくことが知られている．グラフ上のイジング模型の計算は，グラフの森や閉路における表現を持つマトロイド（グラフ的マトロイド）に関する Tutte 多項式の計算と等価である．

本論文では，将来的にマトロイドの表現可能性の理論をイジング模型上の分配関数の計算に応用することを念頭に，マトロイドの表現可能性に関する研究と，イジング分配関数計算に関するアルゴリズムの開発を行った．

第一に，我々は，表現可能マトロイドと向き付け可能マトロイドという二つのマトロイドのクラスに関する禁止マイナーについての話題を扱う．

まず，我々は向き付け可能マトロイドに関する禁止マイナーに関し調べる．向き付け可能性は，順序体上ベクトル空間における表現可能性の拡張と考えることができる概念である．我々は，向き付け可能マトロイドに関する階数 3 の禁止マイナーが，7 以上の任意の要素数に存在することを，二つの新たな禁止マイナーの無限族 YM^1_{3n} と F_{3n-2} を構成することによって示す．そして，これら二つの無限族に関し，YM^1_{3n} は標数が 2 の体上ベクトル空間での一次独立関係による表現を持たないこと，F_{3n-2} は標数が 2 でない体上ベクトル空間での一次独立関係による表現を持たないことを示す．

次に，向き付け可能マトロイドと表現可能マトロイドの和集合と積集合に関する禁止マイナーによる特徴づけについて調べる．和集合に関しては，F_{3n-2} を基に，新たな階数 3 のマトロイドの無限族 F^+_{3n-2} を構成し，それが和集合に関する禁止マイナーになっていることを示す．これにより，和集合に関して，階数が高々定数以下のものに限定してさえ，有限個の禁止マイナーによる特徴づけが不可能であるという結果が得られる．積集合に関しては，向き付け可能マトロイドと，標数 0 の体上ベクトル空間で表現可能なマトロイドの積集合について，階数 3 の禁止マイナーが存在することを，具体的に禁止マイナーを構成することにより示す．さらに，それら禁止マイナーが向き付け可能であることを示す．これらにより，和集合の場合と同様，積集合に関しても，階数が高々定数以下のものに限定してさえ，有限個の禁止マイナーによる特徴づけが不可能であること，そして，マトロイド一般ではなく，向き付け可能マトロイドの中に限定してさえ，標数 0 の体上ベクトル空間における表現可能性が有限個の禁止マイナーにより特徴づけられないことが分かる．

最後に，有理数体 $\mathbb{Q}$ 上のベクトル空間における表現可能性に関して調べる．x を $\mathbb{Q}$ の代数的元とし，その最小多項式の次数が 2 であるとしたとき，$\mathbb{Q}[x]$ 上表現可能マトロイドの中でさえ，$\mathbb{Q}$ に対する階数 3 の禁止マイナーが無限個存在することを示す．これは，単拡大し

た $\mathbb{Q}[x]$ 体上表現可能マトロイドの中でさえ，そしてさらに，階数を高々定数以下のものに限定してさえ，$\mathbb{Q}$ 体上表現可能マトロイド関する特徴づけが困難であることを示している．

第二に，我々はグラフ上イジング模型の分配関数の計算に関し, 最も一般的に場合に関する二種類の高速計算アルゴリズムを開発する．これらのアルゴリズムはグラフ分解上での動的計画法に基づいたものである．一つ目のアルゴリズムは，枝分解というグラフ分解を用いたもので，グラフ G に対し，最適な枝分解（幅を $bw(G)$ とする）が得られている場合に，$O^*(2^{\frac{\omega}{2}bw(G)})$ 時間で分配関数を厳密に計算するアルゴリズムである（ここで ω は行列乗算に関する定数で，現在 2.37287 未満であることが知られている）．二つ目のアルゴリズムは，階数分解というグラフ分解を用いたものである．このアルゴリズムは，グラフ G に対し G の枝上に新たな点を挿入したグラフ $\tilde{G}$ の最適な階数分解（幅を $rw(\tilde{G})$ とする）が得られているときに，$O^*(4^{rw(\tilde{G})})$ 時間で分配関数を厳密に計算するアルゴリズムである．またこのアルゴリズムは，$\sqrt{n} \times \sqrt{n}$ の正方格子上イジング模型に対しては，$O(n2^{\sqrt{n}})$ 時間で分配関数を計算するアルゴリズムとなっている．また二つ目のアルゴリズムと関係して，我々は，計算機科学で普遍的なデータ構造である二分決定図を用いたアルゴリズムを構築し，その実行時間を線形階数幅を用いて解析する．このアルゴリズムは，イジング分配関数を計算する既存の量子アルゴリズムを二分決定図を用いて古典的に模倣するものである．

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Chapter 1

Introduction

1.1 Matroids

1.1.1 Representability of Matroids

Matroids were originally introduced in 1935 by Whitney [84] in an attempt to axiomize linearly independent and dependent relations of vectors in vector spaces. Let E be a finite set of vectors in a vector space over a field, and $\mathcal{I}$ be the collection of all linearly independent subsets of E. Whitney focused on the following two properties which the independent subsets of E always satisfy:

(I-1) For $I \in \mathcal{I}$, any subset of I is again contained in $\mathcal{I}$.

(I-2) For $I_1, I_2 \in \mathcal{I}$, if $|I_1| < |I_2|$, then there exists $e \in I_2 \setminus I_1$ such that $I_1 \cup \{e\} \in \mathcal{I}$.

Whitney adopted these two properties as an abstraction of linearly independent relation of vectors. A matroid on a finite set E is defined by a family of the subsets of E which satisfies the above conditions (I-1) and (I-2); the finite set E is called the *ground set*. The axiom system consisting of (I-1) and (I-2) is called the *independent axiom system*. Apart from the independent axiom system, many other equivalent axiom systems for matroids are also known, three of which were presented by Whitney in the initial paper [84]. Each axiom system describes the combinatorial structures of vectors in vector spaces from different perspectives.

Being equipped with many equivalent axiom systems, matroids have become known to be so fundamental underlying structures in discrete mathematics and combinatorial optimization. While matroids are so universal a structure to provide a unified way to deal with seemingly-different objects and problems, matroids are in a certain sense a roughly-abstracted structure. There exist many matroids which cannot be underlying structures of the concrete mathematical objects. Therefore, it is intrinsically important to investigate the relation between matroids and its representation by some mathematical objects.

Now we will briefly review what kinds of mathematical objects can be described in the framework of matroids. As already mentioned, matroids cover the linearly independent relation of vectors in the vector space over the field $\mathbb{F}$. A matroid is $\mathbb{F}$-*representable* or *representable over* $\mathbb{F}$ if it can be recovered from linearly independent relation by some finite set of vectors in the vector space over $\mathbb{F}$. While all linearly independent relations in vector spaces over fields have some matroid as an underlying structures, there are matroids which are not representable over any field. The classical example of matroids not representable over any field is the non-Pappus matroid, which is explained in Section 5.2. Therefore, for any field $\mathbb{F}$, the class of $\mathbb{F}$-representable matroids is a proper subclass of matroids. *Regular matroids* are the matroids represented by the column vectors of totally unimodular matrices over the real field $\mathbb{R}$. Totally unimodular matrices are fundamentally important in combinatorial optimization problems. The class of regular matroids is known to equal to the class of matroids representable over any field [71].

More generally, matroids can represent the geometries over division rings, i.e. the algebraic structures same as fields except that the commutativity of mutiplications is not assumed. The linear independent relations of vectors in the left vector spaces over division rings can be described as matroids. In the same way as $\mathbb{F}$-representable matroids for a field $\mathbb{F}$, $\mathbb{D}$-representable matroids can be defined for a division ring $\mathbb{D}$. While the non-Pappus matroid is not representable over any field, the non-Pappus matroid is known to be representable over some division ring [34]. On the other hand, matroids not representable over any division ring are known; the Vámos matroid in [73] is a well-known one.

Also in graphs, you can find many types of matroid structures. In a given undirected graph with an edge set E, the family of the subsets of E which form forests always satisfies the condition (I-1) and (I-2). Therefore, any edge set of graphs has a matroid as an underlying structure. Matroids which can be represented by the forests in an undirected graphs in the above manner are called *graphic matroids*. The class of graphic matroids is known to be a proper subclass of the class of regular matroids. The inclusion relation between the classes which appeared until now is Figure 1.1.

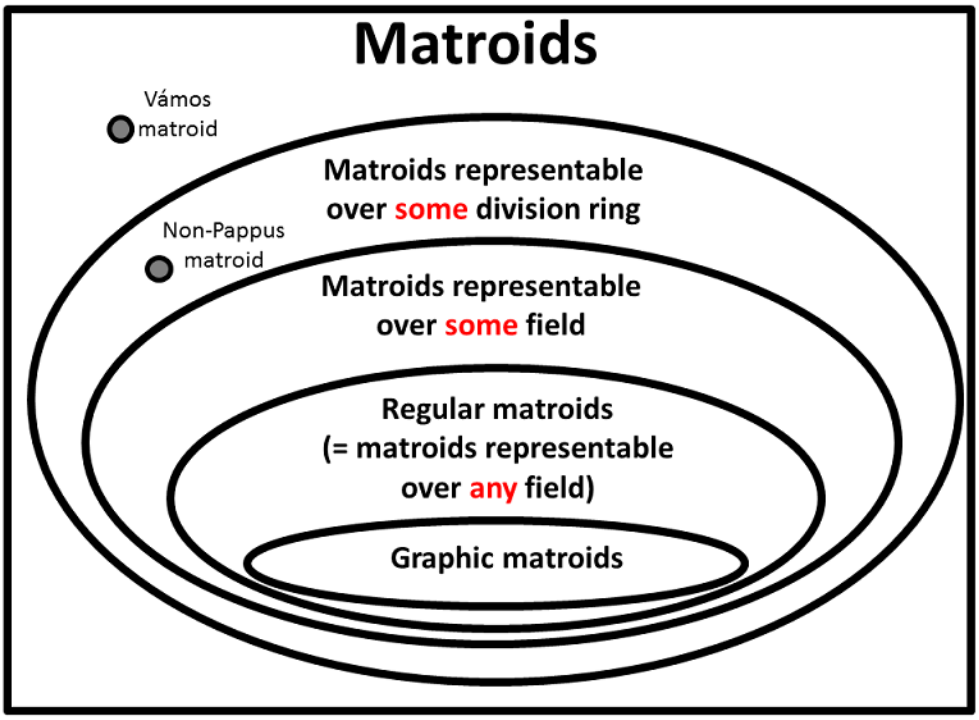

Figure 1.1: The inclusion relation of some classes of matroids.

Matroids arise from graphs in the different way. The matchings in the bipartite graphs with the partition (V_1, V_2) of vertices have matroids as an underlying structure [57]. Let $\mathcal{I}$ be the collection of the subsets of V_1 which consists of the endpoints of some matching. Then $(V_1, \mathcal{I})$ is a matroid. The matroids represented by bipartite matching in this way are called *transversal matroids*. Note that, different graphic matroids whose ground set correspond to edges, the ground set of transversal matroids is vertices. In a directed graph with a given set of source vertices, the set of vertices connected by vertex-disjoint paths from source vertices yields a matroid on the set of vertices; the matroids which are obtained in this way are called *gammoid* [56]. Furthermore, the matroid structures are found in field theory itself. Let $\mathbb{F}$ be a field, and $\mathbb{G}$ an extension field of $\mathbb{F}$. For a finite set E of elements of $\mathbb{G}$, the set of algebraically independent subset of E satisfies the independent axiom system, therefore a matroid [78].

As seen above, matroids exist behind many mathematical objects in different ways. One of the central topics in matroid theory is the problem of characterizing matroids which have the concrete representation. The class of graphic matroids was completely characterized by listing excluded minors [72], i.e. forbidden pattern of matroids minimal with respect to some operations called *matroid-minor operations*. For the representability over some finite fields, the complete characterization is obtained by listing excluded minors; the characterization of $GF(2)$-representability, $GF(3)$-representability, and $GF(4)$-representability was completed by [71], [3, 63] and [20] respectively. For regular matroids, the complete characterization by excluded minors [71] and by decomposition into small pieces of matroids [64] are known.

On the other hand, though the representability over the real field $\mathbb{R}$ was listed as a central open problem in [84], there is no known complete characterization of the representability over infinite fields, as well as $\mathbb{R}$. The existing line of the researches which tackle the characterization of representability over infinite fields is to extend the existing axiom systems by adding some other sentences. However, it is shown that, under some types of logic language, an infinite number of sentences in some types of logic is required to completely characterize the representability over the infinite field [74, 46, 47].

Outside the matroid theory itself, the representation theory of matroids has recently been attracting the considerable attention and yielding the interesting results in other research areas. Network coding is one of the areas in which the matroid representation creates insightful consequences. In the series of papers [14, 15, 16], the unachievability of network coding capacity and the non-representability of matroids over fields are connected, in which the Vámos matroid played a key role. In addtion, being originally motivated by the code theory, the multilinear representability over fields are introduced in [67] as a generalization of the representability over fields. This multilinear representability is utilized to analyze the optimality of network coding in [60].

1.1.2 Orientability of Matroids

The notion which is closely related to the representability over fields, in particular ordered fields, is *orientability of matroids*. Orientability and *oriented matroids* were introduced by Bland and Las Vergnas [6], and by Folkman and Lawrence [18] independently. Oriented matroids provide the combinatorial models to deal with the edge direction of directed graphs and the right-hand and left-hand orientatons of bases in vector spaces over ordered fields. Oriented matroids are obtained by assigning the signs $+$ and $-$ with the bases of matroids, i.e. the maximal independent sets. These signs on matroids enable us to have insightful perspectives to a wide range of areas such as operation research, polytope theory, topology and so on [5].

A matroid is called *orientable*, if it can be extended into oriented matroids. While not all matroids are orientable, for any ordered field $\mathbb{F}$, $\mathbb{F}$-representable matroids are always orientable. In addition, in the following sense, orientability of matroids can be regarded as a natural extension of the representability over the real field $\mathbb{R}$. Identifying the real vector space of dimension r as the real projective space of dimension $r-1$, $\mathbb{R}$-representable matroids are defined as matroids which have the representation by a finite set of points in the real projective space. Then it can be easily known by the duality between points and hyperplanes in projective spaces that the representation by the points is equivalent to the representation by the hyperplanes over the real projective space. Hence $\mathbb{R}$-representable matroids exactly equal to matroids which have the representation by the hyperplanes in the real projective space. The *topological representation theorem* [18] states that orientable matroids coincide with the matroids represented by the pseudohyperplanes in the real projective space, i.e. the homeomorphic image of the hyperplanes in the real projective space. In this sense, the orientability of matroids can be regarded as a natural generalization of the representability over the real field $\mathbb{R}$.

1.1.3 Excluded Minors' Characterization of Matroid Properties

While how to completely characterize the class of matroids, equivalently the matroid properties, is often hard to tackle, the approach to list the excluded minors sometimes gives succinct and clear characterization.

In graph theory, the minor operations consisting of the vertex deletion, edge deletion and edge contraction hold a crucial role in extracting the intrinsic structures of graphs. The planarity of graph, i.e. embeddability of graph into the plane without any edge-crossing, is one of the most well-known examples to which the graph-minor perspectives give the clear characterization. While neither the complete graph K_5 nor the complete bipartite graph $K_{3,3}$ is planar, any graph-minor operation transforms them into planar graphs. The graphs K_5 and $K_{3,3}$ are the forbidden patterns minimal with respect to the minor operations. Such a minimal forbidden pattern is called an excluded minor (for planarity in this case). The *Wagner-Kuratowski theorem* [81, 39], states that the excluded minors for planarity are only K_5 and $K_{3,3}$; a graph is planar if and only if it cannot be transformed by minor operations into either K_5 or $K_{3,3}$. Therefore the planarity of graphs is completely characterized by just two excluded minors. Among the graph-minor results, Robertson and Seymour established the *graph minor theorem* or *Robertson-Seymour theorem*, a monumental result in graph theory spanning from 1983 to 2004 [58].

Theorem 1.1 (Graph Minor Theorem [58]). *Any minor-closed class of graphs has just a finite number of excluded minors.*

This theorem is a splendid generalization of Wagner-Kuratowski theorem. This finiteness can lead to a succinct characterization for any minor-closed classes of graphs.

A matroid is a combinatorial abstraction for graphs and geometries over vector spaces, and the minor operations of matroids are naturally extended from the minor operations of graphs. Characterizing the minor-closed class of matroids by listing excluded minors has been also considered. For some classes of matroids, the complete characterizations by a finite list of excluded minors are known. For example, the class of graphic matroids, a minor-closed class, can be completely characterized by just five excluded minors [72]. The number of excluded minors for the class of regular matroids, also a minor-closed class, is also known to be three [71].

As the class of $\mathbb{F}$-representable matroids is minor-closed, the topic of excluded minors has been investigated for $\mathbb{F}$-representable matroids. Tutte [71], Bixby-Seymour [3, 63] and Geelen-Gerards-Kapoor [20] proved that $GF(2)$, $GF(3)$ and $GF(4)$-representable matroids are characterized by a finite list of excluded minors, respectively. As a generalization of these results, there is the well-known conjecture on the representability over fields, called *Rota's conjecture* [59].

Conjecture 1.2 (Rota's conjecture [59]). *For any fixed finite field $\mathbb{F}$, the class of $\mathbb{F}$-representable matroids has just a finite number of excluded minors*

Geelen, Gerards and Whittle announced in 2013 that this 45-year-old conjecture is affirmatively settled [21, 87].

On the other hand, it is known that the characterization by excluded minors for the representability over infinite fields is quite difficult. For any fixed infinite field $\mathbb{F}$ of characteristic 0, there exist infinitely many excluded minors for the class of $\mathbb{F}$-representable matroids [40]. For an infinite field in general, the infinite family of excluded minors for the class of $\mathbb{F}$-representable matroids are also constructed in [55].

Orientable matroids have been also investigated from the perspective of excluded minors. In the paper [6] founding the oriented matroids, it is shown that the class of orientable matroids has an infinite number of excluded minors. Furthermore, there exist infinitely many excluded minors of rank 3 [86], which implies, even considering the class of orientable matroids of at most fixed rank, the complete characterization by a finite list of excluded minors is impossible.

1.2 Ising Model and Ising Partition Function

1.2.1 Ising Model and Ising Partition Function

Ising model was introduced in 1925 by [35] as a mathematical model to describe ferromagnetism in statistical physics. The problems on Ising model became known to provide bridges between several different areas such as statistical physics, quantum computation, discrete mathematics, combinatorial optimization and so on. Ising model is defined as an undirected graph $G = (V, E)$ equipped with the values on its vertices and edges, where V and E are the set of vertices and edges respectively. The value M_v for each vertex $v \in V$ manifests the influence of external magnetic field on the particle, and the value J_e for each edge $e \in V$ the interaction between the incident particles.

There are two relevant problems on Ising model.

The first one is optimization problem. On the Ising model, a spin state $\sigma \in \{+1, -1\}^V$ is obtained by associating either an up-spin $+1$ or down-spin -1 with each vertex. The Hamiltonian $H(\sigma)$, representing the energy of a spin state σ, is defined as the quadratic function with the coefficient from the values on vertices and edges of Ising model. Finding its minimum-energy state is known to be equivalent to the Quadratic Unconstrained Binary Optimization problem (QUBO), and MAX-CUT problem on a graph. While this problem is polynomial-time solvable for planar underlying graphs without external magnetic field, i.e. $M_v = 0$ for any $v \in V$, it is NP-hard in general [1]. Due to this hardness of exact computation, it is a heart of target problems for approximability through semidefinite-programming approach, unique games conjecture and so on. More recently, a quantum annealing machine finding a small-energy state physically sheds new light on this interaction [8].

The second one is a counting problem, with which we deal in this thesis. Given an Ising model, the Ising partition function Z is the sum of $\mathrm{e}^{-\beta H(\sigma)}$ over all spin configurations where $\beta = \frac{1}{kT}$ for Boltzman constant k and a given temperature parameter T. The Ising partition function itself is fundamental in statistical physics in that the probability of finding a spin state σ is given by $\frac{\mathrm{e}^{-\beta H(\sigma)}}{Z}$. Exact computation of the Ising partition function is

- #P-complete in general [37],
- NP-hard even on planar graphs **with** external magnetic field [1], and
- polynomial-time solvable on planar graphs **without** external magnetic field [1].

The third polynomial-time solvable result is achieved via counting the number of matchings, or Pfaffian computation developed in [43]. The approximability of the Ising partition function has also been investigated, and the classification between approximable and inapproximable cases has been being established. The well approximated cases in the framework of FPTAS or FPRAS are pinned down via Markov-Chain Monte-Carlo (MCMC) method, or rapid mixing of a Markov chain [36]. The so-called tree-threshold is proved to be a boundary of tractable and intractable cases [68, 69]. There has also been shown an approximate computation of the partition function by a quantum annealing machine [80].

The Ising partition function also interacts with other fields such as the graph theory and matroid theory via *Tutte polynomial*. Tutte polynomial $T(x, y)$, or *Tutte-Whitney polynomial*, is the two-variable polynomial obtained from a given undirected graph G. This polynomial is known to contain other graph invariant polynomials as the special cases, e.g. chromatic polynomial, flow polynomial, Jones polynomial and so on (see e.g. [82]). For an Ising model with no magnetic field and the constant interaction, i.e. $M_v = 0$ for each $v \in V$ and $J_e = J_f$ for each $e, f \in E$, the computation of Ising partition function is equivalent to the evaluation of Tutte polynomial along with the hyperbola $(x-1)(y-1) = 2$. Note that, while these cases corresponding to Tutte polynomial might seem to be quite restricted cases, they contain the important instances rigorously researched in statistical physics. Since Tutte polynomial for a given graph is completely and solely determined by the graphic matroid behind the graph, Tutte polynomial for a matroid is naturally extended from Tutte polynomial for a graph. Through Tutte polynomial for a matroid, the Ising model on a graph is generalized into the Ising model on a matroid which is researched in physics in a connection with random cluster model and Potts model (see e.g. [70]). The approximability and inapproximability of the partition function of the Ising model on matroids have been investigated recently; while the certificates of inapproximability on $GF(2)$-representable matroids are presented in [22], there exists FPRAS on regular matroids [23].

1.2.2 Parameterized Complexity to Compute Ising Partition Function

Recently, designing efficient exact exponential algorithms for computationally hard problems has been rigorously tackled (see [19] as the standard reference of this field). It is natural to consider that there exists a graduation of computational difficulties among problems known to be computationally hard, e.g. NP-hard problems. In the field of the exact exponential algorithms, the notion of *Fixed-Parameter Tractable*, or *FPT*, is of fundamental importance to finely classify the intrinsic difficulties of computationally hard problem. Introducing the parameter p which is not an input to the problem, the problem admits FPT with respect to the parameter p, if there exist a polynomial-time algorithm to solve any instance whose parameter p is at most fixed constant.

For the Ising partition function, exact exponential algorithms have been developed, and some of them are FPT algorithms with respect to tree-widths of graphs, etc., under the general framework of computing the Tutte polynomial. Furthermore, exact computation of the partition function has the corresponding counterpart in the measurement-based quantum computation, and via the correspondence a classical simulation of quantum computation yields an exact exponential algorithm [75, 76, 77, 7]. There are the following exact exponential algorithms with currently best or better time complexities, in which O-star notation O^* represents time complexity ignoring a polynomial factor from the big O notation:

(1) $O(n^{3/2}4^{\sqrt{n}})$-time algorithm for $\sqrt{n} \times \sqrt{n}$ square grid graph with external magnetic field [62],

(2) $O(4^{10.74\sqrt{n}})$-time algorithm for planar graphs [62, 61],

(3) $O^*(tw(G)^{O(tw(G))})$-time algorithm for a graph G with tree-width $tw(G)$, given an optimal tree decomposition [49],

(4) $O^*(pw(G)^{O(pw(G))})$-time algorithm for a graph G with path-width $pw(G)$, given an optimal path decomposition [62]

(5) $O^*(2^n)$-time algorithm even for dense graphs with n vertices [4],

(6) $O^*(c^{rw(\tilde{G})})$-time algorithm for a graph G where $\tilde{G}$ is the subdivided graph of G and $rw(\tilde{G})$ the rank-width of $\tilde{G}$ [76]

There are several things to be mentioned about the above results.

The algorithm in (6) uses a tree tensor network to represent a quantum graph state. A tree tensor network is the data structure well-researched in quantum computation and information theory to represent quantum states, e.g. in [65]. The size of tree tensor network in (6) is $\Theta(8^{rw(\tilde{G})})$, which makes the constant c in (6) be ≥ 8. In addition, this algorithm exploits a theory for quantum graph states and stabilizer states, and is therefore not a straightforward graph algorithm.

The algorithms in (1) and (2) are based on an algorithm to compute the Tutte polynomial for planar graphs, and the algorithms in (3), (4) and (5) are that for the Tutte polynomial of general graphs. Therefore the target of these algorithms in (1), (2), (3), (4) and (5) are the special case of Ising model as mentioned before, i.e. Ising models with no magnetic field and the constant interaction, while the algorithm in (6) can compute the partition function of the general case of Ising model.

The algorithms in (1), (2), (3), (4) and (6) exploit the techniques of the graph decomposition which has the origin in graph minor theory. In (1), (2), (3) and (4), the tree decomposition and its special case called the path decomposition are utilized. The tree tensor network which is utilized in (6) has a counterpart in graph theory called the rank decomposition which is introduced in [54] and intensively investigated from the viewpoint of both graphs and matroids in [50, 51, 52, 53].

The result of (5) is called a *vertex-exponential* time, i.e. the number of edges does not affect the exponential parts. The algorithm in (5) is better in magnitude in the perspective of the FPT algorithms than the algorithms in (3) and (4) when the tree width $tw(G)$ and the path width $pw(G)$ are $\Theta(n)$. This $O^*(2^n)$-time is best possible within a constant factor in the exponent except points on $(x-1)(y-1) = 1$ and $y = 0, \pm 1$ under the computational complexity assumption called the *Exponential Time Hypothesis (ETH)* [10]. When parametrized by clique-width, the graph coloring problem is known to be $W[1]$-*hard*. Since the evaluation of the Tutte polynomial contains the graph coloring problem as a special case, the evaluation of the Tutte polynomial along $y = 0$ is also $W[1]$-*hard* in general. In fact, there have been known no FPT algorithm for the Tutte polynomial with the clique width as a parameter. Since the rank-width is at most the clique width, these also hold for the rank-width. In contrast to these results, the result (6) implies, although the constant c is not specific, the Tutte polynomial along $(x-1)(y-1) = 2$, i.e. the partition function, admits an algorithm which is parameter-exponential (single exponential c^p for constant c and parameter p), in terms of the rank-width as mentioned above.

1.3 Our Contributions

1.3.1 Excluded Minors' Characterization of Orientable Matroids and Matroids Representable over Infinite Field

In this thesis, the excluded minors' characterization for the class of orientable matroids and the classes of matroids representable over fields is further investigated. From now, the term "representability" means the representability by vector spaces over fields, unless otherwise specified.

First we construct two new infinite families YM^1_{3n} and F_{3n-2} of the excldued minors of rank 3 for the class of orientable matroids. The infinite family YM^1_{3n} will be constructed from the excluded minor YM^1_9 for the class of the orientable matroids, which is recently obtained by computational enumeration in [44]. The other family F_{3n-2} will be constructed from the Fano matroid F_7 which is known to be the excluded minor for the class of orientable matroids smallest with respect to both the rank and number of elements. Then the representability of these two families will be investigated, which shows that YM^1_{3n} and F_{3n-2} are also the excluded minors for the union of the class of orientable matroids and the classes of matroids representable over some field.

Second we investigate the excluded minors' characterization for the union and intersection of the class of orientable matroids and the classes of matroids representable over fields. In general, even for two classes of matroids with infinitely many excluded minors, there is a possibility that the number of excluded minors becomes finite by taking the union and intersection of these classes. Thefore, while both the class of orientable matroids and the class of $\mathbb{F}$-representable matroids for an infinite field $\mathbb{F}$ are known to have infinitely many excluded minors, it is not necessarily impossible to characterize these union and intersection by a finite list of excluded minors. In this thesis, we show that there exist infinitely many excluded minors of rank 3 for both the union and the intersection. To show this, we explicitly construct infinitely many excluded minors.

For the union, being based on the infinite family F_{3n-2} of excluded minors of rank 3 for the class of orientable matroids, we construct the infinite family F^+_{3n-2} of excluded minors of rank 3 for the union of the class of orientable matroids and the classes of matroids representable over fields.

For the intersection, being based on the non-Pappus matroid, we construct the infinite family NP_{3n} of excluded minors of rank 3 for the intersection of the class of orientable matroids and the classes of matroids representable over fields of characteristic 0. In addition, it will be also shown that the matroids in the constructed infinite family are also orientable. This implies that the representability over fields of characteristic 0 cannot be characterized by a finite list of excluded minors even within the class of orientable matroids.

Thirds we investigate the excluded minors' characterization for the class of $\mathbb{Q}$-representable matroids within the class of $\mathbb{Q}[x]$-representable matroids, where $\mathbb{Q}$ is the rational field and $\mathbb{Q}[x]$ is an single-extension field of $\mathbb{Q}$ by an element algebraic over $\mathbb{Q}$. While the class of $\mathbb{Q}$-representable matroids has infinitely many excluded minors, it is unknown whether the number of excluded minors within $\mathbb{Q}[x]$-representable matroids is finite or infinite. As mentioned before, matroids are in a sense so roughly-abstracted notion that the gap as a combinatorial model between matroids and concrete geometric settings is wide. Therefore, narrowing the matroids in general into the class of the class of $\mathbb{Q}[x]$-representable matroids, we will investigate the gap between the class of $\mathbb{Q}$-representable matroids and the class of $\mathbb{Q}[x]$-representable matroids from the matroid-minor perspective. We show that there exist infinitely many excluded minors of rank 3 for the class of $\mathbb{Q}$-representable matroids within the class of $\mathbb{Q}[x]$-representable matroids, when the degree of the minimal polynomial of the algebraic element x is 2. This implies that the gap still remains wide even between these seemingly-close classes.

1.3.2 Fast Parameterized Algorithm to Compute Ising Partition Function on Graphs

We develop efficient exact exponential algorithms for the Ising partition function of the Ising model with arbitrary external magnetic field and interaction, i.e. the most general cases. Our algorithms use dynamic programming on two types of the graph decompositions: the branch decomposition and rank decomposition. We give the following algorithms:

(I) $O^*(2^{\frac{\omega}{2}bw(G)})$-time algorithm for a graph G, given an optimal branch decomposition of G of width $bw(G)$, when two $N \times N$ matrices can be multiplied in $O(N^\omega)$ time,

(II) $O^*(4^{rw(\tilde{G})})$-time algorithm for a graph G, given an optimal rank decomposition of $\tilde{G}$ of width $rw(\tilde{G})$ whose rank decomposition of the subdivided graph $\tilde{G}$ of G, where $\tilde{G}$ is the subdivided graph of G.

(III) $O(n2^{\sqrt{n}})$-time algorithm for an $\sqrt{n} \times \sqrt{n}$ square grid,

In the algorithm (I), the tables in the dynamic programming are updated in the form of the matrix multiplication, which is the reason why the matrix multiplication constant ω appears. Regarding the constant ω, the current best bound is known to be less than 2.37287 [42]). It should be noted that the result in (II) has a larger time complexity in magnitude than the algorithm in (I), yet is interesting from the viewpoint of quantum computation. As touched before, the algorithm in [76] exploits the data structure called the tree tensor networks. The tree tensor networks are equivalent to the rank decomposition in the graph theory. Since the algorithm in [76] utilizes the theory of quantum graph states and stabilizer states, it is not a purely graph-theoretical algorithm. On the other hand, our algorithm using the rank decomposition is a purely graph-theoretical algorithm with the smaller time complexity. Furthermore, applying our algorithm into the Ising mode on the square grids, the algorithm (III) is obtained. Note that the square grids are one of the most rigorously-studied instances of Ising model in statistical physics. A special case of the result (II) and (III) in terms of Binary Decision Diagram (BDD) is discussed in [26].

