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BOSE-FERMI TRANSMUTATION IN THREE DIMENSIONAL FIELD THEORY

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December, 1990

Acknowledgement

The author would like to express his sincere thanks to Prof. K. Igi for his kind supervision, encouragement and reading the manuscript. Many useful discussions and collaboration with Dr. C. Itoi and Mr. H. Mukaida are greatly acknowledged. He also thanks to Mr. S. Matsuno for illuminative discussions on the quantum Hall effect. He is grateful to Prof. T. Eguchi, Prof. H. Kawai and his colleagues of theory group of Univ. of Tokyo for discussions and encouragement. He expresses his thanks to Prof. C. Iso for reading the manuscript.

Abstract

In (2+1)-dimensions if particles are coupled with the Chern-Simons (CS) gauge field the particles acquire magnetic fluxes and become composite objects of the charge and the flux. Statistics of such particles is changed because of the Aharanov-Bohm phase. When the attached flux is $2\pi m$ (m is an odd integer), fermions become bosons and vice versa. This is called the bose-fermi transmutation. In this thesis we clarify this transmutation both in a non-relativistic case and in a relativistic case. In the relativistic case the Chern-Simons gauge field gives an effect not only on the statistics of the particles but also on its spin. We also discuss one of the most important applications of the bose-fermi transmutation to the fractional quantum Hall effect (FQHE), which is a "macroscopic" quantum effect in a two-dimensional system of electrons subjected to a strong magnetic field.

Contents

1 Introduction

Two different quantum statistics of identical particles are known, Bose-Einstein statistics and Fermi-Dirac statistics. At low temperatures the statistics play an important role in its dynamics. For example bosons can condense into a "macroscopic" wave function (bose condensation). On the other hand the Fermi statistics assures the stability of matters. Then natural questions arise.

Are there exotic statistics which will give new exotic phenomena?

Can we transmute the statistics by interactions?

Recently in (2+1) dimensions many studies have been done for these questions. It was shown that particles with fractional statistics, called anyons, exist and they can be constructed from bosons by interacting with the Chern-Simons (CS) gauge field. Historically Leinaas and Mylheim[1] proposed a theoretical possibility of particles with the fractional statistics in two-spacial dimensions. Later Wilczek and Zee[2] showed that a soliton in $O(3)-\sigma$ model with the Hopf term has a fractional statistics and a fractional spin. These works were followed by many papers [3,4].

These studies were partly stimulted by the two-dimensional condensed matter phydics. A system of electrons in an effectively two dimensional space have been studied extensively and many interesting physics were discovered. Quantum Hall effect and High-T_c superconductivity are two of them. These phenomena are believed to be closely connected with the fractional statistics and the statistical transmutation [1].

Here we briefly explain why the statistics of a charged boson is transmuted to the fractional statistics by interacting with the Chern-Simons gauge field. The Chern-Simons gauge field does not have an ordinary kinetic term. Its action is given by

$$S_{\mathrm{CS}}=\frac{e^2}{8\pi J}\int \epsilon^{\mu\nu\lambda}a_\mu\partial_\nu a_\lambda\ \mathrm{d}^3x. \tag{1.1}$$

If a charged boson current J^μ is coupled with this gauge field, the equations of motion for the gauge field a is

$$eJ^\mu=\frac{e^2}{4\pi J}\epsilon^{\mu\nu\lambda}\partial_\nu a_\lambda. \tag{1.2}$$

[1]Of course the mechanism of High-T_c superconductivity is not yet discovered and nobody knows what the truth is.

The configuration of the gauge field is determined completely by this equations and there are no independent degrees of freedom for the gauge field. In particular the $\mu=0$ component relates the magnetic flux b with the source density ρ :

$$b(x)=\frac{4\pi J}{e}\rho(x). \tag{1.3}$$

After integrating the gauge field the particle becomes a composite object of a charge e and a flux $4\pi J/e$. Therefore when we interchange such two particles, an Aharanov Bohm phase $(=2\pi J)$ [2] appears and the statistics of particles are changed.

For general J the phase is fractional and the system describes anyons. The dynamics of anyons were first studied by Laughlin [5]. In the random phase approximation (RPA) he showed that a system of anyons behaves as a superconductor. This work was followed by many people [6] as a candidate of the High-T_c superconductivity.

If J is an integer the phase is 0 (mod 2π) and statistics does not change.

If J is a half integer ($J\in \mathrm{Z}+1/2$) the original bosonic particle becomes a fermion and vice versa. This is called the bose-fermi transmutation. The bose-fermi transmutation has the following three interesting properties:

(1) In the relativistic case, not only the statistics but the spin is transmuted. For this purpose we must treat the self-interaction induced by the Chern-Simons gauge field carefully.

(2) The bose-fermi transmutation has an important application to the fractional quantum Hall effect (FQHE).

(3) The bose-fermi transmutation can be a mechanism of making anyons. This sounds strange but it is well known that the vortex excitation of the FQHE is an anyon. A charged particle acquires magnetic fluxes through the CS gauge field and the statistics is changed. On the other hand a particle with a unit flux also couples with the CS gauge field and acquires a fractional charge. As a result it becomes an anyon.

[2]Thinking naively it is $4\pi J$ because not only a charge moves around a flux but also a flux move around a charge. It is indeed true if we can get rid of a contribution from the gauge field itself. The gauge field itself contributes to the phase, however, and the true phase is a half of the naive value[7]. We come back to this problem later in chapter 2.

The purpose of this paper is to clarify the above three properties of the bose-fermi transmutation.

(1) was pointed out by Polyakov [8]. There are some unclear points and problems, however. In chapter 3, we prove the Polyakov's conjecture exactly and extend it to the field theories. (2) was discussed by Girvin[34] and there are many papers that have followed. In chapter 4, we review it shortly and also present a new interpretation of the FQHE. The FQHE is shown to be a many-body system of electrons in a two-dimensional "phase" space. For such a system, the CS gauge field changes the symplectic structure on the phase space besides changing the statistics of the particles.

In the following we explain (1) and (2) in more details.

(1) In the non-relativistic case the effective interactions by the Chern-Simons gauge field between particles are topological. That is, the interactions depend only on the linkings of the particles and not on the continuous relative coordinates between particles. In this sense the CS gauge field gives an effect only on changing the statistics. Here we neglected the self-energy corrections. Several authors studying the CS theory as a topological field theory use the point splitting regularization in order to extract topological invariant informations of a knot from the self-energy [11,12]. Polyakov [8](and also [10]) evaluated the self-energy more carefully and claimed that the CS gauge field transmutes also the spin of particles. He showed that a new term proportional to the torsion of a path is induced from the self-energy. Since the torsion term contains a higher derivative of the particle's position the particle may acquire a new degree of freedom [9]. This can be identified with the spin degree of freedom. He suggests that the propagator of a charged scalar particle effectively becomes that of a Dirac particle. From the relativistic view point his claim is plausible because it is consistent with the spin-statistics theorem. In Polyakov's paper, however, many problems still remain. The self-energy should be treated more carefully. Also the measure in the functional integral over random paths is not clearly defined. Further the transmutation of the spin and the statistics in a quantum field theory cannot be fully understood in his language. To solve these problems is the purpose of chapter 2 [13].

(2) Fractional quantum Hall effect is a kind of a "macroscopic" quantum effect

in an effectively two dimensional system of electrons subjected to a strong magnetic field [24]. Soon after the discovery by Tsui et al.[25], Laughlin proposed trial wave functions of the incompressible quantum liquid for the ground state and the excited states [26]. Despite its qualitative and also quantitative successes, there are many unsolved problems: e.g., whether there is a phase transition, or what stabilizes the Laughlin wave function ...etc. One of these is a fundamental one;

$$\text{What is indeed the FQHE?} \tag{1.4}$$

or

$$\text{Can we make a Ginzburg-Landau theory for the FQHE?} \tag{1.5}$$

For these questions one answer was proposed that the FQHE is a new kind of a bose-condensation [34]. Superfluidity can be thought of as a bose condensation of He-4 which is a bose particle. Superconductivity is also a bose-condensation of bound states of electrons called Cooper pairs, which is of course a boson. In two-spacial dimensions, as we discuss in chapter 2 and 3, fermions can be bosonized by coupling with the Chern-Simons gauge field or in other words by attaching fluxes to particles. Such bosonized fermions can be bose-condensed. All qualitative features can be explained from this picture. Although it is unfortunately only a phenomenology and there are no accepted microscopic derivations, Ginzburg-Landau picture will be very powerful when we find new phenomena such as the flux quantization or Josephson effect. These are discussed in chapter 4.

Also another interpretation of the FQHE is presented. If we restrict the Hilbert space of the electrons on the lowest Landau level, a system of electrons subjected to a strong magnetic field is equivalent to a many-body system on a two dimensional "phase" space. Only a little attention has been taken to it before. If we couple the CS gauge field to such a system, not only the statistics of particles but also the symplectic structure on the phase space is changed. The symplectic structure determines the number of states of particles. We give a new interpretation of the FQHE from this picture.

The organization of this paper is as follows. In chapter 2 we discuss the transmutation of statistics in a non-relativistic case. We prove the equivalence of the grand canonical partition function of charged bosonic particles and that of free fermions. In chapter 3 we treat the self-energy corrections carefully and show that in a relativistic case a charged particle acquires a spin degree of freedom. We also discuss a functional integral for relativistic spinning particles. In chapter 4 we apply the bose-fermi transmutation to the FQHE. In chapter 5 we summarize the results of this thesis and discuss future problems.

2 Non-relativistic case

In this chapter we discuss the bose-fermi transmutation in a non-relativistic case. The coefficient J of the Chern-Simons action (1.1) is restricted to $J \in \mathrm{Z} + 1/2$. At these values of the coefficients bosons are transmuted into fermions and vice versa. In the canonical formalism bosons are quantized by commutation relations and fermions by anti-commutation relations. It can be shown that the commutation relations are changed to the anti-commutation relations by the Chern-Simons gauge field [4]. For generalizations to anyons or to the relativistic case, however, the path-integral formalism is more convenient than the canonical formalism. In this chapter we prove the following equivalence of grand canonical partition functions in terms of the path-integral quantizations.

We show that a grand canonical partition function of non-relativistic bosonic particles coupled with the Chern-Simons gauge field $< Z_{\mathrm{B}}(a) >_{\mathrm{CS}}$ is equal to that of free non-relativistic fermionic particles Z_{F} :

$$< Z_{\mathrm{B}}(a) >_{\mathrm{CS}} \;=\; < \mathrm{tr}_{\mathrm{B}}\, e^{-\beta(H_a-\mu N)} >_{\mathrm{CS}} \tag{2.1}$$

$$Z_{\mathrm{F}} \;=\; \mathrm{tr}_{\mathrm{F}}\, e^{-\beta(H_0-\mu N)} \tag{2.2}$$

where

$$H_a = \frac{1}{2m}(p-a)^2 - a_0 \;\; , \;\; H_0 = \frac{p^2}{2m} \tag{2.3}$$

and

$$< \cdots >_{\mathrm{CS}} = \int \mathcal{D}a\; e^{iS_{\mathrm{CS}}}(\cdots). \tag{2.4}$$

To show the above equality it is convenient to express the partition functions in terms of summation over random paths.

At first let's review how to derive a path-integral representation of a grand canonical partition function of free non-relativistic bosonic particles:

$$Z_{\mathrm{B}} = \mathrm{tr}_{\mathrm{B}}\, e^{-\beta(H_0-\mu N)}. \tag{2.5}$$

The trace tr_{B} is taken over symmetric N particle states ($N \geq 0$). The partition function can be rewritten by the following path-integral form:

$$Z_{\rm B} = \sum_N \frac{e^{\mu\beta N}}{N!}\int \sum_{\sigma\in S_N} < x_1...x_N|\exp(-\beta\sum_i \frac{p_i^2}{2m})|x_{\sigma_1}...x_{\sigma_N} > d^2x_1...d^2x_N$$
$$= \sum_N \frac{e^{\mu\beta N}}{N!}\sum_{\sigma\in S_N}\int \mathcal{D}x_1...\mathcal{D}x_N e^{-S} \tag{2.6}$$

where

$$S = \int_0^\beta dt \sum_i \frac{m\dot{x}_i^2}{2}, \tag{2.7}$$

and whose boundary conditions are

$$x_i(\beta) = x_{\sigma_i}(0) \quad (i = 1,..,N). \tag{2.8}$$

Any permutation in S_N can be factorized into cycles with c_1 1-cycles, c_2 2-cycles ... which satisfies $\sum_{\nu=1}^\infty \nu c_\nu = N$. For example a permutation

$$\sigma = (124)(36)(57) \in S_7 \tag{2.9}$$

has c_2=2 2-cycles and c_3=1 3-cycle. The number of permutations with c_ν ν-cycles is

$$M(c_\nu) = \frac{N!}{\prod_\nu \nu^{c_\nu}c_\nu!}. \tag{2.10}$$

Therefore the partition function can be expressed in terms of a path integral with a periodic boundary condition with a period $\nu\beta$

$$h_\nu \equiv \int_{\rm periodic}\mathcal{D}x\exp(-\int_0^{\nu\beta}\frac{m\dot{x}^2}{2}dt) \tag{2.11}$$

as

$$Z_{\rm B} = \sum_N \frac{(e^{\mu\beta})^N}{N!}\sum_{\{c_\nu\}} M(c_\nu)\prod_\nu (h_\nu)^{c_\nu}$$
$$= \prod_\nu \sum_{c_\nu=0}^\infty \frac{(e^{\mu\beta})^{\nu c_\nu}}{\nu^{c_\nu}c_\nu!}(h_\nu)^{c_\nu} = \prod_\nu \exp(\frac{(e^{\mu\beta})^\nu h_\nu}{\nu})$$
$$= \exp(\sum_{\nu=1}^\infty \frac{(e^{\mu\beta})^\nu h_\nu}{\nu}) \equiv e^{W_{\rm B}}. \tag{2.12}$$

This form is easily understood since W_B contains only connected paths with a period $\nu\beta$.

Of course the same formula can be derived directly from a field theoretic expression of the partition function

$$Z_{\rm B} = \int \mathcal{D}\phi^\dagger\mathcal{D}\phi\, e^{-\int_0^\beta \phi^\dagger(\partial+H_0-\mu)\phi\, dtd^2x}. \tag{2.13}$$

Next almost same formula can be derived for free fermions. In this case the trace $\rm tr_F$ is taken over anti-symmetric states;

$$Z_{\rm F} = {\rm tr_F}\, e^{-\beta(H_0-\mu N)}$$
$$= \sum_N \frac{e^{\mu\beta N}}{N!}\int d^2x_1...d^2x_N \sum_{\sigma\in S_N}(-1)^{\epsilon(\sigma)}$$
$$\times < x_1...x_N|\exp(-\beta\sum_i \frac{p_i^2}{2m})|x_{\sigma_1}...x_{\sigma_N} > \tag{2.14}$$

where

$$(-1)^{\epsilon(\sigma)} = \prod_\nu (-1)^{(\nu+1)c_\nu} = (-1)^N(-1)^{\sum c_\nu}. \tag{2.15}$$

Therefore the grand canonical partition function for free fermions is

$$Z_{\rm F} = \sum_N \frac{(-e^{\mu\beta})^N}{N!}\sum_{\{c_\nu\}} M(c_\nu)\prod_\nu (-h_\nu)^{c_\nu}$$
$$= \exp(-\sum_{\nu=1}^\infty \frac{(-1)^\nu(e^{\mu\beta})^\nu h_\nu}{\nu}) \equiv e^{W_{\rm F}}. \tag{2.16}$$

The difference from the bosonic case is a minus sign $(-1)^{\nu+1}$ for a path with a period $\nu\beta$.

