Studies on Discrete and q-Discrete Nonlinear Integrable Systems

離散型およびq−離散型非線形可積分系の研究

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THESIS

博士論文

Studies on Discrete and q-Discrete Nonlinear Integrable Systems

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CONTENTS

Chapter1. Introduction

1.1. Introduction

Development of computers enables us to solve numerically even complicated nonlinear differential equations which do not possess analytical solutions. It should be noted, however, that simple discretization of given continuous equation does not always preserve the original properties, or sometimes it causes even numerically induced chaos. A simple but famous example is the discretization of the logistic equation[1],

$$\frac{dy}{dx} = y(1-y), \tag{1.1.1}$$

whose general solution is given by

$$y = \frac{Ce^x}{1+Ce^x}. \tag{1.1.2}$$

One possible discretization of eq.(1.1.1) is given by

$$\frac{y_{n+1} - y_{n-1}}{2\epsilon} = y_n(1-y_n), \tag{1.1.3}$$

where ϵ is the lattice parameter. It is known that numerical solution of eq.(1.1.3) is chaotic, even though that of original differential equation is not. Another discretization of eq.(1.1.1) is given by

$$\frac{y_{n+1} - y_n}{\epsilon} = y_n(1-y_{n+1}) \tag{1.1.4}$$

whose solution does not show chaotic behavior.

More practical example is the case of the nonlinear Shrödinger equation[2],

$$i\phi_t + \phi_{xx} + 2|\phi|^2\phi = 0 \ , \tag{1.1.5}$$

which is one of the typical nonlinear integrable systems. One simple discretization is given by

$$i\phi_{n,t} + \frac{\phi_{n+1} - 2\phi_n + \phi_{n-1}}{\epsilon^2} + 2|\phi_n|^2\phi_n = 0, \tag{1.1.6}$$

whose numerical solutions behave chaotically. However, if we take another discretization ,

$$i\phi_{n,t} + \frac{\phi_{n+1} - 2\phi_n + \phi_{n-1}}{\epsilon^2} + |\phi_n|^2\,(\phi_{n+1} + \phi_{n-1}) = 0, \tag{1.1.6}$$

then its numerical solution seems to have good property. Actually, it is known that eq.(1.1.6) is "integrable" as a differential-difference equation.

These examples shows the importance of discretization. In the above cases, we know the properties of the solutions of given continuous equation, and we could check whether the numerical solution are true to original continuous one or not. In most cases, however, we do not have the information of the solutions of the original equation. If the numerical solutions behave chaotically, we cannot make out that this phenomena is essential to the original equation or it is induced by bad discretization at a glance. Hence, the studies of discrete system itself should be done more systematically. In general, however, it is quite difficult to seek for "good discretization", and we have to consider it case by case. It is then reasonable to study at first the case of nonlinear integrable systems, which may be considered to be the easiest case.

Besides the standard discretization, there exist a special kind of discretization. It is called "q-discrete" and has been studied from quite different point of view, namely, q-analogue of special functions[3]. We consider the Bessel function,

$$J_\nu(x) = \sum_{k=0}^{\infty} \frac{x^{\nu+2k}}{(\nu+k)!k!2^{\nu+2k}}\,, \tag{1.1.7}$$

as an example. Here we focus on the case of ν being an integer. Even if we replace an integer n by so-called q-integer,

$$[n] = \frac{1-q^n}{1-q}\,, \tag{1.1.8}$$

still many of the properties of the Bessel function are preserved. Note that $[n]$ tends to n in the limit $q \to 1$.

The q-Bessel function is given by

$$J_{q,\nu}(x) = \sum_{k=0}^{\infty} \frac{x^{\nu+2k}}{\{2\nu+2k\}!\{2k\}!}, \tag{1.1.9}$$

where

$$\{2n\} = [2n][2n-2]\cdots[2]\ . \tag{1.1.10}$$

We have the contiguity relations for the Bessel function,

$$\left(\frac{d}{dx} + \frac{n}{x}\right) J_\nu(x) = J_{\nu-1}(x)\ , \tag{1.1.11a}$$

$$\left(\frac{d}{dx} - \frac{n}{x}\right) J_\nu(x) = -J_{\nu+1}(x)\ . \tag{1.1.11b}$$

Those for the q-Bessel function are given by

$$\left(q^n \delta_{q,x} + \frac{[n]}{x}\right) J_{q,\nu} = J_{q,\nu-1}(x), \tag{1.1.12a}$$

$$\left(q^{-n} \delta_{q,x} + \frac{[-n]}{x}\right) J_{q,\nu} = -J_{q,\nu+1}(x), \tag{1.1.12b}$$

where $\delta_{q,x}$ is the q-difference operator defined by

$$\delta_{q,x} f(x) = \frac{f(x) - f(qx)}{(1-q)x}. \tag{1.1.13}$$

We also note that $\delta_{q,x}$ reduces to $\frac{d}{dx}$ in the limit $q \to 1$. We see that the forms of the contiguity relations are preserved by the replacement $n \to [n]$. Besides the Bessel function, it is known that most of special functions have their q-analogue, which have the similar properties to the original ones. The reason has been left unknown until quite recently. After the discovery of quantum groups[4], it has been revealed that q-special functions appear in the representation theory of quantum groups, while ordinary special functions does in that of classical groups[5]. Moreover, q-difference equations appear in the theory of solvable lattice models in statistical mechanics, which has a close relationship with quantum groups. Hence, studies of nonlinear q-difference equations arises as an interesting and important problem.

So far we have used the word "integrability" without definition. Let us now define the notion of integrability. We start from the case of the Hamilton systems of N degree of freedom. Suppose we have the following Hamiltonian,

$$H(q,p,t) = H(q_1,\cdots,q_N,p_1,\cdots,p_N,t). \tag{1.1.13}$$

If there exist N independent integrals $I_1, \cdots, I_N$ such that they are involutive,

$$\{I_i, I_j\} = 0 \quad \text{for } i \neq j \ , \tag{1.1.14}$$

then this system is completely integrable, namely, we can solve the initial value problem in principle (Liouville-Arnold). Let us next proceed to the case of dynamical system of infinite degrees of freedom. Typical example is given by the Korteweg-de Vries (KdV) equation,

$$u_t + 6uu_x + u_{xxx} = 0, \tag{1.1.15}$$

which was the first equation recognized as the nonlinear integrable system of infinite degrees of freedom. It has been shown that this equation can be solved exactly by means of the so-called inverse scattering method[6,7]. The finding was a great breakthrough for mathematical sciences, since no such result has been known for nonlinear partial differential equations before this discovery. In this method, the Lax formalism[8] plays a crucial role. First we consider the following linear system;

$$L\psi = \lambda\psi, \quad \psi_t = B\psi, \tag{1.1.16a}$$

$$L = \partial_x^2 - u, \qquad B = 4\partial_x^3 - 6u\partial_x - 3u_x \ . \tag{1.1.16b}$$

Compatibility condition to eq.(1.1.16) with $\lambda_t = 0$ gives the Lax equation,

$$L_t = [B, L], \tag{1.1.17}$$

which recovers the KdV equation (1.1.15). Conversely, it is proved that if L and B satisfy the Lax equation, then λ does not depend on t. Operators L and B are called the Lax pair. Once given equation is rewritten in this form, we can reduce the initial value problem to solving some linear integral equation. In this sense, the KdV equation (1.1.15) is integrable. Moreover, it has also been shown that the KdV equation admits quite wide class of exact solutions called soliton solutions, which describe the interaction of particle-like solitary waves.

After the discovery of the inverse scattering method, it has been revealed that many equations can be solved by using this method, and extensive studies have been done from various points of view. Among them, there is a way to obtain N-soliton solution systematically by elementary calculations, which is called Hirota's direct method[7,9]. Let us take the KdV eq. (1.1.15) again as an example. We first apply the dependent variable transformation,

$$u = 2\frac{d^2}{dx^2}\log \tau, \tag{1.1.18}$$

on eq.(1.1.15). Then it yields the quadratic equation,

$$(D_x^4 + D_x D_t)\tau \cdot \tau = 0 \ , \tag{1.2.19}$$

where D_x etc. are Hirota's bilinear operators defined by

$$D_x^n \tau \cdot \tau = (\partial_x - \partial_y)^n \tau(x)\tau(y)|_{x=y}. \tag{1.1.20}$$

We call eq.(1.1.19) the bilinear form, and the dependent variable the τ function. To obtain N-soliton solution, we adapt a perturbational technique. Namely, we put

$$\tau = 1 + \epsilon f_1 + \epsilon^2 f_2 + \cdots, \tag{1.1.21}$$

and take f_1 as the sum of exponential functions,

$$f_1 = \sum_{k=1}^{N} e^{p_k x + q_k t}. \tag{1.1.22}$$

Remarkably, it is shown by simple calculations that we can truncate the perturbation by N terms, which yields N-soliton solution. Moreover, it is shown that τ function is expressed in terms of determinant. This fact is a key to reveal the algebraic structures behind the nonlinear integrable systems. We will discuss this point more in the next section. Here we only remark that this method plays an essential role throughout this thesis.

Several essential properties are now extracted from "integrable equations" through such studies, and the followings are accepted as the definitions of integrability for continuous systems;

(1) Existence of the Lax pair,

(2) Existence of an infinite number of conserved quantities or symmetries,

(3) Existence of N-soliton solution,

(4) Existence of bi-Hamilton structure,

(5) Satisfaction of Painlevé property.

It is believed (not proved in rigorous sense) that the properties listed above are all equivalent.

Here we give a brief comment on the Painlevé property, which is an important notion in the chapter 3. This property can be stated as "nonexistence of movable branch point ". For the ordinary differential equations of second order, it is known that there are essentially six nonlinear equations which possess this property[10]. These equations are called the Painlevé equations. Curiously, the Painlevé equations often appear in the theory of integrable systems in wider sense, e.g. solvable lattice models, conformal field theory, etc. Painlevé property is involved in integrability through the conjecture proposed by Ablowitz et.al[11], "A nonlinear partial differential equation is solvable by the inverse scattering method only if every nonlinear ordinary differential equation obtained by exact reduction possess the Painlevé property."

So far, we have discussed the integrability for continuous systems. To consider (q-) discretization preserving integrability, we have to clarify the notion of integrability for (q-)discrete systems. Unfortunately, the notion is not established yet at this moment. The main reason is that successful studies of discretization have not been done so much yet, which is because of the difficulty in analysis of the discrete systems in itself, especially from the analytical point of view. This situation is symbolically expressed in ref. [12]: "Intelligent space-time discretization of integrable systems is a notoriously difficult problem...".

Hirota noticed the importance of discretization already in 70's, and developed a systematic method of discretization based on the direct method[13,14]. Several pioneer works have also been done[15,16], but the number of the successful examples have not been enough to establish "integrability". We need much more studies from various points of view in order to extract common, essential properties from "integrable equations".

Now the situation is changing. Quispel et.al have proposed a large family of "integrable mappings" (ordinary difference equations)[17]. Stimulated by this study, Grammaticos et.al have succeeded to extract common property called "singularity confinement"from them, which may be considered as the discrete analogue of the Painlevé property, Moreover, they have proposed this notion as a criterion to detect integrable discrete systems[18]. They have also presented the discrete version of the Painlevé equations by using this criterion[19]. Singularity confinement provides us with a powerful and useful method to detect "integrability", which is quite different from already known ones. Although its mathematical meaning or validity have not been established yet, it may be possible to get further understandings for discrete nonlinear integrable systems by investigating its essence or comparing the results with those obtained through other methods.

Now we are in the position to explain the theme and purpose of this thesis. The first theme is to extend the method of discretization developed by Hirota to q-discrete system, and present "integrable q-discrete systems". In particular, we take the two-dimensional Toda lattice(2DTL) equation and the two-dimensional Toda Molecule(2DTM) equations as examples, and present the q-2DTL and q-2DTM equations. The reasons why we take these equations are as follows;

(1) These equations are typical examples of integrable systems. Moreover, Toda equations seem to appear more often in other fields such as the conformal field theory or theory of solvable lattice models, than other integrable systems such as KdV equation etc.

(2) There is a close relationship between several types of Toda equations and special functions. Relation between q-discretized equations and q-special functions may be a check of validity of our method.

(3) The discrete 2DTL equation proposed by Hirota[20] has a similar form to the original 2DTL equation, while other discretized equations such as the discrete KdV equation

are not. Similar situation is expected to the q-discrete case.

The second purpose is to investigate the solution of the discrete Painlevé equations. As mentioned above, these equations have been proposed by the method based on the singularity confinement, which is quite different from other methods. Now the studies on these equations is developed actively, and several properties has been discussed. For example, Lax pairs of some of them has been obtained. However, that these Lax pairs are still formal, since almost no further result has been obtained from such Lax pairs. On the other hand, the Lax pairs are originally used to solve the initial value problem or to obtain various exact solutions. Hence, it may be a good test for the validity of singularity confinement to examine their solutions. Moreover, it is known that the Painlevé equations, except for that of the first kind, admit particular solutions expressed by special functions. It is expected that discrete Painlevé equations admit those expressed by discrete analogue of special functions.

Our plan is as follows; In the section 1.2, we give a review of the direct method which is a key throughout this thesis. In the chapter 2, we discuss the q-discretization of Toda equations. In the chapter 3, we study the solution of the second and third discrete Painlevé equations. Moreover, based on the study of solutions, we can show that the discrete Painlevé III equation can be regarded as the q-discrete system by a slight modification. We also propose a q-difference analogue of the Painlevé III equation. The chapter 4 is devoted to the concluding remarks.

1.2. Direct Method

In this section we discuss one method to obtain the solution of nonlinear integrable systems, called the direct method. This method is convenient not only for obtaining exact solutions, but also for investigation of the algebraic structure, and in particular, for the extension of the integrable systems. We demonstrate the essence of this method by taking the 2DTL and 2DTM equations as examples.

