博士論文

On cognate integrable structure for three-point functions in AdS₅/CFT₄ correspondence

（AdS₅/CFT₄ 対応における 3 点関数の可解構造について）

西村 拓也

On cognate integrable structure for three-point functions in $\mathbf{AdS_5/CFT_4}$ correspondence

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Abstract

The discovery of the AdS/CFT correspondence, which is an unprecedented duality between a conformal field theory in d-dimension and a string theory on the maximally symmetric curved background with a negative cosmological constant, had a huge impact on theoretical physics. Although enormous studies have been done for two decades, we remain to reach the fundamental understanding for the underlying mechanism behind the duality. In this doctoral dissertation, to deepen our understanding of the duality, we shall study the three-point functions in the context of $\mathrm{AdS}_5/\mathrm{CFT}_4$ using the integrability techniques since the three-point functions are quite important observables describing the dynamics.

The organization of this thesis is as follows. The part I is devoted to an introduction and reviews concerning the results obtained by the author. The two- and three-point functions are reviewed in chapter 3 and chapter 4, respectively. We will also make few comments on other observables in chapter 5.

The part II contains the author's main results. In chapter 6, we will present a novel formalism in which the tree-level three-point functions in the so-called SU(2) subsector are studied and the result which has not been discussed in the literatures can be obtained. Furthermore, this novel construction allows us to derive the non-trivial identities so-called monodromy relations, which can be regarded as a collection of the Ward-Takahashi identities reflecting the hidden integrable structure.

In chapter 7, we generalize the results of the SU(2) sector to the entire PSU(2,2|4) sector.

Finally, in chapter 8 we develop a new method of computing three-point functions in the SU(2) sector in the semi-classical regime at weak coupling, which closely parallels the strong coupling analysis. The structure threading two disparate regimes is the monodromy relation. As a result, compact semi-classical formulas are obtained for a general class of three-point functions at weak coupling including the ones whose semi-classical behaviors were not known before and it turns out that the results perfectly match with those in the strong coupling regime, after taking the Frolov-Tseytlin limit.

The part III is devoted to the summary and conclusions.

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Part I

Introduction and Review

Chapter 1

Overture -introduction-

1.1   Toward a quantum theory of gravity

The discovery of the gravitational force dates back to the era of Newton and it is most fundamental and familiar force to us since then. The theory of general relativity, which was established by Einstein, describes macroscopic gravitational phenomena in a remarkably precise way, and there is no room for doubt that it is the most successful theory of nineteen century's physics, together with the quantum mechanics. Recently, we finally found the only missing piece of the theory, namely, the gravitational wave [1] and the discovery put the validity of general relativity on a firm basis.

However, there are several enigmatic phenomena such as black hole physics, which are considered to involve with quantum natures of the gravity. In order to demystify such mysterious physics, many studies aiming to quantize the theory of classical gravity have been done, nevertheless we have not succeeded in shedding a light on the quantum aspects of gravity.

Among many attempts to reveal the underlying quantum features, string theory is one of the most prominent candidates for a consistent theory of quantum gravity. String theory was originally an effective theory of strong nuclear force or hadron physics. Unfortunately, it was soon abandoned in favor of quantum chromodynamics (QCD) since the consistency of string theory requires that the space-time dimension is higher than four, and the spectrum contains an "unphysical" massless spin two particle. However, after a while, it was realized that such a massless spin 2 particle can be identified with a quantum of the gravitational field, namely, graviton and it suggests that string theory would contain a quantum gravity.

As indicated by its name, a fundamental constituent of the theory is not a zero-dimensional particle but a one-dimensional extended object, namely, a string. By quantizing the motion of a single string on a fixed background, it turns out that there are

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infinitely many different types of particles corresponding to the various vibration modes of the string. In particular, a massless spin 2 particle and a tower of infinitely many massive higher spin particles appear. Moreover, it should be noticed that the interaction of these infinitely many particles are neatly summarized into the splitting and joinning process of strings. In other words, the interactions among infinitely many particles are severely constrained and governed by the geometrical feature and dynamics of two-dimensional surface so-called world-sheet, which is a natural generalization of particle's world-line. These properties make string theory distinguished from ordinary quantum field theories.

Although string theory seems quite different from usual quantum field theory, the connection between string theory and QCD was "rediscovered". In 1973, 't Hooft showed that the Feynman diagrams of $SU(N_c)$ gauge theory, which is a natural generalization of QCD, can be classified according to their topology of graphs and only a subclass of diagrams (planar graphs) contribute in the large N_c limit [2]. Furthermore, he showed that the classification of the Feynman diagrams of $SU(N_c)$ gauge theory precisely matches a perturbative (genus) expansion of closed string theory with string coupling constant $g_{st} \sim 1/N_c$. Therefore, there was a naive expectation that perturbative string theory can be described in terms of large N_c gauge theory.

In 1997, Maldacena proposed a unprecedented duality between string theory and gauge theory which is so-called AdS/CFT correspondence [3], and many theoretical physicists took the above mentioned expectation seriously. AdS/CFT correspondence states that string theory on $d + 1$ dimensional AdS space-time is equivalent to d dimensional conformal field theory. One of the most remarkable feature of this duality is that there is a possibility that a certain class of quantum field theories, which does not include the gravity at all, gives a concrete and well-defined definition for quantum theory of gravity. Indeed, once we assume the validity of AdS/CFT duality, then we can study string theory, which is a prominent candidate of quantum gravity, using the "dictionary" of AdS/CFT correspondence. Such a feature that a higher dimensional gravitational theory can be described by a lower dimensional theory without gravity is called the holographic principle. The holographic principle was suggested to be a property that consistent quantum gravity should satisfy, and AdS/CFT correspondence is the first realization of the holographic principle in a quantitative manner. Hence, the understanding of the underlying mechanism of the AdS/CFT correspondence would play a crucial role to reveal the nature of quantum gravity.

Toward a consistent theory of quantum gravity, one of the most conservative approaches is to prove that string theory is the unique theory as the UV completion of general relativity. In this bottom up approach, we only assume the necessary properties

such as symmetry, unitarity, asymptotically well UV behaviors and so on. With such essential assumptions, we try to "derive" the string theory. The idea is essentially revival of the S-matrix bootstrap program and partially successful in certain cases [6–9].

On the other hand, one of the most radical approaches for quantum gravity is to exploit the AdS/CFT duality and explore the quantum features of gravity. For example, some try to construct bulk local operators defined in the AdS space in terms of boundary conformal field theory (CFT) operators in order to probe the inside of a black hole. However, it would be better to study the duality itself in detail since we have not yet fully understood what kind of observables are dual to each other and how to describe the dynamics of one theory from a view point of the other theory through the duality.

Although innumerable studies have been made and there are enomous results support AdS/CFT correspondence, we have not yet reached the fundamental and essential understanding of this highly non trivial duality. One of the reasons that make the analysis difficult is the nature of "strong-weak duality". As we will explain later, the parameter of the perturbative expansion on the gauge theory side is inverse of that of string theory and vice versa. In other words, the weak coupling region of one theory corresponds to the strong coupling region of the other theory. Therefore, we need to solve the problem non-perturbatively at least in one of the two theories in order to test the duality.

The most well studied example of AdS/CFT correspondence is the duality between Type IIB string theory on $AdS_5 \times S^5$ background and four-dimensional maximal supersymmetric Yang-Mills theory ($\mathcal{N} = 4$ SYM), which is sometimes denoted as $\mathrm{AdS}_5/\mathrm{CFT}_4$. In that case, due to the high symmetry of the both theory, there exists a hidden structure that is called integrability. By using integrability, major progress has been made in spectrum problem of $\mathrm{AdS}_5/\mathrm{CFT}_4$ and it has been recognized that integrability is a useful tool to study $\mathrm{AdS}_5/\mathrm{CFT}_4$ [19].

Recently, there are some attempts to determine the three-point functions of $\mathcal{N} = 4$ SYM theory and its holographic dual objects from integrability. From a CFT point of view, three-point functions are fundamental building blocks since higher-point functions are uniquely determined from two- and three-point functions by recursively using the operator product expansion (OPE). More importantly, three-point functions holographically describe the interaction of three string states, hence they should reflect the dynamics of the system and have more non-trivial information than two-point functions or spectrum. Therefore, we would expect that we can deepen our understanding of the mechanism of AdS/CFT correspondence by studying the three-point functions.

In this thesis, we will summarize the recent development of $\mathrm{AdS}_5/\mathrm{CFT}_4$ and integrability, especially focusing on studies of three-point functions of $\mathcal{N} = 4$ SYM theory at
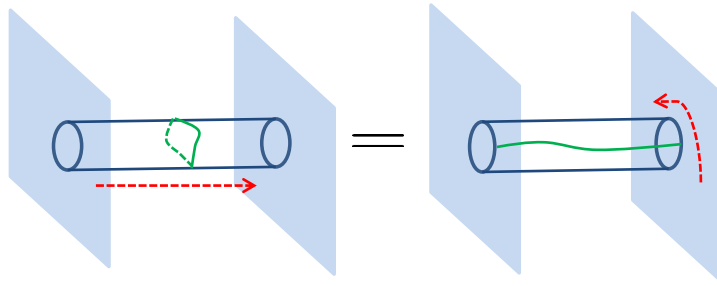
Figure 1.2.1: Open-closed duality states that the closed string propagation diagram (left) is equivalent to the loop diagram of the open string stretched between two branes (right).

weak coupling. The rest of the introduction section is divided into three parts. In section 1.2, we will discuss the picture behind the AdS/CFT correspondence and give a heuristic "derivation" for the duality. In section 1.3, we will give a more detailed "dictionary" for AdS_5/CFT_4 and emphasize the motivations for studying correlation functions. We shall also briefly introduce the notion of integrability and analyticity in section 1.4. The outline of this thesis is shown in section 1.5.

1.2   Discovery of AdS/CFT

In physics, it often happens that two seemingly different systems are actually the two different descriptions of the same system. In other words, we can obtain two different descriptions starting from one theory. AdS/CFT correspondence is a highly non-trivial example of such a duality. String theory is not a theory of strings but the theory of strings and branes, which are the hyper-surfaces on which strings end. There are two types of strings, namely, closed strings and open strings. Closed strings form loops while the ends of open strings are attached to the branes. They interact via joining and splitting. The important fact is that if one quantizes the strings propagating on a space-time, gravitons and gauge bosons naturally appear as the massless modes of the closed strings and the open strings, respectively. Furthermore, the loop diagram of the open string can be reinterpreted as the propagation of the closed string between two branes. (See figure 1.2.1.) Both descriptions are equivalent and this is known as the open-closed duality.

To clarify the picture behind the AdS/CFT, let us consider the scattering of an electron and a heavy proton. In QED, this process can be described in terms of Feynman diagrams (figure 1.2.2). The alternative description is the effective potential created by the heavy
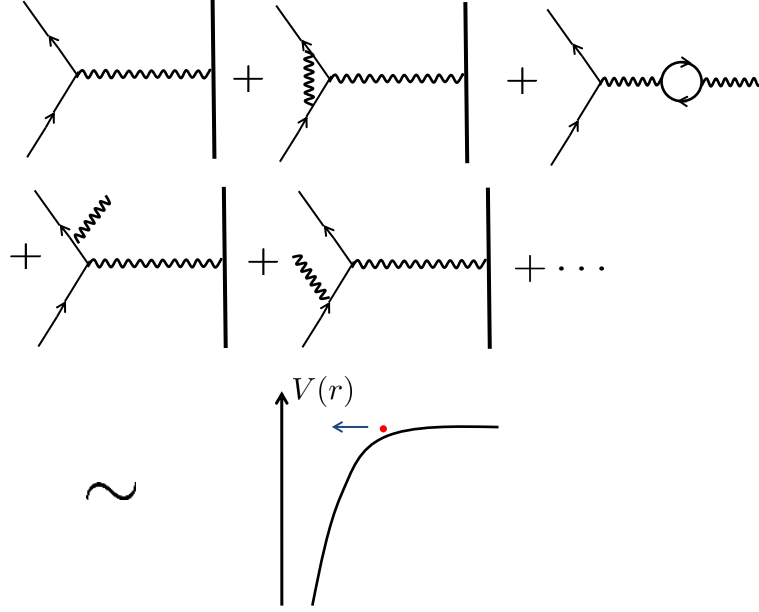
Figure 1.2.2: Feynman diagrams for scattering of an electron and a proton. They interact via exchanging of photons. The next to leading contributions consist of the diagrams such as vertex corrections, the photon self energy correction and the bremsstrahlung. These contributions can be replaced with the effective potential.

proton. Since the proton is heavy and back-reactions are negligible, we can regard that the electron moves in a non-trivial background potential created by the proton and forget it completely. The effective potential is computed by the summation of Feynman diagrams and it is given by

$$V(r) = -\frac{\alpha}{r}\left(1 + \frac{\alpha}{4\sqrt{\pi}}\frac{e^{-2m_e r}}{(m_e r)^{3/2}} + \cdots\right),\tag{1.2.1}$$

where r is the distance between the two charged particles, α is the Coulomb charge and m_e is the mass of the electron. Here, we have two descriptions for the scattering of an electron and a proton. One is the picture that the electron and the proton are in the vacuum and they interact via exchange of virtual photons as shown in figure 1.2.2. The other picture is that the electron moves not in the vacuum but in the effective potential (figure 1.2.2). The effect of the proton is traded by the non-trivial background by summing up all Feynman diagrams. This simple example gives an intuition for AdS/CFT, that is a more complicated example for the above "duality".

Strings and branes are the analogue of the electron and the proton, respectively, in the above example. Let us consider the scattering of a closed string and a brane (figure 1.2.3).
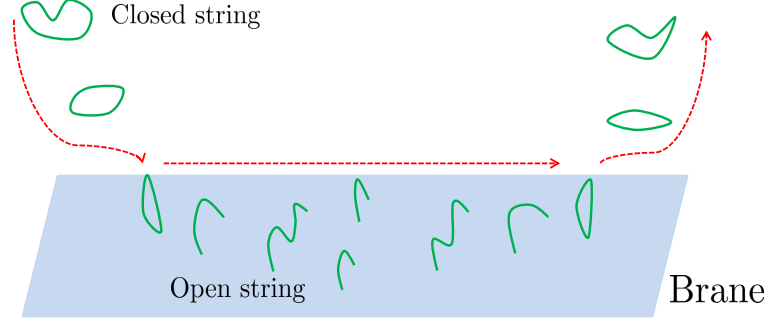
Figure 1.2.3: The ends of open strings are attached to the brane. This figure shows the process that a closed string splits into two open strings which propagate on the brane and the closed string is emitted from the brane again.
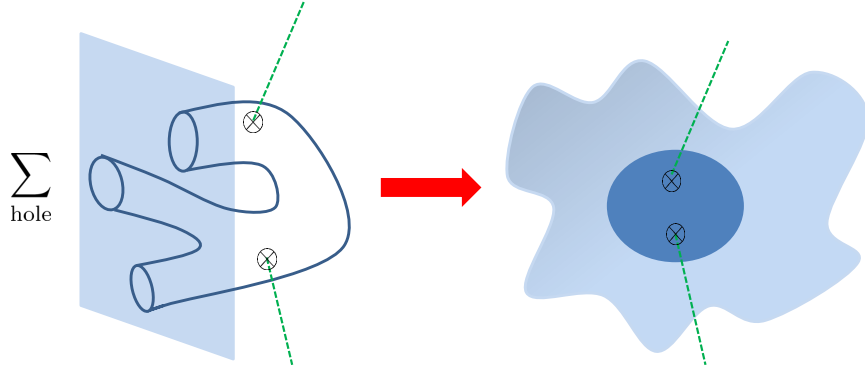
Figure 1.2.4: The summation over world-sheet topology with holes can be replaced with the non-trivial background geometry. The figure showed the example that genus 0 world-sheet with two closed string vertex insertions.

The interaction between the closed string and the brane is carried by open strings. Then, there are two possible perspectives on this situation as in the case for the scattering of an electron and a proton. One is the picture that the closed string and the brane interact via propagations of open strings on the brane. The other picture is that the closed string propagates on a non-trivial background by summing up all diagrams involving open strings (figure 1.2.4). The effect of the brane is traded by the non-trivial background on which the closed string propagates.

AdS$_5$/CFT$_4$

We now apply the above picture to the situation where there are N coincident D3-branes in Type IIB string theory, which is defined on the ten dimensional flat Minkowski space. In the low energy limit, the effective action for this system is given by

$$S = S_{brane} + S_{bulk} + S_{int} \,, \tag{1.2.2}$$

where S_{brane} describes the low energy effective action (LEEA) for the massless modes of the open string on the brane, namely the action for $\mathcal{N} = 4\ U(N)$ SYM theory in 4d, S_{bulk} describes the LEEA for the massless modes of the closed string in the bulk, which is the ten dimensional type IIB supergravity action and S_{int} describes the interaction term between the branes and the closed strings in the bulk. To decouple the bulk interactions from the brane, we take the low energy limit as $\alpha' \to 0$, keeping all the other dimensionless parameters fixed. Then, S_{int} and the higher derivative terms drop from the action, hence we have two decoupled systems[1]:

$$\text{4d } \mathcal{N} = 4\ U(N) \text{ SYM} \oplus \text{10d free Type IIB SUGRA} \,. \tag{1.2.3}$$

This is indeed the first picture that the closed strings are the "electrons" and the branes are the "protons".

Then, summing up the all diagrams involving open strings, which is equivalent to sum up the world-sheet topologies with holes, leads to the back-reaction to the space-time geometry. This is nothing but the second picture and the closed strings propagate on a curved background. The back-reacted geometry is a solution of the Type IIB SUGRA and it is known to be the black 3-brane solution:

$$ds^2 = \left(1 + \frac{R^4}{r^4}\right)^{-1/2} \eta_{\mu\nu} dx^\mu dx^\nu + \left(1 + \frac{R^4}{r^4}\right)^{1/2} (dr^2 + r^2 d\Omega_{[5]}) \,,\ R^4 = 4\pi g_{st} N \alpha'^2 \,. \tag{1.2.4}$$

We must take a decoupling limit as we did in the case for the first picture. Here, we take the low energy limit keeping the dimensionless parameter α'/r^2 fixed. This can be understood as follows. For a distant observer, there are two kinds of low energy excitations. One is the bulk massless modes of the closed strings in the flat Minkowski space (figure 1.2.5). The other are the *all excitations of the closed strings near the horizon* since they are seen as the low energy excitations for the distant observer due to the large red-shift (figure 1.2.5). By introducing the parameter $z = r/R^2$ and taking the near

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[1]SUGRA stands for the supergravity.

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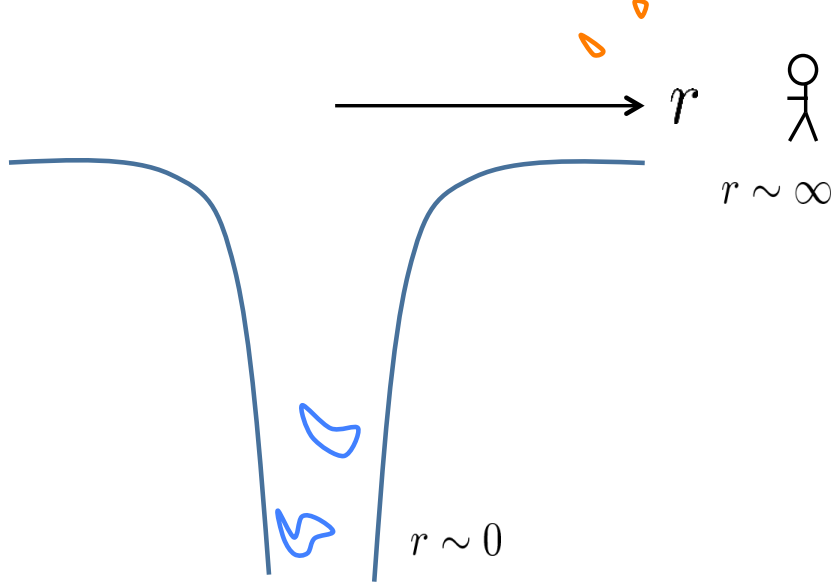
Figure 1.2.5: For a distant observer, all the excitations of the closed string are seen as the low energy excitations due to the large red-shift. The near horizon ($r \sim 0$) geometry is $AdS_5 \times S^5$.

horizon limit, the metric (1.2.4) becomes

$$ds^2 = R^2 \left(\frac{dz^2 + \eta_{\mu\nu}dx^\mu dx^\nu}{z^2} + d\Omega_{[5]} \right) . \qquad (1.2.5)$$

This is the metric of the $AdS_5 \times S^5$ in the Poincare patch and we have obtained it as the near horizon geometry. To summarize, we have the following two decoupled systems

$$\text{Type IIB superstring on } AdS_5 \times S^5 \oplus \text{ 10d free Type IIB SUGRA} . \qquad (1.2.6)$$

By equating (1.2.3) and (1.2.6), we reach the following conjecture

$$\text{4d } \mathcal{N} = 4 \ U(N) \text{ SYM} = \text{Type IIB superstring on } AdS_5 \times S^5 . \qquad (1.2.7)$$

Here, let us stress the importance and the remarkable features of this unprecedented duality.

- **Strong-weak duality**

  As we will see in the next subsection in detail, AdS/CFT correspondence is an example of the strong/weak duality. In other words, the perturbative regime for one theory is mapped to the non-perturbative regime for the other theory. This

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strong/weak nature is one of the main reasons that make it difficult to understand the underlying mechanism of the AdS/CFT. However, once we assume that the duality holds, it is possible to study the strongly coupled gauge theory via classical gravity. Conversely, the highly quantum string in AdS background can be understood from free $\mathcal{N} = 4$ SYM from AdS/CFT correspondence.

- **Holography**

  AdS/CFT is the almost unique example that realizes the idea of the holographic principle quantitatively. The holographic principle that the quantum gravity defined in the bulk should be encoded in the boundary theory without gravity is considered to be a criteria that the consistent theory of quantum gravity must obey. Indeed, AdS/CFT correspondence states that the string theory, which is a candidate of the quantum gravity, in AdS background is equivalent to the field theory defined on the flat Minkowski space, which is the boundary of the AdS space-time.

- **Non-perturbative definition for quantum gravity**

  Up to now, string theory is only defined via perturbative series and thus it only has a perturbative formulation. However, AdS/CFT correspondence provides a possibility that a class of quantum field theories give a non-perturbative definition for the string theory, at least in the AdS background. Furthermore, it may be possible to shed light on the problems of the quantum gravity such as the information paradox. In fact, if the string theory is completely described by field theories, the information must be preserved since the unitarity is manifest in usual field theories. Therefore, it would be quite interesting to explore the possibility to reveal the mysteries of quantum gravity from quantum field theory using AdS/CFT duality.

1.3 Dictionary of AdS/CFT

The more detailed correspondence of the parameters or physical quantities between $\mathcal{N} = 4$ SYM theory and type IIB string theory on $AdS_5 \times S^5$ background are summarized in table 1.3.1. The important point is that the 'tHooft coupling λ which is the effective coupling constant in the large N gauge theory is related to the radius R of the AdS. Since the coefficient in front of the action for the string is proportional to R^2/α', one finds that the coupling constant for the string sigma model is given by $1/\sqrt{\lambda}$ from table 1.3.1. Furthermore, the loop expansion parameter g_{st} for the string theory is in proportion to the inverse of the rank of the gauge group $1/N_c$ and thus the loop expansion of the string theory corresponds to the large N_c expansion of the gauge theory. In the large N_c limit,

$\mathcal{N} = 4\ SU(N_c)$ SYM	Type IIB superstring in $AdS_5 \times S^5$
'tHooft coupling $\lambda = g_{YM}^2 N_c$	$\lambda = \frac{R^4}{\alpha'^2}$
Number of colors N_c	$N_c = \frac{\lambda}{g_{st}} = \frac{R^4}{\alpha'^2} \frac{1}{g_{st}}$
Single trace operators $\mathcal{O}_i(x)$	Single string states
Conformal dimension Δ	AdS Energy E
$\langle \mathcal{O}_i(x_1) \mathcal{O}_j(x_2) \rangle$	Propagation of a string on AdS
$\langle \mathcal{O}_i(x_1) \mathcal{O}_j(x_2) \mathcal{O}_k(x_3) \rangle$	Interaction of three strings
$\langle \mathcal{O}_{i_1}(x_1) \ldots \mathcal{O}_{i_n}(x_n) \rangle$	Splitting and joining of string states

Table 1.3.1: AdS/CFT dictionary relevant for this thesis. The important quantities are correlation functions of gauge invariant operators of $\mathcal{N} = 4$ SYM theory, that correspond to the dynamics of string states. (See figure 1.3.1 and 1.3.2.)

only planar diagrams whose topology is a sphere will contribute to the calculation of the observables and there is no quantum loop effects for the dual string theory at the leading order. We will concentrate on the leading order of the large N_c expansion or genus expansion.

Let us see the parameter region where each theory can be treated perturbatively.

- $\lambda \to 0, N_c \to \infty$

  In this region, the perturbative expansion of the gauge theory in 'tHooft coupling λ is valid. On the other hand, since the tension of the string $1/\alpha' \sim \sqrt{\lambda}$ goes to zero and all the massive modes of the string are degenerate to massless, quantum fluctuations for them are not negligible at all. Such a highly quantum string is called tensionless string and regarded as a quite complicated gauge theory with huge gauge symmetry accompanied by infinitely many massless gauge particles with higher spin [10].

- $\lambda \to \infty, N_c \to \infty$

  In this region, the gauge theory is strongly coupled and the perturbative expansion is of no use at all while the dual theory can be described by classical supergravity. The string tension becomes infinite as $\sqrt{\lambda} \to \infty$, hence all the massive modes decouple from the system and only the massless modes survive, whose low energy effective theory is the supergravity.

  Moreover, the strings with large quantum numbers such as energy or spins are
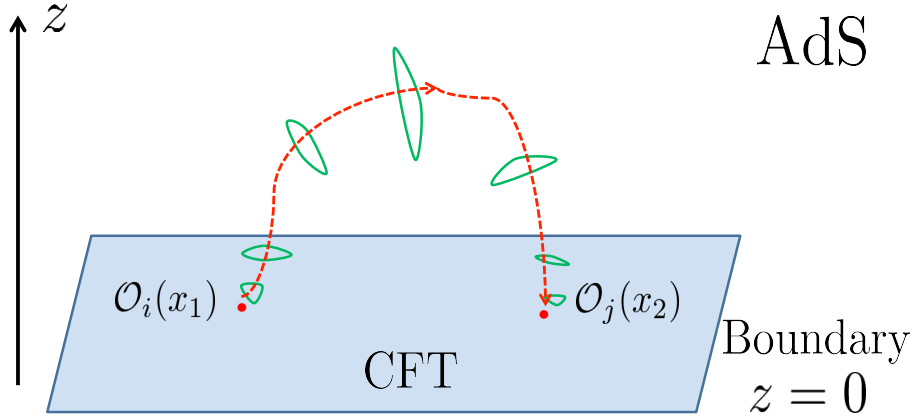
Figure 1.3.1: 1-body propagation of strings is holographically described by the two-point functions of the CFT living on the boundary of the AdS. The boundary operator has the same charges of the global symmetry such as spins.

described by the classical solution as saddles of the path integral:

$$\int \mathcal{D}X e^{-S[X]} \prod_{i=1}^{n} \mathcal{V}_i[X] \approx e^{-S[X^*]} \prod_{i=1}^{n} \mathcal{V}_i[X^*] \,, \tag{1.3.1}$$

$$\frac{\delta S[X^*]}{\delta X} = \sum_{i=1}^{n} \frac{\delta \log \mathcal{V}_i[X^*]}{\delta X} \,. \tag{1.3.2}$$

Since contributions from the vertex operators of string states are comparable to the action, they are regarded as the source terms of the equation of motion.

How to test the duality?

Generally, if we are given a duality between two seemingly different systems, how to investigate the duality? Some important physical quantities to test the duality are spectra and interactions.

- **Two-point function (Spectrum)**

  In quantum field theory, the information of the spectrum is encoded in two-point functions. In particular, it is known that $\mathcal{N} = 4$ SYM theory is a quantum field theory with local scale invariance, namely, conformal field theory (CFT). Conformal invariance of the theory uniquely determines the two-point functions of CFT from the scaling properties of the operators. For example, two-point functions for scalar

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operators with no Lorentz spins are given by

$$\langle \mathcal{O}_i(x_1)\mathcal{O}_j(x_2)\rangle = \frac{\delta_{ij}}{|x-y|^{2\Delta_i}} \,, \tag{1.3.3}$$

where Δ_i is the conformal dimension of the local operator $\mathcal{O}_i(x)$ and the operator transforms under the scale transformation as $\mathcal{O}_i(x) \to \Lambda^{\Delta_i}\mathcal{O}_i(\Lambda x)$.

In the holographic description, two-point functions are mapped to the spectra for strings propagating on AdS background. (See figure 1.3.1.) The energies of the strings E_i are equal to the conformal dimensions Δ_i of the corresponding local operators $\mathcal{O}_i(x)$.

- **Three-point function (Interaction)**

  One of the remarkable features of the CFT is the *operator product expansion* (OPE) which allows us to express the product of any two local operators in terms of the summation over the complete set of local operators as follows[2]

$$\mathcal{O}_i(x)\mathcal{O}_j(y) = \sum_k C_{ijk}|x-y|^{-\Delta_i-\Delta_j+\Delta_k}\mathcal{O}_k(y)\,, \tag{1.3.4}$$

  where C_{ijk} is the structure constant. With the structure constant, conformal invariance also uniquely fixes three-point functions. For example, three-point functions for three scalar operators are given by

$$\langle \mathcal{O}_i(x_1)\mathcal{O}_j(x_2)\mathcal{O}_k(x_3)\rangle = \frac{C_{ijk}}{|x_1-x_2|^{\Delta_i+\Delta_j-\Delta_k}|x_2-x_3|^{\Delta_j+\Delta_k-\Delta_i}|x_3-x_1|^{\Delta_k+\Delta_i-\Delta_j}}\,. \tag{1.3.5}$$

  It should be emphasized that the holographic dual of the three-point function is the S-matrix of the three string states as shown in the figure 1.3.2.

Similarly, higher-point functions correspond to the interactions of multi string states via splitting and joining and they are determined from the two- and three-point functions through recursive application of the OPE. Therefore, the set of conformal dimensions $\{\Delta_i\}$ (spectrum) and the set of structure constants $\{C_{ijk}\}$ for any local operators $\mathcal{O}(x)$ are fundamental building blocks of the CFT and thus the set of data $\{\Delta_i; C_{ijk}\}$ is called CFT data. From a different point of view, given such a set of the data, they define a CFT via OPE. Suppose that we succeed in characterizing the CFT data which has the dual string theory, then it may be possible to describe the dynamics of strings on

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[2]Here, we consider the operators with no Lorentz spins for simplicity. Generally, the space-time dependence is more complicated for operators with Lorentz indices. However, conformal invariance can fixes it as well.
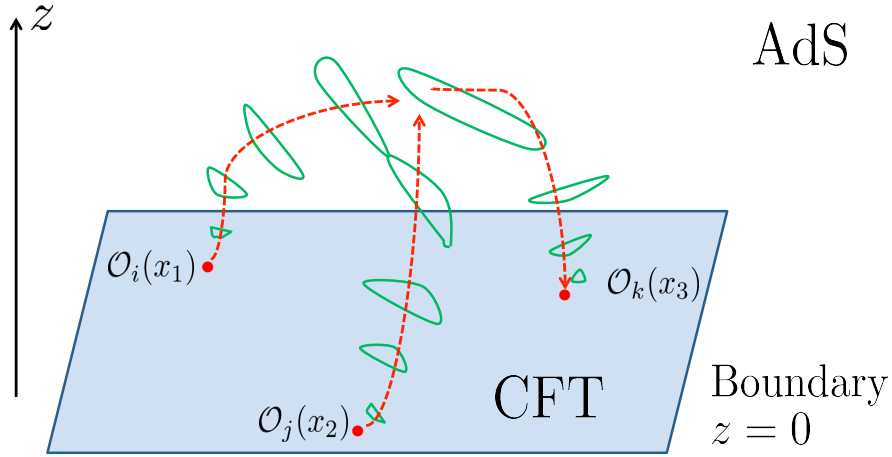
Figure 1.3.2: Interactions of three strings are holographically described by the three-point functions of the CFT living on the boundary of the AdS.

