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学 位 論 文

Soliton Phenomena in Inhomogeneous Systems

(不均一系におけるソリトン現象)

平成 4 年 12 月博士 (理学) 申請

東京大学大学院理学系研究科
物理学専攻
飯塚 剛

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論 文 内 容 の 要 旨

ソリトン現象は、ミクロ系マクロ系問わず、自然界においていたるとこで観測される。特に一次元、二次元の非線形、分散力学系においては、弱非線形近似を行うことによって、コルトヴェーグ・ド・フリース (KdV) 方程式や非線形シュレディンガー (NLS) 方程式といった、代表的なソリトン方程式に極めて一般的に帰着される事が分かっている。これを可能にするのが逓減摂動法である。これは特異摂動の一種であり、波動について次の2つの仮定にもとずく；

a) その振幅は小さいが有限。(非線形効果の最低次までは考慮に入れる。)

b) それは (あるいはその包絡波は)、時間的空間的にゆっくり変化する。

摂動の最低次の部分から、波動の非線形効果、分散効果の競合を表す非線形偏微分方程式が得られる。この手法は、複雑な非線形波動系から、扱いやすいモデル方程式を得るために非常に有用である。この方程式が、多くの場合、ソリトン方程式になっているのである。

ここで系に不均一性が存在したらどうなるのか？ これは、自然界におけるソリトン現象を解明するのに重要且つ興味深い問題の一つである。結論から言うと、上で述べたような競合方程式に帰着させるのは、一般的に困難である。なぜなら、仮定 b) により波の特徴的波長はある大きい値に決まっており、初期条件としてこの波長をもつ波を与えても、不均一性の特徴的な長さによっては別のフーリエ成分が励起される。従って、もはや仮定 b) が成立できなくなるからである。この場合は、元の複雑な方程式にたちにもどって、議論するしか方法はないであろう。ところが次の2種類の不均一系は、逓減摂動法の適用が可能である。つまり、モデル方程式の段階でソリトンに与えられる不均一性効果を議論することができる。

I) 不均一性の特徴的な長さが、波長に比べ非常に長いとき。

II) 不均一領域が、波長に比べ非常に短い領域に局在しているとき。

I) のとき不均一性は、ゆっくり (slowly) 変化している。これについては、最近の発展を含めて Part I で取り上げる。II) については、ソリトンの散乱現象を重点に、Part II で議論する。論文では、非線形力学系で基本的な非線形格子を主に考えたが、同様の手法は (均一系のアナロジーにより)、流体、プラズマ、非線形光学系、など多くの系に適用できると考えられる。

PartI

I) のとき、さらに不均一性を2種類に分ける。

1) 有限不均一；不均一性を表す関数を、$f(x)$ とする。f の揺らぎΔfが fの特徴的な値、たとえは平均値 $\langle f \rangle$ と同程度の場合。

$$\Delta f \sim \langle f \rangle$$

2) 弱不均一；揺らぎΔfが小さい場合。

$$\Delta f \ll \langle f \rangle$$

1) の場合、1 次元モデル方程式より、ソリトン分裂、増減幅現象などが起こり得る事がわかっている。2) の場合は、KdV 方程式や、NLS 方程式などの厳密に解けるモデル方程式になり、ソリトン分裂、増減幅現象等は起こらず、ソリトンの変形のみ起こる事がわかった。その表式は、解析的に求まる。2) の場合の応用例として、ランダム系を考えた。ソリトンの統計的振る舞いは、random walk と同じものである事が分かった。　I) の不均一系は、不安定系、2 次元系に対してもモデル方程式が得られる。

<u>PartII</u>

　1 次元で、局在部分が一ヶ所のときを考える。このときその不均一性によって入射ソリトンの散乱が起こる、つまり透過、反射波が発生する。

　この際、不均一部分で分けられる左右の領域は、それぞれ均一系と考えられるので、通常の逓減摂動法を適用する。但し、左側には群速度が互いに逆な入射反射波が混在するが、独立な（空間発展の）ソリトン方程式を満たす。右側には透過波があるだけとして, これも同様（空間発展の）のソリトン方程式に従う。これら 3 つの波は局在部の接続条件として、関係付けられている。特に入射波が与えられれば、その点における反射透過波のダイナミックスがわかる。よってソリトン方程式を解く事によって、任意の点における、反射透過波が構成できる事になる。

　この方法によって、質量分布に不連続性のある非線形格子での、KdV ソリトンおよびNLS ソリトンの散乱を解析した。また　不純物を含む非線形格子でも同様の解析をした。但し KdV ソリトンの場合、格子波は長波長なので不純物は、他に比べ非常に重くなければならない。これは、いわば物体（重い不純物）と波動（格子波）の相互作用の一つの形態、と考える事ができる。

　さらにこの手法は、質量分布に不連続面のある 2 次元非線形格子にも拡張できる。このとき、空間発展のソリトン方程式は、カドモチェフ・ペトビアシビリ (KP) となる。線状ソリトンの散乱は、反射屈折現象の非線形効果による影響を与えている。

　本論文は、2 章での有限不均一 KdV 方程式の導出と 6 章での KdV ソリトンの散乱を除きすべて、著者および共同研究者のオリジナルである。とくに、不均一分散関係式の導入、弱不均一性の概念の提出、ソリトンのランダム歩行的描像、ＮＬＳソリトンの散乱、物体とソリトンの相互作用、および反射屈折の非線形効果は新しい研究と考えられる。

Doctor Thesis

Soliton Phenomena in Inhomogeneous Systems

Takeshi Iizuka

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Department of Physics, Faculty of Science,
University of Tokyo,
Hongo 7-3-1, Bunkyo-Ku, Tokyo 113, Japan.

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CONTENTS

Chapter 1

Introduction

Preface

Since the discovery of soliton phenomena (Zabusky and Kruskal 1965) for the Korteweg-de Vries (KdV) equation, soliton physics has been one of the most active fields in nonlinear science. The KdV equation was presented to describe the water surface wave in the last century(Korteweg and de Vries 1895). At the same time one soliton solution was analytically found. The model equation includes both nonlinear and dispersive effects whose competition causes the generation of solitary waves. The notion of the soliton has been developed since the initial value problem of the KdV equation was solved by the inverse scattering method (Gardner, Greene, Kruskal and Miura 1967). It is believed that the particle-like stability of solitons is a reflection of the complete integrability of the KdV equation.

The KdV equation is derived from a wide class of one-dimensional nonlinear dispersive systems such as fluid, anharmonic lattice and plasma. For any case, a balance between the nonlinear effect and the dispersion is collected by a perturbational method. A systematic method to obtain the simple model equations from complex nonlinear systems was presented (Taniuti et al. 1974), which is known as the **reductive perturbation method**. In applying this method, we assume that the amplitude of the wave is small but finite (weakly nonlinear approximation) and that the wave is slowly varying (weakly dispersive approximation). To take these assumptions into account, transformations of dependent and independent variables are introduced by use of a smallness parameter. Applying the perturbation method with respect to the smallness parameter, the KdV equation or its families are quite generally derived.

The reductive perturbation method is applicable also to other situations such as two-dimensional systems, modulations of monochromatic waves and a coupling of multi-waves. In most cases the obtained equations are completely integrable equations (soliton equations) such as the Kadmotsev-Petviashivili(KP) equation, the Nonlinear Schrödinger (NLS) equation and the three-wave interaction equation. In this sense, soliton equations and soliton phenomena are ubiquitous in nonlinear dispersive wave systems. In fact solitons are observed in many stages of the nature. This is one of the reasons why the notion of soliton is so important in nonlinear science.

It should be noted that the soliton equations are derived mostly from homogeneous nonlinear systems. However, in real physical systems, effects of the **inhomogeneities** on the wave propagations are often important. In general, to treat the inhomogeneity in the frame of the reductive perturbation method (weakly nonlinear and dispersive approximation) seems to be difficult. But the following two types of inhomogeneities are possibly tractable and have been studied extensively. One is the 'slow inhomogeneity' which will be discussed in the part I. The other is 'localized inhomogeneity' which causes scatterings of the nonlinear waves. The part II is devoted to a study of this type of inhomogeneities.

Throughout this thesis inhomogeneous anharmonic lattices are mainly investigated as typical examples of inhomogeneous nonlinear systems. It should be mentioned that this choice does not lose the generality of the study, because anharmonic lattices contains essential features of nonlinear wave systems. That is, many important soliton equations are derived from anharmonic lattices in the similar way as that for most other nonlinear systems.

Introduction Related to Part I

Based on the recent results, nonlinear wave propagations in slowly inhomogeneous media are considered in part I. The recent studies will be reviewed shortly later. Let us start from a historical review of the studies on slow inhomogeneity in order to make clear the theme of chapter I. If the functions which represent the inhomogeneity of a system are slowly varying in space, it is called **slow inhomogeneity**. Shallow water wave traveling over an uneven bottom is one of the interesting examples and has been studied by many researchers (for instance Peregrine 1967, Madsen and Mei 1969, Kakutani 1971a, Johnson 1973). In these papers, it is assumed that the bottom is slowly varying such that the characteristic length of the bottom change is much longer than the wave length of the shallow water. In such case the reductive perturbation method is applicable and a model equation has been presented(Kakutani 1971a);

$$u_\sigma + c_1 u_\xi u + c_2 u_{\xi\xi\xi} - c_3 B_\sigma u = 0, \tag{1.1}$$

where $u(\sigma,\xi)$ is related to the surface elevation. Here and hereafter subscripts indicate partial differentiations with the respect to the corresponding variables. Independent variable σ indicates a space coordinate. Variable ξ refers to the coordinate running with the local velocity of the linear wave. Coefficients c_1, c_2 and c_3 are functions of the bottom surface $B(\sigma)$. The last term can be removed by a transformation of independent variables(Johnson 1973, Grimshaw 1979a). Moreover applying another transformation of dependent and independent variables, eq.(1.1) is rewritten as (Ono 1972),

$$u_\tau + 6u_\xi u + u_{\xi\xi\xi} + \nu(\tau)u = 0. \tag{1.2}$$

Variable ξ is same as that in (1.1) and τ a deformed space coordinate. Function $\nu(\tau)$ in the last term is related to the depth change. If the bottom is flat, this function disappears and (1.2) reduces to the well known KdV equation. Equation (1.1) and (1.2) represent competitions among effects of the nonlinearity, dispersion and inhomogeneity.

From (1.1) or (1.2), damping, growing and disintegration of a KdV soliton due to a slope of the bottom are discussed(Ono 1974a, S.Watanabe and N.Yajima 1984). For instance, a mild step in the bottom causes fissions of a soliton. A formula which gives the number of disintegrated solitons has been obtained (Ono 1972, Johnson 1973). Recently, Ono studied eq.(1.2) and discussed fissions of a incident soliton by means of the solitary wave approximation (Ono 1992).

The same type of equation as (1.1) or (1.2) is derived from a wide class of nonlinear inhomogeneous systems. Two typical examples other than the shallow water have been presented. One is the magneto-acoustic waves in which plasma density and magnetic field are inhomogeneous (Kakutani 1971b, Asano and Ono 1971). The other is anharmonic lattice where the distributions of masses and springs are not uniform (Ono 1972).

In the case of a slowly inhomogeneous system, we can also apply the reductive perturbation method to the nonlinear modulations of a monochromatic waves. The envelope u is, in general, shown to obey a NLS equation with a purely imaginary potential;

$$iu_\tau + u_{\xi\xi} + 2|u|^2 u + i\nu(\tau)u = 0, \tag{1.3}$$

where τ is related to a space coordinate. In the homogeneous limit, function $\nu(\tau)$ vanishes and (1.3) reduces to the ordinary NLS equation. Equation (1.3) is equivalent to the variable coefficient NLS equation (Grimshaw 1979b). By use of the reductive perturbation method, the model equation (1.3) is derived from the inhomogeneous KdV equation (1.2) (Ono 1974a), ion-acoustic systems in plasma with the x-dependent density and temperature (Leclert, Charles, Karney, Bers and Kaup 1979), and deep water waves over an uneven bottom (Ono 1991b). In deriving the inhomogeneous NLS equation (1.3), it should be noted that the carrier wave is not a simple plane wave any more, but slowly deformed due to the inhomogeneity. In the paper of Leclert et al., effect of a linearly increasing temperature is considered. For this inhomogeneity, $\nu(\tau)$ is proportional to the inverse of τ and (1.2) becomes the cylindrical NLS equation which is also completely integrable. One of the aims of chapter 2 is to obtain (1.3) from inhomogeneous anharmonic lattices.

Recently, a class of slow inhomogeneity—**weak inhomogeneity** is introduced (Wadati 1990, Ono 1991a, Iizuka and Wadati 1992a). If functions which represent inhomogeneities of a system are slowly varying and their fluctuations are much smaller than their mean amplitudes, we term the system weakly inhomogeneous system. For this class of inhomogeneous systems, the reductive perturbation method has been shown to be applicable. If the fluctuations are comparable to the characteristic magnitudes, the inhomogeneity is considered to be finite. (Equations (1.1~3) are typical model equations for systems of slow and finite inhomogeneities.) By the weakly nonlinear and dispersive approximation, the following model equation is obtained form a wide class of weakly inhomogeneous nonlinear media;

$$u_\tau + 6u_\xi u + u_{\xi\xi\xi} + \nu(\tau)u_\xi = 0. \tag{1.4}$$

Dependent variable τ and a function $\nu(\tau)$ represent a scaled space coordinate and inhomogeneity respectively. Variable ξ is the coordinate running with a constant velocity. Equation (1.4) is recently derived from inhomogeneous anharmonic lattice (Wadati 1990, Iizuka, Nakao and Wadati 1992) and shallow water wave over an uneven bottom (Ono 1991a).

As will be discussed in chapter 2, equation (1.4) is reduced to the KdV equation by a transformation of independent variables. Thus, equation (1.4) is completely integrable (Wadati 1983). One soliton solution of this KdV equation offers an analytical expression of the **deformation of a soliton**. It is found that the soliton never disintegrate or emit the radiation under weak inhomogeneity. This is a remarkable property of the weak inhomogeneity as compared with that of finite inhomogeneity.

Similarly to the case of slow and finite inhomogeneity, the reductive perturbation method is applicable to the nonlinear modulations of the monochromatic waves under the weak inhomogeneity. Recently a model equation is obtained from weakly inhomogeneous anharmonic lattices (Iizuka, Nakao and Wadati 1991,1992);

$$iu_\tau + u_{\xi\xi} + 2|u|^2 u + i\nu(\tau)u_\xi + \mu(\tau)u = 0. \tag{1.5}$$

Functions $\nu(\tau)$ and $\mu(\tau)$ are real and related to the inhomogeneity. The fourth and fifth terms in l.h.s. of (1.5) can be removed by transformations of independent variables and the phase of u. Thus, (1.5) is reduced to NLS equation and a deformation of the NLS soliton is analytically discussed as is for the KdV case. Equation (1.5) is also derived from shallow water wave (Ono 1991b). Similar to the case of the finite inhomogeneity, the carrier wave is deformed due to the weak inhomogeneity(Iizuka, Nakao and Wadati 1991,1992).

The purpose of part I is to investigate the effects of the slow inhomogeneity on the nonlinear wave propagations. Part I consists of the following chapters.

In chapter 2 the inhomogeneous dispersion relations (for one dimensional system) are introduced. They suggest how to introduce transformations of independent variables. And next the reductive perturbation method is applied to slowly inhomogeneous systems. As examples, anharmonic lattices whose mass distribution is slowly varying are considered. Two types of waves —slowly changing waves and modulations of monochromatic wave— and two types slow inhomogeneities —finite and weak inhomogeneities— are separately investigated. For all cases model equations which are presented above appear.

In chapter 3 the model equation are applied to the random lattice systems. The mass distributions are supposed to belong to a class of the weak inhomogeneity. The the mass distribution is assumed, for simplicity, to be a Gaussian white noise, which causes the random walk of solitons. Statistical behaviors of KdV soliton, modified KdV soliton and NLS soliton are analytically discussed (Wadati 1990, Iizuka, Nakao and Wadati 1991,1992).

Chapter 4 is devoted to an unstable system with slow inhomogeneities. The Rayleigh-Taylor instability problem (Iizuka and Wadati 1990), where the bottom is slowly varying, is studied. For the homogeneous case, by use of the reductive perturbation method, the unstable nonlinear Schrödinger (UNS) equation has been obtained at the critical wave number (T.Yajima and Wadati 1990ab, Iizuka, Wadati and T.Yajima 1991, Wadati, Iizuka and T.Yajima 1991, Wadati, T.Yajima and Iizuka 1991). This suggests that the UNS equation is ubiquitous in unstable nonlinear systems. The aim of this chapter is to present a model equation which is referred to as the inhomogeneous UNS equation.

A two-dimensional slowly inhomogeneous system— shallow water waves over a two dimensional uneven bottom (Iizuka and Wadati 1992b)— is investigated in chapter 5. For water waves in straits of varying depth and width, a generalized KP equation has been derived (David, Levi and Winternitz 1987,1989). Here, the model equations (1.1),(1.2) and (1.4) are extended to a two-dimensional case. This extension is referred to as the **inhomogeneous KP equation**. Analytical forms of deformations of the line soliton due to some kinds of weak depth changes are shown.

Introduction Related to Part II

The main theme of part II is investigation on nonlinear systems in which inhomogeneities are sufficiently localized. Let us define the **localized inhomogeneity**. If the inhomogeneity of a one-dimensional system is localized in a point or in a much shorter range than the wave length, it is called localized inhomogeneity. In higher dimensional systems, we also use the term 'localized', for instance, for an interface and boundary of the medium. Due to this type of inhomogeneities, there occurs **scattering of nonlinear waves**. That is for an incident wave, there appears reflected and transmitted waves. The main purpose of chapterII is to investigate scattering of soliton due to some kinds of the localized inhomogeneities.

In the application of the weakly nonlinear approximation to these systems, divide the medium are divided into two regions. Between them the inhomogeneous point (or range) exists. In one region incident and reflected waves propagate. Spatial evolutions of these two waves are determined by the reductive perturbation method in the assumption that the group velocities of them are opposite. In general the two waves obey independent soliton equations respectively. In the other region only the transmitted wave propagates. Spatial evolution of this wave is again determined by the reductive perturbation method. The direction of the group velocity is the same as that of incident waves.

The incident, reflected and transmitted waves should be connected at the point of the inhomogeneity (for long waves, we may regard the narrow range of the inhomogeneity as a point). This condition of the connection depends on the property of the system. Due to the condition, the motions of the reflected and transmitted waves are determined at the point from that of the incident wave. Thus, solving the spatial evolution equations for the reflected and transmitted waves respectively, we can construct the two waves from an incident wave(Fig.1.1). In the case that the there exist more than one localized inhomogeneities, if they are separated enough to each other, the method is also effective.

This idea was presented firstly by N.Yajima (N.Yajima 1975). He considered a discontinuous nonlinear lattice— there is a discontinuity of the mass distribution— and discussed scatterings of KdV soliton due to the discontinuity. The reflection and transmission of the incident soliton are analytically formulated by use of the

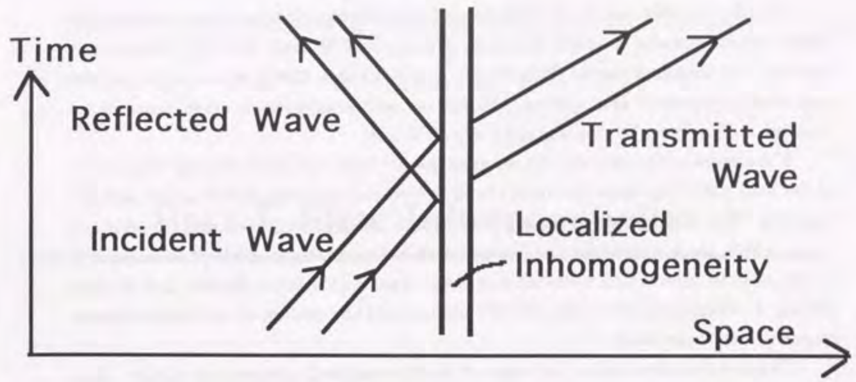

Figure 1.1: For a incident wave we can construct the reflected and transmitted waves.

inverse scattering method. Scatterings of the KdV soliton in plasma due to sheath ions are also investigated (T.Watanabe, Kanamori and N.Yajima 1989).

Moreover, shallow water waves over the localized inhomogeneities of the bottom have been investigated; over a barrier (Sugimoto, Hosokawa and Kakutani 1987) or over a step (Sugimoto, Nakajima and Kakutani 1987). The incident, reflected and transmitted waves are connected by the edge layer theory based on the matched asymptotic method (Sugimoto and Kakutani 1984, Sugimoto, Kusaka and Kakutani 1987). Then scatterings of the KdV soliton are analytically investigated.

This idea is also applicable to the modulations of carrier waves. Recently, scatterings of envelope soliton in anharmonic lattice due to the discontinuity of its mass distribution are investigated analytically (Iizuka and Wadati 1992c). This is the first study which treat the scatterings of the NLS soliton from the localized inhomogeneity. The same analysis has been done for the system with a mass impurity (Iizuka and Wadati 1992d).

The aim of part II is to analyze scatterings of lattice solitons due to some kinds of localized inhomogeneities. The outline of PartII is the followings.

In chapter 6, nonlinear lattices with a discontinuity in its mass distribution are investigated(Iizuka and Wadati 1992c). Fissions and reflections of the KdV and NLS solitons are discussed.

Chapter 7 is devoted to modulations of monochromatic wave in an anharmonic lattice which contain a single impurity (Iizuka and Wadati 1992d). We do not consider the localized mode (Nagahama and N.Yajima 1989) which is one of the important subject in this system. Reflection and transmission coefficients of an incident NLS soliton by the impurity are obtained.

If the mass of the impurity in an anharmonic lattice is much heavier than those of the host particles, there occurs a coupling between the long lattice waves and the impurity. The long lattice waves are divided into incident, reflected and transmitted waves which are governed by the independent KdV equations. This phenomenon is interpreted as interaction between nonlinear wave and matter (Iizuka and Wadati 1992e). In chapter 8, scattering of KdV soliton and the motion of the matter(heavy impurity) are calculated.

Chapter 9 is devoted to the case of two-dimensional anharmonic lattice. It is assumed that there is an infinite interface line of the mass distribution (Iizuka and Wadati 1993). The incident, reflected and transmitted lattice waves obey independent KPII equations. It is known that the KPII equation has stable line soliton solutions. Scatterings of an incident line soliton due to the interface is investigated, the results of which are interpreted as properties of nonlinear reflection and refraction phenomena.

Part I. Slow Inhomogeneity

Chapter 2

Model Equations

We present some model equations of nonlinear waves propagating in slowly inhomogeneous media. At first, inhomogeneous dispersion relations are introduced. From equation of motion for inhomogeneous and anharmonic lattices, we obtain the model equations by applying the reductive perturbation method. The equation represents a competition among effects of nonlinearity, dispersion and the inhomogeneity.

2.1 Inhomogeneous Dispersion Relations

Homogenous Case

Let us consider a wave propagation in a one-dimensional homogeneous and lossless system. We denote a wave field by $\phi(x,t)$, where x and t are space and time variables respectively. In this section we assume that the amplitude of the wave is infinitesimal such that the nonlinear effect is neglected. Using the plane wave solution $\phi = \exp i(kx - \omega t)$, we have the dispersion relation;

$$\omega = \omega(k). \tag{2.1.1}$$

Note that the dispersion relation (2.1.1) is directly related to the operator relation by the transformations;

$$\omega \Longleftrightarrow i\frac{\partial}{\partial t}, \tag{2.1.2a}$$

$$k \Longleftrightarrow -i\frac{\partial}{\partial x}. \tag{2.1.2b}$$

The operator relation is nothing but the linearized partial differential equation which governs $\phi(x,t)$.

Assume that the Taylor expansion of (2.1.1) around $k=0$ (**long wave limit**) is given by

$$\omega = vk - ak^3 + \cdots \qquad \text{(KdV type dispersion)}, \tag{2.1.3}$$

where v is the group velocity $d\omega/dk$(at $k=0$) and a is real constant. Quadratic term of k does not exist because of the invariance of the system under $x \leftrightarrow -x$ and $t \leftrightarrow -t$. We sometimes call (2.1.3) a weak dispersion. We introduce transformations of independent variables;

$$\begin{cases} \xi = \varepsilon(x - vt), \\ \tau = \varepsilon^3 at, \end{cases} \tag{2.1.4}$$

or

$$\begin{cases} \xi = \varepsilon(\frac{x}{v} - t), \\ \tau = \varepsilon^3 \frac{a}{v^4}x, \end{cases} \tag{2.1.5}$$

where ε is a smallness parameter. For both transformations, the operator relation (2.1.3) becomes

$$\partial_\tau + \partial_\xi^3 = 0, \tag{2.1.6}$$

where we have neglected higher order terms of ε. This is nothing but the linear part of the KdV equation. In fact, if we collect the lowest nonlinear term, we often obtain the KdV equation;

$$\phi_\tau + \phi_{\xi\xi\xi} + c\phi\phi_\xi = 0. \tag{2.1.7}$$

In this sense, we may call (2.1.3) KdV type dispersion.

The transformation (2.1.4) (Galilei type transformation) is useful when we consider the time evolution for a given initial value $\phi(x,0) = \phi(\xi,0)$, since initial value problem of the KdV equation is solved analytically (Gardner, Greene, Kruskal and Miura 1967). In the same reason the transformation (2.1.5) is useful if we are interested in the spatial evolution for a given boundary value at $x=0$; $\phi(0,t) = \phi(\xi,0)$ (Asano and Ono 1971). In addition, equation (2.1.5) is suitable for slowly inhomogeneous systems or locally inhomogeneous systems. That is, the inhomogeneity is expressed as a function of τ.

The similar discussion is effective for modulations of monochromatic wave (**short wave**). We assume that the spatial Fourier spectrum distributes around a narrow range around $k = k_0$. Thus we consider the Taylor expansion of the dispersion relation (2.1.1) around $k = k_0$;

$$\omega = \omega_0 + v(k - k_0) + a(k - k_0)^2 + \cdots \quad \text{(Schrödinger type dispersion)}, \tag{2.1.8}$$

where $\omega_0 = \omega(k_0)$. By a transformation of the dependet variable

$$\phi = \psi e^{i(k_0 x - \omega_0 t)}, \tag{2.1.9}$$

we can remove k_0 and ω_0 in the relation (2.1.8). We introduce the transformations of independent variables;

$$\begin{cases} \xi = \varepsilon(x - vt), \\ \tau = \varepsilon^2 at, \end{cases} \tag{2.1.10}$$

or

$$\begin{cases} \xi = \varepsilon(\frac{x}{v} - t), \\ \tau = \varepsilon^2 \frac{a}{v^3} x, \end{cases} \tag{2.1.11}$$

where $v = \mathrm{d}\omega/\mathrm{d}k(k = k_0)$ is the group velocity. With both transformations the operator relation at the lowest order of ε reduce to

$$i\partial_\tau + \partial_\xi^2 = 0. \tag{2.1.12}$$

which is nothing but the operator relation of the free Schrödinger equation. In this sense we term (2.1.8) Schrödinger type dispersion. We should note that when $v = 0$ the transformation (2.1.11) is meaningless. Throughout this thesis we assume that the group velocity is not very small (including zero). Collecting the lowest nonlinear effect, we often obtain the NLS equation

$$i\psi_\tau + \psi_{\xi\xi} + c\,|\psi|^2\,\psi = 0. \tag{2.1.13}$$

Initial value problem of the NLS equation has been solved analytically (Zakharov and Shabat 1972). Thus, similarly to the case of long wave limit, Galilei type transformation (2.1.10) is useful if an initial wave is given. For boundary value problems, (2.1.11) is suitable. In particular the latter is useful for the problem of light emission into a nonlinear optical fiber (Hasegawa 1989). Similarly to the case of long wave, transformation (2.1.11) is also useful for slowly inhomogeneous systems and locally inhomogeneous systems.

Taking the above discussions into account, we shall extend the dispersion relation to an inhomogeneous system and discuss the linear part of model equations.

Slow and Finite Inhomogeneity

In this subsection we consider **slow and finite inhomogeneities**; functions which represent the inhomogeneities of a system are slowly varying functions of x. We neglect nonlinearity and assume that the wave field $\phi(x,t)$ obeys a linear partial differential equation

$$i\partial_t\phi = \omega(-i\partial_x, \delta x)\phi, \tag{2.1.14}$$

where δx is a product of x with smallness parameter δ which represents the rapidness of the change in inhomogeneities. Using the relations (2.1.2), equation (2.1.14) is interpreted as a dispersion relation which explicitly depends on x;

$$\omega = \omega(k, \delta x). \tag{2.1.15}$$

We call (2.1.15) **inhomogeneous dispersion relation**. As was considered in the homogeneous case, we shall investigate the long wave and the short wave separately.

For the long wave limit we consider the Taylor expansion of (2.1.15) around $k = 0$;

$$\omega = v(\delta x)k - a(\delta x)k^3 - i\delta b(\delta x) + O(k^5, \delta^2), \tag{2.1.16}$$

where $v(\delta x)$ is the local group velocity. The term $-i\delta b(\delta x)$ appears because of non-commutability of $k = -i\partial_x$ and x. We introduce a transformation of independent variables by use of a smallness parameter $\varepsilon = \delta^{1/3}$;

$$\begin{cases} \xi = \varepsilon(\int \frac{\mathrm{d}x}{v(\delta x)} - t), \\ \sigma = \delta x = \varepsilon^3 x. \end{cases} \tag{2.1.17}$$

The operator relation (2.1.16) reduces to a linear partial differential equation at $O(\varepsilon^3)$;

$$\phi_\sigma + \frac{a(\sigma)}{(v(\sigma))^4}\phi_{\xi\xi\xi} + \frac{b(\sigma)}{v(\sigma)}\phi = 0. \tag{2.1.18}$$

This represents the linear terms of the model equation (1.1). By collecting the lowest part of the nonlinearity we generally obtain (1.1). In place of (2.1.17) we introduce another transformation of independent variables;

$$\begin{cases} \xi = \varepsilon(\int \frac{\mathrm{d}x}{v(\delta x)} - t), \\ \tau = \varepsilon^3 \int \frac{a(\delta x)}{(v(\delta x))^4}\mathrm{d}x, \end{cases} \tag{2.1.19}$$

which is an extension of (2.1.5). With this choice, we have

$$\phi_\tau + \phi_{\xi\xi\xi} + \nu(\tau)\phi = 0, \tag{2.1.20}$$
$$\nu(\tau) = \frac{bv^3}{a},$$

which has the same form as the linear part of the model equation (1.2). If we consider the lowest nonlinearity and introduce a transformation of the dependent variable $\phi(\xi, \tau)$, we obtain in general the inhomogeneous KdV equation (1.2).

For the short wave case (finite wave number k), we take the Taylor expansion of (2.1.15) around $\omega = \omega_0$;

$$\omega = \omega_0 + v(\delta x)(k - k_0) + a(\delta x)(k - k_0)^2 - i\delta b(\delta x) + \cdots, \tag{2.1.21}$$

where the last term of r.h.s. appears due to the same reason as that for (2.1.16). The local wave number $k_0 = k_0(\delta x)$ is defined by solving $\omega_0 = \omega(k, \delta x)$ with respect to k. By introducing ψ as

$$\phi = \exp i(\int k_0(\delta x)\mathrm{d}x - \omega_0 t) \cdot \psi, \tag{2.1.22}$$

we reduce the dispersion relation (or operator relation of ϕ) (2.1.21) to

$$i\psi_t + iv(\delta x)\psi_x + a(\delta x)\psi_{xx} + i\delta b(\delta x)\psi = 0. \tag{2.1.23}$$

The carrier wave, $\exp(\cdots)$, in (2.1.22) is a **deformed monochromatic wave** due to the inhomogeneity. Introducing a smallness parameter $\varepsilon = \delta^{1/2}$ and the following transformation of independent variables to (2.1.23)

$$\begin{cases} \xi = \varepsilon(\int \dfrac{\mathrm{d}x}{v(\delta x)} - t), \\ \tau = \varepsilon^2 \int \dfrac{a(\delta x)}{\left(v(\delta x)\right)^3}\mathrm{d}x, \end{cases} \tag{2.1.24}$$

we obtain at $O(\varepsilon^2)$;

$$i\psi_\tau + \psi_{\xi\xi} + i\nu(\tau)\psi = 0, \tag{2.1.25}$$
$$\nu(\tau) = \frac{bv^2}{a}.$$

Note that (2.1.24) is a generalization of (2.1.11), This is similar to the linear part of (1.3). The nonlinear term of (1.3) is collected by taking the lowest nonlinearity into account.