Let us first consider the 2DTL equation,

$$\frac{\partial V_n}{\partial x} = V_n(J_n - J_{n+1}) \,, \tag{1.2.1a}$$

$$\frac{\partial J_n}{\partial y} = V_{n-1} - V_n \,, \tag{1.2.1b}$$

$$n \in Z. \tag{1.2.1c}$$

The key idea of this method is to apply suitable dependent variable transformation and reduce the original equation to some quadratic equation called the bilinear form. In this case, we introduce the new dependent variable τ_n by

$$V_n = \frac{\tau_{n-1}\tau_{n+1}}{\tau_n^2} \,, \qquad J_n = \frac{\partial}{\partial x} \log \frac{\tau_{n-1}}{\tau_n}, \tag{1.2.2}$$

which yields the bilinear form,

$$\frac{\partial^2 \tau_n}{\partial x \partial y}\tau_n - \frac{\partial \tau_n}{\partial x}\frac{\partial \tau_n}{\partial y} = \tau_{n+1}\tau_{n-1} - \tau_n^2. \tag{1.2.3}$$

The essential point is that eq.(1.2.3) is nothing but some identity of determinant (Plücker relation). In fact, the solution of eq.(1.2.3) is given by

$$\tau_n = \begin{vmatrix} f_n^{(1)} & f_{n+1}^{(1)} & \cdots & f_{n+N-1}^{(1)} \\ f_n^{(2)} & f_{n+1}^{(2)} & \cdots & f_{n+N-1}^{(2)} \\ \vdots & \vdots & \cdots & \vdots \\ f_n^{(N)} & f_{n+1}^{(N)} & \cdots & f_{n+N-1}^{(N)} \end{vmatrix}, \tag{1.2.4a}$$

where $f_n^{(k)}$, $k = 1, \cdots, N$, are arbitrary functions satisfying "dispersion relation",

$$\partial_x f_n^{(k)} = -f_{n+1}^{(k)} \,, \qquad \partial_y f_n^{(k)} = f_{n-1}^{(k)} \,. \quad k = 1, \cdots, N. \tag{1.2.4b}$$

Let us prove that eq.(1.2.4) really give the solution of the bilinear form (1.2.3). For the purpose, we introduce a notation,

$$\tau_n = |0, 1, \cdots, N-2, N-1|, \tag{1.2.5}$$

where "k" is a column vector given by

$$''k'' = \left.\begin{pmatrix} f_{n+k}^{(1)} \\ f_{n+k}^{(2)} \\ \vdots \\ f_{n+k}^{(N)} \end{pmatrix}\right\} N \quad . \tag{1.2.6}$$

By using the dispersion relation (1.2.4b), we have

$$\partial_x \tau_n = -\,|0, \cdots N-2, N|, \tag{1.2.7}$$

$$\partial_y \tau_n = |-1, 1, \cdots N-2, N-1|, \tag{1.2.8}$$

$$\begin{aligned} \partial_x \partial_y \tau_n = &-|-1, 1, \cdots, N-2, N| \\ &-|0, 1, \cdots N-2, N-1|, \end{aligned} \tag{1.2.9}$$

$$\tau_{n+1} = |1, \cdots N-1, N|, \tag{1.2.10}$$

$$\tau_{n-1} = |-1, 0, \cdots N-2|. \tag{1.2.11}$$

It is important that "shifted" determinants can be expressed by the differentiations or the shifts of suffix of τ. Now we consider the following identity of determinant,

$$0 = \left|\begin{array}{cc:ccc:ccc:cc} -1 & 0 & 1 & \cdots & N-2 & & \emptyset & & N-1 & N \\ \hdashline -1 & 0 & & \emptyset & & 1 & \cdots & N-2 & N-1 & N \end{array}\right|. \tag{1.2.12}$$

Applying the Laplace expansion on the right hand side, we obtain

$$\begin{aligned} 0 = &|-1, 0, \cdots, N-2|\ |1, \cdots, N-2, N-1, N| \\ &-|-1, 1, \cdots, N-2, N-1|\ |0, \cdots, N-2, N| \\ &+|-1, 1, \cdots, N-2, N|\ |0, \cdots, N-2, N-1|\ , \end{aligned} \tag{1.2.13}$$

which can be rewritten in terms of τ by using eqs.(1.2.5) and (1.2.7)-(1.2.11) as

$$0 = \tau_{n-1}\tau_{n+1} + \partial_y \tau_n\ \partial_x \tau_n + \left(-\partial_x \partial_y \tau_n - \tau_n\right)\ \tau_n. \tag{1.2.14}$$

Equation (1.2.14) is nothing but the bilinear form of the 2DTL equation (1.2.3) itself. Thus we have proved eq.(1.2.4) gives the solution of eq.(1.2.3). Here we remark that N-soliton solution is obtained from eq.(1.2.4) by choosing $f_n^{(k)}$ as the sum of two exponential functions,

$$f_n^{(k)} = p_k^n \exp\,(p_k x + \frac{1}{p_k} y) + q_k^n \exp\,(q_k x + \frac{1}{q_k} y), \tag{1.2.15}$$

where p_k and q_k are arbitrary constants.

We next introduce the 2DTM equation,

$$\frac{\partial V_n}{\partial x} = V_n(J_n - J_{n+1})\ , \tag{1.2.16a}$$

$$\frac{\partial J_n}{\partial y} = V_{n-1} - V_n\ , \tag{1.2.16b}$$

$$V_0 = V_M = 0\ . \tag{1.2.16c}$$

Note that only the difference from the 2DTL equation is the boundary condition (1.2.16c). We apply the same dependent variable transformation as (1.2.2), but we slightly modify the decoupling to obtain the bilinear form,

$$\frac{\partial^2 \tau_n}{\partial x \partial y}\tau_n - \frac{\partial \tau_n}{\partial x}\frac{\partial \tau_n}{\partial y} = \tau_{n+1}\tau_{n-1}, \tag{1.2.17a}$$

$$\tau_{-1} = \tau_{M+1} = 0. \tag{1.2.17b}$$

The solution of eq.(1.2.17) is given by the following determinant,

$$\tau_n = \begin{vmatrix} f(x,y) & \partial_x f(x,y) & \cdots & \partial_x^{n-1} f(x,y) \\ \partial_y f(x,y) & \partial_x\partial_y f(x,y) & \cdots & \partial_x^{n-1}\partial_y f(x,y) \\ \vdots & \vdots & \ddots & \vdots \\ \partial_y^{n-1} f(x,y) & \partial_x\partial_y^{n-1} f(x,y) & \cdots & \partial_x^{n-1}\partial_y^{n-1} f(x,y) \end{vmatrix}. \tag{1.2.18}$$

The boundary condition is satisfied if we put

$$f(x,y) = \sum_{k=1}^{M} f_k(x) g_k(y), \tag{1.2.19}$$

where $f_k(x)$ and $g_k(y)$ are arbitrary functions of x and y, respectively. Note that the structures of the τ functions of the 2DTL equation and the 2DTM equations are quite different. In the former case, the lattice site n appears in the suffix of the most upleft entry of its τ function while in the latter case, it appears as the size of its τ function.

It is possible to prove that eq.(1.2.18) actually gives the solution of eq.(1.2.17) in similar manner to the case of the 2DTL equation except that we use the Jacobi identity instead of the Plücker relation. Before going to the proof, we comment on the Jacobi identity. Let D be a determinant, and $D\begin{pmatrix} i \\ j \end{pmatrix}$ is the determinant with i-th column and j-th column being neglected. Then the Jacobi identity is given by

$$D\begin{pmatrix} i \\ j \end{pmatrix} D\begin{pmatrix} k \\ l \end{pmatrix} - D\begin{pmatrix} i \\ l \end{pmatrix} D\begin{pmatrix} k \\ j \end{pmatrix} = D\, D\begin{pmatrix} i & k \\ j & l \end{pmatrix}. \tag{1.2.20}$$

Let us put $\tau_{n+1} = D$. Then we have

$$\tau_n = D\begin{pmatrix} n+1 \\ n+1 \end{pmatrix}, \tag{1.2.21}$$

$$\tau_{n-1} = D\begin{pmatrix} n & n+1 \\ n & n+1 \end{pmatrix}, \tag{1.2.22}$$

$$\partial_x \tau_n = D\begin{pmatrix} n \\ n+1 \end{pmatrix}, \tag{1.2.23}$$

$$\partial_x \tau_n = D\begin{pmatrix} n+1 \\ n \end{pmatrix}, \tag{1.2.24}$$

$$\partial_x \partial_y \tau_n = D\begin{pmatrix} n \\ n \end{pmatrix}. \tag{1.2.25}$$

Now it is easily checked that eq.(1.2.17a) is nothing but the Jacobi identity (1.2.20) of the case $i = j = n$, $k = l = n + 1$. Thus we have proved that eq.(1.2.18) really gives the solution of the bilinear form of the 2DTM equation (1.2.17). We note that the Jacobi identity can be regarded as a special case of the Plücker relation.

We have demonstrated the essence of the direct method. It is known that most of the nonlinear integrable systems have such structure that their solution is expressed by some determinant or Pfaffian[21,22]. We remark that Sato noticed this fact for the Kadomtsev-Petviashivilli equation and developed a grand theory which gives the unified view to nonlinear integrable systems and mathematical meaning to the direct method[23,24]. Moreover, it is possible to extend the class of integrable systems based on this fact. One of such extensions is the main theme of this thesis, namely, extensions to discrete or q-discrete system. Some other results have been reported in ref.[25-28].

Chapter 2.
q-Discretization of the Nonlinear Integrable Systems

In this chapter, we propose a method to extend nonlinear integrable systems to q-discrete systems, based on the bilinear formalism. In fact, this method is quite similar to the method of discretization. In the section 2.1, we demonstrate the method of discretization by taking the 2DTL equation as an example. In the section 2.2, we propose q-discrete versions of the 2DTM and 2DTL equations, and their reductions to cylindrical Toda equations. For the q-2DTM equation, Bäcklund transformation and Lax pair are also discussed. Moreover, for the q-2DTL equation, it is shown that a reduced equation admits the solution expressed by the Casorati determinant whose entries are the q-Bessel functions.

2.1 Discretization of the Nonlinear Integrable Systems

In this section we give a review of the method to discretize a given nonlinear integrable system preserving integrablity proposed by Hirota[13,14]. We demonstrate the case of the 2DTL equation as an example. We have the 2DTL equation again,

$$\frac{\partial V_n}{\partial x} = V_n(J_n - J_{n+1}) \ , \tag{2.1.1a}$$

$$\frac{\partial J_n}{\partial y} = V_{n-1} - V_n \ , \tag{2.1.1b}$$

$$n \in Z, \tag{2.1.1c}$$

which is reduced to the bilinear form,

$$\frac{\partial^2 \tau_n}{\partial x \partial y}\tau_n - \frac{\partial \tau_n}{\partial x}\frac{\partial \tau_n}{\partial y} = \tau_{n+1}\tau_{n-1} - \tau_n^2 \ , \tag{2.1.2}$$

through the dependent variable transformation,

$$V_n = \frac{\tau_{n-1}\tau_{n+1}}{\tau_n^2}\ , \qquad J_n = \frac{\partial}{\partial x}\log\frac{\tau_{n-1}}{\tau_n}. \tag{2.1.3}$$

The solution of eq.(2.1.2) is given by the Casorati determinant,

$$\tau_n = \begin{vmatrix} f_n^{(1)} & f_{n+1}^{(1)} & \cdots & f_{n+N-1}^{(1)} \\ f_n^{(2)} & f_{n+1}^{(2)} & \cdots & f_{n+N-1}^{(2)} \\ \vdots & \vdots & \cdots & \vdots \\ f_n^{(N)} & f_{n+1}^{(N)} & \cdots & f_{n+N-1}^{(N)} \end{vmatrix}\ , \tag{2.1.4a}$$

$$\partial_x f_n^{(k)} = -f_{n+1}^{(k)}\ , \qquad \partial_y f_n^{(k)} = f_{n-1}^{(k)}\ , \quad k = 1, \cdots, N\ , \tag{2.1.4b}$$

Hirota has proposed a systematic method of discretization of nonlinear integrable systems based on the direct method. The key point is that we discretize a given equation on the level of the τ function, instead of discretizing itself. This procedure is illustrated in fig.2.1.

$$\begin{tikzcd}
\text{Continuous Integrable System} \arrow[d] \arrow[rr, Rightarrow, "???"] & & \text{Discrete Integrable System} \\
\text{Dep. Var. Transf.} \arrow[d] & & \text{Dep. Var. Transf.} \arrow[u] \\
\text{Bilinear Form} \arrow[d] & & \text{Bilinear Form} \arrow[u] \\
\text{Solution: } \tau \text{ Function} \arrow[rr, Rightarrow, "\text{Discretization}"'] & & \text{Solution: } \tau \text{ Function} \arrow[u]
\end{tikzcd}$$

Figure 2.1. Procedure of discretization

Following the procedure illustrated in fig. 2.1, we first discretize the τ function (2.1.4) as

$$\tau_n(l,m) = \begin{vmatrix} f_n^{(1)}(l,m) & f_{n+1}^{(1)}(l,m) & \cdots & f_{n+N-1}^{(1)}(l,m) \\ f_n^{(2)}(l,m) & f_{n+1}^{(2)}(l,m) & \cdots & f_{n+N-1}^{(2)}(l,m) \\ \vdots & \vdots & \cdots & \vdots \\ f_n^{(N)}(l,m) & f_{n+1}^{(N)}(l,m) & \cdots & f_{n+N-1}^{(N)}(l,m) \end{vmatrix} , \tag{2.1.5a}$$

$$\Delta_l f_n^{(k)} = \frac{f_n^{(k)}(l+1,m) - f_n^{(k)}(l,m)}{a} = -f_{n+1}^{(k)} , \tag{2.1.5b}$$

$$\Delta_m f_n^{(k)} = \frac{f_n^{(k)}(l,m+1) - f_n^{(k)}(l,m)}{b} = f_{n-1}^{(k)} , \quad k = 1, \cdots, N , \tag{2.1.5c}$$

where a and b are arbitrary constants which play a role of lattice interval. Let us next construct the bilinear form for $\tau_n(l,m)$. For the purpose, we introduce a notation,

$$\tau_n(l,m) = |0, 1, \cdots, N-2, N-1|, \tag{2.1.6}$$

where "k" stands for

$$"k" = \begin{pmatrix} f_{n+k}^{(1)}(l,m) \\ f_{n+k}^{(2)}(l,m) \\ \vdots \end{pmatrix} .$$

Using eqs.(2.1.5b) and (2.1.5c), we have

$$\begin{aligned} \tau_n(l-1,m) &= |0_{l-1}, \cdots N-2_{l-1}, N-1_{l-1}| \\ &= |0, \cdots N-2, N-1_{l-1}|, \end{aligned} \tag{2.1.7}$$

$$\begin{aligned} \tau_n(l,m-1) &= |0_{m-1}, \cdots N-2_{m-1}, N-1_{m-1}| \\ &= |0, \cdots N-2, N-1_{m-1}|, \end{aligned} \tag{2.1.8}$$

$$\begin{aligned} -a\tau_n(l-1,m) &= |0_{l-1}, \cdots, N-2_{l-1}, a \times N-1_{l-1}| \\ &= |0, \cdots N-2, N-2_{l-1}|, \end{aligned} \tag{2.1.9}$$

$$\begin{aligned} b\tau_n(l,m-1) &= |b \times 0_{m-1}, 0, \cdots, N-1_{m-1}| \\ &= |1_{m-1}, 1, \cdots N-1|, \end{aligned} \tag{2.1.10}$$

$$(1+ab)\tau_n(l-1,m-1) = |0_{m-1}, 1, \cdots, N-2, N-1_{l-1}| \tag{2.1.11}$$

Now we consider the following identity of determinant,

$$0 = \left| \begin{array}{cc:ccc:c:cc} 0 & 0_{m-1} & 1 & \cdots & N-2 & \emptyset & N-1 & N-1_{l-1} \\ \hdashline 0 & 0_{m-1} & & \emptyset & & 1 \ \cdots \ N-2 & N-1 & N-1_{l-1} \end{array} \right| . \tag{2.1.12}$$