1.4 Organization of This Thesis

In Chapter 2, the preliminaries of this thesis are given. They cover the basics of matroids, the matter on graph decompositions and the definition of Ising model and Ising partition function. In Chapter 3, the excluded minors for the class of orientable matroids are investigated. Initially, the known excluded minors for the class of orientable matroids are briefly reviewd. Then the main theorem in the chapter which states the existance of excluded minors of rank 3 in every number of elements is proven, by explicitly constructing two infinite families of excluded minors. In Chapter 4, we investigate the excluded minors' characterization for the union of the class of orientable matroids and the classes of representable matroids. First we briefly review the situation on the excluded minors for the union. Then it is shown that there exist infinitely many excluded minors of rank 3 for the union, by modifying the infinite family of excluded minors for the class of orientable matroids constructed in Chapter 3. In Chapter 5, we investigate the excluded minors' characterization for the intersection of the class of orientable matroids and the classes of matroids representable over infinite fields of characteristic 0. First we briefly review the situation on the excluded minors for the union, and the classical theorem on projective geometry, called the Pappus' hecagon theorem, which will play a crucial role in the chapter. Finally it is shown that there exist infinitely many excluded minors of rank 3 for the intersection, by explicitly constructing infinitely many excluded minors. In Chapter 6, we investigate the excluded minors' characterization for the class of $\mathbb{Q}$-representable matroids within the class of $\mathbb{Q}[x]$-representable matroids. It is shown that, even within the class of $\mathbb{Q}[x]$-representable matroids, the number of the excluded minors for $\mathbb{Q}$-representable matroids remains infinite, by constructing the excluded minors form the matrices over $\mathbb{Q}[x]$. In Chapter 7, we develop the algorithms to compute Ising partition function, exploiting two types of graph decomposition: the branch decomposition and rank decomposition. The algorithm is based on the dynamic programming on the graph decompositions. In Chapter 8, we conclude this thesis by reviewing our results and listing the relavant open problems.

Chapter 2

Preliminaries

2.1 Matroid Basics

2.1.1 Two Axiom Systems of Matroids and Basic Terminologies

Matroids have several equivalent axiom systems. The axiom systems used in this thesis are the *base axiom system* and *hyperplane axiom system*.

Let E be a finite set. In both the base axiom system and hyperplane axiom system, matroids are defined as a pair of the finite set E and the families of the subsets of E. The finite set E is called *ground set*.

First we introduce the base axiom system.

Definition 2.1 (Matroid by the Base Axiom System)**.** *A matroid M is a pair $(E, \mathcal{B})$ where E is a finite set and $\mathcal{B}$ a set of subsets of E satisfying the following two conditions.*

(B-1) $\mathcal{B}$ is not empty.

(B-2) For any $B_1, B_2 \in \mathcal{B}$ and any $e \in B_1 \setminus B_2$, there exists $f \in B_2 \setminus B_1$ such that $(B_1 \setminus \{e\}) \cup \{f\}$.

The set of two conditions (B-1) and (B-2) is the base axiom system. An element contained in $\mathcal{B}$ is called a base of the matroid M.

Now some basic terminologies and properties of matroids are presented here. Let $M = (E, \mathcal{B})$ be a matroid. It can be shown by an elementary discussion that each base of a given matroid has the same cardinality. The cardinality of a base of the matroid M is called *rank* of the matroid. A subset of $S \subset E$ is *independent*, if S is included in some base. Note that a base is a maximal independent set with respect to set inclusion. A subset of $S \subset E$ is *dependent*, if S is not independent. For a subset $S \subset E$, the rank $r(S)$ of S is defined of the cardinality of a maximal independent set included in S. The elementary discussion can show that the cardinality of maximal independent sets in a given $S \subset E$ is same. Therefore the rank of S is well-defined. A subset $S \subset E$ is spanning, if $r(S) = r(E)$. A maximal non-spanning subset $S \subset E$ is called a *hyperplane*. Note that, for a hyperplane S, $r(S) = r(E) - 1$. A minimal dependent set with respect to set inclusion is called a *circuit*. An element $e \in E$ is called *loop*, if $\{e\}$ is dependent. Two distinct elements $e, f \in E$ are called *parallel*, if $\{e, f\}$ is dependent. The matroid M is called *simple*, if it does not have any loop and any parallel elements. Given two matroids $M_1 = (E_1, \mathcal{B}_1)$ and $M_2 = (E_2, \mathcal{B}_2)$ the matroid M_1 is *isomorphic* to the matroid

M_2, if there exists a bijection ϕ from E_1 to E_2 such that $S_1 \subset E_1$ is contained in $\mathcal{B}_1$ if and only if $\phi(S_1)$ is contained in $\mathcal{B}_2$. In this thesis, we identify the isomorphic matroids.

Now we present typical examples of matroids called *uniform matroids.* Let n and m be positive integers such that $m \leq n$. Let E_n be the n-element set, and $\mathcal{B}_{m,n}$ the set of m-element subset of E_n. Since $\mathcal{B}_{m,n}$ trivially satisfies the base axiom system, $(E_n, \mathcal{B}_{m,n})$ is a matroid. We denote the matroid $(E_n, \mathcal{B}_{m,n})$ by $U_{m,n}$. The independent sets of $U_{m,n}$ are the subsets of E_n whose cardinality is at most m. The dependent sets of $U_{m,n}$ are the subsets of E_n whose cardinality is at least $m+1$. The circuits of $U_{m,n}$ are the $(m+1)$-element subsets of E_n. The hyperplanes of $U_{m,n}$ are the $(m-1)$-element subsets of E_n. The rank of $U_{m,n}$ is m.

Second, the hyperplanes axiom system is introduced.

Definition 2.2 (Matroid by the Hyperplane Axiom System)**.** *A matroid M is a pair $(E, \mathcal{H})$ where E is a finite set and $\mathcal{H}$ a set of subsets of E satisfying the following two conditions.*

(H-1) $\mathcal{H}$ is not empty.

(H-2) For any $H_1 \neq H_2 \in \mathcal{H}$.

(H-3) For any $H_1, H_2 \in \mathcal{H}$ and $e \in E \setminus (H_1 \cup H_2)$, there exists $H_3 \in \mathcal{H}$ such that $(H_1 \cap H_2) \cup \{e\} \subset H_3$.

The set of three conditions (H-1), (H-2) and (H-3) is the hyperplane axiom system. An element contained in $\mathcal{H}$ is called a hyperplane of the matroid M.

The base axiom system and the hyperplane axiom system are equivalent in the sense that a matroid by one axiom system can be transformed into a matroid by the other axiom system, and vice versa. First we describe the transformation from a matroid by the base axiom system into a matroid by the hyperplane axiom system. Let $M = (E, \mathcal{B})$ be a matroid by the base axiom system. Then, form the set $\mathcal{H}$ by collecting the maximal subsets of E which do not include any base of M. It can be shown that $\mathcal{H}$ satisfies the hyperplane axioms (H-1), (H-2) and (H-3). Therefore, a pair $(E, \mathcal{H})$ is a matroid by the hyperplane axiom system. Conversely, the transformation from a matroid by the hyperplane axiom system into a matroid by the base axiom system is obtained as follows. Let $M = (E, \mathcal{H})$ be a matroid by the hyperplane axiom system. Form the set $\mathcal{B}$ by collecting the minimal subsets of E which are not contained in any hyperplane. The set $\mathcal{B}$, being constructed in the above way, satisfies the base axioms (B-1) and (B-2). Therefore, a pair $(E, \mathcal{B})$ is a matroid by the base axiom system. By these discussion, the two axiom systems are equivalent. Hence we can freely choose one of them, depending on situations. Note that matroids have other different axiom systems, and they are equivalent in the above sense.

Note that the term "hyperplane" in matroids by the base axiom system and by the hyperplane axiom system are compatible through the above transformation. The other terms in matroids by the base axiom system are also naturally defined in matroids by the hyperplane axiom system via the above transformation in a compatible manner.

Finally, we introduce the dual matroid M^* for a given matroid M. Let $M = (E, \mathcal{B})$ be a matroid by base axiom system. Then define $\mathcal{B}^*$ as follows.

- $\mathcal{B}^* = \{E \setminus B \mid B \in \mathcal{B}\}$

Then it is known that $\mathcal{B}^*$ satisfies (B-1) and (B-2). Therefore $M^* = (E, \mathcal{B}^*)$ is also a matroid. This matroid M^* is called the *dual matroid* of M.

2.1.2 Geometric Representation of Matroids

Matroid is given in a set-theoretic form. Therefore it is often difficult to understand the whole structure of a given matroid. To intuitively grasp the structure of a matroid of rank at least 4, the *geometric representation of matroids* is frequently used in the literature of matroids.

Now we introduce the geometric representation of simple rank-3 matroids, which often appear in this thesis. Let $M = (E, \mathcal{H})$ be a rank-3 matroid by the hyperplane axiom system. The geometric representation of M consists of two kinds of objects, and these objects will be drawn on the plane. The first object is a point. Each point bijectively corresponds to each element in E. These points are distinctly drawn on the plane. The second object is a line segment. This line segment is not necessarily straight. Each line segment bijectively corresponds to each hyperplane of at least three elements in $\mathcal{H}$. Consider each line segment l and the corresponding hyperplane H_l. The line segment l is drawn on the plane in a way to connect all the points which correspond to the elements contained in the hyperplane H_l.

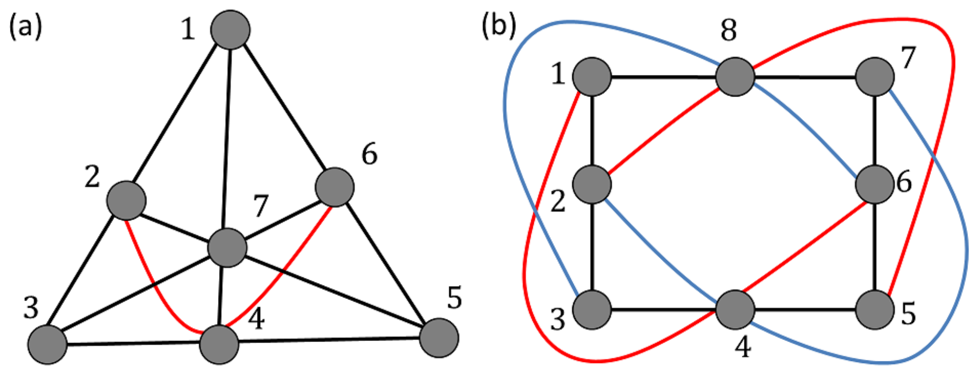

Figure 2.1: The geometric representations of (a) the Fano matroid (b) the MacLane matroid

Now we present the examples of the geometric representation. Let F_7 be the matroid $(E_1, \mathcal{H}_1)$ where $E_1 = \{1, 2, ..., 7\}$ and $\mathcal{H}_1 = \{\{1,2,3\}, \{1,4,7\}, \{1,5,6\}, \{2,4,6\}, \{2,5,7\}, \{3,4,5\}, \{3,6,7\}\}$ Let ML_8 be the matroid $(E_2, \mathcal{H}_2)$ where $E_2 = \{1, 2, ..., 8\}$ and $\mathcal{H}_2 = \{\{1,2,3\}, \{1,4,6\}, \{1,5\}, \{1,7,8\}, \{2,4,7\}, \{2,5,8\}, \{2,6\}, \{3,4,5\}, \{3,6,8\}, \{3,7\}, \{4,8\}, \{5,6,7\}\}$.

The F_7 and ML_8 are called the *Fano matroid* and *McLane matroid* respectively, and often appears as typical and important examples in matroid theory. The geometric representations of the Fano matroid and MacLane matroid are Figure 2.1(a) and (b) respectively. Note that two-element hyperplane is not drawn in the geometric representation.

Conversely, given a geometric representation of a simple matroid, we can recover the matroid as follows. Regard the set E of the points in the geometric representation as the ground set of the matroid. Form $\mathcal{H}$ by collecting the non-collinear two-point sets and the maximal sets of collinear points.

2.2 Representability of Matroids

The base axiom system and the hyperplane axiom system capture combinatorial aspects of vector spaces and projective spaces respectively.

The base axiom is an abstraction of linearly independent relations in vector spaces over fields. Any finite set of vectors in a vector space over a field $\mathbb{F}$ has a matroid as an underlying structure. Now we explain how we can obtain a matroid structure from a finite set of vectors in a vector space. Let E be a finite set of vectors in a given vector space over a field $\mathbb{F}$. Form a family $\mathcal{B}$ by collecting the subsets of E which are bases in the vector space. Since the family $\mathcal{B}$ satisfies the base axiom, $(E, \mathcal{B})$ is a matroid.

The hyperplane axiom is an abstraction of incidence relations of points and hyperplanes in projective spaces. Let E be a finite set of points in the projective space over a given field $\mathbb{F}$. Form a family $\mathcal{H}$ by collecting the maximal subsets of E which does not span the whole projective space. The family $\mathcal{H}$ satisfies the hyperplane axiom, hence $(E, \mathcal{H})$ is a matroid. Therefore any finite set of vectors in the vector space, and any finite set of points in the projective space, have a matroid as an underlying structure.

For a field $\mathbb{F}$, the dimension-$(r-1)$ projective space over $\mathbb{F}$ is obtained by regarding vectors in the dimension-r vector space over $\mathbb{F}$ as points in the dimension-$(r-1)$ projective space over $\mathbb{F}$. From this perspective, we can grasp the intuition of the transformation between the matroid by the base axiom system and the matroid by the hyperplane axiom system, which is described in Section 2.1.1.

For a field $\mathbb{F}$, a rank-r matroid M on the ground set E is *representable* over $\mathbb{F}$ or $\mathbb{F}$-*representable*, if there exists a finite set S of vectors in the vector space of dimension r over $\mathbb{F}$ and a bijection $\phi : S \to E$ such that a subset $T \subset S$ forms a base in the vector space if and only if $\phi(T)$ is a base of the matroid M, or equivalently, if there exists a set S of points in the projective space of dimension $r-1$ over $\mathbb{F}$ and a bijection $\phi : S \to E$ such that a set $T \subset S$ forms a maximal non-spanning set in the projective space if and only if $\phi(T)$ is a hyperplane of the matroid M. Not all matroids arise from a finite set of vectors in a vector space, or equivalently a finite set of points in a projective space in the above way.

2.3 Orientability of Matroids

In this section, we introduce *oriented matroids* and *orientability of matroids*. Oriented matroids can be defined by many equivalent axiom systems. Here we adopt the definition by *chirotope* as follows. See [5] about the other ways to define oriented.

Definition 2.3 (Oriented Matroid). *Let $M = (E, \mathcal{B})$ be a rank-r matroid by the bases axiom system. The map $\chi : E^r \to \{-1, 0, 1\}$ is called chirotope, if it satisfies the following three conditions.*

(O-1) χ is not identically zero

(O-2) *χ is alternating, i.e. for any permutation σ and $x_1, ..., x_r \in E$, $\chi(x_1, ..., x_r) = \chi(x_{\sigma(1)}, ..., x_{\sigma(r)})$*

(O-3) *For any $x_1, ..., x_r, y_1, ..., y_r \in E$, if $\chi(y_i, x_2, ..., x_r) \cdot \chi(y_1, ..., y_{i-1}, x_1, y_{i+1}, ..., y_r) \geq 0$ holds for each $i \in \{1, ..., r\}$, then $\chi(x_1, ..., x_r) \cdot \chi(y_1, ..., y_r) \geq 0$*

Then, oriented matroid is an ordered pair $OM = (M, \chi)$.

For a given oriented matroid $OM = (M, \chi)$, the matroid M is called *underlying matroid* of the oriented matroid OM. While all oriented matroids has their underlying matroids, the converse is not always true; there exist matroids with which no chirotope can be assigned. A matroid on which there exists at least one chirotope is called *orientable*. It is known that, for any ordered field $\mathbb{F}$, $\mathbb{F}$-representable matroid is orientable. On the other hand, there exist a matroid which is representable over some field but not orientable. For example, the Fano matroid and MacLane matroid are these examples; the Fano matroid is representable over $GF(2)$ but not orientable, and the MacLane matroid is representable over $GF(3)$ but not orientable.

2.4 Matroid Minor

In this section, we introduce *minor operations of matroids* and *excluded minors* of matroids. The minor operations of matroids consist of two operations: *deletion* and *contraction*.

Let M be a matroid $(E, \mathcal{B})$ by the base axiom system, and e an element of E. Then define the two set $\mathcal{B}'$ and $\mathcal{B}''$ as follows:

- $\mathcal{B}' = \{B \mid e \notin B \in \mathcal{B}\}$.
- $\mathcal{B}'' = \{B \setminus e \mid e \in B \in \mathcal{B}\}$.

It can be routinely checked that both $\mathcal{B}'$ and $\mathcal{B}''$ satisfy the base axiom system on the finite set $E \setminus \{e\}$. Then the matroid $(E \setminus \{e\}, \mathcal{B}')$ is denoted by $M \setminus \{e\}$ or $M \setminus e$. The operation to construct $M \setminus \{e\}$ from M is called *deletion at e*. The matroid $(E \setminus \{e\}, \mathcal{B}'')$ is denoted by $M/\{e\}$ or M/e. The operation to construct $M/\{e\}$ from M is called *contraction at e*. The contraction is equivalently defined from the deletion and taking a dual as follows: $M/e = (M^* \backslash)^*$.

Note that the deletion and contraction of a matroid is a natural extension of the edge-deletion and contraction of a graph. For a given graph G with the edge set E, the set $\mathcal{B}$ of spanning forests of F always satisfies the base axiom system. For an edge $e \in E$, construct a graph $G \setminus e$ and G/e by deleting and contracting the edge e respectively. Then the sets of the spanning forests of $G \setminus e$ and G/e coincides with $\mathcal{B}'$ and $\mathcal{B}''$ respectively.

Let M and N be a matroid. The matroid N is called a *minor* of the matroid M if M can be transformed into N by sequence of deletions and contractions.

Let M be a matroid on the ground set E, and S and T disjoint subsets of E. Now we consider to delete S and contract T from the matroid M. It is known that the order of deletion and contraction does not affect the result. Hence, once the deleted and contracted subsets of the ground set is designated, the matroid which is obtained after minor operations is uniquely determined. For the deleted subset

S and contracted subset T, we denote the resulted matroid by $M \setminus S/T$ or $M/T \setminus S$. If S or T is empty, then the notations $M \setminus T$ or $M \setminus S$ are used respectively.

Let $\mathcal{C}$ be a class of matroids. The class $\mathcal{C}$ is *minor-closed*, if, for any matroid in $\mathcal{C}$, any minor of the matroid is also in $\mathcal{C}$. For a minor-closed class $\mathcal{C}$ of matroids, a matroid M is an *excluded minor* for $\mathcal{C}$, if M is not in $\mathcal{C}$ and any proper minor of M is in $\mathcal{C}$.

2.5 Graph Decomposition

Let $G = (V, E)$ be a graph. Now we consider another graph D called *a decomposition tree* for G. To avoid confusion, a vertex and edge of a decomposition tree is called *a bag* and *arc*, respectively. The decomposition tree consists of three types of bags:

- *a root bag* with degree 2,
- *an inner bag* with degree 3, and
- *a leaf bag* with degree 1.

A decomposition tree has an unique root. Each arc of a decomposition tree is directed from the root to leaves. For each bag b, an in-coming and out-going arc are called *an in-arc* and *out-arc* of b, respectively. A bag incident with b by an in-arc and out-arc is called with a parent and child of b, respectively. An arc is represented by a ordered pair $a = (u, v)$ of two bags where u is a parent of v. This arc a is an in-arc of v and out-arc of u. Note that a root bag has two children, an inner bag has one parent and two children, and a leaf bag has one parent.

Branch Decomposition

A *branch decomposition* for a graph $G = (V, E)$ is a pair $\delta = (D, \phi)$ where D is a decomposition tree and ϕ is a bijection from leaves of D to E.

For a bag b, its in-arc (b', b) induces two subtree T_b and $T_{b'}$ containing b and b', respectively. Mapping leaves of two subtrees by ϕ^{-1} induces a partition $(E_b, E_{b'})$ of E. We also denote $E_{b'} (= E \setminus E_b)$ by $\overline{E_b}$. For a root b, $(E_b, \overline{E_b})$ is defined as $(E, \emptyset)$.

For a branch decomposition δ and its bag b, the set $\mathrm{mid}(b)$ denotes a set of vertices incident with both E_b and $\overline{E_b}$. *A width* $bw(G, \delta, b)$ *of a bag* b is $|\mathrm{mid}(b)|$. *A width* $bw(G, \delta)$ *of* δ is a maximum $bw(G, \delta, b)$ among all bags b. *A branch width* $bw(G)$ *of* G is minimum $bw(G, \delta)$ among all possible branch decomposition δ for G.

Rank Decomposition

A *rank decomposition* for a graph $G = (V, E)$ is a pair $\delta = (D, \phi)$ where D is a decomposition tree and ϕ is a bijection from leaves of D to V. A rank decomposition (D, ϕ) is *linear* if the decomposition tree D is also a catapillar tree.

For a bag b, its in-arc (b', b) induces two subtree T_b and $T_{b'}$ containing b and b', respectively. Mapping leaves of two subtrees by ϕ^{-1} induces a partition $(V_b, V_{b'})$ of V. We also denote $V_{b'} (= V \setminus V_b)$ by $\overline{V_b}$, respectively. For a root b, $(V_b, \overline{V_b})$ is defined as $(V, \emptyset)$.

Let A be an adjacency matrix of G. For disjoint subsets $S, T \subseteq V$, consider its submatrix $A(S,T)$ whose rows and columns correspond to S and T, respectively. Denote the rank of $A(S,T)$ over $GF(2)$ by $\mathrm{rank}(A(S,T))$. For a bag b, a matrix A_b denotes $A(V_b, \overline{V_b})$. For a rank decomposition δ, *A width* $rw(G,\delta,b)$ *of a bag* b is $\mathrm{rank}(A_b)$. *A width* $rw(G,\delta)$ *of* δ is a maximum $rw(G,\delta,b)$ among all bags b. *A rank width* $rw(G)$ is a minimum $rw(G,\delta)$ among all possible rank decompositions δ for G.

2.6 Ising Model and Ising Partition Function

Ising model is an undirected graph $G = (V,E)$ equipped with the following values:

- the external magnetic field weight M_v for each vertex $v \in V$, and
- the interaction energy weight J_e for each edge $e \in E$.

A spin configuration $\sigma = (\sigma_v)_{v\in V}$ is a vector in $\{\pm 1\}^V$ where -1 and $+1$ correspond to down and up spin at a vertex respectively. For notational convenience, we denote a spin configuration by a subset of V as follows. For $S \subseteq V$, we regard S as a vector $(S_v)_{v\in V}$ such that $S_v = 1$ if $v \in S$, -1 otherwise.

Then, the Hamiltonian $H(S)$ is defined by

$$H(S) = -\sum_{e=\{u,v\}\in E} J_e S_u S_v - \sum_{v\in V} M_v S_v$$

Finally, the partition function $Z(G,\beta)$ where a given parameter β is defined by

$$Z(G,\beta) = \sum_{S\subseteq V} \exp(-\beta H(S)).$$

Chapter 3

Minimal Non-Orientable Matroids of Rank 3

3.1 Overview of this Chapter

In this chapter, we investigate the excluded minors of rank 3 for the class of orientable matroids. Note that our results which are presented in this chapter are published in [32]. Conventionally, an excluded minor for orientable matroids is called a *minimal non-orientable matroid*. We follow this convention in this chapter. The main goal of this section is to prove the following theorem.

Theorem 3.1 ([32]). *For every $m \geq 7$, there exists at least one minimal non-orientable matroid of rank 3 with m elements.*

Before the proof of Theorem 3.1, the known minimal non-orientable matroids will be reviewed in Section 3.2. First, we review typical minimal non-orientable matroids of small size: the Fano matroid F_7 and the MacLane matroid ML_8. Second, the result of the exhaustive computational enumeration of minimal non-orientable matroids of small size by [44] will be presented. Especially, we focus on two minimal non-orientable matroid YM_9^1 and YM_9^2 with 9 elements, which is newly found in [44]. They are found as the third smallest minimal non-orientable matroids of rank 3 with respect to the number of elements after the Fano matroid with 7 elements and the MacLane matroid with 8 elements. Finally, we overview the existing the infinite families of minimal non-orientable matroids [6, 86, 17, 66]. In the proof of Theorem 3.1, the Fano matroid F_7, the matroid YM_9^1 and the construction method in [86] play crucial roles.

In Section 3.3, we define two new infinite families of matroids of rank 3: YM_{3n}^1 and F_{3n-2}. These two families are shown in the later sections to be minimal non-orientable matroids. The YM_{3n}^1 is matroids of rank 3 with $3n$ elements for an integer $n \geq 3$. The F_{3n-2} is matroids of rank 3 with $3n-2$ elements for an integer $n \geq 3$. There is the already-known infinite family Z_{3n-1} [86] of minimal non-orientable matroids of rank 3 with $3n-1$ elements ($n \geq 3$), which will be reviewed in Section 3.2. Together with these three families, the proof of Theorem 3.1 will be completed.

To prove that a given matroid is minimal non-orientable matroids, we have to prove two statements. The first one is that the matroid is not orientable, i.e. non-orientability. The second one is that any proper minor of the matroid is orientable, i.e. minimality. In Section 3.4, the non-orientability of the matroids in our two families YM_{3n}^1 and F_{3n-2} will be proven. In Section 3.5, the minimality of the matroids in our two families YM_{3n}^1 and F_{3n-2} will be proven.

In Section 3.6, the representability of our two families YM^1_{3n} and F_{3n-2} is investigated. We mainly show the following statements: (i) YM^1_{3n} is not representable over fields whose characteristic is not 2, and (ii) F_{3n-2} is not representable over fields whose characteristic is 2. This results on the representability shows that the families YM^1_{3n} and F_{3n-2} are also excluded minors for the union of the class of orientable matroids and the classes of matroids representable over some fields, as explained in Section 4.1. Furthermore, the representability of the two matroid YM^1_9 and YM^2_9 is investigated in more detailed way.

3.2 Known Minimal Non-orientable Matroids

In this section, we review the known minimal non-orientable matroids.

First we review the minimal non-orientable matroids of the small rank with the small number of elements. There are two well-known minimal non-orientable matroids of rank 3: the Fano matroid F_7 with 7 elements and the MacLane matroid ML_8 with 8 elements. Since all matroids of rank 2 and all matroids with at most 6 elements are orientable, the Fano matroid F_7 is the smallest minimal non-orientable matroid with respect to both rank and the number of elements.

Recently Matsumoto, Moriyama, Imai and Bremner gave the following complete enumeration of minimal non-orientable matroids of small rank and element number as in Table 3.2 [44]. This enumeration shows the Fano matroid F_7 and the MacLane matroid ML_8 are the unique rank 3 minimal non-orientable matroids with 7 and 8 elements respectively. Following on the Fano matroid F_7 and MacLane matroid ML_8, there exist just two rank 3 minimal non-orientable matroids of 9 elements. We denote these two minimal non-orientable matroids by YM^1_9 and YM^2_9. Geometric representations of the two matroids are shown in Figure 3.1. Note that YM^2_9 was also listed in [2] as M_9, an excluded minor for $GF(5)$-representable matroids.

$\|E\|$	7	8	9	10	11	12
$r=3$	1	1	2	23	$1,458$	$397,240$
$r=4$	1	10	$8,481$	$*$	$*$	$*$

Table 3.1: Enumeration of minimal non-orientable matroids; r and $|E|$ are the rank and number of element respectively [44]

Second, the infinite families of minimal non-orientable matroids are reviewed. Bland and Las Vergnas [6] showed there exist infinitely many minimal non-orientable matroids, i.e. the class of orientable matroids cannot be characterized by a finite list of excluded minors. They showed it by explicitly constructing the infinite family of minimal non-orientable matroids; the infinite family consists of matroids of rank $r(\geq 4)$ with $2r$ elements.

Ziegler [86] sharpened the result by Bland and Las Vergnas by constructing another infinite family of minimal non-orientable matroids of rank 3 with $3n-1$ elements for every $n \geq 3$. This result shows that, even considering the class of orientable matroids of rank at most r, the class cannot be characterized by a finite number of excluded minors. The family starts from the MacLane matroid ML_8.

Rather the above two infinite families, two other infinite families of minimal non-orientable matroids have been constructed as follows. Flórez and Forge [17]

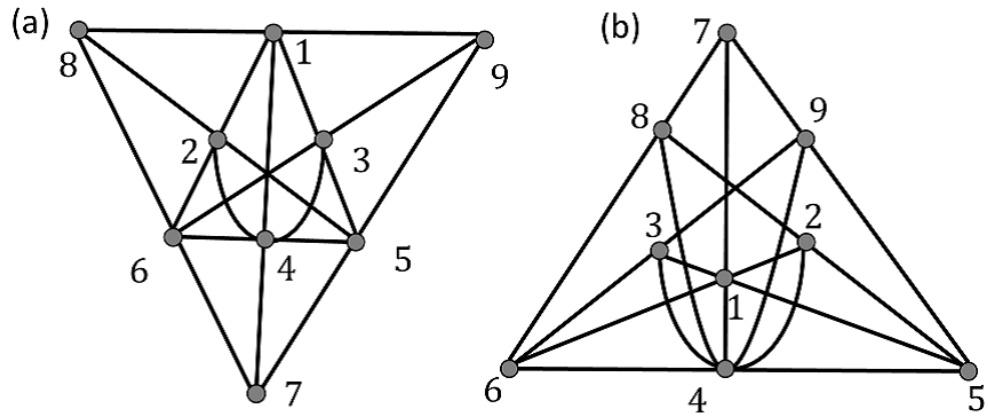

Figure 3.1: Minimal non-orientable matroids of rank 3 with 9 elements [44]: (a) YM_9^1 and (b) YM_9^2 [2]

constructed an infinite family of minimal non-orientable matroids of rank 3 with $2n$ elements for every $n \geq 4$. For any finite field $\mathbb{F}$, this family includes a minimal non-orientable matroid which is representable over $\mathbb{F}$. This also starts from the MacLane matroid ML_8. Da Silva [66] proposed a method to generate minimal non-orientbale matroids of rank $r(\geq 4)$ with $2r$ elements, and expanded the infinite family proposed by Bland and Las Vergnas [6].

Table 3.2 summarizes the rank and number of elements of known minimal non-orientable matroids.

$\lvert E\rvert$	7	8	9	10	11	12	13	14	15	$\cdots$
$r=3$	$\surd$	$\surd$	$\surd$	$\surd$	$\surd$	$\surd$		$\surd$		$\cdots$
$r=4$	$\surd$	$\surd$	$\surd$							$\cdots$
$r=5$		$\surd$	$\surd$	$\surd$						$\cdots$
$r=6$			$\surd$			$\surd$				$\cdots$
$r=7$				$\surd$				$\surd$		$\cdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$

Table 3.2: Rank and number of elements of known minimal non-orientable matroids

3.3 Definition of Two New Families of Matroids: YM_{3n}^1 and F_{3n-2}

In this section, we define two new families of matroids of rank 3.

The first infinite family consists of the following matroids YM_{3n}^1 for $n \geq 3$. This family starts from YM_9^1, which is one of minimal non-orientable matroids of rank 3 with 9 elements. Figure 3.1 (a) and Figure 3.2 show geometric representations of YM_9^1, YM_{12}^1 and YM_{15}^1.

Definition 3.2. *For every $n \geq 3$, YM_{3n}^1 is a simple matroid $(E_n^1, \mathcal{H}_n^1)$ where $E_n^1 := \{1, \cdots, 3n\}$ and $\mathcal{H}_n^1$ is the collection of the following subsets of E_n^1:*

[2] In the geometric representation in [44], the hyperplane $\{2, 3, 4\}$ is missing.

- $\{1, 4, ..., 3n-2\}$, $\{2, 5, ..., 3n-1\}$, *and* $\{3, 6, ..., 3n\}$
- $\{1, 2, 6\}$, $\{1, 3, 5\}$, $\{2, 3, 4\}$, $\{4, 5, 6\}$, *and* $\{1, 3n-1, 3n\}$
- $\{3k, 3k+1, 3k+2\}$ *and* $\{3k-1, 3k+1, 3k+3\}$ *for* $k = 2, 3, ..., n-1$
- *all two-element subsets not contained in any above subset.*

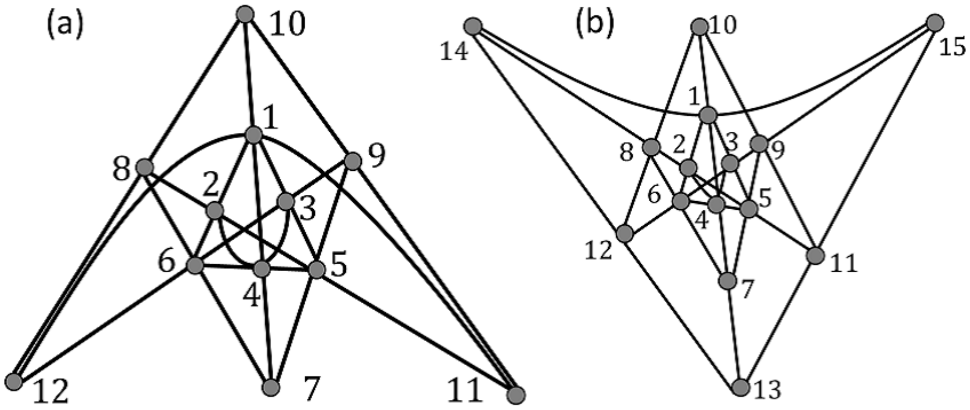

Figure 3.2: Geometric representations of (a) YM^1_{12} and (b) YM^1_{15}

The second infinite family consists of the following matroids F_{3n-2} for $n \geq 3$. This family starts from the Fano matroid F_7. Figure 3.3 shows geometric representations of F_7, F_{10} and F_{13}.

