学位論文

Universal Structures in Noncritical Strings and Two-Dimensional Quantum Gravity

(非臨界次元上の弦と２次元量子重力における普遍的構造)

平成３年１２月博士（理学）申請

東京大学大学院理学系研究科

物理学専攻

〈福間 將文〉

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Ph.D. Thesis

Universal Structures in Noncritical Strings and Two-Dimensional Quantum Gravity

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24 December 1991

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Chapter 1

Introduction

One of the most important problems in elementary particle physics is to construct the quantum theory of gravitation in a consistent way. There have been the following two approaches to the quantum gravity.

One is the string theory. Actually it is known that there is a massless spin-two particle (in space-time) in the excitation mode of closed strings, and thus we can interpret it as a graviton [1]. However, the string theory is defined consistently only in the critical dimension higher than four (26 for bosonic strings and 10 for superstrings). Thus, in order to describe our real world, we must consider the story that the space-time where the strings live are reduced to our four-dimensional world through some dynamical effects. However, it is also known that the string theory has infinitely many possible classical vacua, so that we must consider seriously the non-perturbative structure of the string theory in order to find the stable vacuum corresponding to our world.

Another is a rather traditional approach, where we take the gravity theory as a usual field theory. However, since the gravity theory is known to be non-renormalizable in more than two space-time dimensions, it is impossible to define the quantum gravity within the framework of the conventional perturbative theory. Thus we must develop other non-perturbative frameworks. If we recall here that the non-perturbative definition of gauge theory is given by the lattice gauge theory, it seems natural to try to construct the lattice gravity theory. In our case, however, we do not know the universal classes of the quantum gravity or the corresponding continuum theory even perturbatively, and thus, we cannot convince ourselves that we have the correct answer when we define the lattice gravity. Thus it is helpful to check whether our definition of the lattice gravity successfully works in two dimensions where the continuum theory is defined well. If the correctness is checked in two dimensions, by further investigating the two-dimensional quantum gravity with this approach we might be able to get more insight into the universal structure of the quantum gravity in higher dimensions as well as in two dimensions. Moreover, since the two-dimensional gravity is equivalent to noncritical strings mathematically [2], more investigation on the two-dimensional quantum gravity will possibly give us more information about the non-perturbative structure of the string theory.

Conventionally we have two definitions of the lattice gravity. One is using Regge calculus, and another is using dynamical triangulation method (DTM). Recent study of the lattice gravity shows that the DTM actually yields the correct continuum (Liouville) theory in a continuum limit in two dimensions. Thus the group including me exclusively used the DTM and tried to reveal the universal structure of the two-dimensional quantum gravity. In particular, we showed the explicit behaviour of the operators, and found the universal equation which describes the renormalization group flows connecting some criticalities of the two-dimensional quantum gravity.

The present thesis is intended to describe the development of two-dimensional quantum gravity and noncritical strings, mainly on the basis of our three papers [11][25][26]. I here try not to restrict our discussions to the special features of two-dimensional theories.

In chapter 2, we first explain the DTM and then show that its continuum limit actually reproduces the known result of the continuum theory.

In chapter 3, on the basis of the DTM, we construct the operators of the two-dimensional quantum gravity and derive the relations among their correlation functions, the Schwinger-Dyson equation. There we first treat the pure gravity and then generalize our result to the system where the gravity is coupled to arbitrary minimal conformal matters. In particular, we show that the Schwinger-Dyson equation of the system where the gravity is coupled to (p, q) minimal conformal matters has

the form of the vacuum condition on the square root of the partition function with source terms, which is a τ function of the p-reduced KP hierarchy, with respect to the W_p algebra constructed from the oscillators realized by the multiplication and the differentiation of the sources.

In chapter 4, we investigate the algebraic structure of the above Schwinger-Dyson equation. In particular we prove the equivalence between the Schwinger-Dyson equation and the Douglas equation, and show that the square root of the partition function with source terms is actually a τ function. This chapter is rather mathematical, and the results obtained there reflect the speciality in two dimensions.

In chapter 5, we investigate more on the operator content. In particular, we show that some of the operators constructed in chapter 3 are redundant, that is, there are some operators which can be eliminated by some redefinitions of sources. We further consider the unitarity preserving renormalization group flows.

In chapter 6, we reconsider the topological gravity on the basis of the Kontsevich model, which gives us a good exercise of the method developed in chapter 4.

Chapter 7 is devoted to the conclusion of the present paper and the outlook for higher dimensional gravities.

Chapter 2

Dynamical Triangulation Method

In this chapter, we consider the lattice regularization of quantum gravity. So far have been considered two kinds of lattice regularization: one is *Regge calculus* (RC) and another is *dynamical triangulation method* (DTM).

In RC, fixing the topology of the lattice, we vary the length of each link to represent the fluctuation of metric. In this case it is known that the action given in RC actually yields the classical Einstein-Hilbert action in a proper continuum limit.

In DTM, on the other hand, we replace the summation of all the fluctuation geometry by that of all the possible simplicial decomposition of a manifold, setting the length of all links to be the same. This regularization is supposed to hold the general invariance of gravity theory from the beginning, that is, even before taking a continuum limit, since the ways of metric fluctuation of any two local patches of the manifold are the same after the summation.

Although the relation between the above two lattice approaches to gravity is still not established, we will exclusively use the latter (DTM). This is because a great progress in understanding quantum gravity has recently been made by using this approach, and because DTM is known to be exactly solvable in two dimensions and to yield in a continuum limit with fixed topology the same result with that of a continuum (Liouville) theory. To see this, we first review the continuum theory in section 2.1. Then in section 2.2 we realize DTM in terms of a matrix model and show the equivalence between its continuum limit and the continuum (Liouville) theory.

2.1 Summary of Continuum Theory

In what follows we consider closed two-dimensional (2D) surfaces with their genus fixed to h. Then the partition function $Z_h(t)$ of 2D gravity coupled to conformal matters with central charge c is given as follows:

$$Z_h(t) = \int \frac{\mathcal{D}g_{\mu\nu}}{\mathrm{Vol(Diff)}} Z_{\mathrm{M},h}[g] \exp\{-t_0 \int d^2z\sqrt{g}\}. \tag{2.1.1}$$

Here $Z_{\mathrm{M},h}[g]$ is the partition function of the conformal field theory on a Riemann surface with metric $g_{\mu\nu}$ and genus h, and t_0 (*resp.* t) is a bare (*resp.* renormalized) cosmological constant. The exact form of this partition function was given in [3] (for $h = 0$ by using light-cone gauge) and [4] (for arbitrary h by using conformal gauge), and is

$$Z_h(t) = \text{const.}\, t^{\eta(c,h)} \tag{2.1.2}$$

with

$$\begin{aligned} \eta(c,h) &= \frac{25 - c + \sqrt{(1-c)(25-c)}}{12}(1-h) \\ &\equiv \hat{\eta}(c)\cdot(1-h), \end{aligned} \tag{2.1.3}$$

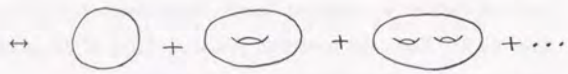
where $\gamma(c,h) \equiv 2 - \eta(c,h)$ is called the string susceptibility.

The above consideration is restricted to fixed topology. Now we would like to sum up all the topologies (genuses). Of course, this is not necessary *a priori* from the gravity theoretic point of view, because we still do not know whether we must take into account the topology change of space-time or not. However, as we will see in the next chapter, we encounter the case in which some configuration of space-time can be regarded as the result of a topology change. Moreover, from the string theoretic point of view this summation corresponds to that of all perturbation series of a partition function of non-critical strings. Thus let us go on this project.

The total partition function can be described schematically as follows:

$$\begin{aligned} Z(t) &\equiv \sum_{h=0}^{\infty} a_h Z_h(t) \quad (a_h\text{: constant}) \\ &= b_0 t^{\hat{\eta}} \;+\; b_1 t^0 \;+\; b_2 t^{-\hat{\eta}} \;+\; \cdots \end{aligned}$$

$$(b_h\text{: constant}) \qquad (2.1.4)$$



and now we need to determine the coefficients a_h (or b_h). So far they have been determined as satisfying the factorization property, which is usually required from the unitarity principle,

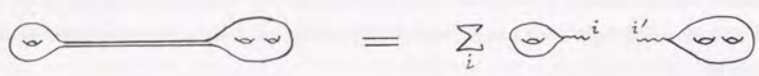


fig. 2.1

However, in order to carry out this program we have to know the explicit behaviour of fields near the boundary of the moduli space of Riemann surfaces, and so it is impossible pragmatically.

In the next section, we will see that the topology summation comes up naturally if we formulate the DTM through matrix models, and that those coefficients a_h are uniquely determined there. Moreover, analyses at low genuses show that the coefficients determined by the matrix model do satisfy the factorization property [5].

2.2 Dynamical Triangulation Method and the Matrix Model

In this section, we restrict our consideration to the pure gravity ($c = 0$). Then, the partition function of the continuum theory

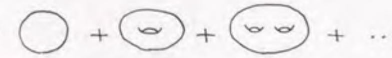
$$Z_h(t) = \int \frac{\mathcal{D}g_{\mu\nu}}{\mathrm{Vol(Diff)}} \exp\{-t_0 \int d^2z\sqrt{g}\} \qquad (2.2.1)$$

has the string susceptibility $\gamma = 2 - \eta$ with

$$\eta(c = 0,\, h) = \frac{5}{2}(1 - h). \qquad (2.2.2)$$

Thus the topology summation in this case requires us to determine the coefficients b_h in the following expansion:

$$Z_h(t) \;=\; b_0 t^{5/2} \;+\; b_1 t^0 \;+\; b_2 t^{-5/2} \;+\; \cdots \qquad (2.2.3)$$



The partition function in the DTM corresponding to (2.2.1) is defined by

$$Z_h^{\mathrm{lat}}(\lambda) = \sum_{\substack{G\text{: connected graph;}\\ \chi(G)=2(1-h)}} \lambda^{A(G)}, \qquad (2.2.4)$$

where $A(G)$ denotes the number of triangles in a graph G and $\chi(G)$ its Euler number. Note that $-\ln\lambda$ corresponds to a cosmological constant (bare one on the lattice). Although at first sight it seems impossible to analytically carry out this summation, we can do it if we rewrite this summation in terms of matrix models. This solvability is what is special in two dimensions.

Let $\phi = ({\phi^i}_j)$ be an $N \times N$ hermitian matrix. Then the partition function of the one-matrix model corresponding to the pure gravity is defined as follows:

$$W^{\mathrm{lat}}(N, \lambda) \equiv \int \mathcal{D}\phi \, \exp\{-N \mathrm{tr}\, V(\phi)\} \Big/ \int \mathcal{D}\phi \, \exp\{-N \mathrm{tr}\, \phi^2/2\}\,, \qquad (2.2.5)$$

where

$$\mathcal{D}\phi \;\equiv\; \prod_{i=1}^{N} d{\phi^i}_i \cdot \prod_{i>j} d(\mathrm{Re}{\phi^i}_j) \cdot \prod_{i>j} d(\mathrm{Im}{\phi^i}_j), \qquad (2.2.6)$$

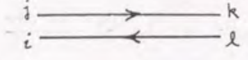
$$V(\phi) \;=\; \frac{1}{2}\phi^2 - \frac{\lambda}{3}\phi^3. \qquad (2.2.7)$$

Then the relation between $Z_h^{\mathrm{lat}}(\lambda)$ and $W^{\mathrm{lat}}(N, \lambda)$ is established in the following way: First, if we expand the partition function with respect to the coupling λ, we obtain the following Feynman rule:

<u>propagator</u>

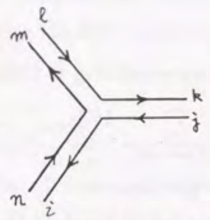
$$\begin{aligned} \left\langle {\phi^i}_j {\phi^k}_l \right\rangle_0 \;&\equiv\; \int \mathcal{D}\phi\, {\phi^i}_j {\phi^k}_l \exp\{-N \mathrm{tr}\, \phi^2/2\} \Big/ \int \mathcal{D}\phi \, \exp\{-N \mathrm{tr}\, \phi^2/2\} \\ &=\; \frac{1}{N}\delta^i_l \delta^j_k, \end{aligned} \qquad (2.2.8)$$

i.e.



$$= \frac{1}{N}\delta^i_l \delta^j_k, \tag{2.2.9}$$

<u>vertex</u>



$$= \lambda N \delta^i_j \delta^k_l \delta^m_n, \tag{2.2.10}$$

and Feynman diagrams are obtained by connecting the above pieces with the direction of all the arrows preserved (fig. 2.2).

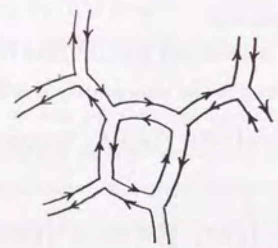


fig. 2.2

Thus, taking into account the fact that each loop contributes a factor $\mathrm{tr}\, 1 = N$, we can easily know that the (N, λ)-dependence of each Feynman graph is

$$N^{-\#(\text{propagator})}(\lambda N)^{\#(\text{vertex})} N^{\#(\text{loop})}. \tag{2.2.11}$$

We then consider the dual graphs of the above, in which the operation of connecting three-point vertices with preserving the direction of arrows are replaced by that of

patching triangles with keeping their orientations (see fig. 2.3),

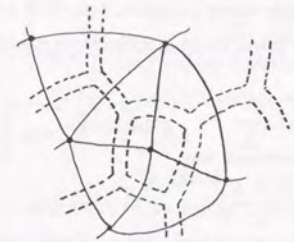


fig. 2.3

and, corresponding to (2.2.11), each dual graph G gives

$$N^{-\#(\text{edge})}(\lambda N)^{\#(\text{triangle})} N^{\#(\text{vertex})} = \lambda^{A(G)} N^{\chi(G)}. \tag{2.2.12}$$

Thus we have the partition function of connected graphs, $Z^{\text{lat}}(N, \lambda)$, as

$$\begin{aligned} Z^{\text{lat}}(N,\lambda) &= \ln W^{\text{lat}}(N,\lambda) \\ &= \sum_{G:\text{connected graphs}} \lambda^{A(G)} N^{\chi(G)} \\ &= \sum_{h=0}^{\infty} N^{2(1-h)} Z_h^{\text{lat}}(\lambda). \end{aligned} \tag{2.2.13}$$

That is, to obtain $Z_h^{\text{lat}}(\lambda)$ given in (2.2.4) we only have to compute the coefficient of the $1/N$ expansion of the matrix model.

A few comments are now in order:

(1) To be precise, we have to multiply eqs. (2.2.11) and (2.2.12) by a statistical factor of each graph. However, it is unity for almost all of the diagrams with sufficiently many triangles and indeed we are interested in such diagrams in a continuum limit. Thus we will neglect the statistical factor in what follows.

(2) So far we have considered only triangular decomposition of a two-dimensional manifold. However we can also consider decomposition with other polygon. Moreover, for n-gonal decomposition we can still use the matrix model by replacing its potential $V(\phi)$ (eq. (2.2.7)) by

$$V_n(\phi) = \frac{1}{2}\phi^2 - \frac{\lambda}{n}\phi^n. \tag{2.2.14}$$

As commented above, among the graphs in the partition function we are interested in those with sufficiently many polygon, *i.e.*, with sufficiently large area. The contribution from them in n-gonal decomposition was calculated for several n's in ref. [6]. The result is

$$\begin{aligned} Z_h^{\mathrm{lat}}(\lambda) &\sim \sum_{A:\text{sufficiently large}} A^{-(5/2)(1-h)-1} \exp\left\{-\frac{\lambda_c-\lambda}{\lambda_c}A\right\} \\ &\sim \left(\frac{\lambda_c-\lambda}{\lambda_c}\right)^{(5/2)(1-h)}, \end{aligned} \tag{2.2.15}$$

where the exponent of A is universal in the sense that it is independent of the n, while λ_c depends on it. In particular, for square decomposition ($n=4$) we have

$$\lambda_c = \frac{1}{12}. \tag{2.2.16}$$

Thus the contribution to $Z^{\mathrm{lat}}(N,\lambda) = \sum_{h\geq 0} N^{2(1-h)} Z_h^{\mathrm{lat}}(\lambda)$ is

$$Z^{\mathrm{lat}}(N,\lambda) = \sum_{h=0}^{\infty} \text{const.}\ N^{2(1-h)} \left(\frac{\lambda_c-\lambda}{\lambda_c}\right)^{(5/2)(1-h)} + (\text{regular function of } \lambda_c-\lambda). \tag{2.2.17}$$

The second term on the right hand side is due to the contribution from the graphs with not so many triangles, and thus should not remain in a continuum limit.

Now we consider the continuum limit of this matrix model. We first introduce the lattice spacing a, the link length of polygon, and make the following transformation:

$$A = \frac{1}{a^2} A_{\mathrm{phys}} \tag{2.2.18}$$

$$\sum_A = \frac{1}{a^2}\int_0^\infty dA_{\mathrm{phys}}, \tag{2.2.19}$$

where A_{phys} represents the "physical area" $\int d^2z\sqrt{g}$ of a manifold with metric $g_{\mu\nu}$. If we denote the renormalized cosmological constant by t, then, by looking at the right hand side of (2.2.15), we find that the coupling constant λ should be set as

$$t = \frac{\lambda_c-\lambda}{\lambda_c}\frac{1}{a^2} \tag{2.2.20}$$

or

$$\lambda(a) = \lambda_c(1-a^2t) \tag{2.2.21}$$

in order to take a correct continuum limit.[1] Thus, for small values of a we obtain the following result

$$Z_h^{\mathrm{lat}}(\lambda) \sim a^{5(1-h)} t^{(5/2)(1-h)} + (\text{regular function of } a^2 t), \tag{2.2.22}$$

$$Z^{\mathrm{lat}}(N,\lambda) \sim \sum_{h=0}^{\infty} \text{const.}\ \left(N\cdot a^{5/2}\right)^{2(1-h)} t^{(5/2)(1-h)} + (\text{non-universal part}), \tag{2.2.23}$$

which surely agrees with that of the continuum theory (eq. (2.2.3)). Here it should be noted that for spheres (h=0) and for tori (h=1) the contribution from the non-universal part (second term in (2.2.22)) dominates. Thus in order to obtain a correct continuum theory these must be removed before taking the limit $a \to 0$.

What is surprising in the above analysis is that we can take into account all the topologies at the same time if we fine-tune the coupling N as

$$N a^{5/2} \sim \text{const.}, \tag{2.2.24}$$

and so we set the coupling as

$$N(a) = \alpha^{-1/2} a^{-5/2}. \tag{2.2.25}$$

Then (2.2.23) is rewritten in the following form:

$$Z^{\mathrm{lat}}(N(a),\lambda(a)) \sim \sum_{h=0}^{\infty} \text{const.}\ \alpha^{h-1} t^{(5/2)(1-h)} + (\text{non-universal part}), \tag{2.2.26}$$

from which the parameter α can be interpreted as the renormalized string coupling constant. Genus expansion corresponds to the expansion around $\alpha = 0$ or to the expansion around $t = \infty$, since α and t always appear in the continuum theory in the combination of $t^{5/2}/\alpha$. In the followings we will set $\alpha = 1$ unless otherwise stated.

The analytic calculation of (2.2.5) with the *double scaling limit* (2.2.21) and (2.2.25) was carried out in refs. [7]. Their result is surprisingly simple: the free energy in the

[1] Since we have fixed the area of polygon to unity so far, for n-gonal decomposition with $n \neq 4$ we have to divide the right hand side of (2.2.18) by the area of a regular n-gon with link length unity. However, it is merely reduced to the redefinition of λ and the finite renormalization of t. Thus we will neglect the factor in the following.

continuum limit given as

$$\begin{aligned} Z^{\mathrm{lat}}(N,\lambda) & \left(=\ln W^{\mathrm{lat}}(N,\lambda)\right) \\ & \xrightarrow{a\to 0} \exp\{F(t)+(\text{non-universal part})\} \end{aligned} \tag{2.2.27}$$

is represented in the form

$$F(t) \;=\; \int_t^\infty dt'\,(t'-t)f(t'), \tag{2.2.28}$$

and the connected two-point function of cosmological terms

$$f(t)=\frac{d^2}{dt^2}F(t)=\langle\,\sigma_0\sigma_0\,\rangle_c \tag{2.2.29}$$

satisfies the Painlevé equation of the first kind,

$$f^2+\frac{1}{3}\frac{d^2f}{dt^2}=t. \tag{2.2.30}$$

The above discovery gave rise to the intensive study of two-dimensional quantum gravity with the matrix model approach. The first few problems to be solved are to construct operators other than the cosmological term, to find their relations, and to generalize the result of pure gravity to the systems where matters exist. In the process of the study, some new important discoveries are done, including the relation of the one-matrix model with the KdV hierarchy [8][9], and the generalization to the multi-matrix model and its relation with the KP hierarchy [10]. However, all these analyses are based on the orthogonal polynomial method so that their physical and geometrical meaning are obscured. In the next chapter, by turning back to the original idea of DTM, we explicitly construct the physical operators, derive their relations (Schwinger-Dyson equation) and generalize the result obtained there to the system where gravity is coupled to minimal conformal matters.

Chapter 3

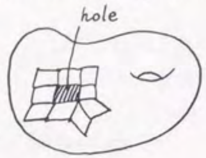
Schwinger-Dyson Equation of 2D Quantum Gravity

In this chapter, on the basis of DTM, we construct the operators of 2D quantum gravity and derive the relations among their correlation functions (Schwinger-Dyson equation) [11][12]. As will be shown in the next chapter, all the results derived in this section can be completely rederived from the Douglas equation which is based on the orthogonal polynomial method. However, the latter approach lacks the physical and geometrical meaning because of the use of rather algebraic manipulation, and can be applied only in two dimensions. Thus we use the DTM directly, hoping to be able to do the same thing in higher dimensions.

3.1 How to Construct Higher Dimensional Operators

In this section, we show the prescription for constructing the operators in quantum gravity. In what follows, we restrict our consideration to the pure gravity (central charge of matter fields is zero) and to the DTM with square decomposition.

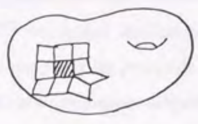
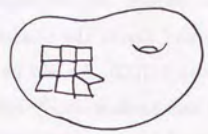
We first note that the cosmological term $\int d^2z\sqrt{g}$ corresponds to the operation of making a hole in the square decomposition. In fact, the summation of punctured graphs (graphs with one square removed to make a hole on a manifold) is equivalent to that of unpunctured graphs with the multiplication of the number of squares to take into account which square should be regarded as the one to be removed:

$\sum_G$ 

$= \sum_G A(G)$ 

(3.1.1)

Thus, the expectation value of the operation of making a hole is nothing but that of the area of graphs, which is interpreted as the one-point function of the cosmological term $\sigma_0 = \int d^2z\sqrt{g}$:

$\sum_G$  $\sum_G A(G)$ 

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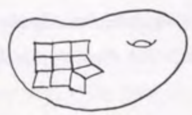
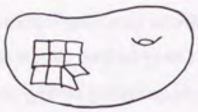
$\sum_G$  $\sum_G$ 

$$\sim \left\langle \int d^2z\sqrt{g} \right\rangle \tag{3.1.2}$$

From the above discussion, we understand that the smallest hole (a square hole) becomes the cosmological term σ_0 in a continuum limit ($A \to \infty$ or $a \to 0$). Moreover, every p-gonal hole in the square decomposition should also become the cosmological term σ_0 in the same limit. That is, if we denote the operation of making a p-gonal hole by W_p, all the W_p's give the same leading terms in the continuum limit ($a \to 0$):

$$\begin{aligned} W_4 &\xrightarrow{a\to 0} \text{const.}\, a^{k^{(4)}}\sigma_0 + \cdots, = \text{const.}\, a^{k_0}\sigma_0 + \cdots \\ W_p &\xrightarrow{a\to 0} \text{const.}\, a^{k^{(p)}}\sigma_0 + \cdots = \text{const.}\, a^{k_0}\sigma_0 + \cdots \\ & k^{(4)} = k^{(p)} \equiv k_0. \end{aligned} \tag{3.1.3}$$

However, by taking a suitable linear combination of the W_p's, we can eliminate the leading contribution from each W_p so that we make higher dimensional operators:

$$\begin{aligned} W_4 - \text{const.}\, W_6 &\xrightarrow{a\to 0} \text{const.}\, a^{k_1}\sigma_1 + \cdots \\ W_4 - \text{const.}\, W_6 + \text{const.}\, W_8 &\xrightarrow{a\to 0} \text{const.}\, a^{k_2}\sigma_2 + \cdots \\ &\vdots \\ &(k_0 < k_1 < k_2 < \cdots) \end{aligned} \tag{3.1.4}$$

These manipulations can be explicitly carried out in the framework of the matrix model [9] if we note that the operator W_p is written as[1]

$$W_p \sim \mathrm{tr}\,\phi^p. \tag{3.1.5}$$

In the present paper, however, we treat all these higher dimensional operators at the same time by considering the generating function of the W_p's, and systematically construct the above operators σ_0, σ_1, σ_2, $\cdots$ as the coefficients in the expansion of this generating function with respect to physical loop length. Before plunging into this calculation, we derive the relations among the correlation functions of the W_p's (discrete Schwinger-Dyson equation) in the next section.

3.2 Discrete Schwinger-Dyson Equation

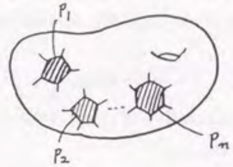
In the following, we will multiply a graph by $1/N$ every time when we make a hole and will denote by W_p the operation of making a p-gonal hole accompanied with this multiplication.[2] Thus in the matrix model the W_p is represented as $\mathrm{tr}\,\phi^p/N$.

[1] In fact, the insertion of $\mathrm{tr}\,\phi^p$ into the correlation function of the matrix model yields Feynman diagrams with a p-point vertex, which can be regarded in the dual graphs as graphs with a p-gonal hole.

[2] In fact, the Euler number χ of a simplicial manifold with boundaries are given by

$$\chi = \#(\text{vertex}) - \#(\text{edge}) + \#(\text{surface}) - \#(\text{boundary}). \tag{3.2.1}$$

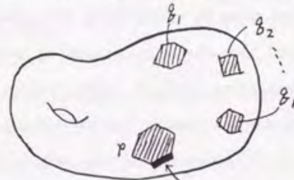
Furthermore we define their connected correlation functions as

$$\langle W_{p_1} W_{p_2} \cdots W_{p_n} \rangle_c^{\text{lat}} = \sum_G$$  $$/ Z^{\text{lat}}(N,\lambda) \quad (3.2.2)$$

$$Z^{\text{lat}}(N,\lambda) = \sum_G$$  $$(3.2.3)$$

The discrete Schwinger-Dyson equation is the relation among the correlation functions above, and can be derived in the DTM in the following way.

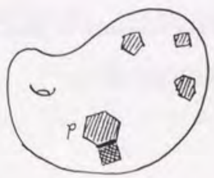
We first consider the correlation function with $(K+1)$ holes of a p-gon, a q_1-gon, $\cdots$ and a q_K-gon:

$$\langle W_p W_{q_1} W_{q_2} \cdots W_{q_K} \rangle_c^{\text{lat}} \cdot Z^{\text{lat}} = \sum_G$$  $$(3.2.4)$$

Then we take notice of a special edge of the p-gonal hole in graphs and classify the summation into four types according to its nature:

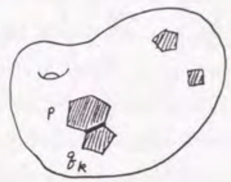
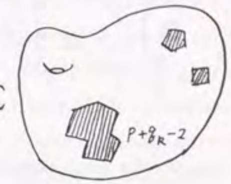
$$\sum_G = \sum_{G^{(1)}} + \sum_{G^{(2)}} + \sum_{G^{(3)}} + \sum_{G^{(4)}} . \quad (3.2.5)$$

(1) ordinary case, *i.e.*, there is a square attached to this edge: In this case we can remove the square and we have

$$\sum_{G^{(1)}}$$  $$= \lambda \sum_G$$  $$. \quad (3.2.6)$$

Here we have multiplied the graphs on the right hand side by λ because they have squares fewer than those on the left hand side by one.

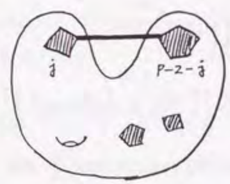
(2) the edge is attached to the other hole: We have q_k ways of attaching the edge to a q_k-gonal hole $(k = 1, \cdots, K)$, thus by removing the edge itself we have

$$\sum_{G^{(2)}}$$  $$= \frac{1}{N^2} \sum_{k=1}^{K} q_k \sum_G$$  $$. \quad (3.2.7)$$

Here the multiplication by $1/N^2$ is required to adjust the difference of the Euler numbers of both hand sides.[3]

(3) and (4) correspond to the case when the edge is attached to the other edge of the p-gonal hole itself. We can further classify this into two cases according to whether the simplicial manifold obtained after removing the edge is connected or not .

(3) connected: In this case, by removing the edge we obtain[4]

$$\sum_{G^{(3)}}$$  $$= \sum_{j=0}^{p-2} \sum_G$$  $$. \quad (3.2.8)$$

---

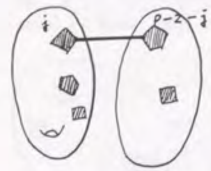
[3]In fact,

$$\begin{aligned}
\Delta\chi &= \chi_{\text{left}} - \chi_{\text{right}} \\
&= \Delta(\#(\text{vertex})) - \Delta(\#(\text{edge})) + \Delta(\#(\text{surface})) - \Delta(\#(\text{boundary})) \\
&= 0 - (+1) + 0 - (+1) \\
&= -2.
\end{aligned}$$

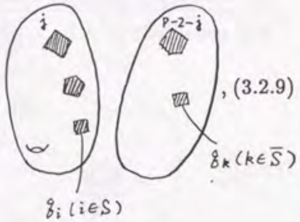
[4]In both equations (3.2.8) and (3.2.9) the Euler numbers of the graphs on both hand sides are the same, since $\Delta\chi = 0 - (+1) + 0 - (-1) = 0$.

(4) <u>disconnected</u>: In this case we have a choice of distributing the other holes to the connected components, and we have

$$\sum_{G^{(4)}}$$



$$= \sum_{j=0}^{p-2} \sum_{S\subset\{1,2,\cdots,K\}} \sum_{G}$$



$$, \tag{3.2.9}$$

where $\bar{S}$ denotes the complementary set of S.

