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Non-stationary Navier-Stokes Equations with Mixed Boundary Conditions

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Abstract. In this paper we are concerned with the initial boundary value problem of the 2, 3-D Navier-Stokes equations with mixed boundary conditions including conditions for velocity, static pressure, stress, rotation and Navier slip condition together. Under a compatibility condition at the initial instance it is proved that for the small data there exists a unique solution on the given interval of time. Also, it is proved that if a solution is given, then there exists a unique solution for small perturbed data satisfying the compatibility condition. Our smoothness condition for initial functions in the compatibility condition is weaker than one in such a previous result.

1. Introduction

For the Navier-Stokes equations

$$-\nu\Delta v + (v\cdot\nabla)v + \nabla p = f, \;\; \nabla\cdot v = 0 \quad \text{in } \Omega\subset R^l, l = 2, 3;$$

and

$$\frac{\partial v}{\partial t} - \nu\Delta v + (v\cdot\nabla)v + \nabla p = f, \;\; \nabla\cdot v = 0 \quad \text{in } \Omega$$

different natural and artificial boundary conditions are considered. For example on solid walls, homogeneous Dirichlet condition $v = 0$ is often used. On a free surface a Neumann condition $2\nu\varepsilon(v)n - pn = 0$ may be useful. Here and in what follows $\varepsilon(v)$ denotes the so-called strain tensor with the components $\varepsilon_{ij}(v) = \frac{1}{2}(\partial_{x_i} v_j + \partial_{x_j} v_i)$ and n is the outward normal unit vector. For simulations of flows in the presence of rough boundaries, the Navier

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slip-with-friction boundary conditions $v \cdot n = 0$, $(\nu\varepsilon_{n\tau}(v) + \alpha v_\tau)|_{\Gamma_5} = 0$ is also used, where $\varepsilon_{n\tau}(v)$ and v_τ are, respectively, the tangent components of $\varepsilon(v)n$ and v. Combination of the condition $v_n = 0$ and the tangential component of the friction (slip condition for uncovered fluid surfaces) or the condition $v_\tau = 0$ and the normal component of the friction (condition for in/out-stream surfaces) are frequently used. At the outlet of a channel "do nothing" condition $\nu\frac{\partial v}{\partial n} - pn = 0$, i.e. the outlet boundary condition, is also used. Rotation boundary condition has been fairly extensively studied over the past several years. Also, for inlet or outlet of flow one deals with the static pressure p or total pressure (Bernoulli's pressure) $\frac{1}{2}|v|^2+p$. For papers dealing with the problems mentioned above one can refer to Introduction of [13].

In practice we deal with mixture of some kind of boundary conditions. For a channel flow a mixture of Dirichlet condition $v = 0$ on the wall and "do nothing" condition on the outlet is used. But for a channel flow with a rough boundary surface a mixture of Dirichlet condition, the Navier slip-with-friction boundary condition and "do nothing" condition may be used. For a flow in a vessel with in/outlet a mixture of Dirichlet condition $v = 0$ on the wall and pressure conditions on the inlet/outlet is used. But for the flow in a vessel with in/outlet and a free surface a mixture of Dirichlet condition, a Neumann condition $2\nu\varepsilon(v)n - pn = 0$ and pressure conditions may be used.

There are vast literatures for the Stokes and Navier-Stokes problems with mixed boundary conditions and several variational formulations are used for them, but two possible examples above are not considered except [13]. With exception [13] mixtures of boundary conditions for Navier-Stokes equations may be divided into three groups according to what bilinear form is used for a variational formulation (for more concrete one can refer to Introduction of [13]).

To include more different boundary conditions together, in [13] the relations among strain, rotation, normal derivative of vector field and shape of boundary surface are obtained and applied to the stationary and non-stationary Navier-Stokes problems with mixture of seven kinds of boundary conditions. However, for the non-stationary Navier-Stokes problems we only were concerned with a mixed boundary condition including total pressure (not static pressure), total stress (not stress) and so on. Thus, in this paper we will study the non-stationary problems with a mixed boundary condition including static pressure (not total pressure), stress (not total stress) and so on.

On the other hand, when one of static pressure (instead of total pressure), stress (instead of total stress) or the outlet boundary condition is given on a portion of boundary, for the initial boundary value problems of the Navier-Stokes equations existence of a unique local-in-time solution and a unique solution on a given interval for small given data (in what follows we call it a solution for small data) are proved. In the mathematical point of view the main difficulty of such problems consists in the fact that in priori estimation the inertial term is not canceled, and in the mechanical point of view it is in the fact that the kinetic energy of fluid is not controlled by the data and uncontrolled “backward flow” can take place at the portion of boundary(cf. preface in [2]).

The Navier-Stokes equations with mixture of Dirichlet condition and stress condition are studied. In [15] under smoothness condition and a compatibility condition of data at the initial instance existence of a unique local-in-time solution to the 3-D Navier-Stokes equations is studied. In [2] for the Navier-Stokes equations on the polyhedral domain with mixture of Dirichlet condition, Navier slip condition and stress condition a local-in-time solution is studied. Here smoothness of solutions to the corresponding steady Stokes problem is used essentially.

The Navier-Stokes equations and the Boussinesq equations with mixture of Dirichlet condition and the outlet boundary condition are studied. For 2-D Navier-Stokes equations a local-in-time solution in [7] and a solution for small data in [8] are studied. Here also smoothness of solutions to the corresponding 2-D steady linear problem is important. For the Boussinesq systems a local-in-time strong solutions in [3] on 2-D channel and in [6] on 3-D channel are studied. Here smoothness of solutions to the corresponding steady linear problems, respectively, in [5] and [1] is the key. In [14] it is proved that if under a compatibility condition at initial instance there exists a unique solution, then so does for small perturbed data. This result shows that under the compatibility condition there exists a unique solution for small data. In [20] for the Boussinesq equations it is proved that under a compatibility condition there exists a unique local-in-time solution, which is similar to the result in [15]. Smoothness of initial function in the compatibility condition of [14] is stronger than one in [15] and [20].

The 2-D Navier-Stokes equations with mixture of Dirichlet condition and pressure is studied. In [16] existence of a unique solution for small data is proved.

The Navier-Stokes equations with mixture of Dirichlet condition, outlet condition and tangent stress condition is studied. In [4] existence of a unique solution for small perturbation data of the given solution is studied. Here also smoothness of solutions to the corresponding steady linear problem is the key.

In the present paper as a continuation of [13], we are concerned with the non-steady Navier-Stokes equations with mixed boundary conditions involving conditions for Dirichlet, static pressure, rotation, stress and normal derivative of velocity together. Owing to the relations among strain, rotation, normal derivative of velocity and shape of boundary surface obtained in [13] (Theorems 2.1, 2.2), we can consider all these boundary conditions together.

In general, the solution of the Stokes problem with mixed boundary conditions has singularities on the intersections of surfaces for different boundary conditions and the leading singular exponent of the solution is a function of the intersection angle (cf. [19]). For the problem with Dirichlet condition and "do nothing" condition if the intersection angle is $\pi/2$, then under some conditions for data the solution belongs to $H^2(\Omega)$ (cf. [5]). For the problem with Dirichlet condition and stress conditions, for similar results refer to subsection 5.5 of [17] and section 10.3 of [18]. In our case the boundary conditions are more complicated than others, and there is no result for smoothness of solutions to the corresponding steady linear problems. Thus, we prove existence of a unique weak solution for small data under a compatibility condition at initial instance. We also prove that if a solution smooth as in [14] is given, then under the compatibility condition for the small perturbed data there exists a unique solution.

We are concerned with two problems distinguished according to boundary conditions. Using relations among strain, rotation, normal derivative of vector field and shape of boundary surface, which are obtained in [13], we reflect all these boundary conditions into variational formulations for problems.

This paper consists of 5 sections. In the end of this section the method in this paper is compared with another one.

In Section 2, the formulations of problems and some results for definitions of weak solutions are given. According to bilinear forms used for variational formulations for problems, the involved boundary conditions are slightly different. Thus, difference between our two problems is explained (Remark 2.1).

In Section 3, first, for the Navier-Stokes problems with seven kinds of boundary conditions a variational formulation, which is based on the bilinear form

$$a(v,u) = 2\Sigma_{i,j}(\varepsilon_{ij}(v), \varepsilon_{ij}(u))_{L_2(\Omega)} \quad \text{for } v,u \in \mathbf{H}^1(\Omega), \tag{1.1}$$

is given. Next, by a transformation of the unknown function, the problem is reduced to an equivalent problem in which the linear main operator is positive definite. Then, studying properties of linear operator differential equations and using a local diffeomorphism theorem of nonlinear operator, we prove that under a compatibility condition similar to one in [14], [15], [20] there exists a unique solution for small data (Theorem 3.8).

In Section 4 for the Navier-Stokes problems with six kinds of boundary conditions, which is a little different from one in Section 3, a variational formulation based on the bilinear form

$$a(v,u) = (\nabla v, \nabla u)_{\mathbf{L}_2(\Omega)} \quad \text{for } v,u \in \mathbf{H}^1(\Omega) \tag{1.2}$$

is given. Also, by a transformation of the unknown function, the problem is reduced to another equivalent problem in which the linear main operator is positive definite. The result similar to one in Section 3 is obtained (Theorem 4.2).

Section 5 is considered in comparison with [14] rather than practical models. Existence of a unique solution for the small data perturbed from a given solutions is proved under a compatibility condition (Theorem 5.7).

The compatibility conditions in Sections 3, 4 and 5 are similar to one in [15], [20] and [14]. In point of view of smoothness of the initial functions, the conditions are the same with one in [15], [20] concerning with local-in-time solutions (cf. Remark 3.4), but weaker than one in [14] concerning with solutions for small data as our case (cf. Remarks 4.2, 5.2). In [14] the main results for the nonlinear problem as perturbation of a linear problem is obtained by a local diffeomorphism theorem relying on the properties of the corresponding linear problem, and so is it in our paper.

Then, let us consider why smoothness of the initial functions in our compatibility conditions is weaker than one in [14].

Let $\mathbf{H}^k = (W_2^k(\Omega))^l$ be a Sobolev spaces on Ω with dimension l, V be a divergence-free subspace of $\mathbf{H}^1$ satisfying appropriate boundary conditions, H - the closure of V in $(L^2(\Omega))^l$, V^*- the adjoint space of V, $V^{r_0}(\Omega) = V \cap \mathbf{H}^{r_0}(\Omega)$, where $r_0 > l/2$, $\mathscr{X} = \{w \in L_2(0,T;V); w' \in L_2(0,T;V), w'' \in L_2(0,T;V^*)\}, \mathscr{Y} = \{w \in L_2(0,T;V^*); w' \in L_2(0,T;V^*)\}$ and $A : V \to V^*$- the Stokes operator.

Considering a linear problem

$$\begin{cases} u'(t) + Au(t) = f(t), \\ u(0) = \varphi, \end{cases}$$

in [14] the author proved the fact that a map $u \to \{u(0), Lu \equiv u' + Au\}$ is linear continuous one-to-one from $\mathcal{X} = \{u \in \mathscr{X} : u(0) \in V^{r_0}(\Omega)\}$ onto $\mathcal{Y} = \{[\varphi, h] : \varphi \in V^{r_0}(\Omega), h \in \mathscr{Y}, h(0) - A\varphi \in H\}$ (Theorem 3.1 in [14]). Then, starting from this fact, the author studied a nonlinear problem

$$\begin{cases} u'(t) + (A+B)u(t) = f(t), \\ u(0) = \varphi, \end{cases} \tag{1.3}$$

where $B : V \to V^*$ is defined by $\langle Bu, v\rangle = \langle (u \cdot \nabla)u, v\rangle$ for $u, v \in V$. To this end, it was proved that the inverse of a nonlinear map $u \to \{u(0), \tilde{L}u \equiv u' + (A+B)u\}$ is one-to-one from a neighborhood of $0_{\mathcal{Y}}$ onto a neighborhood of $0_{\mathcal{X}}$. From this fact the author obtained that under the compatibility condition $f(0) - A\varphi \in H, \varphi \in V^{r_0}(\Omega)$ and smallness of data, there exists a unique solution to (1.3).

However, we prove that for a modified operator A a map $u \to \{u'(0), Lu \equiv u' + Au\}$ is linear continuous one-to-one from $\mathscr{X}$ onto $H \times \mathscr{Y}$ (Lemmas 3.3, 4.3). Then using this fact, we prove that for a modified operator $B(t)$ the inverse of a nonlinear map $u \to \{u'(0), \tilde{L}u \equiv u' + (A+B(t))u\}$ is one-to-one from a neighborhood of $0_{H\times\mathscr{Y}}$ onto a neighborhood of $0_{\mathscr{X}}$. By this under the compatibility condition $f(0) - A\varphi - B(0)\varphi \in H$ without $\varphi \in V^{r_0}(\Omega)$ and smallness of data, we get existence of unique solution to (1.3) (Theorems 3.8, 4.2, 5.7, 5.8). Since $B(0)\varphi = (\varphi \cdot \nabla)\varphi$, for $\varphi \in V^{l/2}(\Omega)$ we get $B(0)\varphi \in H$, and so our condition is weaker than one in [14].

2. Problems and Preliminaries

Throughout this paper we will use the following notation.