Definition 3.3. *For every* $n \geq 3$, F_{3n-2} *is a simple matroid* $(E^2_n, \mathcal{H}^2_n)$ *where* $E^2_n := \{1, \cdots, 3n-2\}$ *and* $\mathcal{H}^2_n$ *is the collection of the following subsets of* E^2_n*:*

- $\{1, 4, ..., 3n-2\}$
- $\{3k-1, 3k+1, 3k+3\}$ *and* $\{3k, 3k+1, 3k+2\}$ *for* $k = 1, 2, ..., n-2$
- $\{1, 3k-1, 3k\}$ *for* $k = 1, 2, ..., n-1$
- $\{a, 3n-4, 3n-2\}$ *and* $\{b, 3n-3, 3n-2\}$ *where* $a = 2$, $b = 3$ *if* n *is odd, otherwise* $a = 3$, $b = 2$
- *all two-element subsets not contained in any above subset.*

Note that it can be confirmed by routinely checking the hyperplane axiom system that YM^1_{3n} and F_{3n-2} are actually matroids, i.e. $\mathcal{H}^1_n$ and $\mathcal{H}^2_n$ satisfy the hyperplane axiom system.

For the proof of Theorem 3.1, we show the following two theorems.

Theorem 3.4 ([32])**.** *For every* $n \geq 3$, YM^1_{3n} *is minimal non-orientable.*

Theorem 3.5 ([32])**.** *For every* $n \geq 3$, F_{3n-2} *is minimal non-orientable.*

The proofs of Theorem 3.4 and Theorem 3.5 are divided into two parts. Firstly, we prove in Proposition 3.11 and Proposition 3.12 that YM^1_{3n} and F_{3n-2} are non-orientable for every $n \geq 3$, respectively. Secondly, we also prove their minimality

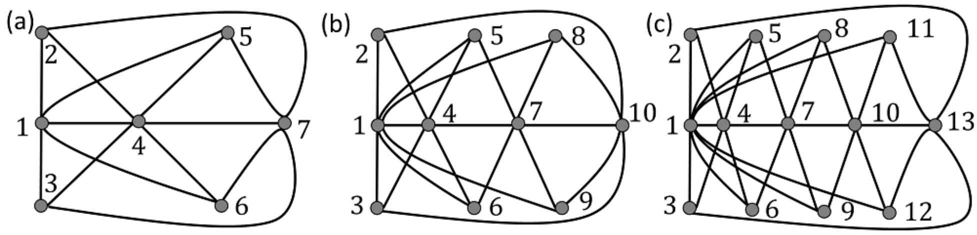

Figure 3.3: Geometric representations of (a) F_7 (the Fano matroid), (b) F_{10}, and (c) F_{13}

in Proposition 3.13 and Proposition 3.14 stating that any proper minor of YM^1_{3n} and F_{3n-2} is orientable. Combining the four propositions, we obtain Theorem 3.4 and Theorem 3.5. Finally, we derive Theorem 3.1 from the two theorems and the existence of the infinite family of minimal non-orientable matroids of rank 3 on the ground set E such that $|E| = 3n - 1$ for $n \geq 3$ constructed by Ziegler [86].

3.4 Non-orientability of YM^1_{3n} and F_{3n-2}

In this section, we prove in Proposition 3.11 and Proposition 3.12 that YM^1_{3n} and F_{3n-2} are non-orientable for every $n \geq 3$, respectively.

3.4.1 Review of Ziegler's proof on non-orientability

In the proofs of Proposition 3.11 and Proposition 3.12, we follow the proof on non-orientability of Ziegler's infinite family of minimal non-orientable matroids. The Ziegler's family consists of the matroid Z_{3n-1} for $n \geq 3$ in Figure 3.4 (a). Here we review Ziegler's proof on non-orientability of the infinite family. To prove non-orientability of Z_{3n-1}, Ziegler introduced another matroid Z'_{3n+1} for $i \geq 3$ in Figure 3.4 (b).

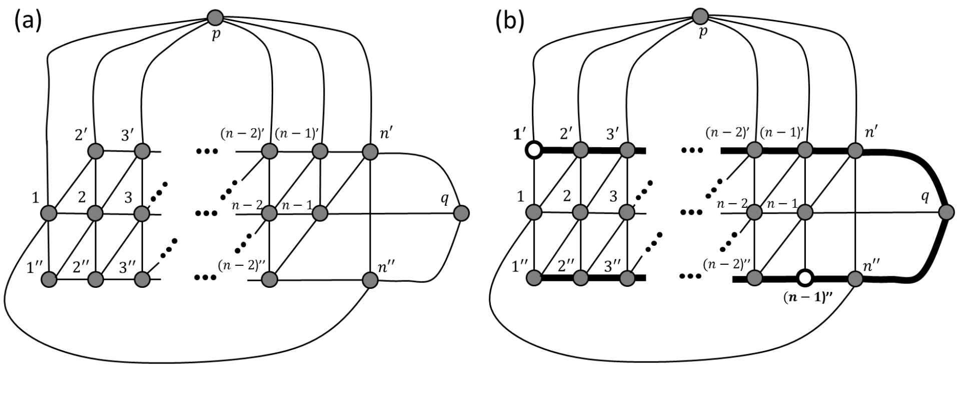

Figure 3.4: Geometric representations of (a) Z_{3n-1} and (b) Z'_{3n+1}

We know immediately by the following Theorem 3.6 [25] that Z_{3n-1} is orientable if and only if Z'_{3n+1} is orientable. Given a matroid on the ground set E, an element

$p \in E$ is called *reducible* if p is contained in at most two hyperplanes of size at least three.

Theorem 3.6 ([25]). *Let M be a simple matroid of rank 3 on the ground set E, and $p \in E$ a reducible element. Then M is orientable if and only if $M \setminus \{p\}$ is orientable.*

The elements $1'$ and $(n-1)''$ of Z'_{3n+1} (see two white points in Figure 3.4 (b)) are reducible. Furthermore, Z_{3n-1} is the deletion of the two reducible element $\{1', (n-1)''\}$ from Z'_{3n+1}. Therefore, Ziegler showed the non-orientability of Z_{3n-1} by proving the non-orientability of Z'_{3n+1}.

Recall that a given matroid M of rank r is said to be orientable if M is the underlying matroid of some oriented matroid. By the topological representation theorem [18], every simple oriented matroid can be represented by a pseudohyperplane arrangement. Particularly if $r = 3$, a pseudohyperplane is called a pseudoline. A pseudoline is a simple closed curve in the real projective plane RP^2. A peudoline arrangement is a finite collection of pseudolines in RP^2 satisfying the following two properties:

(i) any two pseudolines intersect in exactly one point, and

(ii) the intersection of all of the pseudolines is empty.

We say that a pseudoline arrangement $\{L_e\}_{e \in E}$ represents a matroid $(E, \mathcal{H})$ if the following condition is satisfied: $H \in \mathcal{H}$ if and only if the pseudolines L_e for $e \in H$ intersect at one point in the arrangement and there exist no other pseudolines intersecting at the point. Therefore, given an orientable matroid $M = (E, \mathcal{H})$ of rank 3, there exists at least one pseudoline arrangement representing M. Note that one orientable matroid may have more than one representations by a pseudoline arrangement. On the other hand, if a given matroid is non-orientable, there exists no pseudoline arrangement representing the matroid.

Ziegler proved the non-orientability of Z'_{3n+1} for $n \geq 3$ by showing that there exist no pseudoline arrangements representing Z'_{3n+1} [86]. The essential idea is that, supposing that Z'_{3n+1} is orientable, there exists a pseudoline arrangement $A(Z'_{3n+1})$ which includes the following grid structure. Let $M = (E, \mathcal{H})$ be a matroid of rank 3. For two distinct hyperplanes $H_1, H_2 \in \mathcal{H}$, if $H_1 \cap H_2$ consists of a one-element set $\{e\}$ then we say that M has a *partial grid*, denoted by a triple $\pi = (\{e\}, H_1 \setminus \{e\}, H_2 \setminus \{e\})$. The paper [86] mentioned the following fact on partial grids.

Fact 3.7. *Let $M = (E, \mathcal{H})$ be an orientable matroid of rank 3. If M has a partial grid $\pi = (\{e\}, H_1 \setminus \{e\}, H_2 \setminus \{e\})$, there exists a pseudoline arrangement $\mathcal{A}(M) = \{L_e\}_{e \in E}$ which includes the following grid structure $G(\pi)$:*

- *the pseudoline L_e is at infinity,*
- *the pseudolines in $\{L_f \mid f \in H_1 \setminus e\}$ are parallel to the x-axis, and*
- *the pseudolines in $\{L_f \mid f \in H_2 \setminus e\}$ are parallel to the y-axis*

The matroid Z'_{3n+1} has a partial grid $\omega = (\{q\}, \{1', 2', ..., n'\}, \{1'', 2'', ..., n''\})$ (see bold lines of Figure 3.4 (b)). If Z'_{3n+1} is orientable, a pseudoline arrangement

$\mathcal{A}(Z'_{3n+1})$ would have to include the grid structure $G(\omega)$. Ziegler drew a contradiction by showing that the other pseudolines cannot be added to $G(\omega)$ satisfying the hyperplanes of Z'_{3n+1}, which means that there is no pseudoline arrangement including $G(\omega)$.

3.4.2 Non-orientability of YM^1_{3n}

We use Fact 3.7 to show the non-orientability of the first infinite family consisting of YM^1_{3n} for $n \geq 3$.

Firstly, we define the following matroid YM'^1_{3n+1} which is constructed by adding one reducible element 0 to YM^1_{3n}. Figure 3.5 shows geometric representations of YM'^1_{13} and YM'^1_{16}.

Definition 3.8. *For every $n \geq 3$, YM'^1_{3n+1} is a simple matroid $(E^3_n, \mathcal{H}^3_n)$, where $E^3_n := \{0, 1, ..., 3n\} = \{0\} \cup E^1_n$ and $\mathcal{H}^3_n$ is the collection of the following subsets of E^3_n:*

- $\{1, 4, ..., 3n-2\}$, $\{0, 2, 5, ..., 3n-1\}$, *and* $\{0, 3, 6, ..., 3n\}$
- $\{1, 2, 6\}$, $\{1, 3, 5\}$, $\{2, 3, 4\}$, $\{4, 5, 6\}$, *and* $\{1, 3n-1, 3n\}$
- $\{3k, 3k+1, 3k+2\}$ *and* $\{3k-1, 3k+1, 3k+3\}$ *for* $k = 2, 3, ..., n-1$
- *all two-element subsets not contained in any above subset.*

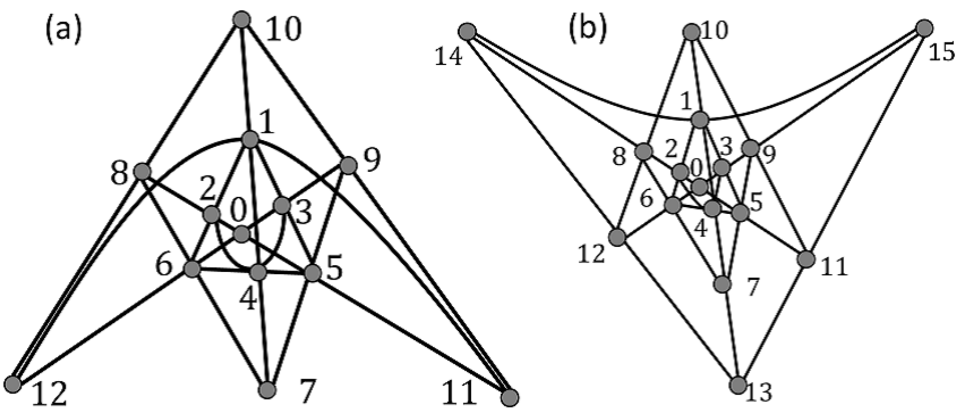

Figure 3.5: Geometric representations of (a) YM'^1_{13} and (b) YM'^1_{16}

Note again that it can be confirmed by routinely checking the hyperplane axiom system that YM'^1_{3n} is actually a matroid.

The element 0 of YM'^1_{3n+1} is reducible, and YM^1_{3n} is the deletion of the reducible element 0 from YM'^1_{3n+1}. Theorem 3.6 implies the following corollary.

Corollary 3.9 ([32])**.** *For every $n \geq 3$, YM^1_{3n} is orientable if and only if YM'^1_{3n+1} is orientable.*

Next we use Fact 3.7 to prove Proposition 3.10 on non-orientability of YM'^1_{3n+1} for $n \geq 3$.

Proposition 3.10 ([32])**.** *For every $n \geq 3$, YM'^1_{3n+1} is non-orientable.*

Proof. From Definition 3.8 of ${YM'}^1_{3n+1}$, there exist two distinct hyperplanes $\{0, 2, 5, ..., 3n-1\}(=: H_1)$ and $\{0, 3, 6, ..., 3n\}(=: H_2)$ whose intersection consists of a one-element set $\{0\}$. Therefore ${YM'}^1_{3n+1}$ has a partial grid $\lambda = (\{0\}, H_1 \setminus \{0\}, H_2 \setminus \{0\})$.

Suppose that ${YM'}^1_{3n+1}$ is orientable for every $n \geq 3$. There exists a pseudoline arrangement representing ${YM'}^1_{3n+1}$. Furthermore, Fact 3.7 implies that there exists a pseudoline arrangement $\mathcal{A}({YM'}^1_{3n+1})$ representing ${YM'}^1_{3n+1}$ which includes the grid structure $G(\lambda)$. The grid structure $G(\lambda)$ consists of the following pseudolines:

- the pseudoline L_0 at infinity
- the pseudolines $\{L_f \mid f \in H_1 \setminus \{0\}\}$ parallel to the x-axis
- the pseudolines $\{L_f \mid f \in H_2 \setminus \{0\}\}$ parallel to the y-axis.

There are $(n!)^2$ possible types of $G(\lambda)$ regarding permutations of the n pseudolines parallel to the x-axis and those of the n pseudolines parallel to the y-axis.

We now add the other pseudolines $E^1_n \setminus (H_1 \cup H_2)$ to $G(\lambda)$. Here is the following crucial observation. The existence of the hyperplanes $\{1, 4, ..., 3n-2\}$, $\{3n-4, 3n-2, 3n\}$ and $\{3n-3, 3n-2, 3n-1\}$ implies that the pseudolines L_{3i-2} for every $i = 1, 2, ..., n$, the three pseudolines L_{3n-4}, L_{3n-2} and L_{3n}, and the three pseudolines L_{3n-3}, L_{3n-2} and L_{3n-1} intersect at one point on the xy plane, respectively. We denote the three points by ω_0, ω_1 and ω_2 respectively, and let $\omega := \{\omega_0, \omega_1, \omega_2\}$. Note that the pseudoline L_{3n-2} passes through the three points of ω. We consider how the three points of ω are arranged on the xy-plane. The number of orderings of their x-coordinates, $x(\omega_0)$, $x(\omega_1)$ and $x(\omega_2)$, is equal to $6(:= 3!)$ and so is that of orderings of their y-coordinates, $y(\omega_0)$, $y(\omega_1)$ and $y(\omega_2)$, where $x(\tau)$ and $y(\tau)$ are the x-coordinate and the y-coordinate of a point τ on the xy-plane. Therefore the three points of ω can be arranged in 36 possible types. Figure 3.6 shows two possible types of the three points of ω. In every possible type, if either $x(\omega_0)$ or $y(\omega_0)$ takes the middle value of the x-coordinates or y-coordinates of the three points of ω, ω is said to satisfy *middle condition.* There exist 16 types satisfying the middle condition out of 36 possible types. Note that ω in both cases of Figure 3.6 satisfies the middle condition. Then, the following fact is observed.

> If ω satisfies the middle condition, the pseudoline L_{3n-2} would intersect twice with at least one of the pseudolines L_{3n-4}, L_{3n-3}, L_{3n-1} and L_{3n}. (3.1)

Finally, we show that ω always satisfies the middle condition. By (3.1), this implies that the pseudoline arrangement $\mathcal{A}({YM'}^1_{3n+1})$ never exists i.e., ${YM'}^1_{3n+1}$ is non-orientable. For every $k = 2, 3, ..., n-1$, we use the following notation:

- the intersection θ^k_1 of the two pseudolines L_{3k-1} and L_{3n}
- the intersection θ^k_2 of the two pseudolines L_{3k} and L_{3n-1}
- the set $S_k = \{\omega_0, \theta^k_1, \theta^k_2\}$ of points. (Note that $S_{n-1} = \omega$)

The two-element sets $\{3k-1, 3n\}$ and $\{3k, 3n-1\}$ for every $k = 2, 3, ..., n-2$ are hyperplanes of the fourth type in Definition 3.8. It follows that no pseudolines other than L_{3k-1} and L_{3n} and those other than L_{3k} and L_{3n-1} pass through θ^k_1 and θ^k_2, respectively. When $k = n-1$, the two-element sets $\{3(n-1)-1, 3n\}$

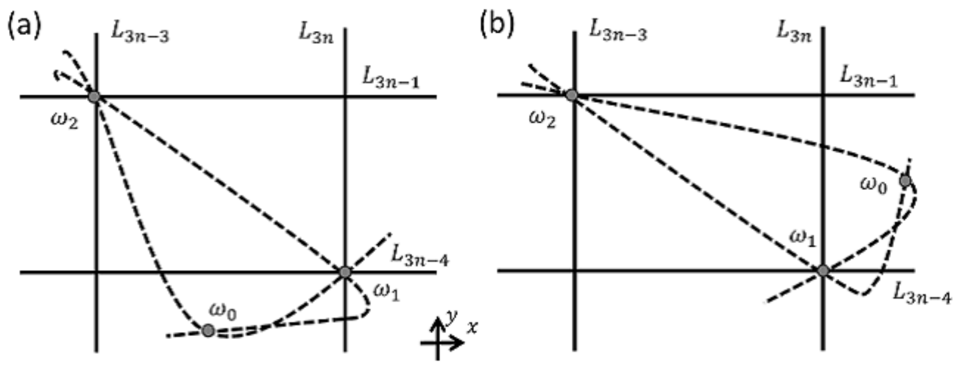

Figure 3.6: The pseudoline L_{3n-2} when ω satisfies the middle condition (a) $x(\omega_0)$ takes the middle value (b) $y(\omega_0)$ takes the middle value

and $\{3(n-1), 3n-1\}$ are included in the hyperplanes $\{3n-4, 3n-2, 3n\}$ and $\{3n-3, 3n-2, 3n-1\}$, respectively. Note that the points θ_1^{n-1} and θ_2^{n-1} coincide with the points ω_1 and ω_2, respectively. We generalize the middle condition on ω for the three points of S_k. Given the three points of S_k, if either $x(\omega_0)$ or $y(\omega_0)$ takes the middle value of the x-coordinates or y-coordinates of the three points of S_k, S_k is said to satisfy the middle condition.

We show the following claim by induction in k.

$$S_k \text{ satisfies the middle condition for } k = 2, 3, ..., n-1. \tag{3.2}$$

Since $S_{n-1} = \omega$, (3.1) and (3.2) lead to a contraction to the assumption that ${YM'}^1_{3n+1}$ is orientable, which completes the proof of this proposition.

Now we show the base case $k = 2$. The existence of hyperplanes of the second type in Definition 3.8 implies that the following three pseudolines intersect at one point respectively: $\{L_1, L_2, L_6\}$, $\{L_1, L_3, L_5\}$, $\{L_2, L_3, L_4\}$, $\{L_4, L_5, L_6\}$, and $\{L_1, L_{3n-1}, L_{3n}\}$. Figure 3.7 shows four possible types of arranging the eight pseudolines. When $k = 2$, the points θ_1^2 and θ_2^2 are the intersection of L_5 and L_{3n} and that of L_6 and L_{3n-1}, respectively. Hence S_2 satisfies the middle condition in the four types. By symmetry, every possible type of arrangements is reduced into one of the arrangements in Figure 3.7. Therefore S_2 always satisfies the middle condition. Therefore, the base case, (3.2) for $k = 2$, has been proven.

Next we move into the induction step. Assume that (3.2) holds for some k. Hereafter, we show that (3.2) for $k+1$ also holds. Figure 3.8 shows two possible types of arranging the three points of S_k satisfying the middle condition such that (a) $x(\theta_2^k) < x(\omega) < x(\theta_1^k)$ and $y(\omega) < y(\theta_1^k) < y(\theta_2^k)$,and (b) $x(\theta_2^k) < x(\omega) < x(\theta_1^k)$ and $y(\omega) < y(\theta_2^k) < y(\theta_1^k)$, respectively. Hereafter, we show that S_{k+1} satisfies the middle condition in these two types. By symmetry, every other possible type satisfying the middle condition is reduced into these two types in Figure 3.8.

The point θ_1^{k+1} is on the pseudoline L_{3n}. In the first possible type of S_k i.e., $x(\theta_2^k) < x(\omega) < x(\theta_1^k)$ and $y(\omega) < y(\theta_1^k) < y(\theta_2^k)$, there are the following four cases on the y-coordinate of θ_1^{k+1}: (i) $y(\theta_1^{k+1}) < y(\omega)$, (ii) $y(\omega) < y(\theta_1^{k+1}) < y(\theta_1^k)$, (iii) $y(\theta_1^k) < y(\theta_1^{k+1}) < y(\theta_2^k)$ and (iv) $y(\theta_2^k) < y(\theta_1^{k+1})$. In the third case, the pseudoline L_{3k+2} is arranged between the pseudolines L_{3n-1} and L_{3k-1} in Figure 3.8 (a). As the point θ_2^{k+1} is on the pseudoline L_{3n-1}, we obtain the relation

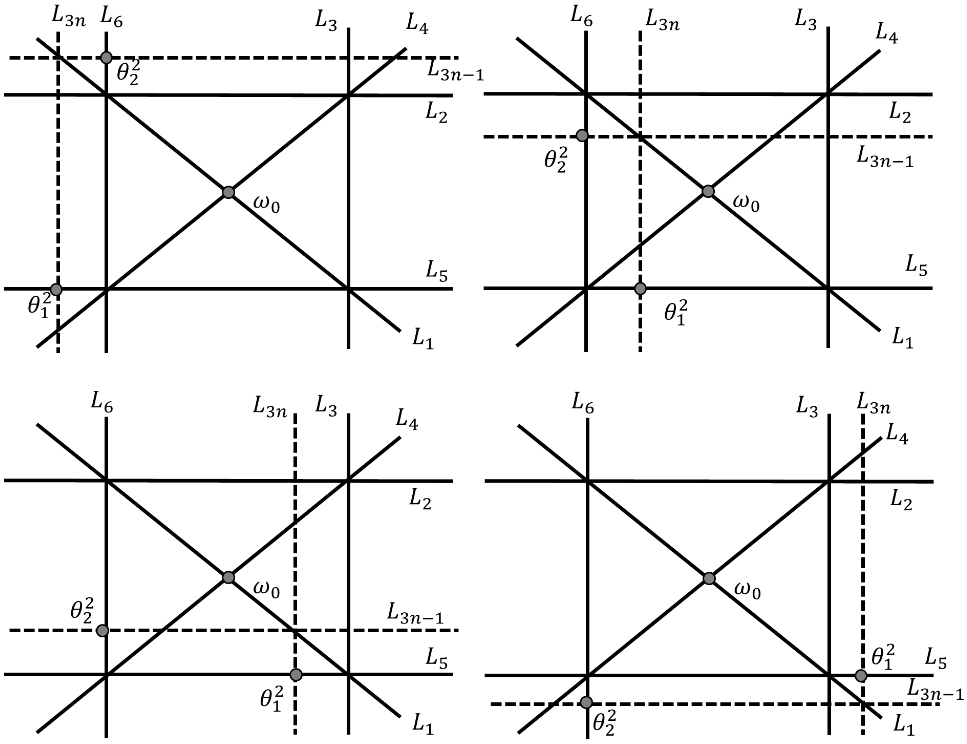

Figure 3.7: Four possible types of arranging the eight pseudolines L_1, L_2, L_3, L_4, L_5, L_6, L_{3n-1} and L_{3n}

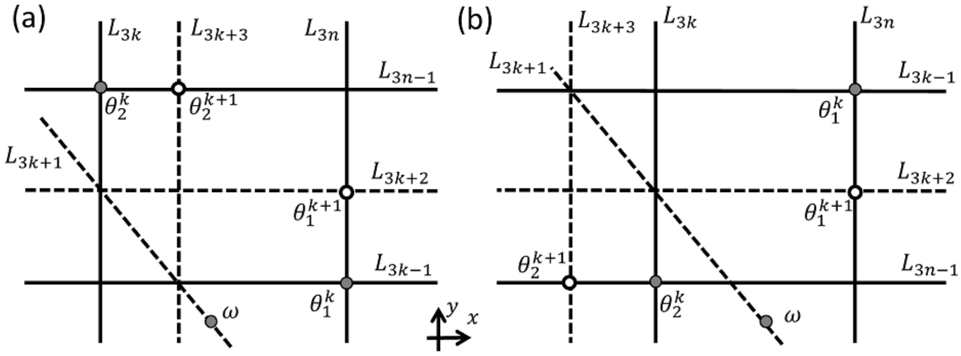

Figure 3.8: Two examples of positions of S_k (a) $x(\theta'_k) < x(\theta) < x(\theta_k)$ and $y(\theta) < y(\theta_k) < y(\theta'_k)$ (b) $x(\theta'_k) < x(\theta) < x(\theta_k)$ and $y(\theta) < y(\theta'_k) < y(\theta_k)$

$y(\omega) < y(\theta_1^{k+1}) < y(\theta_2^{k+1})$. The existence of the hyperplane $\{3k, 3k+1, 3k+2\}$ implies that the pseudolines L_{3k}, L_{3k+1} and L_{3k+2} intersect at one point on the xy-plane. As the pseudoline L_{3k+1} also passes through the point ω, the pseudoline L_{3k+1} is determined to intersect with the pseudoline L_{3k-1} at a point π such that $x(\theta_2^k) < x(\pi) < x(\omega)$. The pseudoline L_{3k+3} also passes through the point π, because of the existence of the hyperplane $\{3k-1, 3k+1, 3k+3\}$. It follows that the x-coordinate of L_{3k+3} is between $x(\theta_2^k)$ and $x(\omega)$. As the point θ_2^{k+1} is the intersection between L_{3n-1} and L_{3k+3}, $x(\theta_2^{k+1}) < x(\omega)$. It follows that the $x(\omega)$ is the largest of the x-coordinates of the three points in S_{k+1}. Therefore S_{k+1} satisfies the middle condition. As the proofs for the other three cases are similar to that of the third case, we omit them. There are also the following four cases on the y-coordinate of θ_1^{k+1} in the second possible type of S_k i.e., $x(\theta_2^k) < x(\omega) < x(\theta_1^k)$ and $y(\omega) < y(\theta_2^k) < y(\theta_1^k)$: (i) $y(\theta_1^{k+1}) < y(\omega)$, (ii) $y(\omega) < y(\theta_1^{k+1}) < y(\theta_2^k)$, (iii) $y(\theta_2^k) < y(\theta_1^{k+1}) < y(\theta_1^k)$ and (iv) $y(\theta_1^k) < y(\theta_1^{k+1})$. Figure 3.8 (b) shows the third case. As the proofs for the four cases are also similar to that of the third case in the first possible type of S_k, we omit them. Therefore, we conclude that S_{k+1} satisfies the middle condition in every case. This shows (3.2) for $k+1$, and completes the proof. □

We obtain the following proposition from Corollary 3.9 and Proposition 3.10.

Proposition 3.11 ([32]). *For every $n \geq 3$, YM_{3n}^1 is non-orientable.*

3.4.3 Non-orientability of F_{3n-2}

We use Fact 3.7 to show the non-orientability of the second infinite family consisting of F_{3n-2} for $n \geq 3$.

Proposition 3.12 ([32]). *For every $n \geq 3$, F_{3n-2} is non-orientable.*

Proof. From Definition 3.3 of F_{3n-2}, there exist two distinct hyperplanes $\{1, 4, ..., 3n-2\}(=: H_1)$ and $\{1, 2, 3\}(=: H_2)$ whose intersection consists of a one-element set $\{1\}$. Therefore F_{3n-2} has a partial grid $\mu = (\{1\}, H_1 \setminus \{1\}, H_2 \setminus \{1\})$.

Suppose that F_{3n-2} is orientable for every $n \geq 3$. There exists a pseudoline arrangement representing F_{3n-2}. Furthermore, Fact 3.7 implies that there exists a pseudoline arrangement $\mathcal{A}(F_{3n-2})$ representing F_{3n-2} which includes the grid structure $G(\mu)$. The grid structure $G(\mu)$ consists of the following pseudolines:

- the pseudoline L_1 at infinity
- the pseudolines $\{L_f \mid f \in H_1 \setminus \{1\}\}$ parallel to the x-axis
- the pseudolines $\{L_f \mid f \in H_2 \setminus \{1\}\}$ parallel to the y-axis.

There are $2((n-1)!)$ possible types of $G(\mu)$ regarding permutations of the $n-1$ pseudolines parallel to the x-axis and those of the two pseudolines parallel to the y-axis. By symmetry, we assume without loss of generality that the x-coordinate of the pseudoline L_3 is greater than that of the pseudoline L_2.

We now add the other pseudolines $E_n^2 \setminus (H_1 \cup H_2)$ to $G(\mu)$. Here is the following crucial observation. The existence of the hyperplanes $\{1, 3k-1, 3k\}$ for every $k = 1, 2, ...n-1$ implies that the two pseudolines L_{3k-1} and L_{3k} intersect on the pseudoline L_1, the line at infinity. Therefore, the following fact is obtained.

> For $k = 1, 2, ...n-1$, the two pseudoline L_{3k-1} and L_{3k} do not intersect on the xy-plane. (3.3)

We begin to show that the assumption that F_{3n-2} is orientable draws a contradiction that the pseudolines L_{3n-4} and L_{3n-3} intersect on the xy-plane.

The existence of the hyperplanes $\{3k-1, 3k+1, 3k+3\}$ and $\{3k, 3k+1, 3k+2\}$ for every $k = 1, 2, ...n-1$ implies that the three pseudolines $\{L_{3k-1}, L_{3k+1}, L_{3k+3}\}$ and $\{L_{3k}, L_{3k+1}, L_{3k+2}\}$ intersect at one point on the xy-plane. For every $k = 1, 2, ..., n-2$, we use the following notation:

- the intersection α_k of L_{3k-1}, L_{3k+1} and L_{3k+3}
- the intersection β_k of L_{3k}, L_{3k+1} and L_{3k+2}.

Hereafter, we show the following claim.

$$\text{The pseudolines } L_{3n-4} \text{ and } L_{3n-3} \text{ intersect on the } xy\text{-plane.} \tag{3.4}$$

This claim contradicts the fact (3.3), and therefore the assumption that F_{3n-2} is orientable. Hence, to prove Proposition 3.12, it suffices to prove the claim (3.4).

As a first step to prove (3.4), we prove the following claim by induction in $k = 1, 2, ..., n-2$.

For $k = 1, 2, ..., n-2$, $x(\alpha_k) < x(\beta_k)$ if k is odd, otherwise $x(\beta_k) < x(\alpha_k)$, where $x(\tau)$ and $y(\tau)$ are the x-coordinate and the y-coordinate of a point τ on the xy-plane. (3.5)

For the base case $k = 1$, $x(\alpha_1) < x(\beta_1)$ from our assumption that the x-coordinate of the pseudoline L_3 is greater than that of the pseudoline L_2 (see Figure 3.9 (a)). For the inductive steps, we first show the cases when k is odd. Assume that $x(\alpha_k) < x(\beta_k)$ for some odd k (see Figure 3.9 (b)). The relation $x(\alpha_k) < x(\beta_k)$ implies that the x-coordinate of the pseudoline L_{3k+2} is greater than the pseudoline L_{3k+3} on the pseudoline L_{3k+1}. From the above observation, the pseudolines L_{3k+3} and L_{3k+2} do not intersect on the xy plane. Therefore, the x-coordinate of the pseudoline L_{3k+2} is also greater than the pseudoline L_{3k+3} on the pseudoline $L_{3(k+1)+1}$, which proves $x(\beta_{k+1}) < x(\alpha_{k+1})$. As the proof when k is even is similar, we omit it. This is the end of the proof of (3.5).

