インスタントンとモノポールの上の
荒木・工藤作用素

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ARAKI-KUDO OPERATIONS ON INSTANTONS AND MONOPOLES

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§1. Introduction

Let $SU(2) \to P_k \to S^4$ be the principal $SU(2)$ bundle of degree $c_2(P_k) = k$ and let $\mathcal{I}_k$ (resp. $\mathcal{A}_k$) be the set of anti-self-dual connections (resp. $SU(2)$ connections) over P_k. The restricted gauge group (consisting automorphisms which are the identity on the base point $\infty \in S^4$) acts on $\mathcal{I}_k$ and $\mathcal{A}_k$. We define M_k and $\mathcal{B}_k$ to be the orbit space of $\mathcal{I}_k$ and $\mathcal{A}_k$ by the restricted group respectively. M_k is called the framed moduli space of instantons of degree k and $\mathcal{B}_k$ the framed moduli space of $SU(2)$ connections of degree k. Let

$$i : M_k \to \mathcal{B}_k \tag{1.1}$$

be the inclusion.

It is known [2] that $\mathcal{B}_k$ is homotopically equivalent to $\Omega^3_k S^3$, the space of all based continuous maps from S^3 to itself of degree k. So, we can define a loop sum and homology operations in $\bigoplus_k H_*(\mathcal{B}_k; \mathbf{Z}_p)$ (p a prime number). In the case $p = 2$, the homology operations are usually called the Araki-Kudo operations.

Recently Boyer and Mann [4] introduced a loop sum and homology operations in $\bigoplus_k H_*(M_k; \mathbf{Z}_p)$ which are compatible with i_* and thus constructed new homology classes in $H_*(M_k; \mathbf{Z}_p)$.

In another direction, Hattori [8] computed $H^*(M_2; \mathbf{Z})$ and $H^*(M_2; \mathbf{Z}_2)$.

In the first half of this paper, we combine these results to obtain further homological information about M_2.

Next we consider the monopoles. Recall that the moduli space of monopoles of monopole number k is homeomorphic to $F_k^*(S^2, S^2)$, the space of based holomorphic maps of degree k from S^2 to itself [7]. More generally, we consider $F_k^*(S^2, \mathbf{C}P^m)$, the space of based holomorphic maps of degree k from S^2 to $\mathbf{C}P^m$. Any element of $F_k^*(S^2, \mathbf{C}P^m)$ is clearly an element of $\Omega_k^2\mathbf{C}P^m$, the space of all based continuous maps from S^2 to $\mathbf{C}P^m$ of degree k. Let

$$j : F_k^*(S^2, \mathbf{C}P^m) \to \Omega_k^2\mathbf{C}P^m \tag{1.2}$$

be the inclusion. Segal [10] showed that j is a homotopy equivalence up to dimension $k(2m-1)$.

As in the case of M_k, Boyer and Mann also introduced a loop sum and homology operations in $\bigoplus_k H_*(F_k^*(S^2, \mathbf{C}P^m); \mathbf{Z}_p)$ which are compatible with j_* [5] and thus constructed new homology classes in $\bigoplus_k H_*(F_k^*(S^2, \mathbf{C}P^m); \mathbf{Z}_p)$.

In the second half of this paper, we consider the case $p=2$. Then our main result is stated as follows. The homology classes constructed by Boyer and Mann generate the homology groups when k and m satisfy one of the following conditions. (i) $k=2$ and $m \geq 1$. (ii) $k=3$ and $m \geq 2$. (iii) $m \geq k+1$.

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I. SOME REMARKS ON THE HOMOLOGY OF MODULI SPACE OF INSTANTONS OF DEGREE 2

§2. Known results

In this section, we recall the results about M_2. First we state a result of Boyer and Mann [**4**].

THEOREM 2.1. *$H_*(M_2;\mathbf{Z}_2)$ contains the following homology classes.*

q	1	2	3
$H_q(M_2;\mathbf{Z}_2)$	$z_1 * [1]$	$z_1^2\ z_2 * [1]$	$Q_1(z_1)\ z_1 * z_2\ z_3 * [1]$

4	5	6
$Q_2(z_1)\ z_2^2\ z_1 * z_3$	$Q_1(z_2)\ z_2 * z_3\ Q_3(z_1)$	$z_3^2\ Q_2(z_2)$

7	8	9
$Q_3(z_2)\ Q_1(z_3)$	$Q_2(z_3)$	$Q_3(z_3)$

*Here z_q $(q = 1,2,3)$ are the generators of $H_q(M_1;\mathbf{Z}_2)$ (it is well known that M_1 is diffeomorphic to $SO(3)\times\mathbf{R}^5$ [**3**]),* [1] the generator of $H_0(M_1;\mathbf{Z}_2)$ and Q_i $(i = 1,2,3)$ the Araki-Kudo operations defined by Boyer and Mann.

Next we state the results of Hattori [**8**].

THEOREM 2.2. *The cohomology groups of M_2 with $\mathbf{Z}$ coefficients are given by the following table. (We follow the notations of* [**8**]*).*

q	1	2	3	4	5	6
$H^q(M_2;\mathbf{Z})$	0	$\mathbf{Z}_2$	$\mathbf{Z}_2$	$\mathbf{Z}_3 \oplus \mathbf{Z}_4$	$\mathbf{Z}_2$	$\mathbf{Z}_2$
generators		β	γ	$p^*z\quad \delta$	$\beta\gamma$	$\beta\delta$

7	8	9
$\mathbf{Z} \oplus \mathbf{Z}_2$	0	$\mathbf{Z}_2$
$\tilde{\pi}_! v\quad \gamma\delta$		$\beta\gamma\delta$

THEOREM 2.3. *The cohomology groups of M_2 with $\mathbf{Z}_2$ coefficients are given by the following table.*

q	1	2	3	4	5	6	7	8	9
$H^q(M_2;\mathbf{Z}_2)$	$\mathbf{Z}_2$	$\mathbf{Z}_2\oplus\mathbf{Z}_2$	$\mathbf{Z}_2\oplus\mathbf{Z}_2$	$\mathbf{Z}_2\oplus\mathbf{Z}_2$	$\mathbf{Z}_2\oplus\mathbf{Z}_2$	$\mathbf{Z}_2\oplus\mathbf{Z}_2$	$\mathbf{Z}_2\oplus\mathbf{Z}_2$	$\mathbf{Z}_2$	$\mathbf{Z}_2$
generators	u	$u^2 \quad v$	$u^3 \quad uv$	$w \quad u^2v$	$uw \quad u^3v$	$u^2w \quad vw$	$u^3w \quad uvw$	u^2vw	u^3vw

The choice of the elements u and v will be specified later.

§3. MAIN RESULTS

We first study the following problem. Do the elements of Theorem 2.1 generate $H_*(M_2;\mathbf{Z}_2)$?

PROPOSITION 3.1. *The elements of Theorem 2.1 generate $H_*(M_2;\mathbf{Z}_2)$ and the following relations hold.*

(1) $Q_1(z_1)+z_1*z_2+z_3*[1]=0$

(2) $Q_2(z_1)=z_1*z_3$

(3) $Q_1(z_2)+z_2*z_3+Q_3(z_1)=0.$

PROOF: Direct computations show that each element of Theorem 2.1 is non-trivial and differs to each other in $H_*(\mathcal{B}_2;\mathbf{Z}_2)$ except for $i_*Q_2(z_1)=i_*(z_1*z_3)$. So, by using Theorem 2.3, we see that the elements of Theorem 2.1 generate $H_*(M_2;\mathbf{Z}_2)$ and there must be one relation for $q=3,4$ and 5.

[4, Proposition 9.10] shows that there are the following relations.