Let Ω be a connected bounded open subset of R^l, $l = 2, 3$. $\partial\Omega \in C^{0,1}$, $\partial\Omega = \cup_{i=1}^N \overline{\Gamma}_i, \Gamma_i \cap \Gamma_j = \varnothing$ for $i \neq j$, $\Gamma_i \in C^2$ for $i = 2, 3$. For Problems I and II stated below we assume, respectively, $\Gamma_7 \in C^2$ and $\Gamma_5 \in C^2$. Let $n(x)$ and $\tau(x)$ be, respectively, outward normal and tangent unit vectors at x in $\partial\Omega$. When X is a Banach space, $\mathbf{X} = X^l$ and $\mathbf{X}^*$ is the dual of $\mathbf{X}$. Let $W^k_\alpha(\Omega)$ be Sobolev spaces, $H^k(\Omega) = W^k_2(\Omega)$, and so $\mathbf{H}^1(\Omega) = \{H^1(\Omega)\}^l$. $Q = \Omega \times (0, T)$, $\Sigma_i = \Gamma_i \times (0, T)$, $0 < T < \infty$.

An inner product and a norm in the space $\mathbf{L}_2(\Omega)$ are denoted, respectively, by $(\cdot\,,\cdot)$ and $\|\cdot\|$; and $\langle\cdot\,,\cdot\rangle$ means the duality pairing between a Sobolev space X and its dual one. Also, $(\cdot\,,\cdot)_{\Gamma_i}$ is an inner product in the $\mathbf{L}_2(\Gamma_i)$ or $L_2(\Gamma_i)$; and $\langle\cdot\,,\cdot\rangle_{\Gamma_i}$ means the duality pairing between $\mathbf{H}^{\frac{1}{2}}(\Gamma_i)$ and $\mathbf{H}^{-\frac{1}{2}}(\Gamma_i)$ or between $H^{\frac{1}{2}}(\Gamma_i)$ and $H^{-\frac{1}{2}}(\Gamma_i)$. The inner product and norms in R^l, respectively, are denoted by $(\cdot\,,\cdot)_{R^l}$ and $|\cdot|$. Sometimes $a \cdot b$ is used for inner product in R^l between a and b. When $\mathbf{X}$ is a Banach space, the zero element of $\mathbf{X}$ is denoted by $0_\mathbf{X}$ and $\mathbb{O}_M(0_\mathbf{X})$ means M-neighborhood of $0_\mathbf{X}$.

In this paper for the Navier-Stokes problem

$$(2.1)\qquad \begin{cases} \frac{\partial v}{\partial t} - \nu\Delta v + (v\cdot\nabla)v + \nabla p = f, & \text{in } Q,\\ \nabla\cdot v = 0, \\ v(0) = v_0 \end{cases}$$

we are concerned with the problems I and II, which are distinguished according to boundary conditions. Problem I is one with the boundary conditions

$$(2.2)\qquad \begin{aligned}
&(1)\quad v|_{\Gamma_1} = h_1,\\
&(2)\quad v_\tau|_{\Gamma_2} = 0,\ -p|_{\Gamma_2} = \phi_2,\\
&(3)\quad v_n|_{\Gamma_3} = 0,\ \operatorname{rot} v \times n|_{\Gamma_3} = \phi_3/\nu,\\
&(4)\quad v_\tau|_{\Gamma_4} = h_4,\ (-p + 2\nu\varepsilon_{nn}(v))|_{\Gamma_4} = \phi_4,\\
&(5)\quad v_n|_{\Gamma_5} = h_5,\ 2(\nu\varepsilon_{n\tau}(v) + \alpha v_\tau)|_{\Gamma_5} = \phi_5,\\
&(6)\quad (-pn + 2\nu\varepsilon_n(v))|_{\Gamma_6} = \phi_6,\\
&(7)\quad v_\tau|_{\Gamma_7} = 0,\ (-p + \nu\frac{\partial v}{\partial n}\cdot n)|_{\Gamma_7} = \phi_7,
\end{aligned}$$

and Problem II is one with the conditions

$$
(2.3)\qquad
\begin{aligned}
&(1)\quad v|_{\Gamma_1} = h_1,\\
&(2)\quad v_\tau|_{\Gamma_2} = 0,\ -p|_{\Gamma_2} = \phi_2,\\
&(3)\quad v_n|_{\Gamma_3} = 0,\ \operatorname{rot} v \times n|_{\Gamma_3} = \phi_3/\nu,\\
&(4)\quad v_\tau|_{\Gamma_4} = h_4,\ (-p + 2\nu\varepsilon_{nn}(v))|_{\Gamma_4} = \phi_4,\\
&(5)\quad v_n|_{\Gamma_5} = h_5,\ 2(\nu\varepsilon_{n\tau}(v) + \alpha v_\tau)|_{\Gamma_5} = \phi_5,\\
&(6)\quad (-pn + \nu\frac{\partial v}{\partial n})|_{\Gamma_7} = \phi_7,
\end{aligned}
$$

together $\Gamma_6 = \varnothing$. Here and in what follows $u_n = u\cdot n$, $u_\tau = u-(u\cdot n)n$, $\varepsilon(v) = \{\varepsilon_{ij}(v)\}$, $\varepsilon_n(v) = \varepsilon(v)n$, $\varepsilon_{nn}(v) = (\varepsilon(v)n, n)_{R^l}$, $\varepsilon_{n\tau}(v) = \varepsilon(v)n - \varepsilon_{nn}(v)n$, and h_i, ϕ_i are functions or vector functions of x, t defined on $\Gamma_i \times (0, T)$.

Remark 2.1. The condition (6) of (2.3) (with $\phi_7 = 0$) is "do nothing" condition, but (7) of (2.2) (with $\phi_7 = 0$) is rather different from "do nothing" condition and we can not unify two problems.

First, let us consider why (7) of (2.2) is not changed with (6) of (2.3). In Section 3 relying on the bilinear form (1.1) and integrating by parts $(-\nu\Delta v + \nabla p, u)$, we get boundary integral $(-2\nu(\varepsilon(v)n, u)_{\partial\Omega} + (p, u \cdot n)_{\partial\Omega}$. Then, in order to reflect the boundary conditions into Formulation 3.1, using $v_\tau = 0$ or $v_n = 0$ and applying Theorems 2.1 or 2.2, we transform the boundary integrals on $\Gamma_i, i = 2, 3, 7$. (cf. (3.1)-(3.3)). Concretely, under conditions $v_\tau|_{\Gamma_7} = 0$ we have

$$
(2.4)\qquad \big\langle -pn + \nu\frac{\partial v}{\partial n}, u\big\rangle_{\Gamma_7} \quad \forall u \text{ with } u_\tau = 0.
$$

Usually, $v_\tau = 0$ does not imply $\frac{\partial v}{\partial n} \cdot \tau = 0$, but by virtue of the conditions $u_\tau = 0$ and (7) of (2.2) we have

$$
(2.5)\qquad
\begin{aligned}
\big\langle -pn + \nu\frac{\partial v}{\partial n}, u\big\rangle_{\Gamma_7} &= \big\langle -p + \nu\frac{\partial v}{\partial n}n, u_n\big\rangle_{\Gamma_7}\\
&= \langle \phi_7, u_n\rangle_{\Gamma_7} \quad \forall u \text{ with } u_\tau = 0.
\end{aligned}
$$

Thus, substituting $\big\langle -pn+\nu\frac{\partial v}{\partial n}, u\big\rangle_{\Gamma_7}$ with $\langle \phi_7, u_n\rangle_{\Gamma_7}$, we reflect the boundary condition (7) of (2.2) into Formulation 3.1.

Changing (7) of (2.2) by $(-pn + \nu\frac{\partial v}{\partial n})|_{\Gamma_7} = \phi_7$ with a vector ϕ_7 and substituting $\big\langle -pn + \nu\frac{\partial v}{\partial n}, u\big\rangle_{\Gamma_7}$ with $\langle \phi_7, u\rangle_{\Gamma_7}$, we can come to a formal variational formulation. But when a solution v is smooth enough, trying to convert from the formal variational formulation to the original problem, we come to

$$(-pn+\nu\frac{\partial v}{\partial n},u)_{\Gamma_7}=\langle\phi_7,u\rangle_{\Gamma_7}\quad\forall u,u_\tau=0 \tag{2.6}$$

on Γ_8. If we have (2.6) without $u_\tau=0$, then from (2.6) we can get $-pn+\nu\frac{\partial v}{\partial n}=\phi_7$ on Γ_8. But owing to $u_\tau=0$ we only get $(-pn+\nu\frac{\partial v}{\partial n},n)_{\Gamma_7}=<\phi_7,n>_{\Gamma_7}$. This shows that the formal variational formulation is not equivalent to the original condition on Γ_7 and equivalent to $(-p+\nu\frac{\partial v}{\partial n}n)|_{\Gamma_7}=\phi_7\cdot n$. (Thus, (7) of (3.3) in [13] was corrected. See Erratum to: [13].)

Similarly, relying on the form (1.2), we can reflect "do nothing" condition into Formulation 4.1, but can not do (6) of (2.2). (cf. (4.1)-(4.3)). Therefore, two problems are not unified.

"Do nothing" boundary condition results from variational formulation based on (1.2) and does not have a real physical meaning, but is rather used in truncating large physical domains to smaller computational domains by assuming parallel flow. If the flow is parallel in a near the boundary, then (7) of (2.2) is same with "do nothing" condition.

For variational formulations of Problems I, II we need the following.

Let Γ be a surface (curve for $l=2$) of C^2 and v be a vector field of C^2 on a domain of R^l near Γ. In what follows the surface is a piece of boundary of 3-D or 2-D bounded connected domains, and so we can assume the surface is oriented.

THEOREM 2.1 (Theorem 2.1 in [13]). *Suppose that $v\cdot n|_\Gamma=0$. Then, on the surface Γ the followings hold.*

$$(\varepsilon(v)n,\tau)_{R^l}=\frac{1}{2}(rot\,v\times n,\tau)_{R^l}-(S\tilde{v},\tilde{\tau})_{R^{l-1}}, \tag{2.7}$$

$$(rot\,v\times n,\tau)_{R^l}=\left(\frac{\partial v}{\partial n},\tau\right)_{R^l}+(S\tilde{v},\tilde{\tau})_{R^{l-1}}, \tag{2.8}$$

$$(\varepsilon(v)n,\tau)_{R^l}=\frac{1}{2}\left(\frac{\partial v}{\partial n},\tau\right)_{R^l}-\frac{1}{2}(S\tilde{v},\tilde{\tau})_{R^{l-1}}, \tag{2.9}$$

where S is the shape operator of the surface Γ for $l = 3$, i.e.

$$S = \begin{pmatrix} L & K \\ M & N \end{pmatrix},$$
$$L = \left(e_1, \frac{\partial n}{\partial e_1}\right)_{R^l}, \; K = \left(e_2, \frac{\partial n}{\partial e_1}\right)_{R^l},$$
$$M = \left(e_1, \frac{\partial n}{\partial e_2}\right)_{R^l}, \; N = \left(e_2, \frac{\partial n}{\partial e_2}\right)_{R^l},$$

and the curvature of Γ for $l = 2$. Here e_i are the unit vectors in a local curvilinear coordinates on Γ and $\tilde{v}, \tilde{\tau}$ are expressions of the vectors v, τ in the coordinate system.

Theorem 2.2 (Theorem 2.2 in [13]). *If $v_\tau|_\Gamma = 0$ and $div v = 0$, then on the surface Γ the following holds.*

$$(\varepsilon(v)n, n)_{R^l} = \left(\frac{\partial v}{\partial n}, n\right)_{R^l} = -(k(x)v, n)_{R^l}$$

where $k(x) = div n(x)$.

Remark 2.2 (cf. [13]). $k(x) = \operatorname{div} n(x) = Tr(S(x)) = 2 \times$ mean curvature.

If Γ is a piece of $\partial\Omega$, then since $\partial\Omega \in C^{0,1}$ and $\Gamma \in C^2$, elements of S belong to $C(\bar{\Gamma})$ and so does $k(x)$.

3. Existence of a Unique Solution to Problem I

We use the following notation.

$\mathbf{V} = \{u \in \mathbf{H}^1(\Omega) : \operatorname{div} u = 0,\ u|_{\Gamma_1} = 0,\ u_\tau|_{\Gamma_2 \cup \Gamma_4 \cup \Gamma_7} = 0,\ u_n|_{\Gamma_3 \cup \Gamma_5} = 0\}$ and $\mathbf{V}_{\Gamma 237}(\Omega) = \{u \in \mathbf{H}^1(\Omega) : \operatorname{div} u = 0,\ u_\tau|_{\Gamma_2 \cup \Gamma_7} = 0,\ u_n|_{\Gamma_3} = 0\}$. Denote by H the completion of $\mathbf{V}$ in the space $\mathbf{L}_2(\Omega)$. Through this paper $\widetilde{\mathbf{V}} = \{u \in \mathbf{H}^1(\Omega) : \operatorname{div} u = 0\}$.