AdS background via correlation functions of CFT. This is a complete realization of the holographic principle and gives us some clues for the quantum theory of gravity if the above scenario is true.

In spite of the fundamental importance of the correlation functions in the context of AdS/CFT, the analysis has been only limited to the BPS (supergravity) sector [4,5] since AdS/CFT is a strong/weak duality and one necessarily relies on the non-renormalization property of the BPS observables so that one obtains the results that do not depend on the coupling. However, in order to understand the mechanism of the AdS/CFT, it is necessary to extract the dynamical information and study physical quantities which are not protected by symmetry.

Integrability based approach

Fortunately, in the case of AdS_5/CFT_4, it turned out that we have the powerful tool to study the duality which is so-called integrability. Indeed, the spectrum problem of AdS_5/CFT_4 was almost solved by using integrability such as spin chain techniques or the Bethe ansatz method [19]. The blessing of the integrability in spectrum problem leads us to the expectation that the integrability is of use to study the higher-point functions as well. In particular, three-point functions are fundamental quantities in CFT, as already explained. Further, they correspond to the interactions of three strings in the dual string theory and thus the important dynamical information should be encoded in three-point functions. Therefore, it is of special importance to study them based on the integrability

in the context of AdS/CFT. Although integrability is a specific structure to $\mathrm{AdS}_5/\mathrm{CFT}_4$, it will play an important role as a first step to reveal the underlying mechanism of the AdS/CFT if we can extract the universal structures that are common to the CFTs with holographic dual descriptions.

1.4   Integrability and analyticity

We should refer to the idea of the integrability and power of the analyticity. The well-known definition of the classical integrability in finite dimensional system is that proposed by the Liouville. If the system with n degrees of freedom possesses n independent conserved charges F_i ($i = 1, \ldots n$), commutating with each other $\{F_i, F_j\} = 0$ and the Hamiltonian $\{H, F_i\} = 0$, the system is called Liouville integrable. In Liouville integrable systems, it is possible to construct canonical variables satisfying $\{x_i, x_j\} = \{p_i, p_j\} = 0$, $\{x_i, p_j\} \sim \delta_{ij}$. In other words, one can reduce the system to a set of decoupled one-dimensional systems and thus it is sometimes called separation of variables (SoV). Furthermore, the motions of these separated variables can be linearized by performing an appropriate canonical transformation. The resulting canonical pair of variables turns out to be (F_i, ϕ_i), where F_i's are the action variables, namely, the conserved charges, and ϕ_i's are the angle variables conjugate to the action variables.

Although the definition of the Liouville integrability is clear, actual construction for conserved charges and the corresponding action-angle variables is difficult in general. However, a more practical method is known and involves the Lax pair. If the equation of motion can be converted into the following form,

$$\dot{L}(x) = [M(x), L(x)], \quad \forall x \in \mathbb{C}, \tag{1.4.1}$$

where (L, M) is a pair of matrices called the Lax pair depending on the spectral parameter x, then we can immediately see that $H_n = \mathrm{Tr}(L^n)$ is conserved. Or equivalently, the information of the conserved charges is encoded in the characteristic equation of the form

$$\det(y - L(x)) = 0, \quad (x, y) \in \mathbb{C}^2, \tag{1.4.2}$$

since the above mentioned charges appear once one expands the defining equation of the complex curve (1.4.2). Put it differently, the conserved charges can be extracted from as moduli of the curve (1.4.2). The curve is called the *spectral curve* in the context of the integrable system.
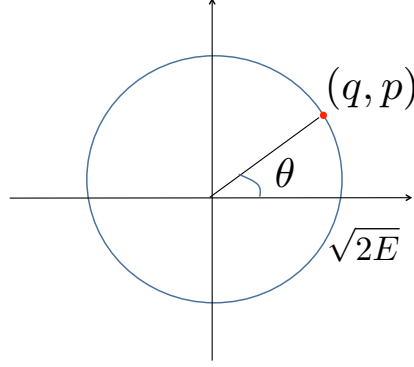
Figure 1.4.1: The phase space of the harmonic oscillator is represented. The energy is the only conserved charge and it is a modulus of the spectral curve. The angle variable is nothing but the angle defined by $(q, p) = (\sqrt{2E}\cos\theta, \sqrt{2E}\sin\theta)$.

Harmonic oscillator

For example, let us consider the one-dimensional harmonic oscillator, which is the simplest example of the integrable systems. The equation of motion is

$$\dot{q} = p, \quad \dot{p} = -q. \tag{1.4.3}$$

It is obvious that the separated variables are (q, p) themselves and it is well-known that the action-angle variables are given as

$$S = 2E = \oint p\, dq, \tag{1.4.4}$$

$$\theta = \int^q \frac{dx}{\sqrt{2E - x^2}} = \arccos\left(\frac{q}{\sqrt{2E}}\right). \tag{1.4.5}$$

To clarify the role of the Lax pair, it should be noted that the equation of motion is rewritten as follows

$$\dot{L}(x) = \left[\frac{\sigma_3}{2}, L(x)\right], \quad L(\lambda) = \begin{pmatrix} ix & q + ip \\ q - ip & -ix \end{pmatrix}, \tag{1.4.6}$$

with the third Pauli matrix σ_3. The characteristic equation is given by

$$x^2 + y^2 - 2E = 0, \quad E = \frac{1}{2}(p^2 + q^2). \tag{1.4.7}$$

Hence, in the present case, the spectral curve is just the defining equation for the circle and its radius is given in terms of the energy, which is the only conserved charge of the harmonic oscillator.

Furthermore, we can extract the separation of variables and obtain the action-angle variables using this formulation. The necessary ingredient is the eigenvector of $L(x)$: [3]

$$L(x)\vec{\psi}_\pm(x) = y_\pm(x)\vec{\psi}_\pm(x) . \tag{1.4.8}$$

Here, $y_\pm(x) = \pm\sqrt{2E - x^2}$ are eigenvalues of the Lax matrix and $\vec{\psi}_\pm(x)$ are the corresponding eigenvectors. The eigenvectors $\psi_{pm}(x)$ are often normalized as $\vec{\psi} \cdot \vec{n} = 1$ using the normalization vector. In particular, we choose $\vec{n} = (1, 1)$ and $\vec{\psi}_+$ behaves as

$$\vec{\psi}_+(x) \propto (q^2 - x^2 - ipq + ix\sqrt{q^2 + p^2 - x^2})^{-1/2} . \tag{1.4.9}$$

Therefore, it turns out that q can be obtained as a pole of the normalized eigenvector x_{pole} and the conjugate variable is also obtained as the corresponding eigenvalue $p = y(x_{\text{pole}})$. With this in mind, the action-angle variables are

$$S = \oint y(x)dx , \tag{1.4.10}$$

$$\theta = \int^{x_{\text{pole}}} \frac{\partial y(x)}{\partial S}dx = \int^{x_{\text{pole}}} \frac{dx}{y} . \tag{1.4.11}$$

The action variable is given as a period integral of the spectral curve and the angle variable is expressed in terms of the integral of the holomorphic 1-form, whose end point of the integral involves the pole of the eigenvector. The construction of the action-angle variables in the harmonic oscillator is rather trivial, however, the construction is almost the same even in the case of more generic integrable systems and they can be described by the language of the algebraic curve.

In the case of 1+1 dimensional field theories, the Lax equation (1.4.1) is generalized to the form of a flat connection:

$$[\partial_\sigma + A_\sigma(x), \partial_\tau + A_\tau(x)] = 0 . \tag{1.4.12}$$

Due to the flat connection with the spectral parameter, the monodromy matrix[4], which is analogous to Wilson loop object, can be defined

$$\Omega(x) = P\exp\left[\int_C A(x)\right] , \tag{1.4.13}$$

with $A(x) = A_\sigma(x)d\sigma + A_\tau(x)d\tau$. With the monodromy matrix, the analogue of the spectral curve (1.4.2) and the eigenvector equation (1.4.8) can be defined. Similarly, the construction for the SoV's and the action-angle variables is parallel.

---

[3] $y_\pm(x)$ is sometimes called *quasi-momentum*.
[4] In a mathematical language, it should be called holonomy matrix. However, we follow the convention in the integrable system context.

Factorization

So far, we have discussed the classical integrability in general set up. It is quite natural to ask how to extend the notion of integrability to quantum systems. The generally acceptable definition remains to be uncovered. However, we believe that the factorization or reduction of complicated systems or observables to simpler building blocks is an important aspect of the integrability even at the quantum level. Actually, multi-particle S-matrices in integrable massive field theories in 1+1 dimension are expected to be factorized into a product of two-body scatterings. Another example is a factorization of the wave function in the SoV basis. Since the classical separated variables commute with each other $\{x_i, x_j\} = 0$, their quantum analogue should satisfy $[\hat{x}_i, \hat{x}_j] = 0$. Therefore, we expect that the wave function in this basis is reduced to a one-dimensional problem: $\langle x_1, \ldots, x_n | \psi \rangle = \prod_{i=1}^{n} \psi(x_i)$.

Analyticity

The important aspect of the integrability we wish to stress is the analyticity concerning the spectral parameter. It goes without saying that analyticity has played a quite important role in theoretical physics. It is well known that analyticity (holomorphy) is used to derive non-perturbative results in supersymmetric gauge theories such as non-renormalization theorems [11]. In particular, the low energy effective action of four dimensional $\mathcal{N} = 2$ theory is completely described in terms of a single holomorphic function, namely, the prepotential, which can be completely determined using a Riemann surface called the Seiberg-Witten curve and period integrals on it [12]. The relation with integrable system is pointed out in [13] and the Seiberg-Witten curve is identified with a spectral curve of an integrable system.

Furthermore, the BPS spectrum in a certain class of $\mathcal{N} = 2$ theories is another interesting example. The Kontsevich-Soibelman wall crossing formula can be understood as a smoothness condition of the metric on the moduli space of vacua, and the construction of "good" coordinates of the moduli space is done by using the Hitchin equations [15]. The Hitchin systems are obtained as BPS equations from the dimensional reduction of $d = 5$ super Yang-Mills theory on a Riemann surface C. The Hitchin system is known to be integrable and the flat connection defined from the Hitchin system is necessary to construct the coordinate of the moduli space, combined with the exact WKB analysis [15]. Note also that the construction is originally done by solving a Riemann-Hilbert problem associated with the flat connection in [14] and the thermodynamic Bethe ansatz (TBA) equation appears.

These examples seem to suggest that there exist a close relation between the integrability of the system and an analyticity or global behaviors in the underlying analytical parameter space. In other words, the existence of the analytical (spectral) parameter would be a manifestation of the underlying integrability. Actually, the analyticity shall play quite significant role in our calculations.

1.5 The organization of this thesis

The organization of this thesis is divided into three parts.

- **Review**

  First, we summarize the basic facts about the AdS_5/CFT_4 in section 2. We will review $\mathcal{N} = 4$ SYM theory and type IIB string on $AdS_5 \times S^5$ background in section 2.1 and section 2.2, respectively.

  The section 3 is devoted to the review of the development of the spectrum problem in AdS_5/CFT_4 and we will see how integrability works in the problem. Since innumerable works has been done, we will briefly overview the history of the development in section 3.1, then we explain the spectrum problem for the gauge theory and string theory based on the integrability in section 3.2, section 3.3, respectively. Then, in section 3.4, the spectrum of both theory is compared in the semiclassical limit. We will end the review of the spectrum problem with comments on the non-perturvative framework (section 3.5) and lessons extracted from it (section 3.6).

  The chapter 4 will be devoted to the studies of three-point functions and integrability, especially focusing on the weak coupling regime. We start with a brief review of developments of the three-point function (section 4.1) and then we shall explain how integrability technique works in the case of a certain subsectors of $\mathcal{N} = 4$ SYM theory (section 4.2), together with the recent non-perturbative work (section 4.3).

  In addition, two sections are appended to explain the power of integrality for other observables such as null polygonal Wilson loop (section 5.1) and quark-anti-quark potential (section 5.2).

- **Main result**

  The main part of this thesis is organized by three chapterss and they are based on the results of the author [136, 138, 139] in collaboration with Yoichi Kazama and Shota Komatsu.

In chapter 6, we give a novel construction for the three-functions in the SU(2) sector based on the singlet projector and derive the so-called monodromy relation, which can be regarded as a kind of the Schwinger-Dyson equation or a collection of Ward identites.

In chapter 7, we generalize the result of the chapter 6 to the full $\mathfrak{psu}(2,2|4)$ sector.

Finally, in chapter 8 we shall discuss the semi-classical three-point functions in the SU(2) sector both at the weak and strong couplings. In the semi-classical limit, the monodromy relation plays an quite important role and exactly the same structure for the three-point function appears. As a result, it reduces the problem to a set of functional equations, which can be solved as the Riemann-Hilbert problem, combined with the analyticity.

- **Conclusion**

  We conclude this thesis with the summary and prospects for future works in section 9.1 and section 9.2, respectively.
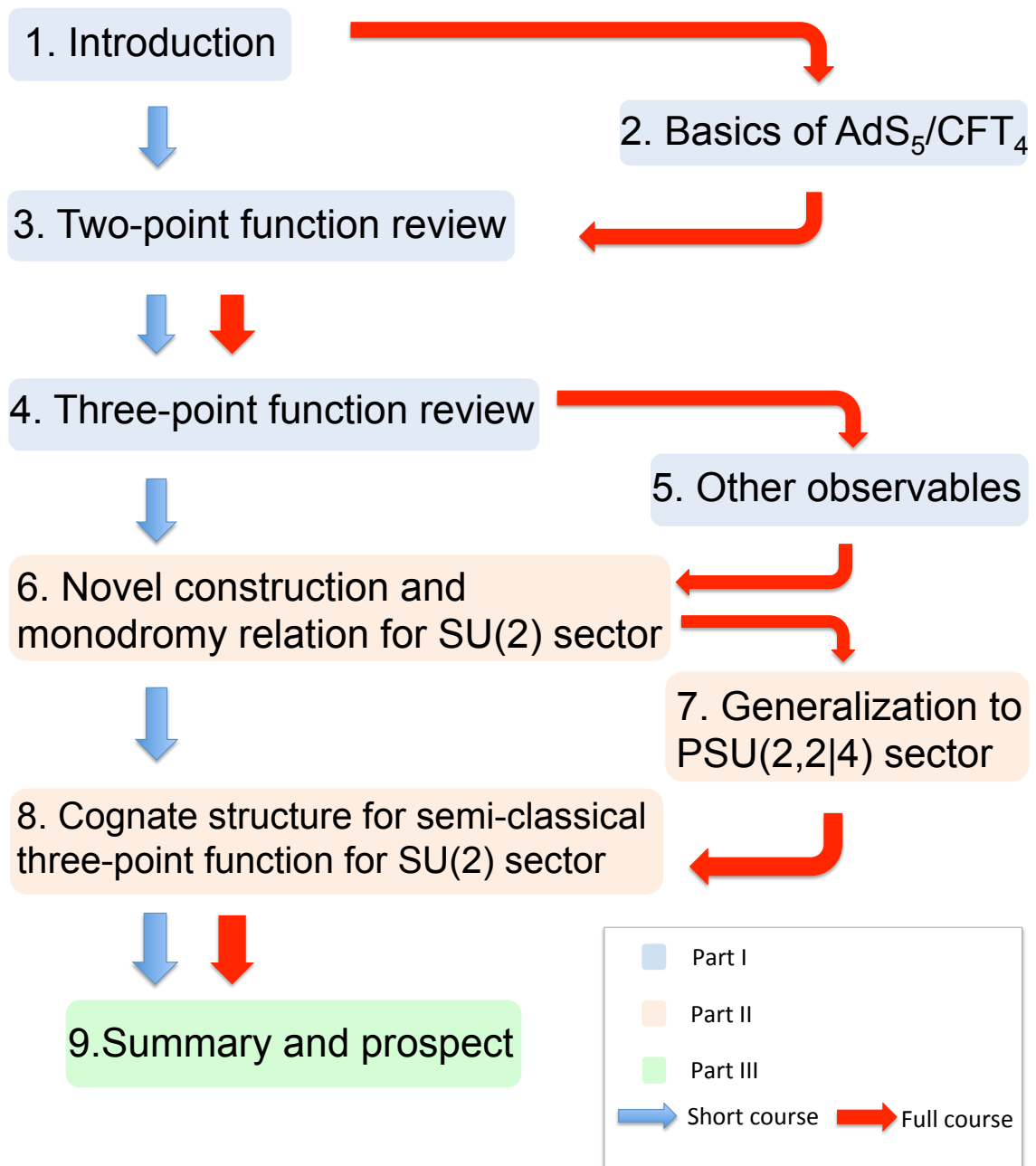
Figure 1.5.1: A guide of the thesis. An ultra short course is 3.1→3.6→4.1→4.2→8→9.

Chapter 2

Basics of AdS$_5$/CFT$_4$

In this chapter, we summarize the basic facts on the AdS$_5$/CFT$_4$ correspondence. The section 2.1 and 2.2 are devoted to the review of $\mathcal{N} = 4$ super Yamg-Mills theory and Type IIB string theory on $AdS_5 \times S^5$ background, respectively.

2.1 $\mathcal{N} = 4$ super Yang-Mills theory

In this section, we will review $\mathcal{N} = 4$ SYM theory, which is the gauge theory with maximal number of supercharges in four dimension. The most remarkable feature of $\mathcal{N} = 4$ SYM theory is that it is a conformal field theory and thus has no inherent mass scale in the theory, even at quantum level. Combined with the supersymmetry, the conformal symmetry is enhanced to the superconformal symmetry, which is supergroup $PSU(2, 2|4)$ in this theory. Therefore, the spectrum of the $\mathcal{N} = 4$ SYM theory can be classified according to the unitary irreducible representation of $PSU(2, 2|4)$. We will see the action and symmetry of the theory in 2.1.1. Then, the classification of gauge invariant composite operators in the theory is discussed in 2.1.2. The correlation functions of them are of particular importance in AdS/CFT.

2.1.1 The action and symmetry

The action for the $\mathcal{N} = 4$ SYM theory in four dimension can be obtained by the dimensional reduction from the $\mathcal{N} = 1$ SYM theory in ten dimension, whose action is give by

$$S = \frac{1}{2\tilde{g}_{YM}^2} \int d^{10}x \operatorname{Tr} \left[-\frac{1}{2} F_{MN} F^{MN} + \bar{\Psi} \Gamma^M D_M \Psi \right], \qquad (2.1.1)$$

where A_M ($M = 0, \ldots, 9$) is a vector field (gluon) in ten dimension, Ψ is a Majorana-Weyl fermion with 16 components, and Γ^M are ten dimensional gammma matrix. All the fields

are in adjoint representation of gauge group $SU(N)$. The covariant derivatives D_M and the field strength is defined as

$$D_M = \partial_M - i[A_M, \ ], \quad F_{MN} = \partial_M A_N - \partial_N A_M + [A_M, A_N].$$ (2.1.2)

By dimensional reduction, six components of the vector field become scalar fields while the 16 components of the spinor field split into four copies of right and left Weyl fermion in four dimension as follows

$$
\begin{aligned}
A_M \ (M = 0, \ldots, 3) &\longrightarrow A_\mu \ (\mu = 0, \ldots, 3) \\
A_M \ (M = 4, \ldots, 9) &\longrightarrow \phi_i \ (I = 1, \ldots, 6) \\
\Psi_A \ (A = 1, \ldots, 16) &\longrightarrow \bar{\psi}_{\dot{\alpha}a}, \psi_\alpha^a, \quad a = 1, \ldots, 4, \quad \alpha, \dot{\alpha} = 1, 2.
\end{aligned}
$$ (2.1.3)

In terms of these four dimensional fields, the dimensional reduced action is

$$
S = \frac{1}{2g_{YM}^2} \int d^4x \left[-\frac{1}{2} F_{\mu\nu} F^{\mu\nu} + D_\mu \phi_I D^\mu \phi^I - \frac{1}{2} \sum_{I,J} [\phi_I, \phi_J]^2 \right.
$$
$$
\left. + i\bar{\psi}_a \gamma^\mu D_\mu \psi^a + \frac{1}{2} \sigma_{ab}^I \psi^a [\phi_I, \psi^b] + \frac{1}{2} \sigma_I^{ab} \bar{\psi}_a [\phi^I, \bar{\psi}^b] \right],
$$ (2.1.4)

where σ^I is the chiral components of the gamma matrices in six dimension.[1] It turns out that this action has the manifest $SO(1,3) \times SO(6)$ symmetry. The origin of the $SO(6)$ symmetry is the rotation for the dimensionally reduced "inner space" and it is nothing else than the R-symmetry in four dimension. The indices α and $\dot{\alpha}$ are the spinor indices of the four dimensional Lorentz group and $I = 1, \ldots, 6$ for scalar fields are the label of vector representation for the R-symmetry. The raised a index for ψ_α^a and the lower index for $\psi_{\dot{\alpha}a}$ are the indices for the fundamental representation **4** and the anti-fundamental representation $\bar{\mathbf{4}}$ of $SO(6) \cong SU(4)$ respectively. From the action, the dimension of the fields are $[A_\mu] = [\phi^I] = 1, \quad [\psi^a] = 3/2$.

Since the action of $\mathcal{N} = 4$ SYM theory has no mass parameter, it is scale invariant at least classical level. Furthermore, the 1-loop β-function for this theory vanishes. To see this, note that the one-loop β-function for the gauge coupling of $SU(N)$ gauge theory is given by

$$
\beta = -\frac{g_{YM}^3}{16\pi^2} \left(\frac{11}{3} N - \frac{1}{6} \sum_i C_i - \frac{1}{3} \sum_j \tilde{C}_j \right),
$$ (2.1.5)

where the first summation is taken over all real scalar fields with quadratic Casimir C_i and the second summation is taken over all Weyl fermions with quadratic Casimir $\tilde{C}_j$. All the

---

[1]Since the fermionic part of the action is irrelevant to our purpose, we do not write down its properties explicitly. See details.

Casimirs for the fields in $\mathcal{N} = 4$ SYM are N since they are in the adjoint representation. The theory contains six real scalar fields and eight Weyl fermions, hence the 1-loop β-function vanishes. There are some arguments that the β-function for this theory vanishes to all-loop in perturbation theory. This implies that $\mathcal{N} = 4$ SYM theory is scale invariant even at quantum level and it is considered to be a conformal field theory (CFT).

The symmetry of $\mathcal{N} = 4$ SYM theory is actually larger supergroup that contains the conformal symmetry, R-symmetry and supersymmetry as a subgroup. Let us first recall the definition of conformal symmetry. An infinitesimal space-time transformation $\delta x^\mu = \epsilon \xi^\mu$ is generated by the current $j^\mu = T^{\mu\nu}\xi_\nu$ which is defined from the energy-momentum tensor and the vector field ξ^μ. If this transformation is a symmetry of the theory, the current j^μ is conserved. This condition is equivalent to the following equation

Space-time symmetry and its generator	Conformal Killing vector
Space-time translation P_μ	$\xi_\mu = c_\mu$ (const.)
Lorentz transformations $M_{\mu\nu}$	$\xi_\mu = \omega_{\mu\nu}x^\nu \; (\omega_{\mu\nu} = -\omega_{\nu\mu})$
Dilatation D	$\xi_\mu = \Lambda x_\mu$
Special conformal transformation (SCT) K_μ	$\xi_\mu = 2c_\nu x^\nu x_\mu - x_\nu x^\nu c_\mu$

Table 2.1.1: The table shows all the solutions of conformal Killing equation and the corresponding symmetry generators.

for the vector field ξ^μ

$$\partial_\mu \xi_\nu + \partial_\nu \xi_\mu = \frac{1}{2}\partial_\rho \xi^\rho \eta_{\mu\nu}\,. \tag{2.1.6}$$

This is so-called conformal Killing equation and its solution ξ^μ is called conformal Killing vector. The conformal Killing vectors form a Lie algebra with respect to the commutator of the vector field $[\zeta, \xi]^\mu = \zeta^\nu \partial_\nu \xi^\mu - \xi^\nu \partial_\nu \zeta^\mu$. All the solutions for the conformal Killing equation are given in table 2.1.1. These generators satisfy the following conformal algebra

$$[M_{\mu\nu}, M_{\rho\sigma}] = i(\eta_{\nu\rho}M_{\mu\sigma} + \eta_{\mu\sigma}M_{\mu\rho} - \eta_{\mu\rho}M_{\nu\sigma} - \eta_{\nu\sigma}M_{\mu\rho})\,, \tag{2.1.7}$$

$$[M_{\mu\nu}, P_\lambda] = -i(\eta_{\mu\lambda}P_\nu - \eta_{\lambda\nu}P_\mu)\,, \quad [M_{\mu\nu}, K_\lambda] = -i(\eta_{\mu\lambda}K_\nu - \eta_{\lambda\nu}K_\mu)\,, \tag{2.1.8}$$

$$[D, M_{\mu\nu}] = 0\,, \quad [D, P_\mu] = -iP_\mu\,, \quad [D, K_\mu] = +iK_\mu\,, \tag{2.1.9}$$

$$[P_\mu, K_\nu] = 2i(M_{\mu\nu} - \eta_{\mu\nu}D)\,. \tag{2.1.10}$$

The commutation relations (2.1.7) are indeed Lorentz algebra and (2.1.8) implies that the generator of the translation P_μ and special conformal transformation K_μ transform as vectors under the Lorentz transformation. The dilatation operator D is a scalar under the Lorentz transformation. From the rest of the commutation relations, it turns out that

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we can assign a charge with respect to the dilatation operator to each generator. For instance, P_μ has $+1$ charge since it is a operator with dimension 1.[2] Furthermore, K_μ has -1 charge, hence P_μ and K_μ are identified with the raising operator and the lowering operator respectively. Since the charges of $M_{\mu\nu}$ and D are zero due to the commutativity with D, the charge of the commutator between P_μ and K_μ is zero as expected.

These algebra can be converted into a simple commutation relation as

$$[J_{AB}, J_{CD}] = i(\eta_{AC} J_{BD} + \eta_{BD} J_{AC} - \eta_{BC} J_{AD} - \eta_{AD} J_{BC}) \ \ (A, B = 0, \ldots, 5), \quad (2.1.11)$$

where we have defined new generators as

$$J_{\mu\nu} = M_{\mu\nu} \,, \quad J_{\mu,4} = \frac{1}{2}(P_\mu + K_\nu) \,, \quad J_{\mu,5} = \frac{1}{2}(K_\mu - P_\nu) \,, \quad J_{45} = D \,, \quad (2.1.12)$$

with the metric $\eta_{AB} = \mathrm{diag}(\eta_{\mu\nu}, +1, -1) = \mathrm{diag}(-1, +1, +1, +1, +1, -1)$. Therefore, we can see that the conformal group in four dimension is equivalent to the rotation group that preserves the metric η_{AB}, namely, $SO(2, 4)$. [3] As we will see in section 2.2, this is the same symmetry as the isometry of the AdS_5 space.

We now consider the structure of the full symmetry of the $\mathcal{N} = 4$ theory by the heuristic argument. The theory also possesses maximal supersymmetry in four dimension and thus we have eight super charges Q_α^a ($\alpha = 1, 2; a = 1, \ldots, 4$) and their conjugate charges $\bar{Q}_{\dot{\alpha}a}$ ($\dot{\alpha} = 1, 2; a = 1, \ldots, 4$). Under the R-symmetry, the supercharges Q_α^a, $\bar{Q}_{\dot{\alpha}a}$ transform as anti-fundamental $\bar{\mathbf{4}}$ and fundamental $\mathbf{4}$ respectively. The R-charges R^{IJ} commute with all the generators of the conformal algebra. The anti-commutators between the supercharges are proportional to the generators of the translation as $\{Q, \bar{Q}\} \propto P$, hence the dimensions of the supercharges are $[Q] = 1/2$. Since K_μ is the lowering operator for the dilatation operator, $[K_\mu, Q]$ has the dimension $-1/2$. However, such a operator with dimension $-1/2$ does not appear neither in the conformal algebra nor in the supersymmetry algebra. Therefore, the closure of the whole symmetry algebra requires that there exist spinor charges S_α^a, $\bar{S}_{\dot{\alpha}a}$ whose dimensions are $[S] = -1/2$. These generators are called superconformal charges and they are "square root" of the generators of the special conformal transformation $\{S, \bar{S}\} \propto K$. The action of the raising operator P_μ on S should be the generators with dimension $1/2$, and thus we have $[P, S] = Q$. The remaining (anti-)commutation relations involve both the supercharges and superconformal charges such as $\{Q, S\}$. Since the dimension of these commutators are zero, they should be a linear combination of the bosonic generators with dimension zero such as $M_{\mu\nu}$, R_{IJ}

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[2]One can easily find this from the differential representation $P_\mu \sim \partial_\mu$.

[3]Generally, the conformal group in d space-time dimension is equivalent to $SO(2, d)$. The mapping of generators are given by $M_{\mu\nu} = J_{\mu\nu}, D = J_{dd+1}, P_\mu = J_{\mu d} - J_{\mu d+1}, K_\mu = J_{\mu d} + J_{\mu d+1}$.

and D. Note that the Lorentz indices or the R-symmetry indices for these generators are anti-symmetric or singlet. Thus one finds that the non-vanishing anti-commutators are those of $\{Q, S\}$ or $\{\bar{Q}, \bar{S}\}$. The complete set of the algebra is given in the appendix A.

Actually the symmetry algebra of $\mathcal{N} = 4$ SYM theory is $\mathfrak{psu}(2, 2|4)$ super Lie algebra. The structure of the algebra is schematically expressed as follows

$$\left(\begin{array}{c|c} J_{AB} & Q_{\alpha a}, \bar{S}_{\dot{\alpha} a} \\ \hline \bar{Q}^a_{\dot{\alpha}}, S^a_\alpha & R^{IJ} \end{array}\right). \tag{2.1.13}$$

The diagonal components of the algebra are bosonic generators of $PSU(2, 2|4)$, namely, the generators of $SO(2, 4) \times SO(6) \cong SU(2, 2) \times SU(4)$. The off diagonal components are fermionic generators and they transform as fundamental or anti-fundamental representation under the bosonic symmetry. For example, the right top off-diagonal components $Q_{\alpha a}, \bar{S}_{\dot{\alpha} a}$ transform $(\mathbf{4}, \bar{\mathbf{4}})$ and $(\bar{\mathbf{4}}, \bar{\mathbf{4}})$ under $SU(2, 2) \times SU(4)$ respectively.

