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Finite Generation of Adjoint Ring for Log Surfaces

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Abstract. We prove the finite generation of the adjoint ring for $\mathbb{Q}$-factorial log surfaces over any algebraically closed field.

Contents

1. Introduction

In this paper, we prove

Theorem 1.1 (Main Theorem). *Let $\pi : X \to U$ be a proper morphism from a normal $\mathbb{Q}$-factorial surface to a variety over an algebraically closed field and let $\Delta^\bullet = (\Delta_1, \cdots, \Delta_n)$ be an n-tuple of boundary $\mathbb{Q}$-divisors. Then the adjoint ring of $(\pi, \Delta^\bullet)$*

$$\mathcal{R}(\pi, \Delta^\bullet) = \bigoplus_{(m_1, \cdots, m_n) \in (\mathbb{Z}_{\geq 0})^n} \pi_* \mathcal{O}_X(\lfloor \sum_{i=1}^{n} m_i(K_X + \Delta_i) \rfloor)$$

is a finitely generated $\mathcal{O}_U$-algebra.

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As a corollary, we have

COROLLARY 1.2. *Let $\pi : X \to U$ be a proper morphism from a normal surface to a variety over an algebraically closed field and let $\Delta^\bullet = (\Delta_1, \cdots, \Delta_n)$ be an n-tuple of $\mathbb{Q}$-divisors such that (X, Δ_i) is log canonical for every $1 \leq i \leq n$. Then the adjoint ring $\mathcal{R}(\pi, \Delta^\bullet)$ of $(\pi, \Delta^\bullet)$ is a finitely generated $\mathcal{O}_U$-algebra.*

Theorem 1.1 is a generalization of [Ft, Corollary 1.5] and [FT, Corollary 1.3]. In higher dimension, the finite generation of the adjoint ring is known by Birkar, Cascini, Hacon and M$^{\mathrm{c}}$Kernan [BCHM] when $\pi : X \to U$ is a projective morphism of normal quasi-projective varieties over the complex number field and $\Delta^\bullet = (A + B_1, \cdots, A + B_n)$, where $A \geq 0$ is a general π-ample $\mathbb{Q}$-divisor and B_i is an effective $\mathbb{Q}$-divisor such that $(X, A + B_i)$ is divisorial log terminal for any i. We emphasize that (X, Δ_i) is not necessarily log canonical in Theorem 1.1.

Let us summarize the history of Theorem 1.1. In [Ft], Takao Fujita established Theorem 1.1 under the assumption that X is nonsingular, U is a point, and $n = 1$. More precisely, he proved that the positive part of the Zariski decomposition of $K_X + \Delta$ is semi-ample by using the notion of Sakai minimality when X is a nonsingular projective surface and Δ is a boundary $\mathbb{Q}$-divisor on X, that is, a $\mathbb{Q}$-divisor on X whose coefficients are in $[0, 1]$. As an easy consequence, he obtained the above mentioned special case of Theorem 1.1. In [F2], Osamu Fujino established the minimal model program (MMP) and the abundance theorem for $\mathbb{Q}$-factorial log surfaces and log canonical surfaces over an algebraically closed field of characteristic zero in full generality. Theorem 1.1 with $n = 1$ in characteristic zero follows immediately from the minimal model program and the abundance theorem for $\mathbb{Q}$-factorial log surfaces [F2]. We note that Fujino's approach based on [F1] heavily depends on vanishing theorems of Kodaira type. Therefore, we can not directly apply his arguments in positive characteristic. Fortunately, in [T], Hiromu Tanaka generalized the results in [F2] for $\mathbb{Q}$-factorial log surfaces and log canonical surfaces in positive characteristic. Consequently, we know that Theorem 1.1 holds true over any algebraically closed field when $n = 1$.

As we mentioned above, we are now able to use the minimal model program and the abundance theorem for $\mathbb{Q}$-factorial log surfaces, which need

not be log canonical. In this paper, we prove Theorem 1.1 in full generality by using the minimal model program and the abundance theorem for $\mathbb{Q}$-factorial log surfaces established in [F2] and [T] (see also [FT]). We will use Shokurov's ideas in [S] in order to reduce Theorem 1.1 to the case when $K_X + \Delta_i$ is semi-ample over U for every i. Note that we have to use the minimal model program and the abundance theorem for $\mathbb{R}$-divisors to carry out Shokurov's ideas.

The contents of this paper are as follows. In Section 2, we recall the definition of log surfaces and collect some other basic definitions and notations. In Section 3, which is the main part of this paper, we discuss a certain cone decomposition of the cone of pseudo-effective divisors. See Lemma 3.1 for details. In Section 4, we reduce the proof of Theorem 1.1 to the case where $K_X + \Delta_i$ is semi-ample for every i. To carry it out, we use some basic properties of graded rings and rational polytopes, which are rather technical. In Section 5, we complete the proof of Theorem 1.1 and Corollary 1.2. In Section 6, which is Appendix, we collect some basic results on graded rings and rational polytopes used in Section 4 for the reader's convenience.

Throughout this paper, we work over an algebraically closed field of any characteristic.

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2. Notations and Definitions

In this section we collect some notations and definitions. Let k be an algebraically closed field. A *variety* is a separated integral scheme of finite type over k. Let X be a normal variety and let $\pi : X \to U$ be a morphism from X to a variety U.

(1) $\mathrm{WDiv}_{\mathbb{R}}(X)$ is the $\mathbb{R}$-vector space with canonical basis given by the prime divisors of X.

(2) An $\mathbb{R}$-divisor D on X is $\mathbb{Q}$-*Cartier* (resp. $\mathbb{R}$-*Cartier*) if D is a $\mathbb{Q}$-linear (resp. an $\mathbb{R}$-linear) combination of Cartier divisors.

(3) X is $\mathbb{Q}$-*factorial* if every Weil divisor is $\mathbb{Q}$-Cartier.

(4) Two $\mathbb{R}$-divisors D and D' on X are *$\mathbb{Q}$-linearly equivalent* (resp. *$\mathbb{R}$-linearly equivalent*), denoted by $D \sim_{\mathbb{Q}} D'$ (resp. $D \sim_{\mathbb{R}} D'$), if $D - D'$ is a $\mathbb{Q}$-linear (resp. an $\mathbb{R}$-linear) combination of principal divisors.

(5) Two $\mathbb{R}$-divisors D and D' on X are *$\mathbb{Q}$-linearly equivalent over U* (resp. *$\mathbb{R}$-linearly equivalent over U*), denoted by $D \sim_{\mathbb{Q},U} D'$ (resp. $D \sim_{\mathbb{R},U} D'$), if there exists a $\mathbb{Q}$-Cartier (resp. an $\mathbb{R}$-Cartier) divisor E on U such that $D - D' \sim_{\mathbb{Q}} \pi^*E$ (resp. $D - D' \sim_{\mathbb{R}} \pi^*E$).

(6) Two $\mathbb{R}$-divisors D and D' on X are *numerically equivalent over U* (or *π-numerically equivalent*) if $D - D'$ is $\mathbb{R}$-Cartier and $(D - D') \cdot C = 0$ for every proper curve C on X contained in a fiber of π.

(7) An $\mathbb{R}$-divisor D on X is *pseudo-effective over U* (or *π-pseudo-effective*) if D is π-numerically equivalent to the limit of effective $\mathbb{R}$-divisors modulo numerically equivalence over U.

(8) An $\mathbb{R}$-Cartier divisor D on X is *nef over U* (or *π-nef*) if $(D \cdot C) \geq 0$ for every proper curve C on X contained in a fiber of π.

(9) An $\mathbb{R}$-divisor D is *semi-ample over U* (or *π-semi-ample*) if D is an $\mathbb{R}_{\geq 0}$-linear combination of semi-ample Cartier divisors over U, or equivalently, there exists a morphism $f : X \to Y$ to a variety over U such that D is $\mathbb{R}$-linearly equivalent to the pullback of an ample $\mathbb{R}$-divisor over U.