By Theorems 2.1 and 2.2 we have that for $v \in \mathbf{H}^2(\Omega) \cap \mathbf{V}_{\Gamma 237}(\Omega)$ and

$u \in \mathbf{V}$

$$
\begin{aligned}
-(\Delta v, u) &= 2(\varepsilon(v), \varepsilon(u)) - 2(\varepsilon(v)n, u)_{\cup_{i=2}^{7}\Gamma_i} \\
&= 2(\varepsilon(v), \varepsilon(u)) + 2(k(x)v, u)_{\Gamma_2} - (\operatorname{rot} v \times n, u)_{\Gamma_3} \\
&\quad + 2(S\tilde{v}, \tilde{u})_{\Gamma_3} - 2(\varepsilon_n(v), u)_{\cup_{i=4}^{7}\Gamma_i} \\
&= 2(\varepsilon(v), \varepsilon(u)) + 2(k(x)v, u)_{\Gamma_2} - (\operatorname{rot} v \times u, u)_{\Gamma_3} \\
&\quad + 2(S\tilde{v}, \tilde{u})_{\Gamma_3} - 2(\varepsilon_{nn}(v), u \cdot n)_{\Gamma_4} - 2(\varepsilon_{n\tau}(v), u)_{\Gamma_5} \\
&\quad - 2(\varepsilon_n(v), u)_{\Gamma_6} - \left(\frac{\partial v}{\partial n}, u\right)_{\Gamma_7} + (k(x)v, u)_{\Gamma_7}.
\end{aligned} \tag{3.1}
$$

Also, for $p \in H^1(\Omega)$ and $u \in \mathbf{V}$ we have

$$
(\nabla p, u) = (p, u \cdot n)_{\cup_{i=2}^{7}\Gamma_i} = (p, u \cdot n)_{\Gamma_2} + (p, u \cdot n)_{\Gamma_4} + (pn, u)_{\Gamma_6 \cup \Gamma_7}, \tag{3.2}
$$

where the fact that $u_n|_{\Gamma_3 \cup \Gamma_5} = 0$ was used.

Let

$$
\begin{aligned}
\mathscr{X} &= \{w \in L_2(0,T;\mathbf{V}); w' \in L_2(0,T;\mathbf{V}), w'' \in L_2(0,T;\mathbf{V}^*)\}, \\
&\qquad \|w\|_{\mathscr{X}} = \|w\|_{L_2(0,T;\mathbf{V})} + \|w'\|_{L_2(0,T;\mathbf{V})} + \|w''\|_{L_2(0,T;\mathbf{V}^*)}, \\
\mathscr{Y} &= \{w \in L_2(0,T;\mathbf{V}^*); w' \in L_2(0,T;\mathbf{V}^*)\}, \\
&\qquad \|w\|_{\mathscr{Y}} = \|w\|_{L_2(0,T;\mathbf{V}^*)} + \|w'\|_{L_2(0,T;\mathbf{V}^*)}, \\
\mathscr{W} &= \{w \in L_2(0,T;\widetilde{\mathbf{V}}); w' \in L_2(0,T;\widetilde{\mathbf{V}}), w'' \in L_2(0,T;\widetilde{\mathbf{V}}^*)\}, \\
&\qquad \|w\|_{\mathscr{W}} = \|w\|_{L_2(0,T;\tilde{\mathbf{V}})} + \|w'\|_{L_2(0,T;\tilde{\mathbf{V}})} + \|w''\|_{L_2(0,T;\tilde{\mathbf{V}}^*)}.
\end{aligned}
$$

Here and in what follows w' means the derivative of $w(t)$ with respect to t.

For Problem I, we use the following assumptions.

Assumption 3.1. *$f, f' \in L_2(0,T;\mathbf{V}^*)$, $\phi_i, \phi_i' \in L_2(0,T;H^{-\frac{1}{2}}(\Gamma_i))$, $i = 2,4,7$, $\phi_i, \phi_i' \in L_2(0,T;\ \mathbf{H}^{-\frac{1}{2}}(\Gamma_i)), i = 3,5,6$, $\alpha_{ij} \in L_\infty(\Gamma_5)$, where α_{ij} are components of the matrix α, and $\Gamma_1 \neq \emptyset$.*

Assumption 3.2. *There exists a function $U \in \mathscr{W}$ such that*

$$
divU = 0,\ U|_{\Gamma_1} = h_1,\ U_\tau|_{\Gamma_2 \cup \Gamma_7} = 0,\ U_n|_{\Gamma_3} = 0,\ U_\tau|_{\Gamma_4} = h_4,\ U_n|_{\Gamma_5} = h_5.
$$

Also, $U(0,x) - v_0 \in \boldsymbol{V}$.

REMARK 3.1. In practical situations $h_4, h_5 = 0$, and in the cases if for every fixed t $h_1(t,x) \in H_{00}^{\frac{1}{2}}(\Gamma_1)$, $\int_{\Gamma_1} h_1(t,x)\cdot n\,dx = 0$ and $\|h_1(t,x)\|_{H^{\frac{1}{2}}(\Gamma_1)}$ is smooth enough with respect to t, then there exists such a function U.

Taking (3.1) and (3.2) into account, we get the following variational formulation for Problem I:

FORMULATION 3.1. *Find v such that*

$$\begin{aligned}
&v - U \in L_2(0,T;\mathbf{V}),\\
&v(0) = v_0,\\
&\langle v', u\rangle + 2\nu(\varepsilon(v),\varepsilon(u)) + \langle (v\cdot\nabla)v, u\rangle + 2\nu(k(x)v,u)_{\Gamma_2}\\
&\qquad + 2\nu(S\tilde{v},\tilde{u})_{\Gamma_3} + 2(\alpha(x)v,u)_{\Gamma_5} + \nu(k(x)v,u)_{\Gamma_7}\\
&\qquad = \langle f,u\rangle + \sum_{i=2,4,7}\langle \phi_i, u_n\rangle_{\Gamma_i} + \sum_{i=3,5,6}\langle \phi_i, u\rangle_{\Gamma_i} \quad \textit{for all } u \in \boldsymbol{V}.
\end{aligned} \tag{3.3}$$

Taking Assumption 3.2 into account, put $v = \overline{z} + U$. Then, we have the following problem equivalent to Formulation 3.1:

Find $\overline{z}$ such that

$$\begin{aligned}
&\overline{z} \in L_2(0,T;\mathbf{V}),\\
&\overline{z}(0) = \overline{z}_0 \equiv v_0 - U(0) \in \mathbf{V},\\
&\langle \overline{z}', u\rangle + 2\nu(\varepsilon(\overline{z}),\varepsilon(u)) + \langle (\overline{z}\cdot\nabla)\overline{z}, u\rangle + \langle (U\cdot\nabla)\overline{z}, u\rangle + \langle (\overline{z}\cdot\nabla)U, u\rangle\\
&\qquad + 2\nu(k(x)\overline{z},u)_{\Gamma_2} + 2\nu(S\tilde{\overline{z}},\tilde{u})_{\Gamma_3} + 2(\alpha(x)\overline{z},u)_{\Gamma_5}\\
&\qquad + \nu(k(x)\overline{z},u)_{\Gamma_7}\\
&= -(U',u) - 2\nu(\varepsilon(U),\varepsilon(u)) - \langle (U\cdot\nabla)U,u\rangle - 2\nu(k(x)U,u)_{\Gamma_2}\\
&\qquad - 2\nu(S\tilde{U},\tilde{u})_{\Gamma_3} - 2(\alpha(x)U,u)_{\Gamma_5} - \nu(k(x)U,u)_{\Gamma_7} + \langle f,u\rangle\\
&\qquad + \sum_{i=2,4,7}\langle \phi_i, u_n\rangle_{\Gamma_i} + \sum_{i=3,5,6}\langle \phi_i, u\rangle_{\Gamma_i} \qquad \text{for all } u \in \mathbf{V}.
\end{aligned} \tag{3.4}$$

Now, define an operator $A_0 : \mathbf{V} \to \mathbf{V}^*$ by

$$\begin{aligned}
\langle A_0 y, u\rangle =& 2\nu(\varepsilon(y),\varepsilon(u)) + 2\nu(k(x)y,u)_{\Gamma_2} + 2\nu(S\tilde{y},\tilde{u})_{\Gamma_3}\\
&+ 2(\alpha(x)y,u)_{\Gamma_5} + \nu(k(x)y,u)_{\Gamma_7} \quad \text{for all } y,u \in \mathbf{V}.
\end{aligned} \tag{3.5}$$

Lemma 3.1. *$\exists\delta>0$, $\exists k_0\geq 0$; $\langle A_0u,u\rangle\geq\delta\|u\|^2_{\mathbf{V}}-k_0\|u\|^2_H$ for all $u\in\mathbf{V}$.*

Proof. By Korn's inequality

$$2\nu(\varepsilon(u),\varepsilon(u))\geq\beta\|u\|^2_{\mathbf{V}}\quad\exists\beta>0,\text{for all } u\in\mathbf{V}. \tag{3.6}$$

By Remark 2.2 and Assumption 3.1, there exists a constant M such that

$$\|S(x)\|_\infty,\ \|k(x)\|_\infty,\|\alpha(x)\|_\infty\leq M,$$

and so there exists a constant c_0 (depending on β) such that

$$\begin{aligned}&|2\nu(k(x)z,z)_{\Gamma_2}+2\nu(S\tilde{z},\tilde{z})_{\Gamma_3}+\nu(k(x)z,z)_{\Gamma_7}+2(\alpha(x)y,u)_{\Gamma_5}|\\&\qquad\leq\frac{\beta}{2}\|z\|^2_{\mathbf{H}^1(\Omega)}+c_0\|z\|^2_H\,dt\quad\text{for all } z\in\mathbf{V}\end{aligned} \tag{3.7}$$

((cf. Theorem 1.6.6 in [9] or (1), p. 258 in [11])). Put $\delta=\frac{\beta}{2}$, $k_0=c_0$. Then, by (3.6), (3.7) we come to the asserted conclusion. □

Remark 3.2. In process of proof of Lemma 3.1, we see that if $\Gamma_i=\emptyset$, $i=2,3,7$, or these are unions of pieces of planes (segments in case of 2-D)and $\Gamma_5=\emptyset$ or $\alpha(x)=0$, then we can take $k_0=0$.

When $k_0>0$, if k_0 is not small enough, then the operator defined by (3.5) is not positive, and so let us transform the unknown function to get a positive operator A in (3.9) bellow. Now, let k_0 be the constant in Lemma 3.1 and put $z=e^{-k_0t}\overline{z}$. Then, since $e^{-k_0t}\overline{z}'=z'+k_0z$, we get the following problem equivalent to problem (3.4):

Find z such that

$$
\begin{aligned}
&z \in L_2(0,T;\mathbf{V}),\\
&z(0) = v_0 - U(0) \in \mathbf{V},\\
&\langle z'(t), u\rangle + 2\nu(\varepsilon(z(t)), \varepsilon(u)) + e^{k_0 t}\langle (z(t)\cdot\nabla)z(t), u\rangle\\
&\qquad + \langle (U(t)\cdot\nabla)z(t), u\rangle + \langle (z(t)\cdot\nabla)U(t), u\rangle + k_0(z(t), u)\\
&\qquad + 2\nu(k(x)z(t), u)_{\Gamma_2} + 2\nu(S\tilde{z}(t), \tilde{u})_{\Gamma_3} + 2(\alpha(x)z(t), u)_{\Gamma_5}\\
&\qquad + \nu(k(x)z(t), u)_{\Gamma_7}\\
&= e^{-k_0 t}\Big[- (U'(t), u) - 2\nu(\varepsilon(U(t)), \varepsilon(u)) - \langle (U(t)\cdot\nabla)U(t), u\rangle\\
&\qquad - 2\nu(k(x)U(t), u)_{\Gamma_2} - 2\nu(S\tilde{U}(t), \tilde{u})_{\Gamma_3} - 2(\alpha(x)U(t), u)_{\Gamma_5}\\
&\qquad - \nu(k(x)U(t), u)_{\Gamma_7} + \langle f(t), u\rangle + \sum_{i=2,4,7}\langle \phi_i(t), u_n\rangle_{\Gamma_i}\\
&\qquad + \sum_{i=3,5,6}\langle \phi_i(t), u\rangle_{\Gamma_i}\Big] \quad \text{for all } u \in \mathbf{V}.
\end{aligned}
\tag{3.8}
$$

Define operators A, $A_U(t) : \mathbf{V} \to \mathbf{V}^*$ by

$$
\langle Av, u\rangle = \langle A_0 v, u\rangle + (k_0 v, u) \quad \text{for all } v, u \in \mathbf{V}, \tag{3.9}
$$

$$
\begin{aligned}
\langle A_U(t)v, u\rangle &= \langle (U(t,x)\cdot\nabla)v, u\rangle\\
&\qquad + \langle (v\cdot\nabla)U(t,x), u\rangle \quad \text{for all } v, u \in \mathbf{V},
\end{aligned}
\tag{3.10}
$$

where A_0 is the operator by (3.5) and k_0 is one in Lemma 3.1. Since $U \in \mathscr{W}$, we have $U \in C\big([0,T]; \mathbf{H}^1(\Omega)\big)$ and so such a definition is well. Then, the operator A is positive definite, and this fact is used in future.

