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# On the Polynomiality of the Multiplicative Penalty Function Method for Linear Programming and Related Inscribed Ellipsoids

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**SUMMARY** The paper proves the polynomiality of the multiplicative penalty function method for linear programming proposed by Iri and Imai(2). This is accomplished by considering ellipsoids determined by the Hessian at an interior point and centered at the point, and showing that, for any interior point, there is such an ellipsoid contained in the feasible region in which the penalty function is well approximated by a linear function determined by the gradient at the point.

## 1. Introduction

Linear programming is a general optimization method, including many discrete optimization methods as special cases. Recently, a nonlinear approach has been shown to be efficient for large-scale linear programming problems, and the so-called interior method is now recognized as a powerful method for linear programming.

The multiplicative penalty function method, proposed by Iri and Imai(2), is an interior method which minimizes the convex multiplicative penalty function defined for a given linear programming problem with inequality constraints by the Newton method. In Ref.( 2 ), the local quadratic convergence of the method was shown, while the global convergence property was left open. Zhang and Shi* proved the global linear convergence of the method under an assumption that line search can be performed rigorously.

This paper shows, in a compact way, how well the multiplicative penalty function at any interior point may be approximated by a linear function defined by the gradient on an ellipsoid determined by the Hessian at the point. This result, combined with a proposition in Ref.( 2 ), implies that the number of main iterations in the multiplicative penalty function method is $O(m^2L)$, thus showing the polynomiality of the method. Iri(1) further investigated the method, especially properties of the Newton direction, with changing the penalty parameter, and gives a better bound. However, the result of this paper may be still of theoretical interest, since it provides a large ellipsoid contained in the feasible region, and gives a good estimate of the function along any direction, not restricted to the Newton direction.

## 2. Preliminaries

We consider the following linear programming problem :

$$\min \boldsymbol{c}^T\boldsymbol{x}$$

$$\text{s.t. } A\boldsymbol{x} \geqq \boldsymbol{b}$$

where $\boldsymbol{c}, \boldsymbol{x} \in \boldsymbol{R}^n$, $\boldsymbol{b} \in \boldsymbol{R}^m$ and $A \in \boldsymbol{R}^{m\times n}$. In the sequel, we assume the following (cf. Ref.( 2 )) :

( 1 ) The feasible region $X=\{\boldsymbol{x} \mid A\boldsymbol{x} \geqq \boldsymbol{b}\}$ is bounded.
( 2 ) The interior Int $X$ of the feasible region $X$ is not empty.
( 3 ) The minimum value of $\boldsymbol{c}^T\boldsymbol{x}$ is zero.

Consider the multiplicative penalty function for this linear programming problem :

$$F(\boldsymbol{x})=(\boldsymbol{c}^T\boldsymbol{x})^{m+1}/\prod_{i=1}^{m}(\boldsymbol{a}_i^T\boldsymbol{x}-b_i) \quad (\boldsymbol{x} \in \text{Int } X)$$

where $\boldsymbol{a}_i^T \in \boldsymbol{R}^n$ is the $i$-th row vector of $A$. This function is introduced in Ref.( 2 ).Under these assumptions, when $F(\boldsymbol{x}) \to 0$, the distance between $\boldsymbol{x}$ and the set of optimum solutions converges to zero. The multiplicative penalty function method directly minimizes the penalty function $F(\boldsymbol{x})$ by the Newton method, starting from some initial interior point.

Define $\boldsymbol{\eta}=\boldsymbol{\eta}(\boldsymbol{x})$ and $H=H(x)$ for $\boldsymbol{x} \in \text{Int } X$ by

$$\boldsymbol{\eta}\left(\boldsymbol{x} \equiv \frac{\nabla F(x)}{F(x)} = \nabla(\log F(\boldsymbol{x})\right)$$

$$=(m+1)\frac{\boldsymbol{c}}{\boldsymbol{c}^T\boldsymbol{x}} - \sum_{i=1}^{m}\frac{a_i}{\boldsymbol{a}_i^T\boldsymbol{x}-b_i}$$

$$B(\boldsymbol{x}) \equiv \nabla^2(\log F(\boldsymbol{x})) = \nabla\boldsymbol{\eta}(\boldsymbol{x})$$