Weak Inomogeneity

Next, we consider a **weakly inhomogeneous** system; functions which represent the inhomogeneities of the system are slowly changing and their fluctuations are much smaller than their characteristic magnitude. Due to the weakness of the inhomogeneity, the dispersion relation (2.1.15) is divided into two parts;

$$\omega = \tilde{\omega}(k) + \delta'\Omega(k, \delta x), \tag{2.1.26}$$

where δ' and δ are smallness parameters which represent the weakness and rapidness of the inhomogeneities respectively. The inhomogeneous dispersion relation is also considered as a partial differential equation for the wave $\phi(x, t)$ (see (2.1.2)).

In the long wave limit, we take the Taylor expansion of (2.1.26) around $k = 0$;

$$\omega = vk - ak^3 + \delta' b(\delta x)k + O(k^5, \delta\delta'), \tag{2.1.27}$$

where $b = (\partial\Omega/\partial k)$ at $k = 0$. The group velocity v is defined as $v = \mathrm{d}\tilde{\omega}/\mathrm{d}k$ at $k = 0$. Note that if the last term is neglected, (2.1.27) is equivalent to the KdV type dispersion (2.1.3). We assume that δ' and δ are expressed by a smallness parameter ε;

$$\begin{cases} \delta' = \varepsilon^2, \\ \delta = \varepsilon^3. \end{cases} \tag{3.1.28}$$

If we employ the transformation (2.1.5), we obtain from (2.1.27);

$$\phi_\tau + \phi_{\xi\xi\xi} + f(\tau)\phi_\xi = 0, \tag{2.1.29a}$$
$$f(\tau) = \frac{v^2}{a}b(\tau). \tag{2.1.29b}$$

Again, by considering the lowest nonlinear effect we generally obtain the inhomogeneous KdV equation (1.4).

For the monochromatic wave modulations, similarly to the previous cases, we consider the Taylor expansion of dispersion relation (2.1.26) around $k = k_0$;

$$\omega = \tilde{\omega}_0 + \delta'\Omega_0(\delta x) + v(k - k_0) + a(k - k_0)^2$$
$$+ \delta' \frac{\partial\Omega(k_0, \delta x)}{\partial k}(k - k_0) + O((k - k_0)^3, \delta\delta'), \tag{2.1.30}$$

where $\tilde{\omega}_0 = \tilde{\omega}(k_0)$ and $\Omega_0(\delta x) = \Omega(k_0, \delta x)$. We should remark that unlike the case of (2.1.21), k_0 does not depend on x. That is k_0 is defined from a given constant $\tilde{\omega}_0$ or *vice versa*. We suppose that δ' and δ are expressed by a smallness parameter ε;

$$\begin{cases} \delta' = \varepsilon, \\ \delta = \varepsilon^2. \end{cases} \tag{2.1.31}$$

We introduce the transformations of independent variables (2.1.11) and a new dependent variable ψ;

$$\phi = \exp i\big(k_0 x - \tilde{\omega}_0 t - \varepsilon \int \mathrm{d}x \frac{\Omega(k_0, \tau)}{v}\big) \cdot \psi, \tag{2.1.32}$$

where $\exp(\cdots)$ is a deformed monochromatic wave due to the weak inhomogeneity. We have defined the group velocity v by $\mathrm{d}\tilde{\omega}/\mathrm{d}k$ at $k = k_0$. Then, considering (2.1.30), we obtain a linear equation for ψ;

$$i\psi_\tau + \psi_{\xi\xi} + if(\tau)\psi_\xi + g(\tau)\psi = 0, \tag{2.1.33a}$$

$$f(\tau) = \frac{v^2}{a}\frac{\partial\Omega(k_0, \delta x)}{\partial k} - 2\Omega_0, \tag{2.1.33b}$$

$$g(\tau) = \frac{v\Omega_0(\tau)}{a}\frac{\partial\Omega(k_0, \delta x)}{\partial k} - \big(\Omega_0(\tau)\big)^2, \tag{2.1.33c}$$

where we have employed transformation (2.1.11). Similarly to the former cases , we can derive (1.5) by taking nonlinear effect into account.

2.2 Finite and Slow Inhomogeneity

In this and the next sections, we shall investigate inhomogeneous anharmonic lattices; the mass distribution is not uniform. Making reference to the previous section, we apply the reductive perturbation method to this system. We shall derive model equations (1.2)~(1.5) (and inhomogeneous modified KdV equations).

A force due to the spring between two adjacent particles is common and assumed to be

$$F = -K(\Delta + \alpha\Delta^2 + \beta\Delta^3 + \cdots), \tag{2.2.1}$$

where Δ is the elongation of the spring and K is the spring constant. Let m_j and y_j be the mass and displacement of j-th particle respectively. The equation of motion for the j-th particle is given by

$$\begin{aligned} m_j\ddot{y}_j = {} & K[y_{j+1} - y_j + \alpha(y_{j+1} - y_j)^2 + \beta(y_{j+1} - y_j)^3 + \cdots] \\ & - K[y_j - y_{j-1} + \alpha(y_j - y_{j-1})^2 + \beta(y_j - y_{j-1})^3 + \cdots], \end{aligned} \tag{2.2.2}$$

where dots on y_j indicate derivatives with respect time t. Since we are interested in weakly nonlinear waves, we have neglected higher order terms in (2.2.2). The mass distribution m_j is slowly changing. Since the derivations of the model equations for finite inhomogeneities and those for the weak inhomogeneities are essentially different, we investigate the two cases separately.

In the present section the fluctuation of the mass distribution Δm_j is assumed to be comparable to the mean mass $\langle m_j \rangle$;

$$\Delta m_j \sim \langle m_j \rangle, \quad \text{finite inhomogeneity}. \tag{2.2.3}$$

Because the inhomogeneity is slow, we express the mass distribution by a continuum variable $x = jh$; (h is the lattice constant) and a function $G(\delta x)$;

$$m_j = \langle m_j \rangle G(\delta x), \tag{2.2.4}$$

where δ is a smallness parameter.

Long Wave Limit

At first we shall study the long wave limit. We employ the continuum approximation $y_j(t) = y(x, t)$ and consider the Taylor expansions;

$$y_{j\pm 1} = y \pm hy_x + \frac{h^2}{2}y_{xx} \pm \frac{h^3}{6}y_{xxx} + \frac{h^4}{24}y_{xxxx} \pm \cdots. \tag{2.2.5}$$

Substitution of (2.2.5) into (2.2.2) yields a inhomogeneous Boussinesq equation

$$G(\delta x)y_{tt} = \frac{K}{\langle m_j \rangle}(h^2 y_{xx} + \frac{h^4}{12}y_{xxxx} + 2\alpha h^3 y_x y_{xx}), \quad (\text{for } \alpha \neq 0) \tag{2.2.6a}$$

$$G(\delta x)y_{tt} = \frac{K}{\langle m_j \rangle}(h^2 y_{xx} + \frac{h^4}{12}y_{xxxx} + 3\beta h^4 y_x^2 y_{xx}), \quad (\text{for } \alpha = 0) \tag{2.2.6b}$$

where higher order dispersion and nonlinear terms are neglected.

1)Case of $\alpha \neq 0$.

Let us define a new smallness parameter $\varepsilon = \delta^{1/3}$. For $\alpha \neq 0$, we introduce the following transformations;

$$\xi = \varepsilon\sqrt{\frac{K}{\langle m_j \rangle}}\big(\int \frac{\mathrm{d}x}{v(\delta x)} - t\big), \tag{2.2.7a}$$

$$\tau = \varepsilon^3 \frac{1}{24h}\int \mathrm{d}x G^{3/2}, \tag{2.2.7b}$$

$$u(\xi, \tau) = \frac{4}{\varepsilon}\alpha G^{-1/2} y_\xi, \tag{2.2.8}$$

where $v(\delta x)$ is the **local acoustic velocity**;

$$v(\delta x) = \sqrt{\frac{h^2 K}{G(\delta x)\langle m_j \rangle}}. \tag{2.2.9}$$

As a result, an inhomogeneous KdV equation is derived at $O(\varepsilon^5)$;

$$u_\tau + u_{\xi\xi\xi} + 6uu_\xi + \frac{3}{4}\big(\log G(\tau)\big)_\tau u = 0. \tag{2.2.10}$$

This is nothing but the model equation (1.2), which is ubiquitous for nonlinear

2)Case of $\alpha = 0$ and $\beta \neq 0$.

long waves traveling over slow and finite inhomogeneities. For the case $\alpha = 0$ ($\beta \neq 0$) we employ transformations (2.2.7) and introduce a new dependent variable;

$$u(\xi, \tau) = \sqrt{6\,|\beta|}\,y_\xi. \tag{2.2.11}$$

A closed equation, in this case, is an inhomogeneous modified KdV (mKdV) equation;

$$u_\tau + u_{\xi\xi\xi} + 6\lambda u^2 u_\xi + \frac{1}{4}\big(\log G(\tau)\big)_\tau u = 0, \tag{2.2.12}$$

where $\lambda = \mathrm{sgn}(\beta)$.

Equations (2.2.10) and (2.2.12) are not integrable. However, by numerical or approximate approach to the model equation, we observe damping, growing and fission of the KdV or mKdV soliton.

Monochromatic Wave Modulations

We shall consider the modulations of monochromatic waves. As discussed in Chapter 1, we expect that the monochromatic waves (carrier wave) are deformed due to the inhomogeneities. The inhomogeneous dispersion relation is given by

$$\omega^2 G(\delta x) = \frac{2K}{\langle m_j \rangle}\big(1 - \cos k(\delta x)h\big), \tag{2.2.13}$$

where G has been introduced in (2.2.4). We define the carrier wave F as

$$F = \exp i\Big(\int k(\delta x)\mathrm{d}x - \omega t\Big). \tag{2.2.14}$$

Here, the **local wave number** $k(\delta x)$ is given by solving (2.2.13) for a given ω. We see that, F in (2.2.14) is a solution of linearized equation of motion (2.2.2) up

to $O(\delta)$. Using this carrier wave, we expand the displacements y_j by a smallness parameter $\varepsilon = \delta^{1/2}$;

$$y_j = \sum_{n=1}^{\infty}\sum_{l=-n}^{n} \varepsilon^n F^l u^{(n,l)}(\xi, \sigma), \tag{2.2.15}$$

where we have introduced dimensionless independent variables;

$$\xi = \varepsilon\sqrt{\frac{\langle m_j \rangle}{K}}\Big(\int \frac{\mathrm{d}x}{v(\sigma)} - t\Big), \tag{2.2.16a}$$

$$\sigma = \frac{\varepsilon^2}{h}x \tag{2.2.16b}$$

The **local group velocity** $v(\sigma)$ will be defined shortly later.

Keeping (2.2.4) in mind, we apply (2.2.15) and (2.2.16) to the equation of motion (2.2.2) and equate the coefficients of $\varepsilon^n F^l$. At $(n, l) = (1, 1)$ we confirm the inhomogeneous dispersion relation (2.2.13). The local group velocity $v(\sigma)$ is obtained in $(n, l) = (2, 1)$;

$$v(\sigma) = \frac{Kh\sin k(\sigma)h}{\langle m_j \rangle \omega G(\sigma)} = \left(\frac{\partial \omega}{\partial k}\right)_\sigma. \tag{2.2.17}$$

Relations among the dependent variables

$$u^{(2,2)} = i\alpha\frac{\sin k(\sigma)h}{1 - \cos k(\sigma)h}\big(u^{(1,1)}\big)^2, \tag{2.2.18}$$

$$u_\xi^{(1,0)} = -8\alpha\frac{v(\sigma)}{h}\sqrt{\frac{K}{\langle m_j \rangle}}\left|u^{(1,1)}\right|^2, \tag{2.2.19}$$

are derived in $(n, l) = (2, 2)$ and $(n, l) = (3, 0)$ respectively. For the latter we have assumed that the lattice wave $u^{(1,1)}$ is sufficiently localized.

Collecting the above results, we obtain a closed equation for $u^{(1,1)}$ in $(n, l) = (3, 1)$;

$$iu_\sigma^{(1,1)} + p(\sigma)u_{\xi\xi}^{(1,1)} + q(\sigma)\left|u^{(1,1)}\right|^2 u^{(1,1)} + ir(\sigma)u^{(1,1)} = 0, \tag{2.2.20}$$

where the coefficient functions are

$$p(\sigma) = -\frac{h^2\langle m_j \rangle}{4K}\tan\frac{k(\sigma)h}{2}\big(v(\sigma)\big)^{-2}, \tag{2.2.21a}$$

$$q(\sigma) = \tan\frac{k(\sigma)h}{2}\Big\{8\alpha^2 + (8\alpha^2 - 12\beta)\sin^2\frac{k(\sigma)h}{2}\Big\}, \tag{2.2.21b}$$

$$r(\sigma) = \frac{1}{2}\tan\frac{k(\sigma)h}{2}\frac{hk'(\sigma)\cos k(\sigma)h}{1 - \cos k(\sigma)h}. \tag{2.2.21c}$$

Prime on $k(\sigma)$ in (2.2.21c) indicates differentiation with respect to σ. Hereafter, we shall assume that $p(\sigma)q(\sigma) > 0$. By a further transformation of independent and dependent variables;

$$\xi \to \xi, \tag{2.2.22a}$$

$$\sigma \to \tau = \int^{\sigma} p(\sigma')\mathrm{d}\sigma', \tag{2.2.22b}$$

$$u^{(1,1)} \to U = \sqrt{\frac{q(\sigma)}{2p(\sigma)}}u^{(1,1)}, \tag{2.2.22c}$$

we reduce (2.2.20) to

$$iU_\tau + U_{\xi\xi} + 2|U|^2 u + i\nu(\tau)U = 0, \tag{2.2.23a}$$

$$\nu(\tau) = \frac{1}{2}\left(\log\frac{2p(\tau)}{q(\tau)}\right)_\tau + \frac{r(\tau)}{p(\tau)}. \tag{2.2.23b}$$

This is the NLS equation with a pure imaginary potential (1.3), which is not integrable. We have derived (1.3) from short waves in lattices for the first time. As seen from the derivation, the model equation appears in a wide class of slowly inhomogeneous systems. Due to the last term in (2.2.23a), we expect damping, growing and fission of a NLS soliton.

2.3 Weak Inhomogeneity

Let us consider the equation of motion (2.2.2) again. We shall be interested in weakly inhomogeneous systems. Then, the fluctuation of the mass distribution Δm_j is much smaller than the mean mass;

$$\Delta m_j \ll \langle m_j \rangle, \quad \text{weak inhomogeneity.} \tag{2.3.1}$$

Here, m_j is a slowly changing function of a lattice site j. Therefore we may assume that m_j takes a form

$$m_j = \langle m_j \rangle \big(1 + \delta' \eta(\delta x)\big), \tag{2.3.2}$$

where $x = jh$. Smallness parameters δ' and δ measure weakness and variation of the inhomogeneity respectively. Relations (2.2.2) and (2.3.2) is our starting point.

Long Wave Limit

1)Case of $\alpha \neq 0$.

For slowly varying waves, we may employ the continuum approximation $y_j = y(x,t)$. Similarly to the previous case, we have an inhomogeneous Boussinesq equation;

$$(1 + \delta'\eta(\delta x))y_{tt} = \frac{K}{\langle m_j \rangle}(h^2 y_{xx} + \frac{1}{12}h^4 y_{xxxx} + 2\alpha h^3 y_x y_{xx} + \cdots). \tag{2.3.3}$$

The smallness parameters are assumed to satisfy (2.1.28). We change independent variables from x, t into ξ, τ;

$$\xi = \frac{1}{h}\varepsilon(x - vt), \tag{2.3.4a}$$

$$\tau = \frac{1}{24h}\varepsilon^3 x, \tag{2.3.4b}$$

where a constant $v = h\sqrt{K/\langle m_j \rangle}$ is the acoustic velocity. A new dependent variable $u(\xi,\tau)$ is introduced as

$$u(\xi,\tau) = \frac{4\alpha}{\varepsilon}y_\xi. \tag{2.3.5}$$

Transformations (2.3.4) and (2.3.5) are equivalent to (2.2.7) and (2.2.8) respectively in the homogeneous limit $G \to 1$. Substituting (2.3.2), (2.3.4) and (2.3.5) into (2.3.3) we have a inhomogeneous KdV equation (Wadati 1990, Iizuka, Nakao and Wadati 1992);

$$u_\tau + 6uu_\xi + u_{\xi\xi\xi} - 12\eta(\tau)u_\xi = 0, \tag{2.3.6}$$

which is in the same form as (1.4). This type of equation is possibly a canonical model equation for a large class of nonlinear weakly inhomogeneous systems. For an example, shallow water waves over a weakly uneven bottom obey (2.3.6) (Ono 1991a).

We can simplify (2.3.6) by a further transformation of independent variables;

$$X = \xi + 12\int_0^\tau \eta(\tau')\mathrm{d}\eta', \tag{2.3.7a}$$

$$T = \tau, \tag{2.3.7b}$$

by which we get

$$u_T + 6uu_X + u_{XXX} = 0. \tag{2.3.8}$$

This is the K-dV equation and is exactly solvable. Since (2.2.8) has soliton solutions, the notion of soliton is also valid in the inhomogeneous system. It is interesting that the soliton solution of (2.3.8) gives a **deformed soliton** in the original frame x, t. Thus, in the weakly inhomogeneous media, the KdV soliton never fuses nor fissions. This is an essentially different property of the weak inhomogeneity from that of the finite and slow inhomogeneity.

If we set $x = 0$ in (2.3.4) and (2.3.7),

$$X = \xi = -\frac{1}{h}\varepsilon v t, \tag{2.3.9a}$$

$$T = \tau = 0. \tag{2.3.9b}$$

This implies that if $y(0,t)$ is given, we know $u(X,0)$. The initial value problem of (2.3.8) can be solved by the inverse scattering method (Gardner,Greene Kruskal and Miura 1967). Using the result, we can construct $y(x,t)$ from $y(0,t)$ up to the lowest order of ε.

2)Case of $\alpha = 0$ and $\beta \neq 0$.

In the continuum approximation, the equation of motion is expressed as

$$(1+\delta'\eta(\delta x))y_{tt} = \frac{K}{\langle m_j\rangle}(h^2 y_{xx} + \frac{h^4}{12}y_{xxxx} + 3\beta h^4 (y_x)^2 y_{xx} + \cdots). \tag{2.3.10}$$

We use the variables ξ and τ in (2.3.4) and define $u(\xi,\tau)$ as

$$u(\xi,\tau) = \sqrt{6|\beta|}y_\xi. \tag{2.3.11}$$

We apply the reductive perturbation method and the result is an inhomogeneous mKdV equation;

$$u_\tau + 6\lambda u^2 u_\xi + u_{\xi\xi\xi} - 12\eta(\tau)u_\xi = 0, \tag{2.3.12}$$

where $\lambda = \mathrm{sgn}(\beta)$.

Similarly to the former case we can transform (2.3.12) to the modified KdV equation by (2.3.7)

$$u_T + 6\lambda u^2 u_X + u_{XXX} = 0. \tag{2.3.13}$$

One soliton solution of (2.3.13) for $\lambda > 0$ represents a deformed soliton in the original coordinates x and t. Since the modified KdV equation is exactly solvable (Wadati 1973), we can construct $y(x,t)$ from $y(0,t)$ up to ε.

Monochromatic Wave Modulations

For the modulations of monochromatic waves, we assume that the smallness parameters δ and δ' in (2.3.2) satisfy (2.1.31). Therefore the inhomogeneous dispersion relation is given by

$$\omega^2 = \frac{2K}{\langle m_j\rangle(1+\varepsilon\eta(\varepsilon^2 x))}(1-\cos kh), \tag{2.3.14}$$

$$\begin{aligned}\omega &= \sqrt{\frac{2K(1-\cos kh)}{\langle m_j\rangle}}(1+\varepsilon\eta(\varepsilon^2 x))^{-1/2} \\ &= \sqrt{\frac{2K(1-\cos kh)}{\langle m_j\rangle}}(1-\frac{\varepsilon}{2}\eta(\varepsilon^2 x) + \frac{3}{8}\varepsilon^2\eta(\varepsilon^2 x)^2 + \cdots).\end{aligned} \tag{2.3.15}$$

We introduce the following dimensionless variables;

$$\xi = \varepsilon\frac{x - vt}{h}, \tag{2.3.16a}$$

$$\tau = \varepsilon^2\frac{x}{h}, \tag{2.3.16b}$$

where a constant v is to be determined later. With reference to the (2.1.32), we choose the carrier as

$$F = \exp i\big(kx - \omega t + \frac{1}{\varepsilon}\frac{1-\cos kh}{\sin kh}\int \eta(\tau)\mathrm{d}\tau\big). \tag{2.3.17}$$

Note that k and ω in (2.3.17) correspond to k_0 and ω_0 in (2.1.32) respectively. Then, constants k and ω satisfy the homogeneous dispersion relation,

$$\omega^2 = \frac{2K}{\langle m_j\rangle}(1-\cos kh). \tag{2.3.18}$$

We expand y_j as

$$y_j = y(x,t) = \sum_{n=1}^{\infty}\sum_{l=-n}^{n}\varepsilon^n F^l u^{(n,l)}, \tag{2.3.19}$$

where $u^{(n,l)}$ are functions of ξ and τ.

Substituting (2.3.2) and (2.3.16)~(2.3.19) into (2.2.2) and equating the term of the order $\varepsilon^n F^l$ we have the following equations. In the order εF, we confirm the dispersion relation (2.3.18), and in the order $\varepsilon^2 F$ we see that a constant v is the group velocity,

$$v = \frac{\mathrm{d}\omega}{\mathrm{d}k} = \frac{h\omega\sin kh}{2(1-\cos kh)}. \tag{2.3.20}$$

In the orders of $\varepsilon^2 F^2$ and ε^3, we see that $u^{(2.2)}$ and $u_\xi^{(1.0)}$ are related to $u^{(1.1)}$ as follows,

$$u^{(2.2)} = \frac{i\alpha \sin kh}{1-\cos kh}(u^{(1.1)})^2, \tag{2.3.21}$$

$$u_\xi^{(1.0)} = -8\alpha \left|u^{(1.1)}\right|^2. \tag{2.3.22}$$

And finally in the order $\varepsilon^3 F$, we obtain a closed nonlinear evolution equation for $u^{(1.1)}$,

$$\begin{aligned} &iu_\tau^{(1.1)} + pu_{\xi\xi}^{(1.1)} + q\left|u^{(1.1)}\right|^2 u^{(1.1)} \\ &- i\frac{1}{1+\cos kh}\eta(\tau)u_\xi^{(1.1)} - \frac{1}{2}\frac{\cos kh(1-\cos kh)^2}{\sin^2 kh}\eta(\tau)^2 u^{(1.1)} = 0, \end{aligned} \tag{2.3.23}$$

where the constant coefficients p and q are

$$p = -\frac{1}{4}\tan\frac{kh}{2}, \tag{2.3.24}$$

$$q = \tan\frac{kh}{2}\{8\alpha^2 + (8\alpha^2 - 12\beta)\sin^2\frac{kh}{2}\}. \tag{2.3.25}$$

This is essentially the same as the equation (1.5) which may be canonical in weakly inhomogeneous media.

Now we suppose $pq > 0$ and then we have a condition for kh;

$$(\sin\frac{kh}{2})^{-2} < \frac{3\beta}{2\alpha^2} - 1. \tag{2.3.26}$$

Again we apply the following transformations to (2.3.23),

$$\begin{aligned} X &= \sqrt{\frac{q}{2p}}(\xi + \frac{1}{1+\cos kh}\int_0^\tau \eta(\tau')d\tau') \\ &= \sqrt{\frac{q}{2p}}(\varepsilon\frac{x - V_g t}{h} + \frac{1}{1+\cos kh}\int_0^\tau \eta(\tau')d\tau') \end{aligned} \tag{2.3.27}$$

$$T = \frac{q}{2}\tau = \frac{q\varepsilon^2}{2h}x, \tag{2.3.28}$$

$$U(X,T) = u^{(1.1)}(\xi,\tau)\exp\frac{i}{2}(\eta(\tau))^2\frac{\cos kh(1-\cos kh)^2}{\sin^2 kh} \tag{2.3.29}$$

and obtain the NLS equation (Iizuka, Nakao and Wadati 1991,1992),

$$iU_T + U_{XX} + 2|U|^2 U = 0. \tag{2.3.30}$$

The soliton solution of (2.3.30) also describes deformations of a soliton in the original frame. Since the NLS equation is exactly solvable (Zakharov and Shabat 1972), we have $y(x,t)$ from $y(0,t)$ up to ε as for the case of KdV equation.

2.4 Conclusions and Discussion

In chapter 2, four model equations for nonlinear waves in slowly inhomogeneous media are presented. We have introduced inhomogeneous dispersion relations which suggest us how to apply the reductive perturbation method in deriving the model equations. As a simple but basic application, we have considered nonlinear lattice where the mass distribution is not uniform. The transformations of independent variables introduced in lattice waves are essentially same as those introduced in 2.1. (For instance, (2.1.24) corresponds to (2.2.7).)

For the finite and slow inhomogeneity we have obtained equations (2.2.10,12) and (2.2.20) for long and short waves respectively. We should note that group velocity or acoustic velocity are slowly varying in space. The wave number and carrier wave are deformed due to the inhomogeneity. The model equations are not integrable. There, fission, damping and growing of (m)KdV or NLS solitons are expected. Extensive analysis for the model equations is a future problem.

For the weak inhomogeneity we have derived (2.3.6,12) and (2.3.23) for long waves and modulations of monochromatic waves respectively. The group velocity, acoustic velocity, and wave number are considered to be constants. However, the carrier wave must be deformed due to the weak inhomogeneity. The model equations are transformed to the soliton equations such as the KdV, mKdV and NLS equations. Therefore we can analytically investigate nonlinear waves in weakly inhomogeneous media. For instance one soliton solution offers an exact expression of deformation of soliton.

Chapter 3

Random Systems

As applications of chapter 2 we investigate nonlinear wave propagations in random systems. The randomness is assumed to belong to the class of the weak inhomogeneity. The function η in (2.3.2) is chosen to be a Gaussian white noise. Using (2.3.8), (2.3.13) and (2.3.30), statistical behaviors of KdV, mKdV and NLS solitons are respectively investigated.

3.1 Introduction to Chapter 3

One of the most important inhomogeneities is the randomness. Recently soliton propagations in the Toda lattice with randomly distributed two-kinds of masses were investigated numerically (Ishiwata, Okada, S.Watanabe and H.Tanaca 1990). It was found that the amplitude of soliton decays as n^{-p}, where n is the step length of the propagation of the front soliton and that the exponent p depends on the initial amplitude. In this simulation the full nonlinearity is included; in other words the weakly nonlinear approximation is not supposed. And the degree of the inhomogeneity is not always small.

As for the weakly nonlinear waves in random system, we should remark the followings. If the inhomogeneity in a nonlinear media is <u>*not slowly varying*</u>, we cannot formally apply the reductive perturbation method as was done in the previous chapter. However, in this situation, two works are reported on water waves over a random bottom. First , it is assumed that the characteristic length of the depth

change is much smaller than the characteristic wave length (N.Yajima 1972). Second, the two length is supposed to be comparable (Kawahara 1976). Applying the averaging method they derived coupled nonlinear equations and observed changes in phase velocities and damping of amplitudes. For either type of inhomogeneity, the obtained nonlinear equation are too complicated for further analysis.

As an tractable case, stochastic KdV equation has been proposed (Wadati 1983). There, statistical behaviors of the KdV solitons for a Gaussian white noise are investigated. It has been shown that for a long time index p is $3/2$. (In this case the step length n is a continuum variable.)

Similar approach for random systems is also done by using the the model equations (2.3.6), (2.3.12) and (2.3.23) (Wadati 1990, Iizuka, Nakao and Wadati 1991). In the investigations it has been found that $p = 1/2$. The theme of the present chapter is to review these analysis. We choose the inhomogeneity $\eta(\tau)$ in the model equations as random functions. That is the mass distribution of the lattice is random, while its fluctuation is small. For convenience, we suppose that $\eta(T)$ is the Gaussian and the white noise defined by

$$\langle \eta(\tau)\eta(\tau')\rangle = 2\theta\delta(\tau-\tau'), \tag{3.1.1}$$

where θ is a constant and $\langle\ \ \rangle$ indicates the ensemble average over the random mass distribution. In this chapter we study statistical behaviors of one-soliton solutions. In particular, we shall be interested in the asymptotic behaviors of the 'averaged' solitons. For the purpose we use a Fourier transformation method and a formula (Appendix 3A);

$$\begin{aligned}&\langle \exp i\int_0^{\tau}(J\eta(\tau')+L\eta(\tau')^2)d\tau'\rangle\\ &=\exp\{-(J^2\theta+2L^2\theta\langle\eta^2\rangle)\tau\}\exp\{iL\langle\eta^2\rangle\tau\},\end{aligned} \tag{3.1.2}$$

which is derived from (3.1.1) (Iizuka, Nakao and Wadati 1991). Again we shall discuss three cases separately.

3.2 Statistical Behavior of the KdV Solitons

It is known that the KdV equation (2.3.8) has one-soliton solution

$$u(X,T) = 2\kappa^2\mathrm{sech}^2(\kappa X - 4\kappa^3 T - \kappa X_0), \tag{3.2.1}$$

with a real parameter κ. Fourier transform of the soliton solution is given by

$$\begin{aligned}\hat{u}(k,T) &= \int_{-\infty}^{\infty} dX u(X,T)e^{-ikX} \\ &= 4\kappa\frac{\pi k}{2\kappa}\frac{1}{\sinh(\pi k/2\kappa)}\exp[-ik(4\kappa^2T+X_0)]. \end{aligned} \tag{3.2.2}$$

Using the formula (3.1.2) the average of the soliton solution is expressed as

$$\begin{aligned}\langle u(X,T)\rangle &= \langle\int_{-\infty}^{\infty}\frac{dk}{2\pi}\hat{u}(k,T)e^{ikX}\rangle \\ &= \int_{-\infty}^{\infty}\frac{dk}{2\pi}\hat{u}(k,T)e^{i\xi k}\langle e^{12ik\int_0^T d T\eta(T)}\rangle \\ &= \int_{-\infty}^{\infty}\frac{dk}{2\pi}\hat{u}(k,T)\exp(i\xi k-144k^2\theta T). \end{aligned} \tag{3.2.3}$$

For a large T, we may apply a saddle point method to have

$$\langle u(X,T)\rangle = \frac{2\kappa}{\pi}\sqrt{\frac{\pi}{a}}\exp(-s^2/4a), \tag{3.2.4}$$

where s and a stand respectively for

$$s = \xi - 4\kappa^2\tau - \xi_0, \tag{3.2.5}$$

$$a = 144\theta\tau + \frac{\pi^2}{24\kappa^2}. \tag{3.2.6}$$

This shows that 'averaged' soliton $\langle u(X,T)\rangle$ deforms during the propagation and that when the initial amplitude $|-2\kappa^2|$ is large it approaches rapidly to the asymptotic form,

$$\langle u(X,T)\rangle = \frac{\kappa}{6}\sqrt{\frac{1}{\pi\theta}}T^{-1/2}\exp[-(\xi-4\kappa^2\tau-\xi_0)^2/(576\theta\tau)]. \tag{3.2.7}$$

In terms of the original variables we explain the result. Let the incident soliton at $x=0$ be

$$y_t(0,t) = -\kappa^{1/2}\varepsilon v \operatorname{sech}^2(\kappa\varepsilon vt/h). \tag{3.2.8}$$

The statistical behavior for a large x is expressed as

$$\langle y_t(x,t)\rangle = A\frac{1}{x^{1/2}}\exp(-B(x-Ct)^2/x), \tag{3.2.9}$$

where constants A, B and C are given by

$$A = -\kappa v(h/6\varepsilon\pi\theta)^{1/2}, \tag{3.2.10}$$

$$B = \frac{1}{24}(1-\frac{1}{6}\kappa^2\varepsilon^2)^2\ \theta\varepsilon h, \tag{3.2.11}$$

$$C = v(1+\kappa^2\varepsilon^2/6). \tag{3.2.12}$$

Equation (3.2.9) suggests that after a long propagation the amplitude of a soliton decreases as $x^{-1/2}$ and the width increases as $x^{1/2}$. This reminds us the distribution function of the **random walk**. Thus we may interpret that the center of mass of the soliton moves randomly around $x = Ct$.