Applying the Laplace expansion on the right hand side of eq.(2.1.12), we have

$$\begin{aligned} 0 &= |0_{m-1}, 0, \cdots, N-2| \times |1, \cdots, N-2, N-1, N-1_{l-1}| \\ &- |0_{m-1}, 1, \cdots, N-2, N-1| \times |0, 1, \cdots, N-2, N-1_{l-1}| \\ &+ |0_{m-1}, 1, \cdots, N-1_{l-1}| \times |0, 1, \cdots N-2, N-1|, \end{aligned} \tag{2.1.13}$$

which can be expressed in terms of τ by using eqs.(2.1.7)-(2.1.11) as

$$\begin{aligned} 0 =& -ab\tau_{n-1}(l, m-1)\tau_{n+1}(l-1, m) - \tau_n(l, m-1)\tau_n(l-1, m) \\ &+(1+ab)\tau_n(l-1, m-1)\tau_n(l, m), \end{aligned} \tag{2.1.14}$$

or

$$\begin{aligned} 0 =& -ab\tau_{n-1}(l+1, m)\tau_{n+1}(l, m+1) - \tau_n(l+1, m)\tau_n(l, m+1) \\ &+(1+ab)\tau_n(l, m)\tau_n(l+1, m+1). \end{aligned} \tag{2.1.15}$$

Equation (2.1.15) can be rewritten as

$$\begin{aligned} &\Delta_l\Delta_m\tau_n(l, m) \cdot \tau_n(l, m) - \Delta_l\tau_n(l, m) \cdot \Delta_m\tau_n(l, m) \\ &= \tau_{n-1}(l+1, m)\tau_{n+1}(l, m+1) - \tau_n(l+1, m+1)\tau_n(l, m). \end{aligned} \tag{2.1.16}$$

Thus we have obtained a discrete version of the bilinear form (2.1.2). Now we introduce the dependent variable transformation by

$$V_n(x, y) = \frac{\tau_{n+1}(l+1, m)\tau_{n-1}(l, m+1)}{\tau_n(l, m+1)\tau_n(l, m)} , \tag{2.1.17a}$$

$$J_n(x, y) = \frac{1}{a}\left\{ (1+ab) \ \frac{\tau_n(l+1, m)\tau_{n-1}(l, m)}{\tau_n(l, m)\tau_{n-1}(l+1, m)} \ - 1 \right\} . \tag{2.1.17b}$$

Then we obtain the discrete 2DTL equation,

$$\Delta_l V_n(l,m) = J_n(x,y)V_n(l+1,m) - J_{n+1}(l,m)V_n(l,m) \ , \tag{2.1.18a}$$

$$\Delta_m J_n(l,m) = V_{n-1}(l+1,m) - V_n(l,m+1) \ . \tag{2.1.18b}$$

Discretization of the 2DTM equation can be done by similar procedure, namely, simply by replacing differential operators in the τ function (1.2.18) by difference operators, and introducing suitable dependent variable transformation[20]. It is also possible to construct the Lax pair and the conserved quantities for discrete 2DTM equation.

2.2 q-Discretization of the Two-dimensional Toda Molecule Equation

In this section, we extend the method of discretization discussed in the previous section to q-discretization, and propose q-discrete version of the 2DTM and 2DTL equations. From our point of view, the difference between discrete system and q-discrete system is that lattice intervals of the latter depend on independent variables while they are constants in the former. This difference might cause difficulties when we construct shift operators of τ functions. Despite of such difference, we show that it is possible to perform q-discretization in similar manner.

2.2a q-Discrete Two-dimensional Toda Molecule Equation and Its Solution

We propose a system given by

$$\delta_{q^\alpha,x} V_N(x,y) = V_N(q^\alpha x,y)J_N(x,q^\beta y) - V_N(x,y)J_{N+1}(x,y) \ , \tag{2.2.1a}$$

$$\delta_{q^\beta,y} J_N(x,y) = V_{N-1}(q^\alpha x,y) - V_N(x,y) \ , \tag{2.2.1b}$$

$$V_0(x,y) = V_M(x,y) = 0 \ , \tag{2.2.1c}$$

where $\delta_{q^\alpha,x}$ and $\delta_{q^\beta,y}$ are the q-difference operators defined by

$$\delta_{q^\alpha,x} f(x) = \frac{f(x) - f(q^\alpha x)}{(1-q)x}\,, \qquad \delta_{q^\beta,y} f(y) = \frac{f(y) - f(q^\beta y)}{(1-q)y}\,. \tag{2.2.2}$$

The operators $\delta_{q^\alpha,x}$ and $\delta_{q^\beta,y}$ tend to $\alpha\dfrac{\partial}{\partial x}$ and $\beta\dfrac{\partial}{\partial y}$ in the limit $q \to 1$, respectively. In this limit, eqs.(2.2.1) are reduced to the 2DTM equation eqs.(1.2.16). We call eqs.(2.2.1) the q-discrete 2DTM equation.

Equations (2.2.1) are transformed into the bilinear form,

$$\begin{aligned} &\delta_{q^\alpha,x}\, \delta_{q^\beta,y} \tau_N(x,y) \cdot \tau_N(x,y) - \delta_{q^\alpha,x} \tau_N(x,y) \delta_{q^\beta,y} \tau_N(x,y) \\ &\quad = \tau_{N+1}(x,y)\ \tau_{N-1}(q^\alpha x, q^\beta y)\,, \end{aligned} \tag{2.2.3}$$

through the dependent variable transformations,

$$J_N(x,y) = \frac{1}{(1-q)x}\left\{\frac{\tau_{N-1}(x,y)\tau_N(q^\alpha x,y)}{\tau_{N-1}(q^\alpha x,y)\tau_N(x,y)} - 1\right\}, \tag{2.2.4a}$$

$$V_N(x,y) = \frac{\tau_{N+1}(x,y)\tau_{N-1}(x,q^\beta y)}{\tau_N(x,y)\tau_N(x,q^\beta y)}\,. \tag{2.2.4b}$$

A solution of the bilinear form (2.2.3) is given by the two-directional Wronski-type determinant,

$$\tau_N(x,y) = \begin{vmatrix} f(x,y) & \delta_{q^\alpha,x} f(x,y) & \cdots & \delta_{q^\alpha,x}^{N-1} f(x,y) \\ \delta_{q^\beta,y} f(x,y) & \delta_{q^\alpha,x}\delta_{q^\beta,y} f(x,y) & \cdots & \delta_{q^\alpha,x}^{N-1}\delta_{q^\beta,y} f(x,y) \\ \vdots & \vdots & \ddots & \vdots \\ \delta_{q^\beta,y}^{N-1} f(x,y) & \delta_{q^\alpha,x}\delta_{q^\beta,y}^{N-1} f(x,y) & \cdots & \delta_{q^\alpha,x}^{N-1}\delta_{q^\beta,y}^{N-1} f(x,y) \end{vmatrix}, \tag{2.2.5}$$

where $f(x,y)$ is chosen to satisfy the boundary condition (2.2.1c) as eq.(1.2.19).

Let us prove that eq. (2.2.5) really gives the solution of the bilinear form (2.2.3). We have for example,

$$\begin{aligned} \tau_N(q^{-\alpha}x,y) &= \begin{vmatrix} f(q^{-\alpha}x,y) & \delta_{q^\alpha,x} f(q^{-\alpha}x,y) & \cdots & \delta_{q^\alpha,x}^{N-1} f(q^{-\alpha}x,y) \\ \delta_{q^\beta,y} f(q^{-\alpha}x,y) & \delta_{q^\alpha,x}\delta_{q^\beta,y} f(q^{-\alpha}x,y) & \cdots & \delta_{q^\alpha,x}^{N-1}\delta_{q^\beta,y} f(q^{-\alpha}x,y) \\ \vdots & \vdots & \ddots & \vdots \\ \delta_{q^\beta,y}^{N-1} f(q^{-\alpha}x,y) & \delta_{q^\alpha,x}\delta_{q^\beta,y}^{N-1} f(q^{-\alpha}x,y) & \cdots & \delta_{q^\alpha,x}^{N-1}\delta_{q^\beta,y}^{N-1} f(q^{-\alpha}x,y) \end{vmatrix} \\ &= \begin{vmatrix} f(x,y) & \delta_{q^\alpha,x} f(x,y) & \cdots & \delta_{q^\alpha,x}^{N-1} f(q^{-\alpha}x,y) \\ \delta_{q^\beta,y} f(x,y) & \delta_{q^\alpha,x}\delta_{q^\beta,y} f(x,y) & \cdots & \delta_{q^\alpha,x}^{N-1}\delta_{q^\beta,y} f(q^{-\alpha}x,y) \\ \vdots & \vdots & \ddots & \vdots \\ \delta_{q^\beta,y}^{N-1} f(x,y) & \delta_{q^\alpha,x}\delta_{q^\beta,y}^{N-1} f(x,y) & \cdots & \delta_{q^\alpha,x}^{N-1}\delta_{q^\beta,y}^{N-1} f(q^{-\alpha}x,y) \end{vmatrix}. \end{aligned} \tag{2.2.6}$$

In deriving the second determinant from the first, we have subtracted $(k+1)$-th column multiplied by $(1-q)q^{-\alpha}x$ from k-th column for $k=1,\cdots N-1$, to confine the shift of the independent variable x to the most right column of the determinant. Moreover, we note that $\delta_{q^\alpha,x}^{N-1}f(q^{-\alpha}x,y)$ means $\delta_{q^\alpha,x}^{N-1}f(x,y)|_{x\to q^{-\alpha}x}$. Multiplying N-th column by $(1-q)q^{-\alpha}x$ and adding $(N-1)$-th column to N-th column in the second determinant of eq.(2.2.6), we get

$$
\begin{aligned}
&(1-q)q^{-\alpha}x\ \tau_N(q^{-\alpha}x,y) \\
&=\begin{vmatrix}
f(x,y) & \delta_{q^\alpha,x}f(x,y) & \cdots & \delta_{q^\alpha,x}^{N-2}f(q^{-\alpha}x,y) \\
\delta_{q^\beta,y}f(x,y) & \delta_{q^\alpha,x}\delta_{q^\beta,y}f(x,y) & \cdots & \delta_{q^\alpha,x}^{N-2}\delta_{q^\beta,y}f(q^{-\alpha}x,y) \\
\vdots & \vdots & \ddots & \vdots \\
\delta_{q^\beta,y}^{N-1}f(x,y) & \delta_{q^\alpha,x}\delta_{q^\beta,y}^{N-1}f(x,y) & \cdots & \delta_{q^\alpha,x}^{N-2}\delta_{q^\beta,y}^{N-1}f(q^{-\alpha}x,y)
\end{vmatrix}.
\end{aligned}
\tag{2.2.7}
$$

Similarly, we obtain

$$
\begin{aligned}
&(1-q)^2q^{-(\alpha+\beta)}xy\ \tau_N(q^{-\alpha}x,q^{-\beta}y) \\
&=\begin{vmatrix}
f(x,y) & \cdots & \delta_{q^\alpha,x}^{N-2}f(x,y) & \delta_{q^\alpha,x}^{N-2}f(q^{-\alpha}x,y) \\
\vdots & \ddots & \vdots & \vdots \\
\delta_{q^\beta,y}^{N-2}f(x,y) & \cdots & \delta_{q^\alpha,x}^{N-2}\delta_{q^\beta,y}^{N-2}f(x,y) & \delta_{q^\alpha,x}^{N-2}\delta_{q^\beta,y}^{N-2}f(q^{-\alpha}x,y) \\
\delta_{q^\beta,y}^{N-2}f(x,q^{-\beta}y) & \cdots & \delta_{q^\alpha,x}^{N-2}\delta_{q^\beta,y}^{N-2}f(x,q^{-\beta}y) & \delta_{q^\alpha,x}^{N-2}\delta_{q^\beta,y}^{N-2}f(q^{-\alpha}x,q^{-\beta}y)
\end{vmatrix}
\end{aligned}
\tag{2.2.8}
$$

Applying Jacobi's identity on eq.(2.2.8) with N replaced by $N+1$, we obtain

$$
\begin{aligned}
&\tau_N(q^{-\alpha}x,q^{-\beta}y)\tau_N(x,y)-\tau_N(q^{-\alpha}x,y)\tau_N(x,q^{-\beta}y) \\
&=(1-q)^2q^{-(\alpha+\beta)}xy\ \tau_{N+1}(q^{-\alpha}x,q^{-\beta}y)\tau_{N-1}(x,y)\ ,
\end{aligned}
\tag{2.2.9}
$$

which is nothing but the bilinear form (2.2.3) with x and y replaced by $q^{-\alpha}x$ and $q^{-\beta}y$, respectively. Thus we have proved that eq.(2.2.5) gives the solution of eq.(2.2.3).

We now discuss a reduction of the q-2DTM equation. Putting $xy=r^2$ and $\alpha=\beta=2$, and imposing the condition that $\tau_N(x,y)$ depends only on r, we find that the bilinear form (2.2.3) and its solution (2.2.5) are reduced to

$$
(\frac{1}{r}\delta_{q,r}+q\delta_{q,r}^2)\tau_N(r)\cdot\tau_N(r)-\{\delta_{q,r}\tau_N(r)\}^2=\tau_{N+1}(r)\tau_{N-1}(q^2r)\ , \tag{2.2.10}
$$

and

$$\tau_N(r) = q^{-\frac{2}{3}N(N-1)(N-2)}\ r^{-N(N-1)}$$
$$\times \begin{vmatrix} f(r) & r\delta_{q,r}f(r) & \cdots & (r\delta_{q,r})^{N-1}f(r) \\ r\delta_{q,r}f(r) & (r\delta_{q,r})^2 f(r) & \cdots & (r\delta_{q,r})^N f(r) \\ \vdots & \vdots & \ddots & \vdots \\ (r\delta_{q,r})^{N-1}f(r) & (r\delta_{q,r})^N f(r) & \cdots & (r\delta_{q,r})^{2N-2}f(r) \end{vmatrix}, \qquad (2.2.11)$$

respectively. Equation (2.2.10) tends to the cylindrical Toda molecule (cTM) equation[29],

$$(\frac{1}{r}\frac{\partial}{\partial r} + \frac{\partial^2}{\partial r^2})\tau_N(r)\cdot\tau_N(r) - \{\frac{\partial \tau_N(r)}{\partial r}\}^2 = \tau_{N+1}(r)\tau_{N-1}(r)\ , \qquad (2.2.12)$$

in the limit $q \to 1$, and hence we call eq.(2.2.10) the q-cTM equation. Note that eq.(2.2.10) is transformed to

$$\delta_{q,r}V_N(r) = qJ_N(qr)V_N(qr) - J_{N+1}(r)V_N(r)\ , \qquad (2.2.13a)$$

$$(q\delta_{q,r} + \frac{1}{r})J_N(r) = V_N(r) - V_{N-1}(qr)\ , \qquad (2.2.13b)$$

$$V_0(r) = V_M(r) = 0\ , \qquad (2.2.13c)$$

through the dependent variable transformations,

$$V_N(r) = \frac{\tau_{N-1}(qr)\tau_{N+1}(r)}{\tau_N(r)\tau_N(qr)}\ , \qquad (2.2.14a)$$

$$J_N(r) = \frac{1}{(1-q)r}\left\{\frac{\tau_N(qr)\tau_{N-1}(r)}{\tau_N(r)\tau_{N-1}(qr)} - 1\right\}. \qquad (2.2.14b)$$