Finally, we show the claim (3.4), by using the relation between $x(\alpha_k)$ and $y(\beta_k)$ for $k = 1, 2, ..., n-2$ which is obtained by (3.5). When n is even, there exist the two hyperplanes $\{3, 3n-4, 3n-2\}$ and $\{2, 3n-3, 3n-2\}$. Let γ and γ' be the intersection of the pseudolines L_3, L_{3n-4} and L_{3n-2} and that of the pseudolines L_2, L_{3n-3} and L_{3n-2}, respectively. The assumption on the pseudolines L_2 and L_3 implies $x(\gamma') < x(\gamma)$. Here, as $n-2$ is also even, $x(\beta_{n-2}) < x(\alpha_{n-2})$. The pseudoline L_{3n-4} passes through both β_{n-2} and γ, and the pseudoline L_{3n-3} passes through both α_{n-2} and γ' (see Figure 3.9 (d)). It follows that the pseudolines L_{3n-4} and L_{3n-3} intersect on the xy-plane. When n is odd, there exist the two hyperplanes $\{3, 3n-3, 3n-2\}$ and $\{2, 3n-4, 3n-2\}$. Let γ and γ' be the intersection of the pseudolines L_3, L_{3n-3} and L_{3n-2} and that of the pseudolines L_2, L_{3n-4} and L_{3n-2}, respectively. Then the proof is similar to the case when k is even, and we omit it. This is the end of the proof of (3.4).

As mentioned before, (3.4) contradicts (3.3) stating that the pseudolines L_3 and L_4 do not intersect on the xy-plane. Therefore we conclude that F_{3n-2} for every $n \geq$ is non-orientable. □

3.5 Minimality of YM^1_{3n} and F_{3n-2}

In this section, we prove any proper minor of YM^1_{3n} and F_{3n-2} is orientable in Proposition 3.13 and Proposition 3.14 respectively.

Proposition 3.13 ([32]). *For every $n \geq 3$, any proper minor of YM^1_{3n} is orientable.*

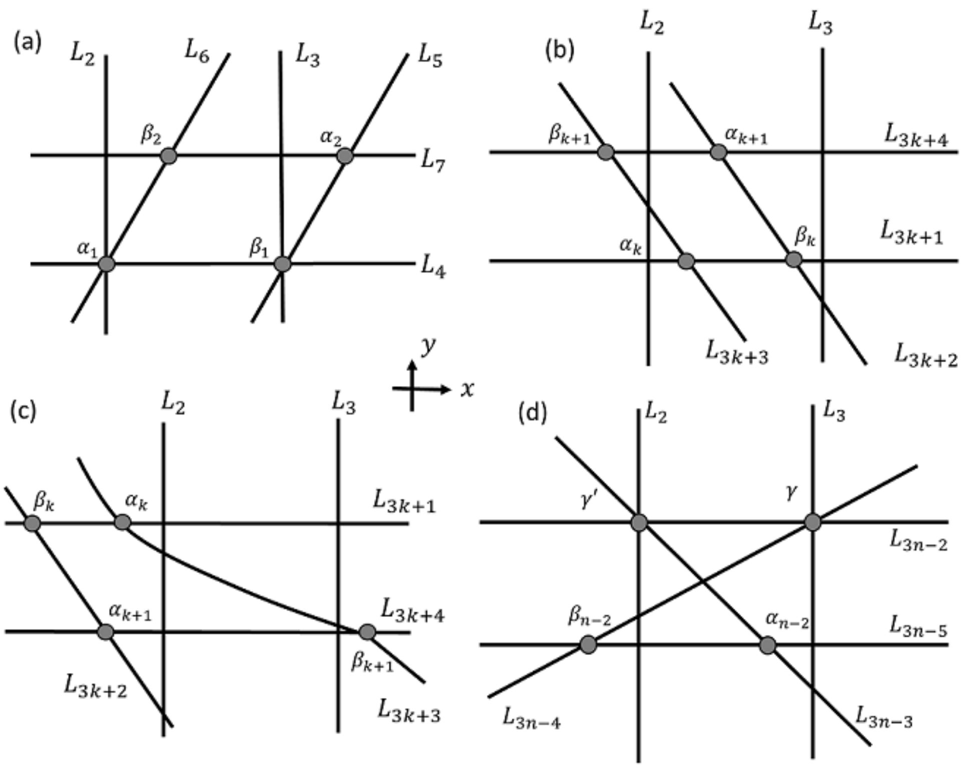

Figure 3.9: (a)-(c) The position of $\alpha_k, \beta_k, \alpha_{k+1}, \beta_{k+1}$: (a) The case $k = 1$ (b) The case k odd (c) the case k even. (d) The position of $\alpha_{n-2}, \beta_{n-2}$ and γ, γ'

Proof. Any contraction of a simple matroid decreases the rank of the matroid. As the matroid YM^1_{3n} is a simple matroid of rank 3, the rank of any contraction is equal to 2. Note that all matroids of rank 2 are orientable. Therefore, any contraction of YM^1_{3n} is orientable.

We prove that, for any $e \in E^1_n$, the deletion $YM^1_{3n} \setminus \{e\}$ is orientable. Firstly we take one element of $\{3k-2, 3k-1, 3k\}$ for some $k = 3, ..., n$ as the deletion element e. The deletion of e from YM^1_{3n} has two reducible elements $\{3k-2, 3k-1, 3k\} \setminus \{e\}(=: s_0)$. By Theorem 3.6, $YM^1_{3n} \setminus \{e\}$ is orientable if and only if $YM^1 \setminus s_0$. When $n \geq 4$, let s_i for every $i = 1, ..., n-3$ be a set $\{3(k-i)-2, 3(k-i)-1, 3(k-i)\}$ if $i \leq k-3$ and a set $\{3(i+3)-2, 3(i+3)-1, 3(i+3)\}$ otherwise. From the hyperplanes of Definition 3.2, any element of s_i is reducible in the deletion of $YM^1_{3n} \setminus (s_0 \cup ... \cup s_{i-1})$ for every $i = 1, ..., n-3$. We delete such reducible elements sequentially. Finally we obtain the matroid $M = YM^1_{3n} \setminus (s_0 \cup ... \cup s_{n-2})$ on the ground set $\{1, 2, 3, 4, 5, 6\}$. As any matroid of rank 3 with at most 6 element is orientable, M is orientable. From Theorem 3.6, $YM^1_{3n} \setminus \{e\}$ is also orientable.

Next, we take one of the other elements $\{1, 2, 3, 4, 5, 6\}$ as e. As their proofs are similar to the above one, we omit them. Therefore we conclude that any deletion of YM^1_{3n} is orientable. This completes the proof. $\square$

Proposition 3.14 ([32]). *For every $n \geq 3$, any proper minor of F_{3n-2} is orientable.*

Proof. As the proof on contractions of F_{3n-2} is the same as that of YM^1_{3n}, we omit it.

We prove that, for any $e \in E_n^2$, the deletion $F_{3n-2} \setminus \{e\}$ is orientable. Firstly we take $3n-2$ as e. For any $k \in \{8, ..., 3n-2\}$, the element $k-1$ is reducible in $F_{3n-2} \setminus \{k, k+1, ..., 3n-2\}$. We delete such reducible elements sequentially to obtain the matroid $M = F_{3n-2} \setminus \{7, 8, ..., 3n-2\}$. By Theorem 3.6, M is orientable if and only if $F_{3n-2} \setminus \{3n-2\}$ is orientable. As the number of elements in M is equal to 6, M is orientable. Consequently $F_{3n-2} \setminus \{3n-2\}$ is orientable.

Next, we take 1, 2 or 3 as e. For any $k \in \{e+1, e+2, ..., 3n-2\}$, the element k is reducible in $F_{3n-2} \setminus \{e, e+1, ..., k-1\}$. We delete such reducible elements sequentially to obtain the matroid $M = F_{3n-2} \setminus \{e, e+1, ..., 3n-2\}$. By Theorem 3.6, M is orientable if and only if $F_{3n-2} \setminus \{e\}$ is orientable. As the number of elements in M is at most 2, M is orientable. Consequently $F_{3n-2} \setminus \{e\}$ is orientable.

Finally, we take one element of $\{3k-2, 3k-1, 3k\} (=: s_0)$ for some $k = 2, ..., n-1$ as the deletion element e. The deletion of e from F_{3n-2} has at least one reducible element e' in $s_0 \setminus \{e\}$. The other element $e'' \in s_0 \setminus \{e, e'\}$ is also reducible in $F_{3n-2} \setminus \{e, e'\}$. By Theorem 3.6, $F_{3n-2} \setminus \{e\}$ is orientable if and only if $F_{3n-2} \setminus s_0$. Then let s_i for every $i = 1, 2, ..., n-2$ be a set $\{s_i(1) = 3(k+i)-2, s_i(2) = 3(k+i)-1, s_i(3) = 3(k+i)\}$ if $i \leq n-k-1$, and a set $\{s_i(1) = 3(i-n+k)+1, s_i(2) = 3(i-n+k)+2, s_i(3) = 3(i-n+k)+3\}$ otherwise. For each $k \in \{1, 2, ..., n-2\}$, the elements $s_i(2)$ and $s_i(3)$ of s_{k+1} are reducible in $F_{3n-2} \setminus (s_0 \cup s_1 \cup ... \cup s_k)$, and the element $s_i(1)$ of s_{k+1} is also reducible in $F_{3n-2} \setminus (s_0 \cup s_1 \cup ... \cup s_k \cup \{s_i(2), s_i(3)\})$. Therefore, Theorem 3.6 also implies that $F_{3n-2} \setminus (s_0 \cup s_1 \cup ... \cup s_k)$ is orientable if and only if $F_{3n-2} \setminus (s_0 \cup s_1 \cup ... \cup s_{k+1})$ is orientable. Consequently, $F_{3n-2} \setminus \{e\}$ is orientable if and only if $M' = F_{3n-2} \setminus (s_0 \cup s_1 \cup ... \cup s_{n-2})$. As the number of elements in M' is equal to one, M' is orientable. Therefore we conclude that any deletion of F_{3n-2} is orientable. This completes the proof. $\square$

Combining with Proposition 3.11 and Proposition 3.12, Proposition 3.13 and Proposition 3.14 yield Theorem 3.4 and Theorem 3.5, respectively.

3.6 Representability of YM_{3n}^1, F_{3n-2} and YM_9^2

In this section, we prove Proposition 3.16 and Proposition 3.17 on our two infinite families of minimal non-orientable matroids of rank 3. Moreover, we also show representability of two minimal non-orientable matroids of rank 3 with 9 elements, YM_9^1 and YM_9^2, in Proposition 3.18. From Proposition 3.16 and Proposition 3.17, we can know that YM_{3n}^1 and F_{3n-2} are also excluded minors for the union of the class of orientable matroids and the classes of matroid representable over some fields, the details of which are skipped in this section and explained in Section 4.1.

First, we introduce Theorem 3.15, which is a key theorem to prove the three propositions. Let M be a matroid on a ground set E, and $B (\subset E)$ a base of M. For any $e \in E \setminus B$, it is known that there exists an unique circuit $C(B, e)$ in $B \cup \{e\}$, called a *fundamental circuit*. We construct a bipartite graph $G(M, B)$ following the fundamental circuits. The graph $G(M, B)$ has two disjoint sets of vertices, B and $E \setminus B$. For $i \in B$ and $j \in E \setminus B$, there exists an edge (i, j) in $G(\mathcal{M}, B)$ if and only if $i \in C(B, j)$. Note that $G(M, B)$ is uniquely determined for B. Supposing that M is representable over a field $\mathbb{F}$, there exists a matrix representation A, which is a $|B| \times |E|$ matrix over $\mathbb{F}$. Note that every row and column of A is indexed by the elements of B and E respectively. Let $A(i, j)$ be the entry of 0 in the row indexed by $i \in B$ and in the column indexed by $j \in E$. Any matrix representation A is changed by elementary row operations into a $|B| \times |E|$ matrix A_B which also represents M such that for $i, j \in B$, $A_B(i,j) = 1$ if $i = j$, and 0 otherwise. The matrix A_B is called a matrix representation *normalized on B*.

Generally M has more than one matrix representations A_B. Theorem 3.15 states that there always exists a matrix representation A_B corresponding to $G(M, B)$.

Theorem 3.15 ([55]). *For a matroid M and a base B of M, let $F = \{(r_1, c_1), \ldots, (r_k, c_k) \mid r_i \in B, c_i \in E\}$ be a set of edges consisting of a spanning forest of $G(M, B)$. If M is representable over a field $\mathbb{F}$, then there is a matrix representation A_B of M over $\mathbb{F}$ such that $A_B(r_i, c_i) = \theta_i$ for every $i = 1, \ldots, k$, where θ_i is a non-zero value over $\mathbb{F}$.*

We use Theorem 3.15 as a key theorem to prove Proposition 3.16 and Proposition 3.17 on our two infinite families of minimal non-orientable matroids.

Proposition 3.16 ([32]). *For every $n \geq 3$, YM^1_{3n} is not representable over fields of characteristic 2.*

Proof. Assume that YM^1_{3n} is representable over a field of characteristic 2. For the base $B = \{1, 5, 6\}$, we consider $G(YM^1_{3n}, B)$ and the spanning tree with the following edges (see Figure 3.10).

- the edges connecting 1: $(1, 2)$, $(1, 3)$ and $(1, i)$ for every $i = 7, 8, \ldots, 3n$
- the edge connecting 5: $(4, 5)$
- the edges connecting 6: $(2, 6)$ and $(4, 6)$

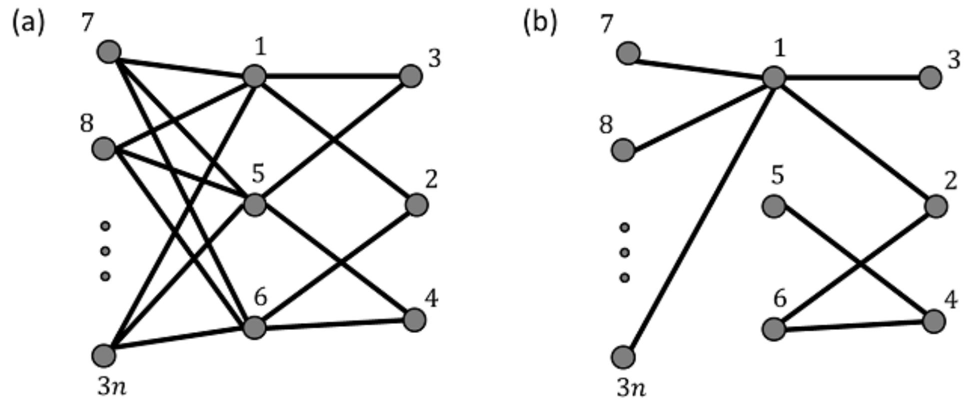

Figure 3.10: (a) $G(YM_{3n-2}, \{1, 5, 6\})$ (b) A spanning tree of $G(YM_{3n}, \{1, 5, 6\})$

By Theorem 3.15, there is a matrix representation A_B corresponding to the above spanning tree:

- the entries in the row 1: $A_B(1, 2) = 1$, $A_B(1, 3) = 1$ and $A_B(1, i) = 1$ for every $i = 7, 8, \ldots, 3n$
- the entry in the row 5: $A_B(5, 4) = 1$

- the entries in the row 6: $A_B(6,2) = 1$ and $A_B(6,4) = 1$

Since $\{1, 4, ..., 3n-2\}$, $\{2, 5, ..., 3n-1\}$ and $\{3, 6, ..., 3n\}$ are hyperplanes of YM^1_{3n}, the submatrices whose column vectors correspond to the hyperplanes have rank 2. It follows that $A_B(5, 3i-2) = A_B(6, 3i-2)$ for every $i = 7, 8, ..., 3n$, $A_B(6, 3i-1) = 1$ and $A_B(5, 3i) = 1$ for every $i = 3, 4, ..., n$. Therefore the matrix representation A_B has the following form where every variable s_i, t_i and r_i for every $i = 3, ..., n$ is non-zero. Note that the most upper and the left numbers are indices of the rows and columns of A_B respectively.

$$\begin{array}{c} \\ 1 \\ 5 \\ 6 \end{array}
\begin{array}{c}
\begin{array}{cccccccccccccc} 1 & 5 & 6 & 4 & 2 & 3 & \cdots & 3i-2 & 3i-1 & 3i & \cdots & 3n-2 & 3n-1 & 3n \end{array} \\
\begin{pmatrix}
1 & 0 & 0 & 0 & 1 & 1 & & 1 & 1 & 1 & & 1 & 1 & 1 \\
0 & 1 & 0 & 1 & 0 & 1 & \cdots & s_i & t_i & 1 & \cdots & s_n & t_n & 1 \\
0 & 0 & 1 & 1 & 1 & 0 & & s_i & 1 & r_i & & s_n & 1 & r_n
\end{pmatrix}
\end{array}$$

Hereafter, we show that the columns $3n-1$ and $3n$ of A_B have to be the same. This contradicts the fact that YM^1_{3n} is simple.

First we prove $t_i = r_i$ by induction in $i = 3, 4, ..., n$. Since $\{6, 7, 8\}$ and $\{5, 7, 9\}$ are hyperplanes, the submatrices whose column vectors correspond to the two hyperplanes of YM^1_{3n} have rank 2. Therefore, we obtain $s_3 = t_3$ and $s_3 = r_3$. It follows $t_3 = r_3$, which is the base case $i = 3$. Next, assuming that $t_k = s_k$ for some $k = 3, 4, ..., n-1$, we prove that $t_{k+1} = s_{k+1}$. Since $\{3k, 3k+1, 3k+2\}$ and $\{3k-1, 3k+1, 3k+3\}$ are hyperplanes of YM^1_{3n}, the submatrices whose column vectors correspond to the two hyperplanes have rank 2. Note that $1 = -1$ and $2 = 0$ as the characteristic of the field is equal to 2. Then we obtain the following two equations:

$$s_k t_{k+1} + s_{k+1}(s_k + t_{k+1}) = 1 \text{ and } s_k r_{k+1} + s_{k+1}(s_k + r_{k+1}) = 1$$

By adding their left-hand and right-hand sides, we obtain the equation $(s_k + s_{k+1})(t_{k+1} + r_{k+1}) = 0$. Since the elements $3k-2$ and $3k+1$ are not parallel, $s_k \neq s_{k+1}$. Hence $t_{k+1} = r_{k+1}$. This is the end of the proof of the equation $t_i = r_i$ for $i = 3, 4, ..., n$.

The existence of the hyperplane $\{1, 3n-1, 3n\}$ implies that the submatrix whose column vectors are the hyperplane has rank 2, which follows $t_n r_n = 1$. The equation system $t_n = r_n$ and $t_n r_n = 1$ gives the solution $t_n = r_n = 1$, which implies that the columns $3n-1$ and $3n$ of A_B are the same. This contradicts the fact that YM^1_{3n} is simple. Therefore, we conclude that YM^1_{3n} is not representable over a field of characteristic 2. $\square$

Proposition 3.17 ([32]). *For every $n \geq 3$, F_{3n-2} is not representable over fields of characteristic other than 2.*

Proof. The Fano matroid F_7 is known to be representable over only a field of characteristic 2.

Assume that F_{3n-2} for $n \geq 4$ is representable over some field of characteristic other than 2. For a base $B = \{2, 3, 4\}$, we consider $G(F_{3n-2}, B)$ and the its spanning tree with the following edges (see Figure 4.4).

- the edges connecting 2: $(1, 2)$

- the edges connecting 3: $(1,3)$, $(3,5)$
- the edges connecting 4: $(4,i)$ for every $i = 5, 6, ..., 3n-2$ (see Figure 4.4)

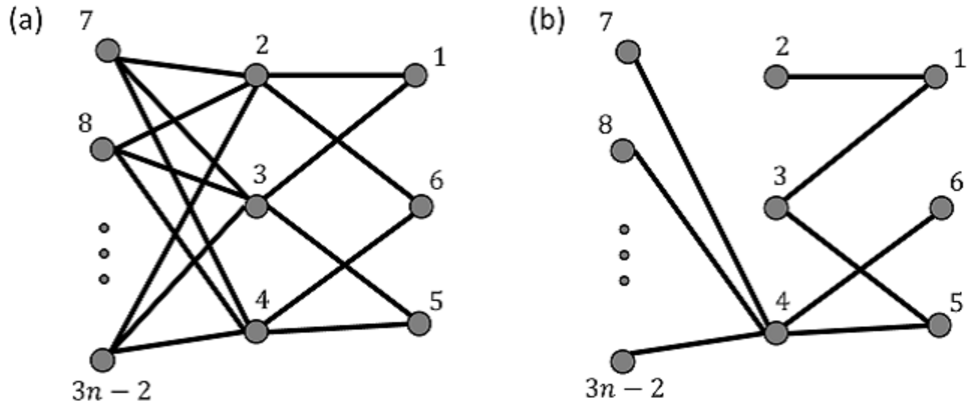

Figure 3.11: (a) $G(F_{3n-2}, \{2,3,4\})$ (b) A spanning tree of $G(F_{3n-2}, \{2,3,4\})$

By Theorem 3.15, there is a matrix representation A_B corresponding to the above spanning tree:

- the entry in the row 2: $A_B(2,1) = 1$
- the entries in the row 3: $A_B(3,1) = A_B(3,5) = 1$
- the entries in the row 4: $A_B(4,i) = 1$ for every $i = 5, 6, ..., 3n-2$

Since $\{1, 4, ..., 3n-2\}$ and $\{1,5,6\}$ are hyperplanes of F_{3n-2}, the submatrices whose column vectors correspond to the hyperplanes have rank 2. It follows that $A_B(2,6) = -1$ and $A_B(2, 3i-2) = A(3, 3i-2)$ for every $i = 3, 4, ..., n$. Therefore the matrix representation A_B has the following form where every variable p_i for every $i = 3, ..., n$ and every q_i, r_i, s_i and t_i for every $i = 3, ..., n-1$ is non-zero. Note that the most upper and the left numbers are indices of the rows and columns of A_B respectively.

$$\begin{array}{c|c} & \begin{array}{cccccccccccc} 2 & 3 & 4 & 1 & 5 & 6 & \cdots & 3i-2 & 3i-1 & 3i & \cdots & 3n-2 \end{array} \\ \begin{array}{c} 2 \\ 3 \\ 4 \end{array} & \begin{pmatrix} 1 & 0 & 0 & 1 & 0 & -1 & & p_i & q_i & s_i & & p_n \\ 0 & 1 & 0 & 1 & 1 & 0 & \cdots & p_i & r_i & t_i & \cdots & p_n \\ 0 & 0 & 1 & 0 & 1 & 1 & & 1 & 1 & 1 & & 1 \end{pmatrix} \end{array}$$

Hereafter we show that the columns 8 and 9 of A_B have to be the same. This contradicts the fact that F_{3n-2} is simple.

Since $\{1, 3i-1, 3i\}$ is the hyperplane of F_{3n-2}, the submatrix whose column vectors are the hyperplanes has rank 2, which implies the following equation for every $i = 3, 4, ..., n-1$:

$$q_i - r_i - s_i + t_i = 0 \tag{3.6}$$

Since $\{3i-3, 3i-2, 3i-1\}$ and $\{3i-4, 3i-2, 3i\}$ are the hyperplanes of F_{3n-2}, the following two equations are also obtained for $j = 4, 5, ..., n$:

$$p_j(q_j - r_j - s_{j-1} + t_{j-1}) - q_j t_{j-1} + r_j s_{j-1} = 0,$$
$$p_j(-s_j + t_j + q_{j-1} + r_{j-1}) - q_{j-1} t_j + r_{j-1} s_j = 0.$$

We add the left-hand and right-hand sides of these two equations, and apply the equations (3.6) for $i = j(< n-1), j+1$ to the sum. We obtain the following equation for $j = 4, 5, ..., n-1$:

$$q_j t_{j-1} + q_{j-1} t_j - r_j s_{j-1} - r_{j-1} s_j = 0 \tag{3.7}$$

Finally we prove that the following claim by induction in $i = 3, 4, ..., n-1$ holds:

$$q_i = t_i \text{ if } i \text{ is odd, and } r_i = s_i \text{ otherwise} \tag{3.8}$$

We consider the base case $i = n-1$. Assume that n is even, therefore $n-1$ is odd. Since $\{2, 3n-4, 3n-2\}$ and $\{3, 3n-3, 3n-2\}$ are hyperplanes of F_{3n-2}, the submatrices whose column vectors correspond to the two hyperplanes have rank 2. Therefore we obtain $r_{n-1} = p_n$ and $s_{n-1} = p_n$. It follows $r_{n-1} = s_{n-1}$, which is the base case $i = n-1$. Then assume that n is odd, therefore $n-1$ is even. Since $\{2, 3n-3, 3n-2\}$ and $\{3, 3n-4, 3n-2\}$ are hyperplanes of F_{3n-2}, the submatrices whose column vectors correspond to the two hyperplanes have rank 2. Therefore we obtain $q_{n-1} = p_n$ and $t_{n-1} = p_n$. As a result, we have completed the base case.

Then we consider the inductive step. Assuming the following equation:

$$q_k = t_k \text{ for some odd } k = 3, ..., n-2 \tag{3.9}$$

Then we show $r_{k-1} = s_{k-1}$. The assumption $q_k = t_k$ and the equation (3.6) for $i = k$ implies the following equation:

$$s_k = 2q_k - r_k \tag{3.10}$$

We substitute the following three equation into t_{k-1}, t_k and s_k in the equation (3.7) for $i = k$ respectively:

- $t_{k-1} = r_{k-1} + s_{k-1} - q_{k-1}$ from the equation (3.6) for $i = k-1$
- $t_k = q_k$ from the equation (3.9)
- $s_k = 2q_k - r_k$ from the equation (3.10)

Then we obtain the equation $(r_{k-1} - s_{k-1})(q_k - r_k) = 0$. If $q_k = r_k$, then the equation $q_k = r_k = s_k = t_k$ is obtained from the equation (3.8) for $i = k$ and the equation (3.9). This means that the element $3k-1$ and $3k$ are parallel, a contradiction. Hence we obtain $r_{k-1} = s_{k-1}$. Assuming $r_k = s_k$ for some even $k = 4, ..., n-1$, we can show $q_{k-1} = t_{k-1}$. We omit it here. This is the end of the proof of the equation (3.8).

Since $\{6, 7, 8\}$ are hyperplanes, the submatrix whose column vectors correspond to the hyperplane $\{6, 7, 8\}$ of YM^1_{3n} have rank 2, which implies the following equation:

$$p_3(q_3 - r_3) + p_3 - r_3 = 0 \tag{3.11}$$

Since $\{5,7,9\}$ are hyperplanes, the submatrix whose column vectors correspond to the hyperplane $\{6,7,8\}$ of YM^1_{3n} have rank 2, which implies the following equation:

$$p_3(t_3 - s_3) + s_3 - p_3 = 0 \tag{3.12}$$

We add the left-hand and right-hand sides of the equations (3.11) and (3.12), and then obtain the following equation:

$$p_3(q_3 - r_3 - s_3 + t_3) - r_3 + s_3 = 0 \tag{3.13}$$

From the equation (3.6) for $i = 3$, we obtain $q_3 - r_3 - s_3 + t_3$, with which we obtain the following equation from the equation (3.13):

$$s_3 = r_3 \tag{3.14}$$

Furthermore, from the equation (3.8) for $i = 3$, we obtain $q_3 = t_3$, which, together with the equation (3.6) for $i = 3$ and the equation (3.14), implies the following:

$$r_3 = t_3 \tag{3.15}$$

After all, the equation (3.8) for $i = 3$, the equation (3.14) and (3.15) implies $q_3 = r_3 = s_3 = t_3$, which means that the element 8 and 9 are parallel, a contradiction. This completes the proof.

□

Finally, we investigate the representability of YM^1_9 and YM^2_9.

Proposition 3.18 ([32])**.** *Let $\mathbb{F}$ be a field and* 1 *be its unity.*
(1) YM^1_9 *are representable over $\mathbb{F}$ if and only if $\mathbb{F}$ has characteristic other than* 2*, and contains a root of equation $x^2 = -1$.*
(2) YM^2_9 *are representable over $\mathbb{F}$ if and only if $\mathbb{F}$ has characteristic* 2*, and is not $GF(2)$.*
(3) YM^1_9 *is an excluded minor over $GF(2^m)$ for $m \geq 3$.*

Proof. (1) First, suppose the former, and prove the latter. Considering the fundamental circuit concerning a base $\{1,2,3\}$, the representation matrix of YM^1_9 over F is as below.

$$\begin{array}{c}
\begin{array}{ccccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \end{array} \\
\begin{pmatrix}
1 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 \\
0 & 1 & 0 & 1 & 0 & y_6 & y_7 & y_8 & y_9 \\
0 & 0 & 1 & z_4 & z_5 & 0 & z_7 & z_8 & z_9
\end{pmatrix}
\end{array}$$

The number above each column signifies the element of YM^1_9 corresponding to each column vector. z_4 and z_5 can be fixed as 1 by Theorem 3.15.

Since $\{1,8,9\}$ is a circuit and hyperplane, rank of the matrix consisting of columns 1, 8 and 9 is 2; $y_9 = ky_8$ and $z_9 = kz_8$ where k is non-zero element of F. Similarly, since $\{1,4,7\}, \{2,5,8\}$ and $\{3,6,9\}$ are circuits, the constraints between each variables are determined: $z_7 = y_7 z_4$ from a circuit $\{1,4,7\}$, $z_5 = y_8$ from

$\{2,5,8\}$ and $y_6 = y_9$ from $\{3,6,9\}$. With the above observation, the representation matrix become below.

$$\begin{array}{c} \begin{array}{ccccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \end{array} \\ \begin{pmatrix} 1 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 0 & ky_8 & y_7 & y_8 & ky_8 \\ 0 & 0 & 1 & 1 & 1 & 0 & y_7 & 1 & k \end{pmatrix} \end{array}$$

Then, next three equations: $-ky_8 = 1$ from a circuit $\{4,5,6\}$, $y_7 - ky_8 = (1-k)y_7y_8$ from a circuit $\{6,7,8\}$ and $(k-1)y_7 = (y_7-1)ky_8$ from a circuit $\{5,7,9\}$. From the first and second equations, $y_7 = -k$. From the third equation together with $y_7 = -k$, $k^2+1 = 0$. This shows that $\mathbb{F}$ must have a root of $x^2+1 = 0$. Besides, if $\mathbb{F}$ has characteristic 2, k must be 1, which 8 and 9 is parallel; a contradiction.

Next we suppose the latter, and prove the latter. Let k be a root of $x^2+1 = 0$. Then the following matrix represents YM_9^1; it is a routine to check that all submatrix corresponding to hyperplanes is rank 2 and to bases rank 3.

$$\begin{array}{c} \begin{array}{ccccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \end{array} \\ \begin{pmatrix} 1 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 0 & -1/k & -k & -1/k & -1 \\ 0 & 0 & 1 & 1 & 1 & 0 & -k & 1 & k \end{pmatrix} \end{array}$$

(2) The proof is similar with (1). Let $t \neq 1$. The representation of YM_9^2 is following.

$$\begin{array}{c} \begin{array}{ccccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \end{array} \\ \begin{pmatrix} 1 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 0 & 1 & t & 1/t & 1 \\ 0 & 0 & 1 & 1 & 1 & 0 & t & 1 & 1/t \end{pmatrix} \end{array}$$

(3) Proposition 3.13 implies that YM_9^1 is not representable over fields of characteristic 2. Hence all we need to prove is that all proper minor of YM_9^1 is representable over $GF(2^m)$ for $m \geq 3$. Note that any contraction $YM_9^1 \,/\, p$ for an element p has rank 2. A simple matroid of rank 2 is representable over $GF(2^n)$ if and only if it does not have $U_{2^m,2^m+2}$ as a minor [55]. Hence, for any $p \in E_3^1$ $YM_9^1 \,/\, p$ is representable over $m \geq 3$ since $9 \leq 2^m+2$ for $m \geq 3$.