(10) For a real number α, its *round down* is the largest integer which is not greater than α. It is denoted by $\lfloor \alpha \rfloor$. If $D = \sum \alpha_i D_i$ is an $\mathbb{R}$-divisor and the D_i are distinct prime divisors, then the *round down* of D, denoted by $\lfloor D \rfloor$, is $\sum \lfloor \alpha_i \rfloor D_i$.

(11) An $\mathbb{R}$-divisor D on X is a *boundary $\mathbb{R}$-divisor* if D is effective and whose coefficients are not greater than one.

(12) Let K_X be the canonical divisor on X and let V be a finite dimensional affine subspace of $\mathrm{WDiv}_{\mathbb{R}}(X)$. Then we define $\mathcal{E}_U(V)$ as

$$\mathcal{E}_U(V) = \{\Delta \in V \mid K_X + \Delta \text{ is } \pi\text{-pseudo-effective}\}.$$

(13) A pair (X, Δ), where X is a normal variety and Δ is an effective $\mathbb{R}$-divisor on X, is said to be *log canonical* if $K_X + \Delta$ is $\mathbb{R}$-Cartier and for any proper birational morphism $f : Y \to X$ from a normal variety Y, every coefficient of $K_Y - f^*(K_X + \Delta)$, where K_Y is the canonical divisor on Y such that $f_*K_Y = K_X$, is greater than or equal to -1.

(14) Let $D^\bullet = (D_1, \cdots, D_n)$ be an n-tuple of $\mathbb{Q}$-divisors on X. Then we define the sheaf of $\mathcal{O}_U$-algebra $\mathcal{R}(\pi, D^\bullet)$ as

$$\mathcal{R}(\pi, D^\bullet) = \bigoplus_{(m_1, \cdots, m_n) \in (\mathbb{Z}_{\geq 0})^n} \pi_* \mathcal{O}_X(\lfloor \sum_{i=1}^{n} m_i(K_X + D_i) \rfloor)$$

and call it *adjoint ring* of $(\pi : X \to U, D^\bullet)$. We note that $\mathcal{R}(\pi, D^\bullet)$ is a *finitely generated $\mathcal{O}_U$-algebra* if there is an affine open covering $\{V_i = \operatorname{Spec} A_i\}_{i \in I}$ of U such that $\mathcal{R}(\pi, D^\bullet)\,|_{V_i}$ is the sheaf associated to a finitely generated A_i-algebra for every $i \in I$.

Next, we recall the definition of log surfaces.

Definition 2.1 (Log surfaces). Let X be a normal surface and let Δ be a boundary $\mathbb{R}$-divisor on X such that $K_X + \Delta$ is $\mathbb{R}$-Cartier. Then the pair (X, Δ) is called a *log surface*.

Finally, we recall the definition of weak log canonical models and minimal models of log surfaces.

Definition 2.2 (cf. Definition 3.6.1 [BCHM], Definition 3.6.7 [BCHM]). Let $\pi : X \to U$ and $\pi' : Y \to U$ be projective morphisms from a normal surface to a variety. Let $f : X \to Y$ be a birational morphism of normal surfaces over U and let D be an $\mathbb{R}$-Cartier divisor on X such that f_*D is also $\mathbb{R}$-Cartier. Then f is *D-non-positive* (resp. *D-negative*) if $E = D - f^*f_*D$ is an effective f-exceptional divisor (resp. an effective f-exceptional divisor and the support of E contains supports of all f-exceptional divisors). Let (X, Δ) be a log surface and $f : X \to Y$ be a birational morphism over U. Then f is a *weak log canonical model* (resp. *minimal model*) of (X, Δ) over U if f is $(K_X + \Delta)$-non-positive (resp. $(K_X + \Delta)$-*negative*) and $K_Y + f_*\Delta$ is nef over U.

REMARK 2.3. Let $\pi : X \to U$, $\pi' : Y \to U$ and $f : X \to Y$ be as above and let V be a finite dimensional affine subspace in $\mathrm{WDiv}_{\mathbb{R}}(X)$. Then we can easily check that the set

$$\{\Delta \in V \mid f \text{ is a weak log canonical model of } (X, \Delta) \text{ over } U\}$$

is a closed convex subset in V.

3. Cone Decomposition

In this section, we discuss a certain cone decomposition of the cone of pseudo-effective divisors by using the minimal model program and the abundance theorem for $\mathbb{Q}$-factorial log surfaces. This cone decomposition will play a crucial role in Section 4. For the definition of rational polytopes and their faces, see Definition B.1.

LEMMA 3.1 (cf. [S]). *Let $\pi : X \to U$ be a projective morphism from a normal $\mathbb{Q}$-factorial surface onto a quasi-projective variety. Let V be a finite dimensional affine subspace of $\mathrm{WDiv}_{\mathbb{R}}(X)$, which is defined over $\mathbb{Q}$, and let V' be the set of the boundary $\mathbb{R}$-divisors on X contained in V. Let $\mathcal{C}$ be a rational polytope in V'. Then there are finitely many proper birational morphisms $f_i : X \to Y_i$ over U and finitely many rational polytopes W_i such that $\mathcal{C} \cap \mathcal{E}_U(V) = \cup_i W_i$ and if $\Delta \in W_i$, then f_i is a weak log canonical model of (X, Δ) over U. In particular, $\mathcal{C} \cap \mathcal{E}_U(V)$ is also a rational polytope.*

PROOF. Without loss of generality, we may assume that $\mathcal{C}$ spans V by replacing V with the span of $\mathcal{C}$. We proceed by induction on the dimension of $\mathcal{C}$.

If $\dim \mathcal{C} = 0$, then we may assume that $\{D\} = \mathcal{C} \cap \mathcal{E}_U(V)$. Then, by [FT, Theorem 1.2], there exists a minimal model $f : X \to Y$ of (X, D) over U. By Definition 2.2, a minimal model of (X, D) over U is a weak log canonical model of (X, D) over U. Thus f and $W = \{D\}$ satisfy the conditions of the lemma. So we may assume that $\dim \mathcal{C} > 0$.

We show the assertion in the lemma assuming that there is an $\mathbb{R}$-divisor $\Delta_0 \in \mathcal{C} \cap \mathcal{E}_U(V)$ such that $K_X + \Delta_0 \sim_{\mathbb{R},U} 0$. We first show that there is a $\mathbb{Q}$-divisor Δ' in $\mathcal{C} \cap \mathcal{E}_U(V)$ such that $K_X + \Delta' \sim_{\mathbb{Q},U} 0$. Indeed, we may write $K_X + \Delta_0 = \sum_{i=1}^{k} \alpha_i(f_i) + \sum_{j=1}^{l} \beta_j \pi^* F_j$ for some $\alpha_i, \beta_j \in \mathbb{R}$, principal divisors (f_i) and Cartier divisors F_j on U. Let T be the finite dimensional

$\mathbb{R}$-vector space in $\mathrm{WDiv}_{\mathbb{R}}(X)$ spanned by (f_i) and π^*F_j. Then T is defined over $\mathbb{Q}$. Therefore, the set

$$\{\Delta \in \mathcal{C} \mid K_X + \Delta \in T\}$$

is a rational polytope and contains Δ_0. In particular, this set is a non-empty rational polytope. Then a rational point Δ' in this set satisfies $K_X + \Delta' \sim_{\mathbb{R},U} 0$, and so $\Delta' \in \mathcal{E}_U(V)$. Since $K_X + \Delta'$ is a $\mathbb{Q}$-divisor, we obtain $K_X + \Delta' \sim_{\mathbb{Q},U} 0$. By replacing Δ_0 with Δ', we may assume that Δ_0 is a $\mathbb{Q}$-divisor and $K_X + \Delta_0 \sim_{\mathbb{Q},U} 0$. Pick $D \in \mathcal{C}$ with $D \neq \Delta_0$. Then there is a divisor D' on the boundary of $\mathcal{C}$ such that $D - \Delta_0 = \lambda(D' - \Delta_0)$ for some $0 < \lambda \leq 1$. Then

$$K_X + D = \lambda(K_X + D') + (1-\lambda)(K_X + \Delta_0) \sim_{\mathbb{R},U} \lambda(K_X + D').$$

In particular, $K_X + D$ is pseudo-effective over U if and only if $K_X + D'$ is pseudo-effective over U. Moreover, the pairs (X, D) and (X, D') have the same weak log canonical model over U by [BCHM, Lemma 3.6.9]. Let $\partial\mathcal{C}$ be the boundary of $\mathcal{C}$. Since $\partial\mathcal{C}$ consists of finitely many rational polytopes, there are finitely many proper birational morphisms $f_i : X \to Y_i$ over U and finitely many rational polytopes W_i' such that $\partial\mathcal{C} \cap \mathcal{E}_U(V) = \cup_i W_i'$ and if $D \in W_i'$, then f_i is a weak log canonical model of (X, D) over U. Let W_i be the cone spanned by Δ_0 and W_i'. Then f_i and W_i satisfy the conditions of the lemma. So we are done.