Define an operator $B(t) : \mathbf{V} \to \mathbf{V}^*$ and $F(t) \in V^*$ by

$$
\langle B(t)v, u\rangle = e^{k_0 t}\langle (v\cdot\nabla)v, u\rangle \quad \text{for all } v, u \in \mathbf{V}, \tag{3.11}
$$

$$
\begin{aligned}
\langle F(t), u\rangle = e^{-k_0 t}\Big[&- (U'(t), u) - 2\nu(\varepsilon(U)(t), \varepsilon(u))\\
&- \langle (U(t)\cdot\nabla)U(t), u\rangle - 2\nu(k(x)U(t), u)_{\Gamma_2}\\
&- 2\nu(S\tilde{U}(t), \tilde{u})_{\Gamma_3} - 2(\alpha(x)U(t), u)_{\Gamma_5}\\
&- \nu(k(x)U(t), u)_{\Gamma_7} + \langle f(t), u\rangle\\
&+ \sum_{i=2,4,7}\langle \phi_i(t), u_n\rangle_{\Gamma_i} + \sum_{i=3,5,6}\langle \phi_i(t), u\rangle_{\Gamma_i}\Big] \quad \text{for all } u \in \mathbf{V}.
\end{aligned}
\tag{3.12}
$$

Then, (3.8) is written by

$$
\begin{aligned}
&z \in L_2(0,T;\mathbf{V}),\\
&z(0) = v_0 - U(0) \in \mathbf{V},\\
&z'(t) + (A + A_U(t) + B(t))\, z(t) = F(t).
\end{aligned}
\tag{3.13}
$$

Now, define operators $L, \widetilde{A}_U, L_U, \widetilde{B} : \mathscr{X} \to \mathscr{Y}$, $C : \mathscr{X} \times \mathscr{X} \to \mathscr{Y}$ and $F \in \mathscr{Y}$ by

$$
\begin{aligned}
&\langle (Lz)(t), u\rangle = \langle z'(t), u\rangle + \langle Az(t), u\rangle \quad \text{for all } z \in \mathscr{X}, \text{for all } u \in \mathbf{V},\\
&\langle (\widetilde{A}_U z)(t), u\rangle = \langle A_U(t) z(t), u\rangle \quad \text{for all } z \in \mathscr{X}, \text{for all } u \in \mathbf{V},\\
&\langle (L_U z)(t), u\rangle = \langle z'(t), u\rangle + \langle (A + A_U(t)) z(t), u\rangle\\
&\qquad\qquad\qquad\qquad \text{for all } z \in \mathscr{X}, \text{for all } u \in \mathbf{V},\\
&\langle (\widetilde{B} z)(t), u\rangle = \langle B(t) z(t), u\rangle \quad \text{for all } z \in \mathscr{X}, \text{for all } u \in \mathbf{V},\\
&\langle C(w,z)(t), u\rangle = e^{k_0 t}\langle (w(t)\cdot\nabla) z(t), u\rangle + e^{k_0 t}\langle (z(t)\cdot\nabla) w(t), u\rangle\\
&\qquad\qquad\qquad\qquad \text{for all } w, z \in \mathscr{X}, \text{for all } u \in \mathbf{V},\\
&(F)(t) = F(t).
\end{aligned}
\tag{3.14}
$$

LEMMA 3.2. *C is a bilinear continuous operator such that $\mathscr{X} \times \mathscr{X} \to \mathscr{Y}$. Under Assumptions 3.1 and 3.2, $\widetilde{A}_U$ is a linear continuous operator such that $\mathscr{X} \to \mathscr{Y}$ and $F \in \mathscr{Y}$.*

PROOF. Obviously, C is bilinear. When $w \in \mathscr{X}$,

$$
w \in L_\infty(0,T;\mathbf{V}), \quad \|w\|_{L_\infty(0,T;\mathbf{V})} \le c\big[\|w\|_{L_2(0,T;\mathbf{V})} + \|w'\|_{L_2(0,T;\mathbf{V})}\big]
$$

and by virtue of Hölder inequality and the imbedding theorem

$$
\begin{aligned}
&\big|e^{k_0 t}\langle (w\cdot\nabla) z, u\rangle + e^{k_0 t}\langle (z\cdot\nabla) w, u\rangle\big|\\
&\qquad \le c(\|w\|_{\mathbf{L}_3}\|\nabla z\|_{\mathbf{L}_2}\|u\|_{\mathbf{L}_6} + \|z\|_{\mathbf{L}_3}\|\nabla w\|_{\mathbf{L}_2}\|u\|_{\mathbf{L}_6}) \le c\|w\|_{\mathbf{V}}\|z\|_{\mathbf{V}}\|u\|_{\mathbf{V}}\\
&\qquad\qquad\qquad \text{for all } w, z, u \in \mathbf{V}.
\end{aligned}
$$

Thus,

$$
\|C(w,z)\|_{L_2(0,T;\mathbf{V}^*)} \le c\|w\|_{L_\infty(0,T;\mathbf{V})}\|z\|_{L_2(0,T;\mathbf{V})} \le c\|w\|_{\mathscr{X}} \cdot \|z\|_{\mathscr{X}}. \tag{3.15}
$$

Also, since

$$|\langle C(w,z)'(t),u\rangle| = e^{k_0 t}\big|k_0\langle (w\cdot\nabla)z,u\rangle + k_0\langle (z\cdot\nabla)w,u\rangle + \langle (w'\cdot\nabla)z,u\rangle + \langle (w\cdot\nabla)z',u\rangle + \langle (z'\cdot\nabla)w,u\rangle + \langle (z\cdot\nabla)w',u\rangle\big|,$$

taking (3.15) into account we have

$$\begin{aligned}\|C(w,z)'\|_{L_2(0,T;\mathbf{V}^*)} &\le c\|C(w,z)\|_{L_2(0,T;\mathbf{V}^*)} \\ &\quad + c\big[\big(\|w'\|_{L_2(0,T;\mathbf{V})} + \|w\|_{L_\infty(0,T;\mathbf{V})}\big) \\ &\qquad \times \big(\|z'\|_{L_2(0,T;\mathbf{V})} + \|z\|_{L_\infty(0,T;\mathbf{V})}\big)\big] \\ &\le c\|w\|_{\mathscr{X}}\cdot\|z\|_{\mathscr{X}}.\end{aligned} \tag{3.16}$$

(3.15) and (3.16) imply

$$\|C(w,z)\|_{\mathscr{Y}} \le c\|w\|_{\mathscr{X}}\cdot\|z\|_{\mathscr{X}}. \tag{3.17}$$

By the same argument above, we have

$$\|\widetilde{A}_U z\|_{\mathscr{Y}} \le c\|U\|_{\mathscr{W}}\cdot\|z\|_{\mathscr{X}}. \tag{3.18}$$

By Assumption 3.1, Remark 2.2 and the trace theorem, we can see that $F \in \mathscr{Y}$. □

Lemma 3.3. *The operator $\overline{L}$ defined by $\overline{L}z = (z'(0), Lz)$ for $z \in \mathscr{X}$ is a linear continuous one-to-one operator from $\mathscr{X}$ onto $H \times \mathscr{Y}$.*

Proof. The linearity of $\overline{L}$ is obvious. The fact $z \in \mathscr{X}$ implies that $z' \in C([0,T];H)$, $\|z'\|_{C([0,T];H)} \le c\|z\|_{\mathscr{X}}$, and so we see that a map $z \in \mathscr{X} \to z'(0) \in H$ is continuous.

Clearly, $\|z'\|_{\mathscr{Y}} \le c\|z\|_{\mathscr{X}}$. Also, by Assumption 3.1, Remark 2.2 and the trace theorem,

$$|\langle Av,u\rangle| \le c\|v\|_{\mathbf{V}}\cdot\|u\|_{\mathbf{V}} \quad \text{for all } v,u \in \mathbf{V}. \tag{3.19}$$

Formula (3.19) implies that the mapping $z \in \mathscr{X} \to Az \in \mathscr{Y}$ is continuous. Therefore, $\overline{L}$ is continuous.

Let us show that $\overline{L}$ is a one-to-one and surjective operator from $\mathscr{X}$ onto $H \times \mathscr{Y}$.

First, let us prove that this operator is injective. For this, it is enough to prove that the inverse image of $(0_H, 0_{\mathscr{Y}}) \in H \times \mathscr{Y}$ by the operator $\overline{L}$ is $0_{\mathscr{X}}$. By Lemma 3.1 and (3.9), we get

$$\langle Av, v \rangle \geq \delta \|v\|_{\mathbf{V}}^2 \quad \exists \delta > 0, \text{for all } v \in \mathbf{V}. \tag{3.20}$$

By (3.19), (3.20) for any $q \in \mathbf{V}^*$ there exists a unique solution $y \in V$ to the following problem

$$Ay = q. \tag{3.21}$$

Let $z \in \mathscr{X}$ be the inverse image of $(0_H, 0_{\mathscr{Y}}) \in H \times \mathscr{Y}$ by $\overline{L}$. Then since $z'(0) = 0_H$, putting $t = 0$ in the first equation of (3.14) we get

$$\langle Az(0), u \rangle = 0 \quad \text{for all } u \in \mathbf{V},$$

where $z(0) = z(0, x)$. This means that $z(0)$ is a unique solution to (3.21) for $q = 0_{\mathbf{V}^*}$, i.e. $z(0) = 0_{\mathbf{V}}$. Putting $w = z'$, we get $w(0) = z'(0) = 0_H$. Taking $Lz = 0$ into account and differentiating the first equation of (3.14), we have

$$\langle w'(t), u \rangle + \langle Aw(t), u \rangle = 0 \quad \text{for all } u \in \mathbf{V}. \tag{3.22}$$

The operator A in (3.22) satisfies all conditions of Theorem 1.1, ch. 6 in [12]. Thus, for problem (3.22) with an initial condition $w(0) \in H$ there exists a unique solution w such that $w \in L_2(0, T; \mathbf{V})$, $w' \in L_2(0, T; \mathbf{V}^*)$. Since $w(0) = 0_H$, we have $w = 0$, which means $z = 0_{\mathscr{X}}$ since $z(0) = 0_{\mathbf{V}}$.

Let us prove that $\overline{L}$ is surjective. Let $(w_0, g) \in H \times \mathscr{Y}$. Since $g \in \mathscr{Y}$, we have $g(0) \in \mathbf{V}^*$. Then, by (3.19) and (3.20), there exists a unique solution $z_0 \in \mathbf{V}$ to problem

$$Az_0 = g(0) - w_0. \tag{3.23}$$

Let us consider problem

$$\begin{cases} w' + Aw = g', \\ w(0) = w_0. \end{cases} \tag{3.24}$$

There exists a unique solution w such that $w \in L_2(0,T;\mathbf{V})$, $w' \in L_2(0,T;\mathbf{V}^*)$ to problem (3.24) (cf. Theorem 1.3 of ch. 6 in [12]). Now, put

$$z = z_0 + \int_0^t w(s)\,ds, \tag{3.25}$$

where z_0 is the solution to (3.23). Then, $z' = w$ and $z \in \mathscr{X}$. Integrating two sides of the first one in (3.24) from 0 to t and using (3.25), we have

$$\begin{aligned}&\langle w(t), u\rangle + \langle Az(t), u\rangle - [\langle w_0, u\rangle + \langle Az_0, u\rangle]\\ &\qquad = \langle g(t), u\rangle - \langle g(0), u\rangle \quad \text{for all } u \in \mathbf{V}.\end{aligned} \tag{3.26}$$

Taking (3.23), (3.25) into account, from (3.26) we get

$$\langle z'(t), u\rangle + \langle Az(t), u\rangle = \langle g(t), u\rangle \quad \text{for all } u \in \mathbf{V}. \tag{3.27}$$

This means that $z \in \mathscr{X}$ defined by (3.25) is the inverse image of $(w_0, g) \in H \times \mathscr{Y}$ by the operator $\overline{L}$, i.e. $\overline{L}$ is surjective. Therefore, $\overline{L}$ is an epimorphism. □

Lemma 3.4. *Under Assumption 3.2, let $\|U(0,x)\|_{\widetilde{\mathbf{V}}}$ be small enough. The operator $\overline{L}_U$ defined by $\overline{L}_U z = (z'(0), L_U z)$ for $z \in \mathscr{X}$ is a linear continuous one-to-one operator from $\mathscr{X}$ onto $H \times \mathscr{Y}$.*

Proof. When $z \in \mathscr{X}$, $z \in C([0,T];\mathbf{V})$ and

$$\|z\|_{C([0,T];\mathbf{V})} \le c\big[\|z\|_{L_2(0,T;\mathbf{V})} + \|z'\|_{L_2(0,T;\mathbf{V})}\big].$$

By virtue of this fact and Lemma 3.2, the operator $\overline{A}_U \in (\mathscr{X} \to H \times \mathscr{Y})$ defined by $\overline{A}_U z = (0_H, \widetilde{A}_U z)$ is continuous. Thus, the operator $\overline{L}_U$ defined on $\mathscr{X}$ is linear continuous.