When a gauge field couples, only a slight modification is necessary. The path-integral representation of a transition amplitude

$$< x_1...x_N|\exp(-\beta\sum_i H_a(p_i,x_i))|x_{\sigma_1}...x_{\sigma_N} > \tag{2.17}$$

includes an expectation value of the Polyakov loops, which wrap a time torus. The grand canonical partition function of charged bosons coupled with the Chern Simons gauge field $< Z_{\rm B}(a) >_{\rm CS}$ is written by using a replaced $h_\nu(a)$

$$h_\nu(a) \equiv \int_{\rm periodic}\mathcal{D}x\exp(-\int_0^{\nu\beta}\frac{m\dot{x}^2}{2}dt)\exp(i\int_0^{\nu\beta}a_\mu\dot{x}^\mu dt) \tag{2.18}$$

as

$$< Z_{\rm B}(a) >_{\rm CS}=< e^{W_{\rm B}(a)} >_{\rm CS}=< \exp(\sum_{\nu=1}^{\infty}\frac{(e^{\mu\beta})^{\nu}h_{\nu}(a)}{\nu}) >_{\rm CS}\,. \tag{2.19}$$

Now let's prove the equivalence of (2.16) and (2.19). For proof of the equivalence it is enough to show both that there appears an extra minus sign $(-1)^{\nu+1}$ for each path with a period $\nu\beta$

$$< \exp(i\int_0^{\nu\beta} a_\mu \dot{x}^\mu {\rm d}t) >_{\rm CS}= (-1)^{\nu+1} \tag{2.20}$$

and that there are no interactions between two disconnected paths. The expectation value of a Polyakov loop with a period $\nu\beta$ (2.20) can be calculated by using classical solutions of (1.2) in the Coulomb gauge

$$\begin{aligned} a_0^{cl}(x) &= 2J\int {\rm d}^2y\frac{\epsilon_{ij}(x-y)_j}{|x-y|^2}J_i(y) \\ a_i^{cl}(x) &= 2J\int {\rm d}^2y\frac{\epsilon_{ij}(x-y)_j}{|x-y|^2}J_0(y). \end{aligned} \tag{2.21}$$

The left hand side of the eq.(2.20) becomes

$$\begin{aligned} &< \exp(i\int_0^{\nu\beta} a_\mu(x(t))\dot{x}^\mu(t){\rm d}t) >_{\rm CS} \\ &= \exp(\frac{i}{2}\int_0^{\nu\beta} a_\mu^{cl}\dot{x}^\mu(t){\rm d}t) \equiv e^{iP[x(t)]}. \end{aligned} \tag{2.22}$$

Note the 1/2 factor in front of $\int_0^{\nu\beta} a_\mu^{cl}\dot{x}^\mu dt$. This arises because the Lagrangian includes not only the minimal coupling term $a_\mu \dot{x}^\mu$ but also the Chern-Simons term itself. This is the reason why the true A-B phase is a half of a naive A-B phase, which I commented in the footnote in page 5. At each time there are ν particles described by $x(t+l\beta)$, $0\le l\le \nu-1$. The source currents which contribute to the classical solutions (2.21) are

$$J_\mu(y) = \sum_{l=1}^{\nu-1}\delta^{(2)}(y-x(t+l\beta))\ \dot{x}_\mu(l+t\beta). \tag{2.23}$$

Here we dropped the self-energy part($l=0$). In the next chapter this self-energy becomes important when we consider the transmutation of "spin".
Inserting (2.21) and (2.23) into (2.22) the phase factor $P[x(t)]$ becomes

$$P[x(t)] = J\sum_{l=1}^{\nu-1}\int_0^{\nu\beta}\frac{\epsilon_{ij}(x(t)-x(t+l\beta))_j}{|x(t)-x(t+l\beta)|^2}\ (\dot{x}(t)-\dot{x}(t+l\beta))_i\ {\rm d}t. \tag{2.24}$$

This can be evaluated easily by using a unit vector

$$y_i^{(l)}(t) \equiv \frac{(x(t)-x(t+l\beta))_i}{|x(t)-x(t+l\beta)|} \in S^1 \tag{2.25}$$

as

$$P[x(t)] = J\sum_{l=1}^{\nu-1}\int_0^{\nu\beta}\epsilon_{ij}\ \dot{y}_i^{(l)}(t)y_j^{(l)}(t){\rm d}t \equiv J\sum_{l=1}^{\nu-1} I^{(l)}. \tag{2.26}$$

The integral $I^{(l)}$ is a topological invariant which is quantized by 2π. These invariants represent winding numbers of $y_i^{(l)}$ around a circle S^1. From an identity

$$y_i^{(l)}(t) = -y_i^{(\nu-l)}(t+l\beta), \tag{2.27}$$

it is easy to show

$$I^{(l)} = I^{(\nu-l)}, \quad 1\le l\le \nu-1 \tag{2.28}$$

and if ν is an even integer

$$I^{(\nu/2)} \in 2\pi\times(2{\rm Z}+1). \tag{2.29}$$

Therefore the phase factor has the following form:
$P[x(t)] \in 2\pi J(\nu-1)\times(2{\rm Z}+1)$.
Since we are considering the cases $J\in {\rm Z}+1/2$, (2.20) is proved;

$$< \exp(i\int_0^{\nu\beta} a_\mu\dot{x}^\mu {\rm d}t) >_{\rm CS}= e^{iP[x(t)]} = (-1)^{\nu-1}. \tag{2.30}$$

It is also easy to prove that there are no interactions between two disconnected paths. Now we have proved the equality of the grand canonical partition functions of non-relativistic particles;

$$< Z_{\rm B}(a) >_{\rm CS}= Z_{\rm F}. \tag{2.31}$$

Note that we have neglected the self-energy part because we are interested in only the transmutation of "statistics".

The system coupled with the Chern-Simons gauge field has a symmetry

$$J\to J+1. \tag{2.32}$$

This is because we have neglected the self-energy. In the next chapter we take account of the self-energy and the symmetry is lost.

3 Relativistic case

In the previous chapter the equality of the grand canonical partition function of the charged bosons and that of free fermions is shown in the non-relativistic case. In this chapter we will show the same equality in a relativistic case[13]. There are many similarities. Many differences exist, however, and the proof of the latter is much more non-trivial. The most different point is that we must consider not only the transmutation of statistics but also that of spin. As shown in chapter 2, the effective interaction between particles is "topological" in the sense that it depends only on the way of linkings of particles. Evaluatin of the self-energy correction is more subtle becuause it needs some regularization. Some people have used the point-splitting regularization in order to extract a topologically invariant information (self-linkings) from the self-energy. More careful treatment reveals that the self-energy has more informations than the topological invariant one. It will be shown in this chapter that the expectation value of a Wilson line is not quantized as it was in the previous chapter and is related to the so called "spin factor". The spin factor is a necessary tool when we describe a spinning particle in terms of a bosonic path integral. As a result particles coupled with the Chern-Simons gauge field become to acquire a spin degree of freedom. Another difference is that we must treat the measure of the functional integral carefully so as to keep the relativistic invariance.

In this chapter, by clarifying these points, we show the following three equalities: A dressed bosonic propagator by the Chern-Simons gauge field is equal to a Dirac propagator, a partition function of charged scalars is equal to that of free fermions, and an N-point correlation function of currents of charged scalars is equal to that of free fermions. The first one is proved by considering summation over random paths with fixed boundary conditions and the latter two are proved by considering summation over random closed paths with periodic boundary conditions.

This chapter is organized as follows. In section 1 we calculate the expectation value of a Wilson line and show how to deal with the self-energy. In section 2 the spin factor is introduced and an important relation between the self-energy and the spin factor is

derived. In section 3 a reparametrization invariant formulation of a path integral of a spinning particle is given. In this section we prove that a bosonic path integral with the spin factor describes a spinning particle. We then show that a dressed bosonic propagator by the Chern-Simons gauge field becomes a Dirac one in the long distance limit. In section 4 we discuss the bose-fermi transmutation at the level of the second quantized field theory and show the equality of partition functions and that of the N-point functions of currents.

The metric is Euclidean in this chapter.

3.1 Gauss-Linking Number and Self-Energy

In this section we study the expectation value of a Wilson line in a relativistic case. We must treat the self-energy correction carefully, which is neglected in the previous chapter.

We consider a charged scalar field coupled with the Chern-Simons gauge field. The field theoretical action is

$$S = S_{\rm matt} - iS_{\rm CS}, \tag{3.1}$$

where

$$S_{\rm matt} = \int d^3x \{|(\partial_\mu + ia_\mu)\phi|^2 + m^2|\phi|^2\}.$$

As in the non-relativistic case, the partition function can be represented by path-integrals over random paths of charged particles.[3] Here we consider the following path integral

$$\sum_{\rm P} e^{-mL({\rm P})} K({\rm P}), \tag{3.2}$$

where summation is taken over closed paths P, $L(\mathrm{P})$ is the sum of their lengths and

$$K({\rm P}) = \int \mathcal{D}a\; e^{i\oint_{\rm P} a_\mu dx^\mu} e^{iS_{\rm CS}}. \tag{3.3}$$

As we want to keep relativistic covariance in this chapter, we use the covariant gauge instead of the Coulomb gauge. The functional averaging over a_μ is performed as follows:

$$K({\rm P}) = e^{\frac{iJ}{2}I}, \tag{3.4}$$

where

$$I = -\int_0^1 ds \int_0^1 dt\; \dot{x}_\mu(s)\dot{y}_\nu(t)\epsilon^{\mu\nu\lambda}\frac{x_\lambda(s) - y_\lambda(t)}{|x(s) - y(t)|^3}. \tag{3.5}$$

Here we have used a two point function of the Chern-Simons gauge field in the covariant gauge

$$< a_\mu(x)a_\nu(y) >= iJ\epsilon^{\mu\nu\lambda}\frac{x^\lambda - y^\lambda}{|x-y|^3}. \tag{3.6}$$

[3]The details are shown in section 3.4

Now we fix the particle number at n. Then the expectation value of the Wilson lines is

$$K({\rm P}) = \exp(iJ \sum_{1\le i<j\le n} \Psi_{ij} + \frac{iJ}{2}\sum_{i=1}^{n}\Psi_{ii}), \tag{3.7}$$

where

$$\Psi_{ij} = -\int_0^1 ds \int_0^1 dt\; \dot{x}^i_\mu(s)\dot{x}^j_\nu(t)\frac{x^i_\lambda(s) - x^j_\lambda(t)}{|x^i(s) - x^j(t)|^3}\epsilon^{\mu\nu\lambda}. \tag{3.8}$$

$x^i(s)$ denotes a position vector of the i-th path. The first part of the exponentiate of $K(\mathrm{P})$ represents the effective interactions between particles and is shown to be quantized by $4\pi J$. The second one is the self-energy which we neglected in the previous chapter. Evaluation of the self-energy is subtle because the form of the self-energy looks singular at $s = t$. Several authors studying the Chern-Simons theory as a topological field theory use the point-splitting regularization (or the loop splitting regularization) in order to extract topologically invariant information of a knot from the self-energy [11,12]. In our treatment we use another regularization. The self-energy part is shown to be related to the spin factor introduced in the next section.

First Ψ_{ij} $(i \neq j$ and $i = j)$ can be rewritten in terms of a unit vector $\mathrm{e}(s,t)$, pointing from $\mathrm{x}^j(t)$ to $\mathrm{x}^i(s)$,

$$\mathrm{e}(s,t) = \frac{\mathrm{x}^i(s) - \mathrm{x}^j(t)}{|\mathrm{x}^i(s) - \mathrm{x}^j(t)|} \in S^2, \tag{3.9}$$

as

$$\Psi_{ij} = \int_0^1 ds \int_0^1 dt [\partial_s \mathrm{e} \times \partial_t \mathrm{e}] \cdot \mathrm{e}. \tag{3.10}$$

The integrand is the surface element of a sphere where $\mathrm{e}(s,t)$ lies. This form makes it easy to evaluate Ψ_{ij} for both cases $i = j$ and $i \neq j$.

For $i \neq j$, the vector $\mathrm{e}(s,t)$ satisfies periodic boundary conditions:

$$\mathrm{e}(s+1,t) = \mathrm{e}(s,t+1) = \mathrm{e}(s,t). \tag{3.11}$$

Therefore, the integral represents the winding number from the torus to the sphere and is quantized by 4π, an area of a unit sphere. This is called the Gauss-linking number;

$$\Psi_{ij} \in 4\pi\mathrm{Z} \quad \text{for } i \neq j. \tag{3.12}$$

For $i = j$, we must take care of the boundary conditions of $\mathrm{e}(s,t)$. For convenience, we change the variables from (s,t) to (u,t) by $u = s - t \in [0,1]$. At $u = 0$, we define $\mathrm{e}(u,t)$ by taking the following limit. Then $\mathrm{e}(u,t)$ becomes the tangent vector at t,

$$\mathrm{e}(u=0,t) = \lim_{\epsilon\to+0} \mathrm{e}(\epsilon,t) = \mathrm{e}(t) \equiv \frac{\dot{\mathrm{x}}(t)}{|\dot{\mathrm{x}}(t)|}. \tag{3.13}$$

At $u = 1$, $\mathrm{e}(u,t)$ becomes

$$\mathrm{e}(u=1,t) = \lim_{\epsilon\to+0} \mathrm{e}(1-\epsilon,t) = -\mathrm{e}(t). \tag{3.14}$$

Therefore $\mathrm{e}(u,t)$ satisfies an anti-periodic boundary condition for u

$$\mathrm{e}(0,t) = -\mathrm{e}(1,t) = \mathrm{e}(t). \tag{3.15}$$

Of course it satisfies a periodic boundary condition for t

$$\mathrm{e}(u,t+1) = \mathrm{e}(u,t). \tag{3.16}$$

Note that singular parts in the self-energy (see (3.8)) drop out of this expression because of the ϵ-tensor in eq.(3.10)[14,12]. Because of an anti-periodic boundary condition (3.15), the contribution to the integral Ψ_{ii} from the boundaries at $u = 0$ and $u = 1$ does not cancel each other and Ψ_{ii} depends on the boundary value $\mathrm{e}(t)$ of $\mathrm{e}(u,t)$. Therefore it is not quantized.

At $J = 1/2$ the Gauss-linking number Ψ_{ij} $(i \neq j)$, which is quantized by 4π, does not contribute to $K(\mathrm{P})$ and the expectation value of the wilson lines can be written as

$$K(\mathrm{P}) = \exp\left(\frac{iJ}{2}\sum_{i=1}^{N}\Psi_{ii}\right). \tag{3.17}$$

This means that there are no interactions except the self-energy. Therefore the path integral (3.2) of a charged bosonic particle coupled with the Chern-Simons gauge field is given by

$$\sum_{\mathrm{P}} e^{-mL+i\frac{J}{2}\Psi[\mathrm{e}]}, \tag{3.18}$$

where Ψ_{ii} is abbreviated to Ψ for simplicity. This form is closely related to a functional integral with the spin factor which describes a propagation of a spinning particle in

three dimensions, as shown in section 3.3. In the next section we clarify the relation between the self-energy Ψ and the spin factor Φ obtained from the $SU(2)$ coherent state.

Here we comment how the self-energy depends on the self-linking of a path. As we show in Appendix 1, the self-energy $\Psi[\mathrm{e}]$ changes 8π if linkings of a path (self-linking) changes. Therefore at $J = 1/2$ dependence on the self-linking also does not contribute to the expectation value of a Wilson line.

3.2 Self-Energy and Spin Factor

In this section we show the relation between the self-energy correction Ψ_{ii} (3.10) and the spin factor.