(i) $i_*(Q_1(z_1)+z_1*z_2+z_3*[1])=0$

(ii) $i_*(Q_2(z_1)+z_1*z_3)=0.$

By using Cartan formula and Adem relation [**6**], we also see the following relation.

(iii) $i_*(Q_1(z_2) + z_2 * z_3 + Q_3(z_1)) = 0$.

Now by using Theorem 2.3, we see that the relations (i) - (iii) imply the relations (1) - (3) in Proposition 3.1.

Next we shall study the Kronecker products of the elements of Theorems 2.1 and 2.3. On account of Proposition 3.1, we can take a basis of $H_q(M_2; \mathbf{Z}_2)$ for $q = 3, 4$ and 5 as follows.

$q = 3 \quad Q_1(z_1) \quad z_1 * z_2$

$q = 4 \quad z_2^2 \qquad\quad z_1 * z_3$

$q = 5 \quad Q_1(z_2) \quad z_2 * z_3$.

THEOREM 3.2. *The Kronecker products of the elements of Theorems 2.1 and 2.3 are given by the following table.*

q	*1*	*2*
Kronecker products	$< u, z_1 * [1] >= 1$	$< u^2, z_1^2 >= 0 \quad < v, z_1^2 >= 1$ $< u^2, z_2 * [1] >= 1 \quad < v, z_2 * [1] >= 0$

3	*4*
$< u^3, Q_1(z_1) >= 0 \quad < uv, Q_1(z_1) >= 1$	$< w, z_2^2 >= 1 \quad < u^2v, z_2^2 >= 0$
$< u^3, z_1 * z_2 >= 1 \quad < uv, z_1 * z_2 >= 1$	$< w, z_1 * z_3 >= 0 \quad < u^2v, z_1 * z_3 >= 1$

5	*6*
$< uw, Q_1(z_2) >= 1 < u^3v, Q_1(z_2) >= 0$	$< u^2w, z_3^2 >= 0 < vw, z_3^2 >= 1$
$< uw, z_2 * z_3 >= 1 < u^3v, z_2 * z_3 >= 1$	$< u^2w, Q_2(z_2) >= 1 < vw, Q_2(z_2) >= 0$

7	*8*
$< u^3w, Q_3(z_2) >= 1 < uvw, Q_3(z_2) >= 0$	$< u^2vw, Q_2(z_3) >= 1$
$< u^3w, Q_1(z_3) >= 0 < uvw, Q_1(z_3) >= 1$	

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$< u^3vw, Q_3(z_3) >= 1$

In the above table we define v by $< v, z_1^2 >= 1$, $< v, z_2 * [1] >= 0$. Note that $< u^2, z_1^2 >= 0$, $< u^2, z_2 * [1] >= 1$. We also define w by $< w, z_2^2 >=$

1, $< w, z_1 * z_3 >= 0$. *Note that* $< u^2v, z_2^2 >= 0$, $< u^2v, z_1 * z_3 >= 1$.

PROOF: Let $\Delta : M_k \to M_k \times M_k$ be the diagonal. Then we can easily show the following relations.

$$
\begin{aligned}
(3.3) \qquad \Delta_* z_1 &= z_1 \otimes [1] + [1] \otimes z_1 \\
\Delta_* z_2 &= z_2 \otimes [1] + z_1 \otimes z_1 + [1] \otimes z_2 \\
\Delta_* z_3 &= z_3 \otimes [1] + z_2 \otimes z_1 + z_1 \otimes z_2 + [1] \otimes z_3.
\end{aligned}
$$

The following relation is known [**6**].

$$\Delta_* Q_j(a) = \sum_{r,s} Q_{j-r}(a'_s) \otimes Q_r(a''_s)$$

where $\Delta_* a = \sum_s a'_s \otimes a''_s$. Theorem 3.2 follows easily from these facts.

Next we shall study the integral classes. On account of Theorem 2.2, there exists an element σ which generates $\mathbf{Z}_4$ in $H_3(M_2; \mathbf{Z})$ and τ which generates $H_7(M_2; \mathbf{Z}) = \mathbf{Z}$. Let

$$j_* : H_*(M_2; \mathbf{Z}) \to H_*(M_2; \mathbf{Z}_2)$$

be the mod 2 reduction. Then we have the following

THEOREM 3.4. $j_*\sigma = z_3 * [1]$, $j_*\tau = Q_3(z_2)$.

PROOF: Let $\{E^r_*\}$ be the mod 2 homology Bockstein spectral sequence of M_2. The following Nishida relation is known [**6**].

$$\beta Q^j(a) = (j-1)Q^{j-1}(a)$$

where β is the Bockstein operation. By using Nishida relation, we compute E^2_* as follows.

q	1	2	3	4	5	6	7	8	9
E^2_q	0	0	$z_3 * [1]$	z_2^2	0	0	$Q_3(z_2)$	0	0

Theorem 3.4 follows easily from this table.

Next as an application of Proposition 3.1 and Theorem 3.2, we shall prove the following

THEOREM 3.5. *The elements of Theorem 2.2 satisfy the following relations.*

(1) $\beta^2 = 2\delta$

(2) $\delta^2 = 0$

(3) $\gamma^2 = \beta\delta$.

Note that Theorem 3.5 completely determines the ring structure of $H^*(M_2; \mathbf{Z})$.

PROOF: (1) is shown in [8]. As $H^8(M_2; \mathbf{Z}) = 0$, (2) follows. We shall prove (3). Let

$$j_* : H^*(M_2; \mathbf{Z}) \to H^*(M_2; \mathbf{Z}_2)$$

be the mod 2 reduction. All we need to show in order to prove (3) is the fact $j_*\gamma^2 \neq 0$. Let u, v, w be the elements in Theorem 2.3. We see $j_*\gamma$ equals to either u^3 or uv or $u^3 + uv$. We shall show that $j_*\gamma = u^3$ cannot occur.

ASSERTION 3.6. $u^4 = 0$, $v^2 = w$.

In fact, in the same way as the proof of Theorem 3.2, we see the following Kronecker products.

$$< u^4, z_2^2 >= 0, < u^4, z_1 * z_3 >= 0$$

$$< v^2, z_2^2 >= 1, < v^2, z_1 * z_3 >= 0.$$

ASSERTION 3.7. $j_*\beta = u^2$.

In fact, $j_*\beta = Sq^1u = u^2$.

Now suppose $j_*\gamma = u^3$. The table in Theorem 2.2 shows that $j_*(\beta\gamma) \neq 0$. But by Assertions 3.6 and 3.7, we have

$$j_*(\beta\gamma) = (j_*\beta)(j_*\gamma) = u^2u^3 = 0.$$

This is a contradiction. Hence $j_*\gamma$ equals to either uv or u^3+uv. Anyway

$$(j_*\gamma)^2 = u^2v^2 = u^2w \neq 0.$$

This completes the proof of (3).

REMARK 3.8. *Whether $\gamma^2 = 0$ or not is left unknown in* [8].

Now by using the above results, we can completely determine $H^*(M_2; \mathbf{Z}_2)$.

THEOREM 3.9. $H^*(M_2; \mathbf{Z}_2) = \mathbf{Z}_2[u, v]/(u^4, v^4)$ *and* $Sq^1v = uv$. *Note that the $\mathcal{A}(2)$-module structure of $H^*(M_2; \mathbf{Z}_2)$ is completely determined.*

PROOF: The ring structure follows from Theorem 2.3 and Assertion 3.6. By using Theorem 3.2 and the following Kronecker products, we can easily prove $Sq^1v = uv$.

$$< Sq^1v, Q_1(z_1) >= 1, < Sq^1v, z_1 * z_2 >= 1.$$

§4. APPENDIX

The proof of [4, Proposition 9.5] seems incomplete. By using Theorem 2.3, we shall give an explicit proof of this proposition.