As in Lemma 3.5 of [14] it is proved that the operator $\widetilde{A}_U \in (\mathscr{X} \to \mathscr{Y})$ is compact. Thus, $\overline{A}_U \in (\mathscr{X} \to H \times \mathscr{Y})$ is also compact. Since $\overline{L}_U = \overline{L} + \overline{A}_U$, by virtue of Theorem 3.4 in [14] and Lemma 3.3 we know that in order to prove that the operator $\overline{L}_U$ is one-to-one from $\mathscr{X}$ onto $H \times \mathscr{Y}$ it is enough to prove that $\overline{L}_U$ is one-to-one from $\mathscr{X}$ into $H \times \mathscr{Y}$. To prove the last fact it is enough to show that the inverse image of $(0_H, 0_{\mathscr{Y}})$ by $\overline{L}_U$ is $0_{\mathscr{X}}$. By Hölder inequality and imbedding theorem

$$\begin{aligned}&\big|\big\langle (U(t,x)\cdot\nabla)v, v\big\rangle + \big\langle (v\cdot\nabla)U(t,x), v\big\rangle\big|\\ &\qquad \le K_0\|v\|_{\mathbf{V}}\|U(t,x)\|_{\mathbf{H}^1}\|v\|_{\mathbf{V}}.\end{aligned} \tag{3.28}$$

Thus, if $\|U(0,x)\|_{\widetilde{V}}$ is so small that $\|U(0,x)\|_{\mathbf{H}^1} \leq \frac{\delta}{2K_0}$, then (3.19), (3.20) and (3.28) imply

$$\begin{aligned} &\big|\big\langle \big(A + A_U(0)\big)v, u\big\rangle\big| \leq c\|v\|_{\mathbf{V}} \cdot \|u\|_{\mathbf{V}}, \\ &\big\langle \big(A + A_U(0)\big)v, v\big\rangle \geq \frac{\delta}{2}\|v\|^2_{\mathbf{V}} \quad \text{for all } v, u \in \mathbf{V}. \end{aligned} \tag{3.29}$$

By (3.29) for any $q \in \mathbf{V}^*$ there exists a unique solution $y \in V$ to

$$(A + A_U(0))y = q. \tag{3.30}$$

Let $z \in \mathscr{X}$ be the inverse image of $(0_H, 0_{\mathscr{Y}})$ by $\overline{L}$. Then $z'(0) = 0_H$, and putting $t = 0$ from the third one in (3.14) we get

$$\big\langle \big(A + A_U(0)\big)z(0), u\big\rangle = 0 \quad \text{for all } u \in \mathbf{V},$$

where $z(0) = z(0,x)$. This means that $z(0)$ is the unique solution to (3.30) with $q = 0_{\mathbf{V}^*}$, i.e. $z(0) = 0_{\mathbf{V}}$. Therefore, $z \in \mathscr{X}$ satisfies

$$\begin{cases} z'(t) + \big(A + A_U(t)\big)z(t) = 0, \\ z(0) = 0_{\mathbf{V}}. \end{cases} \tag{3.31}$$

Now, making duality pairing with $z(t)$ on two sides of

$$z'(t) + Az(t) = -A_U(t)z(t)$$

and taking (3.20) into account and using Gronwall's inequality, we can prove $z = 0_{\mathscr{X}}$ as in Lemma 3.8 of [14]. It is finished to prove the Lemma. □

LEMMA 3.5. *Under Assumption 3.2 the operator T defined by $Tz = \big(z'(0), (L_U + \widetilde{B})z\big)$ for $z \in \mathscr{X}$ is continuously differentiable, $T(0_{\mathscr{X}}) = (0_H, 0_{\mathscr{Y}})$ and the Frechet derivative of T at $0_{\mathscr{X}}$ is $\overline{L}_U$.*

PROOF. It is easy to verify that $T(0_{\mathscr{X}}) = (0_H, 0_{\mathscr{Y}})$. Since the operator L_U is linear, its Frechet derivative is the same with itself. Therefore, if $\widetilde{B}$ is continuously differentiable, then so is T.

For any $w, z \in \mathscr{X}$,

$$\big(\widetilde{B}(w + z) - \widetilde{B}w\big)(t) = e^{k_0 t}\big(w(t) \cdot \nabla\big)z(t) + e^{k_0 t}\big(z(t) \cdot \nabla\big)w(t) + (\widetilde{B}z)(t).$$

By (3.17), we get

$$\lim_{\|z\|_{\mathscr{X}}\to 0}\frac{\|\widetilde{B}z\|_{\mathscr{Y}}}{\|z\|_{\mathscr{X}}}\le \lim_{\|z\|_{\mathscr{X}}\to 0}\frac{c\|z\|^2_{\mathscr{X}}}{\|z\|_{\mathscr{X}}}=0.$$

Then, put

$$C(w,z)(t)\equiv e^{k_0t}\big(w(t)\cdot\nabla\big)z(t)+e^{k_0t}\big(z(t)\cdot\nabla\big)w(t)=(\widetilde{B}'_w z)(t).$$

By Lemma 3.2 $\widetilde{B}'_w\in(\mathscr{X}\to\mathscr{Y})$ is continuous, and it is the Frechet derivative of $\widetilde{B}$ at w and also continuous with respect to w. Thus, T is continuously differentiable. Also from the formula above we can see that the Frechet derivative of $\widetilde{B}$ at $0_{\mathscr{X}}$ is zero. Therefore, the Frechet derivative of T at $0_{\mathscr{X}}$ is $\overline{L}_U$. □

Let us consider problem

$$\big(A+A_U(0)+B(0)\big)u=q. \tag{3.32}$$

Lemma 3.6. *Assume that $\|U(0,x)\|_{\tilde{\mathbf{V}}}$ is small enough. If the norm of $q\in V^*$ is small enough, then there exists a unique solution to* (3.32) *in some $\mathcal{O}_M(0_{\mathbf{V}})$.*

Proof. Since $\|U(0,x)\|_{\tilde{\mathbf{V}}}$ is small enough, by (3.29), for any fixed $z\in\mathbf{V}$ there exists a unique solution to problem

$$\big(A+A_U(0)\big)w=q-B(0)z. \tag{3.33}$$

On the other hand,

$$|\langle B(0)w_1-B(0)w_2,u\rangle|\le KM\|w_1-w_2\|_{\mathbf{V}}\cdot\|u\|_{\mathbf{V}} \quad \text{for all } w_i\in\mathcal{O}_M(0_{\mathbf{V}}),\ \text{ for all } u\in\mathbf{V}. \tag{3.34}$$

Owing to (3.29) the solution w to (3.33) is estimated as follows

$$\|w\|_{\mathbf{V}}\le\frac{2}{\delta}\big(\|q\|_{\mathbf{V}^*}+\|B(0)z\|_{\mathbf{V}^*}\big)\le\frac{2}{\delta}\big(\|q\|_{\mathbf{V}^*}+KM^2\big).$$

Thus, if $\|q\|_{\mathbf{V}^*}$ and M are small enough, then the operator $(z\to w)$ maps $\mathcal{O}_M(0_{\mathbf{V}})$ into itself and by (3.34) this operator is strictly contract. Therefore, in $\mathcal{O}_M(0_{\mathbf{V}})$ there exists a unique solution to (3.33). Thus, we come to the asserted conclusion. □

For proof of unique existence of a solution to Problem I, we use the following

PROPOSITION 3.7 (cf. Theorem 4.1.1 in [10]). *Let X, Y be Banach spaces, $\mathcal{G}$ an open set in X, $f : X \to Y$ continuously differentiable on $\mathcal{G}$. Let the derivative $f'(a)$ be an isomorphism of X onto Y for $a \in \mathcal{G}$. Then there exist neighborhoods $\mathcal{U}$ of a, $\mathcal{V}$ of $f(a)$such that f is injective on $\mathcal{U}$, $f(\mathcal{U}) = \mathcal{V}$.*

One of main results of this paper is the following

THEOREM 3.8. *Suppose that Assumptions 3.1 and 3.2 hold. Assume that $\|U\|_{\mathcal{W}}$ and the norms of f, f', ϕ_i, ϕ_i' in the spaces where they belong to are small enough.*

If

$$w_0 \equiv F(0) - (A + A_U(0) + B(0))z_0 \in H, \tag{3.35}$$

where $z_0 = v_0 - U(0, \cdot)$, and $\|w_0\|_H$ is small enough, then there exists a unique solution to (3.3) *in the space $\mathcal{W}$.*

PROOF. First, let us prove existence of a solution.
If $\|U\|_{\mathcal{W}}$ and the norms of f, f', ϕ_i, ϕ_i' in the spaces they belong to are small enough, then $\|F\|_{\mathcal{Y}}$ is also small enough. By virtue of Lemmas 3.4, 3.5 and Proposition 3.7, for any $R_1 > 0$ small enough if $\|F\|_{\mathcal{Y}}$, R are small enough and $w_1 \in \mathcal{O}_R(0_H)$, there exists a unique $z \in \mathcal{O}_{R_1}(0_{\mathcal{X}})$ such that

$$z'(t) + \big(A + A_U(t) + B(t)\big)z(t) = F(t), \quad z'(0) = w_1 \in \mathcal{O}_R(0_H). \tag{3.36}$$

Putting $t = 0$ in (3.36), we get

$$F(0) - \big(A + A_U(0) + B(0)\big)z(0) = w_1 \in \mathcal{O}_R(0_H).$$

On the other hand, if $\|U\|_{\mathcal{W}}$ is small enough, then so is $\|U(0, x)\|_{\tilde{\mathbf{V}}}$. Thus, when $\|F(0) - w_1\|_{V^*}$ is small enough, by Lemma 3.6 there exists a unique solution $z_0 \in \mathcal{O}_{R_2}(0_V)$ for some $R_2 > 0$ to

$$\big(A + A_U(0) + B(0)\big)z_0 = F(0) - w_1. \tag{3.37}$$

Since $\|z(0)\|_V \leq c\|z\|_{\mathscr{X}}$, we can choose R_1 such that $z(0) \in \mathcal{O}_{R_2}(0_V)$, and we have $z(0) = z_0$. Therefore, if $\|F\|_{\mathscr{Y}}$ is small enough, $F(0) - \big(A + A_U(0) + B(0)\big)z_0$ belongs to H and its norm is small enough, then $z \in \mathscr{X}$, the solution to (3.36), is a solution to problem

$$
\begin{cases} z'(t) + \big(A + A_U(t) + B(t)\big)z(t) = F(t), \\ z(0) = z_0. \end{cases} \tag{3.38}
$$

By definitions of $A, A_U(t), B(t), F$, the solution z of (3.38) is also a solution to (3.8) which is equivalent to (3.3). Thus, $e^{k_0 t}z + U \in \mathscr{W}$ is a solution to (3.3).

Second, let us prove uniqueness.

Let v_1, v_2 be two solutions to (3.3) corresponding to the same data. Putting $\overline{w} = v_1 - v_2$, we have

$$
\begin{aligned}
&\overline{w} \in L_2(0,T;\mathbf{V}),\\
&\overline{w}(0) = 0,\\
&\langle \overline{w}', u\rangle + 2\nu(\varepsilon(\overline{w}), \varepsilon(u)) + \langle (v_1 \cdot \nabla)\overline{w}, u\rangle + \langle (\overline{w} \cdot \nabla)v_2, u\rangle\\
&\qquad + 2\nu(k(x)\overline{w}, u)_{\Gamma_2} + 2\nu(S\tilde{\overline{w}}, \tilde{u})_{\Gamma_3} + 2(\alpha(x)\overline{w}, u)_{\Gamma_5}\\
&\qquad + \nu(k(x)\overline{w}, u)_{\Gamma_7} = 0 \quad \text{for all } u \in \mathbf{V}.
\end{aligned} \tag{3.39}
$$

Putting $w = e^{-k_0 t}\overline{w}$, where k_0 is the constant in Lemma 3.1, we get $e^{-k_0 t}\overline{w}' = w' + k_0 w$. Then, we have

$$
\begin{aligned}
&w \in L_2(0,T;\mathbf{V}),\\
&w(0) = 0,\\
&\langle w', u\rangle + 2\nu(\varepsilon(w), \varepsilon(u)) + \langle (v_1 \cdot \nabla)w, u\rangle + \langle (w \cdot \nabla)v_2, u\rangle + k_0(w,u)\\
&\qquad + 2\nu(k(x)w, u)_{\Gamma_2} + 2\nu(S\tilde{w}, \tilde{u})_{\Gamma_3} + 2(\alpha(x)w, u)_{\Gamma_5}\\
&\qquad + \nu(k(x)w, u)_{\Gamma_7} = 0 \quad \text{for all } u \in \mathbf{V},
\end{aligned} \tag{3.40}
$$

which is equivalent to (3.39). By Lemma 3.1,

$$
\begin{aligned}
2\nu(\varepsilon(w), \varepsilon(w)) + k_0(w,w) + 2\nu(k(x)w, w)_{\Gamma_2} + 2\nu(S\tilde{w}, \tilde{w})_{\Gamma_3}&\\
+ 2(\alpha(x)w, w)_{\Gamma_5} + \nu(k(x)w, w)_{\Gamma_7} \geq \delta\|w\|^2_{\mathbf{V}}&.
\end{aligned}
$$

Taking it into account, we can prove $w = 0_{\mathscr{X}}$ as in Lemma 3.8 of [14]. Thus, uniqueness of a solution is proved, and we finished proof of the theorem. □