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* Zhang S. and Shi M. : "On Polynomial Property of Iri-Imai's New Algorithm for Linear Programming", Manuscript (1988).

$$H(\boldsymbol{x})\equiv\frac{\nabla^2 F(\boldsymbol{x})}{F(\boldsymbol{x})}$$

We often consider the quadratic form $\boldsymbol{\xi}^T B(\boldsymbol{x})\boldsymbol{\xi}$ and $\boldsymbol{\xi}^T H(\boldsymbol{x})\boldsymbol{\xi}$ for $\boldsymbol{\xi}$, which are expressed as follows (cf. Ref. (2)).

$$\boldsymbol{\xi}^T B(\boldsymbol{x})\boldsymbol{\xi}=-(m+1)\tilde{c}(\boldsymbol{x},\boldsymbol{\xi})^2+\sum_{i=1}^{m}\tilde{a}_i(\boldsymbol{x},\boldsymbol{\xi})^2 \quad (1)$$

$$\boldsymbol{\xi}^T H(\boldsymbol{x})\boldsymbol{\xi}=m(m+1)(\tilde{c}(\boldsymbol{x},\boldsymbol{\xi})-\bar{a}(\boldsymbol{x},\boldsymbol{\xi}))^2+\sum_{i=1}^{m}(\tilde{a}_i(\boldsymbol{x},\boldsymbol{\xi})-\bar{a}(\boldsymbol{x},\boldsymbol{\xi}))^2 \quad (2)$$

where

$$\tilde{a}_i(\boldsymbol{x},\boldsymbol{\xi})=\frac{\boldsymbol{a}_i^T\boldsymbol{\xi}}{\boldsymbol{a}_i^T\boldsymbol{x}-b_i}$$

$$\bar{a}(\boldsymbol{x},\boldsymbol{\xi})=\frac{1}{m}\sum_{i=1}^{m}\tilde{a}_i(\boldsymbol{x},\boldsymbol{\xi})$$

$$\tilde{c}(\boldsymbol{x},\boldsymbol{\xi})=\frac{\boldsymbol{c}^T\boldsymbol{\xi}}{\boldsymbol{c}^T\boldsymbol{x}}$$

As is seen from Eq. (2) and the assumptions, $H(\boldsymbol{x})$ is positive definite, and $F(\boldsymbol{x})$ is strongly convex.

## 3. Ellipsoids and Polynomiality

At an interior point $\boldsymbol{x}$, we can consider an ellipsoid $E(\boldsymbol{x}, r)(r\geqq 0)$ defined by the Hessian, which is positive definite, and centered at $\boldsymbol{x}$ as follows:

$$E(\boldsymbol{x}, r)=\{\boldsymbol{x}' \mid (\boldsymbol{x}'-\boldsymbol{x})^T H(\boldsymbol{x})\ (\boldsymbol{x}'-\boldsymbol{x})\leqq r^2\}$$

[Lemma 3.1] If $\boldsymbol{x}+\boldsymbol{\xi}\in E(\boldsymbol{x}, r)$, $|\tilde{a}_i(\boldsymbol{x},\boldsymbol{\xi})|<\sqrt{2}r$.
(Proof) From Eq. (2), we have

$$m(m+1)(\tilde{c}(\boldsymbol{x},\boldsymbol{\xi})-\bar{a}(\boldsymbol{x},\boldsymbol{\xi}))^2+\sum_{i=1}^{m}(\tilde{a}_i(\boldsymbol{x},\boldsymbol{\xi})-\bar{a}(\boldsymbol{x},\boldsymbol{\xi}))^2\leqq r^2 \quad (3)$$

Note that the second term in the left side is $m$ times the variance of $\tilde{a}_i(\boldsymbol{x},\boldsymbol{\xi})(i=1,\cdots,m)$. For $a\equiv\max_i\{\tilde{a}_i(\boldsymbol{x},\boldsymbol{\xi})\}-\min_i\{\tilde{a}_i(\boldsymbol{x},\boldsymbol{\xi})\}$, we can easily see

$$\sum_{i=1}^{m}(\bar{a}(\boldsymbol{x},\boldsymbol{\xi})-\tilde{a}_i(\boldsymbol{x},\boldsymbol{\xi}))^2\geqq\frac{a^2}{2}.$$

