博士論文

論文題目　Calabi–Yau 3-folds in Grassmannians and their I-functions
（グラスマン多様体に含まれる3次元カラビ・ヤウ多様体とそれらの I-関数）

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Calabi–Yau 3-folds in Grassmannians and their I-functions

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Introduction

In this thesis, we call a smooth projective variety Y with trivial canonical bundle and $H^i(Y, \mathcal{O}_Y) = 0$ $(i = 1, \ldots, \dim Y - 1)$ *Calabi–Yau manifold.* Calabi–Yau manifolds of dimension three have been attracting attentions of algebraic geometers and theoretical physicists for the last decades since the discovery of mirror symmetry.

Let Y be a quintic Calabi–Yau 3-fold, i.e. it is defined by a degree five homogeneous polynomial in $\mathbb{P}^4$. Greene and Plesser constructed another Calabi–Yau 3-fold which is so-called mirror of Y. To describe it, let f_ψ be the following one-parameter family of quintic polynomials:

$$f_\psi = z_0^5 + z_1^5 + z_2^5 + z_3^5 + z_4^5 - 5\psi z_0 z_1 z_2 z_3 z_4,$$

where $[z_0 : \cdots : z_4]$ is the homogeneous coordinate of $\mathbb{P}^4$. A finite group $G = (\mathbb{Z}/5\mathbb{Z})^{\oplus 3} \subset SL(5, \mathbb{C})$ acts on the zero locus of f_ψ, and the mirror Y_ψ^* is constructed by a crepant resolution of $\{f_\psi = 0\}/G$ for general ψ. The mirror Calabi–Yau 3-folds Y_ψ^* define a one-parameter family $\{Y_x^*\}_{x \in \mathbb{P}^1}$ over $\mathbb{P}^1$ with $x = (5\psi)^{-5}$.

The quintic Y defines the so-called A-model Yukawa coupling by

$$K_{ttt} = 5 + \sum_{d=1}^{\infty} d^3 N_d q^d$$

where $N_d = \langle\ \rangle_{0,0,d}$ are Gromov–Witten invariants of Y and $q = e^t$. On the other hand, the mirror family $\{Y_x^*\}_{x \in \mathbb{P}^1}$ defines B-model Yukawa coupling. Let Ω_x be a nowhere vanishing holomorphic three-form on Y_x^*. We define the B-model Yukawa coupling by

$$\int_{Y_x^*} \Omega_x \wedge (\nabla_{\frac{d}{dx}})^3 \Omega_x,$$

where ∇ is the Gauss–Manin connection of the family. Mirror symmetry predicts that these two functions are related by the so-called mirror map which is defined by the period integrals.

The period integrals of the mirror quintic are annihilated by

$$P = \theta^4 - 5x(5\theta + 1)(5\theta + 2)(5\theta + 3)(5\theta + 4) \quad \left(\theta := x\frac{d}{dx}\right), \tag{1}$$

which we call Picard–Fuchs operator. The B-model Yukawa coupling and the mirror map are determined from the solutions of the Picard–Fuchs equation $P\omega(x) = 0$.

Candelas–de la Ossa–Green–Parks [CdlOGP91] calculated the B-model Yukawa coupling and the mirror map, and predicted the number of rational curves on the quintic Y for arbitrary degree d by mirror symmetry.

This physical prediction has been solved mathematically, based on the theory of Kontsevich's stable map, by Givental [Giv96] and Lian–Liu–Yau [LLY97]. In particular, Givental has introduced the so-called I-function as a fundamental object to study. In this thesis, we study the I-functions of Calabi–Yau 3-folds with second Betti number one which are given by zero loci of general sections of homogeneous vector bundles on Grassmannians. In the first half of this thesis, we will make a complete list of the pairs $(G(k,n),\mathcal{E})$ such that the zero locus of a general section of $\mathcal{E}$ gives a Calabi–Yau 3-fold in $G(k,n)$.

In his formulation, Givental introduced two functions: I-function and J-function. For the quintic Calabi–Yau 3-fold Y, I-function can be written as

$$I_Y = e^{tH}\sum_{d=0}^{\infty}\frac{\prod_{m=1}^{5d}(5H+m)}{\prod_{m=1}^{d}(H+m)^5}e^{dt}$$

where H is the restriction of the hyperplane class of $\mathbb{P}^4$, hence, I_Y takes values in $H^{\mathrm{even}}(Y,\mathbb{C})$. We write $I_Y=\sum_{i=0}^{3}I_iH^i$. Then I_0,I_1,I_2,I_3 form a basis of the solutions of (1). On the other hand, J-function of Y is defined by

$$J_Y = e^{\tau H}+\frac{H^2}{5}e^{\tau H}\sum_{d=1}^{\infty}n_d d^3\sum_{m=1}^{\infty}\frac{e^{\tau kd}}{(H+kd)^2}$$

where n_d is the so-called BPS numbers which are related to the Gromov–Witten invariants by $N_d=\sum_{k|d}k^{-3}n_{d/k}$. The numbers n_d coincide with the numbers of rational curves of degree d on Y for lower d. These I_Y and J_Y functions are related by

$$J_Y(\tau)=\frac{I_Y(t)}{I_0(t)}\quad\left(\tau=\frac{I_1(t)}{I_0(t)}\right)$$

where the change of coordinate $\tau=\tau(t)$ is the mirror map. This relation reproduces the calculation of Candelas *et al.* More generally the same construction holds for Calabi–Yau 3-folds which are complete intersection of nef line bundles on Gorenstein Fano toric varieties by [BB94] and [Giv98].

One of our motivations, in the second half of this thesis, is to generalize the above calculation beyond the toric complete intersection Calabi–Yau 3-folds. There are several constructions of Calabi–Yau 3-folds; for example, we have Calabi–Yau 3-folds in the following forms:

(i) Complete intersections of line bundles in homogeneous spaces or Fano varieties,

(ii) Zero loci of general sections of globally generated vector bundles on Fano manifolds,

(iii) Smoothings of singular Calabi–Yau varieties,

(iv) Quotients of Calabi–Yau manifolds by their finite symmetry,

(v) Coverings of Fano varieties ramified along their subvarieties.

For most of the above cases, we do not know how to obtain their mirror manifolds. However for the case of (i), i.e. Calabi–Yau 3-folds which are obtained by complete intersections of nef line bundles $\mathcal{L}_i$ on smooth Fano manifolds Z, when quantum cohomologies of Z are well-studied, we can obtain I-functions for $Y = s^{-1}(0)$ with general $s \in H^0(Z, \bigoplus_i \mathcal{L}_i)$ without their mirror manifolds. In order to calculate the I-functions I_Y, we only need to use the quantum Lefschetz theorem by Coates–Givental [CG07].

For the case (ii), we do not know a closed formula of the I-functions of Calabi–Yau 3-folds in general. However Bertram–Ciocan-Fontanine–Kim–Sabbah have given a conjectural formula which expresses the I-function when Z are given by geometric invariant theory quotients and the vector bundles $\mathcal{E}$ are induced from some representations. They have proved their conjectural formula for partial flag manifolds of type A, in particular, for Grassmannians. We set their results at the starting point of our calculations for Calabi–Yau 3-folds which are given by zero loci of vector bundles on Grassmannians.

On Grassmannians, we have natural vector bundles, so-called homogeneous vector bundles which are made of the universal sub-bundles or universal quotient bundles. In contrast to projective spaces, Grassmannians have a number of low rank vector bundles which do not decompose into line bundles. Hence we can find many pairs $(G(k,n), \mathcal{E})$ which give rise to Calabi–Yau 3-folds that are not complete intersections of line bundles.

In a similar context, Küchle has classified Fano 4-folds which are given by zero loci of general sections of globally generated homogeneous vector bundles [Küc95]. In [IIM1] we have given a classification of Calabi–Yau 3-folds which are given by zero loci of general sections of globally generated homogeneous vector bundles on Grassmannians, and have generalized the Küchle's list. This classification result is the main result in the first half of this thesis (see Main result 1.1.10).

For most of pairs $(G(k,n), \mathcal{E})$ in our list, we determine the I-functions of the corresponding Calabi–Yau 3-folds and also differential operators which annihilate the I-functions. For our calculation, we have to restrict our attentions to Calabi–Yau 3-folds with second Betti number one. This is because we only recover twisted J-functions from the I-functions. The twisted J-functions are closely related to the J-functions of the Calabi–Yau 3-folds Y. However some information may be lost when $H^2(G(k,n), \mathbb{C})) \to H^2(Y, \mathbb{C})$ is not surjective. The condition that Y has second Betti number one ensures that the I-function recovers the J-function of Y. The resulting Picard–Fuchs operators are presented in Appendix C. These are the main results in the second half of this thesis (see Main result 2.1.1).

The organization of this thesis is as follows:

In Chapter 1, we consider Calabi–Yau 3-folds which are zero loci of general sections of globally generated homogeneous vector bundles on Grassmannians. We classify all possible pairs $(G(k,n), \mathcal{E})$, up to natural identifications, which give rise to Calabi–Yau 3-folds in $G(k,n)$. Although [IIM1] argues higher dimensional Calabi–Yau manifolds, we only treat the classification of Calabi–Yau 3-fold in this thesis. First we recall some well-known facts about homogeneous vector bundles. Using the formulas of the rank and first Chern class of homogeneous vector bundle, we solve linear equations corresponding to dimensional condition and Calabi–Yau condition. We will determine the topological invariants of Calabi–Yau 3-fold in our list.

In Chapter 2, we turn our attentions to calculating the I-functions of Calabi–Yau 3-folds with $b_2 = 1$ in our list. Firstly, we recall the definition of the J-function especially for Calabi–Yau 3-folds with $b_2 = 1$. Secondly, we apply the abelian/nonabelian correspondence for the Calabi–Yau 3-folds with $b_2 = 1$ which are contained in Proposition 1.1.19. Lastly, we deal with the Calabi–Yau 3-fold which is isomorphic to a zero locus of a general section of the vector bundle on the determinantal nets of conics. We do the calculation under the assumption that Conjecture A.2.4 holds for determinantal nets of conics.

In Chapter 3, we give another calculation of the I-function based on the quantum Lefschetz theorem and the localization formula of twisted Gromov–Witten invariants. First we describe a general procedure to determine the $\mathcal{E}$-twisted J-function from the calculation of the $\mathcal{E}'$-twisted Gromov–Witten invariants where $\mathcal{E} = \mathcal{E}' \oplus \mathcal{H}$ and $\mathcal{H}$ is a direct sum of ample line bundles. Next we apply the general procedure to calculate the $\mathcal{E}$-twisted J-function of No. 25 in Table 1.1.

In Appendix A, we summarize the abelian/nonabelian correspondence. We give necessary data to apply the abelian/nonabelian correspondence for each example.

In Appendix B, we review the calculation of the twisted Gromov–Witten invariants by using the localization formula due to Kontsevich [Kon95]. We follow the notation of Liu–Sheshmani [LS15] which is a nice reference for these topics.

In Appendix C, we collect the Picard–Fuchs operators which are calculated in this thesis.

In Appendix D, we calculate integral symplectic monodromy matrices of selected Picard–Fuchs operators which are derived in Chapter 2 and 3.

Chapter 1 is based on [IIM1] and Chapter 2 and 3 are based on [IIM2]. Throughout of this thesis, we always consider a variety over $\mathbb{C}$.

Acknowledgments. I would like to express my deep gratitude to Professor Shinobu Hosono for worthwhile suggestions and warm encouragement. I am also grateful to Professor Hiromichi Takagi for encouragement. I would like to thank Doctors Atsushi Ito and Makoto Miura for interesting discussions at weekly seminars. I would like to thank Doctor Fumihiko Sanda for helpful explanations of abelian/nonabelian correspondence. I would like to thank Tatsuki Kuwagaki for useful comments at the seminars.

Contents

Chapter 1

Complete intersection Calabi–Yau 3-folds in Grassmannians by homogeneous vector bundles

In this chapter we study Calabi–Yau 3-folds which are given as the vanishing loci of general sections of homogeneous vector bundles over Grassmannians.

1.1 Classification

The method of our classification is similar to Küchle [Küc95]. Küchle classified Fano 4-folds Z which are given as general complete intersections of globally generated homogeneous vector bundles on Grassmannians. Hence general anti-canonical hypersurface of Z gives a Calabi–Yau 3-fold Y. However there exist Calabi–Yau 3-folds in Grassmannians which do not factor through Küchle's Fano 4-folds. Therefore we set up some linear equalities for Calabi–Yau 3-folds, which represent the dimensional condition and Calabi–Yau condition.

We classify pairs of the Grassmannian $G(k, n)$ $(2 \leq k \leq n-2)$ and homogeneous vector bundle $\mathcal{E}$ which is a globally generated homogeneous vector bundle over $G(k, n)$, satisfying

$$\begin{aligned} k(n-k) - \operatorname{rank}(\mathcal{E}) &= 3, \\ c_1(\mathcal{E}) &= n, \end{aligned}$$

and a zero locus of a general section of $\mathcal{E}$ gives a Calabi–Yau 3-fold.

In the following we will summarize classical results of homogeneous vector bundle on Grassmannian. We describe them by applying Schur functor to universal vector bundles on Grassmannian. The rank or first Chern class of homogeneous vector bundle are calculated by the formulas of Schur module. The classification problem is translated into solving linear equalities with respect to k, n and the decreasing sequences of non-negative integers corresponding to Schur modules.

1.1.1 Homogeneous vector bundles

Let $V = \mathbb{C}^n$ be an n-dimensional complex vector space. Let $G(k,n)$ be the Grassmannian of k-dimensional subspaces of V. Let $G = SL(n,\mathbb{C})$ be the special linear group. We fix a subgroup of G

$$P = \left\{ \begin{pmatrix} A & B \\ O & C \end{pmatrix} \;\middle|\; A \in GL(k,\mathbb{C}),\ C \in GL(n-k,\mathbb{C}),\ \det(A)\det(C) = 1 \right\},$$

which is called a parabolic subgroup of G. There exists an isomorphism $G(k,n) \cong G/P$. By using this isomorphism, we define a homogeneous vector bundle on $G(k,n)$.

Definition 1.1.1. Let $\mathcal{E}$ be a vector bundle on $G(k,n)$. We say that $\mathcal{E}$ is a homogeneous vector bundle if and only if there exist a rational representation $\rho : P \to GL(E)$ of the parabolic subgroup P on a vector space E such that

$$\mathcal{E} \cong G \times_P E.$$

Here we give examples of homogeneous vector bundles on Grassmannians.

Example 1.1.2 (universal sub-bundle)**.** Let $\mathcal{S}$ be the universal sub-bundle on $G(k,n)$. The fiber over $[W] \in G(k,n)$ is W. There exists a natural action of G on $\mathcal{S}$ which maps the fiber on $[W]$ to the fiber on $[gW]$ by $w \mapsto gw$ where $g \in G$.

Example 1.1.3 (universal quotient bundle)**.** Let $\mathcal{Q}$ be the universal quotient bundle on $G(k,n)$. The fiber over $[W] \in G(k,n)$ is V/W. Similarly G acts on $\mathcal{Q}$ which maps the fiber on $[W]$ to the fiber on $[gW]$ by $[v] \mapsto [gv]$ where $[v]$ is the equivalence class of $v \in V$ in V/W.

It is well-known that any homogeneous vector bundle on $G(k,n)$ is obtained by applying Schur functor to these vector bundles, i.e.

Fact 1.1.4. *Let $\lambda = (\lambda_1, \dots, \lambda_k)$ $(\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_k \geq 0)$ and $\mu = (\mu_1, \dots, \mu_{n-k})$ $(\mu_1 \geq \mu_2 \geq \cdots \geq \mu_{n-k} \geq 0)$ be sequences of decreasing non-negative integers. Let Σ^λ and Σ^μ be the Schur functors corresponding to λ and μ, respectively. Then any irreducible homogeneous vector bundle is isomorphic to*

$$\Sigma^\lambda \mathcal{S}^* \otimes \Sigma^\mu \mathcal{Q} \otimes \mathcal{O}(d) \tag{1.1}$$

for some λ, μ and $d \in \mathbb{Z}$.

Since $\Sigma^{(\lambda_1+t,\dots,\lambda_k+t)}\mathcal{S}^* = \Sigma^{(\lambda_1,\dots,\lambda_k)}\mathcal{S}^* \otimes \mathcal{O}(t)$ for $t \geq 0$, the expression of (1.1) is not unique. In the following we assume $\lambda_k = 0$ and $\mu_{n-k} = 0$ to avoid this ambiguity. Under this normalization, we have the following statement:

Fact 1.1.5. *Let $\mathcal{E} = \Sigma^\lambda \mathcal{S}^* \otimes \Sigma^\mu \mathcal{Q} \otimes \mathcal{O}(d)$ be an irreducible homogeneous vector bundle. The vector bundle $\mathcal{E}$ is globally generated, i.e. the natural map $H^0(G(k,n),\mathcal{E}) \otimes \mathcal{O}_{G(k,n)} \to \mathcal{E}$ is surjective if and only if $d \geq 0$.*

We recall the formulas of the rank and first Chern class of an irreducible homogeneous vector bundle on $G(k,n)$.

Proposition 1.1.6. *Let $\mathcal{E} = \Sigma^\lambda \mathcal{S}^* \otimes \Sigma^\mu \mathcal{Q} \otimes \mathcal{O}(d)$ be an irreducible homogeneous vector bundle on $G(k,n)$. Then*

$$\operatorname{rank}(\mathcal{E}) = \prod_{1 \le i_1 < i_2 \le k} \frac{i_2 - i_1 + \lambda_{i_1} - \lambda_{i_2}}{i_2 - i_1} \prod_{1 \le j_1 < j_2 \le n-k} \frac{j_2 - j_1 + \mu_{j_1} - \mu_{j_2}}{j_2 - j_1}, \tag{1.2}$$

$$c_1(\mathcal{E}) = \operatorname{rank}(\mathcal{E}) \left(\frac{|\lambda|}{k} + \frac{|\mu|}{n-k} + d \right), \tag{1.3}$$

where $|\lambda| = \sum_{i=1}^{k} \lambda_i$ and $|\mu| = \sum_{j=1}^{n-k} \mu_j$.

We are interested in a zero loci of general sections of globally generated homogeneous vector bundles. There is a Bertini type result for globally generated vector bundles.

Theorem 1.1.7 (cf. [Muk92])**.** *Let X be a smooth algebraic variety and $\mathcal{E}$ be a globally generated vector bundle on X. Let $s \in \Gamma(X, \mathcal{E})$ be a general global section of $\mathcal{E}$. Let Z_s be a zero locus of the section s. If Z_s is not empty, then Z_s is smooth of dimension* $\dim X - \operatorname{rank}(\mathcal{E})$.

By using this result, it is enough to find globally generated homogeneous vector bundles which have appropriate ranks and first Chern classes. The analysis of whether a zero locus of a general section s is empty or not can be done via the calculation of $H^0(Z_s, \mathcal{O}_{Z_s})$.

Next we present some examples of the zero loci of general sections of globally generated homogeneous vector bundles on Grassmannians.

Example 1.1.8 (Symplectic Grassmannian)**.** Let $V = \mathbb{C}^{2n}$ be a $2n$-dimensional vector space. Let $G(k, 2n)$ be a Grassmannian where $k \le n$. Let $\mathcal{E} = \wedge^2 \mathcal{S}^*$. The space of global section of $\mathcal{E}$ is isomorphic to $\wedge^2 V^*$. Let $s \in \Gamma(G(k,2n), \mathcal{E})$ be a global section corresponding to $\omega \in \wedge^2 V^*$. The zero locus of this section Z_s is equal to $\{[W] \in G(k,2n) \mid \omega|_W = 0\}$. From this, if we pick the two form ω as maximal rank $2n$, the zero locus coincides with the symplectic Grassmannian $IG(k, 2n)$.

Example 1.1.9 (Orthogonal Grassmannian)**.** Under the above setting, we take $\mathcal{E} = \operatorname{Sym}^2 \mathcal{S}^*$. Similarly for Example 1.1.8, the zero locus Z_s of a general section $s \in \Gamma(G(k,2n), \mathcal{E})$ is $\{[W] \in G(k,2n) \mid \omega|_W = 0\}$ where $\omega \in \operatorname{Sym}^2 V^*$ corresponds to s. We consider a general section s, this implies that ω has maximal rank $2n$. If $k < n$, the zero locus Z_s coincides with orthogonal Grassmannian $OG(k, 2n)$. The case $k = n$ is exceptional. In this case, there exist two families of isotropic n-dimensional subspaces in V. Then the zero locus Z_s is not connected.

1.1.2 List of Calabi–Yau 3-folds

We are interested in Calabi–Yau 3-folds Y which are given by the zero loci of general sections of globally generated homogeneous vector bundles on Grassmannians. Our main result is summarized as follows:

Main result 1.1.10. *There are only* 33 *pairs of $G(k,n)$ and globally generated homogeneous vector bundles $\mathcal{E}$ on $G(k,n)$ for each of which the zero locus of a general section gives a smooth Calabi–Yau* 3*-fold Y (see Table* 1.1*). Among the* 33 *pairs,* 22 *pairs gives rise to Calabi–Yau* 3*-folds of second Betti number one.*

No	$G(k,n)$	vecter bundle	H^3	$c_2.H$	c_3	$h^{1,1}$	Küchle	Database
1	$G(2,4)$	$\mathcal{O}(4)$	8	56	-176	1		6
2	$G(2,5)$	$\mathcal{O}(1)\oplus\mathcal{O}(2)^{\oplus 2}$	20	68	-120	1	(b2)	25
3		$\mathcal{O}(1)^{\oplus 2}\oplus\mathcal{O}(3)$	15	66	-150	1	(b1)	24
4		$S^*(1)\oplus\mathcal{O}(2)$	24	72	-116	1		29
5		$\Lambda^2 Q(1)$	25	70	-100	1		101
6	$G(2,6)$	$\mathcal{O}(1)^{\oplus 4}\oplus\mathcal{O}(2)$	28	76	-116	1	(b6)	26
7		$S^*(1)\oplus\mathcal{O}(1)^{\oplus 3}$	33	78	-102	1	(b5), I	198
8$*$		$\mathrm{Sym}^2 S^*\oplus\mathcal{O}(1)\oplus\mathcal{O}(2)$	40	88	-128	2	(b4)	
9$*$		$\mathrm{Sym}^2 S^*\oplus S^*(1)$	48	84	-92	2		
10		$Q(1)\oplus\mathcal{O}(1)$	42	84	-98	1	(b3), II	27
11$*$		$\Lambda^3 Q\oplus\mathcal{O}(3)$	18	72	-162	2		
12	$G(2,7)$	$\mathcal{O}(1)^{\oplus 7}$	42	84	-98	1	(b7), III	27
13		$\mathrm{Sym}^2 S^*\oplus\mathcal{O}(1)^{\oplus 4}$	56	92	-92	1	(b8), V	212
14$*$		$(\mathrm{Sym}^2 S^*)^{\oplus 2}\oplus\mathcal{O}(1)$	80	80	-32	8	(b9)	
15		$\Lambda^4 Q\oplus\mathcal{O}(1)\oplus\mathcal{O}(2)$	36	84	-120	1	(b10)	185
16		$S^*(1)\oplus\Lambda^4 Q$	42	84	-98	1		27
17	$G(2,8)$	$\Lambda^5 Q\oplus\mathcal{O}(1)^{\oplus 3}$	57	90	-84	1	(b11), VI	186
18		$\mathrm{Sym}^2 S^*\oplus\Lambda^5 Q$	72	96	-72	1		unknown
19	$G(3,6)$	$\mathcal{O}(1)^{\oplus 6}$	42	84	-96	1	(c1), IV	28
20		$\Lambda^2 S^*\oplus\mathcal{O}(1)^{\oplus 2}\oplus\mathcal{O}(2)$	32	80	-116	1	(c2)	42
21		$S^*(1)\oplus\Lambda^2 S^*$	42	84	-96	1		28
22	$G(3,7)$	$\mathrm{Sym}^2 S^*\oplus\mathcal{O}(1)^{\oplus 3}$	128	128	-128	1	(c4)	3
23		$(\Lambda^2 S^*)^{\oplus 2}\oplus\mathcal{O}(1)^{\oplus 3}$	61	94	-86	1	(c6), VII	124
24		$(\Lambda^3 Q)^{\oplus 2}\oplus\mathcal{O}(1)$	72	96	-74	1	(c3), IX	unknown
25		$\Lambda^2 S^*\oplus\Lambda^3 Q\oplus\mathcal{O}(1)^{\oplus 2}$	66	96	-84	1	(c5), VIII	unknown
26$*$	$G(3,8)$	$(\mathrm{Sym}^2 S^*)^{\oplus 2}$	384	0	0	9		
27$*$		$\mathrm{Sym}^2 S^*\oplus(\Lambda^2 S^*)^{\oplus 2}$	176	128	-64	2		
28		$(\Lambda^2 S^*)^{\oplus 4}$	92	104	-64	1		unknown
29$*$		$\Lambda^3 Q\oplus\mathcal{O}(1)^{\oplus 2}$	102	108	-84	2	(c7)	
30$*$	$G(4,8)$	$\mathrm{Sym}^2 S^*\oplus\mathcal{O}(1)^{\oplus 3}$	256	256	-256	2	(d1)	
31$*$		$(\Lambda^2 S^*)^{\oplus 2}\oplus\mathcal{O}(2)$	48	96	-128	4		
32$*$	$G(4,9)$	$\mathrm{Sym}^2 S^*\oplus\Lambda^2 S^*\oplus\mathcal{O}(1)$	384	192	-128	4	(d2)	
33$*$	$G(5,10)$	$(\Lambda^2 S^*)^{\oplus 2}\oplus\mathcal{O}(1)^{\oplus 2}$	120	180	-220	5	(d3)	

Table 1.1: The list of globally generated homogeneous vector bundles whose zero loci give rise to Calabi–Yau 3-folds. The numbers with $*$ correspond to Calabi–Yau 3-folds of second Betti number greater than one.

Now we explain this table. The second column indicates the type of a Grassmannian. The third column indicates a globally generated homogeneous vector bundle on Grassmannian whose general section gives a Calabi–Yau 3-fold Y. We denote H the restriction of the Schubert divisor of the Grassmannian to Y. Then H^3 is the degree of Y. We denote c_2 and c_3 the second and third Chern classes of Y, respectively. In particular c_3 is the Euler number of Y. The column $h^{1,1}$ represents the $(1,1)$-Hodge number of Y. The column Küchle indicates the corresponding label in Küchle's list of Fano 4-fold [Küc95, Theorem 3.1] with the splitting of vector bundle as $\mathcal{E} = \mathcal{F} \oplus \mathcal{O}(d)$ where $d \geq 1$. The last column indicates the corresponding number in electronic database [vEvS] of Calabi–Yau operators which are calculated in subsequent chapters.

Remark 1.1.11. We impose the following conditions for $G(k,n)$ and homogeneous vector bundle $\mathcal{E}$ on it to avoid the obvious duplication in the table:

(i) Considering the duality $G(k,n) \cong G(n-k,n)$, the following two homogeneous vector bundles correspond to each other under this isomorphism:

$$\Sigma^\lambda \mathcal{S}^* \otimes \Sigma^\mu \mathcal{Q} \otimes \mathcal{O}(d) \text{ on } G(k,n),$$
$$\Sigma^\lambda \mathcal{Q} \otimes \Sigma^\mu \mathcal{S}^* \otimes \mathcal{O}(d) \text{ on } G(n-k,n).$$

In this table we only list $G(k,n)$ satisfying $2k \leq n$.

(ii) Let s be a global section of $\mathcal{S}^*$ corresponding to $\varphi \in V^*$. The zero locus of s on $G(k,n)$ coincides with

$$\{[W] \in G(k,n) \mid \varphi|_W = 0\},$$

then it is isomorphic to $G(k,n-1)$. Moreover the restriction of the homogeneous vector bundle $\mathcal{E}$ on $G(k,n)$ to this set is also the homogeneous vector bundle on $G(k,n-1)$. Hence if $\mathcal{E}$ contains $\mathcal{S}^*$ in its direct summand, i.e. $\mathcal{E} = \mathcal{E}' \oplus \mathcal{S}^*$, then the zero locus of a general section of $\mathcal{E}$ on $G(k,n)$ is the same as the zero locus of the induced section of $\mathcal{E}'$ on $G(k,n-1)$. The same thing is true for the universal quotient bundle $\mathcal{Q}$ by the duality. Hence we assume that $\mathcal{E}$ does not contain $\mathcal{S}^*$ and $\mathcal{Q}$ in its direct summand.