REMARK 3.3. Let us consider more precisely the condition that $F(0)-\big(A+A_U(0)+B(0)\big)z_0$ belongs to H and its norm is small enough. By (3.9)∼(3.14) we have

$$
\begin{aligned}
&\big\langle F(0)-\big(A+A_U(0)+B(0)\big)z_0,u\big\rangle =\\
&\Big[-(U'(0,x),u)-2\nu(\varepsilon(U(0,x)),\varepsilon(u))-\langle (U(0,x)\cdot\nabla)U(0,x),u\rangle\\
&\quad -2\nu(k(x)U(0,x),u)_{\Gamma_2}-2\nu(S\tilde{U}(0),\tilde{u})_{\Gamma_3}-2(\alpha(x)U(0,x),u)_{\Gamma_5}\\
&\quad -\nu(k(x)U(0,x),u)_{\Gamma_7}+\langle f(0),u\rangle\\
&\quad +\sum_{i=2,4,7}\langle\phi_i(t),u_n\rangle_{\Gamma_i}+\sum_{i=3,5,6}\langle\phi_i(t),u\rangle_{\Gamma_i}\Big]\\
&-\big[2\nu(\varepsilon(z_0),\varepsilon(u))+2\nu(k(x)z_0,u)_{\Gamma_2}+2\nu(S\tilde{z}_0,\tilde{u})_{\Gamma_3}\\
&\qquad\qquad +2(\alpha(x)z_0,u)_{\Gamma_5}+\nu(k(x)z_0,u)_{\Gamma_7}\big]\\
&-\big[\langle (U(0,x)\cdot\nabla)z_0,u\rangle+\langle (z_0\cdot\nabla)U(0,x),u\rangle\big]\\
&-\langle (z_0,\nabla)z_0,u\rangle\quad \text{for all } u\in\ \mathbf{V}.
\end{aligned}
\tag{3.41}
$$

Taking into account the fact that $U(0,x)+z_0=v_0$, $U'(0,x)\in\mathbf{L}_2(\Omega)$ and its norm is small enough, from (3.41) we can see that the condition mentioned above is equivalent to the condition $\overline{w}_0\in\mathbb{O}_R(0_H)$ for $R>0$ small enough, where $\overline{w}_0$ is defined by

$$
\begin{aligned}
\langle\overline{w}_0,u\rangle\equiv{}&\langle f(0),u\rangle+\sum_{i=2,4,7}\langle\phi_i(0,x),u_n\rangle_{\Gamma_i}+\sum_{i=3,5,6}\langle\phi_i(0,x),u\rangle_{\Gamma_i}\\
&-\big[2\nu(\varepsilon(v_0),\varepsilon(u))+2\nu(k(x)v_0,u)_{\Gamma_2}+2\nu(S\tilde{v}_0,\tilde{u})_{\Gamma_3}\\
&+2(\alpha(x)v_0,u)_{\Gamma_5}+\nu(k(x)v_0,u)_{\Gamma_7}+\langle (v_0\cdot\nabla)v_0,u\rangle\\
&+k_0(v_0,u)\big]\quad\text{for all } u\in\mathbf{V}.
\end{aligned}
\tag{3.42}
$$

REMARK 3.4. If $\Gamma_i=\varnothing, i=2\sim 5,7$, then the problem is reduced to one in [15] where a local-in-time solution was studied. In this case $k_0=0$ (cf. Remark 3.2), and the condition (3.35) is the same with (25) in [15]. And our condition for U is also the same with one in [15].

4. Existence of a Unique Solution to Problem II

Let $\mathbf{V}_1=\{u\in\mathbf{H}^1(\Omega):\operatorname{div}u=0,\ u|_{\Gamma_1}=0,\ u_\tau|_{(\Gamma_2\cup\Gamma_4)}=0,\ u_n|_{(\Gamma_3\cup\Gamma_5)}=0\}$ and $\mathbf{V}_{\Gamma2-5}(\Omega)=\{u\in\mathbf{H}^1(\Omega):\operatorname{div}u=0,\ u_\tau|_{(\Gamma_2\cup\Gamma_4)}=0,\ u_n|_{(\Gamma_3\cup\Gamma_5)}=0\}$. Denote by H_1 the completion of $\mathbf{V}_1$ in the space $\mathbf{L}_2(\Omega)$.

By Theorems 2.1 and 2.2, for $v \in \mathbf{H}^2(\Omega) \cap \mathbf{V}_{\Gamma 2-5}(\Omega)$ and $u \in \mathbf{V}_1$ we have that

$$
\begin{aligned}
-(\Delta v, u) &= (\nabla v, \nabla u) - \left(\frac{\partial v}{\partial n}, u\right)_{\partial\Omega} \\
&= (\nabla v, \nabla u) + (k(x)v, u)_{\Gamma_2} - (\operatorname{rot} v \times n, u)_{\Gamma_3} + (S\tilde{v}, \tilde{u})_{\Gamma_3} \\
&\quad - (\varepsilon_n(v), u)_{\Gamma_4} - 2(\varepsilon_n(v), u)_{\Gamma_5} - (S\tilde{v}, \tilde{u})_{\Gamma_5} \\
&\quad - \left(\frac{\partial v}{\partial n}, u\right)_{\Gamma_7} \\
&= (\nabla v, \nabla u) + (k(x)v, u)_{\Gamma_2} - (\operatorname{rot} v \times n, u)_{\Gamma_3} + (S\tilde{v}, \tilde{u})_{\Gamma_3} \\
&\quad - (\varepsilon_{nn}(v), u \cdot n)_{\Gamma_4} - 2(\varepsilon_{n\tau}(v), u)_{\Gamma_5} - (S\tilde{v}, \tilde{u})|_{\Gamma_5} \\
&\quad - \left(\frac{\partial v}{\partial n}, u\right)_{\Gamma_7}.
\end{aligned}
\tag{4.1}
$$

Also, for $p \in H^1(\Omega)$ and $u \in \mathbf{V}_1$ we get

$$
(\nabla p, u) = (p, u \cdot n)_{\cup_{i=2}^{7}\Gamma_i} = (p, u \cdot n)_{\Gamma_2} + (p, u \cdot n)_{\Gamma_4} + (pn, u)_{\Gamma_7}, \tag{4.2}
$$

where the fact that $u \cdot n|_{\Gamma_3 \cup \Gamma_5} = 0$ was used.

Let

$$
\begin{aligned}
\mathscr{X}_1 &= \{w \in L_2(0, T; \mathbf{V}_1); w' \in L_2(0, T; \mathbf{V}_1), w'' \in L_2(0, T; \mathbf{V}_1^*)\}, \\
&\qquad \|w\|_{\mathscr{X}_1} = \|w\|_{L_2(0,T;\mathbf{V}_1)} + \|w'\|_{L_2(0,T;\mathbf{V}_1)} + \|w''\|_{L_2(0,T;\mathbf{V}_1^*)}, \\
\mathscr{Y}_1 &= \{w \in L_2(0, T; \mathbf{V}_1^*); w' \in L_2(0, T; \mathbf{V}_1^*)\}, \\
&\qquad \|w\|_{\mathscr{Y}_1} = \|w\|_{L_2(0,T;\mathbf{V}_1^*)} + \|w'\|_{L_2(0,T;\mathbf{V}_1^*)}.
\end{aligned}
$$

Unlike problem I, for problem II we do not require the condition $v_\tau|_{\Gamma_7} = 0$, and so instead of Assumptions 3.1 and 3.2, we use the following assumptions.

ASSUMPTION 4.1. *Assumption 3.1 holds with* $\phi_7, \phi_7' \in L_2(0, T; \boldsymbol{H}^{-\frac{1}{2}}(\Gamma_7))$ *instead of* $\phi_7, \phi_7' \in L_2(0, T; H^{-\frac{1}{2}}(\Gamma_7))$

ASSUMPTION 4.2. *There exists a function* $U \in \mathscr{W}$ *such that*

$$
\mathit{div} U = 0,\ U|_{\Gamma_1} = h_1,\ U_\tau|_{\Gamma_2} = 0,\ U_n|_{\Gamma_3} = 0,\ U_\tau|_{\Gamma_4} = h_4,\ U_n|_{\Gamma_5} = h_5,
$$

where $\mathscr{W}$ *is the same as in the previous section. Also,* $U(0, x) - v_0 \in \boldsymbol{V}_1$.

Taking into account (4.1), (4.2), we get the following variational formulation for Problem II:

FORMULATION 4.1. *Find v such that*

$$
\begin{aligned}
&v - U \in L_2(0,T;\mathbf{V}_1),\\
&v(0) = v_0,\\
&\langle v', u\rangle + \nu(\nabla v, \nabla u) + \langle (v\cdot\nabla)v, u\rangle + \nu(k(x)v, u)_{\Gamma_2}\\
&\qquad + \nu(S\tilde{v}, \tilde{u})_{\Gamma_3} + 2(\alpha(x)v, u)_{\Gamma_5} - \nu(S\tilde{v}, \tilde{u})_{\Gamma_5}\\
&\quad = \langle f, u\rangle + \sum_{i=2,4}\langle \phi_i, u_n\rangle_{\Gamma_i} + \sum_{i=3,5,7}\langle \phi_i, u\rangle_{\Gamma_i} \quad \textit{for all } u \in \mathbf{V}_1.
\end{aligned} \tag{4.3}
$$

Taking into account Assumption 4.2 and putting $v = \overline{z} + U$, we get the following problem which is equivalent to Formulation 4.1:

Find $\overline{z}$ such that

$$
\begin{aligned}
&\overline{z} \in L_2(0,T;\mathbf{V}_1),\\
&\overline{z}(0) \equiv v_0 - U(0) \in \mathbf{V}_1,\\
&\langle \overline{z}', u\rangle + \nu(\nabla\overline{z}, \nabla u) + \langle(\overline{z}\cdot\nabla)\overline{z}, u\rangle + \langle (U\cdot\nabla)\overline{z}, u\rangle + \langle(\overline{z}\cdot\nabla)U, u\rangle\\
&\qquad + \nu(k(x)\overline{z}, u)_{\Gamma_2} + \nu(S\tilde{\overline{z}}, \tilde{u})_{\Gamma_3} + 2(\alpha(x)\overline{z}, u)_{\Gamma_5} - \nu(S\tilde{\overline{z}}, \tilde{u})_{\Gamma_5}\\
&\quad = -\langle U', u\rangle - \nu(\nabla U, \nabla u) - \langle (U\cdot\nabla)U, u\rangle - \nu(k(x)U, u)_{\Gamma_2}\\
&\qquad - \nu(S\tilde{U}, \tilde{u})_{\Gamma_3} - 2(\alpha(x)U, u)_{\Gamma_5} + \nu(S\tilde{U}, \tilde{u})_{\Gamma_5} + \langle f, u\rangle\\
&\qquad + \sum_{i=2,4}\langle\phi_i, u_n\rangle_{\Gamma_i} + \sum_{i=3,5,7}\langle \phi_i, u\rangle_{\Gamma_i} \quad \text{for all } u \in \mathbf{V}_1.
\end{aligned} \tag{4.4}
$$

Define an operator $A_{01} : \mathbf{V}_1 \to \mathbf{V}_1^*$ by

$$
\begin{aligned}
\langle A_{01}y, u\rangle = \nu(\nabla y, \nabla u) + \nu(k(x)y, u)_{\Gamma_2} + \nu(S\tilde{y}, \tilde{u})_{\Gamma_3}&\\
+ 2(\alpha(x)y, u)_{\Gamma_5} - \nu(S\tilde{y}, \tilde{u})_{\Gamma_5} \quad \text{for all } y, u \in \mathbf{V}_1&.
\end{aligned} \tag{4.5}
$$

By virtue of the same argument used to prove Lemma 3.1 we get

LEMMA 4.1. $\exists\delta > 0$, $\exists k_1 \geq 0$*:*

$$\langle A_{01}u, u\rangle \geq \delta\|u\|^2_{\boldsymbol{V}_1} - k_1\|u\|^2_{H_1} \quad \textit{for all } u \in \mathbf{V}_1.$$

Putting $z = e^{-k_1 t}\overline{z}$, where k_1 is the constant in Lemma 4.1, and using the fact that $e^{-k_1 t}\overline{z}' = z' + k_1 z$, we get the following problem which is equivalent to (4.4):

Find z such that

$$
\begin{aligned}
&z \in L_2(0,T;\mathbf{V}_1),\\
&z(0) = z_0 \equiv v_0 - U(0) \in \mathbf{V}_1,\\
&\langle z'(t), u\rangle + \nu(\nabla z, \nabla u) + e^{k_1 t}\langle (z(t)\cdot\nabla)z(t), u\rangle + \langle (U(t)\cdot\nabla)z(t), u\rangle\\
&\qquad + \langle (z(t)\cdot\nabla)U(t), u\rangle + k_1(z(t), u) + \nu(k(x)z(t), u)_{\Gamma_2}\\
&\qquad + \nu(S\tilde{z}(t), \tilde{u})_{\Gamma_3} + 2(\alpha(x)z(t), u)_{\Gamma_5} - \nu(S\tilde{z}(t), \tilde{u})_{\Gamma_5}\\
&= e^{-k_1 t}\Big[- (U'(t), u) - \nu(\nabla U, \nabla u) - \langle (U(t)\cdot\nabla)U(t), u\rangle\\
&\qquad - \nu(k(x)U(t), u)_{\Gamma_2} - \nu(S\tilde{U}(t), \tilde{u})_{\Gamma_3} - 2(\alpha(x)U(t), u)_{\Gamma_5}\\
&\qquad - \nu(S\tilde{U}(t), \tilde{u})_{\Gamma_5} + \langle f(t), u\rangle + \sum_{i=2,4}\langle \phi_i(t), u_n\rangle_{\Gamma_i}\\
&\qquad + \sum_{i=3,5,7}\langle \phi_i(t), u\rangle_{\Gamma_i}\Big] \quad \text{for all } u \in \mathbf{V}_1.
\end{aligned}
\tag{4.6}
$$

Define operators A_1, $A_{1U}(t)$ by

$$
\langle A_1 v, u\rangle = \langle A_{01} v, u\rangle + (k_1 v, u) \quad \text{for all } v, u \in \mathbf{V}_1,
\tag{4.7}
$$

$$
\begin{aligned}
\langle A_{1U}(t)v, u\rangle = \langle (U(t,x)\cdot\nabla)v, u\rangle&\\
+ \langle (v\cdot\nabla)U(t,x), u\rangle& \quad \text{for all } v, u \in \mathbf{V}_1,
\end{aligned}
\tag{4.8}
$$

where A_{01} is one defined in (4.5). $U \in \mathcal{W}$ implies $U \in C\big([0,T];\mathbf{H}^1(\Omega)\big)$, and such definitions have meaning. Also, define an operator $B_1(t) : V_1 \to V_1^*$ by

$$
\langle B_1(t)v, u\rangle = e^{k_1 t}\langle (v\cdot\nabla)v, u\rangle \quad \text{for all } v, u \in \mathbf{V}_1.
\tag{4.9}
$$

Define an element $F_1 \in \mathcal{Y}_1$ by

$$
\begin{aligned}
\langle F_1(t), u\rangle = e^{-k_1 t}\Big[&- \langle U'(t), u\rangle - \nu(\nabla U(t), \nabla u)\\
&- \langle (U(t)\cdot\nabla)U(t), u\rangle - \nu(k(x)U(t), u)_{\Gamma_2} - \nu(S\tilde{U}(t), \tilde{u})_{\Gamma_3}\\
&- 2(\alpha(x)U(t), u)_{\Gamma_5} + \nu(S\tilde{U}(t), \tilde{u})_{\Gamma_5} + \langle f, u\rangle\\
&+ \sum_{i=2,4}\langle \phi_i, u_n\rangle_{\Gamma_i} + \sum_{i=3,5,7}\langle \phi_i, u\rangle_{\Gamma_i}\Big] \quad \text{for all } u \in \mathbf{V}_1.
\end{aligned}
\tag{4.10}
$$

Now, in the same way as Theorem 3.8 we can prove the following theorem which is one of the main results of this paper.