Dividing the above relations by Z^{lat}, we thus have the following *discrete Schwinger-Dyson equation*:

$$\begin{aligned}
\left\langle W_p \prod_{j=1}^{K} W_{q_j} \right\rangle_c^{\mathrm{lat}} &= \lambda \left\langle W_{p+2} \prod_{j=1}^{K} W_{q_j} \right\rangle_c^{\mathrm{lat}} \\
&+ \frac{1}{N^2} \sum_{j=1}^{K} q_j \left\langle \prod_{k=1}^{j-1} W_{q_k} W_{q_j+p-2} \prod_{k=j+1}^{K} W_{q_k} \right\rangle_c^{\mathrm{lat}} \\
&+ \sum_{j=0}^{p-2} \left\langle W_j W_{p-2-j} \prod_{k=1}^{K} W_{q_k} \right\rangle_c^{\mathrm{lat}} \\
&+ \sum_{j=0}^{p-2} \sum_{S\subset\{1,\cdots,K\}} \left\langle W_j \prod_{i\in S} W_{q_i} \right\rangle_c^{\mathrm{lat}} \left\langle W_{p-2-j} \prod_{k\in\bar{S}} W_{q_k} \right\rangle_c^{\mathrm{lat}} .
\end{aligned} \tag{3.2.10}$$

Of course, this equation can be derived from the matrix model with the potential

$$V(\phi) = \frac{1}{2}\phi^2 - \frac{\lambda}{4}\phi^4 \tag{3.2.11}$$

by using the invariance of the (not necessarily connected) correlation function

$$\left\langle W_p \prod_{j=1}^{K} W_{q_j} \right\rangle^{\mathrm{lat}},$$

under the change of the integration variable

$$\phi \mapsto \phi + \delta\phi. \tag{3.2.12}$$

3.3 Continuum Schwinger-Dyson Equation and the Virasoro Constraint

Although we have derived the discrete Schwinger-Dyson (S-D) equation in the previous section, what we really would like to have is the relations among the correlation functions of the *scaling operators*, the operators constructed from a suitable linear combination of the W_p's, as explained in sec. 3.1. Our strategy to this problem is what follows: We first introduce the generating functions of the correlation functions of the W_p's, and rewrite the discrete S-D equation in terms of these generating functions. Then, by expanding them with respect to physical loop length l, we obtain the continuum S-D equation at last. Surprisingly, the equation obtained in this way has a very simple form, the Virasoro constraint on (the square root of) the partition function with source terms [11][12].

First we introduce the generating function of the connected K-point function as

$$G^{(K)}(z_1, z_2, \cdots, z_K; N, \lambda) \equiv \sum_{p_i \geq 0} z_1^{-p_1-1} \cdots z_K^{-p_K-1} \langle W_{p_1} \cdots W_{p_K} \rangle_c^{\mathrm{lat}}. \tag{3.3.1}$$

Then the discrete S-D equation is rewritten into the following form by using these generating function:

$$\begin{aligned}
&(z - \lambda z^3) G^{(K+1)}(z, z_1, \cdots, z_K) + (\lambda z^2 - 1)\delta_{K,0} \\
&\quad + \lambda \sum_{n_i \geq 0} z_1^{-n_1-1} \cdots z_K^{-n_K-1} \left\{ z \left\langle W_1 \prod_{j=1}^{K} W_{n_j} \right\rangle_c^{\mathrm{lat}} + \left\langle W_2 \textstyle\prod_{j=1}^{K} W_{n_j} \right\rangle_c^{\mathrm{lat}} \right\} \\
= \; & G^{(K+2)}(z, z, z_1, \cdots, z_K) \\
&\quad + \sum_{n=0}^{K} \sum_{\substack{S_1=\{i_1,\cdots,i_n\} \\ S_2=\{i_{n+1},\cdots,i_K\} \\ S_1\cap S_2=\emptyset \\ S_1\cup S_2=\{1,2,\cdots,K\}}} G^{(n+1)}(z, z_{i_1}, \cdots, z_{i_n}) G^{(K-n+1)}(z, z_{i_{n+1}}, \cdots, z_{i_K}) \\
&\quad + \frac{1}{N^2} \sum_{j=1}^{K} \frac{\partial}{\partial z_j} \left\{ \frac{G^{(K)}(z_1, \cdots, z_K) - G^{(K)}(z_1, \cdots, z_{j-1}, z, z_{j+1}, \cdots, z_K)}{z_j - z} \right\},
\end{aligned} \tag{3.3.2}$$

Here for $K = 0$ the third term on the left-hand side is $\lambda \langle W_2 \rangle_c^{\mathrm{lat}}$ and the last term on the right-hand side should read 0.

Now let us consider the critical behavior of the partition functions. Substituting the relations (2.2.16), (2.2.21) and (2.2.25):

$$\begin{aligned} \lambda &= \frac{1}{12}(1 - a^2 t) \\ N &= a^{-5/2} \end{aligned} \tag{3.3.3}$$

into the one-point function, it behaves for large p as [6]:

$$\langle W_p \rangle^{\mathrm{lat}} \sim a^{5/2}(2\sqrt{2})^p f^{(1)}(pa, t). \tag{3.3.4}$$

Here $f^{(1)}(l,t)$ is a smooth function of l and t. Since $l = pa$ is nothing but physical loop length of the p-gonal hole, $f^{(1)}(l,t)$ can be interpreted as a loop amplitude of a macroscopic loop. Substituting this equation into the definition of the one-point generating function we have

$$G^{(1)}(z; N, \lambda) \sim a^{5/2} \sum_{p:\text{sufficiently large}} z^{-p-1}(2\sqrt{2})^p f_1(l = pa, t), \tag{3.3.5}$$

and thus its convergence radius is known to be $1/z_c = 1/2\sqrt{2}$. Since we are interested in the contribution from the graphs with large area, we should pick up the contribution from the terms with sufficiently large p in eq. (3.3.5). This can be done easily by considering the behavior near the convergence radius. Thus we set[5]

$$\frac{1}{z} = \frac{1}{2\sqrt{2}} e^{-a\xi}, \tag{3.3.6}$$

that is,

$$z = 2\sqrt{2}e^{a\xi}. \tag{3.3.7}$$

Then eq. (3.3.5) behaves as

$$\begin{aligned} G^{(1)}(z; N, \lambda) &\sim a^{3/2} \int_0^\infty dl f^{(1)}(l,t) e^{-l\xi} \\ &\equiv a^{3/2} g^{(1)}(\xi, t), \end{aligned} \tag{3.3.8}$$

i.e., $g^{(1)}(\xi,t)$ is the Laplace transform of the macroscopic loop amplitude $f^{(1)}(l,t)$. Since $l = pa$ is a finite quantity with definite scale dimension (-1), the coefficients

[5]If we set $1/z = (1/2\sqrt{2})\exp(-a^k \xi)$, then we only have trivial results for k other than 1, as is easily seen in the following analysis.

in the expansion of $f^{(1)}(l,t)$ with respect to l and hence those in the expansion of $g^{(1)}(\xi,t)$ with respect to $1/\xi$ should give the one-point functions of the scaling operators σ_0, σ_1, σ_2, $\cdots$ given in sec. 3.1. Similarly, after substituting $z_i = 2\sqrt{2}e^{a\xi_i}$ and eqs. (3.3.3) into eq. (3.3.1), it behaves in the limit $a \to 0$ as

$$G^{(K)}(z_1, \cdots, z_K; N, \lambda) = a^{3K/2} g^{(k)}(\xi_1, \cdots, \xi_K, t), \tag{3.3.9}$$

and the coefficients in the expansion of $g^{(K)}(\xi,t)$ with respect to $1/\xi_i$ should give the K-point functions of the scaling operators.

We, however, need a careful treatment for the non-universal parts in the one- and two-point correlation functions, which come from the contribution of the spherical topology. Since in the correct continuum limit the one- and two-point correlation functions on the sphere behave as $t^{5/2-1+\Delta}$ and $t^{5/2-2+\Delta_1+\Delta_2}$ $(\Delta, \Delta_1, \Delta_2 \geq 0)$, respectively, the non-universal parts of the matrix model are identified as the pieces of G_1 and G_2 with order smaller than $t^{3/2}$ and $t^{1/2}$, respectively (see eq. (2.2.22)). Using the results of the large-N analysis [6], we obtain

$$\begin{aligned} G^{(1)}_{\mathrm{non}}(2\sqrt{2}\exp(a\zeta)) &= \frac{\sqrt{2}}{3} - \sqrt{2}a\zeta + a^{3/2}\left(\frac{8}{3}\zeta^{3/2} - \frac{1}{4}t\zeta^{-1/2}\right) + O(a^2) \\ G^{(2)}_{\mathrm{non}}(2\sqrt{2}\exp(a\zeta_1), 2\sqrt{2}\exp(a\zeta_2)) &= \frac{a^3}{32}(\zeta_1\zeta_2)^{-1/2}\left(\zeta_1^{1/2} + \zeta_2^{1/2}\right)^{-2} + O(a^{7/2}) \end{aligned} \tag{3.3.10}$$

Then the continuum S-D eq. are obtained by substituting

$$\begin{aligned} G^{(1)}(2\sqrt{2}\exp(a\zeta)) &= G^{(1)}_{\mathrm{non}}(2\sqrt{2}\exp a\zeta) + a^{3/2}g^{(1)}(\zeta, t), \\ G^{(2)}(2\sqrt{2}\exp(a\zeta_1), 2\sqrt{2}\exp(a\zeta_2)) &= G^{(2)}_{\mathrm{non}}(2\sqrt{2}\exp(a\zeta_1), 2\sqrt{2}\exp(a\zeta_2)) \\ &\quad + a^3 g^{(2)}(\zeta_1, \zeta_2, t), \qquad (3.3.11) \\ G^{(K)}(2\sqrt{2}\exp(a\zeta_1), \cdots, 2\sqrt{2}\exp(a\zeta_K)) &= a^{3K/2}g^{(K)}(\zeta_1, \cdots, \zeta_K, t) \quad (K \geq 3) \end{aligned}$$

into eq. (3.3.2). Examining the power behavior in ζ of thus obtained continuum S-D eq., we see that the $g^{(K)}$'s have the following expansion.

$$g^{(K)}(\zeta_1, \cdots, \zeta_K, t) = \sum_{n_i \geq 0} \zeta_1^{-n_1-3/2} \cdots \zeta_K^{-n_K-3/2} g_{n_1,\cdots,n_K}(t). \tag{3.3.12}$$

Here $g_{n_1,\cdots,n_K}$ can be regarded as the connected K-point function of K operators $\sigma_{n_1}, \cdots, \sigma_{n_K}$:

$$g_{n_1,\cdots,n_K} = \langle \sigma_{n_1} \cdots \sigma_{n_K} \rangle_c. \tag{3.3.13}$$

For example, comparing the powers of ζ in the equation yielded by the $K=0$ case of eq. (3.3.2)

$$
\begin{aligned}
-\frac{t^2}{16}\zeta^{-1}a^3+\left(-\frac{16}{3}\zeta^{3/2}+\frac{1}{2}t\zeta^{-1/2}\right)g^{(1)}(\zeta,t)a^3-\frac{1}{9}(1-ta^2)+\frac{1}{12}\langle W_2\rangle_c^{\mathrm{lat}}& \\
=a^3\{g^{(1)}(\zeta,t)\}^2+\frac{1}{128}\zeta^{-2}a^3+g^{(2)}(\zeta,\zeta,t)a^3&,
\end{aligned}
$$

we obtain the following relations among the g_n's.

$$
\begin{aligned}
\langle W_2\rangle_c^{\mathrm{lat}} &= \frac{4}{3}(1-ta^2)+64a^3g_0(t) \quad (\text{from } O(\zeta^0)) \\
128g_1(t) &= -\frac{3}{2}t^2 \quad (\text{from } O(\zeta^{-1})) \\
128g_2(t)-12tg_0(t) &= -\frac{3}{16} \quad (\text{from } O(\zeta^{-2})) \\
128g_n(t)-12tg_{n-2}(t) &= -24\sum_{m=0}^{n-3}\{g_{m,n-3-m}(t)+g_m(t)g_{n-3-m}(t)\} \\
&\quad (\text{from } O(\zeta^{-n})) \quad (n\geq 3)
\end{aligned}
\tag{3.3.14}
$$

The first equation of (3.3.14) is uninteresting in the continuum limit, while the other equations express the insertions of the operator σ_n in terms of those of lower dimensional operators. Performing similar analyses for the other values of K in eq. (3.3.2), we obtain the following equations which express the insertion of σ_n in the presence of other operators.

σ_1 insertion

$$
\begin{aligned}
&128g_{1,k_1,\cdots,k_m}=-\sum_{j=1}^{m}\frac{3}{2}(2k_j+1)g_{k_1,\cdots,k_{j-1},(k_j-1),k_{j+1},\cdots,k_m}(1-\delta_{k_j,0}) \\
&\qquad (m\geq 1 \text{ and at least one of the } k_i\text{'s is non-zero. }) \\
&128g_1=-\frac{3}{2}t^2 \\
&128g_{1,0}=\frac{3}{8}t \\
&128g_{1,0,0}=-\frac{3}{64} \\
&g_{1,\underbrace{0,\cdots\cdots,0}_{\text{more than 2}}}=0
\end{aligned}
\tag{3.3.15}
$$

σ_2 insertion

$$
\begin{aligned}
128g_{2,k_1,\cdots,k_m}-12tg_{0,k_1,\cdots,k_m} &= -\sum_{j=1}^{m}\frac{3}{2}(2k_j+1)g_{k_1,\cdots,k_m} \quad (m\geq 1) \\
128g_2-12tg_0 &= -\frac{3}{16}
\end{aligned}
\tag{3.3.16}
$$

σ_{n+2} insertion $(n\geq 1)$

$$
\begin{aligned}
&128g_{(n+2),k_1,\cdots,k_m}-12tg_{n,k_1,\cdots,k_m} \\
&\quad =-\sum_{j=1}^{m}\frac{3}{2}(2k_j+1)g_{k_1,\cdots,k_{j-1},(k_j+n),k_{j+1},\cdots,k_m} \\
&\qquad -24\sum_{r=0}^{n-1}\{g_{r,n-1-r,k_1,\cdots,k_m}+\sum_{\substack{S\subseteq\{k_1,\cdots,k_m\}\\ S=\emptyset \text{ allowed}}}g_{r,s}g_{n-1-r,\bar{S}}\} \\
&\qquad\qquad (\bar{S} \text{ is the complementary set of } S)
\end{aligned}
\tag{3.3.17}
$$

Since the operator σ_0 corresponds to the cosmological term, we demand that

$$
g_{0,k_1,\cdots,k_m}=-\frac{1}{8}\frac{\partial}{\partial t}g_{k_1,\cdots,k_m}, \tag{3.3.18}
$$

which is consistent with the equations (3.3.15). By introducing the generating function g and its exponential τ

$$
g(\mu_0,\mu_1,\cdots) = \sum_{n_i\geq 0}\frac{\mu_0^{n_0}}{n_0!}\frac{\mu_1^{n_1}}{n_1!}\cdots g_{\underbrace{0,\cdots,0}_{n_0},\underbrace{1,\cdots,1}_{n_1},\cdots} \tag{3.3.19}
$$

$$
\tau(\mu_0,\mu_1,\cdots) = \exp g(\mu_0,\mu_1,\cdots) \tag{3.3.20}
$$

we can assemble eqs. (3.3.15)-(3.3.18) in the following form.

$$
\begin{aligned}
&128\frac{\partial\tau}{\partial\mu_1}=-\sum_{k=1}^{\infty}\frac{3}{2}(2k+1)\mu_k\frac{\partial\tau}{\partial\mu_{k-1}}-\frac{3}{2}t^2\tau+\frac{3}{8}t\mu_0\tau-\frac{3}{128}\mu_0^2\tau, \\
&128\frac{\partial\tau}{\partial\mu_2}-12t\frac{\partial\tau}{\partial\mu_0}=-\sum_{k=0}^{\infty}\frac{3}{2}(2k+1)\mu_k\frac{\partial\tau}{\partial\mu_k}-\frac{3}{16}\tau, \\
&128\frac{\partial\tau}{\partial\mu_{n+2}}-12t\frac{\partial\tau}{\partial\mu_n}=-\sum_{k=0}^{\infty}\frac{3}{2}(2k+1)\mu_k\frac{\partial\tau}{\partial\mu_{k+n}}-24\sum_{r=0}^{n-1}\frac{\partial^2\tau}{\partial\mu_r\partial\mu_{n-1-r}}. \\
&\qquad (n\geq 1)
\end{aligned}
\tag{3.3.21}
$$

In order to make these equations looking simpler, we redefine the sources μ_k and the operators σ_k as

$$\mu_k \mapsto 2^{k+3}\mu_k \tag{3.3.22}$$

$$\sigma_k \mapsto 2^{-k-3}\sigma_k. \tag{3.3.23}$$

Then eq. (3.3.21) is rewritten into

$$\begin{aligned}
&\left[\frac{1}{2}(\mu_0 - t)^2 + \sum_{k=1}^{\infty}(2k+1)\mu_k\frac{\partial}{\partial\mu_{k-1}} + \frac{8}{3}\frac{\partial}{\partial\mu_1}\right]\tau(\mu) = 0\\
&\left[\sum_{k=0}^{\infty}(2k+1)\mu_k\frac{\partial}{\partial\mu_k} - t\frac{\partial}{\partial\mu_0} + \frac{8}{3}\frac{\partial}{\partial\mu_2} + \frac{1}{8}\right]\tau(\mu) = 0\\
&\left[\sum_{k=0}^{\infty}(2k+1)\mu_k\frac{\partial}{\partial\mu_{k+n}} - t\frac{\partial}{\partial\mu_n} + \frac{8}{3}\frac{\partial}{\partial\mu_{n+2}} + \sum_{r=0}^{n-1}\frac{\partial^2}{\partial\mu_r\partial\mu_{n-1-r}}\right]\tau(\mu) = 0\\
&\qquad (n \geq 1).
\end{aligned} \tag{3.3.24}$$

Thus, by further transforming the source variables as

$$\begin{aligned}
t_0 &= \mu_0 - t\\
t_1 &= \mu_1\\
t_2 &= \mu_2 + \frac{8}{15}\\
t_l &= \mu_l \quad (l \geq 3),
\end{aligned} \tag{3.3.25}$$

the continuum S-D eq. (3.3.24) can be expressed as a formal vacuum condition of the Virasoro algebra with central charge 1 on the partition function $\tau(t_k)$ [11][12]:

$$\mathbf{L}_n\tau(t_k) = 0 \quad (n = -1, 0, 1, \cdots). \tag{3.3.26}$$

Here the Virasoro generators are defined by

$$\begin{aligned}
\mathbf{L}_n =& \sum_{k+l=-n-1}\left(k+\frac{1}{2}\right)\left(l+\frac{1}{2}\right)t_k t_l + \sum_{k-l=-n}\left(k+\frac{1}{2}\right)t_k\frac{\partial}{\partial t_l}\\
&+\frac{1}{4}\sum_{k+l=n-1}\frac{\partial}{\partial t_k}\frac{\partial}{\partial t_l} + \frac{1}{16}\delta_{n,0}
\end{aligned} \tag{3.3.27}$$

and satisfy

$$[\mathbf{L}_n, \mathbf{L}_m] = (n-m)\mathbf{L}_{n+m} + \frac{1}{12}(n^3 - n)\delta_{n+m,0}. \tag{3.3.28}$$

In particular, eqs. (3.3.24) correspond to $\mathbf{L}_{-1}\tau = 0$, $\mathbf{L}_0\tau = 0$ and $\mathbf{L}_n\tau = 0$ $(n \geq 1)$, respectively.

Finally, we make a rather subtle comment on the normalization of the free energy $g = \ln\tau$ [11]. In the above argument we have considered $g(\mu_0, \cdots)$ as the free energy with source terms $\mu_0\sigma_0 + \mu_1\sigma_1 + \cdots$. However, since the original ϕ^4 matrix model (3.2.11) has an accidental symmetry of changing the sign of ϕ, the operators $W_n = (\frac{1}{N})\mathrm{tr}\,\phi^n$ with odd n do not have a good continuum limit. For example, for large values of n and m the two-point functions behave as $(2\sqrt{2})^{-2}\langle W_{2m+1}W_{2n+1}\rangle_c^{\mathrm{lat}} \simeq \langle W_{2m}W_{2n}\rangle_c^{\mathrm{lat}} \simeq \langle W_{2m+1}W_{2n-1}\rangle_c^{\mathrm{lat}}$, while $\langle W_{2m+1}W_{2n}\rangle_c^{\mathrm{lat}} = 0$. The simplest way to avoid this unnaturalness is to consider only even operators under the change of the sign of ϕ. Since we have not distinguished odd operators $\sigma_{(o)}$ from even operators $\sigma_{(e)}$, the quantity $g = \ln\tau$ we have considered is symbolically written as

$$\begin{aligned}
\ln\tau &= \ln\frac{\int\mathcal{D}\phi\exp\left\{-S + \frac{\mu}{2}(\sigma_{(e)} + \sigma_{(o)})\right\}}{\int\mathcal{D}\phi\exp\{-S\}}\\
&= \sum_n\frac{1}{n!}\left(\frac{\mu}{2}\right)^n\left\langle(\sigma_{(e)} + \sigma_{(o)})^n\right\rangle_c.
\end{aligned}$$

Using the fact that

$$\left\langle\sigma_{(e)}^k\sigma_{(o)}^\ell\right\rangle_c \simeq \begin{cases}\left\langle\sigma_{(e)}^{k+\ell}\right\rangle_c & \text{for even } \ell\\ 0 & \text{for odd } \ell,\end{cases} \tag{3.3.29}$$

we have

$$\begin{aligned}
\ln\tau &= \sum_n\frac{1}{n!}\left(\frac{\mu}{2}\right)^n\frac{1}{2}2^n\left\langle\sigma_{(e)}^n\right\rangle_c\\
&= \frac{1}{2}\ln\frac{\int\mathcal{D}\phi\exp\left\{-S + \mu\sigma_{(e)}\right\}}{\int\mathcal{D}\phi\exp(-S)}\\
&\equiv \frac{1}{2}F.
\end{aligned} \tag{3.3.30}$$

Thus we learn that the quantity $g = \ln\tau$ we have considered is one half of the free energy F of the correct continuum limit. Therefore if we regard $\tau(\mu_0, \mu_1, \cdots)$ as the square root of the partition function $Z(\mu_0, \mu_1, \cdots) = \exp F(\mu_0, \mu_1, \cdots)$ having the correct continuum limit:

$$\tau(\mu_0, \mu_1, \cdots) = \sqrt{Z(\mu_0, \mu_1, \cdots)}, \tag{3.3.31}$$

then the S-D eq. (3.3.26) gives the partition function $Z(\mu_0, \mu_1, \cdots)$ correctly [11].

3.4 Interpretation of the Schwinger-Dyson Equation

In this section, we give the physical and geometrical interpretation of the continuum S-D equation derived in the previous section.

First we note that to recover the string coupling constant α we only have to replace the (shifted) source t_k and the free energy by

$$t_k \mapsto \frac{1}{\alpha} t_k \tag{3.4.1}$$

$$F \mapsto \alpha^2 F. \tag{3.4.2}$$

Then the $\mathbf{L}_n \tau = 0$ condition in the S-D eq. implies that free energy $F(\mu)$ satisfies the following equation:

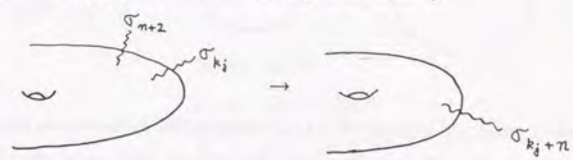
$$\begin{aligned} -\frac{8}{3}\frac{\partial F}{\partial \mu_{n+2}} + t\frac{\partial F}{\partial \mu_n} &= \sum_{k \geq 0}(2k+1)\mu_k \frac{\partial F}{\partial \mu_{k+n}} \\ &\quad + \sum_{r=0}^{n-1}\left[\frac{\alpha^2}{2}\frac{\partial^2 F}{\partial \mu_r \partial \mu_{n-1-r}} + \frac{1}{4}\frac{\partial F}{\partial \mu_r}\frac{\partial F}{\partial \mu_{n-1-r}}\right]. \end{aligned} \tag{3.4.3}$$

Then, by expanding this equation with respect to μ_l and α and by comparing the coefficients of $\mu_{k_1} \cdots \mu_{k_l}$, we obtain the following relation among the correlation function

$$\begin{aligned} &-\frac{8}{3}\left\langle \sigma_{n+2} \prod_{j=1}^{m} \sigma_{k_j} \right\rangle_c^{(h)} + t\left\langle \sigma_n \prod_{j=1}^{m} \sigma_{k_j} \right\rangle_c^{(h)} \\ &= \sum_{j=1}^{m}(2k_j+1)\left\langle \sigma_{k_j+n} \prod_{i(\neq j)} \sigma_{k_i} \right\rangle_c^{(h)} \\ &\quad + \frac{1}{2}\sum_{r=0}^{n-1}\left\langle \sigma_r \sigma_{n-1-r} \prod_{j=1}^{m} \sigma_{k_j} \right\rangle_c^{(h-1)} \\ &\quad + \frac{1}{4}\sum_{h_1+h_2=h}\sum_{r=0}^{n-1}\sum_{S \subset \{k_1, \cdots, k_m\}} \left\langle \sigma_r \prod_{i \in S} \sigma_i \right\rangle_c^{(h_1)} \left\langle \sigma_{n-1-r} \prod_{j \in \bar{S}} \sigma_j \right\rangle_c^{(h_2)}, \end{aligned} \tag{3.4.4}$$

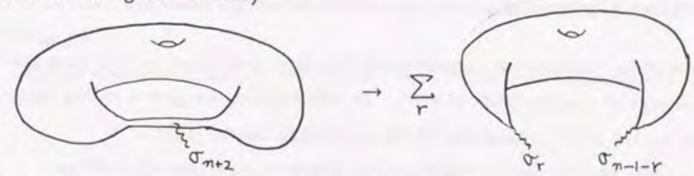
where $\langle \cdots \rangle_c^{(h)}$ denotes the connected correlation function with genus h. Note that this has the same form with that of the discrete S-D eq. (3.2.10).[6]

The equation (3.4.4) can be interpreted in the following way if we set the cosmological constant t to zero.[7] We consider the insertion of the operator σ_{n+2} in the correlation function $\left\langle \prod_{j=1}^{m} \sigma_{k_j} \right\rangle_c$. Then this operator fuses with the other operator σ_{k_j} into the new one σ_{k_j+n} (first term on the right hand side):



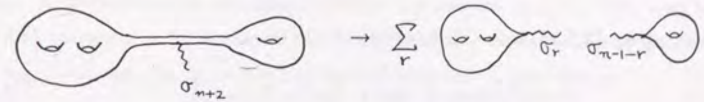
(3.4.5)

This can be regarded as the coefficient of the short distance expansion of local fields. Besides this, the operator σ_{n+2} suffers two kinds of effects from the boundary of the moduli space of Riemann surfaces. One is when a non-trivial cycle is pinched (second term on the right hand side)



(3.4.6)

and another is when a trivial cycle is pinched (third term on the right hand side)



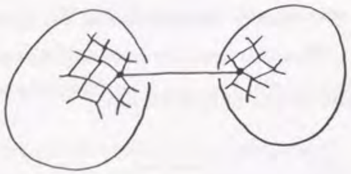
(3.4.7)

This interpretation is similar to the logic given in sec. 3.2 where we derived the discrete S-D equation.

[6] The matrix model is known to satisfy the Virasoro constraint before taking a continuum limit [13].

[7] When $t = 0$, there are no dimensionful parameters in the theory, and thus we can take it as a conformal field theory.

A comment is in order. We have considered the DTM in a rather wild way in the sense that we included the following graph in the simplicial decomposition of a manifold:



(3.4.8)

which is usually excluded because of its inhomogeneity. However, as we have seen so far, such graphs are essential to "topology changing." This point should be kept in mind when we consider the topology changing problem in higher dimensions on the basis of the DTM.

3.5 Other Criticalities of the One-Matrix Model

In the preceding sections our consideration has been restricted to the pure gravity (central charge of matter fields is zero). In this section, we give a rather intuitive discussion on the other criticalities of the one-matrix model [11].

Let us write the function τ satisfying the Virasoro constraint (3.3.26) as

$$\tau^2 = \int \mathcal{D}\phi \exp\left\{\sum_{k=0}^{\infty} t_k \sigma_k\right\}. \tag{3.5.1}$$

Substituting eq. (3.3.25) into this equation, we have

$$\tau^2 = \int \mathcal{D}\phi \exp\left\{-t\sigma_0 + \frac{8}{15}\sigma_2 + \sum_{k=0}^{\infty} \mu_k \sigma_k\right\}. \tag{3.5.2}$$

On the other hand, from the discussion given at the end of sec. 3.3, this should be equal to the partition function $Z(\mu)$ with source terms $\sum_{k=0}^{\infty} \mu_k \sigma_k$:

$$Z(\mu) = \int \mathcal{D}\phi \exp\left\{-S + \sum_{k=0}^{\infty} \mu_k \sigma_k\right\}. \tag{3.5.3}$$

Thus, by comparing this equation with eq. (3.5.2), we can identify the action S of the pure gravity with

$$S = t\sigma_0 - \frac{8}{15}\sigma_2. \tag{3.5.4}$$

Now we know that the pure gravity can be realized by the shift of the source variables (3.3.25), it is easy to see that other criticalities than the pure gravity can be realized by taking a shift such as

$$\begin{aligned} t_0 &= \mu_0 - t \\ t_k &= \mu_k + \text{const.} \\ t_l &= \mu_l \quad (l \neq 0,\, k). \end{aligned} \tag{3.5.5}$$

In fact, the calculation in the next section shows that this gives the continuum limit of the Kazakov series of the matrix model ($k = 1, 2, 3, \cdots$), which is supposed to describe the system of quantum gravity coupled to $(p,q) = (2, 2k-1)$ conformal matters.[8] In particular $k = 1$ corresponds to the topological gravity originally introduced by Witten [5].

Before closing this section, we relabel the operators and their sources as follows:

$$x_{2l+1} = -t_l \tag{3.5.6}$$

$$\mathcal{O}_{2l+1} = -\sigma_l, \tag{3.5.7}$$

to make transparent the extension of the results obtained so far to the general (p,q) case. In particular, the cosmological term is expressed as $\sigma_0 = -\mathcal{O}_1$, and the shift (3.5.5) corresponds to the expansion of the partition function $Z = \tau^2$ around $x_1 = t$, $x_{2k+1} = \text{const.}$, $x_{\text{others}} = 0$. Then the S-D equation is rewritten as

$$\mathrm{L}_n \tau(x) = 0 \tag{3.5.8}$$

[8] Here (p,q) is a set of co-prime positive integers which specifies the central charge of conformal matters as

$$c = 1 - \frac{(p-q)^2}{6pq}.$$

with

$$2\mathrm{L}_n = \frac{1}{2}\sum_{k+l=-2n} klx_kx_l + \sum_{k-l=-2n} kx_l\partial_l + \frac{1}{2}\sum_{k+l=2n}\partial_k\partial_l + \frac{1}{8}\delta_{n,0}, \qquad (3.5.9)$$

where k and l run over positive odd integers.

3.6 Formal Solution of the Schwinger-Dyson Equation

It does not seem easy to solve the continuum S-D eq. (3.5.8), since they are coupled equations containing infinitely many variables. In fact, while for $k = 1$ in the Kazakov series ($(p,q) = (2,1)$ gravity) all the correlation functions can be analytically calculated from the S-D equation (see below), it is in general difficult for $k \geq 2$ to explicitly calculate the correlation functions only with the S-D equation. However, as is shown in the next chapter by using another method, we can prove that the function $\tau(x)$ is a τ function of the KdV hierarchy, and thus we can solve the S-D equation directly [11]. The logic here for solving the S-D equation might seem unnatural if we recall that the S-D equation should be a system of equations which determine the physical system completely. Thus the fact that $\tau(x)$ is a τ function should be proved only within the framework of the S-D equation. This still remains as an open problem.

