学 位 論 文

Self-Consistent Approaches to the Localization Transition

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Contents

List of papers submitted for the requirement of the degree

1. T. Kawarabayashi and M. Suzuki, *A Self-Consistent Approach to Localization Transition*, Physica A 190 (1992) 252.

2. T. Kawarabayashi and M. Suzuki, *Decay-Rate of the Green's Function in a Random Potential on the Bethe Lattice and a Criterion for Localization*, submitted to J. Phys. A: Math. Gen.

List of papers added in references

1. T. Kawarabayashi and M. Suzuki, *Absence of Vector Chiral Order in the Two-Dimensional Antiferromagnetic Heisenberg Model*, Phys. Lett. A 143 (1990) 17.

2. T. Kawarabayashi and M. Suzuki, *Super-Effective-Field Theory of Scalar Chiral Orders in the Two-Dimensional Heisenberg Model*, J. Phys. Soc. Jpn. 60 (1991) 825.

3. T. Kawarabayashi and M. Suzuki, *Chiral Orders in the Two-Dimensional Heisenberg Model*, J. Magnetism and Magnetic Materials 104-107 (1992) 929.

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Chapter I

Introduction

1.1 Introduction

Since the original work by Anderson [1] on the absence of diffusion in disordered media, many studies have been made on this subject [2-7]. It is probably fair to say, however, that the nature of the Anderson transition is not yet fully understood from the statistical mechanical point of view. For example, it is believed that the lower critical dimensionality is two, but the upper critical dimensionality is not yet known. Discrepancies among the values of the critical exponents in the three-dimensional system obtained by several different methods have not yet been resolved. One of the problems is how to define the Anderson transition. There are several criteria [2, 3] to distinguish localized states from extended states, but rigorous relationships between these criteria have not yet been established though they seem to be equivalent to each other.

Usually, extended states (metallic regions) are characterized by the non-zero dc-conductivity at zero temperature and localized states (insulating regions) are characterized by a vanishing dc-conductivity. In the localized states, electrons are localized within a region whose linear-size is given by the localization length ξ and the correlation of eigenfunctions of electrons decays exponentially in a large distance. The localization length ξ is defined by

$$\xi \equiv \lim_{\omega \to 0} \sqrt{\frac{D(\omega)}{-\mathrm{i}\omega}} \tag{1.1.1}$$

or by

$$\xi^{-1} \equiv -\lim_{L\to\infty} \frac{1}{L} \ln |G(0, L; E)|, \tag{1.1.2}$$

where $D(\omega)$ is the diffusion constant and $G(0, L; E)$ is the single particle Green's function of electrons. As is obvious from the definition (1.1.1), the quantity ξ^2 is a linear function of the dielectric constant ϵ, namely $\xi^2 \propto \epsilon - 1$. Thus the divergence of the localization length is directly related to the divergence of the dielectric constant. It can be considered that these two definitions (1.1.1) and (1.1.2) of the localization length are equivalent from the physical point of view. To the best of my knowledge, however, a rigorous relationship between these two definitions is not yet obtained.

The Anderson localization transition is the phase transition between the extended states and the localized states. It is considered that the conductivity at zero temperature should vanish continuously as the transition point is approached from the metallic region and also the localization length should diverge as approached from the insulating region. The critical energy E_c at which the localization length ξ diverges and the dc-conductivity becomes vanishing is called the mobility edge. Critical exponents for the divergence of the localization length and for the emergence of the conductivity at the mobility edge E_c are defined as

$$\xi(E) \approx (E - E_c)^{-\nu}, \quad \text{and} \quad \sigma(E) \approx (E_c - E)^s, \tag{1.1.3}$$

respectively. From the statistical mechanical point of view, the localization transition is characterized by these exponents. It is believed that there are three different universality classes for these exponents corresponding to the orthogonal, unitary and symplectic symmetries of the system. In the case in which the Hamiltonian is represented by a real and symmetric matrix, the system belongs to the orthogonal symmetry class. The system having no time reversal symmetry belongs to the unitary class. The system with an external magnetic field has no time reversal symmetry and thus it belongs to the unitary class. The system with spin-dependent scattering amplitudes invariant under time reversal belongs to the symplectic ensemble. Such a system is realized in the system with a spin-orbit interaction. It is considered that in the symplectic case the localization transition occurs even in the two-dimensional system.

Experimentally, it was reported that in the measurement for AlGaAs:Si samples the exponents were estimated as $s \approx 1$ and $2\nu = 1.9 \sim 3.7$ [8] whereas for Si:P samples they were estimated as $s = 0.51 \pm 0.05$ and $2\nu = 1.15 \pm 0.15$ [9].

1.2 Several Approaches to Localization (Review)

1) Scaling Theory

The scaling theory of the conductivity is proposed by Abrahams et al. in 1979 [10]. They considered the dimensionless conductance $g(L)$ of the finite system L^d and assumed that the conductance $g(L')$ in the system of the size $L' = \alpha L$ is given as a function of α and $g(L)$:

$$\frac{g(L')}{g(L)} = f(g(L), \alpha). \tag{1.2.1}$$

By taking the limit as $\alpha \to 1$, the well-known differential equation

$$\frac{\mathrm{d} \ln g(L)}{\mathrm{d} \ln L} = \beta(g) \tag{1.2.2}$$

is derived from the above scaling assumption. Analyzing the asymptotic behavior of the function $\beta(g)$ in the two limits as $g \to \infty$ and as $g \to 0$, it is predicted that the lower critical dimensionality is two; all the states are localized in one and two dimensions and the localization transition occurs only in higher than two dimensions. The temperature-dependence of the conductivity and the negative magnetoresistance have also been explained within the framework of this theory [11-14].

2) Field Theoretical Approaches

The study of localization using a field theoretical model known as the nonlinear σ model was discussed by several authors [15-23]. In this approach the nonlinear σ model is obtained as an effective lagrangian for a generating functional for the Green's functions of electrons. There are several ways to derive the nonlinear σ model [16-20]. However, the following approximations are involved. One is to neglect higher order fluctuations around the saddle point; only the second order fluctuations are taken into account. Another is that the longitudinal fluctuations of the field theoretical variables are neglected. Once we take the nonlinear σ model as an effective model for localization, we can apply the renormalization group method which has been established in the field theory. The $\varepsilon(\equiv d-2)$ expansion for the nonlinear σ model was carried out by Hikami [21-23] and Wegner [24] and the critical exponents for the conductivity and for the localization length were estimated. For example, in the three-dimensional system, with the computation up to the five-loop order for the orthogonal ensemble, the critical exponent ν for the localization length ξ is obtained as [23]

$$\xi(E) \sim (E - E_c)^{-\nu}, \quad \nu \approx 0.730. \tag{1.2.3}$$

In spite of the approximations involved in the derivation of the nonlinear σ model, this model found to be effective to explain the various aspect of the localization transition such as the absence of singularity in the density of states and the existence of three different universality classes. One of the shortcomings would be that it is difficult to get more accurate values of the critical exponents by this method because the ε expansion is an asymptotic expansion . Recently, a possible way to avoid this difficulty was studied by Hikami [23].

The nonlinear σ model on the Bethe lattice was investigated by Efetov [25, 26]. He calculated the asymptotic form of the density-density correlation function in the

limit of low frequencies and of large distances and obtained the critical exponents and mobility edges. In the case of the Bethe lattice, the exponent ν for the localization length is given by $\nu = 1$ and this value is common to the three different symmetries [26].

3) Numerical Finite Size Scaling Method

The numerical finite-size scaling method has been applied to the localization transition by using the quasi-one-dimensional systems [27-37]. In the quasi-one-dimensional system ($M \times M \times \infty$), it is shown that the inverse of the localization length λ_M^{-1} defined by

$$\lambda_M^{-1} \equiv -\lim_{L\to\infty} \frac{1}{L} \ln |G(0, L; E)| \tag{1.2.4}$$

is positive and self-averaging; it does not depend on the configuration of random potentials in the limit as $L \to \infty$. The decay-rate of the Green's function can be obtained by the transfer matrix method [27]. By considering a sufficiently long system, the value of λ_M^{-1} can be obtained accurately. In usual finite systems, however, it is difficult to estimate the localization length of the system since the size of the system is limited. In quasi-one-dimensional systems, the characteristic length of the system can be obtained rather easily and this is a main advantage to consider such systems. The critical exponent for the localization length and the mobility edges are obtained by assuming the scaling form

$$\frac{\lambda_M}{M} = f(\frac{\xi}{M}) \tag{1.2.5}$$

where ξ is the localization length in the limit as $M \to \infty$ [27]. The mobility edge trajectories for various distribution of site energies are also discussed by Bulka et al. [30, 31]. In the case of the three-dimensional system with the box distribution of site energies, the exponent ν is estimated as

$$\nu = 1.4 \pm 0.2 \quad (M \leq 13) \tag{1.2.6}$$

for the orthogonal ensemble [32]. In the three-dimensional system with a strong magnetic field, the exponent ν is obtained as $\nu = 1.35 \pm 0.15$ by Ohtsuki et al. [34]. The two-dimensional system with the symplectic symmetry is also investigated by several authors [35-37].

4) Self-Consistent or Mean-Field Approaches

Self-consistent or mean-field approaches to the localization transition have also been tried [38-41]. A self-consistent theory of Anderson localization was proposed by Vollhardt and Wölfle [39]. On the basis of the standard diagrammatic expansion for the conductivity, they derived a self-consistency integral equation for the diffusion constant $D(\omega)$. This theory yields a vanishing conductivity for $d \leq 2$, which is consistent with the scaling theory by Abrahams et al. [10]. In the case of $d = 3$ the exponent (ν) for the localization length and that (s) for the conductivity are given by $s = \nu = 1$. This self-consistent approximation can also be related to the scaling theory by Abrahams et al. [39].

Thompson and Matsubara [40] proposed another self-consistent approach to localization. Using decoupling approximations as those in the case of the coherent potential approximation [42, 43], they derived self-consistency equations for the vertex functions associated with the product of the two single particle Green's functions. The self-consistency equation by Vollhardt and Wölfle was recovered in the limit of weak disorder.

A mean-field approach in the real space was proposed by Sornette and Souillard [41]. It also reproduces the self-consistency equation for the diffusion constant obtained by Vollhardt and Wölfle.

5) Mathematical Approaches

Mathematical approaches to localization have been extensively performed especially in one-dimensional systems [3, 6, 44-48, 58]. These methods have also been

developed for higher dimensional systems [49-62]. One of the most important results from these approaches is that the lower bound for the exponent ν for the localization length is given by

$$\nu \geq 2/d \tag{1.2.7}$$

where d denotes the dimensionality of the system [56, 57]. The values of the exponent ν obtained by the ε expansion (1.2.3) and by the numerical finite-size scaling method (1.2.6) both satisfy the above condition.

1.3 Motivations and Organization of the Thesis

In the three-dimensional system, there are considerable discrepancies among the values of the exponents obtained by several different methods. Since the ε expansion is an asymptotic one, it is rather difficult to make an extrapolation to the true critical exponent. The result obtained by the finite-size scaling method appears quite reliable. However, this also suffers from the small-size effect of the systems. In these situations, it may be dangerous to rely on the results obtained by a single approach. The results obtained by one method, for example, by the finite-size scaling, should be re-examined by another method. If the results are confirmed by different methods, they would be more reliable.

The obvious order parameter for the Anderson transition is the dc-conductivity. It is, however, difficult to develop a statistical mechanical theory of this transition by taking the conductivity as an order parameter. On the other hand, from a similarity between the localization transition and the thermodynamic phase transition in magnetic systems, we may consider, as an obvious analogy, that the localization length in the localized states corresponds to the correlation length in the paramagnetic phase. The correlation length diverges as the critical temperature is approached from the

paramagnetic region and its divergence yields that of the magnetic susceptibility. Similarly, the localization length diverges as we approach the mobility edge from the localized region and its divergence yields that of the dielectric constant. Namely, the localized states (extended states) correspond to the temperature-region higher (lower) than the critical temperature T_c. One of the difficult problems in treating the conductivity as an order parameter is that it is not clear what quantity in the thermodynamic phase transition would correspond to the conductivity.

In the present thesis, instead of treating the conductivity, we discuss the localization length ξ and define the mobility edge by the condition that $\xi(E_c) = \infty$. Our aim here is to evaluate the critical exponents to determine the universality classes and the upper critical dimensionality. To discuss the upper critical dimensionality, we have to know how the localization transition behaves in high dimensions. We thus need the mean-field theory which becomes exact in the limit of high dimensionality. In classical spin systems, such a mean-field theory is well-known as the Weiss approximation.

On the basis of these considerations, we adopt self-consistent or mean-field approaches to the localization transition. We show that qualitatively good results for the mobility edge trajectories are obtained by a self-consistent approximation. Moreover, we obtain a criterion which is expected to hold better in higher dimensions by using exact results on the Bethe lattice. However, we have not yet succeeded in obtaining the critical exponents in the two or three-dimensional system.

The present thesis is organized as follows: In chapter 2, we propose a self-consistent approximation in analogy with mean-field approximations in classical spin systems. Mobility edge trajectories in the three-dimensional system with various distribution of site energies are obtained by this approximation. The results are discussed in comparison with those obtained by the finite-size scaling method.

In chapter 3, we show that in the tight-binding Anderson model on the Bethe lattice the exponential decay-rate of the Green's function can be obtained for arbitrary energies and arbitrary disorder. Analytic results for the Lorentzian distribution of site energies are presented. On the basis of these results, a criterion for localization on the corresponding real lattice is proposed. The results for the Lorentzian distribution of site energies are explicitly described. Relations to the previous works [72-74] are also discussed.

In chapter 4, the exponential decay of the eigenfunction on the Bethe lattice is discussed. The intuitive argument for the exponential decay of the eigenfunction near the boundary is proposed.

Chapter 5 is devoted to the summary and future problems.

Chapter II

A Self-Consistent Approximation for the Localization Transition in the Three-Dimensional System

2.1 Introduction

Recently the mean-field theory, which was not considered to be suitable to study non-classical critical behavior, turns out to be applicable also to the study of critical exponents by using the coherent-anomaly method (CAM) [63, 64]. The basic idea of the CAM is to extrapolate the true critical temperatures and critical exponents from some systematic series of generalized mean-field approximations [63]. This method has been successfully applied to classical spin systems and spin glasses [65, 66]. There are several kinds of series of approximations and the most elementary approximations are Weiss-like approximations [64]. Taking these recent developments in statistical mechanics into considerations, we have tried to apply these new concepts to the problem of the Anderson transition. The main problem here is how to construct a systematic series of approximations of the Anderson transition. In the next section we propose a possible example of such approximations in analogy with the Weiss-like approximations in classical spin systems. Our argument is based on the analogy between the localization length and the correlation length. This analogy has appeared also in the rigorous argument about the lower bound of the critical exponent of the localization length [54, 56, 57, 60].

In the present chapter we discuss the mobility edge trajectory, i. e. the phase diagram of the Anderson transition. Mobility edge trajectories in the three-dimensional systems for several kinds of distributions of the random potential have been discussed in detail by Bulka et al. [30, 31]. Their results show that for the box distribution and for the Gaussian one there exist extended states outside the unperturbed band and suggest that there are two different physical mechanisms of localization, namely quantum interferences and potential localization. Since the mean-field approximation is suitable to describe phase diagrams, it is expected that for this problem our self-consistent approximation works well.

2.2 Model and Approximation

We neglect the Coulomb interactions between electrons and consider only the one-particle problem in a random potential. We adopt here the tight-binding Anderson Hamiltonian,

$$H = -t \sum_{<i,j>} C_i^\dagger C_j + \sum_i V_i C_i^\dagger C_i \tag{2.2.1}$$

where $C_i^\dagger$ (C_i) denotes a creation (annihilation) operator of an electron on the site i and t denotes the nearest neighbor hopping amplitude. The site energies $\{V_i\}$ are distributed independently and the distribution function of the site energies is denoted by $P(V)$. Using the Green's function, we define the localization length $\xi(E)$ as follows:

$$\begin{aligned} \xi(E)^{-1} &\equiv \lim_{|x-y|\to\infty} -\frac{1}{2|x-y|} \ln\langle |G(x,y;E)|^2 \rangle_{av} \\ G(x,y;E) &\equiv \langle x|(E-H)^{-1}|y\rangle. \end{aligned} \tag{2.2.2}$$

Here the bracket $\langle\ \ \rangle_{av}$ denotes an average over the distribution of the site energies. The mobility edge E_c is determined by the condition that

$$\xi(E_c) = \infty \tag{2.2.3}$$

and the critical exponent ν is defined as follows:

$$\xi(E) \approx (E - E_c)^{-\nu} \quad \text{near} \quad E \sim E_c. \tag{2.2.4}$$

In eq.(2.2.2), we have adopted the geometrical average rather than the arithmetic average of the Green's function to define the localization length. In one-dimensional disordered chains, the inverse of the localization length defined by eq.(2.2.2) is nothing but the Lyapunov exponent which is known to be self-averaging [6].

Let us briefly review the Weiss-like approximations. As an example, we consider Ising spin systems. Thus we assume the original Hamiltonian

$$H_{\text{Ising}} = -J \sum_{<i,j>} S_i S_j, \qquad S_i = \pm 1, \tag{2.2.5}$$

where J denotes the nearest neighbor interaction. Following Suzuki et al. [64], we consider an isotropic cluster Ω and the following effective Hamiltonian

$$H_{\text{eff}} = -J(\sum_{<i,j>} S_i S_j + \Lambda \sum_{k \in \partial\Omega} z_k S_k), \quad i, j \in \Omega \quad . \tag{2.2.6}$$

Here Λ denotes the effective field and z_k denotes the number of outer bonds of the site k on the boundary of the cluster Ω. We require the following self-consistency condition [64]

$$\langle S_0 \rangle_{\text{eff}} = \Lambda \tag{2.2.7}$$

where the bracket denotes a thermal average for H_{eff} and S_0 denotes the spin in the center of the cluster. Expanding the left hand side of eq.(2.2.7) with respect to Λ, we get the equation [64]

$$\frac{J}{k_B T} \sum_{j \in \partial\Omega} z_j \langle S_0 S_j \rangle_\Omega = 1 \tag{2.2.8}$$

to determine the critical temperature. Here the bracket $\langle\ \ \rangle_\Omega$ denotes a thermal average with respect to the Hamiltonian in the cluster Ω. The left hand side of eq.(2.2.8) is a sum of the correlation functions between the center spin and the boundary spins. This correlation between the center and the boundaries is closely related to the concept of symmetry breaking because it represents the effect of boundary conditions on the center spin.