(iii) Let $s \in \Gamma(G(k+1,2k+1), \wedge^2 \mathcal{S}^*)$ be a global section corresponding to $\omega \in \wedge^2 V^*$. We choose s to be general, which means that the rank of ω is equal to $2k$. Let us take v from the kernel of ω. Then the zero locus of s is isomorphic to

$$\{[W] \in G(k, V/\langle v\rangle) \mid \overline{\omega}|_W = 0\}$$

where $\overline{\omega}$ is an element of $\wedge^2(V/\langle v\rangle)^*$ induced from ω. Moreover the restriction of the homogeneous vector bundle on $G(k+1,2k+1)$ to $G(k, V/\langle v\rangle)$ is also homogeneous. By the duality, we can also assume that $\mathcal{E}$ does not contain $\wedge^2 \mathcal{Q}$ in its direct summand for the case of $G(k,2k+1)$.

(iv) Let $\mathcal{E}$ be a globally generated homogeneous vector bundle on $G(k,2k)$ which contains $\wedge^2 \mathcal{S}^*$ or $\wedge^2 \mathcal{Q}$ in its direct summand. Using the duality (i), we assume that $\mathcal{E}$ contains $\wedge^2 \mathcal{S}^*$ in its direct summand and denote it by $\mathcal{E} = \mathcal{E}' \oplus \wedge^2 \mathcal{S}^*$. Then the zero locus of a general section of $\mathcal{E}$ on $G(k,2k)$ is same as the zero locus of a general section of $\mathcal{E}'|_{IG(k,2k)}$ on $IG(k,2k)$. Since $\mathcal{S}^*$ and $\mathcal{Q}$ are isomorphic on $IG(k,2k)$, it is enough to consider homogeneous vector bundles $\mathcal{E}'$ which are direct sums of Schur modules of $\mathcal{S}^*$.

1.1.3 Possibilities of irreducible summands

Let us decompose $\mathcal{E}$ into irreducible summands $\mathcal{E} = \bigoplus_{l=1}^{r} \mathcal{F}_l$ and consider Grassmannians $G(k,n)$ and homogeneous vector bundles $\mathcal{E} = \bigoplus_{l=1}^{r} \mathcal{F}_l$ which satisfy the following two conditions:

$$k(n-k) - \sum_{l=1}^{r} \operatorname{rank}(\mathcal{F}_l) = 3, \tag{1.4}$$

$$n - \sum_{l=1}^{r} c_1(\mathcal{F}_l) = 0, \tag{1.5}$$

where $\operatorname{rank}(\mathcal{F}_l)$ and $c_1(\mathcal{F}_l)$ are given by the formulas (1.2) and (1.3).

First we restrict possible types of the irreducible summands of $\mathcal{E}$. There exists the following result:

Lemma 1.1.12. *Suppose $G(k,n)$ and $\mathcal{E} = \bigoplus_{l=1}^{r} \mathcal{F}_l$ satisfy the conditions* (1.4) *and* (1.5). *Then for any l, either $\mathcal{F}_l = \Sigma^\lambda \mathcal{S}^* \otimes \mathcal{O}(d)$ or $\mathcal{F}_l = \Sigma^\mu \mathcal{Q} \otimes \mathcal{O}(e)$ holds.*

Proof. Suppose $\Sigma^\lambda \mathcal{S}^* \otimes \Sigma^\mu \mathcal{Q} \otimes \mathcal{O}(d)$ is a direct summand of $\mathcal{E}$ such that both λ and μ are not zero. Since $\operatorname{rank}(\Sigma^\lambda \mathcal{S}^*) \geq k$ and $\operatorname{rank}(\Sigma^\mu \mathcal{Q}) \geq n-k$, the rank of $\mathcal{E}$ is not less than $k(n-k)$. This contradicts the assumption (1.4). □

Lemma 1.1.13. *Let $G(k,n)$ satisfy $2k+1 < n$. Take $\mathcal{E} = \wedge^2 \mathcal{Q}$ and a general global section $s \in \Gamma(G(k,n), \mathcal{E})$. Then the zero locus of s is empty. The same thing is true for $\mathcal{E} = \operatorname{Sym}^2 \mathcal{Q}$ and $2k+1 \leq n$.*

Proof. By using the duality $G(k,n) \cong G(n-k,n)$, it is enough to consider the zero locus of a general section s of $\wedge^2 \mathcal{S}^*$ on $G(n-k,n)$. Take $\omega \in \wedge^2 V^*$ corresponding to s. Then the zero locus of s coincides with

$$\{[W] \in G(n-k,n) \mid \omega|_W = 0\}.$$

But the dimensions of isotropic subspaces of V are at most

$$\begin{cases} \dfrac{n}{2} & \text{(if } n \text{ is even)} \\[2ex] \dfrac{n+1}{2} & \text{(if } n \text{ is odd)} \end{cases}$$

for general ω. Then the zero locus of s is empty from the assumption. □

Remark 1.1.14. There exist infinitely many solutions of (1.4) and (1.5) if we do not care whether the zero locus is empty or not. For example,

$$\mathcal{E} = \wedge^2 \mathcal{Q} \oplus \mathcal{O}(1)^{\oplus k-4} \oplus \mathcal{O}(5) \text{ on } G(k,3k)$$

gives solution of (1.4) and (1.5) for each $k \geq 4$.

Using these results and Remark 1.1.11, we only consider the following homogeneous vector bundles as irreducible summands of $\mathcal{E}$:

$$\Sigma^\lambda \mathcal{S}^* \otimes \mathcal{O}(d) \qquad \text{except for } \mathcal{O} \text{ and } \mathcal{S}^*. \tag{1.6}$$

$$\Sigma^\mu \mathcal{Q} \otimes \mathcal{O}(e) \qquad \begin{cases} \text{except for } \mathcal{O}, \mathcal{Q}, \wedge^2\mathcal{Q}, \text{ and } \mathrm{Sym}^2\mathcal{Q} & \text{if } n \geq 2k+1 \\ \text{except for } \mathcal{O} \text{ and } \mathcal{Q} & \text{if } n = 2k. \end{cases} \tag{1.7}$$

Next we show the following result:

Lemma 1.1.15. *Let us fix the Grassmannian $G(k,n)$. Then the numbers of globally generated homogeneous vector bundles $\mathcal{E}$ satisfying* (1.4) *and* (1.5) *are finite.*

Proof. Let $\mathcal{E}$ be a homogeneous vector bundle satisfying (1.4) and (1.5). Then the irreducible summands of $\mathcal{E}$ are of the forms in the (1.6) or (1.7). It is enough to show that the values λ_1, d, μ_1, and e of the irreducible summands of $\mathcal{E}$ are bounded from above.

First we prove that there exists an upper bound of λ_1. From (1.2), we have

$$\mathrm{rank}(\Sigma^\lambda \mathcal{S}^* \otimes \mathcal{O}(d)) \geq \frac{k-1+\lambda_1}{k-1}.$$

Since $\mathcal{E}$ satisfies (1.4), the rank of the irreducible summands must be lower than $k(n-k)-3$. Then we have

$$\frac{k-1+\lambda_1}{k-1} \leq k(n-k)-3.$$

Since we have fixed k and n, this inequality implies that λ_1 is bounded from above. Similarly we can also prove that μ_1 is bounded from above.

Next we show that there exist upper bounds of d and e. We assume that $\Sigma^\lambda \mathcal{S}^* \otimes \mathcal{O}(d)$ is an irreducible summand of $\mathcal{E}$. Taking the first Chern class, we have

$$c_1(\Sigma^\lambda \mathcal{S}^* \otimes \mathcal{O}(d)) = \mathrm{rank}(\Sigma^\lambda \mathcal{S}^*)d + c_1(\Sigma^\lambda \mathcal{S}^*) \leq n.$$

This inequality implies

$$d \leq \frac{n - c_1(\Sigma^\lambda \mathcal{S}^*)}{\mathrm{rank}(\Sigma^\lambda \mathcal{S}^*)},$$

where the right hand side only depends on k, n, and λ. Since the choices of λ are finite and k, n are fixed, the possible values of d is also finite. The same thing is true for e. $\square$

Using this result, it is enough to prove that the possibilities of the k and n which have a non-trivial solution of the equations (1.4) and (1.5) are finite.

1.1.4 The cases $\mathcal{E} = \bigoplus_l \Sigma^{\lambda_l} \mathcal{S}^* \otimes \mathcal{O}(d_l)$ or $\bigoplus_l \Sigma^{\mu_l} \mathcal{Q} \otimes \mathcal{O}(e_l)$

Let $\mathcal{E}$ be a globally generated homogeneous vector bundle on $G(k,n)$ which satisfies the conditions (1.4) and (1.5). In this subsection we assume that $\mathcal{E}$ has the one of the following forms:

$$\mathcal{E} = \bigoplus_{l=1}^{r} \Sigma^{\lambda_l} \mathcal{S}^*(d_l) \text{ or } \bigoplus_{l=1}^{r} \Sigma^{\mu_l} \mathcal{Q}(e_l), \tag{1.8}$$

where we use the notation $\mathcal{F}(d) := \mathcal{F} \otimes \mathcal{O}(d)$ for a vector bundle $\mathcal{F}$. Using the duality $G(k,n) \cong G(n-k,n)$, we consider $\mathcal{E} = \bigoplus_{l=1}^r \Sigma^{\lambda_l}\mathcal{S}^*(d_l)$ on $G(k,n)$. Note that we do not impose the condition $2k \leq n$ in this subsection. First we prove the following result:

Lemma 1.1.16. *If there exists a globally generated homogeneous vector bundle $\mathcal{E} = \bigoplus_{l=1}^r \Sigma^{\lambda_l}\mathcal{S}^*(d_l)$ on $G(k,n)$ satisfying the equations* (1.4) *and* (1.5)*, the inequality*

$$n \leq 2k + \frac{6}{k}$$

holds.

Proof. Suppose $\mathcal{E} = \bigoplus_{l=1}^r \Sigma^{\lambda_l}\mathcal{S}^*(d_l) = \bigoplus_{l=1}^r \mathcal{F}_l$ satisfies the equations (1.4) and (1.5). Subtracting (1.4) from k times of (1.5), we obtain

$$\begin{aligned} k^2 + 3 &= \sum_{l=1}^r (kc_1(\mathcal{F}_l) - \operatorname{rank}(\mathcal{F}_l)) \\ &= \sum_{l=1}^r \operatorname{rank}(\mathcal{F}_l)(|\lambda_l| + kd_l - 1). \end{aligned}$$

Since $|\lambda_l| + kd_l - 1 \geq 1$ for each l, we have

$$k^2 + 3 \geq \sum_{l=1}^r \operatorname{rank}(\mathcal{F}_l) = k(n-k) - 3,$$

where the last equality follows from the assumption that $\mathcal{E}$ satisfies (1.4). The claimed result follows from this inequality. ☐

Lemma 1.1.17. *Suppose $k \geq 7$. Let $\mathcal{F} = \Sigma^\lambda\mathcal{S}^*(d)$ be an irreducible globally generated homogeneous vector bundle on $G(k,n)$ satisfying*

$$c_1(\mathcal{F}) \leq 2k. \tag{1.9}$$

Then $\mathcal{F}$ is contained in the followings:

$$\mathcal{O}(i)\ (1 \leq i \leq 2k), \quad \mathcal{S}^*(1), \quad \wedge^2\mathcal{S}^*, \quad \operatorname{Sym}^2\mathcal{S}^*, \quad \wedge^{k-1}\mathcal{S}^*, \quad \wedge^{k-1}\mathcal{S}^*(1). \tag{1.10}$$

Proof. We find all irreducible globally generated homogeneous vector bundles $\Sigma^\lambda\mathcal{S}^*(d)$ which satisfy $c_1(\Sigma^\lambda\mathcal{S}^*(d)) \leq 2k$. Let $l(\lambda) := \max\{i \mid \lambda_i \neq 0\}$ be the length of the partition λ.

Suppose $2 \leq l(\lambda) \leq k-2$ and $|\lambda| \geq 5$. Since

$$c_1(\Sigma^\lambda\mathcal{S}^*(d)) \geq \frac{k(k-1)}{2}\frac{5}{k} = \frac{5}{2}(k-1),$$

$c_1(\Sigma^\lambda\mathcal{S}^*(d))$ is greater than $2k$ for $k \geq 7$. We can check that only $\wedge^2\mathcal{S}^*$ satisfy the condition among the remaining cases $\wedge^4\mathcal{S}^*(d), \Sigma^{(2,1,1)}\mathcal{S}^*(d), \Sigma^{(3,1)}\mathcal{S}^*(d), \Sigma^{(2,2)}\mathcal{S}^*(d), \wedge^3\mathcal{S}^*(d), \Sigma^{(2,1)}\mathcal{S}^*(d)$, and $\wedge^2\mathcal{S}^*(d)$ in the ranges $|\lambda| \leq 4$.

Suppose $l(\lambda) = 1$, i.e. $\Sigma^\lambda \mathcal{S}^*(d)$ has the forms $\mathrm{Sym}^b \mathcal{S}^*(d)$ for some $b \geq 1$. Since

$$c_1(\mathrm{Sym}^b \mathcal{S}^*(d)) = \binom{b+k-1}{k-1} \frac{b+kd}{k}$$

is monotonously increase with respect to d and b, we see that the solutions of $c_1 \leq 2k$ are $\mathcal{S}^*(1)$ and $\mathrm{Sym}^2 \mathcal{S}^*$.

Suppose $l(\lambda) = k-1$ and $\lambda \neq (1, \dots, 1, 0)$. From the formula (1.2), there exists $\mu = (2, \dots, 2, 1, \dots, 1, 0)$ satisfying

$$\mathrm{rank}(\Sigma^\lambda \mathcal{S}^*(d)) \geq \mathrm{rank}(\Sigma^\mu \mathcal{S}^*) = (k-i)\binom{k+1}{i},$$

where $i = \#\{j \mid \mu_j = 2\} \neq 0$. We can estimate

$$(k-i)\binom{k+1}{i} \geq \begin{cases} k^2 - 1 & (i = 1) \\ \dfrac{(k+1)k}{2} & (2 \leq i \leq k-1) \end{cases}$$

for each value i. Summarizing the result, we have

$$c_1(\Sigma^\lambda \mathcal{S}^*(d)) \geq \frac{(k+1)k}{2},$$

and this inequality shows that $\Sigma^\lambda \mathcal{S}^*(d)$ does not satisfies the condition. We can check that $\wedge^{k-1}\mathcal{S}^*$ and $\wedge^{k-1}\mathcal{S}^*(1)$ only satisfy the condition among the remaining cases $\wedge^{k-1}\mathcal{S}^*(d)$. □

Lemma 1.1.18. *Suppose $k \geq 7$. Then there does not exist globally generated homogeneous vector bundle of the form $\mathcal{E} = \bigoplus_{l=1}^r \Sigma^{\lambda_l} \mathcal{S}^*(d_l)$ which satisfies (1.4) and (1.5).*

Proof. Suppose $\mathcal{E} = \bigoplus_{l=1}^r \Sigma^{\lambda_l} \mathcal{S}^*(d_l)$ satisfies (1.4) and (1.5). Using Lemma 1.1.16, the irreducible summands of $\mathcal{E}$ satisfy the inequality (1.9). Then $\mathcal{E}$ is a direct sum of the vector bundles in (1.10). We collect the rank and first Chern class of these vector bundles as follows:

$\mathcal{F}$	$\mathcal{O}(i)$	$\mathcal{S}^*(1)$	$\wedge^2\mathcal{S}^*$	$\mathrm{Sym}^2\mathcal{S}^*$	$\wedge^{k-1}\mathcal{S}^*$	$\wedge^{k-1}\mathcal{S}^*(1)$
$\mathrm{rank}(\mathcal{F})$	1	k	$\frac{k(k-1)}{2}$	$\frac{(k+1)k}{2}$	k	k
$c_1(\mathcal{F})$	i	$k+1$	$k-1$	$k+1$	$k-1$	$2k-1$

We divide the following two cases:

(i) The case $\mathcal{E}$ contains $\wedge^2\mathcal{S}^*$ or $\mathrm{Sym}^2\mathcal{S}^*$ in its direct summand. By Lemma 1.1.13, it is enough to consider the cases $n = 2k$ or $2k-1$.

Suppose $n = 2k$. It is easy to check that $\mathcal{E}$ does not contain both $\wedge^2\mathcal{S}^*$ and $\mathrm{Sym}^2\mathcal{S}^*$. Since $c_1(\mathcal{E}) = 2k$, we can estimate the rank of $\mathcal{E}$ by

$$\mathrm{rank}(\mathcal{E}) \leq k^2 - k + 2.$$

However this inequality is contradict to $\operatorname{rank}(\mathcal{E}) = k^2 - 3$ for $k \geq 7$.

Suppose $n = 2k - 1$. From Lemma 1.1.13, the homogeneous vector bundle $\mathcal{E}$ does not contain $\mathrm{Sym}^2\mathcal{S}^*$. Since $c_1(\mathcal{E}) = 2k - 1$, $\mathcal{E}$ coincides with one of the followings:

$$(\wedge^2\mathcal{S}^*)^{\oplus 2} \oplus \mathcal{O}(1), \quad \wedge^2\mathcal{S}^* \oplus \wedge^{k-1}\mathcal{S}^* \oplus \mathcal{O}(1), \quad \wedge^2\mathcal{S}^* \oplus \bigoplus_{1\leq l\leq r} \mathcal{O}(d_l)\ (d_1 + \cdots + d_r = k).$$

The rank of the second and third cases are bounded from above by $k^2/2 + k/2 + 1$, and this contradict to $\operatorname{rank}(\mathcal{E}) = k^2 - k - 3$. The rank of the first case is $k^2 - k + 1$, and this does not give the solution for any $k \geq 7$.

(ii) The case $\mathcal{E}$ does not contain $\wedge^2\mathcal{S}^*$ and $\mathrm{Sym}^2\mathcal{S}^*$ in its direct summand. We have upper bounds of the rank of $\mathcal{E}$

$$\operatorname{rank}(\mathcal{E}) \leq \begin{cases} n+2 & (2k-2 \leq n \leq 2k) \\ n+1 & (k+2 \leq n \leq 2k-3) \end{cases}$$

from the condition $c_1(\mathcal{E}) = n$. We can check that both cases are less than $k(n-k) - 3$ for $k \geq 7$. □

We have the following result:

Proposition 1.1.19. *Let $\mathcal{E}$ be a globally generated homogeneous vector bundle on $G(k,n)$ satisfying* (1.4), (1.5). *If $\mathcal{E}$ has the form* (1.8), *then $\mathcal{E}$ is contained in the following list:*

$G(2,4)$	$\mathcal{O}(4)$
$G(2,5)$	$\mathcal{O}(1) \oplus \mathcal{O}(2)^{\oplus 2}$, $\mathcal{O}(1)^{\oplus 2} \oplus \mathcal{O}(2)$, $\mathcal{S}^*(1) \oplus \mathcal{O}(2)$, $\wedge^2\mathcal{Q}(1)$
$G(2,6)$	$\mathcal{O}(1)^{\oplus 4} \oplus \mathcal{O}(2)$, $\mathcal{S}^*(1) \oplus \mathcal{O}(1)^{\oplus 3}$, $\mathrm{Sym}^2\mathcal{S}^* \oplus \mathcal{O}(1) \oplus \mathcal{O}(2)$, $\mathcal{S}^*(1) \oplus \mathrm{Sym}^2\mathcal{S}^*$, $\mathcal{Q}(1) \oplus \mathcal{O}(1)$, $\wedge^3\mathcal{Q} \oplus \mathcal{O}(3)$
$G(2,7)$	$\mathcal{O}(1)^{\oplus 7}$, $\mathrm{Sym}^2\mathcal{S}^* \oplus \mathcal{O}(1)^{\oplus 4}$, $(\mathrm{Sym}^2\mathcal{S}^*)^{\oplus 2} \oplus \mathcal{O}(1)$, $\wedge^4\mathcal{Q} \oplus \mathcal{O}(1) \oplus \mathcal{O}(2)$
$G(2,8)$	$\wedge^5\mathcal{Q} \oplus \mathcal{O}(1)^{\oplus 3}$
$G(3,6)$	$\mathcal{O}(1)^{\oplus 6}$, $\wedge^2\mathcal{S}^* \oplus \mathcal{O}(1)^{\oplus 2} \oplus \mathcal{O}(2)$, $\mathcal{S}^*(1) \oplus \wedge^2\mathcal{S}^*$
$G(3,7)$	$\mathrm{Sym}^2\mathcal{S}^* \oplus \mathcal{O}(1)^{\oplus 3}$, $(\wedge^2\mathcal{S}^*)^{\oplus 2} \oplus \mathcal{O}(1)^{\oplus 3}$, $(\wedge^3\mathcal{Q})^{\oplus 2} \oplus \mathcal{O}(1)$
$G(3,8)$	$(\mathrm{Sym}^2\mathcal{S}^*)^{\oplus 2}$, $\mathrm{Sym}^2\mathcal{S}^* \oplus (\wedge^2\mathcal{S}^*)^{\oplus 2}$, $(\wedge^2\mathcal{S}^*)^{\oplus 4}$, $\wedge^3\mathcal{Q} \oplus \mathcal{O}(1)^{\oplus 2}$
$G(4,8)$	$\mathrm{Sym}^2\mathcal{S}^* \oplus \mathcal{O}(1)^{\oplus 3}$, $(\wedge^2\mathcal{S}^*)^{\oplus 2} \oplus \mathcal{O}(2)$
$G(4,9)$	$\mathrm{Sym}^2\mathcal{S}^* \oplus \wedge^2\mathcal{S}^* \oplus \mathcal{O}(1)$
$G(5,10)$	$(\wedge^2\mathcal{S}^*)^{\oplus 2} \oplus \mathcal{O}(1)^{\oplus 2}$

Proof. Combining Lemma 1.1.16 and Lemma 1.1.18, it is sufficient to consider the equations (1.4) and (1.5) for the following Grassmannians:

$$G(2,n)\ (4 \leq n \leq 7), \quad G(3,n)\ (5 \leq n \leq 8), \quad G(4,n)\ (6 \leq n \leq 9),$$
$$G(5,n)\ (7 \leq n \leq 11), \quad G(6,n)\ (8 \leq n \leq 13).$$

Since the possible types of the vector bundles $\Sigma^\lambda\mathcal{S}^*(d)$ are finite by Lemma 1.1.15, we can enumerate all solutions of (1.4) and (1.5). □

1.1.5 Other cases of $\mathcal{E}$

As for the other cases, it is enough to consider the vector bundles which contain the vector bundle of the form

$$\Sigma^\lambda \mathcal{S}^*(d) \oplus \Sigma^\mu \mathcal{Q}(e) \tag{1.11}$$

where both λ and μ are not zero. Since the rank and first Chern class of $\Sigma^\lambda \mathcal{S}^*(d)$ coincide with those of $\Sigma^\lambda \mathcal{Q}(d)$ when $n = 2k$, it is sufficient to consider the cases $2k+1 \leq n$. The following result enables us to reduce the ranges of k and n solving the equations (1.4) and (1.5):

Lemma 1.1.20. *We assume that $k \geq 7$ or $n-k \geq 7$ holds. Then there does not exist homogeneous vector bundle which contains* (1.11) *in its irreducible summand and satisfying* (1.4) *and* (1.5).

Proof. We fix k and n satisfying

$$n-k \geq 7 \text{ and } 2k+1 \leq n.$$

Using $n-k \geq 7$ and Lemma 1.1.17, the Schur module of $\mathcal{Q}$ in (1.11) must be one of the followings:

$\mathcal{F}$	$\mathcal{Q}(1)$	$\wedge^{n-k-1}\mathcal{Q}$	$\wedge^{n-k-1}\mathcal{Q}(1)$
$\mathrm{rank}(\mathcal{F})$	$n-k$	$n-k$	$n-k$
$c_1(\mathcal{F})$	$n-k+1$	$n-k-1$	$2n-2k-1$

Note that we do not contain $\wedge^2\mathcal{Q}$ and $\mathrm{Sym}^2\mathcal{Q}$ in the table since $2k+1 \leq n$.

Let $\mathcal{V}_\mathcal{Q}$ be a vector bundle in this table. Then we can estimate

$$n - c_1(\mathcal{V}_\mathcal{Q}) \leq k+1.$$

It is sufficient to consider the vector bundles $\mathcal{V}$ which contain the non-trivial Schur module of $\mathcal{S}^*$ in its direct summand and $\mathcal{V}_\mathcal{Q} \oplus \mathcal{V}$ satisfies the equations (1.4) and (1.5). Since $c_1(\mathcal{V}) \leq k+1$, it is sufficient to find the irreducible homogeneous vector bundles which satisfy $c_1(\mathcal{F}) \leq k+1$. We do this analysis for each value of k.

The case $k = 2$. Homogeneous vector bundles, which are Schur modules of $\mathcal{S}^*$, have the following form:

$$\mathrm{Sym}^a \mathcal{S}^*(b) \quad (a, b \geq 0).$$

Then

$$\mathcal{O}(i) \ (1 \leq i \leq 3), \quad \mathcal{S}^*(1), \quad \mathrm{Sym}^2\mathcal{S}^* \tag{1.12}$$

are all irreducible globally generated homogeneous vector bundles, which satisfy $c_1(\mathcal{F}) \leq 3$. Let $\mathcal{V}$ be a direct sum of the vector bundles (1.12) which satisfies $c_1(\mathcal{V}) \leq 3$. It is obvious that

$$\mathrm{rank}(\mathcal{V}) \leq 3,$$

so $\mathrm{rank}(\mathcal{V}_\mathcal{Q} \oplus \mathcal{V}) \leq n+1$. This means that $\mathcal{V}_\mathcal{Q} \oplus \mathcal{V}$ does not satisfy the equation (1.4). Then there do not exist solutions of (1.4) and (1.5) under the claimed setting.

Similarly we study the case $k = 3$.

The case $k = 3$. Irreducible globally generated homogeneous vector bundles which satisfy $c_1 \leq 4$ are

$$\mathcal{O}(i)\ (1 \leq i \leq 4), \quad \mathcal{S}^*(1), \quad \mathrm{Sym}^2\mathcal{S}^*, \quad \wedge^2\mathcal{S}^*.$$

Then we can estimate the rank of $\mathcal{V}$ by $\mathrm{rank}(\mathcal{V}) \leq 6$. Since we assume $n \geq 10$, the inequalities

$$3(n-3) - 3 - \mathrm{rank}(\mathcal{V}_{\mathcal{Q}} \oplus \mathcal{V}) \geq 2n - 15 > 0$$

show that there do not exist homogeneous vector bundles with the claimed properties.

The case $k \geq 4$ (resp. $k \geq 5$). Similarly we can enumerate irreducible globally generated homogeneous vector bundles which satisfy $c_1 \leq 5$ (resp. $c_1 \leq 6$), and we can check that there do not exist solutions $\mathcal{E} = \mathcal{V}_{\mathcal{Q}} \oplus \mathcal{V}$ of (1.4) and (1.5).

The case $k \geq 6$. We find all irreducible globally generated homogeneous vector bundles $\Sigma^\lambda\mathcal{S}^*(d)$ which satisfy $c_1(\Sigma^\lambda\mathcal{S}^*(d)) \leq k+1$. Similar to Lemma 1.1.17, we can check that the irreducible globally generated homogeneous vector bundles whose first Chern classes are not greater than $k+1$ are contained in the following table:

$\mathcal{F}$	$\mathcal{O}(i)$	$\mathcal{S}^*(1)$	$\mathrm{Sym}^2\mathcal{S}^*$	$\wedge^2\mathcal{S}^*$	$\wedge^{k-1}\mathcal{S}^*$	$\wedge^{n-k-1}\mathcal{Q}$
$\mathrm{rank}(\mathcal{F})$	1	k	$\frac{k(k+1)}{2}$	$\frac{k(k-1)}{2}$	k	$n-k$
$c_1(\mathcal{F})$	i	$k+1$	$k+1$	$k-1$	$k-1$	$n-k-1$

We remark that $\wedge^{n-k-1}\mathcal{Q}$ is occurred only if $n = 2k+1$ or $n = 2k+2$.

Let $\mathcal{V}$ be a direct sum of these vector bundles with $c_1(\mathcal{V}) \leq k+1$. We have the estimate

$$\mathrm{rank}(\mathcal{V}) \leq \frac{k(k+1)}{2}.$$

Therefore, we have

$$\begin{aligned} k(n-k) - 3 - \mathrm{rank}(\mathcal{V}_{\mathcal{Q}} \oplus \mathcal{V}) &= (k-1)(n-k) - 3 - \mathrm{rank}(\mathcal{V}) \\ &\geq \frac{1}{2}k(k-1) - 4, \end{aligned}$$

which means that there do not exist homogeneous vector bundles with the claimed properties. □

Proposition 1.1.21. *Let $\mathcal{E}$ be a globally generated homogeneous vector bundle on $G(k,n)$ which contains a vector bundle of the form* (1.11) *in its direct summand. Then $G(k,n)$ and $\mathcal{E}$ are one of the followings:*

$G(2,7)$	$\mathcal{S}^*(1) \oplus \wedge^4\mathcal{Q}$
$G(2,8)$	$\mathrm{Sym}^2\mathcal{S}^* \oplus \wedge^5\mathcal{Q}$
$G(3,7)$	$\wedge^2\mathcal{S}^* \oplus \wedge^3\mathcal{Q} \oplus \mathcal{O}(1)^{\oplus 2}$

Proof. By using Lemma 1.1.20, we only need to consider the equations (1.4) and (1.5) for the following range of k and n:

$$\{(k,n) \mid 2 \le k \le 5,\ 2k+1 \le n \le k+6\}. \tag{1.13}$$

We can find all solutions of (1.4) and (1.5) in the range (1.13) by Lemma 1.1.15. □

1.2 Calculations of the topological invariants

Let us take $G(k,n)$ and $\mathcal{E}$ which are contained in Table 1.1. Taking a general global section $s \in \Gamma(G(k,n), \mathcal{E})$, we consider the zero locus of s, which we denote by Z_s. Due to Theorem 1.1.7, it is a smooth Calabi–Yau 3-fold if it is not empty set. We determine that Z_s is not empty set for in Table 1.1 by calculating the cohomology group $H^0(Z_s, \mathcal{O}_{Z_s})$. Moreover we determine Hodge numbers of Z_s and some topological invariants $(H^3, c_2.H, e(Z_s))$ by the standard way (see [Küc95, Section 4]).