2.2b. Bäcklund Transformation and Lax Pair

By using the fact that the solution of the q-2DTM equation (2.2.1) is given by (2.2.5), we here propose the Bäcklund transformation. It is written by

$$\delta_{q^\beta,y}\ \tau_N(x,y)\cdot\tau'_N(x,y) - \tau_N(x,y)\ \delta_{q^\beta,y}\ \tau'_N(x,y)$$
$$= -\tau_{N+1}(x,y)\tau'_{N-1}(x,q^\beta y)\ , \qquad (2.2.15a)$$

$$\delta_{q^\alpha,x}\ \tau_N(x,y)\cdot\tau'_{N-1}(x,y) - \tau_N(x,y)\ \delta_{q^\alpha,x}\ \tau'_{N-1}(x,y)$$
$$= \tau_{N-1}(q^\alpha x,y)\tau'_N(x,y)\ , \qquad (2.2.15b)$$

which transforms a solution of the q-2DTM equation,

$$\tau_N(x,y) = \begin{vmatrix} f(x,y) & \delta_{q^\alpha,x} f(x,y) & \cdots & \delta_{q^\alpha,x}^{N-1} f(x,y) \\ \delta_{q^\beta,y} f(x,y) & \delta_{q^\alpha,x}\delta_{q^\beta,y} f(x,y) & \cdots & \delta_{q^\alpha,x}^{N-1}\delta_{q^\beta,y} f(x,y) \\ \vdots & \vdots & \ddots & \vdots \\ \delta_{q^\beta,y}^{N-1} f(x,y) & \delta_{q^\alpha,x}\delta_{q^\beta,y}^{N-1} f(x,y) & \cdots & \delta_{q^\alpha,x}^{N-1}\delta_{q^\beta,y}^{N-1} f(x,y) \end{vmatrix}, \tag{2.2.16}$$

to another solution,

$$\tau'_N(x,y) = \begin{vmatrix} \delta_{q^\alpha,x} f(x,y) & \delta_{q^\alpha,x}^2 f(x,y) & \cdots & \delta_{q^\alpha,x}^{N} f(x,y) \\ \delta_{q^\alpha,x}\delta_{q^\beta,y} f(x,y) & \delta_{q^\alpha,x}^2\delta_{q^\beta,y} f(x,y) & \cdots & \delta_{q^\alpha,x}^{N}\delta_{q^\beta,y} f(x,y) \\ \vdots & \vdots & \ddots & \vdots \\ \delta_{q^\alpha,x}\delta_{q^\beta,y}^{N-1} f(x,y) & \delta_{q^\alpha,x}^2\delta_{q^\beta,y}^{N-1} f(x,y) & \cdots & \delta_{q^\alpha,x}^{N}\delta_{q^\beta,y}^{N-1} f(x,y) \end{vmatrix}. \tag{2.2.17}$$

In other words, eqs. (2.2.15a) and (2.2.15b) are the identities for the determinants (2.2.16) and (2.2.17). This fact is shown by the Plücker relation as follows.

Let us prove the second equation (2.2.15b). First, we introduce notations

$$\tau_N(x,y) = |0,1,\cdots,N-1| , \tag{2.2.18}$$

$$\tau'_N(x,y) = |1,2,\cdots,N| . \tag{2.2.19}$$

Namely, the number "k" in eqs.(2.2.18) and (2.2.19) means a column vector

$$\text{“}k\text{”} = \left.\begin{pmatrix} \delta_{q^\alpha,x}^k f(x,y) \\ \delta_{q^\alpha,x}^k \delta_{q^\beta,y} f(x,y) \\ \vdots \\ \delta_{q^\alpha,x}^k \delta_{q^\beta,y}^{N-1} f(x,y) \end{pmatrix}\right\} N . \tag{2.2.20}$$

Then we have

$$\tau_{N-1}(x,y) = |0,1,\cdots,N-2,\phi| , \tag{2.2.21}$$

$$\tau_N(q^{-\alpha}x,y) = |0,1,\cdots,N-2,N-1_{q^{-\alpha}x}| , \tag{2.2.22}$$

$$\tau'_{N-1}(q^{-\alpha}x,y) = |1,2,\cdots,N-2,N-1_{q^{-\alpha}x},\phi| , \tag{2.2.23}$$

and

$$(1-q)q^{-\alpha}x\ \tau'_N(q^{-\alpha}x,y) = |1,2,\cdots N-1,N-1_{q^{-\alpha}x}| , \tag{2.2.24}$$

where

$$\text{“}N-1_{q^{-\alpha}x}\text{”} = \begin{pmatrix} \delta_{q^\alpha,x}^{N-1} f(q^{-\alpha}x,y) \\ \delta_{q^\alpha,x}^{N-1}\delta_{q^\beta,y} f(q^{-\alpha}x,y) \\ \vdots \\ \delta_{q^\alpha,x}^{N-1}\delta_{q^\beta,y}^{N-1} f(q^{-\alpha}x,y) \end{pmatrix} , \tag{2.2.25}$$

and

$$\phi = \left.\begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix}\right\} N , \tag{2.2.26}$$

which is inserted to equalize the size of the determinant.

We now consider an identity of $2N \times 2N$ determinant,

$$0 = \left|\begin{array}{c:ccc:ccc:ccc} 0 & 1 & \cdots & N-2 & & \emptyset & & N-1 & N-1_{q^{-\alpha}x} & \phi \\ \hdashline 0 & & \emptyset & & 1 & \cdots & N-2 & N-1 & N-1_{q^{-\alpha}x} & \phi \end{array}\right| . \tag{2.2.27}$$

Applying the Laplace expansion to the right-hand side, we obtain an identity (the Plücker relation),

$$\begin{aligned} 0 =& |0,1,\cdots,N-2,N-1_{q^{-\alpha}x}|\,|1,\cdots,N-2,N-1,\phi| \\ &-|0,1,\cdots,N-2,N-1|\,|1,\cdots,N-2,N-1_{q^{-\alpha}x},\phi| \\ &-|1,\cdots,N-2,N-1,N-1_{q^{-\alpha}x}|\,|0,1,\cdots,N-2,\phi| , \end{aligned} \tag{2.2.28}$$

or equivalently,

$$\begin{aligned} &\tau_N(q^{-\alpha}x,y)\tau'_{N-1}(x,y) - \tau_N(x,y)\tau'_{N-1}(q^{-\alpha}x,y) \\ &\quad = (1-q)q^{-\alpha}x\ \tau_{N-1}(x,y)\tau'_N(q^{-\alpha}x,y) , \end{aligned} \tag{2.2.29}$$

which is nothing but eq.(2.2.15b) with x replaced by $q^{-\alpha}x$. Thus we have completed the proof. The first equation (2.2.15a) is proved in a similar way.

It is possible in general to construct the Lax pair from the Bäcklund transformation. Following the method developed by Hirota *et. al.*[14], we derive the Lax pair for the q-2DTM equation (2.2.1) from eqs.(2.2.15). Introducing ψ by

$$\tau'_N(x,y) = \tau_N(x,y)\psi_{N+1}(x,y) , \tag{2.2.30}$$

we have from eqs.(2.2.15),

$$\delta_{q^\beta,y}\psi_{N+1}(x,y) = V_N(x,y)\psi_N(x,q^\beta y)\ , \tag{2.2.31a}$$

$$\delta_{q^\alpha,x}\psi_N(x,y) = -J_N(x,y)\psi_N(x,y) - \psi_{N+1}(x,y)\ . \tag{2.2.31b}$$

Let us define two matrices L and R by

$$L(x,y) = \begin{pmatrix} 0 & & & \\ V_1(x,y) & 0 & & \emptyset \\ & \ddots & \ddots & \\ \emptyset & & & \\ & & V_{M-1}(x,y) & 0 \end{pmatrix}\ , \tag{2.2.32}$$

$$R(x,y) = -\begin{pmatrix} J_1(x,y) & 1 & & & \\ & J_2(x,y) & 1 & \emptyset & \\ & & \ddots & \ddots & \\ \emptyset & & & J_{M-1}(x,y) & 1 \\ & & & & J_M(x,y) \end{pmatrix}\ . \tag{2.2.33}$$

Then eqs.(2.2.31) are rewritten as

$$\delta_{q^\beta,y}\Psi(x,y) = L(x,y)\Psi(x,q^\beta y)\ , \qquad \delta_{q^\alpha,x}\Psi(x,y) = R(x,y)\Psi(x,y)\ , \tag{2.2.34}$$

where

$$\Psi(x,y) = \begin{pmatrix} \psi_1(x,y) \\ \vdots \\ \psi_M(x,y) \end{pmatrix}\ . \tag{2.2.35}$$

The compatibility condition of the linear system (2.2.34) yields

$$\delta_{q^\alpha,x}L(x,y) - \delta_{q^\beta,y}R(x,y) = R(x,y)L(x,y) - L(q^\alpha x,y)R(x,q^\beta y)\ , \tag{2.2.36}$$

which recovers the q-2DTM equation (2.2.1). Consequently, eqs.(2.2.32) and (2.2.33) give the Lax pair of the q-2DTM equation.

2.3 q-Discretization of the Two-Dimensional Toda Lattice Equation

In this section we propose a system given by

$$\delta_{q^\alpha,x}V_n(x,y) = J_n(x,y)V_n(q^\alpha x,y) - J_{n+1}(x,y)V_n(x,y)\ , \tag{2.3.1a}$$

$$\delta_{q^\beta,y}J_n(x,y) = V_{n-1}(q^\alpha x,y) - q^\beta V_n(x,q^\beta y)\ , \tag{2.3.1b}$$

which reduces to the 2DTL equation in the limit $q \to 1$. We call eqs.(2.3.1) the q-discrete 2DTL equation. Equations (2.3.1) are transformed to bilinear form,

$$\begin{aligned}&\{\delta_{q^\alpha,x}\delta_{q^\beta,y}\tau_n(x,y)\}\,\tau_n(x,y)-\{\delta_{q^\alpha,x}\tau_n(x,y)\}\,\{\delta_{q^\beta,y}\tau_n(x,y)\}\\&=\tau_{n+1}(x,q^\beta y)\tau_{n-1}(q^\alpha x,y)-\tau_n(q^\alpha x,q^\beta y)\tau_n(x,y),\end{aligned}\tag{2.3.2}$$

through the dependent variable transformation,

$$V_n(x,y)=\frac{\tau_{n+1}(q^\alpha x,y)\tau_{n-1}(x,q^\beta y)}{\tau_n(x,q^\beta y)\tau_n(x,y)}\,,\tag{2.3.3a}$$

$$J_n(x,y)=\frac{1}{(1-q)x}\left\{\{1+(1-q)^2xy\}\,\frac{\tau_n(q^\alpha x,y)\tau_{n-1}(x,y)}{\tau_n(x,q^\beta y)\tau_{n-1}(q^\alpha x,y)}\,-1\right\},\tag{2.3.3b}$$

and the solution of eq.(2.3.2) is given by

$$\tau_n(x,y)=\begin{vmatrix} f_n^{(1)}(x,y) & f_{n+1}^{(1)}(x,y) & \cdots & f_{n+N-1}^{(1)}(x,y)\\ f_n^{(2)}(x,y) & f_{n+1}^{(2)}(x,y) & \cdots & f_{n+N-1}^{(2)}(x,y)\\ \vdots & \vdots & \cdots & \vdots\\ f_n^{(N)}(x,y) & f_{n+1}^{(N)}(x,y) & \cdots & f_{n+N-1}^{(N)}(x,y)\end{vmatrix}\,,\tag{2.3.4}$$

where $f_n^{(k)}$、$k=1,\cdots,N$, satisfy the dispersion relation

$$\delta_{q^\alpha,x}f_n^{(k)}(x,y)=-f_{n+1}^{(k)}(x,y)\,,\qquad \delta_{q^\beta,y}f_n^{(k)}(x,y)=f_{n-1}^{(k)}(x,y)\,.\tag{2.3.5}$$

In the following, let us prove that eqs(2.3.4) and (2.3.5) give the solution of the bilinear form of the q-2DTL equation (2.3.3). We introduce a notation

$$\tau_n(x,y)=|0,1,\cdots,N-1|,\tag{2.3.6}$$

where "j" is a column vector given by

$$\text{"}j\text{"}=\begin{pmatrix} f_{n+j}^{(1)}(x,y)\\ f_{n+j}^{(2)}(x,y)\\ \vdots\\ f_{n+j}^{(N)}(x,y)\end{pmatrix}.\tag{2.3.7}$$

Using the dispersion relation (2.3.5), we have

$$\tau_n(q^{-\alpha}x,y) = |0_{q^{-\alpha}x}, 1_{q^{-\alpha}x}, \cdots N-1_{q^{-\alpha}x}|$$
$$= |0,\cdots,N-2,N-1_{q^{-\alpha}x}| \,, \tag{2.3.9}$$

$$-(1-q)q^{-\alpha}x\ \tau_n(q^{-\alpha}x,y) = |0,\cdots,N-2,N-2_{q^{-\alpha}x}| \,, \tag{2.3.10}$$

$$\tau_n(x,q^{-\beta}y) = |0_{q^{-\beta}y},1,\cdots,N-1| \,, \tag{2.3.11}$$

$$(1-q)q^{-\beta}y\ \tau_n(x,q^{-\beta}y) = |1_{q^{-\beta}y},1,\cdots,N-1| \,, \tag{2.3.12}$$

$$\{1+(1-q)^2q^{-(\alpha+\beta)}xy\}\ \tau_n(q^{-\alpha}x,q^{-\beta}y) = |0_{q^{-\beta}y},1,\cdots,N-2,N-1_{q^{-\alpha}x}| \,, \tag{2.3.13}$$

where

$$"j_{q^{-\alpha}x}" = \begin{pmatrix} f^{(1)}_{n+j}(q^{-\alpha}x,y) \\ f^{(2)}_{n+j}(q^{-\alpha}x,y) \\ \vdots \\ f^{(N)}_{n+j}(q^{-\alpha}x,y) \end{pmatrix} . \tag{2.3.14}$$

Now we consider the following identity of $2N\times 2N$ determinant,

$$0 = \left| \begin{array}{cc:ccc:ccc:cc} 0 & 0_{q^{-\beta}y} & 1 & \cdots & N-2 & & \emptyset & & N-1 & N-1_{q^{-\alpha}x} \\ \hdashline 0 & 0_{q^{-\beta}y} & & \emptyset & & 1 & \cdots & N-2 & N-1 & N-1_{q^{-\alpha}x} \end{array} \right| . \tag{2.3.15}$$

Applying the Laplace expansion on the right hand side of eq.(2.3.15) and using eqs.(2.3.6), (2.3.9)-(2.3.13), we have

$$\{1+(1-q)^2q^{-(\alpha+\beta)}xy\}\ \tau_n(q^{-\alpha}x,q^{-\beta}y)\ \tau_n(x,y) - \tau_n(q^{-\alpha}x,y)\ \tau_n(x,q^{-\beta}y)$$
$$+(1-q)q^{-\alpha}x\ \tau_{n+1}(q^{-\alpha}x,y)\ (1-q)q^{-\beta}y\ \tau_{n-1}(x,q^{-\beta}y) = 0, \tag{2.3.16}$$

which is nothing but eq.(2.3.2) with x and y replaced by $q^{-\alpha}x$ and $q^{-\beta}y$, respectively. Thus we have proved that eqs.(2.3.4) and (2.3.5) actually gives the solution of eq.(2.3.2).