This kind of argument can be generalized to the case of spin glass systems and then a similar condition of the form

$$(\frac{J}{k_B T_c^{\text{sg}}})^2 \sum_{j \in \partial\Omega} z_j [\langle S_0 S_j \rangle_\Omega^2] = 1 \tag{2.2.9}$$

has been obtained [65, 66] for the spin glass critical temperature T_c^{sg} in the case of the $\pm J$ model. Here the square bracket denotes the average over the bond configurations. In analogy with eq.(2.2.9), we expect that the condition

$$t^2 \sum_{j \in \partial\Omega} z_j \langle |G_\Omega(i, j; E)|^2 \rangle_{av} = 1 \tag{2.2.10}$$

would lead to a good estimation of the mobility edge E_c. Here G_Ω denotes the Green's function in the cluster Ω and i denotes the site in the center of it. We are able to derive this condition from a self-consistent approximation for the quantity $\langle |G(x, y)|^2 \rangle_{av}$ as follows.

Let us consider an isotropic cluster Ω_x whose center is x and radius is R (Fig. 2.1). We assume that the distance $|x|$ from the origin 0 to x is large enough so that the radius R of the cluster is much smaller than the distance $|x|$. Here we introduce a restricted Hamiltonian $H(\partial\Omega_x)$ defined by

$$H(\partial\Omega_x) = -t \sum_{<i,j>} C_i^\dagger C_j + \text{h.c.}, \qquad i \in \partial\Omega_x, \quad j \notin \Omega_x, \quad |i - j| = 1. \tag{2.2.11}$$

This $H(\partial\Omega_x)$ is the Hamiltonian restricted to the boundary of the cluster. Here the following resolvent identity holds for the full Hamiltonian H in eq.(2.2.1) and the

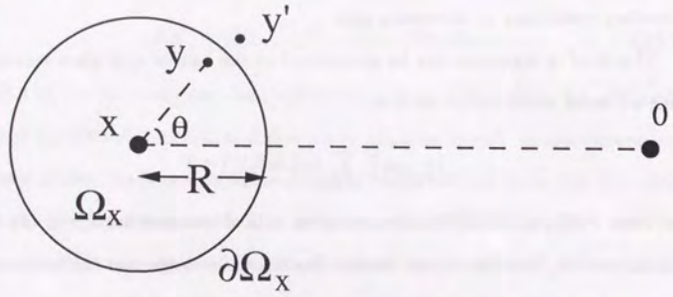

Fig. 2.1: The cluster Ω_x.

above boundary Hamiltonian $H(\partial\Omega_x)$,

$$\begin{aligned} \frac{1}{E-H} &= \frac{1}{E-H_{\partial\Omega_x}} + \frac{1}{E-H_{\partial\Omega_x}} \cdot H(\partial\Omega_x) \cdot \frac{1}{E-H} \qquad (2.2.12) \\ H_{\partial\Omega_x} &\equiv H - H(\partial\Omega_x). \end{aligned}$$

Note that the Hamiltonian $H_{\partial\Omega_x}$ is expressed as

$$H_{\partial\Omega_x} = H(\Omega_x) + H(\overline{\Omega}_x) \qquad (2.2.13)$$

where $H(\Omega_x)$ ($H(\overline{\Omega}_x)$) denotes the Hamiltonian inside (outside) the cluster Ω_x. From eq.(2.2.12), we then obtain the resolvent identity of the Green's function [54, 60]

$$\begin{aligned} G(x,0;E) &= G_{\partial\Omega_x}(x,0;E) - t\sum_{(y,y')} G_{\partial\Omega_x}(x,y;E)G(y',0;E) \qquad (2.2.14) \\ G_{\partial\Omega_x}(z,z';E) &\equiv \langle z|(E-H_{\partial\Omega_x})^{-1}|z'\rangle \end{aligned}$$

where $y \in \partial\Omega_x$, $y' \notin \Omega_x$ and $|y-y'|=1$. Notice that if the boundary Hamiltonian $H(\partial\Omega_x)$ separates z from z', then $G_{\partial\Omega_x}(z,z';E)=0$. Using this property, we obtain the identity that

$$G(x,0;E) = -t\sum_{(y,y')} G_{\partial\Omega_x}(x,y;E)G(y',0;E) \qquad (2.2.15)$$

where $y \in \partial\Omega_x$, $y' \notin \Omega_x$ and $|y-y'|=1$. With this identity, we make the following approximation such that

$$\begin{aligned} \langle|G(x,0;E)|^2\rangle_{av} &= \langle \sum_{(y,y')}\sum_{(z,z')} t^2 G^*_{\partial\Omega_x}(x,y;E)G_{\partial\Omega_x}(x,z;E)G^*(y',0;E)G(z',0;E)\rangle_{av} \\ &\approx t^2 \sum_{(y,y')}\sum_{(z,z')} \langle G^*_{\partial\Omega_x}(x,y;E)G_{\partial\Omega_x}(x,z;E)\rangle_{av}\langle G^*(y',0;E)G(z',0;E)\rangle_{av} \\ &\approx t^2 \sum_{(y,y')} \langle|G_{\partial\Omega_x}(x,y;E)|^2\rangle_{av}\langle|G(y',0;E)|^2\rangle_{av}. \qquad (2.2.16) \end{aligned}$$

In the above, we have replaced the average of the summations of the Green's functions with the summations of the averages of the Green's functions and, in the third line, we also neglect the off-diagonal terms. Under this approximation we get a self-consistency integral equation for the function $\langle|G(z,0)|^2\rangle_{av}$. Since the cluster is isotropic, the function $\langle|G_{\partial\Omega_x}(x,y)|^2\rangle_{av}$ does not depend on y but on $|x-y|=R$, i.e. $\langle|G_{\partial\Omega_x}(x,y;E)|^2\rangle_{av} = \langle|G_{\partial\Omega_x}(|x-y|=R;E)|^2\rangle_{av}$. We then get

$$\langle|G(x,0;E)|^2\rangle_{av} = t^2\langle|G_{\partial\Omega_x}(|x-y|=R;E)|^2\rangle_{av}\sum_{(y,y')}\langle|G(y',0;E)|^2\rangle_{av}, \qquad (2.2.17)$$

where $y \in \partial\Omega_x$, $y' \notin \Omega_x$ and $|y-y'|=1$.

We find that the condition (see Appendix 2.A)

$$t^2\langle|G_{\partial\Omega_x}(|x-y|=R;E)|^2\rangle_{av}\sum_{(y,y')} 1 \le 1 \qquad (2.2.18)$$

has to be satisfied in order that there may exist a solution of eq.(2.2.17) which decays exponentially in large distance, i. e.

$$\langle|G(x,0;E)|^2\rangle_{av} \propto \exp[-\frac{2|x|}{\xi(E)}] \quad \text{as} \quad |x| \to \infty. \qquad (2.2.19)$$

Thus the critical condition which determines the mobility edge E_c is given by

$$t^2\langle|G_{\partial\Omega_x}(|x-y|=R;E_c)|^2\rangle_{av}\sum_{(y,y')}1=1 \tag{2.2.20}$$

in this approximation and the exponent ν becomes $1/2$ (see Appendix 2.A). This value of ν is considered to be that of the mean-field limit [6, 75]. We can easily find that eq.(2.2.20) is equivalent to eq.(2.2.10) if we consider an isotropic cluster.

It should be noted here that the asymptotic form of the left hand side of eq.(2.2.20) should be

$$\sim\exp[-\frac{2R}{\xi(E)}],\quad R\to\infty \tag{2.2.21}$$

in the localized region. We therefore expect that in the localized region the mobility edge E_c in this approximation will vanish if we take the cluster large enough. We further assume the scaling form of the Green's function

$$\langle|G(0,x;E)|^2\rangle_{av}\propto\frac{1}{|x|^{d-2+\eta}}\exp[-\frac{2|x|}{\xi(E)}] \tag{2.2.22}$$

with η satisfying the inequality $0<\eta<1$, as in the case of spin-glass correlation functions [65, 66]. Then we can easily find that the mobility edge E_c in this approximation approaches the correct value as the cluster-size is increased.

2.3 Numerical Calculations

Using eq.(2.2.20) we have estimated the mobility edges for various values of the width of the distribution functions of the site energies and obtained the mobility edge trajectories in the three-dimensional cubic lattice for the box, the Gaussian and the Lorentzian distributions. The method we have used is the exact diagonalization of the Hamiltonian for finite clusters (see Appendix 2.B). The isotropic cluster Ω_x (radius R) we have used is defined by the set of points inside or on the sphere of radius R whose center is x. The averages of the Green's functions were taken over 10^3 samples for the cluster $R=5$ and 3×10^3 samples for $R=2$.

The results are shown in Fig. 2.2, Fig. 2.3 and Fig. 2.4 for the box, the Gaussian and the Lorentzian distributions, respectively. In order to compare our results with the results by Bulka et al. [31] which are obtained by using the finite-size scaling method, their results [31] (dashed lines) are also shown in Figs. 2.2, 2.3 and 2.4. The box, the Gaussian and the Lorentzian distribution functions we have used in the present chapter are given by

$$\begin{aligned}P_B(V) &= \frac{1}{2W_B}\theta(W_B-|V|)\\ P_G(V) &= \frac{1}{(2\pi W_G^2)^{1/2}}\exp(-V^2/2W_G^2)\\ P_L(V) &= \frac{W_L}{\pi(V^2+W_L^2)},\end{aligned} \tag{2.3.1}$$

respectively.

For the box and the Gaussian distributions, the result shows that there is a region in which extended states are enhanced as the width of the distribution of the random potential is increased. On the other hand, no such region exists for the Lorentzian distribution. The critical values of the parameters W_L, W_G and W_B at $E=0$ are estimated as $W_L^c/t\sim3.9(R=2), W_G^c/t\sim6.5(R=2)$ and $W_B^c/t\sim9.4(R=5)$,

respectively. These values obtained even in each single approximation are not far from the results of previous works [31]. However, it should be noted that we have to investigate the R-dependence of these values in several approximations in order to estimate the true critical values in the infinite system.

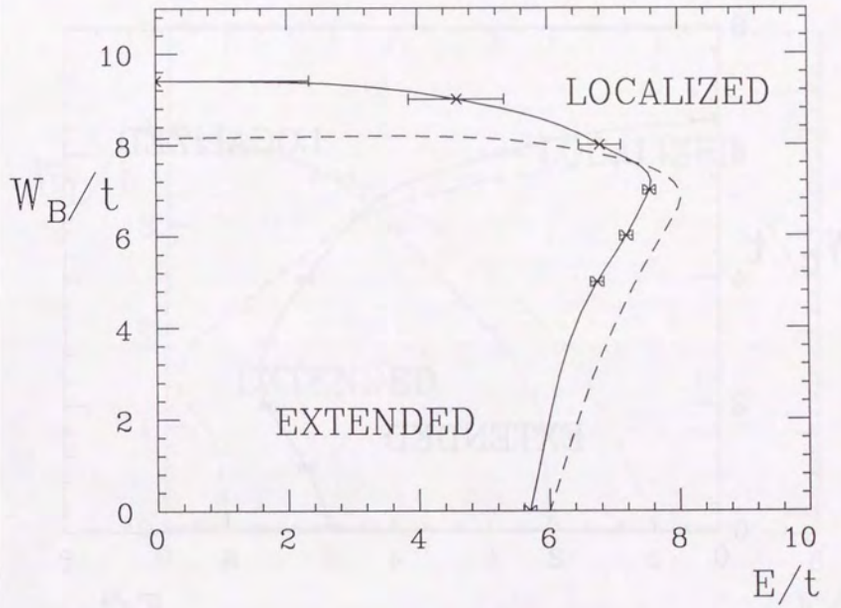

Fig. 2.2: Mobility edge trajectory for the box distribution. The radius of the cluster is $R = 5$ and the number of samples is 10^3. The dashed line represents the result by Bulka et al.

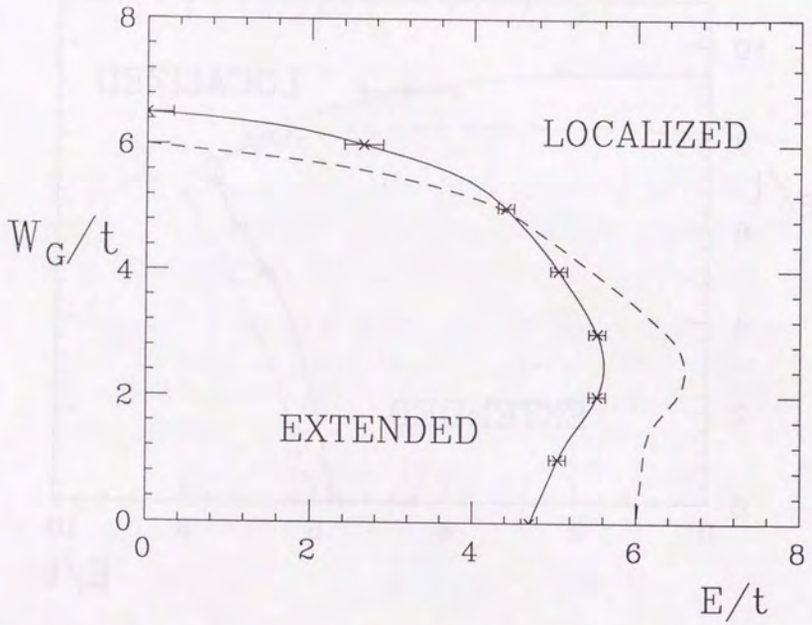

Fig. 2.3: Mobility edge trajectory for the Gaussian distribution. The radius of the cluster is $R = 2$ and the number of samples is 3×10^3. The dashed line represents the result by Bulka et al.

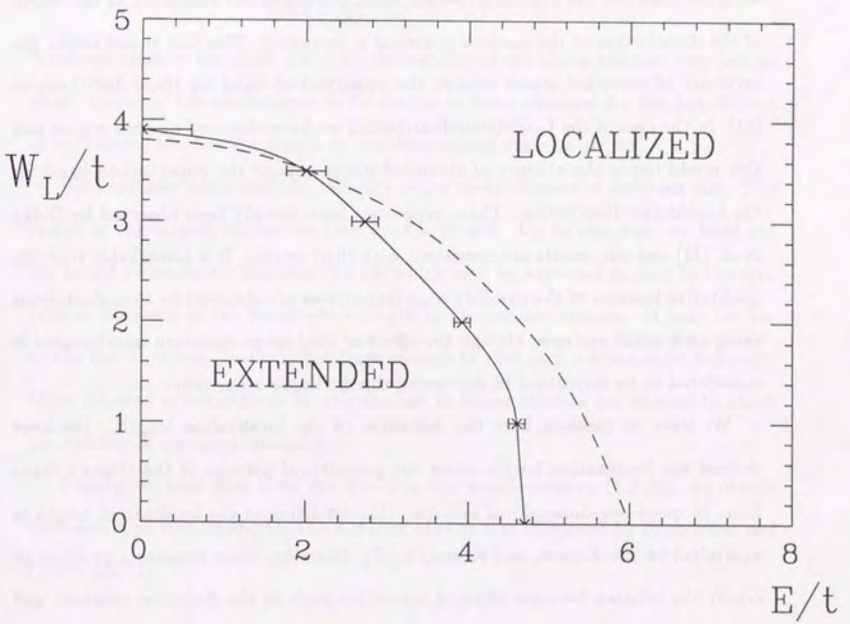

Fig. 2.4: Mobility edge trajectory for the Lorentzian distribution. The radius of the cluster is $R = 2$ and the number of samples is 3×10^3. The dashed line represents the result by Bulka et al.

2.4 Discussions

We have obtained mobility edge trajectories in the three-dimensional cubic lattice for the box, the Gaussian and the Lorentzian distributions. Although our clusters are rather small, our results are qualitatively in good agreement with the results obtained by Bulka et al. [31]. In the case of the Gaussian and the box distributions, we have observed the regions in which extended states are enhanced as the width of the distribution of the random potential is increased. This fact would imply the existence of extended states outside the unperturbed band for these distributions [31]. In the case of the Lorentzian distribution we have observed no such region and this would imply the absence of extended states outside the unperturbed band for the Lorentzian distribution. These properties have already been observed by Bulka et al. [31] and our results are consistent with their results. It is remarkable that the qualitative features of the mobility edge trajectories are obtained by the calculations using such small systems, though the effect of long-range quantum interferences is considered to be important in discussing the Anderson transition.

We have to mention here the definition of the localization length. We have defined the localization length using the geometrical average of the Green's function. In quasi-one-dimensional systems, this definition of the localization length is equivalent to MacKinnon and Kramer's [27]. However, there remains a problem to clarify the relation between physical quantities such as the dielectric constant and the localization length defined by eq.(2.2.2).