1.2.1 Koszul resolutions

Let $i_s : \mathcal{E}^* \to \mathcal{O}_{G(k,n)}$ be a morphism determined by the global section of $\mathcal{E}$. The structure sheaf $\mathcal{O}_{Z_s}$ has the following locally free resolution

$$0 \to \wedge^{\mathrm{rank}(\mathcal{E})}\mathcal{E}^* \to \cdots \to \wedge^2\mathcal{E}^* \to \mathcal{E}^* \to \mathcal{O}_{G(k,n)} \to \mathcal{O}_{Z_s} \to 0 \tag{1.14}$$

for general s. There exists the spectral sequence

$$\mathbb{E}_1^{-q,p} = H^p(G(k,n), \wedge^q\mathcal{E}^*) \Rightarrow H^{p-q}(Z_s, \mathcal{O}_{Z_s}) \tag{1.15}$$

associate with the exact sequence (1.14). Since the E_1-terms of this spectral sequence consists of the cohomology group of homogeneous vector bundles on $G(k,n)$, we can calculate it by using the Bott–Borel–Weil theorem. This enable us to calculate the cohomology groups $H^i(Z_s, \mathcal{O}_{Z_s})$ $(0 \le i \le 3)$. We have checked the following results:

$$\begin{array}{lll} H^0(Z_s,\mathcal{O}_{Z_s}) = H^3(Z_s,\mathcal{O}_{Z_s}) = \mathbb{C}, & H^1(Z_s,\mathcal{O}_{Z_s}) = H^2(Z_s,\mathcal{O}_{Z_s}) = 0 & \text{(except for No. 26, 30)}, \\ H^0(Z_s,\mathcal{O}_{Z_s}) = H^3(Z_s,\mathcal{O}_{Z_s}) = \mathbb{C}^2, & H^1(Z_s,\mathcal{O}_{Z_s}) = H^2(Z_s,\mathcal{O}_{Z_s}) = 0 & \text{(No. 30)}, \\ H^0(Z_s,\mathcal{O}_{Z_s}) = H^3(Z_s,\mathcal{O}_{Z_s}) = \mathbb{C}, & H^1(Z_s,\mathcal{O}_{Z_s}) = H^2(Z_s,\mathcal{O}_{Z_s}) = \mathbb{C}^3 & \text{(No. 26)}. \end{array}$$

These results imply that the zero loci Z_s are not empty set. In particular, except for No. 26 and 30, the zero loci Z_s are Calabi–Yau 3-folds in the strict sense.

Next we calculate the Hodge numbers $h^{1,i}(Z_s) = \dim H^i(Z_s, \Omega^1_{Z_s})$ $(i = 1, 2)$. In particular, we determine $h^{1,1}(Z_s)$ which is the rank of the Picard group of Z_s except for No. 26. We consider the conormal sequence

$$0 \to \mathcal{E}^*|_{Z_s} \to \Omega^1_{G(k,n)}|_{Z_s} \to \Omega^1_{Z_s} \to 0. \tag{1.16}$$

where we use $N^*_{Z_s/G(k,n)} = \mathcal{E}^*$ since Z_s is the zero locus of a general section of the vector bundle $\mathcal{E}$. Using the long exact sequence associate with the short exact sequence

$$\cdots \to H^i(Z_s, \mathcal{E}^*|_{Z_s}) \to H^i(Z_s, \Omega^1_{G(k,n)}|_{Z_s}) \to H^i(Z_s, \Omega^1_{Z_s}) \to H^{i+1}(Z_s, \mathcal{E}^*|_{Z_s}) \to \cdots, \quad (1.17)$$

it is enough to calculate the cohomology groups $H^i(Z_s, \mathcal{E}^*|_{Z_s})$ and $H^i(Z_s, \Omega^1_{G(k,n)}|_{Z_s})$. Since $H^i(Z_s, \mathcal{E}^*|_{Z_s}) = H^i(G(k,n), \mathcal{E}^*)$ is a cohomology group of homogeneous vector bundle on $G(k,n)$, we can calculate it by the Bott–Borel–Weil theorem. To calculate $H^i(Z_s, \Omega^1_{G(k,n)})$, we consider an exact sequence which is obtained by taking $\otimes\Omega^1_{G(k,n)}$ to the Koszul resolution (1.14). Then we have a spectral sequence

$$\mathbb{E}_1^{-q,p} = H^p(G(k,n), \wedge^q \mathcal{E}^* \otimes \Omega^1_{G(k,n)}) \Rightarrow H^{p-q}(Z_s, \Omega^1_{G(k,n)}|_{Z_s}). \quad (1.18)$$

The E_1-terms are also cohomology groups of the homogeneous vector bundles on $G(k,n)$, so we can calculate them.

Example 1.2.1. Here we give the concrete example of the calculation of the cohomology groups described above. We take No. 5 for example, i.e. $\mathcal{E} = \wedge^2\mathcal{Q}(1)$ on $G(2,5)$. The exterior products of $\mathcal{E}^*$ are calculated as follows:

$$\mathcal{E}^* = \mathcal{Q}(-2), \quad \wedge^2\mathcal{E}^* = \wedge^2\mathcal{Q}(-4), \quad \wedge^3\mathcal{E}^* = \mathcal{O}_{G(2,5)}(-5).$$

The followings are the only nontrivial cohomologies among these sheaves:

$$H^0(G(2,5), \mathcal{O}_{G(2,5)}) = \mathbb{C}, \quad H^6(G(2,5), \wedge^3\mathcal{E}^*) = \mathbb{C}.$$

Then the spectral sequence (1.15) is E_1-degenerate and we can check that the cohomology groups $H^i(Z_s, \mathcal{O}_{Z_s})$ are determined as described above. Similarly the locally free resolution of $\Omega^1_{G(2,5)}|_{Z_s}$ can be obtained by tensoring (1.14) with $\Omega^1_{G(2,5)} = S \otimes \mathcal{Q}^*$ (see [Wey03]),

$$\begin{aligned} 0 \to \mathcal{S}^* \otimes \wedge^2\mathcal{Q}(-7) \to \mathcal{S}^* \otimes \Sigma^{(2,2)}\mathcal{Q}(-6) \oplus \mathcal{S}^* \otimes \mathcal{Q}(-5) \to \\ \mathcal{S}^* \otimes \Sigma^{(2,1)}\mathcal{Q}(-4) \oplus \mathcal{S}^*(-3) \to \mathcal{S}^* \otimes \wedge^2\mathcal{Q}(-2) \to \Omega^1_{G(2,5)}|_{Z_s} \to 0, \end{aligned}$$

and the only non-trivial cohomologies of the sheaves in the resolution are

$$\begin{aligned} &H^1(G(2,5), \mathcal{S}^* \otimes \wedge^2\mathcal{Q}(-2)) = V_0, && H^3(G(2,5), \mathcal{S}^* \otimes \Sigma^{(2,1)}\mathcal{Q}(-4)) = V_0, \\ &H^5(G(2,5), \mathcal{S}^* \otimes \mathcal{Q}(-5)) = V_0, && H^6(G(2,5), \mathcal{S}^* \otimes \wedge^2\mathcal{Q}(-7)) = V_{\omega_1+\omega_4}, \end{aligned}$$

where ω_i is the i-th fundamental weight and V_α is the highest weight representation of $SL(5,\mathbb{C})$ with highest weight α. Then the spectral sequence (1.18) is also E_1-degenerate, and

$$\dim H^i(Z_s, \Omega^1_{G(2,5)}|_{Z_s}) = \begin{cases} 0 & (i=0) \\ 1 & (i=1,2) \\ 25 & (i=3) \end{cases}$$

holds. Similarly we obtain

$$\dim H^i(Z_s, \mathcal{E}^*|_{Z_s}) = \begin{cases} 0 & (i \neq 3) \\ 75 & (i = 3). \end{cases}$$

Substituting these results into the exact sequence (1.17), we can check that

$$\dim H^1(Z_s, \Omega^1_{Z_s}) = 1, \quad \dim H^2(Z_s, \Omega^1_{Z_s}) = 51$$

hold.

Remark 1.2.2. There exist some Calabi–Yau 3-folds in Table 1.1 which can not be determined its Hodge numbers by this calculation. However the other description of Calabi–Yau 3-folds in [IIM1] helps to determine them. For example, we can not determine the Hodge numbers of No. 26 by this method. It is isomorphic to the abelian 3-fold by Reid in [Rei72], so it is well-known the Hodge numbers of it.

1.2.2 Calculations by using characteristic classes

In this subsection we calculate the topological invariants

$$H^3, \quad c_2.H, \quad e,$$

where H is the restriction of the Schubert divisor on $G(k,n)$ to Z_s, $c_2 = c_2(Z_s)$ is the second Chern class of Z_s and $e = e(Z_s)$ is the Euler number of Z_s. These invariants are especially important for describing the diffeomorphism classes of Calabi–Yau 3-folds (see [Wal66]). Using the normal bundle sequence

$$0 \to T^1_{Z_s} \to T^1_{G(k,n)} \to \mathcal{E} \to 0,$$

we have $c(Z_s) = c(G(k,n))/c(\mathcal{E})$ where $c(Z_s) = c(T^1_{Z_s})$ and $c(G(k,n)) = c(T^1_{G(k,n)})$ are total Chern classes of these manifolds. Then we can represent the Chern classes of Z_s by the restriction of the cohomology classes on $G(k,n)$. Since the fundamental class of Z_s is $c_{\mathrm{top}}(\mathcal{E})$, the above invariants are given by the integrals on $G(k,n)$, e.g.

$$e = \int_{Z_s} c_3(Z_s) = \int_{G(k,n)} c_3(Z_s) c_{\mathrm{top}}(\mathcal{E}).$$

We can calculate these integrals on $G(k,n)$ by using Schubert calculus.

Example 1.2.3. We continue Example 1.2.1. The top Chern class of $\mathcal{E}$ and the total Chern class of Z_s are given by

$$\begin{aligned} c_3(\mathcal{E}) &= 10\sigma_{(2,1)} + 5\sigma_{(3,0)}, \\ c(Z_s) &= 1 + (2\sigma_{(2,0)} + 4\sigma_{(1,1)})|_{Z_s} - 10\sigma_{(2,1)}|_{Z_s}, \end{aligned}$$

respectively. Then the topological invariants of Z_s are expressed by the integrals

$$
\begin{aligned}
H^3 &= \int_{G(2,5)} \sigma_{(1,0)}^3 (10\sigma_{(2,1)} + 5\sigma_{(3,0)}), \\
c_2.H &= \int_{G(2,5)} (2\sigma_{(2,0)} + 4\sigma_{(1,1)})\sigma_{(1,0)} (10\sigma_{(2,1)} + 5\sigma_{(3,0)}), \\
e &= \int_{G(2,5)} (-10\sigma_{(2,1)})(10\sigma_{(2,1)} + 5\sigma_{(3,0)}),
\end{aligned}
$$

and thus we have obtained the numbers in Table 1.1.

In Appendix D, we present another approach to determine these topological invariants from the so-called conifold period of the mirror family.

Chapter 2

Computing I-functions of Calabi–Yau 3-folds I

In this chapter, we calculate the twisted J-function of the Calabi–Yau 3-folds with second Betti number one in Table 1.1 which have the property that the abelian/nonabelian correspondence are applicable to it.

2.1 J-functions of Calabi–Yau 3-folds with second Betti number one

Let Y be a Calabi–Yau 3-fold with second Betti number one. Let H be the ample generator of the Picard group of Y. Let $\deg(Y) = \int_Y H^3$ be the degree of Y. Let $H^{\text{even}}(Y, \mathbb{C})$ be an even part of the cohomology group of Y. Let us take

$$1,\ H,\ C = H^2/\deg(Y),\ p = H^3/\deg(Y), \tag{2.1}$$

which form a basis of $H^{\text{even}}(Y, \mathbb{C})$. Let $*$ be a small quantum product on $H^{\text{even}}(Y, \mathbb{C})$. It is determined by the following products:

$$\begin{aligned} H * 1 &= H, \\ H * H &= K(q)C, \\ H * C &= p, \\ H * p &= 0, \end{aligned}$$

where $K(q) = \deg(Y) + \sum_{d\geq 1} d^3 N_d e^{d\tau} Q^d$ and $N_d = \langle\ \rangle_{0,0,d}$ are the Gromov–Witten invariants of Y. Let ∇^z be the quantum connection. Then the connection matrix of ∇^z with respect to the basis (2.1) is

$$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & K(q) & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}.$$

Let $J_Y(\tau, z)$ be the small J-function of Y. It is well-known that $J_Y(\tau, z)$ satisfies

$$\frac{d^2}{d\tau^2}\frac{1}{K(q)}\frac{d^2}{d\tau^2}J_Y(\tau, z) = 0 \tag{2.2}$$

by using the relation between $J_Y(\tau, z)$ and the quantum differential systems.

In this thesis, we calculate the I-functions of Y by several methods. Roughly speaking, it has the form

$$I_Y(t, z) = w_0(t)z + w_1(t)H + w_2(t)H^2z^{-1} + w_3(t)H^3z^{-2}$$

and satisfy

$$J_Y(\tau, z) = \frac{I_Y(t, z)}{w_0(t)}$$

where $\tau = w_1(t)/w_0(t)$. It is known that I_Y coincides with period integrals of mirror Calabi–Yau manifold Y^* for many examples. Below we calculate so-called Picard–Fuchs operator

$$P = a_4(x)\theta^4 + a_3(x)\theta^3 + a_2(x)\theta^2 + a_1(x)\theta + a_0(x) \quad (a_i(x) \in \mathbb{Q}[x]), \tag{2.3}$$

which annihilates I_Y, where $x = e^t$ and $\theta = x\frac{d}{dx}$. The quantum differential operator (2.2) and the Picard–Fuchs operator (2.3) are related by the change of coordinate $\tau = w_1(t)/w_0(t)$, which is the so-called mirror map.

In Chapter 2 and Chapter 3, we prove the following result:

Main result 2.1.1. *We calculate I-functions of Calabi–Yau 3-folds with second Betti number one in Table 1.1 except for No. 18. For a Calabi–Yau 3-fold of No. 18, we give a conjectural I-function which is correct if the abelian/nonabelian correspondence holds for determinantal nets of conics. We also calculate Picard–Fuchs operators which annihilate these I-functions.*

In Appendix C, we list the resulting Picard–Fuchs operators for the I-functions of Main result 2.1.1.

2.2 The abelian/nonabelian correspondence for vector bundles on Grassmannians

In this section, we apply the abelian/nonabelian correspondence to homogeneous vector bundles on Grassmannians. We summarize results of the abelian/nonabelian correspondence in Appendix A.

In this section we give $G(k, n)$ by geometric invariant theory quotient $V//_\chi G$ with $V = \mathrm{Mat}_{k\times n}(\mathbb{C})$, $G = GL(k, \mathbb{C})$, and $\chi(g) = \det(g)$ $(g \in G)$.

Definition 2.2.1. Let $\rho : G \to GL(E)$ be a rational representation of G on $\mathbb{C}$-vector space E. This determines a vector bundle on $G(k, n)$ by

$$\mathcal{E} = V^{\mathrm{s}}_\chi(G) \times_G E.$$

We say that $\mathcal{E}$ is induced from the representation of G.

Example 2.2.2. Let $\mathcal{S}$ be the universal sub-bundle on $G(k,n)$. We take the fundamental representation E_0 of G. The map

$$(M,v) \mapsto {}^t vM \quad (M \in V^{\mathrm{s}}_{\chi}(G),\ v \in E_0)$$

defines the isomorphism $\mathcal{E}_0 \cong \mathcal{S}$ of vector bundles.

It is well-known that any irreducible rational representation of $GL(k,\mathbb{C})$ is given by

$$\Sigma^{\lambda} E_0 \otimes (\det E_0)^d$$

for some $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k)$ $(\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_k \geq 0)$ and $d \in \mathbb{Z}$. The dual $\mathcal{S}^*$ of the universal sub-bundle $\mathcal{S}$ is induced by the anti-fundamental representation of G. We can represent vector bundles which are induced from irreducible representation of G by $\Sigma^{\lambda}\mathcal{S}^*(d)$ for some λ and d. The abelian/nonabelian correspondence is applicable to the twisted I-functions with globally generated homogeneous vector bundles of the form

$$\mathcal{E} = \bigoplus_{l=1}^{r} \Sigma^{\lambda_l}\mathcal{S}^*, \tag{2.4}$$

where $\lambda_l = (\lambda_{l,1}, \lambda_{l,2}, \ldots, \lambda_{l,k})$ for $l = 1, 2, \ldots, r$. Now let us collect necessary data for our calculations of I-functions.

(i) (Abelian quotients) Let $T = (\mathbb{C}^*)^k$ be a maximal torus of G consisting of diagonal matrices. Then $V//_{\chi_T} T = \underbrace{\mathbb{P}^{n-1} \times \cdots \times \mathbb{P}^{n-1}}_{k}$.

(ii) (A lift of the cohomology classes) Let σ be the Schubert class in $H^*(G(k,n),\mathbb{C})$. We fix the lift of σ by Schur polynomial of $H_1, \ldots, H_k$ which is described in Example A.1.6.

(iii) (The fundamental Weyl anti-invariant class) Let $W = \mathfrak{S}_k$ be the Weyl group of (G,T). The fundamental anti-invariant class ω [1] is given by $\prod_{1\leq i<j\leq k}(H_i - H_j)$ where H_i is the pullback of the hyperplane class of the i-th factor.

(iv) (Restrictions of the coordinate) Let H be the Schubert divisor of $G(k,n)$. The Weyl invariant lift of H is $H_1 + \cdots + H_k$. We have an isomorphism $\pi^* : H^2(V//_{\chi}G,\mathbb{C}) \cong H^2(V//_{\chi_T}T,\mathbb{C})^W$, which maps $tH \in H^2(V//_{\chi}G,\mathbb{C})$ to $t(H_1 + \cdots + H_k) \in H^2(V//_{\chi_T}T,\mathbb{C})^W$.

(v) (Novikov rings) We have an isomorphism $\mathrm{NE}(V//_{\chi_T}T) \cong (\mathbb{Z}_{\geq 0})^k$ by $C \mapsto (C.H_1, \ldots, C.H_k)$. Then the Novikov ring of $V//_{\chi_T}T$ is isomorphic to $\mathbb{C}[[Q_1, \ldots, Q_k]]$. Similarly we have $\Lambda_{V//_{\chi}G} = \mathbb{C}[[\mathtt{Q}]]$ for the Novikov ring of $V//_{\chi}G$. The homomorphism $\Lambda_{V//_{\chi_T}T} \to \Lambda_{V//_{\chi}G}$ is determined by $Q_i \mapsto (-1)^{k-1}\mathtt{Q}$.

[1] For simplicity of calculation, We choose different ω as described in Appendix A.1.2.

(vi) (Decompositions of vector bundles) The irreducible vector bundle $\mathcal{E}_G = \Sigma^\lambda \mathcal{S}^*$ on $V//_\chi G$ decomposes into a direct sum of line bundles $\mathcal{E}_T$ determined by weights of the irreducible representation of G. For example if $\mathcal{E}_G = \mathcal{S}^*$, then $\mathcal{E}_T = \bigoplus_{i=1}^k \mathcal{O}(0,\dots,0,\overset{i}{\check{1}},0,\dots,0)$.

With the above data, we can apply the abelian/nonabelian correspondence to the vector bundle of the form (2.4) over $G(k,n)$. Also for vector bundles of the form

$$\mathcal{E} = \bigoplus_{i=1}^r \Sigma^{\mu_i} \mathcal{Q}, \tag{2.5}$$

i.e. those constructed by universal quotient bundle $\mathcal{Q}$, we can use the duality $G(k,n) \cong G(n-k,n)$ to reduce (2.5) to the case (2.4) above.

Using either the representation (2.4) or (2.5), we can calculate the twisted I-functions for the Calabi–Yau 3-folds which are contained in Proposition 1.1.19.

Note that we can not apply the abelian/nonabelian correspondence to the following three cases

$$\mathcal{S}^*(1) \oplus \wedge^4 \mathcal{Q}, \quad \mathrm{Sym}^2 \mathcal{S}^* \oplus \wedge^5 \mathcal{Q}, \quad \wedge^2 \mathcal{S}^* \oplus \wedge^3 \mathcal{Q} \oplus \mathcal{O}(1)^{\oplus 2}$$

in this form.

For the first case, we use the following result:

Proposition 2.2.3 ([IIM1]). *Let $Y \subset G(2,7)$ be the zero locus of a general section of $\mathcal{S}^*(1) \oplus \wedge^4 \mathcal{Q}$. Then Z is a flat degeneration of Calabi–Yau 3-folds Y' which are the zero locus of a general section of $\mathcal{O}(1)^{\oplus 7}$ on $G(2,7)$.*

Due to the invariance of Gromov–Witten invariants under the deformation, it is enough to calculate the twisted J-function of $\mathcal{E} = \mathcal{O}(1)^{\oplus 7}$ on $G(2,7)$, which is handled by the abelian/nonabelian correspondence for $G(2,7)$.

We deal with the second case in Section 2.3 and the third case in Section 3.2.

For the Calabi–Yau 3-folds of second Betti number one in Proposition 1.1.19, it is straightforward to determine their I-functions. Since there is no difference in the calculations, we only present the details for a selected example.

Example 2.2.4. Consider a Calabi–Yau 3-fold Y of type No. 7 in Table 1.1. It is given by the zero locus of a general section of $\mathcal{E}_G = \mathcal{S}^*(1) \oplus \mathcal{O}(1)^{\oplus 3}$ on $V//_\chi G = G(2,6)$. It is proved in [IIM1, Lemma 4.3] that Y is isomorphic to a linear section of the Schubert variety $\mathbf{\Sigma}$ in Cayley plane $\mathbb{OP}^2$. It is not easy to calculate the J-function of Y since the Schubert variety $\mathbf{\Sigma}$ is singular. We can not apply quantum Lefschetz theorem to it (see Section 3.1). Then the description of Y as a zero locus of a section of homogeneous vector bundle on Grassmannian is better for the calculation of the I-function of Y.

Let $T = (\mathbb{C}^*)^2$ be a maximal torus of $GL(2,\mathbb{C})$. Then the abelian quotient is $V//_{\chi_T} T = \mathbb{P}^5 \times \mathbb{P}^5$. The decomposition of $\mathcal{E}_T$ is given by $\mathcal{E}_T = \mathcal{O}(2,1) \oplus \mathcal{O}(1,2) \oplus \mathcal{O}(1,1)^{\oplus 3}$. We fix a lift of the Schubert classes $\sigma \in H^*(G(2,6),\mathbb{C})$ by the corresponding Schur polynomials of H_1, H_2 where H_i is the hyperplane class of i-th component.

We introduce affine coordinates t_1, t_2 which correspond to the bases $H_1, H_2 \in H^*(\mathbb{P}^5 \times \mathbb{P}^5, \mathbb{C})$. The $\mathcal{E}_T$-twisted I-function is given by the following formula:

$$I^{\mathcal{E}_T}(t_1, t_2, z) = ze^{(t_1 H_1 + t_2 H_2)/z} \sum_{d_1, d_2 \geq 0} I^{\mathcal{E}_T}_{(d_1, d_2)} Q_1^{d_1} Q_2^{d_2} q_1^{d_1} q_2^{d_2}$$

where $q_i = e^{t_i}$ and

$$I^{\mathcal{E}_T}_{(d_1,d_2)} = \frac{\prod_{m=1}^{2d_1+d_2}(2H_1 + H_2 + mz) \prod_{m=1}^{d_1+2d_2}(H_1 + 2H_2 + mz) \prod_{m=1}^{d_1+d_2}(H_1 + H_2 + mz)^3}{\prod_{m=1}^{d_1}(H_1 + mz)^6 \prod_{m=1}^{d_2}(H_2 + mz)^6}.$$

We also introduce an affine coordinate t on $H^2(G(2,6), \mathbb{C})$ which correspond to the Schubert divisor H. Using (A.11), we have

$$I^{\mathcal{E}_G}(t, z) = \frac{1}{H_1 - H_2} ze^{t(H_1+H_2)/z} \sum_{d=0}^{\infty} (-1)^d \mathtt{Q}^d q^d \sum_{\substack{d_1+d_2=d \\ d_1,d_2 \geq 0}} (H_1 - H_2 + z(d_1 - d_2)) I^{\mathcal{E}_T}_{(d_1,d_2)}$$

where $q = e^t$. We put $\mathtt{Q} = 1$.

Next we expand $I^{\mathcal{E}_G}(t, z)$ with respect to z. We consider that $I^{\mathcal{E}_G}(t, z)$ takes value in $H^*(G(2,6), \mathbb{C})/\mathrm{ann}(e(\mathcal{E}_G))$. Then we obtain $I^{\mathcal{E}_G}(t, z) = w_0 z + w_1 H + O(z^{-1})$ where

$$w_0 = 1 + 7q + 199q^2 + 8359q^3 + \cdots, \tag{2.6}$$

$$w_1 = tw_0 + 21q + \frac{1431}{2}q^2 + \frac{64373}{2}q^3 + \cdots. \tag{2.7}$$

We look for a differential operator of fourth order which annihilates $I^{\mathcal{E}_G}$. We consider a general differential operator with polynomial coefficients of degree at most d,

$$P = \sum_{i=0}^{4} p_i(q)\theta^i, \quad p_i(q) = \sum_{j=0}^{d} a_{i,j} q^j,$$

where $a_{i,j}$ are unknown constants and $\theta = q\frac{d}{dq}$. Applying P to w_0, we obtain many linear equations $\{L_m(a_{ij})\}$ of a_{ij} from the coefficients of q^m $(m = 0, 1, \dots)$. Since we can calculate w_0 for arbitrary degree, we can find the differential operator P which annihilates w_0. We find a same differential operator in [Miu13, Proposition 5.5].

We do the same calculations for each Calabi–Yau 3-folds with second Betti number one in Proposition 1.1.19. Then we have the following result:

Proposition 2.2.5. *We calculate the Picard–Fuchs operators of the $\mathcal{E}$-twisted I-functions where $\mathcal{E}$ is contained in Proposition* 1.1.19 *and the zero locus of a general section of it defines a Calabi–Yau* 3*-fold with second Betti number one. The concrete forms of the Picard–Fuchs operators are in Appendix C.*

2.3 Conjectural abelian/nonabelian correspondence for vector bundles on determinantal nets of conics

In this section we consider the $\mathcal{E}$-twisted I-function for $\mathcal{E} = \mathrm{Sym}^2\mathcal{S}^* \oplus \wedge^5\mathcal{Q}$ on $G(2,8)$. As explained above of Example 2.2.4, the abelian/nonabelian correspondence of Grassmannians can not be applied to this vector bundle $\mathcal{E}$ of the present formulation. To circumvent the situation, we use the fact that the zero locus of a general section of $\wedge^5\mathcal{Q}$ on $G(2,8)$ coincides with determinantal nets of conics N, which is described by the geometric invariant theory quotient. Moreover the restriction of $\mathrm{Sym}^2\mathcal{S}^*$ to N is induced from the representation of the group. If the abelian/nonabelian correspondence is valid for N, we can apply it for $\mathrm{Sym}^2\mathcal{S}^*|_N$ and obtain $\mathrm{Sym}^2\mathcal{S}^*|_N$-twisted I-function on N.

In this section, assuming that Conjecture A.2.4 is valid for N, we determine the Picard–Fuchs operator corresponding to No. 18.

2.3.1 Constructions of the determinantal nets of conics

First we summarize the construction of the determinantal nets of conics in terms of the geometric invariant theory quotient following [EPS87], [Tjø97].