2.1.2 Gauge invariant composite operators

In the previous section, we found that $\mathcal{N} = 4$ SYM theory is conformal invariant and together with four sets of supersymmetries, the symmetry is extended to the superconformal symmetry $PSU(2, 2|4)$. Therefore, the spectra of $\mathcal{N} = 4$ theory can be classified according to the unitary irreducible representations of $PSU(2, 2|4)$. Since the symmetry include the conformal algebra, let us first consider the representation of the conformal algebra as a warm up. The generators of the Cartan subalgebra of the conformal group $SO(2, 4)$ are the dilatation and those of the Lorentz group. Hence, the unitary irreducible representations are characterized by three charges $(\Delta; S_1, S_2)$ with respect to the Cartan generators where Δ is an eigenvalue of the dilatation operator D or conformal dimension and S_1, S_2 are two spins of the Lorentz group. In conformal field theories, since the state-operator correspondence (see figure 2.1.2) asserts that there is a one-to-one correspondence between a local field operator in $\mathbb{R}^4$ and a state defined on a time slice of $\mathbb{R} \times S^3$ via a conformal map,[4] it is sufficient to consider the operator $\mathcal{O}(x)$ with the definite conformal dimension Δ and spins S_1, S_2. Under the scale transformation $x^\mu \to \lambda x^\mu$, $\mathcal{O}(x)$ transforms as $e^{-iD}\mathcal{O}(x)e^{iD}$. On the other hand, it transforms as $\mathcal{O}(x) \to \lambda^{-\Delta}\mathcal{O}(\lambda x)$ by definition. Thus we have

$$[D, \mathcal{O}(x)] = i\left(\Delta + x\frac{\partial}{\partial x}\right)\mathcal{O}(x). \tag{2.1.14}$$

In particular, $\mathcal{O}(0)$ is an eigenstate of the dilatation operator with eigenvalue Δ. As already mentioned, the generators of the translation and special conformal transformation

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[4]By the conformal map, the dilatation operator is mapped to the Hamiltonian on $\mathbb{R} \times S^3$.
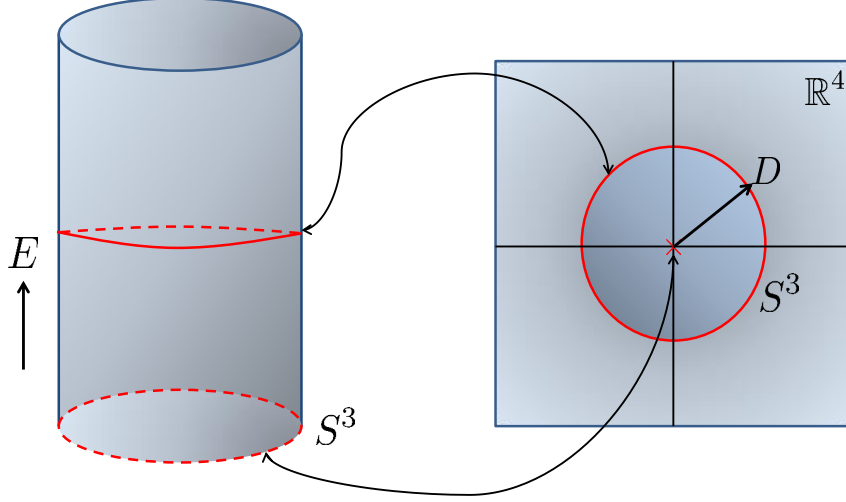
Figure 2.1.1: The figure shows the state-operator correspondence. A state defined at the time slice of $\mathbb{R} \times S^3$ (left) is mapped to a local operator on $\mathbb{R}^4$ (right). The generator of the time shift in $\mathbb{R} \times S^3$, namely the energy E is mapped to the dilatation in $\mathbb{R}^4$.

are the raising and lowering operators for the dilatation operator respectively. Indeed, one can easily find that the operator $[K_\mu, \mathcal{O}(0)]$ has the dimension $\Delta - 1$ from the Jacobi identity and (2.1.9). Since the dimension of the local operator must be positive in an unitary quantum field theory, multiple operations of K_μ to any local operator with finite dimension must annihilate the operator at some point otherwise operators with negative dimension is created and violate the unitarity. Therefore, there should be *conformal primary operators* that satisfy

$$[K_\mu, \mathcal{O}(0)] = 0 \,. \tag{2.1.15}$$

Conformal primary operators are the lowest weight states of the conformal algebra and any descendants are generated by multiple applications of the other generators of the conformal algebra such as P_μ and $M_{\mu\nu}$. The unitary representations are necessarily infinite dimensional since the conformal group $SO(2,4)$ is a non-compact group. For a conformal primary operator, the infinitesimal change under the generic conformal transformation generated by the conformal Killing vector ξ_μ is given by

$$\delta_{\xi_\mu} \mathcal{O}(x) = \left(\xi^\mu \partial_\mu + \frac{\Delta}{4} \partial_\mu \xi^\mu + \frac{1}{2} \partial_\mu \xi_\nu \Sigma^{\mu\nu} \right) \,, \tag{2.1.16}$$

where Σ is a representation matrix of the Lorentz algebra whose highest weight (S_1, S_2) is the same for $\mathcal{O}(x)$. Therefore, unitary irreducible representations of the conformal algebra are characterized by the three charges $(\Delta; S_1, S_2)$.

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We can show that the conformal invariance uniquely fixes the space-time dependence of the two- and three-point functions. For example, let us consider the two-point function of a scalar operator $\langle \mathcal{O}(x)\mathcal{O}(y)\rangle$. The Poincare invariance means that the two-point function only depends on the distance of the two space-time points $|x - y|$. Thus we can set y to be the origin $y = 0$ without loss of generality. From the conformal Ward identity, the two-point function satisfies

$$\langle \delta_{\xi^\mu}\mathcal{O}(x)\mathcal{O}(0)\rangle + \langle \mathcal{O}(x)\delta_{\xi^\mu}\mathcal{O}(0)\rangle = (x^\mu \partial_\mu + 2\Delta)\langle \mathcal{O}(x)\mathcal{O}(0)\rangle\,, \tag{2.1.17}$$

where $\xi^\mu = x^\mu$ is the conformal Killing vector that corresponds to the dilatation and the variation of the field is given by (2.1.14) and this equation can be easily solved to give

$$\langle \mathcal{O}(x)\mathcal{O}(0)\rangle = \frac{C}{|x|^{2\Delta}}\,. \tag{2.1.18}$$

We now consider the representations of the full superconformal algebra, namely, the algebra of $PSU(2,2|4)$. The bosonic symmetry is $SO(2,4) \times SO(6) \cong SU(2,2) \times SU(4)$ and the representations are labeled by 6 charges

$$[\Delta; S_1, S_2; J_1, J_2, J_3]\,, \tag{2.1.19}$$

where Δ, S_1 and S_2 are the charges associated with the conformal symmetry and J_1, J_2 and J_3 are the charges of the $R-$symmetry such as R_{12}, R_{34} and R_{56}. Let us consider the operator $\mathcal{O}(x)$ with definite charges (2.1.19). Since the superconfomal charges satisfy $\{S, \bar{S}\} \propto K$ and have the dimension $-1/2$, commuting the superconformal charges lower the dimension of operators by $1/2$. Therefore, there exist *superconformal primary operators*, which are annihilated by the action of any superconformal charges as follows

$$[S_\alpha^a, \mathcal{O}(0)] = [\bar{S}_{\dot{\alpha}a}, \mathcal{O}(0)] = 0\,, \ (\text{for any } \alpha, \dot{\alpha}, a)\,. \tag{2.1.20}$$

Note that superconformal primary operators are indeed conformal primary operator or satisfy (2.1.15) while conformal operators are not necessarily superconformal operators. The descendant operators are created from the superconformal primaries by the action of the generators of $PSU(2,2|4)$.

In particular, we are interested in a subclass of superconformal operators so-called the *chiral primary operators* whose multiplets are shorten due to the BPS condition

$$[Q_{\alpha a}, \mathcal{O}(0)] = 0 \ (\text{for some } \alpha, a)\,. \tag{2.1.21}$$

Chiral primary operators are annihilated by the superconformal charges and some of the supercharges by definition, and thus it turns out that they should satisfy the following equation from the Jacobi identity

$$[\{Q_{\alpha a}, S_\beta^b\}, \mathcal{O}(0)] = [+(\sigma^{IJ})_a^b \varepsilon_{\alpha\beta} R_{IJ} + i(\sigma^{\mu\nu})\delta_a^b M_{\mu\nu} - iD\varepsilon_{\alpha\beta}\delta_a^b, \mathcal{O}(0)] = 0\,, \tag{2.1.22}$$

where we have used the superconformal algebra. This equation implies that the dimension of the chiral primary operators are determined from their charges. To see this more explicitly, suppose that the operator $\mathcal{O}(x)$ is a scalar operator. It commutes with the Lorentz generators and we have

$$[R_{IJ}, \mathcal{O}(0)](\sigma^{IJ})_a^b = \Delta \delta_a^b \mathcal{O}. \tag{2.1.23}$$

If we choose the Cartan generators of the $SO(6) \cong SU(4)$ to be R_{12}, R_{34} and R_{56}, we can take the corresponding fundamental representation as follows

$$\sigma^{12} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \sigma^{34} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \sigma^{56} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \tag{2.1.24}$$

One can see that a primary operator with the charges $[J; 0, 0; J, 0, 0]$ is annihilated by the supercharges $Q_{\alpha 1}$, $Q_{\alpha 2}$, $\bar{Q}_{\dot\alpha 3}$ and $\bar{Q}_{\dot\alpha 4}$. Since it is annihilated by 8 supercharges, it is a half-BPS operator. By the same argument, it turns out that operators with the charges $[J; 0, 0; 0, J, 0, 0]$ and $[J; 0, 0; 0, 0, J]$ are annihilated by $Q_{\alpha,1}, Q_{\alpha,3}, \bar{Q}_{\dot\alpha 2}, \bar{Q}_{\dot\alpha 4}$ and $Q_{\alpha,1}, Q_{\alpha,4}, \bar{Q}_{\dot\alpha 2}, \bar{Q}_{\dot\alpha 3}$, respectively. However, these states are in the same representation as the operator with the charges $[J; 0, 0; J, 0, 0]$, thus they are in the same $PSU(2, 2|4)$ multiplet as well.

Chiral primary or BPS operators are of particular importance since their conformal dimension have no quantum corrections due to the supersymmetry. In general, the dimension of an operator depends on the coupling constant and it differs from the bare dimension, which is the value of the dimension at zero coupling. The difference from the bare dimension is known to be the anomalous dimension. The anomalous dimensions for chiral primary operators must vanish by the following argument. The multiplets of chiral primary operators are smaller than the other multiplets since they are annihilated by some of the supercharges and a part of the descendants is absent in the multiplets. Therefore, to obtain the anomalous dimensions for chiral primaries, they should form a larger multiplet and get extra degrees of freedom at each level. However, this is impossible since the number of independent operators with a given dimension and charges are finite integer and it cannot be changed continuosly by varying the coupling constant. In other words, the dimensions for the BPS multiplets are uniquely determined from the quantized charges and they are independent of the coupling. Thus, we conclude that the dimensions for chiral primaries do not receive quantum corrections. Moreover, all the operators in the same superconformal multiplet have the same anomalous dimensions. This is because

the generators of the superconformal algebra can only change the dimension of operator by integer or half-integer.

We have discussed the general properties of the operators according to the representations of the superconformal algebra $PSU(2,2|4)$. We will construct such operators from the fundamental fields that appear in the action. The physical observables in gauge theory must be gauge invariant. All the fundamental fields in $\mathcal{N} = 4$ SYM theory are in the adjoint representation of the gauge group, hence the real scalar fields ϕ^I, the Weyl fermions ψ^a_α, $\psi_{\dot\alpha a}$, the covariant derivative D_μ and the field strength $F_{\mu\nu}$ transform adjointly under the gauge transformation as follows

$$\chi(x) \longrightarrow U^\dagger(x)\chi(x)U(x)\,,\ U(x) \in SU(N)\,, \tag{2.1.25}$$

where $\chi(x)$ represents the above fundamental fields and U is a function of space-time and takes values in gauge group. Thus, we find that composite operators with the trace of the product of the fundamental fields are manifestly gauge invariant.

$$\mathcal{O}(x) = \text{Tr}[\chi_1(x)\cdots\chi_n(x)]\,. \tag{2.1.26}$$

This type of composite operators are called the single trace operators. The single trace operators correspond to single string states propagating on the $AdS_5 \times S^5$ background. The multi trace operators which are the products of the single trace operators are also gauge invariant. However, in the large N limit, the correlation functions of the multi trace operators are factorized into the product of the correlation functions of the single trace operators. In particular, the conformal dimension of the multi trace operator is the sum of the dimensions of the single trace operators. These facts mean that there are no bound states of strings and the energy of multi string state is the sum of each single string state. Therefore, we will concentrate on the spectrum of the single trace operators.

Let us construct the superconformal primary operators from the fundamental fields. Note that the action of supercharges on the operators generates the descendants. The supersymmetry transformations for the fundamental fields are schematically given by

$$[Q,\phi] \sim \psi\,,\ \{Q,\psi\} \sim F^+ + [\phi,\phi]\,,$$
$$\{Q,\bar\psi\} \sim D\phi\,,\ [Q,F] \sim D\psi\,. \tag{2.1.27}$$

Hence, the Weyl fermions ψ, the covariant derivative D_μ, the field strength $F_{\mu\nu}$ and the commutators of the scalar fields $[\phi^I,\phi^J]$ do not appear in the superconformal primary operators since they are descendants. In other words, super conformal primary operators are only made up of the scalar fields in a symmetrized way with respect to the indices for

the R-symmetry. To assign the definite R-charges for the scalar fields, let us introduce the following complex linear combinations of the scalar fields

$$Z = \phi^1 + i\phi^2 \,, \quad Y = \phi^3 + i\phi^4 \,, \quad X = \phi^5 + i\phi^6 \,. \tag{2.1.28}$$

These complex scalars have the following R-symmetry charges

$$[R_{12}, Z] = +Z \,, \quad [R_{34}, Y] = +Y \,, \quad [R_{56}, X] = +X \,, \tag{2.1.29}$$

and the complex conjugate fields have the opposite sign of the R-charges. Thus, the single trace operator $\text{Tr}[Z^J]$ has the charges $[J; 0, 0; J, 0, 0]$ and it is a chiral primary operator. Furthermore, the operators $\text{Tr}[X^J]$ and $\text{Tr}[Y^J]$ have the charges $[J; 0, 0; 0, 0, J]$, $[J; 0, 0; 0, J, 0]$, respectively, and are chiral primaries as well. Generally, any operator of the following form is also a half-BPS operator

$$\mathcal{O}(x) = \zeta_{I_1 \ldots I_J} \text{Tr}[\phi^{I_1} \cdots \phi^{I_J}] \,, \tag{2.1.30}$$

where $\zeta_{I_1 \ldots I_J}$ is a completely symmetric tensor and the trace of any two indices vanishes. This operator is belong to the complete symmetric representation of the $SO(6)$ R-symmetry and $\text{Tr}[Z^J]$ is the highest weight state of this representation.

2.2 String on $AdS_5 \times S^5$

In this subsection, we will review the string theory on $AdS_5 \times S^5$ background. $AdS_5 \times S^5$ geometry appears as the near horizon geometry of the black 3-brane solution for Type IIB SUGRA, which is the low energy effective theory of the Type IIB string theory. This "vacuum" of string theory is a product of two maximal symmetric spaces and supported by the 5-form RR-flux. The presence of the RR-flux make it difficult to formulate string theory within the RNS formalism since RR-vertex operators are composed of bi-spinor fields and the non-trivial background of RR-flux leads to a quite complicated action for RNS-string. Thus, the RNS formalism is not suitable for our purpose and the Green-Schwartz formalism, in which the space-time supersymmetry is manifest, seems to be more adequate when the RR-flux is non-vanishing. We will first summarize the basic facts of the AdS space-time and the bosonic string moving on such background in section 2.2.1. Then, we will review the Green-Schwartz formalism based on the coset construction in section 2.2.2.

2.2.1 AdS geometry and Bosonic action

As shown in [16], the type IIB string theory on $AdS_5 \times S^5$ is described as a non-linear sigma model on the supercoset $\frac{PSU(2,2|4)}{SO(1,4) \times SO(5)}$. Although the full action is rather complicated due

to the presence of the fermionic degrees of freedom, the action for the bosonic degrees of freedom is the form of non-linear sigma model with target space $AdS_5 \times S^5 \cong \frac{SO(2,4)}{SO(1,4)} \times \frac{SO(6)}{SO(5)}$ and it is the Polyakov action with the target space metric $G_{\mu\nu}(x)$.

Let us summarize the facts about the bosonic action and AdS space-time. $d + 1$ dimensional AdS space-time is a solution of the Einstein equation with the negative cosmological constant and it is the the maximal symmetric space-time with negative constant curvature. It is defined as a hypersurface in $d + 2$ dimensional flat space with signature $(2, d - 1)$ $\mathbb{R}^{2,d-1}$ as follows

$$-X_{-1}^2 - X_0^2 + X_1^2 + \cdots + X_d^2 = -R^2 \,, \tag{2.2.1}$$

where $X_{-1}, \ldots, X_d$ are the embedding coordinates. The metric of the hypersurface is induced from the flat space metric η_{PQ}

$$ds^2 = \eta_{PQ} dX^P dX^Q \,, \quad \eta_{PQ} = \text{diag}(-1, -1, 1, \ldots, 1) \,. \tag{2.2.2}$$

This expression is manifestly $SO(2, 4)$ invariant. By introducing $X^2 = \sum_{i=1}^d X_i^2$, $Y^2 = X_{-1}^2 + X_0^2$, the defining equation (2.2.1) can be rewritten as $X^2 - Y^2 = -R^2$, thus we can express the metric in terms of "polar coordinates" as follows

$$ds^2 = R^2(-\cosh^2 \rho \, dt^2 + d\rho^2 \sinh^2 \rho \, d\Omega_{d-1}^2) \,, \tag{2.2.3}$$

$$X_{-1} = R \cosh \rho \cos t \,, \quad X_0 = R \cosh \rho \sin t \,, \quad X^2 = R^2 \sinh^2 \rho \,, \tag{2.2.4}$$

where t is the global time coordinate, ρ is the radial coordinate and $d\Omega_{d-1}^2$ is the metric of the unit sphere S^{d-1}. This coordinate patch is called global coordinate since this coordinate system covers the whole AdS_{d+1}. The topology of the AdS_{d+1} is the $S^1 \times \mathbb{R}^d$ and it has the time-like closed circle. Therefore, we usually consider the universal covering space $\widetilde{AdS_{d+1}}$ whose topology is R^{d+1}. Another important coordinate system is the Poincare coordinate. To obtain this expression, let us define new coordinates

$$U = X_{-1} + X_d \,, \quad V = X_{-1} - X_d \,, \quad x^\mu = \frac{X^\mu R}{U} \quad (\mu = 0 \ldots d - 1) \,. \tag{2.2.5}$$

From the defining equation, V can be solved and $V = Ux^2/R^2 + R^2/U$. Putting this into the original metric $ds^2 = \eta_{PQ} dX^P dX^Q = -dU dV + dX_\mu dX^\mu$, we find

$$ds^2 = \frac{U^2}{R^2} dx^\mu dx_\mu + R^2 \frac{dU^2}{U^2} \,. \tag{2.2.6}$$

We further introduce $z = R^2/U$ and the metric becomes

$$ds^2 = R^2 \frac{dz^2 + dx^\mu dx_\mu}{z^2} \,. \tag{2.2.7}$$

This is the metric in the Poincare coordinate patch. Note that the coordinate system (U, x^μ) and (z, x^μ) do not cover the whole AdS_{d+1} since U (z) cannot be distinguished from $-U$ $(-z)$ and should be restricted to the region $U > 0$ $(z > 0)$. Therefore, these coordinate patch cover the half of the AdS space-time.

Let us discuss on the boundary of AdS_{d+1} on which the dual CFT is defined. The boundary of the AdS_{d+1} is defined as the limit $X_i \to \infty$. With the rescaling $X_i \to \lambda X_i$, the embedding equation becomes the following form in the limit $\lambda \to \infty$

$$X_{-1}^2 + X_0^2 = X_1^2 + \cdots + X_d^2 \,. \tag{2.2.8}$$

This equation enjoys the projective invariance $X_i \sim \lambda X_i$ reflecting the dilatation invariance of the boundary CFT. The topology of the boundary is $S^1 \times S^{d-1}$. This can be easily seen by setting the right hand side of the above equation to be 1 by simultaneous rescaling. On the global coordinate, the boundary is defined as the limit $\rho \to \infty$ and the metric becomes $ds^2 = R^2 e^{2\rho}(-dt^2 + d\Omega_{d-1}^2)$. After conformal transformation $ds^2 \to e^{-2\rho} ds^2$, we have

$$ds_{\text{global}}^2 = R^2(-dt^2 + d\Omega_{d-1}^2) \,. \tag{2.2.9}$$

This is the metric of "de Sitter space" and the topology of the boundary is $\mathbb{R} \times S^{d-1}$ after taking the universal covering. It is more convenient to consider the boundary in the Poincare patch. The boundary is defined by the limit $z \to 0$ in this patch. After conformal rescaling $ds^2 \to z^2 ds^2$, the metric becomes

$$ds_{\text{Poincare}}^2 = R^2 \eta_{\mu\nu} dx^\mu dx^\nu \,. \tag{2.2.10}$$

This is the metric of d dimensional flat Minkowski space. Thus, the quantum field theory is naturally defined in this coordinate patch and it is often used in the context of the AdS/CFT correspondence.

We now discuss the bosonic string on $AdS_5 \times S^5$ background. Of course we must include the fermionic degrees of freedom for quantum consistency of the theory, however, is is sufficient to consider the bosonic part of the action for the purpose of constructing classical solutions. Furthermore, this example gives us an intuition about the supersymmetric generalization. The action for the bosonic string is given by

$$S_B = \frac{-1}{4\pi\alpha'} \int d^2\sigma \sqrt{-g} g^{ij} G_{\mu\nu}(X) \partial_i X^\mu \partial_j X^\nu \,, \tag{2.2.11}$$

where g_{ij} is the world-sheet metric and $G_{\mu\nu}$ is the target space metric. After going into

the conformal gauge $g_{ij} \to \gamma_{ab} = \mathrm{diag}(-1,+1)$, the action becomes

$$S = \frac{\sqrt{\lambda}}{2\pi} \int d\tau \int_0^{2\pi} d\sigma (\mathcal{L}_{AdS_5} + \mathcal{L}_{S^5}), \tag{2.2.12}$$

$$\mathcal{L}_{AdS_5} = -\frac{1}{2}\eta_{PQ}\partial_a X^P \partial^a X^Q + \frac{1}{2}\tilde{\Lambda}(X^2 + 1), \tag{2.2.13}$$

$$\mathcal{L}_{S^5} = -\frac{1}{2}\delta_{IJ}\partial_a Y^I \partial^a Y^J + \frac{1}{2}\Lambda(Y^2 - 1), \tag{2.2.14}$$

where X^P and Y^I are the embedding coordinates of the AdS and the sphere respectively. The auxiliary fields $\tilde{\Lambda}$ and Λ are the Lagrange multipliers to impose the constrains $\eta_{PQ}X^P X^Q = -1$ and $Y^I Y_I = 1$ which enforce the string to move on $AdS_5 \times S^5$. Note that we have rescaled the embedding coordinate so that the right hand sides of the constraints such as (2.2.1) become ∓ 1. The radius of the AdS or the sphere is related to the 'tHooft coupling via the following relation: $\sqrt{\lambda} = R^2/\alpha'$. The equations of motion are given as follows

$$(\partial_a \partial^a + \tilde{\Lambda})X^P = 0, \quad (\partial_a \partial^a + \Lambda)Y^I = 0. \tag{2.2.15}$$

By using $\eta_{PQ}X^P X^Q = -1$ and its second order derivative $\eta_{PQ}(\partial_a\partial^a X^P X^Q + \partial_a X^P \partial^a X^Q) = 0$, we can solve the auxiliary field as $\tilde{\Lambda} = -\eta_{PQ}\partial_a X^P \partial^a X^Q$. Similarly, we have $\Lambda = \partial_a Y^I \partial^a Y_I$. Moreover, we have to impose the Virasoro constraints, which are equivalent to the vanishing of the energy-momentum tensor $T_{ab} = \delta S/\delta g^{ab}$. They are given by

$$0 = T_{\tau\tau} + T_{\sigma\sigma} = \eta_{PQ}(\partial_\tau X^P \partial_\tau X^Q + \partial_\sigma X^P \partial_\sigma X^Q) + \partial_\tau Y^I \partial_\tau Y_I + \partial_\sigma Y^I \partial_\sigma Y_I, \tag{2.2.16}$$

$$0 = T_{\tau\sigma} + T_{\sigma\tau} = \eta_{PQ}\partial_\tau X^P \partial_\sigma X^Q + \partial_\tau Y^I \partial_\sigma Y_I. \tag{2.2.17}$$

The equations of motion for the AdS part and the sphere part are decoupled. However, they are coupled via the Virasoro constraints.

The action is manifestly invariant under the global $SO(2,4) \times SO(6)$ transformations. In the following global coordinate system, the global transformations are expressed as the shifts of the angular variables

$$X_1 + iX_2 = \sinh\rho\cos\gamma e^{i\phi_1}, \; X_3 + iX_4 = \sinh\rho\sin\gamma e^{i\phi_2}, \; X_{-1} + iX_0 = \cosh\rho e^{it}, \tag{2.2.18}$$

$$Y_1 + iY_2 = \sin\psi\cos\theta e^{i\varphi_1}, \; Y_3 + iY_4 = \sin\psi\sin\theta e^{i\varphi_2}, \; Y_5 + iY_6 = \cosh\psi e^{i\varphi_3}. \tag{2.2.19}$$

The corresponding Noether currents and charges are given as follows

$$j_{PQ}^a = \sqrt{\lambda}(X_P \partial^a X_Q - X_Q \partial^a X_P), \; S_{PQ} = \int_0^{2\pi} \frac{d\sigma}{2\pi}j_{PQ}^0, \tag{2.2.20}$$

$$j_{IJ}^a = \sqrt{\lambda}(Y_I \partial^a Y_J - Y_J \partial^a Y_I), \; S_{IJ} = \int_0^{2\pi} \frac{d\sigma}{2\pi}j_{IJ}^0. \tag{2.2.21}$$

The natural choice of the Cartan generators for the isometry $SO(2,4) \times SO(6)$ of $AdS_5 \times S^5$ is the following

$$E = S_{-10}, \quad S_1 = S_{12}, \quad S_2 = S_{34}, \tag{2.2.22}$$

$$J_1 = J_{12}, \quad J_2 = J_{34}, \quad J_3 = J_{56}. \tag{2.2.23}$$

E is the so-called AdS energy which is the charge conjugate to the AdS global time t and it generates the shift of t. $S_{1,2}$ and $J_{1,2,3}$ are the AdS and the sphere spins which generate the shifts of angular variables $\phi_{1,2}$ and $\varphi_{1,2,3}$, respectively. The other charges generate the rotations for the axes. Once we identifies the -1-th indice in the generators with the 5-th indice in (2.1.12), we obtain the following map between the generators of the isometry for AdS_5 and the conformal group in four dimension.

$$S_{\mu\nu} \leftrightarrow M_{\mu\nu}, \quad S_{\mu,-1} + S_{\mu,4} \leftrightarrow P_\mu, \quad S_{\mu,-1} - S_{\mu,4} \leftrightarrow K_\mu, \quad S_{-1,4} \leftrightarrow D. \tag{2.2.24}$$

In particular, the AdS energy E corresponds to $\frac{1}{2}(K_0 + P_0)$. The string spectrum is characterized by the six charges

$$[E; S_1, S_2, J_1, J_2, J_3]. \tag{2.2.25}$$

According to the dictionary of the AdS/CFT, this should precisely match with the charges of the dual SYM operator (2.1.19). Thus, we find one of the key proposal of the AdS/CFT,

$$E\left(g_{st}, \frac{R^2}{\alpha'}\right) = \Delta\left(\frac{1}{N}, \lambda\right), \tag{2.2.26}$$

under the identifications of the parameters $4\pi g_{st} = g_{YM}^2 = \lambda/N$, $R^4/\alpha'^2 = \lambda$. This correspondence is rather non-trivial since the AdS energy E does not directly correspond to the dilatation operator D but to $\frac{1}{2}(K_0 + P_0)$. In other words, the conformal multiplets in SYM theory are build upon the vacuum which is annihilated by the action of all the generators of the special conformal transformation K_μ, while the multiplets in the AdS are build upon the vacuum which is annihilated by the lowering operators with respect to the AdS energy. Therefore, the relation between the spectrum of SYM and the string on AdS are "twisted" via the similarity transformation with $\exp[\frac{\pi}{2}S_{04}]$. Note that this is a non-unitary similarity transformation. However, if we consider the *Euclidean AdS*, the two schemes are essentially the same since the above similarity transformation becomes unitary and the two representations are equivalent. Indeed, the boundary of the Euclidean AdS_{d+1} is S^d and it can be mapped to $\mathbb{R}^4$ by a conformal map. Then, the global time coordinate t is mapped to the radial coordinate, which directly corresponds to the dilatation.

2.2.2 Green-Schwartz action and coset construction

Here, we clarify the schematic structure of the action for string on $AdS_5 \times S^5$ to construct the supersymmetric generalization of the bosonic action. For this purpose, it is convenient to recall that the Green-Schwartz action in the ten dimensional flat Minkowski space.

$$
S_{GS}^{\text{flat}} = \frac{1}{2\pi\alpha'} \int_\Sigma d^2\sigma \left[-\frac{1}{2}\sqrt{-g}g^{ij}\eta_{\mu\nu}(\partial_i x^\mu - i\bar{\theta}^I\Gamma^\mu\partial_i\theta_I)(\partial_j x^\nu - i\bar{\theta}^J\Gamma^\nu\partial_j\theta_J) \right.
$$
$$
\left. -i\varepsilon^{ij}s_{IJ}\bar{\theta}^I\Gamma^\mu\partial_j\theta^J\left(\partial_i x_\mu - \frac{1}{2}\bar{\theta}^K\Gamma_\mu\partial_i\theta_K\right) \right], \qquad (2.2.27)
$$

where (x^μ, θ^I) $(I = 1, 2)$ are the coordinates of the target superspace with Majorana-Weyl fermion θ^I and $s_{IJ} = \text{diag}(1, -1)$. This action has the manifest space-time supersymmetry defined as the following transformation

$$
\delta\theta^I = \epsilon^I, \quad \delta x^\mu = i\bar{\epsilon}^I\Gamma^\mu\theta_I, \qquad (2.2.28)
$$

where ϵ is a constant space-time spinor. The first term in (2.2.27) is manifestly invariant under this transformation since it can be written in terms of a SUSY invariant 1-form $\Pi_i^\mu = \partial_i x^\mu - i\bar{\theta}^I\Gamma^\mu\partial_i\theta_I$. The invariance of the second term under the supersymmetry is rather non-trivial since we have to use a Fierz identity of the form $2\bar{\epsilon}\Gamma^\mu\psi_{[1}\bar{\psi}_2\Gamma_\mu\psi_{3]} = 0$ which holds for ten dimensional Majorana-Weyl spinors. The second term is so-called Wess-Zumino term which is necessary for realizing the supersymmetry at the spectrum level. Suppose that the Wess-Zumino term does not appear in the action, then the number of on-shell bosonic degrees of freedom and that of fermionic degrees of freedom are not equal. Indeed, the number of the real components of Majorana-Weyl fermions in ten dimension is 16, which is reduced to 8 by the equation of motion. The action contains two copies of such Majorana-Weyl fermions, thus we have 16 on-shell fermionic degrees of freedom, while the number of on-shell degrees of freedom for bosonic vector variables is 8 after gague fixing world-sheet diffeomorphism. However, the action (2.2.27) has an extra fermionic local symmetry which is known to be κ-symmetry, due to the presence of the Wess-Zumino term. With local fermionic parameters $\kappa^I(\sigma)$ and the projectors $P_\pm$ that satisfy $P_\pm^2 = P_\pm$, the κ-symmetry is defined as

$$
\delta_\kappa\theta_I = \Gamma_\mu\Pi_i^\mu\kappa_I^i, \quad \delta_\kappa x^\mu = i\bar{\theta}^I\Gamma^\mu\delta_\kappa\theta^I, \qquad (2.2.29)
$$
$$
\delta_\kappa(\sqrt{-g}g^{ij}) = -8i\sqrt{-g}(P_+^{ik}\partial_k\bar{\theta}_1\kappa_1^j + P_-^{ik}\partial_k\bar{\theta}_2\kappa_2^j), \qquad (2.2.30)
$$
$$
\kappa_1^i = P_+^{ij}\kappa_j^1, \quad \kappa_2^i = P_-^{ij}\kappa_j^2, \quad P_\pm^{ij} = \frac{1}{2}\left(g^{ij} + \frac{\varepsilon^{ij}}{\sqrt{-g}}\right). \qquad (2.2.31)
$$

The κ-symmetry removes the half of the fermionic degrees of freedom and ensures that the supersymmetry is realized at the physical spectrum level.