The power law x^{-p} for the amplitude is consistent with the numerical results for the random Toda lattice. Recall however that in the latter system the value of the index p depends on the amplitude of the incident soliton. When the amplitude is small, p is close to 1/2. This result completely agrees with our analysis.

3.3 Statistical Behavior of the mKdV Solitons

We assume the λ in (2.3.13) is +1 ($\beta > 0$). One-soliton solution of the modified KdV equation (2.3.13) is given by

$$u(X,T) = \kappa\operatorname{sech}\{\kappa(X-X_0)-\kappa^3T\}. \tag{3.3.1}$$

We employ the same method as that of the former case . Fourier transformation of (3.15) is expressed as

$$\hat{u}(k,T) = \int_{-\infty}^{+\infty} u(X,T)e^{-ikX}dX = \pi\operatorname{sech}(\frac{\pi k}{2\kappa})e^{-ik(\kappa^2T+X_0)}. \tag{3.3.2}$$

Using a formula (3.1.2) with (3.3.1) we obtain

$$\begin{aligned} langleu(X,T)rangle &= \int_{-\infty}^{+\infty}\frac{dk}{2\pi}\hat{u}(k,T)e^{ik\xi-144k^2\theta T} \\ &= \int_{-\infty}^{+\infty}\frac{dk}{2}\operatorname{sech}(\frac{\pi k}{2\kappa})e^{-ik(\kappa^2T+X_0-\xi)-144k^2\theta T}. \end{aligned} \tag{3.3.3}$$

If T is sufficiently large, $\langle u(X,T)\rangle$ is asymptotically given by

$$\langle u(X,T)\rangle = \frac{1}{24}\sqrt{\frac{\pi}{\theta}}T^{-1/2}\exp\{-\frac{(\kappa^2T+X_0-\xi)^2}{576\theta T}\}. \tag{3.3.4}$$

In terms of the original variables, (3.3.4) is written as

$$\langle y_t(x,t)\rangle = \frac{A'}{x^{1/2}}\exp(-B'(x-C't)^2/x), \tag{3.3.5}$$

where constants A', B' and C' are

$$A' = -\frac{v}{h\sqrt{6\beta}}\sqrt{\frac{\pi h}{24\theta}}\varepsilon^{-1/2}, \tag{3.3.6}$$

$$B' = \frac{(1-\kappa^2\varepsilon^4/24)^2}{24\theta h\varepsilon}, \tag{3.3.7}$$

$$C' = \frac{v}{1-\kappa^2\varepsilon^2/24}. \tag{3.3.8}$$

This asymptotic behavior is exactly the same as (3.2.8).

3.4 Statistical Behavior of the NLS Solitons

The NLS equation (2.36) has one-soliton solution,

$$U(X,T) = \sqrt{\Omega}\mathrm{sech}\sqrt{\Omega}(X-wT-X_0)e^{iw(X-X_0)/2-i(w^2/4-\Omega)T}. \tag{3.4.1}$$

Here w, $\Omega(>0)$ and X_0 are real parameters. We evaluate the deformation of solitons to the order of ε. To do this, we begin with calculations of $\langle u^{(1,1)}F\rangle$ and $\langle u^{(1,0)}\rangle$.

Using (3.4.1), we have a form $u^{(1.1)}F$,

$$\begin{aligned} u^{(1.1)}F =& \sqrt{\Omega}\mathrm{sech}\sqrt{\Omega}(X-wT-X_0)\exp\{\frac{iw(X-X_0)}{2} - i(\frac{w^2}{4}-\Omega)T\} \\ &\cdot\exp\{\frac{i}{\varepsilon}\frac{1-\cos kh}{\sin kh}\int_0^\tau \eta(\tau')d\tau'\} \\ &\cdot\exp\{-\frac{i\cos kh(1-\cos kh)^2}{2\sin^3 kh}\int_0^\tau \eta^2(\tau')d\tau'\} \\ &\cdot\exp\{i(kx-\omega t)\}. \end{aligned} \tag{3.4.2}$$

By use of the Fourier transformation we may express $\langle u^{(1.1)}F\rangle$ as

$$\begin{aligned} \langle u^{(1.1)}F\rangle =& \langle\frac{1}{2}\int_{-\infty}^{+\infty} d\zeta\mathrm{sech}(\frac{\pi\zeta}{2\sqrt{\Omega}})\exp i\{\frac{w}{2}(\sqrt{\frac{q}{2p}}\xi - X_0) - \frac{q}{2}(\frac{w^2}{4}-\Omega)\tau\} \\ &\cdot\exp\{i\zeta(\sqrt{\frac{q}{2p}}\xi - \frac{wq}{2}\tau - X_0)\} \\ &\cdot\exp i\{\int_0^\tau (J\eta(\tau') + L\eta^2(\tau'))d\tau'\}\rangle e^{i(kx-\omega t)}, \end{aligned} \tag{3.4.3}$$

where J and L are given by

$$J = (\zeta + \frac{w}{2})\sqrt{\frac{q}{2p}}\frac{1}{1+\cos kh} + \frac{1}{\varepsilon}\frac{1-\cos kh}{\sin kh}, \tag{3.4.4}$$

$$L = -\frac{1}{2}\frac{\cos kh(1-\cos kh)^2}{\sin^3 kh}. \tag{3.4.5}$$

Applying the formula (3.1.2) to (3.4.3), we have

$$\langle u^{(1.1)}F\rangle = \frac{1}{2}\int_{-\infty}^{+\infty} d\zeta\mathrm{sech}(\frac{\pi\zeta}{2\sqrt{\Omega}})\exp\{i(M\zeta+N) + P\zeta^2 + Q\zeta + R\}, \tag{3.4.6}$$

where

$$M = \sqrt{\frac{q}{2p}}\xi - \frac{wq}{2}\tau - X_0, \tag{3.4.7}$$

$$N = kx - \omega t + \frac{w}{2}(\sqrt{\frac{q}{2p}}\xi - X_0) - (\frac{w^2}{4}-\Omega)\frac{q\tau}{2} + L\langle\eta^2\rangle\tau, \tag{3.4.8}$$

$$P = -\frac{q}{2p}\frac{1}{(1+\cos kh)^2}\theta\tau, \tag{3.4.9}$$

$$Q = -2\sqrt{\frac{q}{2p}}\frac{1}{1+\cos kh}(\frac{w}{2}\sqrt{\frac{q}{2p}}\frac{1}{1+\cos kh} + \frac{1-\cos kh}{\varepsilon\sin kh})\theta\tau, \tag{3.4.10}$$

$$R = -(\frac{w}{2}\sqrt{\frac{q}{2p}}\frac{1}{1+\cos kh} + \frac{1-\cos kh}{\varepsilon\sin kh})^2\theta\tau - 2L^2\theta\langle\eta^2\rangle\tau. \tag{3.4.11}$$

For a sufficiently large τ, the asymptotic form of (3.4.8) is

$$\begin{aligned} &\langle u^{(1.1)}F\rangle \\ &= \frac{1}{2}\sqrt{\frac{\pi}{-P}}\mathrm{sech}(-\frac{\pi Q}{4\sqrt{\Omega}P})e^{(M^2-Q^2)/(4P)+R+i(-MQ/(2P)+N)} \\ &= \frac{1}{2}\sqrt{\frac{\pi(1+\cos kh)^2}{q\theta\tau/(2p)}}\mathrm{sech}(-\frac{\pi Q}{4\sqrt{\Omega}P})\cdot\exp\{\frac{-(\sqrt{(q/2p)}\xi - (wq/2)\tau - X_0)^2}{2(q/p)(1+\cos kh)^{-2}\theta\tau}\} \\ &\cdot\exp\{-2L^2\theta\langle\eta^2\rangle\tau\} \\ &\cdot\exp i\{kx - \omega t + (w^2/4+\Omega)q\tau/2 + L\langle\eta^2\rangle\tau\} \\ &\cdot\exp\{-\frac{i}{\varepsilon}\sqrt{\frac{2p}{q}}\sin kh(\sqrt{\frac{q}{2p}}\xi - \frac{wq\tau}{2} - X_0)\}. \end{aligned} \tag{3.4.12}$$

Recalling the relation (2.3.22), we calculate $u_\xi^{(1.0)}$ and $u^{(1.0)}$ as

$$\begin{aligned} u_\xi^{(1.0)} &= -8\alpha\Omega\mathrm{sech}^2\sqrt{\Omega}(X-wT-X_0) \\ &= -4\alpha\int_{-\infty}^{+\infty} d\zeta\frac{\zeta}{\sinh\{\pi\zeta/(2\sqrt{\Omega})\}}e^{-i(wT+X_0)\zeta+i\zeta X}, \end{aligned} \tag{3.4.13a}$$

$$u^{(1.0)} = -8\alpha\sqrt{\frac{2p\Omega}{q}}\tanh[\sqrt{\Omega}(X-wT-X_0)]. \tag{3.4.13b}$$

The average value of $u_\xi^{(1.0)}$ is expressed as

$$\begin{aligned}\langle u_\xi^{(1,0)}\rangle =&\frac{\partial}{\partial\xi}\langle u^{(1,0)}\rangle\\ =&-4\alpha\int_{-\infty}^{+\infty}d\zeta\frac{\zeta}{\sinh\{\pi\zeta/(2\sqrt{\Omega})\}}e^{i(\sqrt{q/(2p)}\xi-wq/2\tau-X_0)\zeta}\\ &\cdot\langle e^{i\sqrt{q/(2p)}\zeta/(1+\cos kh)\int_0^\tau\eta(\tau')d\tau'}\rangle\\ =&-4\alpha\int_{-\infty}^{+\infty}d\zeta\frac{\zeta}{\sinh\{\pi\zeta/(2\sqrt{\Omega})\}}e^{i(\sqrt{q/(2p)}\xi-wq/2\tau-X_0)\zeta}\\ &\cdot\exp\{-\frac{q}{2p(1+\cos kh)^2}\theta\tau\zeta^2\}.\end{aligned}\tag{3.4.14}$$

And we obtain the asymptotic behavior of slowly varying wave $\frac{\partial}{\partial\xi}\langle u^{(1,0)}\rangle$ for a large τ;

$$\begin{aligned}\frac{\partial}{\partial\xi}\langle u^{(1,0)}\rangle =&-\frac{8\alpha\sqrt{\Omega}}{\pi}\sqrt{\frac{\pi(1+\cos kh)^2}{q\theta\tau/(2p)}}\\ &\cdot\exp\{-\frac{p(1+\cos kh)^2}{2q\theta\tau}(\sqrt{\frac{q}{2p}}\xi-\frac{wq}{2}\tau-X_0)^2\}.\end{aligned}\tag{3.4.15}$$

Collecting (3.4.12) and (3.4.15), we find that the asymptotic behavior of $\langle y(x,t)\rangle$ up to the order of ε is given by,

$$\begin{aligned}\langle y(x,t)\rangle =&\varepsilon\{\langle u^{(1,1)}F\rangle+c.c.\}+\varepsilon\langle u^{(1,0)}\rangle\\ =&A_1x^{-1/2}\exp\{-B_1(x-C_1t-x_0)^2/x\}e^{-Dx}\cos(kx-\omega t+Ex+\phi)\\ &+A_2\int_{-\infty}^{\sqrt{B_1}(x-C_1t-x_0)/x^{1/2}}e^{-\chi^2}d\chi,\end{aligned}\tag{3.4.16}$$

where c.c. means the complex conjugate and

$$A_1=\varepsilon\sqrt{\frac{2\pi p(1+\cos kh)^2h}{q\theta}}\operatorname{sech}\{-\frac{\pi}{2\sqrt{\Omega}}(\frac{w}{2}+\frac{1}{\varepsilon}\sqrt{\frac{2p}{q}}\sin kh)\},\tag{3.4.17}$$

$$B_1=\frac{(1+\cos kh)^2}{4\theta h}(1-\varepsilon w\sqrt{pq/2})^2,\tag{3.4.18}$$

$$C_1=\frac{v}{1-\varepsilon w\sqrt{pq/2}}\tag{3.4.19}$$

$$x_0=\frac{X_0h}{\varepsilon\sqrt{q/(2p)}-\varepsilon^2wq/2},\tag{3.4.20}$$

$$D=\frac{\varepsilon^2}{h}2L^2\langle\eta^2\rangle,\tag{3.4.21}$$

$$E=\frac{\varepsilon^2}{h}\{(\frac{w^2}{4}+\Omega)\frac{q}{2}+L\langle\eta_1^2\rangle\},\tag{3.4.22}$$

$$\phi=-\frac{1}{h}(1-\varepsilon w\sqrt{\frac{pq}{2}})\sin kh(x-C_1t-x_0),\tag{3.4.23}$$

$$A_2=-16\varepsilon\alpha\sqrt{\frac{2p\Omega}{\pi q}}.\tag{3.4.24}$$

The result (3.4.16) needs some explanations. The factor $\cos(kx-\omega t+Ex+\phi)$ in the first term means a monochromatic wave. The wave number k is changed to $k+E$ due to the the effects of nonlinearity and inhomogeneity.(see (3.4.22) and (2.3.25)) The factor $x^{-1/2}\exp\{-B_1(x-C_1t-x_0)^2/x\}$ indicates the modulation of the carrier wave. This is the same phenomena as that for slowly varying waves discussed in 3.2 and 3.3. Namely, the random walk of envelope soliton around a mean trajectory $x=C_1t$. The factor e^{-Dx} indicates an exponential damping due to the randomness of the phase. The second term in (3.4.16) i.e. $\varepsilon\langle u^{(1,0)}\rangle$ represents a slowly varying wave which does not suffer the exponential damping.

Since the first term in (3.4.16) decays exponentially in x, we may neglect it for a sufficiently large x. Then, there remains only the slowly varying wave $\varepsilon\langle u^{(1,0)}\rangle$. Differentiating it by t we have

$$\langle y_t\rangle=-A_2\sqrt{B_1}C_1x^{-1/2}e^{-B_1(x-C_1t-x_0)^2/x}.\tag{3.4.25}$$

This has the same form as (3.2.9) and (3.3.5) which indicates the random walk of soliton.

3.5 Conclusion and Discussion

We have investigated the statistical behaviors of solitons in a random lattice by use of model equations (2.3.6), (2.3.12) and (2.3.23). We have assumed that $\eta(\tau)$ in each equation is a Gaussian random white noise defined in (3.1.1). For the first and second cases we have obtained the same results (3.2.9) and (3.3.5) respectively; after a long propagation the amplitude of the soliton decreases as $x^{-1/2}$ and the width increases as $x^{1/2}$. For the third case, (3.4.16) shows that there is an exponential damping term of envelope soliton. For a sufficiently large x, only the term of slow varying wave remains. And then we get the result (3.4.25) which has the same form as (3.2.9). The power law $x^{-1/2}$ for the amplitude is supported by the numerical result for the random Toda lattice ,when the amplitude of the soliton becomes small enough. We remarked that the assumption of the Gaussian white noise is not essential in our theory.

Let us remark that solitons in a weakly inhomogeneous system never damp nor fission but they only deform. Since the averaged solitons (3.2.9), (3.3.5) and (3.4.25) are nothing but the distribution functions of random walk, we may regard the results as **random walks of solitons**. Calculations of higher correlation functions and statistical behaviors of N soliton are interesting problems.

We may choose other ensembles. Analysis of a random system with a long range correlation

$$\langle \eta(\tau)\eta(\tau')\rangle = 2\theta\,|\tau-\tau'|^{-\gamma}\,, \tag{3.5.1}$$

is also very interesting. We expect anomalous diffusion of solitons for (3.5.1). This problem is left for a future study.

Appendix 3A. Derivation of a formula (3.1.2)

We write the definition of the white noise $\eta(\tau)$ again;

$$\langle \eta(\tau)\eta(\tau')\rangle = 2\theta\delta(\tau-\tau'), \tag{3.A.1}$$

where θ is a constants. We shall show that eqs. (3.1.2) are derived from eq.(3.A.1). Since $\eta(\tau)$ is originally defined on the lattice point (see eq.(2.3.2)), we assume that the correlation (3.A.1) does not vanish for $|\tau-\tau'|<\delta$, where δ is the smallness parameter in (2.3.2). Thus we need some care in using (3.A.1). For the purpose, the δ-function in (3.A.1) is modified into a function $\Delta(\tau-\tau')$ such that if $|\tau-\tau'|>\delta$, $\Delta(\tau-\tau')\equiv 0$, and if $|\tau-\tau'|<\varepsilon^2$, $\Delta(\tau-\tau')=$ constant.

We define $\tilde{\eta}$ as

$$\tilde{\eta} = J\eta + L(\eta^2 - \langle\eta^2\rangle). \tag{3.A.2}$$

In terms of the modified δ-function, we have

$$\begin{aligned}\langle\tilde{\eta}(\tau)\tilde{\eta}(\tau')\rangle &= J^2\langle\eta(\tau)\eta(\tau')\rangle + 2L^2\langle\eta(\tau')\eta(\tau)\rangle^2\\ &= 2(\theta J^2 + 4\theta^2L^2\Delta(0))\Delta(\tau-\tau')\\ &= 2(\theta J^2 + 2\theta L^2\langle\eta^2\rangle)\Delta(\tau-\tau')\\ &\equiv 2\tilde{\theta}\Delta(\tau-\tau').\end{aligned} \tag{3.A.3}$$

Equation (3.A.3) indicates that $\tilde{\eta}$ is also a Gaussian white noise. Then we get

$$\begin{aligned}\langle \exp i\int_0^T \tilde{\eta}(\tau')d\tau'\rangle &= \exp(-\tilde{\theta}T)\\ &= \exp -(\theta J^2 - 2\theta L^2\langle\eta^2\rangle)T,\end{aligned} \tag{3.A.4}$$

from which we obtain (3.1.2) directly.

Chapter 4

Unstable Systems

We investigate an unstable system which has an inhomogeneity; the Rayleigh-Taylor instability system with an uneven 'bottom'. At first we introduce Unstable Nonlinear Schrödinger (UNS) equation which is a ubiquitous model for nonlinear unstable systems. Then, inhomogeneous dispersion relations are introduced. Applying the reductive perturbation method, we obtain an inhomogeneous UNS equation.

4.1 The Unstable Nonlinear Schrödinger Equation

The UNS equation (Wadati, Iizuka and T.Yajima 1991 Wadati, T.Yajima and Iizuka 1991) is defined by

$$iu_x + u_{tt} \pm 2\,|u|^2\,u = 0, \tag{4.1.1}$$

where x and t indicate space and time coordinates respectively. It has been shown that (1.1) appears in many physical systems such as capillary waves on the surface of liquid column (Kakutani, Inoue and Kan 1974), two-stream instability in plasma system (T.Watanabe 1969, Yamamoto 1970, T.Yajima and Wadati 1990ab), surface dynamics of the Rayleigh-Taylor instability problem (Iizuka and Wadati 1990).

The exchange of the independent variables x and t gives the ordinary NLS equation and then the Lax pair for (1.1) exists. Thus we can apply the inverse scattering method to the UNS equation. The initial value problem of the UNS equation (1.1) has been solved (T.Yajima and Wadati 1990 for + case, Iizuka, Wadati and T.Yajima 1991 for − case in (4.1.1)). There, attractive soliton collisions are reported.

The ubiquity of the model equation is explained as follows. Let us consider a 1-dimensional wave field $\phi(x,t)$ which satisfies a nonlinear evolution equation. By neglecting nonlinear terms and assuming a plane wave solution $e^{i(kx-\omega t)}$, we have a dispersion relation;

$$k = k(\omega). \tag{4.1.2}$$

As discussed in 2.1, we consider the Taylor expansion of $k(\omega)$ around $\omega = \omega_0$;

$$k - k_0 = \left.\frac{\partial k}{\partial \omega}\right|_{\omega=\omega_0}(\omega-\omega_0) + \frac{1}{2}\left.\frac{\partial^2 k}{\partial \omega^2}\right|_{\omega=\omega_0}(\omega-\omega_0)^2 + \cdots. \tag{4.1.3}$$

If $\partial k/\partial\omega|_{\omega=\omega_0} = 0$, $k_0 = k_c$ is regarded as **critical wave number** because the sign of $(\omega-\omega_0)^2$ changes when k exceeds k_c. Appearance of complex ω indicates the instability of the system.

We introduce dependent variable ψ as

$$\phi = e^{i(k_c x - \omega_0 t)}\psi. \tag{4.1.4}$$

Then, the dispersion relation (4.1.3) which is interpreted as an operator relation through (2.1.2) becomes

$$i\partial_x = \frac{1}{2}\left.\frac{\partial^2 k}{\partial \omega^2}\right|_{\omega=\omega_0}\partial_t^2. \tag{4.1.5}$$

This represents a linear part of the UNS equation. Therefore, the UNS equation is derived as an envelope equation, where the wave number of the carrier wave is near the critical value k_c. We note that for the critical wave number, we need not introduce the moving frame such as (2.1.10,11) but only the scaling transformation of independent variables (4.3.1). To take the nonlinear effect into account, we apply the reductive perturbation method.

In the following, we introduce inhomogeneous dispersion relations around $k = k_c$. Finite and weak inhomogeneities are separately discussed as in 2.1.

For the finite and slow inhomogeneity, k_c depends on x; $k_c = k(\varepsilon^2 x)$. Taylor expansion (4.1.3) at $k = k_c$ is given by

$$k = k(\omega, \varepsilon^2 x) = k_c(\varepsilon^2 x) + a(\varepsilon^2 x)(\omega-\omega_0)^2 + \cdots. \tag{4.1.6}$$

Note that the term which corresponds to last term in (2.1.21) does not appear in this case. By introducing ψ as

$$\phi = \exp i\{\int k_c(\varepsilon^2 x)\mathrm{d}x - \omega t\}\psi, \tag{4.1.7}$$

we obtain an operator relation for Ψ;

$$i\partial_x = a(\varepsilon^2 x)\partial_t^2, \tag{4.1.8}$$

which is transformed to (4.1.5) by a transformation of a space variable x.

If the inhomogeneity is weak, we may express the dispersion relation (4.1.2) as

$$k = \tilde{k}(\omega) + \varepsilon K(\omega, \varepsilon^2 x). \tag{4.1.9}$$

We define the critical wave number k_c by

$$\left.\frac{\mathrm{d}\tilde{k}}{\mathrm{d}\omega}\right|_{k=k_c} = 0. \tag{4.1.10}$$

Then the expansion (4.1.3) becomes

$$k = k_c + \varepsilon K(\omega_c, \varepsilon^2 x) + a(\omega - \omega_0)^2 + \varepsilon \frac{\partial K(\omega_0, \varepsilon^2 x)}{\partial \omega}(\omega - \omega_0) + \cdots, \tag{4.1.11}$$

which is similar to (2.1.30). Again, we introduce ψ as

$$\phi = \exp i\{k_c x - \omega_c t + \varepsilon \int \mathrm{d}x K(\omega_c, \varepsilon^2 x)\}\psi, \tag{4.1.12}$$

and have a linear equation from (4.1.11):

$$i\psi_x - a\psi_{tt} + i\varepsilon \frac{\partial K(\omega_0, \varepsilon^2 x)}{\partial \omega}\psi_t = 0. \tag{4.1.13}$$

This suggests that due to the weak inhomogeneity, the UNS equation (4.1.1) is modified into

$$iu_x + u_{tt} \pm 2|u|^2 u + i\nu(x)u_t = 0. \tag{4.1.14}$$

4.2 The Rayleigh-Taylor Instability Problem

As an example of unstable and inhomogeneous system, we consider the Rayleigh Taylor Instability Problem (Iizuka and Wadati 1990). Let us consider a two dimensional heavy fluid (density ρ) supported by a light fluid. The two fluid are assumed not to mix each other. The coordinates x and y indicate horizontal and vertical positions respectively. The heavy fluid is bounded from the above by a rigid surface

whose shape is slowly changing in the x-direction and expressed as $y = -H(\varepsilon^2 x)$ (Fig.4.1). Constant ε is a smallness parameter. This represents the slow and finite inhomogeneity in this system.

The surface of the fluid is given by $y = \eta(x, t)$, where t indicates time. When the surface is flat, $\eta \equiv 0$. We denote by $\Phi(x, y, t)$ the velocity potential of heavy fluid. The fluids are assumed to be inviscid, incompressible and irrotational.

Between the two fluid there exists surface tension which generates the effect of the stability to the surface. Here and hereafter, we shall assume that the density of the lighter fluid is negligibly smaller than that of the heavy fluid (Malik and Singh 1989). Thus the interface is determined by considering only the motion of the heavy fluid. The fundamental equations for the heavy fluid are

$$\Phi_{xx} + \Phi_{yy} = 0, \qquad -H(x) \le y \le \eta(x, t) \tag{4.2.1}$$

$$\Phi_t + \frac{1}{2}\{(\Phi_x)^2 + (\Phi_y)^2\} = g\eta + \frac{T}{\rho}\frac{\eta_{xx}}{\{1 + (\eta_x)\}^{\frac{3}{2}}}, \quad y = \eta(x, t) \tag{4.2.2}$$

$$\Phi_y = \eta_t + \Phi_x \eta_x, \qquad y = \eta(x, t) \tag{4.2.3}$$

$$\frac{\mathrm{d}H}{\mathrm{d}x}\Phi_x + \Phi_y = 0. \qquad y = -H(\varepsilon^2 x) \tag{4.2.4}$$

The first equation (4.2.1) describes the conditions that the fluid is incompressible and the flows are irrotational. The second equation (4.2.2) is the continuity condition for the pressure that can be derived from the Bernoulli theorem. Here, g and T are acceleration constant of the gravity and coefficient of the surface tension, respectively. The third equation (4.2.3) denotes the free boundary condition at the interface. The last equation (4.2.4) means that the perfect fluid does not penetrate the rigid surface.

Let us consider a linearization of the fundamental equations. The solutions up to $O(\varepsilon^2)$ are given by;

$$\Phi = C\cosh k(H(\varepsilon^2 x) - y)\cos(\int k\mathrm{d}x - \omega t), \tag{4.2.5}$$

$$\eta = -C\frac{k}{\omega}\sin kH(\varepsilon^2 x)\sin(\int k\mathrm{d}x - \omega t), \tag{4.2.6}$$

where local wave number $k = k(\varepsilon^2 x)$ is determined by the following inhomogeneous dispersion relation for a given ω

$$\omega^2 = \left(\frac{T}{\rho}k^3 - kg\right)\tanh kH(\varepsilon^2 x). \tag{4.2.7}$$

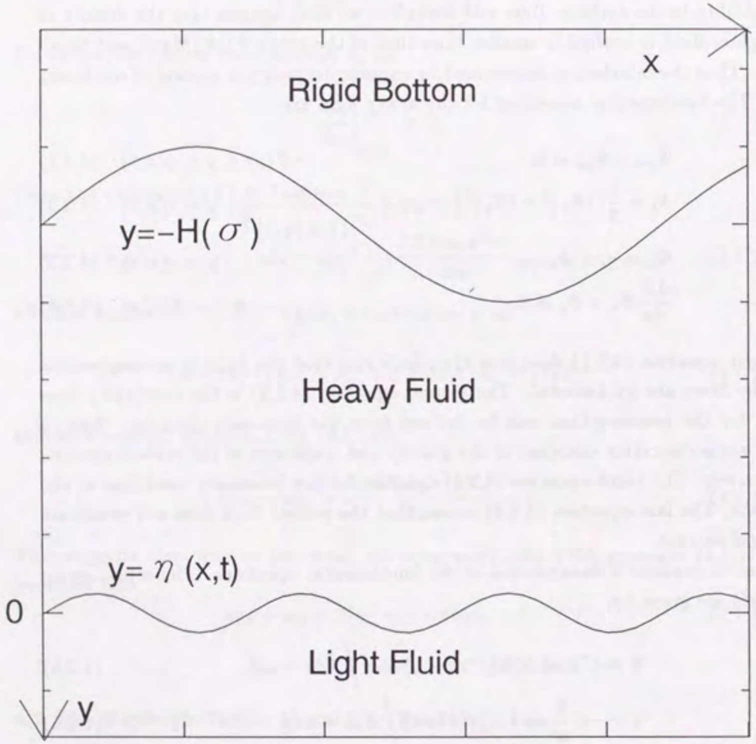

Figure 4.1: System of the Rayleigh Taylor Instability problem. Between the two fluid there exists a surface tension.

The profile of (4.2.7) for the homogeneous case is shown in Fig.4.2.

We find that there is a critical wave number k_c which, fortunately, does not depend on x;

$$k_c = \sqrt{\frac{\rho g}{T}}. \tag{4.2.8}$$

If $k > k_c$, the linearized wave is stable and if $k < k_c$, unstable. When $k = k_c$, $\omega = 0$. Conversely, if we set $\omega = 0$, we obtain from (4.2.7) $k(\varepsilon x) = k_c =$ constant. Therefore, for the critical wave number, the carrier wave F is given by

$$F = \exp i \int k_c \mathrm{d}x = \exp ik_c x. \tag{4.2.9}$$

In the next section we consider nonlinear modulations of the carrier wave (4.2.9).

4.3 The inhomogeneous UNS Equation

The aim of this section is to present a model equation for the nonlinear waves in inhomogeneous and unstable media. The inhomogeneity is assumed to be slow and finite. As an example we consider the Rayleigh-Taylor instability problem. For the purpose we apply the reductive perturbation method to the nonlinear modulation of the monochromatic wave (4.2.9).

We introduce scaling transformations of independent variables

$$\sigma = \varepsilon^2 x, \tag{4.3.1a}$$

$$\tau = \varepsilon t. \tag{4.3.1b}$$

Thus the rigid bottom is expressed as

$$y = -H(\sigma). \tag{4.3.2}$$

We expand the velocity potential $\Phi(x, y, t)$ and the surface elevation $\eta(x, t)$ in terms of the smallness parameter ε and the carrier wave F in (4.2.9);

$$\Phi = \sum_{n=1}^{\infty} \sum_{l=-n}^{l=n} \varepsilon^n F^l \Phi^{(n,l)}(\sigma, \tau, y), \tag{4.3.3}$$

$$\eta = \sum_{n=1}^{\infty} \sum_{l=-n}^{l=n} \varepsilon^n F^l \eta^{(n,l)}(\sigma, \tau). \tag{4.3.4}$$

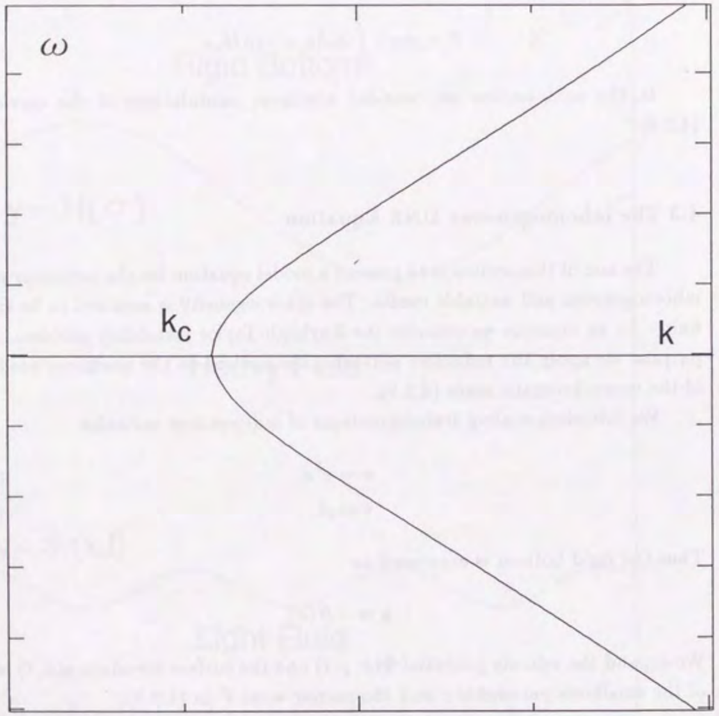

Figure 4.2: Dispersion relation for the Rayleigh Taylor Instability problem (homogeneous case).

We substitute (4.3.1∼4) into fundamental equations (4.2.1∼4) and equate the terms of $\varepsilon^n F^l$.