Let F be a two-dimensional $\mathbb{C}$-vector space and E be a three-dimensional $\mathbb{C}$-vector space. Let $V = H^0(\mathbb{P}^2, \mathcal{O}(1))$ be a $\mathbb{C}$-vector space of linear polynomials on $\mathbb{P}^2$. Choosing a basis of F and E, we identify $\mathrm{Hom}(F, E\otimes V)$ with an affine space of 3×2 matrices having entries in V. The group $GL(3,\mathbb{C})\times GL(2,\mathbb{C})$ acts on $\mathrm{Hom}(F, E\otimes V)$ by

$$(g,h)\cdot M = gMh^{-1}$$

where $(g,h)\in GL(3,\mathbb{C})\times GL(2,\mathbb{C})$ and $M\in \mathrm{Hom}(F, E\otimes V)$. Let $\mathbb{C}^*\cong\{(\lambda I_3, \lambda I_2)\mid \lambda\in\mathbb{C}^*\}$ be the center of $GL(3,\mathbb{C})\times GL(2,\mathbb{C})$. Since $\mathbb{C}^*$ acts on $\mathrm{Hom}(F, E\otimes V)$ trivially, the above action induces the action of $G := GL(3,\mathbb{C})\times GL(2,\mathbb{C})/\mathbb{C}^*$ on $\mathrm{Hom}(F, E\otimes V)$. We define a character $\chi : G\to\mathbb{C}^*$ by

$$[(g,h)]\mapsto (\det g)^2(\deg h)^{-3}$$

where $[(g,h)]$ is the equivalence class of (g,h). The geometric invariant theory quotient

$$N := \mathrm{Hom}(F, E\otimes V)//_{\chi}G$$

is known to define a six-dimensional variety, which parametrizes two-dimensional linear subsystems of conics of determinantal type. Hence there exists a natural embedding

$$j : N \hookrightarrow G(3,6). \tag{2.8}$$

Furthermore the following properties are known:

Proposition 2.3.1 ([EPS87])**.**

(*i*) $\mathrm{Hom}(F, E\otimes V)^{\mathrm{ss}} = \mathrm{Hom}(F, E\otimes V)^{\mathrm{s}}$.

(ii) The action of G on $\mathrm{Hom}(F, E\otimes V)^{\mathrm{s}}$ is fixed-point free.

In particular N is a smooth projective variety.

We construct vector bundles on N. Consider a fundamental representation E of $GL(3,\mathbb{C})$, a fundamental representation F of $GL(2,\mathbb{C})$, and the following induced representations of G:

$$E\otimes(\det E)^{-1}\otimes\det F, \tag{2.9}$$
$$F\otimes(\det E)^{-1}\otimes\det F. \tag{2.10}$$

These define corresponding vector bundles $\mathcal{E}_N$ and $\mathcal{F}_N$ on N.

As is known in [Tjø97], we know $\mathcal{E}_N = j^*\mathcal{S}_{G(3,6)}$ where $\mathcal{S}_{G(3,6)}$ is the universal sub-bundle on $G(3,6)$. As for $\mathcal{F}_N$, we have proved the following results in [IIM1, Proposition 6.1]:

Proposition 2.3.2 ([IIM1])**.** *There exists an embedding $i : N \hookrightarrow G(2,8)$ satisfying*

$$i^*\mathcal{S}_{G(2,8)} = \mathcal{F}_N \tag{2.11}$$

where $\mathcal{S}_{G(2,8)}$ is the universal sub-bundle on $G(2,8)$ and the image $i(N)$ is isomorphic to the zero locus of a general section of $\wedge^5\mathcal{Q}$ on $G(2,8)$.

Identifying N and $i(N)$, we have

$$\mathrm{Sym}^2\mathcal{S}^*_{G(2,8)}|_N = \mathrm{Sym}^2\mathcal{F}^*_N,$$

and use this for our computation of the I-function of No. 18 assuming the validly of the abelian/nonabelian correspondence for the homogeneous vector bundle $\mathrm{Sym}^2\mathcal{F}^*_N$ on N.

2.3.2 Conjectural abelian/nonabelian correspondence

We give the necessary data to apply the abelian/nonabelian correspondence for determinantal nets of conics.

(i) (Abelian quotients) Let $T = (\mathbb{C}^*)^3\times(\mathbb{C}^*)^2/\mathbb{C}^*$ be a maximal torus of G and $W = N(T)/T \cong \mathfrak{S}_3\times\mathfrak{S}_2$ be the Weyl group which acts on $\mathrm{Hom}(F, E\otimes V)$ by permutations of rows and columns. The following fact follows from Proposition 2.3.1:

Lemma 2.3.3. $\mathrm{Hom}(F,E\otimes V)^{\mathrm{ss}}(T) = \mathrm{Hom}(F,E\otimes V)^{\mathrm{s}}(T)$ *for the action of T. Moreover* $\mathrm{Hom}(F,E\otimes V)^{\mathrm{us}} := \mathrm{Hom}(F,E\otimes V)\setminus\mathrm{Hom}(F,E\otimes V)^{\mathrm{s}}(T)$ *is given by*

$$\mathrm{Hom}(F,E\otimes V)^{\mathrm{us}} = \bigcup_{\sigma\in W}\left(\sigma\cdot\mathbb{C}^{12}_{\left(\begin{smallmatrix}0&0\\ *&*\\ *&*\end{smallmatrix}\right)}\cup\sigma\cdot\mathbb{C}^{12}_{\left(\begin{smallmatrix}0&*\\ 0&*\\ *&*\end{smallmatrix}\right)}\right) \tag{2.12}$$

where $\mathbb{C}^{12}_{\left(\begin{smallmatrix}0&0\\ *&*\\ *&*\end{smallmatrix}\right)} = \left\{\left(\begin{smallmatrix}0&0\\ \ell_1&\ell_2\\ \ell_3&\ell_4\end{smallmatrix}\right)\;\middle|\;\ell_1,\dots,\ell_4 \text{ are linear polynomials on } \mathbb{P}^2\right\}$ is the affine subspace in $\mathrm{Hom}(F,E\otimes V)$, and similarly for $\mathbb{C}^{12}_{\left(\begin{smallmatrix}0&*\\ 0&*\\ *&*\end{smallmatrix}\right)}$. Below we put $\mathbb{P} := \mathrm{Hom}(F,E\otimes V)/\!/_{\chi_T}T$ for the abelian quotient by T.

By Lemma 2.3.3, we can check that $(\mathrm{Hom}(F,E\otimes V), G, \chi)$ satisfies Assumption A.1.3.

(ii) (A lift of the cohomology classes) We give the structure of the cohomology ring of N and $\mathbb{P}$. The cohomology ring $H^*(N,\mathbb{Q})$ is generated by $p_i = c_i(\mathcal{E}_N^*)$ $(i=1,2,3)$, $q_j = c_j(\mathcal{F}_N^*)$ $(j=1,2)$ and the relations of these generators are completely known. Note that we can take a basis of $H^*(N,\mathbb{Q})$ as

$$1, q_1, q_1^2, q_2, p_2, q_1^3, q_1q_2, q_1p_2, q_1^4, q_2^2, q_2p_2, q_1q_2^2, q_2^3 \tag{2.13}$$

from [ES89, Theorem 6.9].

Since $\mathbb{P}$ is a smooth toric variety, we can describe combinatorially the cohomology ring of $\mathbb{P}$ (cf. [Ful93]). We denote the Cox coordinates of $\mathbb{P}$ by z_{ij}^{α} with $\ell_{ij} = \sum_{\alpha=0}^{2} z_{ij}^{\alpha} x_{\alpha}$ where $\begin{pmatrix} \ell_{11} & \ell_{12} \\ \ell_{21} & \ell_{22} \\ \ell_{31} & \ell_{32} \end{pmatrix}$ in $\mathrm{Hom}(F, E\otimes V)$. Let H_{ij} be the toric divisor of $\mathbb{P}$ defined by $z_{ij}^{\alpha}=0$ for some α. We have the following result from [Ful93, Section 5.2]:

Proposition 2.3.4. *$H^*(\mathbb{P},\mathbb{Z})$ is generated by H_{ij} $(1\le i\le 3, 1\le j\le 2)$ and presented by*

$$H^*(\mathbb{P},\mathbb{Z}) = \mathbb{Z}[H_{ij} \mid 1\le i\le 3, 1\le j\le 2]/(I_{\mathrm{Lin}} + I_{\mathrm{SR}})$$

where

$$\begin{aligned} I_{\mathrm{Lin}} &= (-H_{11}+H_{12}+H_{31}-H_{32}, -H_{21}+H_{22}+H_{31}-H_{32}),\\ I_{\mathrm{SR}} &= (H_{11}^3H_{12}^3, H_{21}^3H_{22}^3, H_{31}^3H_{32}^3, H_{11}^3H_{21}^3, H_{11}^3H_{31}^3, H_{21}^3H_{31}^3, H_{12}^3H_{22}^3, H_{12}^3H_{32}^3, H_{22}^3H_{32}^3) \end{aligned}$$

are ideals in $\mathbb{Z}[H_{ij} \mid 1\le i\le 3, 1\le j\le 2]$.

Note that the Weyl group $W = \mathfrak{S}_3\times\mathfrak{S}_2$ acts on $H^*(\mathbb{P},\mathbb{C})$ by permutations of $\{H_{ij}\}_{1\le i\le 3, 1\le j\le 2}$.

We define a Weyl invariant lift of the bases in (2.13) by

$$\widetilde{p_i^a q_j^b} = \tilde{p_i}^a \tilde{q_j}^b \in H^*(\mathbb{P},\mathbb{C})^W \tag{2.14}$$

where $\tilde{p}_i = c_i(\mathcal{E}_{N,T}^*)$ $(i=1,2,3)$, $\tilde{q}_j = c_j(\mathcal{F}_{N,T}^*)$ $(j=1,2)$.

(iii) (The fundamental Weyl anti-invariant class) The fundamental Weyl anti-invariant class ω is given by

$$\begin{aligned} \omega &= \prod_{\alpha\in\Phi_+} c_1(\mathcal{L}_\alpha) \\ &= (H_{11}-H_{21})(H_{11}-H_{31})(H_{21}-H_{31})(H_{12}-H_{11}). \end{aligned}$$

(iv) (Restrictions of the coordinate) Let $p_1 = c_1(\mathcal{E}_N^*)$ be a generator of $\mathrm{Pic}(N)$. The lift $\tilde{p}_1$ is given by

$$c_1(\mathcal{E}_{N,T}^*) = H_{11}+H_{12}+H_{21}+H_{22}+H_{31}+H_{32}, \tag{2.15}$$

and this defines an isomorphism $H^2(N,\mathbb{C}) \cong H^2(\mathbb{P},\mathbb{C})^W$ by $tp_1 \mapsto t(H_{11}+H_{12}+H_{21}+H_{22}+H_{31}+H_{32})$.

(v) (Novikov rings) We describe the Novikov ring $\Lambda_{\mathbb{P}}$. Since $\mathbb{P}$ is a toric variety, it is known that $\mathrm{NE}(\mathbb{P})$ is generated by the classes of the torus invariant curves (cf. [CLS11]). We identify a numerical class of a curve with the intersection numbers with divisors H_{ij}, i.e. $C = (C.H_{ij}) \in \mathbb{Z}^6$. Then we have the following description of $\mathrm{NE}(\mathbb{P})$:

$$\begin{aligned}\mathrm{NE}(\mathbb{P}) \cong \{d = (d_{ij}) \in \mathbb{Z}^6 \mid -d_{11} + d_{12} + d_{21} - d_{22} = 0,\ -d_{11} + d_{12} + d_{31} - d_{32} = 0,\\ d_{11} + d_{22} \geq 0,\ d_{21} + d_{32} \geq 0, d_{31} + d_{12} \geq 0,\\ d_{11} + d_{22} + d_{31} \geq 0,\ d_{12} + d_{21} + d_{32} \geq 0\}.\end{aligned}$$

The Novikov ring of $\mathbb{P}$ is a completion of the semi-group ring $\mathbb{C}[\mathrm{NE}(\mathbb{P})]$ by the valuation $v(Q^d) = \sum_{i,j} d_{ij}$.

Since N has $b_2 = 1$, the Novikov ring of N is isomorphic to the formal power series ring $\mathbb{C}[[\mathtt{Q}]]$ generated by one variable $\mathtt{Q}$. It is easy to see that the homomorphism of Novikov rings is given by

$$Q^d \mapsto (-1)^{\sum_{i,j} d_{ij}} \mathtt{Q}^{\sum_{i,j} d_{ij}} \quad (d \in \mathrm{NE}(\mathbb{P}))$$

for Q^d in the Novikov ring $\Lambda_{\mathbb{P}}$.

(vi) (Decompositions of vector bundles) We consider the representations of T given by restricting (2.9) and (2.10). Let $\mathcal{E}^*_{N,T}$, $\mathcal{F}^*_{N,T}$ be the corresponding split vector bundles on $\mathbb{P}$. More precisely

$$\begin{aligned}\mathcal{E}^*_{N,T} &= \mathcal{O}(H_{11} + H_{22}) \oplus \mathcal{O}(H_{21} + H_{32}) \oplus \mathcal{O}(H_{31} + H_{12}),\\ \mathcal{F}^*_{N,T} &= \mathcal{O}(H_{11} + H_{22} + H_{31}) \oplus \mathcal{O}(H_{12} + H_{21} + H_{32}).\end{aligned}$$

By using above data, we calculate $\mathrm{Sym}^2\mathcal{F}^*_N$-twisted I-function. Note that $\mathrm{Sym}^2\mathcal{F}^*_{N,T}$ is given by $\bigoplus_{i=1}^3 \mathcal{O}(D_i)$ where

$$\begin{aligned}D_1 &= 2H_{11} + 2H_{22} + 2H_{31},\\ D_2 &= H_{11} + H_{12} + H_{21} + H_{22} + H_{31} + H_{32},\\ D_3 &= 2H_{12} + 2H_{21} + 2H_{32}.\end{aligned}$$

Let $\tau \in H^2(\mathbb{P}, \mathbb{C})$. From [Giv98], $\mathrm{Sym}^2\mathcal{F}^*_{N,T}$-twisted I-function is given by

$$I^{\mathrm{Sym}^2\mathcal{F}^*_{N,T}}(\tau, z) = ze^{\tau/z} \sum_{d \in \mathrm{NE}(\mathbb{P})} \frac{\prod_{1\leq i\leq 3, 1\leq j\leq 2} \prod_{m=-\infty}^{m=0} (H_{ij} + mz)^3}{\prod_{1\leq i\leq 3, 1\leq j\leq 2} \prod_{m=-\infty}^{m=d_{ij}} (H_{ij} + mz)^3} \prod_{m=1}^{D_1.d} (D_1 + mz) \prod_{m=1}^{D_2.d} (D_2 + mz) \prod_{m=1}^{D_3.d} (D_3 + mz) Q^d e^{\tau.d},$$

which is a function on $H^2(\mathbb{P}, \mathbb{C})$. We write $I^{\mathrm{Sym}^2\mathcal{F}^*_{N,T}} = ze^{\tau/z} \sum_{d\in\mathrm{NE}(\mathbb{P})} I_d Q^d$ for simplicity.

Using above data, we calculate the conjectural form of $\mathrm{Sym}^2\mathcal{F}_N^*$-twisted I-function by (A.11). We have

$$\begin{aligned} I^{\mathrm{Sym}^2\mathcal{F}_N^*}(t,z) &= \frac{1}{\omega}\left(\Big(\prod_{\alpha\in\Phi_+} z\partial_\alpha\Big) I^{\mathrm{Sym}^2\mathcal{F}_{N,T}^*}\right)\Bigg|_{\tau=\sum_{i,j} tH_{ij},\, Q^d=(-1)^{\sum_{i,j} d_{ij}}\mathtt{Q}^{\sum_{i,j} d_{ij}}} \\ &= \frac{1}{\omega} z e^{\sum_{i,j} tH_{ij}/z} \sum_{d\in \mathrm{NE}(\mathbb{P})} (H_{11}-H_{21}+z(d_{11}-d_{21}))(H_{11}-H_{31}+z(d_{11}-d_{31})) \\ &\qquad (H_{21}-H_{31}+z(d_{21}-d_{31}))(H_{12}-H_{11}+z(d_{12}-d_{11}))I_d\mathtt{Q}^d. \end{aligned}$$

By the identification of Weyl anti-invariant classes with Weyl invariant lifts, we consider the $\mathrm{Sym}^2\mathcal{F}_N^*$-twisted I-function taking values in $H^*(N,\mathbb{C})/\mathrm{ann}(e(\mathrm{Sym}^2\mathcal{F}_N^*))$. Expanding $I^{\mathrm{Sym}^2\mathcal{F}_N^*}$ with respect to z, we have

$$I^{\mathrm{Sym}^2\mathcal{F}_N^*}(t,z) = w_0(t)z + w_1(t)p_1 + O(z^{-1}).$$

Now we evaluate $\mathtt{Q}=1$, $q=e^t$, and we obtain

$$\begin{aligned} w_0 &= 1+6q+66q^2+1092q^3+\cdots, \\ w_1 &= tw_0(q)+10q+167q^2+\frac{26746}{3}q^3+\cdots. \end{aligned}$$

We have the following result by searching a differential operator which annihilates w_0:

Proposition 2.3.5. *$I_N^{\mathrm{Sym}^2\mathcal{F}_N^*}$ satisfies the differential equation*

$$\begin{aligned} &\{\theta^4 - 2q\left(2\theta^2+2\theta+1\right)\left(11\theta^2+11\theta+3\right) \\ &\quad + 4q^2(\theta+1)^2\left(76\theta^2+152\theta+111\right) - 144q^3(\theta+1)(\theta+2)(2\theta+3)^2\} I_N^{\mathrm{Sym}^2\mathcal{F}_N^*} = 0. \end{aligned}$$

Remark 2.3.6. We recall the Gopakumar–Vafa invariants n_d which are given by

$$\sum_{d=1}^{\infty} N_d q^d = \sum_{d=1}^{\infty}\sum_{k=1}^{\infty} n_d k^{-3} q^{kd}$$

where N_d ($d\geq 1$) are the Gromov–Witten invariants of Y which is a zero locus of a general section of $\mathrm{Sym}^2\mathcal{F}_N^*$ on N. The Gopakumar–Vafa invariants are related to the number of degree d rational curves on Y. We can read it from the I-function of Y. The results are

$$n_1 = 144, \quad n_2 = 504, \quad n_3 = 3072, \quad \cdots.$$

Tjøtta have derived the (virtual) numbers of lines and conics on Y in [Tjø97, Theorem 4.2 and Theorem 6.5]. The above calculation is consistent with Tjøtta's calculation.

Chapter 3

Computing I-functions of Calabi–Yau 3-folds II

In this chapter, we calculate the $\mathcal{E}$-twisted J-function for the pairs $(G(k,n),\mathcal{E})$ in Table 1.1 when $\mathcal{E}$ decomposes into $\mathcal{E}'\oplus\mathcal{H}$ where $\mathcal{E}'$ is a homogeneous vector bundle and $\mathcal{H}$ is a direct sum of line bundles.

3.1 Reducing to Fano cases

3.1.1 The quantum Lefschetz theorem

Let us recall the following theorem:

Theorem 3.1.1 (Quantum Lefschetz theorem [Kim99], [Lee01], [CG07])**.** *Let X be a smooth projective variety and $\mathcal{E}'$ be a holomorphic vector bundle over X. Let $\mathcal{H}=\bigoplus_{i=1}^{r}\mathcal{L}_i$ be a direct sum of line bundles. We assume that both $\mathcal{E}'$ and $\mathcal{H}$ are globally generated. Let $J^{\mathcal{E}'}(\boldsymbol{t},z)=\sum_{d\in\mathrm{NE}(X)}J_d(\boldsymbol{t},z)Q^d$ be the $\mathcal{E}'$-twisted J-function. We define*

$$I^{\mathcal{H}}_{\mathcal{E}'}(\boldsymbol{t},z):=\sum_{d\in\mathrm{NE}(X)}\prod_{i=1}^{r}\prod_{m=1}^{d.c_1(\mathcal{L}_i)}(c_1(L_i)+mz)J_d(\boldsymbol{t},z)Q^d \tag{3.1}$$

as a modification of $J^{\mathcal{E}'}$. Then $J^{\mathcal{E}'\oplus\mathcal{H}}(\boldsymbol{t},-z)$ and $I^{\mathcal{H}}_{\mathcal{E}'}(\boldsymbol{t},-z)$ generate the same Lagrangian cone in the semi-infinite symplectic space $(H((z^{-1})),\Omega)$ where $H=H^*(X,\mathbb{C})$ and a symplectic form Ω. In particular the same relation as (A.12) holds between $J^{\mathcal{E}'\oplus\mathcal{H}}$ and $I^{\mathcal{H}}_{\mathcal{E}'}$.

We apply this theorem for $X=G(k,n)$, $\mathcal{E}'$, and $\mathcal{H}=\bigoplus_{i=1}^{r}\mathcal{O}(d_i)$. We consider that $I^{\mathcal{H}}_{\mathcal{E}'}$ takes value in $H^*(X,\mathbb{C})/\mathrm{ann}(e(\mathcal{E}'\oplus\mathcal{H}))$, and restrict $\boldsymbol{t}\in H^*(X,\mathbb{C})/\mathrm{ann}(e(\mathcal{E}'\oplus\mathcal{H}))$ to the second cohomology $t\in H^2(X,\mathbb{C})/\mathrm{ann}(e(\mathcal{E}'\oplus\mathcal{H}))$ where $e(-)$ is the Euler class of vector bundles. Our purpose is to find a fourth order differential operator which annihilates $I^{\mathcal{H}}_{\mathcal{E}'}(t,z)$. Suppose we have a differential operator $Q(q,\theta)$ which annihilates $J^{\mathcal{E}'}(t,z)$. If $\dim H^2(X,\mathbb{C})/\mathrm{ann}(e(\mathcal{E}'))=1$, then we can use the following result to determine the differential operator which annihilates $I^{\mathcal{H}}_{\mathcal{E}'}$ from $Q(q,\theta)$:

Lemma 3.1.2 ([BvS95]). *Let us write* $Q(q,\theta)=\sum_{i=0}^{d} q^i Q_i(\theta)$ *where* $Q_i(\theta)\in\mathbb{Q}[\theta]$. *We define*

$$P(q,\theta):=\sum_{i=0}^{d} q^i Q_i(\theta)\prod_{j=1}^{r}\prod_{m=1}^{i}(d_j\theta+m).$$

Then $P(q,\theta)$ *annihilates* $I_{\mathcal{E}'}^{\bigoplus_{i=1}^{r}\mathcal{O}(d_i)}$.

Considering the relation of $J^{\mathcal{E}'}$ and quantum differential systems, we can obtain $Q(q,\theta)$ explicitly. The details are described in the following subsections.

Here we remark that the rank of P is larger than four in general. However we can extract a Picard–Fuchs operator from P after suitable factorization of the operator.

3.1.2 J-functions and quantum differential systems

We summarize basic properties of the J-function. We characterize the J-function as a collection of flat sections of the local system associated with the quantum cohomology. The same formula applies to the $\mathcal{E}'$-twisted J-function when the correlator $\langle\cdots\rangle$ replaced by the $\mathcal{E}'$-twisted correlator $\langle\cdots\rangle^{\mathcal{E}'}$.

Let $T_0=1,T_1,\dots,T_l,T_{l+1},\dots,T_m$ be a homogeneous basis of $H^*(X,\mathbb{C})$ where $T_1,\dots,T_l$ generate $H^2(X,\mathbb{C})$. Let $T^0,\dots,T^m$ be a dual basis of this with respect to the Poincaré pairing.

Definition 3.1.3. Let $\boldsymbol{t}=\sum_{i=0}^{m} t_iT_i\in H^*(X,\mathbb{C})$. We define the J-function of X by

$$J(\boldsymbol{t},z)=z+\boldsymbol{t}+\sum_{n\geq 0}\sum_{d\in \mathrm{NE}(X)}\sum_{a=0}^{m}\frac{Q^d}{n!}\left\langle\frac{T_a}{z-\psi},\underbrace{\boldsymbol{t},\dots,\boldsymbol{t}}_{n}\right\rangle_{0,n+1,d}T^a, \tag{3.2}$$

where $\langle\ \rangle_{0,n+1,d}$ is the genus zero correlator of the quantum cohomology of X.

We consider $\mathcal{M}=H^*(X,\mathbb{C})$ as an affine space $\mathbb{C}^{m+1}$ with coordinate $t_0,\dots,t_m$ by using the basis $T_0,\dots,T_m$. Then the tangent bundle $\mathcal{T}_{\mathcal{M}}$ is a trivial bundle with fiber $H^*(X,\mathbb{C})$. Using the quantum product, we can introduce the connection ∇^z on $\mathcal{T}_{\mathcal{M}}$ by

$$\nabla^z_{\frac{\partial}{\partial t_i}}T_j=z^{-1}T_i*T_j\quad(i=0,\dots,m). \tag{3.3}$$

Due to the associativity of the quantum product, it turns out that the connection ∇^z is flat and defines the corresponding flat sections $s_i(\boldsymbol{t})$ on $\mathcal{M}$. The following proposition is classical results (see [CK99]):

Proposition 3.1.4. *The flat sections* $s_i(\boldsymbol{t})$ $(0\leq i\leq m)$ *are given by*

$$s_i(\boldsymbol{t}):=T_i+\sum_{n\geq 0}\sum_{d\in \mathrm{NE}(X)}\sum_{j=0}^{m}\frac{Q^d}{n!}\left\langle\frac{T_i}{z-\psi},T_j,\underbrace{\boldsymbol{t},\dots,\boldsymbol{t}}_{n}\right\rangle_{0,n+2,d}T^j. \tag{3.4}$$

These sections determine the J*-function by*

$$J(t,z)=z\sum_{i=0}^{m}\langle s_i,1\rangle T^i. \tag{3.5}$$

For our purpose of calculating the $\mathcal{E}$-twisted I-function ($\mathcal{E} = \mathcal{E}' \oplus \mathcal{H}$), it is enough to find the $\mathcal{E}'$-twisted small J-function. Therefore we restrict the full coordinate $\boldsymbol{t}$ to $H^2(X, \mathbb{C})$ by setting $t_0 = t_{l+1} = \cdots = t_m = 0$. Then the quantum connection ∇^z is determined by the small quantum product for divisor classes T_i as follows:

$$\nabla^z_{\frac{\partial}{\partial t_i}} T_j = z^{-1} T_i *_{\text{small}} T_j \quad (i = 1, \dots, l). \tag{3.6}$$

In the following we denote the small quantum product by $*$ for simplicity.

3.1.3 Calculations of the quantum differential equation of $J^{\mathcal{E}'}$

Let $X = G(k, n)$ and fix a basis $T_0, T_1, \dots, T_m$ of $H^*(X, \mathbb{C})$ as before. In particular we fix T_1 to be the Schubert divisor class. By definition of the $\mathcal{E}'$-twisted small quantum product, we have

$$T_i *_{\mathcal{E}'} T_j = \sum_{a=0}^{m} \sum_{d \geq 0} \langle T_i, T_j, T_a \rangle^{\mathcal{E}'}_{0,3,d} e^{dt} T^a. \tag{3.7}$$

Note that the correlators ($\mathcal{E}'$-twisted Gromov–Witten invariants) $\langle T_i, T_j, T_a \rangle^{\mathcal{E}'}_{0,3,d}$ can be non-vanishing only if the degree equality

$$\deg T_i + \deg T_j + \deg T_a = \dim X - \text{rank } \mathcal{E}' + (c_1(X) - c_1(\mathcal{E}'))d \tag{3.8}$$

holds. Since $c_1(X) - c_1(\mathcal{E}') > 0$, in our settings of this section, the degrees d which satisfy (3.8) are finite. This means that the $\mathcal{E}'$-twisted small quantum product is determined by a finite number of $\mathcal{E}'$-twisted Gromov–Witten invariants.

In order to calculate the $\mathcal{E}'$-twisted Gromov–Witten invariants, we use the action of a maximal torus $T = (\mathbb{C}^*)^n$ on $G(k, n)$. This torus action is large enough to calculate $\mathcal{E}'$-twisted Gromov–Witten invariants by the localization formula [Kon95] due to Kontsevich.