The bosonic part of the action is the usual Polyakov action and we can obtain the bosonic action in curved space-time by replacing the background metric as $\eta_{\mu\nu} \to G_{\mu\nu}(x)$. At quadratic order, the fermionic part of the action can be rewritten as follows

$$S_F = \frac{i}{2\pi\alpha'} \int d^2\sigma (\sqrt{-g}g^{ij}\delta_{IJ} - \varepsilon^{ij}s_{IJ})\bar{\theta}^I \rho_i \partial_j \theta^J + O(\theta^4) \,, \qquad (2.2.32)$$

where $\tilde{\rho}_i = \partial_i x^\mu \Gamma_\mu$ is the pull-back of the ten dimensional gamma matrix to the world-sheet. Hence, the natural generalization to the curved background is obtained by the replacement $\tilde{\rho}_i \to \rho_i = \Gamma_A E^A_\mu \partial_i x^\mu$ and $\partial_i \theta^I \to D_i \theta$. Here, E^A_μ is the target space vierbein that satisfies $E^A_\mu E^B_\nu \eta_{AB} = G_{\mu\nu}$ and D_i is the pull-back of the covariant derivative of the target space which is explicitly given by

$$D_i\theta^I = \left(\partial_i + \frac{1}{4}\partial_i x^\mu \omega_\mu^{AB}\Gamma_{AB} + \frac{1}{8 \cdot 5!}\rho_i \Gamma_{\mu_1...\mu_5} F^{\mu_1...\mu_5} \right) \theta^I \,. \qquad (2.2.33)$$

Note that the RR-flux $F^{\mu_1...\mu_5} \sim \varepsilon^{\mu_1...\mu_5}$ appears in the covariant derivative to assure the supersymmetry. Therefore, the Green-Schwartz action in $AdS_5 \times S^5$ background can be guessed from the flat space action (2.2.27) at quadratic order

$$S_{GS} = \frac{1}{2\pi\alpha'} \int_\Sigma d^2\sigma \left[-\frac{1}{2}\sqrt{-g}g^{ij}G_{\mu\nu}(x)\partial_i x^\mu \partial_j x^\nu - i(\sqrt{-g}g^{ij}\delta_{IJ} - \varepsilon^{ij}s_{IJ})\bar{\theta}^I \rho_i \partial_j \theta^J + O(\theta^4) \right] \,.$$
$$(2.2.34)$$

The full Green-Schwartz action on $AdS_5 \times S^5$ background was constructed in [16], based on the coset construction. The guiding principles that determines the action are the following conditions.

- **Bosonic action**

  The bosonic part of the action is the Polyakov action with the $AdS_5 \times S^5$ target space metric.

- **Global $PSU(2,2|4)$ symmetry**

  The isometry group of $AdS_5 \times S^5$ is $SO(2,4) \times SO(6)$. This background geometry possesses two copies of the Killing spinor which satisfies $D_\mu \epsilon^I = 0$. The superisometry group for this background generated by Killing vectors and spinors is $PSU(2,2|4)$ which includes $SO(2,4) \times SO(6)$ as the bosonic subgroup. The action should have the manifest $PSU(2.2|4)$ symmetry.

- **Local κ-symmetry**

  The action should be invariant under the κ-symmetry which is a fermionic local symmetry to realize the space-time supersymmetry at the on-shell spectrum level.

- **Flat space limit**

  In the appropriate flat space limit ($R \to \infty$), the action is reduced to the flat space action (2.2.27).

These conditions uniquely fix the form of the action. In [16], the action is constructed based on the supercoset $\frac{PSU(2,2|4)}{SO(1,4)\times SO(5)}$.

It is convenient to rewrite the flat space action (2.2.27) in terms of the coset language to obtain an intuition. From the coset point of view, the action (2.2.27) describes the sigma model whose target space is G/H where G is the $\mathcal{N}=2$ super Poincare group and H is the ten dimensional Lorentz group $SO(1,9)$. Following the usual coset construction, let us take a representative element for G/H as follows

$$G(x,\theta) = \exp\left[x^\mu P_\mu + \theta^{\alpha I} Q_{\alpha I}\right] , \qquad (2.2.35)$$

where (x^μ, θ^I) ($I=1,2$) are the coordinate of the $\mathcal{N}=2$ superspace. Notice that the generators of the translation P_μ and the supercharges $Q_{\alpha I}$ play the same role as "broken generators" in the coset construction. Hence, the coordinates (x^μ, θ^I) parametrize the "vacuum manifold" G/H and are similar to the pion fields. The left invariant 1-form associated with this space is given by

$$J = G^{-1}dG = L^\mu P_\mu + L^{\alpha I} Q_{\alpha I} . \qquad (2.2.36)$$

Since the generators of the translation mutually commute $[P_\mu, P_\nu] = 0$ and the anti-commutators for the supercharges are in proportion to P_μ, $\{Q,Q\} \propto P$, the left invariant current can be expanded in the basis P, Q and we denote the coefficients as L^μ and $L^{\alpha I}$ respectively. They are explicitly given by

$$L^\mu = dx^\mu - i\bar{\theta}^I \Gamma^\mu d\theta_I , \quad L^{\alpha I} = d\theta^{\alpha I} . \qquad (2.2.37)$$

Since the current J is invariant under the global left multiplications of the super Poincare group, the coefficients L^μ, $L^{\alpha I}$ transform covariantly under the action of G. Hence, they are the fundamental building blocks to construct the invariant action. In fact, the first term in (2.2.27) can be expressed as follows

$$S_K = -\frac{T}{2} \int d^2\sigma \sqrt{-g}\, g^{ij} \eta_{\mu\nu} L_i^\mu L_j^\nu = -\frac{T}{2} \int L^\mu \wedge *L_\mu , \qquad (2.2.38)$$

where $T = \frac{1}{2\pi\alpha'}$ is the string tension and $*$ is the Hodge star operator. Similarly, the second term in (2.2.27) or the Wess-Zumino term can be written in terms of the 1-forms

$$S_{WZ} = T \int_{M_3} h , \quad h = i s_{IJ} L^\mu \wedge \bar{L}^I \Gamma_\mu \wedge L^J , \qquad (2.2.39)$$

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where the boundary of the integration region M_3 is the world-sheet Σ. This term seems to be different from the second term in (2.2.27) at first glance, however, the 3-form h is actually the closed form, $dh = 0$.[5] Therefore, it can be locally written as an exact form $h = d\varpi$ and turns out to be equivalent to the original expression. Although the form ϖ is no more written in terms of the left invariant 1-forms and the invariance under the symmetry is not manifest, the variations with respect to super Poincare transformations generate exact forms. Thus, the total action $S_K + S_{WZ}$ is invariant under the symmetry up to surface terms. Due to the presence of the Wess-Zumino term, the action also enjoys the invariance under the κ-symmetry. It is known that the action of the κ-transformation on the group element $G(x, \theta)$ can be regarded as a particular right multiplication which depends on fermionic parameters.

We next consider the Green-Schrwartz superstring on $AdS_5 \times S^5$. The bosonic part of the target space is given by $AdS_5 \times S^5$ and it is indeed obtained as the following coset space

$$\frac{SU(2,2) \times SU(4)}{SO(1,4) \times SO(5)} \cong \frac{SO(2,4)}{SO((1,4)} \times \frac{SO(6)}{SO(5)} = AdS_5 \times S^5 \qquad (2.2.40)$$

The numerator group is the isometry group and the denominator group is the stability group for $AdS_5 \times S^5$. In fact, there is a one-to-one correspondence for the sphere (AdS) and the coset $SO(d+1)/SO(d)$ $(SO(2, d-1)/SO(1, d-1))$ since we can reach any point on the space starting from a point by the action of isometry group modulo its stability group. For example, any point on the sphere can be reached from the north pole by the action of rotation group modulo its stability group, namely, the rotation around the axes through the north pole. The full target space for the superstring is obtained by replacing the numerator group with the superisometry group $PSU(2, 2|4)$ and thus it is represented by the following supercoset

$$\frac{PSU(2, 2|4)}{SO(1, 4) \times SO(5)}. \qquad (2.2.41)$$

Therefore, it is necessary to consider the algebraic structure of $PSU(2, 2|4)$ together with its subgroup $SO(1, 4) \times SO(5)$. In what follows, many indices will appear, hence, we summarize the notations here following the conventions of [16].

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[5]To prove the closedness of h, it is necessary to use the Fierz identity.

$A, B, C, \ldots = -1, 0, \ldots, 4$	$SO(2,4)$ vector indices
$A', B', C', \ldots = 5, \ldots, 9, 10$	$SO(6)$ vector indices
$\hat{A}, \hat{B}, \hat{C}, \ldots = -1, \ldots, 10$	Pairs of vector indices $(A, A'), (B, B'), (C, C'), \ldots$
$a, b, c, \ldots = 0, \ldots, 4$	$SO(1,4)$ vector indices (AdS_5 tangent space)
$a', b', c', \ldots = 5, \ldots, 9$	$SO(5)$ vector indices (S^5 tangent space)
$\hat{a}, \hat{b}, \hat{c}, \ldots = 0, \ldots, 9$	Pairs of vector indices $(a, a'), (b, b'), (c, c'), \ldots$
$\alpha, \beta, \gamma, \ldots = 1, \ldots, 4$	$SO(1,4)$ spinor indices (AdS_5)
$\alpha', \beta', \gamma', \ldots = 1, \ldots, 4$	$SO(5)$ spinor indices (S^5)
$\eta_{AB} = \mathrm{diag}(-1, -1, 1, 1, 1, 1)$	$SO(2,4)$ invariant tensor
$\eta_{A'B'} = \mathrm{diag}(1, 1, 1, 1, 1, 1)$	$SO(6)$ invariant tensor
$\eta_{ab} = \mathrm{diag}(-1, 1, 1, 1, 1)$	$SO(1,4)$ invariant tensor (metric of AdS_5 tangent space)
$\eta_{ab} = \mathrm{diag}(1, 1, 1, 1, 1)$	$SO(5)$ invariant tensor (metric of S^5 tangent space)

With these notations, the commutation relations for the bosonic generators are simply summarized as follows

$$[J_{\hat{A}\hat{B}}, J_{\hat{C}\hat{D}}] = -\eta_{\hat{A}\hat{C}} J_{\hat{B}\hat{D}} - \eta_{\hat{B}\hat{D}} J_{\hat{A}\hat{C}} + \eta_{\hat{B}\hat{C}} J_{\hat{A}\hat{D}} + \eta_{\hat{A}\hat{D}} J_{\hat{B}\hat{C}}), \qquad (2.2.42)$$

where $\eta_{\hat{A}\hat{B}} = \mathrm{diad}(\eta_{AB}, \eta_{A'B'})$. Note that we take the generators to be anti-hermite. To consider the decomposition under the denominator group, we divide the generators into two classes, namely, "unbroken generators" and "broken generators". The former are composed of the generators of $SO(1,4) \times SO(5)$ and the latter are the remaining generators. We introduce a new notation for these generators as follows

$$P_a := J_{a,-1}, \quad P_{a'} := J_{a',10}. \qquad (2.2.43)$$

Then, the commutation relations are rewritten in terms of $J_{ab}, J_{a'b'}, P_a, P_{a'}$ as follows

$$[J_{ab}, J_{cd}] = -\eta_{ac} J_{bd} + (\text{perm.}), [J_{a'b'}, J_{c'd'}] = -\eta_{a'c'} J_{b'd'} + (\text{perm.}), \qquad (2.2.44)$$

$$[P_a, J_{bc}] = \eta_{ab} P_c - \eta_{ac} P_b, [P'_a, J_{b'c'}] = \eta_{a'b'} P'_c - \eta_{a'c'} P'_b, \qquad (2.2.45)$$

$$[P_a, P_b] = J_{ab}, [P'_a, P'_b] = -J_{a'b'}. \qquad (2.2.46)$$

The first line is the commutation relations for the $SO(1,4) \times SO(5)$ algebra. The second line implies that the generators $P_a, P_{a'}$ transforms as vectors under the rotation of $SO(1,4) \times SO(5)$, hence we will call them "momentum generators", which are the analogue of the generators of the translation in the flat space case. The commutation relations at the last line seem rather strange since the momentum generators do not commute with each other. However, we are now considering the curved space-time and thus the "translations" do not commute with each other. This can be seen from the definition of the momentum operators and they are originally the generators of the higher dimensional rotation group.

We know that $PSU(2,2|4)$ contains 32 fermionic generators. Thus, we should also consider the decomposition of the fermionic generators under $SO(1,4) \times SO(5)$. Note that the spinor representations for $SO(2,4)$ and $SO(6)$ are both 8-dimensional. These spinor representations are split into two copies of the four dimensional spinor representations for $SO(1,4)$ or $SO(5)$, respectively. For example, let us consider the decomposition of $SO(6)$ spinors with respect to $SO(5)$. Suppose that supercharges Q' form the eight dimensional spinor representation: $[J_{A'B'}, Q'] \propto \Gamma_{A'B'}Q'$. Therefore, we obtain the action of the momentum operator $P_{a'} = J_{a',10}$ as $[P_{a'}, Q'] \propto \Gamma_{a'10}Q' = \Gamma_a\Gamma_{10}Q'$. Here, we take a basis of the spinor representation so that Γ^{10} becomes diagonal:

$$Q' = \begin{pmatrix} Q'_1 \\ Q'_2 \end{pmatrix}, \ \Gamma^{10} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \Gamma^{a'} = \begin{pmatrix} 0 & \gamma^{a'} \\ \gamma^{a'} & 0 \end{pmatrix}, \tag{2.2.47}$$

where Q'_1, Q'_2 are five dimensional four components spinors and $\gamma^{a'}$ is the five dimensional gammma matrix. In this basis, Γ^{10} takes the form of the "chirality operator" and the other six dimensional gamma matrices are the form of off-diagonal since Γ^{10} anticommutes with the other gamma matrices. Thus, we find that the actions of the momentum operators become

$$[P_{a'}, Q'_I] \propto \varepsilon_{IJ}\gamma^{a'}Q'_J. \tag{2.2.48}$$

Furthermore, the action of $J_{a'b'}$ on Q'_J is given by

$$[J_{a'b'}, Q'_I] \propto \gamma^{a'b'}Q'_I. \tag{2.2.49}$$

Notice that the action of $J_{a'b'}$ does not change the structure of indices I. In other words, It does not change the chirality with respect to Γ^{10}. Similar discussions hold for the decomposition of $SO(2,4)$ spinors with respect to the subgroup $SO(1,4)$. Hence, we have two sets of the spinor charges $Q_{\alpha\alpha'I}$ ($I = 1,2$) which transform as the bi-spinor representation of $SO(1,4) \times SO(5)$.

The rest of the commutation relations are given by the anti-commutators of the fermionic generators. They should be linear combinations of the bosonic generators and their coefficients are determined by the closedness of the algebra, namely, the Jacobi identities. The anti-commutators of the supercharges are schematically represented as follows

$$\{Q_I, Q_J\} \sim \delta_{IJ}P + \varepsilon_{IJ}J. \tag{2.2.50}$$

The full set of the algebra involving supercharges is given by

$$[Q_I, J_{ab}] = -\frac{1}{2}Q_I\gamma_{ab}\,, \quad [Q_I, J_{a'b'}] = -\frac{1}{2}Q_I\gamma_{a'b'}\,, \tag{2.2.51}$$

$$[Q_I, P_a] = -\frac{i}{2}\varepsilon_{IJ}Q_J\gamma_a\,, \quad [Q_I, P_{a'}] = \frac{1}{2}\varepsilon_{IJ}Q_J\gamma_{a'}\,, \tag{2.2.52}$$

$$\{Q_{\alpha\alpha'}, Q_{\beta\beta'J}\} = \delta_{IJ}(-2iC'_{\alpha'\beta'}(C\gamma^a)_{\alpha\beta}P_a + 2C_{\alpha\beta}(C'\gamma^{a'})_{\alpha'\beta'}P_{a'})$$
$$+ \varepsilon_{IJ}(C'_{\alpha'\beta'}(C\gamma^{ab})_{\alpha\beta}J_{ab} - C_{\alpha\beta}(C'\gamma^{a'b'})_{\alpha'\beta'}J_{a'b'})\,, \tag{2.2.53}$$

where C and C' are the charge conjugation matrices for $SO(1,4)$ and $SO(5)$ respectively and defined as

$$C\gamma^a C^{-1} = \epsilon_C\gamma^{aT}\,, \quad C'\gamma^{a'}C'^{-1} = \epsilon_{C'}\gamma^{a'T}\,, \quad \epsilon_C = \epsilon_{C'} = \pm 1\,. \tag{2.2.54}$$

With these conjugation matrices, the Majorana condition is imposed to the supercharges

$$\bar{Q}_{\alpha\alpha'I} = (Q_I^{\beta\beta'})^\dagger(\gamma^0)_\alpha^\beta\delta_{\alpha'}^{\beta'} = -Q_I^{\beta\beta'}C_{\beta\alpha}C_{\beta'\alpha'}\,. \tag{2.2.55}$$

The most remarkable feature of the algebra is that it admits the following $\mathbb{Z}_4$ decomposition

$$\mathfrak{psu}(2,2|4) = \mathcal{H} + \mathcal{P} + \mathcal{Q}_1 + \mathcal{Q}_2\,, \tag{2.2.56}$$

$$\mathcal{H} : 0\,, \quad \mathcal{Q}_1 : 1\,, \quad \mathcal{P} : 2\,, \quad \mathcal{Q}_2 : 3\,. \tag{2.2.57}$$

where $\mathcal{H}$ is the Lie algebra of the $SO(1,4) \times SO(5)$, $\mathcal{P}$ is made up of the remaining bosonic generators, and $\mathcal{Q}_1$ and $\mathcal{Q}_2$ are two copies of the $(\mathbf{4}, \mathbf{4})$ representation of $\mathcal{H}$. Under this decomposition, each component has the $\mathbb{Z}_4$ charges. This $\mathbb{Z}_4$ decomposition will play an important role when we show the classical integrability of this string sigma model.

We are now ready to write down the Green-Schwartz action in the $AdS_5 \times S^5$ background. Let us take a coset representative and define the left invariant 1-form as follows

$$G(x, \theta) = \exp(x^a P_a + x^{a'}P_{a'})\exp(\theta^I Q_I)\,, \tag{2.2.58}$$

$$J = G^{-1}dG = H + P + Q_1 + Q_2\,, \tag{2.2.59}$$

$$H = \frac{1}{2}L^{ab}J_{ab} + \frac{1}{2}L^{a'b'}J_{a'b'}\,, \quad P = L^a P_a + L^{a'}P_{a'}\,, \quad Q_I = L^{\alpha\alpha'I}Q_{\alpha\alpha'I}\,, \tag{2.2.60}$$

where we have decomposed J according to the $\mathbb{Z}_4$ decomposition of the Lie algebra of $PSU(2,2|4)$. Note that unlike the case of the flat space, the momentum generators no more commute with each other and thus the left invariant current includes the all components of the algebra. The action is the sum of the kinetic term and the Wess-Zumino term as in the flat space. They can be expressed in terms of the components of

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the current as follows

$$S_K = -\frac{\sqrt{\lambda}}{4\pi} \int_\Sigma d^2\sigma \sqrt{-g} g^{ij} (\eta_{ab} L_i^a L_j^b + \eta_{a'b'} L_i^{a'} L_j^{b'}) = \frac{\sqrt{\lambda}}{4\pi} \int_\Sigma \mathrm{Str} P \wedge *P \,, \qquad (2.2.61)$$

$$S_{WZ} = i\frac{\sqrt{\lambda}}{2\pi} \int_{M_3} s_{IJ}(L^a \wedge \bar{L}^I \gamma_a L^J + iL^{a'} \wedge \bar{L}^I \gamma_{a'} L^J)$$

$$= \frac{\sqrt{\lambda}}{2\pi} \int_{M_3} \mathrm{Str}(P \wedge Q_1 \wedge Q_1 - P \wedge Q_2 \wedge Q_2)\,, \qquad (2.2.62)$$

where Str denote the supertrace of the group elements. The Wess-Zumino term can be further converted into the form of the integral over the world-sheet by using the following relation

$$2\mathrm{Str}(P \wedge Q_1 \wedge Q_1 - P \wedge Q_2 \wedge Q_2) = -d\mathrm{Str}(Q_1 \wedge Q_2)\,. \qquad (2.2.63)$$

Therefore, we obtain the total action as follows

$$S = -\frac{\sqrt{\lambda}}{4\pi} \int_\Sigma \mathrm{Str}(P \wedge *P - \kappa Q_1 \wedge Q_2)\,, \qquad (2.2.64)$$

where the relative coefficient κ is fixed to be ± 1 by the κ-symmetry whose transformation properties are almost similar for the case of flat space. The more explicit form of the action and the proof of the invariance under the κ-symmetry are given in [16]. We should emphasize the fact that the Wess-Zumino term can be written in terms of the invariant forms in contrast to the flat space. This is essentially due to the difference of the algebraic structures. Indeed, there exists the inverse of the group invariant tensor η_{AB}^{-1}, $\eta_{AB} = \mathrm{Str}(t_A t_B)$ in the present case, while invariant tensor has no inverse in the flat space. In other words, the Killing form is degenerated in the flat space limit and a part of the forms drops from the action.

Chapter 3

Integrability and AdS/CFT
-spectrum-

This chapter is devoted to the spectrum problem of AdS_5/CFT_4. We will start with the overview of the development of integrability analysis in the spectrum problem (section 3.1). In section 3.2, we calculate the 1-loop dilatation operator and show that it is identified with an integrable spin chain Hamiltonian, which can be diagonalized by using the so-called Bethe ansatz method. We next prove the classical intgerability of the string theory on the $AdS_5 \times S^5$ background and construct a large class of classical solutions based on the so-called finite gap method (section 3.3). Then, we will compare the spectrum in the both theories in the semi-classical limit in section 3.4. Finally, we will review the subsequent developments beyond the perturbative analysis (section 3.5) and extract several lessons (section 3.6).

3.1   Overview of history

In this section, we will briefly summarize the development of the spectrum problem of the AdS/CFT. The main purpose of this section is to give a overview of the highlights of integrability in AdS_5/CFT_4 together with references of original works as well as some of reviews. We should emphasize that the order of references is not necessarily chronological and the list is not complete.

- **Discovery of integrabilty**

  A decade ago, the first integrable structure was discovered in perturbative $\mathcal{N} = 4$ SYM theory by Minahan and Zarembo [20]. Minahan and Zarembo showed that the action of the 1-loop dilatation operator for certain subclass of single trace operators are mapped to the action of Hamiltonian on states in an integrable periodic spin chain with nearest neighbor interactions. The single trace operators are mapped to

the eigenstates of the spin chain Hamiltonian and their conformal dimensions are mapped to the corresponding eigenvalues. Beisert extended this result to all single trace operators at 1-loop level [21]. The diagonalization of the integrable spin chain Hamiltonian can be performed exactly by using Bethe ansatz method which leads to a set of equations so-called the Bethe ansatz equations.

Almost at the same time, Bena, Polchinski, and Roiban proved that string theory on $AdS_5 \times S^5$ is classically integrable [71]. More precisely, they proved that the Metseav-Tseytlin sigma model has an infinite number of conserved charges that ensures the simplification of the model at least classical level.

We will discuss these subjects in section 3.2 and section 3.3.1.

- **Higher loop dilatation operator for $\mathcal{N} = 4$ SYM and spin chain**

Beisert, Kristjansen and Staudacher showed that the integrability in the $SU(2)$ sector, which is the simplest subsector of the $\mathcal{N} = 4$ SYM theory, holds up to 3-loop level [22]. Beyond 1-loop level, the action of the dilatation operator on the single trace operators can be no longer mapped to the Hamiltonian with the nearest neighbor interactions but mapped to a spin chain Hamiltonian with long range interactions.

Serban and Staudacher matched the dilatation operator in the $SU(2)$ sector with higher Hamiltonian in the long range spin chain which is so-called the Inozemtsev model, up to 3-loop level [23]. In the infinite length limit, the model can be diagonalized by the asymptotic Bethe ansatz, which has exponential corrections in the finite length. These results can be also obtained by using the Hubbard model [24, 30]. However, these models do not match the dilatation operator beyond 3-loop level and the integrable model that describes the dilatation operator of $\mathcal{N} = 4$ SYM for all-loop order remains to be known.

- **Classical solution and spectral curve**

Kazakov, Marshakov, Minahan, and Zarembo studied a class of classical solutions that is so-called finite gap solutions for the string moving on $\mathbb{R} \times S^3$ which is a subspace of $AdS_5 \times S^5$ [79]. Finite gap solutions correspond to the solutions whose number of excitations with different modes is finite. The finite gap solutions can be expressed in terms of the language of algebraic curves. Moreover, the semiclassical limit of the Bethe ansatz equation leads to algebraic curves as well, hence it became possible to directly compare the results of the sigma model with the spin chain predictions. The same method was used for other subspaces [80–82] and later for

the whole space [83, 84]. The finite size corrections[1] in the language of spin chains correspond to loop corrections for the sigma model. These were extensively studied by many authors [72–75] and the results are shown to coincide with each other, at least for first few orders where the order of limit problem does not arise. See also [76, 77]. Elegant and convenient methods to calculate the 1-loop fluctuations from algebraic curve were proposed in [99, 100].

- **S-matrix and asymptotic Bethe ansatz**

  Staudacher emphasized the importance of the concept of the S-matrix for magnons, which are fundamental excitations in the spin chain or the gauge fixed string σ-model [27]. In particular, by using the S-matrix and asymptotic Bethe ansatz, one can avoid the difficulties related to the length changing [26], which does not occur in the usual integrable spin chain models.

  Furthermore, it turned out that the S-matrix and asymptotic Bethe ansatz approach are of use to determine the higher-loop spectrum since the symmetry of the theory almost uniquely determines the S-matrix up to an overall phase factor, which is called dressing factor, and the dispersion of the magnons for arbitrary coupling.

  Beisert, Dippel and Staudacher (BDS) conjectured all-loop asymptotic Bethe ansatz equations for the $SU(2)$ sector, which reproduce some of the all-loop features [28]. However, it became soon clear that these equations require the dressing factor to match with the predictions from the strong coupling limit. Beisert and Staudacher extended the BDS equations to the whole sector and proposed all-loop asymptotic Bethe ansatz equations (Beisert-Staudacher equations) [29], which were later re-derived by Beisert assuming that magnons of the spin chain are bi-fundamental representation of the centrally extended $SU(2|2) \times SU(2|2)$ [30]. It was pointed out that these equations seem to be linked with the 1-dimensional Hubbard model [24, 31], not fully clarified yet.

  From a string theory point of view, the magnon dispersion relation proposed by BDS was confirmed by Hofman and Maldacena [85] who considered the strong coupling analogue of the magnon of the spin chain, which is so-called the giant magnon. The giant magnons correspond to the soliton like excitations with momentum of the order of unity. See also [86, 87, 90–92]. The symmetry of excitations in the string sigma model is the same as that of the magnons in the spin chain [32], hence

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[1]These type of finite size corrections are perturbative corrections, namely $1/L$ corrections. Note that the finite size corrections to the spectrum determined from the asymptotic Bethe ansatz are typically the form of e^{-cL} and non-perturbative corrections.

Arutyunov, Frolov and Zamaklar pointed out that the Bethe ansatz equations for the strings are essentially the same as those for the spin chain [33].

- **Dressing phase and crossing equation**

The dressing phase is an overall phase factor of the S-matrix which cannot be determined from the symmetry alone. However, it is crucial for the spectrum to interpolate the results of the strong coupling and the weak coupling. Arutyunov, Frolov and Staudacher found the dressing phase at leading order in the strong coupling [35] in order that the string Bethe ansatz correctly reproduces the results of the weak coupling. This is the first evidence that the dressing phase is not trivial. Hernandez and Lopez determined the next to leading term at the strong coupling [36] by computing 1-loop fluctuations.

Janik proposed an equation that the dressing phase must satisfy [37], which essentially comes from the crossing symmetry of the S-matrix, thus it is called the crossing equation. A solution of the crossing equation was first guessed by Beisert, Hernandez and Lopez (BHL) [38], and Beisert, Eden and Staudacher (BES) showed that this solution has the correct structure at the weak coupling [39].

Dorey, Hofman and Maldacena confirmed that the BHL/BES solution reproduces a right poles/zeros structure of the S-matrix, which arises due to the exchange of pairs of magnon bound states [40]. They also gave a convenient integral representation of the solution. Arutyunov and Frolov first proved that the BHL/BES dressing phase satisfies the crossing equation [41] and Volin showed, conversely, that the solution of crossing equation with minimal number of singularities is actually the BHL/BES solution. The solution of the crossing equation indeed gave the interpolation between the weak coupling and the strong coupling, at least for long length spin chains.

- **Finite size corrections**

The asymptotic Bethe ansatz method is valid without any corrections only in the infinite length limit, $L \to \infty$. For a long but finite length, we must compute the finite size corrections. The leading contributions for the finite size corrections can be computed by the Lüscher's method [46], which is applicable to any massive QFTs. Janik and Lukowsky generalized the Lüscher's method and calculated the finite size corrections in the context of the AdS/CFT [44]. The diagramatic calculations of the Lüscher's corrections give an interpretation that they originate from the interactions between magnons and virtual particles which circulate around the compact direction. If we identifie the compact direction as an Euclidean time, Luüscher

corrections are analogous to instanton effects since they are typically the form of e^{-cL}. Further, Heller, Janik and Lukowsky re-derived the Luüscher' correction by a summation over quadratic fluctuations [47]. Although this approach requires some assumptions, it is quite intuitive and gives a new interpretation on the origin of the Luüscher's corrections. See also [48].

Another important approach for computing the finite size corrections is the thermodynamic Bethe ansatz (TBA) [49], whose first application in the context of AdS/CFT was done by Ambjorn, Janik and Kristjansen [43] and systematic implementation was considered in [56–58]. The TBA approach relies on the equivalence between a finite-size, zero-temperature theory and an infinite-size, finite-temperature mirror theory via a double Wick rotation. The ground state energy is then computed by solving a set of coupled non-linear integral equations, namely the TBA equations [59].

- **Functional equations and quantum spectral curve (QSC)**

  The TBA equations are known to be converted into an elegant form which is so-called the Y-systems. The Y-systems is a system of difference equations which often appears in diverse classical/quantum integrable systems. See [61] for a review of Y-systems and see [60] for a review of applications of the Y-systems in AdS/CFT. We should also comment on the recent work of the *quantum spectral curve* [66]. The quantum spectral curve is proposed as a new formalism, alternative to the TBA-like approach for the spectrum problem. It takes a form of a non-linear matrix Riemann-Hilbert problem and its solutions are called Q-functions which includes the information of all-loop spectrum with finite size corrections. As its name implies, the quantum spectral curve can be regarded as a quantum analogue of the classical spectral curve.