We now prove the general case. Since V' is compact and $\mathcal{C} \cap \mathcal{E}_U(V)$ is closed in V', it is sufficient to prove the lemma for every $\Delta_0 \in \mathcal{C} \cap \mathcal{E}_U(V)$ and a sufficiently small neighborhood $\mathcal{C}_0$ of Δ_0 in $\mathcal{C}$, which is also a rational polytope. By [FT, Theorem 1.2], there exists a minimal model $f_0 : X \to Y_0$ of (X, Δ_0) over U. Since a minimal model of (X, Δ_0) over U is $(K_X + \Delta_0)$-negative, possibly shrinking $\mathcal{C}_0$, we may assume that for any $\Delta \in \mathcal{C}_0$, f_0 is $(K_X + \Delta)$-non-positive. We put $W = f_{0*}(V)$ and $\mathcal{C}' = f_{0*}(\mathcal{C}_0)$. Then $\mathcal{C}' \subset W$ is a rational polytope containing $f_{0*}\Delta_0$ and $\dim \mathcal{C}' \leq \dim \mathcal{C}$.

By [F2, Theorem 8.1] and [T, Theorem 6.7], $K_{Y_0} + f_{0*}\Delta_0$ is semi-ample over U. Then there exists a projective morphism $\phi_0 : Y_0 \to Z_0$ onto a quasi-projective variety Z_0 over U and an ample $\mathbb{R}$-divisor A over U such that $K_{Y_0} + f_{0*}\Delta_0 \sim_{\mathbb{R}} \phi_0^*A$. Since $K_{Y_0} + f_{0*}\Delta_0 \sim_{\mathbb{R},Z_0} 0$, there are finitely many proper birational morphisms $h_i : Y_0 \to Y_i$ over Z_0 and finitely many rational polytopes W_i such that $\mathcal{C}' \cap \mathcal{E}_{Z_0}(W) = \cup_i W_i$ and if $D \in W_i$, then

h_i is a weak log canonical model of (Y_0, D) over Z_0. Possibly shrinking $\mathcal{C}_0$, we may assume that $f_{0*}\Delta_0 \in W_i$ for any i. Let $\phi_i : Y_i \to Z_0$ be the induced morphism. Pick a vertex Δ of W_i. Then $K_{Y_i} + h_{i*}\Delta$ is nef over Z_0 and by [F2, Theorem 8.1] and [T, Theorem 6.7], $K_{Y_i} + h_{i*}\Delta$ is semi-ample over Z_0. Therefore $K_{Y_i} + h_{i*}\Delta + n\phi_i^*A$ is semi-ample over U for a large integer n. In particular $K_{Y_i} + h_{i*}\Delta + n\phi_i^*A$ is nef over U. If $0 < \epsilon < 1/(n+1)$, then

$$h_{i*}(K_{Y_0} + \epsilon\Delta + (1-\epsilon)f_{0*}\Delta_0) \sim_{\mathbb{R}} \epsilon(K_{Y_i} + h_{i*}\Delta) + (1-\epsilon)\phi_i^*A$$

is nef over U. Considering all vertices of all W_i, we may find a sufficiently small neighborhood $\mathcal{C}''$ of $f_{0*}\Delta_0$ in $\mathcal{C}'$, which is a rational polytope, such that if $D' \in \mathcal{C}'' \cap W_i$, then h_i is a weak log canonical model of (Y_0, D') over U. Then $K_{Y_i} + h_{i*}D'$ is nef over U. In particular, $K_{Y_i} + h_{i*}D'$ is pseudo-effective over U. Since h_i is $(K_{Y_0} + D')$-non-positive, $K_{Y_0} + D'$ is also pseudo-effective over U. On the other hand, a pseudo-effective divisor over U is also pseudo-effective over Z_0. Therefore, $\cup_i(\mathcal{C}'' \cap W_i) = \mathcal{C}'' \cap \mathcal{E}_{Z_0}(W) = \mathcal{C}'' \cap \mathcal{E}_U(W)$. Set $\widetilde{\mathcal{C}} = (f_{0*})^{-1}(\mathcal{C}'') \cap \mathcal{C}_0$ and $\widetilde{W_i} = (f_{0*})^{-1}(W_i \cap \mathcal{C}'') \cap \mathcal{C}_0$. Then $\widetilde{\mathcal{C}}$ is a neighborhood of Δ_0 in $\mathcal{C}_0$ and a rational polytope. If $\Delta \in \widetilde{\mathcal{C}} \cap \mathcal{E}_U(V)$, then $f_{0*}\Delta \in \mathcal{C}'' \cap \mathcal{E}_U(W)$. Therefore we have $\Delta \in \cup_i \widetilde{W_i}$. On the other hand, if $\Delta \in \cup_i \widetilde{W_i}$, then $f_{0*}\Delta \in \mathcal{C}'' \cap \mathcal{E}_U(W)$. In particular, $K_{Y_0} + f_{0*}\Delta$ is pseudo-effective over U. Since $\Delta \in \mathcal{C}_0$, f_0 is $(K_X + \Delta)$-non-positive. Therefore $\Delta \in \mathcal{E}_U(V)$. Thus, we see that $\widetilde{\mathcal{C}} \cap \mathcal{E}_U(V) = \cup_i \widetilde{W_i}$. We can also check that if $\Delta \in \widetilde{W_i}$, then $h_i \circ f_0$ is a weak log canonical model of (X, Δ) over U. So we are done. □

4. Reduction to the Special Case

In this section, we reduce Theorem 1.1 to the case where $K_X + \Delta_i$ is semi-ample over U for every $1 \leq i \leq n$.

We put $D_i = K_X + \Delta_i$ for every i. We note that $\pi : X \to U$ is projective since X is a $\mathbb{Q}$-factorial surface (see [F2, Lemma 2.2]). By taking the Stein factorization, we may assume that $\pi_*\mathcal{O}_X = \mathcal{O}_U$. Furthermore, we may also assume that U is an affine variety by the definition of finitely generated $\mathcal{O}_U$-algebras. Set $A = H^0(X, \mathcal{O}_X) = H^0(U, \mathcal{O}_U)$. Then it is sufficient to prove that

$$\mathcal{R}(\pi, \Delta^\bullet) = \bigoplus_{(m_1, \cdots, m_n) \in (\mathbb{Z}_{\geq 0})^n} H^0(X, \mathcal{O}_X(\lfloor \sum_{i=1}^{n} m_i D_i \rfloor))$$

is a finitely generated A-algebra. Let V (resp. $\mathcal{C}$) be the affine subspace (resp. convex hull) in $\mathrm{WDiv}_{\mathbb{R}}(X)$ spanned by $\Delta_1, \cdots, \Delta_n$.

LEMMA 4.1. *To prove Theorem 1.1, we may assume that* $\dim V = \dim \mathcal{C} = n-1$,

PROOF. We prove it with several steps.

Step 1. In this step, we reduce Theorem 1.1 to the case that Δ_i is a vertex of $\mathcal{C}$ for every i and $\Delta_i \neq \Delta_j$ for any $i \neq j$.