PROPOSITION 9.5. [4] $z_i * [1] = Q_i[1]$ $(i = 1, 2, 3)$ *hold in* $H_*(M_2; \mathbf{Z}_2)$.

PROOF: The proof of $z_1 * [1] = Q_1[1]$ is given in [4]. First we shall prove $z_2 * [1] = Q_2[1]$. It is known [6] that $H_2(\mathcal{B}_2; \mathbf{Z}_2) = \mathbf{Z}_2 \oplus \mathbf{Z}_2$ and a basis is $Q_1[1]^2 * [-2]$ and $Q_2[1]$. By Theorem 2.3, $H_2(M_2; \mathbf{Z}_2) = \mathbf{Z}_2 \oplus \mathbf{Z}_2$. Note that z_1^2, $Q_2[1]$ and $z_2 * [1]$ are elements of $H_2(M_2; \mathbf{Z}_2)$.

$$i_* z_1 = Q_1[1] * [-1], i_* z_2 = Q_2[1] * [-1] \tag{4.1}$$

are given in [4, Theorem 8.6]. So, $i_* z_1^2 = Q_1[1]^2 * [-2]$, $i_* Q_2[1] = Q_2[1]$. Hence $i_* : H_2(M_2; \mathbf{Z}_2) \to H_2(\mathcal{B}_2; \mathbf{Z}_2)$ is an isomorphism. As we know $i_*(z_2 * [1]) = Q_2[1]$ by (4.1), $z_2 * [1] = Q_2[1]$ holds.

Next we shall prove $z_3 * [1] = Q_3[1]$. Let $f : SO(3) \to M_2$ be the composite of

$$SO(3) \to M_1 \times 1 \to M_1 \times M_1 \xrightarrow{*} M_2$$

and let $g : SO(3) \to M_2$ be the composite of

$$SO(3) \to S^3 \underset{\mathbf{Z}_2}{\times} \{1 \times 1\} \to S^3 \underset{\mathbf{Z}_2}{\times} \{M_1 \times M_1\} \xrightarrow{\vartheta} M_2$$

here ϑ is defined in [4].

We can easily prove $f_* z_i = z_i * [1]$ and $g_* z_i = Q_i[1]$ for $i = 1, 2$ and 3. We have shown

$$f_* z_1 = g_* z_1, f_* z_2 = g_* z_2. \tag{4.2}$$

By Theorem 2.3, all we need to prove is the following equalities.

$$< u^3, f_* z_3 >=< u^3, g_* z_3 >, < uv, f_* z_3 >=< uv, g_* z_3 > .$$

These equalities follows easily from (3.3) and (4.2).

II. THE MODULO 2 HOMOLOGY OF THE SPACE OF RATIONAL FUNCTIONS

§5. Statement of results

Before we state the main Theorems, we recall some results of [**5**], [**6**] and [**10**]. First we state the result of [**10**].

THEOREM 5.1. *The inclusion*

$$j : F_k^*(S^2, \mathbf{C}P^m) \to \Omega_k^2 \mathbf{C}P^m$$

is a homotopy equivalence up to dimension $k(2m-1)$, i.e. the induced homomorphism $j_* : \pi_q(F_k^*(S^2, \mathbf{C}P^m)) \to \pi_q(\Omega_k^2 \mathbf{C}P^m)$ is bijective for $q < k(2m-1)$ and surjective for $q = k(2m-1)$.

Next we describe the Pontryagin ring structure of $H_*(\Omega^2 \mathbf{C}P^m; \mathbf{Z}_2)$. Let $\tilde{\imath}_{2m-1}$ be the generator of $H_{2m-1}(\Omega_1^2 \mathbf{C}P^m; \mathbf{Z}_2) = \mathbf{Z}_2$ and let $[1]$ be the generator of $H_0(\Omega_1^2 \mathbf{C}P^m; \mathbf{Z}_2)$. Then, according to [**6**], we can state

THEOREM 5.2. $H_*(\Omega^2 \mathbf{C}P^m; \mathbf{Z}_2) = \mathbf{Z}_2[[1], \tilde{\imath}_{2m-1}, Q_{I_l}(\tilde{\imath}_{2m-1})]$
the polynomial algebra over $\mathbf{Z}_2$, under loop sum Pontryagin product, on generators $[1]$, $\tilde{\imath}_{2m-1}$ and $Q_{I_l}(\tilde{\imath}_{2m-1}) = Q_1 Q_1 \ldots Q_1(\tilde{\imath}_{2m-1})$, where I_l has length l and l is an any natural number.

Finally we review some results of [**5**]. If we regard a function belonging to $F_k^*(S^2, \mathbf{C}P^m)$ as a holomorphic function $f : S^2 \to \mathbf{C}P^m$ of degree k such that $f(\infty) = [1, \ldots, 1]$, then $F_k^*(S^2, \mathbf{C}P^m)$ can be described in the following form.

(5.3) $F_k^*(S^2, \mathbf{C}P^m) = \{[p_0(z), \ldots, p_m(z)]$; $p_i(z)$ are monic polynomials of degree k such that there exists no $\alpha \in \mathbf{C}$ which satisfies $p_0(\alpha) = 0, \ldots, p_m(\alpha) = 0.\}$

In the case $m = 1$, we write $\frac{p(z)}{q(z)}$ for $[p_0(z), p_1(z)]$.

Note that $F_1^*(S^2, \mathbf{C}P^m)$ is homotopically equivalent to S^{2m-1} by (5.3).

Let ι_{2m-1} be the generator of $H_{2m-1}(F_1^*(S^2, \mathbf{C}P^m); \mathbf{Z}_2)$. If we start with ι_{2m-1} and compute iterated operations on ι_{2m-1} and loop sums of such elements, we may construct many non-zero homology classes in $H_*(F_k^*(S^2, \mathbf{C}P^m); \mathbf{Z}_2)$. Then by combining Theorems 5.1 and 5.2, the following Theorem is known.

THEOREM 5.4. [5] *Any element ξ of $H_*(F_k^*(S^2, \mathbf{C}P^m); \mathbf{Z}_2)$ with deg ξ $< k(2m-1)$ can be constructed by loop sums and iterated operations on ι_{2m-1}.*

Now we state the main Theorems.

THEOREM 5.5. *The elements constructed by loop sums and iterated operations on ι_1 form a basis of $H_*(F_2^*(S^2, \mathbf{C}P^1); \mathbf{Z}_2)$.*

THEOREM 5.6. *For $m \geq 2$, the elements constructed by loop sums and iterated operations on ι_{2m-1} form a basis of $H_*(F_2^*(S^2, \mathbf{C}P^m); \mathbf{Z}_2)$.*

THEOREM 5.7. *For $m \geq 2$, the elements constructed by loop sums and iterated operations on ι_{2m-1} form a basis of $H_*(F_3^*(S^2, \mathbf{C}P^m); \mathbf{Z}_2)$.*

THEOREM 5.8. *For $m \geq k+1$, the elements constructed by loop sums and iterated operations on ι_{2m-1} form a basis of $H_*(F_k^*(S^2, \mathbf{C}P^m); \mathbf{Z}_2)$.*

The rest of this paper is organized as follows. In §6 we shall give a strategy of proving Theorems 5.6-5.8. In §7 we shall prove Theorems

5.5 and 5.6. In §8 we shall prove Theorem 5.8. In §9 we shall prove Theorem 5.7.

§6. Strategy of proof

We shall give a strategy of proving Theorems 5.6-5.8. A strategy of proving Theorem 5.5 is slightly different. So it will be postponed to §7.

In the following, all homology groups, cohomology groups and compact support cohomology groups have coefficients $\mathbf{Z}_2$.

In order to prove Theorems 5.6-5.8, it will be enough to compute $H_q(F_k^*(S^2, \mathbf{C}P^m))$ for $q \geq k(2m-1)$ by virtue of theorem 5.4.