THEOREM 4.2. *Suppose that Assumptions 4.1 and 4.2 hold. Assume that* $\|U\|_{\mathcal{W}}$ *and the norms of* f, f', ϕ_i, ϕ_i' *in the spaces they belong to are small enough.*

If

$$w_1 \equiv F_1(0) - (A_1 + A_{1U}(0) + B_1(0))z_0 \in H_1, \tag{4.11}$$

where $z_0 = v_0 - U(0,\cdot)$, *and* $\|w_1\|_{H_1}$ *is small enough, then in the space* $\mathcal{W}$ *there exists a unique solution to* (4.3).

REMARK 4.1. By the same argument as Remark 3.3, we can see that the condition (4.11) is equivalent to the condition $\overline{w}_0 \in H_1$, where $\overline{w}_0 \in \mathbf{V}_1^*$ is defined by

$$\begin{aligned} \langle \overline{w}_1, u\rangle = {} & \langle f(0), u\rangle + \sum_{i=2,4} \langle \phi_i(0,x), u_n\rangle_{\Gamma_i} + \sum_{i=3,5,7} \langle \phi_i(0,x), u\rangle_{\Gamma_i} \\ & - \big[\nu(\nabla v_0, \nabla u) + \nu(k(x)v_0, u)_{\Gamma_2} + \nu(S\tilde{v}_0, \tilde{u})_{\Gamma_3} \\ & + 2(\alpha(x)v_0, u)_{\Gamma_5} - \nu(S\tilde{v}_0, \tilde{u})_{\Gamma_5} + \langle (v_0 \cdot \nabla)v_0, u\rangle + k_1(v_0, u)\big] \\ & \qquad \text{for all } u \in \mathbf{V}_1, \end{aligned} \tag{4.12}$$

with k_1 in Lemma 4.1.

REMARK 4.2. If $U \equiv 0$ and $\Gamma_i = \varnothing, i = 2 \sim 5$, then problem (4.3) is reduced to one in [14]. In this case $k_1 = 0$. (cf. Remark 3.2). If $v_0 \in \mathbf{H}^{l/2}(\Omega)$, then $(v_0 \cdot \nabla)v_0 \in \mathbf{L}_2(\Omega)$. Thus, the condition above for $\overline{w}_1$ being in H_1 is the same with one of conditions of Theorems 3.5~3.8 in [14], but we do not demand $v_0 \in \mathbf{H}^{r_0}(\Omega)$, $r_0 > \frac{l}{2}$.

5. Existence of a Unique Solution for Perturbed Data

In [14] it is proved that if a solution satisfying smoothness and a compatibility condition is given, then there exists a unique solution for small perturbed data satisfying the compatibility condition. In this section we get such results for the Problems I and II. In our results the conditions for a given solution is essentially the same with one in [14], but the smoothness condition for the initial functions in the compatibility condition for small perturbed data is weaker than one in [14](cf. Remark 5.2).

Let $\widetilde{\mathbf{V}}^{r_0} = \{u \in \mathbf{H}^{r_0}(\Omega) : \operatorname{div} u = 0\}$, $r_0 > l/2$, and

$$\overline{\mathcal{W}} = \left\{ w \in L_2(0,T;\widetilde{\mathbf{V}}); w' \in L_2(0,T;\widetilde{\mathbf{V}}), w'' \in L_2(0,T;\widetilde{\mathbf{V}}^*), w(0) \in \widetilde{\mathbf{V}}^{r_0} \right\},$$
$$\|w\|_{\mathcal{W}} = \|w\|_{L_2(0,T;\tilde{\mathbf{V}})} + \|w'\|_{L_2(0,T;\tilde{\mathbf{V}})} + \|w''\|_{L_2(0,T;\tilde{\mathbf{V}}^*)} + \|w(0)\|_{\tilde{\mathbf{V}}^{r_0}}.$$

Let us consider Problem I.

Let $W(x,t) \in \overline{\mathcal{W}}$ be a given solution to Problem I. Let v be the solution for the data perturbed except h_i and put $v = \overline{z} + W$. Then, we get a problem for $\overline{z}$:

Find $\overline{z}$ such that

$$\begin{aligned}
&\overline{z} \in L_2(0,T;\mathbf{V}),\\
&\overline{z}(0) = z_0 \equiv v_0 - W(0,x) \in \mathbf{V},\\
&\langle \overline{z}', u\rangle + 2\nu(\varepsilon(\overline{z}), \varepsilon(u)) + \langle (\overline{z}\cdot\nabla)\overline{z}, u\rangle + \langle (W\cdot\nabla)\overline{z}, u\rangle\\
&\qquad + \langle (\overline{z}\cdot\nabla)W, u\rangle + 2\nu(k(x)\overline{z}, u)_{\Gamma_2} + 2\nu(S\tilde{\overline{z}}, \tilde{u})_{\Gamma_3}\\
&\qquad + 2(\alpha(x)\overline{z}, u)_{\Gamma_5} + \nu(k(x)\overline{z}, u)_{\Gamma_7}\\
&\quad = \langle f, u\rangle + \sum_{i=2,4,7} \langle \phi_i, u_n\rangle_{\Gamma_i} + \sum_{i=3,5,6} \langle \phi_i, u\rangle_{\Gamma_i} \quad \text{for all } u \in \mathbf{V},
\end{aligned} \tag{5.1}$$

where z_0, f, ϕ_i are perturbations of corresponding data.

Remark 5.1. Proofs of this section are similar to one in Section 3. Main difference is that we do not assume smallness of $W(0,x)$ unlike $U(0,x)$ in Section 3.

Define an operator $A_{02} : \mathbf{V} \to \mathbf{V}^*$ by

$$\begin{aligned}
\langle A_{02}y, u\rangle =& 2\nu(\varepsilon(y), \varepsilon(u)) + 2\nu(k(x)y, u)_{\Gamma_2} + 2\nu(S\tilde{y}, \tilde{u})_{\Gamma_3}\\
&+ 2(\alpha(x)y, u)_{\Gamma_5} + \nu(k(x)y, u)_{\Gamma_7} + \langle (W(0,x)\cdot\nabla)y, u\rangle\\
&+ \langle (y\cdot\nabla)W(0,x), u\rangle \qquad \text{for all } y, u \in \mathbf{V}.
\end{aligned} \tag{5.2}$$

Lemma 5.1. *There exists $\delta > 0$ and $k_2 \geq 0$ such that*

$$\langle A_{02}u, u\rangle \geq \delta\|u\|^2_{\mathbf{V}} - k_2\|u\|^2_H \quad \textit{for all } u \in \mathbf{V}.$$

Proof. By Korn's inequality

$$2\nu(\varepsilon(u),\varepsilon(u)) \geq \beta\|u\|_{\mathbf{V}}^2 \quad \exists\beta>0, \text{for all } u\in\mathbf{V}. \tag{5.3}$$

By Remark 2.2, there exists a constant M such that

$$\|S(x)\|_\infty,\ \|k(x)\|_\infty, \|\alpha(x)\|_\infty \leq M.$$

Then, there exists a constant c_0 (depend on β) such that

$$\begin{aligned}|2\nu(k(x)u,u)_{\Gamma_2} + 2\nu(S\tilde{u},\tilde{u})_{\Gamma_3} + \nu(k(x)u,u)_{\Gamma_7} + 2(\alpha(x)u,u)_{\Gamma_5}|\\ \leq \frac{\beta}{4}\|u\|_{\mathbf{H}^1(\Omega)}^2 + c_0\|u\|_H^2 \quad \text{for all } u\in\mathbf{V}\end{aligned} \tag{5.4}$$

(cf. Theorem 1.6.6 in [9]).

Let us estimate $\langle (W(0,x)\cdot\nabla)u,u\rangle + \langle (u\cdot\nabla)W(0,x),u\rangle$. Since $W(0,x)\in C(\overline{\Omega})$,

$$\big|\langle (W(0,x)\cdot\nabla)u,u\rangle\big| \leq \frac{\beta}{8}\|u\|_{\mathbf{H}^1(\Omega)}^2 + c_1\|u\|_H^2. \tag{5.5}$$

Taking $\operatorname{div} u = 0$ into account, we get

$$\begin{aligned}\big\langle (u\cdot\nabla)W(0,x),u\big\rangle &= \sum_j\int_\Omega u_j\frac{\partial W(0,x)}{\partial x_j}u\,dx\\ &= \int_{\partial\Omega}(u\cdot n)(W(0,x)\cdot u)\,d\Gamma - \sum_j\int_\Omega u_j\frac{\partial u}{\partial x_j}W(0,x)\,dx.\end{aligned}$$

On the right hand side of the formula above estimating the first term as in (5.4) and applying Hölder inequality in the second term, we have

$$\big|\langle (u\cdot\nabla)W(0,x),u\rangle\big| \leq \frac{\beta}{8}\|u\|_{\mathbf{H}^1(\Omega)}^2 + c_2\|u\|_H^2. \tag{5.6}$$

Putting $\delta = \frac{\beta}{2}$, $k_2 = c_0 + c_1 + c_2$, from (5.3)-(5.6) we get the asserted conclusion. □

Put $z = e^{-k_2 t}\overline{z}$, where k_2 is a constant in Lemma 5.1. Then, $e^{-k_2 t}\overline{z}' = z' + k_2 z$ and we have the following problem which is equivalent to (5.1).

Find z such that

$$
\begin{aligned}
&z \in L_2(0,T;\mathbf{V}),\\
&z(0) = z_0 = v_0 - W(0) \in \mathbf{V},\\
&\langle z'(t),u\rangle + 2\nu(\varepsilon(z)(t),\varepsilon(u)) + e^{k_2 t}\langle (z(t)\cdot\nabla)z(t),u\rangle\\
&\qquad + \langle (W(t)\cdot\nabla)z(t),u\rangle + \langle (z(t)\cdot\nabla)W(t),u\rangle + k_2(z(t),u)\\
&\qquad + 2\nu(k(x)z(t),u)_{\Gamma_2} + 2\nu(S\tilde{z}(t),\tilde{u})_{\Gamma_3} + 2(\alpha(x)z(t),u)_{\Gamma_5}\\
&\qquad + \nu(k(x)z(t),u)_{\Gamma_7}\\
&= e^{-k_2 t}\Big[\langle f(t),u\rangle + \sum_{i=2,4,7}\langle \phi_i(t),u_n\rangle_{\Gamma_i} + \sum_{i=3,5,6}\langle \phi_i(t),u\rangle_{\Gamma_i}\Big]\\
&\qquad\qquad \text{for all } u \in \mathbf{V}.
\end{aligned}
\tag{5.7}
$$

Define operators $A_2, A_W(t) : \mathbf{V} \to \mathbf{V}^*$ by

$$
\begin{aligned}
\langle A_2 y,u\rangle =& 2\nu(\varepsilon(y),\varepsilon(u)) + 2\nu(k(x)y,u)_{\Gamma_2} + 2\nu(S\tilde{y},\tilde{u})_{\Gamma_3}\\
&+ 2(\alpha(x)y,u)_{\Gamma_5} + \nu(k(x)y,u)_{\Gamma_7} + k_2(y,u)\\
&\qquad \text{for all } y,u \in \mathbf{V},
\end{aligned}
\tag{5.8}
$$

$$
\begin{aligned}
\langle A_W(t)v,u\rangle =& \langle (W(t,x)\cdot\nabla)v,u\rangle\\
&+ \langle (v\cdot\nabla)W(t,x),u\rangle \quad \text{for all } v,u \in \mathbf{V},
\end{aligned}
\tag{5.9}
$$

where k_2 is a constant in Lemma 5.1. $W \in \overline{\mathscr{W}}$ implies $W \in C\big([0,T];\mathbf{H}^1(\Omega)\big)$, and such definitions are well.