Suppose that there is an index i such that Δ_i is not a vertex of $\mathcal{C}$ or there are two indices i and j such that $i \neq j$ and $\Delta_i = \Delta_j$. By changing indices, we may write $\Delta_n = \sum_{i=1}^{k}(a_i/q)\Delta_i$ for some $1 \leq k \leq n-1$, $a_i \in \mathbb{Z}_{>0}$ and $q \in \mathbb{Z}_{>0}$ such that $\sum_{i=1}^{k}(a_i/q) = 1$. Then we have $qD_n = \sum_{i=1}^{k} a_i D_i$. By Lemma A.2, it is sufficient to prove the finite generation of

$$\bigoplus_{(m_1, \cdots, m_n) \in (\mathbb{Z}_{\geq 0})^n} H^0(X, \mathcal{O}_X(\lfloor (\sum_{i=1}^{k} m_i a_i D_i + \sum_{i=k+1}^{n-1} m_i D_i + m_n q D_n) \rfloor))$$

as an A-algebra. By Lemma A.3, it is sufficient to prove that

$$\bigoplus_{(m_1, \cdots, \overset{j}{\vee}, \cdots, m_n) \in (\mathbb{Z}_{\geq 0})^{n-1}} H^0(X, \mathcal{O}_X(\lfloor (\sum_{\substack{i=1 \\ i \neq j}}^{k} m_i a_i D_i + \sum_{i=k+1}^{n-1} m_i D_i + m_n q D_n) \rfloor))$$

is a finitely generated A-algebra for every $1 \leq j \leq k$ and moreover we can reduce it to the finite generation of

$$\bigoplus_{(m_1, \cdots, m_{j-1}, m_{j+1}, \cdots, m_n) \in (\mathbb{Z}_{\geq 0})^{n-1}} H^0(X, \mathcal{O}_X(\lfloor (\sum_{i=1, i \neq j}^{n-1} m_i D_i + m_n D_n) \rfloor))$$

as an A-algebra for any $1 \leq j \leq k$ by using Lemma A.2 again. By repeating this discussion, we may assume that Δ_i is a vertex of $\mathcal{C}$ for any i and $\Delta_i \neq \Delta_j$ for any $i \neq j$.

Step 2. Assume that Δ_i is a vertex of $\mathcal{C}$ for every i and $\Delta_i \neq \Delta_j$ for any $i \neq j$. In addition, suppose that $\dim V + 1 < n$. The goal of this step

is to decrease n, the number of the boundary $\mathbb{Q}$-divisors, under the above assumption.

Let $\mathcal{C}_j$ be the convex hull spanned by $\Delta_1, \cdots, \Delta_{j-1}, \Delta_{j+1}, \cdots, \Delta_n$ for every j. We can pick a $\mathbb{Q}$-divisor Δ_{n+1} such that $\Delta_{n+1} \in \mathcal{C}_j$ for any $1 \leq j \leq n$ by Lemma B.2. Set $D_{n+1} = K_X + \Delta_{n+1}$. If

$$\bigoplus_{(m_1, \cdots, m_{n+1}) \in (\mathbb{Z}_{\geq 0})^{n+1}} H^0(X, \mathcal{O}_X(\lfloor \sum_{i=1}^{n+1} m_i D_i \rfloor))$$

is a finitely generated A-algebra, then it is obvious that $\mathcal{R}(\pi, \Delta^{\bullet})$ is also a finitely generated A-algebra. Moreover, by Lemma A.2 and Lemma A.3, it is sufficient to prove that

$$\bigoplus_{(m_1, \cdots, m_{j-1}, m_{j+1}, \cdots, m_{n+1}) \in (\mathbb{Z}_{\geq 0})^{n}} H^0(X, \mathcal{O}_X(\lfloor \sum_{i=1, i \neq j}^{n+1} m_i D_i \rfloor))$$

is a finitely generated A-algebra for any $1 \leq j \leq n$. Since $\Delta_{n+1} \in \mathcal{C}_j$, by using Lemma A.2 and Lemma A.3 again, we can reduce it to the finite generation of

$$\bigoplus_{(m_1, \cdots \overset{j}{\vee} \cdots \overset{j'}{\vee} \cdots, m_{n+1}) \in (\mathbb{Z}_{\geq 0})^{n-1}} H^0(X, \mathcal{O}_X(\lfloor \sum_{i=1, i \neq j, j'}^{n+1} m_i D_i \rfloor))$$

as an A-algebra for any $1 \leq j, j' \leq n$ with $j \neq j'$.

Step 3. By repeating the discussion of Step 1 and Step 2, we may assume that $\dim V + 1 = n$. So we are done. □

By Lemma 4.1, we may assume that $\dim V = \dim \mathcal{C} = n - 1$, or equivalently, any point of V is represented uniquely by the $\mathbb{R}$-linear combination of $\Delta_1, \cdots, \Delta_n$, where the sum of coefficients is equal to one. Next we prove the following lemma. For simplex coverings, see Remark B.4.

LEMMA 4.2. *Suppose that there is a finite* $(n-1)$*-dimensional rational simplex covering* $\{\Sigma_\lambda\}_{\lambda \in \Lambda}$ *of* $\mathcal{C}$ *such that*

(i) $\mathcal{C} = \cup_{\lambda \in \Lambda} \Sigma_\lambda$, *and*

(ii) *if* $\Psi_{\lambda 1}, \cdots, \Psi_{\lambda n}$ *are the vertices of* Σ_λ*, then*

$$R^\lambda = \bigoplus_{(m_1, \cdots, m_n) \in (\mathbb{Z}_{\geq 0})^n} H^0(X, \mathcal{O}_X(\lfloor \sum_{i=1}^n m_i(K_X + \Psi_{\lambda i}) \rfloor))$$

is a finitely generated A*-algebra for any* λ*.*

Then $\mathcal{R}(\pi, \Delta^\bullet)$ *is also a finitely generated* A*-algebra.*

Proof. By hypothesis, for any $\lambda \in \Lambda$ and any $1 \leq i \leq n$, we may write $\Delta_i = \sum_{j=1}^n (a_{\lambda ij}/p)\Psi_{\lambda j}$ and $\Psi_{\lambda i} = \sum_{j=1}^n (b_{\lambda ij}/q)\Delta_j$, where $a_{\lambda ij} \in \mathbb{Z}$, $b_{\lambda ij} \in \mathbb{Z}_{\geq 0}$, $p,\ q \in \mathbb{Z}_{>0}$ and $\sum_{j=1}^n (a_{\lambda ij}/p) = \sum_{j=1}^n (b_{\lambda ij}/q) = 1$. Then

$$\Delta_i = \sum_{j=1}^n \frac{a_{\lambda ij}}{p}\Psi_{\lambda j} = \sum_{j=1}^n \frac{a_{\lambda ij}}{p} \sum_{k=1}^n \frac{b_{\lambda jk}}{q}\Delta_k = \sum_{k=1}^n \Big(\sum_{j=1}^n \frac{a_{\lambda ij} b_{\lambda jk}}{pq} \Big) \Delta_k.$$

Since every Δ_i is represented uniquely by the $\mathbb{R}$-linear combination of $\Delta_1, \cdots, \Delta_n$, where the sum of coefficients is equal to one, we have $\sum_{j=1}^n (a_{\lambda ij} b_{\lambda jk}/pq) = \delta_{ik}$, where δ_{ik} is Kronecker delta.