We have checked numerically that the distribution function of the logarithm of the Green's function in a finite cluster is close to the Gaussian distribution (see Figs. 2.5, 2.6 and 2.7). These results would imply that the distribution of sample means over a sufficiently large number of samples becomes Gaussian and its variance

vanishes as the number of samples goes to infinity, though in the case of the box and the Gaussian distributions, a small difference can be seen between the histogram and the Gaussian with the first and the second moments determined from the numerical data. Apart from a constant C_0, the quantity shown in Figs. 2.5, 2,6 and 2.7 can be related to the inverse localization length by

$$\ln\left(\frac{1}{|\partial\Omega_x|}\sum_{i\in\partial\Omega_x}|G(x,i;E)|^2\right)\sim R\xi^{-1}. \tag{2.4.1}$$

Since our clusters are small ($R\leq 5$), the validity of the above relation may not be clear. However, the results seem to be similar to those obtained for the distribution of the inverse localization length in one-dimensional systems [67, 68].

We have also estimated the mobility edges using clusters of different size. The radius of the largest cluster we have used is $R=5$. Up to this size, we have not yet found a systematic behavior [63, 64] which may be expected to lead to the true critical behavior of the localization length in the infinite system. It may be due to the fact that our cluster is not large enough to find such a systematic behavior. More detailed investigations by calculations in larger clusters are needed to check the validity of our approximation.

Finally, we note that if we set $R=0$ in our approximation (2.2.20), we obtain the same type of criterion for the mobility edge as was discussed by Economou and Cohen [69-71].

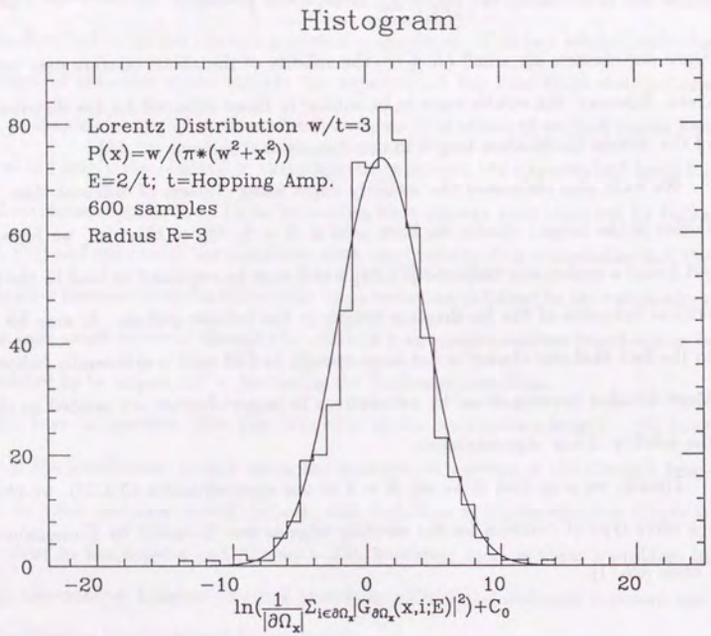

Fig. 2.5: The distribution of the logarithm of the Green's function for the Lorentzian distribution. The solid line is Gaussian with first and second moments determined from the numerical data. Here $R = 3$ and $C_0 = \ln[1.5\pi(2R+1)^2]$.

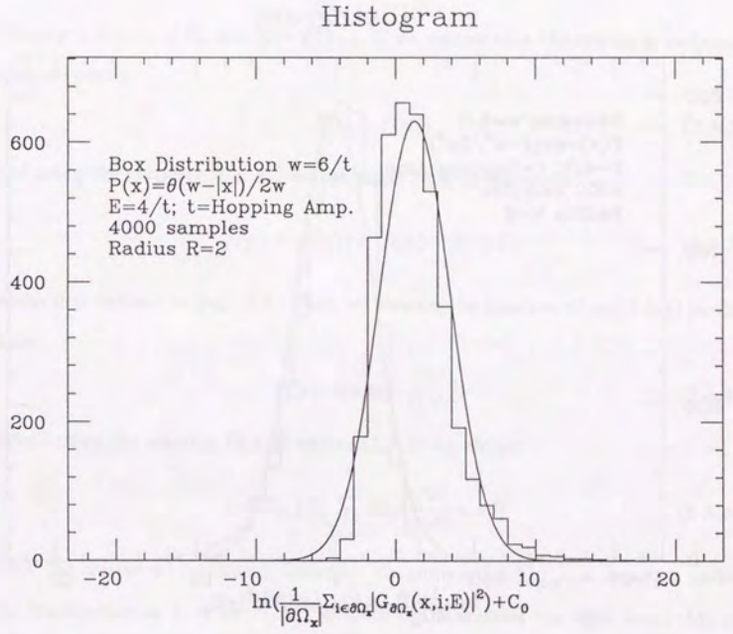

Fig. 2.6: The distribution of the logarithm of the Green's function for the box distribution. The solid line is Gaussian with first and second moments determined from the numerical data. Here $R = 2$ and $C_0 = \ln[1.5\pi(2R+1)^2]$.

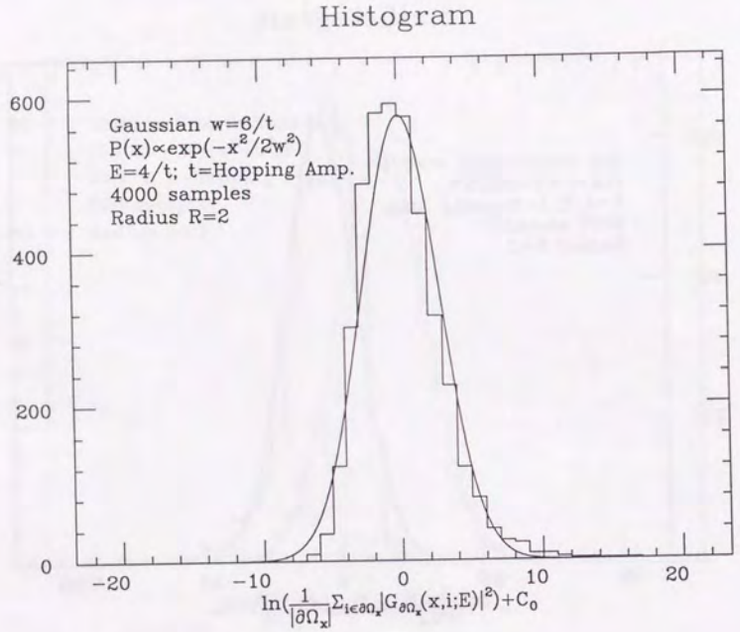

Fig. 2.7: The distribution of the logarithm of the Green's function for the Gaussian distribution. The solid line is Gaussian with first and second moments determined from the numerical data. Here $R = 2$ and $C_0 = \ln[1.5\pi(2R+1)^2]$.

Appendix 2.A : Derivation of (2.2.18)

If we write $f(x) \equiv \langle |G(x,0;E)|^2 \rangle_{av}$, eq.(2.2.17) can be written in the form,

$$\begin{aligned} f(x) &= C_{\Omega_x}(E) \sum_{(y,y')} f(y') \\ C_{\Omega_x}(E) &\equiv t^2 \langle |G_{\partial\Omega_x}(|x-y| = R; E)|^2 \rangle_{av}, \end{aligned} \tag{2.A.1}$$

where $y \in \partial\Omega_x$, $y' \notin \Omega_x$ and $|y - y'| = 1$. If we assume that the system is isotropic, then we obtain

$$f(x) = f(|x|) \tag{2.A.2}$$

and using the property $R \ll |x|$ we can make an approximation

$$f(y') = f(|y'|) \approx f(|x| - R\cos\theta) \tag{2.A.3}$$

where θ is defined in Fig. 2.1. Then we assume the solution of eq.(2.A.1) in the form

$$f(|x|) \propto \exp[-\frac{2|x|}{\xi(E)}]. \tag{2.A.4}$$

Substituting the solution (2.A.4) into eq.(2.A.1) we obtain

$$1 = C_{\Omega_x}(E) \sum_{(y,y')} \exp[\frac{2R}{\xi(E)}\cos\theta]. \tag{2.A.5}$$

Since the cluster we consider is isotropic, the summation $\sum_{(y,y')}$ is invariant under the transformation $\theta \to (\pi - \theta)$. Thus we can transform the right hand side of eq.(2.A.5) as follows:

$$\begin{aligned} 1 &= C_{\Omega_x}(E)\frac{1}{2}\sum_{(y,y')}(\exp[\frac{2R}{\xi(E)}\cos\theta] + \exp[\frac{2R}{\xi(E)}\cos(\pi-\theta)]) \\ &= C_{\Omega_x}(E)\sum_{(y,y')}\cosh(\frac{2R}{\xi(E)}\cos\theta) \\ &= C_{\Omega_x}(E)\{\sum_{(y,y')} 1 + \sum_{n=1}^{\infty}\sum_{(y,y')}\frac{1}{(2n)!}(\frac{2R}{\xi(E)}\cos\theta)^{2n}\}. \end{aligned} \tag{2.A.6}$$

The second term of the last line in eq.(2.A.6) is positive. Therefore we can easily find that the condition

$$C_{\Omega_x}(E) \sum_{(y,y')} 1 \leq 1 \tag{2.A.7}$$

has to be satisfied in order that the localization length ξ may have a real solution. Near the mobility edge (i. e. $R/\xi \ll 1$) we have

$$\xi(E) \propto \frac{R}{(1 - C_{\Omega_x}(E) \sum_{(y,y')} 1)^{1/2}} \tag{2.A.8}$$

and therefore we arrive at the exponent $\nu = 1/2$.

Appendix 2.B : Calculation of the Green's function

Using the eigenvalues and eigenfunctions of the Hamiltonian inside the cluster Ω_x, the Green's function $G_{\partial\Omega_x}$ is expressed as

$$\begin{aligned} G_{\partial\Omega_x}(z, z'; E) &= \sum_n \frac{\psi_n(z)\psi_n^*(z')}{E - E_n} \\ H(\Omega_x)\psi_n &= E_n\psi_n. \end{aligned} \tag{2.B.1}$$

These eigenvalues $\{E_n\}$ and eigenfunctions $\{\psi_n\}$ are obtained numerically by diagonalizing the Hamiltonian $H(\Omega_x)$. As our clusters are rather small, they are not completely isotropic. We thus adopt the following approximation

$$\begin{aligned} &\langle |G_{\partial\Omega_x}(|x-y| = R; E)|^2 \rangle_{av} \sum_{(y,y')} 1 \\ &= \langle |G_{\partial\Omega_x}(|x-y| = R; E)|^2 \sum_{(y,y')} 1 \rangle_{av} \\ &\approx \langle \sum_{(y,y')} |G_{\partial\Omega_x}(x, y; E)|^2 \rangle_{av} \\ &\approx \langle \frac{1}{|\partial\Omega_x|} \sum_{z\in\partial\Omega_x} |G_{\partial\Omega_x}(x, z; E)|^2 \rangle_{av} \sum_{(y,y')} 1 \quad , \end{aligned} \tag{2.B.2}$$

where $y \in \partial\Omega_x$, $y' \notin \Omega_x$ and $|y - y'| = 1$, to get the Green's function from the center to the boundary in the left hand side of eq.(2.2.20). Here $|\partial\Omega_x|$ denotes the number

of the sites on the boundary of the cluster Ω_x. Using this approximation we have calculated the left hand side of eq.(2.2.20) and obtained the results shown in Figs. 2.2, 2.3 and 2.4.

We note that we can also make another approximation of the form

$$\langle |G_{\partial\Omega_x}(|x-y| = R; E)|^2 \rangle_{av} \approx \frac{1}{|\partial\Omega_x|} \sum_{z\in\partial\Omega_x} \langle |G_{\partial\Omega_x}(x, z; E)|^2 \rangle_{av} \tag{2.B.3}$$

in calculating of the left hand side of eq.(2.2.20). In this case, it seems that the small-size effect appears rather seriously compared with the approximation (2.B.2), especially for small clusters. We have thus adopted the approximation (2.B.2) in the present paper. Which approximation should be used remains to be seen in further calculations.

Chapter III

Deacy-Rate of the Green's Function in a Random Potential on the Bethe Lattice and a Criterion for Localization

3.1 Introduction

Besides statistical mechanical approaches, mathematical approaches to the Anderson localization have also been developed in recent years [3, 6, 44-62]. In particular, the one-dimensional case has been studied extensively by this approach [3, 6, 44-48, 51, 58, 62]. Such mathematical methods have also been applied to higher-dimensional systems. The exponential decay of the Green's function for sufficiently large disorder or low energies in multi-dimensional systems was proved first by Fröhlich and Spencer[49]. Their work was followed by Martinelli and Scoppola [50] who proved the absence of an absolutely continuous spectrum in the same range of parameters. The condition for the proof of localization was somewhat relaxed by several authors [51-53]. A new proof for these results was also given by von Dreifus and Spencer[54, 55]. The rigorous lower bound of the critical exponent for the localization length has been obtained by these mathematical approaches [56, 57].

Their crucial assumption for the proof of localization is the exponential decay of the Green's function [51, 58-62]. This exponential decay of the Green's function has been proved in one dimension for arbitrary energies and arbitrarily small disorder using the positivity of the Lyapunov exponent, but in higher dimensions it has been

proved only for sufficiently large disorder or low energies.

The localization transition on the Bethe lattice (the Cayley tree) has also been investigated by several authors [25, 26, 72-76]. The stability of the localized states on the Bethe lattice was discussed first by Abou-Chacra et al. [72, 73]. Some rigorous results for the decay of eigenfunctions are obtained by Kunz and Souillard [74]. It is interesting that the above two groups [72-74] obtained the same criterion for the mobility edge, though their approaches seem to be quite different. The Anderson localization on the Bethe lattice was also investigated by Efetov [25, 26] using the field-theoretical model, i.e. the nonlinear σ model.

The Bethe lattice has no closed path and the sites on its boundary are as numerous as those inside the system. In spite of these specific properties, it can be regarded as an approximate lattice for real lattices. Our main motive for considering such a system is to develop an approximate theory for a real lattice on the level of a mean-field theory by using exact results on the Bethe lattice [72]. In the present chapter, we show that the exponential decay-rate of the Green's function on the Bethe lattice can be obtained by solving a self-consistency nonlinear integral equation for the distribution function of the Green's function. The self-consistency equation is obtained from a recursion formula for the Green's function. Analytic results in the case of the Lorentzian distribution of site energies are explicitly obtained. As far as we know, our result on the Bethe lattice is the first exact estimation of the exponential decay-rate of the Green's function for arbitrary energies and arbitrary disorder, except in one dimension. On the basis of these results we propose a criterion for the localized states, which is exact in the one-dimensional system. The mobility edge trajectory obtained by our criterion in the case of the Lorentzian distribution of site energies seems to be qualitatively in good agreement with the results obtained in the corresponding hypercubic lattice. Relations to previous works

by Abou-Chacra et al. [72, 73] and by Kunz and Souillard [74] are also discussed.

The present chapter is organized as follows: In section 3.2, we introduce a Green's function restricted to one of the branches of the Bethe lattice and derive the self-consistency equation for its distribution function. In section 3.3, the relation between the decay-rate of the Green's function and the distribution function of the restricted Green's function defined in section 3.2 is explained. Analytic results in the case of the Lorentzian distribution of site energies are explicitly presented in section 3.4. A criterion for localization is described with explicit results for the Lorentzian distribution in section 3.5. In section 3.6, the present criterion is discussed in comparison with that of the previous works.

3.2 Recursion Relation for the Green's Function

We consider noninteracting electrons in a random potential on a Bethe lattice with the connectivity K. A Bethe lattice with $K = 2$ is shown in Fig. 3.1. On the Bethe lattice, we can define a set of sites $\{j\}$ whose distance from the origin 0 is n (i.e. $|j| \equiv |j - 0| = n$). We call this set of sites the nth shell and it is denoted by S_n in the following. The nth shell consists of $(K+1)K^{n-1}$ sites. The boundary shell is denoted by S_B, where B is the distance between the origin and the boundary and we adopt the free boundary condition. The lattice constant is taken to be one.

We adopt the tight-binding Anderson model defined by the Hamiltonian:

$$H = -t \sum_{<i,j>} C_i^\dagger C_j + \sum_i V_i C_i^\dagger C_i, \tag{3.2.1}$$

where $C_i^\dagger(C_i)$ is a creation (annihilation) operator of an electron at the site i, t denotes a hopping amplitude between nearest neighbors and site energies are denoted by $\{V_i\}$. The site energies $\{V_i\}$ are distributed independently and its distribution function is denoted by $P(V)$.

The Green's function is defined by

$$G(z, z'; E) \equiv \langle z|(E - H)^{-1}|z'\rangle, \tag{3.2.2}$$

where z and z' denote sites on the Bethe lattice.

Here we introduce an operator $\Gamma_{i,j}$ defined by

$$\Gamma_{i,j} = -t(C_i^\dagger C_j + \text{h.c.}), \tag{3.2.3}$$

for the sites i and j with $|i - j| = 1$ and consider the Hamiltonian $H_{\Gamma_{i,j}}$ defined by

$$H_{\Gamma_{i,j}} \equiv H - \Gamma_{i,j}. \tag{3.2.4}$$

Since there is no hopping amplitude between the site i and the site j in the Hamiltonian $H_{\Gamma_{i,j}}$, the system described by $H_{\Gamma_{i,j}}$ is divided into two parts which do not

interact with each other. The Bethe lattice is accordingly divided into the corresponding two parts. The part including the site i and the other including the site j are denoted by Ω_i and Ω_j, respectively (see Fig. 3.2).

We then define a Green's function $G_{\Gamma_{i,j}}(z,z')$ as

$$G_{\Gamma_{i,j}}(z,z';E) \equiv \langle z|(E-H_{\Gamma_{i,j}})^{-1}|z'\rangle. \tag{3.2.5}$$

It is evident from the definition of the Hamiltonian $H_{\Gamma_{i,j}}$ that $G_{\Gamma_{i,j}}(z,z';E)=0$ if $z\in\Omega_i$ and $z'\in\Omega_j$.