At $(n,l) = (1,0)$ we have relations

$$\eta^{(1,0)} \equiv 0, \tag{4.3.5}$$

$$\Phi_y^{(1,0)} \equiv 0. \tag{4.3.6}$$

At $(n,l) = (1,1)$ we confirm that k_c is the critical value (4.2.8) and that $\Phi^{(1,1)}$ vanishes;

$$\Phi^{(1,1)} \equiv 0. \tag{4.3.7}$$

For $(n,l) = (2,0)$ and $(n,l) = (2,1)$ we have

$$\Phi_\tau^{(1,0)} = g\eta^{(2,0)}, \tag{4.3.8}$$

$$\Phi_y^{(1,0)} \equiv 0, \tag{4.3.9}$$

and

$$\Phi^{(2,0)} = \frac{\cosh k_c\big(y + H(\sigma)\big)}{k_c \sinh kH(\sigma)} \eta_\tau^{(1,1)}, \tag{4.3.10}$$

respectively. In $(n,l) = (2,2)$ we have

$$\eta^{(2,2)} \equiv 0, \tag{4.3.11}$$

$$\Phi^{(2,2)} \equiv 0, \tag{4.3.12}$$

and $(n,l) = (3,0)$

$$\eta^{(2,0)} \equiv 0, \tag{4.3.13}$$

$$\Phi_\tau^{(2,0)} = g\eta^{(3,0)}, \tag{4.3.14}$$

$$\Phi^{(3,0)} \equiv 0. \tag{4.3.15}$$

From (4.3.8) and (4.1.13) we find that $\Phi_\tau^{(1,0)}$ does not depend on τ;

$$\Phi_\tau^{(1,0)} = g\eta^{(2,0)} = f(\sigma). \tag{4.3.16}$$

Thus $f(\sigma)$ is determined from the initial value of η.

In $(n,l) = (3.1)$ we obtain a closed evolution equation for $\eta^{(1,1)}$;

$$\begin{aligned} \eta_{\tau\tau}^{(1,1)} - 2ig\tanh kH(\sigma)\eta_\sigma^{(1,1)} - \frac{3}{2}\frac{T}{\rho}k_c^5 \tanh kH(\sigma)\left|\eta^{(1,1)}\right|^2 \eta^{(1,1)} \\ + f(\sigma)k_c \tanh kH(\sigma)\eta^{(1,1)} = 0. \end{aligned} \tag{4.3.17}$$

Here, the first and the second term correspond to l.h.s and r.h.s. of (4.1.8) respectively. By a further transformation of dependent and independent variables;

$$\xi \to -\int^{\sigma} \frac{\mathrm{d}\sigma'}{2g\tanh kH(\sigma')}, \tag{4.3.18a}$$

$$\tau \to \tau, \tag{4.3.18b}$$

$$U(\xi,\tau) = \sqrt{\frac{3}{2}\frac{T}{\rho}k_c^5 \tanh kH(\sigma)}\exp i\Big(\int f(\sigma)\mathrm{d}\sigma \frac{T}{\rho}k_c\Big)\eta^{(1,1)}, \tag{4.3.19}$$

we reduce (4.3.17) to

$$iU_\xi + U_{\tau\tau} - |U|^2 U + i\nu(\xi)U = 0, \tag{4.3.20a}$$

$$\nu(\xi) = \frac{\mathrm{d}}{\mathrm{d}\xi}\frac{1}{\sqrt{3T/(2\rho)k_c^5 \tanh kH(\sigma)}}. \tag{4.3.20b}$$

This is in the same form as inhomogeneous NLS equation (1.3) or (2.2.23). However the inhomogeneous UNS equation (4.3.20a) is essentially different; in (1.3) or (2.2.23) we consider evolution with respect to $\tau = \varepsilon^j x (j = 2, 3)$ (spatial evolution), while in (4.3.20a) with respect to $\tau = \varepsilon t$ (time evolution). We believe that the model equation (4.3.20a) has many applications for the analysis of wave propagations in unstable and inhomogeneous systems.

We have not considered weak inhomogeneity, because $\partial K/\partial\omega$ in (4.1.13) vanishes for the Rayleigh-Taylor system.

Chapter 5

Two-Dimensional Systems (Shallow Water Waves Upon an Uneven Bottom)

Two-dimensional nonlinear waves in a slowly inhomogeneous system are investigated. As a typical case, three-dimensional shallow water waves over an uneven bottom are considered. Here, the surface waves is two-dimensional. The depth is assumed to be slow in variation. As a model, an inhomogeneous KP equation is proposed;

$$\left(U_\tau + 6U_\xi U + U_{\xi\xi\xi} + A(\tau)U + B(\tau,\zeta)U_\xi\right)_\xi + 3C(\tau)U_{\zeta\zeta} = 0.$$

This model is a two-dimensional extension of (1.2) and (1.4). The "external" functions $A(\tau)$, $B(\zeta,\tau)$ and $C(\tau)$ are related to the depth change. Some reductions of this equation are used to describe deformation of a line soliton due to the depth change. The model equation is valid for a wide class of two-dimensional nonlinear waves in inhomogeneous systems.

5.1 The Inhomogeneous KP Equation

Let us consider water waves over an uneven bottom in three dimensional space (Iizuka and Wadati 1992b). We denote time evolution of the water surface by $z = \eta(x,y,t)$, where z represents the vertical coordinate, x and y the horizontal coordinates and t the time. When the water surface is flat, $\eta(x,y,t) \equiv 0$. The rigid bottom is expressed as $z = -H(x,y)$. Let $\Phi(x,y,t)$, g and ρ be the velocity potential, the gravitational acceleration and the density of the fluid, respectively. Similarly to (4.2.1∼4), fundamental equations for the water waves is given by

$$\Phi_{xx} + \Phi_{yy} + \Phi_{zz} = 0 \quad (\text{for} \quad -H(x,y) \le z \le \eta(x,y,t)), \tag{5.1.1}$$

$$\Phi_t + \frac{1}{2}\{(\Phi_x)^2 + (\Phi_y)^2 + (\Phi_z)^2\} + g\eta = \frac{T}{\rho R} \quad (\text{at} \quad z = \eta(x,y,t)), \tag{5.1.2}$$

$$\Phi_z = \eta_t + \Phi_x\eta_x + \Phi_y\eta_y \quad (\text{at} \quad z = \eta(x,y,t)), \tag{5.1.3}$$

$$H_x\Phi_x + H_y\Phi_y + \Phi_z = 0 \quad (\text{at} \quad z = -H(x,y)), \tag{5.1.4}$$

where T is the surface tension and R^{-1} is the mean curvature of the surface defined by

$$\frac{1}{R} = \left[\{1 + (\eta_x)^2\}\eta_{yy} + \{1 + (\eta_y)^2\}\eta_{xx} - 2\eta_x\eta_y\eta_{xy}\right]\{1 + (\eta_x)^2 + (\eta_y)^2\}^{-\frac{3}{2}}. \tag{5.1.5}$$

The meanings of the equations have been explained just after eqs. (4.2.1∼4). The essential differences are that in the present case the system is three dimensional and that the direction of the gravitational acceleration is opposite.

We shall be interested in weakly nonlinear waves. To this end, we apply the reductive perturbation method. Introducing a smallness parameter ε and a characteristic length l of the depth, we transform the independent variables;

$$\xi = \varepsilon\sqrt{\frac{g}{l}}\Big(\int^x F(\varepsilon^3 l^{-1} x')\mathrm{d}x' - t\Big), \tag{5.1.6a}$$

$$\zeta = \varepsilon^2 l^{-1} y, \tag{5.1.6b}$$

$$\tilde{\tau} = \varepsilon^3 l^{-1} x, \tag{5.1.6c}$$

$$\tilde{z} = l^{-1} z, \tag{5.1.6d}$$

where $F(\varepsilon^3 l^{-1} x) = F(\tilde{\tau})$ will be determined shortly later. New independent variables are chosen to be dimensionless. We expand dependent variables $\eta(x,y,t)$, $\Phi(x,y,z,t)$ and $H(x,y)$ as

$$\eta(x,y,t) = l\Big(\varepsilon^2\eta^{(2)}(\xi,\zeta,\tilde{\tau}) + \varepsilon^4\eta^{(4)}(\xi,\zeta,\tilde{\tau}) + \cdots\Big), \tag{5.1.7a}$$

$$\begin{aligned}\Phi(x,y,z,t) = \sqrt{gl^3}\Big(&\varepsilon\Phi^{(1)}(\xi,\zeta,\tilde{z},\tilde{\tau}) + \varepsilon^3\Phi^{(3)}(\xi,\zeta,\tilde{z},\tilde{\tau}) \\ &+ \varepsilon^5\Phi^{(5)}(\xi,\zeta,\tilde{z},\tilde{\tau}) + \cdots\Big),\end{aligned} \tag{5.1.7b}$$

$$H(x,y) = l\Big(H^{(0)}(\tau) + \varepsilon^2 H^{(2)}(\tau,\zeta)\Big). \tag{5.1.7c}$$

The variation of the surface elevation η in the y-direction is assumed to be much slower than that in the x-direction. The zeroth order inhomogeneity of the bottom, $H^{(0)}$, is assumed to be independent of y. In (5.1.7c) $H^{(0)}$ and $H^{(1)}$ indicate finite and weak inhomogeneity respectively.

We substitute (5.1.6∼7) into equations (5.1.1∼4) and equate the terms of the same order in ε. Within $O(\varepsilon^2)$ we see that $\Phi^{(1)}$ does not depend on $\tilde{z}$ and is given by

$$\Phi^{(1)}_\xi = \eta^{(2)}(\xi,\zeta,\tilde{\tau}). \tag{5.1.8}$$

At $O(\varepsilon^3)$ and $O(\varepsilon^4)$, we have relations as

$$F(\tilde{\tau}) = \frac{1}{\sqrt{glH^{(0)}(\tilde{\tau})}} \equiv \frac{1}{\sqrt{gl}}f(\tilde{\tau}), \tag{5.1.9}$$

$$\Phi^{(3)}_\xi = -\frac{f^2}{2}\eta^{(2)}_{\xi\xi}\tilde{z}^2 - \eta^{(2)}_{\xi\xi}\tilde{z} + \frac{f^2}{2}(\eta^{(2)})^2 + \eta^{(4)} - \kappa f^2\eta^{(2)}_{\xi\xi}, \tag{5.1.10}$$

where $\kappa = T/(\rho g l^2)$ is a dimensionless parameter. If the bottom is uniform, F in (5.1.9) is the inverse of velocity of the linearized wave. A closed equation of $\eta^{(1)}$ is obtained at $O(\varepsilon^5)$:

$$\left\{\eta^{(2)}_{\tilde{\tau}} + \Big(\frac{1}{6f} - \kappa f^3\Big)\eta^{(2)}_{\xi\xi\xi} + \frac{3f^3}{2}\eta^{(2)}_\xi\eta^{(2)} - \frac{f_{\tilde{\tau}}}{2f}\eta^{(2)} + \frac{f^3}{2}H^{(2)}\eta^{(2)}_\xi\right\}_\xi + \frac{1}{2f}\eta^{(1)}_{\zeta\zeta} = 0. \tag{5.1.11}$$

Introducing further new variables τ and $U(\xi,\zeta,\tau)$ by

$$\tau = \int^{\tilde{\tau}} \mathrm{d}\tilde{\tau}\Big(\frac{1}{6f(\tilde{\tau})} - \kappa f^3(\tilde{\tau})\Big), \tag{5.1.12}$$

$$U(\xi,\zeta,\tau) = \Big(\frac{1}{6f} - \kappa f^3\Big)^{-1}\frac{f^3}{4}\eta^{(2)}, \tag{5.1.13}$$

we arrive at

$$\Big(U_\tau + U_{\xi\xi\xi} + 6U_\xi U + A(\tau)U + B(\tau,\zeta)U_\xi\Big)_\xi + C(\tau)U_{\zeta\zeta} = 0, \tag{5.1.14}$$

where the functions $A(\tau)$, $B(\tau,\zeta)$ and $C(\tau)$ are related to the depth change by

$$\begin{aligned} A(\tau) &= -\frac{9}{2}\frac{f_\tau}{f}, \\ B(\tau,\zeta) &= 3f(\tau)^4 H^{(2)}(\tau,\zeta), \\ C(\tau) &= \frac{1}{2f}\Big(\frac{1}{6f} - \kappa f^3\Big)^{-1}. \end{aligned} \tag{5.1.15}$$

This is nothing but the inhomogeneous KP equation. We should remark that this is a two-dimensional extension of the inhomogeneous KdV equations (1.2) and (1.4). We believe that it describes a large class of two-dimensional nonlinear waves in slowly inhomogeneous systems. For instance ion-acoustic waves in varying density and temperature are expected to reduce (5.1.14). We note that if the surface tension is neglected, the coefficient of $U_{\zeta\zeta}$ in (5.1.14) is constant($=3$).

5.2. Deformations of Line Solitons

In this section we shall investigate a simpler but still important case. Namely, we consider the weak inhomogeneity;

$$H^{(0)} \equiv 1 (f \equiv 1), \tag{5.2.1}$$

and assume that bottom variation does not depend on y;

$$H^{(2)} = H^{(2)}(\tau). \tag{5.2.2}$$

Under these assumptions the inhomogeneous KP equation (5.1.14) reduces to

$$\left(U_\tau + U_{\xi\xi\xi} + 6U_\xi U + 3H^{(2)}(\tau)U_\xi\right)_\xi + \frac{3}{(1-6\kappa)}U_{\zeta\zeta} = 0. \tag{5.2.3}$$

Let us introduce a transformation of independent variables

$$X = \xi - 3\int^\tau H^{(2)}(\tau')\mathrm{d}\tau', \tag{5.2.4a}$$

$$Y = \sqrt{\frac{1}{|1-6\kappa|}}\zeta \equiv \lambda\zeta, \tag{5.2.4b}$$

$$T = \tau. \tag{5.2.4c}$$

Then, from (5.2.3), we get the KP equation;

$$(U_T + U_{XXX} + 6UU_X)_X + 3\sigma U_{YY} = 0. \tag{5.2.5}$$

$$\sigma = \mathrm{sgn}(1-6\kappa).$$

From now on we assume that $\sigma = +1$. Recall that $\kappa = T/(\rho g l^2)$ is a dimensionless parameter which is related to the surface tension. It is known that the KP equation (5.2.5) is integrable and has a line soliton solution;

$$U = 2k_1^2\mathrm{sech}^2(k_1X + k_2Y - \omega T - \delta), \tag{5.2.6}$$

$$k_1\omega - 4k_1^4 - 3k_2^2 = 0. \tag{5.2.7}$$

In terms of (ξ, ζ, τ) and (x, y, t), the soliton solution (5.2.6) is expressed as

$$\begin{aligned} U &= 2k_1^2\mathrm{sech}^2\left\{k_1(\xi - 3\int^\tau H^{(2)}(\tau')\mathrm{d}\tau') + k_2\lambda\zeta - \omega\tau\right\}, \\ &= 2k_1^2\mathrm{sech}^2\left\{k_1\varepsilon(l^{-1}x - \sqrt{\frac{g}{l}}t) - 3k_1\int^{\varepsilon^3 x/6l} H^{(2)}(\tau')d\tau' \right. \\ &\quad \left. + k_2\varepsilon^2\lambda\frac{y}{l} - \frac{\omega\varepsilon^3}{6l}x\right\}, \end{aligned} \tag{5.2.8}$$

where we have taken $\delta = 0$ for simplicity. This form suggests that the line soliton undergoes the deformation due to the weak inhomogeneity $H^{(2)}$. In the following two analytically tractable forms of $H^{(2)}$ are investigated.

First, let us consider a case;

$$H^{(2)}(\tau) = a\mathrm{sech}^2\frac{\tau}{d}. \tag{5.2.9}$$

If $a > 0 (a < 0)$, the bottom has a line hollow(heap). The deformed soliton is given by

$$U = 2k_1^2\mathrm{sech}^2\Theta \tag{5.2.10a}$$

$$\Theta = (k_1\varepsilon - \frac{\omega}{6}\varepsilon^3)\tilde{x} - 3k_1 ad\tanh\frac{\varepsilon^3\tilde{x}}{6} + k_2\varepsilon^2\lambda\tilde{y} - k_1\varepsilon\tilde{t}, \tag{5.2.10b}$$

where $\tilde{x} = l^{-1}x$, $\tilde{y} = l^{-1}y$ and $\tilde{t} = \sqrt{g/l}x$ are dimensionless. The waveforms (5.2.10) are shown in Fig.5.1. We see that the soliton is deformed forward(backward) due to the line hollow(heap).

This is a reasonable result because in shallow waters the velocity of the linearized wave is proportional to the square of the depth.

Second, we choose

$$H^{(2)}(\tau) = a\tanh\frac{\tau}{d}. \tag{5.2.11}$$

This indicates a mild step in the bottom. For this inhomogeneity, the phase Θ in (5.2.10.b) is given by

$$\Theta = (k_1\varepsilon - \frac{\omega\varepsilon^3}{6})\tilde{x} - 3k_1 ad\log(\frac{\cosh\varepsilon^3\tilde{x}}{6d}) + k_2\varepsilon^2\lambda\tilde{y} - k_1\varepsilon\tilde{t}. \tag{5.2.12}$$

Deformations of the line soliton are shown in Fig.5.2. We see that the velocity of the soliton is faster in the deeper region and that the line soliton is curved at the step. The latter represents one of the properties nonlinear refraction phenomena.

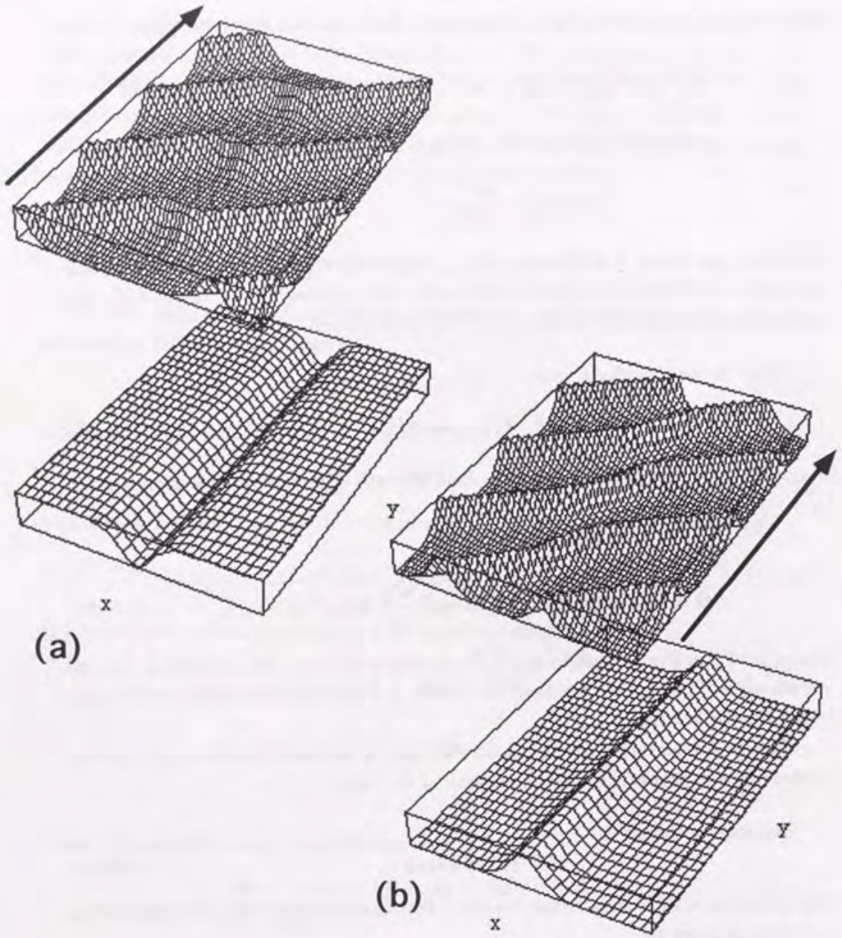

Figure 5.1: Deformation of the line soliton due to a line hollow(a) and a heap(b). The solitons at five different times are superposed. Arrows indicate propagation of the soliton. Lower sheets are the profiles of the bottoms.

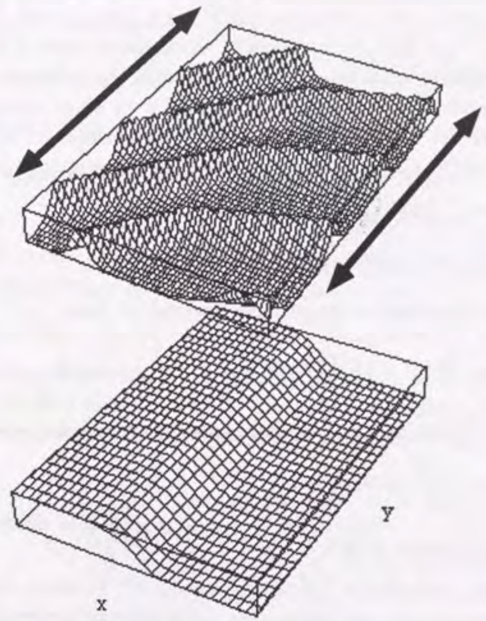

Figure 5.2: Deformation of the line soliton due to a mild step. Both two opposite directions are allowed for propagation of the soliton. We observe a refraction of the soliton.

5.3 Isotropic Scale Inhomogeneity

In this section, we investigate a rather different situation from the above one. In the preceding two sections, we have assumed $H^{(2)}$ is a function of τ and ζ, which implies that $H_x^{(2)} \sim \varepsilon^3$ and $H_y^{(2)} \sim \varepsilon^2$. Namely, the variation scale of the weak inhomogeneity in the x-direction is much larger than that in the y-direction. In the following we shall consider the weak inhomogeneity ($f \equiv 1$) where the two scale lengths are the same order. We term it 'isotropic scale inhomogeneity'. In order to express the situation, $H^{(2)}$ is assumed to be a function of τ and $\varepsilon\zeta$;

$$H^{(2)} = H^{(2)}(\tau, \varepsilon\zeta) = H^{(2)}(\varepsilon^3 l^{-1} x, \varepsilon^3 l^{-1} y), \tag{5.3.1}$$

$$H_x^{(2)} \sim H_y^{(2)} \sim \varepsilon^3. \tag{5.3.2}$$

Again, employing the reductive perturbation method, we have

$$\left(U_\tau + U_{\xi\xi\xi} + 6U_\xi U + 3H^{(2)}(\tau, \varepsilon\zeta)U_\xi\right)_\xi + \frac{3}{1-6\kappa}U_{\zeta\zeta} = 0. \tag{5.3.3}$$

To eliminate the fourth term in (5.3.3), we introduce new independent variables;.

$$X = \xi - 3\int^\tau H^{(2)}(\tau', \varepsilon\zeta)\mathrm{d}\tau', \tag{5.3.4a}$$

$$Y = \lambda\left(\zeta + 18\varepsilon\int^\tau \mathrm{d}\tau' \int^{\tau'} \mathrm{d}\tau" \frac{\partial H^{(2)}(\tau", \varepsilon\zeta)}{\partial(\varepsilon\zeta)}\right), \tag{5.3.4b}$$

$$T = \tau, \tag{5.3.4c}$$

where λ has been defined in (5.2.4b). Using (5.3.4)into (5.3.3), we obtain

$$(U_T + U_{XXX} + 6U_X U)_X + 3\sigma U_{YY} + O(\varepsilon^2) = 0. \tag{5.3.5}$$

This is the KP equation up to $O(\varepsilon^2)$ quantities. In the following we shall neglect $O(\varepsilon^2)$ in (5.3.5). This is not an appropriate approach for solving eq.(5.3.3). But we know that the model equations (5.1.4),(5.2.5) and (5.3.3) are, as surface dynamics, approximations within the error of $O(\varepsilon^2)$. In this sense, removing $O(\varepsilon^2)$ in equation (5.3.5) is permitted.

Similarly to the previous case, one soliton solution of the KP equation (5.3.5) for $\sigma = 1$ is given by

$$\begin{aligned} U &= 2k_1^2 \mathrm{sech}^2(k_1 X + k_2 Y - \omega T) \\ &= 2k_1^2 \mathrm{sech}^2\Theta, \end{aligned} \tag{5.3.6}$$

where

$$\begin{aligned} \Theta &= k_1(\xi - 3\int^\tau H^{(2)}(\tau', \varepsilon\zeta)\mathrm{d}\tau') + k_2\lambda(\zeta + 18\varepsilon\int^\tau \mathrm{d}\tau' \int^{\tau'} \mathrm{d}\tau" \frac{\partial H^{(2)}(\tau", \varepsilon\zeta)}{\partial(\varepsilon\zeta)}) + \omega\tau \\ &= (k_1\varepsilon - \frac{\omega}{6}\varepsilon^3)\tilde{x} + k_2\varepsilon^2\lambda\tilde{y} - k_1\varepsilon\tilde{t} \\ &\quad - 3k_1\int^\tau H^{(2)}(\tau', \varepsilon\zeta)\mathrm{d}\tau' + 18k_2\varepsilon\lambda\int^\tau \mathrm{d}\tau' \int^{\tau'} \mathrm{d}\tau" \frac{\partial H^{(2)}(\tau", \varepsilon\zeta)}{\partial(\varepsilon\zeta)}. \end{aligned} \tag{5.3.7}$$

This describes a deformed line soliton due to the weak inhomogeneity $H^{(2)}$.

As an application, we consider a localized inhomogeneity;

$$H^{(2)}(\tau, \varepsilon\zeta) = a\mathrm{sech}^2\frac{\tau}{d_1}\mathrm{sech}^2\frac{\varepsilon\zeta}{d_2}. \tag{5.3.8}$$

If $a > 0 (a < 0)$ the bottom has a localized hollow(heap). The phase Θ of a deformed line soliton for a simple case $k_2 = 0$ is given by

$$\Theta = (k_1\varepsilon - \frac{\omega}{6}\varepsilon^3)\tilde{x} - 3k_1 a d_1(\tanh\frac{\varepsilon^3\tilde{x}}{6d_1} + 1)\mathrm{sech}^2\frac{\varepsilon^3\tilde{y}}{d_2} - k_1\varepsilon\tilde{t}, \tag{5.3.9}$$

where the integration constant in (5.3.7) is chosen to be a. The wave profile for the phase (5.3.9) is shown in Fig.5.3. Similarly to the case in Section 5.2, the line soliton is deformed forward(backward)due to the hollow(heap).

5.4 Conclusion and Discussion

We have proposed the inhomogeneous KP equation (5.1.14), which describes a three-dimensional shallow water wave traveling over an uneven bottom. The inhomogeneous KP equation is possibly a ubiquitous model for two dimensional nonlinear waves in inhomogeneous systems. If the inhomogeneity of the bottom is weak and depends only on the x-coordinate, equation (5.1.14) reduces to the KP equation. Then, we find that due to the inhomogeneity the deformation of a line soliton occurs during the propagation. As applications, effects of a line hollow, heap and a mild step in the bottom are considered. If the inhomogeneity is weak and the characteristic length of it is common in x and y directions, we obtain the KP equation up to $O(\varepsilon^2)$. Similarly to the former case, the deformation of a line soliton is shown explicitly.

When the inhomogeneity is not weak or B in (5.1.2) depends on ζ, we cannot analytically solve the inhomogeneous KP equation. In this case, we expect not only the deformation of a soliton but also the fission. To investigate this phenomena we need to take numerical or approximate approaches, which is left for a future study.

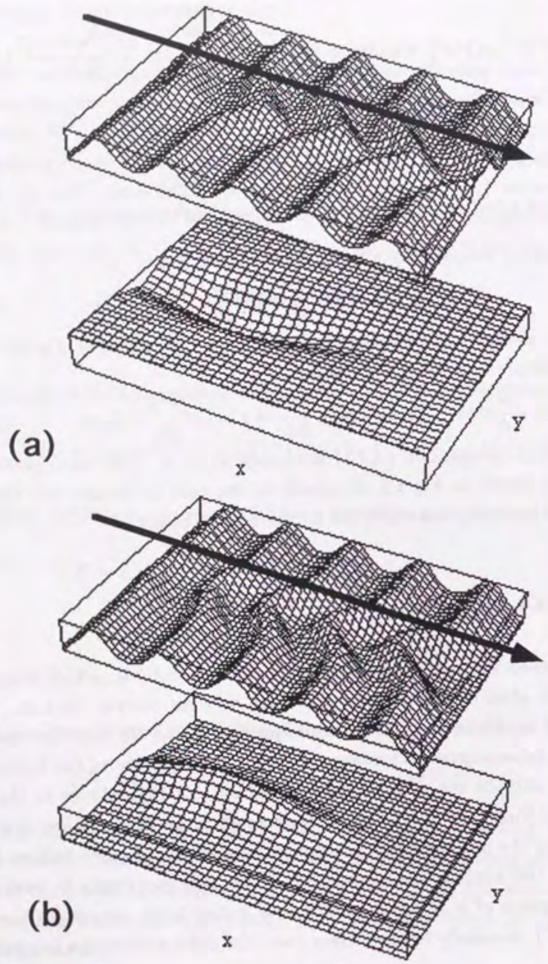

Figure 5.3: Deformation of the line soliton due to a localized hollow(a) and heap(b).

Part II . Scattering of Solitons

Chapter 6

Discontinuous Systems

Scatterings of solitons due to a discontinuity in one-dimensional media are considered. As a model we employ a nonlinear lattice which has a discontinuity in the mass distribution. Two kinds of physically interesting waves, slowly varying waves and carrier wave modulations are studied. The reflected and transmitted waves are constructed from the incident wave analytically. Fission and reflection of incident KdV and NLS solitons due to the discontinuity are observed.

6.1. The Discontinuous Anharmonic Lattice

We shall consider a one-dimensional nonlinear lattice. The mass of j-th particle is M for $j \geq 0$ and m for $j < 0$ (Fig.6.1). We denote by y_j the displacement of the j-th particle.

The equation of motion is given by (Iizuka and Wadati 1992c)

$$m_j \ddot{y}_j = B[y_{j+1} - y_j + \alpha(y_{j+1} - y_j)^2 + \beta(y_{j+1} - y_j)^3] - B[y_j - y_{j-1} + \alpha(y_j - y_{j-1})^2 + \beta(y_j - y_{j-1})^3], \tag{6.1.1a}$$

$$m_j = \begin{cases} m & (j \leq -1) \\ M & (j \geq 0) \end{cases} \tag{6.1.1b}$$

where $B(> 0)$, α and β are constants. Dots indicate differentiations with respect to time t .

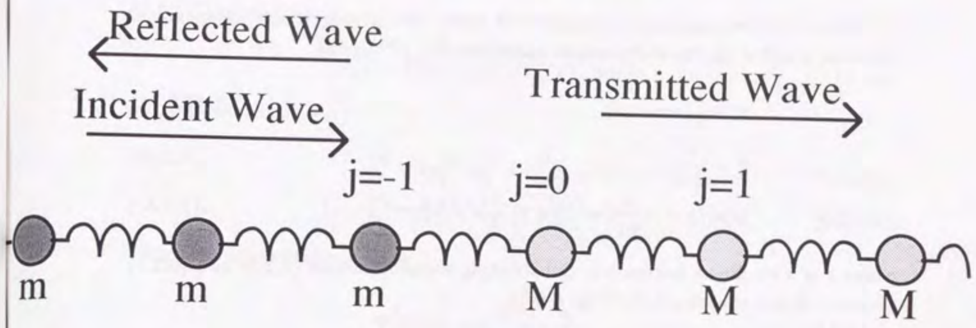

Figure 6.1: Nonlinear lattice with a discontinuity in the mass distribution.

6.2 Slowly Varying Waves

Reductive Perturbation Method

We assume that the displacement y_j is slowly varying function of the site j. Let h denote lattice spacing. We employ the continuum approximation $y_j = y(x, t)$, where $x = jh$. Similarly to case in (2.2.6), the equation of motion (6.1.1) is reduced to a Boussinesq type equation;

$$y_{xx} = \frac{1}{c^2} y_{tt} + \frac{h^2}{12} y_{xxxx} + 2h\alpha y_x y_{xx} + \cdots, \tag{6.2.1}$$

where subscripts x and t represent partial differentiations. The acoustic velocity c is given by

$$c = \begin{cases} c_+ = \sqrt{\dfrac{Kh^2}{M}} & (x > 0), \\ c_- = \sqrt{\dfrac{Kh^2}{m}} & (x < 0). \end{cases} \tag{6.2.2}$$

In the following we consider the weakly nonlinear wave and apply the reductive perturbation method to the regions $x > 0$, and $x < 0$ separately. We assume that $\alpha \neq 0$.

