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論文題目 Cube invariance of higher Chow groups with modulus

（モデュラス付き高次チャウ群のキューブ不変性）

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Cube invariance of higher Chow groups with modulus (summary)

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Abstract

This is a summary of the author's Ph.D thesis [M], in which we study some properties of the higher Chow group with modulus, introduced by F. Binda and S. Saito in [BS]. The higher Chow group with modulus is defined for a pair $\mathscr{X} = (X, D)$, where X is a scheme over a field and D is an effective Cartier divisor on X. It is a common generalization of Bloch's higher Chow group, the Chow group with modulus and the additive higher Chow group. One of the main results in [M] is an invariance property of the higher Chow group with modulus, which generalizes the $\mathbb{A}^1$-homotopy invariance of Bloch's higher Chow group. For this, we need to extend the definition of the higher Chow group with modulus for non-effective Cartier divisor D. Moreover, over a field of positive characteristic p, we prove that the higher Chow group with modulus with p inverted is $\mathbb{A}^1$-homotopy invariant, and is independent of the multiplicity of the modulus divisor. We explain the historical background of the research, state the main results, and provide key ideas of the proofs.

Contents

Throughout this summary, let k be a field, and denote by $\mathbf{Sch}_k$ the category of equidimensional, separated schemes of finite type over k.

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1 Historical Background

1.1 Chow groups

For a scheme $X \in \mathbf{Sch}_k$, the Picard group $\mathrm{Pic}(X)$ of X is a familiar object in algebraic geometry. It is defined as the set of isomorphism classes of line bundles over X. The abelian group structure on $\mathrm{Pic}(X)$ is given by tensor products of line bundles.

The chow group $\mathrm{CH}^r(X)$ of cycles of codimension $r \geq 0$ on X is a generalization of $\mathrm{Pic}(X)$, which is defined as follows:

$$\mathrm{CH}^r(X) := \mathbb{Z}\{\text{ irreducible closed subsets of } X \text{ of codimension } r\,\}/(\text{rational equivalences}).$$

Here, for any set S, we define $\mathbb{Z}S$ as the free abelian group generated on the set S. The elements of $\mathbb{Z}\{\text{ irreducible closed subsets of } X \text{ of codimension } r\ \}$ are called **algebraic cycles** of codimension r on X. We do not provide the precise definition of rational equivalences here (because the more general group will be defined in the following subsections). As a typical example, for any smooth scheme $X \in \mathbf{Sch}_k$, the Chow group of cycles of codimension 1 on X is canonically isomorphic to the Picard group of X:

$$\mathrm{CH}^1(X) \cong \mathrm{Pic}(X),\ D \mapsto \mathcal{L}(D).$$

Here, $\mathcal{L}(D)$ is the line bundle associated to the Cartier divisor D. One of the celebrated results on Chow groups is the following comparison isomorphism with the 0-th K-group $K_0(X)$ for any smooth k-scheme X:

$$K_0(X)_{\mathbb{Q}} \cong \bigoplus_{r\geq 0} \mathrm{CH}^r(X)_{\mathbb{Q}},$$

where the subscript $_{\mathbb{Q}}$ denotes the tensor product $\otimes_{\mathbb{Z}}\mathbb{Q}$.

1.2 Bloch's higher Chow groups

As a generalization of Chow groups, S. Bloch introduced in his paper [B] the q-th **higher Chow group** $\mathrm{CH}^r(X,q)$ of codimension r of X for any non-negative integer q. It is defined as the q-th homology group of a certain chain complex $z^r(X,*)$, which is called the **cycle complex** of X of codimension r. It is indeed a generalization of Chow group because we have a canonical identification $\mathrm{CH}^r(X,0) \cong \mathrm{CH}^r(X)$ for any X and for any r. Originally, the cycle complex $z^r(X,*)$ is defined as the complex associated to a simplicial abelian group, but we make use of a variant, which is called the cubical cycle complex. First, for any X and for any non-negative integers r,q, we define a group $\underline{z}^r(X,q)$ as the free abelian group generated on the following set:

$$\{\text{ irreducible closed subsets of } X \times \mathbb{A}^q \text{ of codimension } r \text{ which satisfy the } \textbf{face condition}\,\}.$$

Here, we say that an irreducible closed subset $V \subset X \times \mathbb{A}^q$ satisfies the face condition if and only if the following condition holds:

- For any face F on $\mathbb{A}^q$, the subset V intersects $X \times F$ properly on $X \times \mathbb{A}^q$, where a **face** on $\mathbb{A}^q$ is a closed subset of the following form for some subset $I \subset \{1,\ldots,q\}$ and for some $\epsilon_i \in \{0,1\},\ i \in I$:

$$\bigcap_{i\in I} \mathbb{A}^{i-1} \times \{\epsilon_i\} \times \mathbb{A}^{q-i}.$$

By the face condition, we can see that for any face $F \subset \mathbb{A}^q$ of codimension 1, the restriction of algebraic cycles along the inclusion $X \times F \subset X \times \mathbb{A}^q$ induces a well-defined morphism of abelian groups

$$\underline{z}^r(X,q) \to \underline{z}^r(X,q-1).$$

By taking an appropriate alternating sum of such morphisms, we obtain the differential map $d_q : \underline{z}^r(X,q) \to \underline{z}^r(X,q-1)$ which satisfies $d_{q+1} \circ d_q = 0$. In fact, $\underline{z}^r(X,*)$ can be equipped with a structure of cubical abelian group, whose degeneracy maps $\underline{z}^r(X,q) \to \underline{z}^r(X,q+1)$ are given by the pullback of cycles along the projections $X \times \mathbb{A}^{q+1} \to X \times \mathbb{A}^q$. In general, for any cubical abelian group $\underline{A}_*$, we can associate a canonical chain complex A_* defined by

$$A_q := \underline{A}_q/(\text{degenerate part}),$$

where the degenerate part is the sum of images of the degeneracy maps. Let $z^r(X,*)$ denote the chain complex associated to the cubical abelian group $\underline{z}^r(X,*)$.