First, we consider a series expansion around the topological gravity ($k = 1$). By setting all the x_i's to zero except for x_1 and x_3, the first two equations of (3.5.8), $n = -1$ and $n = 0$, give

$$\begin{aligned} x_1^2 + 3x_3\partial_1 h &= 0 \\ x_1\partial_1 h + 3x_3\partial_3 h &= -\frac{1}{4}, \end{aligned} \qquad (3.6.1)$$

where $h(x_1, x_3) = 2\ln\tau(x_1, x_3, 0, 0, \cdots)$. These equations determine h uniquely up to an additive constant, giving the free energy of the $k = 1$ Kazakov's model [14]:

$$h = -\frac{x_1^3}{9x_3} - \frac{1}{12}\ln x_3. \qquad (3.6.2)$$

We then consider the insertions of the operators $\mathcal{O}_1, \mathcal{O}_3, \mathcal{O}_5, \cdots$ which correspond to the differentiations with respect to $x_1, x_3, x_5, \cdots$.

$$h_{\underbrace{1,\cdots,1}_{n_1},\underbrace{3,\cdots,3}_{n_3},\underbrace{5,\cdots,5}_{n_5},\cdots}(x_1, x_3) = \partial_1^{n_1}\partial_3^{n_3}\partial_5^{n_5}\cdots 2\ln\tau|_{x_5=x_7=\cdots=0}. \qquad (3.6.3)$$

We obtain the following recursion relations for $h_{p_1,\cdots,p_K}(x_1, x_3)$ by applying $\partial_{p_1}\cdots\partial_{p_K}$ to the S-D eq. (3.5.8) and then setting $x_5 = x_7 = \cdots = 0$.

$\mathcal{O}_1$ insertion

$$\begin{aligned} 3x_3h_{1,p_1,\cdots,p_K} + \sum_{j=1}^{K} p_jh_{p_1,\cdots,p_{j-1},(p_j-2),p_{j+1},\cdots,p_K} &= 0 \\ &(K \geq 1 \text{ and at least one of the } p_j\text{'s is not equal to } 1) \\ 3x_3h_1 &= -x_1^2 \\ 3x_3h_{1,1} &= -2x_1 \\ 3x_3h_{1,1,1} &= -2 \end{aligned}$$

$\mathcal{O}_3$ insertion

$$\begin{aligned} 3x_3h_{3,p_1,\cdots,p_K} + x_1h_{1,p_1,\cdots,p_K} + \sum_{j=1}^{K} p_jh_{p_1,\cdots,p_K} &= 0 \quad (K \geq 1) \\ 3x_3h_3 + x_1h_1 &= -\frac{1}{4} \end{aligned} \qquad (3.6.4)$$

$\mathcal{O}_p$ insertion ($p \geq 5$)

$$\begin{aligned} &3x_3h_{p,p_1,\cdots,p_K} + x_1h_{(p-2),p_1,\cdots,p_K} \\ &+\sum_{j=1}^{K} p_jh_{p_1,\cdots,p_{j-1},(p_j+p-3),p_{j+1},\cdots,p_K} \\ &+\frac{1}{2}\sum_{r+q=p-3}\{h_{r,q,p_1,\cdots,p_K} + \frac{1}{2}\sum_{S\subseteq\{p_1,\cdots,p_K\}} h_{r,S}h_{q,\bar{S}}\} = 0 \end{aligned}$$

This set of equations reduces the $h_{p_1,\cdots,p_K}$'s to those of the form $h_{1,\cdots,1,3,\cdots,3}$, which are determined by eq. (3.6.2). This agrees with the topological field theoretic analysis by Verlinde and Verlinde [15].

In order to solve the S-D eq. (3.5.8) for criticalities other than the topological gravity, we impose the Ansatz that τ is a τ function of the KdV hierarchy (see app.

A), which will be proved in the next chapter. After differentiating with respect to x_1, the first equation of (3.5.8), $\mathbf{L}_{-1}\tau = 0$, becomes

$$x_1 + (3x_3\partial_1^2 \ln\tau + 5x_5\partial_1\partial_3 \ln\tau + \cdots) \;=\; 0. \tag{3.6.5}$$

Then we use the following two facts known from the general analysis of the KP hierarchy. The first one is the identity satisfied by the τ function of the KP hierarchy (see eq. (A.2.17)):

$$\partial_1\partial_m \ln\tau \;=\; (L^m)_{-1}, \tag{3.6.6}$$

where $L = \partial + u_2\partial^{-1} + u_3^{-2} + \cdots$ is the pseudo-differential operator corresponding to the τ function and the symbol $(\;)_{-1}$ means the coefficient of ∂^{-1}. The second fact we use follows from the condition of 2-reduction $(L^2)_- = 0$:

$$(L^{2k-1})_{-1} \;=\; 2R_k[-2u_2] \qquad (k \geq 1), \tag{3.6.7}$$

where $(\;)_-$ stands for the negative power part in ∂ and the R_k's are the coefficients of Gelfand-Dikii's resolvent expansion. Combining (3.6.5), (3.6.6) and (3.6.7), we have

$$\frac{1}{2}x_1 + \sum_{k=1}^{\infty}(2k+1)x_{2k+1}R_k[-2u_2] \;=\; 0, \tag{3.6.8}$$

which is indeed identical to the result in [8][9]. Thus we learn that the equation $\mathbf{L}_{-1}\tau = 0$ is a once-integrated version of this equation.[9] For example, if we set all the x_n's to zero except for x_1 and x_5, eq. (3.6.8) becomes the Painlevé equation for the two-point function of the cosmological terms, $f = 2\,\partial_1^2 \ln\tau = 2u_2$:

$$f^2 + \frac{1}{3}\partial_1^2 f \;=\; x_1. \tag{3.6.9}$$

The scaling dimensions of the operators $\mathcal{O}_i$'s and the string susceptibility exponent [3][4] are easily obtained from the fact that the S-D eq. (3.5.8) preserves the total weight when we define the weight of $\mathcal{O}_j$ as j. First we consider the case of genus zero.

[9]As we will see in the next chapter, the integration constant is uniquely determined by the KP flow.

By setting $x_1 = t$, $x_r = 1$ and the other x_n's $= 0$, an argument similar to that for eq. (3.6.4) gives

$$th_{2n+1} + rh_{2n+r} + \frac{1}{4}\sum_{j+\ell=2n} h_j h_\ell \;=\; 0, \tag{3.6.10}$$

where $h_j(t) \;=\; 2\,\partial_j \ln\tau|_{x_1=t,x_5=1,\text{the other } x\text{'s}=0}$, and we have dropped the unfactorized terms $h_{j,\ell}$ in order to pick up the contribution from the spherical topology. Then we assume that the $h_j(t)$'s have the following power behavior for large t:

$$h_j(t) \;\sim\; t^{1-\gamma+\Delta_j}, \tag{3.6.11}$$

where Δ_j is the scaling dimension of the operator $\mathcal{O}_j$, and γ is the string susceptibility exponent for the spherical topology [3][4]. Demanding that all terms in (3.6.10) have the same power behavior in t, we learn that

$$\begin{aligned} \Delta_{2n+1} + 1 \;&=\; \Delta_{2n+r} \;=\; 1 - \gamma + \Delta_j + \Delta_{2n-j} \\ &\text{for } 1 \leq j \leq 2n-1. \end{aligned} \tag{3.6.12}$$

The last equation means such relations as $\Delta_1 + \Delta_5 = \Delta_3 + \Delta_3$ and $\Delta_1 + \Delta_7 = \Delta_3 + \Delta_5$, from which we see that Δ_j is a linear function of j. Using $\Delta_1 = 0$ which follows from the definition, we obtain

$$\Delta_j \;=\; c(j-1), \tag{3.6.13}$$

where c is some constant. Substituting (3.6.13) back to (3.6.12), we learn

$$\begin{aligned} \Delta_j \;&=\; \frac{j-1}{r-1} \\ \gamma \;&=\; -\frac{2}{r-1}, \end{aligned} \tag{3.6.14}$$

which is the result for $k = (r-1)/2$ in the Kazakov series . Furthermore it is easy to check that the Euler number dependence of the string susceptibility exponent is correctly reproduced, when the unfactorized terms are introduced into (3.6.10) and treated by iteration.

Finally, we give two reinterpretation of the Virasoro generators (3.5.9), which give some clues to the generalization of the S-D eq. to the multi-matrix models. The first one is to express the $\mathbf{L}_n$'s in (3.5.9) as the energy-momentum tensor of a $\mathbf{Z}_2$-twisted free boson. In fact, by introducing

$$\begin{aligned}\phi(z) &= \sum_{\alpha\in \mathbf{Z}+1/2} \frac{a_\alpha}{\alpha} z^{-\alpha} \\ a_{-\alpha} &= \sqrt{2}\alpha x_{2\alpha} \\ a_\alpha &= \frac{1}{\sqrt{2}}\partial_{2\alpha},\end{aligned} \tag{3.6.15}$$

the $\mathbf{L}_n$'s in (3.5.9) are expressed as follows:

$$T(z) = \frac{1}{2} : (\partial_z\phi)^2 : + \frac{1}{16z^2} = \sum_n z^{-n-2}\mathbf{L}_n. \tag{3.6.16}$$

The second reinterpretation is rather formal. We consider the generators of the Virasoro algebra constructed from the variables $x_1, x_2, x_3, \cdots$:

$$\mathcal{L}_n = \frac{1}{2}\sum_{k+l=n} kl x_k x_l + \sum_{k-l=-n} k x_k \partial_l + \frac{1}{2}\sum_{k+l=n}\partial_k\partial_l \tag{3.6.17}$$

If we formally drop the variables with even index and the corresponding derivatives, the $\mathcal{L}$'s in (3.6.17) look similar to the $\mathbf{L}_n$'s in eq. (3.5.9);

$$2\mathbf{L}_n = \mathcal{L}_{2n} + \frac{1}{8}\delta_{n,0} \tag{3.6.18}$$

Although this relation is quite formal, it makes the statement look somewhat plausible that the Virasoro generators (3.5.9) are related to the 2-reduction of the KP hierarchy, that is, the KdV hierarchy.

3.7 Generalization to the Gravity Coupled to Minimal Conformal Matters

As we have seen in the preceding sections, one-matrix models are related to the 2-reduced KP hierarchy. Here we consider the 3-reduction of the KP hierarchy as a straightforward generalization [11]. Then, as a candidate for the S-D equation, it is natural to try the Virasoro algebra obtained from (3.6.17) by formally discarding x_{3k} and ∂_{3k} and picking up only the $3n$-th Virasoro generators. More explicitly, we

consider a system of equations for a function τ of variables $x_1, x_2, x_4, x_5, x_7, x_8, \cdots$, of the following form

$$\begin{aligned}&\mathbf{L}_n\tau = 0 \quad (n = -1, 0, 1, 2, \cdots), \\ &3\mathbf{L}_n = \tfrac{1}{2}\textstyle\sum_{p+q=-3n} pq x_p x_q + \sum_p p x_p \partial_{p+3n} + \tfrac{1}{2}\sum_{p+q=3n}\partial_p\partial_q + \tfrac{1}{3}\delta_{n,0}.\end{aligned} \tag{3.7.1}$$

These $\mathbf{L}_n$'s can be regarded as the generators of the Virasoro algebra for a free complex boson twisted by angle $2\pi/3$. Namely, for a complex boson $\phi(z)$ with mode expansions

$$\begin{aligned}\phi(z) &= \sum_\alpha a_\alpha z^\alpha \frac{1}{\alpha}, \\ \phi^*(z) &= \sum_\alpha a^*_\alpha z^\alpha \frac{1}{\alpha}, \quad \alpha \equiv \frac{1}{3}\ (\mathrm{mod}\ 1) \\ a_\alpha &= \begin{cases} \sqrt{3}\alpha x_{3\alpha} & (\alpha > 0) \\ \frac{1}{\sqrt{3}}\partial_{-3\alpha} & (\alpha < 0)\end{cases} \\ a^*_\alpha &= \begin{cases} \frac{1}{\sqrt{3}}\partial_{3\alpha} & (\alpha > 0) \\ -\sqrt{3}\alpha x_{-3\alpha} & (\alpha < 0),\end{cases}\end{aligned}$$

one can show that the stress-energy tensor is given by

$$T(z) = - : \partial\phi\partial\phi^* : + \frac{1}{9}\frac{1}{z^2} = \sum_n z^{-n-2}\mathbf{L}_n. \tag{3.7.2}$$

Equation (3.7.1), however, is not enough to determine τ uniquely. To see this we expand eq. (3.7.1) around a set of background sources, x_1, x_2 and x_4, as we did in the previous section. Let $\tau = \exp(F/2)$ and

$$h_{p_1,\cdots,p_k}(x_1, x_2, x_4) = \partial_{p_1}\cdots\partial_{p_k} F \mid_{x\text{'s other than } x_1, x_2 \text{ and } x_4 = 0}. \tag{3.7.3}$$

Then (3.7.1) takes a form such as

$$4x_4 h_{3n+1,p_1,\cdots,p_k} + 2x_2 h_{3n-1,p_1,\cdots,p_k} + x_1 h_{3n-2,p_1,\cdots,p_k} + \cdots = 0, \tag{3.7.4}$$

which means that the insertion of the first operator by modulo 3, $\mathcal{O}_{3n+1}$, is reduced to those of lower dimensional operators. Thus, to determine the free energy F completely, we need some extra equations that reduce the insertion of the second operator by modulo 3, $\mathcal{O}_{3n+2}$, to those of lower dimensional operators. Recalling the fact that the above $\mathbf{L}_n$'s generate the Virasoro algebra for a complex boson twisted by angle $2\pi/3$, we are naturally led to try the W_3 algebra [16] as the extra set of equations. That is, in addition to (3.7.1), we impose the following conditions on τ [11][12]

$$\mathbf{W}_n\tau = 0 \quad (n = -2, -1, 0, 1, \cdots), \tag{3.7.5}$$

where $\mathbf{W}_n$ is defined by

$$\mathbf{W}(z) =: (\partial\phi)^3 : + : (\partial\phi^*)^3 := \sum_n z^{-n-3}\mathbf{W}_n. \tag{3.7.6}$$

The explict expression for $\mathbf{W}_n$ is given by

$$\begin{aligned} 3^{\frac{3}{2}}\mathbf{W}_n &= \sum_{p+q+r=-3n} pqrx_px_qx_r + 3\sum_{p+q-r=-3n} pqx_px_q\partial_r + 3\sum_{p-q-r=-3n} px_p\partial_q\partial_r \\ &\quad + \sum_{-p-q-r=-3n} \partial_p\partial_q\partial_r, \end{aligned} \tag{3.7.7}$$

where p, q and r run over positive integers except for multiples of 3. The $\mathbf{W}_n$'s together with the $\mathbf{L}_n$'s generate the W_3 algebra. In particular,

$$\begin{aligned} [\mathbf{L}_n, \mathbf{W}_m] &= (2n-m)\mathbf{W}_{n+m}, \\ [\mathbf{W}_n, \mathbf{W}_m] &= -\frac{1}{10}\delta_{n+m,0}n(n^2-1)(n^2-4) \\ &\quad +(n-m)\left\{\frac{3}{2}(n^2+4nm+m^2)+\frac{27}{2}(n+m)+21\right\}\mathbf{L}_{n+m} \\ &\quad -9(n-m)\mathbf{U}_{n+m}, \end{aligned} \tag{3.7.8}$$

where $\mathbf{U}_n = \sum_{k\le -2}\mathbf{L}_k\mathbf{L}_{n-k} + \sum_{k\ge -1}\mathbf{L}_{n-k}\mathbf{L}_k$. As is clear from these commutation relations, the equations

$$\begin{aligned} \mathbf{L}_n\tau &= 0 \quad (n = -1, 0, \cdots) \\ \mathbf{W}_n\tau &= 0 \quad (n = -2, -1, 0, \cdots) \end{aligned} \tag{3.7.9}$$

form a closed and consistent system. No new conditions with smaller n appear from (3.7.9), because $[\mathbf{L}_{-1}, \mathbf{W}_{-2}] = 0$ and $[\mathbf{W}_{-1}, \mathbf{W}_{-2}]\tau = 0$. When the second equation of (3.7.9) is expanded around the background sources x_1, x_2 and x_4, which corresponds

to what is called the $(3,1)$ topological gravity, we have equations such as

$$\begin{aligned} 16\,(x_4)^2\,h_{3n+2\,p_1,\cdots,p_k} + 8\,x_1x_4h_{3n-1,p_1,\cdots,p_k} + 4\,(x_2)^2h_{3n-2,p_1,\cdots,p_k} & \\ +(x_1)^2h_{3n-4,p_1,\cdots,p_k} + \cdots &= 0, \end{aligned} \tag{3.7.10}$$

which indeed reduce the insertion of the second operator by modulo 3, $\mathcal{O}_{3n+2}$, to those of lower dimensional operators. Due to eqs. (3.7.3) and (3.7.8) all the operator insertions are thus reduced to the insertions of $\mathcal{O}_1, \mathcal{O}_2$ and $\mathcal{O}_4$. Moreover, $\tau(x_1, x_2, x_4, 0, 0, \cdots)$ are uniquely determined and analytically calculated by the constraints $\mathbf{L}_{-1}\tau = 0$, $\mathbf{L}_0\tau = 0$ and $\mathbf{W}_{-2}\tau = 0$. Therefore all the correlation functions are determined uniquely for the theory expanded around this background source x_1, x_2 and x_4. Now we thus know that eq. (3.7.8) does determine the function $\tau(x)$ uniquely so that it deserves to be called the Schwinger-Dyson equation for this generalized system. On the other hand, in the next chapter we generically prove on the basis of the Douglas equation that the square root of the partition function of any matrix model can be expressed as a τ function of the KP hierarchy under some reduction condition. Thus, if we admit the completeness of the S-D equation, it is plausable for the following mathematical theorem to hold:

Conjecture 3.7.1

The following two statements are equivalent.

- *τ satisfies "the vacuum condition" of the W_3 algebra*

$$\begin{cases} \cdot\ \mathbf{L}_n\tau = 0 & (n \ge -1) \\ \cdot\ \mathbf{W}_n\tau = 0 & (n \ge -2) \end{cases}$$

- *τ is a τ function of the Boussinesq hierarchy and satisfies*

$$\mathbf{L}_{-1}\tau = 0. \tag{3.7.11}$$

By using the fact that the $\tau(x)$ is a τ function, one can easily convince oneself that the system (3.7.9) is actually the S-D eq. for some criticalities of the two-matrix model. In order to see this, let us differentiate both sides of $\mathbf{L}_{-1}\tau = 0$ with respect to x_2 and x_1:

$$
\begin{aligned}
2x_1 + \sum_{p=4,5,7,8,10,11,\cdots} p x_p \partial_{p-3}\partial_2 \ln\tau &= 0, \\
2x_2 + \sum_{p=4,5,7,8,10,11,\cdots} p x_p \partial_{p-3}\partial_1 \ln\tau &= 0.
\end{aligned}
\tag{3.7.12}
$$

By setting all source terms other than x_1 and x_{q+3} to zero, we obtain

$$
\begin{aligned}
&2x_1 + (q+3)x_{q+3}\partial_q\partial_2 \ln\tau = 0 \\
&\partial_q\partial_1 \ln\tau = 0,
\end{aligned}
\tag{3.7.13}
$$

where we assume $q \geq 2$. By using the general relations (see eq. (A.2.17))

$$
\begin{aligned}
\partial_1\partial_m \ln\tau &= (L^m)_{-1} \\
\partial_2\partial_m \ln\tau &= 2(L^m)_{-2} + \partial_1(L^m)_{-1}
\end{aligned}
\tag{3.7.14}
$$

between the τ function of the KP hierarchy and the pseudo-differential operator $L = \partial + u\partial^{-1} + \cdots$ that satisfies the equations of motion $\partial_n L = [(L^n)_+, L]$, we obtain from the second eq. of (3.7.13)

$$
(L^q)_{-1} = 0, \tag{3.7.15}
$$

and then the first eq. of (3.7.13) yields

$$
x_1 + (q+3)x_{q+3}(L^q)_{-2} = 0. \tag{3.7.16}
$$

By combining the eqs. (3.7.15) and (3.7.16) and the condition of 3-reduction

$$
(L^3)_- = 0, \tag{3.7.17}
$$

we obtain a differential equation for the second order derivative of the free energy F

$$
f = \partial_1^2 F = 2\,\partial_1^2 \ln\tau = 2u. \tag{3.7.18}
$$

For example, if we consider the case $q = 4$ where $x_1 = t$, $x_7 = \text{const.}$ and the other x_i's equal to zero, then we obtain $v = -\frac{1}{2}u'$ from the eqs. $(L^3)_- = 0$ and $(L^4)_{-1} = 0$ for

$L = \partial + u\partial^{-1} + v\partial^{-2} + \cdots$. By substituting this into (3.7.16), $t + \text{const.}\cdot(L^4)_{-2} = 0$, we find the equation that coincides with the Ising model on random surfaces [17][18][19]

$$
t - \text{const.} \times (f^{(4)} + 9ff'' + \frac{9}{2}(f')^2 + 6f^3) = 0. \tag{3.7.19}
$$

Similaly for the case $q = 2$, where $x_1 = t$ and $x_5 = \text{const.}$, we obtain the equation for the one-matrix model with $k = 2$, that is,

$$
t - \text{const.} \times (f^2 + \frac{1}{3}f'') = 0. \tag{3.7.20}
$$

This indicates that the operator $\mathcal{O}_5$ corresponds to the mass term in the Ising model. For, if so, then setting $x_7 = 0$ and $x_5 \neq 0$ corresponds to setting mass$= \infty$ in the Ising model and the system is reduced to the pure gravity. In the next section, we briefly review the work by Gava and Narain who showed the S-D eq. of the gravity coupled to the Ising model is actually given as the W_3 constraint.

For general values of q, we can derive the Douglas equation for $p = 3$ [10] from eqs. (3.7.15), (3.7.16) and (3.7.17), that is,

$$
[L^3, (L^q)_+] = \text{const.} \tag{3.7.21}
$$

This follows from the facts that

$$
\begin{aligned}
[L^3, (L^q)_+] &= -[L^3, (L^q)_-] = -[\partial^3 + \cdots, (L^q)_{-2}\partial^{-2} + \cdots] \\
&= -(L^q)'_{-2} + O(\partial^{-1})
\end{aligned}
\tag{3.7.22}
$$

and that the left-hand side is a differential operator. Thus we find that the set of eqs. (3.7.15)-(3.7.17) is a once-integrated form of (3.7.21). In this sense the S-D eq. (3.7.9) is a twice-integrated version of (3.7.21).

We can calculate the scaling dimensions of operators and the string susceptibility exponent for the two-matrix models as in the case of one-matrix models [11]. First we note that our S-D eq. (3.7.9) preserve the total weight if we assign a weight r to the variable x_r. Then in the presence of background sources, $x_1 = t$, $x_{q+3} = \text{const.}$ and $x_{\text{others}} = 0$, the scaling dimension Δ_r of the operator $\mathcal{O}_r$ becomes

$$
\Delta_r = \frac{r-1}{q+3-1} = \frac{r-1}{q+2}, \tag{3.7.23}
$$

and the string susceptibility exponent for genus zero is calculated as

$$\gamma = -\frac{2}{q+3-1} = -\frac{2}{q+2}. \qquad (3.7.24)$$

They indeed agree with those of the Ising model on random surfaces [3][4] if we set $q = 4$.

Now we introduce a different way to see the W_3 algebra (3.7.2) and (3.7.6), that is, in terms of $sl(\widehat{3,\mathbf{C}})$ Kac-Moody algebra [11]. Recall that we twisted the complex boson by $2\pi/3$, which corresponds to a rotation in the root space of $sl(3,\mathbf{C})$ by $2\pi/3$ when the complex boson is identified with two real bosons expressing the Cartan subalgebra of $sl(\widehat{3,\mathbf{C}})$. This is also rephrased as a cyclic permutation of the simple roots α_1, α_2 and the lowest root $\alpha_0 = -\alpha_1 - \alpha_2$ of $sl(3,\mathbf{C})$. Therefore we can conclude that the generators of the W_3 algebra considered here are the ones constructed through the Miura transformation from two real bosons with a twist which generates a cyclic permutation of the extended Dynkin diagram of $sl(3,\mathbf{C})$. From the above considerations, we are naturally led to the following conjecture that the S-D eqs. for the other criticalities are obtained by a straightforward generalization to the case of $sl(\widehat{p,\mathbf{C}})$. In fact, this conjecture is generically proved in the next chapter.

Now we consider the following equation as the S-D eq. of the gravity coupled to general (minimal) conformal matters [11][12]:

$$\mathbf{W}^{(k)}_n \tau = 0 \quad (k = 2,3,\cdots,p;\ n \geq -k+1), \qquad (3.7.25)$$

for some integer p. Here the function τ is related to the free energy F by $\tau = \exp(F/2)$, and $\mathbf{W}^{(k)}_n$'s $(k = 2,3,\cdots,p)$ are the generators of the W_p algebra constructed through the Miura transformation from $(p-1)$ bosons with a $\mathbf{Z}_p$-twisted boudary condition which generates a cyclic permutation of the simple roots $\alpha_1,\cdots,\alpha_{p-1}$ and the lowest root $\alpha_0 = -\alpha_1 - \cdots - \alpha_{p-1}$ of $sl(p)$. Explicitly,

$$\begin{aligned} \mathbf{W}^{(k)}(z) &\equiv \sum_{n\in\mathbf{Z}} z^{-n-k}\mathbf{W}^{(k)}_n \\ &= \sum_{j_1<\cdots<j_k} :\prod_{m=1}^{k}(\vec{h}_{j_m}\cdot\partial_z\vec{\phi}): \end{aligned} \qquad (3.7.26)$$

where $\vec{h}_i$'s $(i = 1,\cdots,p)$ are weight vectors of the fundamental representation of $sl(p)$ (in $\mathbf{R}^{p-1}$) with inner product

$$\vec{h}_i \cdot \vec{h} = \delta_{ij} - \frac{1}{p}, \qquad (3.7.27)$$

and $\vec{\phi}$ satisfies the following boundary condition

$$\vec{h}_i \cdot \vec{\phi}(e^{2\pi i}z) = \vec{h}_{i+1}\cdot\vec{\phi}(z) \quad (i = 1,\cdots,p) \qquad (3.7.28)$$
$$(\vec{h}_{p+1} \equiv \vec{h}_1).$$

The rotation (with parity transformation for even value of p) in $\mathbf{R}^{p-1}$ corresponding to the above twist is easily shown to have eigenvalues

$$e^{2\pi i j/p} \quad (j = 1,2,\cdots,p-1). \qquad (3.7.29)$$

This allows a procedure similar to the one described in the preceding sections, where we can take

$$x_i,\ \partial_i \quad (i \neq 0 \ (\mathrm{mod}\ p)) \qquad (3.7.30)$$

as the coefficients in the mode expansion of $\vec{\phi}(z)$. Then, $\mathbf{L}_0$, for instance, has the following form:

$$p\mathbf{L}_0 = \sum_{\substack{j \\ j\neq 0\ (\mathrm{mod}\ p)}} j x_j \partial_j + \text{const.}, \qquad (3.7.31)$$

and the constant is determined to be $(p^2-1)/24$ in order that $\mathbf{L}_{-1}$, $\mathbf{L}_0$ and $\mathbf{L}_1$ make a closed subalgebra without a central term.

Now we demand that $\tau(x)$ is a τ functionof the p-reduced KP hierarchy. This requirement yields predictions on the string susceptibility exponent and the scaling dimensions of operators. In fact, if we insert $\mathcal{O}_1$ and $\mathcal{O}_{p+q}$ as a background, i.e. if we set

$$x_1 = t,\ x_{p+q} = \text{const.},\ x_{\text{others}} = 0, \qquad (3.7.32)$$

then the string susceptibility exponent γ for the spherical topology, and the scaling dimension Δ_r of the operator $\mathcal{O}_r$ $(r \neq 0 \ (\mathrm{mod}\ p))$ can be easily calculated, resulting in [11]

$$\gamma = -\frac{2}{p+q-1}, \tag{3.7.33}$$

$$\Delta_r = \frac{r-1}{p+q-1}. \tag{3.7.34}$$

These critical exponents are nothing but those of the 2D gravity coupled to the (p,q) minimal conformal model, because the gravitationally dressed scaling dimension $\Delta_{r,s}^{(p,q)}$ of (r,s) primary field $\Phi_{r,s}$ on the $(p-1)\times(q-1)$ conformal grid [20] is given by the formula [3][4][17]

$$\Delta_{r,s}^{(p,q)} = \frac{\mid qr - ps \mid -1}{p+q-1}. \tag{3.7.35}$$

Therefore, 2D gravity coupled to the (p,q) minimal conformal model is correctly described by our S-D eq. with background sources at x_1 and x_{p+q}. Moreover, Douglas's result is totally recovered by using the equation $\mathrm{L}_{-1}\tau = 0$ and the general results of the KP hierarchy. Indeed, following the discussion in the previous section, we can show that

$$\begin{aligned} (L^q)_{-1} = \cdots = (L^q)_{-(p-2)} = 0, \\ (L^q)_{-(p-1)} = \text{const.}\cdot x_1, \end{aligned} \tag{3.7.36}$$

which reproduce Douglas's formula [10]

$$\begin{aligned} \left[L^p, (L^q)_+\right] &= \text{const.} \\ (L^p)_- &= 0. \end{aligned} \tag{3.7.37}$$

In the next chapter, we show that the Douglas equation leads in turn to the W_p constraint on a τ function of the p-reduced KP hierarchy.

Note that the minimal conformal models we have considered are of (A_{p-1}, A_{q-1}) type in the ADE classification [21] and correspond to diagonal modular invariants. We can extend our method to the other cases (off-diagonal invariants) with the following speculation: *Let τ be the square root of the partition function of 2D gravity which is coupled to the conformal model of (X_l, A_{q-1}) type in the ADE classification ($X =$ A, D, E). Then, the S-D eq. of the system is given as "the vacuum condition" on τ*

with respect to the W algebra associated with the Lie algebra X. Furthermore, it is equivalent to the condition that the τ is a τ function of the soliton hierarchy associated with the Lie algebra X with a constraint $\mathrm{L}_{-1}\tau = 0$.

3.8 Ising Model on Random Surfaces

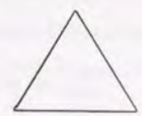
In this section, in order to support the abstract discussion given in the preceding section, we consider the S-D equation of the Ising model on random surfaces, *i.e.* the gravity coupled to $(p,q) = (3,4)$ conformal matter. We follow the work by Gava and Narain [22].

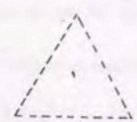
To make the gravity coupled to some matters, we usually assign "colors" to the simplices (triangles) in the DTM. In the case of the Ising model, we use two colors by considering its $\mathbf{Z}_2$ symmetry. We thus adopt the following two-matrix model:

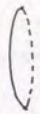
$$Z(N,\lambda,c) = \int \mathcal{D}u\mathcal{D}v \, \exp\{-N\mathrm{tr}\, V(u,v)\} \tag{3.8.1}$$

$$V(u,v) = \frac{1}{2}u^2 + \frac{1}{2}v^2 - cuv - \frac{\lambda}{3}u^3 - \frac{\lambda}{3}v^3. \tag{3.8.2}$$

Here u and v are $N\times N$ hermitian matrices. We expand the partition function perturbatively with respect to the coupling constants c and λ, and write the Feynman rule in the dual graphs for the three-point vertices $\mathrm{tr}\, u^3$ and $\mathrm{tr}\, v^3$ and the two-point vertex $\mathrm{tr}\, uv$ in the following way:

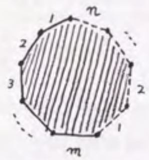
$\mathrm{tr}\, u^3 \quad \longleftrightarrow$  $\times \lambda$ (3.8.3)

$\mathrm{tr}\, v^3 \quad \longleftrightarrow$  $\times \lambda$ (3.8.4)

$$\mathrm{tr}\, uv \quad \longleftrightarrow$$

$$\times\ c. \qquad (3.8.5)$$

Then Feynman diagrams G are obtained in the dual graphs by patching the above surfaces (regarding the graph in eq. (3.8.5) as a two-gonal surface) with straight lines attached to themselves (and the same for dashed lines) and by multiplying it by $N^{\chi(G)}$. Here, if we regard the graphs (3.8.3) and (3.8.4) as representing up-spin and down-spin, respectively, then we can regard the graph (3.8.5) as the interaction of the Ising model.