Pick $(m_1, \cdots, m_n) \in (\mathbb{Z}_{\geq 0})^n$ such that $m = \sum_{i=1}^n m_i > 0$. Then there exists a $\lambda' \in \Lambda$ such that $\sum_{i=1}^n (m_i/m)\Delta_i \in \Sigma_{\lambda'}$. Then

$$\sum_{i=1}^n pqm_i\Delta_i = mpq \sum_{i=1}^n (m_i/m)\Delta_i$$

is uniquely represented by the $\mathbb{R}_{\geq 0}$-linear combination of $\Psi_{\lambda' 1}, \cdots, \Psi_{\lambda' n}$ where the sum of the coefficients is equal to mpq. On the other hand,

$$\sum_{i=1}^n pqm_i\Delta_i = \sum_{i=1}^n pqm_i \sum_{j=1}^n \frac{a_{\lambda' ij}}{p}\Psi_{\lambda' j} = \sum_{j=1}^n \Big(\sum_{i=1}^n q a_{\lambda' ij} m_i \Big) \Psi_{\lambda' j}$$

and $\sum_{i=1}^n q a_{\lambda' ij} m_i \in \mathbb{Z}$. Therefore $\sum_{i=1}^n q a_{\lambda' ij} m_i \in \mathbb{Z}_{\geq 0}$ and so

$$\begin{aligned} &H^0(X, \mathcal{O}_X(\lfloor \sum_{i=1}^n m_i pq(K_X + \Delta_i) \rfloor)) \\ &= H^0(X, \mathcal{O}_X(\lfloor \sum_{j=1}^n q(\sum_{i=1}^n a_{\lambda' ij} m_i)(K_X + \Psi_{\lambda' j}) \rfloor)) \end{aligned}$$

can be identified with the A-module of homogeneous elements of certain degree in $R^{\lambda'}_{(q)}$. Let $\phi_{\boldsymbol{m}\lambda'} : H^0(X, \mathcal{O}_X(\lfloor \sum_{i=1}^n m_i pq(K_X + \Delta_i)\rfloor)) \to R^{\lambda'}_{(q)}$, where $\boldsymbol{m} = (m_1, \cdots, m_n)$, be the natural morphism of A-modules. Similarly, for any $(m'_1, \cdots, m'_n) \in (\mathbb{Z}_{\geq 0})^n$, since

$$\sum_{i=1}^n m'_i q \Psi_{\lambda' i} = \sum_{j=1}^n (\sum_{i=1}^n b_{\lambda' ij} m'_i)\Delta_j,$$

where $\sum_{i=1}^n b_{\lambda' ij} m'_i \in \mathbb{Z}_{\geq 0}$, we get the natural ring homomorphism $\tau_{\lambda'} : R^{\lambda'}_{(q)} \to R$. By the definition of $\phi_{\boldsymbol{m}\lambda'}$ and $\tau_{\lambda'}$, for any $f \in H^0(X, \mathcal{O}_X(\lfloor \sum_{i=1}^n m_i pq(K_X + \Delta_i)\rfloor))$, $\tau_{\lambda'} \circ \phi_{\boldsymbol{m}\lambda'}(f) = f$.

By the hypothesis, $R^{\lambda'}$ is a finitely generated A-algebra. Then $R^{\lambda'}_{(q)}$ is also a finitely generated A-algebra by Lemma A.2. Let $g_{\lambda' 1}, \cdots, g_{\lambda' k_{\lambda'}}$ be the generator of $R^{\lambda'}_{(q)}$. Then there exists an A-polynomial $F \in A[X_1, \cdots, X_{k_{\lambda'}}]$ such that $\phi_{\boldsymbol{m}\lambda'}(f) = F(g_{\lambda' 1}, \cdots, g_{\lambda' k_{\lambda'}})$. Then we have $f = F(\tau_{\lambda'}(g_{\lambda' 1}), \cdots, \tau_{\lambda'}(g_{\lambda' k_{\lambda'}}))$ and so $R_{(pq)}$ is generated by $\tau_{\lambda'}(g_{\lambda' l})$, where $\lambda' \in \Lambda$ and $1 \leq l \leq k_{\lambda'}$. Then the lemma follows from Lemma A.1. □

LEMMA 4.3. *To prove Theorem 1.1, we may assume that each $K_X + \Delta_i$ is semi-ample over U.*

PROOF. Recall that $\mathcal{C}$ is the rational polytope spanned by $\Delta_1, \cdots, \Delta_n$ and V is the finite dimensional affine subspace of $\mathrm{WDiv}_{\mathbb{R}}(X)$ spanned by $\mathcal{C}$ such that $\dim V = n - 1$. By Lemma 3.1, there are finitely many proper birational morphisms $f_i : X \to Y_i$ over U and finitely many rational polytopes W_i such that $\mathcal{C} \cap \mathcal{E}_U(V) = \cup_i W_i$ and if $\Delta \in W_i$, then f_i is a weak log canonical model of (X, Δ) over U. By applying Lemma B.3 to $\mathcal{C}$, $\mathcal{C} \cap \mathcal{E}_U(V)$ and W_i, there is a finite $(n-1)$-dimensional rational simplex covering $\{\Sigma_\lambda\}_{\lambda \in \Lambda}$ of $\mathcal{C}$ such that $\mathcal{C} = \cup_\lambda \Sigma_\lambda$ and for any $\lambda \in \Lambda$, $\mathcal{E}_U(V) \cap \Sigma_\lambda$ is a face of Σ_λ and is contained in W_j for some j. By Lemma 4.2 it is sufficient to prove the case where $\Delta_1, \cdots, \Delta_n$ are vertices of Σ_λ for some λ. By changing indices if necessary, we may assume that $\Delta_1, \cdots, \Delta_k \in \mathcal{E}_U(V)$ and $\Delta_{k+1}, \cdots, \Delta_n \notin \mathcal{E}_U(V)$. We note that if $m_i \geq 0$ for every $1 \leq i \leq n$, then $H^0(X, \mathcal{O}_X(\lfloor \sum_{i=1}^n m_i(K_X + \Delta_i)\rfloor)) \neq 0$ implies that $m_{k+1} = \cdots = m_n = 0$. Indeed, in this case we see that m_i is zero for all i or $\sum_{i=1}^n (m_i/m)\Delta_i$ is in $\mathcal{E}_U(V)$ when $m = \sum_{i=1}^n m_i > 0$. Therefore we may assume that $\Delta_i \in \mathcal{E}_U(V)$ for any $1 \leq i \leq n$. Then we can find j such that $\Delta_i \in W_j$ for any $1 \leq i \leq n$.

Pick a positive integer d such that $d(K_X+\Delta_i)$ and $d(K_{Y_j}+f_{j*}\Delta_i)$ are both Cartier for any $1\leq i\leq n$. Then

$$H^0(X,\mathcal{O}_X(\sum_{i=1}^{n} m_i d(K_X+\Delta_i)))\cong H^0(Y_j,\mathcal{O}_{Y_j}(\sum_{i=1}^{n} m_i d(K_{Y_j}+f_{j*}\Delta_i)))$$

and by Lemma A.2, it is sufficient to prove that

$$\bigoplus_{(m_1,\cdots,m_n)\in(\mathbb{Z}_{\geq 0})^n} H^0(Y_j,\mathcal{O}_{Y_j}(\lfloor\sum_{i=1}^{n} m_i(K_{Y_j}+f_{j*}\Delta_i)\rfloor))$$

is a finitely generated $H^0(Y_j,\mathcal{O}_{Y_j})$-algebra. Since $f_j: X\to Y_j$ is a weak log canonical model of (X,Δ_i) over U for any i, by replacing X with Y_j and $K_X+\Delta_i$ with $K_{Y_j}+f_{j*}\Delta_i$ respectively, we may assume that $K_X+\Delta_i$ is nef over U for any $1\leq i\leq n$. Then by [F2, Theorem 8.1] and [T, Theorem 6.7], $K_X+\Delta_i$ is semi-ample over U for any $1\leq i\leq n$. □

5. Proof of the Main Theorem and Corollary

Now we complete the proof of Theorem 1.1 and Corollary 1.2.

Proof of Theorem 1.1. It is sufficient to prove that

$$\mathcal{R}(\pi,\Delta^{\bullet})=\bigoplus_{(m_1,\cdots,m_n)\in(\mathbb{Z}_{\geq 0})^n} H^0(X,\mathcal{O}_X(\lfloor\sum_{i=1}^{n} m_i D_i\rfloor))$$

is a finitely generated $H^0(X,\mathcal{O}_X)$-algebra, where $D_i=K_X+\Delta_i$ is semi-ample over U for any $1\leq i\leq n$ and U is an affine variety. This follows immediately from the following lemma. □

Lemma 5.1. *Let $\pi: X\to \operatorname{Spec} A$ be a proper morphism from a normal variety to an affine variety and let $D_1,\cdots,D_n$ be π-semi-ample $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisors on X. Then*

$$R=\bigoplus_{(m_1,\cdots,m_n)\in(\mathbb{Z}_{\geq 0})^n} H^0(X,\mathcal{O}_X(\lfloor\sum_{i=1}^{n} m_i D_i\rfloor))$$

is a finitely generated A-algebra.