Definition 1.1. For any $X \in \mathbf{Sch}_k$ and for any non-negative integers r,q, we define the q-th higher Chow group of X of codimension r by

$$\mathrm{CH}^r(X,q) := H_q(z^r(X,*)).$$

It is known that this definition coincides with the one which uses simplicial abelian groups. A remarkable result of Bloch is the following comparison isomorphism for any smooth scheme X and for any non-negative integer q:

$$K_q(X)_{\mathbb{Q}} \cong \bigoplus_{r\geq 0} \mathrm{CH}^r(X,q)_{\mathbb{Q}},$$

where the left hand side denotes Quillen's q-th higher algebraic K-group.

1.3 Higher Chow groups with modulus

In algebraic K-theory, there is an important object called the **relative** K-theroy. It is defined for "pairs of spaces." Precisely speaking, for any scheme X and for any closed subscheme $Z \subset X$, we can define the relative K-group $K_*(X,Z)$ of the pair (X,Z). The relative K-groups measure the difference between the K-groups of X and of Z. As we have seen in the previous subsections, the higher Chow group is closely related to the higher K-theory of schemes. So, it is natural to expect that there exists a kind of "relative" theory of higher Chow groups. The notion of **higher Chow groups with modulus**, which was introduced by F. Binda and S. Saito in their paper [BS], realizes such an expectation in some sense. By technical reasons, the higher Chow groups with modulus are defined for pairs of the form $\mathscr{X} = (X,D)$, where $X \in \mathbf{Sch}_k$ and D is an effective Cartier divisor on X. Similarly to the case of Bloch's higher Chow groups, the q-th higher Chow group with modulus $\mathrm{CH}^r(\mathscr{X},q)$ of $\mathscr{X}$ of codimension r is defined as the q-th homology group of a certain chain complex $z^r(\mathscr{X},*)$, called the cycle complex with modulus. The cycle complex with modulus is defined as the subcomplex of the cycle complex $z^r(\mathscr{X}^\circ,*)$ of the scheme $\mathscr{X}^\circ := X \setminus |D|$ generated by the irreducible closed subsets which satisfy the **modulus condition**. Here, we say that an irreducible closed subset V of the scheme $\mathscr{X}^\circ \times \mathbb{A}^q$ satisfies the modulus condition if and only if it satisfies the following condition $(M)_q$:

$(M)_q$: Regard $\mathbb{A}^q = (\mathbb{A}^1)^q$ as an open subset of the product $(\mathbb{P}^1)^q$ in the natural way. Let $\overline{V}$ be the closure of V in the scheme $X \times (\mathbb{P}^1)^q$ and let $\overline{V}^N \to \overline{V}$ be the normalization morphism. Then, we have the following inequality of Cartier divisors on $\overline{V}^N$:

$$D \times (\mathbb{P}^1)^q|_{\overline{V}^N} \leq X \times F_q|_{\overline{V}^N}.$$

Here, F_q denotes the effective Cartier divisor $\sum_{i=1}^q (\mathbb{P}^1)^{i-1} \times \{\infty\} \times (\mathbb{P}^1)^{q-i}$ on $(\mathbb{P}^1)^q$, and the symbol $|_{\overline{V}^N}$ means the restriction of Cartier divisors to the normal scheme $\overline{V}^N$.

We can check that if an irreducible closed subset $V \subset \mathscr{X}^\circ \times \mathbb{A}^q$ satisfies the modulus condition $(M)_q$, then any irreducible component of the pullback of V along any codimension 1 face $\mathscr{X}^\circ \times \mathbb{A}^{q-1} \hookrightarrow \mathscr{X}^\circ \times \mathbb{A}^q$ satisfies the modulus condition $(M)_{q-1}$. Thus, the differential maps $z^r(\mathscr{X}^\circ,q) \to z^r(\mathscr{X}^\circ,q-1)$ of the cycle complex of the scheme $\mathscr{X}^\circ$ induce differential maps $z^r(\mathscr{X},q) \to z^r(\mathscr{X},q-1)$ on the subgroups. Thus, we obtain a subcomplex $z^r(\mathscr{X},*)$ of $z^r(\mathscr{X}^\circ,*)$.