Now we consider the S-D equation of this two-matrix model, by following the idea explained in sec. 3.2.[10] We define the operators $W_{m,n}$ as the operation of patching the surfaces (3.8.3)-(3.8.5) arond the following disc:



$$\times\ \frac{1}{N}. \qquad (3.8.6)$$

Here, as we have done in sec. 3.2, we multiply the disc by $1/N$, to adjust the Euler number.[11] Then we would like to have the relations among these operators in the closed form. It is in general difficult to write down such equations for general $W_{m,n}$'s. However, as shown in [22] and explained below, for the operators

$$U_m \equiv W_{m,0}, \qquad (3.8.8)$$

their relations are written in the desired form.

[10] Of course, the following equations can also be derived from the S-D eq. of the two-matrix model (3.8.1) and (3.8.2).

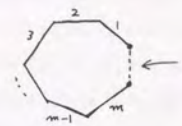
[11] In the above matrix model, this operator is then expressed as

$$W_{m,n} = \frac{1}{N}\mathrm{tr}\,(u^m v^n). \qquad (3.8.7)$$

First we specify an edge of a disc $W_{m,1}$ in a connected correlation function

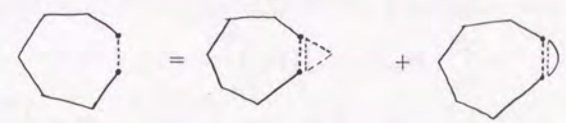
$$\left\langle W_{m,1} \prod_j U_{K_j} \right\rangle_c^{\mathrm{lat}}$$

as



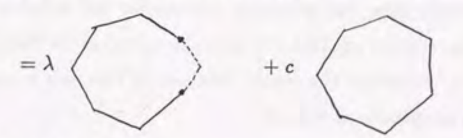
$$(3.8.9)$$

To this edge can be attached only the triangle of type (3.8.4) or the two-gonal surface (3.8.5):



$$. \qquad (3.8.10)$$

In the summation of the graphs this diagram can be replaced by the following one:



$$(3.8.11)$$

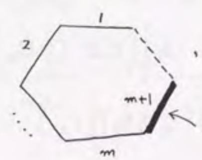
We thus obtain one of the S-D equations,

$$\left\langle W_{m,1} \prod_j U_{k_j} \right\rangle_c^{\mathrm{lat}} = \lambda \left\langle W_{m,2} \prod_j U_{k_j} \right\rangle_c^{\mathrm{lat}} + c\left\langle U_{m+1} \prod_j U_{k_j} \right\rangle_c^{\mathrm{lat}}. \qquad (3.8.12)$$

Next, we take notice of an edge in a disc $W_{m+1,n}$ in the correlation function

$$\left\langle W_{m+1,n} \prod_j U_{k_j} \right\rangle_c^{\mathrm{lat}}$$

as follows:



$$(3.8.13)$$

For this edge can be considered four cases corresponding to the ones which are considered in eqs. (3.2.6)-(3.2.9) as well as the case when two-gonal surface is attached to this edge. Thus we obtain another S-D equation

$$
\begin{aligned}
&\left\langle W_{m+1,n} \prod_j U_{k_j} \right\rangle_c^{\mathrm{lat}} \\
&= c\left\langle W_{m,n+1} \prod_j U_{k_j} \right\rangle_c^{\mathrm{lat}} \\
&\quad +\lambda\left\langle W_{m+2,n} \prod_j U_{k_j} \right\rangle_c^{\mathrm{lat}} \\
&\quad +\frac{1}{N^2}\sum_j k_j\left\langle W_{k_j+m-1,n} \prod_{i(\neq j)} U_{k_i} \right\rangle_c^{\mathrm{lat}} \\
&\quad +\sum_{r=0}^{m-1}\left\langle W_{r,n}U_{m-1-r} \prod_j U_{k_j} \right\rangle_c^{\mathrm{lat}} \\
&\quad +\sum_{r=0}^{m-1}\sum_{S\subset\{k_1,\cdots,k_K\}}\left\langle W_{r,n} \prod_{i\in S} U_{k_i} \right\rangle_c^{\mathrm{lat}}\left\langle U_{m-1-r} \prod_{j\in S} U_{k_j} \right\rangle_c^{\mathrm{lat}}.
\end{aligned}
\tag{3.8.14}
$$

This equation gives the recursion relation on the index n in $W_{m,n}$. Thus by using this we can rewrite eq. (3.8.12) into the equation for the correlation functions only of the U_k's. Although the result obtained in this way is very complicated, it can be expressed essentially as follows:

$$
\left\langle U_n \prod_j U_{k_j} \right\rangle_c^{\mathrm{lat}} \simeq \{(1)+(2)+(3)+(4)\}+(5)+(6)+(7). \tag{3.8.15}
$$

Here $\{(1)+(2)+(3)+(4)\}$ stands for the Virasoro-like part as was given and interpreted in sec. 3.2, while the other parts are essentially new ones which were absent for the one-matrix models:

$$
(5) \simeq \sum_l\sum_m k_l k_m\left\langle U_{k_l+k_m+n-2} \prod_{i(\neq l,m)} U_{k_i} \right\rangle_c^{\mathrm{lat}} \tag{3.8.16}
$$

$$
(6) \simeq \sum_l \sum_{r+s=k_l+n-4} k_l\left[\left\langle U_r U_s \prod_{i(\neq l)} U_{k_i} \right\rangle_c^{\mathrm{lat}} + \cdots\right] \tag{3.8.17}
$$

$$
(7) \simeq \sum_{r+s+t=n-4}\left[\left\langle U_r U_s U_t \prod_j U_{k_j} \right\rangle_c^{\mathrm{lat}} + \cdots\right], \tag{3.8.18}
$$

where $\cdots$ represents the part where the U_r, U_s and U_t factorize from each other. These parts (5), (6) and (7) remind us of the second, third and fourth terms in the generator of the W_3 algebra (3.7.7), respectively, although we should take a suitable linear combination of the U_k's in order to take a correct continuum limit.

The correct continuum S-D equation can be obtained by straightforwardly generalizing our method in sec. 3.3 [22]: First, we introduce the generating functions of the connected correlation functions of the U_k's, and rewrite the discrete S-D equation in terms of them. Then we expand this form of the S-D eq. with respect to physical loop length near the convergence radius of the generating function. Following these steps, Gava and Narain showed [22] that the above discrete S-D equation actually yields the W_3 constraint (3.7.9) in the continuum limit with a background shift at x_1 and x_7.

Chapter 4

Algebraic Structure of the S-D Equation

In this chapter, we investigate the algebraic structure of the S-D equation [25][26]. We first review in section 4.1 that from any matrix model is derived the Douglas equation $[P,Q]=1$ with P and Q differential operators of degree p and q with some positive integers p and q, respectively [10]. After rewriting the Douglas equation in terms of the KP hierarchy in section 4.2, we then show in section 4.3 that the square root of the partition function is a τ function of the KP hierarchy satisfying the following couple of equations, the *string equation:*

$$\begin{aligned}\mathcal{L}_{-p}\tau(x) &= \text{const.}\,\tau(x)\\ \frac{\partial}{\partial x_p}\tau(x) &= \text{const.}\,\tau(x).\end{aligned}$$

After introducing the $W_{1+\infty}$ algebra in section 4.4, we prove in section 4.5 that the string equation for a τ function leads to the $W_{1+\infty}$ constraint, the formal vacuum condition of the $W_{1+\infty}$ algebra, on the τ function. However, this constraint on the τ function is overcomplete, in the sense that the $W_{1+\infty}$ algebra is generated by its W_p subalgebra. We show in section 4.6 how this $W_{1+\infty}$ constraint can be reduced to the W_p constraint which is already discussed in section 3.7. This reduction of the constraint is important since this W_p constraint might be regarded as the S-D equation of the 2D gravity coupled to the (p,q) minimal conformal matters. In particular, one might be able to prove from the W_p constraint on some function that the function is a τ function. This is an open problem as the Virasoro constraint, while for the $(p,1)$ topological gravity we can calculate all the correlation functions analytically, showing

that τ is actually a τ function of p-reduced KP hierarchy. In what follows, we call the 2D gravity coupled to the (p,q) minimal conformal matters *the (p,q) gravity.*

4.1 Douglas Equation

In this section, we give a short review on the Douglas equation [10], restricting ourselves to the two-matrix model explained in sec. 3.8 although we here consider it in its general form.

Let the potentials of $N\times N$ hermitian matrices u and v be

$$\begin{aligned}V_1(u) &= \sum_{k\geq 2}\lambda_k u^k/k \qquad (4.1.1)\\ V_2(v) &= \sum_{k\geq 2}\mu_k v^k/k, \qquad (4.1.2)\end{aligned}$$

respectively. Then the partition function of the two-matrix model with respect to the above potentials is defined, as a function of the coupling constants, by

$$Z^{\text{lat}}(N,\lambda,\mu,c) = \int \mathcal{D}u\mathcal{D}v\,\exp\left\{-N\text{tr}\,(V_1(u)+V_2(v)-cuv)\right\}. \qquad (4.1.3)$$

If we denote the eigenvalues of the matrices u and v by $x_1,\cdots,x_N$ and $y_1,\cdots,y_N$, respectively, then this partition function can be expressed in terms of these eigenvalues in the following way due to Mehta's formula [23]:

$$Z^{\text{lat}}(N,\lambda,\mu,c) = \int\prod_{i=1}^N dx_i\prod_{i=1}^N dy_i\Delta(x)\Delta(y)\exp\left\{-\sum_{i=1}^N w(x_i,y_i)\right\}. \qquad (4.1.4)$$

Here $\Delta(x)$ is the Van der Monde determinant $\prod_{i>j}(x_i-x_j)$ and

$$w(x,y) = N\left(V_1(x)+V_2(y)-cxy\right). \qquad (4.1.5)$$

By introducing the orthonormal polynomials

$$\begin{aligned}\alpha_i(x) &= \frac{1}{\sqrt{h_i}}x^i+\cdots \qquad (4.1.6)\\ \beta_i(y) &= \frac{1}{\sqrt{h_i}}y^i+\cdots \qquad (4.1.7)\end{aligned}$$

as satisfying the following conditions:

$$\int dx\, dy\, e^{-w(x,y)}\alpha_i(x)\beta_j(y) \;=\; \delta_{ij}, \tag{4.1.8}$$

the free energy can then be expressed as

$$\ln Z^{\rm lat}(N,\lambda,\mu,c) \;\sim\; \sum_{i=1}^{N}(N-i)R_i, \qquad R_i \;\equiv\; h_i/h_{i-1}. \tag{4.1.9}$$

Now if we introduce a couple of operators $\hat{P}$ and $\hat{Q}$ by

$$\hat{P} \equiv y, \quad \hat{Q} \equiv -\frac{d}{dy}, \tag{4.1.10}$$

then they obviously satisfy the following relation:

$$\left[\hat{P}, \hat{Q}\right] \;=\; 1. \tag{4.1.11}$$

In the following we consider the continuum limit of this relation.

It is easy to see that their matrix elements have the following form:

$$\begin{aligned} \hat{P}_{ij} &= \int dx\, dy\, e^{-w(x,y)}\alpha_i(x)\, y\, \beta_j(y) \\ &= r_1\delta_{i,j+1} + r_2\delta_{i,j+2} + \cdots, \end{aligned} \tag{4.1.12}$$

$$\begin{aligned} \hat{Q}_{ij} &= \int dx\, dy\, e^{-w(x,y)}\alpha_i(x)\left(-\frac{d}{dy}\right)\beta_j(y) \\ &= N\sum_k \mu_k \left(\hat{P}^{k-1}\right)_{ij} + Nc\int dx\, dy\, e^{-w(x,y)}x\alpha_i(x)\beta_j(y) \\ &= N\sum_k \mu_k \left(\hat{P}^{k-1}\right)_{ij} + s_1\delta_{i,j-1} + s_2\delta_{i,j-2} + \cdots, \end{aligned} \tag{4.1.13}$$

where r_k and s_k are functions of the coupling constants (N,λ,μ,c). Since the first term in eq. (4.1.13) does not contribute to the eq. (4.1.11), we disregard it in the followings:

$$\hat{Q}_{ij} \;\equiv\; s_1\delta_{i,j-1} + s_2\delta_{i,j-2} + \cdots. \tag{4.1.14}$$

Now we consider the scaling limit of these operators, that is, the fine tuning of the coupling constants in these operators as the lattice spacing a goes to zero. As in the one-matrix model, the difference in the index i becomes the differentiation with

respect to the cosmological constant. Thus in the continuum limit the operators $\hat{P}$ and $\hat{Q}$ turn out to be differential operators with respect to the cosmological constant t:

$$\hat{P} - P_c \;\xrightarrow{a\to 0}\; P \;\equiv\; a_p\partial^p + \sum_{n=1}^{p-1} a_n(t)\partial^n \tag{4.1.15}$$

$$\hat{Q} - Q_c \;\xrightarrow{a\to 0}\; Q \;\equiv\; b_q\partial^q + \sum_{n=1}^{q-1} b_n(t)\partial^n, \tag{4.1.16}$$

where P_c and Q_c represent non-universal constants coming from spherical topology which should be subtracted, and ∂ represents the differentiation with respect to t: $\partial = \partial/\partial t$. Due to eq. (4.1.11) these differential operators P and Q satisfy the following *Douglas equation*:

$$[P, Q] \;=\; 1. \tag{4.1.17}$$

Note that if P, Q are solutions of this equation, then so are

$$P' \;=\; cfPf^{-1}, \quad Q' \;=\; \frac{1}{c}fQf^{-1} \tag{4.1.18}$$

with c a t-independent constant and $f = f(t)$ an arbitrary function of t. Thus by using this ambiguity, we can always transform P into the following form:

$$P = \sum_{n=0}^{p} a_n(t)\partial^n, \quad a_p(t) = 1, \quad a_{p-1}(t) = 0. \tag{4.1.19}$$

If P has this form, then we say that P is in the standard form. It can then be shown that the first nontrivial coefficient in P in the standard form, a_{p-2}, is related to the free energy $F(t)$ in the continuum limit via [10]

$$\frac{d^2}{dt^2}F(t) \;=\; \frac{2}{p}a_{p-2}(t). \tag{4.1.20}$$

Furthermore, as is given in app. A, the function a_{p-2} is expressed with a τ function, which will be introduced in sec. 4.2, by the equation

$$a_{p-2} \;=\; p\frac{d^2}{dt^2}\ln\tau, \tag{4.1.21}$$

from which we learn

$$\frac{1}{2}F \;=\; \ln\tau + \text{const.} \tag{4.1.22}$$

As was shown in sec. 3.7, the orders (p, q) of the differential operators P, Q in the Douglas equation are the same as the set of indices specifying the conformal matters the gravity is coupled to. Thus to obtain the theory with higher criticality we must construct higher order differential operators and thus require the finer tuning of the coupling constants in the matrix model, which is consistent with the requirement in usual local field theories. Of course, to do this fine tuning, we must have sufficiently many coupling constants in the matrix model. In fact, in the one-matrix model, the operator $\hat{P}$ has non-zero matrix elements only within at most two lines from the diagonal line, so that the order of the corresponding differential operator P is at most two, no matter how we fine-tune the coupling constants. On the other hand, we expect that the two-matrix model has sufficiently many coupling constants for giving any set (p, q), and this is the case as studied in refs. [24].

4.2 Douglas Equation and the KP Hierarchy

In this section, starting from the two-matrix model, we first generically consider the renormalization group flows which change the criticality (p, q). Then we restrict our consideration to the flows which does not change p, and show that these flows are nothing but the time evolution in the p-reduced KP hierarchy [26].

First we perturb the potential $w(x, y)$ in the two-matrix model (eq. (4.1.5)) as

$$w'(x,y) = w(x,y) + \delta w(x,y), \tag{4.2.1}$$

which causes the changes of the orthonormal polynomials $\alpha_i(x)$ and $\beta_i(y)$ as

$$\alpha_i'(x) = \sum_j(\delta_{ij} + \delta\alpha_{ij})\alpha_j(x), \qquad \beta_i'(y) = \sum_j \beta_j(y)(\delta_{ji} + \delta\beta_{ji}), \tag{4.2.2}$$

where $\delta\alpha_{ij}$ and $\delta\beta_{ji}$ are constants independent of x and y, respectively. Then due to the orthonormality of $\alpha_i(x)$, $\beta_i(y)$ and $\alpha_i'(x)$, $\beta_i'(y)$ we have

$$\begin{aligned} \delta\alpha_{ij} + \delta\beta_{ij} &= \int dx\, dy\, e^{-w(x,y)}\alpha_i(x)\delta w(x,y)\beta_j(y) \\ &\equiv \langle i \,|\, \delta w \,|\, j \rangle . \end{aligned} \tag{4.2.3}$$

By using this relation, we can easily compute the change of the operator $\hat{P}$ as follows:

$$\begin{aligned} \delta\hat{P}_{ij} &= -\langle i \,|\, \delta w \hat{P} \,|\, j \rangle + \sum_k \delta\alpha_{ik} \langle k \,|\, \hat{P} \,|\, j \rangle + \sum_k \langle i \,|\, \hat{P} \,|\, k \rangle\, \delta\beta_{kj} \\ &= -\sum_k \delta\beta_{ik}\hat{P}_{kj} + \sum_k \hat{P}_{ik}\delta\beta_{kj}. \end{aligned} \tag{4.2.4}$$

Thus, setting $\hat{H}_{ij} \equiv -\delta\beta_{ij}$, we obtain the following equation:

$$\delta\hat{P} = \left[\hat{H}, \hat{P}\right]. \tag{4.2.5}$$

Similarly we have

$$\delta\hat{Q} = \left[\hat{H}, \hat{Q}\right]. \tag{4.2.6}$$

Therefore, by writing the above operators in the continuum limit as

$$\hat{P} - P_c \xrightarrow{a\to 0} P \tag{4.2.7}$$

$$\hat{Q} - Q_c \xrightarrow{a\to 0} Q \tag{4.2.8}$$

$$\hat{H} - H_c \xrightarrow{a\to 0} H, \tag{4.2.9}$$

we conclude from eqs. (4.2.5) and (4.2.6) that general renormalization group flows in the continuum theory can be expressed in terms of differential operators in the following way:

$$[P, Q] = 1 \tag{4.2.10}$$

$$\delta P = [H, P] \tag{4.2.11}$$

$$\delta Q = [H, Q]. \tag{4.2.12}$$

Now we consider the flows which does not change the order p of the differential operator P. In this case, we can calculate the explicit form of the flow generator H. First, we assume that the P is in the standard form:

$$P = \partial^p + a_{p-2}\partial^{p-2} + \cdots. \tag{4.2.13}$$

Then we can express this as

$$P = L^p \tag{4.2.14}$$

by using a pseudo-differential operator

$$L = \partial + u\partial^{-1} + \cdots \tag{4.2.15}$$

with $a_{p-2} = pu$. Now the general form of the flow generator H which preserves P in the standard form is given by

$$H = \sum_{n\geq 0} c_n (L^n)_+ , \tag{4.2.16}$$

where c_n is a constant and $(\ )_+$ represents the differential operator part of a pseudo-differential operator.

[proof]
We assume the order of the differential operator H to be N and write H as $H = c_N\partial^N + \cdots$. Since the order of $[H, P]$ is less than that of P, c_N should be constant. Note that $(L^N)_+$ satisfies this condition since $[(L^N)_+, P] = -[(L^N)_-, P] = * \cdot \partial^{p-2} + \cdots$. Thus, by setting $H_1 \equiv H - c_N(L^N)_+ \equiv c_{N-1}\partial^{N-1} + \cdots$, the order of the flow generator in question can be reduced by one. Repeating this procedure, we finally obtain eq. (4.2.16). ∎

Thus, denoting by y_n the parameter corresponding to the flow $(L^n)_+$, we obtain

$$\frac{\partial}{\partial y_n} P = \left[(L^n)_+, P\right] \tag{4.2.17}$$

$$\frac{\partial}{\partial y_n} Q = \left[(L^n)_+, Q\right]. \tag{4.2.18}$$

In particular, since $P = L^p$, we obtain the following KP equation:

$$\frac{\partial}{\partial y_n} L = \left[(L^n)_+, L\right]. \tag{4.2.19}$$

Moreover, since $(L^{mp})_+ = L^{mp}$ for any positive integer m, we have

$$\frac{\partial}{\partial y_{mp}} L = 0 \quad (m = 1, 2, 3, \cdots), \tag{4.2.20}$$

which is nothing but the p-reduction condition on the KP hierarchy. Thus we have shown that the general renormalization group flows which does not change the order

p of the differential operator P is described by the time evolution of the p-reduced KP hierarchy. Furthermore, as is given in app. A, this KP equation is equivalent to the following set of equations, the *Sato equation*:

$$L = W\partial W^{-1} \tag{4.2.21}$$

$$W = 1 + \sum_{n\geq 1} w_n(t;y)\partial^{-n} \tag{4.2.22}$$

$$\frac{\partial}{\partial y_n} W = B_n W - W\partial^n. \tag{4.2.23}$$

Here, B_n is a differential operator which is automatically determined by eq. (4.2.23) as

$$B_n = (W\partial^n W^{-1})_+ = (L^n)_+. \tag{4.2.24}$$

Note that the W operator in eqs. (4.2.21)-(4.2.23) has an ambiguity of right multiplication by a constant pseudo-differential operator $C = 1 + c_1\partial^{-1} + c_2\partial^{-2} + \cdots$:

$$W \mapsto WC. \tag{4.2.25}$$

Our next step is to express the operators P and Q in terms of the W operator introduced above. As for P, eqs. (4.2.14) and (4.2.21) immediately give

$$P = W\partial^p W^{-1}. \tag{4.2.26}$$

Then multiplying eq. (4.2.10) by W^{-1} and W from the left and from the right, respectively, we have

$$[\partial^p, W^{-1}QW] = 1. \tag{4.2.27}$$

From this equation we find that $W^{-1}QW$ should have the following form:

$$W^{-1}QW = \frac{1}{p}\, t\,\partial^{-p+1} + \sum_k d_k(y)\partial^k. \tag{4.2.28}$$

Here, the first term on the right-hand side is a special solution of eq. (4.2.27), while the second term is the general solution of the homogeneous version of eq. (4.2.27) and the $d_k(y)$'s are y-dependent constants. In order to find the y-dependence of $W^{-1}QW$, we consider its y derivatives, $\frac{\partial}{\partial y_n}(W^{-1}QW)$. Using the y-evolution equations given by

$$\begin{aligned}
&\frac{\partial}{\partial y_n}W^{-1} = -W^{-1}B_n + \partial^n W^{-1},\\
&\frac{\partial}{\partial y_n}Q = B_n Q - QB_n,\\
&\frac{\partial}{\partial y_n}W = B_n W - W\partial^n,
\end{aligned} \tag{4.2.29}$$

we obtain

$$\frac{\partial}{\partial y_n}(W^{-1}QW) = [\partial^n, W^{-1}QW]. \tag{4.2.30}$$

Then by substituting (4.2.28) in the right-hand side of this equation, we have the following equation:

$$\frac{\partial}{\partial y_n}(W^{-1}QW) = \frac{n}{p}\partial^{n-p} \quad (n = 1, 2, \cdots), \tag{4.2.31}$$

which together with eq. (4.2.28) determines $W^{-1}QW$ as

$$W^{-1}QW = \frac{1}{p}\left(t\partial^{-p+1} + \sum_{k\geq 1} k(y_k - y_k^{(0)})\partial^{k-p} + \lambda\partial^{-p}\right) + \sum_{k\leq -1}\alpha_k\partial^{k-p}, \tag{4.2.32}$$

where $y_k^{(0)}$, λ and α_k are constants. Since the last term on the right-hand side of this equation can be eliminated by an appropriate redefinition of W given by (4.2.25),[1] we finally obtain [26]

$$Q = \frac{1}{p}W\left(t\partial^{-p+1} + \lambda\partial^{-p} + \sum_{k\geq 1} kx_k\partial^{k-p}\right)W^{-1}, \tag{4.2.33}$$

where we have introduced new variables x defined by $x_k = y_k - y_k^{(0)}$. Note that so far we have not used the conditions that P and Q are differential operators. Therefore the general solution of the flows which does not change the order p of the differential operator P is expressed as (4.2.26) and (4.2.33) for the W operator satisfying the

[1]After such redefinition of W, it has no ambiguity any more and hence the corresponding τ function is fixed up to an overall constant.

Sato equation and the conditions $(P)_- = (Q)_- = 0$ [26]:[2]

$$\begin{aligned}
&\frac{\partial}{\partial x_n}W = B_n W - W\partial^n,\\
&(W\partial^p W^{-1})_- = 0,\\
&\left(W\left\{t\partial^{-p+1} + \lambda\partial^{-p} + \sum_{k\geq 1} kx_k\partial^{k-p}\right\}W^{-1}\right)_- = 0.
\end{aligned} \tag{4.2.34}$$

4.3 The String Equation and the Douglas Equation

In this section, we prove that the Douglas equation (4.2.34) written in terms of pseudo-differential operators is equivalent to the following condition on a τ function of the KP hierarchy [26]:

$$\begin{aligned}
\frac{\partial}{\partial x_p}\tau(x) &= \text{const.}\,\tau(x),\\
\mathcal{L}_{-p}\tau(x) &= \text{const.}\,\tau(x).
\end{aligned} \tag{4.3.1}$$

Here, the x_i's $(i = 1, 2, \cdots)$ stand for the time variables of the KP hierarchy and $\mathcal{L}_n$'s are the Virasoro generators in the following form:

$$\mathcal{L}_n = \frac{1}{2}\sum_{k+l=-n} klx_k x_l + \sum_{k-l=-n} kx_k\frac{\partial}{\partial x_l} + \frac{1}{2}\sum_{k+l=n}\frac{\partial}{\partial x_k}\frac{\partial}{\partial x_l}.$$

To prove this, we first rewrite the Douglas equation in terms of an infinite Grassmannian in subsec. 4.3.1. After introducing τ functions of the KP hierarchy in subsec. 4.3.2, we then prove our assertion in subsec. 4.3.3. The eq. (4.3.1) is called *the string equation* since the $W_{1+\infty}$ (or W_p) constraint on the τ function automatically follows from this equation, as will be shown later in this chapter.

Note that the set of equations (4.3.1) is equivalent to the following weaker conditions which contain three additional constants, a, b and c:

$$\begin{aligned}
&\frac{\partial}{\partial x_p}\tau(x) = a\tau(x)\\
&(\mathcal{L}_{-p} + cx_p)\tau(x) = b\tau(x).
\end{aligned} \tag{4.3.2}$$

In fact, if we require eq. (4.3.2), $c = 0$ follows from $[\mathcal{L}_{-p} + cx_p, \frac{\partial}{\partial x_p}]\tau = 0$.

[2]Equations (4.2.34) were already discussed in ref. [27] from a slightly different point of view. We will see later that the constant λ should be equal to $\frac{-p+1}{2}$ in order for eq. (4.2.34) to have a nontrivial solution.

4.3.1 Infinite Dimensional Grassmannian and the KP Hierarchy

To further rewrite the Douglas equation (4.2.34), we summarize the relation between pseudo-differential operators and the infinite dimensional Grassmannian. For more complete description for it, see app. A.

Let H be the linear space of formal Laurent series:

$$H \;=\; \mathbf{C}[z,z^{-1}] \;=\; \{f(z)=\sum_{n\in\mathbf{Z}} a_n z^n\}, \tag{4.3.3}$$

and H_+ be its subspace of formal Taylor series:

$$H_+ \;=\; \mathbf{C}[z] \;=\; \{f(z)=\sum_{n\in\mathbf{Z}_+} a_n z^n\}. \tag{4.3.4}$$

An infinite dimensional Grassmannian usually called the ***universal Grassmann manifold*** (UGM) is defined as the set of the subspaces of H which is linearly isomorphic to H_+:

$$\mathrm{UGM} \equiv \{V\subset H\,|\,V\simeq H_+\}. \tag{4.3.5}$$

For any pseudo-differential operator U, we construct a corresponding subspace V_U of H, *i.e.* an element of the UGM, as follows:

$$\begin{aligned} U \;\mapsto\; V_U &= [U|_{t=t_0},\,(\partial U)|_{t=t_0},(\partial^2 U)|_{t=t_0},\cdots] \\ &= \text{linear space spanned by } (\partial^k U)|_{t=t_0}\text{'s } (k=0,1,2,\cdots). \end{aligned} \tag{4.3.6}$$

Here, the symbol $|_{t=t_0}$ is defined in the following way. Let S be a pseudo-differential operator (*e.g.* $S=\partial^k U$), and write it in the conventional form, $S=\sum_n S_n(t)\partial^n$. Then $S|_{t=t_0}\in H$ is defined by $S|_{t=t_0}=\sum_n S_n(t_0)z^n$. Although in general the mapping (4.3.6) is not injective, it becomes injective if we restrict the defining domain to the set of pseudo-differential operators having a special form like $U=1+\sum_{n=1}^{\infty}u_n(t)\partial^{-n}$. The following lemma is fundamental [26]:

Lemma 4.3.1

Let U be a pseudo-differential operator and V_U the corresponding subspace of H. Then the following two statements are equivalent:

(a) $$\left\{\sum_{m,n\geq 0} c_{m,n} z^m \left(\frac{d}{dz}\right)^n\right\} V_U \subset V_U. \tag{4.3.7}$$

(b) *There exists a differential operator B such that*[3]

$$U\left\{\sum_{m,n\geq 0} c_{m,n}(t-t_0)^n\partial^m\right\} = BU. \tag{4.3.8}$$

[proof]

The following three statements hold for any pseudo-differential operators X and Y and differential operator B:

$$\begin{aligned} &\text{(i)}\;\; X=Y \iff (\partial^k X)|_{t=t_0}=(\partial^k Y)|_{t=t_0} \;\text{ for } {}^{\forall}k\geq 0,\\ &\text{(ii)}\;\; (X(t-t_0)^n\partial^m)|_{t=t_0} = z^m\left(\frac{d}{dz}\right)^n (X|_{t=t_0}),\\ &\text{(iii)}\;\; (\partial^k BY)|_{t=t_0}=\sum_{l\geq 0}\lambda_{k,l}(B)(\partial^l Y)|_{t=t_0}\quad (k\geq 0),\\ &\qquad \text{where } \lambda_{k,l}(B)=\sum_{r=\max\{0,\,k-l\}}^{k}\binom{k}{r} b^{(r)}_{r+l-k}\\ &\qquad \text{for } B=\sum_{n\geq 0} b_n(t)\partial^n=\sum_{n\geq 0}\left(\sum_{m\geq 0}\frac{1}{m!}b_n^{(m)}(t-t_0)^m\right)\partial^n. \end{aligned}$$

Therefore, by applying (i)~(iii) to eq. (4.3.8), we have

$$\begin{aligned} &\text{(b)} \iff {}^{\exists}B \text{ (differential operator) } s.t.\\ &\qquad \sum_{m,n\geq 0} c_{m,n}z^m\left(\frac{d}{dz}\right)^n(\partial^k U)|_{t=t_0} = \sum_{l\geq 0}\lambda_{k,l}(B)(\partial^l U)|_{t=t_0} \;\text{ for } {}^{\forall}k\geq 0. \end{aligned}$$

On the other hand, since V_U is spanned by $(\partial^k U)|_{t=t_0}$ $(k=0,1,2,\cdots)$, we have

$$\begin{aligned} &\text{(a)} \iff {}^{\exists}\mu_{k,l}\ (k,l\geq 0) \; s.t.\\ &\qquad \sum_{m,n\geq 0} c_{m,n}z^m\left(\frac{d}{dz}\right)^n(\partial^k U)|_{t=t_0} = \sum_{l\geq 0}\mu_{k,l}(\partial^l U)|_{t=t_0} \;\text{ for } {}^{\forall}k\geq 0. \end{aligned}$$

[3]Strictly speaking, B should be regarded as a formal differential operator in the sense that it is given by a formal Taylor series:

$$B=\sum_{n\geq 0}\sum_{m\geq 0}\frac{1}{m!}b_n^{(m)}(t-t_0)^m\partial^n.$$

Actually this is sufficient for the proof of the equivalence between eq. (4.2.34) and (4.3.1).