3.2 Perturbative spectrum of $\mathcal{N} = 4$ SYM

In this subsection, we explicitly compute the 1-loop dilatation operator for the subsector of the $\mathcal{N} = 4$ SYM theory, which is so called the $SO(6)$ sector where the single trace operators are composed of six types of scalar fields. As shown in [20], the 1-loop dilatation operator for this subsector can be identified with an integrable spin chain Hamiltonian with nearest neighbor interactions and each single trace operator corresponds to the eigenstate of this Hamiltonian. In other words, the diagonalization of the 1-loop dilatation operator is equivalent to the diagonalization of the integrable spin chain with nearest neighbor

interactions. In the simplest situation, where the composite operators are composed of two types of complex scalar fields, the problem can be extremely simplified since the spin chain Hamiltonian is reduced to the famous Heisenberg Hamiltonian which is known to be integrable and solved by the Bethe ansatz method. We will explain the meaning of integrability and how Bethe ansatz works to solve the model in section 3.2.2.

3.2.1 1-loop dilatation operator

Before going into calculations, let us recall how to determine the scaling property of composite operators or the dilatation operator in general setting. In quantum field theories, bare composite operators have the UV divergences that come from the products of the same space-time point, therefore we need to introduce field strength renormalization factors and impose some appropriate renormalization conditions so that all the renormalized composite operators are well defined and correlation functions of them are finite. It often happens that there are several composite operators with the same quantum numbers and mass dimensions[2], then such operators can be mixed by quantum corrections. For such a set of operators $\{\mathcal{O}^i\}$, the relation of bare and renormalized operators are the following form

$$\mathcal{O}^i_{bare} = Z^i_j \mathcal{O}^j_{ren} \,, \tag{3.2.1}$$

where we introduce field strength renormalizations $Z^i_j(\Lambda, g)$ that depend on the renormalization scale Λ and a coupling g. Then the renormalized operators satisfy

$$\frac{d\mathcal{O}^i_{ren}}{d\ln\Lambda} = \Gamma^i_j \mathcal{O}^j_{ren} \,, \tag{3.2.2}$$

$$\Gamma^i_j = (Z^{-1})^i_k \frac{dZ^k_j}{d\ln\Lambda} \,. \tag{3.2.3}$$

Γ is the so-called anomalous dimension matrix and the dimensions of composite operators are obtained by diagonalizing the anomalous dimension matrix,

$$\Delta = \Delta^{(0)} + \gamma \,, \tag{3.2.4}$$

with $\Delta^{(0)}$ being the bare dimension and γ being the anomalous dimension, which is an eigenvalue of Γ, arising from quantum corrections. To sum up, the dimensions of operators are obtained by (i) determining the field strength renormalization or the anomalous dimension matrix (3.2.3) and (ii) diagonalizing them.

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[2]We are now considering an operator mixing of conformal or massless field theory. In massive field theories, operators with given dimension can also mixed with operators with lower dimensions.
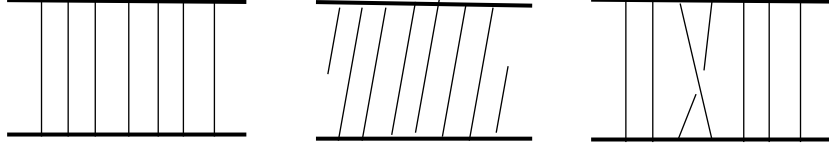
Figure 3.2.1: The tree-level two-point function can be calculated by summing up all possible planar Wick contractions such as the left and the center diagram. There are L cyclic permutated contributions. In the large N limit, non-planar diagrams such as the right graph are negligible.

In the following, we will find the 1-loop dilatation operator for the $SO(6)$ sector by directly calculating the two-point functions of the single trace operators composed of six scalar fields. For this sector, the two-point functions are expressed as follows

$$\langle \mathcal{O}_i(x)\bar{\mathcal{O}}_j(y)\rangle = \frac{\delta_{ij}}{|x-y|^{2\Delta_i}} \, . \tag{3.2.5}$$

Assuming the weak coupling $g_{YM} \ll 1$ and the anomalous dimension is much smaller than the bare dimension $\gamma \ll \Delta^{(0)}$, the correlation function (3.2.5) can be expanded as

$$\langle \mathcal{O}_i(x)\bar{\mathcal{O}}_j(y)\rangle = \frac{\delta_{ij}}{|x-y|^{2\Delta_i^{(0)}}}\left(1 - \gamma_i \ln \Lambda^2 |x-y|^2\right), \tag{3.2.6}$$

where Λ is the UV cut off. Therefore, we can read off the anomalous dimension (matrix) from the 1-loop perturbative calculation of the two-point functions instead of directly determining the field strength renormalizations.

In order to calculate the two-point functions of the $SO(6)$ sector at 1-loop level, recall that propagator of the scalar fields[3] are given by

$$\langle (\phi_I(x))_B^A (\bar{\phi}_J(y))_D^C\rangle_{free} = \frac{\delta^{AC}\delta_{BD}}{4\pi^2|x-y|^2}\delta_{IJ} \, . \tag{3.2.7}$$

Since the operators of the $SO(6)$ sector are only composed of the scalar fields, we can compute the tree-level two-point functions by summing up all possible planar diagrams such as figure 3.2.1 with the free propagator (3.2.7). For example, the tree-level two-point function of the chiral primary operator $\mathrm{Tr}Z^L(x) = Z_{A_2}^{A_1}Z_{A_3}^{A_2}\cdots Z_{A_1}^{A_L}(x)$ is given by

$$\langle \mathrm{Tr}Z^L(x)\mathrm{Tr}\bar{Z}^L(y)\rangle_{tree} = \left(\frac{1}{4\pi^2|x-y|^2}\right)^L \times N^L \times L \, . \tag{3.2.8}$$

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[3]We neglect the U(1) factor $-\frac{1}{N}\delta_B^A\delta_D^C$ since we are interested in the planar limit.
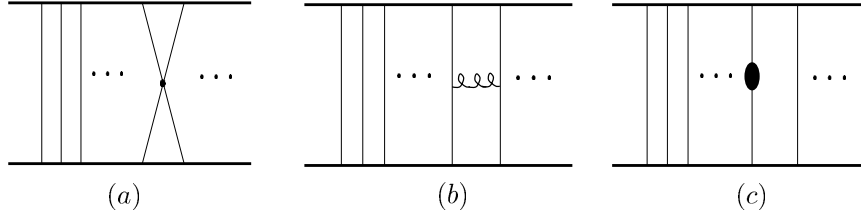
Figure 3.2.2: The diagrams that contribute to the two-point functions of the scalar single trace operators. (a)The diagram of quartic interaction of scalars. (b)The gluon exchanging diagram. (c)The scalar self energy diagrams. The diagrams (b) and (c) are irrelevant for our calculation due to the R-charge conservation.

The factor N^L comes from L factors of $SU(N)$ color indices such as $\delta_A^{A'}\delta_{A'}^A$, which arise due to the contractions of the neighbor fields. The factor L comes from the L ways of contracting the fields such as in the figure 3.2.1. We can also compute the tree-level two-point functions of the composite operator of the form

$$\mathcal{O}_{I_1 I_2 \dots I_L}(x) = \frac{(4\pi)^{L/2}}{\sqrt{C_{I_1 I_2 \dots I_L}} N^{L/2}} \mathrm{Tr}(\phi_{I_1}(x)\phi_{I_2}(x)\cdots\phi_{I_L}(x))\,, \qquad (3.2.9)$$

where $I_1, \dots, I_L$ denote the $SO(6)$ flavor indices and $C_{I_1 I_2 \dots I_L}$ is a symmetry factor introduced so as to normalize the operator properly. The results is

$$\langle \mathcal{O}_{I_1 I_2 \dots I_L}(x)\bar{\mathcal{O}}^{J_1 J_2 \dots J_L}(y)\rangle = \frac{1}{C_{I_1 I_2 \dots I_L}}(\delta_{I_1}^{J_1}\delta_{I_1}^{J_2}\cdots\delta_{I_L}^{J_L} + \text{cycles}) \times \frac{1}{|x-y|^{2L}}\,, \qquad (3.2.10)$$

where "cycles" refers to the $L-1$ cyclic permutations of the J_i indices.

We next consider the two-point function at 1-loop level. 1-loop planar diagrams that contribute to the two-point function of the operator (3.2.9) are summarized in the figure 3.2.2. The only first diagram can change the structure of $SO(6)$ flavor indices and the rest two diagrams don't contribute to the mixing. Therefore it is sufficient to consider the Feynman diagrams containing the quartic scalar vertex and the contribution from the rest of diagrams can be determined by insisting that chiral primaries such as $\mathrm{Tr}Z^L$ have no anomalous dimensions.

The bosonic part of $\mathcal{N}=4$ SYM action is given by

$$S = \frac{1}{2g_{YM}^2}\int d^4x \mathrm{Tr}\left[-\frac{1}{2}F_{\mu\nu}F^{\mu\nu} + D_\mu\phi_I D^\mu\phi^I - \frac{1}{2}\sum_{I,J}[\phi_I,\phi_J]^2\right]\,. \qquad (3.2.11)$$

After we absorb g_{YM} into the fields so that kinetic terms have canonical form, the quartic

term can be rewritten as

$$\frac{g_{YM}^2}{4} \sum_{I,J} \left(\text{Tr} \phi_I \phi_J \phi_J \phi_I - \text{Tr} \phi_I \phi_J \phi_I \phi_J \right). \qquad (3.2.12)$$

Using this vertex, we find the contribution of diagram (a) in figure 3.2.2 includes the following sub correlator

$$\langle (\phi_{I_k} \phi_{I_{k+1}})_C^A(x) \left(i\frac{g_{YM}^2}{4} \int d^4z \sum_{I,J} (\text{Tr} \phi_I \phi_J \phi_J \phi_I - \text{Tr} \phi_I \phi_J \phi_I \phi_J) \right) (\phi^{J_k} \phi^{J_{k+1}})_{C'}^{A'}(y) \rangle$$

$$= i\frac{N\delta^{AA'}\delta_{CC'}}{(4\pi^2)^2)} \frac{g_{YM}^2 N}{64\pi^2} (2\delta_{I_k}^{J_k}\delta_{I_{k+1}}^{J_{k+1}} + 2\delta_{I_k I_{k+1}}\delta^{J_k J_{k+1}} - 4\delta_{I_k}^{J_{k+1}}\delta_{I_{k+1}}^{J_k})$$

$$\times \int \frac{d^4z}{|z-x|^4|z-y|^4}. \qquad (3.2.13)$$

The first two sets of delta functions for flavor indices in the second line of (3.2.13) come from the first vertex in (3.2.12) and the last set of delta functions arises from the second vertex. The coefficients of these delta functions are determined by four possible planar contractions of the two vertices.

The integral in (3.2.13) has a logarithmic divergence as $z \to x$ and $z \to y$, thus we regularize the integral by introducing an UV cut off Λ. After Wick rotating to Euclidean space as $d^4z \to id^4z_E$, the integral is restricted to the region where $|z_E - z| \geq \Lambda^{-1}$ and $|z - y| \geq \Lambda^{-1}$. The integral can be evaluated as follow

$$i \int \frac{d^4z_E}{|z_E - y|^4|z-x|^4} \approx \frac{2i}{|x-y|^4} \int_{\Lambda^{-1}}^{|x-y|} \frac{drd\Omega_3}{r} = \frac{2\pi^2 i}{|x-y|^4} \ln(\Lambda^2|x-y|^2). \qquad (3.2.14)$$

Thus the correlator (3.2.13), which is the subcorrelator in diagram (a) of figure 3.2.2 becomes

$$\frac{N\delta_{CC'}^{AA'}}{(4\pi^2)^2|x-y|^4} \frac{\lambda}{16\pi^2} (2\delta_{I_k}^{J_{k+1}}\delta_{I_{k+1}}^{J_k} - \delta_{I_k I_{k+1}}\delta^{J_k J_{k+1}} - \delta_{I_k}^{J_k}\delta_{I_{k+1}}^{J_{k+1}}) \ln(\Lambda^2|x-y|^2), \qquad (3.2.15)$$

where $\lambda = g_{YM}^2 N$ is the 'tHooft coupling.

There are other possible 1-loop diagrams which contribute to the two-point function of (3.2.9) such as diagrams (b) and (c) in figure 3.2.2. These are nothing but the gluon exchanging diagrams and the self energy diagrams hence we can compute these diagrams explicitly. However, as already mentioned, these diagrams don't change the structure of flavor indices due to the R-charge conservation and thus they give the same flavor structure as the planar tree-level contributions.

Summarizing these results, we finally obtain the 1-loop planar two point function of (3.2.9)

$$\langle \mathcal{O}_{I_1 I_2 \ldots I_L}(x) \bar{\mathcal{O}}^{J_1 J_2 \ldots J_L}(y) \rangle_{1-loop} =$$

$$\frac{1}{|x-y|^{2L}} \left(1 - \frac{\lambda}{16\pi^2} \ln(\Lambda^2 |x-y|^2) \sum_{k=1}^{L} (-2P_{k,k+1} + K_{k,k+1} + 1 + C) \right) \delta_{I_1}^{J_1} \delta_{I_1}^{J_2} \cdots \delta_{I_L}^{J_L}$$

$$+ \text{cycles} \,. \tag{3.2.16}$$

$P_{k,k+1}$ is the exchanging operator that acts on the nearest neighbor flavor indices, and as its name implies, it exchanges the flavor indices of k-th and $k+1$-th sites in the single trace operator. The action of $P_{k,k+1}$ on the delta functions in (3.2.16) is explicitly written as follows

$$P_{k,k+1}(\delta_{I_1}^{J_1} \cdots \delta_{I_k}^{J_k} \delta_{I_{k+1}}^{J_{k+1}} \cdots \delta_{I_L}^{J_L}) = \delta_{I_1}^{J_1} \cdots \delta_{I_k}^{J_{k+1}} \delta_{I_{k+1}}^{J_k} \cdots \delta_{I_L}^{J_L} \,. \tag{3.2.17}$$

$K_{k,k+1}$ is the trace operator that contracts the flavor indices of k-th and $k+1$-th sites of the single trace operator. Its action on the delta functions is

$$K_{k,k+1}(\delta_{I_1}^{J_1} \cdots \delta_{I_k}^{J_k} \delta_{I_{k+1}}^{J_{k+1}} \cdots \delta_{I_L}^{J_L}) = \delta_{I_1}^{J_1} \cdots \delta_{I_k I_{k+1}} \delta^{J_k J_{k+1}} \cdots \delta_{I_L}^{J_L} \,. \tag{3.2.18}$$

Summation over k arises in (3.2.16) since the there are contributions from each nearest neighbor pair of contraction as in figure 3.2.2 or (3.2.13). The constant term C in (3.2.16) comes from the diagrams (b) and (c) in figure 3.2.2 and we will determine this constant later by imposing that the dimensions of chiral primary operators such as $\text{Tr} Z^L$ are indeed protected or their anomalous dimensions are equal to zero.

By comparing (3.2.16) with (3.2.6), we find that the anomalous dimension matrix (3.2.3) for the $SO(6)$ sector at 1-loop level is given by

$$\Gamma = \frac{\lambda}{16\pi^2} \sum_{k=1}^{L} (1 + C - 2P_{k,k+1} + K_{k,k+1}) \,. \tag{3.2.19}$$

The 1-loop anomalous dimensions of composite operators of the $SO(6)$ sector are obtained by diagonalizing Γ. As we will explain in the following, the problem of diagonalizing Γ is equivalent to the problem of diagonalizing the corresponding spin chain Hamiltonian with nearest neighbor interactions.

First, note that all the scalar single trace operators of length L can be identified with the basis of the Hilbert space of the form

$$\mathcal{V}_1 \otimes \mathcal{V}_2 \otimes \cdots \otimes \mathcal{V}_L \,. \tag{3.2.20}$$
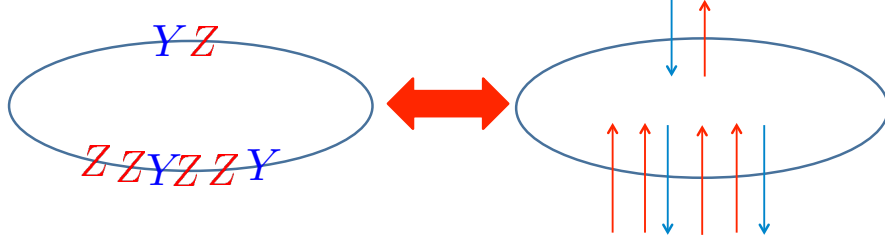
Figure 3.2.3: The dilatation operator for the $SO(6)$ sector can be mapped to the Hamiltonian of an integrable spin chain and the single trace operators are identified with the states of spin chain obeying the periodic boundary condition. In particular, the dilatation operator for the $SU(2)$ sector is mapped to the Heisenberg Hamiltonian and the two kinds of scalar field Z and Y are identified with the up-spin and down-spin respectively.

Here, each Hilbert space $\mathcal{V}_k$ is the vector representation of $SO(6)$. This tensor product space is the same as spin chain Hilbert space with L sites and a $SO(6)$ vector lives on each site. The polarization of spin at each site is identified with the flavor index of the operator at each site in the trace (3.2.3). Due to the cyclicity of the trace, we should restrict the whole Hilbert space to its subspace which is invariant under the cyclic shift such as

$$\mathcal{V}_1 \otimes \mathcal{V}_2 \otimes \cdots \otimes \mathcal{V}_L \to \mathcal{V}_L \otimes \mathcal{V}_1 \otimes \cdots \otimes \mathcal{V}_{L-1} \,. \tag{3.2.21}$$

The action of Γ on the scalar single trace operators can be mapped to the action of a linear operator of the spin chain. Since Γ is hermite and invariant under the shift (3.2.21), Γ can be treated as a spin chain Hamiltonian with nearest neighbor interactions between the spins.[4]

At this point, we determine the constant C in (3.2.19) by computing the action of Γ on the chiral primary operator $\mathrm{Tr}Z^L$. It is completely symmetric and invariant under any permutations of the nearest neighbor fields, hence the action of $P_{k,k+1}$ on $\mathrm{Tr}Z^L$ is trivial for any k : $P_{k,k+1}\mathrm{Tr}Z^L = \mathrm{Tr}Z^L$. Furthermore, $\mathrm{Tr}Z^L$ contains only Z fields, and not $\bar{Z}$, thus $K_{k,k+1}\mathrm{Tr}Z^L = 0$. Therefore, we obtain

$$\Gamma(\mathrm{Tr}Z^L) = \frac{\lambda}{16\pi^2} \sum_{k=1}^{L} (1 + C - 2)\mathrm{Tr}Z^L \,. \tag{3.2.22}$$

However, the anomalous dimension of the chiral primary should be zero, thus we find that

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[4]This is because $P_{k,k+1}$ and $K_{k,k+1}$ in Γ act on nearest neighbor fields.

$C = 1$ and Γ becomes

$$\Gamma = \frac{\lambda}{8\pi^2} \sum_{k=1}^{L} \left(1 - P_{k,k+1} + \frac{1}{2} K_{k,k+1} \right) . \qquad (3.2.23)$$

Notice that if we restrict to the $SU(2)$ subsector, where composite operators are made up of two types of scalar fields, say, Z, Y, the Hamiltonian (3.2.23) is quite simplified. Since there are no conjugate fields such as $\bar{Z}, \bar{Y}$ in the $SU(2)$ sector, the action of the trace operator $K_{k,k+1}$ identically vanishes in this sector and the Hamiltonian becomes

$$\Gamma|_{SU(2)} = \frac{\lambda}{8\pi^2} \sum_{k=1}^{L} (1 - P_{k,k+1}) . \qquad (3.2.24)$$

In fact, the spin chain Hamiltonian (3.2.23) is known to be "integrable" and it can be diagonalized exactly by the Bethe ansatz method. We will discuss this subject in the next subsection.

We have only discussed the 1-loop dilatation operator for the $SO(6)$ sector so far. There are two possible generalizations for this result. One is the extension to higher rank sectors and the other is to include higher loop corrections. We do not go into detail here, but refer the reader to the references.

- **Higher rank sector**

  The 1-loop dilatation operator for the full single trace operators was determined by Beisert in [21]. In general, composite operators composed of all the $\mathcal{N} = 4$ SYM fundamental fields and their covariant derivatives can be mapped to the spin chain whose "spins" at each site of the chain are made up of the elements of $\mathcal{V}$, which is the so-called singleton representation of $PSU(2,2|4)$. The 1-loop dilatation operator is identified with the integrable spin chain Hamiltonian with nearest neighbor interactions.

  Since the Hamiltonian acts on nearest neighbor sites of the spin chain, it involves the tensor product of the two singleton representations. Furthermore, the Hamiltonian or the anomalous dimension operator commutes with the generators of $PSU(2,2|4)$ and thus the Hamiltonian can be written as a sum of the projectors on each irreducible representation which appears in the decomposition of the tensor product of the two singleton representations.

  $$\Gamma = \frac{\lambda}{8\pi^2} \sum_{k=1}^{L} \sum_{j=0}^{\infty} 2h(j) \Pi_{k,k+1}^{(j)} , \qquad (3.2.25)$$

  $$\mathcal{V} \otimes \mathcal{V} = \bigoplus_{j=0}^{\infty} \mathcal{V}_j , \qquad (3.2.26)$$

where $\Pi^{(j)}_{k,k+1}$ projects $\mathcal{V}_k \otimes \mathcal{V}_{k+1}$ onto $\mathcal{V}_j$ and $h(j)$ is the harmonic sum defined by

$$h(j) = \sum_{k=1}^{j} \frac{1}{k} \, . \qquad (3.2.27)$$

As explained in section 2.1.2, all the irreducible representations of $PSU(2,2|4)$ are labeled by the highest weight, which is the set of six charges of the Cartan subalgebra including the bare dimension Δ_0, the two Lorentz spins S_1, S_2, and the three R-charges J_1, J_2, J_3. For example, the singleton representation is labeled by $(1,0,0;1,0,0)$. The highest weight $(\Delta_0, S_1, S_2; J_1, J_2, J_3)$ of $\mathcal{V}_j$ is given by

$$\mathcal{V}_0 : \quad (2,0,0;2,0,0) \qquad (3.2.28)$$

$$\mathcal{V}_1 : \quad (2,0,0;1,1,0) \qquad (3.2.29)$$

$$\mathcal{V}_j : \quad (j, j-2, j-2; 0,0,0) \quad j \geq 2 \, . \qquad (3.2.30)$$

Note that if we consider the $SO(6)$ subsector of the whole space then only $\mathcal{V}_0, \mathcal{V}_1$ and $\mathcal{V}_2$ appear in the decomposition (3.2.26) since the operators of the $SO(6)$ sector have no Lorentz charges. Hence, we recover the $SO(6)$ spin chain Hamiltonian

$$\Gamma|_{SO(6)} = \frac{\lambda}{8\pi^2} \sum_{k=1}^{L} (0 \Pi^{\text{sym}}_{k,k+1} + 2\Pi^{\text{asym}}_{k,k+1} + 3\Pi^{\text{sing}}_{k,k+1} \, , \qquad (3.2.31)$$

$$\Pi^{\text{sym}}_{k,k+1} = \frac{1}{2}(1 + P_{k,k+1}) - \frac{1}{6}K_{k,k+1} \, , \quad \Pi^{\text{asym}}_{k,k+1} = \frac{1}{2}(1 - P_{k,k+1}) \, , \quad \Pi^{\text{sing}}_{k,k+1} = \frac{1}{6}K_{k,k+1} \, . \qquad (3.2.32)$$

where each projector projects $\mathcal{V}_k \otimes \mathcal{V}_{k+1}$ onto the symmetric traceless, antisymmetric, and singlet representations respectively. This is precisely (3.2.23).

- **Higher loop**

  Going beyond 1-loop, one finds that the mixing of composite operators is not restricted to the nearest neighbor one since successive loops in graphs involve non-neighboring fields even in the planar limit. Therefore the higher loop dilatation operator can be mapped to the spin chain Hamiltonian with long range interactions. In [22], the dilatation operator of the $SU(2)$ sector was determined up to 3-loops and it was shown that this dilatation operator is equivalent to the Hamiltonian of the Inozemtsev spin chain, which is a spin chain with long range interactions [23]. Generic properties of long range spin chain and the relation to the dilatation operator of $\mathcal{N} = 4$ SYM are summarized in [25].

  Moreover, the higher-loop corrections to the dilatation operator exhibit a novel feature that operators with the same charges but different lengths can be mixed. In

other words, the action of Hamiltonian changes the length of the spin chain dynamically. Therefore, such a mixing is called dynamical [26]. The simplest example of the dynamical mixing is the mixing between the operator composed of three complex scalars XYZ, which has charges $(3,0,0;1,1,1)$ and the operator composed of two fermions U,V with individual charges $(\frac{3}{2},\frac{1}{2},0;\frac{1}{2},\frac{1}{2},\frac{1}{2})$ and $(\frac{3}{2},-\frac{1}{2},0;\frac{1}{2},\frac{1}{2},\frac{1}{2})$. These operators have the same charges thus the following mixing can occur:

$$\mathrm{Tr}(\dots XYZ\dots) \longleftrightarrow \mathrm{Tr}(\dots UV\dots). \qquad (3.2.33)$$

We comment on the existence of closed sectors under the operator mixing. The $SU(2)$ sector is the set of operators composed of two types of complex scalar fields, say, Z and Y, which have the charges $(1,0,0;1,0,0)$ and $(1,0,0;0,1,0)$ respectively. Thus the elements of the $SU(2)$ sector composed of $L-M$ Z fields and M Y fields have charges $(L,0,0;L-M,M,0)$. Since the operator mixing must preserve the total charges, one finds that the only possible mixing of these operators is the rearrangement of Z and Y in the trace and mixing with other types of field does not occur.[5] Therefore, we conclude that the $SU(2)$ sector is closed under the mixing for any order in perturbation theory.

However, if we add another type of complex scalar field X to the $SU(2)$ sector, the mixing (3.2.33) can occur and we necessarily include two fermions U,V as well so that the mixing closes. This closed sector is called the $SU(2|3)$ sector. The $SO(6)$ sector is not closed sector due to the the mixing with fields with non-zero Lorentz charges. In fact, the smallest closed sector containing the $SO(6)$ sector is the full $PSU(2,2|4)$ sector. However, the $SO(6)$ sector is closed at 1-loop since the mixing outside of the $SO(6)$ sector is dynamical and it cannot occur until two-loop level [26].

Both the $SU(2)$ sector and the $SU(2|3)$ sector are compact, hence, the operators in these sectors are belong to finite dimensional representations of the symmetry group. On the other hand, the set of all single trace operators made up of one type of complex scalar Z and one type of covariant derivative D_{++} with charges $(1,\frac{1}{2},\frac{1}{2};0,0,0)$ form another closed sector with a non-compact symmetry group. This is the $SL(2)$ sector or sometimes called the $SU(1,1)$ sector. In contrast to the compact case, the operators in the $SL(2)$ sector are belong to an infinite dimensional representation since any field with arbitrary number of covariant derivatives does not vanish. The mixing occurs by rearrangement of covariant derivatives among the Z fields.

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[5]One can confirm this by checking the charges of the other fields explicitly.

3.2.2 Bethe ansatz

In this subsection, we will give a brief introduction to the Bethe ansatz, which is known to be a widly applicable method to determine the spectrum of some large class of integrable models. Bethe ansatz was originally suggested by Hans Bethe in 1931 to find the exact eigenvalues and eigenstates of the Heisenberg model, which is a one-dimensional quantum spin chain model with nearest neighbor interactions [68]. Since then, the method has been extended to many other models such as Bose gas, Hubbard model, etc, and the elegant variants of the method including algebraic Bethe ansatz, analytic Bethe ansatz and thermodynamicc Bethe ansatz have been developed.

In the context of spectrum problem of $\mathcal{N} = 4$ SYM theory, the Heisenberg model was "rediscovered" as the 1-loop dilatation operator for the $SU(2)$ sector, where the composite operators are made out of two kinds of complex scalar fields, say, Z and Y. These fields transform as a doublet under the $SU(2)$ symmetry group, hence they are identified with up spin state and down spin state respectively. The dilatation operator (3.2.24) is nothing but the Hamiltonian of the Heisenberg model, which describes the exchanging interactions between nearest neighbor spins. The dilatation operator (3.2.24) can be rewritten in terms of local spin variables that are defined on each site of the spin chain as follows

$$\Gamma|_{SU(2)} = H_{XXX} = \frac{\lambda}{8\pi^2} \sum_{k=1}^{L} \left(\frac{1}{2} - S_k^i S_{k+1}^i \right) , \tag{3.2.34}$$

$$[S_l^i, S_m^j] = i\varepsilon^{ijk}\delta_{lm}s_l^k, \quad \vec{S}_{k+L} = \vec{S}_k . \tag{3.2.35}$$

Coordinate Bethe ansatz

We now explain how to find eigenvalues and eigenstates of the Heisenberg model by following the intuitive discussion by Yang-Yang [69], which is sometimes called coordinate Bethe ansatz. Notice that the total spin $\vec{S} := \sum_k \vec{S}_k$ commutes with the Hamiltonian H_{XXX}, thus the eigenstates of the Hamiltonian are simultaneously eigenstates of the total spin.[6] Due to the negative sigh of the spin-spin interaction term in H_{XXX}, this Hamiltonian is ferromagnetic and the ground state is the state with all the spins aligned in the same direction. It is convenient to choose the vacuum or the ground state as the state with all spins up: $| \uparrow^L \rangle := \underbrace{| \uparrow \rangle \otimes \cdots \otimes | \uparrow \rangle}_{L}$ with the total spin $L/2$. One can easily check that this state is an eigenstate of the Hamiltonian with zero energy. This state corresponds to the chiral primary operator $\text{Tr}Z^L$. The ground state is the highest weight

---

[6]This is not surprising since the dilatation operator and the R-symmetry generators mutually commute and the total spins are the generators of the $SU(2)$ R-symmetry subgroup.

state in the symmetric representation and all the states being belong to this representation, which are obtained by a multiple actions of the lowering operator S^- on $|\uparrow^L\rangle$, are also the eigenstates of the Hamiltonian with zero energy since the total spin commutes with H_{XXX}.[7] Therefore, we need to study excited states with spin lower than $L/2$ in order to find the spectrum of the operators that are not chiral primaries. Let us start with one spin flipped states such as $|\uparrow \ldots \uparrow\overset{k}{\downarrow}\uparrow \ldots \uparrow\rangle$. The permutation operator $P_{l,l+1}$ non-trivially change the state $|\uparrow \ldots \uparrow\overset{k}{\downarrow}\uparrow \ldots \uparrow\rangle$ only if $l = k$ or $l = k + 1$, thus the action of the Hamiltonian is given by

$$
\begin{aligned}
H_{XXX}&|\uparrow \ldots \uparrow\overset{k}{\downarrow}\uparrow \ldots \uparrow\rangle \\
&= \frac{\lambda}{8\pi^2}\left((L-(L-2))|\uparrow \ldots \uparrow\overset{k}{\downarrow}\uparrow \ldots \uparrow\rangle - |\uparrow \ldots \overset{k-1}{\downarrow}\uparrow\uparrow \ldots \uparrow\rangle - |\uparrow \ldots \uparrow\uparrow\overset{k+1}{\downarrow} \ldots \uparrow\rangle\right).
\end{aligned}
\tag{3.2.36}
$$

This form is very reminiscent of the coupled oscillations, hence we are led to the following ansatz of the eigenstates

$$
|p\rangle := \sum_{k=1}^{L} e^{ipk}|\uparrow \ldots \uparrow\overset{k}{\downarrow}\uparrow \ldots \uparrow\rangle.
\tag{3.2.37}
$$

This is actually the eigenstate of the Hamiltonian

$$
H_{XXX}|p\rangle = \varepsilon(p)|p\rangle, \quad \varepsilon(p) := \frac{\lambda}{2\pi^2}\sin^2\frac{p}{2}.
\tag{3.2.38}
$$

The state $|p\rangle$ is a one-magnon state with momentum p. A magnon is the fundamental excitation over the ferromagnetic vacuum and it can be viewed as a quantized spin wave. The momentum of the one magnon state must be quantized so that the state satisfies the periodic boundary condition, thus we have $p = 2\pi n/L$. Furthermore, we must impose the cyclicity condition (3.2.21) that all the states are invariant under the shift $k \to k+1$ since all the single trace operators are invariant under the simultaneous shift of the fields in the trace due to the cyclicity of the trace. Thus, the only allowed one-magnon state is the state with $n = 0$ or $p = 0$ which is belong to the symmetric representation. Therefore, we find that there is no one magnon state that corresponds to a non chiral primary operator.