In proof of Lemma 5.1 it is clear that

$$
\langle A_2 u,u\rangle \geq \frac{3\beta}{4}\|u\|^2_{\mathbf{V}}. \tag{5.10}
$$

Also, by Lemma 5.1

$$
\big\langle \big(A_2 + A_W(0)\big)u,u\big\rangle \geq \frac{\beta}{4}\|u\|^2_{\mathbf{V}}. \tag{5.11}
$$

Define an operator $B_2(t) : V \to V^*$ by

$$
\langle B_2(t)v,u\rangle = e^{k_2 t}\langle (v\cdot\nabla)v,u\rangle \quad \text{for all } v,u \in \mathbf{V}. \tag{5.12}
$$

Define operators $L_2, \widetilde{A}_W, L_{2W}, \widetilde{B}_2 : \mathscr{X} \to \mathscr{Y}$, $C_2 : \mathscr{X} \times \mathscr{X} \to \mathscr{Y}$ and an element $F_2 \in \mathscr{Y}$ by

$$
(5.13)\qquad
\begin{aligned}
&\langle (L_2 z)(t), u\rangle = \langle z'(t), u\rangle + \langle A_2 z(t), u\rangle \\
&\qquad\qquad\qquad \text{for all } z \in \mathscr{X}, \text{for all } u \in \mathbf{V},\\
&\langle (\widetilde{A}_W z)(t), u\rangle = \langle A_W(t) z(t), u\rangle \quad \text{for all } z \in \mathscr{X}, \text{for all } u \in \mathbf{V},\\
&\langle (L_{2W} z)(t), u\rangle = \langle z'(t), u\rangle + \big\langle \big(A_2 + A_W(t)\big) z(t), u\big\rangle \\
&\qquad\qquad\qquad \text{for all } z \in \mathscr{X}, \text{for all } u \in \mathbf{V},\\
&\langle (\widetilde{B}_2 z)(t), u\rangle = \langle B_2(t) z(t), u\rangle \quad \text{for all } z \in \mathscr{X}, \text{for all } u \in \mathbf{V},\\
&\langle C_2(w,z)(t), u\rangle = e^{k_2 t}\langle (w\cdot\nabla) z, u\rangle + e^{k_2 t}\langle (z\cdot\nabla) w, u\rangle \\
&\qquad\qquad\qquad \text{for all } w, z \in \mathscr{X}, \text{for all } u \in \mathbf{V},\\
&\langle (F_2)(t), u\rangle = e^{-k_2 t}\Big[\langle f(t), u\rangle + \sum_{i=2,4,7} \langle \phi_i(t), u_n\rangle_{\Gamma_i} \\
&\qquad\qquad\qquad + \sum_{i=3,5,6} \langle \phi_i(t), u\rangle_{\Gamma_i}\Big] \quad \text{for all } u \in \mathbf{V}.
\end{aligned}
$$

By the argument as Lemma 3.2 we get

Lemma 5.2. *C_2 is a bilinear continuous operator such that $\mathscr{X} \times \mathscr{X} \to \mathscr{Y}$. Under Assumption 3.1 $\widetilde{A}_W$ is a linear continuous operator such that $\mathscr{X} \to \mathscr{Y}$ and $F_2 \in \mathscr{Y}$.*

Using (5.10) instead of (3.20), as Lemma 3.3 we get

Lemma 5.3. *The operator $\overline{L}_2$ defined by $\overline{L}_2 z = (z'(0), L_2 z)$ for $z \in \mathscr{X}$ is a linear continuous one-to-one operator from $\mathscr{X}$ onto $H \times \mathscr{Y}$.*

Now, using (5.11) without assuming the fact that $\|W(0,x)\|_{\tilde{\mathbf{V}}}$ is small enough, as Lemma 3.4 we prove the following

Lemma 5.4. *The operator $\overline{L}_{2W}$ defined by $\overline{L}_{2W} z = (z'(0), L_{2W} z)$ for $z \in \mathscr{X}$ is a linear continuous one-to-one operator from $\mathscr{X}$ onto $H \times \mathscr{Y}$.*

Proof. As Lemma 3.5 in [14] it is proved that the operator $\widetilde{A}_W \in (\mathscr{X} \to \mathscr{Y})$ is compact. Thus, $\overline{A}_W \in (\mathscr{X} \to H \times \mathscr{Y})$ defined by $\overline{A}_W z = \{0_H, \widetilde{A}_W z\}$ is also compact. Since $\overline{L}_{2W} = \overline{L}_2 + \overline{A}_W$, in order to get the asserted conclusion by virtue of Theorem 3.4 in [14] it is enough to prove that $\overline{L}_{2W}$ is one-to-one from $\mathscr{X}$ into $H \times \mathscr{Y}$.

To prove the last fact it is enough to show that the inverse image of $(0_H, 0_{\mathscr{Y}})$ by $\overline{L}_{2W}$ is $0_{\mathscr{X}}$. It is easy to verify that

$$|\langle (A_2 + A_W(0))v, u\rangle| \le c\|v\|_{\mathbf{V}} \cdot \|u\|_{\mathbf{V}} \quad \text{for all } v, u \in \mathbf{V}. \tag{5.14}$$

By (5.11), (5.14) for any $q \in \mathbf{V}^*$ there exists a unique solution $y \in V$ to

$$(A_2 + A_W(0))y = q. \tag{5.15}$$

Let $z \in \mathscr{X}$ be the inverse image of $(0_H, 0_{\mathscr{Y}})$ by $\overline{L}$. Then, $z'(0) = 0_H$, and putting $t = 0$ from the third one in (5.13) we get

$$\langle (A_2 + A_W(0))z(0), u\rangle = 0 \quad \text{for all } u \in \mathbf{V},$$

where $z(0) = z(0, x)$. This means that $z(0)$ is the unique solution to (5.15) with $q = 0_{\mathbf{V}^*}$, i.e., $z(0) = 0_{\mathbf{V}}$. Therefore, $z \in \mathscr{X}$ satisfies

$$\begin{cases} z'(t) + (A_2 + A_W(t))z(t) = 0, \\ z(0) = 0_{\mathbf{V}}. \end{cases} \tag{5.16}$$

Now, using (5.16) and Gronwall's inequality, as in Lemma 3.8 of [14] we can prove $z = 0_{\mathscr{X}}$. It is finished to prove the Lemma. □

By the argument as Lemma 3.5 we get

LEMMA 5.5. *The operator T_2 defined by $T_2 z = \left(z'(0), (L_{2W} + \widetilde{B}_2)z\right)$ for $z \in \mathscr{X}$ is continuously differentiable, $T_2 0_{\mathscr{X}} = (0_H, 0_{\mathscr{Y}})$ and the Frechet derivative of T_2 at $0_{\mathscr{X}}$ is $\overline{L}_{2W}$.*

Let us consider the following problem

$$(A_2 + A_W(0) + B_2(0))u = q. \tag{5.17}$$

Now, using (5.11) without assuming the fact that $\|W(0, x)\|_{\tilde{\mathbf{V}}}$ is small enough, as Lemma 3.6 we can prove

LEMMA 5.6. *If the norm of $q \in V^*$ is small enough, then there exists a unique solution to* (5.17) *in some $\mathcal{O}_M(0_{\mathbf{V}})$.*

Using Lemmas 5.2∼5.6, Proposition 3.7, in the same way as Theorem 3.8 we get

THEOREM 5.7. *Suppose that Assumptions 3.1 holds and the norms of* f, f', ϕ_i, ϕ_i' *in the spaces they belong to are small enough.*

If

$$w_2 \equiv F_2(0) - (A_2 + A_{2W}(0) + B_2(0))z_0 \in H, \tag{5.18}$$

where $z_0 = v_0 - U(0,\cdot)$, *and* $\|w_2\|_H$ *is small enough, then there exists a unique solution to* (5.1) *in the space* $\mathcal{W}$.

REMARK 5.2. By the same argument as Remark 3.3, we can see that the condition (5.18) is equivalent to the condition $\overline{w}_2 \in H_1$, where $\overline{w}_2 \in \mathbf{V}_1^*$ is defined by

$$\begin{aligned}\langle \overline{w}_2, u\rangle = \langle f(0), u\rangle &+ \sum_{i=2,4} \langle \phi_i(0,x), u_n\rangle_{\Gamma_i} + \sum_{i=3,5,7} \langle \phi_i(0,x), u\rangle_{\Gamma_i} \\ &- \big[2\nu(\varepsilon(z_0), \varepsilon(u)) + 2\nu(k(x)z_0, u)_{\Gamma_2} + 2\nu(S\tilde{z}_0, \tilde{u})_{\Gamma_3} \\ &+ 2(\alpha(x)z_0, u)_{\Gamma_5} + \nu(k(x)z_0, u)_{\Gamma_7} + \langle (W(0,x)\cdot\nabla)z_0, u\rangle \\ &+ \langle (z_0\cdot\nabla)W(0,x), u\rangle + k_2(z_0,u) + \langle (z_0\cdot\nabla)z_0, u\rangle\big] \\ &\qquad \text{for all } u \in \mathbf{V}\end{aligned} \tag{5.19}$$

with k_2 in Lemma 5.1.

Let us consider Problem II.

Let $W(x,t) \in \overline{\mathcal{W}}$ be given solution to Problem II. Let v be the solution for the data perturbed except h_i and put $v = \overline{z} + W$. Then, we get a problem for $\overline{z}$:

Find $\overline{z}$ such that

$$\begin{aligned}&\overline{z} \in L_2(0,T;\mathbf{V}_1), \\ &\overline{z}(0) = z_0 \equiv v_0 - W(0,x) \in \mathbf{V}_1, \\ &\langle \overline{z}', u\rangle + \nu(\nabla\overline{z}, \nabla u) + \langle (\overline{z}\cdot\nabla)\overline{z}, u\rangle + \langle (W\cdot\nabla)\overline{z}, u\rangle + \langle (\overline{z}\cdot\nabla)W, u\rangle \\ &\qquad + \nu(k(x)\overline{z}, u)_{\Gamma_2} + \nu(S\tilde{\overline{z}}, \tilde{u})_{\Gamma_3} + 2(\alpha(x)\overline{z}, u)_{\Gamma_5} - \nu(S\tilde{\overline{z}}, \tilde{u})_{\Gamma_5} \\ &\quad = \langle f, u\rangle + \sum_{i=2,4} \langle \phi_i, u_n\rangle_{\Gamma_i} + \sum_{i=3,5,7} \langle \phi_i, u\rangle_{\Gamma_i} \quad \text{for all } u \in \mathbf{V}_1,\end{aligned} \tag{5.20}$$

where z_0, f, ϕ_i are perturbations of corresponding data.

By the same argument as Theorem 5.7 we have

Theorem 5.8. *Suppose that Assumptions 3.1 holds and the norms of* f, f', ϕ_i, ϕ_i' *in the spaces they belong to are small enough. Define an element* $w_3 \in \mathbf{V}_1^*$ *by*

$$
\begin{aligned}
\langle w_3, u\rangle &= \langle f(0), u\rangle + \sum_{i=2,4} \langle \phi_i(0,x), u_n\rangle_{\Gamma_i} + \sum_{i=3,5,7} \langle \phi_i(0,x), u\rangle_{\Gamma_i} \\
&\quad - \big[\nu(\nabla z_0, \nabla u) + \nu(k(x)z_0, u)_{\Gamma_2} + \nu(S\tilde{z}_0, \tilde{u})_{\Gamma_3} \\
&\quad + 2(\alpha(x)z_0, u)_{\Gamma_5} - \nu(S\tilde{z}_0, \tilde{u})_{\Gamma_5} + \langle (W(0,x)\cdot\nabla)z_0, u\rangle \\
&\quad + \langle (z_0\cdot\nabla)W(0,x), u\rangle + k_3(z_0,u) + \langle (z_0\cdot\nabla)z_0, u\rangle\big] \\
&\qquad \text{for all } u \in \mathbf{V}_1,
\end{aligned}
\tag{5.21}
$$

where k_3 *is a constant determined as in Lemma 5.1.*

If $w_3 \in \mathcal{O}_R(0_{H_1})$ *for* $R > 0$ *small enough, then there exists a unique solution to* (5.20) *in the space* $\mathcal{W}$*.*

Remark 5.3. If $\Gamma_i = \varnothing, i = 2 \sim 5$, then problem (5.20) is reduced to one in [14]. If $z_0 \in \mathbf{H}^{l/2}(\Omega)$, then $(z_0\cdot\nabla)z_0, (W(0,x)\cdot\nabla)z_0, (z_0\cdot\nabla)W(0,x) \in \mathbf{L}_2(\Omega)$ and $k_3 z_0 \in \mathbf{L}_2(\Omega)$. Thus, the last 4 terms in the right hand side of (5.21) do not give any effect to the condition for w_3 being in H_1, and so the conditions in the Theorem 5.8 is the same with one of conditions of Theorems 3.5∼3.8 in [14]. Thus, Theorem 5.8 guarantees existence of a unique solution under a condition weaker that one in [14].

Note that putting $W(t,x) \equiv 0$ in Theorems 5.7 and 5.8, we can not get Theorems 3.8 and 4.2, since there $h_i \neq 0$.

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