Let us consider a site $i\in S_n$ and its nearest neighbors $\{i-1,j(1),\ldots,j(K): i-1\in S_{n-1}, j(1),\ldots,j(K)\in S_{n+1}\}$ as shown in Fig. 3.3. Using the resolvent identity, we obtain the following recursion relation for the Green's function (see Appendix 3.A):

$$G_{\Gamma_{i-1,i}}(i,i;E)^{-1} = E-V_i-t^2\sum_{l=1}^{K} G_{\Gamma_{i,j(l)}}(j(l),j(l);E). \tag{3.2.6}$$

We may call $G_{\Gamma_{i-1,i}}(i,i;E)$ the restricted Green's function since it depends only on the freedom restricted in the one of the two parts separated by the bond $(i,i-1)$ between the site i and the site $(i-1)$. It should be noted that the restricted Green's functions $\{G_{\Gamma_{i,j(l)}}(j(l),j(l);E)\}$ in the right hand side of eq.(3.2.6) are completely independent random variables since the corresponding portions of the Bethe lattice do not share any site. It is obvious that for any site $i\in S_m$ there exits an unique site $j\in S_{m-1}$ satisfying the condition that $|i-j|=1$ and in this case the restricted Green's function $G_{\Gamma_{i,j}}(i,i;E)$ thus actually depends only on the site i. This allows us to abbreviate the inverse of the restricted Green's function to

$$y_i \equiv G_{\Gamma_{i,j}}(i,i;E)^{-1} \tag{3.2.7}$$

for the sites i and j with $|i|>|j|$ and $|i-j|=1$. The variable y_i is defined for every

site except $i=0$. We can write eq.(3.2.6) in terms of y_i as

$$y_i = E-V_i-t^2\sum_{l=1}^{K} y_{j(l)}^{-1}. \tag{3.2.8}$$

It should be remarked here that the quantity y_i satisfying the recursion relation (3.2.8) is also related to the decay (or growth) rate of eigenfunctions. The eigenfunction $\psi_E(i)$ is defined as the solution of the Schrödinger equation $H\psi_E=E\psi_E$. We can then easily find that the quantity z_i defined by

$$z_i \equiv -t\psi_E(j)/\psi_E(i), \quad \text{for} \quad \{i,j:|i|>|j|,|i-j|=1\}, \tag{3.2.9}$$

also satisfies the same recursion relation as eq.(3.2.8):

$$z_i = E-V_i-t^2\sum_{l=1}^{K} z_{j(l)}^{-1}. \tag{3.2.10}$$

Note that in the definition (3.2.9) the site j is determined uniquely for fixed i.

Now let us consider the distribution function of the variables $\{y_i\}$, i.e. the inverse of the restricted Green's functions. Here we take E to be real so that the variables $\{y_i\}$ which are determined recursively from the boundary by using the recursion relation (3.2.8) are also real. If we take E to be complex, we have to consider the distribution functions of both the real part and the imaginary part of the Green's function and the argument will be more complicated [72]. If we take the free boundary condition, we have

$$y_m = E-V_m \tag{3.2.11}$$

for any site m in the boundary shell ($m\in S_B$). The distribution function $Q^{(m)}(y_m)$ of y_m is then given by

$$Q^{(m)}(y_m) = P(E-y_m). \tag{3.2.12}$$

We thus obtain that the distribution functions of the variables $\{y_m\}$ in the boundary shell ($y_m\in S_B$) are uniform and are equivalent to the distribution function of site

energies. It is easy to see that if the distributions are uniform in the mth shell, the distributions in the $(m-1)$th shell are then also uniform. The distributions are thus uniform in each shell in the case of the free boundary condition. The distribution at the nth shell is denoted by Q_n in the following. We therefore obtain from eq.(3.2.8) the following nonlinear integral equation for the distribution function $Q_n(y_i)$ for $n \neq 0$:

$$\begin{aligned} Q_n(y_i) &= \int P(V_i)\mathrm{d}V_i \prod_{l=1}^{K} Q_{n+1}(y_{j(l)})\mathrm{d}y_{j(l)}\delta(y_i - E + V_i + t^2\sum_{l=1}^{K} y_{j(l)}^{-1}) \\ &= \int \prod_{l=1}^{K} Q_{n+1}(y_{j(l)})\mathrm{d}y_{j(l)} P(E - y_i - t^2\sum_{l=1}^{K} y_{j(l)}^{-1}), \qquad (3.2.13) \\ & i \in S_n, \quad j(1), \ldots, j(K) \in S_{n+1}. \end{aligned}$$

Next we consider the steady solution $Q(y)$ of eq.(3.2.13). Taking $Q_n = Q_{n+1} \equiv Q$ in eq.(3.2.13), we obtain the following self-consistency nonlinear integral equation

$$Q(y_i) = \int \prod_{l=1}^{K} Q(y_{j(l)})\mathrm{d}y_{j(l)} P(E - y_i - t^2\sum_{l=1}^{K} y_{j(l)}^{-1}) \qquad (3.2.14)$$

for the distribution function $Q(y)$. In different contexts, the nonlinear integral equation of this type has also appeared in the previous papers [72, 74]. The steady solution $Q(y)$ can be obtained analytically when the distribution function of the random potentials $\{V_i\}$ is of the Lorentzian type. Using this solution $Q(y)$, we discuss the exponential decay-rate of the Green's function in the next section.

We have discussed the Green's function $G_{\Gamma_{i-1,i}}(i,i;E)$ which is restricted to one of the branches of the Bethe lattice. In the same way, we can also obtain the distribution Q_0 of the Green's function $G(0,0;E)^{-1}$, where 0 denotes the origin of the Bethe lattice. Since the Green's function $G(0,0;E)$ is expressed in terms of the restricted ones $\{G_{\Gamma_{0,j(l)}}\}$ as

$$G(0,0;E)^{-1} = E - V_0 - t^2\sum_{l=1}^{Z} G_{\Gamma_{0,j(l)}}(j(l),j(l);E), \quad j(1), \ldots, j(Z) \in S_1, \quad (3.2.15)$$

with the coordination number $Z \equiv K+1$, we obtain the following equation

$$Q_0(Y_0) = \int \prod_{l=1}^{Z} Q_1(y_{j(l)})\mathrm{d}y_{j(l)} P(E - Y_0 - t^2\sum_{l=1}^{Z} y_{j(l)}^{-1}) \quad ; \quad Y_0 \equiv G(0,0;E)^{-1} \quad (3.2.16)$$

for the distribution function $Q_0(Y_0)$ at the origin 0. If the distance from the boundary to the origin is sufficiently large, we may replace Q_1 in eq.(3.2.16) by Q. We then finally arrive at

$$Q_0(Y_0) = \int \prod_{l=1}^{Z} Q(y_{j(l)})\mathrm{d}y_{j(l)} P(E - Y_0 - t^2\sum_{l=1}^{Z} y_{j(l)}^{-1}). \qquad (3.2.17)$$

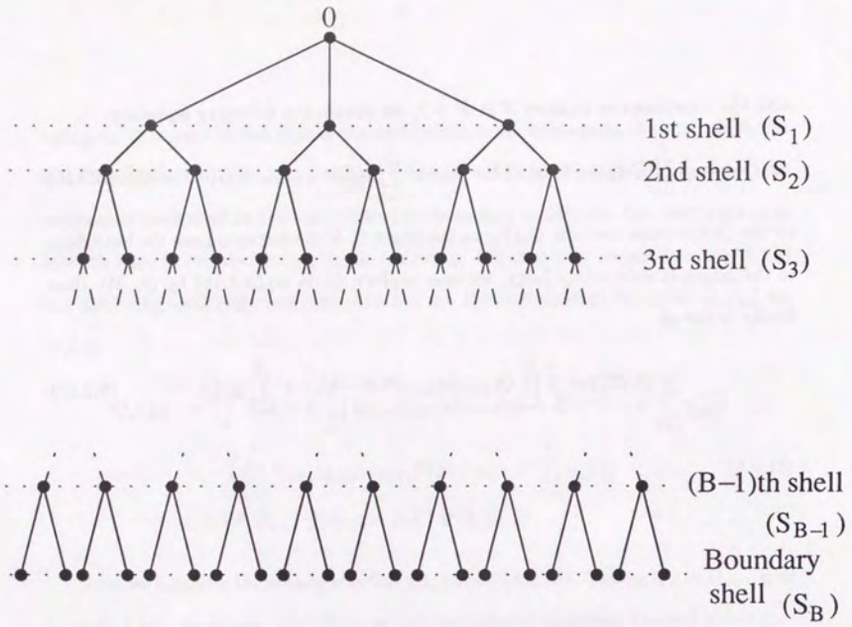

Fig. 3.1: A Bethe lattice with the connectivity $K = 2$. The nth shell is denoted by S_n.

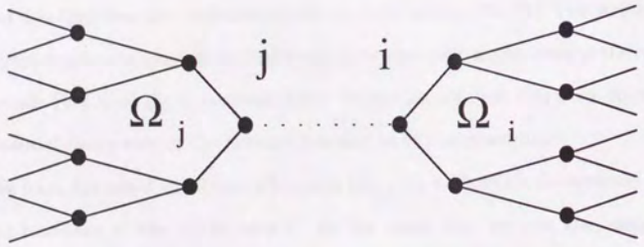

Fig. 3.2: The part Ω_i and the other part Ω_j of the Bethe lattice in the case of $K = 2$, where $i \in \Omega_i$ and $j \in \Omega_j$.

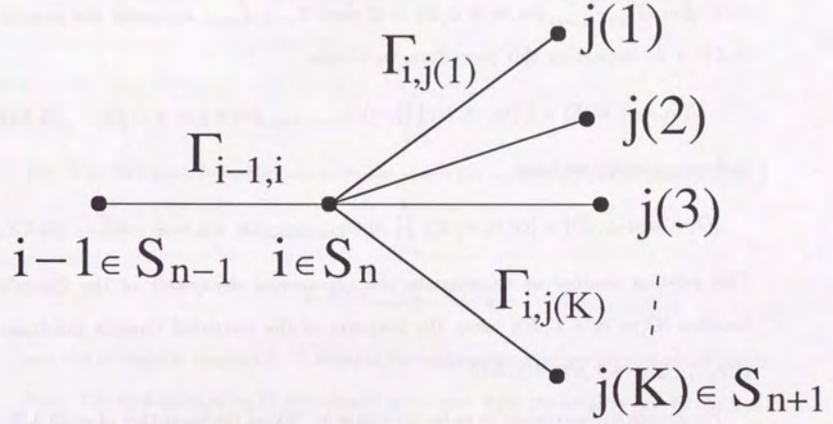

Fig. 3.3: The site $i \in S_n$ and its nearest neighbors $\{(i-1) \in S_{n-1}$ and $j(l) \in S_{n+1}$, $l = 1, \ldots, K\}$, where K denotes the connectivity of the Bethe lattice.

3.3 Exponential Decay-Rate of the Green's function

Let us consider the Green's function $G(m,n;E) \equiv \langle m|(E-H)^{-1}|n\rangle$. On the Bethe lattice, there exists a path from the site m to the site n which is specified by the sites $\{m, m+1, \ldots, n = m+L; L = |m-n|\}$ as shown in Fig. 3.4. Using the resolvent identity (see Appendix 3.A) again, we obtain

$$G(m,m+L;E) = G(m,m+L-1;E)(-t)G_{\Gamma_{m+L-1,m+L}}(m+L,m+L;E). \quad (3.3.1)$$

Note that $G_{\Gamma_{m+L-1,m+L}}(m,m+L;E) = 0$, since $\Gamma_{m+L-1,m+L}$ separates the sites m and $m+L$. Repeating this procedure, we obtain

$$G(m,m+L;E) = G(m,m;E)\prod_{j=1}^{L}(-t)G_{\Gamma_{m+j-1,m+j}}(m+j,m+j;E) \quad (3.3.2)$$

and consequently we have

$$|G(m,m+L;E)| = |G(m,m;E)|\prod_{j=1}^{L}|t||G_{\Gamma_{m+j-1,m+j}}(m+j,m+j;E)|. \quad (3.3.3)$$

This relation enables us to estimate the exponential decay-rate of the Green's function $|G(m,m+L;E)|$ using the property of the restricted Green's functions $\{G_{\Gamma_{m+j-1,m+j}}(m+j,m+j;E)\}$.

For simplicity, we choose m to be the origin 0. Taking the logarithm of eq.(3.3.3), we obtain

$$\frac{1}{L}\ln|G(0,L;E)| = \frac{1}{L}\ln|G(0,0;E)| + \ln|t| + \frac{1}{L}\sum_{j=1}^{L}\ln|G_{\Gamma_{j-1,j}}(j,j;E)|. \quad (3.3.4)$$

We then assume the following conditions:

Condition 1: $\ln|G(0,0;E)| < \infty$ with probability one.

Condition 2: The distribution function of $\ln|y_i| \equiv \ln|G_{\Gamma_{i-1,i}}(i,i;E)^{-1}|$ has finite moments, i.e. $\langle(\ln|y_i|)^2\rangle < \infty$.

Condition 3: $(\langle\ln|y_i|\cdot\ln|y_j|\rangle - \langle\ln|y_i|\rangle\langle\ln|y_j|\rangle) \propto \exp(-a|i-j|)$, for $a > 0$, as $|i-j| \to \infty$.

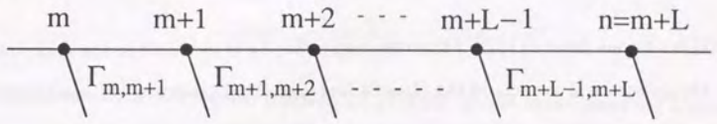

Fig. 3.4: The path from the site m to the site n ($|n-m| = L$) in the case of $K = 2$.

Here we have used the abbreviation

$$y_i \equiv G_{\Gamma_{i-1,i}}(i,i;E)^{-1},$$

and the triangular bracket $\langle\cdots\rangle$ denotes the arithmetic average over random potentials. The first term of eq.(3.3.4) should go to zero with probability one in the limit as $L \to 0$ under the condition 1. Under the conditions 2 and 3, we can apply the law of large numbers to the last term of eq.(3.3.4) though the restricted Green's functions $\{G_{\Gamma_{j-1,j}}(j,j;E)\}$ are not completely independent random variables. We thus obtain

$$\lim_{L\to\infty}\frac{1}{L}\sum_{j=1}^{L}\ln|G_{\Gamma_{j-1,j}}(j,j;E)| = -\langle\ln|y|\rangle, \quad (3.3.5)$$

where $\langle\ln|y|\rangle \equiv \int dy Q(y)\ln|y|$ and therefore the quantity in the left hand side of eq.(3.3.4) is self-averaging; it is independent of the configuration of the random potentials $\{V_i\}$ in the limit as $L \to \infty$. We then finally obtain that the exponential

decay-rate λ_E of the Green's function defined by

$$\lambda_E \equiv -\lim_{L\to\infty}\frac{1}{L}\ln|G(0,L;E)| \tag{3.3.6}$$

is self-averaging and is given by

$$\lambda_E = -\ln|t| + \langle\ln|y|\rangle. \tag{3.3.7}$$

It is obvious from eq.(3.3.7) that the decay-rate λ_E is identical to the decay-rate of the geometrical average of the Green's function. The validity of the conditions 1, 2 and 3 is essential in the present argument. We verify these three conditions in the case of the Lorentzian distribution in the following section and in Appendix 3.B. It is expected that these three conditions are also valid for many other distributions.

It should be remarked here that, in one dimension (K=1), the quantity λ_E is nothing but the Lyapunov exponent Λ_E [51]. It has already been shown that the exponential decay-rate of eigenfunctions is also given by the Lyapunov exponent Λ_E in one-dimensional systems. [47, 48].

If the above three conditions are satisfied and thus the decay-rate λ_E is given by eq.(3.3.7), we immediately obtain

$$\mathrm{Prob}\Big(\frac{1}{L}\ln|G(0,L;E)| \geq -\lambda_E+\varepsilon\Big)\to 0 \quad \text{as} \quad L\to\infty, \tag{3.3.8}$$

where ε is an arbitrary positive number. Here we denote the probability of the event A by $\mathrm{Prob}(A)$. In the same way, we obtain

$$\mathrm{Prob}\Big(\frac{1}{L}\ln|G(0,L;E)| \leq -\lambda_E-\varepsilon\Big)\to 0 \quad \text{as} \quad L\to\infty. \tag{3.3.9}$$

From these two statements, it follows that

$$|G(0,L;E)| \propto \exp(-\lambda_E L) \quad \text{as} \quad L\to\infty \tag{3.3.10}$$

with probability one.