For $x > 0$ we consider the transmitted wave. We introduce new independent variables η and τ and new dependent variables $v^{(1)}$, $v^{(3)}$, $\cdots$ as

$$\eta = \frac{\varepsilon}{h}(x - c_+t), \tag{6.2.3a}$$

$$\tau = \frac{\varepsilon^3}{24h}x, \tag{6.2.3b}$$

$$y(x,t) = -\frac{1}{4\alpha}\{\varepsilon v^{(1)}(\eta,\tau) + \varepsilon^3 v^{(3)}(\eta,\tau) + \cdots\}, \tag{6.2.3c}$$

where ε is a smallness parameter. Substituting transformations (6.2.3) into (6.2.1) we get a closed equation for $v^{(1)}$ in $O(\varepsilon^5)$;

$$v^{(1)}_{\eta\tau} - 6v^{(1)}_\eta v^{(1)}_{\eta\eta} + v^{(1)}_{\eta\eta\eta\eta} = 0. \tag{6.2.4}$$

Equation (6.2.4) is also expressed as

$$\Psi^T_\tau - 6\Psi^T_\eta \Psi^T + \Psi^T_{\eta\eta\eta} = 0 \qquad (\tau > 0) \tag{6.2.5a}$$

$$\Psi^T = v^{(1)}_\eta = -\frac{v^{(1)}_t}{\varepsilon c_+}. \tag{6.2.5b}$$

The transmitted wave $\Psi^T(\eta,\tau)$ is governed by the KdV equation (6.2.5a).

For $x < 0$ we need to consider both the incident and reflected waves. For the purpose we introduce the following transformations of the independent and dependent variables;

$$\xi = \frac{\varepsilon}{h}(x - c_-t), \tag{6.2.6a}$$

$$\bar{\xi} = \frac{\varepsilon}{h}(x + c_-t), \tag{6.2.6b}$$

$$\tau = \frac{\varepsilon^3}{24h}x, \tag{6.2.6c}$$

$$y = -\frac{1}{4\alpha}\{\varepsilon u^{(1)}(\xi,\bar{\xi},\tau) + \varepsilon^3 u^{(3)}(\xi,\bar{\xi},\tau) + \cdots\}. \tag{6.2.6d}$$

Substitution of (6.2.6) into eq.(6.2.1) yields at $O(\varepsilon^3)$,

$$u^{(1)}_{\xi\bar{\xi}} = 0. \tag{6.2.7}$$

Therefore $u^{(1)}$ has a form

$$u^{(1)}(\xi,\bar{\xi},\tau) = u^{(I)}(\xi,\tau) + u^{(R)}(\bar{\xi},\tau). \tag{6.2.8}$$

At $O(\varepsilon^5)$ we can set

$$u^{(3)}_{\xi\bar{\xi}} - 3(u^{(I)}_\xi u^{(R)}_{\bar{\xi}\bar{\xi}} + u^{(I)}_{\xi\xi} u^{(R)}_{\bar{\xi}}) = 0, \tag{6.2.9}$$

and get (Appendix 6.A)

$$u^{(I)}_{\xi\tau} - 6u^{(I)}_\xi u^{(I)}_{\xi\xi} + u^{(I)}_{\xi\xi\xi\xi} = 0, \tag{6.2.10a}$$

$$u^{(R)}_{\bar{\xi}\tau} - 6u^{(R)}_{\bar{\xi}} u^{(R)}_{\bar{\xi}\bar{\xi}} + u^{(R)}_{\bar{\xi}\bar{\xi}\bar{\xi}\bar{\xi}} = 0. \tag{6.2.10b}$$

Using new dependent variables

$$\Psi^I(\xi,\tau) = u^{(I)}_\xi = -\frac{u^{(I)}_t}{\varepsilon c_-}, \tag{6.2.11a}$$

$$\Psi^R(\bar{\xi},\tau) = u^{(R)}_{\bar{\xi}} = \frac{u^{(R)}_t}{\varepsilon c_-}, \tag{6.2.11b}$$

we obtain

$$\Psi^I_\tau - 6\Psi^I \Psi^I_\xi + \Psi^I_{\xi\xi\xi} = 0 \qquad (\tau < 0), \tag{6.2.12a}$$

$$\Psi^R_\tau - 6\Psi^R \Psi^R_{\bar{\xi}} + \Psi^R_{\bar{\xi}\bar{\xi}\bar{\xi}} = 0 \qquad (\tau < 0). \tag{6.2.12b}$$

Thus, the incident wave $\Psi^I(\xi,\tau)$ and reflected wave $\Psi^R(\bar{\xi},\tau)$ are described by independent KdV equations (6.2.12). Similar results are obtained for the shallow water waves (Ablowitz and Segur 1981). The above result means that there is no interaction between the incident and the reflected waves. However, we have assumed the following additional condition. That is the collision time T_c between the two waves is much shorter than the characteristic time T_m of the modulation of the waves. Such situation is realized when the total length of the waves are comparable with characteristic scale of the wave length($\sim \xi$). Then we have $T_c/T_m \sim \varepsilon^{-1}/\varepsilon^{-3} \ll 1$. The same discussion is effective to the carrier wave modulations in the next section and to the following chapters.

Construction of the Transmitted and Reflected Waves

We examine the conditions of the continuity for $y(x,t)$ and $y_x(x,t)$ (up to $0(\varepsilon)$) at $x = 0$. We set

$$f_1(t) = u^{(I)}(-c_-\varepsilon t, 0), \tag{6.2.13a}$$

$$f_2(t) = u^{(R)}(c_-\varepsilon t, 0), \tag{6.2.13b}$$

$$f_3(t) = v^{(1)}(-c_+\varepsilon t, 0). \tag{6.2.13c}$$

Neglecting $O(\varepsilon^2)$ quantities we have

$$f_1(t) + f_2(t) = f_3(t), \tag{6.2.14a}$$

$$-\frac{\dot{f}_1(t)}{c_-} + \frac{\dot{f}_2(t)}{c_-} = -\frac{\dot{f}_3(t)}{c_+}. \tag{6.2.14b}$$

Equations (6.2.14) are rewritten as

$$f_2(t) = \frac{1-\nu}{1+\nu} f_1(t), \tag{6.2.15a}$$

$$f_3(t) = \frac{2}{1+\nu} f_1(t). \tag{6.2.15b}$$

$$\nu = \frac{c_-}{c_+} = \sqrt{\frac{M}{m}} \tag{6.2.15c}$$

From the definitions of $\Psi^I(\xi, \tau)$, $\Psi^R(\bar{\xi}, \tau)$ and $\Psi^T(\eta, \tau)$ we have

$$\Psi^I(\xi, 0) = -\frac{1}{\varepsilon c_-}\dot{f}_1(-\frac{\xi}{c_-}), \tag{6.2.16a}$$

$$\Psi^R(\bar{\xi}, 0) = +\frac{1}{\varepsilon c_-}\dot{f}_2(\frac{\bar{\xi}}{c_-}), \tag{6.2.16b}$$

$$\Psi^T(\eta, 0) = -\frac{1}{\varepsilon c_+}\dot{f}_3(-\frac{\xi}{c_+}). \tag{6.2.16c}$$

Using (6.2.15) in (6.2.16) we arrive at

$$\Psi^R(\bar{\xi}, 0) = \frac{\nu - 1}{\nu + 1}\Psi^I(-\bar{\xi}, 0), \tag{6.2.17a}$$

$$\Psi^T(\eta, 0) = \frac{2\nu}{1+\nu}\Psi^I(\nu\eta, 0). \tag{6.2.17b}$$

Equation (6.2.17) gives the 'initial' condition for the reflected and transmitted waves. Therefore by means of the inverse scattering method for the KdV equation (Gardner, Greene, Kruskal and Miura 1967). *we can construct analytically the reflected wave* $\Psi^R(\bar{\xi}, \tau < 0)$ *and the transmitted wave* $\Psi^T(\eta, \tau > 0)$ *from the incident wave* $\Psi^I(\xi, \tau < 0)$.

Our method is also applicable to the case that $\alpha = 0$ and $\beta \neq 0$. There, we obtain three independent modified KdV equations for the three waves. The reflected and transmitted waves are constructed by the inverse scattering method (Wadati 1973).

N.Yajima (1977) also derived the relation (6.2.17) by a different approach. He employed another type of variable transformation in which small phase shifts θ_1, θ_2 are introduced(see eq.(18) in the paper). The variables ξ_1 and ξ_2 in the paper correspond to ξ and $\bar{\xi}$ respectively. There, it is necessary to get the form of θ_1 and θ_2. However our transformation are simple linear relation. The condition of the continuity in the paper (eq.(25)) is a set of relation at $\tau_2 = 0(t = 0)$, while our condition (6.2.17) is just at the mass interface ($\tau = x = 0$). In this sense our transformation offers clearer condition. Moreover, recall that transformation (6.2.3ab) for $-\infty < x < +\infty$ is useful for the weakly inhomogeneous system (Chapter 2), because the inhomogeneity is represented as a function of τ.

6.3 Carrier Wave Modulations

Reductive Perturbation Method

In this section we investigate transmission and reflection of a modulated carrier wave whose wave length is comparable with the lattice spacing h.

At first we consider the transmitted wave. In the region $j \geq -1$, we expand y_j by a smallness parameter ε and a carrier wave E as

$$y_j = \sum_{n=1}^{\infty}\sum_{|l|\leq n} \varepsilon^n E^l v^{(n,l)}(\eta, \tau) \qquad (j \geq -1), \tag{6.3.1a}$$

$$E = e^{i(Kx - \Omega t)} \qquad (x = jh). \tag{6.3.1b}$$

The wave number K and the angular frequency Ω satisfy a dispersion relation

$$\Omega^2 = \frac{2B}{M}(1 - \cos Kh). \tag{6.3.2}$$

We have introduced new independent variables η and τ;

$$\eta = \frac{\varepsilon}{h}(x - Vt), \tag{6.3.3a}$$

$$\tau = \frac{\varepsilon^2}{h}x \qquad (\tau > 0), \tag{6.3.3b}$$

where V is the group velocity given by

$$V = \frac{\mathrm{d}\Omega}{\mathrm{d}K} = \frac{Bh}{M\Omega}\sin Kh. \tag{6.3.4}$$

Substituting (6.3.1) into the equation of motion (6.1.1) for $j \geq 0$ and comparing the coefficients of $\varepsilon^n E^l$ we have the following relations;

$$\varepsilon E: \qquad \text{dispersion relation (6.3.2)}, \tag{6.3.5a}$$

$$\varepsilon^2 E: \quad \text{group velocity (6.3.4)}, \tag{6.3.5b}$$

$$\varepsilon^2 E^2: \quad v^{(2,2)} = i\alpha \frac{\sin Kh}{1-\cos Kh} (v^{(1,1)})^2, \tag{6.3.5c}$$

$$\varepsilon^3: \quad v_\eta^{(1,0)} = -8\alpha \left| v^{(1,1)} \right|^2. \tag{6.3.5d}$$

At $O(\varepsilon^3 E)$ we find the NLS equation for $v^{(1,1)}$,

$$iv_\tau^{(1,1)} + Pv_{\eta\eta}^{(1,1)} + Q\left| v^{(1,1)} \right|^2 v^{(1,1)} = 0, \tag{6.3.6a}$$

$$P = \frac{1}{2hV}\frac{\mathrm{d}^2\Omega}{\mathrm{d}K^2} = -\frac{1}{4}\tan\frac{Kh}{2}, \tag{6.3.6b}$$

$$Q = 4\tan\frac{Kh}{2}\{2\alpha^2 + (2\alpha^2 - 3\beta)\sin^2\frac{Kh}{2}\}. \tag{6.3.6c}$$

In the region $j \leq 0$ we consider incident and reflected waves. We expand $y_j (j \leq 0)$ by the smallness parameter ε, carrier of the incident wave F, that of the reflected wave G;

$$y_j = \sum_{n=1}^{\infty} \sum_{|l|+|l'| \leq n} \varepsilon^n F^l G^{l'} u^{(n,l,l')}(\xi, \bar{\xi}, \tau) \qquad (j \leq 0), \tag{6.3.7a}$$

$$F = \mathrm{e}^{i(kx - \omega t)}, \tag{6.3.7b}$$

$$G = \mathrm{e}^{-i(kx + \omega t)}. \tag{6.3.7c}$$

The wave number k and the angular frequency ω are related by a dispersion relation;

$$\omega^2 = \frac{2B}{m}(1 - \cos kh). \tag{6.3.8}$$

New independent variables ξ , $\bar{\xi}$ and τ have been introduced by

$$\xi = \frac{\varepsilon}{h}(x - vt), \tag{6.3.9a}$$

$$\bar{\xi} = \frac{\varepsilon}{h}(x + vt), \tag{6.3.9b}$$

where v is the group velocity in the region $j \leq -1$;

$$v = \frac{\mathrm{d}\omega}{\mathrm{d}k} = \frac{Bh}{m\omega}\sin kh. \tag{6.3.10}$$

Substituting (6.3.7) into (6.1.1) for $j \leq -1$ and equating the coefficients of $\varepsilon^n F^l G^{l'}$, we have

$$\varepsilon F, \quad \varepsilon G: \quad \text{dispersion relation (6.3.8)}, \tag{6.3.11a}$$

$$\varepsilon^2 F: \quad u_{\bar{\xi}}^{(1,1,0)} = 0, \tag{6.3.11b}$$

$$\varepsilon^2 G: \quad u_\xi^{(1,0,1)} = 0. \tag{6.3.11c}$$

Equation (6.3.11b) means that $u^{(1.1.0)}$ does not depend on $\bar{\xi}$. Similarly, $u^{(1,0,1)}$ does not depend on ξ. These factors are consistent with that $u^{(1.1.0)} = u^{(1.1.0)}(\xi, \tau)$ indicates the incident wave and $u^{(1.0.1)} = u^{(1,0,1)}(\bar{\xi}, \tau)$ the reflected wave. At the higher orders we have

$$\varepsilon^2 FG^{-1}: \quad u^{(2,1,-1)} = -2i\alpha \frac{\sin kh}{1+\cos kh} u^{(1,1,0)} u^{(1,0,-1)}, \tag{6.3.12a}$$

$$\varepsilon^2 FG: \quad u^{(2,1,1)} = 0, \tag{6.3.12b}$$

$$\varepsilon^2 F^2: \quad u^{(2,2,0)} = i\alpha \frac{\sin kh}{1-\cos kh} \left(u^{(1,1,0)}\right)^2, \tag{6.3.12c}$$

$$\varepsilon^2 G^2: \quad u^{(2,0,2)} = -i\alpha \frac{\sin kh}{1-\cos kh} \left(u^{(1,0,1)}\right)^2, \tag{6.3.12d}$$

$$\varepsilon^3: \quad u_\xi^{(1,0,0)} = -8\alpha \left| u^{(1,1,0)} \right|^2, \tag{6.3.12e}$$

$$u_{\bar{\xi}}^{(1,0,0)} = -8\alpha \left| u^{(1,0,1)} \right|^2. \tag{6.3.12f}$$

At $O(\varepsilon^3 F)$ we have

$$u^{(2,1,0)} = \frac{ig}{\sin kh} \int^{\bar{\xi}} \left| u^{(1,0,1)}(\xi', \tau) \right|^2 \mathrm{d}\xi' \cdot u^{(1,1,0)}(\xi, \tau) + C_1(\xi, \tau), \tag{6.3.13a}$$

$$g = 4\alpha^2(1 - \cos kh)(3 + \cos kh) - 12\beta(1 - \cos kh)^2, \tag{6.3.13b}$$

where $C_1(\xi, \tau)$ is unknown and we obtain the NLS equation for $u^{(1,1,0)}(\xi, \tau)$;

$$iu_\tau^{(1,1,0)} + pu_{\xi\xi}^{(1,1,0)} + q\left| u^{(1,1,0)} \right|^2 u^{(1,1,0)} = 0, \tag{6.3.14a}$$

$$p = \frac{1}{2hv}\frac{\mathrm{d}^2\omega}{\mathrm{d}k^2} = -\frac{1}{4}\tan\frac{kh}{2}, \tag{6.3.14b}$$

$$q = 4\tan\frac{kh}{2}\{2\alpha^2 + (2\alpha^2 - 3\beta)\sin^2\frac{kh}{2}\}. \tag{6.3.14c}$$

Similarly using a unknown function $C_2(\bar{\xi}, \tau)$ we get at $O(\varepsilon^3 G)$

$$u^{(2,0,1)} = -\frac{ig}{\sin kh} \int^{\xi} \left| u^{(1,1,0)}(\xi', \tau) \right|^2 \mathrm{d}\xi' \cdot u^{(1,0,1)}(\bar{\xi}, \tau) + C_2(\bar{\xi}, \tau), \tag{6.3.15}$$

and the NLS equation for $u^{(1,0,1)}$;

$$iu_\tau^{(1,0,1)} - pu_{\bar{\xi}\bar{\xi}}^{(1,0,1)} - q\left| u^{(1,0,1)} \right|^2 u^{(1,0,1)} = 0. \tag{6.3.16}$$

The incident wave $u^{(1,1,0)}(\xi, \tau)$ and the reflected wave $u^{(1,0,1)}(\bar{\xi}, \tau)$ obey independent NLS equations (6.3.14a) and (6.3.16). This situation is similar to the case of slowly varying waves discussed in §2.

Construction of the Transmitted and Reflected Wave

We notice that $y_{j=-1}$ and $y_{j=0}$ are expressed by two different expansions (6.3.1a) and (6.3.7a). Equating the two expressions we have at $j=-1$;

$$
\begin{aligned}
y_{i=-1} &= \sum_{n=1}^{\infty}\sum_{|l|\le n} \varepsilon^n e^{-il(Kh+\Omega t)} v^{(n,l)} \\
&= \sum_{n=1}^{\infty}\sum_{|l|+|l'|\le n} e^{-il(kh+\omega t)} e^{il'(kh-\omega t)} u^{(n,l,l')}, && (6.3.17a)\\
\tau &= -\varepsilon^2, && (6.3.17b)\\
\eta &= -\varepsilon - \frac{\varepsilon V t}{h}, && (6.3.17c)\\
\xi &= -\bar{\xi} = -\varepsilon - \frac{\varepsilon v t}{h}. && (6.3.17d)
\end{aligned}
$$

At $j=0$ we have

$$
\begin{aligned}
y_{i=0} &= \sum_{n=1}^{\infty}\sum_{|l|\le n} \varepsilon^n e^{-il\Omega t} v^{(n,l)} \\
&= \sum_{n=1}^{\infty}\sum_{|l|+|l'|\le n} e^{-il\omega t} e^{-il'\omega t} u^{(n,l,l')}, && (6.3.18a)\\
\tau &= 0, && (6.3.18b)\\
\eta &= -\frac{\varepsilon V t}{h}, && (6.3.18c)\\
\xi &= -\bar{\xi} = -\frac{\varepsilon v t}{h}. && (6.3.18d)
\end{aligned}
$$

Comparing high frequency components in (6.3.17a) and (6.3.18a), we must set (Fig.6.2)

$$\Omega = \omega \tag{6.3.19}$$

which is equivalent to

$$\frac{\sin Kh/2}{\sin kh/2} = \sqrt{\frac{M}{m}}. \tag{6.3.20}$$

Equating the coefficients of $\varepsilon^n e^{-il\omega t}$ in (6.3.17a) and (6.3.18a), we get at $n = l = 1$

$$
\begin{aligned}
e^{-iKh} v^{(1,1)} &= e^{-ikh} u^{(1,1,0)} + e^{ikh} u^{(1,0,1)} \qquad (\tau = 0), && (6.3.21a)\\
v^{(1,1)} &= u^{(1,1,0)} + u^{(1,0,1)} \qquad (\tau = 0), && (6.3.21b)
\end{aligned}
$$

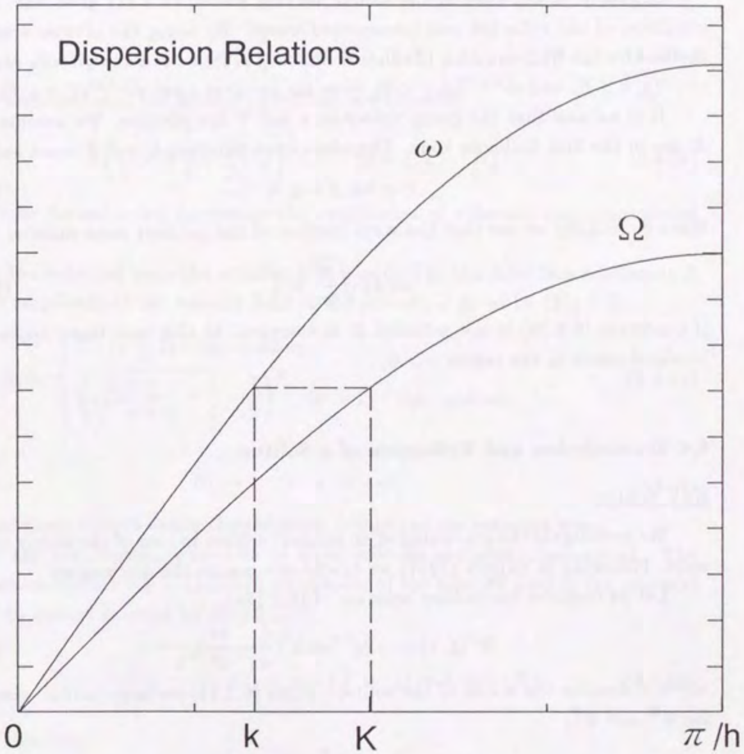

Figure 6.2: Dispersion relations $\omega = \omega(k)$, $\Omega = \Omega(K)$. The two wave numbers k and K are related by $\omega(k) = \Omega(K)$.

where we have used the Taylor expansions of $v^{(1,1)}$, $u^{(1,1,0)}$ and $u^{(1,0,1)}$ in (6.3.17b ~ d) around (6.3.18b~d). Equations (6.3.21) yield

$$v^{(1,1)}(\eta,0)=\frac{e^{-ikh}-e^{ikh}}{e^{-iKh}-e^{ikh}}u^{(1,1,0)}(\frac{v}{V}\eta,0), \tag{6.3.22a}$$

$$u^{(1,0,1)}(\bar{\xi},0)=\frac{e^{-iKh}-e^{-ikh}}{e^{ikh}-e^{-iKh}}u^{(1,1,0)}(-\bar{\xi},0). \tag{6.3.22b}$$

This is an analogue of the relations (6.2.17).

Similarly to the case of the slowly varying waves, (6.3.22) gives the 'initial' condition of the reflected and transmitted waves. By using the inverse scattering method for the NLS equation (Zakharov Shabat 1972), *we can analytically construct* $v^{(1,1)}(\eta,\tau>0)$ *and* $u^{(1,0,1)}(\bar{\xi},\tau<0)$ *from the incident wave* $u^{(1,1,0)}(\xi,\tau<0)$.

It is natural that the group velocities v and V are positive. We assume k and K are in the first Brillouin zone. Therefore wave numbers k and K must satisfy

$$0<kh, Kh<\pi. \tag{6.3.23}$$

From eq.(6.3.20) we see that the wave number of the incident wave satisfies

$$\sin kh\sqrt{\frac{M}{m}}<1. \tag{6.3.24}$$

If condition (6.3.24) is not satisfied K is complex. In this case there appears the localized mode in the region $x>0$.

6.4 Transmission and Reflection of a Soliton

KdV Soliton

We investigate the scattering of an incident soliton in case of the slowly varying wave. Following N.Yajima (1977) we briefly summarize the discussions.

Let us consider one-soliton solution of (6.2.12a);

$$\Psi^I(\xi,\tau)=-2d^{-2}\mathrm{sech}^2(\frac{\xi}{d}-\frac{4\tau}{d^3}), \tag{6.4.1}$$

where d denotes the width of the soliton. From (6.2.17) we have 'initial' condition for Ψ^R and Ψ^T;

$$\Psi^R(\bar{\xi},0)=-2\frac{\nu-1}{\nu+1}d^{-2}\mathrm{sech}^2\frac{\bar{\xi}}{d}, \tag{6.4.2a}$$

$$\Psi^T(\eta,0)=-\frac{4}{\nu(\nu+1)}D^{-2}\mathrm{sech}^2\frac{\eta}{D}, \tag{6.4.2b}$$

$$D=\nu^{-1}d. \tag{6.4.2c}$$

Let us consider the following result for the further analysis. From an initial condition

$$\Psi(X,0)=-AL^{-2}\mathrm{sech}^2\frac{X}{L}\quad(A>0), \tag{6.4.3a}$$

for the KdV equation

$$\Psi_T-6\Psi_X\Psi+\Psi_{XXX}=0, \tag{6.4.3b}$$

we can determine the number N of generated solitons by the inverse scattering method (G.L.Lamb,Jr. 1980). The number N is the maximum integer which satisfies

$$\sqrt{A+\frac{1}{4}}+\frac{1}{2}-N>0, \tag{6.4.4a}$$

and the amplitudes of the generated solitons are given by

$$2\left(\sqrt{A+\frac{1}{4}}+\frac{1}{2}-j\right)^2L^{-2}\quad(j=1,2,\cdots N). \tag{6.4.4b}$$

Using these formulae we determine the amplitudes of reflected and transmitted solitons.

For the reflected wave the number N is 1 or 0. The the reflection coefficient R (relative amplitude of the velocity field in the soliton) is given by (Fig.6.3)

$$R_1=\begin{cases}0\quad(\nu\le 1)\quad \text{no}\quad \text{soliton}\\ \left(\sqrt{2\frac{\nu-1}{\nu+1}+\frac{1}{4}}-\frac{1}{2}\right)^2\quad(\nu>1)\quad\text{one}\quad\text{soliton,}\end{cases} \tag{6.4.5a}$$

where

$$R_1\to 1\quad\text{if}\quad \nu\to+\infty. \tag{6.4.5b}$$

Note that there always exists the radiation (ripple) in the reflected wave.

For the transmitted wave one or more solitons are always generated. The transmission coefficients T_j (relative amplitudes of the velocity field in the solitons) of the j-th soliton is given by (Fig.6.3)

$$T_j=\left(\sqrt{\frac{4}{\nu(\nu+1)}+\frac{1}{4}}+\frac{1}{2}-j\right)^2\nu\quad(j=1,2,\cdots N), \tag{6.4.6a}$$

where T_1 satisfies

$$T_1=1\quad\text{if}\quad\nu=1. \tag{6.4.6b}$$

We remark that in the paper of N.Yajima (1977), the transmission coefficients are defined differently ($=T_j\nu$). In another paper (Iizuka and Wadati 1992c) they are defined by $T_j\nu^{-1}$.

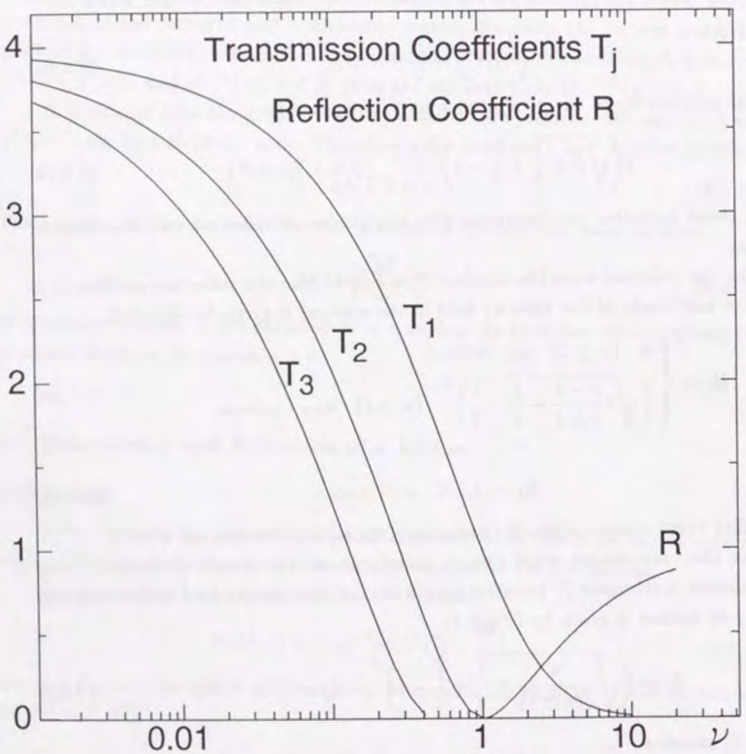

Figure 6.3: The reflection coefficient R_1 and the transmission coefficients $T_1 \sim T_3$ of the KdV soliton.

The velocities of N generated solitons are all distinct. Therefore we observe the fission of the incident soliton. In particular when ν satisfies

$$\frac{4}{\nu(\nu+1)} = N(N+1) \qquad N : \text{integer}, \tag{6.4.7}$$

N transmitted solitons are generated and there occur no radiation. In term of the scattering problem $\Psi^T(\eta)$ is the reflectionless potential.

NLS Soliton

As in the former case we consider the scattering of the incident soliton due to a discontinuity. In the following we denote $u^{(1,1,0)}$, $u^{(1,0,1)}$ and $v^{(1,1)}$ as Φ^I, Φ^R and Φ^T respectively. The evolution equations are given by

$$i\Phi^I_\tau + p\Phi^I_{\xi\xi} + q\left|\Phi^I\right|^2\Phi^I = 0, \tag{6.4.8}$$
$$i\Phi^R_\tau - p\Phi^R_{\bar\xi\bar\xi} - q\left|\Phi^R\right|^2\Phi^R = 0, \tag{6.4.9}$$
$$i\Phi^T_\tau + P\Phi^T_{\eta\eta} + Q\left|\Phi^T\right|^2\Phi^T = 0. \tag{6.4.10}$$

The constants p, q, P and Q have been defined in (6.3.6) and (6.3.14). We assume that $pq > 0$ and $PQ > 0$.

Let us consider one soliton solution of (6.4.8);

$$\Phi^I = \sqrt{\frac{2p}{q}}d^{-1}\mathrm{sech}\left(\frac{\xi}{d} - 2\kappa p\frac{\tau}{d}\right)\exp\left\{i\kappa\xi + i(d^{-2} - \kappa^2)p\tau\right\}, \tag{6.4.11}$$

where d denotes the width of the soliton and κ is a real parameter. From eqs. (6.3.22) we have 'initial' conditions for (6.4.9) and (6.4.10);

$$\Phi^R(\bar\xi, 0) = \sqrt{\frac{2p}{q}}d^{-1}\frac{\mathrm{e}^{-iKh} - \mathrm{e}^{-ikh}}{\mathrm{e}^{ikh} - \mathrm{e}^{-iKh}}\mathrm{sech}\frac{\bar\xi}{d}\mathrm{e}^{-i\kappa\bar\xi}, \tag{6.4.12}$$
$$\Phi^T(\eta, 0) = \sqrt{\frac{2p}{q}}d^{-1}\frac{\mathrm{e}^{-ikh} - \mathrm{e}^{ikh}}{\mathrm{e}^{-iKh} - \mathrm{e}^{ikh}}\mathrm{sech}\frac{v\eta}{Vd}\mathrm{e}^{i\kappa v\eta/V}. \tag{6.4.13}$$

In order to analyze the generation of solitons for initial conditions (6.4.12) and (6.4.13) we use the following results which are obtained by the inverse scattering method (Satsuma and N.Yajima 1974). We solve the nonlinear Schrödinger equation;

$$i\Phi_T + a\Phi_{XX} + b\left|\Phi\right|^2\Phi = 0 \qquad (ab > 0), \tag{6.4.14}$$

for the initial condition;

$$\Phi(X,0) = A\sqrt{\frac{2a}{b}}L^{-1}\mathrm{sech}\frac{X}{L}e^{i\kappa X}. \qquad (6.4.15)$$

The number N of generated soliton is given by the maximum integer which satisfies

$$|A| + \frac{1}{2} - N > 0. \qquad (6.4.16)$$

The velocities of the generated solitons are identically $2\kappa a$. Thus *there emerges bound state of N solitons.* The bound state is unstable because it splits into N solitons under small perturbation. The amplitudes A_j of the j-th soliton is given by

$$A_j = 2\sqrt{\frac{2a}{b}}L^{-1}(|A| + \frac{1}{2} - j). \qquad (6.4.17)$$

1) Reflected wave

Using the above results and (6.4.12) we find that the one or no soliton is generated in the reflected wave;

$$1 > |A|^2 = \frac{1-\cos(K-k)h}{1-\cos(K+k)h} = \begin{cases} > \frac{1}{4} & \text{one soliton} \\ \le \frac{1}{4} & \text{no soliton} \end{cases} \qquad (6.4.18)$$

If a soliton is generated the reflection coefficient of the soliton R (relative amplitude of the envelope soliton) is given by (Fig.6.4)

$$R = 2\left(\sqrt{\frac{1-\cos(K-k)h}{1-\cos(K+k)h}} - \frac{1}{2}\right)$$

$$= \frac{2\,|1-\mu|}{1+\mu} - 1, \qquad (6.4.19a)$$

$$\mu^{-1} = \sqrt{1 + (\frac{m}{M} - 1)\cos^{-2}\frac{kh}{2}} \quad (0 < \mu < +\infty), \qquad (6.4.19b)$$

where we have normalized as,

$$R \to 1 \quad \text{if} \quad \mu \to 0, +\infty. \qquad (6.4.19c)$$

Let us remark that the relation (6.3.24) is satisfied. Substituting $k = 0$ into (6.4.19b) we have $\mu = \nu$ (see eq.(6.2.15c)).