First we briefly review the spin coherent state of $SU(2)$ and the spin factor. The spin coherent state in the spin J representation [15] is defined by

$$|\mathrm{e}>= \exp(-i\theta\frac{(\mathrm{e}_0 \times \mathrm{e})}{|\mathrm{e}_0 \times \mathrm{e}|}\cdot \mathrm{J})|0>, \tag{3.19}$$

where J is an $SU(2)$ generator, $\mathrm{e}_0 = (0,0,1)$, θ is the angle between e and e_0 and $|0>$ denotes the highest weight vector in this representation. These states are parameterized by points on S^2. The spin coherent state has the following three properties:

$$\text{Partition of unity}: \qquad \int d\mathrm{e}|\mathrm{e}><\mathrm{e}| = 1, \tag{3.20}$$

where de is a rotationally invariant measure on a sphere.

$$\text{Inner product}: \qquad <\mathrm{e}+\delta\mathrm{e}|\mathrm{e}>= e^{iJA[\mathrm{e},\mathrm{e}+\delta\mathrm{e},\mathrm{e}_0]} + O((\delta\mathrm{e})^2), \tag{3.21}$$

where $A[\mathrm{e},\mathrm{e}+\delta\mathrm{e},\mathrm{e}_0]$ is the area of a spherical triangle with vertices $\mathrm{e},\mathrm{e}+\delta\mathrm{e}$,and e_0.

$$\text{Expectation value}: \qquad <\mathrm{e}|\mathrm{J}|\mathrm{e}>= J\mathrm{e}. \tag{3.22}$$

By using these properties, we can rewrite the transition amplitude between spin coherent states in terms of a path integral over random paths on S^2:

$$\begin{aligned} <\mathrm{e}_f|T\exp(\int_0^L dt\mathrm{J}\cdot\mathrm{S})|\mathrm{e}_i> &\equiv \lim_{N\to\infty} <\mathrm{e}_f|\prod_{i=1}^{N}(1+\Delta t\mathrm{J}\cdot\mathrm{S}(t_i))|\mathrm{e}_i> \\ &= \int \mathcal{D}e e^{iJ\Phi[e]+J\int_0^L dte\cdot\mathrm{S}}, \end{aligned} \tag{3.23}$$

where S is a c-number source and

$$\Phi[e] \equiv \int_D dudt\mathrm{e}\cdot[\partial_u\mathrm{e}\times\partial_t\mathrm{e}]. \tag{3.24}$$

$\mathrm{e}(u,t)$ is defined by extending the path $\mathrm{e}(t)$ on S^2 as follows: $\mathrm{e}(0,t)$ is some fixed vector which does not depend on t and $\mathrm{e}(1,t) = \mathrm{e}(t)$. If the trajectory of the unit vector $\mathrm{e}(t)$

on S^2 is closed, D is an area on S^2 enclosed by the trajectory. This $\Phi[e]$ is called the spin factor. The above path integral is a phase-space path-integral and the spin factor determines the symplectic structure on S^2. The left hand side of (3.23) is a generating function of N-point functions of the $SU(2)$ generators J. Therefore eq.(3.23) shows that the algebra of the $SU(2)$ generators is given by a path integral over random paths on S^2 with the spin factor. The $SU(2)$ generators J are represented by the vectors $J\mathrm{e}$.

In the remainder of this section we clarify the relation between the self-energy and the spin factor. First we show the following formula for a closed path:

$$\frac{J}{2}\Psi[\mathrm{e}] - J\Phi[\mathrm{e}] = 2\pi J \pmod{4\pi J}, \tag{3.25}$$

where $\Psi[\mathrm{e}]$ stands for the self-energy. The self-energy is defined for a path $\mathrm{X}(t)$ of a particle in three-dimensional space-time. But after all, it depends on the configuration of its tangent vector $\mathrm{e}(t)$ and the self-linkings. On the other hand, the spin factor is a functional of $\mathrm{e}(t)$ by definition. The left-hand side is easily shown to be invariant under continuous deformation of a vector field $\mathrm{e}(u,t)$. The difference of factor 2 in front of $\Psi[\mathrm{e}]$ and $\Phi[\mathrm{e}]$ arises because $\Psi[\mathrm{e}]$ has two boundaries at $u=0$ and $u=1$ but $\Phi[\mathrm{e}]$ has only one boundary at $u=1$. Any path can be deformed to a path on a certain plane by changing $\mathrm{e}(u,t)$ continuously. The resulting path is called the knot diagram of the original path. If the number of the crossing points of the knot diagram is N, $\Psi[\mathrm{e}]$ for this diagram is $4\pi N \pmod{8\pi}$, as shown in appendix A. On the other hand, $\Phi[\mathrm{e}]$ for the knot diagram is determined by a winding number M of the tangent vector $\mathrm{e}(0,t)$ of the knot diagram as $2\Phi[\mathrm{e}] = 4\pi M \pmod{8\pi}$. It can be shown that $N-M$ is an odd number for arbitrary loops. Thus eq.(3.25) has been proved. [4]

Eq.(3.25) shows that locally $J\Psi[\mathrm{e}]/2$ gives the spin J symplectic structure to the field $\mathrm{e}(t)$. The difference by $2\pi J$ from the ordinary spin factor plays an important role in section 3.4.

For a path with fixed boundaries, the relation between the spin factor and the

[4]The same equation to eq.(3.25) was obtained by Coste, Lüscher and Grundberg, Hansson *etal.* [16,14]. They argued that the observable spin of a charged Dirac particle vanishes in the context of Fermi-Bose transmutation. Our results are consistent with those of them.

self-energy is not so simple as that for a closed path. We consider the same quantity:

$$\Delta \equiv \frac{J}{2}\Psi[\mathrm{e}] - J\Phi[\mathrm{e}] \tag{3.26}$$

for a path $\mathrm{X}(t)$, whose boundaries $\mathrm{X}(0)$ and $\mathrm{X}(1)$ and the tangent vectors at these boundaries are fixed. If $\mathrm{X}(t)$ is a path connecting $\mathrm{X}(0)$ and $\mathrm{X}(1)$ straight, both $\Psi[\mathrm{e}]$ and $\Phi[\mathrm{e}]$ are zero and $\Delta = 0$. Under a continuous deformation of a path the variation of Δ does not vanish:

$$\begin{aligned} \delta\Delta &= \frac{J}{2}\int [(\delta \mathrm{e} \times \partial_t \mathrm{e}) \cdot \mathrm{e}]_{s=0}^{s=1} \mathrm{d}t - \frac{J}{2}\int [(\delta \mathrm{e} \times \partial_s \mathrm{e}) \cdot \mathrm{e}]_{t=0}^{t=1} \mathrm{d}s \\ &= J\int [(\delta \mathrm{e} \times \partial_t \mathrm{e}) \cdot \mathrm{e}]_{s=0}^{s=1} \mathrm{d}t. \end{aligned} \tag{3.27}$$

Here we used $\mathrm{e}(s,t) = -\mathrm{e}(t,s)$. In eq.(3.18) the dominant contribution to the path integral comes from those paths whose lengths $L(P)$ satisfy $L(P) - L_0 < 1/m$ where $L_0 = |\mathrm{X}(1) - \mathrm{X}(0)|$. For these paths, as L_0 becomes large $\delta\Delta$ becomes small generally with a power of L_0^{-2}. Therefore in the long distance limit of L_0 where $L_0 \gg 1/m$, the self-energy $J\Psi/2[e]$ can be identified with the spin factor $J\Phi[e]$. By using this fact we show in section 3 that at $J = 1/2$ the dressed propagator of a charged scalar particle becomes that of a spinning particle in the long distance limit. On the other hand, if $L_0 \sim 1/m$ the self-energy is far from the spin factor. In this case, the dressed propagator seems not to be either that of a spinning particle or that of a scalar particle.

Finally, we make a comment on the quantization of J. For ordinary spin factor $\Phi[\mathrm{e}]$, it has a mod 4π ambiguity and J must be an integer or a half-integer in order to define a consistent quantum mechanics for spin. This is nothing but the monopole quantization condition. However, for our $\Psi[\mathrm{e}]$, it has no such ambiguity and J does not have to be quantized. Instead, if J is not an integer or a half-integer, a non-local interactions which depend on linkings of paths remain.

3.3 Functional Integral for Relativistic Spinning Particles

In this section in order to clarify the boson-fermion transmutation we study the bosonic functional integral over random paths for spinning particles in three dimensions. Usually a path integral of spinning particles is given by using Grassmann variables. But in three dimensions it is also given by using a bosonic functional integral with the spin factor introduced in the previous section. We treat such a path integral in a reparametrization invariant manner.

We define the functional integral rigorously for both cases of fixed and closed boundary conditions. First we study the case of paths with fixed boundaries [17]. Path integral weighted by the spin factor (3.34) can be written in a covariant form under local coordinate transformations in one dimension as follows;

$$F(\mathrm{X}_f\mathrm{e}_f|\mathrm{X}_i\mathrm{e}_i) = \int \frac{\mathcal{D}X\mathcal{D}\mathrm{k}\mathcal{D}\mathrm{e}\mathcal{D}h}{V_{\mathrm{Diff}}} e^{-S} \tag{3.28}$$

where

$$S = \int_0^1 \mathrm{d}t\{m_0 h + ik_\mu(\dot{X}^\mu - he^\mu)\} - iJ\Phi[\mathrm{e}]. \tag{3.29}$$

X^μ is a position vector of the particle with boundary conditions:

$$\mathrm{X}(0) = \mathrm{X}_i, \quad \mathrm{X}(1) = \mathrm{X}_f. \tag{3.30}$$

h is an einbein and $\mathrm{e}(t)$ is a field on S^2. The action is invariant under the diffeomorphism

$$t \to f(t) \quad ; f(0) = 0, f(1) = 1 \tag{3.31}$$

if the einbein h transforms as

$$h(t) \to \dot{f}h(f(t)). \tag{3.32}$$

V_{Diff} denotes the volume of this local gauge transformations. By integrating over the multiplier field k, constraints for h and e are obtained:

$$h = \sqrt{\dot{\mathrm{X}}^2}, \quad e^\mu = \frac{\dot{\mathrm{X}}^\mu}{\sqrt{\dot{\mathrm{X}}^2}}. \tag{3.33}$$

Therefore (3.28) is equal to a bosonic path integral with the spin factor which is a higher derivative of X:

$$F(\mathrm{X}_f\mathrm{e}_f|\mathrm{X}_i\mathrm{e}_i) = \sum_{\mathrm{P}} e^{-mL+iJ\Phi}. \tag{3.34}$$

Boundary conditions are given by the coordinate X and its derivatives $\dot{\mathrm{X}}$;

$$\mathrm{X}(t=0)=\mathrm{X}_i, \qquad \mathrm{X}(t=1)=\mathrm{X}_f$$
$$\frac{\dot{\mathrm{X}}(t=0)}{|\dot{\mathrm{X}}(t=0)|}=\mathrm{e}_i, \qquad \frac{\dot{\mathrm{X}}(t=1)}{|\dot{\mathrm{X}}(t=1)|}=\mathrm{e}_f. \tag{3.35}$$

This is quite similar to the functional integral (3.18) which is obtained by integration over the Chern-Simons gauge field. The differnce is that $\Psi/2$ is replaced by Φ. At $J=0$, eq.(3.28) describes a propagation of a free scalar particle (see Appendix 3). The measures in the functional integral should be defined by the gauge invariant manner [18,19]. This can be done on the basis of the following gauge invariant norms:

$$||\delta\varphi||^2=\int_0^1 \mathrm{d}t\ h(t)(\delta\varphi(t))^2 \tag{3.36}$$

for scalar fields $\delta\varphi=\delta X^\mu, \delta e^\mu$, and δk^μ,

$$||\delta h||^2=\int_0^1 \mathrm{d}t\ h(t)(h^{-1}(t)\delta h(t))^2 \tag{3.37}$$

for a 1-form δh,

$$||\delta\xi||^2=\int_0^1 \mathrm{d}t\ h(t)(h(t)\delta\xi(t))^2 \tag{3.38}$$

for a vector field $\delta\xi$.

Here we calculate the explicit form of the functional integral and show that (3.28) describes a propagation of a Dirac particle. In order to perform the h-integral we take the following parameterization of the deformation of h field

$$\delta h=\delta L+L\partial_t\delta\xi, \tag{3.39}$$

where L is a zero mode of h, $L=\int_0^1 \mathrm{d}t h(t)$ and $\delta\xi$ is a deformation by local coordinate transformations. L is the only reparametrization invariant quantity. In this parameterization, the norm of δh is

$$||\delta h||^2=\frac{\delta L^2}{L}+\int_0^1 \mathrm{d}t L(\partial_t\delta\xi)^2, \tag{3.40}$$

and $V_{\mathrm{diff}}=\int\mathcal{D}\xi'$. Therefore the h-integral can be reduced to the zero mode integral as follows (see appendix 2):

$$\int\frac{\mathcal{D}h}{V_{\mathrm{Diff}}}=\int_0^\infty\frac{\mathrm{d}L}{\sqrt{L}}\int\frac{\mathcal{D}h'}{\mathcal{D}\xi'}=\int_0^\infty\frac{\mathrm{d}L}{\sqrt{L}}\det{}'^{\frac{1}{2}}(-L^{-2}\partial_t^2), \tag{3.41}$$

where the prime denotes the modes except the zero mode. In eq.(3.41), L-dependence of the functional determinant can be evaluated by the gauge invariant regularization [19]

$$\det{}'(-L^{-2}\partial_t^2)=\mathrm{const.}Le^{-L/\epsilon\sqrt{\pi}} \tag{3.42}$$

for functions with fixed boundary conditions, where ϵ is a cut-off parameter of ultraviolet divergence. ϵ has the dimension of length in the three-dimensional space. The calculation of this determinant is given in Appendix 2. Then, the gauge fixed functional integral is

$$F=\int_0^\infty \mathrm{d}Le^{-L/2\epsilon\sqrt{\pi}}\int\mathcal{D}\mathrm{X}\mathcal{D}\mathrm{e}\mathcal{D}\mathrm{k}e^{-S[h=L]}. \tag{3.43}$$

In eq.(3.43), we can perform the integral over X under the boundary condition (3.30). If we put

$$\mathrm{X}(t)=(1-t)\mathrm{X}_i+t\mathrm{X}_f+y(t), \tag{3.44}$$

then $y(t)$ satisfies

$$y(0)=y(1)=0. \tag{3.45}$$

Now by inserting (3.44) into (3.29) and integrating over $y(t)$, we get

$$F=\int_0^\infty \mathrm{d}Le^{-L/2\epsilon\sqrt{\pi}}\int\mathcal{D}\mathrm{e}\mathcal{D}\mathrm{k}\delta(L^{-1}\partial_t\sqrt{L}\mathrm{k})e^{-S}, \tag{3.46}$$

where

$$S=\int_0^1 \mathrm{d}t[L(m-i\mathrm{k}\cdot\mathrm{e})-i\mathrm{k}(\mathrm{X}_i-\mathrm{X}_f)]+iJ\Phi[\mathrm{e}].$$

Next by integrating over k except its zero mode and using the formula(3.42) again, one obtains the following expression of the propagator;

$$F=\int_0^\infty \mathrm{d}Le^{-mL}\int \mathrm{d}\mathrm{k}_0e^{i\mathrm{k}_0(\mathrm{X}_f-\mathrm{X}_i)}\int\mathcal{D}\mathrm{e}e^{i\int_0^1 \mathrm{d}tL\mathrm{k}_0\cdot\mathrm{e}+iJ\Phi[\mathrm{e}]} \tag{3.47}$$

where $m=m_0-(\sqrt{\pi}\epsilon)^{-1}$ and k_0 denotes the zero mode of k. The formula (3.23) enables us to rewrite the propagator in the following form if J is a non-zero integer or a half-integer:

$$F(\mathrm{k}_0)=(-iJ^{-1}\mathrm{k}_0\cdot\mathrm{J}+m)^{-1}, \tag{3.48}$$

where J is an $SU(2)$ generator. At $J=1/2$, this is the Dirac propagator [20,21,22].

At $J = 0$ we must evaluate the path integral in a different manner. As we show in Appendix 3, it gives the free bosonic propagator:

$$F(\mathrm{k}_0) = \frac{1}{\mathrm{k}_0^2 + m^2}. \tag{3.49}$$

For an integer or a half-integer $J \geq 1$, it describes a higher spin particle [17,23](see Appendix 4).