3.4 Analytic Results for the Lorentzian Distribution

In this section we consider the case in which the distribution function of site energies is the Lorentzian distribution. The Lorentzian distribution function $P(V)$ of the site energies $\{V_i\}$ is defined by

$$P(V) = \frac{1}{\pi}\frac{\gamma}{V^2+\gamma^2}, \quad \gamma > 0. \tag{3.4.1}$$

In this case, if the distribution function Q_{m+1} of the inverse of the restricted Green's function at the $m+1$th shell is Lorentzian, the distribution Q_m at the mth shell is also Lorentzian. We have already shown in section 3.2 that the distribution function at the boundary shell, Q_B, is given by

$$Q_B(y) = P(E-y), \tag{3.4.2}$$

since we adopt the free boundary condition and therefore Q_B is the Lorentzian distribution. The distribution Q_m at any shell then becomes Lorentzian and thus the distribution Q_j at the jth shell ($j = 1, \ldots, B$) is expressed using the parameters A_j and $W_j > 0$ in the form:

$$Q_j(y) = \frac{1}{\pi}\frac{W_j}{(y-A_j)^2+W_j^2}. \tag{3.4.3}$$

The Fourier transformation of the distribution $Q_j(x)$ is given by

$$\begin{aligned} Q_j[k] &= \exp(-\mathrm{i}A_j k - W_j|k|) \\ \text{where} \quad Q_j[k] &\equiv \int \exp(-\mathrm{i}kx)Q_j(x)\mathrm{d}x. \end{aligned} \tag{3.4.4}$$

Using eq.(3.4.4), we can transform eq.(3.2.13) as

$$\begin{aligned} &\frac{1}{2\pi}\int \exp(\mathrm{i}ky_i)\mathrm{d}k\exp(-\mathrm{i}A_n k - W_n|k|) \\ &= \frac{1}{2\pi}\int \mathrm{d}k\exp\{\mathrm{i}k(E-y_i-t^2\sum_{l=1}^{K} y_{j(l)}^{-1}) - \gamma|k|\}\prod_{l=1}^{K} Q_{n+1}(y_{j(l)})\mathrm{d}y_{j(l)}, \end{aligned} \tag{3.4.5}$$

where $i \in S_n$, $j(1), \ldots, j(K) \in S_{n+1}$. Changing the variable k to $-k$ in the right hand side and integrating with respect to $\{y_{j(l)}\}$, we get

$$\begin{aligned}
&\frac{1}{2\pi}\int \exp(\mathrm{i}ky_i)\mathrm{d}k \exp(-\mathrm{i}A_n k - W_n|k|) \\
&\quad = \frac{1}{2\pi}\int \exp(\mathrm{i}ky_i)\mathrm{d}k \\
&\quad \cdot \exp\{-\mathrm{i}k(E - Kt^2\frac{A_{n+1}}{A_{n+1}^2+W_{n+1}^2}) - (\gamma + Kt^2\frac{W_{n+1}}{A_{n+1}^2+W_{n+1}^2})|k|\}. \quad (3.4.6)
\end{aligned}$$

We thus obtain the recursion relation for the parameters (A_n, W_n) as

$$\begin{aligned}
A_n &= E - Kt^2\frac{A_{n+1}}{A_{n+1}^2+W_{n+1}^2} \\
W_n &= \gamma + Kt^2\frac{W_{n+1}}{A_{n+1}^2+W_{n+1}^2} \qquad (3.4.7)
\end{aligned}$$

or equivalently

$$\omega_n = E + \mathrm{i}\gamma - Kt^2\omega_{n+1}^{-1} \qquad (3.4.8)$$

where $\omega_n \equiv A_n + \mathrm{i}W_n$. Taking $\omega_n = \omega_{n+1} \equiv \omega$ in eq.(3.4.8), we obtain the self-consistency equation

$$\omega = E + \mathrm{i}\gamma - Kt^2\omega^{-1} \qquad (3.4.9)$$

for the parameters (A, W) of the steady distribution $Q(y)$, where $\omega = A + \mathrm{i}W$ and

$$Q(y) = \frac{1}{\pi}\frac{W}{(y-A)^2+W^2}.$$

The physical solution of eq.(3.4.9) is easily obtained as

$$\omega = \frac{1}{2}\Big\{E\Big(\frac{\sqrt{x}+\gamma}{\sqrt{x}}\Big) + \mathrm{i}\Big(\gamma + \sqrt{x}\Big)\Big\}, \qquad (3.4.10)$$

where x is the positive root of

$$x^2 + (E^2 - \gamma^2 - 4Kt^2)x - E^2\gamma^2 = 0. \qquad (3.4.11)$$

The exponential decay-rate λ_E of the Green's function is thus evaluated from eq.(3.3.7) as

$$\begin{aligned}
\lambda_E &= \int \ln|y|Q(y)\mathrm{d}y - \ln|t| \\
&= \ln|\omega| - \ln|t| \\
&= \frac{1}{2}\ln\Big(K\frac{\sqrt{x}+\gamma}{\sqrt{x}-\gamma}\Big) > 0. \qquad (3.4.12)
\end{aligned}$$

In one dimension (K=1), we obtain

$$\lambda_E = \frac{1}{2}\ln\Big(\frac{\sqrt{x}+\gamma}{\sqrt{x}-\gamma}\Big), \qquad (3.4.13)$$

where x is the positive root of $x^2 + (E^2 - \gamma^2 - 4t^2)x - E^2\gamma^2 = 0$ in this case. This is nothing but the Lyapunov exponent in the case of the Lorentzian distribution [3]. It should be noted that λ_E is always positive since $K \geq 1$.

The distribution Q_0 of the Green's function $G(0,0;E)^{-1}$ is also obtained from eq.(3.2.17) as

$$Q_0(Y_0) = \frac{1}{\pi}\frac{W_0}{(Y_0 - A_0)^2 + W_0^2}, \qquad (3.4.14)$$

where the parameters (A_0, W_0) are given by

$$A_0 = E - Zt^2\frac{A}{A^2+W^2}, \quad W_0 = \gamma + Zt^2\frac{W}{A^2+W^2}. \qquad (3.4.15)$$

It should be noted that the event that the variable $Y_0 \equiv G(0,0;E)^{-1}$ is exactly equal to some value $x \in \boldsymbol{R}$ has zero measure since the distribution $Q_0(Y_0)$ is continuous and has no singularity. This guarantees the condition 1 in the previous section. The convergence of the series $\{\omega_n\}$ and the validity of the condition 3 are discussed in Appendix 3.B.

In this case, we can easily show that the condition 2 is satisfied. Since we know that the distribution $Q(y)$ of the inverse of the restricted Green's function is

Lorentzian, the distribution function of $x = \ln|y|$ becomes

$$\begin{aligned} F(x) &\equiv \int \mathrm{d}y Q(y)\delta(x - \ln|y|) \\ &= \frac{W}{\pi}\Big(\frac{\mathrm{e}^x}{(\mathrm{e}^x - A)^2 + W^2} + \frac{\mathrm{e}^x}{(\mathrm{e}^x + A)^2 + W^2}\Big). \end{aligned} \tag{3.4.16}$$

It is easily seen that

$$F(x) \propto \exp(-|x|) \quad \text{as} \quad |x| \to \infty$$

and consequently the distribution $F(x)$ has finite moments.

3.5 A Criterion for Localization

Let us consider the condition that

$$\lambda_S \equiv \lambda_E - \ln K > 0 \tag{3.5.1}$$

as a criterion for localization on the d-dimensional hypercubic lattice ($\boldsymbol{Z}^d$) where $d = (K+1)/2$. It is obvious that this criterion is exact in the one-dimensional systems. Before discussing the meaning of our criterion for localization, we present explicitly the analytic results obtained by our criterion for the Lorentzian distribution.

In the case of the Lorentzian distribution of site energies, we obtain from eq.(3.4.12) that

$$\lambda_S \equiv \lambda_E - \ln K = \frac{1}{2}\ln\Big(\frac{1}{K}\frac{\sqrt{x}+\gamma}{\sqrt{x}-\gamma}\Big). \tag{3.5.2}$$

The condition that $\lambda_S > 0$ is thus reduced to

$$\frac{1}{K}\frac{\sqrt{x}+\gamma}{\sqrt{x}-\gamma} > 1, \tag{3.5.3}$$

where x is the positive root of

$$x^2 + (E^2 - \gamma^2 - 4Kt^2)x - \gamma^2 E^2 = 0. \tag{3.5.4}$$

Direct calculations show that the condition (3.5.3) becomes

$$\frac{E^2}{(K+1)^2} + \frac{\gamma^2}{(K-1)^2} > t^2. \tag{3.5.5}$$

The region where $\lambda_S > 0$ in the $E - \gamma$ plane is thus obtained as shown in Fig. 3.5. The boundary $\lambda_S = 0$ of this region, which is regarded as the mobility edge trajectory in our approximation, is given by

$$\frac{E^2}{(K+1)^2} + \frac{\gamma^2}{(K-1)^2} = t^2. \tag{3.5.6}$$

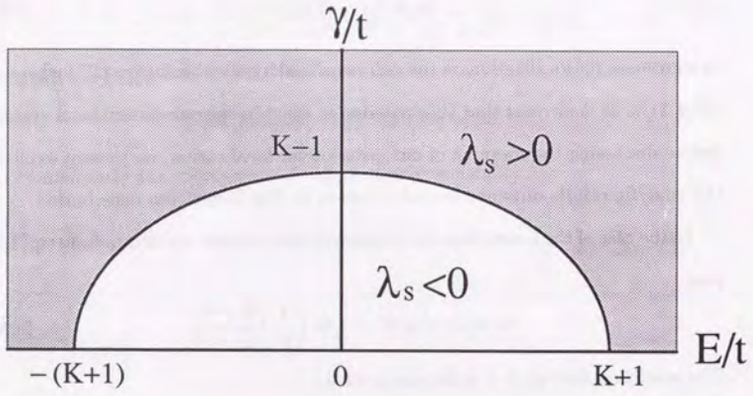

Fig. 3.5: The mobility edge trajectory by our criterion for the Lorentzian distribution $P(V) = \gamma/\pi(V^2 + \gamma^2)$ in the system with the coordination number $Z = K + 1$. Here $\lambda_S = \lambda_E - \ln K$. The region where $\lambda_S > 0$ is represented by the shaded area.

Let us look into this result in detail. First, consider the case $\gamma = 0$, i.e. the regular system. In this case, the mobility edges are given by $E = \pm(K+1)t = \pm Zt$ and consequently we obtain the exact band edges of the d-dimensional ($2d = Z$) hypercubic lattice ($\boldsymbol{Z}^d$) [72, 73]. In the one-dimensional case ($K = 1$), we recover the rigorous result that eigenstates are localized for any E provided that $\gamma > 0$ [3, 6, 44-48, 58-62]. We have no region where $\lambda_S < 0$ in the $E - \gamma$ plane in this case. It should be also remarked that the shape of the boundary is quite close to the mobility edge trajectory obtained numerically in the three-dimensional system ($\boldsymbol{Z}^3$)

[30, 31]. It has been considered that the absence of the extended states outside the unperturbed band is one of the characteristic features of the mobility edge trajectory for the Lorentzian distribution [31] and, in fact, this is clearly seen in our result (Fig. 3.5). We find from eq.(3.5.6) that the critical value γ_c/t for $E = 0$ and $Z = 6$ is given by $K - 1 = 4$. This is close to the value ~ 3.8 [31] obtained numerically in the three-dimensional system with the coordination number 6.

We can define the localization length by the inverse of the decay-rate λ_S:

$$\xi(E) \equiv \lambda_S^{-1} = \left[\frac{1}{2}\ln\left(\frac{1}{K}\frac{\sqrt{x}+\gamma}{\sqrt{x}-\gamma}\right)\right]^{-1}. \tag{3.5.7}$$

The critical exponent ν for the localization length is defined by

$$\xi(E) \approx (E - E_c)^{-\nu}, \quad E \sim E_c \tag{3.5.8}$$

where E_c denotes the mobility edge determined from eq.(3.5.6). We then find that the critical exponent ν is given by $\nu = 1$. This value of the critical exponent is commonly obtained in the previous works of the localization transition on the Bethe lattice [25, 26, 74].

Now, we discuss the physical meaning of our criterion (3.5.1). It can be also expressed in the form:

$$\sum_{j \in S_L} \langle |G(0, j; E)| \rangle_g \propto \exp(-\lambda_S L), \quad \lambda_S > 0, \tag{3.5.9}$$

in the limit as $L \to \infty$, where $\langle \cdots \rangle_g$ denotes the geometrical average over random potentials. Apart from the above geometrical average in eq.(3.5.9), our criterion is thus based on the specific property of the summation of the Green's function over the sites in the Lth shell:

$$C(L) \equiv \sum_{j \in S_L} |G(0, j; E)|. \tag{3.5.10}$$

It is evident, at least in the case of the Lorentzian distribution, that the Green's function between two sites on the Bethe lattice always decays exponentially fast and that it tells us nothing about the localization transition. In the case of regular systems, as is seen from the above results for the Lorentzian distribution, we recover the exact band edges $\pm(K+1)t$ of the corresponding hypercubic lattice instead of the exact band edges $\pm 2\sqrt{K}t$ of the Bethe lattice [73] from the condition that the quantity $C(L)$ should decay exponentially. Namely, the exponential decay of $C(L)$ on the Bethe lattice is equivalent to the exponential decay of the Green's function on the corresponding real lattice in the absence of random potentials. This surface-to-center correlation $C(L)$ plays a crucial role. Although the connection between the behaviour of the quantity $C(L)$ obtained on the Bethe lattice and the properties of the Green's functions in the corresponding real lattice is not clear, we expect that the condition for the exponential decay of $C(L)$ must be an approximate condition for the exponential decay of the Green's function on the corresponding hypercubic lattice even in the presence of random potentials. The characteristic feature of the Bethe lattice is that it has no closed path. The physical meaning of this approximation would then be interpreted as neglecting the effect of interferences between different paths and as taking into account only the effect of transmissions and reflections. Our criterion is therefore expected to work well for large $d(=(K+1)/2)$, where the effect of closed loops becomes less important.

3.6 Discussion

We have shown that the exponential decay-rate of the Green's function in a random potential on the Bethe lattice can be obtained by solving the self-consistency nonlinear integral equation (3.2.14) for the distribution function of the restricted Green's functions. The self-consistency equation (3.2.14) has been derived from the recursion relation for the restricted Green's functions. Analytic results in the case of the Lorentzian distribution of site energies are explicitly obtained. On the basis of these results, we have proposed the criterion (3.5.1) for localization, which is exact in one dimension, and obtained qualitatively good results for the Lorentzian distribution.

It should be noted here that, in general, the decay-rate of the Green's function (or eigenfunctions) defined by

$$-\lim_{L\to\infty}\frac{1}{L}\ln|G(0,L;E)| \quad \left(\text{or} \quad -\lim_{L\to\infty}\frac{1}{L}\ln|\psi_E(0)\psi_E(L)|\right) \tag{3.6.1}$$

is expected to be larger than the decay-rate of its arithmetic average. For example, in one-dimensional systems, the decay-rate defined by eq.(3.6.1) is known as the Lyapunov exponent Λ_E [47, 48, 51]. On the other hand, the decay-rate ξ_E^{-1} at fixed energy E defined in

$$\langle|\psi_E(0)\psi_E(x)|\rangle_a \propto \exp(-\xi_E^{-1}|x|) \quad \text{as} \quad |x|\to\infty, \tag{3.6.2}$$

with the arithmetic average over random potentials $\langle\cdots\rangle_a$, was also estimated by Delyon et al. [45] and their estimation of the decay-rate ξ_E^{-1} is indeed smaller than the Lyapunov exponent Λ_E, i.e. $\Lambda_E > \xi_E^{-1}$ [45]. They proved that the spectrum is a pure point-spectrum under the condition that ξ_E^{-1} is positive [44, 45]. The condition that ξ_E^{-1} is positive is, however, more strict than the condition that Λ_E is positive. It is already known that the localization in one dimension is proved

under the condition that the Lyapunov exponent is positive [51, 58-62]. Note that the present exponential decay-rate λ_E of the Green's function is considered as a generalization of the ordinary Lyapunov exponent. In fact, it is proved that even in higher dimensions ($\boldsymbol{Z}^d$) the spectrum is a pure point-spectrum under the condition that

$$\text{Prob}\{\sum_{j\in\partial\Lambda(L)} |G(0,j;E)| \propto \exp(-aL), a>0\} \to 1 \quad \text{as} \quad L\to\infty, \tag{3.6.3}$$

provided that the distribution function of random potentials is absolutely continuous with a bounded density [51, 59, 62]. Here $\Lambda(L)$, 0 and $\partial\Lambda(L)$ denote a finite system with a linear-size L, its center and its boundary, respectively.

On the basis of these considerations, it is expected that our criterion for localization

$$\lambda_S \equiv \lambda_E - \ln K > 0 \tag{3.6.4}$$

yields a better estimate of the mobility edge trajectory in the d-dimensional hypercubic lattice ($d=(K+1)/2$) compared with the criterion obtained by Abou-Chacra et al.[72] and also by Kunz and Souillard [74]. The reason is that their criterion is based on the property of the decay-rate of the arithmetic average of the Green's function or eigenfunctions and is given by (see Appendix 3.C)

$$\lambda_S^a \equiv \lambda_E^a - \ln K > 0, \tag{3.6.5}$$

instead of (3.6.4), with the exponential decay-rate λ_E^a of the arithmetic average of the Green's function defined by

$$\lambda_E^a \equiv -\lim_{L\to\infty}\frac{1}{L}\ln\langle|G(0,L;E)|\rangle_a. \tag{3.6.6}$$

Since it is expected that $\lambda_S \geq \lambda_S^a$, the condition for localization is relaxed in our criterion. Actually, in the case of the Lorentzian distribution, we have shown that

our criterion succeeds in reproducing a typical feature of the mobility edge trajectory in the corresponding hypercubic lattice which was obtained numerically by the finite-size scaling method, namely the absence of the extended states outside the unperturbed band. On the other hand, the criterion (3.6.5) obtained by previous works [72, 74] failed to reproduce this property of the mobility edge trajectory [73]. Moreover the critical value of γ_c obtained by our criterion in the center of the band ($E=0$) for $Z=6$ is close to the values obtained numerically in the three-dimensional systems [31]. The results in the case of the Lorentzian distribution thus suggest that our criterion works well even in the three-dimensional systems.

In the previous section, we have argued that our criterion is expected to work well in high dimensions. It should be also noted, however, that in the case of the localization transition the relationship between critical behavior on the Bethe lattice and that in high dimensions might not be so simple as in the case of percolation or the Ising model.

We expect that our criterion will work well for other distributions though the conditions 1, 2 and 3 are not yet proved. The results for other distributions will be reported elsewhere.

The argument developed in this chapter might be generalized to the case of the generalized cactus tree. If this generalization could be achieved, an application of the coherent-anomaly method might be possible [77, 78].