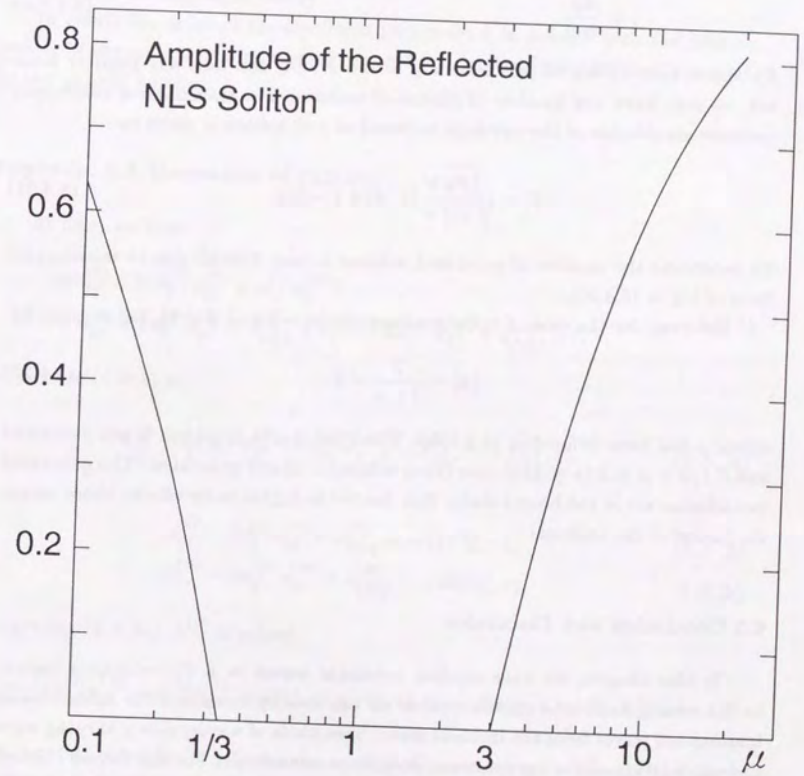

Figure 6.4: Reflection Coefficient of the NLS soliton.

2)Transmitted wave

For the transmitted waves, A in (6.4.15) corresponds to

$$A = \sqrt{\frac{2p}{q}}\sqrt{\frac{Q}{2P}}\frac{V}{v}\frac{\mathrm{e}^{-ikh} - \mathrm{e}^{ikh}}{\mathrm{e}^{-iKh} - \mathrm{e}^{ikh}}, \tag{6.4.20a}$$

$$|A|^2 = \frac{(\gamma - 1)\sin^2(kh/2) - 1}{(\gamma - 1)\sin^2(Kh/2) - 1} \cdot \frac{2\sin^2 Kh}{1 - \cos(k + K)h}, \tag{6.4.20b}$$

$$\gamma = \frac{3\beta}{2\alpha^2}. \tag{6.4.20c}$$

By appropriate choice of parameters γ, k and K, $|A|$ can take any positive number; we may have any number of generated solitons. The transmission coefficients (relative amplitudes of the envelope solitons) of j-th soliton is given by

$$T_j = \sqrt{\frac{Pq}{pQ}}\frac{V}{v}(2|A| + 1 - 2j). \tag{6.4.21}$$

To determine the number of generated solitons is very difficult due to the complex form of $|A|$ in (6.4.20).

However, for the case of cubic nonlinearity ($\alpha = 0$ and $\beta \neq 0$), $|A|$ is given by

$$|A| = \frac{2}{1 + \mu} < 2, \tag{6.4.22}$$

where μ has been defined in (6.4.19b). Therefore if $\mu \geq 3$, soliton is not generated and if $1/3 < \mu < 3$ ($\nu \leq 1/3$), one (two) soliton(s) is(are) generated. The generated two solitons are in the bound state. But due to the higher order effects, there occurs the fission of the solitons.

6.5 Conclusion and Discussion

In this chapter we have studied nonlinear waves in a discontinuous lattice. In the weakly nonlinear approximation we can exactly construct the reflected and transmitted waves from the incident wave. Two kinds of waves; slowly varying wave and modulation of the carrier wave, have been considered. For the former (latter) incident, reflected and transmitted waves are governed by independent KdV (NLS) equations. As an interesting example transmission and reflection of an incident soliton are investigated. For the KdV soliton at most one soliton is generated in the reflected wave and at least one in the transmitted wave. The number of generated

soliton is determined by the mass ratio M/m. For the NLS soliton at most one soliton is generated in the reflected wave. If the nonlinearity of the spring is cubic ($\alpha = 0$), at most two soliton are generated in the transmitted wave. The numbers are determined by a parameter μ defined by (6.4.19b).

Our approach for scattering of KdV soliton has been applied to shallow water waves over a barrier (Sugimoto, Hosokawa and Kakutani 1987) and over a step (Sugimoto, Nakajima and Kakutani 1987). There, the connection conditions are derived from the edge-layer theory.

In partI, the order of the smallness parameter ε is defined from the inhomogeneity of the system, while in partII (except for chapter 8) the order of it is set to be any (small) number.

Appendix 6.A Derivation of (6.2.10)

At $O(\varepsilon)$ we have

$$\begin{aligned} &48u^{(3)}_{\xi\bar{\xi}} - 6(u^{(I)}_{\xi}u^{(R)}_{\bar{\xi}\bar{\xi}} + u^{(I)}_{\xi\xi}u^{(R)}_{\bar{\xi}}) \\ &+ u^{(I)}_{\xi\tau} - 6u^{(I)}_{\xi}u^{(I)}_{\xi\xi} + u^{(I)}_{\xi\xi\xi\xi} + u^{(R)}_{\bar{\xi}\tau} - 6u^{(R)}_{\bar{\xi}}u^{(R)}_{\bar{\xi}\bar{\xi}} + u^{(R)}_{\bar{\xi}\bar{\xi}\bar{\xi}\bar{\xi}} = 0. \end{aligned} \tag{6.A.1}$$

We rewrite (6.A.1) as

$$u^{(3)}_{\xi\bar{\xi}} - \frac{1}{8}(u^{(I)}_{\xi}u^{(R)}_{\bar{\xi}\bar{\xi}} + u^{(I)}_{\xi\xi}u^{(R)}_{\bar{\xi}}) = F(\xi, \tau) + G(\bar{\xi}, \tau), \tag{6.A.2a}$$

$$u^{(I)}_{\xi\tau} - 6u^{(I)}_{\xi}u^{(I)}_{\xi\xi} + u^{(I)}_{\xi\xi\xi\xi} = -48F(\xi, \tau), \tag{6.A.2b}$$

$$u^{(R)}_{\bar{\xi}\tau} - 6u^{(R)}_{\bar{\xi}}u^{(R)}_{\bar{\xi}\bar{\xi}} + u^{(R)}_{\bar{\xi}\bar{\xi}\bar{\xi}\bar{\xi}} = -48G(\bar{\xi}, \tau). \tag{6.A.2c}$$

Integrating (6.A.2a), $u^{(3)}$ is solved;

$$u^{(3)} = \frac{1}{8}(u^{(I)}u^{(R)}_{\bar{\xi}} + u^{(R)}u^{(I)}_{\xi}) + H(\xi, \tau) + I(\bar{\xi}, \tau) + (\int d\xi F)\bar{\xi} + (\int d\bar{\xi} G)\xi. \tag{6.A.3}$$

The last two terms in (6.A.3) are the secular terms which grow linearly in ξ or $\bar{\xi}$. To eliminate them we see $F \equiv G \equiv 0$. Thus we have (6.2.10).

Chapter 7

Effects of Single Impurity

Scatterings of nonlinear lattice waves by a mass impurity are studied. The waves are assumed to be nonlinear modulations of the monochromatic waves. Due to the impurity there appear the incident, reflected and transmitted waves. We show that the three waves are described by independent Nonlinear Schrodinger(NLS) equations respectively. Using the continuity conditions of the waves at the impurity site, we analytically construct the transmitted and reflected waves from the incident wave. As an application, scattering of an incident NLS envelope soliton is investigated. We find that at most one soliton is generated both in the reflected wave and in the transmitted wave.

7.1 NLS Equations

We consider a one-dimensional anharmonic lattice where the springs are identical (Iizuka and Wadati 1992d). The masses are also identical except that a single mass impurity is contained (Fig.7.1).

Denoting the displacement and mass of j-th particle by y_j and m_j respectively, we write the equation of motion for the j-th particle;

$$\begin{aligned} m_j \ddot{y}_j &= B\left[(y_{j+1}-y_j)+\alpha(y_{j+1}-y_j)^2+\beta(y_{j+1}-y_j)^3+\cdots\right] \\ &\quad - B\left[(y_j-y_{j-1})+\alpha(y_j-y_{j-1})^2+\beta(y_j-y_{j-1})^3+\cdots\right], \end{aligned} \tag{7.1.1}$$

where $B(>0)$, α and β are constants and dots on dynamical variables indicate differentiations with respect to time t. While B is the spring constant, α and β are

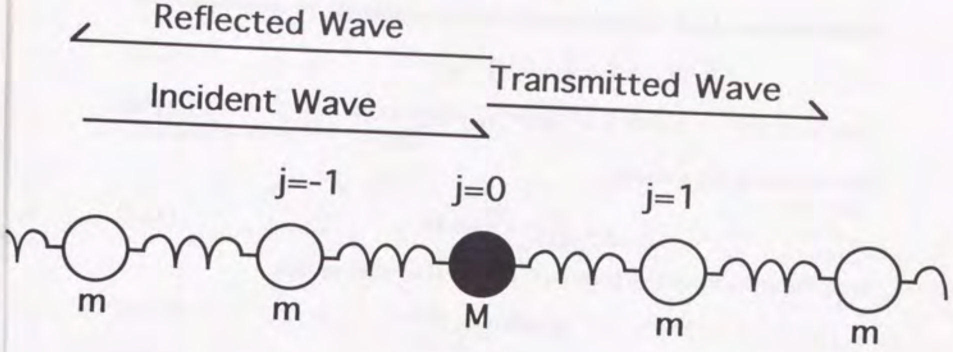

Figure 7.1: Nonlinear lattice which contain a mass impurity.

related to the coefficients of quadratic and cubic nonlinearities in the spring. We assume that $|\alpha|^2+|\beta|^2 \neq 0$. The impurity is located at $j=0$;

$$\begin{cases} m_j = m & (j \neq 0) \\ m_0 = M. \end{cases} \tag{7.1.2}$$

We consider nonlinear modulations of the monochromatic waves assuming that their magnitudes are small. We shall follow the method which was developed in 6.3. At first we investigate the equation of motion for $j>0$. This corresponds to the transmitted wave. Displacements y_j for $j \geq 0$ are expanded in terms of a smallness parameter ε and a carrier wave E as

$$y_j = \sum_{n=1}^{\infty} \sum_{|l|\leq n} \varepsilon^n E^l v^{(n,l)}(\xi,\tau) \qquad (j \geq 0), \tag{7.1.3a}$$

$$E = e^{i(kx-\omega t)} \qquad (x = jh), \tag{7.1.3b}$$

where h is the lattice constant. Since y_j is real we have

$$v^{(n,-m)} = (v^{(n,m)})^*, \tag{7.1.4}$$

where $*$ indicates complex conjugation. The wave number k and the angular frequency ω are related by a dispersion relation

$$\omega^2 = \frac{2B}{m}(1-\cos kh). \tag{7.1.5}$$

Transformations of independent variables are introduced as

$$\xi = \frac{\varepsilon}{h}(x - vt), \tag{7.1.6a}$$

$$\tau = \frac{\varepsilon^2}{h}x \qquad (\tau \geq 0), \tag{7.1.6b}$$

where v is the group velocity:

$$v = \frac{\mathrm{d}\omega}{\mathrm{d}k} = \frac{Bh}{m\omega}\sin kh. \tag{7.1.7}$$

To set the group velocity v positive, it is natural to suppose that

$$0 < kh < \pi, \tag{7.1.8a}$$

$$\omega > 0. \tag{7.1.9b}$$

We substitute the expansion (7.1.3) into the equation of motion (7.1.1) for $j \geq 1$ and equate the coefficients of $\varepsilon^n E^m$. Then we have

$$v^{(2,2)} = i\alpha\frac{\sin kh}{1-\cos kh}(v^{(1,1)})^2, \tag{7.1.9a}$$

$$v_\xi^{(1,0)} = -8\alpha\left|v^{(1,1)}\right|^2. \tag{7.1.9b}$$

At $O(\varepsilon^3 E)$ i.e. the order of $\varepsilon^3 E$, we see that $v^{(1,1)}$ satisfies the NLS equation;

$$iv_\tau^{(1,1)} + pv_{\xi\xi}^{(1,1)} + q\left|v^{(1,1)}\right|^2 v^{(1,1)} = 0, \quad (\tau \geq 0) \tag{7.1.10a}$$

$$p = \frac{1}{2hv}\frac{\mathrm{d}^2\omega}{\mathrm{d}k^2} = -\frac{1}{4}\tan\frac{kh}{2}, \tag{7.1.10b}$$

$$q = 4\tan\frac{kh}{2}\{2\alpha^2 + (2\alpha^2 - 3\beta)\sin^2\frac{kh}{2}\}. \tag{7.1.10c}$$

Next we consider the equation of motion (7.1.1) for $j \leq 0$. Let E be carrier wave of the transmitted wave and G that of the reflected wave. We expand $y_j (j \leq 0)$ by the smallness parameter ε, E and G;

$$y_j = \sum_{n=1}^{\infty}\sum_{|l|+|l'|\leq n} \varepsilon^n E^l G^{l'} u^{(n,l,l')}(\xi,\bar{\xi},\tau) \qquad (j \leq 0), \tag{7.1.11a}$$

$$G = \mathrm{e}^{-i(kx+\omega t)}, \tag{7.1.11b}$$

Similarly to eq.(7.1.4), $u^{(n,l,l')}$ satisfy real conditions

$$u^{(n,-l,l')} = u^{(n,l,-l')} = (u^{(n,l,l')})^*. \tag{7.1.12}$$

A new independent variable $\bar{\xi}$ is introduced by

$$\bar{\xi} = \frac{\varepsilon}{h}(x + vt). \tag{7.1.13}$$

Substituting (7.1.11) into the equation of motion (7.1.1) for $j \leq -1$ and comparing the coefficients of $\varepsilon^n E^l G^{l'}$, we have

$$u_{\bar{\xi}}^{(1,1,0)} = u_\xi^{(1,0,1)} = 0. \tag{7.1.14a}$$

$$u^{(2,1,-1)} = -2i\alpha\frac{\sin kh}{1+\cos kh}u^{(1,1,0)}u^{(1,0,-1)}, \tag{7.1.14b}$$

$$u^{(2,1,1)} = 0, \tag{7.1.14c}$$

$$u^{(2,2,0)} = i\alpha\frac{\sin kh}{1-\cos kh}\left(u^{(1,1,0)}\right)^2, \tag{7.1.14d}$$

$$u^{(2,0,2)} = -i\alpha\frac{\sin kh}{1-\cos kh}\left(u^{(1,0,1)}\right)^2, \tag{7.1.14e}$$

$$u_\xi^{(1,0,0)} = -8\alpha\left|u^{(1,1,0)}\right|^2, \tag{7.1.14f}$$

$$u_{\bar{\xi}}^{(1,0,0)} = -8\alpha\left|u^{(1,0,1)}\right|^2. \tag{7.1.14g}$$

Equation (7.1.14a) implies that $u^{(1,1,0)}$ and $u^{(1,0,1)}$ do not depend on $\bar{\xi}$ and ξ respectively. It is thus obvious that $u^{(1,1,0)}(\xi, \tau \leq 0)$ describes a modulation of the transmitted wave and $u^{(1,0,1)}(\bar{\xi}, \tau \leq 0)$ that of the reflected wave. At $O(\varepsilon^3 E)$ and $O(\varepsilon^3 G)$ we have

$$u^{(2,1,0)} = \frac{ig}{\sin kh}\int^{\bar{\xi}}\left|u^{(1,0,1)}(\xi',\tau)\right|^2 \mathrm{d}\xi' \cdot u^{(1,1,0)}(\xi,\tau) + C_1(\xi,\tau), \tag{7.1.15a}$$

$$u^{(2,0,1)} = -\frac{ig}{\sin kh}\int^{\xi}\left|u^{(1,1,0)}(\xi',\tau)\right|^2 \mathrm{d}\xi' \cdot u^{(1,0,1)}(\bar{\xi},\tau) + C_2(\bar{\xi},\tau), \tag{7.1.15b}$$

$$g = 4\alpha^2(1-\cos kh)(3+\cos kh) - 12\beta(1-\cos kh)^2, \tag{7.1.15c}$$

where $C_1(\xi,\tau)$ and $C_2(\bar{\xi},\tau)$ are unknown functions. At last we obtain the NLS equations for $u^{(1,1,0)}(\xi,\tau)$ and $u^{(1,0,1)}(\bar{\xi},\tau)$;

$$iu_\tau^{(1,1,0)} + pu_{\xi\xi}^{(1,1,0)} + q\left|u^{(1,1,0)}\right|^2 u^{(1,1,0)} = 0, \quad (\tau \leq 0) \tag{7.1.16a}$$

$$iu_\tau^{(1,0,1)} - pu_{\bar{\xi}\bar{\xi}}^{(1,0,1)} - q\left|u^{(1,0,1)}\right|^2 u^{(1,0,1)} = 0, \quad (\tau \leq 0) \tag{7.1.16b}$$

where real constants p and q are given in eqs. (7.1.10b) and (7.1.10c). We find that the incident wave $u^{(1,1,0)}$, the transmitted wave $v^{(1,1)}$ and the reflected wave $u^{(1,0,1)}$ are governed by independent NLS equations (7.1.16a),(7.1.10a) and (7.1.16b).

7.2. Construction of the Transmitted and Reflected Waves

In this section the conditions of the continuity at site $j = 0$ are considered. There are two conditions which relate the incident, transmitted and reflected waves at $\tau = 0$. First, we notice that both expansions (7.1.3a) and (7.1.11a) are available for $j = \tau = 0$;

$$y_0 = \sum_{n=1}^{\infty} \sum_{|l| \le n} \varepsilon^n e^{-il\omega t} v^{(n,l)}(\xi, 0) = \sum_{n=1}^{\infty} \sum_{|l|+|l'| \le n} \varepsilon^n e^{-il\omega t} e^{-il'\omega t} u^{(n,l,l')}(\xi, \bar{\xi}, 0). \quad (7.2.1)$$

Comparing the coefficient of $\varepsilon e^{-i\omega t}$, we have

$$v^{(1,1)}(\xi, 0) = u^{(1,1,0)}(\xi, 0) + u^{(1,0,1)}(\bar{\xi}, 0). \quad (7.2.2)$$

The second condition comes from the equation of motion at the impurity site $j = 0$. We substitute (7.1.3) into l.h.s. of (7.1.1) for $j = 0$. For the first term of r.h.s. of (7.1.1) we use the expansion (7.1.3) and for the second term we use the expansion (7.1.11). Comparing the coefficients of $\varepsilon e^{-i\omega t}$, we obtain

$$\begin{aligned} -\frac{M}{B}\omega^2 v^{(1,1)}(\xi, 0) =& (e^{ikh} - 1) v^{(1,1)}(\xi, 0) + (e^{-ikh} - 1) u^{(1,1,0)}(\xi, 0) \\ &+ (e^{ikh} - 1) u^{(1,0,1)}(\bar{\xi}, 0). \end{aligned} \quad (7.2.3)$$

The set of relations (7.2.2) and (7.2.3) constitutes the condition of continuity in the lowest order of ε. It is rewritten as

$$v^{(1,1)}(\xi, 0) = -\frac{i \sin kh}{1 - e^{ikh} - \frac{M}{2B}\omega^2} u^{(1,1,0)}(\xi, 0), \quad (7.2.4a)$$

$$u^{(1,0,1)}(\bar{\xi}, 0) = -\frac{1 - \cos kh - \frac{M}{2B}\omega^2}{1 - e^{ikh} - \frac{M}{2B}\omega^2} u^{(1,1,0)}(-\bar{\xi}, 0). \quad (7.2.4b)$$

Note that when $\tau = 0$, $\xi = -\bar{\xi}$. That is, if the incident wave $u^{(1,1,0)}(\xi, \tau)$ is given, we obtain 'initial' conditions $v^{(1,1)}(\xi, 0)$ and $u^{(1,0,1)}(\bar{\xi}, 0)$ through eqs.(7.2.4). We have shown that $v^{(1,1)}(\xi, \tau)$ and $u^{(1,0,1)}(\bar{\xi}, \tau)$ are governed by the NLS equations (7.1.10a) and (7.1.16a) respectively. Initial value problem of the NLS equation is solved by the inverse scattering method. Therefore the reflected wave $u^{(1,0,1)}(\bar{\xi}, \tau \le 0)$ and the transmitted wave $v^{(1,1)}(\xi, \tau \ge 0)$ are constructed analytically from the incident wave.

7.3. Transmission and Reflection of the Incident Soliton

We shall investigate the scattering of the incident envelope soliton due to the impurity. For convenience we denote $u^{(1,1,0)}$, $u^{(1,0,1)}$ and $v^{(1,1)}$ as Φ^I, Φ^R and Φ^T respectively. The evolution equations are

$$i\Phi^I_\tau + p\Phi^I_{\xi\xi} + q\left|\Phi^I\right|^2 \Phi^I = 0, \quad (\tau \le 0) \quad (7.3.1)$$

$$i\Phi^R_\tau - p\Phi^R_{\bar{\xi}\bar{\xi}} - q\left|\Phi^R\right|^2 \Phi^R = 0, \quad (\tau \le 0) \quad (7.3.2)$$

$$i\Phi^T_\tau + p\Phi^T_{\xi\xi} + q\left|\Phi^T\right|^2 \Phi^T = 0. \quad (\tau \ge 0) \quad (7.3.3)$$

The constants p and q have been defined in (7.1.10b) and (7.1.10c). In this section we assume that $pq > 0$, which is equivalent to

$$2\alpha^2 + (2\alpha^2 - 3\beta)\sin^2\frac{kh}{2} < 0. \quad (7.3.4)$$

One-soliton solution of (7.3.1) is given by

$$\Phi^I = \sqrt{\frac{2p}{q}} \mathrm{sech}(\xi - 2\kappa p \tau) \exp\{i\kappa\xi + i(1 - \kappa^2)p\tau\}, \quad (7.3.5)$$

where κ is a real parameter. A more general expression of one-soliton solution contains another parameter(L in (6.4.16)), but the results does not change. From eqs.(7.2.4) we get 'initial' conditions of (7.3.2) and (7.3.3);

$$\Phi^T(\xi, 0) = -\frac{i \sin kh}{1 - e^{ikh} - \frac{M}{2B}\omega^2} \sqrt{\frac{2p}{q}} \mathrm{sech}\xi e^{i\kappa\xi} \equiv A^T \sqrt{\frac{2p}{q}} \mathrm{sech}\xi e^{i\kappa\xi}, \quad (7.3.6)$$

$$\Phi^R(\bar{\xi}, 0) = -\frac{1 - \cos kh - \frac{M}{2B}\omega^2}{1 - e^{ikh} - \frac{M}{2B}\omega^2} \sqrt{\frac{2p}{q}} \mathrm{sech}\bar{\xi} e^{-i\kappa\bar{\xi}} \equiv A^R \sqrt{\frac{2p}{q}} \mathrm{sech}\bar{\xi} e^{-i\kappa\bar{\xi}}. \quad (7.3.7)$$

We discuss the generation of solitons in the transmitted and reflected waves. We shall review the results mentioned in (6.4.14~17). Let us solve the Nonlinear Schrödinger equation;

$$i\Phi_T + a\Phi_{XX} + b\left|\Phi\right|^2 \Phi = 0 \qquad (ab > 0), \quad (7.3.8)$$

for the initial condition;

$$\Phi(X, 0) = A\sqrt{\frac{2a}{b}} \mathrm{sech} X e^{i\kappa X}. \quad (7.3.9)$$

The number of generated soliton, $N > 0$, is the maximum integer which satisfies

$$|A| + \frac{1}{2} - N > 0. \tag{7.3.10}$$

The velocities of the generated solitons are identically $2\kappa a$ and the amplitude A_j of the j-th soliton is given by

$$A_j = 2\sqrt{\frac{2a}{b}}(|A| + \frac{1}{2} - j). \tag{7.3.11}$$

In the present case, A^T and A^R correspond to A in (7.3.9). Their absolute values are expressed as

$$|A^T| = (\sigma + 1)^{-\frac{1}{2}}, \tag{7.3.12}$$

$$|A^R| = (\sigma^{-1} + 1)^{-\frac{1}{2}}, \tag{7.3.13}$$

$$\sigma = (1 - \frac{M}{m})^2 \tan^2 \frac{kh}{2}, \tag{7.3.14}$$

where (7.1.5) and (7.1.8) have been considered. Note that σ can be taken as any positive value by choosing appropriate m, M and kh. Using the formulae (7.3.10) and (7.3.11) we find the followings. If $0 < \sigma < 3(\sigma > 3)$, one(no) soliton is generated in the transmitted wave. If $\sigma > 1/3(0 < \sigma < 1/3)$, one(no) soliton is generated in the reflected wave. The relative amplitude of the transmitted soliton T and that of the reflected soliton R are given by (see Fig.7.2)

$$T = \begin{cases} 2(|A^T| - \frac{1}{2}) = \dfrac{2}{\sqrt{\sigma+1}} - 1, & 0 < \sigma < 3 \\ 0, & \sigma > 3 \end{cases} \tag{7.3.15}$$

$$R = \begin{cases} 2(|A^R| - \frac{1}{2}) = \dfrac{2}{\sqrt{\sigma^{-1}+1}} - 1, & \sigma > \frac{1}{3} \\ 0, & 0 < \sigma < \frac{1}{3} \end{cases} \tag{7.3.16}$$

where we have normalized as

$$T \to 1 \quad (\sigma \to 1 \quad \text{or} \quad M \to m), \tag{7.3.17}$$

$$R \to 1 \quad (\sigma \to +\infty \quad \text{or} \quad M \to +\infty). \tag{7.3.18}$$

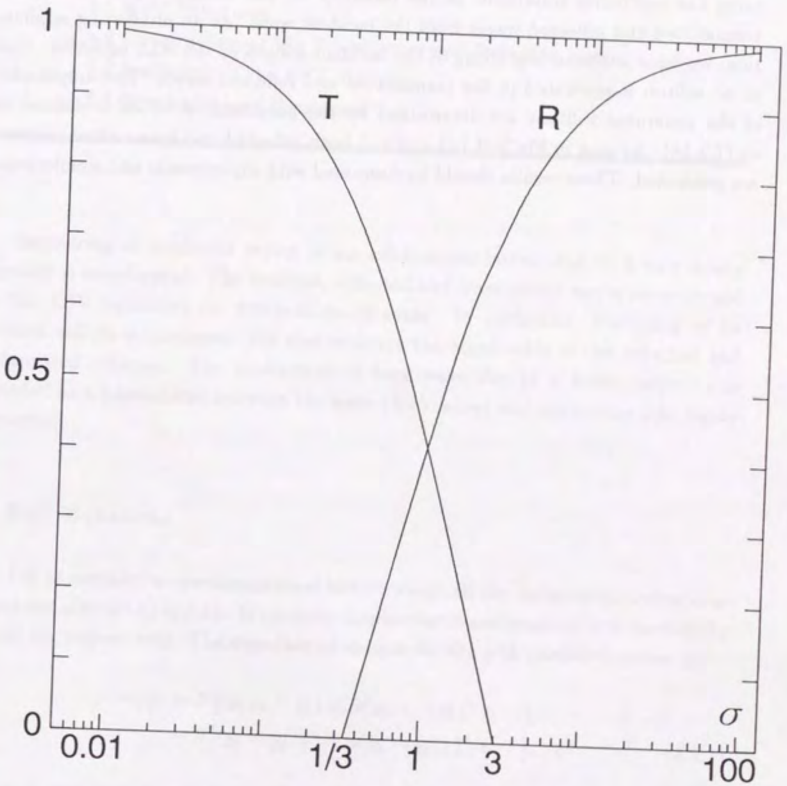

Figure 7.2 Relative amplitudes of the transmitted(T) and reflected(R) solitons. A parameter σ is defined in eq.(7.3.14).

7.4. Conclusion and Discussion

In this chapter we have investigated scatterings of nonlinear waves in one-dimensional lattice due to a mass impurity. The lattice waves we considered are nonlinear modulations of the monochromatic waves. The incident, transmitted and reflected waves are governed by the Nonlinear Schřodinger (NLS) equations. By using the continuity conditions at the impurity we can analytically construct the transmitted and reflected waves from the incident wave. As an interesting application, we have analyzed scattering of the incident soliton of the NLS equation. One or no soliton is generated in the transmitted and reflected waves. The amplitudes of the generated solitons are determined by the parameter σ which is defined in eq.(7.3.14). As seen in Fig.2, if $1/3 < \sigma < 3$ both reflected and transmitted solitons are generated. These results should be compared with experiments and simulations.

Chapter 8

Interaction of Wave and Matter

Scattering of nonlinear waves in an anharmonic lattice due to a very heavy impurity is investigated. The incident, reflected and transmitted waves are governed by the KdV equations on different coordinates. In particular, scattering of an incident soliton is analyzed. We also evaluate the amplitudes of the reflected and transmitted solitons. The scatterings of long wave due to a heavy impurity is regarded as a interactions between the wave (KdV wave) and the matter (the heavy impurity).

8.1 KdV Equations

Let us consider a one-dimensional lattice where all the anharmonic spring constants are identical(Fig.8.1). We denote displacement and mass of j-th particle by y_j and m_j respectively. The equation of motion for the j-th particle is given by

$$
\begin{aligned}
m_j \ddot{y}_j &= B\big[(y_{j+1} - y_j) + \alpha(y_{j+1} - y_j)^2 + \cdots\big] \\
&\quad - B\big[(y_j - y_{j-1}) + \alpha(y_j - y_{j-1})^2 + \cdots\big],
\end{aligned}
\tag{8.1.1}
$$

where B and $\alpha(\neq 0)$ are constants and the dot indicates a differentiation with respect to time t.

We have neglected the higher nonlinear terms because $y_j - y_{j-1}$ are assumed to be small. We assume that there is a very heavy impurity at the site $j = 0$ while

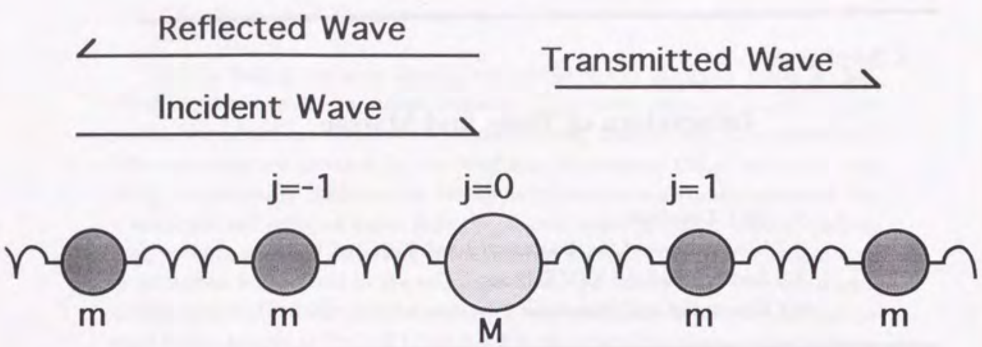

Figure 8.1: Nonlinear lattice which contains a very heavy impurity at site $j = 0$.

the others have the same mass;

$$\begin{cases} m_j = m \quad (j \neq 0), \\ m_0 = M = \dfrac{2m}{\varepsilon\mu}, \end{cases} \tag{8.1.2}$$

where ε is a small fixed constant. It indicates the amplitude of the lattice waves which is related to the nonlinearity (see eqs. (8.14c) and (8.18d)). Constant μ is $O(1)$; the order of ε^0. Thus the parameter μ is determined by the mass ratio and the amplitude of the waves (Iizuka and Wadati 1992e).