Definition 1.2. For any effective modulus pair $\mathscr{X} = (X,D)$ and for any non-negative integers r,q, we define the q-th **higher Chow group with modulus** of $\mathscr{X}$ of codimension r by

$$\mathrm{CH}^r(\mathscr{X},q) := H_q(z^r(\mathscr{X},*)).$$

The higher Chow group with modulus is a common generalization of the following classical objects:

(i) Bloch's higher Chow group: $\mathrm{CH}^r((X,\emptyset),q) \cong \mathrm{CH}^r(X,q)$,

(ii) additive higher Chow group: $\mathrm{CH}^r((X\times\mathbb{A}^1, m\cdot\{0\},q)) \cong \mathrm{TCH}^{r+1}(X,q;m),\ m\geq 1$, and

(iii) Chow group with modulus: $\mathrm{CH}^r(\mathscr{X},0) \cong \mathrm{CH}^r(X|D)$.

The identification (i) is by definition of cycle complexes with modulus. The identification (ii) is also by definition, where the right hand side is called the **additive higher Chow group** of a scheme X, which was first introduced by Bloch and H. Esnault in their paper [BE] to capture the information of "additive" objects via algebro-cycle-theoretic methods, and was generalized later by K. Rülling and by J. Park. A remarkable result by Rülling, proven in his Ph.D thesis [Ru, Theorem 4.19], is the following comparison theorem with the big de Rham-Witt group of L. Hesselhold and I. Madsen (with the assumption that the characteristic of the base field is away from 2):

$$\mathrm{TCH}^r(\mathrm{Spec}(k),r;m) \cong \mathbb{W}_m\Omega_k^{r-1}.$$

As a partial generalization, A. Krishna and Park proved in their paper [KP15, Theorem 7.12] the following functorial isomorphism:

$$\mathrm{TCH}^1(\mathrm{Spec}(R),1;m) \cong \mathbb{W}_m(R), \quad \cdots(*)$$

where R is an integral k-domain which is a UFD. Finally, the identification (iii) is a non-trivial fact proven in [BS, Theorem 3.3]. The right hand side of (iii) is called the **Chow group with modulus**, which was introduced as a (partial) generalization of the relative Picard group $\mathrm{Pic}(X,D)$. It is used in the paper [KeS] by M. Kerz and Saito, in which they construct the higher dimensional class field theory of schemes over a finite field (which can treat wild ramifications).

Recently, R. Iwasa and W. Kai announced that there exists a canonical map from the relative higher K-group to the (hypercohomology version of) higher Chow group with modulus under some relevant conditions. It is expected that the map gives a comparison isomorphism, at least after it is tensored by $\mathbb{Q}$.

2 Statements of Main Results

Motivation of research

The starting point of the author's research is the following fundamental result, which is called "$\mathbb{A}^1$-homotopy invariance," proven by Bloch in [B, Theorem 2.1]:

Theorem 2.1. *For any scheme $X\in\mathbf{Sch}_k$ and for any non-negative integers r,q, the pullback of algebraic cycles along the first projection $X\times\mathbb{A}^1\to X$ induces an isomorphism of groups*

$$\mathrm{CH}^r(X,q) \cong \mathrm{CH}^r(X\times\mathbb{A}^1,q).$$

As we have seen in the previous section, the higher Chow group is generalized to higher Chow group with modulus by Binda and Saito. However, by the isomorphism $(*)$ in the previous section, we can see that the higher Chow group with modulus does not satisfy $\mathbb{A}^1$-homotopy invariance. So, it is natural to ask the following question:

(1) How far is the higher Chow group with modulus from $\mathbb{A}^1$-homotopy invariance?

(2) Is there an appropriate invariance property for the higher Chow group with modulus which generalizes $\mathbb{A}^1$-homotopy invariance?

In the following, we state the main results of the paper.

2.1 Preliminaries

For later use, we introduce a generalized notion of modulus pairs:

Definition 2.2. We call a pair $\mathscr{X} = (X, D)$ a **modulus pair** if X is a k-scheme in $\mathbf{Sch}_k$ and if D is a Cartier divisor on X. We say that $\mathscr{X} = (X, D)$ is **effective** if and only if D is an effective Cartier divisor. For any modulus pair $\mathscr{X}$, we write $\mathscr{X}^\circ := X \setminus |D|$, where $|D|$ denotes the support of D.

Note that the effectivity of the Cartier divisors D is not used essentially in the definition of the cycle complex with modulus $z^r(\mathscr{X}, *)$. So, it is natural to expect that we can define higher Chow groups with modulus for any (not necessarily effective) modulus pair.

Proposition 2.3. *For any modulus pair $\mathscr{X}$ and for any non-negative integer r, we can define the cycle complex $z^r(\mathscr{X}, *)$ by using completely the same procedure in the case of effective modulus pairs. We define the q-th higher Chow group with modulus $\mathrm{CH}^r(\mathscr{X}, q)$ of $\mathscr{X}$ of codimension r as the q-th homology group*

$$\mathrm{CH}^r(\mathscr{X}, q) := H_q(z^r(\mathscr{X}, *)).$$

We introduce the notion of products of modulus pairs as follows:

Definition 2.4. For modulus pairs $\mathscr{X} = (X, D), \mathscr{Y} = (Y, E)$, we define the **product modulus pair** of $\mathscr{X}$ and $\mathscr{Y}$ by

$$\mathscr{X} \times \mathscr{Y} := (X \times Y, D \times Y + X \times E).$$

We can check the following fact:

Proposition 2.5. *For any modulus pairs $\mathscr{X}, \mathscr{Y}$, the product $\mathscr{X} \times \mathscr{Y}$ is a modulus pair. Moreover, we have*

$$(\mathscr{X} \times \mathscr{Y})^\circ = \mathscr{X}^\circ \times \mathscr{Y}^\circ.$$

2.2 $\mathbb{A}^1$-homotopy invariance in positive characteristics

In the following, we give an answer to the question (1) over any field of **positive** characteristic. Throughout this subsection, we assume that the characteristic of the base field k is positive, and denote it by p. As an evidence of the fact that the higher Chow group with modulus is not $\mathbb{A}^1$-homotopy invariant, we have recalled the comparison isomorphism with the big de Rham-Witt group $\mathbb{W}_m\Omega^r$, which is a typical example of non-homotopy invariant object. Here, note that this object is annihilated by p^d for sufficiently large d. By this observation, it is possible to expect that the non-homotopy invariant part of the higher Chow group with modulus is annihilated by a sufficiently large power of the characteristic p. Now, we have the first main result:

Theorem 2.6. *For any modulus pair $\mathscr{X} = (X, D)$ over k and for any non-negative integers r, q, the pullback of algebraic cycles along the first projection $\mathscr{X}^\circ \times \mathbb{A}^1 \to \mathscr{X}^\circ$ induces an isomorphism of abelian groups*

$$\mathrm{CH}^r(\mathscr{X}, q) \otimes_{\mathbb{Z}} \mathbb{Z}[1/p] \cong \mathrm{CH}^r(\mathscr{X} \times \mathbb{A}^1, q) \otimes_{\mathbb{Z}} \mathbb{Z}[1/p].$$

Here, $\mathbb{A}^1$ on the right hand side of the isomorphism is naturally identified with the modulus pair $(\mathbb{A}^1, \emptyset)$, where $\emptyset$ is the trivial Cartier divisor on $\mathbb{A}^1$. In other words, $\mathscr{X} \times \mathbb{A}^1 = (X \times \mathbb{A}^1, D \times \mathbb{A}^1)$.

This theorem states that the higher Chow group with modulus is $\mathbb{A}^1$-homotopy invariant if we invert the characteristic of the base field, as expected. Then, it is natural to ask the following question:

- Is $\mathrm{CH}^r(\mathscr{X}, q) \otimes_{\mathbb{Z}} \mathbb{Z}[1/p]$ isomorphic to Bloch's higher Chow group of the scheme $\mathscr{X}^\circ = X \setminus |D|$?

In fact, the answer to the question is NO, even if the Cartier divisor D is effective. So, it is an important task to provide a precise description of $\mathrm{CH}^r(\mathscr{X}, q) \otimes_{\mathbb{Z}} \mathbb{Z}[1/p]$. For this, we introduce the following variant of the higher Chow group with modulus:

Definition 2.7. For any modulus pair $\mathscr{X} = (X, D)$ and for any non-negative integers r, q, we define the subgroup $z^r(\mathscr{X}, q)'$ of $z^r(\mathscr{X}^\circ, q)$ as the one generated by irreducible closed subsets V of $\mathscr{X}^\circ \times \mathbb{A}^q$ which satisfy the following **naïve modulus condition** $(M)'_q$:

$(M)'_q$ Let $\overline{V}$ be the closure of V in $X \times \mathbb{A}^q$ and let $\overline{V}^N \to \overline{V}$ be the normalization morphism. Then, we have the following inequality of Cartier divisors on the normal scheme $\overline{V}^N$:

$$D \times \mathbb{A}^q|_{\overline{V}^N} \leq 0.$$

Then, the differential maps of the cycle complex $z^r(\mathscr{X}^\circ, *)$ induce maps $z^r(\mathscr{X}, q)' \to z^r(\mathscr{X}, q-1)'$. Thus, we obtain a subcomplex $z^r(\mathscr{X}, *)'$ of $z^r(\mathscr{X}^\circ, *)$. We define the q-th **naïve higher Chow group with modulus** of $\mathscr{X}$ of codimension r by

$$\mathrm{CH}^r(\mathscr{X}, q)' := H_q(z^r(\mathscr{X}, *)').$$

Now, we are ready to give a description of $\mathrm{CH}^r(\mathscr{X}, q) \otimes_{\mathbb{Z}} \mathbb{Z}[1/p]$:

Theorem 2.8. *For any modulus pair $\mathscr{X}$ over k and for any non-negative integers r, q, the identity map on $\mathscr{X}^\circ$ induces an isomorphism of abelian groups*

$$\mathrm{CH}^r(\mathscr{X}, q) \otimes_{\mathbb{Z}} \mathbb{Z}[1/p] \cong \mathrm{CH}^r(\mathscr{X}, q)' \otimes_{\mathbb{Z}} \mathbb{Z}[1/p].$$

If we consider effective modulus pairs, we have the following corollary, which states that the higher Chow group with modulus with p inverted is independent of the multiplicity of the effective Cartier divisors:

Corollary 2.9. *For any effective modulus pairs $\mathscr{X} = (X, D), \mathscr{X}' = (X, D')$ with $|D| = |D'|$ and for any non-negative integers r, q, the identity map on $\mathscr{X}^\circ$ induces an isomorphism of abelian groups*

$$\mathrm{CH}^r(\mathscr{X}, q) \otimes_{\mathbb{Z}} \mathbb{Z}[1/p] \cong \mathrm{CH}^r(\mathscr{X}', q) \otimes_{\mathbb{Z}} \mathbb{Z}[1/p].$$

Proof. This follows immediately from Theorem 2.8, noting that the naïve modulus condition $(M)'_q$ is equivalent to the condition that the closure $\overline{V}$ does not intersect with the support of D if D is effective. □

In the case that $r = \dim X$, $q = 0$ and X is a proper normal integral scheme over k, Corollary 2.9 is already known in [BCKS, Corollary 2.22 (2)].