We have discussed here the decay-rate of the Green's function with the real energy $E\in\boldsymbol{R}$. We have to analyze, however, the property of the Green's function with the complex energy $E+i\eta\in\boldsymbol{C}$ to discuss the properties of physical quantities such as the dielectric constant and the electric conductivity. The exponential decay-rate of the Green's function with the complex energy $|G(i,j;E+i\eta)|$ remains to be seen in further investigations.

Appendix 3.A : Derivation of eq.(3.2.6)

The resolvent identity states that

$$\begin{aligned}\frac{1}{E-H_\Gamma} &= \frac{1}{E-H_{\Gamma+\Gamma'}-\Gamma'} \\ &= \frac{1}{E-H_{\Gamma+\Gamma'}}+\frac{1}{E-H_{\Gamma+\Gamma'}}\Gamma'\frac{1}{E-H_\Gamma},\end{aligned} \tag{3.A.1}$$

where $H_\Gamma \equiv H-\Gamma$ and $H_{\Gamma+\Gamma'} \equiv H-\Gamma-\Gamma'$. Here let us choose Γ and Γ' as

$$\Gamma = \Gamma_{i-1,i} \quad \text{and} \quad \Gamma' = \sum_{l=1}^{K}\Gamma_{i,j(l)}, \tag{3.A.2}$$

(see Fig. 3.3). We then obtain the following identity

$$\begin{aligned}G_{\Gamma_{i-1,i}}(i,i;E) &= G_{\Gamma_{i-1,i}+\sum_l\Gamma_{i,j(l)}}(i,i;E) \\ &\quad +G_{\Gamma_{i-1,i}+\sum_l\Gamma_{i,j(l)}}(i,i;E)(-t)\sum_{l=1}^{K}G_{\Gamma_{i-1,i}}(j(l),i;E).\end{aligned} \tag{3.A.3}$$

In the same way, we get

$$G_{\Gamma_{i-1,i}}(j(l),i;E) = G_{\Gamma_{i-1,i}+\sum_l\Gamma_{i,j(l)}}(j(l),j(l);E)(-t)G_{\Gamma_{i-1,i}}(i,i;E). \tag{3.A.4}$$

Note that

$$G_{\Gamma_{i-1,i}+\sum_l\Gamma_{i,j(l)}}(j(l),j(l);E) = G_{\Gamma_{i,j(l)}}(j(l),j(l);E)$$

and

$$G_{\Gamma_{i-1,i}+\sum_l\Gamma_{i,j(l)}}(i,i;E) = (E-V_i)^{-1}.$$

Substituting eq.(3.A.4) into eq.(3.A.3), we obtain

$$G_{\Gamma_{i-1,i}}(i,i;E) = \frac{1}{E-V_i}\{1+t^2\Big(\sum_{l=1}^{K}G_{\Gamma_{i,j(l)}}(j(l),j(l);E)\Big)G_{\Gamma_{i-1,i}}(i,i;E)\}, \tag{3.A.5}$$

and we thus finally arrive at eq.(3.2.6):

$$G_{\Gamma_{i-1,i}}(i,i;E)^{-1} = E-V_i-t^2\sum_{l=1}^{K}G_{\Gamma_{i,j(l)}}(j(l),j(l);E). \tag{3.A.6}$$

Appendix 3.B: Convergence of the series $\{\omega_n\}$

The recursion relation is given by

$$\omega_n = E+\mathrm{i}\gamma - Kt^2\omega_{n+1}^{-1}. \tag{3.B.1}$$

The above parameter ω_n of the distribution on the boundary $n=B$ is given by (see eq.(3.4.2))

$$\omega_B = E+\mathrm{i}\gamma. \tag{3.B.2}$$

Since γ is positive, we immediately have

$$\mathrm{Im}\ \omega_m > \gamma > 0 \tag{3.B.3}$$

for any m. Consequently the right hand side of the recursion relation is well-defined for any n.

Let us introduce the following new variables $\{\alpha_i\}$ defined in

$$\omega_i = t\frac{\alpha_{i-1}}{\alpha_i}, \quad i=1,\ldots,B. \tag{3.B.4}$$

We then find that the recursion relation for $\{\alpha_i\}$ becomes

$$\alpha_{i-2} = \frac{E+\mathrm{i}\gamma}{t}\alpha_{i-1} - K\alpha_i. \tag{3.B.5}$$

The initial condition should be chosen as

$$\alpha_B = 1, \quad \alpha_{B-1} = (E+\mathrm{i}\gamma)/t \tag{3.B.6}$$

so that

$$\omega_B \equiv t\frac{\alpha_{B-1}}{\alpha_B} = E+\mathrm{i}\gamma. \tag{3.B.7}$$

The recursion relation (3.B.5) can be expressed in the form

$$\begin{pmatrix}\alpha_{i-2}\\ \alpha_{i-1}\end{pmatrix} = T\begin{pmatrix}\alpha_{i-1}\\ \alpha_i\end{pmatrix} \tag{3.B.8}$$

where

$$T = \begin{pmatrix} (E+\mathrm{i}\gamma)/t & -K \\ 1 & 0 \end{pmatrix}, \quad \begin{pmatrix} \alpha_{B-1} \\ \alpha_B \end{pmatrix} = \begin{pmatrix} \omega_B \\ 1 \end{pmatrix}. \tag{3.B.9}$$

The eigenvalues and eigenvectors of the matrix T are obtained as

$$T \begin{pmatrix} \lambda_\pm \\ 1 \end{pmatrix} = \lambda_\pm \begin{pmatrix} \lambda_\pm \\ 1 \end{pmatrix} \tag{3.B.10}$$

and

$$\lambda_\pm = \frac{1}{2t}\Big(E \pm \frac{\gamma E}{\sqrt{x}} + \mathrm{i}(\gamma \pm \sqrt{x})\Big) \tag{3.B.11}$$

where x is the positive solution of

$$x^2 + (E^2 - \gamma^2 - 4Kt^2)x - E^2\gamma^2 = 0. \tag{3.B.12}$$

Notice that the initial vector is expressed as

$$\begin{aligned} \begin{pmatrix} \omega_B \\ 1 \end{pmatrix} &= c_+ \begin{pmatrix} \lambda_+ \\ 1 \end{pmatrix} + c_- \begin{pmatrix} \lambda_- \\ 1 \end{pmatrix} \qquad (3.B.13) \\ c_+ &= \frac{\omega_B - \lambda_-}{\lambda_+ - \lambda_-}, \quad c_- = \frac{-(\omega_B - \lambda_+)}{\lambda_+ - \lambda_-}. \end{aligned}$$

We thus obtain

$$\begin{aligned} \begin{pmatrix} \alpha_{B-L} \\ \alpha_{B-L+1} \end{pmatrix} &= T^{L-1} \begin{pmatrix} \omega_B \\ 1 \end{pmatrix} \\ &= c_+\lambda_+^{L-1} \begin{pmatrix} \lambda_+ \\ 1 \end{pmatrix} + c_-\lambda_-^{L-1} \begin{pmatrix} \lambda_- \\ 1 \end{pmatrix} \qquad (3.B.14) \end{aligned}$$

and therefore

$$\begin{aligned} \omega_{B-L} &= t\frac{\alpha_{B-L}}{\alpha_{B-L+1}} \\ &= t\frac{c_+\lambda_+^L + c_-\lambda_-^L}{c_+\lambda_+^{L-1} + c_-\lambda_-^{L-1}} \\ &= t\frac{c_+\lambda_+ + c_-\lambda_-(\lambda_-/\lambda_+)^{L-1}}{c_+ + c_-(\lambda_-/\lambda_+)^{L-1}}. \qquad (3.B.15) \end{aligned}$$

Since the absolute values of the eigenvalues λ_+ and λ_- are given by

$$|\lambda_+|^2 = K\frac{\sqrt{x}+\gamma}{\sqrt{x}-\gamma} \quad \text{and} \quad |\lambda_-|^2 = K\frac{\sqrt{x}-\gamma}{\sqrt{x}+\gamma}, \tag{3.B.16}$$

we have $|\lambda_+| \geq |\lambda_-|$ provided that $\gamma > 0$. We thus finally obtain that for $\gamma > 0$

$$\begin{aligned} \lim_{L\to\infty} \omega_{B-L} &= t\lambda_+ \\ &= \frac{1}{2}\Big(E + \frac{\gamma E}{\sqrt{x}} + \mathrm{i}(\sqrt{x}+\gamma)\Big) \\ &= \omega \qquad (3.B.17) \end{aligned}$$

where $B > L$. Note that ω_{B-L} converges to ω exponentially fast. This means that the distribution function at the shell sufficiently far from the boundary can be regarded as the steady distribution Q.

Next, we consider the correlation between y_i, $i \in S_{m-|i-j|}$ and y_j, $j \in S_m$. We show in the following that the distribution y_i under the restriction that the value of y_j is fixed, which is denoted by $Q(y_i; y_j)$, converges exponentially fast to $Q(y_i)$ as $|i-j| \to \infty$. This leads to the condition 3 in section 3.3. Since the distribution is known to be Lorentzian, the convergence of the distribution is equivalent to the convergence of the parameters (A_i, W_i). We assume that the sites i and j are sufficiently far from the boundary so that the distribution of y_k around them is considered as $Q(y_k)$. In order to fix the variable y_j, we take the distribution $Q(y_j)$ as

$$Q(y_j) = \delta(y_j - x), \quad x \in \boldsymbol{R}, \tag{3.B.18}$$

i.e. $A_j = x$ and $W_j = 0$. We consider the path from j to i which is specified by the sites $\{j, j+1, \ldots, j+L=i; |i-j| = L\}$ where $j+n \in S_{m-n}$ as shown in Fig. 3.6 and the distributions under the condition $y_j = x$ are denoted by

$$Q(y_{j+m}; y_j = x) = \frac{1}{\pi}\frac{\tilde{W}_{j+m}}{(y_{j+m} - \tilde{A}_{j+m})^2 + \tilde{W}_{j+m}^2}, \quad \tilde{\omega}_{j+m} \equiv \tilde{A}_{j+m} + \mathrm{i}\tilde{W}_{j+m}. \tag{3.B.19}$$

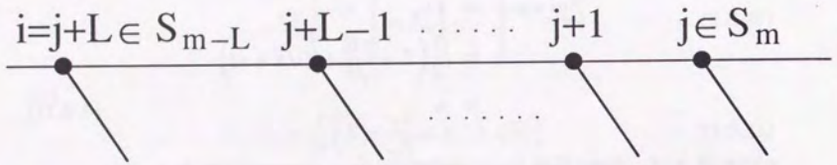

Fig. 3.6: The path from the site $j \in S_m$ to the site $i \in S_{m-L}$ ($|i-j| = L$) in the case of $K = 2$.

We then get the recursion relation

$$\tilde{\omega}_{j+n+1} = E + \mathrm{i}\gamma - (K-1)t^2\omega^{-1} - t^2\tilde{\omega}_{j+n}^{-1}, \tag{3.B.20}$$

where the initial condition is $\tilde{\omega}_j = x \in \boldsymbol{R}$. Using the property (3.4.9) of ω, we get

$$\tilde{\omega}_{j+n+1} = \omega + t^2\omega^{-1} - t^2\tilde{\omega}_{j+n}^{-1}. \tag{3.B.21}$$

Now we again introduce the following new variables $\{\tilde{\alpha}_i\}$ defined in

$$\tilde{\omega}_{j+n} = t\frac{\tilde{\alpha}_{j+n+1}}{\tilde{\alpha}_{j+n}}. \tag{3.B.22}$$

The recursion relation for $\{\tilde{\alpha}_i\}$ is then expressed as

$$\begin{pmatrix} \tilde{\alpha}_{j+n+2} \\ \tilde{\alpha}_{j+n+1} \end{pmatrix} = T' \begin{pmatrix} \tilde{\alpha}_{j+n+1} \\ \tilde{\alpha}_{j+n} \end{pmatrix}, \tag{3.B.23}$$

where T' and the initial vector are given by

$$T' = \begin{pmatrix} (\omega + t^2\omega^{-1})/t & -1 \\ 1 & 0 \end{pmatrix}, \quad \text{and} \quad \begin{pmatrix} \tilde{\alpha}_{j+1} \\ \tilde{\alpha}_j \end{pmatrix} = \begin{pmatrix} x \\ 1 \end{pmatrix}, \tag{3.B.24}$$

respectively. The eigenvalues and eigenvectors of the matrix T' are obtained as

$$T' \begin{pmatrix} \tilde{\lambda}_\pm \\ 1 \end{pmatrix} = \tilde{\lambda}_\pm \begin{pmatrix} \tilde{\lambda}_\pm \\ 1 \end{pmatrix} \tag{3.B.25}$$

where $\tilde{\lambda}_+ = \omega/t, \quad \tilde{\lambda}_- = \tilde{\lambda}_+^{-1}$. We then find that

$$\begin{aligned} \begin{pmatrix} \tilde{\alpha}_{j+L} \\ \tilde{\alpha}_{j+L-1} \end{pmatrix} &= (T')^{L-1} \begin{pmatrix} \tilde{\alpha}_{j+1} \\ \tilde{\alpha}_j \end{pmatrix} \\ &= \tilde{c}_+\tilde{\lambda}_+^{L-1} \begin{pmatrix} \tilde{\lambda}_+ \\ 1 \end{pmatrix} + \tilde{c}_-\tilde{\lambda}_-^{L-1} \begin{pmatrix} \tilde{\lambda}_- \\ 1 \end{pmatrix} \end{aligned} \tag{3.B.26}$$

where $\{\tilde{c}_\pm\}$ are given by

$$\tilde{c}_+ = \frac{x - \tilde{\lambda}_-}{\tilde{\lambda}_+ - \tilde{\lambda}_-}, \quad \text{and} \quad \tilde{c}_- = \frac{-(x - \tilde{\lambda}_+)}{\tilde{\lambda}_+ - \tilde{\lambda}_-}. \tag{3.B.27}$$

Note that $\tilde{c}_+ \neq 0$ for $\gamma > 0$, since $\mathrm{Im}\omega > 0$. We can easily prove that $|\tilde{\lambda}_+| \geq |\tilde{\lambda}_-|$ and we thus find that $\tilde{\omega}_{j+L}$ converges to ω

$$\lim_{L\to\infty} \tilde{\omega}_{j+L} = \omega \tag{3.B.28}$$

exponentially fast by using the same argument.

Appendix 3.C : Relation to Previous Works

In order to see the relation to previous works [72, 74], let us consider the quantity defined by

$$C(L) \equiv \sum_{j \in S_L} |G(0, j; E)|, \tag{3.C.1}$$

where 0 denotes the origin of the Bethe lattice.

First, we consider the arithmetic average of the quantity $C(L)$

$$\langle C(L) \rangle_a = \sum_{j \in S_L} \langle |G(0, j; E)| \rangle_a, \tag{3.C.2}$$

where the triangular bracket $\langle \cdots \rangle_a$ denotes the arithmetic average with respect to the distribution of the random potentials $\{V_i\}$. Since averaged quantities are uniform in our system, eq.(3.C.2) can be further simplified as

$$\sum_{j \in S_L} \langle |G(0, j; E)| \rangle_a = (K+1) K^{L-1} \langle |G(0, L; E)| \rangle_a, \tag{3.C.3}$$

where the site L in the right hand site of this equation denotes a site in the Lth shell and the path from the origin 0 to the site L is parametrized by the sites $\{0, 1, 2, \ldots, L\}$ as in section 3.3. From eq.(3.3.3) we get

$$\begin{aligned} \langle |G(0, L; E)| \rangle_a &= \langle |G(0,0)| \prod_{m=1}^{L} |t| |G_{\Gamma_{m-1,m}}(m, m; E)| \rangle_a \\ &= \langle |y_0|^{-1} \prod_{m=1}^{L} |t| |y_m|^{-1} \rangle_a \end{aligned} \tag{3.C.4}$$

where $m \in S_m$, $y_0 = G(0,0;E)^{-1}$ and $y_m = G_{\Gamma_{m-1,m}}(m, m; E)^{-1}$. It is then easy to see that the exponential decay-rate of the arithmetic average of the Green's function is given as

$$-\lim_{L \to \infty} \frac{1}{L} \ln \langle |G(0, L; E)| \rangle_a = -\ln \lambda_{\max}(E), \tag{3.C.5}$$

where $\lambda_{\max}(E)$ denotes the largest eigenvalue of the transfer operator T defined by

$$T(x; y) \equiv \int \frac{|t|}{|y|} \prod_{i=1}^{K-1} Q(y_i) \mathrm{d}y_i P(E - x - t^2 y^{-1} - t^2 \sum_{i=1}^{K-1} y_i^{-1}). \tag{3.C.6}$$

We thus obtain that the condition for the exponential decay of the arithmetic average of the summation of the Green's function $\langle C(L) \rangle_a$ is given by $K \lambda_{\max}(E) < 1$.

The properties of the operator T in (3.C.6) and its eigenvalues were discussed by Kunz and Souillard [74]. They discussed the property of the correlation of eigenfunctions instead of the Green's function and found that the decay of the arithmetic average of the correlation of eigenfunctions at energy E was determined by the largest eigenvalue of the same transfer operator T above defined [74]. Namely,

$$-\lim_{L \to \infty} \frac{1}{L} \ln \langle |\psi_E(0) \psi_E(L)| \rangle_a = -\ln \lambda_{\max}(E). \tag{3.C.7}$$

Here ψ_E is the solution of the Schrödinger equation $H\psi = E\psi$ and the bracket $\langle \cdots \rangle_a$ again denotes the arithmetic average over random potentials. As a result, we find that the decay-rate of the arithmetic average of the Green's function is identical to that of the correlation of eigenfunctions. The reason why we have obtained the same decay-rate for both the Green's function and the correlation function is that the ratio of the eigenfunction z_i defined by

$$z_i \equiv -t \psi_E(j) / \psi_E(i), \quad \text{for} \quad \{i, j : |i| > |j|, |i - j| = 1\}$$

and the restricted Green's function

$$G_{\Gamma_{i-1,i}}(i, i; E)^{-1}$$

satisfy the same recursion relation as mentioned in section 3.2. Note that the correlation of eigenfunctions is also expressed by the product of the variables $\{z_i\}$ as

$$|\psi_E(0) \psi_E(L)| = |\psi_E(L)|^2 \prod_{i=1}^{L} \frac{|z_i|}{|t|} \tag{3.C.8}$$

(compare with eq.(3.3.3) in section 3.3). They claimed that they proved localization under the condition $K \lambda_{\max}(E) < 1$, namely the exponential decay of the summation

of the correlation functions over the sites in the Lth shell [74]:

$$\langle C_\psi(L)\rangle_a \equiv \sum_{j\in S_L} \langle|\psi_E(0)\psi_E(j)|\rangle_a \propto \exp(-aL), \quad a>0 \tag{3.C.9}$$

in the limit as $L\to\infty$.