Let $M = (M_{ij})$ be a connection matrix of ∇^z with respect to the fixed basis, i.e.

$$\nabla^z_{\frac{d}{dt}} T_j = T_1 *_{\mathcal{E}'} T_j = \sum_{i=0}^{m} M_{ij} T_i. \tag{3.9}$$

The condition that a section $s = \sum_{i=0}^{m} s_i(t) T_i$ is flat is equivalent to

$$\sum_{i=0}^{m} \frac{ds_i(t)}{dt} T_i = T_1 *_{\mathcal{E}'} \sum_{i=0}^{m} s_i(t) T_i, \tag{3.10}$$

which entails the system of linear differential equations

$$\frac{ds_i(t)}{dt} = \sum_{j=0}^{m} M_{ij} s_j(t) \quad (i = 0, 1, \dots, m). \tag{3.11}$$

In order to find a differential operator which annihilates $J^{\mathcal{E}'}$, it is enough to find a differential operator which annihilates $s_m(t)$ for any flat section $s = \sum_{i=0}^{m} s_i(t) T_i$. Using (3.11) repeatedly, we express the higher derivatives $\frac{d^j s_m(t)}{dt^j}$ $(j = 0, 1, \ldots, m+1)$ of $s_m(t)$ by suitable linear combinations of $s_0(t), \ldots, s_m(t)$, i.e.

$$\frac{d^j s_m(t)}{dt^j} = \sum_{i=0}^{m} C_{ij}(q) s_i(t), \tag{3.12}$$

where $C = (C_{ij})$ is a matrix of size $(m+1) \times (m+2)$ with entries in $\mathbb{C}[q]$. This matrix has a kernel

$${}^t(f_0(q), f_1(q), \ldots, f_{m+1}(q)) := {}^t(-\det C_1, \det C_2, \ldots, (-1)^{m+2} \det C_{m+2})$$

where C_i is the matrix obtained from C by deleting the i-th column. Evaluating the both side of (3.12) with this kernel, we obtain

$$\left(\sum_{i=0}^{m+1} f_i(q) \frac{d^i}{dt^i} \right) s_m(t) = 0. \tag{3.13}$$

This is the operator $Q(q, \theta)$ which annihilates $J^{\mathcal{E}'}$.

3.2 The case of $\mathcal{E} = \wedge^2 \mathcal{S}^* \oplus \wedge^3 \mathcal{Q} \oplus \mathcal{O}(1)^{\oplus 2}$ on $G(3,7)$

Here we apply the method described above for the vector bundles $\mathcal{E} = \mathcal{E}' \oplus \mathcal{H}$ in Table 1.1. Although we only present the detail of the calculations for No. 25 in Table 1.1, the same method applies for other cases, too.

The Calabi–Yau 3-fold of No. 25 is given by the zero locus of a general section of $\mathcal{E} = \wedge^2 \mathcal{S}^* \oplus \wedge^3 \mathcal{Q} \oplus \mathcal{O}(1)^{\oplus 2}$ on $X = G(3,7)$. Let us denote $\mathcal{E}' = \wedge^2 \mathcal{S}^* \oplus \wedge^3 \mathcal{Q}$. We will determine all two point $\mathcal{E}'$-twisted Gromov–Witten invariants. Let $T = (\mathbb{C}^*)^7$ be the maximal torus which naturally acts on $G(3,7)$. We use this T-action to compute $\mathcal{E}'$-twisted Gromov–Witten invariants.

Let $s_i = c_i(\mathcal{Q})$ $(1 \leq i \leq 4)$ be the i-th Chern classes of $\mathcal{Q}$. After some algebra, it is easy to see that we can take a basis of the $H^*(X, \mathbb{C})/\mathrm{ann}(e(\mathcal{E}'))$ as

$$T_0, T_1, \ldots, T_7 = 1, s_1, s_1^2, s_2, s_1^3, s_1 s_2, s_1^4, s_1^5. \tag{3.14}$$

We can calculate the twisted Poincaré pairing $(\alpha, \beta)_{e(\mathcal{E}')} := \int_{G(3,7)} \alpha \cup \beta \cup e(\mathcal{E}')$ as follows:

$$\int_{G(3,7)} s_1^5 e(\mathcal{E}') = 66, \quad \int_{G(3,7)} s_1^3 s_2 e(\mathcal{E}') = 36, \quad \int_{G(3,7)} s_1 s_2^2 e(\mathcal{E}') = 20.$$

Let $T^0, \ldots, T^7$ be the dual basis of $T_0, \ldots, T_7$ with respect to $(\,,\,)_{e(\mathcal{E}')}$. The $\mathcal{E}'$-twisted small quantum product $*_{\mathcal{E}'}$ is defined by

$$T_i *_{\mathcal{E}'} T_j = \sum_{k=0}^{7} \sum_{d \geq 0} \langle T_i, T_j, T_k \rangle_{0,3,d}^{\mathcal{E}'} q^d T^k.$$

To determine the quantum connection we need to calculate the $\mathcal{E}'$-twisted small quantum product with divisor class T_1. Using the several formulas of twisted Gromov–Witten invariants (see [CK99]), it turns out that it is enough to calculate the following two point $\mathcal{E}'$-twisted Gromov–Witten invariants $\langle T_i, T_j \rangle_{0,2,d}^{\mathcal{E}'}$:

Lemma 3.2.1. *For $\mathcal{E}'$-twisted Gromov–Witten invariants, we have*

$$\begin{aligned}
&\langle s_1^5 \rangle_{0,1,1}^{\mathcal{E}'} = 264, && \langle s_1^2, s_1^4 \rangle_{0,2,1}^{\mathcal{E}'} = 594, && \langle s_2, s_1^4 \rangle_{0,2,1}^{\mathcal{E}'} = 330,\\
&\langle s_1^3, s_1^3 \rangle_{0,2,1}^{\mathcal{E}'} = 744, && \langle s_1^3, s_1 s_2 \rangle_{0,2,1}^{\mathcal{E}'} = 408, && \langle s_1 s_2, s_1 s_2 \rangle_{0,2,1}^{\mathcal{E}'} = 224,\\
&\langle s_2, s_2, s_1 s_2 \rangle_{0,3,1}^{\mathcal{E}'} = 176, && \langle s_1^3, s_1^5 \rangle_{0,2,2}^{\mathcal{E}'} = 2376, && \langle s_1^4, s_1^4 \rangle_{0,2,2}^{\mathcal{E}'} = 4356,\\
&\langle s_1^5, s_1 s_2 \rangle_{0,2,2}^{\mathcal{E}'} = 1320.
\end{aligned}$$

Proof. Since $\mathcal{E}'$ is an equivariant vector bundle with respect to T-action on $G(3,7)$, we can calculate these numbers by using of localization formula given by Theorem B.4.2. □

Lemma 3.2.2. *Any three point $\mathcal{E}'$-twisted Gromov–Witten invariants are determined uniquely from the numbers in Lemma* 3.2.1 *by WDVV relations.*

Proof. Recall $\mathcal{E}'$-twisted Gromov–Witten invariants satisfy WDVV relations, which are quadratic relations among them. More precisely, for any $0 \leq i,j,k,l \leq 7$ and $d \geq 0$, we have

$$\sum_{m=0}^{d} \sum_{a=0}^{7} \langle T_i, T_j, T_a \rangle_{0,3,m}^{\mathcal{E}'} \langle T^a, T_k, T_l \rangle_{0,3,d-m}^{\mathcal{E}'} = \sum_{m=0}^{d} \sum_{a=0}^{7} \langle T_i, T_l, T_a \rangle_{0,3,m}^{\mathcal{E}'} \langle T^a, T_j, T_k \rangle_{0,3,d-m}^{\mathcal{E}'}. \tag{3.15}$$

We can verify directly that WDVV relations determine any other three point $\mathcal{E}'$-twisted Gromov–Witten invariants if we are given the numbers in Lemma 3.2.1. □

Proposition 3.2.3. *The connection matrix of $\nabla^z_{\frac{d}{dt}}$ with respect to the basis* (3.14) *is given by*

$$\nabla^z_{\frac{d}{dt}} = z\frac{d}{dt} - \begin{pmatrix}
0 & 4q & 0 & 0 & 72q^2 & 40q^2 & 0 & 396q^3\\
1 & 0 & 9q & 5q & 0 & 0 & 132q^2 & 0\\
0 & 1 & 0 & 0 & 8q & 4q & 0 & 0\\
0 & 0 & 0 & 0 & 6q & 4q & 0 & 132q^2\\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 1 & 0 & 0 & \frac{33}{2}q & 0\\
0 & 0 & 0 & 0 & 1 & \frac{6}{11} & 0 & 4q\\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0
\end{pmatrix}.$$

This determines a quantum differential operator $Q(q,\theta)$ which annihilates $J^{\mathcal{E}'}$.

Proof. From the definition of the quantum connection, we have

$$\nabla^z_{\frac{d}{dt}} T_i = T_1 *_{\mathcal{E}'} T_i = \sum_{j=0}^{7} \sum_{d\geq 0} \langle T_1, T_i, T_j \rangle^{\mathcal{E}'}_{0,3,d} q^d T^j.$$

Using the three point twisted Gromov–Witten invariants $\langle T_1, T_i, T_j \rangle^{\mathcal{E}'}_{0,3,d}$ in Lemma 3.2.2, it is straightforward to obtain the matrix M representing $T_1 *_{\mathcal{E}'}$ (see (3.9)). If we have a connection matrix of $\nabla^z_{\frac{d}{dt}}$, the claimed differential operator $Q(q,\theta)$ follows from (3.13). □

Proposition 3.2.4. *The I-function of the Calabi–Yau 3-fold of type No. 25 satisfies the following differential equation:*

$$\begin{aligned}\{121\theta^4 &- 11q(434\theta^4 + 820\theta^3 + 685\theta^2 + 275\theta + 44) \\ &+ q^2(7841\theta^4 + 10916\theta^3 + 3133\theta^2 - 2486\theta - 1320) \\ &- 4q^3(1488\theta^4 + 4092\theta^3 + 6761\theta^2 + 5511\theta + 1694) \\ &- 32q^4(136\theta^4 + 1460\theta^3 + 4556\theta^2 + 5245\theta + 2017) \\ &+ 256q^5(\theta+1)^2(40\theta^2 + 212\theta + 237) - 4096q^6(\theta+1)^2(\theta+2)^2\} I^{\mathcal{E}}_X = 0.\end{aligned}$$

Proof. We define a differential operator

$$P(q,\theta) := \sum_{d=0}^{6} q^d \prod_{m=1}^{d} (\theta+m)^2 Q_d(\theta).$$

It follows from Lemma 3.1.2 that $P(q,\theta)$ annihilates $I^{\mathcal{O}(1)^{\oplus 2}}_{\mathcal{E}'}$. The differential operator $P(q,\theta)$ is of order 12, but we find the following factorization in $\mathbb{Q}[q][\frac{d}{dq}]$:

$$P = \theta^2(\theta-1)^2 \frac{1}{r(q)} R_4(q,\theta) S_4(q,\theta), \tag{3.16}$$

where $S_4(q,\theta)$ is the claimed Picard–Fuchs operator. Since $J^{\mathcal{E}'\oplus\mathcal{O}(1)^{\oplus 2}}$ corresponds to the flat sections of a local system of rank 4, the I-function $I^{\mathcal{O}(1)^{\oplus 2}}_{\mathcal{E}'}$ satisfies the fourth order differential equation which is given by $S_4(q,\theta)$. □

Remark 3.2.5. In the factorization (3.16), $r(q)$ and $R_4(q,\theta)$ are given as follows:

$$\begin{aligned} r(q) &= 32768q^5 - 225280q^4 + 619520q^3 - 851840q^2 + 585640q - 161051, \\ R_4(q,\theta) &= 103968q^2(8q-11)^3\theta^4 - 415872q^2(8q-11)^2(20q-11)\theta^3 \\ &\quad + 19q(8q-11)\left(12957696q^3 - 11215424q^2 + 2923360q - 113135\right)\theta^2 \\ &\quad - 57q\left(56033280q^4 - 48468992q^3 + 5056832q^2 - 3620320q + 1244485\right)\theta \\ &\quad + 2\left(958169088q^5 + 892286208q^4 - 67694176q^3 - 91902888q^2 + 27891105q - 161051\right).\end{aligned}$$

Appendix A

Summary of abelian/nonabelian correspondence

A.1 Set up

A.1.1 Geometric invariant theory quotients

Let G be a reductive algebraic group over $\mathbb{C}$. Let $V = \mathrm{Spec}\, \mathbb{C}[x_1, \ldots, x_N]$ be an affine space of dimension N over $\mathbb{C}$. We consider a rational action of G on V determined by $\rho : G \to GL(V, \mathbb{C})$. We assume that the action of G is effective, i.e. $\ker \rho = \{e\}$. Let $\chi : G \to \mathbb{C}^*$ be a character on G. From these data, we obtain

$$V//_\chi G := \mathrm{Proj} \bigoplus_{n=0}^{\infty} R_n^\chi, \tag{A.1}$$

where $R_n^\chi = \{f \in \mathbb{C}[x_1, \ldots, x_N] \mid g^* f = \chi(g)^n f\}$.

Let $V^{\mathrm{ss}}(G)$ and $V^{\mathrm{s}}(G)$ be the semi-stable and stable locus of V with respect to the G-action and the character χ, which are given by

$$V_\chi^{\mathrm{ss}}(G) = \{x \in V \mid {}^\exists f \in R_n^\chi \text{ such that } f(x) \neq 0\}, \tag{A.2}$$

$$V_\chi^{\mathrm{s}}(G) = \{x \in V \mid x \in V^{\mathrm{ss}}(G),\ \mathrm{Stab}_x(G) \text{ is finite and } Gx \subset V^{\mathrm{ss}}(G) \text{ is closed}\}. \tag{A.3}$$

We will give some examples of the geometric invariant theory quotient.

Example A.1.1 (Grassmannian $G(k, n)$)**.** Let $V = \mathrm{Mat}_{k\times n}(\mathbb{C})$ be the space of $k \times n$ matrices with entries in $\mathbb{C}$. It is an affine space isomorphic to $\mathbb{C}^{k(n-k)}$. Let $G = GL(k, \mathbb{C})$ be the general linear group which acts on V by left multiplication. We define a character χ by $g \mapsto \det g$. The geometric invariant theory quotient $V//_\chi G = G(k, n)$ is the Grassmannian of k-dimensional subspaces in $\mathbb{C}^n$.

Example A.1.2 (toric variety)**.** Let us take $T = (\mathbb{C}^*)^r$ and an affine space $V = \mathbb{C}^d$ with coordinates $x_1, \ldots, x_d$. We denote by $\mathcal{X}(T) = \mathbb{Z}^r$ the weight lattice of T, and define the action of T on V by $(x_1, \ldots, x_d) \mapsto (t^{m_1} x_1, \ldots, t^{m_d} x_d)$ $(m_1, \ldots, m_d \in \mathcal{X}(T))$. We assume that

$m_1, \dots, m_d$ generate strictly convex maximal dimensional cone in $\mathcal{X}(T)_{\mathbb{R}}$. Let $\chi : T \to \mathbb{C}^*$ be a character on T which is identified to the point of $\mathcal{X}(T)$. When $V^{\mathrm{ss}}_{\chi}(T) \neq \emptyset$, $V//_{\chi}T$ is a projective variety over $\mathbb{C}$. The semi-stable locus $V^{\mathrm{ss}}_{\chi}(T)$ is written by

$$V^{\mathrm{ss}}_{\chi}(T) = \bigcup_{i_1,\dots,i_k \text{ satisfies } (*)} V(x_{i_1}x_{i_2}\cdots x_{i_k})^c \tag{A.4}$$

where $V(x_{i_1}x_{i_2}\cdots x_{i_k})^c = V \setminus V(x_{i_1}x_{i_2}\cdots x_{i_k})$ and $(*)$ is a condition that the cone $C_{i_1\dots i_k}$ which is generated by $m_{i_1}, \dots, m_{i_k}$ in $\mathcal{X}(T)_{\mathbb{R}}$ contains χ.

We will consider a nonabelian reductive algebraic group G and fix a maximal torus $T \subset G$. Let us consider a rational representation of G on V. This action induces corresponding action of T on V. Let $\chi : G \to \mathbb{C}^*$ be a character on G. The restriction of χ to T determines the character on T. Then we obtain two different geometric invariant theory quotients, $V//_{\chi}G$ and $V//_{\chi_T}T$. In this thesis we assume the following conditions for G, T, V and χ:

Assumption A.1.3. The semi-stable points and stable points coincide, i.e.

$$V^{\mathrm{ss}}_{\chi}(G) = V^{\mathrm{s}}_{\chi}(G), \quad V^{\mathrm{ss}}_{\chi_T}(T) = V^{\mathrm{s}}_{\chi_T}(T).$$

The stabilizer group of G at any point of $V^{\mathrm{s}}_{\chi}(G)$ is trivial. Similarly we assume the same condition for T and $V^{\mathrm{s}}_{\chi_T}(T)$. We assume that the codimension of $V^{\mathrm{us}}(G) := V \setminus V^{\mathrm{ss}}(G)$ in V is larger than or equal to two. Moreover there is no T-invariant polynomial on V except the constant.

From this assumption the geometric invariant theory quotients $V//_{\chi}G$ and $V//_{\chi_T}T$ are smooth projective variety. The abelian/nonabelian correspondence is a relation of the Gromov–Witten invariants of these different spaces.

A.1.2 General settings

Let us give necessary settings in order to state the abelian/nonabelian correspondence. We fix G, T, V and χ as above. From the definition of the semi-stable points, there is a natural inclusion $V^{\mathrm{ss}}_{\chi}(G) \subset V^{\mathrm{ss}}_{\chi_T}(T)$. This inclusion induces an open embedding $i : U = V^{\mathrm{ss}}_{\chi}(G)/T \hookrightarrow V^{\mathrm{ss}}_{\chi_T}(T)/T\ (= V//_{\chi_T}T)$. On the other hand there is a natural projection $\pi : U = V^{\mathrm{ss}}_{\chi}(G)/T \to V^{\mathrm{ss}}_{\chi}(G)/G\ (= V//_{\chi}G)$. This projection induces a fibration with a fiber G/T. Together with these morphisms, we obtain the following diagram:

$$\begin{array}{ccc} V//_{\chi_T}T = V^{\mathrm{s}}_{\chi_T}(T)/T & \xleftarrow[i]{} & U = V^{\mathrm{s}}_{\chi}(G)/T \\ & & \downarrow{\scriptstyle \pi} \\ & & V//_{\chi}G = V^{\mathrm{s}}_{\chi}(G)/G \end{array}.$$

By using this diagram, we will relate the cohomology classes of $V//_{\chi}G$ and $V//_{\chi_T}T$.

Lift of cohomology classes

Let $W = N(T)/T$ be the Weyl group of G which acts on U and $V//_{\chi_T}T$ compatibly. We consider the Weyl invariant part of the cohomology groups $H^*(U,\mathbb{C})^W$ and $H^*(V//_{\chi_T}T,\mathbb{C})^W$. The following facts are known:

Fact A.1.4 ([ES89], [Kir05]). *The projection* π *induces an isomorphism*

$$\pi^* : H^*(V//_\chi G,\mathbb{C}) \xrightarrow{\cong} H^*(U,\mathbb{C})^W. \tag{A.5}$$

The induced homomorphism

$$(\pi^*)^{-1} \circ i^* : H^*(V//_{\chi_T}T,\mathbb{C})^W \to H^*(V//_\chi G,\mathbb{C}) \tag{A.6}$$

is surjective.

By using this homomorphism, we define the lift of the cohomology class $\alpha \in H^*(V//_\chi G,\mathbb{C})$ to $H^*(V//_{\chi_T}T,\mathbb{C})^W$.

Definition A.1.5. Let $\alpha \in H^*(V//_\chi G,\mathbb{C})$. We say $\tilde{\alpha} \in H^*(V//_{\chi_T}T,\mathbb{C})^W$ is a lift of α when

$$\pi^*\alpha = i^*\tilde{\alpha} \tag{A.7}$$

holds. In other words α is an image of $\tilde{\alpha}$ by $(\pi^*)^{-1} \circ i^*$.

Note that any $\alpha \in H^*(V//_\chi G,\mathbb{C})$ has a lift. But there is no canonical choice for them. The ambiguity of the choice of a lift of α will be described later.

Example A.1.6. Let us consider $G(k,n)$ and $(\mathbb{P}^{n-1})^k$. Let σ_λ be the Schubert class of $G(k,n)$ corresponding to the Young diagram $\lambda = (\lambda_1,\ldots,\lambda_k)$ $(n-k \geq \lambda_1 \geq \cdots \geq \lambda_k \geq 0)$. In this case we find that the lift of σ_λ is given by the Schur polynomial of type λ

$$s_\lambda(H_1,\ldots,H_k) = \frac{\det((H_i^{k-j+\lambda_j})_{1\leq i,j\leq k})}{\det((H_i^{k-j})_{1\leq i,j\leq k})},$$

where H_i is a pull back of the hyperplane class of $\mathbb{P}^{n-1}$ of the i-th factor.

Weyl anti-invariant classes

Let us consider the subspace of $H^*(V//_{\chi_T}T,\mathbb{C})$ which consists of an anti-invariant elements with respect to the action of the Weyl group W.

Definition A.1.7. Let α be an element of $H^*(V//_{\chi_T}T,\mathbb{C})$. We say that the α is a Weyl anti-invariant class if it satisfies

$$\sigma^*\alpha = (-1)^{l(\sigma)}\alpha \tag{A.8}$$

for any $\sigma \in W$ where $l(\sigma)$ is the length of σ. We denote the space of Weyl anti-invariant class by $H^*(V//_{\chi_T}T,\mathbb{C})^a$.

Let us introduce a fundamental Weyl anti-invariant class ω. Let Φ be a root system associated to (G, T). We fix a decomposition $\Phi = \Phi_+ \amalg \Phi_-$ where Φ_+ (resp. Φ_-) is a set of positive (resp. negative) roots. We denote by L_α the one-dimensional representation corresponding to α. It determines the line bundle $\mathcal{L}_\alpha$ on $V//_{\chi_T}T$. By using these line bundles, we define ω as follows:

Definition A.1.8. We define

$$\omega = \sqrt{\frac{(-1)^{|\Phi_+|}}{|W|}} \prod_{\alpha \in \Phi_+} c_1(\mathcal{L}_\alpha) \in H^*(V//_{\chi_T}T, \mathbb{C})^a$$

which is called the fundamental Weyl anti-invariant class.

The following fact is known:

Fact A.1.9. *Any $x \in H^*(V//_{\chi_T}T, \mathbb{C})^a$ is given by a product of some Weyl invariant class y and ω, i.e.*

$$x = y \cup \omega \quad ({}^\exists y \in H^*(V//_{\chi_T}T, \mathbb{C})^W).$$

Note that y is not necessary unique.

Example A.1.10. We continue Example A.1.6. In this example the Weyl group $W = \mathfrak{S}_k$ is the symmetric group of order k which acts on $H^*((\mathbb{P}^{n-1})^k, \mathbb{C}) = \mathbb{C}[H_1, \dots, H_k]/(H_1^n, \dots, H_k^n)$ via the permutations of $H_1, \dots, H_k$. Weyl anti-invariant classes can be represented by anti-symmetric polynomials $f(H_1, \dots, H_k) \in \mathbb{C}[H_1, \dots, H_k]$. By Definition A.1.8 the fundamental Weyl anti-invariant class is given by

$$\omega = \sqrt{\frac{(-1)^{\frac{k(k-1)}{2}}}{k!}} \prod_{1 \le i < j \le k} (H_i - H_j).$$

The Fact A.1.9 is nothing but the elementary result that any anti-symmetric polynomial is a product of a symmetric polynomial and the fundamental anti-symmetric polynomial.

Martin's integration formula

Let us recall the result of Martin [Mar00] which relates integrations over $V//_\chi G$ and those over $V//_{\chi_T}T$.

Theorem A.1.11 (Martin)**.** *Let V, G, T and χ be as above. Let ω be the fundamental Weyl anti-invariant class. Let $\alpha \in H^*(V//_\chi G, \mathbb{C})$ and choose a lift $\tilde{\alpha} \in H^*(V//_{\chi_T}T, \mathbb{C})^W$ of α. Then*

$$\int_{V//_\chi G} \alpha = \int_{V//_{\chi_T}T} \tilde{\alpha} \cup \omega^2 \qquad \text{(A.9)}$$

holds. In particular, the calculation of integrals over nonabelian quotient $V//_\chi G$ reduces to the calculation of the corresponding integrals over abelian quotient $V//_{\chi_T}T$.

Example A.1.12 ($G(2,n)$ and $\mathbb{P}^{n-1}\times\mathbb{P}^{n-1}$)**.** Let H be the Schubert divisor of $G(2,n)$. Let us calculate the degree of H, i.e. $\int_{G(2,n)} H^{2(n-2)}$, via Martin's integration formula. We choose a lift of H by H_1+H_2. From Theorem A.1.11, we have

$$\begin{aligned}\int_{G(2,n)} H^{2(n-2)} &= \int_{\mathbb{P}^{n-1}\times\mathbb{P}^{n-1}} (H_1+H_2)^{2(n-2)}\frac{(-1)}{2}(H_1-H_2)^2 \\ &= -\frac{1}{2}\binom{2n-4}{n-3}+\binom{2n-4}{n-2}-\frac{1}{2}\binom{2n-4}{n-1} \\ &= \frac{1}{n-1}\binom{2n-4}{n-2}.\end{aligned}$$

We obtain a description of the kernel of the map $(\pi^*)^{-1}\circ i^* : H^*(V//_{\chi_T}T,\mathbb{C})^W \to H^*(V//_{\chi}G,\mathbb{C})$ from the following corollary of the Martin's integration formula:

Corollary A.1.13 ([Mar00])**.** *The kernel of the map* $(\pi^*)^{-1}\circ i^* : H^*(V//_{\chi_T}T,\mathbb{C})^W \to H^*(V//_{\chi}G,\mathbb{C})$ *coincides with* $\ker(\cup\omega)=\{x\in H^*(V//_{\chi_T}T,\mathbb{C})^W \mid x\cup\omega=0\}$*. In particular, there exists the following exact sequence:*

$$0\to\ker(\cup\omega)\to H^*(V//_{\chi_T}T,\mathbb{C})^W\to H^*(V//_{\chi}G,\mathbb{C})\to 0.$$

From this exact sequence, the cohomology group $H^*(V//_{\chi}G,\mathbb{C})$ *is naturally identified with the Weyl anti-invariant classes* $H^*(V//_{\chi_T}T,\mathbb{C})^a$*.*

Remark A.1.14. The isomorphism $H^*(V//_{\chi_T}T,\mathbb{C})^a\cong H^*(V//_{\chi}G,\mathbb{C})$ is defined by the exact sequence. Specifying a splitting morphism determines the embedding of $\mathbb{C}$-vector space $H^*(V//_{\chi}G,\mathbb{C})\hookrightarrow H^*(V//_{\chi_T}T,\mathbb{C})^W$. However it does not preserve the ring structure.

Example A.1.15. Let us consider $G(2,4)$ and $\mathbb{P}^3\times\mathbb{P}^3$. Let $s_i=c_i(\mathcal{Q})$ $(i=1,2)$ be the i-th Chern class of the universal quotient bundle $\mathcal{Q}$. We fix the lifts $\tilde{s}_1=H_1+H_2$ and $\tilde{s}_2=H_1H_2$ as described in Example A.1.6. There is a relation $s_1^3=2s_1s_2$ in $H^*(G(2,4),\mathbb{C})$. However the equation

$$(H_1+H_2)^3-2(H_1+H_2)H_1H_2=H_1^3+H_1^2H_2+H_1H_2^2+H_2^3$$

shows that the lift does not preserves the product. The difference of these two products is in $\ker(\cup\omega)$.

Homomorphism of Novikov rings

Let $\mathrm{NE}(V//_{\chi_T}T)$ be the semi-group generated by the classes of effective curves in $N_1(V//_{\chi_T}T)$. The Novikov ring of $V//_{\chi_T}T$ is defined by the completion of the semi-group ring $\mathbb{C}[\mathrm{NE}(V//_{\chi_T}T)]$. Similarly, we define the corresponding Novikov ring for $V//_{\chi}G$. We denote Novikov ring of $V//_{\chi_T}T$ (resp. $V//_{\chi}G$) by $\Lambda_{V//_{\chi_T}T}$ (resp. $\Lambda_{V//_{\chi}G}$).

We consider a homomorphism $\mathrm{Pic}(V//_{\chi}G)_{\mathbb{R}}\to\mathrm{Pic}(V//_{\chi_T}T)^W_{\mathbb{R}}$ which is naturally defined since the codimension of $V^{\mathrm{us}}(G)$ in V is greater than one. We also consider the composition $\mathrm{Pic}(V//_{\chi}G)_{\mathbb{R}}\to\mathrm{Pic}(V//_{\chi_T}T)^W_{\mathbb{R}}\hookrightarrow\mathrm{Pic}(V//_{\chi_T}T)_{\mathbb{R}}$. Taking the dual of the ample cones in

$\mathrm{Pic}(V//_\chi G)_\mathbb{R}$ and $\mathrm{Pic}(V//_{\chi_T} T)_\mathbb{R}$ and restricting them to integral classes, we have the homomorphism of semi-groups $\mathrm{NE}(V//_{\chi_T} T) \to \mathrm{NE}(V//_\chi G)$. Let $\tilde{\beta}$ maps to β by this homomorphism. We define a ring homomorphism $\Lambda_{V//_{\chi_T} T} \to \Lambda_{V//_\chi G}$ by

$$Q^{\tilde{\beta}} \mapsto (-1)^{\epsilon(\tilde{\beta})} Q^\beta$$

where $\epsilon(\tilde{\beta}) = \int_{\tilde{\beta}} \sum_{\alpha \in \Phi_+} c_1(\mathcal{L}_\alpha)$. It is known that $(-1)^{\epsilon(\tilde{\beta})}$ depends on β, then we can write the above map as $Q^{\tilde{\beta}} \mapsto (-1)^{\epsilon(\beta)} Q^\beta$.