We give a further comment on Theorem 2.6, 2.8, Corollary 2.9: in general, for any $\mathbb{Z}$-module A, we can define the "higher Chow group with modulus with coefficients in A." All the results in this subsection can be generalized for higher Chow groups with modulus with coefficients in any $\mathbb{Z}[1/p]$-module A. If we take $A = \mathbb{Z}[1/p]$, then we recover the results in this subsection.

2.3 General invariance property: the cube invariance

In the following, we consider an arbitrary base field k, which is not necessarily of positive characteristic. We provide an answer to the question (2). In other words, we formulate an invariance property of the higher Chow groups with modulus and prove it. For this, we consider the following non-effective modulus pair:

$$\overline{\square}^{(-1)} := (\mathbb{P}^1, -\{\infty\}).$$

We call it the **minus-cube**. Note that, for any modulus pair $\mathscr{X} = (X, D)$, we have

$$\mathscr{X} \times \overline{\square}^{(-1)} = (X \times \mathbb{P}^1, D \times \mathbb{P}^1 - X \times \{\infty\}), \quad (\mathscr{X} \times \overline{\square}^{(-1)})^\circ = \mathscr{X}^\circ \times \mathbb{A}^1.$$

Now, we have the following result:

Theorem 2.10. *For any modulus pair $\mathscr{X} = (X, D)$ over k and for any non-negative integers r, q, the pullback of algebraic cycles along the first projection $\mathscr{X}^\circ \times \mathbb{A}^1 \to \mathscr{X}^\circ$ induces an isomorphism of abelian groups:*

$$\mathrm{CH}^r(\mathscr{X}, q) \cong \mathrm{CH}^r(\mathscr{X} \times \overline{\square}^{(-1)}, q).$$

If the Cartier divisor D is trivial, then this isomorphism coincides with the $\mathbb{A}^1$-homotopy invariance of Bloch's higher Chow groups.

3 Sketch of Proofs

In this section, we sketch the proofs of Theorem 2.6, 2.8 and 2.10. First, we reduce 2.6 to another invariance property Theorem 3.1 below, which is similar to 2.10. Next, we explain the strategy of proving 2.10 and 3.1. Finally, we explain that 2.6 implies 2.8.

3.1 Reducing Theorem 2.6 to another invariance property

We reduce Theorem 2.6 to the following theorem:

Theorem 3.1. *Let k be a field of positive characteristic p. Then, for any modulus pair $\mathscr{X} = (X, D)$ over k, for any non-negative integers r, q and for any positive integer n, the pullback of algebraic cycles along the first projection $\mathscr{X}^\circ \times \mathbb{A}^1 \to \mathscr{X}^\circ$ induces an isomorphism of abelian groups:*

$$\mathrm{CH}^r(\mathscr{X}, q) \otimes_{\mathbb{Z}} \mathbb{Z}[1/p] \cong \mathrm{CH}^r(\mathscr{X} \times \overline{\square}^{(-n)}, q) \otimes_{\mathbb{Z}} \mathbb{Z}[1/p],$$

where $\overline{\square}^{(-n)} := (\mathbb{P}^1, -n \cdot \{\infty\})$.

Proposition 3.2. *Theorem 3.1 implies Theorem 2.6.*

This proposition is an immediate consequence of the following lemma:

Lemma 3.3. *For any modulus pair $\mathscr{X}$ over k and for any non-negative integers r, q, the natural inclusions of complexes $z^r(\mathscr{X} \times \overline{\square}^{(-n)}, *) \hookrightarrow z^r(\mathscr{X} \times \mathbb{A}^1, *)$ induce an isomorphism of abelian groups:*

$$\varinjlim_{n \geq 1} \mathrm{CH}^r(\mathscr{X} \times \overline{\square}^{(-n)}, q) \cong \mathrm{CH}^r(\mathscr{X} \times \mathbb{A}^1, q).$$

Proof. To prove the lemma, it suffices to prove the isomorphism

$$\bigcup_{n \geq 1} z^r(\mathscr{X} \times \overline{\square}^{(-n)}, q) \cong z^r(\mathscr{X} \times \mathbb{A}^1, q).$$

Let V be an irreducible closed subset of $\mathscr{X}^\circ \times \mathbb{A}^1 \times \mathbb{A}^q$. Let $\overline{V}$ be the closure of V in $X \times \mathbb{A}^1 \times (\mathbb{P}^1)^q$ and let $\overline{V}^N \to \overline{V}$ be the normalization morphism. Assume that we have the following inequality of Cartier divisors on the normal scheme $\overline{V}^N$:

$$D \times \mathbb{A}^1 \times (\mathbb{P}^1)^q|_{\overline{V}^N} \leq X \times \mathbb{A}^1 \times F_q|_{\overline{V}^N}.$$