The above two statements are equivalent, since for any set of numbers $\mu_{k,l}$ $(k, l \geq 0)$ we can construct the following formal differential operator B that satisfies $\lambda_{k,l}(B) = \mu_{k,l}$:

$$B = \sum_{n\geq 0}\sum_{m\geq 0} \frac{1}{m!} b_n^{(m)}(t-t_0)^m \partial^n, \quad b_n^{(m)} = \sum_{k=\max\{0,m-n\}}^{m} (-1)^{m-k} \binom{m}{k} \mu_{k,k+n-m}. \quad \blacksquare$$

We thus can reexpress the pseudo-differential form of the Douglas equation (4.2.34) in the form of the infinite dimensional Grassmannian as [26]

$$z^p V_{W(x)} \subset V_{W(x)},$$
$$\left(z^{-p+1}\frac{d}{dz} + \frac{-p+1}{2} z^{-p} + \sum_{k\geq 1} k(x_k + \delta_{k,1}t_0) z^{k-p} + cz^{-p} \right) V_{W(x)} \subset V_{W(x)}. \quad (4.3.9)$$

4.3.2 τ Functions of the KP Hierarchy

In order to express the τ functions in a compact form, we introduce fermionic operators $\psi_n^\dagger$, ψ_m $(n, m \in \mathbf{Z})$ satisfying the following anticommutation relations

$$\{\psi_n^\dagger, \psi_m\} = \delta_{n\,m}, \ \{\psi_n^\dagger, \psi_m^\dagger\} = 0, \ \{\psi_n, \psi_m\} = 0, \quad (4.3.10)$$

and define the zero-particle state $|-\infty\rangle$ as the state that satisfies

$$\psi_n |-\infty\rangle = 0 \quad ({}^\forall n \in \mathbf{Z}). \quad (4.3.11)$$

We then make a correspondence between a vector $\xi(z)$ in $H = \mathbf{C}[z, z^{-1}] = \{f(z) = \sum_{n\in\mathbf{Z}} a_n z^n\}$ and a fermionic operator $\psi[\xi]^\dagger$ in the following way:

$$\xi(z) = \sum_{n\in\mathbf{Z}} \xi_n z^n \ \leftrightarrow \ \psi[\xi]^\dagger = \sum_{n\in\mathbf{Z}} \xi_n \psi_n^\dagger. \quad (4.3.12)$$

Furthermore, we associate a subspace $V = [\xi^{(0)}(z), \xi^{(1)}(z), \cdots]$ of H with a decomposable multi-fermion state $|g\rangle$ via[4]

$$V = [\xi^{(0)}(z), \xi^{(1)}(z), \cdots] \ \mapsto \ |g\rangle = \psi[\xi^{(0)}]^\dagger \psi[\xi^{(1)}]^\dagger \cdots |-\infty\rangle. \quad (4.3.13)$$

[4]A multi-fermion state $|g\rangle$ is called decomposable if it can be expressed in the form of the right-hand side of eq. (4.3.13). This state is nothing but the one that can be written as a Slater determinant.

Obviously this correspondence between the UGM and the set of all decomposable states is one-to-one up to an overall factor. We denote the subspace V associated with the decomposable state $|g\rangle$ by V_g. We further define the vacuum as the state of Dirac sea filled up to $n = 0$ from $n = +\infty$ and denote it by $|0\rangle$:

$$|0\rangle \equiv \psi_0^\dagger \psi_{+1}^\dagger \psi_{+2}^\dagger \cdots |-\infty\rangle. \quad (4.3.14)$$

This state satisfies $\psi_n |0\rangle = 0$ $(n < 0)$ and $\psi_n^\dagger |0\rangle = 0$ $(n \geq 0)$.

The τ functions of the KP hierarchy are now defined as follows. First, we introduce the normal ordering for fermionic operators by

$$:\psi_n^\dagger \psi_m: \equiv \begin{cases} \psi_n^\dagger \psi_m & (n < 0) \\ -\psi_m \psi_n^\dagger & (n \geq 0), \end{cases} \quad (4.3.15)$$

and define the current operators J_n as

$$J_n \equiv \sum_k :\psi_{n+k}^\dagger \psi_k: \quad (n \in \mathbf{Z}), \quad (4.3.16)$$

or equivalently

$$J(z) = \sum_{n\in\mathbf{Z}} J_n z^{-n-1} =: \psi^\dagger(z)\psi(z):, \quad (4.3.17)$$

where $\psi(z) = \sum_{n\in\mathbf{Z}} \psi_n z^n$ and $\psi^\dagger(z) = \sum_{n\in\mathbf{Z}} \psi_n^\dagger z^{-n-1}$.[5] Then, the τ function associated with a decomposable state $|g\rangle$ is defined as a function of infinitely many variables $x = (x_1, x_2, x_3, \cdots)$:[6]

$$\tau(x) = \langle 0 | \exp\left(-\sum_{n\geq 1} x_n J_n \right) |g\rangle. \quad (4.3.18)$$

In particular, for the pseudo-differential operator W in the Sato equation, we introduce the corresponding τ function, following the prescriptions given in (4.3.6) and

[5]These fermion fields satisfy the following OPE:

$$\psi(z)\psi^\dagger(w) \sim \frac{1}{z-w}.$$

[6]Hereafter we will consider only neutral states, that is, $J_0 |g\rangle = 0$.

(4.3.13):

$$\begin{aligned} W \mapsto V_W &= [W|_{t=t_0}, (\partial W)|_{t=t_0}, (\partial^2 W)|_{t=t_0}, \cdots] \\ &\equiv [\xi_W^{(0)}(z), \xi_W^{(1)}(z), \xi_W^{(2)}(z), \cdots] \\ \mapsto |g_W\rangle &= \psi[\xi_W^{(0)}]^\dagger \psi[\xi_W^{(1)}]^\dagger \psi[\xi_W^{(2)}]^\dagger \cdots |-\infty\rangle . \end{aligned}$$

Note that the function $\tau(x)$ given above can be also interpreted as the bosonic coherent representation of the state $|g\rangle$. In fact, if we introduce a free boson $\phi(z)$ via

$$\partial\phi(z) = J(z) = :\psi^\dagger(z)\psi(z): \tag{4.3.19}$$

or conversely

$$\psi^\dagger(z) = :e^{\phi(z)}: \tag{4.3.20}$$

$$\psi(z) = :e^{-\phi(z)}:, \tag{4.3.21}$$

then $\langle 0|\exp(-\sum_{n\geq 1} x_n J_n)$ is nothing but the coherent state of the free boson. Thus, the following relations hold:

$$\langle 0|e^{-\sum_{n\geq 1} x_n J_n} J_m = \begin{cases} -\partial_m \langle 0|e^{-\sum_{n\geq 1} x_n J_n} & (m>0) \\ -|m| x_{|m|} \langle 0|e^{-\sum_{n\geq 1} x_n J_n} & (m<0). \end{cases} \tag{4.3.22}$$

Noting that the $J(x)$ in eq. (4.3.17) is a local fermion bilinear operator, we further introduce another such operator:

$$T(z) = \sum_{n\in Z} L_n z^{-n-2} = \frac{1}{2}(:\partial\psi^\dagger(z)\psi(z): - :\psi^\dagger(z)\partial\psi(z):). \tag{4.3.23}$$

Since $T(z)$ is bosonized into $T(z) = (1/2):(\partial\phi)^2:$, we obtain from (4.3.22)

$$\langle 0|e^{-\sum_{n\geq 1} x_n J_n} L_n = \mathcal{L}_n \langle 0|e^{-\sum_{n\geq 1} x_n J_n}, \tag{4.3.24}$$

where

$$\mathcal{L}_n \equiv \frac{1}{2}\sum_{k+l=-n} kl x_k x_l + \sum_{k-l=-n} k x_k \partial_l + \frac{1}{2}\sum_{k+l=n} \partial_k \partial_l, \tag{4.3.25}$$

satisfying the Virasoro algebra with central charge $c=1$:

$$[\mathcal{L}_n, \mathcal{L}_m] = (n-m)\mathcal{L}_{n+m} + \frac{1}{12}(n^3-n)\delta_{n+m,0}. \tag{4.3.26}$$

[*remark*][25]

Rigorously speaking, the multi-fermion state $|g\rangle$ given in eq. (4.3.13) is, in general, ill-defined. In fact, the τ function for the (2,1) topological gravity has the form (see eq. (3.6.2))

$$\tau(x_1, x_3, 0, 0, \cdots) = \text{const.}\, x_3^{-\frac{1}{24}} \exp[-\frac{x_1^3}{18x_3}], \tag{4.3.27}$$

so that we can not set $x=0$. However, since we can expand it around a generic background source $x = x^0$, the state $|g\rangle$ becomes well-defined if we define the τ function by

$$\tau(x) = \langle 0|e^{-\sum_{n\geq 1}(x_n + x_n^0)J_n}|g\rangle . \tag{4.3.28}$$

Furthermore, if we introduce J_n' by

$$J_n' \equiv e^{\sum_m x_m^0 J_m} J_n e^{-\sum_m x_m^0 J_m} = \begin{cases} J_n & (n>0) \\ J_n - n x_{-n}^0 & (n<0), \end{cases} \tag{4.3.29}$$

then we have

$$\langle 0|e^{-\sum_{n\geq 1}(x_n+x_n^0)J_n} J_m' = \begin{cases} -\partial_m \langle 0|e^{-\sum_{n\geq 1}(x_n+x_n^0)J_n} & (m>0) \\ -|m| x_{|m|} \langle 0|e^{-\sum_{n\geq 1}(x_n+x_n^0)J_n} & (m<0), \end{cases} \tag{4.3.30}$$

which allows us to still regard $\langle 0|\exp(-\sum_{n\geq 1}(x_n + x_n^0)J_n)$ as the bosonic coherent state. Although we will formally set $x^0 = 0$ hereafter, the generalization to the cases $x^0 \neq 0$ is easily carried out by replacing the operator O which acts on the state $|g\rangle$ by

$$O' \equiv e^{\sum_m x_m^0 J_m} O\, e^{-\sum_m x_m^0 J_m}. \tag{4.3.31}$$

For example, as for the Virasoro generators in eq. (4.2.34), it holds that

$$\mathcal{L}_n \tau(x) = \langle 0|e^{-\sum_m (x_m + x_m^0)J_m} L_n' |g\rangle \tag{4.3.32}$$

for

$$L_n' = e^{\sum_m x_m^0 J_m} L_n\, e^{-\sum_m x_m^0 J_m}. \tag{4.3.33}$$

4.3.3 Equivalence between the String Equation and the Douglas Equation

As is shown in the previous subsection, both of the differential operators $\partial/\partial x_n$ and $\mathcal{L}_n$ can be expressed as the mode coefficients of local fermion bilinear operators. In this section, we first investigate how such operator act on a decomposable state $|g\rangle$ [25] and then prove the equivalence between the string equation and the Douglas equation [26].

First we consider fermion bilinear operators of the general form:

$$O = \sum_{n,m} :\psi_n^\dagger O_{nm}\psi_m: . \tag{4.3.34}$$

The crucial point is that bilinear operators can be regarded as one-body operators in terms of the first quantization. In fact, one can construct the one-body operator o that corresponds to a fermion bilinear operator O by taking a commutator with the fermion operators ψ_n:

$$[\psi_n, O] = \sum_m O_{nm}\psi_m, \tag{4.3.35}$$

which means that the one-body operator o maps a one-body wave function $f(z) = \sum_n a_n z^n \in H$ to

$$of(z) \equiv \sum_n (\sum_m O_{nm}a_m) z^n. \tag{4.3.36}$$

In particular, the one-body operators corresponding to the current J_n and the energy-momentum tensor L_n are as follows [25][28]:

$$\begin{aligned} J_n &\leftrightarrow w_n^{(1)} = z^n, \\ L_n &\leftrightarrow w_n^{(2)} = -[z^{n+1}\frac{d}{dz} + \frac{n+1}{2}z^n]. \end{aligned} \tag{4.3.37}$$

Furthermore, if we define the τ function as (4.3.28), then this expression for w_n must be replaced by

$$w_n' = \exp(\sum_m x_m^0 z^m)\, w_n \exp(-\sum_m x_m^0 z^m), \tag{4.3.38}$$

that is,

$$\begin{aligned} w_n'^{(1)} &= z^n, \\ w_n'^{(2)} &= -[z^{n+1}\frac{d}{dz} + \frac{n+1}{2}z^n + \sum_{k\geq 1} k x_k^0 z^{k+n}]. \end{aligned} \tag{4.3.39}$$

Conversely, if a one-body operator o is given, we can construct a fermion bilinear operator O uniquely up to an ambiguity of additive constant caused by the operator ordering. Furthermore, if fermion bilinear operators O_1 and O_2 correspond to one-body operators o_1 and o_2, respectively, then the commutator of O_1 and O_2 corresponds to the commutator of o_1 and o_2:

$$\begin{aligned} & O_1 \leftrightarrow o_1, \quad O_2 \leftrightarrow o_2 \\ \Longrightarrow\ & [O_1, O_2] \leftrightarrow [o_1, o_2]. \end{aligned} \tag{4.3.40}$$

Thus we have [25]

Lemma 4.3.1

If a set of one-body operator forms a Lie algebra w, then the set of corresponding fermion bilinear operators forms a central extension of w.

Furthermore, for any value of ϵ the action of $\exp(\epsilon O)$ on a decomposable multi-fermion state $|g\rangle$ is represented over the UGM as

$$e^{\epsilon O}|g\rangle \leftrightarrow e^{\epsilon o}V_g = [e^{\epsilon o}\xi^{(0)}(z), e^{\epsilon o}\xi^{(1)}(z), \cdots], \tag{4.3.41}$$

where $V_g = [\xi^{(0)}(z), \xi^{(1)}(z), \cdots]$ is the subspace of H corresponding to $|g\rangle$. From this fact, it is obvious that the following lemma holds [25]:

Lemma 4.3.2

$$O|g\rangle = \text{const.}|g\rangle \iff oV_g \subset V_g \tag{4.3.42}$$

This lemma will be frequently used below.

As a corollary of Lemma 4.3.2, the string equation (4.3.1)

$$\begin{aligned} \partial_p \tau(x) &= \text{const.}\,\tau(x) \\ \mathcal{L}_{-p}\tau(x) &= \text{const.}\,\tau(x), \end{aligned}$$

or equivalently

$$\begin{aligned} J_p \,|\, g\,\rangle &= \text{const.}\,|\, g\,\rangle && (4.3.43) \\ L_{-p}\,|\, g\,\rangle &= \text{const.}\,|\, g\,\rangle\,, && (4.3.44) \end{aligned}$$

is proved to be equivalent to [26]

$$\begin{aligned} w_p^{(1)} V_g &= z^p V_g \,\subset\, V_g, && (4.3.45) \\ w_{-p}^{(2)} V_g &= -[z^{-p+1}\frac{d}{dz} + \frac{-p+1}{2} z^{-p}]\, V_g \,\subset\, V_g. && (4.3.46) \end{aligned}$$

Moreover, if we consider the τ function with the background $x_k^0 = x_k + \delta_{k,1} t_0$, then from eqs. (4.3.39) we conclude that the string equation (4.3.1) is equivalent to the Douglas equation in terms of the infinite dimensional Grassmannian (4.3.9) [26]:

$$\begin{aligned} & z^p V_g \subset V_g, \\ & \left(z^{-p+1}\frac{d}{dz} + \frac{-p+1}{2} z^{-p} + \sum_{k\geq 1} k(x_k + \delta_{k,1} t_0) z^{k-p} + c z^{-p} \right) V_g \subset V_g, \qquad (4.3.47) \end{aligned}$$

where we have added the term cz^{-p} in the second eq. using the freedom shown in eq. (4.3.2). This is nothing but the assertion we promised to prove.

4.4 $W_{1+\infty}$ Algebra

Before proving the $W_{1+\infty}$ constraint on the τ function which satisfies the string equation (4.3.1), we introduce the $w_{1+\infty}$ algebra and its central extension, the $W_{1+\infty}$ algebra, pointing out that the KP hierarchy has them as a fundamental symmetry. The following three sections completely follow ref. [25].

As we have seen in the previous section, fermion bilinear operators such as the J_n's and the L_n's play essential roles in the two-dimensional gravity. We call operators

$W(z)$ local fermion bilinear of spin k if they have the following form:

$$W(z) = \sum_{j=0}^{k-1} c_j : \partial^j \psi^\dagger(z) \partial^{k-1-j}\psi(z) :, \qquad (4.4.1)$$

where c_j are complex numbers. In particular, $J(z)$ and $T(z)$ are local fermion bilinear operators of spin 1 and 2, respectively:

$$\begin{aligned} J(z) &= \sum_{n\in\mathbf{Z}} J_n z^{-n-1} =: \psi^\dagger(z)\psi(z) :, \\ T(z) &= \sum_{n\in\mathbf{Z}} L_n z^{-n-2} = \frac{1}{2}(: \partial\psi^\dagger(z)\psi(z) : - : \psi^\dagger(z)\partial\psi(z) :). \qquad (4.4.2) \end{aligned}$$

We also call W_n local fermion bilinear of spin k if it is the n-th mode of such an operator:

$$W(z) = \sum_n W_n z^{-n-k}. \qquad (4.4.3)$$

For these local fermion bilinear operators, the corresponding one-body operators become local differential operators with respect to z [25]. In fact, for an operator having the form

$$\begin{aligned} W_n &= \oint \frac{dw}{2\pi i} w^{n+k-1} W(w), \\ W(w) &=: \partial^j \psi^\dagger(w) \partial^{k-1-j}\psi(w) :, \qquad (4.4.4) \end{aligned}$$

the commutator with $\psi(z)$ gives

$$\begin{aligned} [\psi(z), W_n] &= -\oint_z \frac{dw}{2\pi i} w^{n+k-1} \psi(z) : \partial^j\psi^\dagger(w)\partial^{k-1-j}\psi(w) : \\ &= (-1)^j (\frac{d}{dz})^j z^{n+k-1} (\frac{d}{dz})^{k-1-j} \psi(z). \qquad (4.4.5) \end{aligned}$$

This equation indicates that the corresponding one-body operator is given by[7]

$$\begin{aligned} w_n &= (-1)^j (\frac{d}{dz})^j z^{n+k-1} (\frac{d}{dz})^{k-1-j} \\ &= (-1)^j \sum_{l=0}^{j} \binom{j}{l} [n+k-1]_l \, z^{n+k-1-l} (\frac{d}{dz})^{k-1-l}, \qquad (4.4.6) \end{aligned}$$

[7] If we define the τ function as (4.3.28), then this expression for w_n must be replaced by $w_n' = \exp(\sum_m x_m^0 z^m)\, w_n \exp(-\sum_m x_m^0 z^m)$.

where $[m]_n \equiv m!/(m-n)!$. In particular, the one-body operators corresponding to the current J_n and the energy-momentum tensor L_n are given as in eqs. (4.3.37):

$$\begin{aligned} J_n &\leftrightarrow w_n^{(1)} = z^n, \\ L_n &\leftrightarrow w_n^{(2)} = -[z^{n+1}\frac{d}{dz} + \frac{n+1}{2}z^n]. \end{aligned}$$

If we take into account all modes of all the local fermion bilinear operators, they generate a Lie algebra called the $W_{1+\infty}$ algebra. From eq. (4.4.6) it is easily seen that the corresponding one-body operators form a Lie algebra of differential operators that is spanned by

$$z^n(\frac{d}{dz})^l \quad (n \in \mathbf{Z},\ l \in \mathbf{Z}_+ = \{0, 1, 2, \cdots\}), \tag{4.4.7}$$

which we call the $w_{1+\infty}$ algebra. Obviously the $W_{1+\infty}$ algebra is a central extension of the $w_{1+\infty}$ algebra. In fact, the central charge c of the Virasoro subalgebra of $W_{1+\infty}$ is 1, whereas that of $w_{1+\infty}$ is 0.

As we have seen in the previous section, the string equation (4.3.1):

$$\begin{aligned} \partial_p \tau(x) &= \text{const.}\,\tau(x), \\ \mathcal{L}_{-p}\tau(x) &= \text{const.}\,\tau(x), \end{aligned}$$

is equivalent in terms of the infinite Grassmannian to

$$w_p^{(1)} V_g = z^p V_g \subset V_g, \tag{4.4.8}$$

$$w_{-p}^{(2)} V_g = -[z^{-p+1}\frac{d}{dz} + \frac{-p+1}{2}z^{-p}]\, V_g \subset V_g. \tag{4.4.9}$$

Therefore the subspace V_g is invariant under any differential operator that is constructed by repeatedly taking products and linear combinations of $w_p^{(1)}$ and $w_{-p}^{(2)}$. In other words, if we define $r^+(p)$ as the associative subalgebra of differential operators that is generated by $w_p^{(1)}$ and $w_{-p}^{(2)}$, we have

$$o V_g \subset V_g \quad \text{for } {}^\forall o \in r^+(p). \tag{4.4.10}$$

Since the commutator of $w_p^{(1)}$ and $w_{-p}^{(2)}$ is a c-number

$$[w_p^{(1)}, \frac{1}{p} w_{-p}^{(2)}] = 1, \tag{4.4.11}$$

and the differential operators $(w_p^{(1)})^n (w_{-p}^{(2)})^l$ $(n,\ l = 0, 1, 2, \cdots)$ are all linearly independent to each other, any element of $r^+(p)$ is uniquely expressed as

$$\sum_{n,l=0}^{\infty} c_{nl} (w_p^{(1)})^n (w_{-p}^{(2)})^l, \tag{4.4.12}$$

where the c_{nl}'s are complex numbers. Thus we have the following theorem [25][28]:

Theorem 4.4.1

Let $\tau(x)$ be a τ function of the KP hierarchy that satisfies the string equations of the (p, q) gravity (4.3.43) and (4.3.44). Then the corresponding element V_g of the UGM satisfies

$$r^+(p) V_g = V_g, \tag{4.4.13}$$

where $r^+(p) = \{\sum_{n,l=0}^{\infty} c_{nl}(w_p^{(1)})^n (w_{-p}^{(2)})^l,\ c_{nl} \in \mathbf{C}\}$.

As is shown in the next section, we can reinterpret eq. (4.4.13) in terms of the τ function by using Lemma 4.3.2 again. There we will see that the structure of the $W_{1+\infty}$ algebra arises in a natural way. Here, before going back to the τ function, we analyze the Lie algebraic structures of the set $r^+(p)$ of one-body operators that appears in eq. (4.4.13).

First we introduce the following notations for $p \geq 2$:

$$r = w_{1+\infty} = \{\sum_{n\in\mathbf{Z}} \sum_{l\in\mathbf{Z}_+} c_{nl} z^n (-\frac{d}{dz})^l\}, \tag{4.4.14}$$

$$r^+ = w_{1+\infty}^+ = \{\sum_{n\in\mathbf{Z}_+} \sum_{l\in\mathbf{Z}_+} c_{nl} z^n (-\frac{d}{dz})^l\}, \tag{4.4.15}$$

$$r(p) = w_{1+\infty}(p) = \{\sum_{n\in\mathbf{Z}} \sum_{l\in\mathbf{Z}_+} c_{nl} (w_p^{(1)})^n (\frac{1}{p} w_{-p}^{(2)})^l\}, \tag{4.4.16}$$

$$r^+(p) = w_{1+\infty}^+(p) = \{\sum_{n\in\mathbf{Z}_+} \sum_{l\in\mathbf{Z}_+} c_{nl} (w_p^{(1)})^n (\frac{1}{p} w_{-p}^{(2)})^l\}. \tag{4.4.17}$$

Here, r, r^+, $r(p)$ and $r^+(p)$ are, as a set, identical to $w_{1+\infty}$, $w_{1+\infty}^+$, $w_{1+\infty}(p)$ and $w_{1+\infty}^+(p)$, respectively. We introduce, however, different symbols for them in order to indicate whether we regard them as associative algebras or Lie algebras. In other words, when we call them r, we consider not only commutators but also products as differential operators.[8] In addition to the trivial relations

$$\begin{aligned} & r^+ \subset r, \quad r^+(p) \subset r(p) \subset r \quad \text{(as associative subalgebras)}, \\ & w_{1+\infty}^+ \subset w_{1+\infty}, \quad w_{1+\infty}^+(p) \subset w_{1+\infty}(p) \subset w_{1+\infty} \quad \text{(as Lie subalgebras)}, \end{aligned} \tag{4.4.18}$$

we have the following isomorphisms, which indicate that the $w_{1+\infty}$ algebra has an infinite-fold self-similar structure [25]:

Lemma 4.4.2

$$\begin{aligned} & r(p) \cong r, \quad r^+(p) \cong r^+ \quad \textit{(as associative algebras)}, \\ & w_{1+\infty}(p) \cong w_{1+\infty}, \quad w_{1+\infty}^+(p) \cong w_{1+\infty}^+ \quad \textit{(as Lie algebras)}. \end{aligned} \tag{4.4.19}$$

[proof]

As is clear from the definition, r is generated by $w_n^{(1)} = z^n$ ($n \in \mathbf{Z}$) and $w_{-1}^{(2)} = -d/dz$, and the structure of r is completely specified by the following relations among them:

$$w_n^{(1)} = (w_1^{(1)})^n, \tag{4.4.20}$$

$$[w_1^{(1)}, w_{-1}^{(2)}] = 1. \tag{4.4.21}$$

On the other hand, $r(p)$ is generated by $w_{np}^{(1)} = z^{np}$ ($n \in \mathbf{Z}$) and $w_{-p}^{(2)}$ with the relations

$$w_{np}^{(1)} = (w_p^{(1)})^n, \tag{4.4.22}$$

$$[w_p^{(1)}, \frac{1}{p} w_{-p}^{(2)}] = 1. \tag{4.4.23}$$

Therefore $r(p)$ is isomorphic to r as an associative algebra through the following mapping:

$$\begin{aligned} \bar{w}_1^{(1)} \equiv w_p^{(1)} & \mapsto w_1^{(1)}, \\ \bar{w}_{-1}^{(2)} \equiv \frac{1}{p} w_{-p}^{(2)} & \mapsto w_{-1}^{(2)}. \end{aligned} \tag{4.4.24}$$

[8] In fact, the so-called lone-star product of the $W_{1+\infty}$ algebra [29] is nothing but the usual product in r, when we translate the set of fermion bilinear operators, $W_{1+\infty}$, into the set of corresponding one-body operators, $w_{1+\infty}$.

Furthermore, since $r^+(p)$ is generated by $w_p^{(1)}$ and $w_{-p}^{(2)}$, and r^+ is generated by $w_1^{(1)}$ and $w_{-1}^{(2)}$, these two associative algebras are isomorphic under the above mapping. ∎

To express this isomorphism more explicitly, we note the following relation:

$$\begin{aligned} \bar{w}_{-1}^{(2)} &= \frac{1}{p} w_{-p}^{(2)} = -\frac{1}{p}\left(z^{-p+1}\frac{d}{dz} + \frac{-p+1}{2} z^{-p} \right) \\ &= z^{(p-1)/2} \left(-\frac{d}{d\lambda} \right) z^{-(p-1)/2}, \end{aligned} \tag{4.4.25}$$

$$\begin{aligned} \bar{w}_1^{(1)} &= w_p^{(1)} = z^p \\ &= z^{(p-1)/2} (\lambda) z^{-(p-1)/2}, \end{aligned} \tag{4.4.26}$$

where $\lambda = z^p$. Then we see that the $w_{1+\infty}(p)$ algebra and the $w_{1+\infty}$ algebra are related via

$$w_{1+\infty}(p) = z^{(p-1)/2} \left(w_{1+\infty} |_{\text{with } z \text{ replaced by } \lambda} \right) z^{-(p-1)/2}. \tag{4.4.27}$$

In order to see how $w_{1+\infty}(p)$ is imbedded in $w_{1+\infty}$ more explicitly, we introduce the following basis of the $w_{1+\infty}$ algebra:

$$w_n^{(k)} = (-1)^{k-1} \sum_{l=0}^{k-1} \frac{1}{l!} \frac{([k-1]_l)^2}{[2k-2]_l} [n+k-1]_l \, z^{n+k-1-l} (\frac{d}{dz})^{k-1-l}, \tag{4.4.28}$$

which corresponds to the standard basis of the $W_{1+\infty}$ algebra [29]:[9]

$$\begin{aligned} W^{(k)}(z) &= \sum_n W_n^{(k)} z^{-n-k} \\ &= \frac{(k-1)!}{2^{k-1}(2k-3)!!} \sum_{j=0}^{k-1} (-1)^j \binom{k-1}{j}^2 : \partial^{k-1-j} \psi^\dagger(z) \partial^j \psi(z) : . \end{aligned} \tag{4.4.29}$$

Then by using eq. (4.4.27) the corresponding basis of $w_{1+\infty}(p)$ is expressed as [25]

$$\bar{w}_n^{(k)} = (-1)^{k-1} \sum_{l=0}^{k-1} \frac{1}{l!} \frac{([k-1]_l)^2}{[2k-2]_l} [n+k-1]_l \, z^{(p-1)/2} \lambda^{n+k-1-l} (\frac{d}{d\lambda})^{k-1-l} z^{-(p-1)/2}. \tag{4.4.30}$$

[9] Here, $(-1)!! \equiv 1$

The right-hand side of this equation can be expressed as a linear combination of the $w_n^{(k)}$'s. For example,

$$\tilde{w}_n^{(1)} = w_{np}^{(1)}, \quad (4.4.31)$$

$$\tilde{w}_n^{(2)} = \frac{1}{p} w_{np}^{(2)}, \quad (4.4.32)$$

$$\tilde{w}_n^{(3)} = \frac{1}{p^2}\{w_{np}^{(3)} + \frac{1}{12}(p^2-1)w_{np}^{(1)}\}, \quad (4.4.33)$$

$$\tilde{w}_n^{(4)} = \frac{1}{p^3}\{w_{np}^{(4)} + \frac{7}{20}(p^2-1)w_{np}^{(2)}\}. \quad (4.4.34)$$

As a final remark in this section, we point out the following properties of $w_{1+\infty}^+$ and $w_{1+\infty}^+(p)$, which we use in the next section [25]:

Lemma 4.4.3

Any element of $w_{1+\infty}^+$ can be expressed as a commutator of two elements of $w_{1+\infty}^+$. The same statement holds for $w_{1+\infty}^+(p)$.