We next discuss the reduction of the q-2DTL equation. Similar to the case of q-2DTM equation, we impose the condition that τ depends only on $xy=r^2$, and $\alpha=\beta=2$, then the bilinear form (2.3.2) is reduced to

$$\left\{\left(\frac{1}{r}\delta_{q,r}+q\delta^2_{q,r}\right)\tau_n(r)\right\}\tau_n(r) - \left\{\delta_{q,r}\tau_n(r)\right\}^2 = \tau_{n+1}(qr)\tau_{n-1}(qr) - \tau_n(q^2r)\tau_n(r). \tag{2.3.17}$$

Equation (2.3.17) reduces to the cylindrical Toda Lattice (cTL) equation[30] in the limit $q \to 1$,

$$\{(\frac{1}{r}\frac{\partial}{\partial r} + \frac{\partial^2}{\partial r^2})\tau_n(r)\}\tau_n(r) - \{\frac{\partial}{\partial r}\tau_n(r)\}^2 = \tau_{n+1}(r)\tau_{n-1}(r) - \tau_n(r)^2 \qquad (2.3.18)$$

We call (2.3.17) the q-cTL equation. It is known that cTL equation (2.3.18) has a solution expressed by the Casorati determinant whose entry is the Bessel function.

Let us discuss the solution of eq.(2.3.17). We impose the above constraint on the solution of q-2DTL equation (2.3.4), (2.3.5). We put

$$f_n^{(k)}(x,y) = (\frac{y}{x})^{(n+p_k)/2}\phi_{n+p_k}(r) \qquad (2.3.19)$$

where p_k is arbitraty constants. Substituting eq.(2.3.19) into (2.3.3)(note that $\alpha = \beta = 2$), we have the equation for $\phi_{n+p_k}(r)$,

$$(q^{-(n+p_k)}\delta_{q,r} + \frac{[-(n+p_k)]}{r})\phi_{n+p_k}(r) = -\phi_{n+p_k+1}(r)\ , \qquad (2.3.20)$$

$$(q^{n+p_k}\delta_{q,r} + \frac{[n+p_k]}{r})\phi_{n+p_k}(r) = \phi_{n+p_k-1}(r)\ , \qquad (2.3.21)$$

where $[n]$ is the q-integer,

$$[n] = \frac{1-q^n}{1-q}. \qquad (2.3.22)$$

Equations (2.3.20) and (2.3.21) are nothing but the contiguity relations of the q-Bessel function. Moreover, noticing that q-2DTL equation is invariant if we replace $\tau_n(x,y)$ by $x^{c_1 n+c_2}y^{c_3 n+c_4}\tau_n(x,y)$, where c_1-c_4 are arbitrary constants, we find that

$$\tau_n(r) = \begin{vmatrix} J_{q,n+p_1}(r) & J_{q,n+p_1+1}(r) & \cdots & J_{q,n+p_1+N-1}(r) \\ J_{q,n+p_2}(r) & J_{q,n+p_2+1}(r) & \cdots & J_{q,n+p_2+N-1}(r) \\ \vdots & \vdots & \cdots & \vdots \\ J_{q,n+p_N}(r) & J_{q,n+p_N+1}(r) & \cdots & J_{q,n+p_N+N-1}(r) \end{vmatrix}, \qquad (2.3.23)$$

where $J_{q,k}(r)$ is the q-Bessel function of degree k, gives the solution of the q-cTL equation (2.3.17). This result recovers that for cTL equation in the limit $q \to 1$.

Chapter3. Solutions of the Discrete Painlevé Equations

In this chapter we investigate the solutions of the discrete Painlevé equations. Before going to this subject, we first give a brief review on the Painlevé equations and the discrete Painlevé equations in the sections 3.1 and 3.2, respectively. In the sections 3.3 and 3.4, we discuss the solutions of the discrete Painlevé II and III equations, respectively. In the section 3.4, we propose a q-difference analogue of the Painlevé III equation based on the results obtained in the section 3.3.

3.1 Painlevé Equations

As mentioned in the chapter 1, there are several definitions for integrability, all of which are believed to be almost equivalent. Painlevé property is one of them, which can be expressed as "nonexistence of movable singular points except for poles". This property looks somewhat curious, and it is not obvious to understand what this property means for the integrability. One intuitive explanation is as follows. For linear ordinary equations, every singular points are determined by the coefficients, which means they have no movable singular points. For nonlinear equations, however, this property is lost. For example, even the simple equation,

$$\frac{dy}{dz} = -y^2 , \tag{3.1.1}$$

do not have this property. In fact, the general solution of eq.(3.1.1) is written as

$$y = \frac{1}{z - C}, \tag{3.1.2}$$

where C is a constant, which means that eq.(3.1.1) has a movable pole. Hence, Painlevé property can be considered to be a detector of comparably good nonlinear ordinary differential equation. But the reason why this property is deeply connected to the integrability might not be understood fully yet.

It is known for the ordinary differential equations of second order that there are essentially only six equations which possess Painlevé property[10], These equations are called the Painlevé equations $\mathrm{P_I} - \mathrm{P_{VI}}$, first four of which are written as

$$\mathrm{P_I}: \quad \frac{d^2w}{dx^2} = 6y^2 + x \, , \tag{3.1.3}$$

$$\mathrm{P_{II}}: \quad \frac{d^2w}{dx^2} = 2w^3 + xw + \alpha \, , \tag{3.1.4}$$

$$\begin{aligned} \mathrm{P_{III}}: \quad \frac{d^2w}{dx^2} &= \frac{1}{w}\left(\frac{dw}{dx}\right)^2 - \frac{1}{x}\frac{dw}{dx} + \frac{1}{x}(\alpha w^2 + \beta) \\ &\quad + \gamma w^3 + \frac{\delta}{w} \, , \end{aligned} \tag{3.1.5}$$

$$\begin{aligned} \mathrm{P_{IV}}: \quad \frac{dw^2}{dx^2} &= \frac{1}{2w} + \frac{3}{2}w^3 + 4xw^2 \\ &\quad + 2(x^2 - \alpha)w + \frac{\beta}{w} \, . \end{aligned} \tag{3.1.6}$$

Now the Painlevé equations are of very common occurrence in the theory of integrable systems. Nonlinear evolution equations, integrable through inverse scattering method, have been shown to possess one-dimensional (similarity) reductions that are just the Painlevé equations. For example, the modified KdV equation,

$$u_t - 6u^2 u_x + u_{xxx} = 0 \, , \tag{3.1.7}$$

is reduced to $\mathrm{P_{II}}$ (3.1.4) through the similarity reduction,

$$u(x,t) = \frac{w(z)}{(3t)^{1/3}}, \quad z = \frac{x}{(3t)^{2/3}}. \tag{3.1.8}$$

This feature of integrable partial differential equations eventually evolved into an integrability criterion[12]. In this sense, the Painlevé equations are the most fundamental nonlinear integrable systems.

The Painlevé equations have many aspects. Among them, let us summarize the properties of their solutions. It is known in general that the solutions of the Painlevé equations cannot be expressed by those of linear equations. In this sense, their solutions are called the Painlevé transcendents. However,they admits the solutions written by special functions for some special values of parameters. For example, P_{II}

$$w_{xx} - 2w^3 + 2xw + \alpha = 0 \ , \tag{3.1.9}$$

has a solution for $\alpha = -(2N+1)$ [31,32],

$$w = \left(\log \frac{\tau_{N+1}}{\tau_N}\right)_x \ , \tag{3.1.10}$$

where τ_N is given by an $N \times N$ Wronskian of the Airy function,

$$\tau_N = \begin{vmatrix} Ai & \frac{d}{dx} Ai & \cdots & \left(\frac{d}{dx}\right)^{N-1} Ai \\ \frac{d}{dx} Ai & \left(\frac{d}{dx}\right)^2 Ai & \cdots & \left(\frac{d}{dx}\right)^N Ai \\ \vdots & \vdots & \cdots & \vdots \\ \left(\frac{d}{dx}\right)^{N-1} Ai & \left(\frac{d}{dx}\right)^N Ai & \cdots & \left(\frac{d}{dx}\right)^{2N-2} Ai \end{vmatrix} . \tag{3.1.11}$$

Note that Ai is the Airy function satisfying

$$\frac{d^2}{dx^2} Ai = x Ai \ . \tag{3.1.12}$$

We also note that we have slightly changed the scale from the original P_{II} for the simplicity of the expression of the solutions. The other Painlevé equations except for P_I have similar solutions. The corresponding special functions are the Bessel function for P_{III}, the Hermite-Weber function for P_{IV}, the confluent hypergeometric function for P_V and hypergeometric function for P_{VI}, respectively. In this sense, the Painlevé transcendents can be regarded as nonlinear versions of the special functions.

Let us finally remark on the direct relationship with Toda molecule equation[31], which is one of the most typical nonlinear integrable systems. The Painlevé equations can be rewritten in the Hamilton formalism. For example, let us demonstrate the case of P_{II}. It is possible to rewrite eq.(3.1.4) as the equations of motion for the following Hamiltonian,

$$H_{II}(q,p,t;\alpha) = \frac{1}{2}p^2 - (q^2 + \frac{1}{2}x)p - \frac{1}{2}(2\alpha+1)q \ . \tag{3.1.13}$$

The equation of motion for this Hamiltonian are given by

$$\frac{dq}{dx} = p - (q^2 + \frac{1}{2}x), \tag{3.1.14}$$

$$\frac{dp}{dx} = 2qp + \frac{1}{2}(2\alpha + 1), \tag{3.1.15}$$

from which we obtain P_{II} for q by eliminating p. It is known that the Hamiltonian for each Painlevé equation can be expressed by log derivative of some regular function on C, which is called the τ function. For P_{II}, the τ function is defined through the Hamiltonian by

$$H_{II}(q, p, x; \alpha) = \frac{d}{dx}\log\, \tau(x; \alpha)\ . \tag{3.1.16}$$

Remarkably, it can be shown that the τ function satisfies the bilinear form of the Toda molecule equation,

$$\frac{d^2\tau(x;\alpha)}{dx^2}\tau(x;\alpha) - \left(\frac{d\tau(x;\alpha)}{dx}\right)^2 = \tau(x;\alpha+1)\tau(x;\alpha-1). \tag{3.1.17}$$

The τ functions for other Painlevé equations satisfy the Toda molecule equation for P_I,P_{IV}, the cylindrical Toda molecule equation for P_{III}, P_V and the spherical Toda molecule equation for P_{VI}, respectively.

3.2 Discrete Painlevé Equations

In the chapter 2, we have discussed one method for discretization or q-discretization of given nonlinear integrable systems, which is based on the guiding principle, "discretization preserving the structure of the τ function". Grammaticos et.al have proposed another method based on a conjecture called "singularity confinement", which can be stated as follows: " The movable singularities of integrable mappings are confined i.e. they are canceled one after a finite number of steps. Moreover the memory of the initial condition is not lost whenever a singularity is crossed."

Using this criterion, they have proposed the discrete analogue of the Painlevé equations P_I - P_V, the first four of which are written as

$$\mathrm{dP_I}: \quad w_{n+1} + w_n + w_{n-1} = \frac{an+b}{w_n} + c, \tag{3.2.1}$$

$$\mathrm{dP_{II}}: \quad w_{n+1} + w_{n-1} = \frac{(an+b)w_n + c}{1-w_n^2}, \tag{3.2.2}$$

$$\mathrm{dP_{III}}: \quad w_{n+1}w_{n-1} = \frac{\alpha w_n^2 + \beta q^{2n} w_n + \gamma q^{4n}}{w_n^2 + \delta w_n + \alpha}, \tag{3.2.3}$$

$$\begin{aligned}\mathrm{dP_{IV}}: \quad & w_{n+1}w_{n-1} + w_n(w_{n+1} + w_{n-1}) \\ &= \frac{-(an+b)w_n^3 + (d - \frac{1}{4}(an+b)^2)w_n^2 + m}{w_n^2 + (an+b)w_n + (c + \frac{1}{4}(an+b)^2)}.\end{aligned} \tag{3.2.4}$$

Indeed, the discrete Painlevé equations reduce to corresponding continuous ones by taking suitable continuous limits. We note that $\mathrm{dP_I}$ and $\mathrm{dP_{II}}$ appear in the theory of two-dimensional quantum gravity.[33-35]. We here summarize known properties of the discrete Painlevé equations.

(1) Lax pairs are known for $\mathrm{dP_I}$, $\mathrm{dP_{II}}$ and $\mathrm{dP_{III}}$ [36].

For example, the Lax pair of $\mathrm{dP_I}$ is given as follows; We consider the following linear equations,

$$h\frac{d}{dh}\Phi_n = L_n(h)\Phi_n, \tag{3.1.5a}$$

$$\Phi_{n+1} = M_n(h)\Phi_n, \tag{3.1.5b}$$

where L_n and M_n are matrices given by

$$L_n(h) = \begin{pmatrix} \kappa & v_2 & 1 \\ h & \lambda & v_3 \\ hv_1 & h & \mu \end{pmatrix}, \tag{3.1.6a}$$

$$M_n(h) = \begin{pmatrix} d_1 & 1 & 0 \\ 0 & 0 & 1 \\ h & 0 & 0 \end{pmatrix}, \tag{3.1.6b}$$

respectively. We note that all the variables in eqs.(3.1.6) depend on n except for the parameter h. Compatibility condition to eqs.(3.1.5) is given by

$$h\frac{d}{dh}M_n(h) = L_{n+1}(h)M_n(h) - M_n(h)L_n(h), \tag{3.1.7}$$

which recovers dP_I (3.1.1) for $v_2 = w_n$. The Lax pairs for other equations are obtained by choosing suitable matrices. It should be noted, however, that we need slightly modified formulation for dP_{III}. Namely, we have to choose the following linear q-difference- difference system instead of eq.(3.1.5),

$$\Phi_n(qh) = L_n(h)\Phi_n(h), \tag{3.1.8a}$$

$$\Phi_{n+1}(h) = M_n(h)\Phi_n(h). \tag{3.1.8b}$$

This implies that dP_{III} has a different nature from other equations. We shall discuss this point in the section 3.4.

(2) Coalescence cascade [17]

It is known for the continuous Painlevé equations that the "lower" equations can be obtained from the "higher" ones through suitable limiting procedure involving dependent variable and free parameters in the equations. The same situation is known for the discrete Painlevé equations.

(3) Bäcklund transformations has been obtained for dP_{II} [37].

These results seems to imply that the discrete Painlevé equations are "good" equations, but also seems not to be sufficient to state it, since almost no result has been obtained on their solutions yet. This observation leads us to the study of solutions of the discrete Painlevé equations, which may be a good check of the validity of the singularity confinement. Moreover, it is expected that their solutions can be expressed in terms of the discrete analogue of special functions, which may be interesting from the viewpoint of the theory of special functions. We show the results of this problem for the case of dP_{II} and dP_{III} in the following sections.