Let us filter $F_k^*(S^2, \mathbf{C}P^m)$ by the closed subspaces

$$F_k^*(S^2, \mathbf{C}P^m) = X_k \supset X_{k-1} \supset \cdots \supset X_1 \tag{6.1}$$

where

$$X_n = \{[p_0(z), \ldots, p_m(z)] \in F_k^*(S^2, \mathbf{C}P^m) \; ; \; p_0(z) \text{ has at most } n \text{ distinct zeros.}\} \tag{6.2}$$

Let H_c^* be the compact support cohomology group. Assume that we have some informations about $H_c^*(X_{n-1})$ and $H_c^*(X_n - X_{n-1})$, then we obtain new informations about $H_c^*(X_n)$ by using the following compact support cohomology exact sequence of the pair (X_n, X_{n-1}).

$$\cdots \to H_c^q(X_n - X_{n-1}) \to H_c^q(X_n) \to H_c^q(X_{n-1}) \to H_c^{q+1}(X_n - X_{n-1}) \to \ldots \tag{6.3}$$

Moreover assume that we have some informations about $H_c^*(X_{n+1} - X_n)$, then we obtain new informations about $H_c^*(X_{n+1})$ by using the compact support cohomology exact sequence of the pair (X_{n+1}, X_n).

We repeat this process. Then finally we obtain new informations about $H_c^*(F_k^*(S^2, \mathbf{C}P^m))$ which can be converted to those of $H_*(F_k^*(S^2, \mathbf{C}P^m))$ by the Poincaré duality. In particular if k and m are taken to be in Theorems 5.6-5.8, then we can determine $H_q(F_k^*(S^2, \mathbf{C}P^m))$ for $q \geq k(2m-1)$.

§7. Proof of Theorems 5.5 and 5.6.

First we prove Theorem 5.6 by using the strategy given in §6. Note that in degrees greater than or equal to $4m-2$, the elements constructed by loop sums and iterated operations are given by ι_{2m-1}^2 and $Q_1(\iota_{2m-1})$ (which are non-trivial by Theorem 5.2). Hence it will be enough to show the following proposition in order to prove Theorem 5.6.

Proposition 7.1. $H_q(F_2^*(S^2, \mathbf{C}P^m)) = \begin{cases} \mathbf{Z}_2 & q = 4m-2, 4m-1 \\ 0 & q \geq 4m. \end{cases}$

Proof: We filter $F_2^*(S^2, \mathbf{C}P^m)$ as given in §6.

Lemma 7.2. *X_1 is homeomorphic to $\mathbf{C} \times \mathbf{C}^m \times (\mathbf{C}^m)^*$.*

In fact if $[p_0(z), \ldots, p_m(z)]$ belongs to X_1 and $p_0(z)$ has a multiple root α, then $p_i(z)$ $(1 \leq i \leq m)$ are completely determined by giving $p_i(\alpha)$, $p_i'(\alpha)$ which are arbitrary except for the constraint $(p_1(\alpha), \ldots, p_m(\alpha)) \neq (0, \ldots, 0)$.

Let $\tilde{C}_n$ be the space of ordered distinct n-tuples in $\mathbf{C}$.

LEMMA 7.3. *$X_2 - X_1$ is the quotient of $\{(\mathbf{C}^m)^* \times (\mathbf{C}^m)^*\} \times \tilde{C}_2$ by a free action of the symmetric group Σ_2.*

In fact if $[p_0(z), \dots, p_m(z)]$ belongs to $X_2 - X_1$ and $p_0(z)$ has roots $\alpha_1,\ \alpha_2$, then $p_i(z)$ $(1 \le i \le m)$ are completely determined by giving $p_i(\alpha_1),\ p_i(\alpha_2)$ which are arbitrary except for the constraint $(p_1(\alpha_1), \dots, p_m(\alpha_1)) \neq (0, \dots, 0)$ and $(p_1(\alpha_2), \dots, p_m(\alpha_2)) \neq (0, \dots, 0)$.

Note that X_1 is homotopically equivalent to S^{2m-1} by Lemma 7.2. Hence we see $H^q(X_1) = 0$ for $q \ge 2m$. Note also that $\dim_{\mathbf{R}} X_1 = 4m+2$. Hence by the Poincaré duality, we see

$$H_c^q(X_1) = 0 \quad \text{for} \quad q \le 2m+2. \tag{7.4}$$

Note that $X_2 - X_1$ is homotopically equivalent to $(S^{2m-1})^2 \underset{\Sigma_2}{\times} S^1$ by Lemma 7.3. We consider the Serre spectral sequence of the fiber bundle

$$(S^{2m-1})^2 \to (S^{2m-1})^2 \underset{\Sigma_2}{\times} S^1 \to S^1. \tag{7.5}$$

As $H^{2(2m-1)}((S^{2m-1})^2) = \mathbf{Z}_2$, the action of $\pi_1(S^1)$ on $H^{2(2m-1)}((S^{2m-1})^2)$ is trivial. By using this fact, spectral sequence argument shows

$$H^q(X_2 - X_1) = \begin{cases} \mathbf{Z}_2 & q = 4m-2, 4m-1 \\ 0 & q \ge 4m. \end{cases} \tag{7.6}$$

Note that $\dim_{\mathbf{R}} X_2 = 4m+4$. Hence by using (7.6) and the Poincaré duality, we see

$$H_c^q(X_2 - X_1) = \begin{cases} \mathbf{Z}_2 & q = 5, 6 \\ 0 & q \le 4. \end{cases} \tag{7.7}$$

By using (7.4) and (7.7), the compact support cohomology exact sequence of the pair (X_2, X_1) shows

$$H_c^q(X_2) = \begin{cases} \mathbf{Z}_2 & q = 5, 6 \\ 0 & q \leq 4. \end{cases} \tag{7.8}$$

Proposition 7.1 follows easily from (7.8) by the Poincaré duality.

Next we shall prove Theorem 5.5. We write F_k^* for $F_k^*(S^2, \mathbf{C}P^1)$. Let [1] be the generator of $H_0(F_1^*)$. Then the elements constructed by loop sums and iterated operations are given by $\iota_1 * [1], \iota_1^2$ and $Q_1(\iota_1)$ (which are non-trivial by Theorem 5.2). Hence it will be enough to show the following proposition in order to prove Theorem 5.5.

PROPOSITION 7.9. $H_q(F_2^*) = \begin{cases} \mathbf{Z}_2 & q = 0, 1, 2, 3 \\ 0 & q \geq 4. \end{cases}$

PROOF: Note that $\pi_1(F_2^*) = \mathbf{Z}$ by Theorem 5.1. Hence if we follow the proof of Theorem 5.6 in order to prove Theorem 5.5, we will encounter some difficulties. So we first consider the universal covering of F_2^*.

We define

$$R : F_2^* \to \mathbf{C}^* \tag{7.10}$$

as follows. Let $\frac{p(z)}{q(z)}$ be an element of F_2^* and let α_1, α_2 be the roots of $p(z)$, β_1, β_2 be the roots of $q(z)$. Then $R(\frac{p(z)}{q(z)})$ is defined by $\prod_{i,j}(\alpha_i - \beta_j)$.

Let Y_2 be $R^{-1}(1)$. Then it is known that (7.10) is a fiber bundle with simply connected fiber Y_2 **[10]**.

First we shall compute $H^*(Y_2)$. We define the closed subspace Y_1 of Y_2 by

$$Y_1 = \left\{ \frac{p(z)}{q(z)} \in Y_2 \ ; \ q(z) \text{ has a multiple root.} \right\}$$

LEMMA 7.11. *Y_1 is homeomorphic to $\mathbf{C}^2 \amalg \mathbf{C}^2$.*

In fact if $\frac{p(z)}{q(z)}$ belongs to Y_1 and $q(z)$ has a multiple root β, then $p(z)$ is completely determined by giving $p(\beta), p'(\beta)$ which are arbitrary except for the constraint $R\left(\frac{p(z)}{q(z)}\right) = p(\beta)^2 = 1$.