Now, let $\overline{\overline{V}}$ be the closure of V in the larger scheme $X \times \mathbb{P}^1 \times (\mathbb{P}^1)^q$, and let $\overline{\overline{V}}^N \to \overline{\overline{V}}$ be the normalization morphism. Then, there exists a positive integer n_0 such that for any $n \geq n_0$, we have the following inequality of Cartier divisors on the normal scheme $\overline{\overline{V}}^N$:

$$D \times \mathbb{P}^1 \times (\mathbb{P}^1)^q|_{\overline{\overline{V}}^N} \leq X \times \mathbb{P}^1 \times F_q|_{\overline{\overline{V}}^N} + n \cdot X \times \{\infty\} \times (\mathbb{P}^1)^q|_{\overline{\overline{V}}^N},$$

or equivalently,

$$D \times \mathbb{P}^1 \times (\mathbb{P}^1)^q|_{\overline{\overline{V}}^N} - n \cdot X \times \{\infty\} \times (\mathbb{P}^1)^q|_{\overline{\overline{V}}^N} \leq X \times \mathbb{P}^1 \times F_q|_{\overline{\overline{V}}^N}.$$

Noting that $\mathscr{X} \times \overline{\square}^{(-n)} = (X \times \mathbb{P}^1, D \times \mathbb{P}^1 - n \cdot X \times \{\infty\})$, this inequality concludes the proof. $\square$

3.2 Sketch of Proofs of Theorem 2.10 and 3.1

Befor starting the proof, we introduce the following definition:

Definition 3.4. Let $\mathscr{X} = (X, D)$ be a modulus pair over a field k, and let w be a finite set of closed subsets of the scheme $\mathscr{X}^\circ$. For any non-negative integers r, q, we define an abelian subgroup $z^r_w(\mathscr{X}, q)$ of $z^r(\mathscr{X}, q)$ as the one generated by irreducible closed subsets V of $\mathscr{X}^\circ \times \mathbb{A}^q$ which belong $z^r(\mathscr{X}, q)$ and satisfy the following additional condition:

- For any $W \in w$ and for any face F of $\mathbb{A}^q$, the subset V intersects $W \times F$ properly in $\mathscr{X}^\circ \times \mathbb{A}^q$.

Then, we can check that the differentials of the complex $z^r(\mathscr{X}, *)$ restrict to maps $z^r_w(\mathscr{X}, q) \to z^r_w(\mathscr{X}, q-1)$. Thus, we obtain a subcomplex $z^r_w(\mathscr{X}, *)$ of $z^r(\mathscr{X}, *)$.

First, we prove Theorem 2.10. Let k be a field of any characteristic, and let $\mathscr{X} = (X, D)$ be a modulus pair over k. By imitating the proof of $\mathbb{A}^1$-homotopy invariance of Bloch's higher Chow group as in [B, Theorem 2.1], we are reduced to prove the following two lemmas:

Lemma 3.5. *Set $w := \{\, \mathscr{X}^\circ \times \{0,1\}\, \}$. Then, for each $\epsilon \in \{0,1\}$, the pullback of algebraic cycles along the inclusion morphism $\mathscr{X}^\circ \cong \mathscr{X}^\circ \times \{\epsilon\} \hookrightarrow \mathscr{X}^\circ \times \mathbb{A}^1$ induces a map of complexes*

$$i^*_\epsilon : z^r_w(\mathscr{X} \times \overline{\square}^{(-1)}, *) \to z^r(\mathscr{X}, *),$$

and the maps i^*_0, i^*_1 are homotopic.

Proof. The proof of this lemma is easy and formal, noting the fact that there is a canonical identification

$$\underline{z}^r_w(\mathscr{X} \times \overline{\square}^{(-1)}, q) = \underline{z}^r(\mathscr{X}, q+1)$$

by definition for any non-negative integer q, which implies that for any $V \in z^r_w(\mathscr{X} \times \overline{\square}^{(-1)}, q)$, the cycle V itself gives the homotopy between $i^*_0 V$ and $i^*_1 V$. This finishes the proof. $\square$

Lemma 3.6. *Set $w := \{\, \mathscr{X}^\circ \times \{0,1\}\, \}$. Then, the natural inclusion map of complexes*

$$\iota : z^r_w(\mathscr{X} \times \overline{\square}^{(-1)}, *) \hookrightarrow z^r(\mathscr{X} \times \overline{\square}^{(-1)}, *)$$

is a quasi-isomorphism.