Next, we study the integral over closed paths, which is necessary in section 4. In this case, there are some technical differences from the previous case. For a closed path, X and the diffeomorphism have zero modes and the evaluation of the functional determinant should be modified. The functional integral over closed paths is defined by

$$W = \int \frac{\mathcal{D}X\mathcal{D}\mathrm{k}\mathcal{D}\mathrm{e}\mathcal{D}h}{V_{\mathrm{Diff}}V_{\mathrm{space}}} e^{-S}, \tag{3.50}$$

where S is given by eq.(3.29) and V_{space} is the volume of the three-dimensional space. The divergence due to the zero mode integral of X can be removed by the factor $V_{\mathrm{space}} = \int d\mathrm{X}_0$ in eq.(3.50). The norm of $\delta\mathrm{X}$ is (see (3.36))

$$\|\delta\mathrm{X}\|^2 = \int_0^1 \mathrm{d}t\ L(\delta\mathrm{X}_0)^2 + \|\delta\mathrm{X}'\|^2 \tag{3.51}$$

and hence

$$\int \frac{\mathcal{D}X}{V_{\mathrm{space}}} = \int \mathcal{D}X' L^{\frac{3}{2}}. \tag{3.52}$$

Decomposing the $\xi(t)$ into the zero mode ξ_0 and the other modes $\xi'(t)$, the norm for ξ becomes

$$\|\delta\xi\|^2 = \int_0^1 \mathrm{d}t\ L(L\delta\xi)^2 = L^3(\delta\xi_0)^2 + \|\delta\xi'\|^2. \tag{3.53}$$

and the volume of the diffeomorphism V_{Diff} is factorized as follows:

$$V_{\mathrm{Diff}} = \int L^{\frac{3}{2}} \mathrm{d}\xi_0 \int \mathcal{D}\xi'. \tag{3.54}$$

The h- integral can be reduced to the zero mode L- integral as

$$\begin{aligned} \int \frac{\mathcal{D}h}{V_{\mathrm{Diff}}} &= \int_0^\infty \frac{\mathrm{d}L}{\sqrt{L}} \frac{\mathcal{D}h'}{\mathcal{D}\xi'} \frac{1}{\int \mathrm{d}\xi_0 L^{\frac{3}{2}}} \\ &= \int_0^\infty \frac{\mathrm{d}L}{\sqrt{L}} \det{}'^{\frac{1}{2}}(-L^{-2}\partial_t^2) \frac{1}{\int \mathrm{d}\xi_0 L^{\frac{3}{2}}} \\ &= \int_0^\infty \frac{\mathrm{d}L}{L} e^{-L/2\epsilon\sqrt{\pi}} \frac{1}{\int \mathrm{d}\xi_0}. \end{aligned} \tag{3.55}$$

Here the L- dependence of the functional determinant for a closed path is

$$\det{}'(-L^{-2}\partial_t^2) = \mathrm{const.}L^2 e^{-L/\epsilon\sqrt{\pi}}. \tag{3.56}$$

The calculation is given in Appendix 5. The difference of this functional determinant from the fixed boundary case is due to the existence of twice the number of modes in this case. The gauge fixed functional integral becomes

$$W = \int_0^\infty \frac{\mathrm{d}L}{L} e^{-L/2\epsilon\sqrt{\pi}} \int L^{\frac{3}{2}} \mathcal{D}\mathrm{X}'\mathcal{D}\mathrm{k}\mathcal{D}\mathrm{e} e^{-S[h=L]} \frac{1}{\int \mathrm{d}\xi_0}. \tag{3.57}$$

One can perform the integral over X' and k' easily and get

$$W = \int_0^\infty \frac{\mathrm{d}L}{L} \int \mathcal{D}\mathrm{e}\mathrm{dk}_0 e^{-S}, \tag{3.58}$$

where $S = \int_0^1 \mathrm{d}t L(m - i\mathrm{k}_0 \cdot \mathrm{e}) + iJ\Phi[\mathrm{e}]$. The formula (3.23) for the $SU(2)$ coherent states enables us to rewrite W in the operator formalism

$$\begin{aligned} W &= \mathrm{Tr}' \int_0^\infty \frac{\mathrm{d}L}{L} e^{-L(-iJ^{-1}\hat{\mathrm{p}}\cdot\mathrm{J}+m)} \\ &\sim -\mathrm{Tr}'\log(-iJ^{-1}\mathrm{p}\cdot\mathrm{J} + m) \end{aligned} \tag{3.59}$$

Eq. (3.59) is employed to represent the partition function of the Dirac field in the next section. In eq.(3.59), the trace is taken over the representation of $SU(2)$ and the operator p

$$\mathrm{Tr}'\hat{O} = \sum_\alpha \int \mathrm{dp} < \mathrm{p}|\hat{O}_{\alpha\alpha}|\mathrm{p} > \frac{1}{< \mathrm{p}|\mathrm{p} >}, \tag{3.60}$$

where $\hat{\mathrm{p}}|\mathrm{p} >= \mathrm{p}|\mathrm{p} >$ and α is an index of the $SU(2)$ representation. Note $< \mathrm{p}|\mathrm{p} >= V_{\mathrm{space}}(2\pi)^{-3}$.

Lastly we comment on the dressed propagator by the Chern-Simons gauge field at $J = 1/2$. The dressed propagator is described by (3.18) with fixed boundary conditions. As we showed in section 3.2, the following relation between the self-energy Ψ and the spin fctor Φ holds in the long distance limit:

$$\frac{J}{2}\Psi[\mathrm{e}] - J\Phi[\mathrm{e}] \sim 0. \tag{3.61}$$

Therefore in this limit the path integral (3.18) which describes propagation of the charged scalars becomes equal to the path integral (3.34) which describes propagation of spinning particles. In the next section we prove the equality of the partition functions. This equality holds exactly. On the other hand, as we have seen, the equality of the propagators holds only in the long distance limit. This may be because the dressed boson by the Chern-Simons gauge field is extended by quantum effect but the true reason is not evident at present.

3.4 Bose-Fermi Transmutation in the Second-Quantized Theories

In this section we investigate the correspondence of bosons and fermions at the level of the relativistic quantum field theories. We show that the partition function of a relativistic boson theory coupled with the Chern-Simons field is identical with that of a free relativistic fermion theory. The action for the boson theory is given by (3.1). Hereafter the value of J is fixed at $1/2$. Let us start with describing how to express the partition function Z_B of the boson theory in terms of summation over random paths. Z_B can be rewritten as follows:

$$\begin{aligned} < Z_{\mathrm{B}} >_{\mathrm{CS}} &\equiv \int \frac{\mathcal{D}a_\mu \mathcal{D}\bar{\phi}\mathcal{D}\phi}{V_{\mathrm{gauge}}} e^{-S} \\ &= \int \frac{\mathcal{D}a_\mu}{V_{\mathrm{gauge}}} \det{}^{-1}(-D_\mu D_\mu + m^2) e^{iS_{CS}} \\ &\equiv < e^{W_{\mathrm{B}}} >_{\mathrm{CS}} \\ &= < \sum_n \frac{1}{n!}(W_{\mathrm{B}})^n >_{\mathrm{CS}} \end{aligned} \tag{3.62}$$

where $D_\mu \equiv \partial_\mu + ia_\mu$ and

$$\begin{aligned} W_{\mathrm{B}} &\equiv -\mathrm{Tr}\log(-D^2 + m^2) \\ &= \lim_{\epsilon \to 0} \mathrm{Tr} \int_\epsilon^\infty \frac{\mathrm{d}L}{L} e^{-L(-D^2+m^2)}. \end{aligned} \tag{3.63}$$

Let us represent W_{B} as a sum over random closed paths. By inserting the identity

$$\int |\mathrm{X} >< \mathrm{X}| \mathrm{dX} = 1, \tag{3.64}$$

W_{B} can be written as

$$W_{\mathrm{B}} = \int_\epsilon^\infty \frac{\mathrm{d}L}{L} \int \prod_{j=1}^N \mathrm{dX}_j < \mathrm{X}_j | e^{-\Delta L(-D^2+m^2)} | \mathrm{X}_{j-1} >, \tag{3.65}$$

where $\Delta L \equiv L/N$ and $\mathrm{X}_0 = \mathrm{X}_N$. The infinitesimal kernels are easily evaluated:

$$\begin{aligned} < \mathrm{X}_j | e^{-\Delta L(-D^2+m^2)} | \mathrm{X}_{j-1} > &\sim \int \frac{\mathrm{dp}}{(2\pi)^3} e^{-\Delta L(i\mathrm{p}\cdot\dot{\mathrm{X}}_j + (\mathrm{p}+\mathrm{a})^2 + m^2)} \\ &= \left(\frac{1}{4\pi\Delta L}\right)^{\frac{3}{2}} e^{-\Delta L(\frac{1}{4}(\dot{\mathrm{X}}_j)^2 - i\mathrm{a}\cdot\dot{\mathrm{X}}_j) + m^2}, \end{aligned} \tag{3.66}$$

where $\dot{X}_j \equiv (X_j - X_{j-1})/\Delta L$. When we take the limit $\Delta L \to 0$, dL/L can be regarded as the measure of einbein in the proper-time gauge as we have seen in section 3.3. From (3.55) and (3.66) we obtain

$$\begin{aligned} W_{\mathrm{B}} &= \int_{\epsilon}^{\infty} \frac{\mathrm{d}L}{L} \mathcal{D}\mathrm{X} e^{-\int_0^L (\frac{1}{4}\dot{\mathrm{X}}^2 + m^2)\mathrm{d}t} e^{i\int_0^L \mathbf{a}\cdot\dot{\mathrm{X}}\mathrm{d}t} \\ &= \int_{\epsilon}^{\infty} \frac{\mathrm{d}L}{L} \mathcal{D}\mathrm{X} e^{-\int_0^1 (\frac{1}{4L}\dot{\mathrm{X}}^2 + m^2 L)\mathrm{d}t} e^{i\int_0^1 \mathbf{a}\cdot\dot{\mathrm{X}}\mathrm{d}t} \\ &= \int \frac{\mathcal{D}h\mathcal{D}\mathrm{X}}{V_{\mathrm{Diff}}} e^{-\int_0^1 (\frac{1}{4h}\dot{\mathrm{X}}^2 + m^2 h)\mathrm{d}t} e^{i\int_0^1 \mathbf{a}\cdot\dot{\mathrm{X}}\mathrm{d}t} \\ &= \int \frac{\mathcal{D}\mathrm{X}}{V_{\mathrm{Diff}}} e^{-m\int_0^1 \sqrt{\dot{\mathrm{X}}^2}\mathrm{d}t} e^{i\int_0^1 \mathbf{a}\cdot\dot{\mathrm{X}}\mathrm{d}t}. \end{aligned} \tag{3.67}$$

We have used the formula

$$\int_0^{\infty} \frac{\mathrm{d}x}{\sqrt{x}} e^{-a^2 x - b^2/x} = \frac{\sqrt{\pi}}{a} e^{-2ab} \tag{3.68}$$

when the integral $\mathcal{D}h$ is performed in the last equality in eq. (3.67). Thus, we can express the partition function Z_{B} in the form of a particle-number series. That is, W_{B} gives the contribution to Z_{B} from the one-particle sector, and $(W_{\mathrm{B}})^n$ in (3.62) gives that from the n-particle sector.

Now let's take an average of the CS gauge field. The n-particles sector is given (see (3.7)) by

$$\begin{aligned} < (W_{\mathrm{B}})^n >_{\mathrm{CS}} &= \int \prod_{i=1}^{n} \frac{\mathcal{D}\mathrm{X}^i}{V_{\mathrm{Diff}} V_{\mathrm{space}}} \exp\{-m \int_0^1 \sum_{i=1}^{n} \sqrt{\dot{\mathrm{X}}^{i2}} \mathrm{d}t \\ &\quad + \frac{iJ}{2} \sum_{i=1}^{n} \Psi_{ii}[\mathrm{e}] + iJ \sum_{i<j} \Psi_{ij}[\mathrm{e}]\} \end{aligned} \tag{3.69}$$

where e is the unit tangent vector along the loop. As we have seen in section 3.1, Ψ_{ij} is the Gauss-linking number between the i-th and the j-th particle, which takes a value of 4π times an integer. Therefore it does not contribute to $(W_{\mathrm{B}})^n$ at $J = 1/2$. The self-energy $\Psi_{ii}[\mathrm{e}]$ is related to the spin factor Φ in eq.(3.25). As a result we get

$$Z_{\mathrm{B}} = < e^{W_{\mathrm{B}}} >_{\mathrm{CS}} \equiv e^{W_{\mathrm{F}}}, \tag{3.70}$$

where W_{F} is defined by

$$W_{\mathrm{F}} = -\int \frac{\mathcal{D}\mathrm{X}}{V_{\mathrm{diff}} V_{\mathrm{space}}} e^{-m\int_0^1 \sqrt{\dot{\mathrm{X}}^2}\mathrm{d}t + iJ\Phi[\mathrm{e}]}. \tag{3.71}$$

Note that the minus sign in the right-hand-side in eq.(3.71). This corresponds to the minus sign factor for a fermion loop. It originates from the difference $2\pi J$ between the self-energy and the spin factor in eq.(3.25). W_{F} is nothing but the amplitude for the spinning particle with the periodic boundary condition as we have shown in section 3:

$$\begin{aligned} W_{\mathrm{F}} &= -\mathrm{tr} \int_{\epsilon}^{\infty} \frac{\mathrm{d}L}{L} \int \mathrm{dp} e^{L(i\mathrm{p}\cdot\sigma - m)} \\ &= \mathrm{tr} \int \mathrm{dp} \log(-i\mathrm{p}\cdot\sigma + m) \\ &= \mathrm{Tr}' \log(-\partial\cdot\sigma + m). \end{aligned} \tag{3.72}$$

tr is a trace of the spin indices. Now we get the final result

$$< Z_{\mathrm{B}} >_{CS} = e^{W_{\mathrm{F}}} = \det(-\partial\cdot\sigma + m) \equiv Z_{\mathrm{F}}. \tag{3.73}$$

The right hand side is precisely equal to the partition function for a free fermi theory. Thus we have obtained a new result which is concerned with the bose-fermi transmutation in the relativistic quantum field theories.

We can derive a similar relation between the bose and the fermi theories in the case that an external gauge field is coupled to the matter fields. That is, the following relation also holds:

$$\begin{aligned} < Z_{\mathrm{B}}[A] >_{\mathrm{CS}} &\equiv < \det{}^{-1}((\partial - i\boldsymbol{A} - ia)^2 + m^2) >_{\mathrm{CS}} \\ &= \det((\partial - i\boldsymbol{A})\cdot\sigma + m), \\ &\equiv Z_{\mathrm{F}}[A] \end{aligned} \tag{3.74}$$

where A_μ is an external gauge field. From this identity the equality of the N-point correlation function of the currents of the charged bosons and free fermions are shown.

In this section, we have shown an equality between the partition functions of charged scalars and that of free fermions in the language of the functional integrals over random closed paths. Also the equality of the N-point correlation function of the currents of charged scalars and free fermions are shown. By these results and the result in the previous section that a dressed scalar propagator becomes a Dirac propagator, a system of charged scalars coupled with the Chern-Simons gauge field and that of free fermions are shown to be equivalent.

4 Application to the Fractional Quantum Hall Effect

In this chapter we apply the bose-fermi transmutation of non-relativistic particles to the fractional quantum Hall effect (FQHE). As we briefly review in section 4.1, FQHE is a kind of a "macroscopic" quantum effect in an effectively two-dimensional system of electrons subjected to a strong magnetic field [24]. Soon after the discovery by Tsui et al.[25], Laughlin proposed trial wave functions of the incompressible quantum liquid for the ground state and the excited states [26]. His wave functions are characterized by its incompressibility and the properties of the quasi-particle (hole) excitations. These are reviewed in section 4.1.

Although his approach is quite successful both qualitatively and quantitatively it seems that something essential has not yet been understood or discovered. One of these is the existence of the order parameter. Recently it was found that the FQHE can be explained from the idea of a new type of bose condensation. As we showed in chapter 2 and 3, fermion can be bosonized by interacting with the Chern-Simons gauge field or in other words by attaching magnetic fluxes to the particles. The idea is that the FQHE is a bose condensation of this bosonized electrons. These are reviewed in section 4.2. If the FQHE is really a bose condensation, a macroscopic wave function (order parameter field) should exist and an essentially new phenomena, which is related to the existence of the order parameter, should be found.