We think of $\mathbf{C}^*$ as $\{(\xi_1, \xi_2) \in (\mathbf{C}^*)^2 \ ; \ \xi_1 \xi_2 = 1\}$.

LEMMA 7.12. *$Y_2 - Y_1$ is the quotient of $\mathbf{C}^* \times \tilde{C}_2$ by a free action of the symmetric group Σ_2.*

In fact if $\frac{p(z)}{q(z)}$ belongs to $Y_2 - Y_1$ and $q(z)$ has roots β_1, β_2, then $p(z)$ is completely determined by giving $p(\beta_1), p(\beta_2)$ which are arbitrary except for the constraint $R\left(\frac{p(z)}{q(z)}\right) = p(\beta_1)p(\beta_2) = 1$.

We define the involution τ on $S^1 \times S^1$ by

$$(z, w)\tau = \left(\frac{1}{z}, -w\right) \qquad (z, w) \in S^1 \times S^1.$$

Then by Lemma 7.12, we see that $Y_2 - Y_1$ is homotopically equivalent to $S^1 \times S^1/\tau$. Note that $S^1 \times S^1/\tau$ is Klein's bottle.

Now by Lemma 7.11 and the Poincaré duality, we see

$$H_c^q(Y_1) = \begin{cases} \mathbf{Z}_2 \oplus \mathbf{Z}_2 & q = 4 \\ 0 & \text{otherwise.} \end{cases} \tag{7.13}$$

By Lemma 7.12 and the Poincaré duality, we see

$$(7.14) \qquad H_c^q(Y_2 - Y_1) = \begin{cases} \mathbf{Z}_2 & q = 4, 6 \\ \mathbf{Z}_2 \oplus \mathbf{Z}_2 & q = 5 \\ 0 & \text{otherwise.} \end{cases}$$

Note that $H^1(Y_2) = 0$. (In fact Y_2 is simply connected). Hence by the Poincaré duality, we see

$$(7.15) \qquad H_c^5(Y_2) = 0.$$

Now by using the compact support cohomology exact sequence of the pair (Y_2, Y_1), we see by (7.13)-(7.15) that

$$(7.16) \qquad H_c^q(Y_2) = \begin{cases} \mathbf{Z}_2 & q = 4, 6 \\ 0 & \text{otherwise.} \end{cases}$$

By the Poincaré duality, we see

$$(7.17) \qquad H^q(Y_2) = \begin{cases} \mathbf{Z}_2 & q = 0, 2 \\ 0 & \text{otherwise.} \end{cases}$$

We consider the Serre spectral sequence of (7.10). As $H^2(Y_2) = \mathbf{Z}_2$, the action of $\pi_1(\mathbf{C}^*)$ on $H^2(Y_2)$ is trivial. By using this fact, spectral sequence argument shows Proposition 7.9.

As a Corollary of Theorem 5.5, we shall determine the $\mathcal{A}(2)$-module structure of $H^*(F_2^*)$. Note that $\{[2], \iota_1 * [1], \iota_1^2, Q_1(\iota_1)\}$ form the basis of $H_*(F_2^*)$ by Theorem 5.5. Let $u \in H^1(F_2^*)$ be the dual of $\iota_1 * [1]$ and $v \in H^2(F_2^*)$ be the dual of ι_1^2. Then we have the following

COROLLARY 7.18. *$H^*(F_2^*) = \bigwedge(u, v)$, the exterior algebra over $\mathbf{Z}_2$ on generators u and v. $Sq^1 v = uv$.*

PROOF: Note that the following relation holds in $H_1(F_2^*)$ by Theorem 5.1.

$$Q_1[1] = \iota_1 * [1]. \tag{7.19}$$

Let $\Delta : F_k^* \to F_k^* \times F_k^*$ be the diagonal. Then the following relations are well known [**6**].

$$\Delta_* Q_1(a) = \sum_s \left\{ Q_1(a_s') \otimes (a_s'')^2 + (a_s')^2 \otimes Q_1(a_s'') \right\} \tag{7.20}$$

where $\Delta_* a = \sum_s a_s' \otimes a_s''$.

(7.21) (Nishida relation) $\beta Q^j(a) = (j-1)Q^{j-1}(a)$

where β is the Bockstein operation.

Then the ring structure is proved by observing the following Kronecker products.

$$< u^2, \iota_1^2 >= 0, \quad < uv, Q_1(\iota_1) >= 1.$$

The fact $Sq^1 v = uv$ is proved by observing the following Kronecker product.

$$< Sq^1 v, Q_1(\iota_1) >= 1.$$

§8. PROOF OF THEOREM 5.8.

We prove Theorem 5.8 by using the strategy given in §6. We filter $F_k^*(S^2, \mathbf{C}P^m)$ as given in §6.

In general $X_n - X_{n-1}$ has one component for each partition of k into n pieces. Let $k = \nu_1 + \cdots + \nu_n$ be one of such partitions. We shall study the component which corresponds to this partition. Let $\mu_1, \ldots, \mu_s$ be the numbers distinct to each other which appear among the ν_i. We can assume μ_1 appears with multiplicity i_1, μ_2 appears with multiplicity i_2, $\ldots$, μ_s appears with multiplicity i_s so that $i_1 + \cdots + i_s = n$.

We define the subgroup G of Σ_n to be $G = \Sigma_{i_1} \times \Sigma_{i_2} \times \cdots \times \Sigma_{i_s}$. Then by the same argument as the proof of Lemma 7.3, we see the following

LEMMA 8.1. *The component which corresponds to the partition* $k = \nu_1 + \cdots + \nu_n$ *as above is homotopically equivalent to* $(S^{2m-1})^n \underset{G}{\times} \tilde{C}_n$.

By using Lemma 8.1, we shall show the following

PROPOSITION 8.2. $H_c^q(X_{k-1}) = 0 \quad$ *for* $\quad q \leq 2m + k - 2$.

PROOF: We shall admit the following Lemma for a moment.

LEMMA 8.3. $H_c^q(X_n - X_{n-1}) = 0 \quad$ *for* $\quad q \leq n + 2m(k-n) - 1$.

Now we see by Lemma 8.3

$$H_c^q(X_1) = 0 \quad \text{for} \quad q \leq 2m(k-1)$$

and

$$H_c^q(X_2 - X_1) = 0 \quad \text{for} \quad q \leq 2m(k-2) + 1.$$

Hence by using the compact support cohomology exact sequence of the pair (X_2, X_1), we see

$$H_c^q(X_2) = 0 \quad \text{for} \quad q \leq 2m(k-2) + 1.$$

If we repeat this process, we can inductively prove the following fact.

$$H_c^q(X_n) = 0 \quad \text{for} \quad q \leq n + 2m(k-n) - 1.$$

In particular we see

$$H_c^q(X_{k-1}) = 0 \quad \text{for} \quad q \leq 2m + k - 2.$$

PROOF OF LEMMA 8.3: By Lemma 8.1, each component of $X_n - X_{n-1}$ is homotopically equivalent to $(S^{2m-1})^n \underset{G}{\times} \tilde{C}_n$ where G is a subgroup of Σ_n. Note that $\dim_{\mathbf{R}}((S^{2m-1})^n \underset{G}{\times} \tilde{C}_n) = 2mn + n$. Hence we see

$$H^q(X_n - X_{n-1}) = 0 \quad \text{for} \quad q \geq 2mn + n + 1. \tag{8.4}$$

Note that $\dim_{\mathbf{R}} X_n = 2km + 2n$. Hence by the Poincaré duality, we see

$$\begin{aligned} H_c^q(X_n - X_{n-1}) = 0 \quad \text{for} \quad q &\leq (2km + 2n) - (2mn + n + 1) \\ &= n + 2m(k-n) - 1. \end{aligned}$$

Next by using Proposition 8.2, we shall show the following

PROPOSITION 8.5. $H^q(X_k) \simeq H^q(X_k - X_{k-1})$ *for* $q \geq k(2m-1)$.