Sketch of Proof. The proof of this lemma is technical and complicated, so we give only a sketch. Fix a non-negative integer q, and take any irreducible closed subset V of $\mathscr{X}^\circ \times \mathbb{A}^1 \times \mathbb{A}^q$ which belongs to $z^r(\mathscr{X} \times \overline{\square}^{(-1)}, q)$. We would like to find a homotopy cycle $\mathcal{H}$ on $\mathscr{X}^\circ \times \mathbb{A}^1 \times \mathbb{A}^q \times \mathbb{A}^1$ which belongs to $z^r(\mathscr{X} \times \overline{\square}^{(-1)}, q+1)$, where the last affine line $\mathbb{A}^1$ is the "time axis of moving" whose parameter is denoted by t, such that the fiber $\mathcal{H}_0$ over $t = 0$ is equal to V and the fiber $\mathcal{H}_\tau$ over $t = \tau \neq 0$ belongs to the subgroup $z^r_w(\mathscr{X} \times \overline{\square}^{(-1)}, q)$. Bloch's proof of $\mathbb{A}^1$-homotopy invariance allows us to find a cycle $\mathcal{H}$ which satisfies all the conditions as above except that $\mathcal{H}$ belongs to $z^r(\mathscr{X} \times \overline{\square}^{(-1)}, q+1)$. Precisely speaking, the homotopy cycle $\mathcal{H}$ is constructed over a purely transcendental field extension K/k, but we do not care about the point because it is not essential in this sketch. The problem is that we do not know whether the closure of $\mathcal{H}$ in $X \times \mathbb{P}^1 \times (\mathbb{P}^1)^q \times \mathbb{P}^1$ satisfies the modulus condition around the infinity point of the last projective line. The key idea to overcome this problem is to pushforward the cycle $\mathcal{H}$ by the finite endomorphism

$$\mathrm{id}_{\mathscr{X}^\circ \times \mathbb{A}^1 \times \mathbb{A}^q} \times \rho^d : \mathscr{X}^\circ \times \mathbb{A}^1 \times \mathbb{A}^q \times \mathbb{A}^1 \to \mathscr{X}^\circ \times \mathbb{A}^1 \times \mathbb{A}^q \times \mathbb{A}^1,$$

where d is a positive integer and $\rho^d : \mathbb{A}^1 \to \mathbb{A}^1$ is the finite morphism which sends the parameter t to t^d. Note that ρ^d is naturally extended to a finite endomorphism $\overline{\rho}^d : \mathbb{P}^1 \to \mathbb{P}^1$, which satisfies that $(\overline{\rho}^d)^*\{\infty\} = d \cdot \{\infty\}$. Hence, using [KP, Lemma 2.2], we can prove that if d is sufficiently large, then the cycle $\mathcal{H}^{(d)}$ obtained as the pushforward of $\mathcal{H}$ by the morphism $\mathrm{id}_{\mathscr{X}^\circ\times\mathbb{A}^1\times\mathbb{A}^q}\times\rho^d$ satisfies the modulus condition and hence belongs to $z^r(\mathscr{X}\times\overline{\square}^{(-1)},q+1)$. Noting the map $\mathrm{id}_{\mathscr{X}^\circ\times\mathbb{A}^1\times\mathbb{A}^q}\times\rho^d$ is a finite map, we can easily check that $\mathcal{H}^{(d)}$ satisfies the condition that the fiber $\mathcal{H}_1^{(d)}$ over $t=1$ belongs to $z^r_w(\mathscr{X}\times\overline{\square}^{(-1)},q)$. However, since $(\rho^d)^*(\{0\})=d\cdot\{0\}$, the fiber $\mathcal{H}_0^{(d)}$ is equal to $d\cdot V$. So, to obtain a right homotopy cycle, we should take the difference $\mathcal{H}^{(d+1)}-\mathcal{H}^{(d)}$. This finishes the proof. $\square$

Thus, we finished the sketch of the proof of Theorem 2.10.

Next, we sketch the proof of Theorem 3.1. Let k be a field of **positive** characteristic p. Similarly to the case of Theorem 2.10, we are reduced to prove the following two lemmas:

Lemma 3.7. *Set $w:=\{\,\mathscr{X}^\circ\times\{0,1\}\,\}$. Then, for each $\epsilon\in\{0,1\}$ and for any positive integer n, the pullback of algebraic cycles along the inclusion morphism $\mathscr{X}^\circ\cong\mathscr{X}^\circ\times\{\epsilon\}\hookrightarrow\mathscr{X}^\circ\times\mathbb{A}^1$ induces a map of complexes*

$$i_\epsilon^*: z^r_w(\mathscr{X}\times\overline{\square}^{(-n)},*)\otimes_{\mathbb{Z}}\mathbb{Z}[1/p]\to z^r(\mathscr{X},*)\otimes_{\mathbb{Z}}\mathbb{Z}[1/p],$$

and the maps i_0^*, i_1^* are homotopic.

Lemma 3.8. *Set $w:=\{\,\mathscr{X}^\circ\times\{0,1\}\,\}$. Then, for any positive integer n, the natural inclusion map of complexes*

$$\iota: z^r_w(\mathscr{X}\times\overline{\square}^{(-n)},*)\hookrightarrow z^r(\mathscr{X}\times\overline{\square}^{(-n)},*)$$

is a quasi-isomorphism.

The proof of Lemma 3.8 is completely the same as that of Lemma 3.6. So, it suffices to prove Lemma 3.7.