The bose-condensation approach succeeded in explaining all qualitative features, but there are no well-accepted microscopic derivations and, moreover, there are some unclear points. In particular, it is not obvious in these theories whether the constraint for electrons being in the lowest Landau level is correctly imposed or not. In section 4.3, by noticing an equivalence of a many-body electron-system in the lowest Landau level and that on a two-dimensional *phase* space, we propose a new interpretation of the FQHE from a view point of a phase-space path-integral. When we fill as many electrons as possible in a two-dimensional *phase* space, the uncertainty relation makes each electron occupy an area $\Delta p \cdot \Delta q = 2\pi\hbar$. It is resposible for the maximum electron

density $n_B = eB/2\pi\hbar$ in each Landau level. If some mechanism increases each occupied area to $2\pi m\hbar$ (where m is a positive odd integer), the maximum electron density becomes one m-th. This state will correspond to the fractional quantum Hall (FQH) state with a filling factor $\nu = 1/m$. We will show, in this section, that the Chern-Simons gauge field realizes this mechanism. Our approach has some similarities with the bose-condensation approach. Both of them, in particular, make use of the Chern-Simons gauge field. The basic ideas are, however, quite different. The projection on the lowest Landau level is our starting point.

This chapter is organized as follows. In section 4.1 we briefly review the FQHE and the Laughlin wave function. In section 4.2 we show how qualitative features of the FQHE are derived from the bose condensation picture. In section 4.3 we interpret the FQHE from the view point of a phase-space path-integral. It will be shown that the FQHE is a many body problemof elctrons on a two-dimensional "phase space". The key concept is a symplectic structure on it.

4.1 Laughlin Wave functions

(1) Notations

In a two dimensional system of electrons subjected to a uniform magnetic field, energy levels are split into Landau levels. Without impurities and interactions one-particle Hamiltonian is given by

$$H_0 = \frac{\pi^2}{2m} \equiv \frac{1}{2m}(\mathrm{p} - e\mathrm{A})^2. \tag{4.1}$$

The commutation relations for π's are given by

$$[\pi^i, \pi^j] = ieB\epsilon^{ij} \quad (i, j = 1, 2). \tag{4.2}$$

We set $\hbar = 1$. Therefore H_0 describes a harmonic oscillator

$$H_0 = \omega_c(a^\dagger a + 1/2) \tag{4.3}$$

where ω_c is a cyclotron frequency $\omega_c = eB/m$ and the annihilation operator is defined by

$$a = \frac{1}{\sqrt{2eB}}(\pi_x + i\pi_y). \tag{4.4}$$

In the symmetric gauge

$$\mathrm{A} = (-\frac{By}{2}, \frac{Bx}{2}) \tag{4.5}$$

the annihilation operator a is represented as

$$a = -i\sqrt{\frac{2}{eB}}(\partial_{\bar{z}} + \frac{eB}{4}z). \tag{4.6}$$

Here z is a holomorphic coordinate:$z = x + iy$. States in the lowest Landau level satisfy $a\Psi = 0$ and generally they have the following form:

$$\Psi(\bar{z}, z) = f(z)\, e^{-\frac{eB}{4}|z|^2} \tag{4.7}$$

where $f(z)$ is a holomorphic function. Next we define the guiding center coordinates [27]:

$$X^i = x^i + \frac{1}{eB}\epsilon^{ij}\pi^j. \tag{4.8}$$

the quantized σ_{xy} are odd integers. It is called the odd denominator rule. The FQHE must be essentially due to a many body effect because there are many degenerate states if ν is not an integer.

(3) Laughlin's picture

Now we explain the Laughlin's picture of the incompressible quantum liquid. He proposed a trial ground state wave function for $\nu = 1/m$ [26];

$$\Psi_m = \prod_{i<j}^{N}(z_i - z_j)^m\, e^{-\frac{eB}{4}\sum|z_i|^2}. \tag{4.14}$$

m must be an odd integer because of the anti-symmetry of the wave function. By the technique of the plasma analogy [26], it can be shown that the filling factor of this wave function is indeed $\nu = 1/m$. In particular, $\Psi_{m=1}$ is nothing but a single Slater determinant of a completely filled state(Vandermonde determinant):

$$\Psi_1 = \begin{vmatrix} 1 & z_1 & z_1^2 & \dots \\ 1 & z_2 & \dots & \\ 1 & \dots & & \\ \dots & & & \end{vmatrix} e^{-\frac{eB}{4}\sum|z_i|^2}. \tag{4.15}$$

The most essential property of Ψ_m is that there are no components with relative angular momentum less than m between any pair of particles. We will comment on it later.

He also proposed wave functions of quasi-particle and quasi-hole excitations. Quasi-hole excitation at z is created by inserting a flux quantum at z;

$$|m; z> \equiv \Psi_{m,z} = \prod_{i=1}^{N}(z_i - z)\Psi_m. \tag{4.16}$$

By the plasma analogy the deficiency of the density of electrons at z is shown to be $\delta\rho = -1/m$. This can be also shown by calculating the following Berry's phase [30]. When we move a charged particle in a magnetic field along some loop C, a phase proportional to the flux enclosed by the loop C appears. In the case of the above quasi-hole it is evaluated as

$$\gamma_0 = i\oint_C <m; z|\frac{\partial}{\partial z}|m; z> \mathrm{d}z = \frac{e\Phi}{m}. \tag{4.17}$$

These variables commute with π and therefore they commute with H_0. The guiding center coordinates are constants of motion. From the commutation relations for X's

$$[X^i, X^j] = \frac{-i}{eB}\epsilon^{ij} \tag{4.9}$$

the number of states of each Landau level per unit area is obtained:

$$n_B = \frac{eB}{2\pi}. \tag{4.10}$$

Filling factor ν is defined by

$$\nu \equiv \frac{n}{n_B} \tag{4.11}$$

where n is the density of electrons.

(2) Experiments

Here we shortly comment on the experiments. In a two-dimensional system under a magnetic field, the conductivity tensor σ for a pure system takes the following form:

$$\sigma = \begin{pmatrix} 0 & -\frac{ne}{B} \\ \frac{ne}{B} & 0 \end{pmatrix} = \begin{pmatrix} 0 & -\nu\frac{e^2}{2\pi} \\ \nu\frac{e^2}{2\pi} & 0 \end{pmatrix}, \tag{4.12}$$

where the conductivity tensor σ is defined by

$$\mathrm{J} = \sigma \mathrm{E}. \tag{4.13}$$

In 1980 von Klitzing discovered that there are plateaus for σ_{xy} at $\nu = 1, 2, \ldots$ and at these values of ν, σ_{xx} vanishes [28]. This phenomena is called the integer quantum Hall effect(IQHE).[5] The quantization of σ_{xy} is quite accurate($\sim$ 0.02 ppm). It is very curious that such an exact quantization occurs in a dirty material. Roughly speaking the IQHE is due to the splitting of the Landau levels and is believed to be a manifestation of the transport properties of a non-interacting electron system. In 1982 more curious phenomenon was discovered [25]. In nearly impurity free samples at very low temperatures σ_{xy} becomes to have plateaus at rational fillings such as $\nu = 1/3,\ 2/3,\ 2/5,\ 2/7\ldots$. The important observation is that the denominators of all

[5] Two dimensional electron system under a strong magnetic field had been extensively studied by Japanese physicists [29]. Unfortunately, however, the exact quantization of σ_{xy} was missed.

This should be identified with $e^*\Phi$ where e^* is a charge of this quasi-hole. Therefore the quasi-hole has a fractional charge e/m. The statistics of this quasi-hole is also known from similar Berry's phase when we move one quasi-hole around another one. A state with two quasi-holes at z and w is

$$|m; z, w> = \prod_i^N (z_i - z)(z_j - w)\ \Psi_m. \tag{4.18}$$

The Berry's phase is shown to be

$$\gamma_1 = i\oint_C < m; z, w|\frac{\partial}{\partial z}|m; z, w > \mathrm{d}z = \gamma_0 - \frac{2\pi}{m}, \tag{4.19}$$

where C is a loop around w. This shows that the statistics of the quasi-hole is fractional $\theta = \pi/m$. If $m = 1$, $\theta = \pi$ and the quasi-particle is a fermion. (These two results can be interpreted differently from a view point of a phase space path integral in section 4.3.)

One of the important conclusions of Laughlin's picture is that quasi-particles(holes) have a fractional charge and a fractional statistics. Another important conclusion is its incompressibility. The Laughlin wave function is very stable and all excitations should have a gap. Low energy excitations are usually described by collective modes, which are density fluctuations. As Girvin et al. showed [31] collective mode spectrum has gap at $k = 0$ and magneto-roton minimum at finite k. It is reminiscent of the Feynmann's collective mode spectrum of He-4 [32]. In that case, however, the spectrum has gapless linear dispersion at $k \sim 0$.

The Laughlin wave function can explain the FQHE at $\nu = 1/m$ very well but it cannot explain plateaus at other filling factors, such as 2/5, 2/7. These are explained by the "hierarchy" mechanism.

(4) Hierarchy

In order to explain plateaus at filling factors whose numerators are not 1, "hierarchy" mechanism was proposed [33]. When the density of electrons is increased(or decreased) from $\nu = 1/m$, quasi-particles(holes) are created. Effective interactions of these quasi-particles(holes) are repulsive and they are subjected to a strong magnetic field. Therefore these quasi-particles(holes) can be condensed into Laughlin wave functions again.

By noting that quasi-particles(holes) have fractional charge $-e/m$ $(+e/m)$ and statistics $-\pi/m$ $(+\pi/m)$, generalized Laughlin wave functions for quasi-particles(holes) are written as

$$\Psi_p = \prod_{i<j}(w_i - w_j)^{p\mp 1/m}\, e^{-\frac{|e^*|B}{4}\sum|w_i|^2} \tag{4.20}$$

where w_i's are coordinates of quasi-particles(holes), $e^* = \pm e/m$ and p is an even integer. By the plasma analogy again the density of the quasi-particles(holes) of Ψ_p is given by

$$\rho' = \frac{|e^*|B}{2\pi}\frac{1}{p\mp\frac{1}{m}}. \tag{4.21}$$

Therefore the density of electrons is

$$\begin{aligned}\rho &= \frac{eB}{2\pi m} \pm \frac{1}{m}\frac{|e^*|B}{2\pi}\frac{1}{p\mp\frac{1}{m}} \\ &= \frac{eB}{2\pi}\frac{1}{m\mp\frac{1}{p}} \\ &\equiv \frac{eB}{2\pi}\nu.\end{aligned} \tag{4.22}$$

For $m = 3$ and $p = 2$, $\nu = 2/5$ and $2/7$.

$(quasi)^2$-particles(holes) of Ψ_p are also constructed as (4.16). When these $(quasi)^2$-particles(holes) are condensed into a Laughlin state, next hierarchical FQHE at the following filling factors can be explained.

$$\nu = \frac{1}{m\mp\frac{1}{p\mp\frac{1}{p'}}}. \tag{4.23}$$

This procedure can be iterated.

It is astonishing that the Laughlin wave function are so successful. There are many unsolved problems, however, and the FQHE is not yet completely understood. In the next section a bose-condensation picture of the FQHE is given.

4.2 Ginzburg-Landau theory of FQHE

Judging from the fact that σ_{xy} is quantized quite exactly and there are very little energy dissipations ($\sigma_{xx} \sim 0$), the QHE seems to be a macroscopic quantum effect such as superfluidity or superconductivity. These two phenomena (superfluidity and superconductivity) are well understood because we have Ginzburg-Landau theories for them. For the FQHE Laughlin's picture of the incompressible quantum liquid is quite successful but its essence is still unclear:

$$\text{Is there an order parameter for FQHE?} \tag{4.24}$$

On analogy of the superfluidity Girvin et al.[34] discovered that Laughlin's ground state has a kind of off-diagonal long-range order (ODLRO) related to a bose-condensation of composite objects of a charge and a flux. This work was followed by Zhang et al.[35] and Read[36], who constructed an effective theory of FQHE as a system coupled with the Chern-Simons gauge field. The hierarchical extension was also done[37,38].

Here we briefly derive qualitative properties of FQHE from the bose condensation picture. Fermions in two-spacial dimensions can be bosonized by coupling with the Chern-Simons gauge field. The following Lagrangian of a bose field ϕ describes electrons in a magnetic field;

$$\begin{aligned}\mathcal{L} &= \phi^*(i\partial_0 + e(A_0 + a_0))\phi - \mu|\phi|^2 - \frac{1}{2M^*}\phi^*(\mathbf{p} - e(\mathbf{A}+\mathbf{a}))^2\phi - V(|\phi|^2) \\ &\quad + \frac{e^2}{4\pi m}\epsilon_{\mu\nu\lambda}a^\mu\partial^\nu a^\lambda,\end{aligned} \tag{4.25}$$

where ϕ is a boson field, A is a real electro-magnetic gauge field and a is a statistical gauge field. Equations of motion for a is

$$-\frac{e^2}{2\pi m}\epsilon_{\mu\nu\lambda}\partial^\nu a^\lambda = ej^\mu \tag{4.26}$$

where j^μ is a matter current. The boson field ϕ couples with both the uniform magnetic field B and the statistical magnetic field b. The fluxes of the latter are attached to the particles. Now assume that the boson field has a vacuum expectation value (bose condensation). In order to make the vacuum energy finite two magnetic fields must cancel:

$$B = -b = \frac{2\pi m}{e}\rho \tag{4.27}$$

The second equality is the equation of motion for the statistical gauge field(4.26). This equation determines the density of the electrons by $\nu = 1/m$. The Goldstone boson χ, which is the phase of the boson field is eaten by the gauge field a and there is no massless mode.

Next let's consider the quasi-particle excitations. In this picture vortices are identified with quasi-particles(holes). If the Goldstone boson field is singular it cannot be gauged away. This is because the Chern-Simons action is not invariant under singular gauge transformations. In order to know how a point-like vortex couples with the gauge field, we write the bose field ϕ as

$$\phi = \sqrt{\rho}\, e^{i(\chi+\chi_s)} \tag{4.28}$$

where χ_s is a singular part of the phase. When there are N vortices at x_i $(i = 1,,,N)$ with strength ϵ_i $(\epsilon = \pm 1)$, χ_s is written as

$$\chi_s = \sum_{i=1}^{N} \epsilon_i\, \theta(x - x_i) \tag{4.29}$$

where θ is an angle. As the flux quantization in the superconductivity the statistical gauge field has a singularity there;

$$\mathrm{a} \to \mathrm{a} + \frac{\nabla\chi_s}{e} \tag{4.30}$$

and the change of the statistical flux around one vortex is

$$\delta\Phi = \oint \frac{\nabla\chi_s}{e} = \frac{2\pi}{e}. \tag{4.31}$$

By the equations of motion for a (4.26) this change of flux means the deficiency of the electron density at the vortex;

$$\delta\rho = -\frac{e}{2\pi m}\delta\Phi = -\frac{1}{m}. \tag{4.32}$$

Therefore the quasi-particle(hole) has a fractional charge.