[proof]

For any element of $w_{1+\infty}^+$, we have

$$\sum_{k,l\in \mathbf{Z}_+} c_{k,l} z^k (\frac{d}{dz})^l = [\frac{d}{dz}, \sum_{k,l\in \mathbf{Z}_+} \frac{c_{k,l}}{k+1} z^{k+1} (\frac{d}{dz})^l], \quad (4.4.35)$$

which proves the assertion for $w_{1+\infty}^+$. The same assertion holds for $w_{1+\infty}^+(p)$ because it is isomorphic to $w_{1+\infty}^+$. ∎

4.5 $W_{1+\infty}$ Constraint

In the preceding sections we have investigated the structure of the string equation in terms of the UGM. In this section we will rewrite eq. (4.4.13) as a set of differential equations for $\tau(x)$. This section also follows ref. [25] completely.

As we have seen in the previous section, if a set of one-body operators is given, we can construct a corresponding set of fermion bilinear operators up to the operator ordering ambiguities. One way to fix the ambiguities is to introduce the usual normal

ordering for the fermion operators (4.3.15). In fact, the $W_{1+\infty}$ algebra can be regarded as the set of fermion bilinear operators constructed from $w_{1+\infty}$ through this normal ordering. Similarly, we define $W_{1+\infty}^+$ as the subset of $W_{1+\infty}$ that is constructed from $w_{1+\infty}^+$ by using the same normal ordering. As is clear from eq. (4.4.6), $W_{1+\infty}^+$ is spanned by $W_n^{(k)}$'s satisfying $n \geq -k+1$. A crucial property of the $W_{1+\infty}^+$ is the following [25]:

Lemma 4.5.1

$W_{1+\infty}^+$ forms a Lie algebra without a central term. In other words, $W_{1+\infty}^+$ closes under the commutator.

[proof]

Since $w_{1+\infty}^+$ is a Lie algebra, it is obvious that $W_{1+\infty}^+$ is a Lie algebra with possible central terms. In order to show that the central terms vanish, we consider the c-number term of the operator product expansion between fermion bilinear operators $W^{(k)}(z)$ and $W^{(l)}(w)$ of spin k and l, respectively:

$$W^{(k)}(z)W^{(l)}(w) = \text{const.}\ \frac{1}{(z-w)^{k+l}} + (\text{fermion bilinear operators}), \quad (4.5.1)$$

which means that the c-number term of the commutator $[W_n^{(k)}, W_m^{(l)}]$ is given by

$$\text{const.}\ [n+k-1]_{k+l-1}\, \delta_{n+m,0} = \text{const.}\ (n+k-1)\cdots(n-l+1)\, \delta_{n+m,0}. \quad (4.5.2)$$

Then it is easy to see that one of the factors in eq. (4.5.2) vanishes if $n \geq -k+1$ and $m \geq -l+1$. Thus we find that $W_{1+\infty}^+$ closes under the commutator. ∎

We then define $W_{1+\infty}(p)$ and $W_{1+\infty}^+(p)$ as the sets of fermion bilinear operators that are constructed from $w_{1+\infty}(p)$ and $w_{1+\infty}^+(p)$, respectively. Here again we have the operator ordering ambiguities for the 0-th modes of the fermion bilinear operators. We can, however, show the following [25]:

Lemma 4.5.2

There exists such a proper definition of the 0-th modes that $W_{1+\infty}^+(p)$ closes under the commutator. Furthermore, any element of $W_{1+\infty}^+(p)$ can be expressed as a commutator of two elements of $W_{1+\infty}^+(p)$.

[proof]

The isomorphisms (4.4.27) between $w_{1+\infty}$ and $w_{1+\infty}(p)$ can be regarded as a conformal mapping from z to $\lambda = z^p$:

$$\psi'(\lambda) = \left(\frac{d\lambda}{dz}\right)^{-1/2} \psi(z). \tag{4.5.3}$$

Therefore, if we fix the operator ordering by introducing the new normal ordering as the subtraction of the singular part in the (p-sheeted) λ plane, such as

$$\overset{\circ}{\circ}\psi'^{\dagger}(\lambda')\psi'(\lambda)\overset{\circ}{\circ} = \psi'^{\dagger}(\lambda')\psi'(\lambda) - \frac{1}{\lambda' - \lambda}, \tag{4.5.4}$$

then the structure of the c-number terms of the operator product expansions for $W_{1+\infty}(p)$ in the λ plane is exactly the same as that of $W^+_{1+\infty}$ in the z plane. Thus the same argument as in Lemma 4.4.3 leads to the closedness of $W^+_{1+\infty}(p)$ under the commutator. The latter assertion follows immediately from this fact and Lemma 4.4.3. ∎

As is well known, the difference between the λ plane normal ordering and the usual normal ordering can be calculated as the Schwarzian terms associated with the transformation $z \mapsto \lambda = z^p$. For example, the generators of the $W_{1+\infty}(p)$ corresponding to (4.4.31)-(4.4.34) take the following forms:

$$\begin{aligned}
\bar{W}^{(1)}_n &= W^{(1)}_{np}, && (4.5.5)\\
\bar{W}^{(2)}_n &= \frac{1}{p}\{W^{(2)}_{np} + \frac{1}{24}(p^2-1)\delta_{n,0}\}, && (4.5.6)\\
\bar{W}^{(3)}_n &= \frac{1}{p^2}\{W^{(3)}_{np} + \frac{1}{12}(p^2-1)W^{(1)}_{np}\}, && (4.5.7)\\
\bar{W}^{(4)}_n &= \frac{1}{p^3}\{W^{(4)}_{np} + \frac{7}{20}(p^2-1)W^{(2)}_{np} + \frac{7}{960}(p^2-1)^2\delta_{n,0}\}. && (4.5.8)
\end{aligned}$$

Note the appearance of the additional c-number correction terms compared to (4.4.31)-(4.4.34). It is obvious from the construction that they satisfy the commutation relations of the $W_{1+\infty}$ algebra with central charge $c = p$.

After the rather lengthy preparation given above, we can finally prove that a τ function of the p-reduced KP hierarchy satisfies the vacuum condition of the $W_{1+\infty}(p)$ algebra when it obeys the string equation (4.3.1):

$$\begin{aligned}
\partial_p\tau(x) &= \text{const.}\,\tau(x), && (4.5.9)\\
\mathcal{L}_{-p}\tau(x) &= \text{const.}\,\tau(x), && (4.5.10)
\end{aligned}$$

or equivalently

$$\begin{aligned}
J_p\,|\,g\,\rangle &= \text{const.}\,|\,g\,\rangle, && (4.5.11)\\
L_{-p}\,|\,g\,\rangle &= \text{const.}\,|\,g\,\rangle. && (4.5.12)
\end{aligned}$$

In Theorem 4.4.1, we found that these equations are equivalent to

$$r^+(p)V_g = w^+_{1+\infty}(p)V_g = V_g. \tag{4.5.13}$$

By using Lemma 4.3.2 again, this means that $|\,g\,\rangle$ is a simultaneous eigenstate of $W^+_{1+\infty}(p)$:

$$O\,|\,g\,\rangle = \text{const.}\,|\,g\,\rangle \quad \text{for } {}^{\forall}O \in W^+_{1+\infty}(p). \tag{4.5.14}$$

Furthermore, Lemma 4.5.2 asserts that all of these constants vanish. In fact, any element O of $W^+_{1+\infty}(p)$ can be expressed as

$$O = [O_1, O_2], \quad O_1, O_2 \in W^+_{1+\infty}(p). \tag{4.5.15}$$

Therefore, we have $O\,|\,g\,\rangle = O_1O_2\,|\,g\,\rangle - O_2O_1\,|\,g\,\rangle = 0$, because $|\,g\,\rangle$ is a simultaneous eigenstate of $W^+_{1+\infty}(p)$.

We thus have proved the following theorem [25][28].

Theorem 4.5.3

Let $\tau(x)$ be a τ function of the KP hierarchy that satisfies the string equation (4.5.9) and (4.5.10). Then the decomposable fermion state $|\,g\,\rangle$ corresponding to the τ function satisfies

$$O\,|\,g\,\rangle = 0 \quad \textit{for } {}^{\forall}O \in W^+_{1+\infty}(p), \tag{4.5.16}$$

that is,

$$\bar{W}^{(k)}_n\,|\,g\,\rangle = 0 \quad (k = 1, 2, \cdots; n \geq -k+1). \tag{4.5.17}$$

This theorem can be restated in terms of bosons as follows [25]. By using the bosonization rule (4.3.19)-(4.3.21), the normal ordering of two fermion operators

$$:\partial^k\psi^\dagger(w)\partial^l\psi(z): \tag{4.5.18}$$

is expanded in powers of $w-z$ as

$$\begin{aligned}
&=\partial_w^k\partial_z^l\left(\frac{1}{w-z}:e^{\phi(w)-\phi(z)}:-\frac{1}{w-z}\right)\\
&=\sum_{j=1}^{\infty}\sum_{r=0}^{j-1-k}(-1)^r\binom{l}{r}\frac{1}{j\cdot(j-1-k-r)!}(w-z)^{j-1-k-r}\partial^{l-r}P^{(j)}(z),
\end{aligned} \tag{4.5.19}$$

where $P^{(j)}(z)$ is defined by

$$P^{(j)}(z)\equiv :e^{-\phi(z)}\partial_z^j e^{\phi(z)}:, \tag{4.5.20}$$

and $:\ :$ stands for the normal ordering for bosonic operators. Considering the coefficient of $(w-z)^0$ in eq. (4.5.19), we have

$$:\partial^k\psi^\dagger(z)\partial^l\psi(z):=\sum_{j=k+1}^{k+l+1}\frac{1}{j}(-1)^{j-1-k}\binom{l}{j-1-k}\partial^{k+l-j+1}P^{(j)}(z). \tag{4.5.21}$$

Thus, by substituting this expression into eq. (4.4.29), we obtain the bosonic realization of the $W_{1+\infty}$ algebra [25]:

$$W^{(k)}(z)=\sum_{l=0}^{k-1}\frac{(-1)^l}{(k-l)\cdot l!}\frac{([k-1]_l)^2}{[2k-2]_l}\partial^l P^{(k-l)}(z), \tag{4.5.22}$$

the first few of which are

$$\begin{aligned}
W^{(1)}(z) &= J(z), && (4.5.23)\\
W^{(2)}(z) &= \frac{1}{2}:J(z)^2:, && (4.5.24)\\
W^{(3)}(z) &= \frac{1}{3}:J(z)^3:, && (4.5.25)\\
W^{(4)}(z) &= \frac{1}{4}[:J(z)^4:+\frac{2}{5}:J(z)\partial^2J(z):-\frac{3}{5}:(\partial J(z))^2:]. && (4.5.26)
\end{aligned}$$

Then by using the equations such as (4.5.5)-(4.5.8), the generators of $W_{1+\infty}(p)$ are expressed as follows:

$$\bar{W}_n^{(1)} = J_{np}, \tag{4.5.27}$$

$$\begin{aligned}
p\bar{W}_n^{(2)} &= \frac{1}{2}\sum_{a+b=np}:J_aJ_b:+\frac{1}{24}(p^2-1)\delta_{n,0}, && (4.5.28)\\
p^2\bar{W}_n^{(3)} &= \frac{1}{3}\sum_{a+b+c=np}:J_aJ_bJ_c:+\frac{1}{12}(p^2-1)J_{np}, && (4.5.29)\\
p^3\bar{W}_n^{(4)} &= \frac{1}{4}\sum_{a+b+c+d=np}:J_aJ_bJ_cJ_d:-\frac{1}{4}\sum_{a+b=np}(a+1)(b+1):J_aJ_b:\\
&\quad+\frac{1}{20}(np+2)(np+3)\sum_{a+b=np}:J_aJ_b:+\frac{7}{40}(p^2-1)\sum_{a+b=np}:J_aJ_b:\\
&\quad+\frac{7}{960}(p^2-1)^2\delta_{n,0}, && (4.5.30)\\
&\vdots
\end{aligned}$$

Hence, eq. (4.5.17) is rewritten as a set of differential equations for the τ function:

$$\bar{\mathcal{W}}_n^{(k)}\tau(x)=0 \quad (k=1,2,3,\cdots;n\geq -k+1), \tag{4.5.31}$$

where the differential operator $\bar{\mathcal{W}}_n^{(k)}$ is obtained by replacing all the J_n's in the $\bar{W}_n^{(k)}$ by

$$\mathcal{J}_n\equiv\begin{cases}-\partial_n & (n>0)\\ -|n|x_{|n|} & (n<0).\end{cases} \tag{4.5.32}$$

Note that the first equation $-\bar{\mathcal{W}}_n^{(1)}\tau(x)=\frac{\partial}{\partial x_{np}}\tau(x)=0$ shows that the $\tau(x)$ is a τ function of the p-reduced KP hierarchy [25]. As was discussed in sec. 3.7 it is expected that a function $\tau(x)$ satisfying eq. (4.5.31) automatically becomes a τ function of the KP hierarchy, since eq. (4.5.31) determines the function $\tau(x)$ completely at least for the case of topological fields.

4.6 W_p Constraint

We have shown that a τ function of the KP hierarchy under the conditions (4.5.9) and (4.5.10) is a τ function of the p-reduced KP hierarchy which satisfies the vacuum condition of the $W_{1+\infty}(p)$. However, in the expressions (4.5.30)-(4.5.32) there appear redundant variables for the p-reduced KP hierarchy, that is, J_{np} ($n\in\mathbf{Z}$). In this section we show that after the elimination of these redundant variables the $W_{1+\infty}$

algebra with central charge $c = p$ is reduced to the W_p algebra with $c = p-1$. This section also follows ref. [25] completely.

As we have seen in the previous section, the generators of $W_{1+\infty}(p)$ have simple forms when they are expressed on the p-sheeted λ plane which is the image space of the conformal transformation $z \mapsto \lambda = z^p$. More explicitly, we first define the operators $W_n^{(k)}(\lambda)$ on the p-sheeted λ plane by eq. (4.5.22) with z replaced by λ [25]:

$$W^{(k)}(\lambda) = \sum_{l=0}^{k-1} \frac{(-1)^l}{(k-l)\cdot l!} \frac{([k-1]_l)^2}{[2k-2]_l} \partial_\lambda^l P^{(k-l)}(\lambda), \tag{4.6.1}$$

$$P^{(j)}(\lambda) = {}^\circ_\circ e^{-\phi(\lambda)} \partial_\lambda^j e^{\phi(\lambda)} {}^\circ_\circ = {}^\circ_\circ (\partial_\lambda + J(\lambda))^j {}^\circ_\circ\, 1. \tag{4.6.2}$$

Here, $\partial_\lambda \phi(\lambda) = J(\lambda) = \frac{1}{p}\sum_n \lambda^{-n/p-1} J_n$, and ${}^\circ_\circ\ {}^\circ_\circ$ stands for the "minimal" normal ordering on the p-sheeted λ plane, by which we mean the following procedure:

$$\begin{aligned}
{}^\circ_\circ J(\lambda) {}^\circ_\circ &= J(\lambda), \\
{}^\circ_\circ J(\lambda)^2 {}^\circ_\circ &= \lim_{\lambda' \to \lambda} \left\{ J(\lambda')J(\lambda) - \frac{1}{(\lambda' - \lambda)^2} \right\}, \\
{}^\circ_\circ J(\lambda)^3 {}^\circ_\circ &= \lim_{\substack{\lambda'' \to \lambda \\ \lambda' \to \lambda}} \{ J(\lambda'')J(\lambda')J(\lambda) \\
&\qquad - \frac{1}{(\lambda'' - \lambda')^2} J(\lambda) - \frac{1}{(\lambda'' - \lambda)^2} J(\lambda') - \frac{1}{(\lambda' - \lambda)^2} J(\lambda'') \right\}, \\
&\ \vdots
\end{aligned}$$

Since we are considering the p-sheeted λ plane, each value of λ corresponds to p different points, which we denote by $\lambda_1, \cdots, \lambda_p$. Then the generators of $W_{1+\infty}(p)$ are expressed as

$$\bar{W}^{(k)}(\lambda) = \sum_n \bar{W}_n^{(k)} \lambda^{-n-1} = W^{(k)}(\lambda_1) + W^{(k)}(\lambda_2) + \cdots + W^{(k)}(\lambda_p). \tag{4.6.3}$$

Note that this expression gives a single-valued function of λ because the right-hand

side is invariant under the transformation: $\lambda \mapsto \exp[2\pi i]\lambda$, which generates a cyclic permutation of the λ_i's.

In order to investigate further the structure of the $W_{1+\infty}(p)$ constraint, we introduce the elementary symmetric polynomials of $J(\lambda_1), \cdots, J(\lambda_p)$ as [25]

$$S^{(k)}(\lambda) = \sum_{1 \le i_1 < i_2 < \cdots < i_k \le p} J(\lambda_{i_1}) \cdots J(\lambda_{i_k}) \quad (1 \le k \le p). \tag{4.6.4}$$

Here, the ordering of the operators on the right-hand side need not be specified because we have for two different points λ and λ' on the p-sheeted λ plane

$$[J(\lambda'), J(\lambda'')] = 0 \quad \text{if } |\lambda'| = |\lambda''|. \tag{4.6.5}$$

Furthermore it is apparent that the $S^{(k)}$'s are single-valued functions of λ for the same reason as in eq. (4.6.3). Next we introduce another type of product for two local operators on the single-sheeted λ plane as

$$(O_1(\lambda),\, O_2(\lambda)) = \oint_\lambda \frac{d\lambda'}{2\pi i} \frac{O_1(\lambda')O_2(\lambda)}{\lambda' - \lambda}. \tag{4.6.6}$$

Although this product is neither commutative nor associative, it plays a crucial role in the following argument.

For any set S of local operators on the single-sheeted λ plane, we can construct an algebra $R[S]$ of operators by repeatedly taking λ-derivatives, linear combinations and the products (4.6.6). Then it is expected that the following statement holds [25], although we do not have a complete proof:

Lemma 4.6.1 (conjecture)

$$R\left[\{ S^{(k)}(\lambda)\,;\, k = 1, 2, \cdots, p \} \right] = R\left[\{ \bar{W}^{(k)}(\lambda)\,;\, k = 1, 2, \cdots \} \right]. \tag{4.6.7}$$

In other words,

$$\bar{W}^{(l)} \in R\left[\{ S^{(k)}(\lambda)\,;\, k = 1, 2, \cdots, p \} \right] \tag{4.6.8}$$

and

$$S^{(l)} \in R\left[\{ \bar{W}^{(k)}(\lambda)\,;\, k = 1, 2, \cdots \} \right]. \tag{4.6.9}$$

This equation becomes obvious if we ignore all terms including λ-derivatives of $J(\lambda)$. In fact, we have

$$\bar{W}^{(k)}(\lambda)=\frac{1}{k}\sum_{i=1}^{p}\ {}^{\circ}_{\circ}J(\lambda_i)^k{}^{\circ}_{\circ}\ +\text{ terms including }\partial J(\lambda),\ \partial^2 J(\lambda),\cdots, \tag{4.6.10}$$

and the product (4.6.6) is equal to the normal ordering up to terms with derivatives of $J(\lambda)$, *e.g.*,

$$\begin{aligned}&\left(S^{(k)}(\lambda),\ S^{(l)}(\lambda)\right)\\ &\quad= {}^{\circ}_{\circ}S^{(k)}(\lambda)S^{(l)}(\lambda){}^{\circ}_{\circ}\ +(\text{ terms including }\partial J(\lambda),\ \partial^2 J(\lambda),\cdots).\end{aligned}\tag{4.6.11}$$

Therefore, if we keep terms only with the form of polynomials of $J(\lambda)$, the statement of Lemma 4.6.1 is nothing but the fundamental theorem for symmetric polynomials. The remarkable fact is that the use of the product (4.6.6) gives eq. (4.6.7) exactly. As is shown in app. B in ref. [25], we have checked this for operators of spin $k=1,\ 2,\ \cdots,\ 6$. The explicit relations between the $\bar{W}^{(k)}$'s and $S^{(k)}$'s are given by, for example,

$$\begin{aligned}&\bar{W}^{(1)}=S^{(1)},\\ &2\bar{W}^{(2)}=(S^{(1)},\ S^{(1)})-2S^{(2)},\\ &3\bar{W}^{(3)}=(S^{(1)},\ (S^{(1)},\ S^{(1)}))-3(S^{(1)},\ S^{(2)})+3S^{(3)},\\ &4\bar{W}^{(4)}+\frac{3}{5}\,\partial^2\bar{W}^{(2)}\\ &=\left\{\tfrac{1}{2}(S^{(1)},\ (S^{(1)},\ (S^{(1)},\ S^{(1)})))+\tfrac{1}{2}((S^{(1)},\ S^{(1)}),\ (S^{(1)},\ S^{(1)}))\right\}\\ &\quad-\left\{2(S^{(1)},\ (S^{(1)},\ S^{(2)}))+((S^{(1)},\ S^{(1)}),\ S^{(2)})+(S^{(2)},\ (S^{(1)},\ S^{(1)}))\right\}\\ &\quad+4(S^{(1)},\ S^{(3)})+2(S^{(2)},\ S^{(2)})-4S^{(4)}.\end{aligned}\tag{4.6.12}$$

Assuming the correctness of Lemma 4.6.1, we have [25]

Theorem 4.6.2

The vacuum condition of the $W_{1+\infty}$ algebra

$$\bar{W}^{(k)}_n\,|\,g\,\rangle=0\quad(k=1,\ 2,\ 3,\cdots;\ n\geq -k+1) \tag{4.6.13}$$

is equivalent to

$$S^{(k)}_n\,|\,g\,\rangle=0\quad(k=1,\ 2,\ 3,\cdots,p;\ n\geq -k+1), \tag{4.6.14}$$

where $S^{(k)}(\lambda)=\sum_n\lambda^{-n-k}S^{(k)}_n$.

[proof]

We introduce such an abbreviation as $U^{(k)}\,|\,g\,\rangle=0$, if a state $|\,g\,\rangle$ is annihilated by the modes of spin k operator $U^{(k)}(\lambda)$ with $n\geq -k+1$, that is,

$$U^{(k)}_n\,|\,g\,\rangle=0\ \ (\text{for } n\geq -k+1), \tag{4.6.15}$$

where $U^{(k)}=\sum_n\lambda^{-n-k}U^{(k)}_n$. Obviously, we have

$$\begin{aligned}&\text{(i)}\ \ U^{(k)}\,|\,g\,\rangle=0\ \Rightarrow\ \left(\partial^l U^{(k)}\right)|\,g\,\rangle=0,\\ &\text{(ii)}\ \ U^{(k)}\,|\,g\,\rangle=0,\ V^{(k)}\,|\,g\,\rangle=0\ \Rightarrow\ \left(aU^{(k)}+bV^{(k)}\right)|\,g\,\rangle=0.\end{aligned}\tag{4.6.16}$$

Furthermore, since the n-th mode of the product of operators $U^{(k)}$ and $V^{(l)}$ of spin k and l, respectively, is given by

$$\left(U^{(k)},\ V^{(l)}\right)_n=\sum_{q\geq k+n}U^{(k)}_{n-q}V^{(l)}_q+\sum_{r\geq -k+1}V^{(l)}_{n-r}U^{(k)}_r \tag{4.6.17}$$

and the left-hand side is an operator of spin $k+l$, we have

$$\text{(iii)}\ \ U^{(k)}\,|\,g\,\rangle=0,\ V^{(l)}\,|\,g\,\rangle=0\ \Rightarrow\ \left(U^{(k)},\ V^{(l)}\right)|\,g\,\rangle=0. \tag{4.6.18}$$

Lemma 4.6.1 together with (i), (ii) and (iii) proves the statement of this theorem. ∎

Finally we show that the condition (4.6.14) becomes the vacuum condition of the W_p algebra after the elimination of the variables J_{np} $(n\in\mathbf{Z})$ which are redundant for the p-reduced KP hierarchy [25]. In order to eliminate these variables, we first expand $S^{(k)}(\lambda)$ as a sum of operators with the usual normal ordering for the oscillators J_n. For example, we rewrite $S^{(k)}(\lambda)$ as

$$S^{(1)}(\lambda)\ =\ \sum_{i=1}^{p}J(\lambda_i), \tag{4.6.19}$$

$$
\begin{aligned}
S^{(2)}(\lambda) &= \sum_{1\le i<j\le p} J(\lambda_i)J(\lambda_j) \\
&= \sum_{1\le i<j\le p} :J(\lambda_i)J(\lambda_j): + \sum_{1\le i<j\le p} \langle J(\lambda_i)J(\lambda_j)\rangle, \qquad (4.6.20)\\
S^{(3)}(\lambda) &= \sum_{1\le i<j<k\le p} J(\lambda_i)J(\lambda_j)J(\lambda_k) \\
&= \sum_{1\le i<j<k\le p} :J(\lambda_i)J(\lambda_j)J(\lambda_k): + \sum_{1\le i<j<k\le p} \{\langle J(\lambda_i)J(\lambda_j)\rangle\, J(\lambda_k) \\
&\quad + \langle J(\lambda_i)J(\lambda_k)\rangle\, J(\lambda_j) + \langle J(\lambda_j)J(\lambda_k)\rangle\, J(\lambda_i)\}\,, \qquad (4.6.21)
\end{aligned}
$$

where $: \;:$ is the usual normal ordering for the J_n's (recall that $J(\lambda) = \frac{1}{p}\sum_n \lambda^{-n/p-1} J_n$), and $\langle\ \rangle$ is the usual vacuum expectation value, $\langle J_n J_m \rangle = n\,\delta_{n+m,0}\,\theta(n)$. We then define $\mathbf{W}^{(k)}(\lambda)$ by formally setting J_{np} ($n \in \mathbf{Z}$) to 0 in this normal ordered form of $S^{(k)}(\lambda)$. Note that $\mathbf{W}^{(1)}(\lambda)$ vanishes identically, because $S^{(1)}(\lambda) = \sum_n \lambda^{-n-1} J_{np}$ contains only the variables to be dropped. As is shown in app. C in ref. [25], these $\mathbf{W}^{(k)}(\lambda)$'s are identified with the generators of the W_p algebra with Virasoro central charge $c = p-1$ that is to be constructed from $\mathbf{Z}_p$-twisted free bosons. Furthermore we have [25]

Lemma 4.6.3

The condition $S_n^{(k)}\,|\,g\,\rangle = 0$ ($k = 1, 2, \cdots, p\,;\, n \ge -k+1$) is equivalent to

$$
\left\{
\begin{array}{ll}
J_{np}\,|\,g\,\rangle = 0 & (n \ge 0) \\
\mathbf{W}_n^{(k)}\,|\,g\,\rangle = 0 & (k = 2, \cdots, p\,;\, n \ge -k+1).
\end{array}
\right. \qquad (4.6.22)
$$

[proof]

First we note that $S_n^{(k)}$ can be expanded with respect to J_{np} ($n \in \mathbf{Z}$) as

$$
S_n^{(k)} = \mathbf{W}_n^{(k)} + \sum_{l=1}^{k} \sum_{n_1,\cdots,n_l \in \mathbf{Z}} C^{(k-l)}_{n-(n_1+\cdots+n_l)}[n_1, \cdots, n_l] : J_{n_1 p} \cdots J_{n_l p} : . \qquad (4.6.23)
$$

Here the operators $C^{(k-l)}_{n-(n_1+\cdots+n_l)}[n_1,\cdots,n_l]$ do not contain J_{np} ($n \in \mathbf{Z}$), and are obtained from $S_n^{(k)}$ by repeatedly taking commutators with the J_{np}'s ($n \in \mathbf{Z}$) and then setting J_{np} ($n \in \mathbf{Z}$) to 0:

$$
C^{(k-l)}_{n-(n_1+\cdots+n_l)}[n_1,\cdots,n_l] = \text{const.}\ [J_{-n_l p}, [\cdots, [J_{-n_1 p}, S_n^{(k)}]\cdots]]\Big|_{J_{np}=0\ (n\in\mathbf{Z})} . \qquad (4.6.24)
$$

This implies that

$$
C^{(k-l)}_{n-(n_1+\cdots+n_l)}[n_1,\cdots,n_l] = \text{const.}\ \mathbf{W}^{(k-l)}_{n-(n_1+\cdots+n_l)}, \qquad (4.6.25)
$$

because we have

$$
[J_{np}, S_m^{(k)}] = [S_n^{(1)}, S_m^{(k)}] = (p+1-k)\, n\, S_{m+n}^{(k-1)}, \qquad (4.6.26)
$$

which follows from

$$
S^{(1)}(\lambda') S^{(k)}(\lambda) = \frac{p+1-k}{(\lambda'-\lambda)^2} S^{(k-1)}(\lambda) + (\text{regular terms}). \qquad (4.6.27)
$$

The assertion follows immediately. ▮

Thus combining Theorem 4.6.2 and Lemma 4.6.3, we obtain the following theorem [25]:

Theorem 4.6.4

Let $\tau(x)$ be a τ function of the KP hierarchy that satisfies the string equation (4.5.9) and (4.5.10). Then the corresponding state $|\,g\,\rangle$ satisfies the conditions of the p-reduction

$$
J_{np}\,|\,g\,\rangle = 0 \quad (n \ge 1) \qquad (4.6.28)
$$

and the vacuum condition of the W_p algebra

$$
\mathbf{W}_n^{(k)}\,|\,g\,\rangle = 0 \quad (k = 2, 3, \cdots, p\,;\, n \ge -k+1). \qquad (4.6.29)
$$

By reinterpreting the oscillators J_n's as the differential operators $\mathcal{J}_n$'s acting on $\tau(x)$ (see eq. (4.5.32)), the set of equations (4.6.28) and (4.6.29) are rewritten in the form of differential equations for $\tau(x)$. Thus we have shown, using the infinite Grassmannian structure of 2D quantum gravity, that the string equation naturally leads to the W_p constraint on a τ function.[10]

[10] See [30] where some analyses are made on the basis of Hirota's bilinear equation.

Chapter 5

More Investigation on the Operator Content

The system of the 2D gravity coupled to $(p,*)$ conformal field is described in a unified way by a τ function of the KP hierarchy that satisfies the string equation (4.3.1). For example, the 2D gravity coupled to (p,q) conformal field is realized by setting $x_1 = t$, $x_{p+q} =$ const. and $x_i = 0$ $(i \neq 1, p+q)$ [11][12] (see sec. 3.7). In this sense the string equation (4.3.1) or the S-D equation (3.7.25) gives a universal description for various values of q, while in order to vary p one has to change the form of the equation itself. In particular, the equivalence between the (p,q) gravity and the (q,p) gravity is not manifest in this formalism, nevertheless they should describe the same theory. On the other hand, in the Douglas equation (4.1.17) p and q appear in a symmetric manner, although the structure of the operators is not transparent. The aim of this chapter is to give a one-to-one correspondence between the (p,q) and (q,p) gravities by using the relation between the two formulations established in the preceding chapter.

In section 5.1, using this equivalence between the string equation and the Douglas equation, we consider the p-q duality and explicitly write down the transformation rules of the operators under the exchange of p and q. Section 5.2 is devoted to physical interpretation of our results. There, by using the transformation rules, the operators $\mathcal{O}_k$ $(k = 0 \bmod p$ or q, $k < p+q)$ in the (p,q) gravity are shown to be redundant in the sense that their sources can be eliminated by a redefinition of the sources of lower dimensional operators. This result is consistent with the BRST cohomological analysis by Lian and Zuckerman [31]. Finally, we apply this analysis to the minimal

unitary case, the $(p,p+1)$ gravity, and consider the renormalization group flows from the $(p,p+1)$ gravity to the $(p-1,p)$ gravity. This chapter completely follows ref. [26].