3.3 Solutions of the Discrete Painlevé-II Equation

3.3.a Solutions of $\mathrm{dP_{II}}$

In this section we consider the Casorati determinant solutions of $\mathrm{dP_{II}}$,

$$w_{n+1} + w_{n-1} = \frac{(\alpha n + \beta) w_n + \gamma}{1 - w_n^2}, \tag{3.3.1}$$

where α, β and γ are parameters. Before going to the determinant solutions, let us seek for a simple one. The first step is to consider the discrete Riccati equation,

$$w_{n+1} = \frac{a_n w_n + b_n}{c_n w_n + d_n}, \tag{3.3.2}$$

which was proposed by Hirota[38]. In this case, it is easy to show that if w_n satisfies the following equation,

$$w_{n+1} = \frac{w_n - (an + b)}{1 + w_n}, \tag{3.3.3}$$

then it gives a solution of eq.(3.3.1) with the constraint $\gamma = -\alpha/2$. In fact, we have from eq.(3.3.3)

$$w_{n-1} = \frac{w_n + (an - a + b)}{1 - w_n}. \tag{3.3.4}$$

Adding eqs.(3.3.3) and (3.3.4), we obtain

$$w_{n+1} + w_{n-1} = \frac{(2an - a + 2b + 2) w_n - a}{1 - w_n^2}, \tag{3.3.5}$$

which is a special case of eq.(3.3.1). Now we put

$$w_n = \frac{F_n}{G_n}. \tag{3.3.6}$$

Substituting eq.(3.3.6) into eq.(3.3.3) and assuming that the numerators and the denominators of both sides of eq.(3.3.3) to be equal, respectively, we have

$$F_{n+1} = F_n - (an + b) G_n, \tag{3.3.7a}$$

$$G_{n+1} = G_n + F_n \ . \tag{3.3.7b}$$

Eliminating F_n from eqs.(3.3.7a) and (3.3.7b), we see that G_n satisfies

$$G_{n+2} - 2G_{n+1} + G_n = -(an+b)G_n \ , \tag{3.3.8}$$

which is considered to be the discrete version of the Airy equation and has a solution given by the discrete analogue of the Airy function. By means of the solution, w_n is expressed as

$$w_n = \frac{G_{n+1}}{G_n} - 1 \ . \tag{3.3.9}$$

It is possible to construct a series of solutions expressed by the discrete analogue of the Airy function. We here give the result, leaving the derivation in the next section. We consider the τ function,

$$\tau_N^n = \begin{vmatrix} A_n & A_{n+2} & \cdots & A_{n+2N-2} \\ A_{n+1} & A_{n+3} & \cdots & A_{n+2N-1} \\ \vdots & \vdots & \ddots & \vdots \\ A_{n+N-1} & A_{n+N+1} & \cdots & A_{n+3N-3} \end{vmatrix} , \tag{3.3.10}$$

where A_n satisfies

$$A_{n+2} = 2A_{n+1} - (pn+q)A_n \ . \tag{3.3.11}$$

We can show that τ_N^n satisfies the following bilinear forms,

$$\tau_{N+1}^{n-1}\tau_{N-1}^{n+2} = \tau_N^{n-1}\tau_N^{n+2} - \tau_N^n\tau_N^{n+1} \ , \tag{3.3.12}$$

$$\tau_{N+1}^{n+2}\tau_N^{n+1} - 2\ \tau_{N+1}^{n+1}\tau_N^{n+2} + (pn+q)\ \tau_{N+1}^n\tau_N^{n+3} = 0 \ , \tag{3.3.13}$$

and

$$\tau_{N+1}^{n+1}\tau_{N-1}^{n+2} = -(p(n+2N)+q)\ \tau_N^{n+2}\tau_N^{n+1} + (pn+q)\ \tau_N^n\tau_N^{n+3} \ . \tag{3.3.14}$$

Applying the dependent variable transformation as

$$w_n = \frac{\tau_{N+1}^{n+1}\tau_N^n}{\tau_{N+1}^n\tau_N^{n+1}} - 1 \ , \tag{3.3.15}$$

we obtain a special case of $\mathrm{dP_{II}}$,

$$w_{n+1} + w_{n-1} = \frac{(2pn + (2N-1)p + 2q)\ w_n - (2N+1)p}{1 - w_n^2}\ . \qquad (3.3.16)$$

We note that eq.(3.3.5) and its solution is recovered by putting $p = a$, $q = b + 1$, and $N = 0$. We also remark that eq.(3.3.15) reduces to eq.(3.1.9) with $\alpha = -(2N+1)$ if we choose $p = -\epsilon^3$, $q = 1$, $w_n = \epsilon w$ and $n = \frac{x}{\epsilon}$, and take the limit of $\epsilon \to 0$.

3.3.b Derivation of the Results

In this section we show that eq.(3.3.10) really gives the solution of eq.(3.3.16) through the dependent variable transformation (3.3.14).

First, let us prove that the τ function (3.3.10) satisfies the bilinear forms (3.3.12)-(3.3.14). For the purpose we show that eqs.(3.3.12)-(3.3.14) reduce to the Jacobi identity or the Plücker relations. Before doing so, let us remind of the the Jacobi identity again. Let D be some determinant, and $D\begin{pmatrix} i \\ j \end{pmatrix}$ be the determinant with the i-th row and the j-th column removed from D. Then the Jacobi identity is given by

$$D\begin{pmatrix} i \\ j \end{pmatrix}\ D\begin{pmatrix} k \\ l \end{pmatrix} - D\begin{pmatrix} i \\ l \end{pmatrix}\ D\begin{pmatrix} k \\ j \end{pmatrix} = D\ D\begin{pmatrix} i & k \\ j & l \end{pmatrix}\ . \qquad (3.3.17)$$

It is easily seen that eq.(3.3.12) is nothing but the Jacobi identity. In fact, taking τ_{N+1}^{n-1} as D, and putting $i = j = 1$, $k = l = N+1$, we find that eq.(3.3.12) reduces to eq.(3.3.17). Hence it is shown that eq.(3.3.10) satisfies eq.(3.3.12).

Let us next prove eq.(3.3.13). Notice that τ_N^n is rewritten as

$$\tau_N^n = \begin{vmatrix} A_n & \cdots & A_{n+2N-4} & 2A_{n+2N-3} - (p(n+2N-4)+q)A_{n+2N-4} \\ A_{n+1} & \cdots & A_{n+2N-3} & 2A_{n+2N-2} - (p(n+2N-3)+q)A_{n+2N-3} \\ \vdots & \cdots & \vdots & \vdots \\ A_{n+N-1} & \cdots & A_{n+3N-5} & 2A_{n+3N-4} - (p(n+3N-5)+q)A_{n+3N-5} \end{vmatrix}$$
$$= \begin{vmatrix} A_n & \cdots & A_{n+2N-4} & 2A_{n+2N-3} \\ A_{n+1} & \cdots & A_{n+2N-3} & 2A_{n+2N-2} - pA_{n+2N-3} \\ \vdots & \cdots & \vdots & \vdots \\ A_{n+N-1} & \cdots & A_{n+3N-5} & 2A_{n+3N-4} - (N-1)pA_{n+3N-5} \end{vmatrix}$$

$$= \begin{vmatrix} A_n & 2A_{n+1} & \cdots & 2A_{n+2N-3} \\ A_{n+1} & 2A_{n+2} - pA_{n+1} & \cdots & 2A_{n+2N-2} - pA_{n+2N-3} \\ \vdots & \vdots & \cdots & \vdots \\ A_{n+N-1} & 2A_{n+N} - (N-1)pA_{n+N-1} & \cdots & 2A_{n+3N-4} - (N-1)pA_{n+3N-5} \end{vmatrix}$$

$$= 2^{N-1} \begin{vmatrix} B_n^{(0)} & A_{n+1} & \cdots & A_{n+2N-3} \\ \vdots & \vdots & \cdots & \vdots \\ B_n^{(N-1)} & A_{n+N} & \cdots & A_{n+3N-4} \end{vmatrix} , \tag{3.3.18}$$

where $B_n^{(k)}$, $k = 0, 1, \cdots$, are given by

$$B_n^{(0)} = A_n, \qquad B_n^{(k)} = A_{n+k} + \frac{kp}{2} B_n^{(k-1)} \quad \text{for } k \geq 1 \ . \tag{3.3.19}$$

Similarly, we have

$$(pn+q)\tau_N^n = 2^{N-1} \begin{vmatrix} A_{n+1} & B_{n+2}^{(0)} & A_{n+3} & \cdots & A_{n+2N-3} \\ A_{n+2} & B_{n+2}^{(1)} & A_{n+4} & \cdots & A_{n+2N-2} \\ \vdots & \vdots & \vdots & \cdots & \cdots \\ A_{n+N} & B_{n+2}^{(N-1)} & A_{n+N+2} & \cdots & A_{n+3N-4} \end{vmatrix} . \tag{3.3.20}$$

Let us introduce the notations as

$$\text{“}j\text{”} = \begin{pmatrix} A_{n+j} \\ A_{n+j+1} \\ \vdots \end{pmatrix} , \qquad \text{“}j'\text{”} = \begin{pmatrix} B_{n+j}^{(0)} \\ B_{n+j}^{(1)} \\ \vdots \end{pmatrix} , \qquad \phi = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix} . \tag{3.3.21}$$

For example, τ_N^n and $(pn+q)\tau_N^n$ are rewritten by

$$\begin{aligned} \tau_N^n &= |0, 2, \cdots, 2N-2| = |0, 2, \cdots, 2N-2, \phi| \\ &= 2^{N-1}|0', 1, 3, \cdots, 2N-3| , \\ (pn+q)\tau_N^n &= 2^{N-1}|1, 2', 3, 5, \cdots, 2N-3| = 2^{N-1}|1, 2', 3, \cdots, 2N-3, \phi| , \end{aligned}$$

respectively. Now consider the following identity of $(2N+2) \times (2N+2)$ determinant,

$$0 = \left| \begin{array}{cc:ccc:ccc:cc} -1 & 0' & 1 & \cdots & 2N-5 & & \emptyset & & 2N-3 & \phi \\ \hdashline -1 & 0' & & \emptyset & & 1 & \cdots & 2N-5 & 2N-3 & \phi \end{array} \right| . \tag{3.3.22}$$

Applying the Laplace expansion on the right hand side of eq.(3.3.22), we obtain

$$\begin{aligned} 0 =& |-1,0',1,\cdots,2N-5| \times |1,\cdots,2N-5,2N-3,\phi| \\ &-|-1,1,\cdots,2N-5,2N-3| \times |0',1,\cdots 2N-5,\phi| \\ &+|-1,1,\cdots,2N-5,\phi| \times |0',1,\cdots,2N-5,2N-3| \, , \end{aligned} \tag{3.3.23}$$

which is nothing but the special case of the Plücker relations. Equation (3.3.23) is rewritten by using eqs.(3.3.18) and (3.3.20) as

$$0 = (p(n-2)+q)\ \tau_N^{n-2}\ \tau_{N-1}^{n+1} - 2\ \tau_N^{n-1}\ \tau_{N-1}^{n} + \tau_N^{n}\ \tau_{N-1}^{n-1} \, , \tag{3.3.24}$$

which is essentially the same as eq.(3.3.13).

We next prove that eq.(3.3.14) holds. We have the following equation similar to eqs.(3.3.18) and (3.3.20);

$$(p(n+2N)+q)\tau_N^{n+2} = -|2,\cdots,2N-2,2N+2| + 2^{N-1}|2',3,\cdots,2N-3,2N+1| \, . \tag{3.3.25}$$

Then the right hand side of eq.(3.3.14) is rewritten as

$$\begin{aligned} &|2,\cdots,2N-2,2N+2| \times |1,3,\cdots,2N-1| \\ &-2^{N-1}|2',3,\cdots,2N-3,2N+1| \times |1,3,\cdots,2N-1| \\ &+2^{N-1}|1,2',3,\cdots,2N-5,2N-3| \times |3,5,\cdots 2N-1,2N+1| \, . \end{aligned} \tag{3.3.26}$$

From the identity

$$\begin{aligned} 0 =& \left|\begin{array}{cc:ccc:c:cc} 1 & 2' & 3 & \cdots & 2N-3 & \emptyset & 2N-1 & 2N+1 \\ \hdashline 1 & 2' & & \emptyset & & 3 \ \cdots \ 2N-3 & 2N-1 & 2N+1 \end{array}\right| \\ =& |1,2',3,\cdots,2N-3| \times |3,5,\cdots 2N-5,2N-1,2N+1| \\ &- |1,3,\cdots 2N-3,2N-1| \times |2',3,\cdots,2N-3,2N+1| \\ &+ |1,3,\cdots,2N-3,2N+1| \times |2',3,\cdots 2N-3,2N-1| \, , \end{aligned} \tag{3.3.27}$$

the second and third terms of eq.(3.3.26) yield

$$\begin{aligned}
&-2^{N-1}|1,3,\cdots,2N-3,2N+1| \times |2',3,\cdots,2N-3,2N-1| \\
=&-|1,3,\cdots,2N-3,2N+1| \times |2,4,\cdots,2N-2,2N| \ . \qquad (3.3.28)
\end{aligned}$$

Hence, eq.(3.3.14) is reduced to

$$\begin{aligned}
&|2,4,\cdots,2N-2,2N+2| \times |1,3,\cdots,2N-3,2N-1| \\
&-|1,3,\cdots,2N-3,2N+1| \times |2,4,\cdots,2N-2,2N| \\
&=|1,3,\cdots,2N-1,2N+1| \times |2,4,\cdots,2N-2| \ , \qquad (3.3.29)
\end{aligned}$$

which is again nothing but the Jacobi identity (3.3.17). In fact, taking $D = |1,3,\cdots,2N-1,2N+1|$, $i=1$, $j=N+1$, $k=N$ and $l=N+1$, we see that eq.(3.3.17) is the same as eq.(3.3.29). This completes the proof that the τ function (3.3.10) satisfies the bilinear forms (3.3.12)-(3.3.14).

Finally, let us derive eq.(3.3.16) from the bilinear forms (3.3.12)-(3.3.14). We introduce the dependent variables by

$$v_N^n = \frac{\tau_{N+1}^n}{\tau_N^n} \ , \qquad u_N^n = \frac{\tau_N^n \tau_N^{n+3}}{\tau_N^{n+1}\tau_N^{n+2}} \ . \qquad (3.3.30)$$

Then eqs. (3.3.12)-(3.3.14) are rewritten as

$$v_N^{n-1} = v_{N-1}^{n+2}\left(1 - \frac{1}{u_N^{n-1}}\right) \ , \qquad (3.3.31)$$

$$v_N^{n+2} - 2v_N^{n+1} + (pn+q)u_N^n v_N^n = 0 \ , \qquad (3.3.32)$$

$$v_N^{n+1} = v_{N-1}^{n+2}\left(-(p(n+2N)+q) + (pn+q)u_N^n\right) \ , \qquad (3.3.33)$$

respectively. Eliminating u_N and v_{N-1} from eqs.(3.3.31)-(3.3.33) and introducing w_n by

$$w_n = \frac{v_N^{n+1}}{v_N^n} - 1 \ , \qquad (3.3.34)$$

we obtain eq.(3.3.16).