PROOF. By Lemma A.2, we may assume that D_i is Cartier and free over $\operatorname{Spec} A$ for any $1 \leq i \leq n$. Thus we may assume that D_i is base point free for any $1 \leq i \leq n$. Then there is a surjective morphism $\overset{r_i}{\oplus}\mathcal{O}_X \to \mathcal{O}_X(D_i)$ for some positive integer r_i for every i. Suppose that $n \geq 2$. Set $\mathcal{E} = \mathcal{O}_X(D_1) \oplus \cdots \oplus \mathcal{O}_X(D_n)$. Then there is a surjective morphism $\overset{r}{\oplus}\mathcal{O}_X \to \mathcal{E}$, where $r = \sum_{i=1}^{n} r_i$. Let $p : \boldsymbol{P}_X(\mathcal{E}) \to X$ be the projective bundle over X associated to $\mathcal{E}$. Then there is a natural surjective morphism $p^*\mathcal{E} \to \mathcal{O}_{\boldsymbol{P}_X(\mathcal{E})}(1)$. Therefore there exists a surjective morphism $\overset{r}{\oplus}\mathcal{O}_{\boldsymbol{P}_X(\mathcal{E})} \to \mathcal{O}_{\boldsymbol{P}_X(\mathcal{E})}(1)$ and so $|\mathcal{O}_{\boldsymbol{P}_X(\mathcal{E})}(1)\ |$ is base point free. Moreover, there is a canonical isomorphism of the graded $\mathcal{O}_X$-algebras

$$\bigoplus_{l\in\mathbb{Z}_{\geq 0}} p_*\mathcal{O}_{\boldsymbol{P}_X(\mathcal{E})}(l) \cong \bigoplus_{l\in\mathbb{Z}_{\geq 0}} \left(\bigoplus_{m_1+\cdots+m_n=l} \mathcal{O}_X(\sum_{i=1}^{n} m_i D_i) \right).$$

Therefore

$$R \cong \bigoplus_{l\in\mathbb{Z}_{\geq 0}} H^0(\boldsymbol{P}_X(\mathcal{E}), \mathcal{O}_{\boldsymbol{P}_X(\mathcal{E})}(l))$$

as A-algebras. So we may assume that $n = 1$ and then the lemma is clear. □

Let us prove Corollary 1.2.

PROOF OF COROLLARY 1.2. Let $f : Y \to X$ be a resolution of X. Then we may write

$$K_Y + \Gamma_i = f^*(K_X + \Delta_i) + E_i,$$

where $\Gamma_i \geq 0$ and $E_i \geq 0$ have no common components, $f_*\Gamma_i = \Delta_i$ and E_i is f-exceptional. Then Γ_i is a $\mathbb{Q}$-divisor for any i and

$$\mathcal{R}(\pi, \Delta^\bullet) \cong \mathcal{R}(\pi \circ f, \Gamma^\bullet)$$

as graded $\mathcal{O}_U$-algebras. Moreover, by the definition of log canonical pairs, Γ_i is a boundary $\mathbb{Q}$-divisor for every i. Therefore we can reduce it to Theorem 1.1. □

6. Appendix

In this section we collect some basic results for the reader's convenience.

Appendix A. Graded Ring

In this part, let k be an algebraically closed field and let A be a finitely generated k-algebra such that A is an integral domain. Let

$$R = \bigoplus_{(a_1, \cdots, a_n) \in (\mathbb{Z}_{\geq 0})^n} R_{(a_1, \cdots, a_n)}$$

be a graded A-algebra such that R is an integral domain with $R_{(0, \cdots, 0)} = A$ and let $R_{(d)}$ be the d-th truncation of R. More precisely,

$$R_{(d)} = \bigoplus_{(a_1, \cdots, a_n) \in (\mathbb{Z}_{\geq 0})^n} R_{(da_1, \cdots, da_n)}.$$

For any $\boldsymbol{a} \in (\mathbb{Z}_{\geq 0})^n$, we identify $R_{\boldsymbol{a}}$ with a homogeneous part of R by the natural inclusion $R_{\boldsymbol{a}} \hookrightarrow R$ of A-modules. For any $f \in R_{\boldsymbol{a}}$, we define the *degree* of f as $\deg(f) = \boldsymbol{a}$.

We introduce two well known results. For the proofs, see [ADHL, Proposition 1.1.2.4] and [ADHL, Proposition 1.1.2.5].

Lemma A.1. *Suppose that there are $x_1, \cdots, x_r \in R$ such that $R_{(d)}$ is a A-subalgebra of $R' = A[x_1, \cdots, x_r]$ for some positive integer d. Then R is a finitely generated A-algebra.*

Lemma A.2 (cf. [ADHL, Corollary 1.1.2.6]). *For any*

$$\boldsymbol{d} = (d_1, \cdots, d_n) \in (\mathbb{Z}_{>0})^n,$$

the graded ring

$$R_{[\boldsymbol{d}]} = \bigoplus_{(a_1, \cdots, a_n) \in (\mathbb{Z}_{\geq 0})^n} R_{(d_1 a_1, \cdots, d_n a_n)}$$

is a finitely generated A-algebra if and only if R is a finitely generated A-algebra.

We close this part with the following technical result.

Lemma A.3. *Let $\boldsymbol{e}_1, \cdots, \boldsymbol{e}_n$ be the canonical basis of $\mathbb{Z}^n$. We set $\boldsymbol{e} = \sum_{i=1}^{m} \boldsymbol{e}_{j_i}$, where $j_i \neq j_{i'}$ for $i \neq i'$. Then the following conditions are equivalent:*

(i) *The ring*

$$\overline{R} = \bigoplus_{(a_1, \cdots, a_n, b) \in (\mathbb{Z}_{\geq 0})^{n+1}} R_{(a_1, \cdots, a_n)+b\boldsymbol{e}}$$

is a finitely generated A-algebra.

(ii) *For any i such that $1 \leq i \leq m$, the ring*

$$\overline{R}^i = \bigoplus_{(a_1, \cdots, a_{j_i-1}, a_{j_i+1}, \cdots, a_n, b) \in (\mathbb{Z}_{\geq 0})^{n}} R_{(a_1, \cdots, a_{j_i-1}, 0, a_{j_i+1}, \cdots, a_n)+b\boldsymbol{e}}$$

is a finitely generated A-algebra.

PROOF. For any homogeneous element $f \in R_{(a_1, \cdots, a_{j_i-1}, 0, a_{j_i+1}, \cdots, a_n)+b\boldsymbol{e}}$, we put $\deg_i(f) = (a_1, \cdots, a_{j_i-1}, a_{j_i+1}, \cdots, a_n, b) \in (\mathbb{Z}_{\geq 0})^n$ and we will call it the degree of f in $\overline{R}^i$. Similarly, for any homogeneous element $f \in R_{(a_1, \cdots, a_n)+b\boldsymbol{e}}$ we put $\mathrm{Deg}(f) = (a_1, \cdots, a_n, b) \in (\mathbb{Z}_{\geq 0})^{n+1}$ and we will call it the degree of f in $\overline{R}$. Note that $\overline{R}^i$ can be identified with the A-subalgebra of $\overline{R}$ generated by all homogeneous elements of $\overline{R}$ whose j_i-th component is zero. If $\overline{R}$ is generated as an A-algebra by finitely many elements of $\overline{R}$, which can be assumed to be homogeneous elements, then $\overline{R}^i$ is generated as an A-algebra by the generator of $\overline{R}$ whose j_i-th component is zero. Thus $\overline{R}^i$ is also a finitely generated A-algebra.