Next let us consider the following two magnon states or the states with two down spins

$$
|p_1, p_2\rangle = \sum_{k_1 < k_2} e^{ip_1 k_1 + ip_2 k_2}|\ldots \overset{k_1}{\downarrow} \ldots \overset{k_2}{\downarrow} \ldots\rangle + e^{i\phi}\sum_{k_1 > k_2} e^{ip_1 k_1 + ip_2 k_2}|\ldots \overset{k_2}{\downarrow} \ldots \overset{k_1}{\downarrow} \ldots\rangle.
\tag{3.2.39}
$$

---

[7]Generally, all the states in the same representation have the same eigenvalue of H_{XXX}.

If we assume that $p_1 > p_2$, we can view $|p_1, p_2\rangle$ as a scattering state of two magnons. The first term is the incoming wave while the second term is the outgoing wave and the phase $e^{i\phi}$ is the S-matrix of the scattering. Since the interaction only occurs between two adjacent magnons, the eigenvalues of the Hamiltonian for two-magnon states are the sum of the eigenvalue of each magnon. When the two magnons are next to each other, the interaction non-trivially changes the state thus we need to adjust the coefficients in front of each magnon state in (3.2.39) in order to ensure that $|p_1, p_2\rangle$ is actually an eigenstate. By considering the action of Hamiltonian on all the possible adjacent two down spins states, we find that the phase $e^{i\phi}$ must satisfy the following equation

$$
e^{ip_2}(2 - e^{-ip_1} - e^{ip_2}) + e^{ip_1}(2 - e^{ip_1} - e^{-ip_2})e^{i\phi}
$$
$$
= (4 - e^{-ip_1} - e^{ip_1} - e^{-ip_2} - e^{ip_2})(e^{ip_2} + e^{ip_1}e^{i\phi}), \tag{3.2.40}
$$

and this can be solved as

$$
S(p_2, p_1) := e^{i\phi} = -\frac{e^{ip_1 + ip_2} - 2e^{ip_2} + 1}{e^{ip_1 + ip_2} - 2e^{ip_1} + 1}. \tag{3.2.41}
$$

We now consider the periodicity condition. If we transport the magnon with p_1 around the spin chain once, the state will get an extra phase $e^{ip_1 L} \times e^{i\phi}$. The first factor comes from the transportation around the circle, and the second factor comes from the phase shift of the scattering by the other magnon. However, the state must be invariant, thus we have the quantization condition $e^{ip_1 L}e^{i\phi} = 1$. On the other hand, the cyclicity condition enforces the total momentum to be zero: $p_1 + p_2 = 0$. With $p_2 = -p_1$, we readily see that the phase (3.2.41) is simplified as $e^{i\phi} = e^{-ip_1}$. Combined with these results, we obtain the possible quantized momenta $p_1 = -p_2 = 2\pi n/(L-1)$ and the eigenvalues of the two-magnon state

$$
H_{XXX}|p_1, p_2\rangle = \gamma|p_1, p_2\rangle,
$$
$$
\gamma = \varepsilon(p_1) + \varepsilon(p_2) = \frac{\lambda}{\pi^2}\sin^2\frac{\pi n}{L-1}. \tag{3.2.42}
$$

Notice that the elastic scattering of two identical particles in 1+1 dimensions can at most exchange their momenta due to the energy and momentum conservation. Thus it is not surprising that the ansatz of the wave function for two-magnon states (3.2.39) is actually correct. However, the final set of momenta is in general not a simple rearrangement of the initial set of momenta when three or more particles involve. On the other hand, for integrable systems, this is indeed the case. The integrability of the system means that in addition to the momentum $Q_1 := \hat{P}$ and the energy $Q_2 := \hat{H}$, there exists a family of mutually commuting conserved charges Q_n. For example, if we consider the scattering
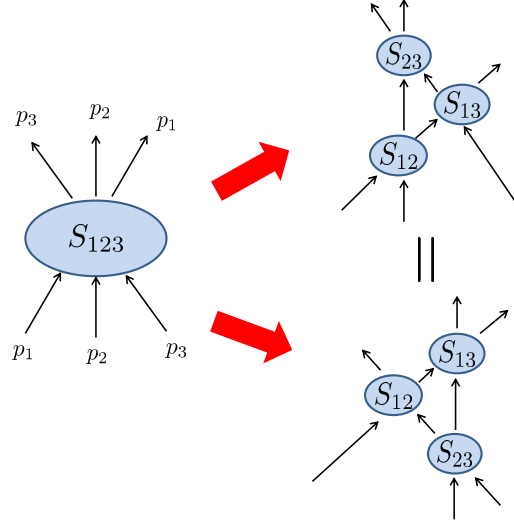
Figure 3.2.4: In an integrable system, n-body scatterings of elementary excitations are usually factorized into the successive 2-body scatterings. The figure shows a example of the 3-body scattering. The consistency condition for the factorized scattering is shown in the right side. It is so-called Yang-Baxter equation.

of three particles in a theory where not only the momentum $Q_1 = p_1 + p_2 + p_3$ and the energy $Q_2 = p_1^2 + p_2^2 + p_3^2$ are conserved, but also an extra charge $Q_3 = p_1^3 + p_2^3 + p_3^3$ is, then the conservation of these charges ensures that the final set of the momenta is just a reshuffling of the original set of momenta

$$\{p_1, p_2, p_3\} = \{p_1', p_2', p_3'\}. \tag{3.2.43}$$

In the similar way, the existence of higher conserved charges would imply that a scattering of particles results in a state with some other momenta, which must be a rearrangement of momenta before the scattering. In other words, scattering is effectively *factorized* and reduced to a sequence of several two-body scatterings since the momenta of particles will be simply given by a permutation after the scattering (figure 3.2.4). We will confirm later that the Heisenberg model indeed has a family of mutually commuting conserved charges by explicitly constructing it in the framework of the algebraic Bethe ansatz.

The ansatz for the wave function for M-magnon states can be generalized as follow

$$|p_1, \ldots, p_M\rangle = \sum_{1 \le n_1 < \ldots < n_M \le L} \Psi(n_1, \ldots, n_M)| \ldots \overset{n_1}{\downarrow} \ldots \overset{n_2}{\downarrow} \ldots \overset{n_M}{\downarrow} \ldots \rangle,$$

$$\Psi(n_1, \ldots, n_M) = \sum_{\sigma \in P_M} A(\sigma) \prod_{k=1}^{M} e^{ip_{\sigma(k)} n_k}, \tag{3.2.44}$$

69

where the summation runs over all $M!$ permutations of $(1, \ldots, M)$. The coefficient of each plane wave satisfies the following relation

$$\frac{A(\ldots, j, i, \ldots)}{A(\ldots, i, j, \ldots)} = S(p_j, p_i), \qquad (3.2.45)$$

where $S(p, p')$ is the two-body S-matrix given in (3.2.41). We usually normalize the wave function (3.2.44) so that $A(1, \ldots, M) = 1$. The states of the form (3.2.44) are called Bethe states. Since the system is a circle with finite length L, the momenta of magnons satisfy the quantization condition which comes from the periodicity of the wave function

$$e^{ip_k L} \prod_{l \neq k}^{M} S(p_l, p_k) = 1. \qquad (3.2.46)$$

This is called the Bethe ansatz equation. The physical meaning of the Bethe ansatz equation is the following. If we carry a magnon with momentum p_k around the circle, the free propagation phase $e^{ip_k L}$ times the phase shift that comes from the scattering by the other magnons must be trivial. The Bethe states whose momenta satisfy the Bethe ansatz equation (3.2.46) are called *on-shell* otherwise *off-shell*. Furthermore, we must impose the cyclicity condition (3.2.21) since we are dealing with the states that correspond to the single trace operators and we have to take into account the cyclicity of the trace. This condition impose the total momentum of magnons is zero modulo $2\pi n$:

$$\prod_{k=1}^{M} e^{ip_k} = 1. \qquad (3.2.47)$$

With the momenta satisfying the Bethe ansatz equation and the trace condition, the energy of the M-magnon state is given by the sum of the energies of individual magnons:

$$E = \sum_{i=1}^{M} \varepsilon(p_i). \qquad (3.2.48)$$

We must stress that the fact that a simple ansatz for the wave function (3.2.44) actually diagonalizes the Hamiltonian is quite non-trivial and a miracle at first glance. In fact, in a generic spin chain model, many body scatterings do not factorize into a product of two-body scatterings, thus the set of momenta of multi magnon states will not be conserved. However, for the case of the Heisenberg model, the S-matrix is factorized into a product of the two-body ones due to the existence of higher conserved charges. Hence, the ansatz (3.2.44) is correct and the problem is reduced to solve the set of equations for the momenta of magnons.

Before explaining the algebraic Bethe ansatz, it is convenient to introduce the rapidity, which is related to the momentum as follows:

$$u := \frac{1}{2} \cot \frac{p}{2}, \quad e^{ip} = \frac{u + i/2}{u - i/2}. \tag{3.2.49}$$

In terms of the corresponding rapidities, we can rewrite the S-matrix as

$$S(u_k, u_l) = \frac{u_k - u_l + i}{u_k - u_l - i}, \tag{3.2.50}$$

and the Bethe ansatz equation (3.2.46) becomes very simple form

$$\left(\frac{u_k + i/2}{u_k - i/2} \right)^L \prod_{l \neq k} \frac{u_l - u_k + i}{u_l - u_k - i} = 1 \iff \left(\frac{u_k + i/2}{u_k - i/2} \right)^L = \prod_{l \neq k} \frac{u_k - u_l + i}{u_k - u_l - i}. \tag{3.2.51}$$

The rapidities that satisfy the Bethe ansatz equations are called *Bethe roots*. With the Bethe roots, the energy of the M-magnon state is given by

$$E = \frac{\lambda}{8\pi^2} \sum_{k=1}^{M} \frac{1}{u_k^2 + \frac{1}{4}}. \tag{3.2.52}$$

Algebraic Bethe ansatz

We will now explain the algebraic Bethe ansatz, which is a more elegant method to solve the model. The coordinate Bethe ansatz gives us a clear physical picture that the Bethe states (3.2.44) are excited states of quanta of spin wave over the ferromagnetic vacuum and the quantization condition for the momenta of magnons leads to the Bethe ansatz equations (3.2.51). However, the coordinate Bethe ansatz itself does not explain why the ansatz indeed provides the correct answer, or equivalently, why the many-body scattering factorizes into two-body scatterings. As already mentioned, the algebraic Bethe ansatz explicitly gives a tower of mutually commuting charges, hence we expect that conservation of these higher charges reduces the many-body scattering to the two-body ones.

The basic ingredient in the framework of the algebraic Bethe ansatz is the so-called Lax operator acting on the product of the spin-chain Hilbert space $\mathcal{H}$ and an auxiliary vector space. In the case of the Heisenberg spin chain with L sites, $\mathcal{H}$ is the tensor product of L copies of a two-dimensional vector space, consisting of the up-spin state $|\uparrow\rangle$ and the down-spin state $|\downarrow\rangle$ at each site, and the auxiliary space has the structure of $\mathbb{C}^2$. The Lax operator $L_n(u)$ acting on the n-th site is then given by

$$L_n(u) \equiv u\mathbf{1} + i \sum_{i=x,y,z} S_n^i \sigma^i = \begin{pmatrix} u + iS_n^z & iS_n^- \\ iS_n^+ & u - iS_n^z \end{pmatrix}, \tag{3.2.53}$$

where S_n^i are the local spin operators[8] and u is a arbitrary complex parameter which is called spectral parameter. Lax operator plays the role of a "connection" of the spin chain. Going around the spin chain, we define the monodromy matrix $\Omega(u)$ as

$$\Omega(u) \equiv L_1(u - \theta_1) \cdots L_L(u - \theta_L) \equiv \begin{pmatrix} A(u) & B(u) \\ C(u) & D(u) \end{pmatrix} \qquad (3.2.54)$$

$$= u^L \mathbf{1} + i u^{L-1} \left(\sum_{i=x,y,z} S^i \sigma^i + i \sum_{j=1}^{L} \theta_j \right) + \mathcal{O}(u^{L-2}). \qquad (3.2.55)$$

Here $S^i = \sum_n S_n^i$ are the total spin operators and we have introduced the inhomogeneity parameters $\boldsymbol{\theta} = \{\theta_1, \ldots, \theta_L\}$ at each site, which preserve the integrability. They are sometimes necessary for avoiding certain degeneracies in the intermediate steps and are also useful for other purposes[9].

The entries of the monodromy matrix $A(u) \ldots D(u)$ are the operators that act on spin chain Hilbert space $\mathcal{H}$. Due to the complexity of the expression of $A(u) \ldots D(u)$ in terms of local spin variables, the actions of these operators on the states of $\mathcal{H}$ are quite complicated and non-local, however, they satisfy rather simple exchange relations, which we call Yang-Baxter algebra [120]. The derivation of Yang-Baxter algebra involves technical calculations thus we don't discuss here.[10] See appendix B for details. The most important consequence of the underlying algebra is that the trace of the monodromy matrix with respect to the auxiliary space, which is called the transfer matrix, is a generating function of conserved charges. In fact, the transfer matrices commute with each other for different values of spectral parameter,

$$T(u) := \mathrm{Tr}\,\Omega(u) = A(u) + D(u), \qquad (3.2.56)$$

$$[T(u), T(v)] = 0, \quad \forall u, v \in \mathbb{C}. \qquad (3.2.57)$$

If we expand the transfer matrix in power of u, one find that the coefficients of u mutually commute due to (3.2.57). Furthermore, one can show that the Hamiltonian is obtained from the logarithmic derivative of the transfer matrix as follows:

$$H_{XXX} = \frac{\lambda}{8\pi^2} \left(L - i\frac{d}{du} \log T(u)|_{u=i/2} \right). \qquad (3.2.58)$$

Therefore, the logarithm of the transfer matrix generates a tower of mutually commuting

---

[8]We define $S_n^{\pm}$ as $S_n^{\pm} \equiv S^x \pm iS^y$.

[9]Although the physical meaning of the inhomogeneity parameters in the context of the three point functions has not been fully clarified, they are useful in generating loop corrections from the tree-level contributions [118, 119].

[10]We shall only recall a necessary portion of this algebra later when we need them.

conserved charges including the spin chain Hamiltonian,

$$\log T(u + i/2) = \sum_{n=0}^{\infty} Q_n u^n \,, \qquad (3.2.59)$$

where Q_1 is the total momentum, Q_2 is the Hamiltonian and the others are higher conserved charges we were looking for. Although the expansion (3.2.59) is an infinite series, the number of independent conserved charges should be finite since the number of degrees of freedom for the spin chain is finite. Indeed, if we consider the expansion of $T(u)$ around $u = 0$, the number of conserved is finite since $T(u)$ is a polynomial in u. Hence, the expression (3.2.59) is somehow redundant [11]. The existence of higher conserved charges Q_n explains the factorization of multi-particle scattering or the reason why the ansatz (3.2.44) ought to work.

Since the Hamiltonian is obtained from the transfer matrix as (3.2.58), we can consider the following eigenvalue problem instead of diagonalizing the Hamiltonian directly:

$$T(u)|\Psi_i\rangle = t_i(u)|\Psi_i\rangle \,, \qquad (3.2.60)$$

where $t_i(u)$ is the eigenvalue of the transfer matrix. Notice that the eigenstate $|\Psi_i\rangle$ does not depend on the spectral parameter since the transfer matrices commute with each other for different values of it. In other words, the eigenstate $|\Psi_i\rangle$ diagonalizes all the conserved charges including the Hamiltonian at the same time.

The coordinate Bethe ansatz tell us that such eigenstates are multi-magnon states with the quantized momenta that satisfy the Bethe ansatz equations. Therefore, we need to construct a multi-magnon state in the algebraic Bethe ansatz in order to solve the eigenvalue problem (3.2.60). Here, the Yang-Baxter algebra play an important role again. The algebra reveals that off-diagonal components of monodromy matrix $B(u)$ and $C(u)$ are identified with the creation and annihilation operator for the magnon respectively. This allows one to regard the Hilbert space $\mathcal{H}$ as the Fock space build over the ferromagnetic vacuum $|\uparrow^L\rangle$, which is spanned by the M-magnon states of the form

$$|\boldsymbol{u}\rangle = B(u_1) \cdots B(u_M)|\uparrow^L\rangle \,. \qquad (3.2.61)$$

Once we identify u_i's with the rapidities of the magnons according to (3.2.49), it turns out that this state is the same as the Bethe state (3.2.44) up to a normalization, thus we shall call (3.2.61) the Bethe state as well. If rapidities u_i are Bethe roots, then $|\boldsymbol{u}\rangle$ satisfies (3.2.60), and is an on-shell Bethe state.

---

[11]The advantage of the expansion (3.2.59) is that all the charges generated in this way are local. Of course, we can obtain conserved charges by expanding any functional of $T(u)$ around any point u^*.

We now calculate the action of $T(u) = A(u) + D(u)$ on the Bethe state (3.2.61) and derive the conditions that rapidities should satisfy in order that the Bethe state is on-shell. For this purpose, we need to find (i) the exchange relation between $T(u)$ and $B(u)$, and (ii) the action of $T(u)$ on the vacuum $|\uparrow^L\rangle$. With these knowledge, we can compute the action of $T(u)$ by consecutively using the exchanging relation and taking $T(u)$ to the neighbor of the vacuum.

The action of $A(u), C(u), D(u)$ on the vacuum $|\uparrow^L\rangle$ becomes very simple. Indeed, from the definition of the Lax operator L_n (3.2.53), the action of it on the vacuum becomes an upper triangle matrix

$$L_n(u - \theta_n)|\uparrow^L\rangle = \begin{pmatrix} u - \theta_n + i/2 & iS_n^- \\ 0 & u - \theta_n - i/2 \end{pmatrix}|\uparrow^L\rangle \qquad (3.2.62)$$

since the raising operator S_n^+ annihilates the vacuum. Thus we find that the action of the monodromy matrix on the vacuum becomes also an upper triangle form

$$\begin{aligned} \Omega(u)|\uparrow^L\rangle &= L_1(u - \theta_1) \cdots L_L(u - \theta_L)|\uparrow^L\rangle \\ &= \begin{pmatrix} u - \theta_1 + i/2 & iS_1^- \\ 0 & u - \theta_1 - i/2 \end{pmatrix} \cdots \begin{pmatrix} u - \theta_L + i/2 & iS_L^- \\ 0 & u - \theta_L - i/2 \end{pmatrix}|\uparrow^L\rangle \\ &= \begin{pmatrix} \prod_{k=1}^L (u - \theta_k + i/2)|\uparrow^L\rangle & B(u)|\uparrow^L\rangle \\ 0 & \prod_{k=1}^L (u - \theta_k - i/2)|\uparrow^L\rangle \end{pmatrix}. \end{aligned} \qquad (3.2.63)$$

From this expression, we can read

$$A(u)|\uparrow^L\rangle = Q_{\boldsymbol{\theta}}^+|\uparrow^L\rangle, \quad D(u)|\uparrow^L\rangle = Q_{\boldsymbol{\theta}}^-|\uparrow^L\rangle, \quad C(u)|\uparrow^L\rangle = 0, \qquad (3.2.64)$$

$$\langle\uparrow^L|A(u) = Q_{\boldsymbol{\theta}}^+|\uparrow^L\rangle, \quad \langle\uparrow^L|D(u) = Q_{\boldsymbol{\theta}}^-|\uparrow^L\rangle, \quad \langle\uparrow^L|B(u) = 0, \qquad (3.2.65)$$

where we also write down the action of $A(u)$, $D(u)$ and $B(u)$ on the dual pseudo-vacuum $\langle\uparrow^L|$ which satisfies $\langle\uparrow|B(u) = 0$ and $\langle\uparrow^L|\uparrow^L\rangle = 1$. $Q_{\boldsymbol{\theta}}$ functions are defined as[12]

$$Q_{\boldsymbol{\theta}}(u) := \prod_{k=1}^L (u - \theta_k), \quad Q_{\boldsymbol{\theta}}^{\pm}(u) := \prod_{k=1}^L \left(u - \theta_k \pm \frac{i}{2} \right). \qquad (3.2.66)$$

The exchanging relations we will use are the following:

$$[B(u), B(v)] = 0, \qquad (3.2.67)$$

$$A(u)B(v) = \frac{u - v - i}{u - v}B(v)A(u) + \frac{i}{u - v}B(u)A(v), \qquad (3.2.68)$$

$$D(u)B(v) = \frac{u - v + i}{u - v}B(v)A(u) - \frac{i}{u - v}B(u)A(v). \qquad (3.2.69)$$

---

[12] As in these definitions, each $+$ (respectively $-$) superscript on a function signifies that its argument is shifted by $+\frac{i}{2}$ (respectively $-\frac{i}{2}$). According to this convention, $Q_{\boldsymbol{\theta}}^{++}(u)$ means $Q_{\boldsymbol{\theta}}(u + i)$, etc. When $\theta_k = 0$, the functions $Q_{\boldsymbol{\theta}}^{\pm}(u)$ are often referred to as $a(u)$ (for $+$) and $d(u)$ (for $-$).

For example, the action of $A(u)$ on generic Bethe state can be computed by pushing it through $B(u_i)$'s using exchanging relation (3.2.68) and the result is given by

$$A(u)|\boldsymbol{u}\rangle = Q_{\boldsymbol{\theta}}^+(u) \prod_{j=1}^M \frac{u - u_j - i}{u - u_j}|\boldsymbol{u}\rangle$$

$$+ \sum_{k=1}^M \alpha_k(u; \{u_i\}) B(u)B(u_1)\cdots B(u_{k-1})B(u_{k+1})\cdots B(u_M)|\uparrow^L\rangle. \qquad (3.2.70)$$

Notice that the extra terms, one of whose rapidities are exchanged with u, appear due to the second term of the exchanging relation (3.2.68). One can determine the coefficient of these extra terms $\alpha_k(u, \{u_k\})$ by the following argument. First, we arrange all the B's so that the leftmost creation operator is $B(u_k)$. It is possible to do so due to (3.2.67). Then, we exchange $A(u)$ with $B(u_k)$ by using the second term of the exchanging relation (3.2.68) as follows:

$$A(u)B(u_k) \prod_{j\neq k} B(u_j)|\uparrow^L\rangle = \frac{i}{u - u_k}B(u)A(u_k)\prod_{j\neq k} B(u_j)|\uparrow^L\rangle + \cdots. \qquad (3.2.71)$$

However, the first term is already the form we want, hence we will continue to move $A(u_k)$ until it hits the vacuum using only the first term of (3.2.68) in each step. Finally we find

$$\alpha_k(u, \{u_k\}) = \frac{i}{u - u_k} \prod_{j\neq k}^M \frac{u_k - u_j - i}{u_k - u_j} Q_{\boldsymbol{\theta}}^+(u_k). \qquad (3.2.72)$$

In the same way the action of $D(u)$ is given by

$$D(u)|\boldsymbol{u}\rangle = Q_{\boldsymbol{\theta}}^-(u) \prod_{j=1}^M \frac{u - u_j + i}{u - u_j}|\boldsymbol{u}\rangle$$

$$+ \sum_{k=1}^M \delta_k(u; \{u_i\}) B(u)B(u_1)\cdots B(u_{k-1})B(u_{k+1})\cdots B(u_M)|\uparrow^L\rangle, \qquad (3.2.73)$$

where

$$\delta_k(u, \{u_k\}) = -\frac{i}{u - u_k} \prod_{j\neq k}^M \frac{u_k - u_j + i}{u_k - u_j} Q_{\boldsymbol{\theta}}^-(u_k). \qquad (3.2.74)$$

Therefore, we find that $|\boldsymbol{u}\rangle$ is an eigenstate of $T(u) = A(u) + D(u)$ if we choose the rapidities so that the coefficients of extra terms α_k and δ_k cancel with each other. More precisely, Bethe state $|\boldsymbol{u}\rangle$ is on-shell if and only if the following equations for rapidities are satisfied:

$$\alpha_k - \delta_k = 0 \iff \prod_{l=1}^L \left(\frac{u_k - \theta_l + \frac{i}{2}}{u_k - \theta_l - \frac{i}{2}} \right) = \prod_{j\neq k}^M \frac{u_k - u_j + i}{u_k - u_j - i}. \qquad (3.2.75)$$

In that case, the eigenvalue $t_{\boldsymbol{u}}(u)$ of transfer matrix is given by

$$t_{\boldsymbol{u}}(u) = Q_{\boldsymbol{\theta}}^+(u)\frac{Q_{\boldsymbol{u}}^{--}(u)}{Q_{\boldsymbol{u}}(u)} + Q_{\boldsymbol{\theta}}^-(u)\frac{Q_{\boldsymbol{u}}^{++}(u)}{Q_{\boldsymbol{u}}(u)}\,, \qquad (3.2.76)$$

where $Q_{\boldsymbol{u}}(u)$ is called Q-function and defined as

$$Q_{\boldsymbol{u}}(u) := \prod_{i=1}^{M}(u - u_i)\,. \qquad (3.2.77)$$

In homogeneous limit $\theta_l \to 0$, the on-shell conditions (3.2.75) reproduce the Bethe ansatz equations (3.2.51), which are the quantization condition for the momenta of magnons. From a different point of view, (3.2.75) can be interpreted as the pole-free condition for the eigenvalue of the transfer matrix. In fact, for generic rapidities, the right hand side in (3.2.76) have poles at the zeros of Q-function or at the position of u_i's. However, if u_i's are Bethe root, $t_{\boldsymbol{u}}(u)$ is actually an analytic function on u-plane since (3.2.75) can be rewritten as

$$Q_{\boldsymbol{\theta}}^+(u_k)Q_{\boldsymbol{u}}^{--}(u_k) + Q_{\boldsymbol{\theta}}^-(u_k)Q_{\boldsymbol{u}}^{++}(u_k) = 0\,, \qquad (3.2.78)$$

and the poles from denominator and the zeros from numerator cancel with each other.

Let us comment on the Q-function. The Q-function can be regarded as a representation for the wave function of the spin chain. In this picture, Bethe roots are the positions of nodes, which are the zeros of the wave function. Furthermore, (3.2.76) is the "Shrödinger equation" for the Q-function, which is sometimes called Baxter equation and the Bethe ansatz equations are consistency conditions for the nodes required from the Baxter equation.

It is important to consider the global $SU(2)$ charge of the Bethe state. From the definition of monodromy matrix (3.2.55), we find that asymptotic behavior of $A(u)\ldots D(u)$ in the limit $u \to \infty$ are $A(u)/u^L \sim 1+i(1+S^z)/u$, $B(u)/u^L \sim iS^-/u$, $C(u)/u^L \sim iS^+/u$, and $D(u)/u^L \sim 1+i(1-S^z)/u$. Then we can read transformation properties of $A(u)\cdots D(u)$ under the global $SU(2)$ generators by considering the Yang-Baxter algebra in the limit $v \to \infty$ with u held fixed. For example, from the exchanging relation (3.2.68), we can read

$$[S^z, B(u)] = -B(u)\,. \qquad (3.2.79)$$

This implies that $B(u)$ lower the spin by unit and create a down spin excitation over up spin vacuum $|\uparrow^L\rangle$ as expected. Further, from the exchanging relation between $B(u)$ and $C(v)$, we obtain $[S^+, B(u)] = A(u) - D(u)$. By the same calculation for the action of $T(u)$

on the Bethe state, it turns out that raising operator S^+ annihilates the Bethe states if they are on-shell. In other words, if the Bethe state $|\boldsymbol{u}\rangle = \prod_{i=1}^{M} B(u_i)|\uparrow^L\rangle$ is on-shell, it is the highest weight state with spin $\frac{L}{2} - M$. Notice that off-shell Bethe states with M-magnon excitations also have spin $\frac{L}{2} - M$, however, they are generally direct sum of the states being belong to various representations and they are not the highest weight state.

Let us end this section with the discussion on the completeness of the Bethe states. The number of independent solutions for the Bethe ansatz equation (3.2.51) is discussed in [70], and it is expected to be

$$Z_{L,M} = \begin{pmatrix} L \\ M \end{pmatrix} - \begin{pmatrix} L \\ M-1 \end{pmatrix}. \tag{3.2.80}$$

Since the on-shell Bethe states are the highest weight states, the multiplets of the Bethe states are obtained by acting the lowering operator S^- on the on-shell Bethe states. The number of these states are

$$\sum_{M=0}^{L/2} \left(2 \left(\frac{L}{2} - M \right) + 1 \right) Z_{L,M} = 2^L. \tag{3.2.81}$$

This is the same as the dimension of the spin chain Hilbert space $\mathcal{H}$, hence we have a complete set of the Hilbert space of the form $(S^-)^n \prod_{i=1}^{M} B(u_i)|\uparrow^L\rangle$ where u_i's are Bethe roots and $n = 0, \ldots, L - 2M$. Notice that all the descendant states have the same energy as the highest weight state since the global symmetry generators commute with the transfer matrix. Moreover, the descendants are actually obtained from generic Bethe states by sending some of rapidities to infinity,[13]

$$(S^-)^n \prod_{i=1}^{M} B(u_i)|\uparrow^L\rangle = \lim_{w_1,\ldots,w_n \to \infty} \prod_{k=1}^{n} \frac{B(w_k)}{iw_k^{L-1}} \prod_{i=1}^{M} B(u_i)|\uparrow^L\rangle. \tag{3.2.82}$$

and it is enough to consider the Bethe states.

3.3 Semiclassical spectrum of string σ-model

In this section, we will discuss on the classical integrability of string on $AdS_5 \times S^5$ and a certain general class of classical solutions which are so-called finite gap solutions. As we mentioned in section 2.2, the quantization of the string on such a background is extremely difficult since the full action is quite complicated due to the presence of the non-trivial curvature and the RR flux in contrast to the case of the flat space where the action is

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[13]One can see this from $B(u) \sim iS^- u^{L-1} + \mathcal{O}(u^{L-2})$.

free. Therefore, we will consider the semiclassical spectrum as a starting point for the perturbative treatment of the σ-model with respect to the "Planck constant" $\frac{1}{\sqrt{\lambda}}$. The WKB approximation allows us to approximate semiclassical states with large charges such as angular momentum by the corresponding classical solutions.