3.4 Solutions of the Discrete Painlevé-III Equation

3.4.a Solutions of $\mathrm{dP_{III}}$

In this section we investigate the solutions of $\mathrm{dP_{III}}$,

$$w_{n+1}w_{n-1} = \frac{\alpha w_n^2 + \beta\lambda^n w_n + \gamma\lambda^{2n}}{w_n^2 + \delta w_n + \alpha} , \tag{3.4.1}$$

where α, β, γ, δ, and λ are parameters. Similar to the previous section, let us start from the discrete Riccati equation. It is shown that if w_n satisfies

$$w_{n+1} = -\frac{aw_n + \lambda^n}{w_n + d} , \tag{3.4.2}$$

with a constraint

$$d + \frac{a}{\lambda} = \beta , \tag{3.4.3}$$

then w_n gives the particular solution of eq.(3.4.1). Putting

$$w_n = \frac{F_n}{G_n} , \tag{3.4.4}$$

we have

$$w_n = \frac{G_{n+1}}{G_n} + d , \tag{3.4.5}$$

$$G_{n+2} + (a-d)G_{n+1} + (\lambda^n - ad)G_n = 0 . \tag{3.4.6}$$

By the analogy with the continuous case, it is expected that eq.(3.4.6) is a discrete version of Bessel equation, although it is not trivial to see the correspondence between eq.(3.4.6) and the Bessel equation. Hence we should consider another parameterization to identify the linearized equation with the discrete Bessel equation. Instead of eq.(3.4.6), we take

$$J_\nu(n+2) - (q^\nu - q^{-\nu})J_\nu(n+1) + \{1 + (1-q)^2 q^{2n}\}J_\nu(n) = 0 . \tag{3.4.7}$$

We can show that if $J_\nu(n)$ satisfies eq.(3.4.7), then

$$w_n = \frac{J_\nu(n+1)}{J_\nu(n)} - q^\nu \tag{3.4.8}$$

gives the solution of $\mathrm{dP_{III}}$ (3.4.1) with the parameters

$$\alpha = -1,\ \beta = (q^{\nu} - q^{-\nu-2})(1-q)^2,\ \gamma = \frac{(1-q)^4}{q^2},\ \delta = q^{\nu} - q^{-\nu}, \tag{3.4.9}$$

where $q = \lambda^2$. Continuous limit of eq.(3.4.7) is given by putting $q = 1+\epsilon$, $n = \frac{z}{\epsilon}$ and taking the limit of $\epsilon \to 0$ to yield

$$\frac{d^2 J_\nu}{dz^2} + (\mathrm{e}^{2z} - \nu^2) J_\nu = 0\ . \tag{3.4.10}$$

Equation (3.4.10) reduces to the Bessel equation if we put $x = \mathrm{e}^z$. In this sense, eq.(3.4.7) is regarded as the discrete Bessel equation.

It is possible to extend this result to $N \times N$ determinant solution. Similar to the previous section, we here give only the results, leaving the derivation in the next section. We consider the following τ function,

$$\tau_N^\nu(n) = \begin{vmatrix} J_\nu(n) & J_\nu(n+2) & \cdots & J_\nu(n+2N-2) \\ \vdots & \vdots & \cdots & \vdots \\ J_\nu(n+N-1) & J_\nu(n+N+1) & \cdots & J_\nu(n+3N-3) \end{vmatrix}, \tag{3.4.11}$$

where $J_\nu(n)$ is a solution of eq.(3.4.7). We note that this τ function has a similar structure to that of $\mathrm{dP_{II}}$. We can show that $\tau_N^\nu(n)$ satisfies the following five bilinear forms,

$$\begin{aligned} &\tau_{N+1}^\nu(n)\tau_N^{\nu+1}(n+1) - q^{-\nu-N}\tau_{N+1}^\nu(n+1)\tau_N^{\nu+1}(n) \\ &\quad = -(1-q)q^{n+2N}\tau_{N+1}^{\nu+1}(n)\tau_N^\nu(n+1)\ , \end{aligned} \tag{3.4.12}$$

$$\begin{aligned} &\tau_{N+1}^{\nu+1}(n)\tau_N^\nu(n) - q^{\nu-N+1}\tau_{N+1}^{\nu+1}(n+1)\tau_N^\nu(n) \\ &\quad = (1-q)q^{n+2N}\tau_{N+1}^\nu(n)\tau_N^{\nu+1}(n+1)\ , \end{aligned} \tag{3.4.13}$$

$$\begin{aligned} &\tau_N^\nu(n)\tau_N^{\nu+1}(n+3) - q^{-\nu-N}\tau_{N+1}^\nu(n+1)\tau_N^{\nu+1}(n+2) \\ &\quad = -(1-q)q^{n}\tau_{N+1}^{\nu+1}(n)\tau_N^\nu(n+3)\ , \end{aligned} \tag{3.4.14}$$

$$\begin{aligned} &\tau_N^{\nu+1}(n)\tau_N^\nu(n+3) - q^{\nu-N+1}\tau_{N+1}^{\nu+1}(n+1)\tau_N^\nu(n+2) \\ &\quad = (1-q)q^{n}\tau_{N+1}^\nu(n)\tau_N^{\nu+1}(n+3)\ , \end{aligned} \tag{3.4.15}$$

$$\begin{aligned} &\tau_{N+1}^\nu(n+2)\tau_N^{\nu+1}(n+1) - q^{2N}(q^{\nu-N} + q^{\nu+N})\tau_{N+1}^\nu(n+1)\tau_{N+1}^{\nu+1}(n+2) \\ &+ q^{4N}\{1 + (1-q)^2 q^{2n}\}\tau_{N+1}^\nu(n)\tau_N^\nu(n+3) = 0\ . \end{aligned} \tag{3.4.16}$$

Applying the dependent variable transformation

$$w_n = \frac{\tau_{N+1}^{\nu}(n+1)\tau_N^{\nu+1}(n)}{\tau_{N+1}^{\nu}(n)\tau_N^{\nu+1}(n+1)} - q^{\nu+N}\ , \tag{3.4.17}$$

we obtain the special case of $\mathrm{dP_{III}}$

$$w_{n+1}w_{n-1} = \frac{\alpha w_n^2 + \beta q^{2n} w_n + \gamma q^{4n}}{w_n^2 + \delta w_n + \alpha} \tag{3.4.18}$$

with the parameters

$$\alpha = -q^{4N}, \quad \beta = \left(q^{\nu+N} - q^{-\nu-N-2}\right) q^{8N}(1-q)^2,$$

$$\gamma = q^{2(6N-1)}\,(1-q)^4, \quad \delta = \left(q^{\nu-N} - q^{-\nu+N}\right) q^{2N}.$$

3.4.b Derivation of the Results

In this section we show that the τ function (3.4.11) actually gives the solution of eq.(3.4.18) through the dependent variable transformation (3.4.17). We first note that eq.(3.4.18) is factorized as

$$w_{n+1}w_{n-1} = -\frac{\left(q^{2N}w_n + (1-q)^2 q^{\nu+7N+2n}\right)\left(q^{2N}w_n - (1-q)^2 q^{-\nu+5N-2+2n}\right)}{\left(w_n + q^{\nu+N}\right)\left(w_n - q^{-\nu+3N}\right)}. \tag{3.4.19}$$

Substituting eq.(3.4.17) into eq.(3.4.19) and combining suitable factors, we see that eq.(3.4.18) is decomposed into the bilinear forms (3.4.12-16).

We next prove that eqs.(3.4.12-16) holds. It is possible to reduce them to the Plücker relation. Let us first construct the shift operator of the τ function,

$$\tau_N^{\nu}(n) = \begin{vmatrix} J_\nu(n) & J_\nu(n+1) & \cdots & J_\nu(n+N-1) \\ \vdots & \vdots & \cdots & \vdots \\ J_\nu(n+2N-2) & J_\nu(n+2N-1) & \cdots & J_\nu(n+3N-3) \end{vmatrix}, \tag{3.4.20}$$

where $J_\nu(n)$ is the solution of the discrete Bessel equation (3.4.7) and satisfies the following contiguity relations,

$$J_\nu(n+1) - q^{-\nu}J_\nu(n) = -(1-q)q^n J_{\nu+1}(n), \tag{3.4.21}$$

$$J_\nu(n+1) - q^{\nu}J_\nu(n) = (1-q)q^n J_{\nu-1}(n). \tag{3.4.22}$$

Using eqs.(3.4.21) and (3.4.22), it is possible to construct the shift operators for "one-skipped direction". For this purpose, we introduce the notation,

$$\tau_N^\nu(n) = |0_\nu, 1_\nu, \cdots, N-2_\nu, N-1_\nu|, \tag{3.4.23}$$

where k_ν stands for the column which starts from $J_\nu(n+k)$. Now let us construct the shift operators. By adding $(i+1)$-th column multiplied by $-q^{-\nu}$ to i-th column and using eq.(3.4.21) for $i = 1 \sim N-1$, we have

$$\begin{aligned}
\tau_N^\nu(n) =& \{-(1-q)\}^{N-1} q^{\frac{(2n+N-2)(N-1)}{2}} \times \\
&\begin{vmatrix} J_{\nu+1}(n) & \cdots & J_{\nu+1}(n+N-2) & J_\nu(n+N-1) \\ q^2 J_{\nu+1}(n+2) & \cdots & q^2 J_{\nu+1}(n+N) & J_\nu(n+N+1) \\ \vdots & \cdots & \vdots & \vdots \\ q^{2N-2} J_{\nu+1}(n+2N-2) & \cdots & q^{2N-2} J_{\nu+1}(n+3N-4) & J_\nu(n+3N-3) \end{vmatrix} \\
=& \{-(1-q)\}^{N-1} q^{\frac{(2n+3N-2)(N-1)}{2}} \times \\
&\begin{vmatrix} J_{\nu+1}(n) & \cdots & J_{\nu+1}(n+N-2) & J_\nu(n+N-1) \\ J_\nu(n+2) & \cdots & J_{\nu+1}(n+N) & q^{-2} J_\nu(n+N+1) \\ \vdots & \cdots & \vdots & \vdots \\ J_{\nu+1}(n+2N-2) & \cdots & J_{\nu+1}(n+3N-4) & q^{-2N+2} J_\nu(n+3N-3) \end{vmatrix} \\
=& \{-(1-q)\}^{N-1} q^{\frac{(2n+3N-2)(N-1)}{2}} \quad |0_{\nu+1}, \cdots, N-2_{\nu+1}, N-1'_\nu| ,
\end{aligned}$$

where k'_ν stands for

$$k'_\nu = \begin{pmatrix} J_\nu(n+k) \\ q^{-2} J_\nu(n+k+2) \\ q^{-4} J_\nu(n+k+4) \\ \vdots \end{pmatrix} . \tag{3.4.24}$$

Hence we obtain

$$|0_\nu, \cdots, N-2_\nu, N-1'_{\nu-1}| = \{-(1-q)\}^{-N+1} q^{-\frac{(2n+3N-2)(N-1)}{2}} \quad \tau_N^{\nu-1}(n). \tag{3.4.25}$$

We next construct another shift operator. We add $(i+1)$-th column multiplied by $q^{-\nu}$ to i-th column for $i = 1 \sim N-2$, subtract $(N-1)$-th column from N-th column multiplied by $q^{-\nu}$, and divide N-th column by $(1-q)q^{n+N-1}$. Then we have

$$\tau_N^\nu(n) = \{-(1-q)\}^{N-2} q^{\frac{(2n+N-3)(N-2)}{2}+(2+4+\cdots+2N-2)} \times \begin{vmatrix} \cdots & J_{\nu+1}(n+N-3) & J_\nu(n+N-2) & J_\nu(n+N-1) \\ \cdots & J_{\nu+1}(n+N-1) & q^{-2}J_\nu(n+N) & q^{-2}J_\nu(n+N+1) \\ \cdots & \vdots & \vdots & \vdots \\ \cdots & J_{\nu+1}(n+3N-5) & q^{-2N+2}J_\nu(n+3N-4) & q^{-2N+2}J_\nu(n+3N-3) \end{vmatrix}$$

$$= \{-(1-q)\}^{N-2} q^{\frac{(2n+N-3)(N-2)}{2}+N(N-1)+\nu} (1-q)q^{n+N-1} \times \begin{vmatrix} \cdots & J_{\nu+1}(n+N-3) & J_\nu(n+N-2) & J_{\nu+1}(n+N-2) \\ \cdots & J_{\nu+1}(n+N-1) & q^{-2}J_\nu(n+N) & J_{\nu+1}(n+N) \\ \cdots & \vdots & \vdots & \vdots \\ \cdots & J_{\nu+1}(n+3N-5) & q^{-2N+2}J_\nu(n+3N-4) & J_{\nu+1}(n+3N-4) \end{vmatrix}$$

$$= \{-(1-q)\}^{N-1} q^{\frac{(2n+3N-2)(N-1)}{2}+\nu} \times |0_{\nu+1}, \cdots, N-2_{\nu+1}, N-2'_\nu|.$$

Hence we get

$$|0_\nu, \cdots, N-2_\nu, N-2'_{\nu-1}| = \{-(1-q)\}^{-N+1} q^{-\frac{(2n+3N-2)(N-1)}{2}-\nu+1} \quad \tau_N^{\nu-1}(n) \qquad (3.4.26)$$

By using eq.(3.4.22) instead of (3.4.21), similar calculations give the following shift operators;

$$|0_\nu, \cdots, N-2_\nu, N-1'_{\nu+1}| = \{-(1-q)\}^{-N+1} q^{-\frac{(2n+3N-2)(N-1)}{2}} \quad \tau_N^{\nu+1}(n), \qquad (3.4.27)$$

$$|0_\nu, \cdots, N-2_\nu, N-2'_{\nu+1}| = \{-(1-q)\}^{-N+1} q^{-\frac{(2n+3N-2)(N-1)}{2}+\nu+1} \quad \tau_N^{\nu+1}(n) \qquad (3.4.28)$$

Now we show that τ function (3.4.20) satisfies the bilinear forms (3.4.12)-(3.4.15). Here we note that the shift operators in "two-skipped" direction are needed to derive eq.(3.4.16). First we derive eq.(3.4.12). For the purpose we consider the identity of determinant,

$$0 = \left| \begin{array}{c|ccc|ccc|ccc} 0 & 1 & \cdots & N-1 & & \emptyset & & N & N'_{\nu-1} & \phi_1 \\ \hdashline 0 & & \emptyset & & 1 & \cdots & N-1 & N & N'_{\nu-1} & \phi_1 \end{array} \right|, \tag{3.4.29}$$

where

$$\phi_1 = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix}. \tag{3.4.30}$$

Applying the Laplace expansion on the right hand side, we have

$$\begin{aligned} 0 &= |0, \cdots, N-1, N'_{\nu-1}| \times |1, \cdots, N-1, N, \phi_1| \\ &- |0, \cdots, N-1, N| \times |1, \cdots, N-1, N'_{\nu-1}, \phi_1| \\ &- |1, \cdots, N-1, N, N'_{\nu-1}| \times |0, \cdots, N-1, \phi_1| \end{aligned}$$

This identity is rewritten in terms of τ by using eqs.(3.4.25-28) as

$$\begin{aligned} &\tau^{\nu-1}_{N+1}(n)\tau^{\nu}_{N}(n+1) - q^{-\nu-N+1}\tau^{\nu-1}_{N+1}(n+1)\tau^{\nu}_{N}(n) \\ &= -(1-q)q^{n+2N}\tau^{\nu}_{N+1}(n)\tau^{\nu-1}_{N}(n+1) , \end{aligned} \tag{3.4.31}$$

which is essentially the same as eq.(3.4.12). Equation (3.4.13) is derived from the same identity as eq.(3.4.29) except that $N'_{\nu-1}$ is replaced by $N'_{\nu+1}$.