As was pointed out by Kunz and Souillard [74], the condition that $K\lambda_{\max}(E)<1$ is equivalent to the condition for the stability of the localized states obtained by Abou-Chacra et al. [72]. Abou-Chacra et al. considered that the energy E lies in the localized region when the imaginary part of the self-energy vanishes as the imaginary part of the energy goes to zero and they investigated the distribution function of the self-energy. The distribution function of the real part of the self-energy they considered is directly related to the distribution Q we consider in the present chapter, but they did not mention the decay of the Green's function. They discussed the stability of the desired solution of the distribution function of the self-energy for the localized state. We then find that their condition for the stability of the localized states [72] is understood as the condition for $\langle C(L)\rangle_a \propto \exp(-aL), a>0$.

The criterion for the mobility edge was thus given by $K\lambda_{\max}(E)=1$ in the previous works by Abou-Chacra et al. [72] and by Kunz and Souillard [74]. It should be emphasized that this criterion for the mobility edge is based on the property of the arithmetic average of the Green's function or eigenfunctions.

Chapter IV

Exponential Decay of Eigenfunctions in a Random Potential on the Bethe Lattice

4.1 Introduction

We have discussed the exponential decay-rate of the Green's function in the previous chapter. In this chapter, we discuss the exponential decay of eigenfunctions by analyzing their distribution functions. Eigenfunctions are defined as the solutions of the Schrödinger equation $H\psi_E = E\psi_E$. Properties of eigenfunctions must be related to that of the Green's function. In one-dimensional systems, it has already been shown that the exponential decay-rate of eigenfunctions is also given by the Lyapunov exponent Λ_E [47,48]. Some rigorous results for the arithmetic average of the correlation of eigenfunctions on the Bethe lattice were obtained by Kunz and Souillard [74]. We partially follow their argument and discuss the distribution function of the ratio $\psi_E(i-1)/\psi_E(i)$ of the eigenfunction ψ_E. Notations used in this chapter are the same as in chapter 3.

4.2 Change of Variables in the Probability Density

The Schrödinger equation $H\psi_E = E\psi_E$ is explicitly given by

$$-t\sum_{j:|j-i|=1}\psi_E(j)+V_i\psi_E(i)=E\psi_E(i) \tag{4.2.1}$$

for the tight-binding Anderson Hamiltonian, where the random potentials $\{V_i\}$ are distributed independently. Since we consider the Bethe lattice, from the Schrödinger equation (4.2.1), we obtain the following relation:

$$z_i = E - V_i - t^2 \sum_{l=1}^{K} z_{j(l)}^{-1}, \tag{4.2.2}$$

for the variables $\{z_k\}$ defined by

$$z_i \equiv -t \frac{\psi_E(j)}{\psi_E(i)} \quad \text{for} \quad \{i, j : |j| < |i|, |i-j| = 1\}. \tag{4.2.3}$$

The variables $\{z_i\}$ describe the decay(or growth)-rate of eigenfunctions which are defined on each site except the origin 0. The number of degrees of freedom is $N-1$, where N denotes the total number of sites of the system. We can thus interpret eq.(4.2.2) as the relation between the variables $\{V_i : i = 0, 1, \ldots, N\}$ and $\{E, z_j : j = 1, 2, \ldots, N\}$.

We consider here the change of variables for the probability density [44, 45, 74]:

$$\prod_i P(V_i) \mathrm{d}V_i = \prod_i P(E - z_i - t^2 \sum_{l=1}^{K} z_{j(l)}^{-1}) J(\{V_i\}, \{E, z_k\}) \mathrm{d}E \prod_j \mathrm{d}z_j, \tag{4.2.4}$$

where $P(V_i)$ denotes the distribution function of random potentials. It is obtained in the case of the Bethe lattice that the Jacobian $J(\{V_i\}, \{E, z_k\})$ is given by [74]

$$\begin{aligned} J(\{V_i\}, \{E, z_k\}) &= \|\alpha\|^2 \\ &= \sum_i \alpha(i)^2 \\ &= 1 + \sum_{i \neq 0} \prod_{0 \to i} t^2 z_j^{-2}. \end{aligned} \tag{4.2.5}$$

Here $\{\alpha(i)\}$ are the eigenfunction defined by

$$\alpha(i) \equiv \frac{\psi_E(i)}{\psi_E(0)}, \quad i \in S_n, \tag{4.2.6}$$

namely, this is the eigenfunction normalized so that $\alpha(0) = 1$. Here $\prod_{0 \to i}$ means the product along the shortest path from the origin to the site i. It is then obvious

that we can not define the Jacobian if the wavefunction ψ_E is not normalizable. The brief derivation of these results on the Bethe lattice is shown in Appendix 4.A. From this change of variables, we can obtain the distribution of the variables $\{z_k\}$. If we consider the distribution $F_{E_0}(z_l)$ of z_l for fixed energy E_0 defined by

$$F_{E_0}(z_l)\mathrm{d}z_l \equiv \int \prod_i P(E - z_i - t^2 \sum_{m=1}^{K} z_{j(m)}^{-1}) J(\{V_i\}, \{E, z_k\}) \delta(E - E_0) \mathrm{d}E \prod_{j \neq l} \mathrm{d}z_j, \tag{4.2.7}$$

the average of a function $f(z_l)$ of z_l for the fixed energy E_0 can be obtained as

$$\begin{aligned} \langle f(z_l) \rangle_{E=E_0} &\equiv \int f(z_l) \delta(E - E_0) \prod_i P(V_i) \mathrm{d}V_i \\ &= \int f(z_l) F_{E_0}(z_l) \mathrm{d}z_l. \end{aligned} \tag{4.2.8}$$

In the following, we explain why the eigenfunction decays exponentially toward the boundary by analyzing the distribution $F_E(z_i)$ of the variables $\{z_i\}$. As an example, we consider the one-dimensional system $[-M, M]$ as shown in Fig. 4.1. We find that the argument below is easily extended to the case of the Bethe lattice.

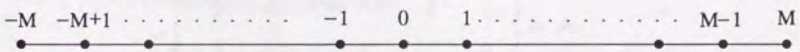

Fig. 4.1: The one-dimensional system $[-M, M]$.

In one dimension, the relation (4.2.2) is explicitly given as

$$\begin{aligned} V_M &= E - z_M \\ V_i &= E - z_i - t^2 z_{i+1}^{-1}, \quad M-1 > i > 0 \\ V_0 &= E - t^2 (z_1)^{-1} - t^2 (z_{-1})^{-1} \end{aligned}$$

$$
\begin{aligned}
V_i &= E - z_i - t^2 z_{i-1}^{-1}, \quad -M+1 < i < 0 \\
V_{-M} &= E - z_{-M}.
\end{aligned} \tag{4.2.9}
$$

The Jacobian $J(\{V_i\},\{E,z_k\})$ is then given by

$$
J(\{V_i\},\{E,z_k\}) = \left|\det\left(\frac{\partial(\{V_i\})}{\partial(\{E,z_k\})}\right)\right| = |\det D| \tag{4.2.10}
$$

where the matrix D is defined as

$$
D \equiv \left(\begin{array}{c|ccc|ccc}
1 & t^2 z_1^{-2} & 0 & \cdots\ 0 & t^2 z_{-1}^{-2} & 0 & \cdots\ 0 \\
\hline
1 & & & & & & \\
\vdots & & A_1 & & & 0 & \\
1 & & & & & & \\
\hline
1 & & & & & & \\
\vdots & & 0 & & & A_2 & \\
1 & & & & & &
\end{array}\right) \tag{4.2.11}
$$

$$
A_1 \equiv \begin{pmatrix}
-1 & t^2 z_2^{-2} & 0 & \cdots & 0 & 0 \\
0 & -1 & t^2 z_3^{-2} & 0 & \cdots & 0 \\
\vdots & 0 & \ddots & \ddots & 0 & 0 \\
0 & \vdots & 0 & \ddots & \ddots & 0 \\
0 & 0 & \vdots & 0 & -1 & t^2 z_M^{-2} \\
0 & 0 & 0 & 0 & 0 & -1
\end{pmatrix},
$$

$$
A_2 \equiv \begin{pmatrix}
-1 & t^2 z_{-2}^{-2} & 0 & \cdots & 0 & 0 \\
0 & -1 & t^2 z_{-3}^{-2} & 0 & \cdots & 0 \\
\vdots & 0 & \ddots & \ddots & 0 & 0 \\
0 & \vdots & 0 & \ddots & \ddots & 0 \\
0 & 0 & \vdots & 0 & -1 & t^2 z_{-M}^{-2} \\
0 & 0 & 0 & 0 & 0 & -1
\end{pmatrix}.
$$

We can easily find that [45]

$$
J(\{V_i\},\{E,z_k\}) = |\det D| = \left|\det\left(\begin{array}{c|c|c}
a_0 & 0 & 0 \\
\hline
* & I & 0 \\
\hline
* & 0 & I
\end{array}\right)\right| = a_0, \tag{4.2.12}
$$

where

$$
\begin{aligned}
a_0 &= 1 + \sum_{i>0}\prod_{j=1}^{i} t^2 z_j^{-2} + \sum_{i<0}\prod_{j=-1}^{i} t^2 z_j^{-2} \\
&= 1 + \sum_{i\neq 0}\left(\prod_{0\to i} t^2 z_j^{-2}\right),
\end{aligned} \tag{4.2.13}
$$

and I denotes the identity matrix. This is the special case of eq.(4.2.5) [45]. We thus find that the probability distribution is expressed as [45]

$$
\begin{aligned}
\prod_i P(V_i)\mathrm{d}V_i &= \sum_r P(E - z_{-M}) \cdot \prod_{i=-M+1}^{-1} P(E - z_i - t^2 z_{i-1}^{-1}) \\
&\quad \cdot P(E - t^2 z_{-1}^{-1} - t^2 z_1^{-1}) \\
&\quad \cdot \prod_{i=1}^{M-1} P(E - z_i - t^2 z_{i+1}^{-1}) \cdot P(E - z_M) \\
&\quad \left(\prod_{0\to r} t^2 z_j^{-2}\right)\mathrm{d}E \prod_{i\neq 0} \mathrm{d}z_i.
\end{aligned} \tag{4.2.14}
$$

We introduce the quantity $I(r)$ defined by

$$
\begin{aligned}
I(r) &\equiv P(E - z_{-M}) \cdot \prod_{i=-M+1}^{-1} P(E - z_i - t^2 z_{i-1}^{-1}) \\
&\quad \cdot P(E - t^2 z_{-1}^{-1} - t^2 z_1^{-1}) \\
&\quad \cdot \prod_{i=1}^{M-1} P(E - z_i - t^2 z_{i+1}^{-1}) \cdot P(E - z_M) \\
&\quad \left(\prod_{0\to r} t^2 z_j^{-2}\right)\mathrm{d}E \prod_{i\neq 0} \mathrm{d}z_i.
\end{aligned} \tag{4.2.15}
$$

The probability density in the left hand site of eq.(4.2.14) is then expressed as

$$
\prod_i P(V_i)\mathrm{d}V_i = \sum_r I(r). \tag{4.2.16}
$$

By changing the variables in eq.(4.2.15) as

$$
t^2 z_j^{-1} = \tilde{z}_{j-1} \tag{4.2.17}
$$

for $j = 1, 2, \ldots, r$, we obtain that

$$\begin{aligned} I(r) &= P(E - z_{-M}) \cdot \prod_{i=-M+1}^{-1} P(E - z_i - t^2 z_{i-1}^{-1}) \\ &\cdot P(E - t^2 z_{-1}^{-1} - t^2 \tilde{z}_0^{-1}) \\ &\cdot \prod_{i=1}^{r-1} P(E - \tilde{z}_i - t^2 \tilde{z}_{i-1}^{-1}) \\ &\cdot P(E - t^2 \tilde{z}_{r-1}^{-1} - t^2 z_{r+1}^{-1}) \\ &\cdot \prod_{i=r+1}^{M-1} P(E - z_i - t^2 z_{i+1}^{-1}) \cdot P(E - z_M) \\ &\mathrm{d}E \prod_{i<0} \mathrm{d}z_i \prod_{0 \le i < r} \mathrm{d}\tilde{z}_i \prod_{i>r} \mathrm{d}z_i. \end{aligned} \tag{4.2.18}$$

In order to get a simpler expression, let us introduce new variables $\{x_i^{(r)}\}$ for $\{i : i \neq r\}$ which are defined by

$$\begin{aligned} x_i^{(r)} &\equiv -t \frac{\psi_E(i+1)}{\psi_E(i)}, \quad \text{for} \quad i < r \\ \text{and} \quad x_i^{(r)} &\equiv -t \frac{\psi_E(i-1)}{\psi_E(i)}, \quad \text{for} \quad i > r. \end{aligned} \tag{4.2.19}$$

Using these variables $\{x_i^{(r)}\}$, $I(r)$ can be expressed for any r as

$$\begin{aligned} I(r) &= P(E - x_{-M}^{(r)}) \prod_{i=-M+1}^{r-1} P(E - x_i^{(r)} - t^2 (x_{i-1}^{(r)})^{-1}) \\ &\cdot P(E - t^2 (x_{r-1}^{(r)})^{-1} - t^2 (x_{r+1}^{(r)})^{-1}) \\ &\prod_{i=r+1}^{M-1} P(E - x_i - t^2 (x_{i+1}^{(r)})^{-1}) \cdot P(E - x_M^{(r)}) \mathrm{d}E \prod_{i \neq r} \mathrm{d}x_i^{(r)} \\ &\equiv \tilde{I}_E(r) \mathrm{d}E \prod_{i \neq r} \mathrm{d}x_i^{(r)}. \end{aligned} \tag{4.2.20}$$

4.3 Exponential Decay of the Eigenfunction ψ_E

In order to discuss the average of the ratio R_k of the eigenfunction ψ_E defined by

$$R_k \equiv \frac{\psi_E(k-1)}{\psi_E(k)}, \tag{4.3.1}$$

we consider a distribution $F_E(R_k)$ of R_k for fixed energy E, which is obtained by integrating out over the variables $\{z_i\}$ except R_k in the probability density (4.2.4). From eq.(4.2.16) and eq.(4.2.20), such a distribution $F_E(R_k)$ is expressed by the summation of contributions from each $I(r)$ as

$$\begin{aligned} F_E(R_k)\mathrm{d}R_k\mathrm{d}E &= \mathrm{d}E \sum_{r<k} \mathrm{d}x_k^{(r)} \int \tilde{I}_E(r) \prod_{i \neq r,k} \mathrm{d}x_i^{(r)} \\ &\quad + \mathrm{d}E \sum_{r \ge k} \mathrm{d}x_{k-1}^{(r)} \int \tilde{I}_E(r) \prod_{i \neq r,k-1} \mathrm{d}x_i^{(r)} \\ &= \mathrm{d}E\mathrm{d}(-tR_k) \sum_{r<k} \int \tilde{I}_E(r) \prod_{i \neq r,k} \mathrm{d}x_i^{(r)} \\ &\quad + \mathrm{d}E\mathrm{d}(-tR_k^{-1}) \sum_{r \ge k} \int \tilde{I}_E(r) \prod_{i \neq r,k-1} \mathrm{d}x_i^{(r)}. \end{aligned} \tag{4.3.2}$$

Note that R_k and $x_k^{(r)}$ is related by

$$\begin{aligned} x_k^{(r)} &= -tR_k \quad \text{for } r < k \\ x_{k-1}^{(r)} &= -tR_k^{-1} \quad \text{for } r \ge k. \end{aligned} \tag{4.3.3}$$

Using this distribution $F_E(R_k)$, the average of some function $f(R_k)$ of R_k for fixed energy E_0 is given by

$$\begin{aligned} \langle f(R_k)\delta(E - E_0) \rangle &\equiv \int f(R_k)\delta(E - E_0) \prod_i P(V_i)\mathrm{d}V_i \\ &= \int f(R_k) F_{E_0}(R_k)\mathrm{d}R_k. \end{aligned} \tag{4.3.4}$$

The distribution $F_E(R_k)$ is expressed as a linear combination of the contributions from $r \ge k$ and those from $r < k$, and we consider them separately in the following.

In order to estimate the contribution from each $I(r)$, we introduce here transfer operators T and T^* defined by

$$T(x;y) \equiv P(E - x - t^2 y^{-1}) \tag{4.3.5}$$

and

$$T^*(x;y) \equiv P(E - t^2 x^{-1} - y), \tag{4.3.6}$$

respectively. Namely, $T(x;y) = T^*(y;x)$. The operation is defined by

$$T \cdot f(x) \equiv \int \mathrm{d}y T(x;y) f(y), \tag{4.3.7}$$

and the functions which we treat in the following are assumed to be nonnegative ($f(x) \geq 0$), since we discuss here the transformation of distribution functions. The following properties of these two operators are crucial to estimate the distribution function of the ratio of the eigenfunction $\psi_E(i-1)/\psi_E(i)$.