Even at the classical level, one need to solve the non-linear equation of motion, thus it seems to be difficult to get exact solutions. However, shortly after the discovery that the 1-loop dilatation operator for a certain sector of $\mathcal{N} = 4$ SYM theory can be exactly diagonalized by using Bethe ansatz, Bena, Polchinski and Roiban proved that the Metsaev-Tseytlin sigma model is classically integrable [71]. The classical integrability means that there are sufficiently many conserved charges which ensure that a "factorization" of dynamical variables occurs. In the case of a finite dimensional system, there exist canonical variables (x_i, p_i) such that $\{x_i, x_j\} = \{p_i, p_j\} = 0, \{x_i, p_j\} = \delta_{ij}$ if the system possesses as many conserved charges as the number of degrees of freedom. This is expected to hold in the case of infinite dimensional system. Indeed, these canonical variables, which we will call "separation of variables (SoV), play an important role in constructing the finite-gap solutions. It turns out that the finite-gap solutions can be expressed in the language of algebraic curves [79]. These curves are essentially determined from the charges of the solutions and they perfectly encode the information of the solutions. We will discus on the finite-gap solution in section 3.3.2.

3.3.1 Classical integrability

We will now show the classical integrability of the string on $AdS_5 \times S^5$. That is, we will construct a family of infinitely many conserved charges.

The key idea is to construct a 1-parameter family of flat current $A(\sigma, \tau; x)$ satisfying $dA + A \wedge A = 0$. x is an arbitrary complex number so-called spectral parameter. Given such a flat connection, the following Wilson loop like object only depends on the homotopy class of the integration contour C due to the flatness condition

$$W(C; x) := P \exp \left[\int_C A(\sigma, \tau; x) \right], \qquad (3.3.1)$$

where P is the path ordering operator. In other words, (3.3.1) is invariant under the continuous deformation of the contour C. We now take the contour to be a specific time slice of the cylinder and define the monodromy matrix

$$\Omega(\tau; x) := P \exp \left[\int_0^{2\pi} d\sigma A_\sigma(\sigma, \tau; x) \right]. \qquad (3.3.2)$$

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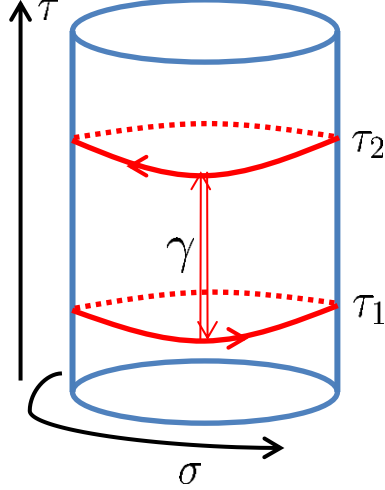
We can show that the monodromy matrices defined at the different time slices only differ by a similarity transformation:

$$\Omega(\tau_2; x) = U\Omega(\tau_1; x)U^{-1}. \tag{3.3.3}$$

This immediately allows us to construct an infinite number of conserved charges since the eigenvalues of the monodromy matrix are independent of τ and the Taylor expansion of the eigenvalues around a particular x generates infinitely many conserved charges.

The proof of (3.3.3) is as follow: Notice that $W(C; x) = 1$ if C is contractible. Then, we take the contour as in the figure 3.3.1 and obtain

$$P \exp\left[\int_{\gamma^{-1}} (A_\sigma d\sigma + A_\tau d\tau)\right] \Omega^{-1}(\tau_2; x) P \exp\left[\int_{\gamma} (A_\sigma d\sigma + A_\tau d\tau)\right] \Omega(\tau_1; x) = 1. \tag{3.3.4}$$

This is exactly the form of (3.3.3) if we identify $U = P \exp\left[\int_{\gamma} (A_\sigma d\sigma + A_\tau d\tau)\right]$.

We now wish to show the classical integrability of the full string on $AdS_5 \times S^5$, namely, the existence of a 1-parameter family of flat connections according to [71]. First, let us remind the reader that the Green-Schwarz superstring on $AdS_5 \times S^5$ can be regarded as a non-linear sigma model with the field taking values in the following coset space:

$$\frac{PSU(2, 2|4)}{SO(1, 4) \times SO(5)}. \tag{3.3.5}$$

The Lie algebra of $PSU(2, 2|4)$ admits $\mathbb{Z}_4$ decomposition as follows:

$$\mathcal{G} = \mathcal{H} + \mathcal{P} + \mathcal{Q}_1 + \mathcal{Q}_2, \tag{3.3.6}$$

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where $\mathcal{H}$ is the Lie algebra of the $SO(1,4) \times SO(5)$, $\mathcal{P}$ is made up of the remaining bosonic generators, and $\mathcal{Q}_1$ and $\mathcal{Q}_2$ are two copies of the $(\mathbf{4},\mathbf{4})$ representations of $\mathcal{H}$. Under this decomposition, each component has the following $\mathbb{Z}_4$ charges

$$\mathcal{H}:0\,,\ \ \mathcal{Q}_1:1\,,\ \ \mathcal{P}:2\,,\ \ \mathcal{Q}_2:3\,. \tag{3.3.7}$$

The Metseav-Tseytlin action is given by

$$S = \frac{\sqrt{\lambda}}{4\pi}\int \mathrm{Str}(J^{(2)}\wedge *J^{(2)} - \kappa J^{(1)}\wedge J^{(3)}) + \Lambda \wedge \mathrm{Str}J^{(2)}\,, \tag{3.3.8}$$

where $J^{(i)}$ $(i = 1,\ldots,4)$ are the $\mathbb{Z}_4$ components of the left invariant current $J = -g^{-1}dg$ according to the above decomposition of the $\mathfrak{psu}(2,2|4)$ algebra. The first term is the standard kinetic term for the coset model and the third term is required from the unimodularity of $PSU(2,2|4)$. The second term is the Wess-Zumino term and its coefficient κ must be equal to 1 or -1 due to the κ-symmetry. However, we will leave it to be indefinite for a while to understand how the presence of the Wess-Zumino term and κ-symmetry works in the following discussion.

Since J is the left invariant current, it satisfies the flatness condition or the Maurer-Cartan equation $dJ = J \wedge J$. The $\mathbb{Z}_4$ grading allows us to decompose it as follow:

$$dJ^{(i)} = \sum_{j+k=i,\mathrm{mod}4} J^{(j)}\wedge J^{(k)}\,. \tag{3.3.9}$$

The equations of motion are given by the Noether current for the global symmetry of the action (3.3.8)

$$d*k = 0\,, \tag{3.3.10}$$

$$k = g^{-1}Kg\,, \ \ K = J^{(2)} + \frac{\kappa}{2}*(J^{(1)} - J^{(3)}) - *\Lambda\,. \tag{3.3.11}$$

Although the equation (3.3.10) is very simple, it is more convenient to rewrite the equations of motion in terms of K since the only left invariant currents have a good $\mathbb{Z}_4$ decomposition. From (3.3.10) and (3.3.11), we find $d*K = J\wedge *K + *K \wedge J$ and the $\mathbb{Z}_4$ decomposition gives,

$$0 = J^{(3)}\wedge *J^{(2)} + *J^{(2)}\wedge J^{(3)} - \kappa(J^{(3)}\wedge J^{(2)} + J^{(2)}\wedge J^{(3)})\,, \tag{3.3.12}$$

$$dJ^{(2)} = J^{(0)}\wedge *J^{(2)} + *J^{(2)}\wedge J^{(0)} - \kappa(J^{(1)}\wedge J^{(1)} - J^{(3)}\wedge J^{(3)})\,, \tag{3.3.13}$$

$$0 = J^{(1)}\wedge *J^{(2)} + *J^{(2)}\wedge J^{(1)} - \kappa(J^{(1)}\wedge J^{(2)} + J^{(2)}\wedge J^{(1)})\,. \tag{3.3.14}$$

Here, we have used (3.3.9) to eliminate $dJ^{(1)}$ and $dJ^{(3)}$ in (3.3.12) and (3.3.14). Note that the equation for the 0-th component is trivial.

To construct a 1-parameter family of flat connections which assure the existence of higher conserved charges or classical integrability, we employ the following ansatz for the form of the flat connection:

$$A(x) = \sum_{i=0}^{3} \alpha_i(x) J^{(i)} + \beta(x)(*J^{(2)} - \Lambda) \,. \tag{3.3.15}$$

We will now determine the coefficients $\alpha_i(x)$ and $\beta(x)$ so that the connection $A(x)$ satisfies $dA - A \wedge A = 0$ if the equations of motion are satisfied. The derivatives of currents such as $dJ^{(i)}$ and $d*J^{(2)}$ can be eliminated by using flatness condition (3.3.9) or the equation of motion (3.3.13). Thus $dA - A \wedge A$ is a linear combination of the current bilinears such as $J^{(0)} \wedge *J^{(2)}$, $J^{(2)} \wedge J^{(3)}$ etc. However, this linear combination is redundant since the equations of motion (3.3.12), (3.3.14) reduce the number of independent current bilinears. With the identities $*a \wedge b + a \wedge *b = 0$, $** = +1$ that hold for 1-forms, we finally obtain the set of equations for $\alpha_i(x)$ and $\beta(x)$

$$\alpha_0 = 1 \,, \tag{3.3.16}$$
$$\alpha_0 - \alpha_1 \alpha_3 = 0 \,, \tag{3.3.17}$$
$$\alpha_2 - \kappa\beta - \alpha_3^2 = 0 \,, \tag{3.3.18}$$
$$\alpha_2 + \kappa\beta - \alpha_1^2 = 0 \,, \tag{3.3.19}$$
$$\alpha_0 + \beta^2 - \alpha_2^2 = 0 \,, \tag{3.3.20}$$
$$\kappa\beta\alpha_3 + \alpha_2\alpha_3 - \alpha_1 = 0 \,, \tag{3.3.21}$$
$$\kappa\beta\alpha_1 - \alpha_1\alpha_2 + \alpha_3 = 0 \,. \tag{3.3.22}$$

This set of equations seems to overdetermined, as we have only five unknown coefficients but there are seven equations. However, the system of equations is highly redundant and there exists a 1-parameter family of the solution. First, notice that the first four equations become

$$\alpha_0 = 1 \,, \alpha_1 = \frac{1}{\alpha_3} \,, \alpha_2 = \frac{1}{2}\left(\alpha_3^2 + \frac{1}{\alpha_3^2}\right) \,, \beta = -\frac{1}{2\kappa}\left(\alpha_3^2 - \frac{1}{\alpha_3^2}\right) \,. \tag{3.3.23}$$

In other words, we can solve the α_1, α_2, and β in terms of α_3. Surprisingly, the last two equations are automatically satisfied if α_i and β satisfy (3.3.23). The remaining constraint gives

$$\left(\frac{1}{\kappa^2} - 1\right)\left(1 - \frac{1}{\alpha_3^4}\right)^2 = 0 \,. \tag{3.3.24}$$

Therefore, if and only if $\kappa = \pm 1$, $A(x)$ indeed satisfies the flatness condition for any function $\alpha_3(x)$. Otherwise, (3.3.24) gives $\alpha_3^4 = 1$ and we obtain a trivial flat connection

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that does not depend on an arbitrary continuous parameter.[14] In the original work [71], $f(x)$ was chosen to be $f(x) = e^x$, however, it turns out that $f(x) = \sqrt{\frac{x-1}{x+1}}$ is a more convenient parameterization in order to consider the spectral curve [84]. Thus we have

$$A(x) = J^{(0)} + \frac{x^2+1}{x^2-1}J^{(2)} - \frac{2x}{x^2-1}(*J^{(2)} - \Lambda) + \sqrt{\frac{x+1}{x-1}}J^{(1)} + \sqrt{\frac{x-1}{x+1}}J^{(3)}. \quad (3.3.25)$$

3.3.2 Finite gap solution and spectral curve

In the previous section, we have seen that a 1-parameter family of flat connections leads to the existence of infinitely many conserved charges. The existence of a sufficient number of conserved charges ensures that the simplification for the dynamics of the world-sheet theory occurs at least classical level.[15] In fact, it turns out that the 1-parameter family of flat connections allows us to construct a very large class of classical solutions which is so-called finite-gap solutions. The key ingredient of the construction for the finite-gap solutions is a complex curve which is known as the spectral curve. The spectral curve is characterized as the curve in the phase space of the string on which the conserved charges are constant. Hence, the spectral curve contains all the information associated with the conserved charges for each classical solution, and each solution uniquely defines the spectral curve. Conversely, given a spectral curve, we can uniquely reconstruct a solution that has the charges determined from the curve since sufficiently many conserved charges uniquely characterize the solution. This is a manifestation of the integrability. We now discuss these topics in this section. First, we will explain the basic notions and generic properties of the spectral curve. Then, we will see how finite-gap solutions are constructed from the spectral curves. For simplicity, we will only consider the classical string solution moving on $\mathbb{R} \times S^3$ which is a subspace of the whole $AdS_5 \times S^5$, according to the discussion of [95, 96]. However, the most of the following discussions hold for the generic case, at least conceptually. The strings moving on this subspace are closely related to the single trace operators in the $SU(2)$ sector of $\mathcal{N} = 4$ SYM. The comparison for both theory will be discussed in the next subsection.

The action for string moving on $\mathbb{R} \times S^3$ in the conformal gauge is given by

$$S = \frac{\sqrt{\lambda}}{4\pi} \int d\sigma d\tau [-(\partial_a X^0)^2 + (\partial_a X^i)^2 + \Lambda(X_i X^i - 1)], \quad (3.3.26)$$

where X^0 is the time coordinate, X^i $(i = 1, \ldots 4)$ are the embedding coordinate and Λ is a Lagrange multiplier that constrains the string to the sphere S^3. Since S^3 is the group

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[14]There is some relation between the κ-symmetry and the integrability since if we chose $\kappa \neq \pm 1$, then the both κ-symmetry and the higher symmetry are broken. However, it remains to be revealed.

[15]Even at quantum level, we can expect that an infinite number of conserved charges reduces any scattering matrix for the excitations of the world-sheet theory to a product of two-body S-matrices.

manifold $SU(2)$, this sigma model can be rewritten in terms of the currents associated with $SU(2)$. We define a field g taking values in $SU(2)$ as follows:

$$g = \begin{pmatrix} Z & Y \\ -\bar{Y} & \bar{Z} \end{pmatrix} \in SU(2) \,, \tag{3.3.27}$$

where $Z = X^1 + iX^2$, $Y = X^3 + iX^4$ and $\bar{Z}, \bar{Y}$ are their complex conjugate. The left invariant current is defined as usual, $j = -g^{-1}dg$. With it, the action becomes

$$S = -\frac{\sqrt{\lambda}}{4\pi} \int \left[\frac{1}{2}\mathrm{Tr}(j \wedge *j) + dX^0 \wedge *dX^0 \right] \,. \tag{3.3.28}$$

In the static gauge, $X^0 = \kappa\tau$,[16] the Virasoro constraints[17] become

$$\frac{1}{2}\mathrm{Tr}j_{\pm}^2 = -\kappa^2 \,, \tag{3.3.29}$$

where $j_{\pm} := j_0 \pm j_1$ are the light-cone components of the currents. In this gauge, the space-time energy of the string is given by

$$\Delta = \frac{\sqrt{\lambda}}{2\pi} \int_0^{2\pi} d\sigma \partial_\tau X^0 = \sqrt{\lambda}\kappa \,. \tag{3.3.30}$$

The action (3.3.28) has a global $SU(2)_L \times SU(2)_R$ symmetry since the target space S^3 has the isometry $SO(4) \cong SU(2)_L \times SU(2)_R$. The action of this symmetry is realized as

$$g \to U_L g U_R \,, \tag{3.3.31}$$

where U_L and U_R are constant group elements of $SU(2)_L$ and $SU(2)_R$ respectively. The Noether current associated with the $SU(2)_R$ symmetry is the left invariant current $j = -g^{-1}dg$ and the Noether current for the $SU(2)_L$ symmetry is the right invariant current $l = -dgg^{-1} = gjg^{-1}$.[18] The corresponding Noether charges are defined as

$$SU(2)_R: \quad Q_R = -\frac{\sqrt{\lambda}}{4\pi} \int_0^{2\pi} d\sigma j_0 \,, \tag{3.3.32}$$

$$SU(2)_L: \quad Q_L = -\frac{\sqrt{\lambda}}{4\pi} \int_0^{2\pi} d\sigma l_0 \,. \tag{3.3.33}$$

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[16]The equation of motion for the time coordinate is simply the wave equation and it can be solved separately. The general solution is the form of $X^0 = \kappa\tau + f^+(\sigma^+) + f^-(\sigma^-)$ where $\sigma^{\pm} = \frac{1}{2}(\tau \pm \sigma)$. With the residual world-sheet diffeomorphism, we can gauge away $f^{\pm}(\sigma^{\pm})$.

[17]The Virasoro constraints are originally $(\partial_{\pm}X^0)^2 = (\partial_{\pm}X^i)^2$.

[18]Notice that the left invariant current transforms under the action of $U_R \in SU(2)_R$ as $j \to U_R^{-1}jU_R$. Similarly, the right invariant current transforms as $l \to U_L^{-1}jU_L$ under the action of $U_L \in SU(2)_L$.

The left invariant current is identically flat by definition and the equations of motion are equivalent to the conservation of the current, thus we have $dj = j \wedge j$ and $d * j = 0$. Therefore, we can construct a 1-parameter family of flat connections as

$$J(x) = \frac{1}{1 - x^2} j - \frac{x}{1 - x^2} * j \,, \tag{3.3.34}$$

where we adopt the convention of sign in [95]. In components, we have

$$J_0(\tau, \sigma; x) = \frac{1}{2} \left(\frac{j_+}{1 - x} + \frac{j_-}{1 + x} \right) \,, \tag{3.3.35}$$

$$J_1(\tau, \sigma; x) = \frac{1}{2} \left(\frac{j_+}{1 - x} - \frac{j_-}{1 + x} \right) \,. \tag{3.3.36}$$

Our particular interest is the monodromy matrix, which generates infinitely many conserved charges. The monodromy matrix is defined as the Wilson loop along with the closed loop winding the world-sheet once,

$$\Omega(\tau, \sigma; x) = P \exp \left[\int_\sigma^{\sigma + 2\pi} d\tilde{\sigma} \frac{1}{2} \left(\frac{j_+(\tau, \tilde{\sigma})}{1 - x} - \frac{j_-(\tau, \tilde{\sigma})}{1 + x} \right) \right] \,. \tag{3.3.37}$$

By virtue of the flatness condition, monodromy matrices defined at the different base points (τ, σ) and (τ', σ') only differ by the conjugation,

$$\Omega(\tau', \sigma'; x) = U \Omega(\tau, \sigma; x) U^{-1} \,, \tag{3.3.38}$$

where $U = P \exp \left[\int_\gamma J(\tau, \sigma; x) \right]$ and γ is any path connecting the two base points. Therefore, eigenvalues of the monodromy matrix do not depend on the world-sheet coordinate at all. Since the monodromy matrix is unimodular $\det \Omega = 1$, due to $\mathrm{Tr} J = 0$, the monodromy matrix can be diagonalized for generic values of x as follows

$$u(\tau, \sigma; x) \Omega(\tau, \sigma; x) u(\tau, \sigma; x)^{-1} \sim \mathrm{diag}(e^{ip(x)}, e^{-ip(x)}) \,, \tag{3.3.39}$$

where $p(x)$ is the so-called *quasi-momentum*. The expansion of quasi-momentum with respect to the spectral parameter x generates a set of infinite number of conserved charges. Thus all the information of the charges for classical solutions is encoded in the corresponding quasi-momentum. In particular, the global charges can be read from the asymptotic behavior around $x = \infty$ and $x = 0$. To see this, recall that the definition of the flat current (3.3.34) and we find $J(x) \to *j/x + O(x^{-2})$ $(x \to \infty)$. Hence, we obtain the following asymptotic behavior of the monodromy matrix around $x = \infty$

$$\Omega(\tau, \sigma; x) = \mathbf{1} - \frac{1}{x} \int_\sigma^{\sigma + 2\pi} d\tilde{\sigma} j_0(\tau, \tilde{\sigma}) + O(x^{-2})$$

$$= \mathbf{1} - \frac{1}{x} \frac{4\pi Q_R}{\sqrt{\lambda}} + O(x^{-2}) \quad (x \to \infty) \,, \tag{3.3.40}$$

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where Q_R is the charge defined in (3.3.32). We are interested in classical solutions with the definite charges. Thus, we assume that the left and right charges (3.3.32), (3.3.33) are diagonal

$$Q_R = \frac{\sigma_3}{2i} R \,, \quad Q_L = \frac{\sigma_3}{2i} L \,, \tag{3.3.41}$$

where R and L are eigenvalues of the Q_R and Q_L, respectively. This assumption is equivalent to consider the "highest weight states" and other solutions are obtained by global transformations. After choosing the branch of the logarithm for the definition of the quasi-momentum so that $p(x) \sim O(1/x)$ ($x \to \infty$), we obtain the asymptotic behavior of the quasi-momentum,

$$p(x) = -\frac{2\pi R}{\sqrt{\lambda}} \frac{1}{x} + O(x^{-2}) \,, \quad (x \to \infty) \,. \tag{3.3.42}$$

This implies that the asymptotic behavior of the quasi-momentum around $x = \infty$ gives the charge of the $SU(2)_R$ symmetry. On the other hand, the charge of the $SU(2)_L$ symmetry can read from the asymptotic behavior around $x = 0$. Notice that the expansion of the flat connection around $x = 0$ is the form of $J(x) = j - x * j + O(x^2)$ ($x \to 0$). With the definition of the left invariant current $j = -g^{-1}dg$ and the right one $l = -dgg^{-1} = gjg^{-1}$, $J(x)$ can be expressed in terms of the right invariant current as $J(x) = -g^{-1}(d + x * l)g$. This is the form of a "gauge transformation" thus we have

$$g(\tau, \sigma)\Omega(\tau, \sigma; x)g(\tau, \sigma)^{-1} = P \exp\left[-x \int *l + O(x^2)\right]$$
$$= \mathbf{1} + \int_{\sigma}^{\sigma+2\pi} d\tilde{\sigma} l_0(\tau, \tilde{\sigma}) + O(x^2)$$
$$= \mathbf{1} - x\frac{4\pi Q_L}{\sqrt{\lambda}} + O(x^2) \quad (x \to 0) \,. \tag{3.3.43}$$

Here we have used the fact that g is periodic in σ, $g(\tau, \sigma + 2\pi) = g(\tau, \sigma)$. Since we now assume that Q_L is diagonal as (3.3.41), we find the following asymptotic behavior of the quasi-momentum around $x = 0$

$$p(x) = 2\pi m + \frac{2\pi L}{\sqrt{\lambda}} x + O(x^2) \,, \quad (x \to 0) \,, \tag{3.3.44}$$

where $m \in \mathbb{Z}$ does not vanish for a generic solution.

We have seen that the asymptotic behavior of the quasi-momentum is closely related to the charges for the global symmetry. To see how we can obtain the other charges, let us consider the spectral curve which is defined as the characteristic equation for the monodromy matrix

$$\Gamma: \quad \Gamma(x, y) = \det\left(y\mathbf{1} - \Omega(\tau, \sigma; x)\right) = 0 \,. \tag{3.3.45}$$

The spectral curve is independent of the world-sheet coordinate (τ, σ) since the monodromy matrices defined at the different base points of the world-sheet only differ by a similarity transformation. Moreover, from (3.3.39) the definition of the spectral curve is equivalent to $(y - e^{ip(x)})(y - e^{-ip(x)}) = 0$. The information of the charges is encoded in the curve via this relation. Roughly, the moduli of the curve, such as the positions of the branch points and the size of the branch cuts determine the charges. We will show this later. Therefore, the analytic structures of the spectral curve, or equivalently those of the quasi-momentum are important to characterize the solutions. The singularities of the curve are characterized as the zeros of the discriminant,

$$\Delta_\Gamma(x) := (e^{ip(x)} - e^{-ip(x)})^2 \,. \tag{3.3.46}$$

Notice that there are infinitely many singular points $\{x_k\}$, at which $e^{2ip(x_k)} = 1$ by the definition, since the quasi-momentum has poles at $x = \pm 1$ and $e^{2ip(x)}$ can take the value 1 infinitely many times near $x = \pm 1$. [19] Indeed, the monodromy matrix around $x = \pm 1$ behaves as follows

$$\Omega(\tau, \sigma; x) = \exp \int_\sigma^{\sigma+2\pi} d\tilde{\sigma} \left[-\frac{1}{2} \frac{j_\pm}{x \mp 1} + O((x \mp 1)^0) \right], \quad (x \to \pm 1) \,. \tag{3.3.47}$$

With the similarity transformation such that $u_0(\tau, \sigma) j_\pm u_0(\tau, \sigma)^{-1} = i\kappa\sigma_3$ [20], we obtain

$$u(\tau, \sigma; x) \Omega(\tau, \sigma; x) u(\tau, \sigma; x)^{-1} = \exp \left[-\frac{i\pi\kappa}{x \mp 1} + O((x \mp 1)) \right], \quad (x \to \pm 1) \,, \tag{3.3.48}$$

which in turn implies the following asymptotic behavior of the quasi-momentum around $x = \pm 1$

$$p(x) = -\frac{\pi\kappa}{x \mp 1} + O((x \mp 1)^0) \quad (x \to \pm 1) \,. \tag{3.3.49}$$

There is another possible choice for the asymptotic behavior near $x = \pm 1$ whose relative signs in front of the poles at $x = +1$ and $x - 1$ to be opposite. The function which has this asymptotic behavior and the same analytic structures for the other singularities as the quasi-momentum is called the *quasi-energy* and denoted as $q(x)$. By definition, it behaves as

$$q(x) = \mp \frac{\pi\kappa}{x \mp 1} + O((x \mp 1)^0) \,, \quad (x \to \pm 1) \,. \tag{3.3.50}$$

The quasi-energy plays an important role in the analysis of the 1-loop fluctuation energy [99, 100].

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[19] In other words, $e^{ip(x)}$ has the essential singularities at $x = \pm 1$.

[20] $u_0(\tau, \sigma)$ is the leading term of the matrix which diagonalizes the monodromy matrix as in (3.3.39): $u(\tau, \sigma; x) = u_0(\tau, \sigma) + O(x \mp 1)$.

Let us return to the discussion of the singularities of the spectral curve. The singularities defined as the zeros of the discriminant (3.3.46) are divided into two types of the singularities according to the degrees of the zeros, namely, *cusp-like* points and *node-like* points. The discriminant around the singular point x_k is locally of the form

$$\Delta_\Gamma(x) \sim (x - x_k)^n \,. \tag{3.3.51}$$

If the degree n is odd, x_k is called cusp. If the degree n is even, x_k is called node. From (3.3.51), the order of the zero x_k of the discriminant determines the behavior of the eigenvalues $e^{\pm ip(x)}$ near x_k as follows

$$e^{\pm ip(x)} = e^{ip(x_k)} \pm \frac{\sqrt{\Delta_\Gamma}}{2} \,. \tag{3.3.52}$$

This indicates that the cusp-like points lead to the branch cuts since going around the cusp-like points change the sigh of $\sqrt{\Delta_\Gamma}$ and interchange $e^{ip(x)}$ with $e^{-ip(x)}$.

The spectral curve cannot be described by an algebraic curve since it has infinitely many singular points due to the essential singularities of $e^{ip(x)}$ at $x = \pm 1$, thus powerful mathematical results cannot be applied to such a curve directly. However, we can improve the situation by introducing the "log-curve", which is defined as the characteristic equation for the logarithmic derivative of the monodromy matrix

$$\hat{\Sigma} : \quad \hat{\Sigma}(x, y) = \det\left(y\mathbf{1} - L(\tau, \sigma; x)\right) = 0 \,, \tag{3.3.53}$$

$$u(\tau, \sigma; x)L(\tau, \sigma; x)u(\tau, \sigma; x)^{-1} := -i\frac{\partial}{\partial x}\log(u(\tau, \sigma; x)\Omega(\tau, \sigma; x)u(\tau, \sigma; x)^{-1}) \,. \tag{3.3.54}$$

The analytical structures for the log-curve is very similar to those of the spectral curve. Indeed, branch cut structures are identical to each other. On the other hand, the log-curve has the better property with respect to node-like singularities. The eigenvalues of the matrix $L(\tau, \sigma; x)$ are $p'(x), -p'(x)$ and they are meromorphic function in x with double poles at $x = \pm 1$. Therefore, the log-curve is an algebraic curve with only finitely many node-like points which are the even order zeros of $p'(x)$. It is well known that for algebraic curves with a finite number of singular points, all the singular points can be removed after performing a redefinition of the variables. In the present case, the log-curve can be turned into the following hyperelliptic curve [95, 97],

$$\Sigma : \quad y^2 = \prod_{i=1}^{K}(x - w_i)(x - z_i) \,, \tag{3.3.55}$$

where K is the number of branch cuts C_i, connecting the branch points w_i and z_i ($i = 1, \ldots, K$). Since hyperelliptic curves with K branch cuts lead to Riemann surfaces with genus $g = K - 1$, the corresponding solutions are called *finite-gap solutions*.
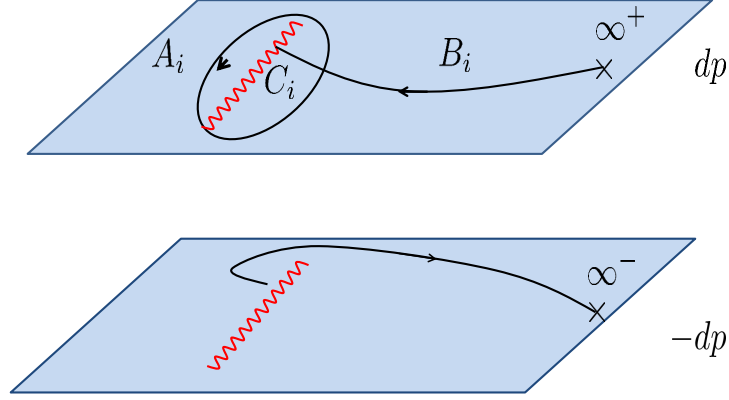
Figure 3.3.1: The definition of the A_i-cycle and the B_i-cycle.