PROOF: By Proposition 8.2, we know

$$H_c^q(X_{k-1}) = 0 \quad \text{for} \quad q \leq 2m + k - 2.$$

Hence by the compact support cohomology exact sequence of the pair (X_k, X_{k-1}), we see

$$H_c^q(X_k) \simeq H_c^q(X_k - X_{k-1}) \quad \text{for} \quad q \leq 2m + k - 2. \tag{8.6}$$

Note that $\dim_{\mathbf{R}} X_k = 2k(m+1)$. Hence by the Poincaré duality, we see

$$(8.7) \quad H^q(X_k) \simeq H^q(X_k - X_{k-1}) \quad \text{for} \quad q \geq 2k(m+1) - (2m+k-2) \\ = 2m(k-1) + k + 2.$$

Note that we assumed $m \geq k+1$. Hence by (8.7), we see

$$H^q(X_k) \simeq H^q(X_k - X_{k-1}) \quad \text{for} \quad q \geq 2mk + k + 2 - 2(k+1) \\ = k(2m-1).$$

PROPOSITION 8.8. *We have the following isomorphism as graded $\mathbf{Z}_2$ vector spaces.*

$$\bigoplus_{q \geq k(2m-1)} H^q(X_k - X_{k-1}) \simeq H^*(\tilde{C}_k/\Sigma_k) \otimes H^{k(2m-1)}((S^{2m-1})^k).$$

PROOF: First note that $X_k - X_{k-1}$ is homotopically equivalent to $(S^{2m-1})^k \underset{\Sigma_k}{\times} \tilde{C}_k$ by Lemma 8.1.

We consider the Serre spectral sequence of the fiber bundle

$$(8.9) \qquad (S^{2m-1})^k \to (S^{2m-1})^k \underset{\Sigma_k}{\times} \tilde{C}_k \to \tilde{C}_k/\Sigma_k.$$

As $H^{k(2m-1)}((S^{2m-1})^k) = \mathbf{Z}_2$, the action of $\pi_1(\tilde{C}_k/\Sigma_k)$ on $H^{k(2m-1)}((S^{2m-1})^k)$ is trivial. Note that $\dim_{\mathbf{R}} \tilde{C}_k/\Sigma_k = 2k$. Then we see the following facts.

$$(8.10) \qquad \bigoplus_{p \leq 2k} E_2^{p,k(2m-1)} \simeq H^*(\tilde{C}_k/\Sigma_k) \otimes H^{k(2m-1)}((S^{2m-1})^k).$$

(8.11) $$E_2^{p,q} = 0 \quad \text{for} \quad p > 2k.$$

(8.12) $$E_2^{p,q} = 0 \quad \text{for} \quad (k-1)(2m-1) < q < k(2m-1)$$
$$\text{or}$$
$$k(2m-1) < q.$$

Note that we assumed $m \geq k+1$. Then by (8.10)-(8.12), we see

(8.13) $$E_2^{p,k(2m-1)} \simeq E_\infty^{p,k(2m-1)} \quad \text{for all} \quad p.$$

If we use the condition $m \geq k+1$ once more, we can easily prove Proposition 8.8.

Now by Propositions 8.5 and 8.8, we see

$$\bigoplus_{q \geq k(2m-1)} H^q(X_k) \simeq H^*(\tilde{C}_k/\Sigma_k) \otimes H^{k(2m-1)}((S^{2m-1})^k).$$

Equivalently

(8.14) $$\bigoplus_{q \geq k(2m-1)} H_q(X_k) \simeq H_*(\tilde{C}_k/\Sigma_k) \otimes H_{k(2m-1)}((S^{2m-1})^k).$$

Hence it will be enough to show the following proposition in order to prove Theorem 5.8.

PROPOSITION 8.15. *The elements of* $\bigoplus_{q \geq k(2m-1)} H_q(X_k)$ *constructed by loop sums and iterated operations correspond bijectively to the elements of* $H_*(\tilde{C}_k/\Sigma_k) \otimes H_{k(2m-1)}((S^{2m-1})^k)$.

PROOF: First we shall study the elements constructed by loop sums and iterated operations. We define $l \in \mathbf{N}$ to be $2^{l+1} > k \geq 2^l$. Let $[s]$ be the generator of $H_0(F_s^*(S^2, \mathbf{C}P^m))$. Then the elements constructed by loop sums and iterated operations are given by the following two types.

$$\iota_{2m-1}^{\alpha_0} * Q_1(\iota_{2m-1})^{\alpha_1} * \cdots * Q_1 \ldots Q_1(\iota_{2m-1})^{\alpha_l} \tag{8.16}$$

$$\iota_{2m-1}^{\alpha_0} * Q_1(\iota_{2m-1})^{\alpha_1} * \cdots * Q_1 \ldots Q_1(\iota_{2m-1})^{\alpha_l} * [s] \tag{8.17}$$

for some $s \in \mathbf{N}$.

LEMMA 8.18. *The degree of an element of type (8.17) is less than $k(2m-1)$. While the degree of an element of type (8.16) is greater than or equal to $k(2m-1)$.*

PROOF: We prove the first half. The second half can be proved similarly. We assume that an element

$$x = \iota_{2m-1}^{\alpha_0} * Q_1(\iota_{2m-1})^{\alpha_1} * \cdots * Q_1 \ldots Q_1(\iota_{2m-1})^{\alpha_l} * [s]$$

of type (8.17) has degree greater than or equal to $k(2m-1)$. As x is an element of $H_*(F_k^*(S^2, \mathbf{C}P^m))$, we have the following fact.

$$s + \alpha_0 + 2\alpha_1 + \cdots + 2^l \alpha_l = k. \tag{8.19}$$

As $\deg x \geq k(2m-1)$, we have the following fact. We write M for $2m-1$.

$$\alpha_0 M + \alpha_1(2M+1) + \alpha_2(4M+3) + \cdots + \alpha_l(2^l M + 2^l - 1) \geq kM. \tag{8.20}$$

Combining (8.19) and (8.20), we see

$$(8.21) \qquad \alpha_0 M + \alpha_1(2M+1) + \alpha_2(4M+3) + \cdots + \alpha_l(2^l M + 2^l - 1)$$
$$\geq sM + \alpha_0 M + 2\alpha_1 M + \cdots + 2^l \alpha_l M.$$

(8.21) is equivalent to

$$(8.22) \qquad \alpha_1 + 3\alpha_2 + \cdots + (2^l - 1)\alpha_l \geq sM.$$

By (8.19), we have the following inequality.

$$(8.23) \qquad \alpha_1 + 3\alpha_2 + \cdots + (2^l - 1)\alpha_l \leq k - s.$$

Combining (8.22) and (8.23), we see $k - s \geq sM$. Hence

$$(8.24) \qquad k \geq s(M+1) = 2ms.$$

Note that we assumed $m \geq k+1$. Hence we see $s = 0$ by (8.24). This is a contradiction. This completes the proof of the first half of Lemma (8.18).

We write ζ_i for $Q_1 \ldots Q_1(\iota_{2m-1})$. Then by Lemma 8.18, the elements of $\bigoplus_{q \geq k(2m-1)} H_q(X_k)$ constructed by loop sums and iterated operations correspond to

$$(8.25) \qquad \left\{ \zeta_0^{\alpha_0} \zeta_1^{\alpha_1} \ldots \zeta_l^{\alpha_l} \,;\, \alpha_i \geq 0,\ \alpha_0 + 2\alpha_1 + \cdots + 2^l \alpha_l = k \right\}.$$

(Note that the elements of (8.25) are linearly independent by Theorem 5.2).