In the following we apply the method used in 6.2. We suppose that displacements y_j are slowly changing with j. Then we may employ the continuum approximation; $y_j(t) = y(x,t)$, where $x = jh$, h being the lattice spacing. The equation of motion at $x \neq 0$ is reduced to a Boussinesq type equation.

$$\frac{1}{c^2} y_{tt} = y_{xx} + \frac{h^2}{12} y_{xxxx} + 2h\alpha y_x y_{xx} + \cdots \quad (\text{at} \quad x \neq 0), \tag{8.1.3a}$$

$$c = \sqrt{\frac{Bh^2}{m}}, \tag{8.1.3b}$$

where the subscripts indicate partial differentiations and c is the sound velocity. Equation (8.1.3) is not valid at $x = 0$; we consider the waves separately in the regions $x > 0$ and $x < 0$. We shall follow the technique given in 6.2. A condition of continuity at $x = 0$ will be discussed in 8.2.

For $x > 0$ we regard the wave as a transmitted wave and express it as $y(x,t) = y^+(x,t)$. We assume that the amplitude of the wave is small and slowly varying. To describe it, we introduce new independent variables ξ, τ and dependent variables $v^{(1)}, v^{(3)}, \cdots$ as

$$\xi = \frac{\varepsilon}{h}(x - ct), \tag{8.1.4a}$$

$$\tau = \frac{\varepsilon^3}{24h} x, \tag{8.1.4b}$$

$$y(x,t) = y^+(x,t) = -\frac{1}{4\alpha}\{\varepsilon v^{(1)}(\xi,\tau) + \varepsilon^3 v^{(3)}(\xi,\tau) + \cdots\}. \tag{8.1.4c}$$

Substitution of (8.1.4) into (8.1.3) gives

$$v^{(1)}_{\xi\tau} - 6v^{(1)}_{\xi} v^{(1)}_{\xi\xi} + v^{(1)}_{\xi\xi\xi\xi} = 0. \tag{8.1.5}$$

Using a new dependent variable

$$\Psi^T(\xi,\tau) = v^{(1)}_{\xi} = -\frac{h v^{(1)}_t}{\varepsilon c}, \tag{8.1.6}$$

we get the KdV equation from (8.1.5)

$$\Psi^T_\tau - 6\Psi^T \Psi^T_\xi + \Psi^T_{\xi\xi\xi} = 0 \quad (\tau > 0). \tag{8.1.7}$$

Note that $\Psi^T(\xi, 0)$ is proportional to the velocity of the impurity (see (8.1.4c) and (8.1.6)).

In the region $x < 0$ we concentrate on incident and reflected waves. Similarly to the previous case, we use the reductive perturbation method. New independent and dependent variables are defined by

$$\xi = \frac{\varepsilon}{h}(x - ct), \tag{8.1.8a}$$

$$\bar{\xi} = \frac{\varepsilon}{h}(x + ct), \tag{8.1.8b}$$

$$\tau = \frac{\varepsilon^3}{24h} x, \tag{8.1.8c}$$

$$y(x,t) = y^-(x,t) = -\frac{1}{4\alpha}\{\varepsilon u^{(1)}(\xi,\bar{\xi},\tau) + \varepsilon^3 u^{(3)}(\xi,\bar{\xi},\tau) + \cdots\}. \tag{8.1.8d}$$

We substitute (8.1.8) into eq.(8.1.3). At $O(\varepsilon^3)$, we have

$$u^{(1)}_{\xi\bar{\xi}} = 0, \tag{8.1.9}$$

which specifies the functional form of $u^{(1)}$ as

$$u^{(1)}(\xi,\bar{\xi},\tau)=u^{(I)}(\xi,\tau)+u^{(R)}(\bar{\xi},\tau). \tag{8.1.10}$$

From the result of Appendix in chapter 6, we have

$$u^{(I)}_{\xi\tau}-6u^{(I)}_{\xi}u^{(I)}_{\xi\xi}+u^{(I)}_{\xi\xi\xi\xi}=0, \tag{8.1.11a}$$

$$u^{(R)}_{\bar{\xi}\tau}-6u^{(R)}_{\bar{\xi}}u^{(R)}_{\bar{\xi}\bar{\xi}}+u^{(R)}_{\bar{\xi}\bar{\xi}\bar{\xi}\bar{\xi}}=0. \tag{8.1.11b}$$

Therefore, using new dependent variables

$$\Psi^{I}(\xi,\tau)=u^{(I)}_{\xi}=-\frac{hu^{(I)}_{t}}{\varepsilon c}, \tag{8.1.12a}$$

$$\Psi^{R}(\bar{\xi},\tau)=u^{(R)}_{\bar{\xi}}=\frac{hu^{(R)}_{t}}{\varepsilon c}, \tag{8.1.12b}$$

equations (8.1.11c,d) reduce to

$$\Psi^{I}_{\tau}-6\Psi^{I}\Psi^{I}_{\xi}+\Psi^{I}_{\xi\xi\xi}=0 \qquad (\tau<0), \tag{8.1.13a}$$

$$\Psi^{R}_{\tau}-6\Psi^{R}\Psi^{R}_{\bar{\xi}}+\Psi^{R}_{\bar{\xi}\bar{\xi}\bar{\xi}}=0 \qquad (\tau<0). \tag{8.1.13b}$$

Thus, the incident and reflected waves are governed by independent KdV equations (8.1.13).

8.2. Construction of Transmitted and Reflected Waves

Even if there is an impurity, the weakly nonlinear wave is not influenced very much as far as we are dealing with a long wave. The collective motion of many particles are rather independent on the single particle motion. However, *when the mass of impurity is large enough, we expect a coupling between the weakly nonlinear wave and the heavy impurity.* In this section we shall explicitly show the existence of this coupling by calculating the conditions of connection. Remark that in chapter 7 we have considered short wave scatterings due to an impurity whose mass is comparable with others.

In this section we connect y^- and y^+ by considering equations of motion for $y_{\pm1}$ and y_0;

$$j=-1;\quad m\frac{\partial^2}{\partial t^2}y^-(-h,t)=K(y_0-y_{-1})-K(y_{-1}-y_{-2})+O(\varepsilon^3)$$

$$=K(y_0-y_{-1})-Kh\frac{\partial}{\partial x}y^-(-h,0)+O(\varepsilon^3), \tag{8.2.1a}$$

$$j=1;\quad m\frac{\partial^2}{\partial t^2}y^+(+h,t)=K(y_2-y_1)-K(y_1-y_0)+O(\varepsilon^3)$$

$$=Kh\frac{\partial}{\partial x}y^+(+h,0)-K(y_1-y_0)+O(\varepsilon^3), \tag{8.2.1b}$$

$$j=0;\quad M\frac{d^2}{dt^2}y_0=K[(y_1-y_0)+\alpha(y_1-y_0)^2]$$

$$+K[(y_0-y_{-1})+\alpha(y_0-y_{-1})^2]. \tag{8.2.1c}$$

Since l.h.s of (8.2.1a) is $O(\varepsilon^3)$, we have

$$y_0-y_{-1}=h\frac{\partial}{\partial x}y^-(-h,t)+O(\varepsilon^3)=h\frac{\partial}{\partial x}y^-(0,t)+O(\varepsilon^3). \tag{8.2.2a}$$

Similarly, we obtain

$$y_1-y_0=h\frac{\partial}{\partial x}y^+(h,t)+O(\varepsilon^3)=h\frac{\partial}{\partial x}y^+(0,t)+O(\varepsilon^3). \tag{8.2.2b}$$

Using

$$y_{-1}=y^-(-h,t)=y^-(0,t)-h\frac{\partial}{\partial x}y^-(0,t)+\cdots$$

$$=\varepsilon(u^{(I)}(\xi,0)+u^{(R)}(\bar{\xi},0))-h\frac{\partial}{\partial x}y^-(0,t)+O(\varepsilon^3), \tag{8.2.3a}$$

$$y_1=y^+(h,t)=y^+(0,t)+h\frac{\partial}{\partial x}y^+(0,t)+\cdots$$

$$=\varepsilon v^{(1)}(\xi,0)+h\frac{\partial}{\partial x}y^+(0,t)+O(\varepsilon^3), \tag{8.2.3b}$$

we rewrite (8.2.2) as

$$y_0=\varepsilon(u^{(I)}(\xi,0)+u^{(R)}(\bar{\xi},0))+O(\varepsilon^3), \tag{8.2.4a}$$

$$y_0=\varepsilon v^{(1)}(\xi,0)+O(\varepsilon^2)=\varepsilon v^{(1)}(-\frac{\varepsilon}{h}ct,0)+O(\varepsilon^3). \tag{8.2.4b}$$

By eliminating y_0 we have

$$v^{(1)}(\xi,0)=u^{(I)}(\xi,0)+u^{(R)}(\xi,0)+O(\varepsilon^2). \tag{8.2.4c}$$

Neglecting $O(\varepsilon^2)$ and taking differentiation in (8.2.4c) we obtain

$$\Psi^{T}(\xi,0)=\Psi^{I}(\xi,0)-\Psi^{R}(\bar{\xi},0), \tag{8.2.5}$$

where we have used (8.1.6) and (8.1.12).

Substituting (8.2.4b) and (8.2.2) into (8.2.1c) and eliminating y_0, we have

$$\frac{2}{\mu}v^{(1)}_{\xi\xi}(\xi,0) = v^{(1)}_{\xi}(\xi,0) - u^{(I)}_{\xi}(\xi,0) - u^{(R)}_{\bar{\xi}}(\bar{\xi},0) + O(\varepsilon^2), \tag{8.2.6}$$

where (8.1.2),(8.1.4) and (8.1.8) are considered. If $O(\varepsilon^2)$ is neglected this is equivalent to

$$\frac{2}{\mu}\Psi^T_{\xi}(\xi,0) = \Psi^T(\xi,0) - \Psi^I(\xi,0) - \Psi^R(\bar{\xi},0), \tag{8.2.7}$$

where (8.1.6) and (8.1.12) have been used.

Relations (8.2.5) and (8.2.7) are the conditions of the continuity. All quantities included in the relations are $O(1)$. The condition (8.2.7) is a differential equation while that of discontinuous system is an algebraic relation (see eq.(6.2.17)). Equation (8.2.7) indicates a coupling between the lattice wave and the heavy impurity. In our method we take a balance among nonlinearity, dispersion and mass ratio (inhomogeneity). If M/m is of order $O(1)$, l.h.s. of (8.2.6) is 0 and there remains no coupling between wave and impurity. In this case $v^{(1)} = u^{(T)}$ obeys a single KdV equation and $u^{(R)}$ does not appear.

We can construct 'initial conditions' $\Psi^T(\xi,0)$ and $\Psi^R(\bar{\xi},0)$ from $\Psi^I(\xi,0)$ by solving the system (8.2.5) and (8.2.7). Eliminating $\Psi^R(\bar{\xi},0)$ we get

$$\Psi^T(\xi,0) - \Psi^I(\xi,0) = \mu^{-1}\Psi^T_{\xi}(\xi,0), \tag{8.2.8}$$

whose solution is given by

$$\Psi^T(\xi,0) = \mu\int_{\xi}^{+\infty}\Psi^I(\xi',0)e^{-\mu(\xi'-\xi)}d\xi'. \tag{8.2.9}$$

Then $\Psi^R(\bar{\xi},0)$ is determined as

$$\Psi^R(\bar{\xi},0) = \Psi^I(-\bar{\xi},0) - \Psi^T(-\bar{\xi},0). \tag{8.2.10}$$

We summarize the above results. If the incident wave $u^{(I)}$ or Ψ^I is given, we can calculate $\Psi^T(\xi,0)$ and $\Psi^R(\bar{\xi},0)$ through (8.2.9) and (8.2.10). Evolutions of the transmitted wave Ψ^T and the reflected wave Ψ^R are described by the KdV equations (8.1.7) and (8.1.13b). Initial value problem of the KdV equation is solved analytically by the inverse scattering method. Therefore we can construct transmitted wave $\Psi^T(\tau>0)$ and reflected wave $\Psi^R(\tau<0)$.

8.3. Scattering of the KdV Solitons

We shall investigate scattering of an incident soliton due to a heavy impurity. The incident wave is chosen to be one soliton solution;

$$\Psi^I(\xi,\tau) = -2\mathrm{sech}^2(\xi-4\tau). \tag{8.3.1}$$

Substituting (8.3.1) into eq.(8.2.9) we have $\Psi^T(\xi,0)$ as

$$\begin{aligned}\Psi^T(\xi,0) &= -2\mu\int_{\xi}^{\infty}\mathrm{sech}^2\xi' e^{-\mu\xi'}d\xi' e^{\mu\xi}\\ &= -4\mu e^{\mu\xi}\int_0^{v} v'^{\mu/2}(1-v')^{-\mu/2}dv'\\ &= -4\mu e^{\mu\xi}B_v(\frac{\mu}{2}+1,-\frac{\mu}{2}+1),\end{aligned} \tag{8.3.2}$$

where v stands for

$$v = \frac{e^{-2\xi}}{e^{-2\xi}+1}, \tag{8.3.3}$$

and $B_v(x,y)$ is the incomplete beta function. Using (8.3.1) and (8.3.2) in (8.2.10) we have $\Psi^R(\bar{\xi},0)$. In Fig.8.2 $\Psi^I(\xi,0)$, $\Psi^T(\xi,0)$ and $\Psi^R(-\xi,0)$ are shown for $\mu=1$(a) and $\mu=0.2$(b).

We can solve the KdV equations (8.1.7) and (8.1.13b) by the inverse scattering method. We concentrate on a problem that how many solitons are generated in the reflected and transmitted waves. If $u(\xi,\tau)$ satisfies the KdV equation

$$u_{\tau} - 6uu_{\xi} + u_{\xi\xi\xi} = 0, \tag{8.3.4}$$

and an initial condition $u(\xi,0)$ is given, we can determine the number of generated solitons and their amplitudes by solving the Schrodinger equation;

$$\frac{\partial^2\Phi(\xi)}{\partial\xi^2} + (\lambda - u(\xi,0))\Phi(\xi) = 0. \tag{8.3.5}$$

Let us denote the bound state energies as

$$\lambda = -l_1, -l_2, \cdots, -l_n \qquad l_1 > l_2 > \cdots > l_n > 0 \tag{8.3.6}$$

The number of the bound states n is equal to that of the generated solitons and their amplitudes are given by $2l_j$. Since the amplitude of the incident soliton is 2, we refer to l_j as 'relative amplitude'. It should be noted that radiations (small oscillations) are also generated in general.

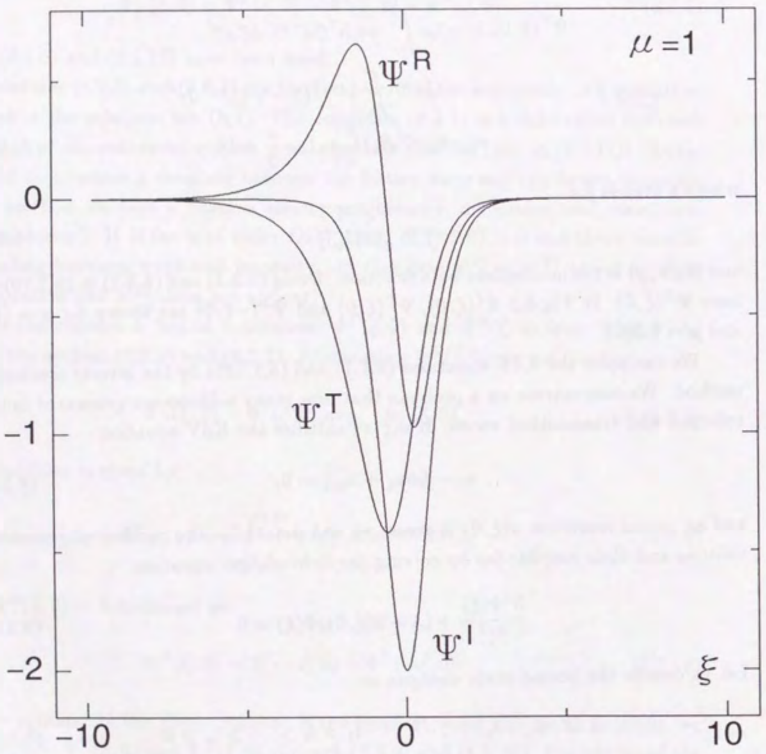

Figure 8.2(a): The incident Ψ^I, transmitted Ψ^T and reflected Ψ^R waves at $x=0(\tau=0)$ for $\mu=1$. Note that $x=0$ corresponds to $\xi=-\varepsilon ct/h$.

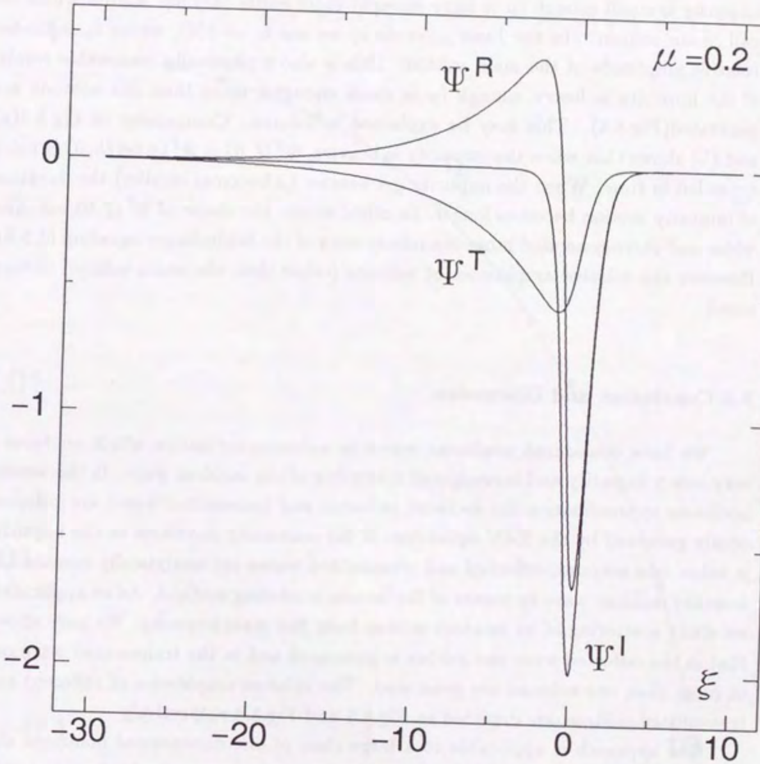

Figure 8.2(b): The incident Ψ^I, transmitted Ψ^T and reflected Ψ^R waves at $x=0(\tau=0)$ for $\mu=0.2$. We observe longer time motion of Ψ^R and Ψ^T due to heavier impurity. Note that $x=0$ corresponds to $\xi=-\varepsilon ct/h$.

Using the above facts we numerically calculate the relative amplitude of generated solitons for the initial condition (8.3.2) and the corresponding $\Psi^R(\bar{\xi}, 0)$. The results are shown in Fig.8.3,4. In the reflected wave we observe only one soliton (Fig.8.3). If the mass of the impurity becomes large (μ becomes small) the amplitude increases. In the limit $M \to \infty(0)$, we have $l_1 \to 1(0)$, which can be explained by a simple argument.
In the transmitted wave we observe at least 3 solitons (Fig.8.4). If the mass of the impurity is small enough (μ is large enough) there exists only one soliton which we call 'main soliton'. In the limit $\mu \to \infty(0)$ we see $l_1 \to 1(0)$, where l_1 indicates relative amplitude of the main soliton. This is also a physically reasonable result. If the impurity is heavy enough (μ is small enough), more than one solitons are generated(Fig.8.4). This may be explained as follows. Comparison of Fig.8.2(a) and (b) shows that when the impurity is heavier, $\Psi^T(\xi, 0) = \Psi^T(-\varepsilon ct/h, 0)$ is more extended in time. When the impurity get heavier (μ becomes smaller) the duration of impurity motion becomes longer. In other words, the shape of $\Psi^T(\xi, 0)$ becomes wider and there generated more discrete spectra of the Schr̈odinger equation (8.3.5). However the relative amplitudes of solitons (other than the main soliton) is very small.

8.4 Conclusion and Discussion

We have considered nonlinear waves in anharmonic lattice which contains a very heavy impurity and investigated scattering of the incident wave. In the weakly nonlinear approximation the incident, reflected and transmitted waves are independently governed by the KdV equations. If the continuity condition at the impurity is taken into account, reflected and transmitted waves are analytically constructed from the incident wave by means of the inverse scattering method. As an application we study scattering of an incident soliton from the mass impurity. We have shown that in the reflected wave one soliton is generated and in the transmitted wave one or more than one solitons are generated. The relative amplitudes of reflected and transmitted solitons are depicted in Fig.8.3 and Fig.8.4 respectively.

Our approach is applicable to a large class of one dimensional nonlinear dispersive systems. We have assumed that the lattice waves are slowly changing. The scattering of long wave due to the heavy impurity is regarded as **interactions of nonlinear wave and matter**. This picture is possibly applicable to a large class of such systems.

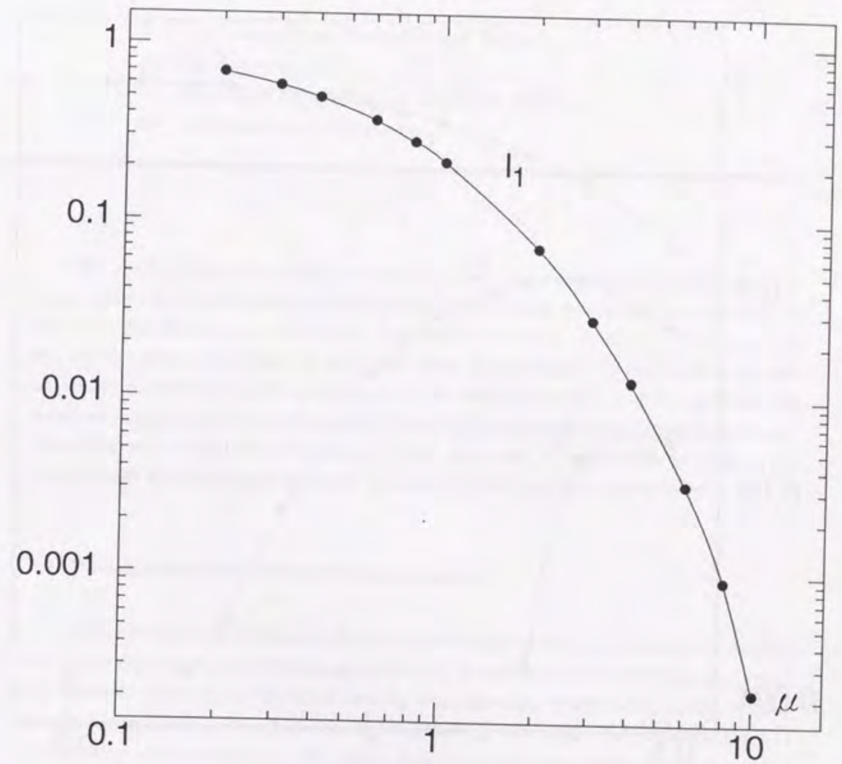

Figure 8.3: Relative amplitude of the reflected soliton. We observe only one soliton.

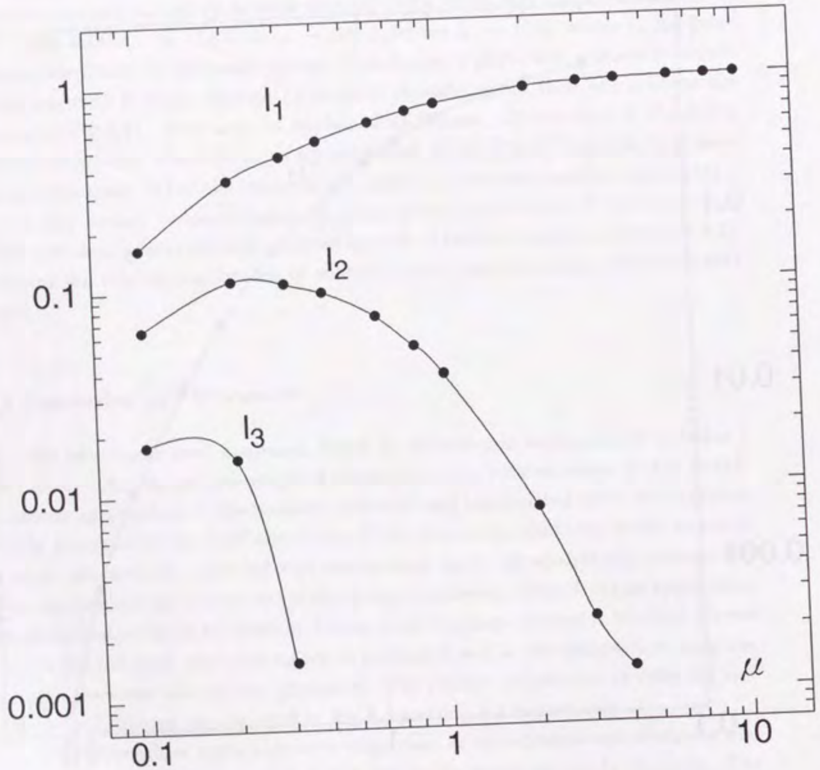

Figure 8.4: Relative amplitude of the transmitted solitons. We observe at least 3 solitons. As μ becomes smaller, more solitons are expected to appear since the width of $\Psi(\xi, 0)$ gets larger.

Chapter 9

Nonlinear Refraction and Reflection Phenomena

We shall propose the notion of nonlinear refraction and reflection phenomena. As a model, a two-dimensional anharmonic lattice which has a line discontinuity in the mass distribution is considered. We study scattering of the KP (line) solitons due to the mass interface. It is shown that the transmitted line soliton breaks up while the reflected soliton does not. The reflection angle is different from the incident angle due to the nonlinearity. These are considered as features of nonlinear refraction and reflection phenomena. Our approach is applicable to other two-dimensional nonlinear systems such as shallow water and ion acoustic wave.

9.1 Two-dimensional Discontinuous Lattices

The investigations in chapter 6~8 are scatterings of the one-dimensional nonlinear waves due to localized inhomogeneities. In the present chapter we aim to extend the analysis into two-dimensional case by introducing a mechanical model; anharmonic triangular lattice which has an interface in the mass distribution.(Fig.9.1) The lattice is anisotropic in the sense that there are two kinds of springs in the lattice. One is assigned on horizontal edges in Fig.9.1 and the other on inclined edges.

The (i, j)-th particle and its neighbors are assigned as shown in Fig.9.2. Denoting the displacement and mass of (i, j)-th particle by $\vec{r}_{i,j}$ and $m_{i,j}$ respectively,

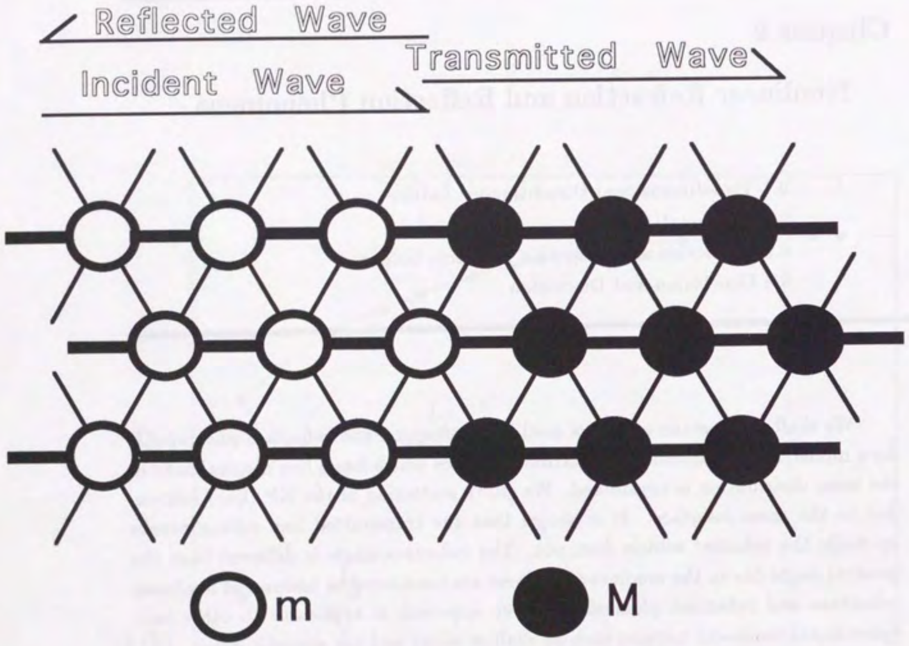

Figure 9.1: Anharmonic triangular lattice with a mass interface. Strong springs are assigned on horizontal edges and weak springs on inclined edges.

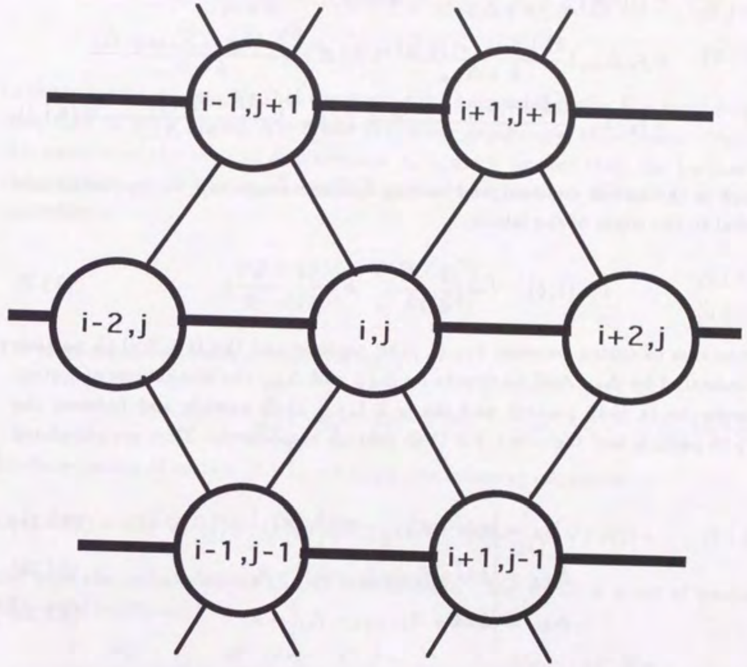

Figure 9.2: Arrangement of the (i, j)-th particle and its neighbors.

we write the equation of motion;

$$m_{i,j}\frac{\mathrm{d}^2\vec{r}_{ij}}{\mathrm{d}t^2}=f_1(\Delta_+)\frac{h\vec{i}+\vec{r}_{i+2,j}-\vec{r}_{i,j}}{h+\Delta_+}+f_1(\Delta_-)\frac{-h\vec{i}+\vec{r}_{i-2,j}-\vec{r}_{i,j}}{h+\Delta_-}$$
$$+f_2(\Delta_{++})\frac{h\vec{l}+\vec{r}_{i+1,j+1}-\vec{r}_{i,j}}{h+\Delta_{++}}+f_2(\Delta_{--})\frac{-h\vec{l}+\vec{r}_{i-1,j-1}-\vec{r}_{i,j}}{h+\Delta_{--}}$$
$$+f_2(\Delta_{+-})\frac{h\vec{k}+\vec{r}_{i+1,j-1}-\vec{r}_{i,j}}{h+\Delta_{+-}}+f_2(\Delta_{+-})\frac{-h\vec{k}+\vec{r}_{i-1,j+1}-\vec{r}_{i,j}}{h+\Delta_{-+}} \quad (9.1.1)$$

Here h is the lattice constant and vectors $\vec{i}$, $\vec{l}$ and $\vec{k}$ are unit vectors which are parallel to the edges of the lattice;

$$\vec{i}=(1,0),\quad \vec{l}=(\frac{1}{2},\frac{\sqrt{3}}{2}),\quad \vec{k}=(\frac{1}{2},-\frac{\sqrt{3}}{2}). \quad (9.1.2)$$

Elongations of spring between the (i,j)-th particle and the $(i\pm 2,j)$-th particle are indicated by $\Delta_\pm$. And we denote by $\Delta_{\pm\pm}$ and $\Delta_{\pm\mp}$ the elongations of springs between the (i,j)-th particle and the $(i\pm 1, j\pm 1)$-th particle and between the (i,j)-th particle and the $(i\pm 1, j\mp 1)$-th particle respectively. They are calculated as

$$\Delta_\pm=\left|\pm h\vec{i}+\vec{r}_{i\pm 2,j}-\vec{r}_{i,j}\right|-h, \quad (9.1.3a)$$
$$\Delta_{\pm\pm}=\left|\pm h\vec{l}+\vec{r}_{i\pm 1,j\pm 1}-\vec{r}_{i,j}\right|-h, \quad (9.1.3b)$$
$$\Delta_{\pm\mp}=\left|\pm h\vec{k}+\vec{r}_{i\pm 1,j\mp 1}-\vec{r}_{i,j}\right|-h. \quad (9.1.3c)$$

The force due to the horizontal spring $f_1(\Delta)$ and that due to the inclined spring $f_2(\Delta)$ are given by

$$f_1(\Delta)=B(\Delta+\alpha\Delta^2+\cdots)\quad (B>0), \quad (9.1.4a)$$
$$f_2(\Delta)=B'(\Delta+\alpha'\Delta^2+\cdots)\quad (B'>0). \quad (9.1.4b)$$

We assume that the mass distribution $m_{i,j}$ is given by

$$m_{i,j}=\begin{cases} m & (i\le -1) \\ M & (i\ge 0)\end{cases}, \quad (9.1.5)$$

which shows the existence of a discontinuous interface.