Proof of Lemma 3.7. Fix any positive integer n. First, noting that the multiplication by p induces an automorphism on the chain complex $z^r(\mathscr{X},*)\otimes_{\mathbb{Z}}\mathbb{Z}[1/p]$, we are reduced to prove that there exists a non-negative integer m such that the morphisms of complexes

$$p^m i_\epsilon^*: z^r(\mathscr{X}\times\overline{\square}^{(-n)},*)\otimes_{\mathbb{Z}}\mathbb{Z}[1/p]\to z^r(\mathscr{X},*)\otimes_{\mathbb{Z}}\mathbb{Z}[1/p],\quad \epsilon\in\{0,1\}$$

are homotopic. Now, take a non-negative integer m such that $n\le p^m$. Set $d:=p^m$ and consider the finite endomorphism $\rho^d:\mathbb{A}^1\to\mathbb{A}^1$, which we introduced in the sketch of the proof of Lemma 3.6. By exactly the same reason as *loc. cit.*, we can check that the pushforward of algebraic cycles along the finite endomorphism $\mathrm{id}_{\mathscr{X}^\circ}\times\rho^d$ induces a well-defined morphism of chain complexes

$$(\mathrm{id}_{\mathscr{X}^\circ}\times\rho^d)_*: z^r(\mathscr{X}\times\overline{\square}^{(-n)},*)\to z^r(\mathscr{X}\times\overline{\square}^{(-1)},*).$$

By Lemma 3.5, we see that the composite maps

$$i_\epsilon^*\circ(\mathrm{id}_{\mathscr{X}^\circ}\times\rho^d)_*: z^r(\mathscr{X}\times\overline{\square}^{(-n)},*)\to z^r(\mathscr{X}\times\overline{\square}^{(-1)},*)\to z^r(\mathscr{X},*),\quad \epsilon\in\{0,1\}$$

are homotopic. So, it suffices to prove for each $\epsilon\in\{0,1\}$ the following equations:

$$i_\epsilon^*\circ(\mathrm{id}_{\mathscr{X}^\circ}\times\rho^d)_*=d\cdot i_\epsilon^*=p^m\cdot i_\epsilon^*.$$

These equations follow easily from the facts that $(\rho^d)^*\{0\}=p^m\cdot\{0\}$ and that $(\rho^d)^*\{1\}=p^m\cdot\{1\}$, where the latter one is a consequence of the following formula which holds only in characteristic p:

$$t^{p^m}-1=(t-1)^{p^m}.$$

This finishes the proof. $\square$

3.3 Sketch of Proof of Theorem 2.8

Let k be a field of positive characteristic p. For any modulus pair $\mathscr{X} = (X, D)$ over k, and for any non-negative integers a, b, define an abelian group $\mathcal{Z}(a, b)$ by

$$\mathcal{Z}(a,b) := \mathbb{Z}\left\{ \begin{array}{l} \text{irreducible closed subsets of } \mathscr{X}^\circ \times \mathbb{A}^a \times \mathbb{A}^b \text{ of codimension } r \text{ which} \\ \text{satisfy the face condition and the modulus condition } (M)_{a,b} \end{array} \right\} \otimes_{\mathbb{Z}} \mathbb{Z}[1/p].$$

Here, we say that an irreducible closed subset V of $\mathscr{X}^\circ \times \mathbb{A}^a \times \mathbb{A}^b$ satisfies the **face condition** if for any face F of $\mathbb{A}^a$ and for any face F' of $\mathbb{A}^b$, the closed subset V intersects $\mathscr{X}^\circ \times F \times F'$ properly in $\mathscr{X}^\circ \times \mathbb{A}^a \times \mathbb{A}^b$. We say that an irreducible closed subset V of $\mathscr{X}^\circ \times \mathbb{A}^a \times \mathbb{A}^b$ satisfies the **modulus condition** $(M)_{a,b}$ if the following condition holds:

$(M)_{a,b}$: Let $\overline{V}$ be the closure of V in the scheme $\mathscr{X}^\circ \times (\mathbb{P}^1)^a \times \mathbb{A}^b$, and let $\overline{V}^N \to \overline{V}$ be the normalization morphism. Then, we have the following inequality of Cartier divisors on the normal scheme $\overline{V}^N$:

$$D \times (\mathbb{P}^1)^a \times \mathbb{A}^b|_{\overline{V}^N} \leq X \times F_a \times \mathbb{A}^b|_{\overline{V}^N}.$$

Here, we regard $\mathbb{A}^a = (\mathbb{A}^1)^a$ as an open subset of $(\mathbb{P}^1)^a$. Recall that $F_a = \sum_{i=1}^a (\mathbb{P}^1)^{i-1} \times \{\infty\} \times (\mathbb{P}^1)^{a-i}$.

Then, we can define a double cubical abelian group $\mathcal{Z}(*, *)$, where each of two cubical structures is defined in the same way as in the case of the cycle complex with modulus and the naïve cycle complex with modulus. After "cutting out" the degenerate part of $\mathcal{Z}(*, *)$, we obtain a double chain complex $\mathcal{Z}(*, *)_0$. Using Theorem 2.6, we can prove that the two types of spectral sequences (converging to the homology groups of the total complex $\mathrm{Tot}\mathcal{Z}(*, *)_0$ of $\mathcal{Z}(*, *)_0$) degenerate on the first pages. By using this fact, we can prove that both $\mathrm{CH}^r(\mathscr{X}, q) \otimes_{\mathbb{Z}} \mathbb{Z}[1/p]$ and $\mathrm{CH}^r(\mathscr{X}, q)' \otimes_{\mathbb{Z}} \mathbb{Z}[1/p]$ are canonically isomorphic to the q-th homology group of the total complex $\mathrm{Tot}\mathcal{Z}(*, *)_0$. This finishes the proof.

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