Next we shall study the elements of $H_*(\tilde{C}_k/\Sigma_k) \otimes H_{k(2m-1)}((S^{2m-1})^k)$. $H_*(\tilde{C}_k/\Sigma_k)$ is described in [**6**]. We follow the notation of [**6**].

PROPOSITION 8.26. *$H_*(\tilde{C}_k/\Sigma_k) = \mathbf{Z}_2\,[\xi_j]\,/I$.*
Where $deg\, \xi_j = 2^j - 1$ and I is the two sided ideal generated by

$$(\xi_{j_1})^{k_1} \ldots (\xi_{j_t})^{k_t} \quad \textit{here} \quad \sum_{i=1}^{t} k_i 2^{j_i} > k.$$

By Proposition 8.26, the basis of $H_*(\tilde{C}_k/\Sigma_k)$ is given as follows.

$$\left\{ \xi_1^{k_1} \xi_2^{k_2} \ldots \xi_l^{k_l} \; ; \; k_i \geq 0, \; 2k_1 + 4k_2 + \cdots + 2^l k_l \leq k \right\}. \tag{8.27}$$

Let $[(S^{2m-1})^k]$ be the fundamental class of $(S^{2m-1})^k$. Then by (8.27), the elements of $H_*(\tilde{C}_k/\Sigma_k) \otimes H_{k(2m-1)}((S^{2m-1})^k)$ correspond to

$$\left\{ \xi_1^{k_1} \xi_2^{k_2} \ldots \xi_l^{k_l} \otimes [(S^{2m-1})^k] \; ; \; k_i \geq 0, \; 2k_1 + 4k_2 + \cdots + 2^l k_l \leq k \right\}. \tag{8.28}$$

We see that (8.25) and (8.28) correspond to each other. This completes the proof of Proposition 8.15 and, consequently, of Theorem 5.8.

§9. PROOF OF THEOREM 5.7.

In order to prove Theorem 5.7, the case we need to consider is $F_3^*(S^2, \mathbf{C}P^2)$ and $F_3^*(S^2, \mathbf{C}P^3)$ by virtue of Theorem 5.8. We shall prove the former. The latter can be proved similarly.

Note that in degrees greater than or equal to 9, the elements constructed by loop sums and iterated operations in $H_*(F_3^*(S^2, \mathbf{C}P^2))$ are given by ι_3^3 and $\iota_3 * Q_1(\iota_3)$ (which are non-trivial by Theorem 5.2). Hence it will be enough to show the following proposition in order to prove Theorem 5.7 in the case $F_3^*(S^2, \mathbf{C}P^2)$.

PROPOSITION 9.1. $H_q(F_3^*(S^2, \mathbf{C}P^2)) = \begin{cases} \mathbf{Z}_2 & q = 9, 10 \\ 0 & q \geq 11. \end{cases}$

We filter $F_3^*(S^2, \mathbf{C}P^2)$ as given in §6. Then by the same argument as the proof of Lemmas 7.2 and 7.3, we see the following Lemmas.

LEMMA 9.2. *X_1 is homotopically equivalent to S^3.*

LEMMA 9.3. *$X_2 - X_1$ is homotopically equivalent to $(S^3)^2 \times S^1$.*

LEMMA 9.4. *$X_3 - X_2$ is homotopically equivalent to $(S^3)^3 \underset{\Sigma_3}{\times} \tilde{C}_3$.*

Note that $\dim_{\mathbf{R}} X_3 = 18$, $\dim_{\mathbf{R}} X_2 = 16$ and $\dim_{\mathbf{R}} X_1 = 14$.

First we compute $H_c^*(X_2)$.

LEMMA 9.5. $H_c^q(X_2) = \begin{cases} \mathbf{Z}_2 & q = 9 \\ 0 & q \leq 8. \end{cases}$

PROOF: By Lemma 9.2 and the Poincaré duality, we see

$$H_c^q(X_1) = 0 \quad \text{for} \quad q \leq 10. \tag{9.6}$$

By Lemma 9.3 and the Poincaré duality, we see

$$H_c^q(X_2 - X_1) = \begin{cases} \mathbf{Z}_2 & q = 9 \\ 0 & q \leq 8. \end{cases} \tag{9.7}$$

Hence Lemma 9.5 follows from the compact support cohomology exact sequence of the pair (X_2, X_1).

Next we compute $H^*(X_3 - X_2)$. Note that $X_3 - X_2$ is homotopically equivalent to $(S^3)^3 \underset{\Sigma_3}{\times} \tilde{C}_3$ by Lemma 9.4. In order to compute $H^*((S^3)^3 \underset{\Sigma_3}{\times} \tilde{C}_3)$, we decompose the covering space

$$\Sigma_3 \to (S^3)^3 \times \tilde{C}_3 \to (S^3)^3 \underset{\Sigma_3}{\times} \tilde{C}_3 \tag{9.8}$$

into the following two covering spaces.

We embed $\mathbf{Z}_3$ in Σ_3 as the alternating group. Note that the following extension holds.

$$1 \to \mathbf{Z}_3 \to \Sigma_3 \to \mathbf{Z}_2 \to 1. \tag{9.9}$$

Then (9.8) is decomposed as follows.

$$\mathbf{Z}_3 \to (S^3)^3 \times \tilde{C}_3 \to (S^3)^3 \underset{\mathbf{Z}_3}{\times} \tilde{C}_3. \tag{9.10}$$

$$\mathbf{Z}_2 \to (S^3)^3 \underset{\mathbf{Z}_3}{\times} \tilde{C}_3 \to (S^3)^3 \underset{\Sigma_3}{\times} \tilde{C}_3. \tag{9.11}$$

As for (9.10), we see

$$H^*((S^3)^3 \underset{\mathbf{Z}_3}{\times} \tilde{C}_3) \simeq H^*((S^3)^3 \times \tilde{C}_3)^{\mathbf{Z}_3}. \tag{9.12}$$

In order to compute (9.12), we need to know $H^*(\tilde{C}_3)$. $H^*(\tilde{C}_3)$ is described in [**6**]. We follow the notation of [**6**].

Proposition 9.13.

(1) $H^1(\tilde{C}_3) = \mathbf{Z}_2 \oplus \mathbf{Z}_2 \oplus \mathbf{Z}_2$ *and a basis is* $\{\alpha^*_{11}\,,\ \alpha^*_{21}\,,\ \alpha^*_{22}\}$.

(2) $H^2(\tilde{C}_3) = \mathbf{Z}_2 \oplus \mathbf{Z}_2$ *and a basis is* $\{\alpha_{11}^*\alpha_{21}^*\ ,\ \alpha_{11}^*\alpha_{22}^*\}$.

(3) $\alpha_{21}^*\alpha_{22}^* = \alpha_{11}^*(\alpha_{21}^* + \alpha_{22}^*)$.

(4) *Let* $\sigma = (2\ 3)(1\ 2)$ *be the generator of* $\mathbf{Z}_3$. *Then*

$$\sigma^*\alpha_{11}^* = \alpha_{22}^*, \quad \sigma^*\alpha_{21}^* = \alpha_{11}^*, \quad \sigma^*\alpha_{22}^* = \alpha_{21}^*.$$

(5) $H^q(\tilde{C}_3) = 0$ *for* $q \geq 3$.