Conversely, suppose that $\overline{R}^i$ is a finitely generated A-algebra for any $1 \leq i \leq m$. For simplicity, suppose that $j_i = i$ for any $1 \leq i \leq m$. Let $g_{i1}, \cdots, g_{ir(i)}$ be a generator of $\overline{R}^i$, where g_{ij} is homogeneous for any $1 \leq j \leq r(i)$, and let

$$\boldsymbol{a}^{ij} = (a_1^{ij}, \cdots, a_{(i-1)}^{ij}, a_{(i+1)}^{ij}, \cdots, a_n^{ij}, b^{ij})$$

be the degree of g_{ij} in $\overline{R}^i$. Then

$$g_{ij} \in R_{(a_1^{ij}+b^{ij}, \cdots, a_{(i-1)}^{ij}+b^{ij}, b^{ij}, a_{(i+1)}^{ij}+b^{ij}, \cdots, a_m^{ij}+b^{ij}, a_{(m+1)}^{ij}, \cdots, a_n^{ij})}.$$

For any $0 \leq s \leq b^{ij}$, let $g_{ij}(s)$ be the element of $\overline{R}$ such that

$$g_{ij}(s) = g_{ij}\left(\in R_{(a_1^{ij}+b^{ij}, \cdots, a_{(i-1)}^{ij}+b^{ij}, b^{ij}, a_{(i+1)}^{ij}+b^{ij}, \cdots, a_m^{ij}+b^{ij}, a_{(m+1)}^{ij}, \cdots, a_n^{ij})}\right),$$

and

$$\begin{aligned}\mathrm{Deg}(g_{ij}(s)) = (&a_1^{ij} + b^{ij} - s, \cdots, a_{(i-1)}^{ij} + b^{ij} - s, b^{ij} - s,\\ &a_{(i+1)}^{ij} + b^{ij} - s, \cdots, a_m^{ij} + b^{ij} - s, a_{(m+1)}^{ij}, \cdots, a_n^{ij}, s)\ .\end{aligned}$$

Then we prove that $\overline{R}$ is generated as an A-algebra by all $g_{ij}(s)$, where $1 \le i \le m$, $1 \le j \le r(i)$ and $0 \le s \le b^{ij}$.

Pick any homogeneous element $f \in \overline{R}$ and let $(a_1, \cdots, a_n, b)$ be the degree of f in $\overline{R}$. Pick an l which satisfies $a_l = \min\{a_i \mid 1 \le i \le m\}$. Without loss of generality, we may assume that $l = 1$. Let f' be the element of $\overline{R}^1$ such that $f' = f$ as an element of $R_{(a_1+b, \cdots, a_m+b, a_{m+1}, \cdots, a_n)}$ and

$$\deg_1(f') = (a_2 - a_1, \cdots, a_m - a_1, a_{m+1}, \cdots, a_n, a_1 + b).$$

By the hypothesis there exists a polynomial $F \in A[X_1, \cdots, X_{r(1)}]$ such that $f' = F(g_{11}, \cdots, g_{1r(1)})$. Taking the homogeneous decomposition, we may assume that for any monomial $\alpha X_1^{t_1} \cdots X_{r(1)}^{t_{r(1)}}$ of F, where $\alpha \in A \setminus \{0\}$,

$$\begin{aligned}\deg_1(\alpha g_{11}^{t_1} \cdots g_{1r(1)}^{t_{r(1)}}) &= \left(\sum_{j=1}^{r(1)} t_j a_2^{1j}, \cdots, \sum_{j=1}^{r(1)} t_j a_n^{1j}, \sum_{j=1}^{r(1)} t_j b^{1j}\right)\\ &= (a_2 - a_1, \cdots, a_m - a_1, a_{m+1}, \cdots, a_n, a_1 + b)\\ &= \deg_1(f')\ .\end{aligned}$$

Then $\sum_{j=1}^{r(1)} t_j b^{1j} = a_1 + b \ge b$. Therefore, for each $1 \le j \le r(1)$ and $1 \le \lambda \le t_j$, we may find $0 \le s_{j\lambda} \le b^{1j}$ such that $\sum_{j=1}^{r(1)} \sum_{\lambda=1}^{t_j} s_{j\lambda} = b$. Then

$$\begin{aligned}&\mathrm{Deg}(\alpha \Pi_{j=1}^{r(1)} \Pi_{\lambda=1}^{t_j} g_{1j}(s_{j\lambda}))\\ &= \left(\sum_{j=1}^{r(1)}\sum_{\lambda=1}^{t_j}(b^{1j} - s_{j\lambda}), \sum_{j=1}^{r(1)}\sum_{\lambda=1}^{t_j}(a_2^{1j} + b^{1j} - s_{j\lambda}), \cdots,\right.\\ &\qquad \left.\sum_{j=1}^{r(1)}\sum_{\lambda=1}^{t_j}(a_m^{1j} + b^{1j} - s_{j\lambda}), \sum_{j=1}^{r(1)} t_j a_{m+1}^{1j}, \cdots, \sum_{j=1}^{r(1)} t_j a_n^{1j}, \sum_{j=1}^{r(1)}\sum_{\lambda=1}^{t_j} s_{j\lambda}\right)\\ &= (a_1, \cdots, a_n, b).\end{aligned}$$

Considering all monomials of F, f is expressed as a polynomial of $g_{ij}(s)$ with coefficients A. Thus, $\overline{R}$ is generated by all $g_{ij}(s)$ as an A-algebra. □

Appendix B. Rational Polytope

In this part, we collect the definition and some basic properties of rational polytopes.

DEFINITION B.1 (Rational polytopes). Let $\mathcal{C}$ be a subset in a finite dimensional $\mathbb{R}$-vector space. Then $\mathcal{C}$ is a *polytope* if $\mathcal{C}$ is compact and the intersection of finitely many half-spaces, or equivalently, the convex hull of finitely many points. $\mathcal{C}$ is a *rational polytope* if $\mathcal{C}$ is a polytope defined by rational half-spaces, or the convex hull of finitely many rational points. $\mathcal{F} \subset \mathcal{C}$ is a *face* if whenever $\sum_{i=1}^{k} r_i \boldsymbol{v}_i \in \mathcal{F}$, where $r_1, \cdots, r_k$ are positive real numbers such that $\sum_{i=1}^{k} r_i = 1$ and $\boldsymbol{v}_1, \cdots, \boldsymbol{v}_k$ belong to $\mathcal{C}$, then $\boldsymbol{v}_1, \cdots, \boldsymbol{v}_k$ belong to $\mathcal{F}$. We call $\mathcal{F}$ *proper face* of $\mathcal{C}$ if $\mathcal{F} \subsetneq \mathcal{C}$.

By the definition, a face of a polytope (resp. rational polytope) is also a polytope (resp. rational polytope).

LEMMA B.2. *Let $\mathcal{C}$ be a rational polytope in $\mathbb{R}^n$ such that $\dim \mathcal{C} = n$ and let $p_1, \cdots, p_m$ be its vertices. For every $1 \leq i \leq m$, let $\mathcal{C}_i$ be the rational polytope spanned by $p_1, \cdots, p_{i-1}, p_{i+1}, \cdots, p_m$. If $m > n+1$, then there exists a rational point p in $\mathcal{C}$ such that $p \in \cap_{i=1}^{m} \mathcal{C}_i$.*

PROOF. We prove this lemma in several steps.

Step 1. In this step we reduce the lemma to the case where $m = n+2$ by the induction on m.

Suppose that the statement is true in the case of m vertices. Then there exists a rational point p in $\mathcal{C}$ such that $p \in \cap_{i=1}^{m} \mathcal{C}_i$. In the case of $m+1$ vertices, by changing indices if necessary, we may assume that $\dim \mathcal{C}_{m+1} = \dim \mathcal{C} = n$. Let $\mathcal{C}_i'$ be the rational polytope spanned by $p_1, \cdots, p_{i-1}, p_{i+1}, \cdots, p_m$ for every $1 \leq i \leq m$. Then we have $\mathcal{C}_i' \subset \mathcal{C}_i$ and $\mathcal{C}_i' \subset \mathcal{C}_{m+1}$ for every $1 \leq i \leq m$. By the induction hypothesis, there is a rational point p such that $p \in \cap_{i=1}^{m} \mathcal{C}_i'$. Thus $p \in \cap_{i=1}^{m} \mathcal{C}_i' \subset \cap_{i=1}^{m+1} \mathcal{C}_i$ and so we are done. Therefore we may assume that $m = n+2$.