Then we consider the deletion: $YM_9^1 \setminus p$. By symmetry, there are three cases to be considered: $p = 1, 4$ or 7. In each case, we give the matrix representation for $YM_9^1 \setminus p$. Hereafter, $f(x)$ is a irreducible polynomial which defines $GF(2^m)$ and $x^{-1}, (x+1)^{-1}$ are the inverse elements of $x, x+1$ over $GF(2^m) = GF(2)[x]/\langle f(x)\rangle$. In each case of $p = 1, 4, 7$, the following matrices represent YM_9^1/p respectively;

$$\begin{array}{cc} & \begin{array}{cccccccc} 8 & 7 & 5 & 6 & 2 & 9 & 4 & 3 \end{array} \\ \begin{array}{c} 8 \\ 7 \\ 5 \end{array} & \begin{pmatrix} 1 & 0 & 0 & 1 & 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 & 0 & 1 & 1 & x^{-1} \\ 0 & 0 & 1 & 0 & 1 & x & x^2+1 & x+1 \end{pmatrix} \end{array} \quad \text{for } p = 1,$$

$$
\begin{array}{c|cccccccc}
 & 8 & 9 & 7 & 1 & 6 & 5 & 2 & 3 \\
\hline
8 & 1 & 0 & 0 & 1 & 1 & 0 & 1 & 1 \\
9 & 0 & 1 & 0 & 1 & 0 & 1 & x^2 & (x+1)^{-1} \\
7 & 0 & 0 & 1 & 0 & 1 & 1+x^{-1} & x^2+1 & 1
\end{array} \quad \text{for } p = 4,
$$

$$
\begin{array}{c|cccccccc}
 & 1 & 5 & 6 & 2 & 3 & 4 & 8 & 9 \\
\hline
1 & 1 & 0 & 0 & 1 & 1 & 0 & 1 & 1 \\
5 & 0 & 1 & 0 & 0 & 1 & 1 & x & 1 \\
6 & 0 & 0 & 1 & 1 & 0 & 1 & 1 & x^{-1}
\end{array} \quad \text{for } p = 7.
$$

Above all, it is proved every proper minor of YM_9^1 is representable over $GF(2^m)$. $\square$

Chapter 4

Excluded Minors of Rank 3 for the Union of the Class of Orientable and the Classes of Representable Matroids

4.1 Introduction

In this section, we investigate the excluded minors' characterization for the union of the class of orientable matroids and the classes of representable matroids. The results which are presented in this chapter are publised in [30, 33].

The union of minor-closed classes is also minor-closed, so excluded minors can be considered for the union of minor-closed classes. In the case of graphs, it is straight-forward that the union of two minor-closed classes of graphs has just a finite number of excluded minors, by graph minor theorem [58] (Theorem 1.1) stating that any minor-closed class of graphs has just a finite number of excluded minors. In the case of matroids, the situation is totally different. Actually, even for the two classes which have one excluded minor respectively, the union of these two classes can have infinitely many excluded minors [79]. It is listed as a research problem to describe excluded minors for the union of minor-closed classes in the N. White's book [83] published in 1986. As of now, there exist a few results on the union of minor-closed classes. In what follows, we review the situations on the excluded minors for the union of classes of matroids.

First, we overview the matters on the union of the classes of matroids representable over finite fields. In [48], they start the research on the uinon of the classes of matroids representable over finite fields. As the inception of the research of these line, they investigate the excluded minors for the union of the class of $GF(2)$-representable matroids and the class of $GF(3)$-representable matroids. Remind that both the class of $GF(2)$-representable matroids and and the class of $GF(3)$-representable matroids have just a finite number of excluded minors; one excluded minor for the class of $GF(2)$-representable matroids, and three excluded minors for the class of $GF(3)$-representable matroids. They showed that there are just seven excluded minors for the union of the two classes. In the same paper [48], the following conjecture is presented as generalization of the above result.

Conjecture 4.1 ([48])**.** *Let $\mathcal{F}$ be a finite family of finite fields. The union of the class of $\mathbb{F}$-representable matroids for $\mathbb{F} \in \mathcal{F}$ has just finitely many excluded minors.*

In the case that the family $\mathcal{F}$ in the above conjecture consists of one field, it coincides with Rota's conjecture itself. Therefore, this conjecture is also generalization of Rota's conjecture 1.2.

Once we focus on other minor-closed classes of matroids, the situation on the number of excluded minors for their union can become chaotic. In the past, the following conjecture called *intertwining conjecture* was proposed by T. H. Brylawski in the N. White's book [83].

Conjecture 4.2 (In [83] by T. H. Brylawski)**.** *For two minor-closed classes of matroids each of which has just a finite number of excluded minors, the union of them always have just a finite number of excluded minors.*

However, this conjecture was refuted by [79]. As mentioned before, it is shown in [79] that there exist two classes of matroids such that, though each class has just one excluded minor, the union of the two classes has infinitely many excluded minors. On the contrary, there is a possibility that there exist two classes of matroids such that, though at least one of the two classes has infinitely many excluded minors, the union of them has just a finite number of excluded minors. As of now, no example of such classes is known.

In this section, we investigate the number of the excluded minors for the union of the following two classes: the class of orientable matroids and the class of representable matroids. Now, review that the number of excluded minors for these classes.

- Orientable matroids have infinitely many excluded minors.
- For any finite field $\mathbb{F}$, $\mathbb{F}$-representable matroids have just a finite number of excluded minors, if Rota's conjecture 1.2 is true.
- For any infinite field $\mathbb{F}$, there are infinitely many excluded minors for $\mathbb{F}$-representable matroids.

Now we overview the situation on the union of the class of orientable matroids and the class of representable matroids. For convenience, we introduce one additional terminology. Let $\mathcal{F}$ be a subclass of the class of fields. A matroid is *representable over* $\mathcal{F}$ or $\mathcal{F}$*-representable* if it can be representable over at least one of the fields in $\mathcal{F}$. Note that the class of $\mathcal{F}$-representable matroids is the union of the classes of $\mathbb{F}$-representable matroids for $\mathbb{F} \in \mathcal{F}$. Bland and Las Vergnas [6] constructed an infinite family of minimal non-orientable matroids, i.e. excluded minors for orientable matroids as mentioned in Section 3.2. Their infinite family consists of one matroid M_r of rank r with $2r$ elements for each integer $r \leq 4$. In addition, they also showed that, if r is a prime number, M_r is not representable over any field. If a minimal non-orientable matroid is not representable over any field, the matroid is also an excluded minor for the union of the class of orientable matroids and the class of $\mathcal{F}$-representable matroids where $\mathcal{F}$ is any subclass of the class of fields. This can be easily seen as follows:

(i) the matroid is neither in the class of orientable matroids nor in the class of $\mathcal{F}$-representable matroids, and

(ii) any proper minor of the matroid is in the class of orientable matroids, and therefore in the union of the class of orientable matroids and the class of $\mathcal{F}$-representable matroids.

Therefore, for a prime r and any subclass $\mathcal{F}$ of the class of fields, M_r is an excluded minor for the union of the class of orientable matroids and the class of $\mathcal{F}$-representable matroids. In summary, the following Theorem 4.3 is obtained from [6]; note that Theorem 4.3 is not explicitly stated in [6].

Theorem 4.3 ([6]). *Let $\mathcal{F}$ be a subclass of the class of fields. There exist infinitely many excluded minors for the union of the class of orientable matroids and the class of $\mathcal{F}$-representable matroids.*

On the other hand, the excluded minors for the union in [6] are the matroids of rank p with $2p$ elements for a prime number p. Therefore, for an positive integer r, the number of the known excluded minors of at most rank r for the union is finite at present. The deletion and contraction of matroids does not increase the rank of the matroid. From this fact, for a minor closed class $\mathcal{C}$ of matroids and an fixed positive integer r, the class of matroids which consists of all matroids of rank at most r in $\mathcal{C}$ is also minor-closed. This means that there remains a possibility that the union of the class of orientable matroids and the class of $\mathcal{F}$-representable matroid can be characterized by a finite number of excluded minors by restricting the rank to at most a fixed positive integer. Unfortunately, we will eradicate the possibility in this chapter. The goal of this section is to prove the following theorem.

Theorem 4.4 ([30, 33]). *Let $\mathcal{F}$ be a subclass of the class of fields. There exist infinitely many excluded minors of rank 3 for the union of the class of orientable matroids and the class of $\mathcal{F}$-representable matroids.*

To prove this theorem, we construct the infinite family of those excluded minors of rank 3 for the class of orientable matroids, which are not representable over any field. These excluded minors for the class of orientable matroids satisfy the above conditions (i) and (ii). Therefore they are the excluded minors for the union of orientable matroids and the class of $\mathcal{F}$-representable matroids where $\mathcal{F}$ is a subclass of the class of fields. In the result, the proof of the Theorem 4.4 will be completed by these construction.

Finally, we mention that, from the non-representability result of YM^1_{3n} and F_{3n-2} in Proposition 3.16 and Proposition 3.17, we can immediately know that the matroids YM^1_{3n} and F_{3n-2} are new examples of excluded minors of rank 3 for the union of the class of orientable matroids and the class of some $\mathcal{F}$-representable matroids as follows.

Corollary 4.5. *Let $\mathcal{F}$ be a subclass of the class of fields of characteristic 2. The matroid YM^1_{3n} is an excluded minors of rank 3 for the union of the class of orientable matroids and the class of $\mathcal{F}$-representable matroids.*

Corollary 4.6. *Let $\mathcal{F}$ be a subclass of the class of fields of characteristic other than 2. The matroid F_{3n-2} is an excluded minors of rank 3 for the union of the class of orientable matroids and the class of $\mathcal{F}$-representable matroids.*

4.2 Definition of the Family F^+_{3n-2} of Matroids

In this section, we define new family F^+_{3n-2} of rank 3 matroids. We obtain Theorem 4.4 by constructing an infinite family of excluded minors of rank 3 for orientable matroids which are not representable over any field. For a given field $\mathbb{F}$, an excluded minor for orientable matroids which is not representable over any field is neither orientable nor representable over $\mathbb{F}$, therefore is not in the union of the class of orientable matroids and the class of $\mathbb{F}$-representable matroids. Furthermore, any minor of such an excluded minor is orientable, therefore This infinite family consists of the following matroids F^+_{3n-2} for every $n \geq 4$. These matroids are similar with the matroids F_{3n-2} which were introduced in [32]. The difference between F^+_{3n-2} and F_{3n-2} is whether a hyperplane $\{2, 5, 8\}$ exists or not. Figure 4.1 and 4.2 show geometric representations of F^+_{3n-2} and F_{3n-2}, respectively.

Definition 4.7. *For every $n \geq 4$, F^+_{3n-2} is a simple matroid $(E_n, \mathcal{H}_n)$ where $E_n := \{1, \cdots, 3n-2\}$ and $\mathcal{H}_n$ is the collection of the following subsets of E_n:*

- $\{1, 4, ..., 3n-2\}$ *and* $\{2, 5, 8\}$
- $\{3k-1, 3k+1, 3k+3\}$ *and* $\{3k, 3k+1, 3k+2\}$ *for* $k = 1, 2, ..., n-2$
- $\{1, 3k-1, 3k\}$ *for* $k = 1, 2, ..., n-1$
- $\{a, 3n-4, 3n-2\}$ *and* $\{b, 3n-3, 3n-2\}$ *where* $a = 2$, $b = 3$ *if* n *is odd, otherwise* $a = 3$, $b = 2$
- *all two-element subsets not contained in any above subset.*

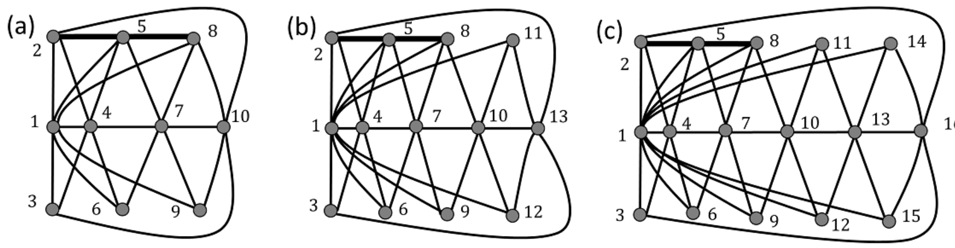

Figure 4.1: Geometric representations of (a) F^+_{10} (b) F^+_{13} and (c) F^+_{16}

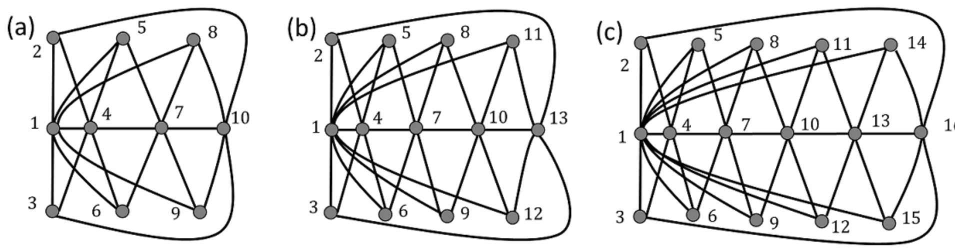

Figure 4.2: Geometric representations of (a) F_{10} (b) F_{13} and (c) F_{16}

For the proof of Theorem 4.4, we show the following theorem.

Theorem 4.8 ([30, 33])**.** *For every $n \geq 4$, F^+_{3n-2} is an excluded minor for orientable matroids and not representable over any field.*

The proof of Theorem 4.8 is divided into three parts. Firstly we prove in Proposition 4.10 that F^+_{3n-2} is not orientable for every $n \geq 4$. Secondly we show in Proposition 4.11 that any proper minor of F^+_{3n-2} is orientable. Lastly we prove in Proposition 4.12 that F^+_{3n-2} is not representable over any field for every $n \geq 4$. Combining the three propositions, we obtain Theorem 4.8.

In proving Proposition 4.11 and Proposition 4.12 we follow the same line of ideas as Proposition 3.14 and Proposition 3.17 respectively. For the proof of Proposition 3.14, we use Theorem 3.6. For the proof of Proposition 3.17, we use Theorem 3.15.

4.3 Non-orientability of F^+_{3n-2}

We prove in Proposition 4.10 that F^+_{3n-2} is non-orientable for every $n \geq 4$. Let $M = (E, \mathcal{B})$ be a matroid where E is the ground set and $\mathcal{B}$ is a set of bases. A subset $S \subset E$ is called a *hyperplane-circuit* if S is simultaneously a hyperplane of M and a circuit of M. For a hyperplane-circuit S of M, the set $\mathcal{B}' := \mathcal{B} \cup \{S\}$ satisfies the base axiom system [55]. We denote the matroid $(E, \mathcal{B}')$ by $M \circ S$. We present an important theorem regarding to the orientability of $M \circ S$.

Theorem 4.9 (Proposition 7.9.1(9)[5]). *Let M be a matroid, and S a hyperplane-circuit of M. If M is orientable, then $M \circ S$ is also orientable. Equivalently, if $M \circ S$ is not orientable, then M is not orientable.*

Theorem 4.9 implies Theorem 4.10 for non-orientability of F^+_{3n-2}.

Proposition 4.10 ([30, 33]). *For every $n \geq 4$, F^+_{3n-2} is not orientable.*

Proof. The subset $\{2, 5, 8\}$ of E_n is a hyperplane-circuit of F^+_{3n-2}. The matroid $F^+_{3n-2} \circ \{2, 5, 8\}$ is isomorphic to F_{3n-2}, which is not orientable for $n \geq 4$ [32]. Therefore Theorem 4.9 implies that F^+_{3n-2} is not orientable. □

4.4 Minimality of F^+_{3n-2}

We show in Proposition 4.11 that any proper minor of F^+_{3n-2} is orientable. For this proof, Theorem 3.6 is used. The outline of by removing reducible elements one by one in Proposition 4.11.

Proposition 4.11 ([30, 33]). *For $n \geq 4$, any proper minor of the matroids F^+_{3n-2} is orientable.*

Proof. Any contraction of a simple matroid decreases the rank of the matroid. Since F^+_{3n-2} is a simple matroid of rank 3, $F^+_{3n-2}/\{e\}$ for any element $e \in E_n$ has rank two. As any matroid of rank 2 is orientable, $F^+_{3n-2}/\{e\}$ for any element $e \in E_n$ is orientable.

Then we prove that, for any element $s_1 \in E_n$, $F^+_{3n-2} \setminus \{s_1\}$ is orientable. We set the other elements $s_2, ..., s_{3n-2}$ depending on s_1.

(i) $s_1 \in \{1, 4, ..., 3n-2\}$

$$s_i = \begin{cases} s_1 + \frac{3i}{2} - \frac{3}{2}, & \text{if } i \text{ is odd and } i \le \frac{6n-2s_1-1}{3} \\ s_1 + \frac{3i}{2} - 1, & \text{if } i \text{ is even and } i \le \frac{6n-2s_1-4}{3} \\ 3n - \frac{3i}{2} - \frac{1}{2}, & \text{if } i \text{ is odd and } \frac{6n-2s_1-1}{3} < i \le 2n-1 \\ 3n - \frac{3i}{2}, & \text{if } i \text{ is even and } \frac{6n-2s_1-4}{3} < i \le 2n-2 \\ 3i - 6n + 2, & \text{if } 2n-1 < i \end{cases}$$

(ii) $s_1 \in \{2, 5, ..., 3n-4\}$

$$s_i = \begin{cases} s_1 + \frac{3i}{2} - \frac{1}{2}, & \text{if } i \text{ is odd and } i \le \frac{6n-2s_1-5}{3} \\ s_1 + \frac{3i}{2} - 1, & \text{if } i \text{ is even and } i \le \frac{6n-2s_1-2}{3} \\ 3n - \frac{3i}{2} + \frac{3}{2}, & \text{if } i \text{ is odd and } \frac{6n-2s_1-5}{3} < i \le 2n-1 \\ 3n - \frac{3i}{2} + 1, & \text{if } i \text{ is even and } \frac{6n-2s_1-2}{3} < i \le 2n \\ 3i - 6n - 1, & \text{if } 2n < i < \frac{s_1+6n+1}{3} \\ 3i - 6n + 2, & \text{if } \frac{s_1+6n+1}{3} \le i \end{cases}$$

(iii) $s_1 \in \{3, 6, ..., 3n-3\}$

$$s_i = \begin{cases} s_1 + \frac{3i}{2} - \frac{3}{2}, & \text{if } i \text{ is odd and } i \le \frac{6n-2s_1-3}{3} \\ s_1 + \frac{3i}{2} - 2, & \text{if } i \text{ is even and } i \le \frac{6n-2s_1}{3} \\ 3n - \frac{3i}{2} - \frac{1}{2}, & \text{if } i \text{ is odd and } \frac{6n-2s_1-3}{3} < i \le 2n-1 \\ 3n - \frac{3i}{2}, & \text{if } i \text{ is even and } \frac{6n-2s_1}{3} < i \le 2n-2 \\ 3i - 6n + 2, & \text{if } 2n-1 < i \end{cases}$$

In the three cases (i), (ii) and (iii), the element s_{i+1} is reducible in the matroid $F^+_{3n-2} \setminus \{s_1, s_2, ..., s_i\}$ for every $i \in \{1, 2, ..., 3n-3\}$. We delete the elements $\{s_1, s_2, ..., s_{3n-8}\}$ one by one. The deletion orders for (i), (ii) and (iii) are along with arrows in Figure 4.3(b), (c) and (d), respectively. Theorem 3.6 implies that $F^+_{3n-2} \setminus \{s_1\}$ is orientable if and only if $F^+_{3n-2} \setminus \{s_1, s_2, ..., s_{3n-8}\}$. Note that any matroid of rank 3 with at most 6 elements is orientable. As the matroid $F^+_{3n-2} \setminus \{s_1, s_2, ..., s_{3n-8}\}$ has just six elements, it is orientable. Therefore, for any element $e \in E_n$, $F^+_{3n-2} \setminus \{s_1\}$ is orientable for any $s_1 \in E_n$. This completes the proof. □

4.5 Non-representability of F^+_{3n-2}

We show in Proposition 4.12 that F^+_{3n-2} is not representable over any field. We use Theorem 3.15 to prove Proposition 4.12 on non-representability of F^+_{3n-2}. The proof of Proposition 4.12 consists of two parts. The first part is for non-representability over a field of characteristic other than 2, and the other part is for non-representability over a field of characteristic 2. The proof on non-representability of F^+_{3n-2} over a field of characteristic other than 2 is analogous to that for F_{3n-2} in Proposition 3.17. Hence, we mainly give the proof of non-representability of F^+_{3n-2} over a field of characteristic 2, which is not treated in Proposition 3.17.

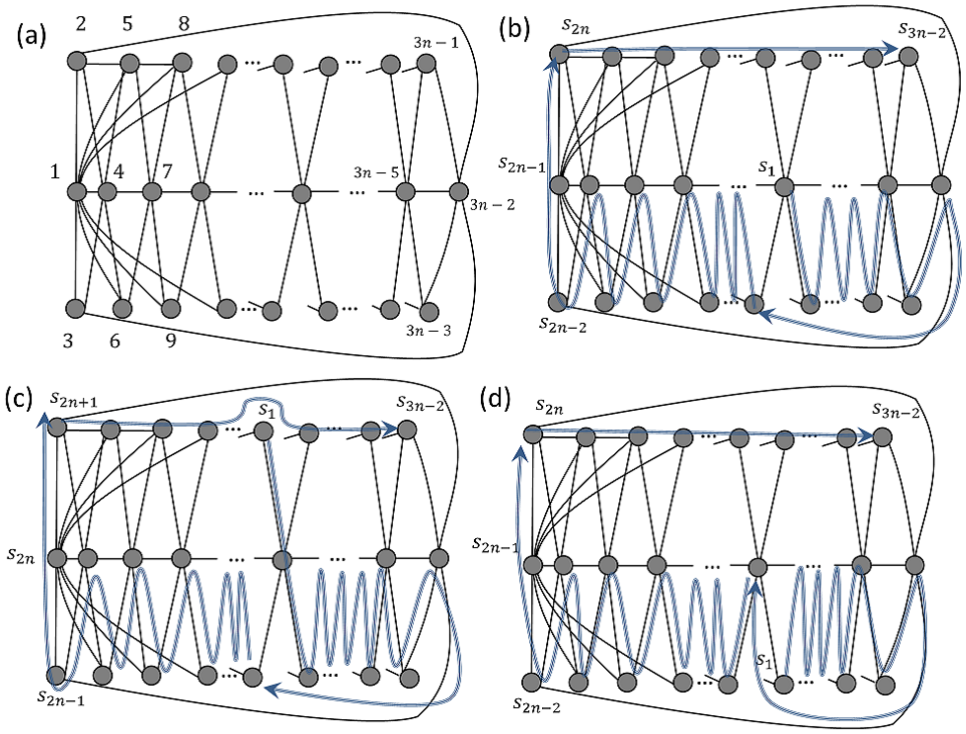

Figure 4.3: (a) A geometric representation of F^+_{3n-2}, and (b), (c) and (d) deletion orders of reducible elements $s_2, s_3, ..., s_{3n-8}$ for (i), (ii) and (iii), respectively

Proposition 4.12 ([30, 33])**.** *For every $n \geq 4$, F^+_{3n-2} is not representable over any field.*

Proof. Firstly we prove non-representability of F^+_{3n-2} over a field of characteristic other than 2. Suppose the contrary: F^+_{3n-2} for $n \geq 4$ is representable over a field $\mathbb{F}$ of characteristic other than 2. For the base $B = \{2, 3, 4\}$, we consider the bipartite graph $G(F^+_{3n-2}, B)$ and a spanning tree of $G(F^+_{3n-2}, B)$ with the following edges as in Figure 4.4(a).

- the edges connecting 2: $(1, 2)$
- the edges connecting 3: $(1, 3)$ and $(3, 5)$
- the edges connecting 4: $(4, i)$ for every $i = 5, 6, ..., 3n-2$ (see Figure 4.4(b))

Theorem 3.15 implies that there is a matrix representation A corresponding to the spanning tree:

- the entry in the row 2: $A(2, 1) = 1$
- the entries in the row 3: $A(3, 1) = A(3, 5) = 1$
- the entries in the row 4: $A(4, i) = 1$ for every $i = 5, 6, ..., 3n-2$

Since $\{1, 4, ..., 3n-2\}$ and $\{1, 5, 6\}$ are hyperplanes of F^+_{3n-2}, the submatrices whose column vectors correspond to the hyperplanes have rank 2. It follows that

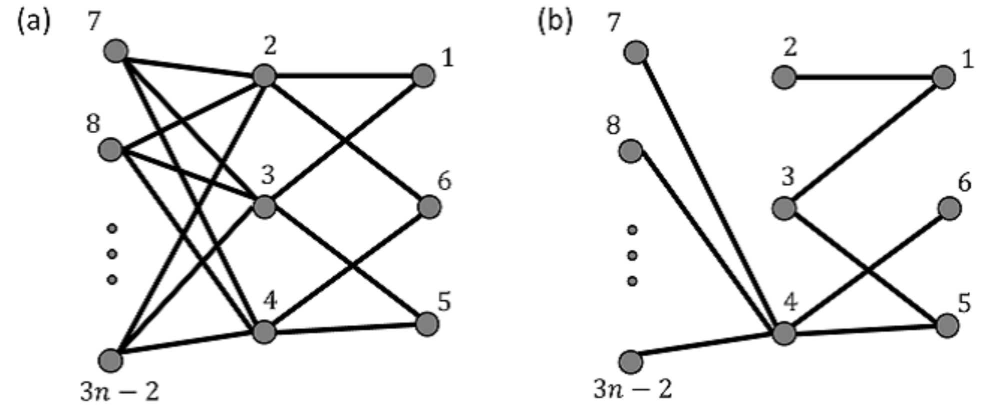

Figure 4.4: (a) $G(F^+_{3n-2}, \{2,3,4\})$ (b) A spanning tree of $G(F^+_{3n-2}, \{2,3,4\})$

$A(2,6) = -1$ and $A(2, 3i-2) = A(3, 3i-2)$ for every $i = 3, 4, ..., n$. Therefore the matrix representation A is the following matrix (4.1) where $p_i \neq 0$ for every $i = 3, ..., n$ and $q_j, r_j, s_j, t_j \neq 0$ for every $j = 3, ..., n-1$. Note that the most upper and the left numbers are indices of the rows and columns of A respectively.

$$\begin{array}{c} \begin{array}{cccccccccccc} 2 & 3 & 4 & 1 & 5 & 6 & \cdots & 3i-2 & 3i-1 & 3i & \cdots & 3n-2 \end{array} \\ \begin{pmatrix} 1 & 0 & 0 & 1 & 0 & -1 & & p_i & q_i & s_i & & p_n \\ 0 & 1 & 0 & 1 & 1 & 0 & \cdots & p_i & r_i & t_i & \cdots & p_n \\ 0 & 0 & 1 & 0 & 1 & 1 & & 1 & 1 & 1 & & 1 \end{pmatrix} \end{array} \tag{4.1}$$

If the matrix (4.1) represents F^+_{3n-2} over a field of characteristic other than 2, the following equation (4.2) has to hold.

$$q_i = t_i \text{ if } i \text{ is odd, and } r_i = s_i \text{ otherwise} \tag{4.2}$$

Assuming that the matroid F_{3n-2} is representable over a field of characteristic other than 2, F_{3n-2} has the same matrix representation (4.1), which follows that the same equation (4.2) would hold. The equation (4.2) also contradicts the simplicity of F^+_{3n-2} by the same argument in Proposition 3.3 [32]. Hence we skip the proof here.

Secondly, we prove non-representability of F^+_{3n-2} over a field of characteristic 2. Suppose the contrary: F^+_{3n-2} is representable over a field $\mathbb{F}$ of characteristic 2. Note that 1 is equal to -1 over the field $\mathbb{F}$, hence the matrix (4.1) takes the following form.

$$\begin{array}{cc} & \begin{array}{cccccccccc} 2 & 3 & 4 & 1 & 5 & 6 & 7 & 8 & 9 & \cdots \end{array} \\ \begin{array}{c} 2 \\ 3 \\ 4 \end{array} & \begin{pmatrix} 1 & 0 & 0 & 1 & 0 & 1 & p_3 & q_3 & s_3 & \\ 0 & 1 & 0 & 1 & 1 & 0 & p_3 & r_3 & q_3 & \cdots \\ 0 & 0 & 1 & 0 & 1 & 1 & 1 & 1 & 1 & \end{pmatrix} \end{array}$$

The existence of the hyperplane $\{2,5,8\} \in \mathcal{H}_n$ implies that the submatrix consisting of the kth column vectors for $k \in \{2,5,8\}$ has rank 2, which in turn implies the following equation:

$$r_3 = 1 \tag{4.3}$$

The existence of the hyperplane $\{6,7,8\} \in \mathcal{H}_n$ also implies that the submatrix consisting of the kth column vectors for $k \in \{6,7,8\}$ has rank 2. Together with the equation(4.3), we obtain the following equation:

$$q_3 = \frac{1}{p_3} \tag{4.4}$$

The existence of the hyperplane $\{5,7,9\} \in \mathcal{H}_n$ also implies that the submatrix consisting of the kth column vectors for $k \in \{5,7,9\}$ has rank 2. Together with the equation (4.4), we obtain the following equation:

$$(1+p_3)(1+s_3) = 0 \tag{4.5}$$

If the value of $1+p_3$ is equal to 0, the submatrix consisting of the kth column vectors for $k \in \{2,5,7\}$ has rank 2. This implies that $\{2,5,7\}$ is a hyperplane of F^+_{3n-2}, which contradicts that $\{2,5,7\}$ is a base of F^+_{3n-2}. Therefore the value of s_3+1 has to be equal to 0. Then the submatrix consisting of the kth column vectors for $k \in \{3,6,9\}$ has rank 2. This implies that $\{3,6,9\}$ is a hyperplane of F^+_{3n-2}, which contradicts that $\{3,6,9\}$ is a base of F^+_{3n-2}. Therefore F^+_{3n-2} is not representable over a field with characteristic 2.

We conclude that F^+_{3n-2} is not representable over any field. This completes the proof. □

The three propositions, Proposition 4.10, Proposition 4.11 and Proposition 4.12 yield Theorem 4.8.

Chapter 5

Excluded Minors of Rank 3 for the Intersection of the Class of Orientable Matroids and the classes of Representable Matroids

5.1 Introduction

In this section, we investigate the excluded minors' characterization for the intersection of the class of orientable matroids and the classes of representable matroids. Note that the results which are included in this chapter are presented in [29] and submitted to [33].

The intersection of minor-closed classes is also minor-closed. Compared to the union, the number of excluded minors for the intersection is easy to handle with. For two minor-closed classes $\mathcal{C}_1$ and $\mathcal{C}_2$ of matroids, let S_1 and S_2 be a set of excluded minors for $\mathcal{C}_1$ and for $\mathcal{C}_2$ respectively. Then it can be easily seen that the set of excluded minors for the intersection of $\mathcal{C}_1$ and $\mathcal{C}_2$ is a subset of $S_1 \cup S_2$. Therefore, if both of $|S_1|$ and $|S_2|$ are finite, the intersection of $\mathcal{C}_1$ and $\mathcal{C}_2$ can be always characterized by a finite number of excluded minors. Then we consider the case that $\mathcal{C}_1$ has infinitely many excluded minors. In this case, there is a possibility that the number of excluded minors for the intersection of $\mathcal{C}_1$ and $\mathcal{C}_2$ becomes finite.

Now we review the situation on the intersection of the class of orientable matroids and the classes of representable matroids. As mentioned before, the class of orientable matroids cannot be characterized by a finite number of excluded minors [6]. There exist several results on the number of excluded minors for the intersection of the class of orientable matroids and the classes of matroids representable over finite fields. It is proven in [6] that the intersection of the class of orientable matroids and the class of $GF(2)$-representable matroids exactly coincides with the class of regular matroids. The class of regular matroids has just three excluded minors [71]. In the results, the intersection of the class of orientable matroids and the class of $GF(2)$-representable matroids can be characterized by a finite number of excluded minors. Furthermore, the intersection of the class of orientable matroids and the class of $GF(3)$-representable matroids exactly coincides with the intersection of the class of $GF(3)$-representable matroids and the class of $GF(5)$-representable matroids [85, 41]. The class of $GF(3)$-representable matroids has just four excluded minors [63, 3]. If Rota's conjecture 1.2 is true as announced in [21], then $GF(5)$-representable matroids has just a finite number of excluded minors. These results imply that the intersection of the class of orientable matroids and the class of $GF(3)$-representable matroids will be also characterized by a finite number of excluded minors.

In this chapter, we investigate the excluded minors' characterization for the intersection of the class of orientable matroids and the classes of matroids representable over fields of characteristic 0. Remind that, for an field F of characteristic 0, the class of $\mathbb{F}$-representable matroids has infinitely many excluded minors [40]. Natural question here is whether, by taking the intersection of the class of orientable matroids and the classes of matroids representable over fields of characteristic 0, the number of excluded minors becomes finite or remains infinite.

By looking closely into [45], we can know that the answer to this question is infinite, though the motivation of the paper [45] is different from one of this chapter. Actually, the following theorem can be derived from [45].

Theorem 5.1 (from [45])**.** *Let $\mathcal{F}$ be a subclass of the class of fields of characteristic 0. There exist infinitely many* **orientable** *excluded minors for the intersection of the class of orientable matroids and the classes of $\mathbb{F}$-representable matroids for $\mathbb{F} \in \mathcal{F}$.*

Here the thing to be mentioned is that the excluded minors for the intersection are also orientable as written in the bold font in Theorem 5.1. This means that, even within the class of orientable matroids, the number of the excluded minors for the intersection is infinite.

Theorem 5.1 can be obtained in the following way. The original statement which is shown in [45] is that, for an infinite field $\mathbb{F}$ and any matroid M representable over $\mathbb{F}$, there exists an excluded minor for the class of $\mathbb{F}$-representable matroids which has the following two properties:

(1) it contains the matroid M as a minor, and

(2) it is not representable over any field.