Now by using (9.12) and Proposition 9.13, we have the following

LEMMA 9.14. $H^q((S^3)^3 \underset{\mathbf{Z}_3}{\times} \tilde{C}_3) = \begin{cases} \mathbf{Z}_2 \oplus \mathbf{Z}_2 & q = 8 \\ \mathbf{Z}_2 & q = 9, 10 \\ 0 & q \geq 11. \end{cases}$

Let $(\mathcal{G}_1)$ be the Gysin exact sequence of (9.11) and let $(\mathcal{G}_2)$ be the compact support cohomology exact sequence of the pair (X_3, X_2). By inspecting $(\mathcal{G}_1)$ and $(\mathcal{G}_2)$, we shall prove Proposition 9.1. We write X for $X_3 - X_2$.

STEP 1. $H^q(X) = 0$ *for* $q \geq 11$.

In fact by the fact $H^q((S^3)^3 \underset{\mathbf{Z}_3}{\times} \tilde{C}_3) = 0$ for $q \geq 11$ (Lemma 9.14), we see $H^q(X) \simeq H^{11}(X)$ for $q \geq 11$ by $(\mathcal{G}_1)$. As X is a finite dimensional manifold, Step 1 holds.

STEP 2. $H^q(X_3) = 0$ *for* $q \geq 11$.

In fact we see $H_c^q(X) = 0$ for $q \leq 7$ by Step 1 and the Poincaré duality. Note that $H_c^q(X_2) = 0$ for $q \leq 8$ (Lemma 9.5). Hence we see $H_c^q(X_3) = 0$ for $q \leq 7$ by $(\mathcal{G}_2)$. By the Poincaré duality, we see $H^q(X_3) = 0$ for $q \geq 11$.

In order to complete the proof of Proposition 9.1, it will be enough to determine $H^9(X_3)$ and $H^{10}(X_3)$ by virtue of Step 2.

STEP 3. $H^{10}(X) = \mathbf{Z}_2$ *and* $H^9(X) \to H^{10}(X)$ *is surjective in* $(\mathcal{G}_1)$.

In fact by the fact $H^{11}(X) = 0$ (Step 1) and $H^{10}((S^3)^3 \underset{\mathbf{Z}_3}{\times} \tilde{C}_3) = \mathbf{Z}_2$ (Lemma 9.14), we can write $(\mathcal{G}_1)$ in the following form.

$$\to H^9(X) \to H^{10}(X) \to \mathbf{Z}_2 \to H^{10}(X) \to 0$$

By the exactness, Step 3 follows.

Before we proceed to Step 4,we shall state a fact about $H^8(X_3)$.

$$H^8(X_3) = 0. \tag{9.15}$$

((9.15) is easily proved by using Theorems 5.1 and 5.2.)

STEP 4. $H^{10}_c(X) = \mathbf{Z}_2$. *Hence* $H^8(X) = \mathbf{Z}_2$.

In fact by the fact $H^8((S^3)^3 \underset{\mathbf{Z}_3}{\times} \tilde{C}_3) = \mathbf{Z}_2 \oplus \mathbf{Z}_2$ (Lemma 9.14), we see $H^8(X) \neq 0$ by $(\mathcal{G}_1)$. Hence $H^{10}_c(X) \neq 0$ by the Poincaré duality. Note that $H^9_c(X_2) = \mathbf{Z}_2$ (Lemma 9.5). Note also that $H^{10}_c(X_3) = 0$ ((9.15) and the Poincaré duality). Hence we see $H^{10}_c(X) = \mathbf{Z}_2$ by $(\mathcal{G}_2)$. By the Poincaré duality, $H^8(X) = \mathbf{Z}_2$.

STEP 5. $H^8_c(X) \simeq H^8_c(X_3), H^9_c(X) \simeq H^9_c(X_3)$.

In fact as $H^7_c(X_2) \simeq H^8_c(X_2) = 0$ (Lemma 9.5), we see $H^8_c(X) \simeq H^8_c(X_3)$ by $(\mathcal{G}_2)$. In $(\mathcal{G}_2)$, we see $H^9_c(X_2) \to H^{10}_c(X)$ is isomorphism by Step 4. Hence we see $H^9_c(X) \simeq H^9_c(X_3)$ by $(\mathcal{G}_2)$.

Step 6. $H^{10}(X_3) = \mathbf{Z}_2$.

In fact by the fact $H^{10}(X) = \mathbf{Z}_2$ (Step 3), we see $H_c^8(X) = \mathbf{Z}_2$ by the Poincaré duality. Hence we see $H_c^8(X_3) = \mathbf{Z}_2$ by Step 5. Then we see $H^{10}(X_3) = \mathbf{Z}_2$ by the Poincaré duality.

Step 7. $H^9(X_3) = \mathbf{Z}_2$.

In fact by the fact $H^8(X) = \mathbf{Z}_2$ (Step 4), $H^9((S^3)^3 \times_{\mathbf{Z}_3} \tilde{C}_3) = \mathbf{Z}_2$ (Lemma 9.14), $H^{10}(X) = \mathbf{Z}_2$ and $H^9(X) \to H^{10}(X)$ is surjective in $(\mathcal{G}_1)$ (Step 3), we can write $(\mathcal{G}_1)$ in the following form.

$$\to \mathbf{Z}_2 \to H^9(X) \to \mathbf{Z}_2 \to H^9(X) \to \mathbf{Z}_2 \to 0$$

By the exactness, we see $H^9(X) = \mathbf{Z}_2$. Hence $H_c^9(X) = \mathbf{Z}_2$ by the Poincaré duality. Then $H_c^9(X_3) = \mathbf{Z}_2$ by Step 5 so $H^9(X_3) = \mathbf{Z}_2$ by the Poincaré duality. This completes the proof of Proposition 9.1 and, consequently, of Theorem 5.7 in the case $F_3^*(S^2, \mathbf{C}P^2)$.

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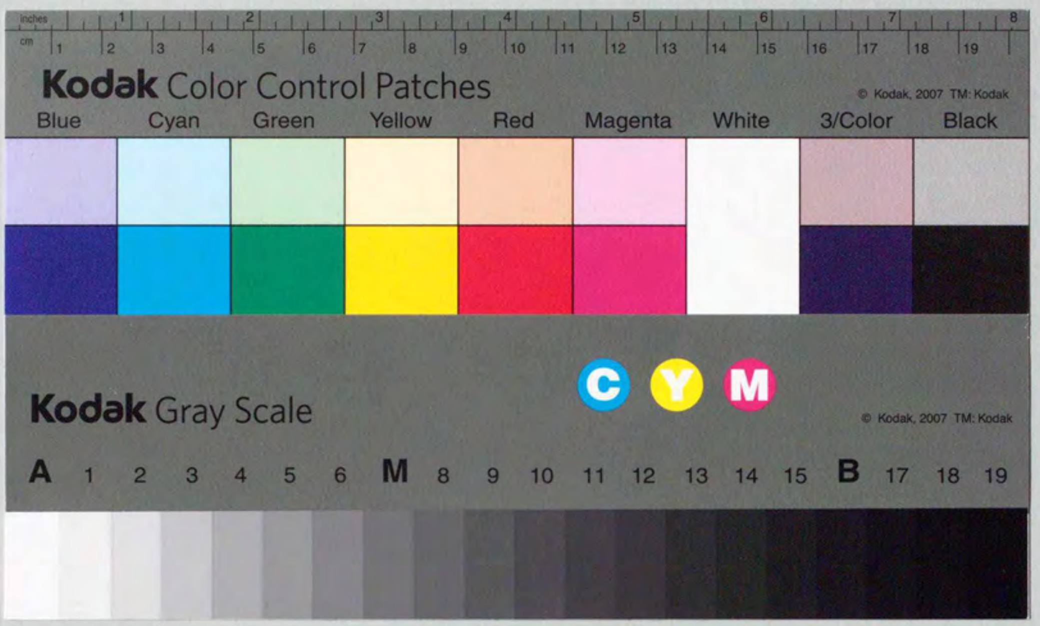
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