By an appropriate affine transformation and changing the indices if necessary, we may assume that $p_1, \cdots, p_n$ are canonical basis of $\mathbb{R}^n$ and p_{n+1} is the origin. Then we may write $p_{n+2} = (a_1, \cdots, a_n)$, where $a_i \in \mathbb{Q}$.

Step 2. We first prove the case where $a_i \geq 0$ for any $1 \leq i \leq n$. In this case, we have $\sum_{i=1}^n a_i > 1$. Indeed, if $\sum_{i=1}^n a_i \leq 1$, then $p_{n+2} = \sum_{i=1}^n a_i p_i + (1 - \sum_{i=1}^n a_i) p_{n+1}$. This contradicts to the hypothesis that $p_1, \cdots, p_{n+2}$ are the vertices of $\mathcal{C}$. Set $a = \sum_{i=1}^n a_i$ and let p be $(a_1/a, \cdots, a_n/a)$. Then p is a rational point in $\mathcal{C}$ and

$$\begin{aligned} p &= \sum_{i=1}^{n} \frac{a_i}{a} p_i \ (\in \mathcal{C}_i \text{ for } i = n+1,\ n+2) \\ &= \Big(\frac{1}{a}\Big) p_{n+2} + \Big(1 - \frac{1}{a}\Big) p_{n+1} \ (\in \mathcal{C}_i \text{ for any } 1 \leq i \leq n). \end{aligned}$$

Thus $p \in \cap_{i=1}^{n+2} \mathcal{C}_i$.

Step 3. We prove the case where $a_i < 0$ for some i. By changing indices if necessary, we may assume that $a_1, \cdots, a_l < 0$ and $a_{l+1}, \cdots, a_n \geq 0$ for some l. Set $a = -\sum_{i=1}^l a_i$ and $b = \sum_{i=l+1}^n a_i$.

If $1 + a \leq b$, then let p be $(0, \cdots, 0, a_{l+1}/b, \cdots, a_n/b)$. Then p is a rational point in $\mathcal{C}$ and

$$\begin{aligned} p &= \sum_{i=l+1}^{n} \Big(\frac{a_i}{b}\Big) p_i \ (\in \mathcal{C}_i \text{ for } i = 1, \cdots, l, n+1, n+2) \\ &= \frac{1}{b} \cdot \Big(-\sum_{i=1}^{l} a_i p_i + p_{n+2}\Big) + \Big(1 - \frac{1+a}{b}\Big) p_{n+1} \\ &\qquad (\in \mathcal{C}_i \text{ for any } l+1 \leq i \leq n). \end{aligned}$$

If $1 + a > b$, then set $p = (0, \cdots, 0, a_{l+1}/(1+a), \cdots, a_n/(1+a))$. Then p is a rational point in $\mathcal{C}$ and

$$\begin{aligned} p &= \sum_{i=l+1}^{n} \Big(\frac{a_i}{1+a}\Big) p_i + \Big(1 - \frac{b}{1+a}\Big) p_{n+1} \\ &\qquad (\in \mathcal{C}_i \text{ for } i = 1, \cdots, l, n+2) \\ &= \frac{1}{1+a}\Big(-\sum_{i=1}^{l} a_i p_i + p_{n+2}\Big) \ (\in \mathcal{C}_i \text{ for any } l+1 \leq i \leq n+1). \end{aligned}$$

Thus, $p \in \cap_{i=1}^{n+2} \mathcal{C}_i$. □

LEMMA B.3. *Let $\mathcal{C}$, $\mathcal{D}$, $\mathcal{D}_1, \cdots, \mathcal{D}_r$ be rational polytopes in $\mathbb{R}^n$ such that $\dim \mathcal{C} = n$ and $\mathcal{C} \supset \mathcal{D} = \cup_{i=1}^r \mathcal{D}_i$. Then there exists a finite n-dimensional rational simplex covering $\{\Sigma_\lambda\}_\lambda$ of $\mathcal{C}$ such that $\mathcal{C} = \cup_\lambda \Sigma_\lambda$ and for any λ, $\mathcal{D} \cap \Sigma_\lambda$ is a face of Σ_λ and contained in $\mathcal{D}_i$ for some $1 \leq i \leq r$.*

REMARK B.4. In Lemma B.3, the word "a simplex covering" means a covering by simplices. In particular it does not mean a triangulation. Similarly, the covering $\mathcal{D} = \cup_{i=1}^r \mathcal{D}_i$ of $\mathcal{D}$ by $\{\mathcal{D}_i\}_{i=1}^r$ need not be a subdivision of $\mathcal{D}$ by $\{\mathcal{D}_i\}_{i=1}^r$.

PROOF OF LEMMA B.3. We prove it by the induction on the dimension of $\mathcal{C}$. In the case of $\dim \mathcal{C} = 0$, the statement is trivial. So we may assume that $\dim \mathcal{C} > 0$.

Since $\mathcal{D}$ is a rational polytope, there are finitely many affine functions $H_1, \cdots, H_k$ such that $\mathcal{D}$ is the intersection of these half spaces $(H_j)_{\geq 0} = \{\boldsymbol{x} \in \mathbb{R}^n \mid H_j(\boldsymbol{x}) \geq 0\}$. Set $(H_j)_{\leq 0} = \{\boldsymbol{x} \in \mathbb{R}^n \mid H_j(\boldsymbol{x}) \leq 0\}$ and consider $\mathcal{C}'_j = (H_j)_{\leq 0} \cap \mathcal{C}$. Note that if $\dim \mathcal{C}'_j < n$ for some j, then $\mathcal{C}'_j$ is a proper face of $\mathcal{C}$. Therefore, if we pick all indices j satisfying the condition that $\dim \mathcal{C}'_j = n$, then we have $\mathcal{C} = (\cup_j \mathcal{C}'_j) \cup \mathcal{D}$.

Pick an index j such that $\dim \mathcal{C}'_j = n$. We note that $\mathcal{C}'_j \cap \mathcal{D}$ is a rational polytope contained in a proper face of $\mathcal{C}'_j$. Fix an interior rational point p of $\mathcal{C}'_j$ and let $\mathcal{F}$ be an $(n-1)$-dimensional face of $\mathcal{C}'_j$. Then $\mathcal{F} \cap \mathcal{D}$ and each $\mathcal{F} \cap \mathcal{D}_i$ is empty or a rational polytope in $\mathbb{R}^{n-1}$ and $\mathcal{F} \supset \mathcal{F} \cap \mathcal{D} = \cup_{i=1}^r \mathcal{F} \cap \mathcal{D}_i$. Therefore $\mathcal{F}$, $\mathcal{F} \cap \mathcal{D}$ and $\mathcal{F} \cap \mathcal{D}_i$, where $1 \leq i \leq r$, satisfy the induction hypothesis. So there is a finite $(n-1)$-dimensional rational simplex covering $\{\Sigma'_{\lambda'}\}_{\lambda'}$ of $\mathcal{F}$ which satisfies the conditions of the lemma. Let $\Sigma''_{\lambda'}$ be the convex hull spanned by p and $\Sigma'_{\lambda'}$. Then $\{\Sigma''_{\lambda'}\}_{\lambda'}$ is a finite n-dimensional rational simplex covering of the convex hull spanned by p and $\mathcal{F}$ satisfying the conditions of the lemma. Considering all $(n-1)$-dimensional faces of $\mathcal{C}'_j$, we may find a finite n-dimensional rational simplex covering of $\mathcal{C}'_j$ satisfying the conditions of the lemma.

Considering all $\mathcal{C}'_j$ and a triangulation of each $\mathcal{D}_i$, where $1 \leq i \leq r$, we get a desired covering. □

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