Hence, $a\leqq\sqrt{2}r$. Since there exist both strictly positive and negative numbers among $\tilde{a}_i(\boldsymbol{x},\boldsymbol{\xi})(i=1,\cdots,m)$, which follows from the assumption that the feasible region $X$ is bounded, this implies $|\tilde{a}_i(\boldsymbol{x},\boldsymbol{\xi})|<\sqrt{2}r$. □

[Theorem 3.1] $E(\boldsymbol{x}, 1/\sqrt{2})$ is contained in the feasible region $X$.
(Proof) For $\boldsymbol{x}+\boldsymbol{\xi}\in E(\boldsymbol{x}, 1/\sqrt{2})$, $|\tilde{a}_i(\boldsymbol{x},\boldsymbol{\xi})|$ is expressed as

$$1>|\tilde{a}_i(\boldsymbol{x},\boldsymbol{\xi})|=\left|\frac{\boldsymbol{a}_i^T\boldsymbol{\xi}}{\boldsymbol{a}_i^T\boldsymbol{x}-b_i}\right|=\left|\frac{\boldsymbol{a}_i^T(\boldsymbol{x}+\boldsymbol{\xi})-b_i}{\boldsymbol{a}_i^T\boldsymbol{x}-b_i}-1\right|$$

and hence

$$\boldsymbol{a}_i^T(\boldsymbol{x}+\boldsymbol{\xi})-b_i>0\quad(i=1,\cdots,m).$$ □

Thus, we can construct, at any interior point, a large ellipsoid contained in the feasible region from the Hessian. Next, let us consider approximating the multiplicative penalty function by a linear function determined by the gradient inside the ellipsoid.
[Theorem 3.2] For $\boldsymbol{x}+\boldsymbol{\xi}\in E(\boldsymbol{x}, \alpha/\sqrt{2})(0<\alpha<1)$,

$$\log F(\boldsymbol{x}+\boldsymbol{\xi})\leqq\log F(\boldsymbol{x})+\boldsymbol{\xi}^T\boldsymbol{\eta}(\boldsymbol{x})+\frac{m\alpha^2}{2(1-\alpha)^2}.$$

(Proof) Since there exists some $\theta(0<\theta<1)$ satisfying

$$\log F(\boldsymbol{x}+\boldsymbol{\xi})=\log F(\boldsymbol{x}+\boldsymbol{\xi}^T\boldsymbol{\eta}(x)+\frac{1}{2}\boldsymbol{\xi}^T B(\boldsymbol{x}+\theta\boldsymbol{\xi})\boldsymbol{\xi},$$

we have only to show that $\boldsymbol{\xi}^T B(\boldsymbol{x}+\theta\boldsymbol{\xi})\boldsymbol{\xi}$ in the last term is bounded by $m\alpha^2/(1-\alpha)^2$.From Eq. (1), we have

$$\boldsymbol{\xi}^T B(\boldsymbol{x}+\theta\boldsymbol{\xi})\boldsymbol{\xi}=-(m+1)\tilde{c}(\boldsymbol{x}+\theta\boldsymbol{\xi},\boldsymbol{\xi})^2+\sum_{i=1}^{m}\tilde{a}_i(\boldsymbol{x}+\theta\boldsymbol{\xi},\boldsymbol{\xi})^2\leqq\sum_{i=1}^{m}\tilde{a}_i(\boldsymbol{x}+\theta\boldsymbol{\xi},\boldsymbol{\xi})^2. \quad (4)$$

Since $\boldsymbol{x}+\theta\boldsymbol{\xi}/\alpha$ is in the feasible region from Theorem 3.1, we have

$$\boldsymbol{a}_i^T(\boldsymbol{x}+\theta\boldsymbol{\xi})-b_i\geqq(1-\alpha)(\boldsymbol{a}_i^T\boldsymbol{x}-b_i).$$