The same result can be shown more formally by inserting (4.30) into the Chern-Simons action[38]. The Chern-Simons action is not invariant under singular gauge transformations and two new terms appear;

$$\begin{aligned}\mathcal{L}_{CS} &\to \frac{e^2}{4\pi m}\epsilon^{\mu\nu\lambda}(a_\mu + \frac{\partial_\mu\chi_s}{e})\partial_\nu(a_\lambda + \frac{\partial_\mu\chi_s}{e}) \\ &= \mathcal{L}_{CS} + \frac{e}{2\pi m}\epsilon^{\mu\nu\lambda}\partial_\nu\partial_\lambda\chi_s \cdot a^\mu + \frac{1}{4\pi m}\epsilon^{\mu\nu\lambda}\partial_\mu\chi_s \cdot \partial_\nu\partial_\lambda\chi_s. \end{aligned} \tag{4.33}$$

The second term is a coupling between the topological current and the statistical gauge field

$$\frac{e}{m}K_\mu a^\mu \tag{4.34}$$

where the topological current is defined by

$$K_\mu(x) = \frac{1}{2\pi}\epsilon_{\mu\nu\lambda}\partial^\nu\partial^\lambda\chi_s. \tag{4.35}$$

This topological current is identified with the quasi-particles (holes) current. To see this it is enough to show

$$\int K_0(x)\mathrm{d}^2x = \sum_{i=1}^{N}\epsilon_i. \tag{4.36}$$

The third term can be shown to be a topological invariant and represent fractional statistics of the vortex. This term corresponds to the Hopf term in $O(3)-\sigma$ model. It can be rewritten by introducing another fictitious gauge field a' as

$$eJ_\mu a'^\mu - \frac{me^2}{4\pi}\epsilon_{\mu\nu\lambda}a'^\mu\partial^\nu a'^\lambda. \tag{4.37}$$

This is easily shown to be equivalent to the third term by integrating over the fictitious gauge field a'. Therefore the third term in (4.33) gives the fractional statistics $\theta = \pi/m$ to the quasi-particles. These are consistent with the Laughlin's picture. The extension to hierarchy is also straightforward.

In this section we reviewed a phenomenological Ginzburg-Landau theory for the FQHE. In this picture the FQHE is a bose-condensation of composite objects of the charge and the flux. There are some unclear points, however. In the next section we give another interpretation of the FQHE, which seems to be related to the bose-condensation picture.

4.3 Phase-Space interpretation of FQHE

For the purpose of understanding the FQHE as a macroscopic quantum effect the Ginzburg-Landau picture of the previous section looks very successful. But it is still unclear whether there is a Josephson-like effect which is a clear proof of the "macroscopic" quantum effect or whether there is a phase transition. Moreover it is only a phenomenology and there are no accepted microscopic derivations. A problem is what is the mass M^* in (4.25). The magnetic field is so strong that all electrons are in the lowest Landau level (L.L.L.) and the mass cannot appear in an effective field theory. Indeed Read[36] showed that the coeficient is determined by the Coulomb potential. In this section we propose a new approach to the FQHE.

At first we show that a system of electrons in a strong magnetic field can be described by a many body system on a two-dimensional phase space not on a two dimensional configuration space. The energies of higher Landau levels are so large that we can project its Hilbert space on the L.L.L. [6] The kinetic energies are degenerate on the L.L.L. and the guiding center coordinates introduced in (4.8) characterize the degenerate states in the L.L.L. Therefore the dynamics we must solve is given by the following Hamiltonian:

$$H = \sum_{a<b} \tilde{V}(I_{ab}) \tag{4.38}$$

where $\tilde{V}$ is a projected potential on the L.L.L. and I_{ab} is defined by

$$I_{ab} = (X_a - X_b)^2 + (Y_a - Y_b)^2. \tag{4.39}$$

This is a relative angular momentum operator between two particles because the coordinates X and Y are canonically conjugate

$$[X_a, Y_b] = \frac{-i}{eB}\delta_{ab}. \tag{4.40}$$

No mass appears in this dynamics. Only the range of the potential and the cyclotron radius can determine the scale of its dynamics. If the potential is repulsive, a many

[6] When we discuss a transport phenomena we cannot neglect higher levels because an electric field mix different Landau levels.

body state with higher relative angular momentum is stabler. As we commented, by expanding the Laughlin wave function with eigenfunctions of a relative angular momentum of any pair of particles there are no components with relative angular momentum less than m. This suggests that the Laughlin state is stable.

When we fill electrons completely in the phase space, the uncertainty relation makes each electron occupy the area $2\pi\hbar$. If some mechanism increases the occupied area to $2\pi m\hbar$ (where m is an odd integer), the electron density becomes one m-th. This state must correspond to the Laughlin state. We will show in this section that the Chern-Simons gauge field realizes the mechanism.

(1) Phase-Space Path-Integral

In order to analyse the dynamics of (4.38) we quantize the system by a phase-space path-integral. First let's consider a one-particle case. We define an annihilation operator b by

$$b \equiv \sqrt{\frac{eB}{2}}(X - iY). \tag{4.41}$$

(Please don't confuse with the statistical magnetic field b.) A coherent state is defined by

$$|\bar{z}> = e^{\sqrt{\frac{eB}{2}}\bar{z}b^\dagger}|0> \tag{4.42}$$

where $b|0> = 0$. This coherent state corresponds to a cyclotron motion whose center is $\bar{z}$. They satisfy the partition of unity:

$$\int |\bar{z}><z|\, e^{-\frac{eB}{2}|z|^2}\, d^2z = 1. \tag{4.43}$$

By inserting this identity into a partition function of a one-particle Hamiltonian, we obtain the following path integral form

$$\begin{aligned} \mathrm{tr}\, e^{-\beta H(a^\dagger, a)} &= \lim_{N\to\infty} \mathrm{tr}(e^{-\frac{\beta H}{N}})^N \\ &= \int \mathcal{D}z \exp(\frac{eB}{2}\int \bar{z}\dot{z}dt - \int_0^\beta H(z,\bar{z})dt). \end{aligned}$$

The path integral is a summation of all closed paths on a two-dimensional "phase" space. When we canonically quantize the above action S, the first term determines the commutation relations of the phase space variables z. We define a symplectic 1-form by

$$A \equiv \frac{-ieB}{2}\bar{z}dz. \tag{4.44}$$

This is a generalization of the canonical symplectic 1-form $A = pdq$. The number of states on the phase space is proportional to the area S. For a canonical symplectic structure $A = pdq$ the number of states in an area S is given by

$$\frac{1}{2\pi\hbar}\int_S F = \frac{1}{2\pi\hbar}\int_S \mathrm{d}p \wedge \mathrm{d}q = \frac{S}{2\pi\hbar}. \tag{4.45}$$

This is the Bohr-Sommerfeld quantization condition. For our symplectic structure it is

$$\frac{1}{2\pi\hbar}\int_S F = \frac{eB}{2\pi\hbar}S \tag{4.46}$$

Hereafter we set $\hbar = 1$. When all states are filled the density of electrons is $n_B = eB/2\pi$. (The density of states is given by the total flux divided by 2π.) The symplectic structure we considered above is uniform on the two-dimensional plane. The above result can be generalized to a non-uniform symplectic structure; i.e., the number of states in an area S is given by the surface integral of the non-uniform symplectic 2-form F:

$$\frac{1}{2\pi\hbar}\int_S F. \tag{4.47}$$

Note that $\int_S F = \oint_C A$ is an imaginary part of the action. It depends only on the geometry of the path C. In this sense it is called "geometric" phase.

(2) Laughlin state and Quasi-particles (holes)

Now we consider a many-particles case. By using the coherent-state path integral the grand canonical partition function can be written as (see chapter 2)

$$\mathrm{tr_F}\ e^{-\beta(H-\mu N)} = \sum_N \frac{(e^{\mu\beta})^N}{N!} \sum_{\sigma\in S_N} (-1)^{\epsilon(\sigma)} \int \mathcal{D}z_1....\mathcal{D}z_N\ e^{-S}, \tag{4.48}$$

where the action S is given by

$$S = \int_0^\beta (-\sum_{a=1}^N \frac{eB}{2}\bar{z}_a\dot{z}_a + \sum_{a<b}^N \tilde{V}(I_{ab}))\mathrm{d}t \tag{4.49}$$

and whose boundary conditions are

$$z_a(\beta) = z_{\sigma(a)}(0). \tag{4.50}$$

As we have shown in chapter 2, the minus sign for fermions $(-1)^{\epsilon(\sigma)}$ can be dropped if we couple the Chern-Simons gauge field instead;

$$\mathrm{tr_F}\ e^{-\beta(H-\mu N)} = \sum_N \frac{(e^{\mu\beta})^N}{N!} \sum_{\sigma\in S_N} \int \mathcal{D}z_1....\mathcal{D}z_N\ e^{-S} < e^{i\int a_\mu \dot{x}^\mu \mathrm{d}t} >_{\mathrm{CS}}. \tag{4.51}$$

The coefficient m of the CS action

$$S_{\mathrm{CS}} = \frac{1}{4\pi m}\int \epsilon_{\mu\nu\lambda} a^\mu \partial^\nu a^\lambda \mathrm{d}^3x \tag{4.52}$$

must be an odd integer.

Classically fluxes $2\pi m$ of the Chern-Simons gauge field a are attached to each particle. These fluxes are responsible for the minus sign $(-1)^{\epsilon(\sigma)}$. Quantum mechanically, however, we must be more careful because the gauge coupling term is a first order derivative in time and deforms the symplectic structure of the system. In other words the canonically conjugate momentum changes. As a result the number of states on the phase space also changes. (Remember (4.47)). In this case the total flux is given by a sum of the external magnetic field and the statistical magnetic flux. Since the statistical fluxes are attached to "particles", the change of the symplectic structure depends on how many particles there are.

Now we set N particles in some area S and then bring another particle there. Of course, due to the Pauli principle, the number of states must be less than $n_B S = (eB/2\pi)S$. Therefore the statistical fluxes attached to particles must be anti-parallel to the uniform magnetic field B. Each particle has $2\pi m$ statistical fluxes. Therefore the density of states of the particle in the area S is given by the following geometric phase:

$$\frac{1}{2\pi}\int_S F + \frac{1}{2\pi}\int_S f = \frac{eB}{2\pi}S - mN. \tag{4.53}$$

This cannot be negative and determines the maximum density of electrons. It is determined by the condition that these two fluxes (4.53) cancel. To summarize, if the

Fermi statistics is substituted by coupling with the CS gauge field with a coefficient m, the density of a maximumly filled state is

$$N_{max} = \frac{eB}{2\pi m} = \frac{n_B}{m}. \tag{4.54}$$

The filling factor of this state is $\nu = 1/m$. This state may be identified with the Laughlin state because coupling with the CS gauge field with a coefficient m is, in a sense, equal to increasing relative angular momentum of any pair of particles by m. Also note that the Laughlin state has no components with relative angular momentum less than m.

It is very curious that we reached different goals from the same starting point. The difference will come from the substitution of the Fermi statistics by coupling to the Chern-Simons gauge field. Even if the substitution is justified in path integral representations, it may not be an equivalent rewriting in the Hamiltonian formalism. The coefficient m is chosen by hands here. It should be determined, of course, by the dynamics itself. This is one of the future problems.

Quasi-particles (holes) are created by inserting a flux quantum of the statistical gauge field in the phase space. As we have seen, the occupied area is determined by the attached fluxes. For example $2\pi m$ fluxes are attached to the electrons and therefore each electron occupies $2\pi m$ area on the phase space. The above quasi-particle (hole) occupies unit area ($=2\pi$) because a unit flux is attached to the quasi-particle (hole). As a result the charge of the quasi-particles (holes) is $-e/m$ ($+e/m$). This rough argument can be confirmed by calculating the "geometric" phase (4.53) when there is one quasi-particle (hole) in the area S. Since quasi-particles (holes) have a unit flux $-2\pi(+2\pi)$ of the statistical gauge field the number of electrons N_{max} of a completely filled state is determined by

$$\frac{eB}{2\pi}S - mN_{max} \mp 1 = 0. \tag{4.55}$$

Therefore

$$N_{max} = \frac{eB}{2\pi m}S \pm \frac{1}{m}. \tag{4.56}$$

This means that if there is a quasi-particle (hole) the total electron number increases (decreases) by 1/m.

Charge and statistics of the quasi-particles (holes) are also derived by the same equation as (4.33). Since quasi-particles (holes) are characterized by their unit flux, the coupling of the quasi-particle (hole) with the statistical gauge field is determined by (4.33).

(3) Hierarchy

Next we interpret the hierarchy by the phase-space path-integral. Let $|m; z>$ be the FQH state of $\nu = 1/m$ with a quasi-particle (hole) at z. These $|m; z>$ are not orthogonal but (over) complete. Therefore we can choose the measure $\mathrm{d}^2 z$ so as to satisfy

$$\int |m; z> < m; z|\ \mathrm{d}^2 z = 1. \tag{4.57}$$

(Herafter we replace $|m; z>$ by $|z>$ for simplicity.) In order to obtain a phase-space path-integral of the quasi-particle (hole) we insert the above partition of unity into the partition function. First let's consider a one-particle case:

$$\begin{aligned} \mathrm{tr}\ e^{-\beta H} &= \lim_{N\to\infty} \mathrm{tr}(e^{-\frac{\beta H}{N}})^N \\ &= \lim_{N\to\infty} \int \mathrm{d}^2 z_1 ... \mathrm{d}^2 z_N < z_N|e^{-\frac{\beta H}{N}}|z_{N-1}> < z_{N-1}|..... \\ &\qquad |z_1> < z_1|e^{-\frac{\beta H}{N}}|z_N> \\ &= \int \mathcal{D}z \exp(-\int_0^\beta (< z|\frac{\mathrm{d}}{\mathrm{d}t}|z> + H)\mathrm{d}t). \end{aligned} \tag{4.58}$$

The number of states of the quasi-particles (holes) in an area S is given by the following "geometric" phase:

$$\gamma_0 = i \oint_C < z|\frac{\partial}{\partial z}|z> \mathrm{d}z. \tag{4.59}$$

(The number of states is $\gamma_0/2\pi$.) For a quasi-hole it is already given in (4.17):

$$\gamma_0 = \frac{eB}{m}S. \tag{4.60}$$

This geometric phase is also obtained from (4.34).

Next let's consider a many-body problem. As is seen in eq.(4.53), the "geometric" phase changes if there are quasi-particles (holes) in the area S. From (4.19) if there is one quasi-hole the number of states decreases 1/m. That is, one quasi-hole occupies at least $2\pi/m$ area on the phase space. Here we couple another Chern-Simons gauge field a' to the quasi-holes. Without changing the statistics, the occupied area by one quasi-hole can be changed as

$$\frac{2\pi}{m} \longrightarrow 2\pi(p+\frac{1}{m}) \tag{4.61}$$

where p is an even integer. The maximum density N'_{max} of the quasi-hole is determined by (4.60) and (4.61):

$$\gamma_0 = 2\pi(p+\frac{1}{m})N'_{max}. \tag{4.62}$$

Therefore the density of the quasi-holes is

$$\rho' = \frac{eB}{2\pi m}\frac{1}{p+\frac{1}{m}} \tag{4.63}$$

and by (4.22) the hierarchical FQH condition is obtained.

In this section we showed that the FQHE can be interpreted from the view point of a many body problem on the two-dimensional phase space. Our interpretation is mere an interpretation at present but I believe that it can be a good starting point for deep understanding of the FQHE.

Relation to the bose-condensation picture initiated by Girvin *et al.* is not evident but there are some similarities. In both cases the density of electrons is determined by the condition that the external magnetic field B is cancelled by the averaged statistical magnetic field b. Also the quasi-particles are described as vortices with a fractional charge and a fractional statistics. Despite these similarities, *the basic ideas are quite different.* In our approach, we start from restricting the Hilbert space to the lowest Landau level and thereby the dynamics are described on the two-dimensional phase space. Only a finite number of electrons can be filled in a finite area on the phase space. For such a system, it is uncertain that we can apply techniques or concepts of usual field theories by which we can deal with systems of infinite degrees of freedom. Rather, we should develop new techniques and concepts of a many-body problem on the *phase* space.

Dynamics of the FQHE are not discusssed in this paper. We must first investigate how the collective modes are described in our approach. It is certainly possible since they are constructed on the ground state. In this paper, we determined the coefficient

of the Chern-Simons action m by hands. It should be determined, of course, by the dymnamics itself. This problem will be difficult since we must treat all states with a generic filling factor. The difficulty is a common one for all approaches. The FQHE is the first example of the dynamics on the phase space, which has not yet been fully discussed and in which some exciting phenomena are expected to be discovered.