A.2 Relations between Frobenius manifolds

A.2.1 Construction of Frobenius manifolds associated to $V//_{\chi_T} T$ and Weyl groups W

We summarize the construction of the Frobenius manifolds, given in [CFKS08], from the action of the Weyl group W on $V//_{\chi_T} T$.

Let $R = \Lambda_{V//_{\chi_T} T}$ be the Novikov ring of $V//_{\chi_T} T$, and set $K = R \otimes H^*(V//_{\chi_T} T, \mathbb{C})$. Let $Z = \mathrm{Spf}(R[[K^\vee]])$. The space Z is a formal scheme over $\mathrm{Spec} R$ with the Frobenius structure determined by $F^{V//_{\chi_T} T}$ which is the Gromov–Witten potential on $V//_{\chi_T} T$. We define another Frobenius manifold M by considering the base change of the homomorphism $\Lambda_{V//_{\chi_T} T} \to \Lambda_{V//_\chi G}$.

We fix a homogeneous basis $T_0 = 1, T_1, \ldots, T_r, T_{r+1}, \ldots, T_m$ of $H^*(V//_\chi G, \mathbb{C})$ where $T_1, \ldots, T_r$ generate the $H^2(V//_\chi G, \mathbb{C})$. We fix a lift $\tilde{T}_i \in H^*(V//_{\chi_T} T, \mathbb{C})^W$ for each i. This defines $\mathbb{C}$-linear embedding

$$H^*(V//_\chi G, \mathbb{C}) \hookrightarrow H^*(V//_{\chi_T} T, \mathbb{C})^W,$$

and we denote the corresponding submanifold of M by N. Let Θ_N be a sheaf of vector field on N. Let $V = H^*(V//_{\chi_T} T, \mathbb{C})^a$ be the Weyl anti-invariant subspace of $H^*(V//_{\chi_T} T, \mathbb{C})$. We define

$$\mathcal{V} = V \otimes \mathcal{O}_N,$$

which is a subsheaf of $\Theta_M|_N$. Let $*$ be the product on the Frobenius manifold M. Let us consider the following homomorphism:

$$\begin{aligned} *\,\omega : \Theta_M|_N &\to \mathcal{V} \\ \xi &\mapsto (\tilde{\xi} * \omega)|_N \end{aligned}$$

where $\tilde{\xi}$ is some extension of $\xi \in \Theta_M|_N$ to M. It is known that the restriction of this homomorphism to Θ_N induces the isomorphism $\Theta_N \cong \mathcal{V}$. We denote the inverse of this isomorphism by

$$*\,\omega^{-1} : \mathcal{V} \to \Theta_N.$$

By using this isomorphism, we introduce the product on Θ_N.

Theorem A.2.1 (Ciocan-Fontanine–Kim–Sabbah [CFKS08]). *The following data define the Frobenius structure on N:*

(i) (*product*) *Let* $\xi, \eta \in \Theta_N$. *We define the product* $\circ$ *on* Θ_N *via*

$$\xi \circ \eta = (\tilde{\xi} * \tilde{\eta} * \omega)|_N * \omega^{-1} \in \Theta_N.$$

(ii) (*metric*) *We define the metric* ${}^{\omega}g$ *on* Θ_N *by*

$${}^{\omega}g(\xi, \eta) = g|_N((\tilde{\xi} * \omega)|_N, (\tilde{\eta} * \omega)|_N)$$

where $g|_N$ *is the restriction of the metric on* M *to* N.

(iii) (*unit vector field*) *We denote* $\partial_{\tilde{t}_0}$ *the vector field corresponding to* $\tilde{T}_0 = 1 \in H^*(V//_{\chi_T}T, \mathbb{C})^W$.

(iv) (*Euler vector field*) *We define the vector field on* Θ_N *by*

$$E = \sum_{i=0}^{m} \left(1 - \frac{\deg(\tilde{T}_i)}{2}\right) \partial_{\tilde{t}_i} + c_1(V//_{\chi_T}T)|_N$$

where $\deg(\tilde{T}_i)$ *is the degree of the cohomology class* $\tilde{T}_i$. *Note that* $c_1(V//_{\chi_T}T)$ *is a Weyl invariant class.*

Example A.2.2. Let us consider $G(2,5)$ and $\mathbb{P}^4 \times \mathbb{P}^4$. We fix the lift as in Example A.1.6. We compare two products, $*$ which is a product on M and $\circ$ which is a product on N defined by Theorem A.2.1. Then

$$\begin{aligned}
\partial_{t_{(1,0)}} * \partial_{t_{(2,1)}}|_N &= H_1^3 H_2 + 2H_1^2 H_2^2 + H_1 H_2^3 - Qt_{(3,0)} H_1 - Qt_{(3,0)} H_2 - Qt_{(3,1)} H_1^2 - Qt_{(3,1)} H_2^2 \\
&\quad - Qt_{(3,2)} H_1^3 - Qt_{(3,2)} H_2^3 - Qt_{(3,3)} H_1^4 - Qt_{(3,3)} H_2^4 + \cdots, \\
\partial_{t_{(1,0)}} \circ \partial_{t_{(2,1)}} &= H_1^3 H_2 + 2H_1^2 H_2^2 + H_1 H_2^3 - Qt_{(3,0)} H_1 - Qt_{(3,0)} H_2 - Qt_{(3,1)} H_1^2 - Qt_{(3,1)} H_2^2 \\
&\quad - Qt_{(3,2)} H_1^3 - Qt_{(3,2)} H_2^3 + Qt_{(3,3)} H_1^3 H_2 + Qt_{(3,3)} H_1^2 H_2^2 + Qt_{(3,3)} H_1 H_2^3 + \cdots \\
&= \tilde{\sigma}_{(3,1)} + \tilde{\sigma}_{(2,2)} - Qt_{(3,0)} \tilde{\sigma}_{(1,0)} - Qt_{(3,1)} \tilde{\sigma}_{(2,0)} + Qt_{(3,1)} \tilde{\sigma}_{(1,1)} \\
&\quad - Qt_{(3,2)} \tilde{\sigma}_{(3,0)} + Qt_{(3,2)} \tilde{\sigma}_{(2,1)} + Qt_{(3,3)} \tilde{\sigma}_{(3,1)} + \cdots,
\end{aligned}$$

where $t_{(i,j)}$ are coordinates on N determined by the lift $\tilde{\sigma}_{(i,j)}$ and $\partial_{t_{(i,j)}}$ are the vector fields on N corresponding to $\tilde{\sigma}_{(i,j)}$.

Example A.2.3. Continue the above example. The calculation

$${}^{\omega}g(\partial_{t_{(2,0)}}, \partial_{t_{(3,1)}}) = -2 - 4Qt_{(3,3)} + \cdots$$

shows that the coordinate $t_{(i,j)}$ is not flat coordinate on N.

We describe a flat coordinate on N explicitly. Let $\tilde{t} = (\tilde{t}_0, \tilde{t}_1, \ldots, \tilde{t}_m)$ be a coordinate on N corresponding to the basis $\tilde{T}_0, \tilde{T}_1, \ldots, \tilde{T}_m$. Let us define

$$\xi_i := (\tilde{T}_i \cup \omega) * \omega^{-1}$$

for $i = 0, 1, \ldots, m$. These ξ_i form a basis of Θ_N and define a coordinate $s = (s_0, s_1, \ldots, s_m)$ on N. It is known that s provides a flat coordinate on N of the Frobenius structure defined by Theorem A.2.1.

For a flat coordinate, we have the so-called Frobenius potential F' which satisfies

$$\partial_i \circ \partial_j = (\partial_i \partial_j \partial_k F')\partial^k$$

where $\partial_i = \frac{\partial}{\partial s_i}$, and ∂^k is the dual basis of ∂_ℓ satisfying ${}^\omega g(\partial^k, \partial_\ell) = \delta^k_\ell$.

Under the above settings, we state the conjecture of the abelian/nonabelian correspondence for Frobenius manifolds:

Conjecture A.2.4 (Ciocan-Fontanine–Kim–Sabbah [CFKS08])**.** *Let $T_0, \ldots, T_m$ be a homogeneous basis of $H^*(V//_\chi G, \mathbb{C})$, and $t = (t_0, \ldots, t_m)$ be the corresponding coordinate on $H^*(V//_\chi G, \mathbb{C})$. Let P be the Frobenius manifold associate to the Gromov–Witten potential of $V//_\chi G$. We fix lifts of T_i to be $\tilde{T}_i \in H^*(V//_{\chi_T} T, \mathbb{C})^W$, and introduce the Frobenius manifold N associated to $V//_{\chi_T} T$ and W with its flat coordinate $s = (s_0, \ldots, s_m)$. Then $\varphi : P \to N$, the isomorphism of formal schemes given by $\varphi^* s_i = t_i$, induces an isomorphism of Frobenius manifolds such that $\varphi^* \xi_i = T_i$ and $\varphi^* F^{V//_\chi G} = F'$ up to quadratic terms.*

A.2.2 Generalization to $(\mathbb{C}^*)^l$-equivariant formulations

If the cohomology group $H^*(V//_\chi G, \mathbb{C})$ is generated by the divisor classes $H^2(V//_\chi G, \mathbb{C})$, Conjecture A.2.4 can be reduced to some identities of the J-functions for each quotients. However the generating property is a strong condition in general; e.g. it does not hold for $G(2,4)$ as we see in

$$\sigma^2_{(1,0)} = \sigma_{(2,0)} + \sigma_{(1,1)}$$

for the Schubert classes. We can remedy the situation by extending every definitions for suitable equivariant cohomologies.

Let us assume that the algebraic torus $S = (\mathbb{C}^*)^l$ acts on V and commutes with the action of G on V. This induces the action of S on the spaces $V//_\chi G$ and $V//_{\chi_T} T$. Let $\mathbb{C}[\lambda_1, \ldots, \lambda_l] = H^*(BS)$ and $\mathbb{C}(\lambda_1, \ldots, \lambda_l)$ be a field of fraction of $H^*(BS)$. The condition that the equivariant cohomology $H^*_S(V//_\chi G, \mathbb{C})$ is generated by the divisor classes $H^2_S(V//_\chi G, \mathbb{C})$ over $\mathbb{C}(\lambda_1, \ldots, \lambda_l)$ is more broader property.

Example A.2.5. Let us consider $G(2,4)$. Let $S = (\mathbb{C}^*)^4$ be a maximal torus of $GL(4, \mathbb{C})$ which consists of diagonal matrices. Let $e_1, \ldots, e_4$ be the standard basis of $\mathbb{C}^4$. Let $0 \subset V_1 \subset V_2 \subset V_3 \subset V_4 = \mathbb{C}^4$ be the standard flag of $\mathbb{C}^4$ where $V_i = \langle e_1, \ldots, e_i \rangle$ $(i = 1, \ldots, 4)$. For each integers $2 \geq \lambda_1 \geq \lambda_2 \geq 0$, we define

$$\Sigma_{(\lambda_1, \lambda_2)} = \{[W] \in G(2,4) \mid \dim(W \cap V_{3-\lambda_1}) \geq 1, \dim(W \cap V_{4-\lambda_2}) \geq 2\}$$

which is the Schubert variety. Since $\Sigma_{(\lambda_1,\lambda_2)}$ is invariant under the action of the S, it defines the equivariant cohomology class $[\Sigma_{(\lambda_1,\lambda_2)}]^S \in H_S^{2(\lambda_1+\lambda_2)}(G(2,4), \mathbb{C})$. We denote the equivariant cohomology associated to the Schubert divisor by $x = [\Sigma_{(1,0)}]^S$. Then

$$[\Sigma_{(2,0)}]^S, [\Sigma_{(1,1)}]^S \in \langle 1, x, x^2, x^3, x^4, x^5 \rangle,$$

where $\langle 1, x, x^2, x^3, x^4, x^5 \rangle$ is a vector space spanned by x^i $(i = 0, \dots, 5)$ over $\mathbb{C}(\lambda_1, \lambda_2, \lambda_3, \lambda_4)$.

The construction of the Frobenius manifold in Section A.2.1 generalizes to the S-equivariant settings. We only need to replace each object with S-equivariant counterpart, e.g. S-equivariant pairing, S-equivariant quantum product, S-equivariant Chern class, etc. As a result, we have the Frobenius manifold N over $R = \Lambda_{V//_\chi G} \otimes \mathbb{C}[\lambda_1, \dots, \lambda_l]$.

Remark A.2.6. We give an explicit form of the corresponding object of the Euler vector field in the S-equivariant setting. In this case it is given by

$$E = \sum_{i=0}^{m} \left(1 - \frac{\deg(\tilde{T}_i)}{2} \right) \partial_{\tilde{t}_i} + c_1^S(V//_{\chi_T} T)|_N + \sum_{j=1}^{l} \lambda_j \partial_{\lambda_j},$$

which is only $\Lambda_{V//_\chi G}$-derivation.

By using the several reconstruction results in [KM94] and [LP04], the following theorem has been proved:

Theorem A.2.7 (Ciocan-Fontanine–Kim–Sabbah [CFKS08])**.** *Let V, G, T, χ and S be as above. Assume that $V//_\chi G$ is a Fano manifold with Fano index larger than or equal two. Moreover we assume that the $H_S^*(V//_\chi G, \mathbb{C})$ is generated by the divisor classes $H^2(V//_\chi G, \mathbb{C})$ after localization with respect to the equivariant parameters. Then the followings are equivalent:*

(*i*) *Conjecture A.2.4 holds for S-equivariant setting.*

(*ii*) *The following identity holds:*

$$J^S_{V//_\chi G}(\tau, z) = \frac{1}{\omega} \left(\left(\prod_{\alpha \in \Phi_+} z\partial_\alpha \right) J^S_{V//_{\chi_T} T}(t, z) \right) \Bigg|_{Q^{\tilde{\beta}} = (-1)^{\epsilon(\beta)} Q^\beta, N} \tag{A.10}$$

where $J^S_{V//_\chi G}$ (resp. $J^S_{V//_{\chi_T} T}$) is the S-equivariant small J-function of $V//_\chi G$ (resp. $V//_{\chi_T} T$).

Example A.2.8. Let us consider $G(2,4)$ and $\mathbb{P}^3 \times \mathbb{P}^3$. Let $S = (\mathbb{C}^*)^4$ be the algebraic torus acting on $G(2,4)$ and $\mathbb{P}^3 \times \mathbb{P}^3$. The S-equivariant small J-function of $\mathbb{P}^3 \times \mathbb{P}^3$ is given by

$$J^S_{\mathbb{P}^3 \times \mathbb{P}^3}(t_1, t_2, z) = z e^{\frac{t_1 H_1 + t_2 H_2}{z}} \sum_{d_1, d_2 \geq 0} \frac{Q^{(d_1, d_2)} e^{d_1 t_1 + d_2 t_2}}{\prod_{i_1=0}^{3} \prod_{m=1}^{d_1} (H_{1,i_1} + mz) \prod_{i_2=0}^{3} \prod_{m=1}^{d_2} (H_{2,i_2} + mz)}$$

where $H_1 = c_1^S(\mathcal{O}(1,0))$, $H_2 = c_1^S(\mathcal{O}(0,1))$ are basis of $H_S^2(\mathbb{P}^3 \times \mathbb{P}^3, \mathbb{C})$ over $\mathbb{C}[\lambda_1, \dots, \lambda_4]$ and H_{1,i_1}, H_{2,i_2} are an elements of $H_S^2(\mathbb{P}^3 \times \mathbb{P}^3, \mathbb{C})$ which are equivariant classes of corresponding toric divisors. Then the right hand side of (A.10) is given by

$$\frac{1}{H_1 - H_2} z e^{\frac{\tau(H_1 + H_2)}{z}} \sum_{d \geq 0} (-1)^d Q^d e^{d\tau} \sum_{\substack{d_1 + d_2 = d \\ d_1, d_2 \geq 0}} \frac{(H_1 - H_2 + z(d_1 - d_2))}{\prod_{i_1=0}^{3} \prod_{m=1}^{d_1} (H_{1,i_1} + mz) \prod_{i_2=0}^{3} \prod_{m=1}^{d_2} (H_{2,i_2} + mz)}$$

where τ is a coordinate on $H_S^2(G(2,4), \mathbb{C})$ corresponding to $c_1^S(\mathcal{O}(1))$.

The equality (A.10) holds for partial flag manifolds of type A due to the following result:

Theorem A.2.9 (Bertram–Ciocan-Fontanine–Kim [BCFK05], [BCFK08]). *Let $F(k_1, \dots, k_r; n)$ $(1 \le k_1 < \cdots < k_r \le n-1)$ be the partial flag manifold parameterizing flags $0 \subset V_1 \subset \cdots \subset V_r \subset \mathbb{C}^n$ where $\dim V_i = k_i$ $(i = 1, \dots, r)$. We consider $F(k_1, \dots, k_r; n)$ as a geometric invariant theory quotient of $V = \prod_{i=1}^r \mathrm{Mat}_{k_i \times k_{i+1}}(\mathbb{C})$ $(k_{r+1} = n)$ with $G = \prod_{i=1}^r GL(k_i, \mathbb{C})$ and appropriate χ. Let $S = (\mathbb{C}^*)^n$ be a maximal torus of $GL(n, \mathbb{C})$. Then the S-equivariant small J-function of $F(k_1, \dots, k_r; n)$ satisfies the equation (A.10).*

It is known that $F(k_1, \dots, k_r; n)$ with $S = (\mathbb{C}^*)^n$ satisfy the assumption of Theorem A.2.7. In particular the abelian/nonabelian correspondence holds for $F(k_1, \dots, k_r; n)$ and $V//_{\chi_T} T$ in the S-equivariant setting.

Remark A.2.10. Taking the non-equivariant limit of S-equivariant cohomology, we obtain the abelian/nonabelian correspondence of $V//_\chi G$ and $V//_{\chi_T} T$ in non-equivariant setting.

In Section 2 we mainly use the abelian/nonabelian correspondence of $G(k, n)$ and $(\mathbb{P}^{n-1})^k$ in non-equivariant setting.

A.2.3 Generalization of Frobenius structures for twisted invariants

Consider the Weyl group W and a Weyl invariant vector bundle $\mathcal{V}_T$ on $V//_{\chi_T} T$. Associated to $\mathcal{V}_T$, we can introduce the Frobenius manifold structure on $H^*(V//_{\chi_T} T, \mathbb{C})$ (see [CFKS08]). Let us consider the $\mathbb{C}^*$-action on $\mathcal{V}_T$ which is given by the fiberwise scalar multiplication. We denote $H^*(B\mathbb{C}^*, \mathbb{Q}) = \mathbb{Q}[\lambda]$. Let $e(-)$ be a $\mathbb{C}^*$-equivariant Euler class. It is given by

$$e(\mathcal{V}) = \sum_{i=0}^{\mathrm{rank}(\mathcal{V})} \lambda^i c_{\mathrm{rank}(\mathcal{V})-i}(\mathcal{V})$$

for any $\mathbb{C}^*$-equivariant vector bundle $\mathcal{V}$. There exists an analogous construction of the Frobenius manifold in Theorem A.2.1 which include the Weyl invariant vector bundle $\mathcal{V}_T$. We note that some modifications are required. The pairing g on $H^*(V//_{\chi_T} T, \mathbb{C})$ is replaced by the $\mathcal{V}_T$-twisted pairing

$$g^{\mathcal{V}_T}(\alpha, \beta) = \int_{V//_{\chi_T} T} \alpha \cup \beta \cup e(\mathcal{V}_T),$$

where $\alpha, \beta \in H^*(V//_{\chi_T} T, \mathbb{C})$. The quantum product $*$ on $H^*(V//_{\chi_T} T, \mathbb{C})$ is replaced by the $\mathcal{V}_T$-twisted quantum product $*^{\mathcal{V}_T}$ which is determined by the $\mathcal{V}_T$-twisted Gromov–Witten invariants

$$\langle \alpha_1, \dots, \alpha_n \rangle_{0,n,d}^{\mathcal{V}_T},$$

where $\alpha_1, \dots, \alpha_n \in H^*(V//_{\chi_T} T, \mathbb{C})$. As a result, we obtain the Frobenius manifold N over $\Lambda_{V//_\chi G} \otimes \mathbb{C}((\lambda^{-1}))$.

Let E be a representation of G. This induce the representation of T by restriction. These determine the vector bundles on $V//_\chi G$ and $V//_{\chi_T} T$ by

$$\mathcal{E}_G := V^s(G) \times_G E,$$
$$\mathcal{E}_T := V^s(T) \times_T E.$$

Since $\mathcal{E}_T$ is invariant under the Weyl group action, it determines the Frobenius manifold N as above. Similarly $\mathcal{E}_G$ determines the Frobenius manifold P by using the $\mathcal{E}_G$-twisted Gromov–Witten invariants of $V//_\chi G$. The following result is known:

Theorem A.2.11 (Ciocan-Fontanine–Kim–Sabbah [CFKS08])**.** *Let V, G, T and χ be as in Assumption A.1.3. Suppose Conjecture A.2.4 is true for these. Then Conjecture A.2.4 is also true for the twisted settings with $\mathcal{E}_G$ and $\mathcal{E}_T$.*

We use this result to construct $\mathcal{E}_G$-twisted I-function for a homogeneous vector bundles $\mathcal{E}_G$ on Grassmannians.

A.3 Constructions of I-functions

A.3.1 Constructions of I-functions of $V//_\chi G$

We recall the definition of the J-functions of $V//_\chi G$ and $V//_{\chi_T} T$.

Definition A.3.1. Let $T_0, T_1, \ldots, T_m$ be a homogeneous basis of $H^*(V//_\chi G, \mathbb{C})$ as before. We denote $\boldsymbol{\tau} = \sum_{i=0}^m \tau_i T_i$. We define the J-function of $V//_\chi G$ by

$$J_{V//_\chi G}(\boldsymbol{\tau}, z) := z + \boldsymbol{\tau} + \sum_{d \in \mathrm{NE}(V//_\chi G)} \sum_{n \geq 0} \sum_{0 \leq a \leq m} \frac{Q^d}{n!} \left\langle \frac{T_a}{z - \psi}, \underbrace{\boldsymbol{\tau}, \ldots, \boldsymbol{\tau}}_{n} \right\rangle_{0,n,d} T^a,$$

where $\langle\ \rangle_{0,n,d}$ is the Gromov–Witten invariants of $V//_\chi G$. Similarly we define the J-function of $V//_{\chi_T} T$ which is a function of $\boldsymbol{t} \in H^*(V//_{\chi_T} T, \mathbb{C})$.

We can construct another function $I_{V//_\chi G}(\boldsymbol{\tau}, z)$, which is the so-called I-function of $V//_\chi G$, from $J_{V//_{\chi_T} T}$ and the Weyl group W. Below we denote by $N \subset H^*(V//_{\chi_T} T, \mathbb{C})^W$ a lift of $H^*(V//_\chi G, \mathbb{C})$ with a basis $\tilde{T}_i$.

Definition A.3.2. We define

$$I_{V//_\chi G}(\tilde{\boldsymbol{t}}, z) := \frac{1}{\omega} \left(\left(\prod_{\alpha \in \Phi_+} z\partial_\alpha \right) J_{V//_{\chi_T} T} \right) \Bigg|_{Q^{\tilde{\beta}} = (-1)^{\epsilon(\beta)} Q^\beta, N}$$

which is a function of $\tilde{\boldsymbol{t}} \in N$ taking its values in $H^*(V//_\chi G, \mathbb{C})((z^{-1}))$.

The following theorem states that we can recover the genus-zero Gromov–Witten theory of $V//_\chi G$ from the I-function:

Theorem A.3.3 (Ciocan-Fontanine–Kim–Sabbah [CFKS08])**.** *Let us denote $\tilde{\boldsymbol{t}} = \sum_{i=0}^m \tilde{t}_i \tilde{T}_i \in N$ and $\boldsymbol{t} = \sum_{i=0}^m t_i T_i \in H^*(V//_\chi G, \mathbb{C})$. Then there exists unique $C_i(\tilde{\boldsymbol{t}}, z) \in \Lambda_{V//_\chi G}[[\boldsymbol{t}, z]]$ $(i = 0, \ldots, m)$ satisfying*

$$J_{V//_\chi G}(\boldsymbol{t}, z) = I_{V//_\chi G}(\tilde{\boldsymbol{t}}, z) + \sum_{i=0}^m C_i(\tilde{\boldsymbol{t}}, z) z\partial_{\tilde{t}_i} I_{V//_{\chi_T} T}(\tilde{\boldsymbol{t}}, z).$$

The coordinate change of $\boldsymbol{t}$ *and* $\tilde{\boldsymbol{t}}$ *can be read from the coefficient of* z^0 *of the* z*-expansion of the right hand side.*

Example A.3.4. Let us consider $G(2,4)$ and $\mathbb{P}^3 \times \mathbb{P}^3$. We restrict the coordinate $\tilde{\boldsymbol{t}} \in N$ to the small parameter space $\tilde{t}(H_1 + H_2) \in H^2(\mathbb{P}^3 \times \mathbb{P}^3, \mathbb{C})^W$. Then we can express explicitly the I-function of $G(2,4)$ by

$$I_{G(2,4)}(\tilde{t}, z) = \frac{1}{H_1 - H_2} z e^{\tilde{t}(H_1+H_2)/z} \sum_{d \geq 0} (-1)^d Q^d e^{d\tilde{t}} \sum_{\substack{d_1+d_2=d \\ d_1,d_2 \geq 0}} \frac{(H_1 - H_2 + z(d_1 - d_2))}{\prod_{m=1}^{d_1}(H_1 + mz)^4 \prod_{m=1}^{d_2}(H_2 + mz)^4}.$$

The z-expansion of this is

$$I_{G(2,4)}(\tilde{t}, z) = z + \tilde{t}(H_1 + H_2) + O(z^{-1}),$$

then Theorem A.3.3 states that

$$J_{G(2,4)}(t, z) = I_{G(2,4)}(t, z).$$

Note that there is no coordinate change in this case. This is a specific case of Hori–Vafa conjecture [HV00] which is solved by [BCFK05].

A.3.2 Constructions of $\mathcal{V}_G$-twisted I-functions

Now we define the $\mathcal{E}_G$-twisted I-function under the above settings. Let $I^{\mathcal{E}_T}(\boldsymbol{\tau}, z)$ be a $\mathcal{E}_T$-twisted I-function defined for the abelian quotient $V//_{\chi_T}T$. We can find a concise of formula of $I^{\mathcal{E}_T}(\boldsymbol{\tau}, z)$ as a function taking its values in (an appropriate completion of) $H^*(V//_{\chi_T}T, \mathbb{C}) \otimes \Lambda_{V//_{\chi_T}T}((z^{-1}))$ in [Giv98]. We denote by ∂_α a vector field on $H^*(V//T, \mathbb{C})$ corresponding to the $c_1(\mathcal{L}_\alpha)$ for $\alpha \in \Phi_+$.

Definition A.3.5 ([CFKS08])**.** We define $\mathcal{E}_G$-twisted I-function by

$$I^{\mathcal{E}_G}(\tilde{\boldsymbol{t}}, z) := \frac{1}{\omega}\bigg(\Big(\prod_{\alpha \in \Phi_+} z\partial_\alpha\Big) I^{\mathcal{E}_T}(\boldsymbol{\tau}, z)\bigg)\bigg|_{Q^{\tilde{\beta}} = (-1)^{\epsilon(\beta)} Q^\beta, N} \tag{A.11}$$

which is a function of $\tilde{\boldsymbol{t}} \in H^*(V//_\chi G, \mathbb{C})$.

To obtain $\mathcal{E}_G$-twisted J-function from $I^{\mathcal{E}_G}$, we use the following theorem:

Theorem A.3.6 ([CFKS08])**.** *Let* $V//_\chi G$ *be a partial flag manifold of type A. Assume that* $\mathcal{E}_G$ *is generated by global sections. Let* $\tilde{\boldsymbol{t}} = (\tilde{t}_0, \tilde{t}_1, \ldots, \tilde{t}_r, \tilde{t}_{r+1}, \ldots, \tilde{t}_m)$ *be a coordinate on* $H^*(V//_\chi G, \mathbb{C})$ *corresponding to a basis* $T_0, T_1, \ldots, T_r, T_{r+1}, \ldots, T_m$ *where* $T_0 = 1$ *and* $T_1, \ldots, T_r$ *form a basis of* $H^2(V//_\chi G, \mathbb{C})$*. Then there exists unique* $C^i(\tilde{\boldsymbol{t}}, z) \in \Lambda_{V//_\chi G}[[\tilde{\boldsymbol{t}}, z]]$ $(i = 0, 1, \ldots, m)$ *satisfying*

$$J^{\mathcal{E}_G}(\boldsymbol{t}, z) = I^{\mathcal{E}_G}(\tilde{\boldsymbol{t}}, z) + \sum_{i=0}^{m} C^i(\tilde{\boldsymbol{t}}, z) z \partial_{\tilde{t}_i} I^{\mathcal{E}_G}(\tilde{\boldsymbol{t}}, z). \tag{A.12}$$

The coordinate change $\boldsymbol{t} = \boldsymbol{t}(\tilde{\boldsymbol{t}})$ *can be read off from the coefficient of* z^0 *of the expansion with respect to* z *of the right hand side.*

In the most of calculations, we restrict the parameter $\tilde{\boldsymbol{t}}$ to the so-called small parameter space, i.e. $\tilde{t}_0 = \tilde{t}_{r+1} = \cdots = \tilde{t}_m = 0$. In the small parameter space, we can find an explicit transformation which express $J^{\mathcal{E}_G}$ in terms of $I^{\mathcal{E}_G}$ in our setting.