5.1 p-q Duality in the (p,q) Gravity

In sec. 4.2, we found that a pair of differential operators (P,Q) satisfying the Douglas equation (4.1.17) is represented as follows by a pseudo-differential operator W that obeys eq. (4.2.34):

$$\begin{aligned} P &= W\partial^p W^{-1}, \\ Q &= \frac{1}{p} W \left(t\partial^{-p+1} + \frac{-p+1}{2}\partial^{-p} + \sum_{k=1}^{p+q} k x_k \partial^{k-p} \right) W^{-1}, \end{aligned} \tag{5.1.1}$$

where W has the form $W = 1 + \sum_{k=1}^{\infty} w_k(t;x)\partial^{-k}$. In this expression we have set P to be in the standard form (4.1.19). Note that Q comes to have the standard form if we use the ambiguity (4.1.18). In fact if we introduce $\bar{P}$ and $\bar{Q}$ defined by

$$\begin{aligned} \bar{P} &= \frac{p}{(p+q)x_{p+q}} e^{at} Q e^{-at}, \\ \bar{Q} &= -\frac{(p+q)x_{p+q}}{p} e^{at} P e^{-at}, \\ a &= \frac{(p+q-1)x_{p+q-1}}{q(p+q)x_{p+q}}, \end{aligned} \tag{5.1.2}$$

then obviously $(-\bar{Q}, \bar{P})$ gives the same solution with (P,Q) as in eq. (4.1.18) and $\bar{P}$ is now in the standard form (4.1.19). Furthermore $(\bar{P},\bar{Q})$ satisfies eq. (4.1.17) with p and q exchanged. Therefore, the general argument given in sec. 4.2 implies that there uniquely exist a pseudo-differential operator $\bar{W}$ and a set of constants $\bar{x}_k$ $(k = 1, 2, \cdots, p+q)$ such that

$$\begin{aligned} \bar{P} &= \bar{W}\partial^q \bar{W}^{-1}, \\ \bar{Q} &= \frac{1}{q} \bar{W} \left(t\partial^{-q+1} + \frac{-q+1}{2}\partial^{-q} + \sum_{k=1}^{p+q} k \bar{x}_k \partial^{k-q} \right) \bar{W}^{-1}, \end{aligned} \tag{5.1.3}$$

where $\bar{W}$ has the form $\bar{W} = 1 + \sum_{k=1}^{\infty} \bar{w}_k(t, \bar{x})\partial^{-k}$ and satisfies the Sato equation. By combining eqs. (5.1.1), (5.1.2) and (5.1.3), we obtain [26]

$$U\partial^q U^{-1} = \frac{1}{(p+q)x_{p+q}}\left(t\partial^{-p+1} + \frac{-p+1}{2}\partial^{-p} + \sum_{k=1}^{p+q} kx_k\partial^{k-p}\right), \tag{5.1.4}$$

$$U\frac{-p}{q(p+q)x_{p+q}}\left(t\partial^{-q+1} + \frac{-q+1}{2}\partial^{-q} + \sum_{k=1}^{p+q} k\bar{x}_k\partial^{k-q}\right)U^{-1} = \partial^p, \tag{5.1.5}$$

where U is defined by

$$U = W^{-1}e^{-at}\bar{W}. \tag{5.1.6}$$

From eq. (5.1.5) we immediately see that

$$\bar{x}_{p+q} = -\frac{q}{p}x_{p+q}, \tag{5.1.7}$$

by comparing the coefficients of ∂^p. In order to make the equations transparent, we introduce the following notations:

$$a_{p+q-k} = \frac{kx_k}{(p+q)x_{p+q}}, \quad \bar{a}_{p+q-k} = \frac{k\bar{x}_k}{(p+q)\bar{x}_{p+q}}, \tag{5.1.8}$$

$$\bar{\partial} = U\partial U^{-1}, \quad \bar{t} = UtU^{-1}. \tag{5.1.9}$$

Obviously $\bar{\partial}$ and $\bar{t}$ satisfy

$$[\bar{\partial}, \bar{t}] = 1 \tag{5.1.10}$$

and have the following forms:

$$\begin{aligned} \bar{\partial} &= \partial + \cdots \\ \bar{t} &= t\partial^0 + \cdots . \end{aligned} \tag{5.1.11}$$

Then the eqs. (5.1.4) and (5.1.5) can be rewritten in a manifestly p-q symmetric form:

$$\begin{aligned} \bar{\partial}^q &= \partial^q\left(1 + \sum_{k=1}^{p+q-1} a_k\partial^{-k}\right) + \frac{1}{(p+q)x_{p+q}}\left(t\partial^{-p+1} + \frac{-p+1}{2}\partial^{-p}\right), \\ \partial^p &= \bar{\partial}^p\left(1 + \sum_{k=1}^{p+q-1} \bar{a}_k\bar{\partial}^{-k}\right) + \frac{1}{(p+q)\bar{x}_{p+q}}\left(\bar{t}\bar{\partial}^{-q+1} + \frac{-q+1}{2}\bar{\partial}^{-q}\right). \end{aligned} \tag{5.1.12}$$

Although to obtain a complete solution of (5.1.12) is not an easy task, a simple power counting shows that for the first $p+q$ terms we can write

$$\begin{aligned} \bar{\partial} &= \partial\left(1 + \sum_{k=1}^{p+q-1} a_k\partial^{-k}\right)^{1/q} + \frac{t}{q(p+q)x_{p+q}}\partial^{2-p-q} + O(\partial^{1-p-q}), \\ \partial &= \bar{\partial}\left(1 + \sum_{k=1}^{p+q-1} \bar{a}_k\bar{\partial}^{-k}\right)^{1/p} + \frac{\bar{t}}{p(p+q)\bar{x}_{p+q}}\bar{\partial}^{2-p-q} + O(\bar{\partial}^{1-p-q}). \end{aligned} \tag{5.1.13}$$

Actually this form is sufficient for solving a_k in terms of $\bar{a}_k$ because we need just $p+q-1$ relations among them. Therefore, the problem is reduced to finding the condition for the following two Laurent series to be inverse functions of each other:

$$\begin{aligned} w &= z\left(1 + \sum_{n=1}^{\infty} a_n z^{-n}\right)^{1/q}, \\ z &= w\left(1 + \sum_{n=1}^{\infty} \bar{a}_n w^{-n}\right)^{1/p}. \end{aligned} \tag{5.1.14}$$

This problem is solved by the following series of manipulations:

$$\begin{aligned} a_n &= \frac{1}{2\pi i}\oint dz z^{n-1}\left(\frac{w}{z}\right)^q \\ &= \frac{1}{2\pi i}\oint dw\frac{dz}{dw}z^{n-1}\left(\frac{z}{w}\right)^{-q} \\ &= \frac{1}{2\pi i}\oint dw w^{n-1}\left(1 + \sum_{m=1}^{\infty} \bar{a}_m w^{-m}\right)^{(n-p-q)/p}\left(1 + \sum_{m=1}^{\infty}(1-m/p)\bar{a}_m w^{-m}\right). \end{aligned} \tag{5.1.15}$$

The last expression is easily evaluated and we finally obtain [26]

$$a_n = -\frac{q}{p}\sum_{l\geq 1}\frac{1}{l}\binom{(n-p-q)/p}{l-1}\sum_{\substack{m_1,\cdots,m_l\geq 1 \\ m_1+\cdots+m_l=n}} \bar{a}_{m_1}\cdots\bar{a}_{m_l}, \tag{5.1.16}$$

where a_k and $\bar{a}_k$ were defined in (5.1.8).[1]

As for the partition function, we can in principle write down the relation between the (p,q) and (q,p) theories by first solving eqs. (5.1.4) and (5.1.5) for U and then

[1]When a_k and $\bar{a}_k$ are related by (5.1.16), we can prove the following identities which generalize eq. (5.1.16):

$$\frac{1}{r}\sum_{l=1}^{\infty}\binom{r/q}{l}A_{s+r}^{(l)} + \frac{1}{s}\sum_{l=1}^{\infty}\binom{s/p}{l}\bar{A}_{s+r}^{(l)} = 0,$$

translating eq. (5.1.6) in terms of τ functions. However, in order to solve U from eqs. (5.1.4) and (5.1.5) we have to keep the higher-order terms in (5.1.13) which we did not need in the derivation of eq. (5.1.16). On the other hand, if we are satisfied with the second derivative of the free energy with respect to the cosmological constant t, the p-q duality relation can easily be obtained. In fact, we have

$$\begin{aligned} P &= \partial^p + pu\partial^{p-2} + \cdots, \quad 2u = \frac{\partial^2}{\partial t^2} \ln Z, \\ \bar{P} &= \partial^q + q\bar{u}\partial^{q-2} + \cdots, \quad 2\bar{u} = \frac{\partial^2}{\partial t^2} \ln \bar{Z}, \end{aligned} \tag{5.1.17}$$

where Z and $\bar{Z}$ are the partition functions of the (p,q) and (q,p) theories, respectively. Then eqs. (5.1.1) and (5.1.2) give the following relation between u and $\bar{u}$ [26]:

$$\bar{u} = u + \frac{(p+q-2)x_{p+q-2}}{q(p+q)x_{p+q}} - \frac{q-1}{2}\left(\frac{(p+q-1)x_{p+q-1}}{q(p+q)x_{p+q}}\right)^2. \tag{5.1.18}$$

In order to see physical consequences of the p-q duality discussed above, we consider the case $(p,q) = (2,3)$ as an example. As was stated in chapter 4, the partition function $Z_{(2,*)}$ of the $(2,*)$ gravity is given by a τ function of the 2-reduced KP hierarchy, τ_2, which satisfies eq. (4.3.1) for $p = 2$:

$$\sqrt{Z_{(2,*)}} = \tau_2(x_1, x_3, x_5, x_7, \cdots). \tag{5.1.19}$$

Here τ_2 depends only on x_k ($k \neq 0 \mod 2$). By setting $x_k = 0$ for $k \geq 7$, we have the partition function of the $(2,3)$ gravity in the $(2,*)$ description and $u(x_1, x_3, x_5)$ is expressed as

$$u(x_1, x_3, x_5) = \frac{\partial^2}{\partial {x_1}^2} \ln \tau_2(x_1, x_3, x_5, \text{other } x_k\text{'s=0}). \tag{5.1.20}$$

Similarly if we regard the $(2,3)$ gravity as a special case of the $(3,*)$ gravity, we have

$$\bar{u}(x_1, x_2, x_4, x_5) = \frac{\partial^2}{\partial {x_1}^2} \ln \tau_3(x_1, x_2, x_4, x_5, \text{other } x_k\text{'s=0}), \tag{5.1.21}$$

---

where r and s are arbitrary integers satisfying $s + r \geq 1$, and

$$A_n^{(l)} = \sum_{\substack{m_1,\cdots,m_l \geq 1 \\ m_1+\cdots+m_l = n}} a_{m_1} \cdots a_{m_l}, \quad \bar{A}_n^{(l)} = \sum_{\substack{m_1,\cdots,m_l \geq 1 \\ m_1+\cdots+m_l = n}} \bar{a}_{m_1} \cdots \bar{a}_{m_l}.$$

where τ_3 is a τ function of the 3-reduced KP hierarchy which satisfies eq. (4.3.1) for $p = 3$ and depends only on x_k ($k \neq 0 \mod 3$). Although u and $\bar{u}$ describe the same theory, they appear to have different structures; u depends on three variables while $\bar{u}$ involves four variables. However, the p-q duality gives a complete one-to-one correspondence between them. In fact, by applying eqs. (5.1.7), (5.1.16) and (5.1.18) to this case, $(p,q) = (2,3)$, we have [26]

$$\bar{u}(\bar{x}_1, \bar{x}_2, \bar{x}_4, \bar{x}_5) = u(x_1, x_3, x_5) - \frac{3}{10}\frac{\bar{x}_3}{\bar{x}_5} + \frac{2}{25}\frac{\bar{x}_4^2}{\bar{x}_5^2}, \tag{5.1.22}$$

where x and $\bar{x}$ are related by

$$\begin{aligned} x_1 &= \bar{x}_1 - \frac{4\bar{x}_2\bar{x}_4}{5\bar{x}_5} - \frac{9\bar{x}_3^2}{20\bar{x}_5} + \frac{18\bar{x}_3\bar{x}_4^2}{25\bar{x}_5^2} - \frac{4\bar{x}_4^4}{25\bar{x}_5^3}, \\ x_2 &= \bar{x}_2 - \frac{6\bar{x}_3\bar{x}_4}{5\bar{x}_5} + \frac{32\bar{x}_4^3}{75\bar{x}_5^2}, \\ x_3 &= \bar{x}_3 - \frac{4\bar{x}_4^2}{5\bar{x}_5}, \\ x_4 &= \bar{x}_4, \\ x_5 &= -\frac{2}{3}\bar{x}_5. \end{aligned} \tag{5.1.23}$$

Since the left-hand side of eq. (5.1.22) does not depend on $\bar{x}_3$, we can restrict the variables so that $\bar{x}_3 = \frac{4}{15}\frac{\bar{x}_4^2}{\bar{x}_5}$ without losing any information and we have [26]

$$\bar{u}(\bar{x}_1, \bar{x}_2, \bar{x}_4, \bar{x}_5) = u(x_1, x_3, x_5), \tag{5.1.24}$$

for the variables satisfying

$$\begin{aligned} x_1 &= \bar{x}_1 - \frac{4\bar{x}_2\bar{x}_4}{5\bar{x}_5}, \\ x_3 &= -\frac{8\bar{x}_4^2}{15\bar{x}_5}, \\ x_5 &= -\frac{2}{3}\bar{x}_5. \end{aligned} \tag{5.1.25}$$

To obtain the relation between Z and $\bar{Z}$ we must integrate the equation (5.1.22) twice and the integration constants are determined by KP flows. A straightforward but long calculation leads to the following relation between the two partition functions

Z and $\bar{Z}$ [26]:

$$\begin{aligned} & \ln\sqrt{Z(x_1,\, x_3,\, x_5)} - \frac{x_1 x_3^3}{25x_5^2} + \frac{3x_3^5}{625x_5^3} + \frac{x_1^2 x_3}{10x_5} + \frac{1}{40}\ln x_5 \\ = & \ln\sqrt{\bar{Z}(\bar{x}_1,\, \bar{x}_2,\, \bar{x}_4,\, \bar{x}_5)} + \frac{2\bar{x}_1^2\bar{x}_4^2}{25\bar{x}_5^2} + \frac{\bar{x}_1\bar{x}_2^2}{5\bar{x}_5} - \frac{16\bar{x}_1\bar{x}_2\bar{x}_4^3}{125\bar{x}_5^3} \\ & + \frac{128\bar{x}_1\bar{x}_4^6}{9375\bar{x}_5^5} - \frac{4\bar{x}_2^3\bar{x}_4}{75\bar{x}_5^2} + \frac{32\bar{x}_2^2\bar{x}_4^4}{625\bar{x}_5^4} - \frac{512\bar{x}_2\bar{x}_4^7}{46875\bar{x}_5^6} \\ & + \frac{4096\bar{x}_4^{10}}{5859375\bar{x}_5^8} + \frac{1}{15}\ln\bar{x}_5. \end{aligned} \tag{5.1.26}$$

Here the functions added to $\ln\sqrt{Z}$ and $\ln\sqrt{\bar{Z}}$ can be regarded as local counterterms to make the $(2,3)$ and $(3,2)$ gravity theories equivalent.

The equations (5.1.24) and (5.1.25) have a rather interesting implication. As was shown in secs. 3.7 and 3.8, the 2D gravity coupled to the Ising model is described by τ_3 if one keeps $\bar{x}_1, \bar{x}_2, \cdots, \bar{x}_7$ and sets other $\bar{x}_k$'s to zero. Then $\bar{x}_2$ and $\bar{x}_5$ are identified with the sources for the spin operator and the mass operator, respectively, and $\bar{x}_1$ and $\bar{x}_4$ with those for the cosmological term and a higher gravitational operator, respectively. With this interpretation, the $(3,2)$ gravity can be regarded as the limit of $\bar{x}_7 \to 0$, or equivalently as the limit of infinite mass, $m \to \infty$ [11] (see sec. 3.7). Therefore eqs. (5.1.24) and (5.1.25) indicate that the four operators corresponding to $\bar{x}_1, \bar{x}_2, \bar{x}_4$ and $\bar{x}_5$, which were independent in the case of the Ising model, are recombined into three independent ones as in eq. (5.1.25). In other words, $\mathbf{Z}_2$-odd operators, which correspond to $\bar{x}_2$ and $\bar{x}_4$, form $\mathbf{Z}_2$-invariant bound state operators in the limit of $m \to \infty$. In general, if we consider a renormalization group flow in the direction of unitary deformation, operators will be recombined rather wildly, and this seems to be the greatest obstacle for constructing a completely universal description of 2D gravity.

5.2 Redundant Operator Problem and the Unitarity Preserving Renormalization Group Flows

First, we consider the problem of redundant operators. In the (p,q) gravity, the operators $\mathcal{O}_k$ ($k = 0 \bmod q$ or $k = 0 \bmod p$) are known to lack physical interpretation. For example, if we describe the $(p,p+1)$ gravity in terms of the S-D equation of $(p,*)$

type, we have $(2p-1)$ sources x_k ($k = 1, \cdots, 2p+1;\ k \neq p,\, 2p$) and they correspond to the ϕ^l operators of the ϕ^{2p-2} theory as

$$\begin{array}{ccccccccc} x_1 & x_2 & \cdots & x_{p-1} & x_{p+1} & x_{p+2} & \cdots & x_{2p-1} & x_{2p+1} \\ \updownarrow & \updownarrow & \cdots & \updownarrow & & \updownarrow & \cdots & \updownarrow & \\ 1 & \phi & \cdots & \phi^{p-2} & & \phi^{p-1} & \cdots & \phi^{2p-4} & \end{array} \tag{5.2.1}$$

and x_{2p+1} should be regarded as a source for the Lagrangian itself. However, $\mathcal{O}_{q=p+1}$ is a rather mysterious operator in the sense that it has no corresponding field in the above list, and indeed it does not appear in the BRST cohomology [31]. In the following, we show that the operators $\mathcal{O}_k$ ($k = 0 \bmod q,\ k < p+q$) in the $(p,*)$ formalism of the (p,q) gravity are redundant in the sense that their sources can be absorbed into the sources of lower dimensional operators through their analytic redefinition [26].[2]

We consider the (p,q) gravity in the $(p,*)$ formalism and apply the p-q duality equation (5.1.18). Since its left-hand side does not depend on $\bar{x}_q$, and eq. (5.1.16) reads as

$$x_k = \bar{x}_k + \bar{x}_{p+q} \times \left(\text{a polynomial of } \frac{\bar{x}_j}{\bar{x}_{p+q}} \text{ for } j > k\right), \tag{5.2.2}$$

the KP flow of the parameter $\bar{x}_q$ causes the change only of the parameters x_j with j less than or equal to q. Thus, by setting $\bar{x}_q$ to a special value, x_q can be set to zero accompanied with an analytic redefinition of $x_1, \cdots, x_{q-1}$:[3]

$$\begin{aligned} & u(x_1, \cdots, x_{p-1}, x_{p+1}, \cdots, x_{q-1}, x_q, x_{q+1}, \cdots, x_{p+q}) \\ & = u(x'_1, \cdots, x'_{p-1}, x'_{p+1}, \cdots, x'_{q-1},\ 0\ , x_{q+1}, \cdots, x_{p+q}) \\ & \quad + (\text{an additional term appearing only for the case } p = 2), \end{aligned} \tag{5.2.3}$$

[2] The same problem has also been considered in ref. [37] and the operator $\mathcal{O}_q$ in the case $p < q$ is interpreted there as a boundary operator.

[3] Equation (5.2.3) can also be obtained by considering the characteristic curve of the S-D equation : $\mathcal{L}_{-p}\tau = 0$ or equivalently,

$$\frac{1}{2}\sum_{k+l=p} k l x_k x_l + \sum_{k=p+1}^{p+q} k x_k \partial_{k-p} \ln\tau = 0.$$

where x'_i $(i = 1, \cdots, q-1)$ have a form such as

$$x'_i = x_i + x_{p+q} \times \left(\text{a polynomial of } \frac{x_j}{x_{p+q}} \text{ for } j > i \right). \tag{5.2.4}$$

Similarly, the sources x_k with $k = q, 2q, \cdots, [(p+q)/q]\,q$ can also be set to zero by an analytic redefinition of lower dimensional sources.[4] Thus, starting from the highest dimensional source, $x_{[(p+q)/q]q}$, we can eliminate all the sources x_k with $k = q, 2q, \cdots, [(p+q)/q]\,q$. In this sense the operators corresponding to these sources are redundant and it is natural to set them to zero from the beginning. In particular, in the minimal unitary case (the $(p, p+1)$ gravity), this procedure ends with the operators which can be interpreted in terms of the Ginzburg-Landau potential. For example, for the case of $(p,q) = (2,3)$ we can easily follow this prescription for eqs. (5.1.22) and (5.1.23), and obtain[5]

$$u(x_1, x_3, x_5) = u(x_1 - \frac{3x_3^2}{10x_5}, 0, x_5) - \frac{x_3}{5x_5}. \tag{5.2.5}$$

The above result agrees with a BRST cohomological analysis by Lian and Zuckerman [31] which excludes these operators $\mathcal{O}_k$ $(k = 0 \mod q)$ as well as $\mathcal{O}_k$ $(k = 0 \mod p)$ as BRST trivial operators, although our analysis is restricted to the first $(p+q)$ operators.

It is now possible to consider the renormalization group flow from $(p, p+1)$ to $(p-1, p)$ as follows [26]. Starting from the $(p, p+1)$ gravity with x_{p+1} eliminated as above, we obtain the $(p, p-1)$ gravity by setting x_{2p+1} to zero. Then the p-q duality maps this gravity to the $(p-1, p)$ gravity and the above prescription eliminates the redundant variable to give a new Ginzburg-Landau potential. Consideration along this line suggests the possibility of constructing a universal description of the minimal unitary series coupled to 2D gravity.

[4] $[a]$ denotes a maximal integer $\leq a$.

[5] At the partition function level, we have the following relation between the new partition functions

$$\begin{aligned} &\ln \sqrt{Z(x_1 - \frac{3x_3^2}{10x_5}, 0, x_5)} + \frac{1}{40} \ln x_5 \\ = \ &\ln \sqrt{\tilde{Z}(\tilde{x}_1 - \frac{4\tilde{x}_2\tilde{x}_4}{5\tilde{x}_5} + \frac{16\tilde{x}_4^4}{125\tilde{x}_5^3}, 0, 0, \tilde{x}_5)} + \frac{1}{15} \ln \tilde{x}_5. \end{aligned}$$

Chapter 6

Topological Gravities

In this chapter, we consider the matrix model which directly gives the partition function of the topological gravities, in the sense that the partition function of the matrix model becomes that of the topological gravity in the large N limit with no use of any other double scaling. First, in section 6.1 we briefly summarize the result on the Kontsevich model whose Feynman diagrams give the cell decomposition of the moduli space of Riemann surfaces with loop boundaries [33]. This matrix model is far different from the matrix models considered so far in its interpretation. In section 6.2 we then introduce the generalized form of the Kontsevich model first given by Kharchev et al., and prove that this generating function is a τ function of the KP hierarchy for every finite N (the size of the matrices) and comes to satisfy the string equation of the $(p,1)$ topological gravity with $p = 2,3,4,\cdots$ in the large N limit [34]. However, this matrix model does not show us what kind of matters the gravity is coupled to, which is now supposed to be "minimal topological matters" [35], as proposed in ref. [36].

6.1 Summary of the Kontsevich Model

The original definition of the topological gravity is given by Witten as follows [5]: Let $\mathcal{M}_{h,s}$ be the moduli space of genus h Riemann surfaces Σ with s marked points $x_1, x_2, \cdots x_s$, and $\bar{\mathcal{M}}_{h,s}$ be its Deligne-Mumford compactification. For $i = 1,2,\cdots,s$ we introduce the complex line bundle $\mathcal{L}_i$ over $\bar{\mathcal{M}}_{h,s}$ whose fiber is the cotangent

space to Σ at x_i. Then the operators of the topological gravity and their correlation functions are defined functionally by

$$\langle \sigma_{n_1} \cdots \sigma_{n_s} \rangle_c \equiv \int_{\bar{\mathcal{M}}_{h,s}} \prod_{i=1}^{s} [c_1(\mathcal{L}_i)]^{n_i} \tag{6.1.1}$$

for non-negative integers n_i, which yield non-vanishing results only for n_i satisfying

$$\sum_{i=1}^{s} n_i = 3h - 3 + s, \tag{6.1.2}$$

since $\dim_{\mathbf{R}} \bar{\mathcal{M}}_{h,s} = 2(3h-3+s)$. For later convenience we rescale the operators σ_n as

$$\sigma_n \mapsto \frac{1}{(2n+1)!!}\sigma_n, \tag{6.1.3}$$

that is,

$$\langle \sigma_{n_1} \cdots \sigma_{n_s} \rangle_c \equiv \prod_{i=1}^{s}(2n_i+1)!! \cdot \int_{\bar{\mathcal{M}}_{h,s}} \prod_{i=1}^{s} [c_1(\mathcal{L}_i)]^{n_i}. \tag{6.1.4}$$

Now we introduce the generating function (free energy) of these connected correlation functions as

$$F(\mu) \equiv \sum_{\{d_i \geq 0\}} \left\langle \sigma_0^{d_0} \sigma_1^{d_1} \cdots \right\rangle_c \frac{\mu_0^{d_0} \mu_1^{d_1} \cdots}{d_0! d_1! \cdots}. \tag{6.1.5}$$

Kontsevich showed [33] that this free energy is expressed by the following matrix model which we will call the *Kontsevich model*:

$$F(\mu) = \ln \lim_{N\to\infty} Z_N \tag{6.1.6}$$

$$Z_N = \frac{\int \mathcal{D}X \exp\left(-\frac{1}{2}\mathrm{tr}\,\Lambda X^2 - \frac{1}{6}\mathrm{tr}\,X^3\right)}{\int \mathcal{D}X \exp\left(-\frac{1}{2}\mathrm{tr}\,\Lambda X^2\right)}. \tag{6.1.7}$$

Here X and Λ are $N \times N$ hermitian matrices with which μ_k's are related as

$$\mu_k = -\frac{1}{2k+1}\mathrm{tr}\,\Lambda^{-(2k+1)}. \tag{6.1.8}$$

These variables $\mu_0, \mu_1, \mu_2, \cdots$ become independent of each other for large N.

As will be proved in the next section within more general framework, $\exp(F)$ is a τ function satisfying the string equation (and thus the Virasoro constraint) with a background shift corresponding to the $k=1$ case in the Kazakov series (see eqs. (3.3.26), (3.3.27) and (3.5.5)):

$$\begin{aligned} e^{F(\mu)} &= \tau(t) && (6.1.9)\\ \mathrm{L}_n\tau(t) &= 0 \quad (n=-1,0,1,2,\cdots) && (6.1.10)\end{aligned}$$

with

$$\begin{aligned} t_0 &= \mu_0\\ t_1 &= \mu_1+\text{const.} \qquad (6.1.11)\\ t_l &= \mu_l \quad (l\geq 2).\end{aligned}$$

6.2 Generalized Kontsevich Model

In this section, we consider the *generalized Kontsevich model* (GKM) which is defined by Kharchev et al. as

$$Z_N^{(V)}(\Lambda) \equiv \int \mathcal{D}X\, e^{-U(\Lambda,X)} \Big/ \int \mathcal{D}X\, e^{-U_2(\Lambda,X)}. \qquad (6.2.1)$$

Here Λ and X are $N\times N$ hermitian matrices and the potentials $U(\Lambda,X)$ and $U_2(\Lambda,X)$ are defined by

$$\begin{aligned} U(\Lambda,X) &\equiv \mathrm{tr}\,[V(\Lambda+X)-V(\Lambda)-V'(\Lambda)X] && (6.2.2)\\ U_2(\Lambda,X) &\equiv \lim_{\epsilon\to 0}\frac{1}{\epsilon^2}U(\Lambda,\epsilon X)\\ &= \text{the quadratic part in } U(\Lambda,X) \text{ with respect to } X && (6.2.3)\end{aligned}$$

for arbitrary analytic function $V(z)$. Note that the GKM for the function

$$V(z) = \frac{1}{6}z^3 \qquad (6.2.4)$$

is nothing but the original Kontsevich model given in the preceding section. In the following we prove after a kind of Miwa transformation that the partition function of

the GKM (6.2.1) is a τ function of the KP hierarchy even for finite N, and that, in particular case where

$$V(z) = \text{const.}\, z^{p+1}, \qquad (6.2.5)$$

the partition function become a τ function of the p-reduced KP hierarchy which satisfies the string equation (and thus the $W_{1+\infty}$ (or W_p) constraint) of the $(p,1)$ topological gravity.