First, let us consider the operator T^*. Since $P(V) \geq 0$ and $\int P(V)\mathrm{d}V = 1$, we obtain that

$$T^* \cdot 1 = 1. \tag{4.3.8}$$

This means that the steady solution of the recursion equation

$$f_{n+1} = T^* f_n \tag{4.3.9}$$

is given by a constant. From this property, we expect that

$$(T^*)^n f(x) \approx \text{constant} \geq 0, \tag{4.3.10}$$

for $n \gg 1$ and for an arbitrary nonnegative function $f(x)(\geq 0)$.

Next, we consider the operator T. The recursion equation for the operator T

$$Q_{n+1} = T \cdot Q_n \tag{4.3.11}$$

is the special case (K=1) of the recursion relation (3.2.13) in section 3.2 in chapter 3. The relation between the steady solution Q of eq.(4.3.11) and the Lyapunov exponent Λ_E is then given by

$$\Lambda_E = \int \ln|y| Q(y)\mathrm{d}y - \ln|t|, \tag{4.3.12}$$

(see section 3.3 in chapter 3). It is also expected that

$$\lim_{n\to\infty} T^n \cdot f = Q, \tag{4.3.13}$$

for the nonnegative function $f(x)(\geq 0)$ with $\int f(x)\mathrm{d}x = 1$.

By using these two operators T and T^*, we can express the contribution from $I(r)$ with $r < k$ (the first term of eq.(4.3.2)) as

$$\int \tilde{I}_E(r) \prod_{i\neq k,r} \mathrm{d}x_i^{(r)} = (T^*)^{k-r-1} T_0 T^{M-1+r} P(x_k^{(r)}) \cdot T^{M-k} P(x_k^{(r)}), \tag{4.3.14}$$

where T_0 is defined by

$$T_0(x;y) \equiv P(E - t^2 x^{-1} - t^2 y^{-1})$$

(see Fig. 4.2(a)). If we consider the case $k - r \gg 1$, it is expected that

$$(T^*)^{k-r-1} \cdot g(x_k^{(r)}) \approx \text{constant} \geq 0. \tag{4.3.15}$$

If the site k is also sufficiently far from the boundary ($M - k \gg 1$), the contribution (4.3.14) is considered to be proportional to the steady distribution Q:

$$\int \tilde{I}_E(r) \prod_{i\neq k,r} \mathrm{d}x_i^{(r)} \propto Q(x_k^{(r)}). \tag{4.3.16}$$

The variable $x_k^{(r)}$ for $k > r$ is related to R_k as

$$x_k^{(r)} \equiv -t\frac{\psi_E(k-1)}{\psi_E(k)} = -tR_k. \tag{4.3.17}$$

If the distribution function of the variable $x_k^{(r)}$ is given by $Q(x_k^{(r)})$, it is then obtained from eq.(4.3.12) that

$$\int \ln|R_k| Q(x_k^{(r)}) \mathrm{d}x_k^{(r)} = \int \ln|y| Q(y) \mathrm{d}y - \ln|t| = \Lambda_E > 0. \tag{4.3.18}$$

This implies that the wavefunction ψ_E decays toward the boundary of the right hand side.

On the other hand, the contribution from $I(r)$ with $r \geq k$ (the second term of eq.(4.3.2)) is given by

$$\int \tilde{I}_E(r) \prod_{i \neq k-1, r} \mathrm{d}x_i^{(r)} = T^{k+M} P(x_k^{(r)}) \cdot (T^*)^{r-1-k} T_0 T^{M-r-1} P(x_k^{(r)}) \tag{4.3.19}$$

(see Fig. 4.2(b)). We then also find that the contribution (4.3.19) is almost proportional to the steady distribution Q under the condition $-M \ll k \ll r$. Note first that $x_{k-1}^{(r)}$ for $r \geq k$ is related to R_k as

$$x_{k-1}^{(r)} \equiv -t \frac{\psi_E(k)}{\psi_E(k-1)} = -t R_k^{-1}. \tag{4.3.20}$$

If the distribution function $x_{k-1}^{(r)}$ is given by $Q(x_{k-1}^{(r)})$, we obtain

$$\int \ln|R_k| Q(x_{k-1}^{(r)}) \mathrm{d}x_{k-1}^{(r)} = -\Lambda_E < 0 \tag{4.3.21}$$

in this case. This implies the exponential decay of eigenfunctions toward the boundary of the left hand side.

From the argument given above, we obtained that in eq.(4.3.2) the first part gives the exponential decay toward the right hand side and the second part gives the one toward the left hand side of the system. As a result, the distribution function $F_E(R_k)$ of the ratio of eigenfunctions

$$R_k \equiv \frac{\psi_E(k-1)}{\psi_E(k)} \tag{4.3.22}$$

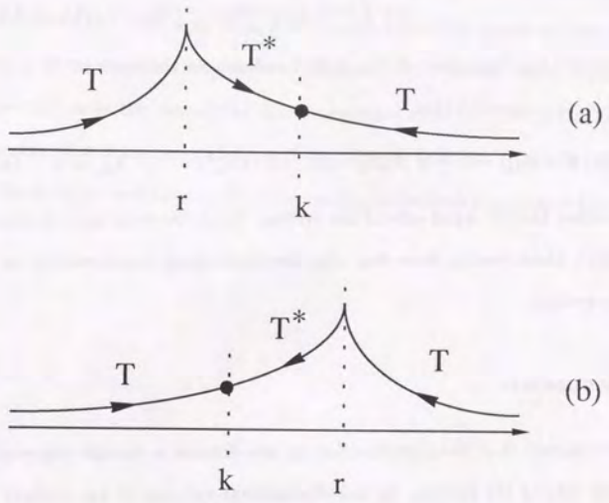

Fig. 4.2:

at a fixed site k can be approximately expressed as a linear combination

$$F_E(R_k)\mathrm{d}R_k \approx c_1 Q(-tR_k)\mathrm{d}(-tR_k) + c_2 Q(-tR_k^{-1})\mathrm{d}(-tR_k^{-1}). \tag{4.3.23}$$

Note that the constants c_1 and c_2 are nonnegative, since they are obtained by repeating integrations of nonnegative functions.

As the site k approaches the boundary of the right hand side, it is expected that the contribution from $\{I(r) : r < k\}$ becomes dominant; the coefficient c_1 approaches some finite constant $c > 0$ whereas $c_2 \to 0$. It is then considered that the distribution $F_E(R_k)$ of R_k may be approximately given by $c \cdot Q(-tR_k)$. Consequently, we get

$$\langle \ln|R_k| \delta(E - E_0) \rangle \equiv \int \ln|R_k| \delta(E - E_0) \prod_i P(V_i) \mathrm{d}V_i$$

$$= \int \ln|R_k| F_{E_0}(R_k) \mathrm{d}R_k$$
$$\to c \int \ln|R_k| Q(-tR_k) \mathrm{d}(-tR_k) = c \cdot \Lambda_{E_0} > 0 \quad (4.3.24)$$

as k approaches the boundary of the right hand side of the system. In a similar manner, it is also expected that

$$\langle \ln|R_k| \delta(E - E_0) \rangle \to c \int \ln|R_k| Q(-tR_k^{-1}) \mathrm{d}(-tR_k^{-1}) = -c \cdot \Lambda_{E_0} < 0 \quad (4.3.25)$$

as k approaches the left hand side of the system. Since the constant c is expected to be positive, these results show that eigenfunctions decay exponentially on both sides of the system.

4.4 Discussion

We have argued that the eigenfunction in one dimension decays exponentially fast on both side of the system. In one-dimensional systems, it has already been shown that the exponential decay-rate of the correlation of eigenfunctions defined by

$$-\lim_{L\to\infty} \frac{1}{L} \ln|\psi_E(0)\psi_E(L)| \quad (4.4.1)$$

is given by the Lyapunov exponent Λ_E [47, 48]. In order to estimate the decay-rate, we have to calculate the constant c, which is not yet done at present. The foregoing argument provides an intuitive understanding on the decay of eigenfunctions on both sides of the system, though it is not rigorous. Furthermore, this argument can be applied straightforwardly to the case of the Bethe lattice. The transfer operators T and T^* in the case of the Bethe lattice should be chosen as

$$T(x;y) \equiv \int \prod_{i=1}^{K-1} Q(y_i) \mathrm{d}y_i P(E - x - t^2 y^{-1} - t^2 \sum_{i=1}^{K-1} y_i^{-1}) \quad (4.4.2)$$

and $T^*(x;y) \equiv T(y;x)$. Here $Q(y)$ is the steady solution of eq.(3.2.14) in section 3.2 in chapter 3. The fundamental properties of the operators T and T^* in the case of

the Bethe lattice are the same as those in the case of one-dimensional systems. On the basis of the same argument, it is expected that the eigenfunctions on the Bethe lattice decay exponentially toward the boundary. On the Bethe lattice it may be expected, as in the case of the one-dimensional systems, that the decay-rate of the eigenfunction coincides with that of the Green's function. However, the estimation of the decay-rate of the eigenfunction on the Bethe lattice remains a future problem.

Appendix 4.A : Derivation of the Jacobian

For simplicity, we consider the case $K = 2$ in the following. First, we consider the change of veriables from $\{V_i\}$ to $\{E, \alpha(j)\}$. Here α is the eigenfunction normalized so that $\alpha(0) = 1$. The relation of these two variables is given by

$$V_i = E - t\frac{\sum_{|j-i|=1}\alpha(j)}{\alpha(i)}. \tag{4.A.1}$$

Let us consider a portion of the Bethe lattice at the boundary as shown in Fig. 4.3. The elements of the Jacobian matrix corresponding to the sites $\{2, 4, 5\}$ are given by

$$\det(-t)^{N-1}\begin{pmatrix} * & * & & 0 & & * \\ \hline 1 & 0\ \cdots\ 0 & \sum\alpha/\alpha(2)^2 & -\alpha(2)^{-1} & -\alpha(2)^{-1} & 0\ \cdots\ 0 \\ 1 & 0\ \cdots\ 0 & -\alpha(4)^{-1} & \alpha(2)/\alpha(4)^2 & 0 & 0\ \cdots\ 0 \\ 1 & 0\ \cdots\ 0 & -\alpha(5)^{-1} & 0 & \alpha(2)/\alpha(5)^2 & 0\ \cdots\ 0 \\ \hline * & * & & 0 & & * \end{pmatrix}. \tag{4.A.2}$$

We can transform this part as

$$= c_0 \det\begin{pmatrix} * & * & & 0 & & * \\ \hline \alpha(2) & 0\ \cdots\ 0 & \sum\alpha/\alpha(2) & -1 & -1 & 0\ \cdots\ 0 \\ \alpha(4) & 0\ \cdots\ 0 & -1 & \alpha(2)/\alpha(4) & 0 & 0\ \cdots\ 0 \\ \alpha(5) & 0\ \cdots\ 0 & -1 & 0 & \alpha(2)/\alpha(5) & 0\ \cdots\ 0 \\ \hline * & * & & 0 & & * \end{pmatrix}$$

$$= c_0 \det\begin{pmatrix} * & * & & 0 & & * \\ \hline b_0 & 0\ \cdots\ 0 & \alpha(1)/\alpha(2) & 0 & 0 & 0\ \cdots\ 0 \\ \alpha(4) & 0\ \cdots\ 0 & -1 & \alpha(2)/\alpha(4) & 0 & 0\ \cdots\ 0 \\ \alpha(5) & 0\ \cdots\ 0 & -1 & 0 & \alpha(2)/\alpha(5) & 0\ \cdots\ 0 \\ \hline * & * & & 0 & & * \end{pmatrix}$$

where $c_0 = \alpha(2)^{-1}\alpha(4)^{-1}\alpha(5)^{-1}(-t)^{N-1}$ and $b_0 = (\alpha(2)^2 + \alpha(4)^2 + \alpha(5)^2)/\alpha(2)$. We can perform the same procedure for the sites $\{3, 6, 7\}$ independently. Repeating these procedures from the boundary to the origin, it is obtained that the Jacobian is given by

$$J(\{V_i\}, \{E, \alpha(j)\}) = |t|^{N-1}||\alpha||^2 \prod_{i\neq 0}\left(|\alpha(i)|^{-1}\frac{|\alpha(i')|}{|\alpha(i)|}\right), \tag{4.A.3}$$

where $||\alpha||^2 \equiv \sum_i |\alpha(i)|^2$ and the site i' is the site uniquely determined for each i by the condition $|i' - i| = 1$ and $|i| > |i'|$.

Next, we consider another change of variables

$$\{\alpha(i)\} \to \{z_i\}, \tag{4.A.4}$$

where z_i is defined by

$$z_i \equiv -t\frac{\alpha(i')}{\alpha(i)} \quad \text{for} \quad \{i, i' : |i| > |i'|, |i - i'| = 1\}. \tag{4.A.5}$$

It is then obtained that the Jacobian $J(\{\alpha(i)\}, \{z_i\})$ is given by

$$J(\{\alpha(i)\}, \{z_i\}) = \prod_{i\neq 0}|\alpha(i)||z_i|^{-1}. \tag{4.A.6}$$

From eq.(4.A.3) and eq.(4.A.6), we finally arrive at

$$J(\{V_i\}, \{E, z_i\}) = ||\alpha||^2. \tag{4.A.7}$$

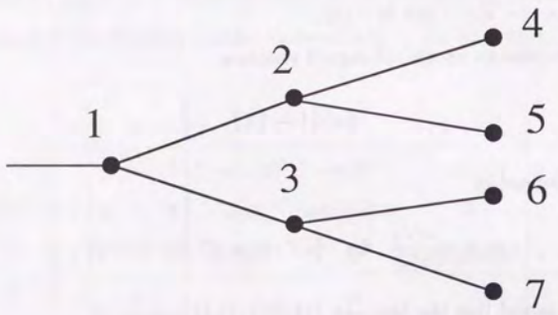

Fig. 4.3: A portion of the Bethe lattice at the boundary.

Chapter V

Summary and Future Problems

In the present thesis, the Anderson localization transition has been investigated by self-consistent approaches.

In chapter 2, we have obtained mobility edge trajectories for several distributions of site energies and the results are qualitatively in good agreement with those obtained by the finite-size scaling method.

In chapter 3, we have shown that in the tight-binding Anderson model on the Bethe lattice the decay-rate of the Green's function can be obtained for arbitrary energies and arbitrary disorder. The explicit results for the Lorentzian distribution of site energies have been presented. Based on these exact results on the Bethe lattice, we have proposed a criterion for localization in the corresponding real systems. Our criterion yields exact band edges in the vanishing limit of disorder and also yields exact results in one dimension. Moreover, this criterion is expected to hold better in higher dimensions. The mobility edge trajectory by this criterion for the Lorentzian distribution of site energies is given by the following elliptic curve

$$\frac{E^2}{(K+1)^2} + \frac{\gamma^2}{(K-1)^2} = t^2, \tag{5.1.1}$$

(see section 3.5) and it is quite consistent with the result obtained by the finite-size scaling method.

In chapter 4, the exponential decay of the eigenfunction near the boundary has been discussed. An intuitive explanation of the decay of the eigenfunction was described.

As was mentioned in chapter 1, one of our motivations for considering the mean-field type approaches is to obtain the value of the critical exponent ν in high dimensions. We have obtained in chapter 3 that the exponent ν is given by $\nu = 1$ on the basis of the argument on the Bethe lattice. To relate the argument on the Bethe lattice with that in high dimensions, we may think of the Bethe lattice as being embedded in the infinite-dimensional hypercubic lattice ($\boldsymbol{Z}^\infty$) [79]. On the Bethe lattice embedded in $\boldsymbol{Z}^\infty$, the euclidean distance between the origin and a site in the Lth shell can be taken as $\sqrt{L}$ instead of L [79]. On the grounds of this consideration, Kunz and Souillard claimed that $\nu = 1$ on the Bethe lattice should be reinterpreted as $\nu = 1/2$ for the d-dimensional systems with a large $d(\gg 1)$ [6, 74, 75], as in the case of percolation [79, 80] or the Ising model [81]. Our result for the exponent ν in chapter 2 is consistent with this interpretation. Using this value of the exponent ν and other ingredients, it is also argued by Kunz and Souillard that the upper critical dimensionality would be four ($d_c = 4$) [6, 75]. This problem, however, has not yet been settled and remains to be studied in future.

As for the mobility edge trajectory, it is given by eq.(5.1.1) for the Lorentzian distribution of site energies. This may suggest that in order to obtain a non-trivial phase diagram of the localization transition in the limit as $K \to \infty$, we should consider a scaling, for example, as

$$t \propto \tilde{t}/\sqrt{K}, \quad E \propto \tilde{E}\sqrt{K}, \quad \gamma \propto \tilde{\gamma}\sqrt{K} \tag{5.1.2}$$

with fixed $\tilde{t}$, $\tilde{E}$ and $\tilde{\gamma}$. It may be also an interesting problem to study the infinite-dimensional limit ($d = \infty$) of the Anderson transition with the above scaling ($K = 2d - 1$).

Our criterion yields the localization transition even in the two-dimensional system. In our approximation in chapter 3, the effect of closed loops was neglected.

This may be the reason why the localization transition occurs in our criterion even for the two-dimensional system.

In the present paper, we have discussed the localization length but have not mentioned the conductivity. It would be also a challenging problem to study the critical exponent for the conductivity near the mobility edge in the extended region using the Kubo formula [82]. We have also not mentioned the critical exponents in the two or three-dimensional systems. Since we have obtained good results for the mobility edge trajectories, our next important problem would be to study the critical exponents.

Actually, it should be made clear whether the decay-rate of the Green's function is self-averaging or not even in the d-dimensional system ($d > 1$). The relation between the decay-rate of the Green's function and physical quantities should also be clarified rigorously.

Recently, mathematical approaches to the localization transition have been applied to multi-dimensional systems. These approaches may give us another perspective of the Anderson localization.

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