Appendix B

Summary of localization formulas

B.1 Preliminaries

Let us recall the following classical result:

Theorem B.1.1 (Atiyah–Bott [AB84]). *Let X be a smooth algebraic variety. Let $T = (\mathbb{C}^*)^l$ be an algebraic torus which acts on X algebraically. Let $F_1, \dots, F_r$ be fixed components of X and $i_j : F_j \hookrightarrow X$ be natural inclusions. Let N_j be a normal bundle on F_j which inherit a natural T-action. For any T-equivariant cohomology class $\alpha^T \in H_T^*(X, \mathbb{C})$, the equality*

$$\int_{[X]^T} \alpha^T = \sum_{j=1}^{r} \int_{[F_j]^T} \frac{i_j^* \alpha^T}{e^T(N_j)}$$

holds.

Example B.1.2. Let $X = \mathbb{P}^2$ and consider the standard torus action $T = (\mathbb{C}^*)^3$ on X. The fixed points of X are $p_0 = [1 : 0 : 0]$, $p_1 = [0 : 1 : 0]$, and $p_2 = [0 : 0 : 1]$. Let us calculate the integral $\int_X H^2$ where H is the hyperplane class of $\mathbb{P}^2$ by Theorem B.1.1. Since H is a first Chern class of $\mathcal{O}(1)$, we see that a corresponding equivariant lift is given by $c_1^T(\mathcal{O}(1))$ which is an equivariant first Chern class of $\mathcal{O}(1)$. Let us calculate the equivariant integral $\int_{[X]^T} c_1^T(\mathcal{O}(1))^2$. Since $c_1^T(\mathcal{O}(1))|_{p_i} = -\lambda_i$, we have

$$\begin{aligned}\int_{[X]^T} c_1^T(\mathcal{O}(1))^2 &= \frac{(-\lambda_0)^2}{(\lambda_1 - \lambda_0)(\lambda_2 - \lambda_0)} + \frac{(-\lambda_1)^2}{(\lambda_0 - \lambda_1)(\lambda_2 - \lambda_1)} + \frac{(-\lambda_2)^2}{(\lambda_0 - \lambda_2)(\lambda_1 - \lambda_2)} \\ &= 1.\end{aligned}$$

This coincides with the expected answer. Second we take a corresponding equivariant class of H as the equivariant fundamental class of the coordinate line $H_0 = \{[0 : x_1 : x_2] \mid x_1, x_2 \in \mathbb{C}\}$. Since $p_0 \notin H_0$, the restriction of $[H_0]^T$ to p_0 is zero. We have

$$\begin{aligned}\int_{[X]^T} ([H_0]^T)^2 &= \frac{(\lambda_0 - \lambda_1)^2}{(\lambda_0 - \lambda_1)(\lambda_2 - \lambda_1)} + \frac{(\lambda_0 - \lambda_2)^2}{(\lambda_0 - \lambda_2)(\lambda_1 - \lambda_2)} \\ &= 1.\end{aligned}$$

Then we obtain the same answer.

Let X be a smooth projective variety. In this appendix we assume the following condition for X:

Assumption B.1.3. For any holomorphic map $f : \mathbb{P}^1 \to X$, the condition

$$H^1(\mathbb{P}^1, f^* T^1_X) = 0$$

holds where T^1_X is a holomorphic tangent bundle on X. We call that X is a convex variety.

This assumption guarantees that $X_{0,n,d}$, the moduli space of genus zero with n-marked points stable maps to X representing the class $d \in H_2(X, \mathbb{Z})$, is at worst orbifold. The homogeneous spaces are example of convex variety since its holomorphic tangent bundle is generated by global sections.

The idea of Kontsevich is to use the orbifold generalization of Theorem B.1.1 to the moduli space $X_{0,n,d}$ with the torus action which is induced from the torus action on X. To have graphical computations for the localization, we impose the following condition for the torus action on X:

Assumption B.1.4. Fixed points of X are isolated, i.e.

$$\Sigma(0) := X^T = \{p_1, \ldots, p_{n_0}\}$$

for some $n_0 \geq 0$. Moreover one-dimensional orbits in X are also isolated, i.e.

$$\begin{aligned}\Sigma(1) :=& \{O \subset X \mid O \text{ is an orbit of the action of } T,\ \dim O = 1\} \\ =& \{O_1, \ldots, O_{n_1}\}\end{aligned}$$

for some $n_1 \geq 0$.

Note that the one-dimensional orbit O_i is isomorphic to $\mathbb{C}^*$ and its closures in X is isomorphic to $\mathbb{P}^1$. We denote the closure of O_i by ℓ_i.

Example B.1.5. Let $X = \mathbb{P}^1 \times \mathbb{P}^1$ and $T = (\mathbb{C}^*)^2 \times (\mathbb{C}^*)^2$ which acts naturally on X. The fixed points and one-dimensional orbits are followings:

$$\begin{aligned}\Sigma(0) &= \{(0,0), (0,\infty), (\infty,0), (\infty,\infty)\}, \\ \Sigma(1) &= \{\{0\} \times \mathbb{C}^*, \{\infty\} \times \mathbb{C}^*, \mathbb{C}^* \times \{0\}, \mathbb{C}^* \times \{\infty\}\}.\end{aligned}$$

More generally every toric manifolds with their natural torus action satisfy Assumption B.1.4 (but does not necessarily become convex variety).

B.2 Twisted Gromov–Witten invariants

Let X be a smooth projective variety and T be an algebraic torus acting on X. Assume that X and T satisfy Assumption B.1.3 and B.1.4. Let $\mathcal{E}$ be a globally generated vector bundle on X. Let $\mathrm{ev}_{n+1} : X_{0,n+1,d} \to X$ be an evaluation map to evaluate at the last marked point and $\pi : X_{0,n+1,d} \to X_{0,n,d}$ be a forgetful map with respect to the last marked point.

$$\begin{array}{ccc} & & \mathcal{E} \\ & & \downarrow \\ X_{0,n+1,d} & \xrightarrow[\mathrm{ev}_{n+1}]{} & X \\ \downarrow{\scriptstyle \pi} & & \\ X_{0,n,d} & & \end{array}$$

We define a vector bundle on $X_{0,n,d}$ by

$$\mathcal{E}_{0,n,d} := \pi_* \mathrm{ev}_{n+1}^* \mathcal{E}.$$

and denote the Euler class of it by $E_{0,n,d}$. For $\alpha_1, \ldots, \alpha_n \in H^*(X, \mathbb{C})$, we define $\mathcal{E}$-twisted Gromov–Witten invariants by the integral

$$\langle \alpha_1, \ldots, \alpha_n \rangle_{0,n,d}^{\mathcal{E}} := \int_{[X_{0,n,d}]} \mathrm{ev}_1^* \alpha_1 \wedge \cdots \wedge \mathrm{ev}_n^* \alpha_n \wedge E_{0,n,d}, \qquad \text{(B.1)}$$

where ev_i is an evaluation map to evaluate at the i-th marked point.

We calculate the integral (B.1) via the localization expressing it as an equivariant integral. Let us take an equivariant lift of α_i, say $\alpha_i^T \in H_T^*(X, \mathbb{C})$, i.e. $\rho(\alpha_i^T) = \alpha_i$ where $\rho : H_T^*(X, \mathbb{C}) \to H^*(X, \mathbb{C})$ is a map taking the non-equivariant limit. Similarly we take an equivariant lift of the class $E_{0,n,d}$, say $E_{0,n,d}^T$. If $\mathcal{E}$ is a T-equivariant vector bundle, it is enough to take the T-equivariant Euler class of $\mathcal{E}_{0,n,d}$. We consider the following T-equivariant Gromov–Witten invariants:

$$\int_{[X_{0,n,d}]^T} \mathrm{ev}_1^* \alpha_1^T \wedge \cdots \wedge \mathrm{ev}_n^* \alpha_n^T \wedge E_{0,n,d}^T, \qquad \text{(B.2)}$$

where ev_i^* is the pull back of the equivariant cohomology class. Since the non-equivariant limit of (B.2) is (B.1), it is sufficient to calculate T-equivariant Gromov–Witten invariants.

Applying Theorem B.1.1 under this orbifold setting, (B.2) is equal to

$$\sum_{j=1}^{r} \int_{[F_j]^T} \frac{i_j^* (\mathrm{ev}_1^* \alpha_1^T \wedge \cdots \wedge \mathrm{ev}_n^* \alpha_n^T \wedge E_{0,n,d}^T)}{e^T(N_{F_j/X_{0,n,d}})}, \qquad \text{(B.3)}$$

where F_j is a fixed component of $X_{0,n,d}$ and i_j is a natural inclusion map $(i = 1, \ldots, r)$. Using Assumption B.1.4, the fixed components are in one to one corresponding to tree graphs satisfying appropriate conditions. Moreover the integrands $e^T(N_{F_j/X_{0,n,d}})$, $i_j^* \alpha_1^T, \ldots, i_j^* \alpha_n^T$ and $i_j^* E_{0,n,d}^T$ are described by the data of the action of T on X.

B.3 Tree graphs parameterizing fixed components

Definition B.3.1. We define the set of decorated graphs $\vec{\Gamma} = (\Gamma, \vec{f}, \vec{d}, \vec{s}) \in G_{0,n}(X, d)$ which consists of the following objects:

(i) (underlying graph) Γ is a compact connected one-dimensional CW complex with genus zero, i.e. Γ is a tree graph. We denote by $V(\Gamma)$ the set of vertices of Γ and by $E(\Gamma)$ the set of one-dimensional cells of Γ. We denote by $F(\Gamma)$ the set of flags in Γ, i.e.

$$F(\Gamma) = \{(v, e) \in V(\Gamma) \times E(\Gamma) \mid v \in \bar{e}\}.$$

(ii) (labeling of the vertices and edges) $\vec{f}$ is a map $V(\Gamma) \amalg E(\Gamma) \to \Sigma(0) \amalg \Sigma(1)$ which maps $v \in V(\Gamma)$ to $p_v \in \Sigma(0)$ and $e \in E(\Gamma)$ to $O_e \in \Sigma(1)$, respectively, satisfying the condition

$$p_v \in \overline{O_e} =: \ell_e$$

for any flag $(v, e) \in F(\Gamma)$.

(iii) (covering degree of the edges) $\vec{d}$ is a map $E(\Gamma) \to \mathbb{Z}_{\geq 1}$ which sends e to positive integer d_e. This assignment satisfies

$$\sum_{e \in E(\Gamma)} d_e[\ell_e] = d \in H_2(X, \mathbb{Z})$$

where $[\ell_e]$ is a homology class of ℓ_e.

(iv) (marked points of the domain curve) $\vec{s}$ is a map $\{1, 2, \ldots, n\} \to V(\Gamma)$, which is defined if $n \geq 1$.

For each $v \in V(\Gamma)$, we define

$$E_v = \{e \in E(\Gamma) \mid (v, e) \in F(\Gamma)\},$$
$$S_v = \{i \mid \vec{s}(i) = v\},$$

and denote numbers of elements of these sets by $\mathrm{val}(v)$ and n_v, respectively. Using these numbers, we divide the vertices into four subsets as follows:

$$V^S(\Gamma) = \{v \in V(\Gamma) \mid \mathrm{val}(v) + n_v \geq 3\},$$
$$V^2(\Gamma) = \{v \in V(\Gamma) \mid \mathrm{val}(v) = 2,\ n_v = 0\},$$
$$V^{1,1}(\Gamma) = \{v \in V(\Gamma) \mid \mathrm{val}(v) = 1,\ n_v = 1\},$$
$$V^1(\Gamma) = \{v \in V(\Gamma) \mid \mathrm{val}(v) = 1,\ n_v = 0\}.$$

We call $V^S(\Gamma)$ stable vertices and $V^2(\Gamma) \amalg V^{1,1}(\Gamma) \amalg V^1(\Gamma)$ unstable vertices.

We associate the fixed component $M_{\vec{\Gamma}}$ of $X_{0,n,d}$ to each $\vec{\Gamma} \in G_{0,n}(X, d)$ as follows: For each $v \in V^S(\Gamma)$, we pick $C_v \in \overline{\mathcal{M}}_{0,n_v+\mathrm{val}(v)}$. We specify the marked points on C_v by $p_{v,m_1}, \ldots, p_{v,m_{n_v}}$

and $q_{v,e_1}, \ldots, q_{v,e_{\mathrm{val}(v)}}$ where $S_v = \{m_1, \ldots, m_{n_v}\}$ and $E_v = \{e_1, \ldots, e_{\mathrm{val}(v)}\}$. We define $f_v : C_v \to X$ which is a constant map satisfying $f_v(C_v) = \{p_v\}$. For each $e \in E(\Gamma)$, we pick $C_e \cong \mathbb{P}^1$ with two marked points $p_{e,v}$ and $p_{e,v'}$ where $p_v, p_{v'} \in \ell_e$. We define $f_e : C_e \to X$ which is a degree d_e morphism onto ℓ_e satisfying $f_e(p_{e,v}) = p_v$, $f_e(p_{e,v'}) = p_{v'}$ and ramified at p_v and $p_{v'}$.

Let us construct a nodal rational curve Σ by gluing these curves. For $v \in V^S(\Gamma)$, we glue the curves C_v and C_{e_i} at the marked points q_{v,e_i} and $p_{e_i,v}$. For $v \in V^2(\Gamma)$, we glue C_{e_1} and C_{e_2} at the points $p_{e_1,v}$ and $p_{e_2,v}$ where $E_v = \{e_1, e_2\}$. The maps f_v ($v \in V^S(\Gamma)$) and f_e ($e \in E(\Gamma)$) are naturally glued and determine the morphism $f : \Sigma \to X$. Then we obtain the stable map which is invariant under the torus action.

Note that the stable maps $f : \Sigma \to X$ depend on the choices of $C_v \in \overline{\mathcal{M}}_{0,n_v+\mathrm{val}(v)}$. Then we obtain the stable maps for each element in the product $\prod_{v \in V^S(\Gamma)} \overline{\mathcal{M}}_{0,n_v+\mathrm{val}(v)}$. We need to divide this space by the automorphism of the stable maps $A_{\vec{\Gamma}}$. There exists the following exact sequence:

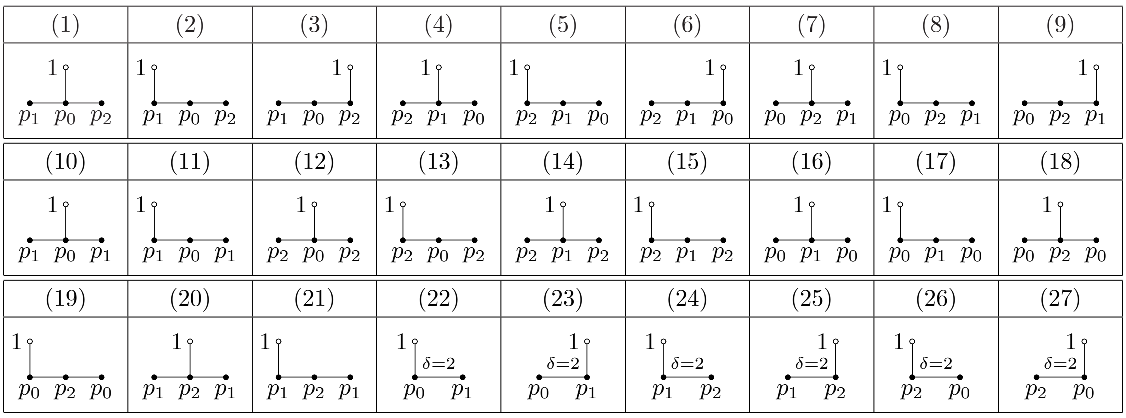
$$0 \to \prod_{e \in E(\Gamma)} \mathbb{Z}_{d_e} \to A_{\vec{\Gamma}} \to \mathrm{Aut}(\Gamma) \to 0,$$

where $\mathbb{Z}_{d_e}$ is a deck transformation of $f_e : C_e \to \ell_e$ and $\mathrm{Aut}(\Gamma)$ is the automorphism group of the decorated graph, i.e. it consists of bijective maps $\varphi_V : V(\Gamma) \to V(\Gamma)$, $\varphi_E : E(\Gamma) \to E(\Gamma)$ preserving the structures of the decorated graph $\vec{\Gamma}$. Then we have the description of the fixed locus

$$M_{\vec{\Gamma}} = \left[\prod_{v \in V^S(\Gamma)} \overline{\mathcal{M}}_{0,n_v+\mathrm{val}(v)} / A_{\vec{\Gamma}} \right]$$

for each decorated graph $\vec{\Gamma}$. Conversely any fixed locus can be represented by some decorated graph $\vec{\Gamma}$. We denote the natural inclusion of $M_{\vec{\Gamma}}$ to $X_{0,n,d}$ by $i_{\vec{\Gamma}}$.

Example B.3.2 ($X = \mathbb{P}^2$, $n = 1$, and $d = 2[\ell] \in H_2(X, \mathbb{Z})$)**.** In this case, the set $G_{0,1}(X, d)$ consists of the following decorated graphs:



where $p_0 = [1:0:0]$, $p_1 = [0:1:0]$, $p_2 = [0:0:1]$ indicate the images of the vertices, and δ is a covering degree of the edges. The decorated graph has non-trivial automorphism $\mathbb{Z}_2$ for (10), (12), (14), (16), (18) and (20).

Next section, we calculate the contribution of each fixed locus corresponding to a decorated graph in the localization formula.

B.4 Contributions of each graphs

Let us take a decorated graph $\vec{\Gamma} \in G_{0,n}(X,d)$ and consider the corresponding fixed locus $M_{\vec{\Gamma}} \subset X^T_{0,n,d}$. We give a formula which expresses the integral

$$\int_{[M_{\vec{\Gamma}}]^T} \frac{i^*_{\vec{\Gamma}}(\alpha_1^T \wedge \cdots \wedge \alpha_n^T)}{e^T(N_{M_{\vec{\Gamma}}/X_{0,n,d}})} \tag{B.4}$$

by the data of the torus action on X.

Remark B.4.1. To calculate the $\mathcal{E}$-twisted Gromov–Witten invariants, we need to incorporate the class $E_{0,n,d}$ in the integrand (B.4). The restriction of the class $E_{0,n,d}$ to the fixed locus $M_{\vec{\Gamma}}$ is contained in $\mathbb{C}[\lambda_1, \ldots, \lambda_l]$, which is a subspace of $H^*_T(M_{\vec{\Gamma}}, \mathbb{C}) = \mathbb{C}[\lambda_1, \ldots, \lambda_l] \otimes H^*(M_{\vec{\Gamma}}, \mathbb{C})$. Then it does not affect the integral (B.4). We calculate it for individual vector bundles.

We introduce some notations to state the precise formula. For each $v \in V(\Gamma)$, we denote by $\boldsymbol{w}(p_v)$ a product of the all weights of the torus action on the tangent space $T_{p_v}X$. For each flag $(v,e) \in F(\Gamma)$, we denote by $\boldsymbol{w}_{(v,e)}$ the weight of the torus action on $T_{p_v}\ell_e$ and define $w_{(v,e)} := \boldsymbol{w}_{(v,e)}/d_e$. For each $e \in E(\Gamma)$ and $d \in \mathbb{Z}_{\geq 1}$, we define $\boldsymbol{h}(e,d)$ as follows: Let v and v' be vertices of e. Let $\{w_{v,1}, \ldots, w_{v,N-1}, w_{v,N} = \boldsymbol{w}_{(v,e)}\}$ (resp. $\{w_{v',1}, \ldots, w_{v',N-1}, w_{v',N} = \boldsymbol{w}_{(v',e)}\}$) be a set of weights of the torus action on $T_{p_v}X$ (resp. $T_{p_{v'}}X$). It is known that these weights have the following properties:

(i) $\boldsymbol{w}_{(v',e)} = -\boldsymbol{w}_{(v,e)}$.

(ii) There exist a permutation $\sigma : \{1, \ldots, N-1\} \to \{1, \ldots, N-1\}$ and $a_i \in \mathbb{Z}_{\geq 0}$ $(i = 1, \ldots, N-1)$ such that
$$w_{v',i} = w_{v,\sigma(i)} - a_i \boldsymbol{w}_{(v,e)}.$$

Note that $a_i \geq 0$ follow from the assumption that X is a convex variety. Using this result, we define
$$\boldsymbol{h}(e,d) := \frac{(-1)^d d^{2d}}{d!^2 \boldsymbol{w}_{(v,e)}^{2d}} \prod_{i=1}^{N-1} \frac{1}{\prod_{j=0}^{da_i} w_{v,i} - j\frac{\boldsymbol{w}_{(v,e)}}{d}}.$$

Theorem B.4.2 (cf. [LS15])**.** *The integral* $(B.4)$ *can be described as follows:*

$$\frac{1}{|A_{\vec{\Gamma}}|} \prod_{e \in E(\Gamma)} \frac{\boldsymbol{h}(e,d_e)}{d_e} \prod_{v \in V(\Gamma)} \left(\boldsymbol{w}(p_v)^{\mathrm{val}(v)-1} \prod_{i \in S_v} \iota^*_{p_v} \alpha_i^T \right) \int_{\overline{\mathcal{M}}_{0,n_v+\mathrm{val}(v)}} \frac{1}{\prod_{e \in E_v}(w_{(v,e)} - \psi_{(v,e)})}$$

where ι_{p_v} *is a natural inclusion* $\{p_v\} \hookrightarrow X$ *and* $\psi_{(v,e)}$ *is a psi-class on* $\overline{\mathcal{M}}_{0,n_v+\mathrm{val}(v)}$ *at* $p_{v,e}$ $(e \in E_v)$*. We use the following conventions for the integrals on* $\overline{\mathcal{M}}_{0,1}$ *and* $\overline{\mathcal{M}}_{0,2}$*:*

$$\int_{\overline{\mathcal{M}}_{0,2}} \frac{1}{(w_1-\psi_1)(w_2-\psi_2)} = \frac{1}{w_1+w_2},$$
$$\int_{\overline{\mathcal{M}}_{0,2}} \frac{1}{w_1-\psi_1} = 1, \qquad \int_{\overline{\mathcal{M}}_{0,1}} \frac{1}{w_1-\psi_1} = w_1,$$

for $v \in V^2(\Gamma) \amalg V^{1,1}(\Gamma) \amalg V^1(\Gamma)$*.*

The integrals in Theorem B.4.2 can be calculated by the following formula [Kon95]:

$$\int_{\overline{\mathcal{M}}_{0,n}} \psi_1^{d_1} \cdots \psi_n^{d_n} = \frac{(n-3)!}{d_1! \cdots d_n!} \quad (n \geq 3,\ d_1, \ldots, d_n \geq 0).$$

Then we can calculate the equivariant integral (B.2) by the localization formula. We do this calculation for twisted Gromov–Witten invariants of Grassmannians in Section 3.2.

Appendix C

List of Picard–Fuchs operators

We list differential operators which annihilate the $\mathcal{E}$-twisted I-functions for the pairs $(X, \mathcal{E})$ corresponding to Calabi–Yau 3-folds with second Betti number one in Table 1.1. In [BCFKvS98], we can find a conjectural mirror family for complete intersection Calabi–Yau 3-folds in Grassmannian by using the conifold transition as well as their Picard–Fuchs operators. To avoid overlap with their list, we omit the Picard–Fuchs operators for complete intersection Calabi–Yau 3-folds in Grassmannian which correspond to No. 1, 2, 3, 6, 12, 19. Except for this omission, for completeness, we include previously known examples in other literatures.

In the list below, we set $\theta = q\frac{d}{dq}$ and Y to be the zero locus of a general section of $\mathcal{E}$ on X.

No. 4 : $X = G(2,5), \mathcal{E} = \mathcal{S}^*(1) \oplus \mathcal{O}(2)$.

$$\theta^4 - 2q(2\theta+1)^2\left(17\theta^2+17\theta+5\right) + 4q^2(\theta+1)^2(2\theta+1)(2\theta+3)$$

It is known that the zero locus Y is a complete intersection of $OG(5,10)$ (see the description of V_{12} in [CCGK16]).

No. 5 : $X = G(2,5), \mathcal{E} = \wedge^2\mathcal{Q}(1)$.

$$\begin{aligned}\theta^4 &- q\left(124\theta^4+242\theta^3+187\theta^2+66\theta+9\right)\\ &+ q^2\left(123\theta^4-246\theta^3-787\theta^2-554\theta-124\right)\\ &+ q^3\left(123\theta^4+738\theta^3+689\theta^2+210\theta+12\right)\\ &- q^4\left(124\theta^4+254\theta^3+205\theta^2+78\theta+12\right) + q^5(\theta+1)^4\end{aligned}$$

It is proved in [IIM1] that the zero locus Y is deformation equivalent to the complete intersection of two $G(2,5)$ in $\mathbb{P}^9$ which is studied in [GP01], [Kan12]. The Picard–Fuchs operator of the Calabi–Yau 3-fold is derived in [Kap13] via conifold transition.

No. 7 : $X = G(2,6), \mathcal{E} = \mathcal{S}^*(1) \oplus \mathcal{O}(1)^{\oplus 3}$**.**

$$\begin{aligned}121\theta^4 - 77q\left(130\theta^4 + 266\theta^3 + 210\theta^2 + 77\theta + 11\right)\\ - q^2\left(321126\theta^4 + 899990\theta^3 + 1037250\theta^2 + 552530\theta + 11198\right)\\ - q^3\left(287230\theta^4 + 741840\theta^3 + 634740\theta^2 + 206250\theta + 1716\right)\\ - 7q^4\left(1135\theta^4 + 2336\theta^3 + 1881\theta^2 + 713\theta + 110\right) - 49q^5(\theta+1)^4\end{aligned}$$

The zero locus Y is isomorphic to a linear section of a minuscule Schubert variety in $\mathbb{OP}^2$ (see [IIM1]), which is studied in [Miu13], [Gal14], [GKM].

No. 10 : $X = G(2,6), \mathcal{E} = \mathcal{Q}(1) \oplus \mathcal{O}(1)$**.** In [Man15], Manivel shows that Y is isomorphic to a general linear section of $G(2,7)$ of codimension 7. Since the I-function $I_X^{\mathcal{E}}$ is same as No. 12, we omit the Picard–Fuchs operator of it.

No. 13 : $X = G(2,7), \mathcal{E} = \mathbf{Sym}^2\mathcal{S}^* \oplus \mathcal{O}(1)^{\oplus 4}$**.**

$$\begin{aligned}49\theta^4 - 14q\left(134\theta^4 + 286\theta^3 + 234\theta^2 + 91\theta + 14\right)\\ - 4q^2\left(3183\theta^4 + 10266\theta^3 + 13501\theta^2 + 8225\theta + 1918\right)\\ - 8q^3\left(2588\theta^4 + 8400\theta^3 + 10256\theta^2 + 5649\theta + 1190\right)\\ - 48q^4\left(256\theta^4 + 848\theta^3 + 1141\theta^2 + 717\theta + 174\right) - 2304q^5(\theta+1)^4\end{aligned}$$

The zero locus Y is a complete intersection of the orthogonal Grassmannian $OG(2,7)$.

No. 15 : $X = G(2,7), \mathcal{E} = \wedge^4\mathcal{Q} \oplus \mathcal{O}(1) \oplus \mathcal{O}(2)$**.**

$$\theta^4 - 6q(2\theta+1)^2\left(3\theta^2 + 3\theta + 1\right) - 108q^2(\theta+1)^2(2\theta+1)(2\theta+3)$$

The zero locus of a general section of $\wedge^4\mathcal{Q}$ is isomorphic to the homogeneous space G_2/P_{long} (see the description of V_{18} in [CCGK16]). Hence Y is a complete intersection of the homogeneous space. The Picard–Fuchs operator is already known by [Kap12] via conifold transition.

No. 16 : $X = G(2,7), \mathcal{E} = \mathcal{S}^*(1) \oplus \wedge^4\mathcal{Q}$**.** It is proved in [IIM1] that Y is deformation equivalent to a linear section of $G(2,7)$. Since Gromov–Witten invariants are deformation invariant, the twisted J-function is the same as No. 12.