We first show that the partition function of the GKM can be rewritten as

$$Z_N^{(V)}(\Lambda) = \left[\det \phi^{(i)}(z_j)\right]_{\substack{i=0,1,\cdots,N-1\\ j=1,2,\cdots,N}} /\Delta(z), \qquad (6.2.6)$$

where $(z_1,z_2,\cdots,z_N)$ are the eigenvalues of Λ, $\Delta(z)$ is their Van der Monde determinant, and $\phi^{(i)}(z)$ $(i=0,1,2,\cdots)$ is defined by

$$\phi^{(i)}(z) \equiv \left[\frac{V''(z)}{\pi}\right]^{1/2} e^{V(z)-zV'(z)}\int dx\, x^i e^{-V(x)+xV'(z)}. \qquad (6.2.7)$$

In fact, if we write eq. (6.2.1) in the following form:

$$Z_N^{(V)}(\Lambda) = Z_1/Z_0, \qquad (6.2.8)$$

then the numerator Z_1 can be easily calculated by the transformation of the matrix valuables

$$A = V'(\Lambda), \quad B = X+\Lambda, \qquad (6.2.9)$$

and by making use of Mehta's formula [23] for two hermitian matrices A and B:

$$\int \mathcal{D}B\, e^{\mathrm{tr}\,[-V(B)+cAB]} = \frac{(\pi/c)^{N(N-1)/2}}{\Delta(a)}\int\prod_{i=1}^N db_i\,\Delta(b)e^{\sum_{i=1}^N[-V(b_i)+ca_ib_i]}, \qquad (6.2.10)$$

where $(a_1,\cdots,a_N)$ and $(b_1,\cdots,b_N)$ are the eigenvalues of the matrices A and B, respectively. We thus obtain

$$\begin{aligned} Z_1 &= e^{\mathrm{tr}\,[V(\Lambda)-\Lambda V'(\Lambda)]}\frac{\pi^{N(N-1)/2}}{\Delta(V'(z))}\int\prod_{i=1}^N dx_i\,\Delta(x)e^{\sum_{i=1}^N[-V(x_i)+x_iV'(z_i)]}\\ &= \frac{\pi^{N^2/2}}{\Delta(V'(z))}\prod_{j=1}^N\left[\frac{1}{V''(z_j)}\right]^{1/2}\left[\det\phi^{(i)}(z_j)\right]_{\substack{i=0,1,\cdots,N-1\\ j=1,2,\cdots,N}}. \qquad (6.2.11)\end{aligned}$$

We can calculate the denominator Z_0 more easily because we only have to transform in the above expression as

$$X \mapsto \epsilon X, \quad V \mapsto \frac{1}{\epsilon^2} V, \tag{6.2.12}$$

and to take the limit $\epsilon \to 0$. We obtain in this way

$$\begin{aligned} Z_0 &= \frac{\pi^{N^2/2}}{\Delta(V'(z))} \prod_{j=1}^{N} \left[\frac{1}{V''(z_j)}\right]^{1/2} \left[\det z_j^i\right]_{\substack{i=0,1,\cdots,N-1 \\ j=1,2,\cdots,N}} \\ &= \frac{\pi^{N^2/2}}{\Delta(V'(z))} \prod_{j=1}^{N} \left[\frac{1}{V''(z_j)}\right]^{1/2} \Delta(z). \end{aligned} \tag{6.2.13}$$

Next, we introduce the following Miwa coordinates

$$\begin{aligned} x_n &= -\frac{1}{n} \mathrm{tr}\, \Lambda^{-n} \\ &= -\frac{1}{n} \sum_{j=1}^{N} z_j^{-n} \quad (n = 1, 2, 3, \cdots) \end{aligned} \tag{6.2.14}$$

in order to establish the relation between $Z_N^{(V)}(\Lambda)$ and τ functions of the KP hierarchy. As in subsec. 4.3.2, we bosonize the fermion fields $\psi(z)$ and $\psi^\dagger(z)$

$$\psi(z) = \sum_n \psi_n z^n \tag{6.2.15}$$

$$\psi^\dagger(z) = \sum_n \psi_n^\dagger z^{-n-1} \tag{6.2.16}$$

by using a scalar boson

$$\phi(z) = q + J_0 \ln z - \sum_{n \neq 0} \frac{J_n}{n} z^{-n} \tag{6.2.17}$$

as

$$\psi(z) = : e^{-\phi(z)} :, \quad \psi(z)^\dagger = : e^{\phi(z)} : . \tag{6.2.18}$$

Here the boson satisfies the following OPE:

$$\phi(z)\phi(w) \sim \ln(z - w), \tag{6.2.19}$$

that is, its mode coefficients satisfy the following commutation relations:

$$\begin{aligned} [\, J_0, q\,] &= 1 \\ [\, J_n, J_m\,] &= n\delta_{n+m,0}. \end{aligned} \tag{6.2.20}$$

Note that $\exp(q)$ raises the fermion number by one. For the normal ordering we use the following definition:

$$: e^{\alpha\phi(z)} : = e^{\alpha q} z^{\alpha J_0} \left(e^{\alpha\phi(z)}\right)_- \left(e^{\alpha\phi(z)}\right)_+, \tag{6.2.21}$$

so that we have the Wick theorem in its conventional form:

$$: e^{\alpha\phi(z)} :: e^{\beta\phi(w)} : = (z - w)^{\alpha\beta} : e^{\alpha\phi(z) + \beta\phi(w)} : . \tag{6.2.22}$$

Using this equation repeatedly, we obtain

$$\begin{aligned} &\psi(z_N) \cdots \psi(z_1) \\ &\quad = : e^{-\phi(z_N)} : \cdots : e^{-\phi(z_1)} : \\ &\quad = \prod_{i>j} (z_i - z_j) : e^{-\sum_{j=1}^N \phi(z_j)} : \qquad (6.2.23) \\ &\quad = \Delta(z) e^{-Nq} (z_1 \cdots z_N)^{-J_0} \left(e^{-\sum_{j=1}^N \phi(z_j)}\right)_- \left(e^{-\sum_{j=1}^N \phi(z_j)}\right)_+ . \qquad (6.2.24) \end{aligned}$$

Thus with respect to the Dirac vacuum

$$|\,0\,\rangle = \psi_0^\dagger \psi_{+1}^\dagger \psi_{+2}^\dagger \cdots |-\infty\,\rangle, \tag{6.2.25}$$

and to the x_n's defined in eq. (6.2.14), we have the following relation:

$$\begin{aligned} \langle 0\,|\, e^{-\sum_n x_n J_n} &= \langle 0\,| \left(e^{-\sum_{j=1}^N \phi(z_j)}\right)_+ \\ &= \langle 0\,| \left(e^{-\sum_{j=1}^N \phi(z_j)}\right)_- \left(e^{-\sum_{j=1}^N \phi(z_j)}\right)_+ \\ &= \frac{1}{\Delta(z)} \langle 0\,|\,(z_1 \cdots z_N)^{J_0} e^{Nq} \psi(z_N) \cdots \psi(z_1) \\ &= \frac{1}{\Delta(z)} \langle -N\,|\,\psi(z_N) \cdots \psi(z_1) \\ &\qquad\qquad (\, J_0\,|-N\,\rangle = (-N)\,|-N\,\rangle\,). \end{aligned} \tag{6.2.26}$$

Thus, for the decomposable multi-fermion state

$$|\,g_N\,\rangle = \sum_{n_1, \cdots, n_N \in \mathbf{Z}} \left(\phi_{n_1}^{(1)} \psi_{n_1}^\dagger\right) \cdots \left(\phi_{n_N}^{(N)} \psi_{n_N}^\dagger\right) |-N\,\rangle, \tag{6.2.27}$$

the corresponding τ function is rewritten as follows:

$$\begin{aligned} \langle 0\,|\, e^{-\sum_{n \geq 1} x_n J_n}\,|\, g_N\,\rangle &= \frac{1}{\Delta(z)} \langle -N\,|\,\psi(z_N) \cdots \psi(z_1)\,|\,g_N\,\rangle \\ &= \frac{1}{\Delta(z)} \left[\det \phi^{(i)}(z_j)\right]_{\substack{i=0,1,\cdots,N-1 \\ j=1,2,\cdots,N}}, \end{aligned} \tag{6.2.28}$$

where

$$\phi^{(i)}(z) \equiv \sum_{n\in\mathbf{Z}} \phi_n^{(i)} z^n. \tag{6.2.29}$$

The last expression has exactly the same form with eq. (6.2.6).

We therefore conclude from the above discussions that the partition function of the GKM with finite N is a τ function specified by the decomposable multi-fermion state (6.2.27) with (6.2.7), or equivalently, specified by an element of the infinite dimensional Grassmannian represented by the subspace

$$U_N = [\phi^{(0)}(z), \phi^{(1)}(z), \cdots, \phi^{(N-1)}(z),\ z^N,\ z^{N+1},\ \cdots] \tag{6.2.30}$$

with (6.2.7). For finite N, this τ function corresponds to an N-soliton solution and only the first N x_n's are algebraically independent. Now we consider the limit $N \to \infty$ which can be taken "smoothly." Note that all the $\phi^{(i)}(z)$'s are not independent even in such a limit since they are related with each other through the equation (6.2.7). In fact one can easily show that they satisfy the following equations:

$$\left[(V''(z))^{-1/2}\frac{d}{dz}(V''(z))^{-1/2} + z\right]\phi^{(i)}(z) = \phi^{(i+1)}(z) \tag{6.2.31}$$

$$V'(z)\phi^{(i)}(z) = \text{linear combination of the } \phi^{(j)}(z)\text{'s}. \tag{6.2.32}$$

Thus, for the element (6.2.30) of the infinite dimensional Grassmannian with $N = \infty$, this equation means

$$\left[(V''(z))^{-1/2}\frac{d}{dz}(V''(z))^{-1/2} + z\right] U_\infty \subset U_\infty \tag{6.2.33}$$

$$V'(z)U_\infty \subset U_\infty. \tag{6.2.34}$$

In particular, if we set

$$V(z) = \frac{1}{p(p+1)} z^{p+1}, \tag{6.2.35}$$

then eqs. (6.2.33) and (6.2.34) are rewritten into the following form:

$$\left[z^{-p+1}\frac{d}{dz} + \frac{-p+1}{2}z^{-p} + z\right] U_\infty \quad , \tag{6.2.36}$$

$$= e^{-z^{p+1}/(p+1)}\left[z^{-p+1}\frac{d}{dz} + \frac{-p+1}{2}z^{-p}\right] e^{z^{p+1}/(p+1)}\, U_\infty \subset U_\infty$$

$$z^p U_\infty \subset U_\infty. \tag{6.2.37}$$

These equations are nothing but the string equation with a background shift at x_{p+1} (see eqs. (4.3.39), (4.3.45) and (4.3.46)), and thus lead to the $W_{1+\infty}$ (or W_p) constraint on the τ function as we have already shown in detail in chapter 4.

Surprisingly, in the GKM we can change the p of the $(p,1)$ gravity freely by changing the form of the function $V(z)$. It might be interesting to try to write down the universal equation which describes the renormalization group flows changing both p and q by making use of the GKM. However, thus obtained equation is not satisfactory since it has no p-q symmetric form.[1]

[1] I thank H. Kawai and K. Ogawa for discussion on this point.

Chapter 7

Conclusion

In the present paper, we have tried to reveal the universal structure of two-dimensional quantum gravity and noncritical strings by using the dynamical triangulation method (DTM) as their non-perturbative definition. In the following, we summarize and criticize our results.

After explaining the DTM in chapter 2, we consider the (p,q) gravity in chapters 3 and 4. There we showed that the Schwinger-Dyson equation of the system is expressed as the W_p constraint on the square root of the partition function with source terms, which is a τ function of the p-reduced KP hierarchy, and proved that the Schwinger-Dyson equation is equivalent to the Douglas equation. Moreover, we showed that the renormalization group flows which does not change the p in (p,q) are nothing but the time evolution of the p-reduced KP hierarchy. However, our Schwinger-Dyson equation does not give us the really universal description of the two-dimensional quantum gravity since it is insensitive to the change of the p in (p,q). Thus we considered in chapter 5 the duality transformation between the (p,q) gravity and the (q,p) gravity as a first step towards the solution. There we showed that some operators we have considered are redundant, and that among the first $(p+q)$ operators the really independent ones are given by $\mathcal{O}_n$ $(n \neq 0 \pmod{p},\ n \neq 0 \pmod{q})$, consistent with the BRST cohomological analysis by Lian and Zuckerman. In chapter 6, as another way to the solution and as an exercise of the methods developed in chapter 4, we reviewed the generalized Kontsevich model which describes all the $(p,1)$ topological

gravities systematically. Although this model actually describes the flow with respect to the p, it is not sufficient for our purpose since it is difficult to extend it to the (p,q) gravity in a manifestly p-q symmetric manner. Thus we might have to conclude that we have not yet reached the final understanding of the universal structure of two-dimensional quantum gravity.

Now we reconsider our results from the string theoretic point of view. Although our results revealed some universal structures of noncritical strings, it gave us no information about the non-perturbative structure of the string theory we were interested in. In fact, whereas we would like to control freely the non-perturbative parameter θ in the partition function (*e.g.*, the first double pole of a solution of the Painlevé equation), we can deal only with the flow parameter of the KP hierarchy which is based on the asymptotic expansion of a τ function. Thus we cannot see the θ in our formalism.

Although there are such shortcomings as above in our Schwinger-Dyson equation, it is plausible that our method and idea explained in the present paper will be efficient in the systematic investigation of higher dimensional quantum gravity based on DTM. Moreover, numerical simulations of the higher dimensional DTM have started recently [38], showing a phase transition between the stable space-time and the unstable space-time, which qualitatively agrees with the result of a continuum theory [39] obtained on the basis of the ϵ expansion of $(2+\epsilon)$-dimensional quantum gravity. Note that if the phase transition is certainly of second order, we then can confirm ourselves that quantum gravity can also be defined as a usual field theory. However, if it is of first order, then we should face the string theory seriously again.

Acknowledgement

It is a pleasure to acknowledge the aid I have received from my teachers and colleagues. First of all, I am very grateful to my adviser Prof. T. Eguchi for his kind advice and continuous encouragement throughout my Ph.D. course, and also to Prof. H. Kawai for stimulating discussions and letting me know the joy of physics through the collaboration. I would like to thank Prof. K. Igi, Prof. K. Fujikawa and Prof. M. Ninomiya for their kind-hearted advice and continuous encouragement, and to Dr. R. Nakayama for stimulating discussions and his kind help through the collaboration. I also wish to thank Prof. T. Yoneya, Prof. N. Sakai, Prof. Y. Kazama, Prof. E. Date, Prof. Y. Kitazawa, Prof. H. Ooguri, Dr. Y. Okamoto, Dr. K. Ogawa, Dr. S. Hosono, Dr. Y. Matsuo and Dr. Y. Yamada for useful discussions. Finally, I also gratefully acknowledge the encouragement and help given by Dr. A. Kato, Dr. T. Kawai, Dr. H. Kunitomo, Dr. S. Mizoguchi, Dr. S. Odake, Dr. S. Iso, Dr. H. Murayama, Mr. T. Dillon, Mr. R. Hays, Mr. R. Nagaoka, Mr. M. Troester, Mr. T. Nakatsu, Mr. T. Kuruma, Mr. T. Sano, Mr. Y. Quano, Mr. S. Ishihara, Mr. Y. Sugawara, Ms. N. Imamura, Ms. N. Takahashi and Ms. Y. Watanabe as well as to other colleagues and secretaries at the University of Tokyo.

Appendix A

Explicit Construction of the τ Functions

In this appendix, we explain a method of constructing a τ function directly from a solution of the KP hierarchy.

A.1 The KP Hierarchy and the Sato Equation

In this section, we define the KP hierarchy and derive the Sato equation which plays an important role when we interpret the KP hierarchy as a dynamical system over an infinite dimensional Grassmann manifold [40][41][42].

First we introduce functions $u_i(t,x)$ $(i = 2, 3, \cdots)$ of infinitely many variables $(t,x) = (t, x_1, x_2, x_3, \cdots)$ and define the Lax operator

$$L \equiv \partial + u_2(t,x)\partial^{-1} + u_3(t,x)\partial^{-2} + \cdots \tag{A.1.1}$$

as a pseudo-differential operator with respect to $\partial \equiv \partial/\partial t$. Here, ∂^{-1} is defined so that $\partial\partial^{-1} = \partial^{-1}\partial = 1$. Explicitly, we have for any function f

$$\partial^k \cdot f \;=\; \sum_{l=0}^{\infty}\binom{k}{l}\frac{\partial^l f}{\partial t^l}\cdot\partial^{k-l}, \quad \binom{k}{l} \equiv \frac{k(k-1)\cdots(k-l+1)}{l!}. \tag{A.1.2}$$

We further introduce the potentials B_n and B_n^C as

$$B_n \equiv (L^n)_+, \quad B_n^C \equiv -(L^n)_-, \tag{A.1.3}$$

where $(\ )_+$ (*resp.* $(\ )_-$) is non-negative (negative) power part of a pseudo-differential operator with respect to ∂. Then the KP hierarchy is defined as the set of the following differential equations:

$$\frac{\partial}{\partial x_n}L = [B_n^C, L] \; (= [B_n, L]). \tag{A.1.4}$$

Note that they satisfy the integrability conditions $\partial^2 L/\partial x_n \partial x_m = \partial^2 L/\partial x_m \partial x_n$. The following theorem is fundamental.

Theorem A.1.1

There exists a pseudo-differential operator W of the form

$$W = 1 + w_1(t,x)\partial^{-1} + w_2(t,x)\partial^{-2} + \cdots \tag{A.1.5}$$

such that

$$L = W\partial W^{-1}, \tag{A.1.6}$$

$$\frac{\partial}{\partial x_n}W = B_n^C W = B_n W - W\partial^n. \tag{A.1.7}$$

The equation (A.1.7) is called the Sato equation.

[proof]

First we introduce a group $\mathcal{G}$ and its Lie algebra $Lie\,\mathcal{G}$ as the set of elements of the following forms:

$$\begin{aligned} \mathcal{G} &\equiv \{1 + v_1(t)\partial^{-1} + v_2(t)\partial^{-2} + \cdots\}, \\ Lie\,\mathcal{G} &\equiv \{b_1(t)\partial^{-1} + b_2(t)\partial^{-2} + \cdots\}. \end{aligned} \tag{A.1.8}$$

Then the B_n^C's can be regarded as the components of a $Lie\,\mathcal{G}$-valued 1-form Ω^C on the space of the x_n's:

$$\Omega^C \equiv \sum_{n\geq 1} B_n^C dx_n. \tag{A.1.9}$$

Then the KP equation (A.1.4) is rewritten as

$$dL = [\Omega^C, L], \quad d \equiv \sum_{n\geq 1} dx_n \frac{\partial}{\partial x_n}. \tag{A.1.10}$$

Furthermore, by using an identity $\frac{\partial}{\partial x_l}B_k^C = ([B_l, B_k^C])_-$, one can easily show that eq. (A.1.10) leads to the zero-curvature condition

$$d\Omega^C = \Omega^C \wedge \Omega^C. \tag{A.1.11}$$

Thus the connection Ω^C is a pure gauge and can be written in the form

$$\Omega^C = -V^{-1}dV, \tag{A.1.12}$$

where V is a $\mathcal{G}$-valued function of the x_n's. It is also easy to see that $d(VLV^{-1}) = 0$, which indicates that the VLV^{-1} has the form

$$VLV^{-1} = \partial + r_2(t)\partial^{-1} + r_3(t)\partial^{-2} + \cdots. \tag{A.1.13}$$

Therefore, VLV^{-1} can be expressed as

$$VLV^{-1} = U\partial U^{-1} \tag{A.1.14}$$

using an element U of $\mathcal{G}$ which depends only on t:

$$U = 1 + p_1(t)\partial^{-1} + p_2(t)\partial^{-2} + \cdots. \tag{A.1.15}$$

Hence, if we denote $V^{-1}U$ by W, that is,

$$W \equiv V(t,x)^{-1}U(t) \equiv 1 + w_1(t,x)\partial^{-1} + w_2(t,x)\partial^{-2} + \cdots, \tag{A.1.16}$$

then the following relations hold:

$$L = W\partial W^{-1}, \tag{A.1.17}$$

$$\Omega^C = -V^{-1}dV = -W dW^{-1} = dW \cdot W^{-1}. \tag{A.1.18}$$

The latter equation $dW = \Omega^C W$ is nothing but the Sato equation

$$\frac{\partial}{\partial x_n}W = B_n^C W = B_n W - W\partial^n. \quad \blacksquare$$

[remark]

From the identity $B_1 = (L)_+ = \partial/\partial t$, we obtain

$$\frac{\partial}{\partial x_1}L = [B_1, L] = \frac{\partial}{\partial t}L. \tag{A.1.19}$$

Thus every function f appearing in the KP equation depends on the variables (t, x) in the following manner:

$$f = f(t + x_1, x_2, x_2', \cdots), \tag{A.1.20}$$

so that we can (and will) set $t = 0$ without loss of generality. Then ∂ is interpreted as a differential operator $\partial/\partial x_1$.

A.2 τ Functions and an Infinite Dimensional Grassmann Manifold

In this section, by using the Sato equation, we show that the KP hierarchy is nothing but a dynamical system over an infinite dimensional Grassmann manifold [40][41][42]. Furthermore we explain that all the unknown functions $u_i(x)$'s in $L = \partial + u_2(t,x)\partial^{-1} + u_3(t,x)\partial^{-2} + \cdots$ can be described in terms of a single function, Hirota's τ function [43]. In what follows, we assume that the functions $w_j(x)$ can be taylor-expanded around a point $x^0 = (x_1^0, x_2^0, \cdots)$. In particular, we restrict ourselves to the case $x^0 = 0$. Generalization to the cases $x^0 \neq 0$ is straightforward.

Let H be the *linear* space consisting of pseudo-differential operators, which is isomorphic to $\mathbf{C}^{\mathbf{Z}}$:

$$H = \{\sum_{k\in\mathbf{Z}} a_k \partial^k\}, \tag{A.2.1}$$

and H_+ be a linear subspace of H consisting of all the *differential* operators. Then all the linear subspaces of H which are linearly isomorphic to H_+ make an infinite dimensional Grassmann manifold, and we denote it by UGM (Universal Grassmann Manifold).[1]

Now we make a mapping from the set of the solutions of the KP equation $\{W\}$ into a set of orbits in the UGM in the following way. First we construct the vectors $\eta^{(n)}(x)$ $(n = 0, 1, 2, \cdots)$ in H as those whose components are the coefficients of the pseudo-differential operators $\partial^n W$:

$$\begin{aligned} \partial^n W &= \sum_{k\in\mathbf{Z}} \eta_k^{(n)}(x)\partial^k, \qquad (A.2.2)\\ \eta^{(n)}(x) &= (\eta_k^{(n)}(x))_{k\in\mathbf{Z}} \end{aligned}$$

[1] For more mathematically complete definitions, see ref. [42].

$$= \begin{bmatrix} \vdots \\ \eta_{-2}^{(n)}(x) \\ \eta_{-1}^{(n)}(x) \\ \eta_0^{(n)}(x) \\ \eta_1^{(n)}(x) \\ \vdots \end{bmatrix}. \tag{A.2.3}$$

Then we define the subspace spanned by these vectors $\eta^{(0)}(x)$, $\eta^{(1)}(x), \cdots$ and denote it by $V(x)$:

$$V(x) = [\eta^{(0)}(x), \eta^{(1)}(x), \cdots]. \tag{A.2.4}$$

Note that we can represent the subspace $V(x)$ as a $\mathbf{Z} \times \mathbf{Z}_+$ matrix

$$\eta(x) = [\eta^{(0)}(x)\, \eta^{(1)}(x)\, \cdots], \tag{A.2.5}$$

Obviously, this matrix has an ambiguity of the right-multiplication of $GL(\mathbf{Z}_+)$, which corresponds to changes of the basis of $V(x)$.

The time (x) evolution of the $V(x)$ in UGM is, in general, determined from the Sato equation (A.1.7). This evolution can be represented in terms of the corresponding matrix in the following form:

$$\eta(x) = e^{-\sum_{n\geq 1} x_n \Lambda^n} \eta(0) C(x), \quad C(x) \in GL(\mathbf{Z}_+), \tag{A.2.6}$$

where $\Lambda = (\Lambda_{kl}) = (\delta_{k,l+1}) \in GL(\mathbf{Z})$. In fact, since the first term on the right-hand side of eq. (A.1.7) has the form $B_n W = \sum_{j\geq 0} b_{n,j}\partial^j W$, it does not change the subspace $V(x)$ itself and gives rise to the factor $C(x)$ in eq. (A.2.6). Hence x-evolution of the $V(x)$ comes only from the second term $-W\partial^n$ in eq. (A.1.7), which is integrated into $\exp[-\sum_n x_n \Lambda^n]$ in the matrix representation. Thus we have seen that the KP hierarchy is nothing but a dynamical system over the UGM whose time evolution is given in a simple form $\exp[-\sum_n x_n \Lambda^n]$.

We next show that a solution of the KP equation, W, which has infinitely many unknown functions $w_j(x)$, can be expressed by a single function, τ function. First let us introduce a matrix $\bar{\eta}(x)$, which also represents the subspace $V(x)$, as

$$\bar{\eta}(x) \equiv e^{-\sum_{n\geq 1} x_n \Lambda^n} \eta(0), \tag{A.2.7}$$

and further we decompose it in the following way:

$$\bar{\eta}(x) = \begin{bmatrix} \bar{\eta}_-(x) \\ \bar{\eta}_+(x) \end{bmatrix}, \tag{A.2.8}$$

$$\bar{\eta}_-(x) \equiv (\eta_k^{(n)}(x))_{k<0,\, n\geq 0}, \quad \bar{\eta}_+(x) \equiv (\eta_r^{(n)}(x))_{k\geq 0,\, n\geq 0}. \tag{A.2.9}$$

Then the τ function corresponding to the subspace $V(x)$ is defined as

$$\tau(x) \equiv \det \bar{\eta}_+(x). \tag{A.2.10}$$

This τ function completely reproduces the solution of the KP equation due to the following theorem:

Theorem A.2.1

Let $\tau(x)$ be the τ function which corresponds to a solution of the Sato equation, $W = \sum_{j\geq 0} w_j(x)\partial^{-j}$. Then it holds that

$$\sum_{j\geq 0} w_j(x) k^{-j} = \tau(x - \epsilon(k^{-1}))/\tau(x), \tag{A.2.11}$$

where $x - \epsilon(k^{-1}) = (x_1 - 1/k,\, x_2 - 1/(2k^2),\, x_3 - 1/(3k^3), \cdots)$.

[proof]

First, if we rewrite $\bar{\eta}(x)$ as

$$\begin{aligned} \bar{\eta}(x) &= \begin{bmatrix} \bar{\eta}_-(x)\bar{\eta}_+(x)^{-1} \\ 1 \end{bmatrix} \bar{\eta}_+(x) \\ &\equiv \rho(x)\bar{\eta}_+(x), \end{aligned} \tag{A.2.12}$$

then the matrix $\rho(x)$ also represents the subspace $V(x)$. Noting that the vector having

the form

$$\begin{array}{cc} \begin{bmatrix} \vdots \\ * \\ * \\ 1 \\ 0 \\ 0 \\ \vdots \end{bmatrix} & \begin{array}{c} \vdots \\ \scriptstyle -2 \\ \scriptstyle -1 \\ \scriptstyle 0 \\ \scriptstyle 1 \\ \scriptstyle 2 \\ \vdots \end{array} \end{array} \tag{A.2.13}$$

does not exist in $V(x)$ except for

$$\eta^{(0)}(x) = \begin{array}{cc} \begin{bmatrix} \vdots \\ w_2(x) \\ w_1(x) \\ 1 \\ 0 \\ 0 \\ \vdots \end{bmatrix} & \begin{array}{c} \vdots \\ \scriptstyle -2 \\ \scriptstyle -1 \\ \scriptstyle 0 \\ \scriptstyle 1 \\ \scriptstyle 2 \\ \vdots \end{array} \end{array}, \tag{A.2.14}$$

we find that the first column vector of the matrix $\rho(x)$ is nothing but $\eta^{(0)}(x)$. Next, if we calculate $\bar{\eta}(x - \epsilon(k^{-1}))$, then we can show from eqs. (A.2.6), (A.2.7) and (A.2.9) that

$$\begin{aligned} \bar{\eta}(x - \epsilon(k^{-1})) &= \exp[\sum_{n\geq 1} \frac{1}{n}(\frac{\Lambda}{k})^n]\, \bar{\eta}(x) \\ &= \sum_{j\geq 0} (\frac{\Lambda}{k})^j \rho(x)\, \bar{\eta}_+(x) \\ &\equiv \begin{bmatrix} \bar{\rho}_-(x,k) \\ \bar{\rho}_+(x,k) \end{bmatrix} \bar{\eta}_+(x). \end{aligned} \tag{A.2.15}$$

Since a straightforward calculation shows that

$$\det \bar{\rho}_+(x,k) = \sum_{j\geq 0} w_j(x) k^{-j}, \tag{A.2.16}$$

we conclude that

$$\begin{aligned}\tau(x-\epsilon(k^{-1})) &= \det\tilde{\eta}_+(x-\epsilon(k^{-1})) \\ &= \det(\tilde{\rho}_+(x,k)\cdot\tilde{\eta}_+(x)) \\ &= (\sum_{j\geq 0} w_j(x)k^{-j})\cdot\tau(x). \quad \blacksquare\end{aligned}$$

The set of equations (A.1.5)-(A.1.7) and (A.2.11) yields useful formulas which express the second derivatives of $\ln\tau$ in terms of the pseudo-differential operator L:

$$\begin{aligned}\frac{\partial^2}{\partial x_1\partial x_n}\ln\tau &= (L^n)_{-1}, \\ \frac{\partial^2}{\partial x_2\partial x_n}\ln\tau &= 2(L^n)_{-2}+\frac{\partial}{\partial x_1}(L^n)_{-1}, \\ \frac{\partial^2}{\partial x_3\partial x_n}\ln\tau &= 3(L^n)_{-3}+3\frac{\partial}{\partial x_1}(L^n)_{-2}+\frac{\partial^2}{\partial x_1^2}(L^n)_{-1}+3\frac{\partial^2}{\partial x_1^2}\ln\tau\cdot(L^n)_{-1}, \\ &\cdots \text{ etc.,}\end{aligned} \tag{A.2.17}$$

where the symbol $(\ )_{-k}$ stands for the coefficient of ∂^{-k}.

A.3 Free Fermion Representation of τ Functions

In this section, we describe the method of expressing τ functions in terms of free fermions.

First, by using the correspondence between UGM and the set of all decomposable states (see eqs. (4.3.12) and (4.3.13)), we construct a multi-fermion state $|\,g\,\rangle$ from the matrix $\eta(0)=[\eta^{(0)}(0)\,\eta^{(1)}(0)\,\cdots]$ which represents the subspace $V(0)$:

$$|\,g\,\rangle = \sum_{n_0,n_1,\cdots}\eta^{(0)}_{n_0}(0)\eta^{(1)}_{n_1}(0)\cdots\psi^\dagger_{n_0}\psi^\dagger_{n_1}\cdots|-\infty\,\rangle. \tag{A.3.1}$$

Then, the multi-fermion state corresponding to the matrix $\tilde{\eta}(x)=\exp[-\sum_n x_n\Lambda^n]\,\eta(0)$ in eq. (A.2.7) is

$$|\,g(x)\,\rangle \equiv e^{-\sum_{n\geq 1}x_nJ_n}\,|\,g\,\rangle, \tag{A.3.2}$$

where $J_n=\sum_k :\psi^\dagger_{n+k}\psi_k:\ (n\in\mathbf{Z})$. Thus

$$\begin{aligned}\tau(x) &= \det\tilde{\eta}_+(x) = \langle 0|g(x)\rangle \\ &= \langle 0\,|\,e^{-\sum_{n\geq 1}x_nJ_n}\,|\,g\,\rangle.\end{aligned} \tag{A.3.3}$$

Note that this state $|\,g\,\rangle$ can be expressed in terms of the Clifford group *Cliff*:

$$|\,g\,\rangle = g\,|\,0\,\rangle, \quad g\in \mathit{Cliff}, \tag{A.3.4}$$

where the Clifford group is the set of the elements of the following form:

$$\mathit{Cliff} = \left\{ g=\exp[\sum_{n,m} :\psi^\dagger_n b_{nm}\psi_m:]\right\}. \tag{A.3.5}$$

Thus the $\tau(x)$ is also written as

$$\tau(x) = \langle 0\,|\,e^{-\sum_{n\geq 1}x_nJ_n}g\,|\,0\,\rangle, \quad g\in\mathit{Cliff}. \tag{A.3.6}$$

In some references, the different sign is used in the expression (A.3.3) in its exponent. We can change this by carrying out the *CP* transformation

$$\psi_n \mapsto \psi^\dagger_{-n-1} \tag{A.3.7}$$

$$\psi^\dagger_n \mapsto \psi_{-n-1}. \tag{A.3.8}$$

In fact, under this mapping, J_n changes its sign; $J_n \mapsto -J_n$, while the vacuum $|\,0\,\rangle$ remains unchanged. Moreover, recalling that τ functions can be calculated by using algebraic relations of fermions alone, we find that $\tau(x)$ is unchanged under the transformation. Thus, denoting the transformed element in *Cliff* by g', the $\tau(x)$ can be reexpressed as follows:

$$\begin{aligned}\tau(x) &= \langle 0\,|\,e^{-\sum_{n\geq 1}x_nJ_n}g\,|\,0\,\rangle = \langle 0\,|\,e^{\sum_{n\geq 1}x_nJ_n}g'\,|\,0\,\rangle \\ &= \langle 0\,|\,e^{\sum_{n\geq 1}x_nJ_n}\,|\,g'\,\rangle \qquad (|\,g'\,\rangle\equiv g'\,|\,0\,\rangle).\end{aligned}$$

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