The future problems are to solve the dynamics and to search new phenomena from these pictures.

5 Conclusions and Discussions

In this paper we studied the bose-fermi transmutation in (2+1) dimensional field theories. Fermions can be bosonized by interacting with the Chern-Simons gauge field or in other words by attaching magnetic fluxes to particles. Also bosons are transmuted to fermions. In chapter 2, we discussed the transmutation in the non-relativistic case. Equivalence of the grand canonical partition functions of charged bosons and free fermions were proved in terms of the path integral language. Since we neglected the self-energy correction in this chapter the CS gauge field gives an effect only on changing the statistics of particles.

In chapter 3 we discussed the transmutation in the relativistic case. We studied the self-energy carefully and showed that it is closely related to the spin factor. The spin factor gives the commutation relations of the $SU(2)$ generators to the tangent vector of a path and therefore a bosonic path integral with the spin factor describes spinning particles. As a result charged bosonic particles dressed by the CS gauge field acquires a spin degree of freedom. By considering the path integrals both with fixed boundary conditions and with closed boundary conditions we proved the following three equalities: The dressed propagator of the charged scalars is equal to the Dirac propagator in the long distance limit, the partition function of the charged scalars is equal to that of the free fermions and the N-point correlation function of currents of charged scalars is equal to that of free fermions. The equalities of the quantities including only internal lines of matters hold exactly such as the latter two equalities. For the equalities of quantities including external lines of matters, however, the treatment of the boundary conditions in the charged boson theory remains to be studied.

Here we comment on the intersection problem. In chapter 2 and 3, we implicitly assumed that particles do not intersect. If they intersect, it becomes difficult to calculate a well-defined expectation value of Wilson lines. I guess that we can neglect intersections when bosons become fermions by the CS gauge field because the fractal dimension of fermions is 1. On the other hand since the fractal dimension of bosons is 2, the intersection problem will not be able to be neglected when fermions become

bosons.

The idea of the bose-fermi transmutation in the two-spacial dimensions is applied to the fractional quantum Hall effect in chapter 4. the FQHE can be interpreted as a bose-condensation of the bosonized electrons by the CS gauge field. All the phenomena are explained from this picture at least qualitatively. Although the approach is desirable for deep understanding of the FQHE, there are no microscopic derivations and, moreover, there are some unclear points. In particular, it is not obvious in this approch whther the constraint for electrons in the lowest Landau level is correctly imposed or not. We gave a new interpretation of the FQHE, using the bose-fermi transmutation and imposing the above constraint. If we project the Hilbert space on the lowest Landau level, a system of electrons subjected to a strong magnetic field is equal to a system on a two-dimensional phase space, in which X and Y coordinates are canonically conjugate. The most important characteristic of an electron-system on the *phase* space is that there is the maximum density for electrons. It corresponds to the density of the state with a filling factor $\nu = 1$. Each electron occupies a unit area $\Delta p \cdot \Delta q = 2\pi\hbar$ on the phase space. We showed that, through substituting the Fermi statistics of electrons by coupling to the Chern-Simons gauge field, the occupied area becomes m-times, $2\pi m\hbar$, and the density of electrons becomes one m-th accordingly. It is curious that we reached differnt goals from the same starting point. I guess that, although the substitution of the Fermi statistics by coupling with the Chern-Simons gauge field is justified in the path integral representations, it may not be an equivalent rewriting if we quantize the action in Hamiltonian formalism. Quasi-particles (-holes) are created by piercing a hole on the phase space and inserting a unit magnetic flux of the Chern-Simons gauge field in it. They occupy a unit area $\pm 2\pi\hbar$ on the phase space, one m-th compared to an electron, and acquire a fractional charge and a fractional statistics. The dynamics of these quasi-particles (-holes) are also described as a many-bodysystem on the two-dimensional *phase* space. The hierarchy is explained straightforwardly from this picture.

There are many unsolved problems in the FQHE. One of them is whether there are new phenomena which is related to the existence of the order parameter. In the superconductivity Josephson effect is such a phenomena. In the FQHE Josephson effect may not exist but I guess an existence of an exotic new phenomena related to the edge states [39]. Another important problem is the dynamics of the FQHE. We have no examples of a many-body problem on the "phase" space. As the discovery of the renormalization group stimulated the study of the Kondo effect and the critical phenomena, some new method has to be developed.

A Appendix

All the following appendices are those for chapter 3.

A.1 Self-Linking Dependence of the Self-Energy

In this appendix we explain how the self-energy (3.10) depends on the self-linking of a path. We consider a difference between the self-energy Ψ_{ii} of two loops almost on a plane (Figure 1 ,2), which are different from each other only at the crossing point. Although the tangent vector $e(t)$ of each loop is almost the same, the self-energy Ψ_{ii} differs by 8π. This can be shown as follows. For each loop, the unit vector $e(s,t)$ lies on a plane except around two points on the (s,t)-parameter space, where $x(s)$ and $x(t)$ lie on the two different lines near the crossing point (Figs.3(a) and 4(a) for loop A and Figs.5(a) and 6(a) for loop B). The integrand of Ψ_{ii} vanishes except around these two points. For loop A, the contribution to the integral Ψ_{ii} from the neighborhood of each point is $+2\pi$ because $e(s,t)$ covers half of the sphere. Figs.3(b) and 4(b) show the direction of the unit vector $e(s,t)$ around these two points. Summing up these two contributions, the value of self-energy Ψ_{ii} is $+4\pi$ for loop A. On the other hand, for loop B, the contribution to Ψ_{ii} from each point is -2π (Fig.5(b) and 6(b)) and Ψ_{ii} becomes -4π. Therefore, the difference of Ψ_{ii} between two paths, whose tangent vectors are almost the same but linkings are different is 8π.

To conclude, the self-energy Ψ_{ii} is a sum of the continuous functional of the tangent vector $e(t)$ and $8\pi Z$ which depends on the linkings of the path. At $J = 1/2$, this self-linking dependence does not cotribute to (3.18).

A.2 $\det'^{1/2}(-L^{-2}\partial_t^2)$ with Fixed Boundaries

In this appendix we calculate the Jacobian factor (3.42)

$$\int \frac{\mathcal{D}h'}{\mathcal{D}\xi'} = \det'^{1/2}(-L^{-2}\partial_t^2) = \text{const.}L\ e^{-L/\epsilon\sqrt{\pi}} \tag{A.1}$$

for functions with fixed boundary conditions [8]. Reparametrization transformations are given by

$$t \to t' = \xi(t) = t - \delta\xi(t) \tag{A.2}$$

where it satisfies fixed boundary conditions

$$\delta\xi(0) = \delta\xi(1) = 0. \tag{A.3}$$

$\delta\xi(t)$ can be expanded as

$$\delta\xi(t) = \sum \delta a_n \ \sin(n\pi t). \tag{A.4}$$

Each $\sin(n\pi t)$ is an eigenfunction of $-L^{-2}\partial_t^2$ with an eigenvalue $\lambda_n = (\frac{n\pi}{L})^2$. The measure of $\|\delta\xi\|^2$ and $\|\delta h'\|^2$ are given (see (3.38) and (3.37)) by

$$\|\delta\xi\|^2 = \int_0^1 L^3(\delta\xi)^2 \mathrm{d}t = L^3 \sum(\delta a_n)^2 \tag{A.5}$$

$$\|\delta h'\| = \int_0^1 L(\partial_t \delta\xi)^2 \mathrm{d}t = L\sum(n\pi\delta a_n)^2. \tag{A.6}$$

Then

$$\begin{aligned} \mathcal{D}\xi &= \prod_n L^{3/2} \mathrm{d}a_n, \\ \mathcal{D}n' &= \prod_n L^{1/2} n\pi \mathrm{d}a_n \end{aligned} \tag{A.7}$$

and therefore

$$\frac{\mathcal{D}h'}{\mathcal{D}\xi} = \prod_n (\frac{n\pi}{L}) = \det{}'^{1/2}(-L^{-2}\partial_t^2). \tag{A.8}$$

Now let's evaluate the determinant for functions with fixed boundary conditions.

We regularize the determinant by the Gaussian regularization;

$$-\ln \det{}'(-L^{-2}\partial_t^2) = \int_{\epsilon^2}^{\infty} \frac{\mathrm{d}\tau}{\tau} \sum_n e^{-\tau\lambda_n}. \tag{A.9}$$

At first we rewrite the summation of n as

$$\begin{aligned} \sum_{n=1}^{\infty} e^{-(\frac{n\pi}{L})^2\tau} &= \frac{1}{2}\sum_{-\infty}^{\infty} e^{-(\frac{n\pi}{L})^2\tau} - \frac{1}{2} \\ &= \frac{1}{2}\int_{-\infty}^{\infty} e^{-(\frac{n\pi}{L})^2\tau x^2} dx + O(\exp(-\frac{c}{\tau})) - \frac{1}{2} \\ &= \frac{L}{2\sqrt{\pi\tau}} + O(\exp(-\frac{c}{\tau})) - \frac{1}{2}. \end{aligned} \tag{A.10}$$

We t en eva uate t e eterm nant y v ng t to t e o ow ng two terms;

$$- \text{og} \quad \text{et}'(- \quad ^{-2} \ ^{2}_{t}) = \int_{(\epsilon \ L)^2}^{\infty} \frac{\tau}{\tau} \sum_n e^{-(n\pi)^2\tau} = \int_{(\epsilon \ L)^2}^{1} + \int_1^{\infty}. \tag{ .11}$$

The second term does not depend on L. By inserting (A.10)

$$\begin{aligned} -\log \det{}'(-L^{-2}\partial_t^2) &\sim \int_{(\epsilon/L)^2}^{1} \frac{\mathrm{d}\tau}{\tau}(\frac{1}{2\sqrt{\pi\tau}} - \frac{1}{2}) + (\text{L-independent term}) \\ &= (\frac{L}{\epsilon\sqrt{\pi}} - \log\frac{L}{\epsilon}) + (\text{L-independent term}). \end{aligned} \tag{A.12}$$

Therefore

$$\det{}'(-L^{-2}\partial_t^2) = \text{const.} L \ e^{-L/\epsilon\sqrt{\pi}}. \tag{A.13}$$

A.3 $J = 0$ Case

In this appendix we evaluate (3.28) when $J = 0$. At first by integrating over k_μ, the field on S^2 e and einbein h are solved by

$$\mathrm{e} = \frac{\dot{\mathrm{X}}}{|\dot{\mathrm{X}}|}, \ h = \sqrt{\dot{\mathrm{X}}^2}. \tag{A.14}$$

Therefore (3.28) becomes

$$F = \int \frac{\mathcal{D}\mathrm{X}}{V_{\mathrm{Diff}}} e^{-m\int_0^1 \sqrt{\dot{\mathrm{X}}^2} dt} \tag{A.15}$$

This can be rewritten by using the einbein h again as

$$F = \int \frac{\mathcal{D}h\mathcal{D}\mathrm{X}}{V_{\mathrm{Diff}}} e^{-\int_0^1 dt(\frac{1}{4h}\dot{\mathrm{X}}^2 + m^2 h)}. \tag{A.16}$$

Here we used the formula (3.68). Then by fixing the gauge of $h(t)$ and using (3.42), F is written by

$$F = \int dL\mathcal{D}\mathrm{X} e^{-\int_0^L dt(\frac{\dot{\mathrm{X}}^2}{4} + m^2)}. \tag{A.17}$$

This functional integral is equaivalent with the following transition kernel (see eq.(3.66)):

$$F = \int_0^{\infty} \mathrm{d}L < \mathrm{X}_f | e^{-L(-\partial^2 + m^2)} | \mathrm{X}_i > = < \mathrm{X}_f | \frac{1}{-\partial^2 + m^2} | \mathrm{X}_i > . \tag{A.18}$$

Therefore (3.28) describes a propagation of a free scalar field at $J = 0$.

A.4 Higher J

As we see in chapter 2, a theory coupled with the Chern Simons gauge field is symmetric under $J \to J+1$. However in a relativistic case we cannot neglect the self-energy correction and the symmetry is broken. In this appendix we comment on what eq.(3.1) describes at $J=1$. For higher J, see [17].

For spin 1 ($J=1$) case, the corresponding field theory is "massive Chern Simons theory":

$$\mathcal{L} = A^{\mu}(\epsilon_{\mu\nu\lambda}\partial^{\nu} + m\delta_{\mu\lambda})A^{\lambda}, \tag{A.19}$$

whose propagator is

$$\frac{-i}{p^2 - m^2}(\epsilon_{\mu\nu\lambda}p_{\lambda} + \frac{i}{m}p^{\mu}p^{\nu} - im\delta^{\mu\nu}). \tag{A.20}$$

This theory describes a free massive scalar. For higher J, eq.(3.1) can be shown to have more than one particle.

A.5 $\det'(-L^{-2}\partial_t^2)$ with Periodic Boundaries

In this appendix we calculate the determinant for functions with periodic boundary conditions (3.56). Reparametrization transformation is given by

$$t \to t' = \xi(t) = t - \delta\xi(t) \tag{A.21}$$

where it satisfies periodic boundary conditions:

$$\delta\xi(0) = \delta\xi(1). \tag{A.22}$$

In this case eigenfunctions of $-L^{-2}\partial_t^2$ are $e^{i2\pi nt}$ with eigenvalues $\lambda_n = (\frac{2\pi n}{L})^2$. Therefore

$$\begin{aligned} -\ln \det'(-L^{-2}\partial_t^2) &= \int_{\epsilon^2}^{\infty}\frac{d\tau}{\tau}\sum_{n\neq 0}e^{-\tau\lambda_n^2} = \int_{(\epsilon/L)^2}^{\infty}\frac{d\tau}{\tau}\sum_{n\neq 0}e^{-\tau(4\pi^2n^2)} \\ &= \int_{(\epsilon/L)^2}^{\infty}\frac{d\tau}{\tau}(\sum_n e^{-\tau(4\pi^2n^2)} - 1). \end{aligned} \tag{A.23}$$

The sum of the above integral is rewritten as

$$\begin{aligned} (\sum_n e^{-\tau(4\pi^2n^2)} - 1) &= \int_{-\infty}^{\infty} e^{-4\pi^2x^2}dx + O(\exp(-\frac{c}{\epsilon})) - 1 \\ &= \frac{1}{2\sqrt{\pi\tau}} + O(\exp(-\frac{c}{\epsilon})) - 1. \end{aligned} \tag{A.24}$$

By inserting it to (A.23) and integrating, we get

$$-\ln \det'(-L^{-2}\partial_t^2) = \frac{L}{\epsilon\sqrt{\pi}} - 2\log(\frac{L}{\epsilon}) + (\text{L-independent term}). \tag{A.25}$$

Therefore the determinant is

$$\det'(-L^{-2}\partial_t^2) = \text{const.}\ (\frac{L}{\epsilon})^2 e^{-\frac{L}{\epsilon\sqrt{\pi}}}. \tag{A.26}$$

References

[1] J. M. Leinaas and J. Myrlheim, *Nouvo Cimento* **37B**(1977)1

[2] F. Wilczek and A. Zee, *Phys. Rev. Lett.*,**51**(1983)2250

[3] Here we list only a few of them.

G. W. Semenoff, *Phys. Rev. Lett.*,**61**(1988)517;

Frölich and Marchetti, *Comm. Math. Phys.*,**121**(1989)177;

G.Dunne, R. Jackiw and C. Trugenberger,*Ann. Phys.*, **194**(1989)197.