Now we consider the case that an infinite field $\mathbb{F}$ is the rational field $\mathbb{Q}$. Closely checking out the construction of excluded minors in [45], we can know that,

(3) for the rational field $\mathbb{Q}$, these excluded minors for $\mathbb{Q}$-representable matroids in [45] are orientable.

These facts (1), (2) and (3) imply Theorem 5.1 in the following way.

(i) since the excluded minors in [45] for the class of $\mathbb{Q}$-representable matroids are not representable over any field (from (2)), they are not in the intersection of the class of orientable matroids and the classes of $\mathbb{F}$-representable matroids for $\mathbb{F} \in \mathcal{F}$, and

(ii) since these excluded minors in [45] for the class of $\mathbb{Q}$-representable matroids are orientable (from (3)) and orientability is a minor-closed property, any proper minor of these excluded minors for $\mathbb{Q}$-representable matroids are orientable, and

(iii) since a field of characteristic 0 contains $\mathbb{Q}$ as a subfield and any proper minor of these excluded minors for $\mathbb{Q}$-representable matroids are $\mathbb{Q}$-representable, any proper minor of these excluded minors for $\mathbb{Q}$-representable matroids are also $\mathbb{F}$-representable for any $\mathbb{F} \in \mathcal{F}$, and

(iv) since, for a $\mathbb{Q}$-representable matroid M, the corresponding excluded minor in (1) has the more elements than M and there exist $\mathbb{Q}$-representable matroids with arbitrarily large number of elements, the number of the excluded minors in [45] for the class of $\mathbb{Q}$-representable matroids is infinite.

The thing to be mentioned again here is that these excluded minors for $\mathbb{F}$-representability are orientable. Therefore, we can also interpret Theorem 5.1 in the following way; even within the class of orientable matroids, the characterization of $\mathbb{F} - \textit{representability}$ by a finite list of excluded minors is impossible (See Figure 5.1). In particular, consider the case that $\mathbb{F}$ is an ordered field. As mentioned in Section 3, the orientability can be regarded as as a natural extension of representability over ordered field, and the class of orientable matroids contains the class of $\mathbb{F}$-representable matroids. Hence the class of orientable matroids seems to be the closely-related to the classes of matroids representable over ordered fields. Nevertheless, there exists a significant gap between orientability and representability over ordered fields from the perspective of the excluded minors' characterization.

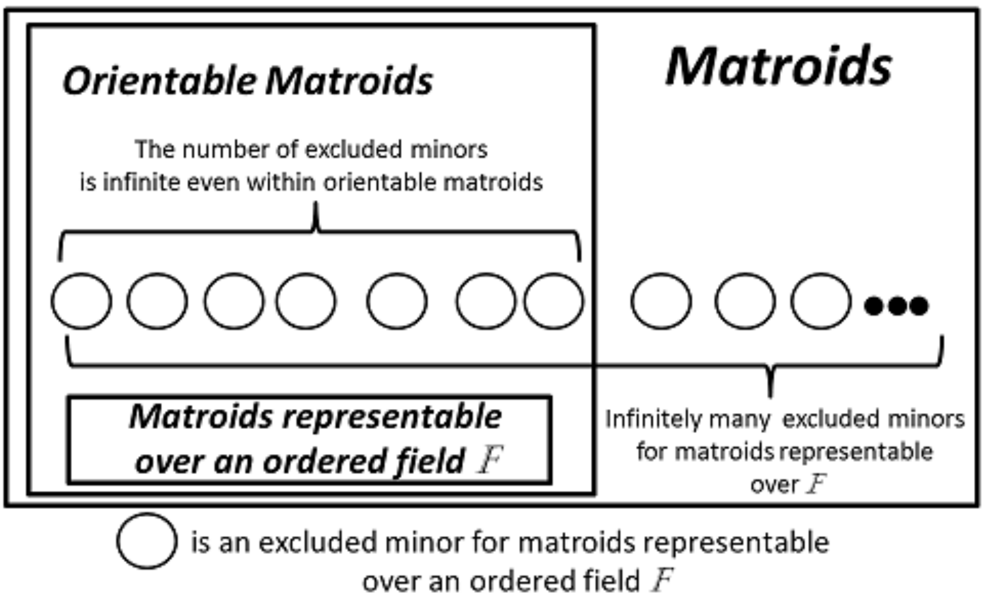

Figure 5.1: On the number of excluded minors for the classes of matroids representable over ordered fields within the class of orientable matroids

In this chapter, we investigate further the difficulty of excluded minors' characterization for the intersection of the class of orientable matroids and the classes of matroids representable over fields of characteristic 0. In [45], the excluded minors for the class of matroids representable over infinite fields consists of a matroid of rank r with $3r-1$ elements for each r. Therefore, for a fixed positive integer r, there exists just a finite number of the known excluded minors for the intersection of rank at most r. Hence, there remains the possibility that, by restricting the rank of the intersection, the characterization by a finite list of excluded minors becomes possible. Unfortunate, we will show that the possibility is refuted. The goal of this section is to prove the following theorem.

Theorem 5.2 ([29, 33]). *Let $\mathcal{F}$ be a subclass of the class of fields of characteristic* 0*. There exist infinitely many* **orientable** *excluded minors of rank* 3 *for the intersection of the class of orientable matroids and the classes of* $\mathbb{F}$*-representable matroids for* $\mathbb{F} \in \mathcal{F}$*.*

We obtain this theorem by constructing an infinite family of orientable excluded minors of rank 3 for matroids representable over a field $\mathbb{F}$ with characteristic 0.

The organization of this chapter is as follows. In Section 5.2, we review the classical theorem called *Pappus' hexagon theorem* and the *non-Pappus matroid* which is related to Papppus' hexagon theorem. The theorem and matroid will be utilized to prove Theorem 5.2 in the later sections. In Section 5.3, we define a new infinite family of matroids which contains the non-Pappus' matroid. The matroids in the family is shown to be orientable in Section 5.4, not representable over any field in Section 5.5. In Section 5.6, any proper minor of the matroids are representable over the rational field $\mathbb{Q}$. In the results, we obtain Theorem 5.2, in the same reasoning as (i), (ii) and (iii).

5.2 Pappus' Hexagon Theorem and Non-Pappus Matroid

In this section, we review Pappus' hexagon theorem and non-Pappus matroid. They play a crucial role in our construction of excluded minors. For a field $\mathbb{F}$, let $PG(3, \mathbb{F})$ be a 2-dimensional projective space over $\mathbb{F}$. We take two sets of three collinear points in $PG(3, \mathbb{F})$, $\{a, b, c\}$ and $\{d, e, f\}$, such that any distinct four points of $\{a, b, c, d, e, f\}$ are non-collinear, as in Figure 5.2 (a). Let g, h and i be intersections of two lines ae and bd, of two lines bf and ce, and of two lines af and cd respectively. Then the Pappus' hexagon theorem states that the three points g, h and i are collinear.

The point configuration of the nine points in Figure 5.2 (a) induces a matroid of rank 3 with 9 elements, called the *Pappus matroid.* Figure 5.2 (b) is a geometric representation of the Pappus matroid. If we change the hyperplane $\{3, 6, 9\}$ of the Pappus matroid into a base, we obtain another matroid, called the *non-Pappus matroid.* Figure 5.2 (c) is a geometric representation of the non-Pappus matroid. While the Pappus matroid is representable over fields whose cardinalities are 4 or more than 6, the non-Pappus matroid is not representable over any field. Actually the Non-Pappus's matroid is also known to be an excluded minor for $\mathbb{F}$-representability for a fixed field $\mathbb{F}$ with at least five cardinalities.

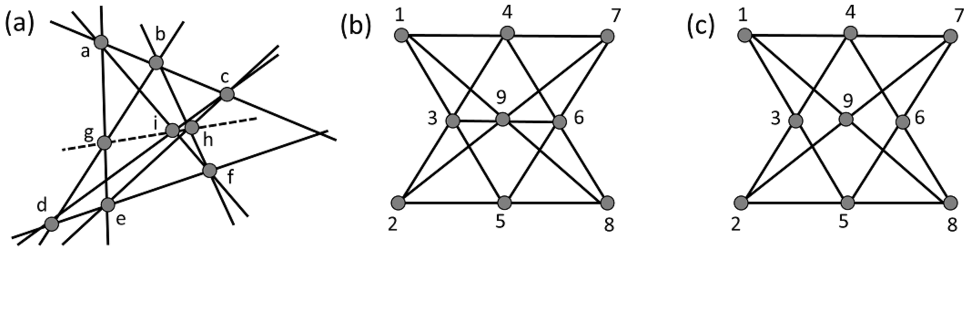

Figure 5.2: (a) An arrangement of the nine points in the Pappus' hexagon theorem, and geometric representations of (b) the Pappus matroid and (c) the non-Pappus matroid

5.3 Definition of the Family NP_{3n} of Matroids

We construct the infinite family from the non-Pappus matroid. The infinite family consists of the following matroids NP_{3n} for $n \geq 3$. Figure 5.3(a), (b) and (c) show geometric representations of NP_9, NP_{12} and NP_{15}, respectively. Note that the matroid NP_9 is the non-Pappus matroid.

Definition 5.3. *For $n \geq 3$, NP_{3n} is a simple matroid on a ground set $E_{3n} = \{1, 2, ..., 3n\}$ with the following set $\mathcal{H}_{3n}$ of hyperplanes:*

- $\{1, 4, ..., 3n-2\}$, $\{2, 5, ..., 3n-1\}$, $\{3, 6, ..., 3n-3\}$,
- $\{1, 3n-1, 3n\}$, $\{2, 3n-2, 3n\}$,
- $\{3k-2, 3k, 3k+2\}$, $\{3k-1, 3k, 3k+1\}$ $(k = 1, 2, ..., n-1)$, *and*
- *all 2-element subsets of E not being contained in the preceding subsets*

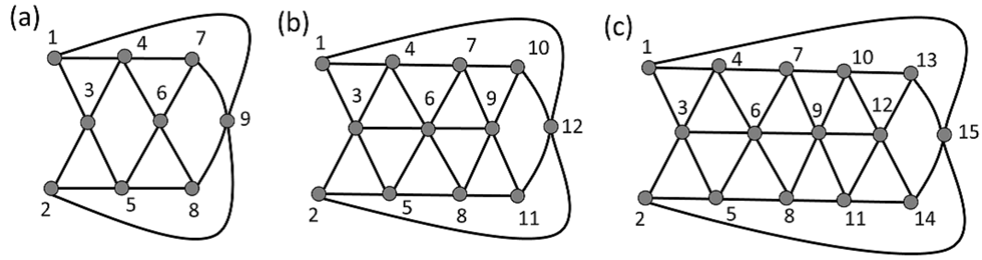

Figure 5.3: Geometric representations of (a) NP_9, (b) NP_{12} and (c) NP_{15}

For the proof of Theorem 5.2, we show the following theorem.

Theorem 5.4 ([29, 33]). *For every $n \geq 3$, NP_{3n} is orientable and minimal non-representable over any field.*

The proof of Theorem 5.4 is divided into three parts. Firstly we prove in Proposition 5.5 that NP_{3n} is orientable for every $n \geq 3$. Secondly we prove in Proposition 5.6 that NP_{3n} is not representable over any field for every $n \geq 3$. Lastly we prove in Proposition 5.7 that any minor of NP_{3n} is representable over $\mathbb{Q}$. Combining the three propositions, we obtain Theorem 5.4.

5.4 Orientability of NP_{3n}

Using Theorem 3.6, we prove in Proposition 5.5 that NP_{3n} is orientable.

Proposition 5.5 ([29, 33]). *For every $n \geq 3$, NP_{3n} is orientable.*

Proof. As any matroid of rank 3 with at most 6 elements is orientable, the deletion $NP_{3n} \setminus \{7, 8, ..., 3n\}$, which has 6 elements, is orientable. Therefore we prove that NP_{3n} is orientable if and only if $NP_{3n} \setminus \{7, 8, ..., 3n\}$ is orientable.

For a fixed $n \geq 3$, let S_i be a set $\{i, i+1, ..., 3n\}$ for every $1 \leq i \leq 3n$, and S_{3n+1} the empty set $\emptyset$. An element i of NP_{3n} is contained in at most two hyperplanes with at least three elements in $NP_{3n} \setminus S_{i+1}$. Therefore the element i is reducible in $NP_{3n} \setminus S_{i+1}$. Theorem 3.6 implies that for any $i \in \{1, 2, ..., 3n+1\}$, $NP_{3n} \setminus S_i$ is orientable if and only if $NP_{3n} \setminus S_{i+1}$ is orientable. It follows that NP_{3n} is orientable if and only if $NP_{3n} \setminus S_7 = NP_{3n} \setminus \{7, 8, ..., 3n\}$ is orientable. This completes the proof. □

5.5 Representability of NP_{3n}

We prove in Proposition 5.6 that NP_{3n} is not representable over any field for every $n \geq 3$.

Proposition 5.6 ([29, 33])**.** *For $n \geq 3$, NP_{3n} is not representable over any field.*

Proof. Suppose the contrary, i.e. NP_{3n} is representable over a filed $\mathbb{F}$. Then consider an embedding of the matroid NP_{3n} into the projective plane over $\mathbb{F}$. We regard the elements of NP_{3n} as the points in the embedding on the projective plane.

Let l be a line on the projective plane which corresponds to the hyperplane $H = \{3, 6, ... , 3n-3\} \in \mathcal{H}_{3n}$. For $i \in \{3, 4, ..., n\}$, let p_i be an intersection of the following two lines: a line through 1 and $3i-1$, and a line through 2 and $3i-2$. Note that p_n is the same as $3n$ and therefore must not be on l as in Figure 5.4 (a). We prove by an induction on i that p_i is on l for $i \in \{3, 4, ..., n\}$. This leads to a contradiction to the fact that p_n, or equivalently $3n$, is not on l.

In the base case $i = 3$, apply the Pappus' hexagon theorem to the following nine points: $1, 2, 3, 4, 5, 6, 7, 8$ and p_3. Then p_3 is collinear with 3 and 6. Hence p_3 is on l. As for an inductive step, assume that p_k is on l. In the same way as the base case, apply the Pappus's hexagon theorem to the following nine points: $1, 2, 3k-2, 3k-1, 3k, 3k+1, 3k+2, p_k$ and p_{k+1}. Then p_{k+1} is collinear with $3k$ and p_k. The line passing $3k$ and p_k is l, and therefore p_{k+1} is also on l as Figure 5.4 (b).

In the result, $3n$ must be on the line l. However the point $3n$ is not on the line l in any embedding of NP_{3n} into the projective plane as mentioned above, since the element $3n$ is not contained in the hyperplane H of NP_{3n}. This is a contradiction. Hence NP_{3n} cannot be embedded into the projective plane over any field, and not representable over any field. This completes the proof.

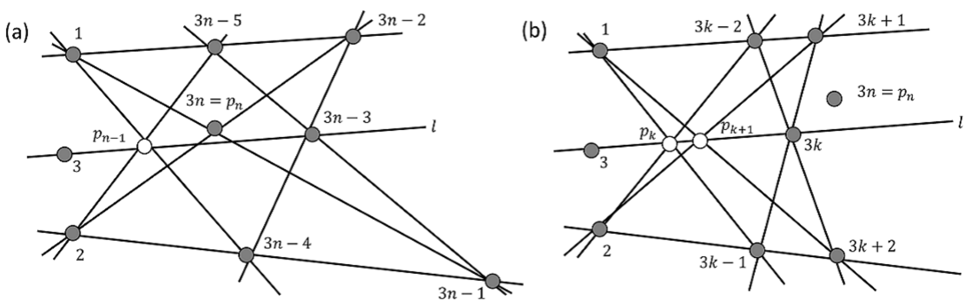

Figure 5.4: (a) The nine points 1, 2, $3k-2$, $3k-1$, $3k$, $3k+1$, $3k+2$, p_k, p_{k+1}, and the line l, and (b) the point $3n = p_n$ and the line l.

□

5.6 Minimality of NP_{3n}

In this section, we prove that any proper minor of NP_{3n} is representable over any field with characteristic 0. Since any field with characteristic 0 contains the rational field $\mathbb{Q}$ as its prime field, it suffices to prove $\mathbb{Q}$-representability.

Proposition 5.7 ([29, 33]). *For $n \geq 3$, any proper minor of NP_{3n} is representable over $\mathbb{Q}$.*

Proof. For $n = 3$, the matroid NP_9 is isomorphic to the non-Pappus matroid whose any proper minor is representable over $\mathbb{Q}$. Therefore we prove for $n \geq 4$ that there exists the representation over $\mathbb{Q}$ of $NP_{3n} \setminus \{e\}$ for any $e \in E_{3n}$, i.e. the point configuration on the projective plane consisting of the $3n-1$ points other than e. By symmetry, it is sufficient to consider the following three cases:

(i) $e = 3n$,

(ii) $e \in \{3, 6, ..., 3n-3\}$, and

(iii) $e \in \{1, 4, ..., 3n-2\}$.

The first condition of the hyperplanes $\mathcal{H}_{3n}$ in Definition 5.3 implies that the points $\{1, 4, ..., 3n-2\}, \{2, 5, ..., 3n-1\}, \{3, 6, ..., 3n-3\}$ are collinear on the projective plane, respectively. For a point p on the projective plane, its x-coordinate and y-coordinate are denoted by $x(p)$ and $y(p)$ respectively. In the point configurations for the three cases (i), (ii) and (iii), we determine the y-coordinates of the $3n-1$ points as follows satisfying the first condition in Definition 5.3. Therefore we show that the point configurations for the three cases (i), (ii) and (iii) satisfy the other conditions of the hyperplanes $\mathcal{H}_{3n}$ in Definition 5.3.

- $y(1) = y(4) = \cdots = y(3n-2) = 1$,
- $y(2) = y(5) = \cdots = y(3n-1) = 1$, and
- $y(3) = y(6) = \cdots = y(3n-3) = 1$.

Firstly, we deal with the case (i). We consider the point configuration consisting of the $3n-1$ points other than e with the following x-coordinates.

(i-1) $x(1) = x(2) = 1$

(i-2) $x(3s-1) = x(3s-2) = 10\, x(3s-4)$ for $s \in \{2, 3, ..., n\}$

(i-3) $x(3s-3) = \frac{x(3s-1)+x(3s-4)}{2}$ for $s \in \{2, 3, ..., n\}$

Now we show that the point configuration represents $NP_{3n} \setminus \{3n\}$. For the purpose, we prove by induction on $s \in \{1, 2, ..., n\}$ that when the points $1, \cdots, 3s-1$ have the above-mentioned x and y-coordinates, the point configuration consisting of the points $1, \cdots, 3s-1$ represents $NP_{3n} \setminus \{3s, 3s+1, ..., 3n\}$. For $s = 1$, the point configuration of the points $\{1, 2\}$ clearly represents $NP_{3n} \setminus \{3, 4, ..., 3n\}$. For some $s \in \{1, 2, ..., n-1\}$, assume that the point configuration consisting of the points $\{1, 2, ..., 3s-1\}$ represents $NP_{3n} \setminus \{3s, 3s+1, ..., 3n\}$. For two distinct points p and q, we denote the line passing through p and q by $[p : q]$. The x-coordinates $x(3s+1)$ and $x(3s+2)$ are so large that the points $3s+1$ and $3s+2$ are not on

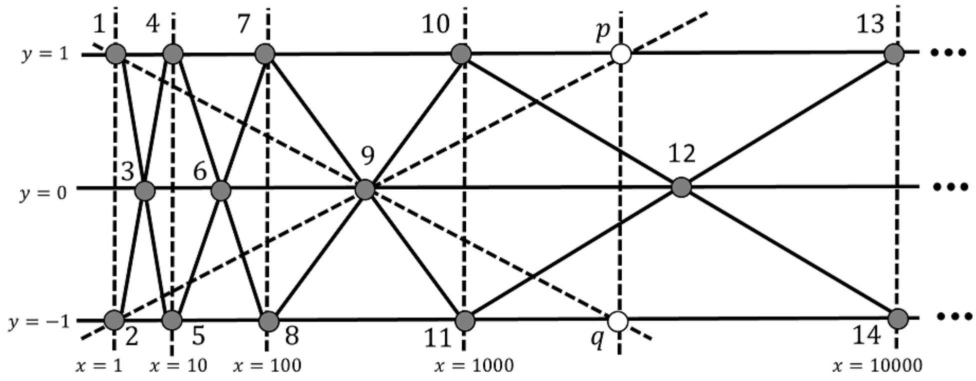

Figure 5.5: The construction of the representation of $NP_{3n} \setminus \{3n\}$

the diagonal line $[3s_1 - 3 : 3s_2 - 1]$ and $[3s_1 - 3 : 3s_2 - 2]$ for any $1 \leq s_1, s_2 \leq s$ respectively. Similarly $x(3s)$ is also so large that the point $3s$ is not on the vertical line $[3s_1 - 1, 3s_2 - 2]$ for any $1 \leq s_1, s_2 \leq s$. Figure 5.5 describes the case $s = 4$. Assume that the point configuration consisting of the points $\{1, 2, ..., 11\}$ represents $NP_{3n} \setminus \{12, 13, 14, ..., 3n\}$. The points p and q are intersections of two lines $[2 : 9]$ and $y = 1$ and of two lines $[1 : 9]$ and $y = -1$. You can check that The x-coordinate of the intersection of two lines $y = 1$ and $[3s_1 - 1 : 3s_2]$ is equal to or less than $x(p)$ for any $1 \leq s_1 \leq 4$ and $1 \leq s_2 \leq 3$. The condition (i-2) makes the x-coordinate of 13 larger than $x(p)$. The same argument holds for the element 14 and the point q. This shows that the elements 1, 2, ..., 13 represent $NP_{3n} \setminus \{14, 15, ..., 3n\}$. Eventually the point configuration consisting of the points $\{1, 2, ..., 3s + 2\}$ satisfies the third condition of the hyperplanes $\mathcal{H}_{3n}$ in Definition 5.3, hence the points represent $NP_{3n} \setminus \{3s + 3, 3s + 4, ..., 3n\}$. Note that the second condition in Definition 5.3 is vacuous. Therefore the points $\{1, 2, ..., 3n - 1\}$ represent $NP_{3n} \setminus \{3n\}$.

Secondly, we deal with the case (ii). We consider the point configuration consisting of the $3n - 1$ points other than $e = 3k$ with the following x-coordinates.

(ii-1) $x(1) = x(2) = 1$

(ii-2) $x(3s - 1) = x(3s - 2) = 10\,x(3s - 5)$ for $s \in \{2, 3, ..., k\}$

(ii-3) $x(3s - 1) - 1 = x(3s - 2) = 10\,x(3s - 5)$ for $s \in \{k + 1, k + 3, ..., n\}$

(ii-4) $x(3s - 3) = \frac{x(3s-1)+x(3s-5)}{2}$ for $s \in \{2, 3, ..., k\} \cup \{k + 2, k + 3, ..., n\}$

(ii-5) $3n$ is on the intersection of $[1 : 3n - 1]$ and $[2 : 3n - 2]$

Note that, for $s \in \{1, 2, ..., n\}$, the slopes of the line $[3s - 1 : 3s - 2]$ depend on the value of s as see Figure 5.6 (a). The slope is ∞ for $s \leq k$ under (ii-2), and -2 for $s \geq k + 1$ under (ii-3). As the x-coordinates $x(3s - 1)$ and $x(3s - 2)$ are sufficiently larger than $x(3s-4)$ and $x(3s-5)$, the diagonal lines $[3s_1 - 1 : 3s_2]$ and $[3s_1 - 2 : 3s_2]$ do not pass through the points $3s - 2$ and $3s - 1$ for any $1 \leq s_1 \leq s - 1$ and $s_2 \in \{1, 2, ..., s - 2\} \setminus \{k\}$. Therefore the $3n - 2$ points other than $e = 3k$ and $3n$ satisfy the third condition of the hyperplanes $\mathcal{H}_{3n}$ in Definition 5.3. Next we consider whether the point $3n$ satisfies the second condition in Definition 5.3. The y-coordinate $y(3n)$ is positive under (ii-5) as in Figure 5.6 (b). The x-coordinates

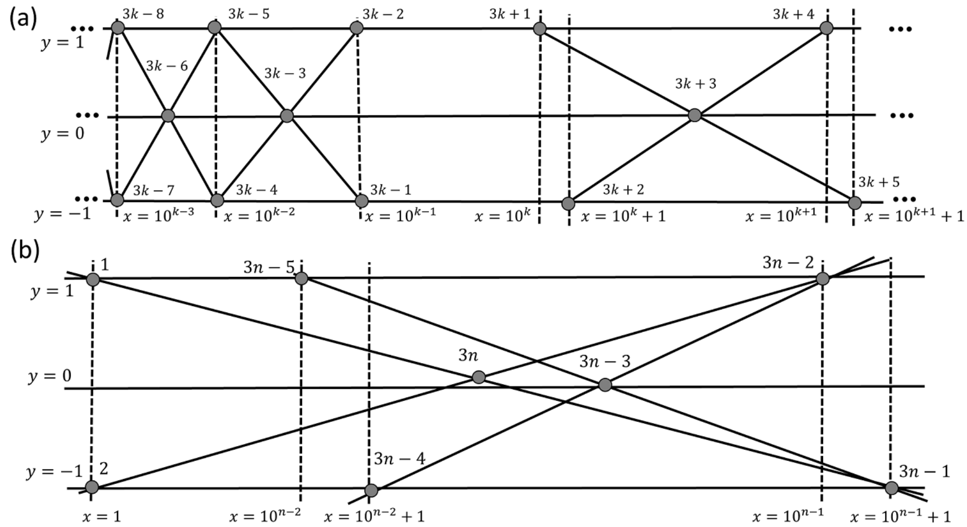

Figure 5.6: The construction of the representation of $NP_{3n} \setminus \{3k\}$ for some $k \in \{1, 2, ..., n-1\}$

$x(3n-2)$ and $x(3n-1)$ are so large compared to $x(3n-5)$ and $x(3n-4)$ under (ii-3) that $x(3n)$ is more than $x(3n-4)$ and less than $x(3n-1)$. Therefore for two distinct points $s_1, s_2 \in \{1, 2, ..., 3n-1\} \setminus \{3k\}$, the point $3n$ is on the line $[s_1 : s_2]$ if and only if $\{s_1, s_2\} = \{1, 3n-1\}$ or $\{2, 3n-2\}$. It follows that the $3n-1$ points other than $e = 3k$ satisfy the second condition of the hyperplanes $\mathcal{H}_{3n}$ in Definition 5.3, i.e. the point configuration consisting of the $3n-1$ points represents $NP_{3n} \setminus \{3k\}$.

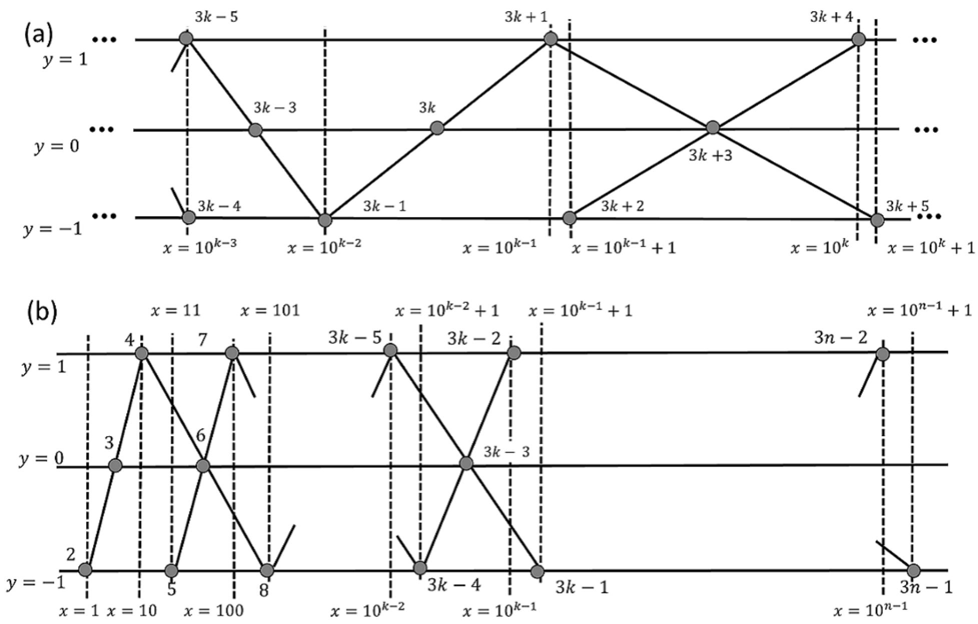

Figure 5.7: The construction of the representation of $NP_{3n} \setminus \{3k-2\}$

Finally, we deal with the case (iii). We consider two types of the point configurations for (iii-a) $k \neq 1, 3n-2$, and (iii-b) $k = 1, 3n-2$ consisting of the $3n-1$ points other than e with the following x-coordinates. Note that we deal with only the case $k = 1$ for (iii-b) by symmetry.

(iii-a-1) $x(1) = x(2) = 1$

(iii-a-2) $x(3s-1) = x(3s-2) = 10\,x(3s-5)$ for $s \in \{2, 3, ..., k-1\}$

(iii-a-3) $x(3k-1) = 10\,x(3k-4)$

(iii-a-4) $x(3s-1) - 1 = x(3s-2) = 10\,x(3s-4)$ for $s \in \{k+1, k+2, ..., n\}$

(iii-a-5) $x(3s-3) = \frac{x(3s-1)+x(3s-5)}{2}$ for $s \in \{2, 3, ..., k\} \cup \{k+2, k+3, ..., n\}$

(iii-a-6) $x(3k) = \frac{x(3k-1)+x(3k+1)}{2}$

(iii-a-7) $3n$ is on the intersection of $[1 : 3n-1]$ and $[2 : 3n-2]$

(iii-b-1) $x(2) = 1$, $x(4) = 10$ and $x(5) = 11$

(iii-b-2) $x(3s-1) - 1 = x(3s-2) = 10\,x(3s-5)$ for $s \in \{3, 4..., n\}$

(iii-b-3) $x(3s-3) = \frac{x(3s-2)+x(3s-4)}{2}$ for $s \in \{2, 3, ..., n\}$

(iii-b-4) $3n$ is on the intersection of $[2 : 3n-2]$ and $x = \frac{9x(3n-2)}{10}$

It is proved similarly to the case (i) that the $3n-2$ points other than $e = 3k-2$ and $3n$ satisfy the third condition of the hyperplanes $\mathcal{H}_{3n}$ in Definition 5.3. Next we consider whether the point $3n$ under the rules (iii-a-8) and (iii-b-4) satisfies the second condition in Definition 5.3. For the case (iii-a), it is proved in the same discussion as the case (ii) that the line $[s : 3n]$ for $s \neq 1, 2, 3n-2, 3n-1$ passes through no element except s and $3n$. Therefore the $3n-1$ points other than $e = 3k-2$ satisfy the second condition of the hyperplanes $\mathcal{H}_{3n}$ in Definition 5.3, For the case (iii-b), the point $3n$ under (iii-b-4) is not in the intersections of the line $[2 : 3n-2]$ and the line $[s_1 : s_2]$ for any different two elements $s_1, s_2 \in \{2, ..., 3n-1\}$. Hence in the both cases (iii-a) and (iii-b), the point configurations consisting of the $3n-1$ points represent $NP_{3n} \setminus \{3k\}$.

After all, there exists the representation over $\mathbb{Q}$ for $NP_{3n} \setminus \{e\}$ for any $e \in E_{3n}$. This complete the proof. $\square$

Chapter 6

Excluded Minors for $\mathbb{Q}$-Representability in Algebraic Extension

6.1 Introduction

In this chapter, we investigate further the representability over the rational field $\mathbb{Q}$. The results which are presented in this chapter are published in [31].

As mentioned before, it is known that the class of $\mathbb{Q}$-representable matroids has infinitely many excluded minors [40]. Now we consider some class which properly contains the class of $\mathbb{Q}$-representable matroids, e.g. the class $\mathcal{C}$ of $\mathbb{R}$-representable matroids. Our question in this chapter is on the number of the excluded minors for the class of $\mathbb{Q}$-representable matroids within the class $\mathcal{C}$. In particular, we consider $\mathcal{C}$ to be algebraic extension fields of $\mathbb{Q}$ (see Figure 6.1). The point configuration $GP(9)$ in Figure 6.2 is listed in [24]. Then it is known that this matroid from this point configuration is known to be not representable over $\mathbb{Q}$ but representable over $\mathbb{Q}[\sqrt{5}]$. Routinely checking the minors of this matroid, we can know that any proper minor of this matroid is $\mathbb{Q}$-representable, which show that this matroid is a $\mathbb{Q}[\sqrt{5}]$-representable excluded minors for the class of $\mathbb{Q}$-representable matroids.