Hence,

$$|\tilde{a}_i(\boldsymbol{x}+\theta\boldsymbol{\xi},\boldsymbol{\xi})|\leqq\frac{1}{1-\alpha}|\tilde{a}_i(\boldsymbol{x},\boldsymbol{\xi})|\leqq\frac{\alpha}{1-\alpha}$$

where the last inequality follows from Lemma 3.1. Combining this with Eq. (4), we obtain the theorem. □

For an interior point $\boldsymbol{x}$, consider $\hat{\boldsymbol{\xi}}$ satisfying

$$H(\boldsymbol{x})\hat{\boldsymbol{\xi}}=-\boldsymbol{\eta}(\boldsymbol{x}).$$

$\hat{\boldsymbol{\xi}}$ is the so-called Newton direction. For this direction, the following is shown in the proof of Proposition 5.2 of the paper by Iri and Imai[(2)] (see also Ref. (1)).
[Lemma 3.2] [(1),(2)] $h\equiv\hat{\boldsymbol{\xi}}^T H(\boldsymbol{x})\hat{\boldsymbol{\xi}}=-\hat{\boldsymbol{\xi}}\boldsymbol{\eta}(\boldsymbol{x})\geqq 1/2$. □

[Theorem 3.3] Suppose $m\geqq 6$. For the Newton direction $\hat{\boldsymbol{\xi}}$, $\boldsymbol{x}+\hat{\boldsymbol{\xi}}/4hm$ is in the feasible region, and

$$\log F\left(\boldsymbol{x}+\frac{\hat{\boldsymbol{\xi}}}{4hm}\right)\leqq\log F(\boldsymbol{x})-\frac{1}{10m}.$$

(Proof) By the definition of $h$ in Lemma 3.2, we have

$$\frac{1}{16h^2m^2}\hat{\xi}^T H(x)\hat{\xi}=\frac{1}{16hm^2}\leqq\frac{1}{8m^2}=\left(\frac{\alpha}{\sqrt{2}}\right)^2$$

with $\alpha=\frac{1}{2m}<1$

Then, from Theorem 3. 2, $x+\hat{\xi}/4hm$ is in the feasible region, and, for $\alpha=1/(2m)$,

$$\log F\left(x+\frac{\hat{\xi}}{4hm}\right)\leqq\log F(x)-\frac{h}{4hm}+\frac{1}{8m(1-1/(2m))^2}\leqq\log F(x)+\frac{1}{m}\left(-\frac{1}{4}+\frac{1}{8(11/12)^2}\right)$$

since $m\geqq 6$. The coefficient of $1/m$ in the last term above is $-0.1012\cdots<-1/10$, and we obtain the theorem. □

By the well-known arguments about the polynomiality of the interior method (e.g., see Ref. (3)), this theorem implies that the multiplicative penalty function method solves a linear programming problem in $O(m^2L)$ iterations where each iteration requires $O(m^3)$ operations on numbers of $L$ bits ($L$ is the size of the input of the problem).

## 4. Concluding Remarks

We have shown that the ellipsoid contained in the feasible region is easily constructed from the Hessian in the multiplicative penalty function method[(2)], and verified the polynomiality of the method. The ellipsoid is determined by not only $\tilde{a}_i(x, \xi)$ but also $\tilde{c}(x, \xi)$, that is, it naturally has connection with the objective function besides the constraints.

To guarantee the reduction of the penalty function value as in Theorem 3. 3, the Newton direction should be used. However, Theorem 3. 2 in this paper holds for any direction, and hence might be used further.

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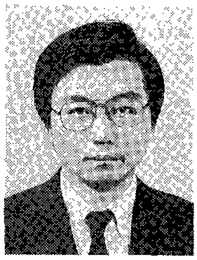
## References

(1) Iri M.: "On the Polynomiality of Iri-Imai Method", Manuscript (1990).
(2) Iri M. and Imai H.: "A Multiplicative Barrier Function Method for Linear Programming", Algorithmica, **1**, pp. 455-482 (1986).
(3) Karmarkar N.: "A New Polynomial-Time Algorithm for Linear Programming", Combinatorica, **4**, pp. 373-395 (1984).



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