Since we are interested in small and slowly varying waves, we introduce the continuum approximation for the regions $i>0$ and $i<0$ separately;

$$\vec{r}_{i,j}(t)=\vec{r}(\vec{x},t)=\vec{r}(x,y,t)=(r_1(x,y,t),r_2(x,y,t)), \quad (9.1.6a)$$
$$x=\frac{ih}{2},\quad y=\frac{\sqrt{3}}{2}jh,\quad \vec{x}=(x,y). \quad (9.1.6b)$$

In the following we suppose that constant of the horizontal spring B is much larger than that of other spring B'. Thus *the lattice is strongly anisotropic.* Due to this anisotropy the vertical displacement r_2 is much smaller than the horizontal displacement r_1. By using a smallness parameter ε, we set the orders of these quantities as

$$B\sim O(1),\quad B'\sim O(\varepsilon^2), \quad (9.1.7a)$$
$$r_1\sim O(\varepsilon),\quad r_2\sim O(\varepsilon^3). \quad (9.1.7b)$$

The slowness of the waves is indicated by

$$\frac{\partial}{\partial x}\sim\frac{\partial}{\partial y}\sim\frac{\partial}{\partial t}\sim O(\varepsilon). \quad (9.1.7c)$$

To the equation of motion (9.1.1), we apply the following expansion

$$\vec{r}(\vec{x}\pm\vec{\delta})-\vec{r}(\vec{x})=\pm(\vec{\delta}\cdot\nabla)\vec{r}(\vec{x})+\frac{1}{2}(\vec{\delta}\cdot\nabla)^2\vec{r}(\vec{x})\pm\frac{1}{6}(\vec{\delta}\cdot\nabla)^3\vec{r}(\vec{x})+\frac{1}{24}(\vec{\delta}\cdot\nabla)^4\vec{r}(\vec{x})+\cdots, \quad (9.1.8)$$

and take the order-relations (9.1.7) into account. The result is a set of partial differential equations;

$$\frac{1}{c(x)^2}\frac{\partial^2 r_1}{\partial t^2}=(1+\frac{B'}{8B})\frac{\partial^2 r_1}{\partial x^2}+\frac{h^2}{12}\frac{\partial^4 r_1}{\partial x^4}+2\alpha h\frac{\partial r_1}{\partial x}\frac{\partial^2 r_1}{\partial x^2}+\frac{3B'}{8B}\frac{\partial^2 r_1}{\partial y^2}, \quad (9.1.9a)$$
$$\frac{1}{c(x)^2}\frac{\partial^2 r_2}{\partial t^2}=\frac{3B'}{4B}\frac{\partial^2 r_1}{\partial x\partial y}. \quad (9.1.9b)$$

In (9.1.9), $O(\varepsilon^6)$ is neglected. Here $c(x)$ is the acoustic velocity in x-direction;

$$c(x)=\begin{cases} c^-=\sqrt{\dfrac{Bh^2}{m}} & (x<0) \\ c^+=\sqrt{\dfrac{Bh^2}{M}} & (x>0)\end{cases}. \quad (9.1.10)$$

The two-dimensional Boussinesq equation (9.1.9a) is a starting point of further analysis. In the next section, we apply the reductive perturbation method to

(9.1.9). The Kadomtsev-Petviashvili (KP) equations are derived for the incident, transmitted and reflected waves. And then, conditions of continuity at $x = 0$ which connect the three waves are presented. In 9.3 we examine refraction and reflection of the line soliton. The explicit results are obtained since the problem is reduced to the initial value problem of the KdV equation.

9.2 KP equations

We shall apply the reductive perturbation method to the regions $x > 0$ and $x < 0$ separately. In $x > 0$ we treat transmitted wave. Let us introduce dimensionless parameter κ as

$$\frac{B'}{B} = \varepsilon^2\kappa, \tag{9.2.1}$$

and transformations of independent and dependent variables as

$$\eta = \frac{\varepsilon}{h}\{x - c^{+}(1 + \frac{\varepsilon^2\kappa}{16})t\}, \tag{9.2.2a}$$

$$\zeta = \sqrt{\frac{2}{3\kappa}}\frac{\varepsilon}{h}y, \tag{9.2.2b}$$

$$\tau = \frac{1}{24h}\varepsilon^3 x, \tag{9.2.2c}$$

$$r_1 = \frac{1}{4\alpha}(\varepsilon u^T(\eta,\zeta,\tau) + \varepsilon^3 u^{(T,3)}(\eta,\zeta,\tau) + \cdots), \tag{9.2.3a}$$

$$r_2 = \frac{\varepsilon^3}{\alpha}(v^T(\eta,\zeta,\tau) + \cdots). \tag{9.2.3b}$$

The relation (9.2.1) indicates the strong anisotropy of the lattice. In (9.2.3a,b) we assume that the vertical displacement r_2 is much smaller than the horizontal displacement r_1. Substituting these transformations to (9.1.9), we have at $O(\varepsilon^5)$

$$u^T_{\eta\tau} + 6u^T_\eta u^T_{\eta\eta} + u^T_{\eta\eta\eta\eta} + 3u^T_{\zeta\zeta} = 0, \quad (\tau > 0) \tag{9.2.4a}$$

$$v^T_{\eta\eta} = \frac{\sqrt{6}}{16}u^T_{\eta\zeta}. \quad (\tau > 0) \tag{9.2.4b}$$

Subscripts mean partial differentiations with respect to the corresponding independent variables. Equation (9.2.4a) is rewritten as

$$U^T_\tau + 6U^T U^T_\eta + U^T_{\eta\eta\eta} + 3u^T_{\zeta\zeta} = 0, \tag{9.2.5a}$$

$$U^T = u^T_\eta. \tag{9.2.5b}$$

Equation (9.2.5) is the KP(II) equation which is well known as two-dimensional extension of the KdV equation. We have shown that the transmitted wave obeys KPII equation (9.2.5a).

In the region $x < 0$ incident and transmitted waves propagate. Similarly to (9.2.2) and (9.2.3), we introduce independent and dependent variables as

$$\xi = \frac{\varepsilon}{h}\{x - c^{-}(1 + \frac{\varepsilon^2\kappa}{16})t\}, \tag{9.2.6a}$$

$$\bar{\xi} = \frac{\varepsilon}{h}\{x + c^{-}(1 + \frac{\varepsilon^2\kappa}{16})t\}, \tag{9.2.6b}$$

$$\zeta = \sqrt{\frac{2}{3\kappa}}\frac{\varepsilon}{h}y, \tag{9.2.6c}$$

$$\tau = \frac{1}{24h}\varepsilon^3 x, \tag{9.2.6d}$$

$$r_1 = \frac{1}{4\alpha}(\varepsilon u^I(\xi,\zeta,\tau) + \varepsilon u^R(\bar{\xi},\zeta,\tau) + \varepsilon^3 u^{(3)}(\xi,\bar{\xi},\zeta,\tau) + \cdots), \tag{9.2.7a}$$

$$r_2 = \frac{\varepsilon^3}{\alpha}(v^I(\xi,\zeta,\tau) + v^R(\bar{\xi},\zeta,\tau) + \cdots). \tag{9.2.7b}$$

Incident and reflected waves are indicated by $u^I(\xi,\zeta,\tau)$ and $u^R(\bar{\xi},\zeta,\tau)$ respectively. Substitution of (9.2.6) and (9.2.7) into (9.1.9) gives at $O(\varepsilon^5)$

$$\begin{aligned} &48u^{(3)}_{\xi\bar{\xi}} + 6(u^I_\xi u^R_{\bar{\xi}\bar{\xi}} + u^R_{\bar{\xi}} u^I_{\xi\xi}) \\ &+ (u^I_{\xi\tau} + 6u^I_\xi u^I_{\xi\xi} + u^I_{\xi\xi\xi\xi} + 3u^I_{\zeta\zeta}) \\ &+ (u^R_{\bar{\xi}\tau} + 6u^R_{\bar{\xi}} u^I_{\bar{\xi}\bar{\xi}} + u^I_{\bar{\xi}\bar{\xi}\bar{\xi}\bar{\xi}} + 3u^I_{\zeta\zeta}) = 0, \end{aligned} \tag{9.2.8a}$$

$$v^I_{\xi\xi} = \frac{\sqrt{6}}{16}u^I_{\xi\zeta}, \tag{9.2.8b}$$

$$v^R_{\bar{\xi}\eta} = \frac{\sqrt{6}}{16}u^R_{\bar{\xi}\zeta}. \tag{9.2.8c}$$

With some functions $F(\xi,\zeta,\tau)$ and $G(\bar{\xi},\zeta,\tau)$, equation (9.2.8a) is decomposed into

$$u^{(3)}_{\xi\bar{\xi}} + \frac{1}{8}(u^I_\xi u^R_{\bar{\xi}\bar{\xi}} + u^R_{\bar{\xi}} u^I_{\xi\xi}) = F(\xi,\zeta,\tau) + G(\bar{\xi},\zeta,\tau), \tag{9.2.9a}$$

$$u^I_{\xi\tau} + 6u^I_\xi u^I_{\xi\xi} + u^I_{\xi\xi\xi\xi} + 3u^I_{\zeta\zeta} = -48F(\xi,\zeta,\tau), \tag{9.2.9b}$$

$$u^R_{\bar{\xi}\tau} + 6u^R_{\bar{\xi}} u^I_{\bar{\xi}\bar{\xi}} + u^I_{\bar{\xi}\bar{\xi}\bar{\xi}\bar{\xi}} + 3u^I_{\zeta\zeta} = -48G(\bar{\xi},\zeta,\tau). \tag{9.2.9c}$$

By integrating (9.2.9a), we have

$$u^{(3)} = -\frac{1}{8}(u^I u^R_{\bar{\xi}} + u^R u^I_\xi) + H(\xi,\zeta,\tau) + I(\bar{\xi},\zeta,\tau) + \bar{\xi}\int F d\xi + \xi \int G d\bar{\xi}, \tag{9.2.10}$$

where $H(\xi,\zeta,\tau)$ and $I(\bar{\xi},\zeta,\tau)$ are arbitrary functions. The last two terms in $u^{(3)}$ are secular terms which grow linearly in $\bar{\xi}$ and ξ respectively. In order to remove these terms we must set

$$F(\xi,\zeta,\tau) \equiv G(\bar{\xi},\zeta,\tau) \equiv 0. \tag{9.2.11}$$

Therefore equations (9.2.9b) and (9.2.9c) respectively reduce to

$$U^I_\tau + 6U^I U^I_\xi + U^I_{\xi\xi\xi} + 3u^I_{\zeta\zeta} = 0 \quad (\tau < 0), \tag{9.2.12a}$$
$$U^I = u^I_\xi, \tag{9.2.12b}$$

and

$$U^R_\tau + 6U^R U^R_{\bar{\xi}} + U^R_{\bar{\xi}\bar{\xi}\bar{\xi}} + 3u^R_{\zeta\zeta} = 0 \quad (\tau < 0), \tag{9.2.13a}$$
$$U^R = u^R_{\bar{\xi}}. \tag{9.2.13b}$$

We see that incident and reflected waves obey independent KPII equations.

In the following we consider continuity conditions of $r_1(x,t)$ at $x=0$ to connect the incident, reflected and transmitted waves. There are two continuity conditions of $r_1(x,t)$ at $x=\tau=0$. The first is the continuity of r_1;

$$r_1(x=+0,t) = r_1(x=-0,t) = 0, \tag{9.2.14}$$

from which we have at the lowest order of ε

$$u^I + u^R = u^T \quad (\tau = 0). \tag{9.2.15}$$

The second is the continuity of $\partial r_1/\partial x$;

$$\frac{\partial r_1}{\partial x}(x=+0,t) = \frac{\partial r_1}{\partial x}(x=-0,t). \tag{9.2.16}$$

Neglecting $O(\varepsilon^2)$, we have

$$-\frac{u^I}{c_-} + \frac{u^R}{c_-} = -\frac{u^T}{c_+}. \tag{9.2.17}$$

From (9.2.15) and (9.2.17) we obtain

$$u^T(\eta,\zeta,0) = \frac{2}{1+\nu}u^I(\nu\eta,\zeta,0), \tag{9.2.18a}$$
$$u^R(\bar{\xi},\zeta,0) = \frac{1-\nu}{1+\nu}u^I(-\bar{\xi},\zeta,0), \tag{9.2.18b}$$

where

$$\nu = \frac{c_-}{c_+} = \sqrt{\frac{M}{m}}. \tag{9.2.19}$$

The same conditions as (9.2.18) have been derived in one-dimensional systems (see eq.(6.2.17)). Taking t-derivative of (9.2.18) we get

$$U^T(\eta,\zeta,0) = \frac{2}{1+\nu}U^I(\nu\eta,\zeta,0), \tag{9.2.20a}$$
$$U^R(\bar{\xi},\zeta,0) = -\frac{1-\nu}{1+\nu}U^I(-\bar{\xi},\zeta,0). \tag{9.2.20b}$$

Conditions (9.2.18) and (9.2.20) are very useful. If the incident wave, u^I and U^I, is given, we find 'initial' conditions for the transmitted and reflected waves; $u^T(\tau=0)$, $U^T(\tau=0)$,$u^R(\tau=0)$ and $U^R(\tau=0)$. Therefore by solving the initial value problem of the KPII equation, the transmitted wave ($u^T(\tau>0)$, $U^T(\tau>0)$) and the reflected wave ($u^R(\tau<0)$, $U^R(\tau<0)$) are analytically constructed if U^I is sufficiently localized. We should note that τ does not stand for time t but it is proportional to x. Thus, solving KPII equations corresponds to constructing spatial evolutions of the reflected and transmitted waves in the x-direction. The above discussion is similar to that in 6.2.

9.3 Refraction and Reflection of a Line Soliton

Using the results of the previous section, we investigate scattering of an incident line soliton due to the interface. It is known that the KPII equation (9.2.12) has a stable line soliton solution;

$$U^I = 2\mathrm{sech}^2(\xi + k\zeta - \omega\tau), \tag{9.3.1a}$$
$$u^I = 2\tanh(\xi + k\zeta - \omega\tau), \tag{9.3.1b}$$
$$\omega - 4 - 3k^2 = 0. \tag{9.3.1c}$$

We assume that the incident wave is given by (9.3.1) in $\tau < 0$. Using (9.3.1) in (9.2.18) and (9.2.20), we have the 'initial' conditions for the transmitted and reflected waves,

$$U^T(\tau=0) = \frac{4\nu}{1+\nu}\mathrm{sech}^2(\nu\eta + k\zeta), \tag{9.3.2a}$$
$$u^T(\tau=0) = \frac{4\nu}{1+\nu}\tanh(\nu\eta + k\zeta), \tag{9.3.2b}$$

$$U^R(\tau=0)=\frac{-2(1-\nu)}{1+\nu}\mathrm{sech}^2(-\bar{\xi}+k\zeta), \tag{9.3.3a}$$

$$u^R(\tau=0)=\frac{2(1-\nu)}{1+\nu}\tanh(-\bar{\xi}+k\zeta). \tag{9.3.3b}$$

Time evolutions of the KPII equations (9.2.5) and (9.2.13) for (9.3.2) and (9.3.3) respectively are determined by reducing them to the KdV equations in the following way.

Let us solve the KPII equation

$$\Psi_{\tilde{T}}+6\Psi\Psi_{\tilde{X}}+\Psi_{\tilde{X}\tilde{X}\tilde{X}}+3\phi_{\tilde{Y}\tilde{Y}}=0, \tag{9.3.4a}$$

$$\Psi=\phi_{\tilde{X}}, \tag{9.3.4b}$$

for initial values;

$$\Psi(\tilde{X},\tilde{Y},0)=a\,\mathrm{sech}^2(b\tilde{X}+c\tilde{Y}), \tag{9.3.5a}$$

$$\phi(\tilde{X},\tilde{Y},0)=\frac{a}{b}\tanh(b\tilde{X}+c\tilde{Y}). \tag{9.3.5b}$$

Noting that the initial values are functions of $\tilde{X}+(c/b)\tilde{Y}$, we introduce transformations of independent variables;

$$X=(\tilde{X}+d\tilde{Y})-3d^2\tilde{T}, \tag{9.3.6a}$$

$$Y=\tilde{Y}-6d\tilde{T}, \tag{9.3.6b}$$

$$T=\tilde{T}, \tag{9.3.6c}$$

where

$$d=\frac{c}{b}. \tag{9.3.7}$$

Applying (9.3.6) to (9.3.4) we have again the KPII equation

$$\Psi_T+6\Psi\Psi_X+\Psi_{XXX}+3\phi_{YY}=0, \tag{9.3.8a}$$

$$\Psi=\phi_X. \tag{9.3.8b}$$

The initial condition (9.3.5) is transformed to

$$\Psi(X,Y,0)=a\,\mathrm{sech}^2(bX), \tag{9.3.9a}$$

$$\phi(X,Y,0)=\frac{a}{b}\tanh(bX), \tag{9.3.9b}$$

which depend only on X. Therefore we can drop Y-dependence of Ψ and ϕ, and the problem is reduced to solve the KdV equation

$$\Psi_T+6\Psi\Psi_X+\Psi_{XXX}=0, \tag{9.3.10}$$

for the initial value (9.3.9a). This problem is solved by the inverse scattering method. We summarize the results concerning with the generation of solitons as was done in 6.2 and 8.2. The number of generated solitons N, is the maximum integer which satisfies

$$\sqrt{A+\frac{1}{4}}+\frac{1}{2}-N>0, \tag{9.3.11}$$

where

$$A=\frac{a}{b^2}. \tag{9.3.12}$$

Width of the j-th soliton, b_j^{-1}, is determined by

$$b_j^{-1}=(\sqrt{A+\frac{1}{4}}+\frac{1}{2}-j)^{-1}b^{-1}\quad(j=1,\cdots,N). \tag{9.3.13}$$

For sufficiently large $|T|$, the asymptotic behavior of Ψ is given by

$$\begin{aligned}\Psi\ \xrightarrow[T\to\pm\infty]{}\ &\sum_{j=1}^{N}2b_j^2\mathrm{sech}^2(b_jX-4b_j^3T)+(\text{radiation}),\\ =&\sum_{j=1}^{N}2b_j^2\mathrm{sech}^2\{b_j(\tilde{X}+d\tilde{Y})-(3d^2b_j+4b_j^3)\tilde{T}\}+(\text{radiation}).\end{aligned} \tag{9.3.14}$$

We apply the above results to our problem.

For the transmitted wave (9.3.2), constants a, b, d and A are

$$a=\frac{4\nu}{1+\nu},\quad b=\nu,\quad d=\frac{k}{\nu},\quad A=\frac{4}{\nu(1+\nu)}. \tag{9.3.15}$$

Thus, from (9.3.13), b_j is given by

$$b_j=(\sqrt{\frac{4}{\nu(1+\nu)}+\frac{1}{4}}+\frac{1}{2}-j)\nu\quad(j=1,\cdots,N). \tag{9.3.16}$$

Since A is chosen to be any positive number, many solitons are generated when ν is small enough. At least one soliton is generated because A is always positive. Using (9.3.14) and transformation (9.2.2), we have the asymptotic form of U^T

$$U^T\ \xrightarrow[\tau\to+\infty]{}\ \sum_{j=1}^{N}2b_j^2\mathrm{sech}^2\Theta_j+(\text{radiation}), \tag{9.3.17a}$$

$$\begin{aligned}\Theta_j=&b_j(\eta+\frac{k}{\nu}\zeta)-(\frac{3k^2}{\nu^2}b_j+4b_j^3)\tau\\ =&\frac{\varepsilon b_j}{h}\{(1-\frac{\varepsilon^2k^2}{8\nu^2}-\frac{\varepsilon^2b_j^2}{6})x+\sqrt{\frac{2}{3\kappa}}\frac{k}{\nu}y-c^+(1+\frac{\varepsilon^2\kappa}{16})t\}.\end{aligned} \tag{9.3.17b}$$

For the reflected wave (9.3.3), constants a, b, d and A are

$$a = A = 2\frac{\nu - 1}{\nu + 1}, \quad b = -1, \quad d = -k, . \tag{9.3.18}$$

From the expression of A and (9.3.11), we find that if $\nu \leq 1$ no soliton is generated and that if $\nu > 1$ only one soliton is generated. In the latter case b_1 is

$$b_1 = -(\sqrt{2\frac{\nu - 1}{\nu + 1} + \frac{1}{4}} - \frac{1}{2}). \tag{9.3.19}$$

Taking account of (9.3.14) and transformation (9.2.6), we get the asymptotic behavior of U^R for $\nu > 1$,

$$U^R \xrightarrow[\tau \to -\infty]{} 2b_1^2 \mathrm{sech}^2 \Theta_1 + (\text{radiation}), \tag{9.3.20a}$$

$$\begin{aligned} \Theta_1 &= b_1(\bar{\xi} - k\zeta) - (3k^2 b_1 + 4b_1^3)\tau) \\ &= \frac{\varepsilon b_1}{h}\{(1 - \frac{\varepsilon^2 k^2}{8} - \frac{\varepsilon^2 b_1^2}{6})x - \sqrt{\frac{2}{3\kappa}}ky + c^-(1 + \frac{\varepsilon^2 \kappa}{16})t\}. \end{aligned} \tag{9.3.20b}$$

From (9.3.1a) and (9.2.6) we have the expressions of the incident soliton in terms of the original independent variables x, y and t;

$$U^I = 2\mathrm{sech}^2 \frac{\varepsilon}{h}\{(1 - \frac{\varepsilon^2 k^2}{8} - \frac{\varepsilon^2}{6})x + \sqrt{\frac{2}{3\kappa}}ky - c^-(1 + \frac{\varepsilon^2 \kappa}{16})t\}. \tag{9.3.21}$$

Profiles of incident, transmitted and reflected line solitons are shown in Fig.9.3(a,b).

It is observed that the refracted line soliton breaks up (Fig. 9.3b), while the reflected line soliton does not. It should be noted that reflection angle is not equal to the incident angle due to the nonlinearity. These are referred to as nonlinear refraction and reflection phenomena.

Next we study the transmission and reflection coefficient of the soliton. Let us consider relations;

$$U^I = u_\xi^I = -\frac{1}{\varepsilon c_-}u_t^I, \tag{9.3.22a}$$

$$U^I = u_{\bar{\xi}}^r = +\frac{1}{\varepsilon c_-}u_t^R, \tag{9.3.22b}$$

$$U^T = u_\eta^T = -\frac{1}{\varepsilon c_+}u_t^T. \tag{9.3.22c}$$

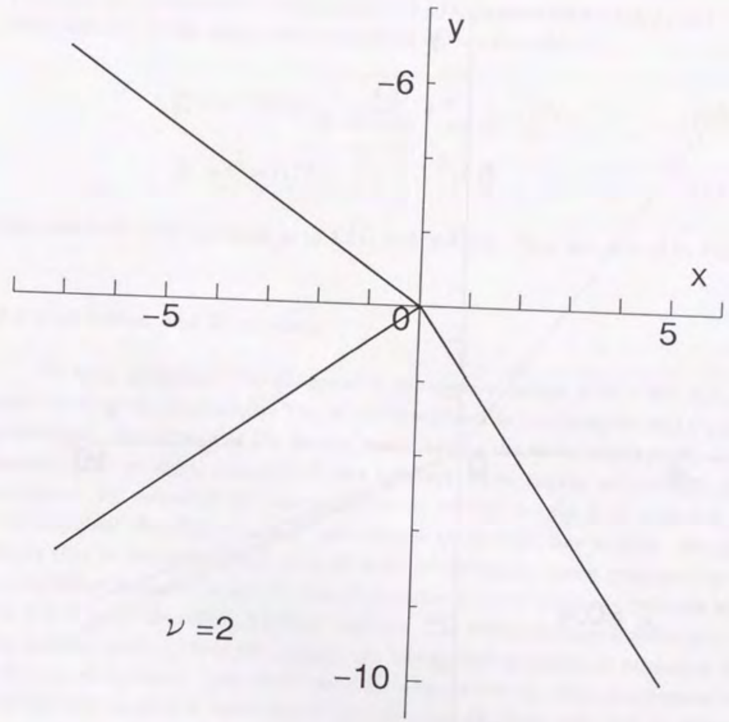

Figure 9.3a: Scatterings of a line soliton for $\nu = 2$, $t = 0$, $\varepsilon = 0.4$, $k = 1$ and $\kappa = 2/3$. The lines indicate the center lines of line solitons. For the transmitted and the reflected waves, the asymptotic forms (9.3.17) and (9.3.20) respectively, are used.

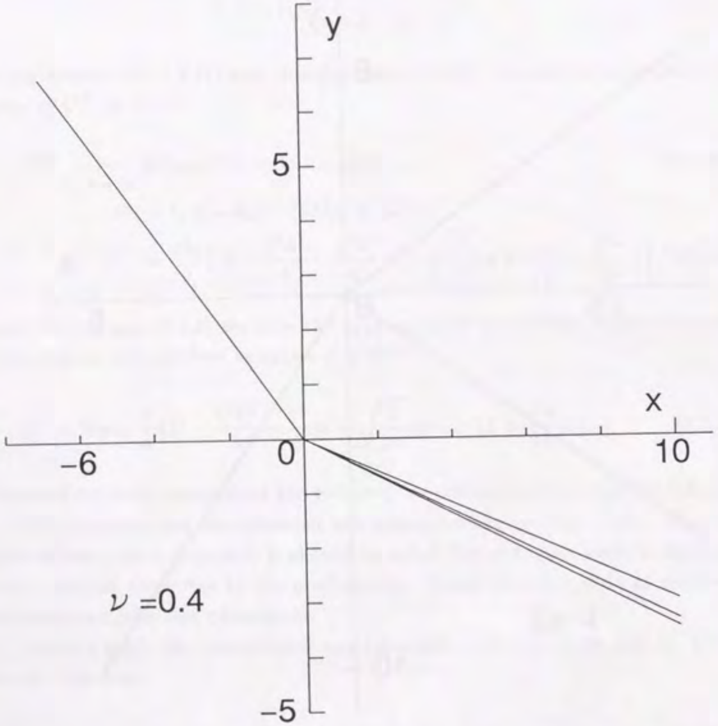

Figure 9.3b: Scatterings of a line soliton for $\nu = 0.4$, $t = 0$, $\varepsilon = 0.4$, $k = 1$ and $\kappa = 2/3$. The lines indicate the center lines of line solitons. No reflected soliton is observed and the transmitted wave fissions into three line solitons.

The relative amplitudes of the reflected and transmitted waves are defined as ratios of the velocity fields;

$$T = \left|\frac{u_t^T}{u_t^I}\right| = \left|\frac{c_+}{c_-}\frac{U^T}{U^I}\right| = \nu^{-1}\left|\frac{U^T}{U^I}\right|, \tag{9.3.23a}$$

$$R = \left|\frac{u_t^R}{u_t^I}\right| = \left|\frac{U^R}{U^I}\right|. \tag{9.3.23b}$$

Therefore the transmission coefficient of the j-th transmitted soliton T_j and reflection coefficient of the single reflected soliton R_1 are given by

$$T_j = \nu^{-1}b_j^2 = (\sqrt{\frac{4}{\nu(1+\nu)} + \frac{1}{4}} + \frac{1}{2} - j)^2\nu, \tag{9.3.24a}$$

$$R_1 = b_1^2 = (\sqrt{2\frac{\nu-1}{\nu+1} + \frac{1}{4}} - \frac{1}{2})^2. \tag{9.3.24b}$$

The results (9.3.24) are same as (6.4.5a) and (6.4.6a). They are plotted in Fig.6.3.

9.4 Conclusion and Discussion

We have introduced two-dimensional anharmonic lattice with a line discontinuity in the mass distribution. The lattice is assumed to be triangular and strongly anisotropic. Scatterings of the lattice waves due to the mass interface are investigated. The incident, transmitted and reflected waves satisfy independent KPII equations. By means of the inverse scattering method for the KdV equation, we have analyzed the refraction and reflection of an incident line soliton. We have shown that in the transmitted wave at least one soliton is always generated and if $\nu < 1$, soliton fission is observed. One(No) soliton is observed in the reflected wave if $\nu > 1$ ($\nu < 1$). the reflected soliton does not. The reflection angle is different from the incident angle. These phenomena are interpreted as nonlinear refraction and reflection phenomena. The whole analysis is applicable to other two-dimensional systems such as shallow water waves traveling over a bottom step and ion acoustic waves with a discontinuity of ion temperature. We expect that **nonlinear refraction and reflection phenomena** will be experimentally observed (Iizuka and Wadati 1993).

Concluding Remarks

In this thesis we have considered weakly nonlinear wave propagations in inhomogeneous systems. The effects of certain kinds of inhomogeneities on solitons have been investigated.

Part I is devoted to the slow inhomogeneity, namely the inhomogeneity of the system is represented by a slowly varying function of space coordinate(x). To model the wave propagations, inhomogeneous KdV equations (1.2~3) and inhomogeneous NLS equations (1.4~5) have been derived from anharmonic lattices(chapter 2). We have introduced the weak inhomogeneity in which the analytic form of soliton deformations are obtained. As an application the deformations of soliton in random systems have been investigated in chapter 3. It is found that statistical behaviors of the KdV, mKdV and NLS solitons are described by random walk of solitons. The model equations are extended to unstable(chapter 4) and two-dimensional(chapter 5) systems.

To the temporally inhomogeneous systems the same method can be applied and moreover if the inhomogeneity is represented by the function of a linear combination of space(x) and time(t) we can obtain the same model equations.

We have considered localized inhomogeneities in Part II and investigated fissions and reflections of the lattice solitons. Scatterings of KdV(chapter 6), NLS (chapter 6) and KP(chapter 9) solitons due to a discontinuity of the mass distribution have been studied. In chapter 7 and chapter 8, scatterings of the NLS soliton and KdV soliton respectively from a single impurity have been investigated. For each case the amplitudes of the generated solitons are calculated.

The main aim of the soliton scattering is to construct the transmitted and reflected waves from the incident wave. However, the method is applicable to the construction of the transmitted and incident waves from the reflected wave, and the construction of the reflected and incident waves from the transmitted wave.

Throughout this thesis, the anharmonic lattices are mainly considered. However, the method is applicable to a large class of nonlinear systems such as fluid, plasma, elastic wire and optical fiber. Our approach is applicable to the inverse problems; we expect the unknown inhomogeneities (e.g. bottom topography in fluid systems and mass distribution in anharmonic lattices) by observing soliton phenomena.

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