The main result of this section is as follows. This theorem implies that, even if we know that a given matroid is not only $\mathbb{R}$-representable but $\mathbb{Q}[\sqrt{2}]$-representable, the excluded minors' characterization of $\mathbb{Q}$-representable matroids is difficult.

Theorem 6.1 ([31])**.** *Let x be an element algebraic over $\mathbb{Q}$ the degree of whose minimal polynomial is* 2*. There exist infinitely many $\mathbb{Q}[x]$-representable excluded minors of rank* 3 *for $\mathbb{Q}$-representability.*

To prove this theorem, we explicitly construct an infinite number of excluded minors for $\mathbb{Q}$-representable matroids which have the corresponding point configuration on the projective space over $\mathbb{Q}[x]$. Let $y^2 + ay + b$ be a minimal polynomial for an algebraic element x. We construct different excluded minors for $c(y^2 + ay + b)$ for each positive integer c, which leads to construction of infinitely many excluded minors since there exist (needless to say) infinitely many integers. This means that, even under the knowledge that a given matroid is not only $\mathbb{Q}[x]$-representable but at most rank-r for any fixed integer r, the excluded minors characterization is still difficult.

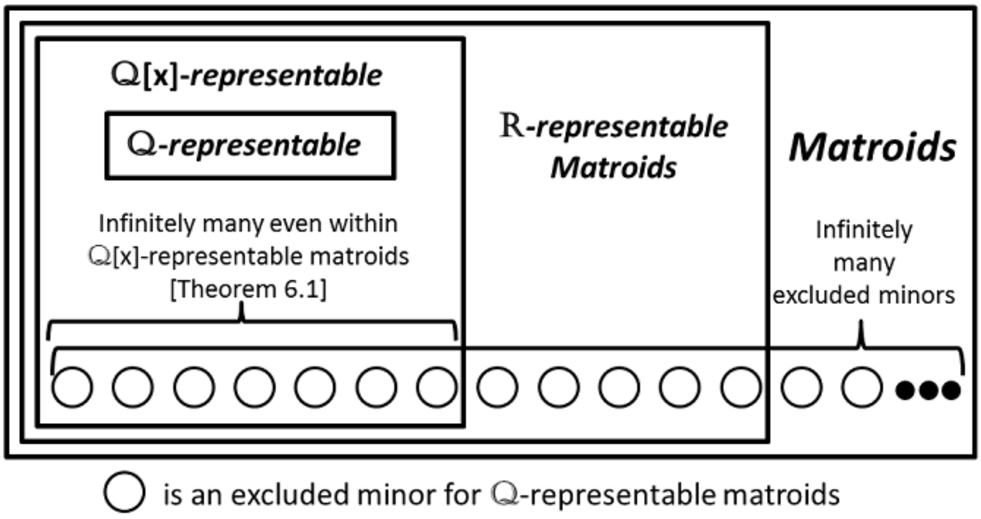

Figure 6.1: On the number of excluded minors for $\mathbb{Q}$-representable matroids

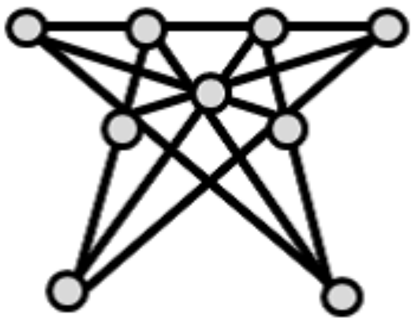

Figure 6.2: The point configuration $GP(9)$ in [24]

6.2 Definition of Matroids $M_{a,b,c}$ for the equation $ay^2 + by + c = 0$

In this section, we define a matroid $M_{a,b,c}$ for the equation $ay^2 + by + c = 0$ where a, b, c are integers, $a > 0$, $c \neq 0$ and y is an indeterminate.

The matroid is defined as the underlying matroid of the matrix $M(a, b, c)$. In the following, let x be a solution of the equation $ay^2 + by + c = 0$. To define the matrix $M(a, b, c)$, we prepare some matrices in what follows.

First, we define the matrix M_0. Let the set E_0 be $\{e_1, e_2, e_3\}$. The column vectors of the matrix M_0 are labeled by the elements in the set E_0 as follows.

$$M_0 = \begin{array}{c} \\ 1 \\ 2 \\ 3 \end{array} \begin{array}{c} \begin{array}{ccc} e_1 & e_2 & e_3 \end{array} \\ \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \end{array}$$

Second, we define the matrix M_1. Let the set E_1 be $\{f_1, f_2, ..., f_{2d+1}\}$, where $d = \max\{a, |c|\},$. The column vectors of the matrix M_1 are labeled by the elements in the set E_1 as follows.

$$\begin{array}{cc} & \begin{array}{cccccccccc} f_1 & f_2 & f_3 & f_4 & f_5 & f_6 & f_7 & f_8 & \cdots & f_{2d+1} \end{array} \\ M_1 = \begin{array}{c} 1 \\ 2 \\ 3 \end{array} & \begin{pmatrix} 1 & 1 & 1 & 0 & 1 & 0 & 1 & 0 & \cdots & 1 \\ 1 & 0 & 1 & 1 & 0 & 1 & 0 & 1 & \cdots & 0 \\ 0 & 1 & 1 & -1 & 2 & -2 & 3 & -3 & \cdots & d \end{pmatrix} \end{array}$$

Third, we define the matrix M_2. Let the set E_2 be $\{g_1, g_2, ..., g_6\}$. The column vectors of the matrix M_2 are labeled by the elements in the set E_2 as follows.

$$\begin{array}{cc} & \begin{array}{cccccc} g_1 & g_2 & g_3 & g_4 & g_5 & g_6 \end{array} \\ M_2 = \begin{array}{c} 1 \\ 2 \\ 3 \end{array} & \begin{pmatrix} 1 & 1 & 1 & 0 & 1 & 1 \\ x & 0 & 1 & 1 & 0 & ax \\ 0 & x & x & -\frac{1}{x} & a & 0 \end{pmatrix} \end{array}$$

Fourth, we define the matrix M_3. If $c < 0$, let the set E_2 be the empty set, and M_3 the 3×0 matrix. If $c > 0$, et the set E_2 be $\{h_1, h_2, h_3\}$. The column vectors of the matrix M_3 are labeled by the elements in the set E_3 as follows.

$$\begin{array}{cc} & \begin{array}{ccc} h_1 & h_2 & h_3 \end{array} \\ M_3 = \begin{array}{c} 1 \\ 2 \\ 3 \end{array} & \begin{pmatrix} 0 & 1 & 1 \\ 1 & -c & 0 \\ 1 & 0 & -c \end{pmatrix} \end{array}$$

Fifth, we define the matrix M_4. If $b = 0$, let the set E_4 be the empty set, and M_4 the 3×0 matrix. If $b > 0$, let the set E_4 be $\{i_1, g_2, ..., i_{2b}\}$. The column vectors of the matrix M_4 are labeled by the elements in the set E_4 as follows.

$$\begin{array}{cc} & \begin{array}{cccccc} i_1 & i_2 & i_3 & i_4 & \cdots & i_{2b} \end{array} \\ M_4 = \begin{array}{c} 1 \\ 2 \\ 3 \end{array} & \begin{pmatrix} 0 & 1 & 0 & 1 & \cdots & 1 \\ 1 & ax+1 & 1 & ax+2 & \cdots & ax+b \\ -\frac{1}{ax} & 0 & -\frac{1}{ax+1} & 0 & \cdots & 0 \end{pmatrix} \end{array}$$

Otherwise, let the set E_4 be $\{i_1, g_2, ..., i_{2|b|+1}\}$. The column vectors of the matrix M_4 are labeled by the elements in the set E_4 as follows.

$$\begin{array}{cc} & \begin{array}{ccccccc} i_1 & i_2 & i_3 & i_4 & i_5 & \cdots & i_{2|b|+1} \end{array} \\ M_4 = \begin{array}{c} 1 \\ 2 \\ 3 \end{array} & \begin{pmatrix} 1 & 0 & 1 & 0 & 1 & \cdots & 1 \\ -1 & 1 & ax-1 & 1 & ax-2 & \cdots & ax+b \\ 1 & -\frac{1}{ax} & 0 & -\frac{1}{ax-1} & 0 & \cdots & 0 \end{pmatrix} \end{array}$$

Finally, we define the matrix M_5. Let the set E_5 be $\{j\}$. The column vectors of the matrix M_5 are labeled by the elements in the set E_5 as follows.

$$M_5 = \begin{array}{c} \\ 1 \\ 2 \\ 3 \end{array} \begin{array}{c} j \\ \begin{pmatrix} 1 \\ ax+b \\ c \end{pmatrix} \end{array}$$

Then, $M(a,b,c)$ is defined as the concatenation of the above six matrices M_0 ,M_1, M_2, M_3, M_4 and M_5. The matroid $M_{a,b,c}$ on the ground set $E = E_0 \cup E_1 \cup E_2 \cup E_3 \cup E_4 \cup E_5$ is the matroid derived from the column vectors of the matrix $M(a,b,c)$.

6.3 $\mathbb{Q}[x]$-representable Excluded Minors for $\mathbb{Q}$-representable Matroids

In this section, we prove the following theorem.

Theorem 6.2. *Let a be a positive integer, b be an integer, c be a non-zero integer and x be a complex number which is the solution of the equation $ay^2 + by + c = 0$. If $x \notin \mathbb{Q}$, then the matroid $M_{a,b,c}$ is an $\mathbb{Q}[x]$-representable matroid and an excluded minor for $\mathbb{Q}$-representable matroids.*

Now we consider the family $\{M_{a,b,c}\}_{a,b,c\in\mathbb{Z},a>0,c\neq 0}$ of matroids. There are an infinite number of distinct matroids in this family, which we can easily known from the fact that the cardinality of the ground sets of the matroids in the family $\{M_{a,b,c}\}_{a,b,c\in\mathbb{Z},a>0,c\neq c}$ can be arbitrarily large. Therefore, Theorem 6.1 can be obtained by proving Theorem 6.2.

Proof. The proof Theorem 6.2 consists of the following three parts.

(1) $M_{a,b,c}$ is representable over $\mathbb{Q}[x]$

(2) $M_{a,b,c}$ is not representable over $\mathbb{Q}$

(3) any proper minor of $M_{a,b,c}$ is representable over $\mathbb{Q}$

The part (1) is straight-forward, since each matroid $M_{a,b,c}$ is obtained as an underlying matroid of the matrix $M_{a,b,c}$ over $\mathbb{Q}[x]$.

Then we move to the proof of the part (2). Now we use Theorem 3.15. Take a base B in the Theorem 3.15 to be a set $E_0 = \{e_1, e_2, e_3\}$, and consider the bipartite graph $G(M_{a,b,c}, B)$. Then we construct the spanning tree in $G(M_{a,b,c}, E_0)$ by collecting, for each $s \in E \setminus E_0$, one edge among the edges incident with s as follows:

- (e_1, s) if the (1,s) element of $M_{a,b,c}$ is non-zero,
- (e_2, s) if the (2,s) element of $M_{a,b,c}$ is non-zero,
- (e_3, s) otherwise.

The matrix A_b in Theorem 3.15 which corresponds the above spanning tree is the concatenation of the following five matrices A_1, A_2, A_3, A_4, A_5.

First, for $d = \max a, |c|$, the matrix A_1 is defined as follows:

$$A_1 = \begin{array}{c} \\ e_1 \\ e_2 \\ e_3 \end{array} \begin{array}{c} \begin{array}{cccccccccc} f_1 & f_2 & f_3 & f_4 & f_5 & f_6 & f_7 & f_8 & \cdots & f_{2d+1} \end{array} \\ \begin{pmatrix} 1 & 1 & 1 & 0 & 1 & 0 & 1 & 0 & \cdots & 1 \\ 1 & 0 & f_{2,3} & 1 & 0 & 1 & 0 & 1 & \cdots & 0 \\ 0 & 1 & f_{3,3} & f_{3,4} & f_{3,5} & f_{3,6} & f_{3,7} & f_{3,8} & \cdots & f_{3,2d+1} \end{pmatrix} \end{array}$$

Second, the matrix A_2 is defined as follows:

$$A_2 = \begin{array}{c} \\ e_1 \\ e_2 \\ e_3 \end{array} \begin{array}{c} \begin{array}{cccccc} g_1 & g_2 & g_3 & g_4 & g_5 & g_6 \end{array} \\ \begin{pmatrix} 1 & 1 & 1 & 0 & 1 & 1 \\ \mathbf{z} & 0 & g_{2,3} & 1 & 0 & g_{6,2} \\ 0 & g_{3,2} & g_{3,3} & g_{4,3} & g_{5,3} & 0 \end{pmatrix} \end{array}$$

Third, if $c < 0$, then the matrix A_3 is the 3×0 matrix. Otherwise, the matrix A_3 is defined as follows:

$$A_3 = \begin{array}{c} \\ e_1 \\ e_2 \\ e_3 \end{array} \begin{array}{c} \begin{array}{ccc} h_1 & h_2 & h_3 \end{array} \\ \begin{pmatrix} 0 & 1 & 1 \\ 1 & h_{2,2} & 0 \\ h_{3,1} & 0 & h_{3,3} \end{pmatrix} \end{array}$$

Fourth, if $b = 0$, then the matrix A_4 is the 3×0 matrix. if $b > 0$, then the matrix A_4 is defined as follows:

$$A_4 = \begin{array}{c} \\ e_1 \\ e_2 \\ e_3 \end{array} \begin{array}{c} \begin{array}{cccccc} i_1 & i_2 & i_3 & i_4 & \cdots & i_{2b} \end{array} \\ \begin{pmatrix} 0 & 1 & 0 & 1 & \cdots & 1 \\ 1 & i_{2,2} & 1 & i_{2,4} & \cdots & i_{2,2b} \\ i_{3,1} & 0 & i_{3,3} & 0 & \cdots & 0 \end{pmatrix} \end{array}$$

Otherwise, the matrix A_4 is defined as follows:

$$A_4 = \begin{array}{c} \\ e_1 \\ e_2 \\ e_3 \end{array} \begin{array}{c} \begin{array}{ccccccc} i_1 & i_2 & i_3 & i_4 & i_5 & \cdots & i_{2|b|+1} \end{array} \\ \begin{pmatrix} 1 & 0 & 1 & 0 & 1 & \cdots & 1 \\ i_{2,1} & 1 & i_{2,3} & 1 & i_{2,5} & \cdots & i_{2,2|b|+1} \\ i_{3,1} & i_{3,2} & 0 & i_{3,4} & 0 & \cdots & 0 \end{pmatrix} \end{array}$$

Finally, the matrix A_5 is defined as follows:

$$A_5 = \begin{array}{c} \\ e_1 \\ e_2 \\ e_3 \end{array} \begin{array}{c} j \\ \begin{pmatrix} 1 \\ j_2 \\ j_3 \end{pmatrix} \end{array}$$

Note that each symbol which is as the elements in the above matrices is indeterminate. Then we assign the indeterminates in A_B with the constant in a way that the matrix derives the matroid $M_{a,b,c}$. For the matrix $M(a, b, c)$ in the Section 6, consider the submatrix which is obtained from the column vectors labeled by e_3, f_1 and f_3:

$$\begin{array}{ccc} e_3 & f_1 & f_3 \end{array}$$
$$\begin{pmatrix} 0 & 1 & 1 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{pmatrix}$$

The rank of this submatrix is 2. Therefore, the set $\{e_2, f_1, f_3\}$ is rank 2 in the matroid $M_{a,b,c}$. This implies that the rank of the following matrix must be 2:

$$\begin{array}{ccc} & f_1 & f_3 \end{array}$$
$$\begin{pmatrix} 0 & 1 & 1 \\ 0 & 1 & f_{2.3} \\ 1 & 0 & f_{3,3} \end{pmatrix}$$

Therefore we obtain the following equation: $f_{2,3} = 1$. By repeating the similar discussion, we can routinely assign the indeterminants in A_B with the constants in the following way:

- If the entry $M(a, b, c)(p.q)$ is rational, then $A_B(p, q) = M(a, b, c)(p, q)$
- Otherwise, $A_B(p, q)$ is the rational form which is obtained by replacing x in $M(a, b, c)(p, q)$ with the indeterminante z;

The examples of the assignment with respect to the second one are as follows:

- $A_B(e_3, g_2) = g_{3,2} = z$ for $A_B(3, g_2) = x$,
- $A_B(e_3, g_4) = g_{3,4} = -\frac{1}{x}$ for $A_B(3, g_4) = -\frac{1}{x}$,
- $A_B(e_2, g_6) = g_{2,6} = ax$ for $A_B(2, g_6) = ax$

Then, for the matrix $M(a, b, c)$, consider its submatrix which is obtained from the column vectors labeled by the following labels: $\{e_1, f_1, g_6\}$ if $b = 0$, $\{e_1, f_1, i_{2b}\}$ if $b > 0$, and $\{e_1, f_1, i_{2b+1}\}$ otherwise.

$$\begin{array}{ccc} 1 & 1 & 1 \end{array}$$
$$\begin{pmatrix} 0 & ax + b & 1 \\ 0 & -c & x \end{pmatrix}$$

The rank of this matrix is s, since x is the solution of the equation $ay^2 + by + c = 0$. Therefore, we must assign the indeterminate z with the solution of the equation $ay^2 + by + c = 0$. The solution of the equation $ay^2 + by + c = 0$ must not be rational. This contradicts the assumption that $M_{a,b,c}$ is representable over $\mathbb{Q}$. Hence, we have completed the proof of the part (2).

Regarding to the part (3), we have to consider the contraction and deletion. Any contraction of the simple matroid decrease the rank of the matroid by one. Furthermore, any rank-2 matroid is known to be $\mathbb{Q}$-representable. Finally, consider the deletion. In the matrix M By routinely checking, any deletion of the matroid make the equation which is obtained from the column vectors e_1, g_3 and j two variable equation, and the degree of each variable is at most one. Since $\mathbb{Q}$ is an infinite field, we can assign some value with each variable satisfying the equation. In the result, we can obtain the representation over $\mathbb{Q}$, which complete the proof of the part (3).

$\square$

Chapter 7

Parameterized Algorithms to Compute Ising Partition Function

7.1 Algorithm Parameterized by Branch Width

We construct an algorithm to compute Ising partition function on a graph G, using a given optimal branch decomposition $\delta = (D, \phi)$. This algorithm is done in $O^*(2^{\frac{\omega}{2} bw(G)})$. Hereafter we identify edges and leaves under ϕ. The algorithm which is constructed here is published in [28].

Now we introduce some notations. For a finite set V, we regard a subset $S \subseteq V$ as a vector $S = (S_v)_{v \in V} \in \{\pm 1\}^V$ as follows: for $v \in V$, $S_v = 1$, if $v \in S$, -1 otherwise. A bag b induces a partition $(E_b, \overline{E_b})$ of E as Section 2.5. A graph G_b denotes an edge-induced subgraph by E_b. By abuse of notation, we denote a vertex set of G_b by V_b. Note the notation V_b in this section differs in its meaning from one in Section 2.5.

First we construct an $O^*(4^{bw(G)})$ algorithm. This algorithm creates a table DP each entry of which is initialized by 0, and then updates the entries iteratively in a bottom-up manner. An entry of this table is specified like DP[b][U] where b is a bag and $U \subseteq \mathrm{mid}(b)$. For a fixed bag b, a set DP[b] of all entries DP[b][U] is called *an array.* Now we introduce one notation: for a bag b,

$$H_b(S) = -(\sum_{\{u,v\} \in E_b} J_e S_u S_v + \sum_{v \in V_b} M_v S_v)$$

This corresponds to the Hamiltonian of a spin configuration S on a graph G_b. An entry DP[b][U] is *valid* if it retains the following value:

$$\sum_{S \subseteq V_b, S \cap \mathrm{mid}(b) = U} \exp(-\beta H_b(S)) \tag{7.1}$$

That is, DP[b][U] retains a sum of $\exp(-\beta H_b(S))$ over all spin configurations on G_b such that their configurations coincide on the vertices in mid(b). Then, an array DP[b] is *valid* if all entries DP[b][U] are valid.

For a leaf l, mid(l) consists of at most two vertices. There we can create a valid array DP[l] in constant time by assigning DP[l][U] with $\exp(-\beta H_l(\{v\}))$ for each $U \subset \mathrm{mid}(l)$.

Let b be a bag which is not a leaf, and b_i $(i = 1, 2)$ be children of b. Supposing that we have already obtained valid entries DP[b_1] and DP[b_2], we show that a valid entry DP[b] can be constructed in $O^*(4^{bw(G)})$ time. First we introduce some notations. From the definition of a branch decomposition, we can observe that $\mathrm{mid}(b) \subseteq \mathrm{mid}(b_1) \cup \mathrm{mid}(b_2)$. A set $\mathrm{mid}(b_1) \cup \mathrm{mid}(b_2)$ is partitioned into (not disjoint) parts as follows: $C = (\mathrm{mid}(b_1) \cup \mathrm{mid}(b_2)) \setminus \mathrm{mid}(b)$, $L = \mathrm{mid}(b_1) \setminus C$ and $R = \mathrm{mid}(b_2) \setminus C$. Then, for $U \subseteq \mathrm{mid}(b)$, define $U_L = U \cap L$, $U_R = U \cap R$ and $U_I = U_L \cap U_R$.

For $C' \subseteq C$, define

$$\mathrm{DP}[b][U]_{C'} = \sum_{\substack{S \subseteq V_b \text{ s.t. } S \cap \mathrm{mid}(b) = U \\ S \cap C = C'}} \exp(-\beta H_b(S)).$$

This value is a sum of terms such that their corresponding spins on $\mathrm{mid}(b_1) \cup \mathrm{mid}(b_2)$ are fixed by U and C'. Then compare $\mathrm{DP}[b][U]_{C'}$ and $\mathrm{DP}[b_1][C' \cup U_L] \times \mathrm{DP}[b_2][C' \cup U_R]$. The latter value is equal to

$$\left[\sum_{\substack{S \subseteq V_{b_1} \text{ s.t. } S \cap L = U_L \\ S \cap C = C'}} \exp(-\beta H_{b_1}(S))\right] \cdot \left[\sum_{\substack{S \subseteq V_{b_2} \text{ s.t. } S \cap R = U_R \\ S \cap C = C'}} \exp(-\beta H_{b_2}(S))\right]$$

Comparing with the former value, we can observe that the latter value overlaps twofold over $L \cap R$ on the part corresponding to magnetic field weights. Therefore, by removing the overlap, we obtain $\mathrm{DP}[b][U]_{C'} = \mathrm{DP}[b_1][C' \cup U_L] \cdot \mathrm{DP}[b_2][C' \cup U_R]/\mathrm{OL}_{C'}$ where $\mathrm{OL}_{C'} = \exp(-\beta(\sum_{v \in L \cap R} M_v (U_I \cup C')_v))$. Then a valid entry is obtained as follows:

$$\mathrm{DP}[b][U] = \sum_{C' \subseteq C} \mathrm{DP}[b_1][C' \cup U_L]\ \mathrm{DP}[b_2][C' \cup U_R]\ /\ \mathrm{OL}_{C'} \tag{7.2}$$

For each $U \subseteq \mathrm{mid}(b)$, a valid entry DP[$b$][$U$] can be computed in $O^*(2^{bw(G)})$ time, and a valid array has $2^{|\mathrm{mid}(b)|} = O(2^{bw(G)})$ entries. Therefore a valid array is therefore constructed in $O^*(4^{bw(G)})$ time.

Then we modify this $O^*(4^{bw(G)})$ time algorithm into $O^*(2^{\frac{\omega}{2} bw(G)})$ one. To this aim, we convert each array into a matrix. From a valid array DP[b_1], construct a $2^{|L|} \times 2^{|C|}$ matrix M_{b_1} whose row and column are indexed by $L' \subseteq L$ and $C' \subseteq C$; the value indexed by L' and C' is $\mathrm{DP}[b][L' \cup C']$. Similarly, from a valid array DP[b_2], construct a $2^{|C|} \times 2^{|R|}$ matrix M_{b_2} whose row and column are indexed by $C' \subseteq C$ and $R' \subseteq R$; the value indexed by C' and R' is $\mathrm{DP}[b][R' \cup C']$.

Then we compute M_b from which a valid array DP[b] can be constructed. First, define $\mathrm{OL}(S) = \exp(-\beta(\sum_{v \in L \cap R} M_v S_v))$. Second, by dividing each entry of M_{b_1} indexed with $L' \subseteq L$ and $C' \subseteq C$ by $\mathrm{OL}(L' \cup C')$, create a matrix M'_{b_1}. Third, compute a multiplication $M_b = M'_{b_1} M_{b_2}$. A row and column of the obtained matrix M_b are indexed by a subset of L and R, respectively. For $U \subseteq \mathrm{mid}(b)$, an entry of M_b indexed by $U_L = U \cap L$ and $U_R = U \cap R$ has a value (7.1), since the multiplication between a row of M'_{b_1} indexed by U_L and a column of M_{b_2} indexed by U_R is same as the computation of (7.2). Therefore, finally, we can construct a valid array by collecting values from M_b.

It is shown in the same way as [13], that the matrix multiplication in the third step, the heaviest part, can be done in $O^*(2^{\frac{\omega}{2}bw(G)})$ time. Summarizing the above discussion, we obtain the following theorem.

Theorem 7.1. *Our algorithm computes Ising partition function in time* $O^*(2^{\frac{\omega}{2}bw(G)})$.

7.2 Algorithm Parameterized by Rank Width

7.2.1 Formulation of Ising Partition Function by Boolean Function

In this section, for a finite set V, we regard its subset $S \subseteq V$ as a vector $(S_v) \in \{0,1\}^V$ in the following way: $S_v = 1$ if $v \in V$, 0 otherwise. Let $G = (V, E)$ be a graph. For a subset $S \subseteq V$, $G(S)$ denotes a vertex-induced subgraph of G by S. $E(S)$ denotes an edge set of $G(S)$.

Consider a Boolean function f_G for a simple and connected graph $G = (V, E)$. An input to this function is a subset $S \subseteq V$, and an output is a parity of $|E(S)|$;

$$f_G(S) = \sum_{\{u,v\} \in E(S)} S_u S_v \quad (\text{over } GF(2)) \tag{7.3}$$

Here we introduce the following problem. Computing Ising partition function is reduced to this problem, which is explained soon later.

Problem 7.2. *For a given graph G with a vertex set V and two real-valued vectors $\boldsymbol{\omega}^{(0)}$, $\boldsymbol{\omega}^{(1)} \in \mathbb{R}^V$ on V, compute the following function:*

$$F(G, \boldsymbol{\omega}^{(0)}, \boldsymbol{\omega}^{(1)}) = \sum_{S \subseteq V} [(-1)^{f_G(S)} \prod_{v \in V} \omega_v^{(S_v)}] \tag{7.4}$$

Given Ising model on a graph $G = (V, E)$ with weights M_v for $v \in V$ and J_e for $e \in E$, computing Ising partition function can be formulated by this Boolean function as follows.

First, take a subdivided graph $\tilde{G} = (\tilde{V}, \tilde{E})$ by inserting a new vertex on each edge of G. Denote a set of newly-introduced vertices by $E' = \{e' \mid e \in E\}$. Then $\tilde{V} = V \cup E'$ and an $\tilde{E} = \{\{v, e'\} \mid e \in E \text{ is incident with } v \in V \text{ in } G\}$. Then, define on $\tilde{V}$ two parameters $\boldsymbol{\omega}^{(0)}$, $\boldsymbol{\omega}^{(1)} \in \mathbb{R}^{\tilde{V}}$ by

$$\begin{aligned} v \in V: &\qquad \omega_v^{(0)} = \exp(-\beta M_v), \qquad \omega_v^{(1)} = \exp(\beta M_v) \\ e \in E: &\qquad \omega_{e'}^{(0)} = \cosh(\beta J_e), \qquad \omega_{e'}^{(1)} = \sinh(\beta J_e) \end{aligned}$$

By the following theorem, Ising partition function on a graph G can be obtained through computing a function F on its subdivided graph $\tilde{G}$.

Theorem 7.3. $Z(G, \beta) = F(\tilde{G}, \tilde{\boldsymbol{\omega}}^{(0)}, \tilde{\boldsymbol{\omega}}^{(1)})$.

Proof. Note that, in this proof, S denotes a vector on $\{0,1\}$ and $\{\pm 1\}$.

$$
\begin{aligned}
F(\tilde{G}, \boldsymbol{\omega}^{(0)}, \boldsymbol{\omega}^{(1)}) &= \sum_{S \in \{0,1\}^{\tilde{V}}} \left[\left(\prod_{v \in V} \omega_v^{(S_v)} \right) \left(\prod_{e=\{u,w\} \in E \text{ in } G} (-1)^{(S_u+S_w)S_{e'}} \omega_{e'}^{(S_{e'})} \right) \right] \\
&= \sum_{S \in \{0,1\}^V} \left(\prod_{v \in V} \omega_v^{(S_v)} \right) \left(\prod_{e=\{u,w\} \in E} \left(\omega_{e'}^{(0)} + (-1)^{(S_u+S_w)} \omega_{e'}^{(1))} \right) \right) \\
&= \sum_{S \in \{\pm 1\}^V} \prod_{v \in V} \exp(\beta M_v S_v) \prod_{e \in E} (\cosh(\beta J_e) + S_u S_v \sinh(\beta J_e)) \\
&= \sum_{S \in \{\pm 1\}^V} \exp(\sum_{v \in V} \beta M_v S_v) \prod_{e \in E} \exp(\beta J_e S_u S_v) \;=\; Z(G, \beta)
\end{aligned}
$$

□

7.2.2 Algorithm by Dynamic Programming on Rank Decomposition

For a graph $G = (V, E)$ equipped with a rank decomposition $\delta = (D, \phi)$, we propose an algorithm to compute Problem 7.2 in $O^*(4^{rw(G,\delta)})$ time. The algorithm which is constructed here is published in [28]. This algorithm is based on dynamic programming on D. Hereafter leaves of D are identified with vertices of G under ϕ.

Here is one crucial observation. To state this, we need some notations. For disjoint subsets $S, T \subseteq V$, a set $E(S, T)$ consists of edges of G such that one end is in S and the other in T. Let A be an adjacency matrix of G. For a given bag b, we identify $w \in V_b$ with a corresponding row vector in A_b (see Section 2.5 about A_b). For $V' \subseteq V_b$, a vector $\boldsymbol{v}_b(V')$ is defined to be $\sum_{w \in V'} w$. $\boldsymbol{v}_b(\emptyset)$ is defined to be a zero vector. From a set V_b of row vectors, we take a set B_b of bases of A_b, i.e. maximal set of independent row vectors. Although B_b is not uniquely determined, it suffices to take one set of bases arbitrarily. Then the observation is as follows.

Lemma 7.4. *Let $V_1, V_2 \subseteq V_b$ such that $\boldsymbol{v}_b(V_1) = \boldsymbol{v}_b(V_2)$, and V' be a subset of $\overline{V_b}$. Then $|E(V_1, V')|$ and $|E(V_2, V')|$ have same parity. Furthermore, for every $V_3 \subseteq V_b$, $\boldsymbol{v}_b(V_3)$ is written as $\sum_{b \in B} b$ for some $B \subseteq B_b$.*

Proof. Let $V_1 \subseteq V_b$ and $\boldsymbol{v}_b(V_1) = (v_w)_{w \in \overline{V_b}}$. Then $v_w = 1$ if w is incident with an odd number of vertices in V_1, 0 otherwise. Therefore $|E(V_1, V_3)|$ is odd if $\sum_{w \in W'} v_w = 1$, even otherwise. The same argument is true for V_2, which has shown the first part.

The second part is obvious because B_b is a set of bases. □

Our algorithm first constructs a table DP for dynamic programming, and iteratively updates it from leaves. An entry of DP is specified by two indices b and B like DP$[b][B]$, where b is a bag and $B \subseteq B_b$. For a fixed bag b, a set DP$[b]$ of all entries DP$[b][B]$ is called *an array.* For a bag b and $B \subseteq B_b$, we call an entry DP$[b][B]$ *valid* if it has the following value:

$$
\sum_{S \subseteq V_b \text{ s.t. } \boldsymbol{v}_b(S) = \boldsymbol{v}_b(B)} [(-1)^{f_{G(V_b)}(S)} \prod_{w \in V_b} \omega_w^{(S_w)}] \tag{7.5}
$$

If all entries of an array DP$[b]$ is valid, the array DP$[b]$ is called *valid.* For a root b_r, we say that an array DP$[b_r]$ is valid if it retains the value (7.4).

Our algorithm constructs valid arrays and entries, updating a table from leaves to the root. First, a valid array DP$[l]$ is constructed for each leave l as follows; DP$[l][\emptyset] = \omega_l^{(0)}$ and DP$[l][\{l\}] = \omega_l^{(1)}$. For an inner bag b of D, assume that a valid array DP$[b_i]$ has been already constructed for each child b_i of b ($i = 1, 2$). Then a valid array for b can be constructed by the following Algorithm 1.