S. D. Deser and A. N. Redlich,*Phys. Rev. Lett.* **61**(1988)1541

A good but old review is

R. Mackenzie and F. Wilczek, *Int.J.Mod.Phys* **A3**(1988)2827

[4] M. Lüscher,*Nucl. Phys.*B**326**(1989)557

T. Matsuyama, *Phys. Lett.*B**228**(1989)99

[5] R. B. Laughlin *Phys. Rev. Lett.* **60**(1988)2677

V. Kalmeyer and R. B. Laughlin, *Phys. Rev. Lett.* **59**(1987)2095

[6] Y. H. Chen, F. Wilczek, E. Witten and B. Halperin *Int. J.Mod.Phys* **b3** (1989)1001

Y. Hosotani and S. Chakrabarty, IASSNS-HEP-89/31

X. G. Wen and A. Zee *Phys.Rev.* **B41**(1990)2401

T. Banks and J. Lyken, Santa Crutz preprint SCIPP 89/12

Y. Kitazawa and H. Murayama, *Phys.Rev.* **B41** (1990) 11101

[7] A. S. Goldhaber, R. Mackenzie and F. Wilczek,

*Mod.Phys.Lett.A*Vol.4(1989)21

[8] A. M. Polyakov, *Mod. Phys. Lett.*,**A3**(1988)325.

[9] C. Itoi,*Phys. Lett.*B**211** (1988)146

[10] C. H. Tze, *Int. Jour. Mod. Phys.*,**A3**(1988)1959.

[11] E. Witten, *Commun. Math. Phys.*, **121**(1989)351.

[12] E. Guadagnini, M. Martellini and Mintchev, "Wilson Lines in Chern-Simons Theory and Link Invariants", CERN preprint CERN-TH.5420/89, IFUP-TH 24/89.

[13] S. Iso, C. Itoi and H. Mukaida, *Nucl.Phys* **B346**(1990)293

[14] J. Grundberg, T. H. Hansson, A. Karlhede and U. Lindström,

*Phys. Lett.*B**218**(1989)321

[15] A. Perelomov, *Generalized Coherent States and Their Applications*, (Springer-Verlag, Berlin, 1986);

E. Fradkin and M. Stone, *Phys. Rev.*,**B38**(1988),7215;

J. R. Klauder and B. S. Skagerstam, *Coherent States, Applications in Physics and Mathematical Physics.* (World Scientific, Singapore, 1985).

[16] A. Coste and M. Lüscher *Nucl. Phys.*B **323**(1989)631

[17] S. Iso, C. Itoi and H. Mukaida, *Phys. Lett.*B **236**(1990)287

[18] A. Cohen, G. Moore, P. Nelson and J. Polchinski,

Nucl. Phys.,B**267**(1986)143.

[19] A. M. Polyakov, *Gauge Fields and Strings*, (Harwood Academic, New York, 1987)

[20] A. Yu. Alekseev and S. L. Shatashvili,*Mod. Phys. Lett.*, **A3**(1988)1551.

[21] P. B. Wiegmann, *Nucl. Phys.*,B**323**(1989)311.

[22] T. Jacobson, *Phys. Lett.* **B216**(1989)150

[23] N. Imai, K. Ishikawa and I. Tanaka, "Quantum Theory of Relativistic Particles with Torsion", Hokkaido University preprint EPHOU-89-003

The refernces below are for the FQHE.

[24] R. E. Prange and S. M. Girvin (eds) The Quntun Hall Effect, Springer-Verlag (1988)

[25] D. C. Tsui, H. L. Strormer and A. C. Gosssard, *Phys. Rev. Lett.* **44**(1982)341

[26] R. B. Laughlin, *Phys. Rev. Lett.* **50**(1983)1385

[27] R. Kubo, S. J. Miyake and N. Hashitsume, Solid State Physics vol.17, 269 F. Seitz and D. Turnbull(eds), Academic Press(1965)

[28] von Klitzing, G. Dorda and M. Pepper, *Phys.Rev.Lett.* 45(1980)495

[29] T. Ando, Y. Matsunoto and Y. Uemura, *J.Phys.Soc.Japan* **36**(1975)279
S. Kawaji, T. Igarashi and J. Wakabayashi, *Prog.Theor.Phys.Suppl.* **57**(1975)176

[30] D. Arovas, J. R. Schrieffer and F. Wilczek, *Phys.Rev.Lett.* **53**(1984)722

[31] S. M. Girvin, A. H. Macdonald and P. M. Platzman, *Phys.Rev.B* **33**(1986)2481

[32] R. P. Feynmann, 1972 Statistical Mechanics (Benjamin reading Mass)

[33] F. D. M. Haldane, *Phys.Rev.Lett.* **51**(1983)605
B. I. Halperin, *Phys.Rev.Lett.* **52**(1984)1583
R. B. Laughlin, *Surface Sience* **141**(1984)11

[34] S. M. Girvin, in ref[24] Chapter 10
S. M. Girvin and A. H. Macdonald, *Phys.Rev.Lett.* **58**(1987)1252

[35] S. C. Zhang, T. H. Hanson and S. Kivelson, *Phys.Rev.Lett.* **62**(1989) 82

[36] N. Read,*Phys.Rev.Lett.* **62**(1989)86

[37] S. Iso in INS-Report INS-T-493 (Feb. 1990) edited by H.Yamamoto

[38] Z. F. Ezawa and A.Iwazaki, Touhoku univ. preprint, TU-352 (Apr.1990)

[39] B. I. Halperin, *Phys.Rev.* **B25**(1982)25
X. G. Wen, ITP-preprint NSF-ITP-89-157

Figure Captions

The figures are used in appendix 1, where we calculate the self-linking dependence of the self-energy.

Fig.1,2 These figures represent two closed loops , whose tangent vectors are almost the same but their linkings are different.

Fig.3(a),4(a) These figures are the enlarged figures of the nearly-intersecting point of loop A. The bold-faced lines represent unit vectors $\mathbf{e}(s,t)$. Only around these vectors, the integrand of Ψ_{ii} does not vanish.

Fig.3(b),4(b) These figures represent how the directions of unit vectors $\mathbf{e}(s,t)$ change around these two vectors of Fig.3(a) and 4(a). At the middle points, the unit vectors are perpendicular to the sheets. The contribution to Ψ_{ii} from the neiborhood of each vector is $+2\pi$. From these two figures we know that the value of the self-energy Ψ_{ii} for the loop A is 4π.

Fig.5,6 These two figures represent the same figures for loop B as Fig.3 and 4 for loop A. The contribution to Ψ_{ii} from the neiborhood of each vector is -2π. Therefore the value of the self-energy is -4π for loop B.

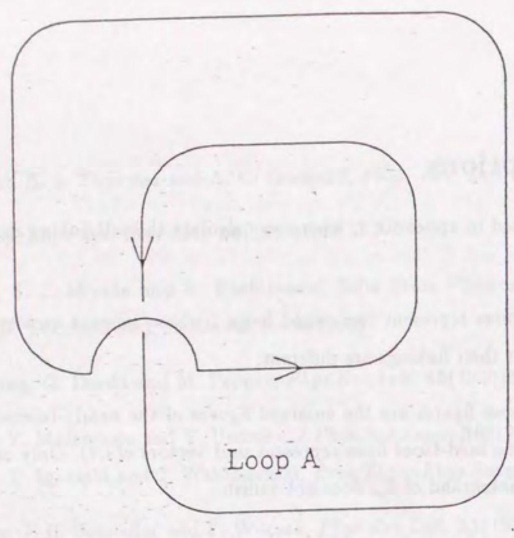

Fig.1

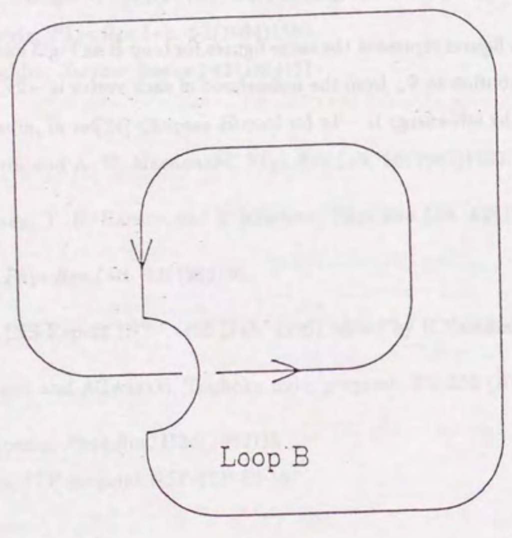

Fig.2

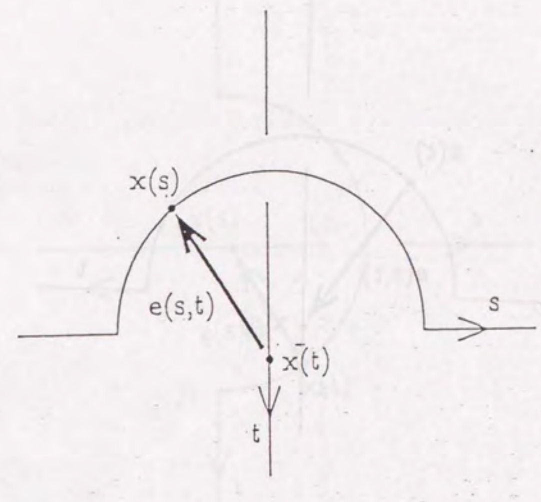

Fig.3(a)

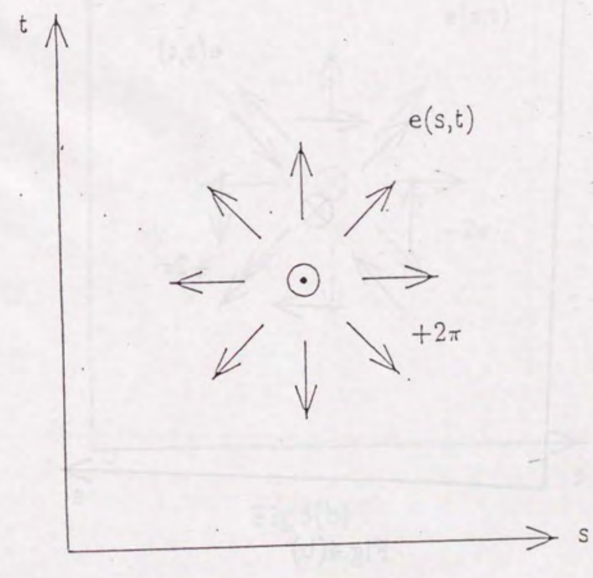

Fig.3(b)

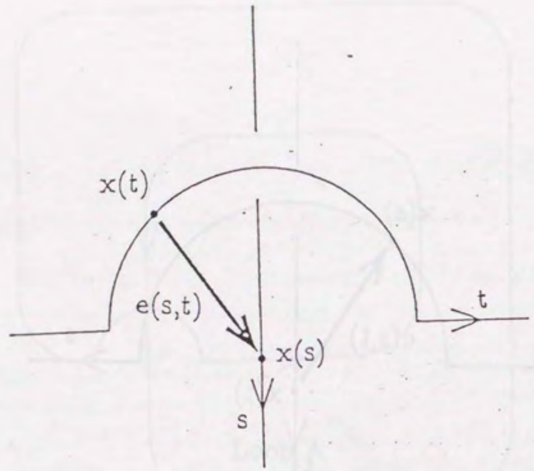

Fig.4(a)

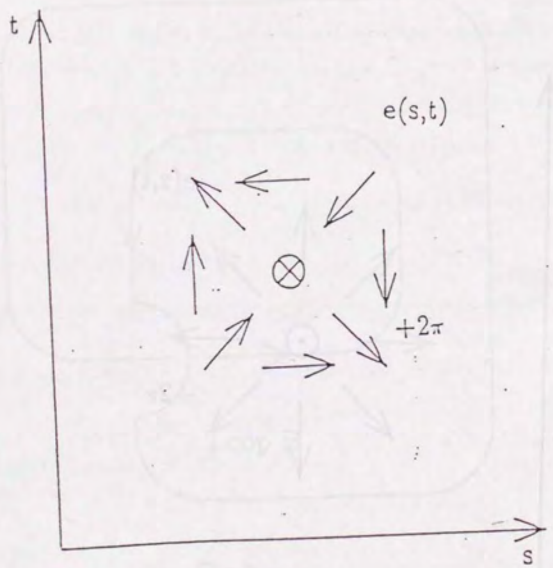

Fig.4(b)

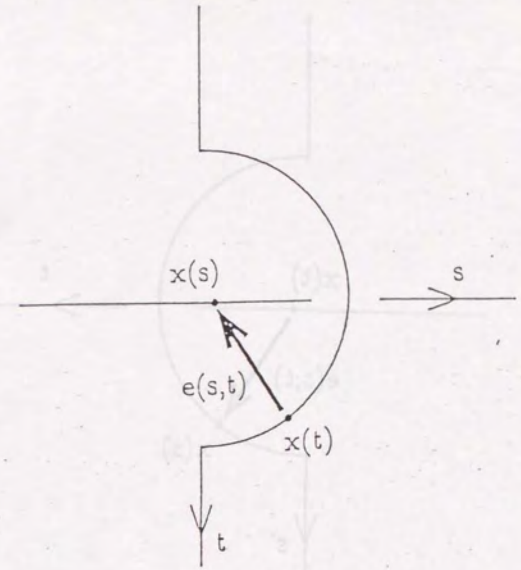

Fig.5(a)

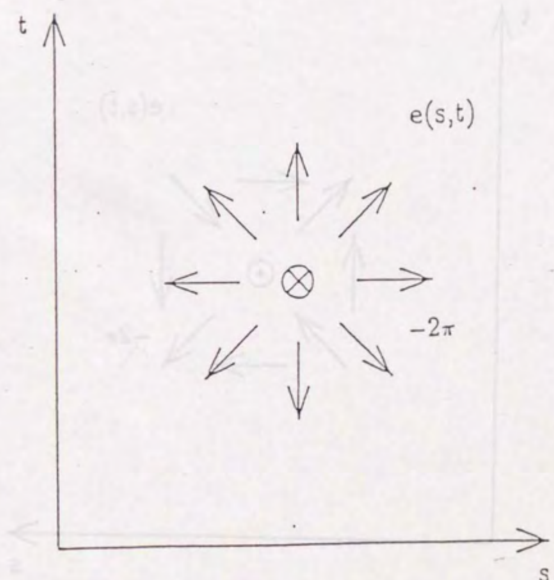

Fig.5(b)

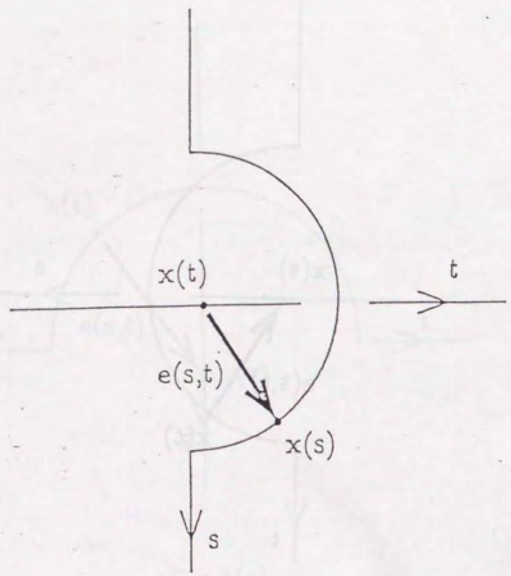

Fig.6(a)

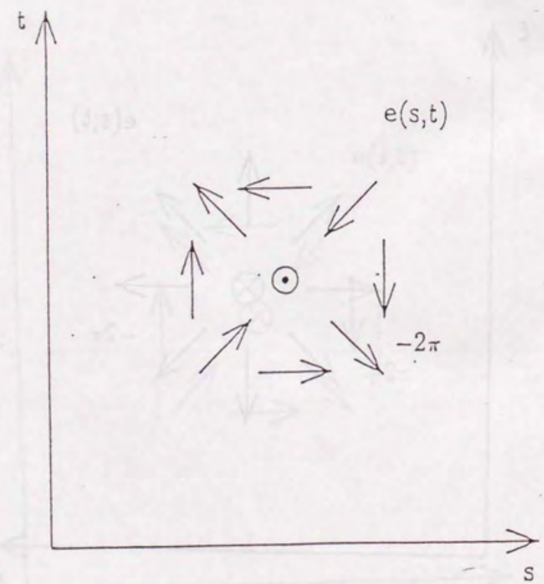

Fig.6(b)