No. 17 : $X = G(2,8), \mathcal{E} = \wedge^5\mathcal{Q} \oplus \mathcal{O}(1)^{\oplus 3}$**.**

$$\begin{aligned}361\theta^4 - 19q\left(700\theta^4 + 1238\theta^3 + 999\theta^2 + 380\theta + 57\right)\\ + q^2\left(-64745\theta^4 - 368006\theta^3 - 609133\theta^2 - 412756\theta - 102258\right)\\ + 27q^3\left(6397\theta^4 + 12198\theta^3 - 11923\theta^2 - 27360\theta - 11286\right)\\ + 729q^4\left(64\theta^4 + 1154\theta^3 + 2425\theta^2 + 1848\theta + 486\right) - 177147q^5(\theta+1)^4\end{aligned}$$

The zero locus Y is isomorphic to a linear section of the determinantal nets of conics which is studied by Tjøtta. Tjøtta has calculated the Picard–Fuchs operator in [Tjø97] by using the direct calculation of the quantum connection of N and using the quantum Lefschetz theorem.

No. 18 : $X = G(2,8), \mathcal{E} = \mathbf{Sym}^2\mathcal{S}^* \oplus \wedge^5\mathcal{Q}$.

$$\theta^4 - 2q\left(2\theta^2+2\theta+1\right)\left(11\theta^2+11\theta+3\right) + 4q^2(\theta+1)^2\left(76\theta^2+152\theta+111\right) - 144q^3(\theta+1)(\theta+2)(2\theta+3)^2$$

The zero locus Y has appeared first in [Tjø97], and the virtual number of lines and conics on Y are computed in [Tjø97]. Our computations in Section 2.3 are consistent to these numbers.

No. 20 : $X = G(3,6), \mathcal{E} = \wedge^2\mathcal{S}^* \oplus \mathcal{O}(1)^{\oplus 2} \oplus \mathcal{O}(2)$.

$$\theta^4 - 8q(2\theta+1)^2\left(3\theta^2+3\theta+1\right) + 64q^2(\theta+1)^2(2\theta+1)(2\theta+3)$$

The zero locus of a general section of $\wedge^2\mathcal{S}^*$ is Lagrangian Grassmannian $LG(3,6)$. Then Y is a complete intersection on the homogeneous space $LG(3,6)$. The above differential operator coincides with the predicted one by [vEvS06].

No. 21 : $X = G(3,6), \mathcal{E} = \mathcal{S}^*(1) \oplus \wedge^2\mathcal{S}^*$. Similarly to No. 16, Y is deformation equivalent to a linear section of $G(3,6)$ by [IIM1]. The Picard–Fuchs operator can be found in [BCFKvS98].

No. 22 : $X = G(3,7), \mathcal{E} = \mathbf{Sym}^2\mathcal{S}^* \oplus \mathcal{O}(1)^{\oplus 3}$. The zero locus Z of a general section of $\mathrm{Sym}^2\mathcal{S}^*$ is isomorphic to orthogonal Grassmannian $OG(3,7)$. Using the natural isomorphisms $OG(3,7) \cong OG(4,8) \cong OG(1,8)$, Z is isomorphic to a smooth quadric hypersurface Q of $\mathbb{P}^7$. Under this isomorphism, the restriction of $\mathcal{O}_{G(3,7)}(1)$ on Z coincides with $\mathcal{O}_Q(2)$. Hence Y is isomorphic to a complete intersection of four quadric hypersurfaces of $\mathbb{P}^7$.

No. 23 : $X = G(3,7), \mathcal{E} = (\wedge^2\mathcal{S}^*)^{\oplus 2} \oplus \mathcal{O}(1)^{\oplus 3}$.

$$\begin{aligned} &3721\theta^4 - 61q\left(3029\theta^4+5572\theta^3+4677\theta^2+1891\theta+305\right) \\ &\quad + q^2\left(12152215\theta^4+34281132\theta^3+42672228\theta^2+25726775\theta+611586\right) \\ &\quad - 81q^3\left(393770\theta^4+1401178\theta^3+2068077\theta^2+1421191\theta+37332\right) \\ &\quad + 6561q^4\left(566\theta^4+2230\theta^3+3356\theta^2+2241\theta+558\right) - 1594323q^5(\theta+1)^4 \end{aligned}$$

No. 24 : $X = G(3,7), \mathcal{E} = (\wedge^3 \mathcal{Q})^{\oplus 2} \oplus \mathcal{O}(1)$**.**

$$\begin{aligned}81\theta^4 - 9q\left(317\theta^4 + 520\theta^3 + 431\theta^2 + 171\theta + 27\right)\\
+ q^2\left(6589\theta^4 - 7616\theta^3 - 316688\theta^2 - 282251\theta - 8370\right)\\
- q^3\left(5521\theta^4 + 213840\theta^3 + 1072230\theta^2 + 1384020\theta + 55782\right)\\
+ q^4\left(219870\theta^4 + 1307520\theta^3 + 1521680\theta^2 + 91940\theta - 39016\right)\\
- 19q^5(\theta+1)\left(293\theta^3 - 7005\theta^2 - 18780\theta - 12535\right)\\
- 361q^6(\theta+1)(\theta+2)\left(137\theta^2 + 357\theta + 223\right) - 6859q^7(\theta+1)(\theta+2)^2(\theta+3)\end{aligned}$$

No. 25 : $X = G(3,7), \mathcal{E} = \wedge^2 \mathcal{S}^* \oplus \wedge^3 \mathcal{Q} \oplus \mathcal{O}(1)^{\oplus 2}$**.**

$$\begin{aligned}121\theta^4 - 11q(434\theta^4 + 820\theta^3 + 685\theta^2 + 275\theta + 44)\\
+ q^2(7841\theta^4 + 10916\theta^3 + 3133\theta^2 - 2486\theta - 1320)\\
- 4q^3(1488\theta^4 + 4092\theta^3 + 6761\theta^2 + 5511\theta + 1694)\\
- 32q^4(136\theta^4 + 1460\theta^3 + 4556\theta^2 + 5245\theta + 2017)\\
+ 256q^5(\theta+1)^2(40\theta^2 + 212\theta + 237) - 4096q^6(\theta+1)^2(\theta+2)^2\end{aligned}$$

No. 28 : $X = G(3,8), \mathcal{E} = (\wedge^2 \mathcal{S}^*)^{\oplus 4}$**.**

$$\begin{aligned}529\theta^4 - 23q\left(850\theta^4 + 1634\theta^3 + 1461\theta^2 + 644\theta + 115\right)\\
+ q^2\left(140191\theta^4 + 504286\theta^3 + 765193\theta^2 + 554484\theta + 160080\right)\\
- 2q^3\left(2255598\theta^4 + 11456682\theta^3 + 23385290\theta^2 + 22035840\theta + 805023\right)\\
+ q^4\left(7390230\theta^4 + 45335640\theta^3 + 110085380\theta^2 + 122651360\theta + 5219792\right)\\
- q^5\left(6834380\theta^4 + 45037340\theta^3 + 115762070\theta^2 + 136581000\theta + 6109137\right)\\
+ q^6\left(3923370\theta^4 + 25589820\theta^3 + 62758690\theta^2 + 69253080\theta + 2877864\right)\\
- 4q^7\left(367740\theta^4 + 2350860\theta^3 + 5584990\theta^2 + 5797480\theta + 221197\right)\\
+ 16q^8\left(20810\theta^4 + 125480\theta^3 + 288140\theta^2 + 293640\theta + 11133\right) - 256q^9(2\theta+3)^4\end{aligned}$$

Appendix D

Determining the topological invariants via monodromy calculations

D.1 Topological invariants

In this section we summarize the calculation of the topological invariants (H^3, $c_2(Y).H$, $e(Y)$) of Calabi–Yau 3-folds Y which are contained in Appendix C by using the monodromy calculation of the I-function of Y. The details can be found in [vEvS06], [HT14].

Let I_Y be a I-function of Y and P be a fourth order differential operator which annihilates I_Y. Let $x \in \mathbb{C}$ be a local coordinate around the maximally unipotent monodromy point. Practically we find a singular point α of P which is nearest to $x = 0$, so-called conifold point. There exists three holomorphic solutions of P around α. Moreover there exists the following solution:

$$w_{\mathrm{cp}}(y)\log(y) + w_{\mathrm{hol}}(y) \tag{D.1}$$

where $y := x - \alpha$ is a local coordinate around α. The function $w_{\mathrm{cp}}(y)$ is determined up to scalar multiplication and we can calculate it for arbitrary degree of y.

We connect $x = 0$ and $x = \alpha$ by the line segment L and we do the analytic continuation of the function $w_{\mathrm{cp}}(y)$ along L. Candelas–de la Ossa–Green–Parks observed the following behavior of the conifold period under this analytic continuation [CdlOGP91]:

$$w_{\mathrm{cp}}(t) = \frac{H^3}{6}t^3 + \frac{c_2(Y).H}{24}t + \frac{e(Y)}{(2\pi i)^3}\zeta(3) + O(q) \tag{D.2}$$

where $2\pi i t$ is the mirror map around $x = 0$ and $q = e^{2\pi i t}$.

It is necessary to determine the function $w_{\mathrm{cp}}(y)$ which behaves similar to (D.2). If we know the degree H^3 of Y, we can read other invariants $c_2(Y).H$ and $e(Y)$ from $w_{\mathrm{cp}}(t)$ by suitably normalizing the coefficient of t^3.

Although actual computation is numerical, we can find reasonable integral invariants in the digits of sufficiently high accuracy. We have checked that the computation of each Picard–Fuchs equation of Calabi–Yau 3-folds with $\rho = 1$ is consistent with our computations in Section 1.2.

D.2 Integral symplectic basis

If I_Y coincides with a period integral of mirror Calabi–Yau 3-folds $\{Y_z^*\}_{z\in\mathbb{P}^1}$, the Picard–Fuchs equation has a basis of solutions whose monodromy matrices are in $Sp(4,\mathbb{Z})$. We do not discussed the existence of the mirror family of the Calabi–Yau 3-folds in our list. But there exists an ansatz of the integral symplectic basis of the Picard–Fuchs equation.

Let $P(x,\theta_x)$ be a Picard–Fuchs operator which has $x=0$ as a maximally unipotent monodromy point. We can take a basis of the local solution of P as follows:

$$\begin{aligned}
&w_0(x),\\
&w_1(x) = w_0(x)\log x + w_1^{\mathrm{hol}}(x),\\
&w_2(x) = w_0(x)(\log x)^2 + 2w_1^{\mathrm{hol}}(x)\log x + w_2^{\mathrm{hol}}(x),\\
&w_3(x) = w_0(x)(\log x)^3 + 3w_1^{\mathrm{hol}}(x)\log x + 3w_2^{\mathrm{hol}}(x)\log x + w_3^{\mathrm{hol}}(x)
\end{aligned}$$

where $w_0(x)$ and $w_i^{\mathrm{hol}}(x)$ $(i=1,2,3)$ are holomorphic and $w_i^{\mathrm{hol}}(0)=0$ $(i=1,2,3)$. We replace the basis of the solution by

$$\Pi(x) := \begin{pmatrix} 1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ \frac{\beta}{24} & a & -\frac{\kappa}{2} & 0\\ \gamma & \frac{\beta}{24} & 0 & \frac{\kappa}{6}\end{pmatrix}\begin{pmatrix} n_0w_0(x)\\ n_1w_1(x)\\ n_2w_2(x)\\ n_3w_3(x)\end{pmatrix}$$

where $\kappa=-H^3$, $\beta=-c_2(Y).H$, $\gamma=-e(Y)(2\pi i)^{-3}\zeta(3)$, $n_j=(2\pi i)^{-j}$ $(j=0,\dots,3)$ and a is an unknown constant. Then $\Pi(x)$ gives the candidate of the solution of P whose monodromy matrices are in $Sp(4,\mathbb{Z})$.

In general it is not easy to do the analytic continuation of the solution of the differential equation. But numerically we can do the analytic continuation along any path by using of suitable computer software. In Section D.3, we carry out the analytic continuation of the basis $\Pi(x)$ and find the candidate of the integral symplectic monodromy matrices.

D.3 Computations for selected examples

We present the monodromy matrices of the basis $\Pi(x)$ which are the solutions of the Picard–Fuchs operator P. We only take the selected Picard–Fuchs operators in Appendix C, but the same calculation works for other operators.

Let P be a Picard–Fuchs operator and $D=\{\alpha_0=0,\alpha_1,\dots,\alpha_k\}$ ($\alpha_i\neq\alpha_j$ for $i\neq j$) be the singular points of P in $\mathbb{C}$. We assume that α_1 is a conifold point which is the nearest to the origin among $D\setminus\{0\}$. We fix the base-point by $q_b:=\alpha_1/2$.

Let us define $r:=\min\{|\alpha_i-\alpha_j| \mid 0\le i<j\le k\}$. For $i=0,1,\dots,k$, we calculate the analytic continuation along the following paths: If q_b and α_i are connected by a line segment to avoid other singular points, we take

$$\gamma_i(t)=\begin{cases} q_b+t(q_i-q_b) & (0\le t\le 1)\\ \alpha_i+(q_i-\alpha_i)e^{2\pi i(t-1)} & (1\le t\le 2)\\ q_i+(t-2)(q_b-q_i) & (2\le t\le 3)\end{cases}$$

where $q_i = \alpha_i + \frac{r}{2}\frac{q_b-\alpha_i}{|q_b-\alpha_i|}$. If there exist some other singular point between q_b and α_i, we deform the path γ_i by

$$e^{2\pi i\epsilon}(\gamma_i(t) - q_b) + q_b$$

for sufficiently small $\epsilon > 0$. We denote the monodromy matrix of $\Pi(x)$ along the path γ_i by M_{α_i}, i.e. it satisfies

$$(\gamma_i)_*\Pi(x) = M_{\alpha_i}\Pi(x)$$

where $(\gamma_i)_*\Pi(x)$ denote the analytic continuation of $\Pi(x)$ along γ_i.

In the rest of this appendix, we present the results of our calculations. We denote P the Picard–Fuchs operator corresponding to each number (see Appendix C).

No.18. $\mathcal{E} = \mathrm{Sym}^2\mathcal{S}^* \oplus \wedge^5\mathcal{Q}$ on $G(2,8)$.

The Riemann scheme of P:

$$\left\{\begin{array}{cccc} 0 & \frac{1}{36} & \frac{1}{4} & \infty \\ \hline 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 3/2 \\ 0 & 1 & 1 & 3/2 \\ 0 & 2 & 1 & 2 \end{array}\right\}$$

The monodromy matrices of $\Pi(x)$ $(a=0)$:

$$M_0 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 36 & 72 & 1 & 0 \\ -20 & -36 & -1 & 1 \end{pmatrix}, \quad M_{\frac{1}{36}} = \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad M_{\frac{1}{4}} = \begin{pmatrix} -17 & -12 & -3 & -11 \\ -6 & 1 & -1 & -3 \\ 0 & 72 & 1 & 12 \\ 36 & 0 & 6 & 19 \end{pmatrix}.$$

No.23. $\mathcal{E} = (\wedge^2\mathcal{S}^*)^{\oplus 2} \oplus \mathcal{O}(1)^{\oplus 3}$ on $G(3,7)$.

The Riemann scheme of P:

$$\left\{\begin{array}{cccccc} 0 & \alpha_1 & \alpha_2 & \alpha_3 & \frac{61}{81} & \infty \\ \hline 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 3 & 1 \\ 0 & 2 & 2 & 2 & 4 & 1 \end{array}\right\}$$

where $\alpha_1, \alpha_2, \alpha_3$ are solutions of $1 - 47q + 200q^2 - 243q^3 = 0$.

The monodromy matrices of $\Pi(x)$ $(a = 1/2)$:

$$M_0 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 31 & 61 & 1 & 0 \\ -18 & -30 & -1 & 1 \end{pmatrix}, \quad M_{\alpha_1} = \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad M_{\alpha_2} = \begin{pmatrix} 22 & -9 & 3 & 9 \\ 7 & -2 & 1 & 3 \\ 21 & -9 & 4 & 9 \\ -49 & 21 & -7 & -20 \end{pmatrix},$$

$$M_{\alpha_3} = \begin{pmatrix} -20 & -6 & -3 & 9 \\ 7 & 3 & 1 & -3 \\ -14 & -4 & -1 & 6 \\ -49 & -14 & -7 & 22 \end{pmatrix}, \quad M_{\frac{61}{81}} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

No.24. $\mathcal{E} = (\wedge^3\mathcal{Q})^{\oplus 2} \oplus \mathcal{O}(1)$ on $G(3,7)$.

The Riemann scheme of P:

$$\left\{\begin{array}{cccccccc} 0 & \alpha_1 & \alpha_2 & \alpha_3 & \alpha_4 & -1 & \frac{9}{19} & \infty \\ \hline 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 1 & 1 & 1 & 1 & 2 \\ 0 & 1 & 1 & 1 & 1 & 1 & 3 & 2 \\ 0 & 2 & 2 & 2 & 2 & 2 & 4 & 3 \end{array}\right\}$$

where $\alpha_1, \alpha_2, \alpha_3, \alpha_4$ are solutions of $1 - 32q - 22q^2 - 136q^3 - 19q^4 = 0$.
The monodromy matrices of $\Pi(x)$ $(a = 0)$:

$$M_0 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 36 & 72 & 1 & 0 \\ -20 & -36 & -1 & 1 \end{pmatrix}, \quad M_{\alpha_1} = \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad M_{\alpha_2} = \begin{pmatrix} 22 & -12 & 3 & 9 \\ 7 & -3 & 1 & 3 \\ 28 & -16 & 5 & 12 \\ -49 & 28 & -7 & -20 \end{pmatrix},$$

$$M_{\alpha_3} = \begin{pmatrix} -20 & -12 & -3 & 9 \\ 7 & 5 & 1 & 3 \\ -28 & -16 & -3 & 12 \\ -49 & -28 & -7 & 22 \end{pmatrix}, \quad M_{\alpha_4} = \begin{pmatrix} 603 & 672 & 84 & -196 \\ -258 & -287 & -36 & 84 \\ 2064 & 2304 & 289 & -672 \\ 1849 & 2064 & 258 & -601 \end{pmatrix},$$

$$M_{-1} = \begin{pmatrix} -111 & -168 & -14 & 28 \\ 56 & 85 & 7 & -14 \\ -672 & -1008 & -83 & 168 \\ -448 & -672 & -56 & 113 \end{pmatrix}, \quad M_{\frac{9}{19}} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

No.25. $\mathcal{E} = \wedge^2\mathcal{S}^* \oplus \wedge^3\mathcal{Q} \oplus \mathcal{O}(1)^{\oplus 2}$ on $G(3,7)$.

The Riemann scheme of P:

$$\left\{\begin{array}{ccccccc} 0 & \alpha_1 & \alpha_2 & \alpha_3 & -1 & \frac{11}{8} & \infty \\ \hline 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 1 & 3 & 2 \\ 0 & 2 & 2 & 2 & 2 & 4 & 2 \end{array}\right\}$$

where $\alpha_1, \alpha_2, \alpha_3$ are solutions of $1 - 39q + 48q^2 - 64q^3 = 0$.
The monodromy matrices of $\Pi(x)$ $(a = 0)$:

$$M_0 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 33 & 66 & 1 & 0 \\ -19 & -33 & -1 & 1 \end{pmatrix}, \quad M_{\alpha_1} = \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad M_{\alpha_2} = \begin{pmatrix} 22 & -9 & 3 & 9 \\ 7 & -2 & 1 & 3 \\ 21 & -9 & 4 & 9 \\ -49 & 21 & -7 & -20 \end{pmatrix}.$$

$$M_{\alpha_3} = \begin{pmatrix} -20 & -9 & -3 & 9 \\ 7 & 4 & 1 & -3 \\ -21 & -9 & -2 & 9 \\ -49 & -21 & -7 & 22 \end{pmatrix}, \quad M_{-1} = \begin{pmatrix} -47 & -66 & -6 & 12 \\ 24 & 34 & 3 & -6 \\ -264 & -363 & -32 & 66 \\ -192 & -264 & -24 & 49 \end{pmatrix}, \quad M_{\frac{11}{8}} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

No.28. $\mathcal{E} = (\wedge^2 \mathcal{S}^*)^{\oplus 4}$ on $G(3,8)$.

The Riemann scheme of P:

$$\left\{\begin{array}{cccccccc} 0 & \alpha_1 & \alpha_2 & \alpha_3 & 1 & \beta_1 & \beta_2 & \infty \\ \hline 0 & 0 & 0 & 0 & 0 & 0 & 0 & 3/2 \\ 0 & 1 & 1 & 1 & 1 & 1 & 1 & 3/2 \\ 0 & 1 & 1 & 1 & 1 & 3 & 3 & 3/2 \\ 0 & 2 & 2 & 2 & 2 & 4 & 4 & 3/2 \end{array}\right\}$$

where $\alpha_1, \alpha_2, \alpha_3$ are solutions of $1 - 34q + 161q^2 - 256q^3 = 0$ and β_1, β_2 are solutions of $23 - 11q + 4q^2 = 0$.

The monodromy matrices of $\Pi(x)$ $(a = 0)$:

$$M_0 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 46 & 92 & 1 & 0 \\ -24 & -46 & -1 & 1 \end{pmatrix}, \quad M_{\alpha_1} = \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad M_{\alpha_2} = \begin{pmatrix} 25 & -18 & 3 & 9 \\ 8 & -5 & 1 & 3 \\ 48 & -36 & 7 & 18 \\ -64 & 48 & -8 & -23 \end{pmatrix},$$

$$M_{\alpha_3} = \begin{pmatrix} -23 & -18 & -3 & 9 \\ 8 & 7 & 1 & -3 \\ -48 & -36 & -5 & 18 \\ -64 & -48 & -8 & 25 \end{pmatrix}, \quad M_1 = \begin{pmatrix} -195 & -28 & -28 & -102 \\ -56 & 1 & -8 & -28 \\ 0 & 196 & 1 & 28 \\ 392 & 0 & 56 & 197 \end{pmatrix},$$

$$M_{\beta_1} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad M_{\beta_2} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Remark D.3.1. In Appendix A.2.3, we define Frobenius manifold associated with $(\mathbb{P}^{n-1})^k$, Weyl group $W = \mathfrak{S}_k$ and Weyl invariant vector bundle $\mathcal{V}_T$. The I-function (A.11) is defined even if $\mathcal{V}_T$ is not related to the vector bundle on $G(k,n)$.

The calculations of this appendix work for the following vector bundles:

$$\text{(a)} \left(\begin{array}{c|ccc} \mathbb{P}^4 & 2 & 0 & 3 \\ \mathbb{P}^4 & 0 & 2 & 3 \end{array}\right), \qquad \text{(b)} \left(\begin{array}{c|ccc} \mathbb{P}^4 & 3 & 1 & 1 \\ \mathbb{P}^4 & 1 & 3 & 1 \end{array}\right), \qquad \text{(c)} \left(\begin{array}{c|ccc} \mathbb{P}^4 & 3 & 1 & 1 \\ \mathbb{P}^4 & 1 & 3 & 1 \\ \mathbb{P}^4 & 1 & 1 & 3 \end{array}\right).$$

We can find reasonable topological invariants for each of them. These invariants coincide with the formal calculation described in Section 1.2.2.

We give the details of calculations of these vector bundles as follows:

(a) The Picard–Fuchs operator:

$$P = \theta^4 - 36q(3\theta + 1)(3\theta + 2)(6\theta + 1)(6\theta + 5)$$

The Riemann scheme of P:

$$\left\{\begin{array}{ccc} 0 & \frac{1}{11664} & \infty \\ \hline 0 & 0 & 1/6 \\ 0 & 1 & 1/3 \\ 0 & 1 & 2/3 \\ 0 & 2 & 5/6 \end{array}\right\}$$

The topological invariants which is read from P:

$$H^3 = 3, \quad c_2(Y).H = 42, \quad e(Y) = -204.$$

The monodromy matrices of $\Pi(x)$ $(a = 0)$:

$$M_0 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 2 & 3 & 1 & 0 \\ -4 & -1 & -1 & 1 \end{pmatrix}, \quad M_{\frac{1}{11664}} = \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

(b) The Picard–Fuchs operator:

$$\begin{aligned} P = {} & 529\theta^4 - 23q\left(3271\theta^4 + 5078\theta^3 + 3896\theta^2 + 1357\theta + 184\right) \\ & + q^2\left(1357863\theta^4 + 999924\theta^3 - 787393\theta^2 - 850862\theta - 205712\right) \\ & - 8q^3\left(775799\theta^4 - 272481\theta^3 - 218821\theta^2 + 176709\theta + 100234\right) \\ & - 976q^4\left(1005\theta^4 - 15654\theta^3 - 36317\theta^2 - 27938\theta - 7304\right) \\ & - 1905152q^5(\theta+1)^2(4\theta+3)(4\theta+5) \end{aligned}$$

The Riemann scheme of P:

$$\left\{ \begin{array}{cccccc} 0 & \alpha_1 & \alpha_2 & \alpha_3 & \frac{23}{244} & \infty \\ \hline 0 & 0 & 0 & 0 & 0 & 3/4 \\ 0 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 3 & 1 \\ 0 & 2 & 2 & 2 & 4 & 5/4 \end{array} \right\}$$

where $\alpha_1, \alpha_2, \alpha_3$ are solutions of $1 - 121q - 113q^2 - 512q^3 = 0$.
The topological invariants which is read from P:

$$H^3 = 23, \quad c_2(Y).H = 62, \quad e(Y) = -74.$$

The monodromy matrices of $\Pi(x)$ $(a = 1/2)$:

$$M_0 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 12 & 23 & 1 & 0 \\ -9 & -11 & -1 & 1 \end{pmatrix}, \quad M_{\alpha_1} = \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad M_{\alpha_2} = \begin{pmatrix} 13 & 3 & 3 & 9 \\ 4 & 2 & 1 & 3 \\ -4 & -1 & 0 & -3 \\ -16 & -4 & -4 & -11 \end{pmatrix},$$

$$M_{\alpha_3} = \begin{pmatrix} -11 & 6 & -3 & 9 \\ 4 & -1 & 1 & -3 \\ 8 & -4 & 3 & -6 \\ -16 & 8 & -4 & 13 \end{pmatrix}, \quad M_{\frac{23}{244}} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

(c) The Picard–Fuchs operator:

$$\begin{aligned}P = 49\theta^4 &- 7q\left(155\theta^4+286\theta^3+234\theta^2+91\theta+14\right)\\&+q^2\left(-161056\theta^4-680440\theta^3-1022610\theta^2-660940\theta-15736\right)\\&+8q^3\left(2625\theta^4+8589\theta^3+9071\theta^2+3759\theta+476\right)\\&-16q^4\left(465\theta^4+1266\theta^3+1439\theta^2+806\theta+184\right)+512q^5(\theta+1)^4\end{aligned}$$

The Riemann scheme of P:

$$\left\{\begin{array}{cccccc} 0 & \frac{1}{32} & \alpha_1 & \alpha_2 & \frac{7}{4} & \infty \\ \hline 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 3 & 1 \\ 0 & 2 & 2 & 2 & 4 & 1 \end{array}\right\}$$

where α_1, α_2 are solutions of $1+11q-q^2=0$.

The topological invariants which is read from P:

$$H^3 = 35, \quad c_2(Y).H = 50, \quad e(Y) = -50.$$

The monodromy matrices of $\Pi(x)$ $(a=1/2)$:

$$M_0 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 18 & 35 & 0 & 0 \\ -10 & -17 & -1 & 1 \end{pmatrix}, \quad M_{\frac{1}{32}} = \begin{pmatrix} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad M_{\alpha_1} = \begin{pmatrix} -9 & -14 & -2 & 4 \\ 5 & 8 & 1 & -2 \\ -35 & -49 & -6 & 14 \\ -25 & -35 & -5 & 11 \end{pmatrix},$$

$$M_{\alpha_2} = \begin{pmatrix} 1 & 10 & 0 & 4 \\ 0 & 1 & 0 & 0 \\ 0 & -25 & 1 & -10 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad M_{\frac{7}{4}} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

By these calculation, we hope that there exist the following rank three globally generated vector bundles on $G(2,5)$:

(a) vector bundle $\mathcal{E}_a$ satisfying $c(\mathcal{E}_a) = 1+5\sigma_{(1,0)}+6\sigma_{(2,0)}+10\sigma_{(1,1)}+12\sigma_{(2,1)}$.

(b) vector bundle $\mathcal{E}_b$ satisfying $c(\mathcal{E}_b) = 1+5\sigma_{(1,0)}+7\sigma_{(2,0)}+11\sigma_{(1,1)}+3\sigma_{(3,0)}+10\sigma_{(2,1)}$.

(c) vector bundle $\mathcal{E}_c$ satisfying $c(\mathcal{E}_c) = 1+5\sigma_{(1,0)}+11\sigma_{(2,0)}+7\sigma_{(1,1)}+15\sigma_{(3,0)}+10\sigma_{(2,1)}$.

Note that the operators (a) and (c) coincide with the Picard–Fuchs operators of the following Calabi–Yau 3-folds:

(a) sextic hypersurface in $\mathbb{P}[1,1,1,1,2]$.

(c) Reye congruence Calabi–Yau 3-fold which is studied in [HT14].

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