Algorithm 1

1: Construct TMP$[b][B_1][B_2]$ for each $B_1 \subseteq B_{b_1}$ and $B_2 \subseteq B_{b_2}$
2: **for** each $B_1 \subseteq B_{b_1}$ and $B_2 \subseteq B_{b_2}$ **do**
3:    **if** $\boldsymbol{v}_{b_1}(B_1)A(V_{b_1}, V_{b_2})\boldsymbol{v}_{b_2}(B_2)^{\mathrm{T}}$ over $GF(2)$ equals to 1 **then**
4:       TMP$[b][B_1][B_2] = (-1)$ DP$[b_1][B_1]$ DP$[b_2][B_2]$
5:    **else**
6:       TMP$[b][B_1][B_2] =$ DP$[b_1][B_1]$ DP$[b_2][B_2]$
7: Take a set B_b of bases such that $B_b \subseteq B_{b_1} \cup B_{b_2}$
8: **for** each $B_1 \subseteq B_{b_1}$ and $B_2 \subseteq B_{b_2}$ **do**
9:    DP$[b][B] =$ DP$[b][B] +$ TMP$[b][B_1][B_2]$ where $B \subseteq B_b$ s.t. $\boldsymbol{v}_b(B) = \boldsymbol{v}_b(B_1 \cup B_2)$

Lemma 7.5. *For each bag b and $B \subseteq B_b$, an array DP$[b][B]$ outputted by Algorithm 1 is valid.*

Proof. First see line 7 in Algorithm 1. Note that a set $B_{b_1} \cup B_{b_2}$ of row vectors of A_b contains a set of bases. This is because the row space of A_b is generated by a following set of vectors: restrictions on $\overline{V_b}$ of row vectors $v_1 \in B_{b_1}$ of A_{b_1} and of row vectors $v_2 \in B_{b_2}$ of A_{b_2}. Therefore the operation in line 7 can be done correctly.

For notational convenience, we adopt the following abbreviation: $g_b(S) = f_{G(V_b)}(S) \prod_{w \in V_b} \omega_w^{(S_w)}$. By assumption, DP$[b_1][B_1]$ and DP$[b_2][B_2]$ are valid for each $B_1 \subseteq B_{b_1}$ and $B_2 \subseteq B_{b_2}$. Therefore, DP$[b_i][B_i]$ has a sum of $g_{b_i}(S)$ over $S \subseteq V_i$ ($i = 1, 2$) such that $\boldsymbol{v}_{b_i}(S) = \boldsymbol{v}_{b_i}(B_i)$. By Lemma 7.4, line 2-6 in Algorithm 1 assign TMP$[b][B_1][B_2]$ with a sum of $g_b(S)$ over $S \subseteq V_b$ satisfying the following condition: $\boldsymbol{v}_{b_1}(S \cap V_{b_1}) = \boldsymbol{v}_{b_1}(B_1)$ and $\boldsymbol{v}_{b_2}(S \cap V_{b_2}) = \boldsymbol{v}_{b_2}(B_2)$. For any $S \subseteq V_b$ satisfying the previous condition, $\boldsymbol{v}_b(S)$ is a same vector, since $\boldsymbol{v}_{b_1}(S \cap V_{b_1})$ and $\boldsymbol{v}_{b_2}(S \cap V_{b_2})$ takes same value on each entry $w \in \overline{V_b}$ Finally, for DP$[b][B]$, we gather DP$[b][B_1][B_2]$ such that $\boldsymbol{v}_b(B) = \boldsymbol{v}_b(B_1 \cup B_2)$ in line 8-9. $\square$

The iteration of Algorithm 1 is $|V|$ time. The heaviest part is line 2-6. This iteration is $2^{|B_1|} \cdot 2^{|B_2|} = O(4^{rw(G,\delta)})$, since $|B_i|$ is at most the width of given rank decomposition. Summarizing the above, we obtain the following theorem.

Theorem 7.6. *For a graph G equipped with rank decomposition δ, our algorithm solves Problem 7.2 in $O^*(4^{rw(G,\delta)})$ time.*

7.2.3 Computing Ising Partition Function for Square Grids

Summarizing the discussion so far, we obtain an algorithm to compute Ising partition function. Given Ising model on $G = (V, E)$ with parameters M_v for $v \in V$ and J_e for $e \in E$, the algorithm consists of three parts.

1. Construct a subdivided graph $\tilde{G}$ with parameter $\tilde{\boldsymbol{\omega}}^{(0)}$ and $\tilde{\boldsymbol{\omega}}^{(1)}$ on $\tilde{V}$.

2. Obtain a rank decomposition δ of $\tilde{G}$.

3. On the decomposition, compute $F(\tilde{G}, \tilde{\boldsymbol{\omega}}^{(0)}, \tilde{\boldsymbol{\omega}}^{(1)})$.

Let t_2 be the number of steps for the 2nd part. Then this algorithm compute Ising partition function in $O(t_2 \ + \ (|V|+|E|)4^{rw(\tilde{G},\delta)})$ time.

Here we investigate further the time complexity for Ising model on a $k \times l$ grid. Let $G_{k,l}$ be a 2-dimensional $k \times l$ grid ($k \leq l$), i.e. a grid with k rows and l columns. Then we consider a special type of the rank decomposition $\delta_{k,l} = (D_{k,l}, \phi)$ for $\tilde{G_{k,l}}$ such that the decomposition tree $D_{k,l}$ is a caterpillar graph and its root bag is placed on the edge of its central path of the caterpillar graph. Then an order on leaves in the decomposition tree $D_{k,l}$ is naturally induced by ordering leaves along its central path from one side to the other side. Then we order vertices of a square grid by row-major order. Finally set the bijection ϕ such that the i-th vertex in the row-major order is mapped into the i-th leaf in the order induced along the central path of the caterpillar decomposition tree. Then you can confirm by elementary linear algebra that this rank decomposition $\delta_{k,l}$ has width k.

Furthermore, using this rank decomposition $\delta_{k,l}$, each iteration of Algorithm 1 is done in $O(2^k)$ time, since one of two arrays has always 2 entries, a constant. Together with these observations, the third step is done in $O(kl2^k)$ time. Hence the following theorem is derived.

Theorem 7.7. *Our algorithm computes Ising partition function for a $k \times l$ grid ($k \leq l$) in $O(kl2^k)$ time.*

Applying this into the square grid with n vertices, the algorithm to compute the Ising model with n vertices in $O(n2^{\sqrt{n}})$ time is obtained.

7.3 Algorithm by Binary Decision Diagram

7.3.1 Graph States and Binary Decision Diagram

In this chapter, we construct the algorithm using Binary Decision Diagram (BDD). This algorithm is obtained by classically simulating the quantum algorithm in [76] with BDD. The results which are presented here are published in [27].

Now we introduce some notations. Consider an undirected graph $G = (V, E)$ with vertex set V and edge set E. We assume it is simple and connected. For $S \subseteq V$, define $E(S)$ to be a set of edges whose two vertices are both in S (i.e., the edge set of a subgraph induced by S).

For the vertex set $V = \{v_1, v_2, \ldots .v_n\}$ ($|V| = n$), consider a n-qubit quantum state $|x_1x_2\ldots x_n\rangle \in \mathbf{C}^{2^n}$ ($x_i \in \{0, 1\}$). For $S \subseteq V$, its characteristic vector is defined as $\chi_S^V = (x_1, x_2, \ldots, x_n)$ with $S = \{i \mid x_i = 1\}$. For a gigen graph G, define a Boolean function $f_G(x_1, \ldots, x_n)$ for a graph G by

$$f_G(\chi_S^V) = \bigoplus_{e=(v_i,v_j)\in E} (x_i \wedge x_j). \tag{7.6}$$

This boolean function f_G maps a subset $S \subset V$, to be precise its characteristic vector, into the parity of the edge number of a subgraph induced by S.

Then, a graph state $|G\rangle$ for graph G is defined as follows.

$$|G\rangle = \frac{1}{2^{|V|/2}} \sum_{S\subseteq V} (-1)^{f_G(\chi_S^V)} |\chi_S^V\rangle \tag{7.7}$$

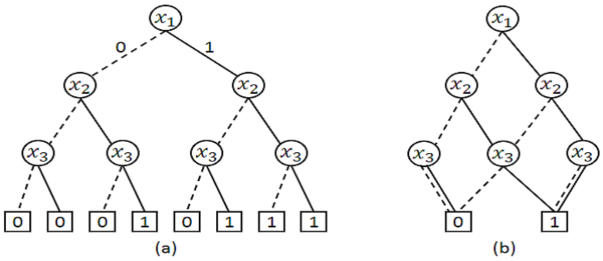

Figure 7.1: For a graph state $|K_3\rangle = (|000\rangle + |001\rangle + |010\rangle - |011\rangle + |100\rangle - |101\rangle - |110\rangle - |111\rangle)$ of a complete graph K_3, (a) a complete binary tree represent it as a Boolean function, and (b) its QOBDD.

From now on we do without normalization factors to make the norm of state to 1, and then we have only to consider signs $\pm$ of computational bases $|x_1x_2\dots x_n\rangle$ ($x_i \in \{0,1\}$). Under the above observation, quantum graph states are indentified with boolean functions behind the underlying graph.

A binary decision diagram (BDD) represents a Boolean function in a compact and operational manner. See [9]. Here we explain it through an example. In Fig.7.1(a), for an ordering of x_1, x_2, x_3, this labeled complete binary tree represents the sign $+$ and $-$ of $|K_3\rangle$ by 0 (false) and 1 (true), respectively. By unifying two nodes in the tree such that trees rooted at each of them are isomorphic, including labels, we derive a BDD, slightly smaller the complete binary tree in this case. A BDD obtained by performing this unifying operation as much as possible is called a quasi-reduced ordered binary decision (QOBDD). A QOBDD is a layered diagram such that i-th level corresponds to x_i for the variable ordering. The width of a level is the number of nodes in the level, and the width of QOBDD (we below write simply as BDD) is the maximum among widths of levels. In Fig.7.1, the BDD width is 3.

7.3.2 BDD of a graph state

We investigate the properties of BDD indentified with quantum graph states by linear rank width.

Lemma 7.8. *For a graph state $|G\rangle$ with its BDD, (i) given a basis $|x_1x_2\dots x_n\rangle$, its sign in $|G\rangle$ can be answered in $O(|V|)$ time, and (ii) given a complete product quantum state $|\alpha\rangle \in (\mathbf{C}^2)^{\otimes n}$, the inner product $\langle\alpha|G\rangle$ can be computed in size proportional to the size of the BDD.*

The former is obvious; given a BDD representation, just trace the BDD from its root to end node. The latter is exactly a necessary operation of computing the partition function of Ising model via a type of graph state [77]. Its proof can be obtained in the same way as Lemma 7.15.

We are interested in investigating BDDs, specifically a QOBDD, of f_G. Let us first consider a variable order $x_1, \dots, x_n$ in BDDs. For $S \subseteq V$, we denote $V - S$ by $\overline{S}$. For $i = 1, \dots, n-1$, define a subset S_i of V to be $\{v_1, v_2, \dots, v_i\}$. For $T \subseteq S_i$, define a Boolean function $f_T^i = f_T^i(x_{i+1}, \dots, x_n)$ of variables $x_{i+1}, \dots, x_n$ by

$$f_T^i(x_{i+1}, \dots, x_n) = f_G(\chi_T^{S_i}, x_{i+1}, \dots, x_n)$$

For a vertex $v \in \overline{S_i}$, define $\delta^i(v)$ to be $\{u \mid u \in S_i,\ (u,v) \in E\}$. For $T \subseteq S$, define $\Gamma_2(T)$ by

$$\Gamma_2(T) = \{v \mid v \in \overline{S_i},\ |T \cap \delta^i(v)|\colon \text{odd}\}$$

Lemma 7.9. *For $i \in \{1, \ldots, n-1\}$ and $T, T' \subseteq S_i$, we have $f^i_T = f^i_{T'}$ iff $(-1)^{|E(T)|} = (-1)^{|E(T')|}$ and $\Gamma_2(T) = \Gamma_2(T')$.*

Proof. First we show the necessary condition. Assume that $f^i_T = f^i_{T'}$. Since $f^i_T(\chi^{\overline{S_i}}_\emptyset) = f^i_{T'}(\chi^{\overline{S_i}}_\emptyset)$, $|E(T)| = |E(T')|$. Then take $\forall u \in \overline{S_i}$. If u is contained in exactly one of $\Gamma_2(T)$ and $\Gamma_2(T')$, then $f^i_T(\chi^{\overline{S_i}}_{\{u\}}) \neq f^i_{T'}(\chi^{\overline{S_i}}_{\{u\}})$. Therefore u is always contained in both $\Gamma_2(T)$ and $\Gamma_2(T')$, or none of $\Gamma_2(T)$ and $\Gamma_2(T')$. In result, the equation $\Gamma_2(T) = \Gamma_2(T')$ holds.

The remaining is the sufficient condition. Assume that $f^i_T \neq f^i_{T'}$ and $(-1)^{|E(T)|} = (-1)^{|E(T')|}$. Then we show that $\Gamma_2(T) \neq \Gamma_2(T')$. Take $U \subset \overline{S_i}$ such that $f^i_T(\chi^{\overline{S_i}}_U) \neq f^i_{T'}(\chi^{\overline{S_i}}_U)$. Since $(-1)^{|E(T)|} = (-1)^{|E(T')|}$, the parities of the numbers of edges between U and T, and between U and T' are different. Therefore there exists $u \in U$ which is contained in exactly one of $\Gamma_2(T)$ and $\Gamma_2(T')$, This means $\Gamma_2(T) \neq \Gamma_2(T')$, which completes the proof. □

The following two lemmas are obvious from Lemma 7.9.

Lemma 7.10. *The number of nonequivalent Boolean functions among f^i_T $(T \subseteq S_i)$ is bounded by $2w_i$ where $w_i = |\{\Gamma_2(T) \mid T \subseteq S_i\}|$.*

Lemma 7.11. *For the vertex order, the width of QOBDD of the Boolean function f_G is bounded by $2\max\{w_i \mid i = 1, \ldots, n-1\}$.*

Same as Section 7.3.1, we denote by $A(S, \overline{S})$ the adjacency matrix of a bipartite subgraph of G induced by a vertex partision $(S, \overline{S})$ and a edge set between S and $\overline{S}$. Then, linear algebra over $GF(2)$ gives the following.

Lemma 7.12. $|\{\Gamma_2(T) \mid T \subseteq S\}| \leq 2^{\mathrm{rank}(A(S,\overline{S}))}$.

This has connection with discussions about defining and analyzing the rank-width and linear rank-width of a graph G. A linear rank-decomposition of G naturally determines a permutation σ on V, and vice versa. Hereafter we redefine a linear rank decomposition and linear rank width in a term of permutation. For any permutation σ on V, consider a vertex oder $v_{\sigma(1)}, v_{\sigma(2)}, \ldots, v_{\sigma(n)}$ and $S'_i = \{v_{\sigma(1)}, v_{\sigma(2)} \ldots, v_{\sigma(i)}\}$, the linear rank-width of graph G with respect to the permutation σ is defined to be

$$lrw(G, \sigma) = \max\{\mathrm{rank}(A(S'_i, \overline{S'_i}) \mid i = 1, \ldots, n-1\}$$

and the linear rank-width G is defined to be

$$lrw(G) = \min\{lrw(G, \sigma) \mid \forall \text{permutation } \sigma\}$$

N.B. $rw(G) \leq lrw(G) \leq pw(G) = O(tw(G) \log n)$. Then, combining the above lemmas and discussion, we have the following theorem.

Theorem 7.13 ([27])**.** *For a graph G, there is a QOBDD representing $|G\rangle$ with width at most $2^{lrw(G)+1}$.*

Finding an optimal permutation σ' attaining $lrw(G)$ in polynomial time of the size of an input graph is left open, as mentioned at the end of introduction. Supposing a permutation σ is given, define

$$w' = \max_i |\{v \mid v \in \overline{S'_i},\ \delta^i(v) \neq \emptyset\}|.$$

Then we obtain the following theorem.

Theorem 7.14 ([27]). *Supposing a graph G and σ are given, the BDD representing $|G\rangle$ can be constructed in $O(w'|V|2^{lrw(G,\sigma)})$ time.*

Proof. For convenience, we call these edges as 0-edge or 1-edge depending on the values the edges retain. In Figure 7.1, the broken edges and the solid edges correspond to 0-edges and 1-edges respectively.

We show the algorithm to construct the BDD representing $|G\rangle$ in Algorithm 2. This algorithm summarizes the discussion in Section 7.3.2. The BDD constructed by this algorithm has no isomorphic nodes and is therefore not reducible, since it excludes all isomorphic nodes in each level (line 24 to 28) by the necessary and sufficient condition for nodes to be isomorphic in Lemma 7.9. Hence the constructed BDD is QOBDD. Recall that QOBDD is uniquely determined by a variable ordering σ. Furthremore, the BDD width of QOBDD on a given variable ordering, i.e. permutation σ, is at most $O(2^{lrw(G,\sigma)})$ by Lemma 7.10, 7.11 and 7.12. Therefore the BDD width of the (QO)BDD constructed by our algorithm is at most $lrw(G, \sigma)$.

Here we investigate the running time of this algorithm. There are three loops which are nested and mainly affects the time complexity of this algorithm: the first loop from line 7 to line 34, the second loop from line 9 to line 34, and the third loop from line 18 to line 21. The number of iteration of the first loop is $|V|$ time. The number of iteration of the second loop in each level is the width of the level of the BDD constructed by this algorithm. Therefore the number is at most $O(2^{lrw(G,\sigma)})$ times by the former discussion of this proof. The number of iteration of the third loops is at most w' times. Note that w' concides with the max value $|\Gamma^i|$ in Algorithm 2 among i. It is moderate to assume that each operation in this algorithm can be done in a constant time under some appropriate computational setting.

In result, this algorithm constructs the (QO)BDD representing $|G\rangle$ in $O(w'|V|2^{lrw(G,\sigma)})$ time.

□

Similar arguments hold for a generalized graph state $|\varphi_{\tilde{G}}\rangle$ in [77] for the Ising partition function.

7.3.3 Computation of Ising partition function via BDD

Now we present an algorithm to compute $F(\tilde{G}, \tilde{\boldsymbol{\omega}}^{(0)}, \tilde{\boldsymbol{\omega}}^{(1)})$. Our algorithm consists of four steps. For a given graph $G = (V, E)$ and weights M_v, J_e, the first step constructs a decorated graph $\tilde{G} = (\tilde{V}, \tilde{E})$ and weighted vectors $\tilde{\boldsymbol{\omega}}^{(0)}, \tilde{\boldsymbol{\omega}}^{(1)}$. The second step obtains a linear rank decomposition of $\tilde{G}$. The third step constructs BDD using the obtained linear rank decomposition. The final step computes $F(\tilde{G}, \tilde{\boldsymbol{\omega}}^{(0)}, \tilde{\boldsymbol{\omega}}^{(1)})$ on this BDD. The running time of the final step is given in the following lemma.

Algorithm 2

Input: A graph $G = (V, E)$ where $V = \{v_1, v_2, ..., v_n\}$, and a permutation σ on n

1: Rename each vertex as follows: $v_i := v_{\sigma(i)}$
2: /* Each L_i is a set of nodes in i-th level. */
3: $L_1 := \cup\{(1, \emptyset, \emptyset)\}$ and $L_{n+1} := \{0, 1\}$
4: **for** $i = 2$ to n **do**
5: $L_i = \emptyset$
6: /* In the following loop, nodes in (i)-level generate nodes in $(i+1)$-th level. */
7: **for** $i = 1$ to n **do**
8: /* In the following loop, each node in i-th level generates two nodes (line 22). If the generated nodes are not isomorphic to any node already in $(i+1)$-th level, then they are added to $(i+1)$-th level. */
9: **for** each $N = (l, \Gamma, T) \in L_i$ **do**
10: $T_0 := T$ and $T_1 := T \cup \{v_i\}$
11: /* l_j and Γ_j corresponds to the first and second conditions in Lemma 7.9. These are used to check isomorphism between nodes in line 24-28. */
12: $l_0 := (1 + (-1)^{|E(T_0)|})/2$ and $l_1 := (1 + (-1)^{|E(T_1)|})/2$
13: $\Gamma_0 := \emptyset$ and $\Gamma_1 := \emptyset$
14: **if** $i \neq 1$ **then**
15: /* The following steps computes the set of the second conditions in Lemma 7.9. For this purpose, it suffices to check the vertices in $\overline{S_i'} = \{v_{i+1}, v_{i+2}, ..., v_n\}$ which are incident with some vertex in $S_i' = \{v_1, v_2, ..., v_i\}$, i.e. Γ^i. */
16: $\overline{S_i'} := \{v_{i+1}, v_{i+2}, ..., v_n\}$
17: $\Gamma^i := \{v \mid v \in \overline{S_i'}, \delta^i(v) \neq \emptyset\}$
18: **for** each $v \in \Gamma^i$ **do**
19: **for** $j = 0$ to 1 **do**
20: **if** $T_j \cap \delta^i(v)$ is odd **then**
21: $\Gamma_j := \Gamma_j \cup \{v\}$
22: Generate two nodes $N_0 = (l_0, \Gamma_0, T_0)$ and $N_0 = (l_1, \Gamma_1, T_1)$
23: /* The following step checks isomorphism between the generated nodes and nodes already in $(i+1)$-level by Lemma 7.9. */
24: **for** $j = 0$ to 1 **do**
25: **if** There exists a node $N' = (l', \Gamma', T')$ in L_{i+1} such that $l' = l_j$ and $\Gamma' = \Gamma_j$ **then**
26: Connect N and N' by a j-edge
27: **else**
28: Connect N and N_j by a j-edge, and $L_{i+1} := L_{i+1} \cup \{N_j\}$
29: **else**
30: **for** $j = 0$ to 1 **do**
31: **if** $l_j = 0$ **then**
32: Connect N and $\{0\} \in L_{i+1}$ by a j-edge
33: **else**
34: Connect N and $\{1\} \in L_{i+1}$ by a j-edge

Lemma 7.15. *Let $G = (V, E)$ be a graph, and $\boldsymbol{\omega}^0, \boldsymbol{\omega}^1 \in \mathbf{R}^V$. Assume that BDD representing f_G whose width is at most k in each level is given. Then $F(G, \boldsymbol{\omega}^{(0)}, \boldsymbol{\omega}^{(1)})$ can be computed on the BDD in $O(k|V|)$ time.*

Proof. For convenience, we fix some notations. We name the vertices as $V = \{v_1, v_2, ..., v_n\}$ $(n = |V|)$ such that i-th level in BDD corresponds to v_i. Note that 1-st level has a root node of BDD, and $(n+1)$-th level has two nodes N_- and N_+ corresponding to the boolean value 0 and 1. We call the edges in BDD as 0-edge or 1-edge in the same way as the proof in Theorem 7.14.

First we assign values to each nodes of BDD. The node in 1-st level is assigned with 1, and other nodes are assigned with 0. We denote the value which a node N of BDD retains by a_N. Then we iteratively do the following procedure from 1-st level to $(n-1)$-th level.

For each node N in i-th level, N has two incident nodes in $(i+1)$-th level. Denote the nodes incident with 0-edge and 1-edge by N_0 and N_1 respectively. Then Update a_{N_j} to be $a_{N_j} + a_N \omega_{v_i}^{(j)}$ $(j = 0, 1)$.

After that, for each node N in n-th level, update a_{N_-} and a_{N_+} to be $a_{N_-} - a_N \omega_{v_n}^{(0)}$ and $a_{N_+} + a_N \omega_{v_n}^{(1)}$ respectively.

After the above all procedures, $F(G, \boldsymbol{\omega}^{(0)}, \boldsymbol{\omega}^{(1)}) = a_{N_-} + a_{N_+}$. In each level, the operations are done at most $2k$ times. Therefore, the total runnning time is $O(k|V|)$. □

Here we analyze the time complexity of our algorithm for a square-grid Ising model, and show that the exponential part of the running time is quite low with respect to linear rank width.

Let G be an $k \times l$ square grid where $k \leq l$. In the first step, the decorated graph $\tilde{G}$ can be constructed in $O(kl)$ time. In the second step, the column-major order on vertices of the decorated square grid naturally gives a linear rank decomposition of width k. This linear rank decomposition, combining with [50] and [38], gives the following bound of linear rank width of the decorated square grid.

Lemma 7.16. *Let G be a $k \times l$ square grid where $k \leq l$.*
Then, $k - 1 \leq lrw(G) \leq lrw(\tilde{G}) \leq k$.

Proof. By Proposition 2.6 and Corollary 2.7 in [50], if a graph H with vertex set V is a vertex minor of a graph H', then, for any $S \subset V$, the cutrank of S in a graph H is at most the cutrank of S in a graph H'. Therefore, $lrw(H) \leq lrw(H')$.

G is a vertex minor of $\tilde{G}$, and a $k \times k$ square grid G' is a vertex minor of G. Furthremore, the rank width of $k \times k$ square grid is $k - 1$ [38]. By definitions, the rank width of a graph does not exceed the linear rank width of the graph. Hence, $k - 1 = rw(G') \leq lrw(G') \leq lrw(G) \leq lrw(\tilde{G})$.

Furthremore, the column-major ordering of vertices of $\tilde{G}$ gives a linear rank decomposition of width k. Hence we obtain $k - 1 \leq lrw(G) \leq lrw(\tilde{G}) \leq k$. □

Therefore, we obtain a linear rank decomposition of $\tilde{G}$ with width at most $lrw(G) + 1$ by simply taking the vertices of $\tilde{G}$ in a column-major order. This can be done in $O(kl)$ time. Using this linear rank decomposition of width at most $lrw(G) + 1$, the third step can construct BDD in $O(k^2 l\, 2^{lrw(G)})$ time by Theorem 7.14 and Lemma 7.16. This BDD has at most $O(2^{lrw(G)})$ width in each level. Therefore the final step can compute $F(\tilde{G}, \tilde{\boldsymbol{\omega}}^{(0)}, \tilde{\boldsymbol{\omega}}^{(1)})$ in $O(kl\, 2^{lrw(G)})$ time by Lemma 7.15.

The heaviest step of our algorithm is the third step. Here we note that big O-star notation O^* represents the time complexity ignoring a polynomial factor of input size from big O notation. Then, the running time of our algorithm is summarized in the following theorem, which corresponds to Theorem 3 in the extended abstract submitted to AQIS2014.

Theorem 7.17 (Theorem 3 in the extended abstract)**.** *Our BDD-based algorithm computes Ising partition function for an Ising model on an $k \times l$ square grid G ($k \leq l$) in $O(k^2 l 2^{lrw(G)}) = O(k^2 l 2^k)$ time, and simultaneously in $O^*(2^{lrw(G)}) = O^*(2^k)$ time.*

Chapter 8

Concluding Remarks

The existing researches on excluded minors of matroids have mainly focused on the matters related to the representability over finite fields. Among them, the Rota's conjecture 1.2, which states that the representability over a finite field is always chacterized by a finite number of excluded minors, has been the most central topic. Compared to the representability over finite fields, there exists quite a small number of literatures on the excluded minors' characterization for the classes which have infinitely many excluded minors, e.g. the class of orientable matroids and the classes of matroids representable over infinite fields. Now that the Rota's conjecture, the 45-year-old conjecture, was announced to be settled and the milestone result on the representability over finite fields was established, it is time to focus closer attention on the other fundamental classes of matroids.

Among the fundamental classes of matroids except the classes of matroids representable over finite fields, the classes of matroids representable over infinite fields and the class of orientable matroids deserve to receive more attention. Regarding the classes of matroids representable over infinite fields, the vector spaces and projective spaces over infinite fields such as the real field and the complex field are among the most natural and fundamental settings of geometry. Furthermore, as mentioned before, the problem of characterizing the representability over the real field is the main motivation in the paper [84] initiating the matroid theory. Regarding the class of orientable matroids, it is introduced to give more detailed description of the combinatorial structures of the vector space or projective over the ordered fields such as the real field. Therefore it is natural to consider the orientability in the context of the representability over infinite fields.

In this thesis, we closely investigated the excluded minors for the class of orientable matroids and for the classes of matroid representable over infinite fields.

In Chapter 3, the excluded minors of rank 3 for the class of orientable matroids are investigated. In the existing researches, there are four infinite families of excluded minors for the class of orientable matroids; the two families of them consists of excluded minors of rank r with $2r$ elements for $r \geq 4$, and the other two families consist of excluded minors of rank 3. Regarding the excluded minors of rank 3, it had not been known whether there exists an excluded minor of rank 3 with m elements for every value of m. We have shown that there exists at least one excluded minor of rank 3 with m elements for every $m \geq 7$. It is known that the class of orientable matroids is closed under taking a dual. Therefore the dual of an excluded minor for the class of orientable matroids is also an excluded minor for the class of orientable matroids. By taking a dual of the excluded minors of rank 3 with $m+3$ elements, we have obtained at least one excluded minors of rank m with $m+3$ elements for every $m \geq 4$. The well-known fact that every rank 2 matroid is orientable implies the matroid of rank m with at most $m+2$ elements is orientable. In the result, for each rank, at least one example of a minimal non-orientable matroid smallest with respect to an number of element is obtained. Then a relevant question is as follows:

Question 8.1. *For each rank $r \geq 4$ and $m \geq r+4$, does at least one minimal non-orientable matroid with m elements exist?*

In Chapter 5 and Chapter 6, for the minor-closed class $\mathcal{C}_1$ of matroids which have infinitely many excluded minors, we consider the other class $\mathcal{C}_2$ and investigate the excluded minors' characterization for the class $\mathcal{C}_1$ within the class $\mathcal{C}_2$. In Chapter 5, the classes $\mathcal{C}_1$ and $\mathcal{C}_2$ are the class of matroids representable over fields of characteristic 0 and the class of orientable matroids respectively. In Chapter 5, the classes $\mathcal{C}_1$ and $\mathcal{C}_2$ are the class of $\mathbb{Q}$-representable matroids and the class of $\mathbb{Q}[x]$-representable matroids respectively. In both chapters, it is shown that the characterization of $\mathcal{C}_1$ within $\mathcal{C}_2$ by a finite list of excluded minors is impossible. As the future work, we would list the following question.

Question 8.2. *Let $\mathcal{C}_1$ be a minor-closed class of matroids with infinitely many excluded minors, e.g. the class of orientable matroids and the classes of matroids representable over infinite fields. Then look for the class $\mathcal{C}_2$ within which the characterization for $\mathcal{C}_1$ by a finite list of excluded minors is possible.*

For example, the class of $GF(2)$-representable matroids is one of the answers of the above question. Let $\mathcal{C}_2$ be the class of $GF(2)$-representable matroids. Then, consider $\mathcal{C}_1$ to be either the class of orientable matroids or the class of matroids representable over a given infinite field of characteristic other than 2. Then the matroids in the class $\mathcal{C}_1$ within $\mathcal{C}_2$ are regular matroids. Hence the class $\mathcal{C}_1$ are characterized by a finite list of excluded minors: the Fano matroid and its dual. Note that, if $\mathcal{C}_1$ is the class of matroids representable over an infinite field of characteristic 2, $\mathcal{C}_1$ contains $\mathcal{C}_2$. Therefore there is nothing to be considered.

Now we list as the candidate of the class $\mathcal{C}_2$ in Question 8.2 the class of matroids of branch width at most r for a fixed constant r. The branch decomposition and branch width of matroids were introduced in [11, 12] as an extension of the branch width of graphs. Regarding the branch width of matroids, the following result is known.

Theorem 8.3 ([12]). *The class of matroids whose branch width is at most r for a fixed constant r is minor-closed.*

Theorem 8.4 ([12]). *The class of matroids whose branch width is at most 2 is characterized by just two excluded minors: $U_{2,4}$ and $M(K_4)$.*

The matroid $M(K_4)$ is a graphic matroid obtained from the 4-vertex complete graph K_4. The class of $GF(2)$-representable matroid is characterized by just one excluded minor: $U_{2,4}$. Therefore the class of matroids whose branch width is at most 2 is contained in the class of $GF(2)$-representable matroid. Therefore the class of matroids whose branch width is at most 2 is one of the answer to Question 8.2. Now our question relevant to Question 8.2 is as follows.

Question 8.5. *Let r be a fixed integer greater than 2. Does the class of matroids whose branch width is at most r work as the answer to Question 8.2?*

The branch decomposition of graphs plays an important role in designing efficient algorithms on graphs. Also for the case of matroids, exploiting the branch decomposition of matroids is one of the promising way to develop efficient algorithms on matroids. Therefore, investigating the representability in a context of branch width and branch decomposition may be of an fundamental importance in the fields of algorithm design. In particular, the research on Question 8.5 may be utilized to to generalize the algorithms to compute Ising partition function on graphs developed in Chapter 7 to the algorithms to compute Ising partition function on some classes of matroids which is wider than the class of graphic matroids.

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