We next derive eq.(3.4.14). We consider the identity of determinant,

$$0 = \left| \begin{array}{c|ccc|ccc|ccc} 0 & 1 & \cdots & N-2 & & \emptyset & & N & N'_{\nu-1} & \phi_2 \\ \hdashline 0 & & \emptyset & & 1 & \cdots & N-2 & N & N'_{\nu-1} & \phi_2 \end{array} \right|, \tag{3.4.32}$$

where

$$\phi_2 = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}.$$

Applying the Laplace expansion on the right hand side of eq.(3.4.32), we have

$$\begin{aligned} 0 &= |0, \cdots, N-1, N'_{\nu-1}| \times |1, \cdots, N-1, N, \phi_2| \\ &- |0, \cdots, N-1, N| \times |1, \cdots, N-1, N'_{\nu-1}, \phi_2| \\ &- |0, \cdots, N-1, \phi_2| \times |1, \cdots, N-1, N, N'_{\nu-1}|, \end{aligned}$$

which can be rewritten in terms of τ again. Here we give a remark on the treatise of the determinant including the column ϕ_2. For example, let us consider $|1,\cdots,N-1,N'_{\nu-1},\phi_2|$. Expanding by $(N+1)$-th column, we have

$$\begin{aligned}
&|1,\cdots,N-1,N'_{\nu-1},\phi_2|\\
=&\begin{vmatrix} J_\nu(n+1) & \cdots & J_\nu(n+N-1) & J_{\nu-1}(n+N) & 1\\ J_\nu(n+3) & \cdots & J_\nu(n+N+1) & q^{-2}J_{\nu-1}(n+N+2) & 0\\ \vdots & \cdots & \vdots & \vdots & \vdots\\ J_\nu(n+2N+1) & \cdots & J_\nu(n+3N-1) & J_{\nu-1}(n+3N) & 0\end{vmatrix}\\
=&(-1)^{N+2}\begin{vmatrix} J_\nu(n+3) & \cdots & J_\nu(n+N+1) & q^{-2}J_{\nu-1}(n+N+2)\\ \vdots & \cdots & \vdots & \vdots\\ J_\nu(n+2N+1) & \cdots & J_\nu(n+3N-1) & q^{-2N}J_{\nu-1}(n+3N)\end{vmatrix}\\
=&q^{-2}(-1)^{N+2}\quad |3,4,\cdots,N+1,N+2'_{\nu-1}|,
\end{aligned}$$

Namely, we need additional factor $(-1)^{N+2}\ q^{-2}$. Taking this remark into account, we obtain

$$\begin{aligned}
&\tau_N^{\nu-1}(n)\tau_N^{\nu}(n+3)-q^{-\nu-N+1}\tau_{N+1}^{\nu-1}(n+1)\tau_N^{\nu}(n+2)\\
&=-(1-q)q^n\tau_{N+1}^{\nu}(n)\tau_N^{\nu-1}(n+3)\ ,
\end{aligned}\tag{3.4.33}$$

which is essentially the same as eq.(3.4.14). Similarly, eq.(3.4.15) is derived from the same identity as (3.4.32) except that the column $N'_{\nu-1}$ is replaced by $N'_{\nu+1}$.

Finally we derive eq.(3.4.16). In this case, we have to consider the shift operators in "two-skipped" direction. We consider the τ function,

$$\begin{aligned}
\tau_N^\nu(n)=&\begin{vmatrix} J_\nu(n) & J_\nu(n+2) & \cdots & J_\nu(n+2N-2)\\ \vdots & \vdots & \cdots & \vdots\\ J_\nu(n+N-1) & J_\nu(n+N+1) & \cdots & J_\nu(n+3N-3)\end{vmatrix}\\
=&\ |0,2,\cdots,2N-4,2N-2|\ .
\end{aligned}\tag{3.4.34}$$

We use the contiguity relation for $J_\nu(n)$,

$$J_\nu(n+2)-(1-q)(1+q^{2\nu})q^{n+\nu+1}J_{\nu+1}(n+1)$$

$$-q^{2\nu}\{1+(1-q)^2q^{2n}\}J_\nu(n)=0\ , \tag{3.4.35}$$

and a technique similar to the derivation of eq.(3.4.25). After some tedious calculations, we obtain the shift operators,

$$\begin{aligned}&|0''_{\nu-1},1,3,\cdots,2N-3|\\&=\{(1-q)q^{n+3\nu-4}\}^{-N+1}q^{-\frac{N(N-1)}{2}}\prod_{k=0}^{N-1}\frac{1}{1+q^{2\nu-2+2k}}\quad\tau_N^{\nu-1}(n),\end{aligned} \tag{3.4.36}$$

$$\begin{aligned}&|1,2''_{\nu-1},3,5,\cdots,2N-3|\\&=\{(1-q)q^{n+3\nu-4}\}^{-N+1}q^{-\frac{N(N-1)}{2}}\prod_{k=0}^{N-1}\frac{1}{1+q^{2\nu-2+2k}}\times\\&\quad q^{2\nu}\{1+(1-q)^2q^{2n}\}\quad\tau_N^{\nu-1}(n),\end{aligned} \tag{3.4.37}$$

where

$$k''_\nu=\begin{pmatrix}K_\nu(n)_0\\K_\nu(n)_1\\\vdots\end{pmatrix}, \tag{3.4.38}$$

and $K_\nu(n)_i$ is defined by the following recursion relation,

$$\begin{cases}K_\nu(n)_i=\dfrac{1}{1+q^{-2\nu+2i}}\left(q^iJ_\nu(n+i)+(1-q^{2i})q^{-\nu-1}K_\nu(n)_{i-1}\right),\quad\text{for } i\ge1,\\[2mm] K_\nu(n)_0=\dfrac{1}{1+q^{-2\nu}}J_\nu(n)\ .\end{cases} \tag{3.4.39}$$

We now consider the identity of determinant,

$$0=\left|\begin{array}{cc:ccc:ccc:cc}0&1''_{\nu-1}&2&4&\cdots 2N-4&&\emptyset&&2N-2&\phi_1\\\hdashline 0&1''_{\nu-1}&&\emptyset&&2&4\ \cdots&2N-4&2N-2&\phi_1\end{array}\right|. \tag{3.4.40}$$

Applying the Laplace expansion to the right hand side of eq.(3.4.40) and using eqs.(3.4.36) and (3.4.37), we get

$$\begin{aligned}&\tau_{N+1}^\nu(n+2)\tau_N^{\nu+1}(n+1)-q^{2\nu+2n}\{1+(1-q)^2q^{2n}\}\tau_{N+1}^\nu(n)\tau_N^{\nu+1}(n+3)\\&\qquad=(1-q)q^{n+N+\nu+1}(q^{2N}+q^{2\nu})\tau_{N+1}^{\nu+1}(n+1)\tau_N^\nu(n+2).\end{aligned} \tag{3.4.41}$$

Combining eqs.(3.4.41) and (3.4.12), we obtain eq.(3.4.16). Thus we have proved that the τ function (3.4.11) really satisfies the bilinear forms (3.4.12-16).

3.5. q-Difference Analogue of the Painlevé-III Equation

We have discussed the solution of dP_{III}, and shown that it is expressed by the discrete Bessel fucntion. It should be noted, however, that eq.(3.4.7) is essentially the same as the q-Bessel equation, which is given by

$$J_\nu(q^2x) - (q^\nu - q^{-\nu})J_\nu(qx) + \{1 + (1-q)^2x^2\}J_\nu(x) = 0 \ , \tag{3.5.1}$$

or

$$\left\{qx^2\delta_x^2 + (1 - [\nu] - [-\nu])x\ \delta_x + (x^2 + [\nu][-\nu])\right\}\ J_\nu(x) = 0 \ . \tag{3.5.2}$$

In fact, eq.(3.4.7) yields eq.(3.5.1) by the replacement $q^n \to x$. This fact seems to imply that dP_{III} is a q-discrete system rather than a discrete system in essence. More precisely, let us consider the following equation instead of dP_{III} (3.4.1),

$$w(qx)w(q^{-1}x) = \frac{\alpha w(x)^2 + \beta x^2 w(x) + \gamma x^4}{w(x)^2 + \delta w(x) + \alpha} \ . \tag{3.5.3}$$

The solutions of eq.(3.5.3) is expressed by

$$\tau_N^\nu(n) = \begin{vmatrix} J_\nu(x) & J_\nu(q^2x) & \cdots & J_\nu(q^{2N-2}x) \\ \vdots & \vdots & \cdots & \vdots \\ J_\nu(q^{N-1}x) & J_\nu(q^{N+1}x) & \cdots & J_\nu(q^{3N-3}x) \end{vmatrix} , \tag{3.5.4}$$

through the dependent variable transformation,

$$w_n = \frac{\tau_{N+1}^\nu(qx)\tau_N^{\nu+1}(x)}{\tau_{N+1}^\nu(x)\tau_N^{\nu+1}(qx)} - q^{\nu+N} \ . \tag{3.5.5}$$

Here, $J_\nu(x)$ is the q-Bessel function of degree ν.

We have mentioned in the section 3.2 that the auxiliary linear system for dP_{III} includes a q-difference equation. It is easily shown that the linear system for eq.(3.5.3) is completely written as a q-difference system,

$$\Phi(x;qh) = L(x;h)\Phi(x;h), \tag{3.5.6a}$$

$$\Phi(qx;h) = M(x;h)\Phi(x;h), \tag{3.5.6b}$$

whose compatibility condition yields

$$M(x;qh)L(x;h) = L(qx;h)M(x;h). \tag{3.5.7}$$

In fact, if we choose the same matrices L and M as those in [36], we obtain eq.(3.5.3). Moreover, putting $w = (1-q)x\ u$ and taking the limit $q \to 1$, eq.(3.5.3) reduces to $\mathrm{P_{III}}$,

$$\frac{d^2u}{dx^2} = \frac{1}{u}\Big(\frac{du}{dx}\Big)^2 - \frac{1}{x}\frac{du}{dx} + \frac{1}{x}(\alpha u^2 + \beta) + \gamma u^3 + \frac{\delta}{u},$$

where $\alpha = 2\nu - 2N$, $\beta = 2\nu + 2N + 2$, $\gamma = 1$, $\delta = -1$. These facts imply that eq.(3.5.3) can be one candidate for q-difference analogue of $\mathrm{P_{III}}$.

Chapter4. Concluding Remarks

In this thesis, we have discussed on two subjects, one is q-discretization of the two-dimensional Toda equations, another is the solutions of the discrete Painlevé equations. We first summarize the main results.

(1) We have succeeded to extend the 2DTL and 2DTM equations to the q-discrete system by using a procedure based on the direct method. We have also discussed reductions of these equations to cylindrical systems. In particular, we have shown that the q-discrete cylindrical Toda lattice equation admits the solution expressed by a determinant whose entries are the q-Bessel functions. Moreover, we have derived the Bäcklund transformation and the Lax pair for the q-2DTM equations. These results may imply the validity of our method.

(2) We have derived the particular solutions of the discrete Painlevé II and III equations, the former of which is expressed by the determinant whose entries are the discrete analogue of the Airy function, and the latter of which is expressed by the determinant whose entries are the discrete analogues of the Bessel function. These results imply the validity of the singularity confinement method. It is noted that, for the case of the discrete Painlevé-III equation, the solution is essentially the same as the q-Bessel function. From such an observation, we have proposed q-difference analogue of the Painlevé III equation.

Let us next give two remarks. First, it is expected that the q-2DTL and the q-2DTM equation are related with the quantum groups. However, the direct relationship between them has not been found yet. Recently, Nagatomo and Koga have constructed the q-difference analogue of the Euler-Poisson-Darboux(EPD) equation[39]. The EPD equation is regarded as a two-dimensional version of the hypergeometric equation, and closely related to the 2DTM equation. Moreover, $SL(2,C)$ acts on the EPD equation as the transformation group. They have shown that the quantum group $A_q(SL(2,C))$ acts on the q-EPD equation and the q-2DTM equation is closely related to it. At present, this relationship between the q-2DTM equation and the quantum group seems not to be direct.

Secondly, it was expected that the τ function of the discrete Painlevé equation satisfy the discrete Toda molecule equation. However, our result shows it is not. Namely, the structure of the τ function is asymmetric, while that of the discrete Toda molecule equation is symmetric. The similar situation occurs for the case of the discrete Painlevé III equation. We do not know yet whether this structure is essential or not.

Before concluding, we give some comments on the future problems.

(1) The physical meanings of the q-2DTL and q-2DTM equations are still open. And the relationship between quantum groups deserves further research.

(2) It is expected that other discrete Painlevé equations admit the solution expressed by the determinants whose entries are discrete or q-difference analogue of special functions. Moreover, it may be expected that they also admit rational solutions.

(3) Quite recently, it has been revealed that the discrete Painlevé equations are not unique[40,41]. Namely, several different discrete equations have been obtained which reduces to the same Painlevé equations in the continuous limit and passes the singularity confinement. Their properties or solutions are still unknown.

(4) As mentioned in the section 3.1, the Painlevé equations can be rewritten in the framework of the Hamilton formalism in which the τ function plays a crucial role. The similar situation is expected to hold for the discrete case. It might be possible to construct the discrete Hamilton structure through the τ function.

Let us finally remind again that we need much more studies from various points of view to establish "integrability" for the discrete and q-discrete systems. We hope that the results in this thesis contribute to the theory of the discrete and q-discrete nonlinear integrable systems in view of fundamental understanding.

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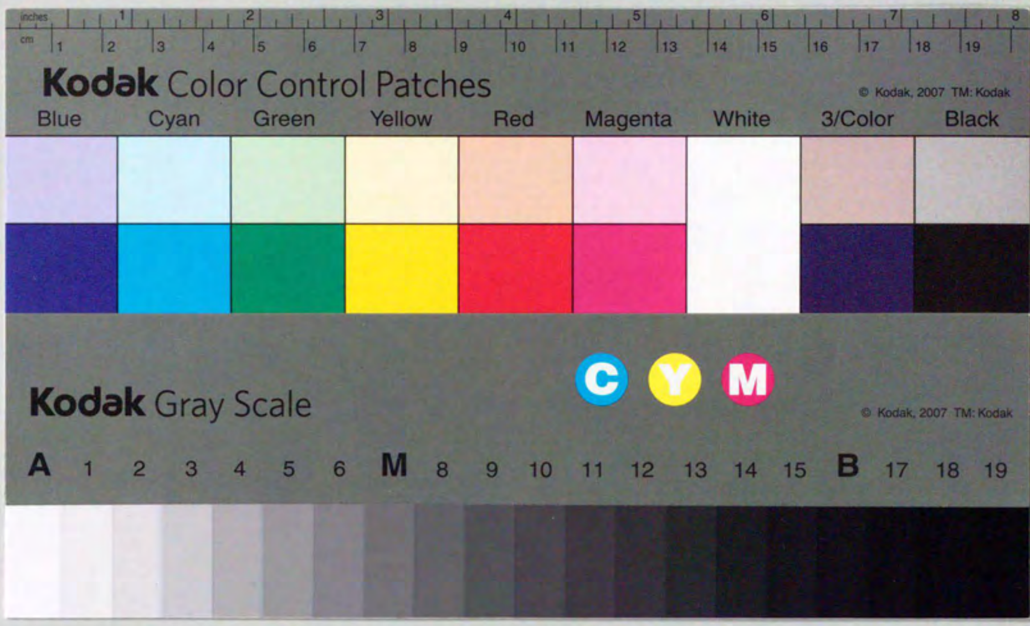
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