Extracting charges

To clarify the relation between the moduli of the curve and the conserved charges, we should explain how $p(x)$ can be determined as a meromorphic function on Σ. First, notice that the hyperelliptic curve Σ is the double cover of the Riemann sphere $\mathbb{CP}^1$. Hence, there are two points in Σ which correspond to $x \in \mathbb{CP}^1$. We will call the sheet of the double cover corresponding to $p(x)$ as the physical sheet, on which the quasi-momentum is single valued by definition. Further, for any $x \in \mathbb{CP}^1$, we will denote the corresponding two points on the sheets as $x^{\pm}$, where x^+ is on the physical sheet. For each cut C_i, we define A_i cycle which encircles the cut on the physical sheet and B_i cycle joining the point ∞^+ and ∞^- through the cut C_i as shown in the figure 3.3.1.[21] Since $p(x)$ is single valued on the physical sheet by definition, integral of dp over the A_i cycle must vanish. Moreover, the integral of dp over the B_i cycle must take values in $2\pi\mathbb{Z}$ due to the asymptotics of the monodromy matrix $\Omega \sim \mathbf{1}$ as $x \to \infty$. Thus we find

$$\int_{A_i} dp = 0 \,, \quad \int_{B_i} dp = 2\pi n_i \,, \tag{3.3.56}$$

where integer n_i is the so-called mode number associated with the cut C_i. The asymptotic behavior near the poles at $x = \pm 1$ can be read from (3.3.49)

$$dp(x^{\pm}) = \mp d\left(\frac{\pi\kappa}{x-1}\right) + O((x-1)^0) \,, (x \to 1) \,, \tag{3.3.57}$$

$$dp(x^{\pm}) = \mp d\left(\frac{\pi\kappa}{x+1}\right) + O((x+1)^0) \,, (x \to -1) \,. \tag{3.3.58}$$

---

[21]Canonical homology basis $\{a_i\} \cup \{b_i\}$ $(i = 1, \ldots, g)$ which has the intersection matrix $(a_i, a_j) = (b_i, b_j) = 0, (a_i, b_j) = \delta_{ij}$ is expressed in terms of A_i B_i as $a_i = A_i$, $b_i = B_i - B_{g+1}$.

where we have used $dp(x^+) = -dp(x^-)$. The A-period conditions (3.3.56) together with the asymptotics (3.3.58) determines the meromorphic differential dp in terms of the moduli of the hyperelliptic curve Σ such as w_i, z_i. The information of the charges can be extracted from the following periods of the meromorphic differential

$$S_i = \frac{1}{2\pi i} \int_{a_i} \alpha \,, \quad \alpha = \frac{\sqrt{\lambda}}{4\pi} \left(x + \frac{1}{x} \right) dp \,, \tag{3.3.59}$$

where $z := x + 1/x$ is the Zhukovsky variable. S_i's are desired conserved charges which are so-called *filling fractions*. We will later see that filling fractions are nothing but the action variables for the motion of string and there exist angle variables ϕ_i that are conjugate to the action variables satisfying $\{S_i, \phi_j\} = \delta_{ij}$. Further, the residue of the meromorphic differential at $x = 0, \infty$ are related to the global charges as follows

$$\text{Res}_{0^+} \alpha = \frac{L}{2} \,, \quad \text{Res}_{\infty^+} \alpha = \frac{R}{2} \,. \tag{3.3.60}$$

By evaluating the integral of α along the contour enclosing all the cuts C_i in two ways, we obtain the following constraint for the filling fractions

$$\sum_{i=1}^{K} S_i = \text{Res}_{0^+} \alpha + \text{Res}_{\infty^+} \alpha = \frac{1}{2}(L - R) \,. \tag{3.3.61}$$

Another constraint comes from the condition that total world-sheet momentum should be equal to zero up to $2\pi\mathbb{Z}$. As shown in [95], this is equivalent to

$$\sum_{i=1}^{K} S_i n_i = 2\pi m \,. \tag{3.3.62}$$

Therefore, we can completely determine the relation between the moduli of the hyperelliptic curve (w_i, z_i) and free parameters $(S_i, n_i; R, L)$ via B-cycle conditions (3.3.56) and the definition of the filling fraction.

The infinite number of singular points $\{x_k\}$ for the spectral curve Γ are specified as the following condition

$$p(x_k) = n_k \pi \,. \tag{3.3.63}$$

One finds that singular points can be seen as degenerated branch cuts, namely the two branch points being at the same point on the sheet by using B-cycle condition $2\pi n_i = \int_{B_i} dp$. Indeed, meromorphic differential dp defines the following integral

$$p(P) = \int_{\infty^+}^{P} dp \,, \quad P \in \Sigma \,. \tag{3.3.64}$$
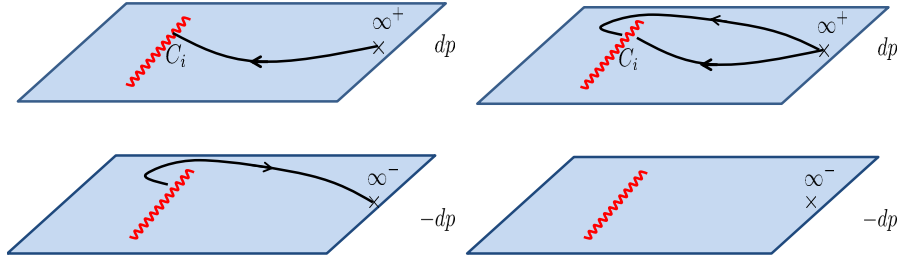
Figure 3.3.2: The contour integral on the second sheet, which is a part of the B_i-cycle (left) can be lifted to the physical sheet with simultaneous flipping the orientation (right).

If we restrict the integration contour to the physical sheet, this is a single valued function due to the vanishing A-periods. Then, for any point $Q^+ \in \Sigma$ on the physical sheet near the cut C_k, the B-periods condition can be rewritten as

$$
\begin{aligned}
2\pi n_k &= \int_{\infty^+}^{Q^+} dp + \int_{Q^+}^{\infty^-} dp \\
&= \int_{\infty^+}^{Q^+} dp - \int_{Q^-}^{\infty^+} dp = p(Q^+) + p(Q^-) \,.
\end{aligned}
\tag{3.3.65}
$$

Here, at the second line, we lifted the integration contour of the second term to the physical sheet (figure 3.3.2). Therefore, if the cut C_k shrinks to a single point x_k, then it becomes a point that satisfies $x_k^+ = x_k^-$ so that $p(x_k^+) + p(x_k^-) = 2\pi n_k$, i.e., $p(x_k) = n_k\pi$.

The spectral curve descriptions for classical solutions can be understood as the generalization of the Fourier mode decomposition of the string, in the sense that for any integer $n_i \in \mathbb{Z}$, there exists a cut C_i with the B-period $\int_{B_i} dp = 2\pi n_i$ or a singular point x_i, at which $p(x_i) = n_i\pi$. The number of cuts corresponds to the number of the excitation modes. Since the singular points are infinitesimally shrunk cuts, turning on the mode with mode number m corresponds to opening up the singular point at which $p(x) = m\pi$ into a finite cut, which in turn introduces a new filling fraction. Therefore, the filling fractions for each cut is the amplitude of the oscillation and the finite-gap solutions are solutions with a finite number of excited modes. Actually, we must consider an infinite-gap solution for a general classical solution. We will discuss this issue at the end of this subsection.

To clarify how the filling fractions are related to classical solutions, we now treat the classical solutions of the string moving on the flat Minkowski background from this point of view. In the conformal gauge, the equations of motion for the embedding coordinates are $\partial_+\partial_- X^\mu(\tau, \sigma)$ ($\mu = 0, \ldots, D$), which are exactly the wave equations with no mass.

Hence, the general solution with the periodic boundary condition can be written as the following Fourier series for the left mover modes and the right mover modes,

$$X^\mu(\tau,\sigma) = x^\mu + p^\mu \tau + i \sum_{n \neq 0} \frac{1}{n} \alpha_n^\mu e^{-in(\tau-\sigma)} + i \sum_{n \neq 0} \frac{1}{n} \tilde{\alpha}_n^\mu e^{-in(\tau+\sigma)} . \quad (3.3.66)$$

Here (x^μ, p^μ) correspond to the motion of the center of mass and the coefficients α_n^μ and $\tilde{\alpha}_n^\mu$ represent the oscillation coordinates for the modes of the left and right mover, respectively. As we will see soon later, the finite-gap solutions in the flat space correspond to the solutions with a finite number of oscillators turned on. Let us consider the time-slice at $\tau = 0$ to follow the dynamics with respect to τ. The configuration for the coordinates $X^\mu(\sigma)$ and their canonical conjugate variables $P^\mu(\sigma) = \dot{X}^\mu(0,\sigma)$ specifies a point of the phase space of the string. The Poisson brackets relations for these variables are

$$\{X^\mu(\sigma), P^\nu(\sigma')\} = \eta^{\mu\nu} \delta(\sigma - \sigma') , \quad (3.3.67)$$

$$\{X^\mu(\sigma), X^\nu(\sigma')\} = \{P^\mu(\sigma), P^\nu(\sigma')\} = 0 . \quad (3.3.68)$$

We change the variables from the coordinates to the oscillators by the Fourier decomposition as usual:

$$X^\mu(\sigma) = x^\mu + i \sum_{n \neq 0} \frac{1}{n} \alpha_n^\mu e^{in\sigma} + i \sum_{n \neq 0} \frac{1}{n} \tilde{\alpha}_n^\mu e^{-in\sigma} , \quad (3.3.69)$$

$$P^\mu(\sigma) = p^\mu + i \sum_{n \neq 0} \alpha_n^\mu e^{in\sigma} + i \sum_{n \neq 0} \tilde{\alpha}_n^\mu e^{-in\sigma} . \quad (3.3.70)$$

From the Poisson brackets for $X^\mu(\sigma)$ and $P^\mu(\sigma)$, one can easily find that oscillator coordinates satisfy the following Poisson brackets:

$$\{\alpha_n^\mu, \alpha_m^\mu\} = \{\tilde{\alpha}_n^\mu, \tilde{\alpha}_m^\mu\} = in\delta_{n+m}\eta^{\mu\nu} , \quad (3.3.71)$$

$$\{\alpha_n^\mu, \tilde{\alpha}_n^\mu\} = 0 , \quad \{x^\mu, p^\nu\} = \eta^{\mu\nu} . \quad (3.3.72)$$

Since the equations of motion in the flat space are linear wave equations, the oscillation modes are manifestly decoupled. Even after the light cone gauge fixing and leaving the only physical degrees of freedom, the remaining transverse coordinates $(x^i, p^i, \alpha_n^i, \tilde{\alpha}_n^i)$ $(i = 1, \ldots, D-1)$ still satisfy the naive Poisson bracket relations[22],

$$\{\alpha_n^i, \alpha_m^j\} = \{\tilde{\alpha}_n^i, \tilde{\alpha}_m^j\} = in\delta_{n+m}\eta^{ij} , \quad (3.3.73)$$

$$\{\alpha_n^i, \tilde{\alpha}_n^j\} = 0 , \quad \{x^i, p^j\} = \delta^{ij} . \quad (3.3.74)$$

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[22]Of course, we must follow the Dirac's prescription for the constrained system, however, the Dirac bracket for the transverse modes becomes the usual Poisson bracket in this case.

Moreover, we can construct the action-angle variables explicitly from the oscillators. The action-angle variable representation is one of the most apparent manifestation of the integrability and it gives us some intuitions for the finite-gap solutions in the non-linear case. The action-angle variables for each transverse mode are defined as

$$\alpha_n^k = \sqrt{nS_n^k}\,e^{i\theta_n^k}\,, \quad \tilde{\alpha}_n^k = \sqrt{n\tilde{S}_n^k}\,e^{i\tilde{\theta}_n^k}\,. \tag{3.3.75}$$

The variables S_n^j and $\tilde{S}_n^j$ are classical analogues of the occupation numbers for each transverse oscillator with the mode number n. These variables are time independent as expected since the occupation numbers are adiabatic invariant. Therefore the action variables are conserved charges. Furthermore, the Poisson brackets for these variables are

$$\{S_n^i, S_m^j\} = \{\tilde{S}_n^i, \tilde{S}_m^j\} = 0\,, \tag{3.3.76}$$

$$\{\theta_n^i, \theta_m^j\} = \{\tilde{\theta}_n^i, \tilde{\theta}_n^j\} = 0\,, \tag{3.3.77}$$

$$\{S_n^i, \theta_m^j\} = \{\tilde{S}_n^i, \tilde{\theta}_m^j\} = \delta_{nm}\delta^{ij}\,. \tag{3.3.78}$$

The third relation implies that the variables θ_n^i are canonical conjugate to the action variables, which are nothing but the angle variables. It turns out that the angle variables evolve linearly in the time τ, while the action variables remain constant

$$\theta_n^\mu(\tau) = \theta_n^\mu(0) - n\tau\,, \quad \tilde{\theta}_n^\mu(\tau) = \tilde{\theta}_n^\mu(0) - n\tau\,. \tag{3.3.79}$$

This immediately reproduces the general solution (3.3.66) for the flat space string.

Reconstruction of solution

We have seen so far how to extract the charges from a given solution. The filling fractions are obtained from the integral of the meromorphic differential dp over the cycles of the spectral curve which is defined by the characteristic equation for the monodromy matrix or its logarithmic derivative. To summarize, we have the following sequence:

$$(\text{Solution}) \underset{(3.3.34)}{\to} J(\tau,\sigma;x) \underset{(3.3.37)}{\to} \Omega(\tau,\sigma;x) \underset{(3.3.53),(3.3.54)}{\to} (\Sigma;dp) \underset{(3.3.59)}{\to} S_i \tag{3.3.80}$$

Conversely, we can reconstruct the solution from the curve and certain analytic properties since sufficiently many charges characterize the solution uniquely. The first key ingredient is the auxiliary linear problem:

$$(d - J(\tau,\sigma;x))\psi(\tau,\sigma;x) = 0\,. \tag{3.3.81}$$

The flatness condition $dJ(\tau,\sigma;x) = J(\tau,\sigma;x)\wedge J(\tau,\sigma;x)$ is equivalent to the compatibility condition for this linear problem. Given a solution for the auxiliary linear problem, the

flat connection can be recovered as follows:

$$J(\tau, \sigma; x) = d\Psi(\tau, \sigma; x)\Psi(\tau, \sigma; x)^{-1}, \qquad (3.3.82)$$

where $\Psi(\tau, \sigma; x)$ is a matrix of two independent eigenvectors as $\Psi(\tau, \sigma; x) = (\psi(x^+), \psi(x^-))$. Furthermore, with the definition for the flat connection $J(\tau, \sigma; x)$ (3.3.34) and the left invariant current $j = -g^{-1}dg$, we can recover the original string variables as[23]

$$g = \sqrt{\det \Psi(0)}\Psi(0)^{-1}. \qquad (3.3.83)$$

where the determinant is the normalization factor which ensures $\det g = 1$. The second key observation is that dynamics of the solution is encoded in the eigenvectors of the monodromy matrix via relation (3.3.39). The eigenvector for the monodromy matrix is called *Baker-Akhiezer vector*. In fact, due to the commutativity $[d - J(\tau, \sigma; x), \Omega(\tau, \sigma; x)] = 0$, these operators are simultaneously diagonalized and the solution of the auxiliary linear problem $\psi(x)$ is proportion to the normalized eigenvector for the monodromy matrix:

$$\psi(\tau, \sigma; x) = \varphi(\tau, \sigma; x)h(\tau, \sigma; x) \qquad (3.3.84)$$

$$\Omega(\tau, \sigma; x)h(\tau, \sigma; x) = e^{ip(x)}h(\tau, \sigma; x), \quad \vec{\alpha} \cdot h = 1, \quad \vec{\alpha} \in \mathbb{C}^2 \qquad (3.3.85)$$

The normalization vector $\vec{\alpha}$ is typically taken to be $\vec{\alpha} = (1, 0)^T$. The third key ingredient are the poles of the normalized eigenvector for the monodromy matrix. It is known that the components of the normalized Baker-Akhiezer vector has $g+1$ poles as a function of x for the case of a genus g curve. As discussed in [96,97], we can extract the dynamical variables which is so-called *Sklyanin's separation of variables* as the poles of Baker-Akhiezer vector $\{\gamma_i\}$ $(i = 1, \ldots, g + 1)$. Indeed, it turns out that variables $\left(z(\gamma_i), \frac{i\sqrt{\lambda}}{4\pi}p(\gamma_i)\right)$, where $z(\gamma) = x_\gamma + 1/x_\gamma$ is the Zhukovsky variable, have the following canonical Poisson bracket relations

$$\{z(\gamma_i), z(\gamma_j)\} = \{p(\gamma_i), p(\gamma_j)\} = 0, \qquad (3.3.86)$$

$$\left\{z(\gamma_i), \frac{i\sqrt{\lambda}}{4\pi}p(\gamma_i)\right\} = \delta_{ij}. \qquad (3.3.87)$$

From this Poisson bracket relations, the filling fractions $S_i = \frac{\sqrt{\lambda}}{4\pi i}\int_{a_i} p\, dz$ are nothing but the action variables. With the extended Abel map defined in [96], the angle variables are obtained by

$$(\theta_i, \theta_\infty) = \left(2\pi \sum_{k=1}^{g+1} \int^{z(\gamma_k)} \omega_i, -2\pi \sum_{k=1}^{g+1} \int^{z(\gamma_k)} \omega_\infty\right) \qquad (3.3.88)$$

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[23]There is arbitrariness multiplying diagonal matrices from the left and right side. We must further impose reality condition and periodicity condition.

where ω_i is the normalized holomorphic 1-form with $\int_{a_i} \omega_j = \delta_{ij}$ and ω_∞ is the normalized Abelian differential of the third kind with residues $\pm \frac{1}{2\pi i}$ at the simple poles $\infty^\pm$. To see that they are actually the angle variables, consider the generating function of the canonical transformation $\left(z(\gamma), \frac{i\sqrt{\lambda}}{4\pi} p(\gamma_i) \right) \to (S_i, \theta_i)$, which is defined as follows

$$W(S_i; z(\gamma_i)) = \frac{\sqrt{\lambda}}{4\pi i} \sum_{k=1}^{g+1} \int^{z(\gamma_k)} p(x) dz . \tag{3.3.89}$$

Notice that the dependence of the filling fractions is encoded implicitly in the functional form of $p(x)$ via the definition of the filling fractions. The angle variable that is conjugate to S_i is obtained by the derivative of the generating function as follows

$$\phi_i = \frac{\partial W}{\partial S_i} . \tag{3.3.90}$$

Notice that we have $\delta_{ij} = \frac{\sqrt{\lambda}}{4\pi i} \int_{a_i} \frac{\partial p(x)}{\partial S_j} dz$ from the definition of the filling fractions $S_i = \frac{\sqrt{\lambda}}{4\pi i} \int_{a_i} p \, dz$. In other words, infinitesimal change of the filling fraction add the one-form whose non-vanishing period integral is only the period for a_i-cycle to $p \, dz$. Such a one-form is in proportion to the normalized holomorphic one-form ω_i, thus we have (3.3.88). The same argument holds for the remaining angle variable θ_∞. The construction of the action-angle variables is the most fundamental manifestation of the integrability since the evolutions for the angle variables are linear:

$$\theta_i = k_i \sigma + w_i \tau . \tag{3.3.91}$$

The reconstruction procedure is as follows. First, the Riemann-Roch theorem uniquely fixes the normalized Baker-Akhiezer vector $h(\tau, \sigma; P)$ as a meromorphic function on the genus g curve with the specified divisor of the poles $\{\gamma_i(\tau, \sigma)\}$. The components are expressed in terms of the Riemann θ-functions [95] and the arguments of the θ-function are the angle variables. This is the non-linear analogue of the Fourier decomposition for the string on the curved background. The remaining scalar function $\varphi(\tau, \sigma; P)$ can be specified from the analytical properties. Note that the solution of the auxiliary linear problem $\psi(\tau, \sigma; x)$ has the essential singularities at $x = \pm 1$ which come from the poles of the flat connection $J(\tau, \sigma; P)$. Together with the initial divisor of the poles for the normalized Baker-Akhiezer vector $\gamma(0, 0)$, the asymptotic behaviors near the essential singularities uniquely determine the $\psi(\tau, \sigma; P)$,[24] which in turn determine the solution via (3.3.83).

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[24]If there are two functions with such properties, their quotient is meromorphic on the curve and the Riemann-Roch theorem tell us that the quotient is a function that is independent of $P \in \Sigma$.

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We conclude this subsection with the discussion for the solutions that cannot be described in terms of the finite-gap solutions. First, notice that singular points can be viewed as the shrunk cuts as explained before. Then, one can construct the filling fractions for these infinitesimal "cuts" as in (3.3.59). However, the filling fraction for the degenerated cut or node-like point x_k is zero

$$S_{x_k} := \frac{1}{2\pi i} \oint_{x_k} \alpha = \frac{\sqrt{\lambda}}{8\pi^2 i} \oint_{x_k} \left(x + \frac{1}{x} \right) dp = 0 \,, \qquad (3.3.92)$$

where the contour only encircles the singular point x_k. S_{x_k} vanishes since the differential α has no residue at x_k. The filling fractions or action variables are the classical analogue of the occupation numbers, hence we conclude that the finite-gap solutions are the solutions with a finite number of excited modes. In this picture, singular points correspond to unexcited modes of the string. Thus, some of the important information encoded in the spectral curve seem to be overlooked by considering the desingularized log-curve Σ instead of the original curve since all the singular points that appear in the spectral curve are resolved in the desingularized curve by construction. Actually, the number of the dynamical variables such as the SoV defined in (3.3.86) or the angle variables (3.3.88) is $g + 1$ for the genus g curve. On the other hand, there should be infinitely many variables representing the motion of string since the phase space of the string is infinite dimensional. In fact, the remaining infinitely many variables are fixed to the singular points and non-dynamical. One can confirm this from a certain degeneration limit of cuts as explained in [95]. Therefore, it is important to bare in mind that we must use the spectral curve with infinitely many singular points when we consider the dynamics which involves all the variables of the string. For example, the reconstruction formula of the string indicates that all the finite-gap solutions correspond to the 2-leg string solutions. However, the 3-leg solutions are relevant to the calculation of the semicclassical three-point functions and we must necessarily consider the "infinite-gap solutions" [110]. Following the Sklyanin's magic recipe [93], the dynamical poles of the normalized Baker-Akhiezer vector along with all the singular points of the spectral curve Γ can be characterized as the zeros of the off diagonal component of the monodromy matrix. This is a convenient characterization since not only the dynamical variables but also infinitely many singular points, at which non-dynamical variables are trapped, can be treated simultaneously.

3.4 Comparison for spectrum

In this subsection, we will compare the spectrum of the gauge invariant composite operators of $\mathcal{N} = 4$ SYM with the spectrum of string on $AdS_5 \times S^5$. According to the

dictionary for the AdS/CFT, the dimensions of the gauge invariant operators are mapped to the energies of the corresponding closed string states propagating on $AdS_5 \times S^5$.

However, the direct comparison for arbitrary operators or states is quite difficult due to the following reasons. First, the exact quantization for string on curved background has not been successful. Second, due to the strong-weak nature of the duality, the perturbative calculations of gauge theory are reliable for small 'tHooft coupling λ, while the calculations of the string theory are valid in the strong coupling regime.

To avoid these difficulties, two classes of operators have been the main subject for the test of the duality. The first class is the chiral primary BPS operators or supergravity massless modes since their dimensions or energies are not renormalized due to the supersymmetry. The second set of spectrum consist of operators with large quantum numbers or semiclassical states. Indeed, by considering semiclassical states, which are given by classical strings using the WKB approximation, it is possible to directly compare with the perturbative gauge theory since the effective expansion parameter becomes λ/L^2 where L is the angular momentum around the equator of S^5 and later identified with the length of the spin chain, and one can take the strong coupling limit keeping λ/L^2 small. We will discuss the semiclassical spectrum in this subsection especially focusing on the structural similarities that appear in this limit.

3.4.1 Semiclassical limit

The first non-trivial test of the spectrum for non-BPS spectrum or stringy modes was done for the so-called BMN operators. The BMN operators are defined as the operators with a small number of "impurities" over the chiral primary operators which are composed of a large number of complex scalar fields such as $\text{Tr} Z^L$ ($L \gg 1$). In the language of spin chain, these operators correspond to the states with a small number of excitations or magnons over the ferromagnetic vacuum of the large length spin chain.

However, the dual string states of the BMN operators are almost point like strings since the dual string states for the chiral primary $\text{Tr} Z^L$ is the point like string states orbiting a geodesic of S^5 with angular momentum L, and the BMN operators have few excitations over such a vacuum. Therefore, to obtain the spectrum of operators that correspond to stretched stringy states, we must consider the long operators with a large number of excitations or impurities. The computation of the anomalous dimensions for such long operators becomes more and more difficult as the size of the operator grows since the degeneracy of the classical dimensions grows exponentially and the mixing of operators becomes complicated. Fortunately, the 1-loop dilatation operator for single trace operators can be mapped to the integrable Hamiltonian of the spin chain and it

can be diagonalized by the Bethe ansatz as discussed in section 3.2. We will consider the spectrum of operators that correspond to semiclassical strings, which are long wave length excitations of spin waves in the spin chain description. More precisely, we will consider the infinite length limit $L \to \infty$ with keeping the ratio to the number of the excitations M/L finite. In this limit, the distribution of the Bethe roots leads to an algebraic curve description which also appear in string theory as the spectral curve, hence we can compare the spectrum of the both theories directly.

For simplicity, we concentrate on the $SU(2)$ subsector where all single trace operators are made up of two kinds of complex scalar fields. The anomalous dimensions of such operators are obtained by diagonalizing the Heisenberg Hamiltonian and the problem is reduced to solve the algebraic equations for the rapidities of the magnons

$$\left(\frac{u_k + i/2}{u_k - i/2} \right)^L = \prod_{l \neq k}^M \frac{u_k - u_l + i}{u_k - u_l - i}, \tag{3.4.1}$$

$$e^{iP} = \prod_{k=1}^M \frac{u_k + i/2}{u_k - i/2} = 1, \tag{3.4.2}$$

where M is the number of the magnons. The first equation is the Bethe ansatz equation that arises as the periodicity of the spin chain wave function and the second equation is the trace condition that comes from the cyclicity of the single trace operators. With the Bethe roots, the anomalous dimension is given by $\gamma = \frac{\lambda}{8\pi^2} \sum_{k=1}^M \frac{1}{u_k^2 + 1/4}$. To consider the limit $L \to \infty$, let us take the logarithm of the Bethe ansatz equation

$$L \log \left(\frac{u_k + i/2}{u_k - i/2} \right) = \sum_{l \neq k}^M \log \frac{u_k - u_l + i}{u_k - u_l - i} - 2\pi i n_k, \tag{3.4.3}$$

where n_i is the mode numbers which specify the branch of the logarithm. We are now interested in the long wave length limit, each Bethe root u_i should scale as $u_i \sim L$ since the momentum of each magnon is $p = i \log \frac{u + i/2}{u - i/2} \sim 1/L$. After rescaling the Bethe roots as $u_i = L x_i$, the Bethe ansatz equation becomes

$$\frac{1}{x_k} = \frac{2}{L} \sum_{l \neq k}^M \frac{1}{x_k - x_l} - 2\pi n_k + O\left(\frac{1}{L} \right). \tag{3.4.4}$$

We assume that the number of the different mode number n_k is finite. Then, the roots with mode number n_k roughly locate around the point $x = 1/2\pi n_k$, with a typical distance between the adjacent roots being order of $\Delta x \sim 1/L$. Thus, the distribution of the roots with the same mode number form a continuous curve in the complex plane in the scaling
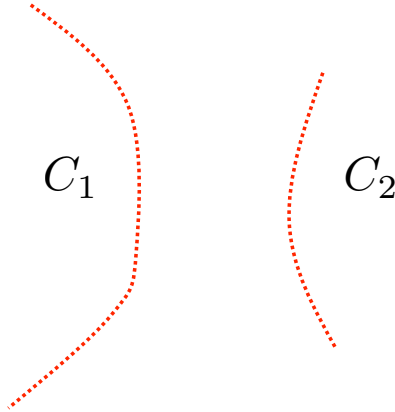
Figure 3.4.1: In the scaling limit $L, M \to \infty$, keeping M/L finite, typical solutions of the Bethe ansatz equations condense in the complex plane and form cuts. The figure shows two cuts case.

limit $L \to \infty$.[25] We denote the condensation of the Bethe roots as C_i accompanying the mode number n_i. Therefore, we are led to introduce the density of the Bethe roots and the resolvent that characterize the distribution for the Bethe roots as follows

$$\rho(x) = \frac{1}{L} \sum_{k=1}^{M} \delta(x - x_k) \,, \tag{3.4.5}$$

$$G(x) = \frac{1}{L} \sum_{k=1}^{M} \frac{1}{x - x_i} = \int_C d\xi \frac{\rho(\xi)}{x - \xi} \,, \tag{3.4.6}$$

where $C = C_1 \cup \ldots \cup C_K$ is the union of the cuts. As seen from the definition of the resolvent, it is an analytic function on the x plane with the cuts C_i on the plane.

The momentum and the energy can be expressed in terms of the resolvent in the scaling limit

$$P = \frac{1}{i} \sum_{k=1}^{M} \log \frac{u_k + i/2}{u_k - i/2} \sim \frac{1}{L} \sum_{k=1}^{M} \frac{1}{x_k} = -G(0) \,, \tag{3.4.7}$$

$$\gamma = \frac{\lambda}{8\pi^2} \sum_{k=1}^{M} \frac{1}{u_k^2 + 1/4} \sim \frac{\lambda}{8\pi^2 L} \frac{1}{L} \sum_{k=1}^{M} \frac{1}{x_k^2} = -\frac{\lambda}{8\pi^2 L} G'(0) \,. \tag{3.4.8}$$

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[25] Actually, it is known that there exist so called string solutions whose separation between the adjacent roots does not scale as $1/L$ and is finite.

Moreover, it turns out that the resolvent is a generating function of the conserved charges of the Heisenberg model since the eigenvalue of the transfer matrix (3.2.76) can be rewritten as

$$\frac{t_{\bm{u}}(Lx + i/2)}{(Lx + i/2)^L} \sim \exp\left(-\frac{i}{L}\sum_{k=1}^{M}\frac{1}{x - x_k}\right). \tag{3.4.9}$$

Hence, if we can determine the resolvent as an analytic function on the complex plane, we can extract the information of all the conserved charges.

Indeed, it is possible to determine the resolvent as a function on a Riemann surface by solving a Riemann-Hilbert problem. To see this, note that the Bethe ansatz equation (3.4.4) is expressed as an integral equation for the density of the Bethe roots,

$$\frac{1}{x} + 2\pi n_i = 2\mathcal{P}\int_C d\xi \frac{\rho(\xi)}{x - \xi} = G(x + i0) + G(x - i0)\,, \quad x \in C_i\,, \tag{3.4.10}$$

where $\mathcal{P}$ denotes the Cauchy's principal value integral. The last equality follows from the following identity

$$G(x + i0) + G(x - i0) = \int_C d\xi \left(\frac{\rho(\xi)}{x + i0 - \xi} + \frac{\rho(\xi)}{x - i0 - \xi}\right) = 2\mathcal{P}\int_C d\xi \frac{\rho(\xi)}{x - \xi}\,. \tag{3.4.11}$$

where we have used $\frac{1}{x \pm i0} = \mathcal{P}\frac{1}{x} \mp \pi\delta(x)$.

To consider the general solution of (3.4.10), we define the quasi-momentum as follows:

$$p(x) = G(x) - \frac{1}{2x}\,. \tag{3.4.12}$$

Then, the equation (3.4.10) can be rewritten in terms of the quasi-momentum

$$p(x + i0) + p(x - i0) = 2\pi n_i\,, \quad x \in C_i\,. \tag{3.4.13}$$

This is exactly the form of a Riemann-Hilbert problem determining the analytic function $p(x)$ from the discontinuity on each cut C_i. This equation appears in string theory side as (3.3.65), however, the singularity structure is slightly different since the (3.4.12) has the only pole at $x = 0$ while the quasi-momentum of string has poles at $x = \pm 1$ and regular at $x = 0$. We will see how the quasi-momentum are determined from these analytic properties. The quasi-momentum is a double valued function on the complex plane, however, it must be a single valued function on the physical sheet of the Riemann surface, which is a complex plane with cuts C_i. In other words, $\int^x dp$ for the meromorphic differential dp is single valued on the physical sheet due to the vanishing A-cycles for the hyperelliptic curve Σ [26]

$$\Sigma: \quad y^2 = \prod_{i=1}^{K}(x - w_i)(x - z_i) = x^{2K} + r_1 x^{2K-1} + \cdots + r_{2K}\,, \tag{3.4.14}$$

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[26]The definition of the cycles are the same in figure 3.3.1.

where (w_i, z_i) denotes the branch points of the cut C_i. Moreover, they should be complex conjugate to each other since the Bethe ansatz equation is invariant under the complex conjugation and the Bethe roots are invariant as well.

Since $p(x)$ has the discontinuities on the cuts C_i, it takes the form of $p(x) = g(x)y + n_i \pi$ for some meromorphic function $g(x)$. Hence, the meromorphic differential dp can be expressed in terms of y and a meromorphic function as follows

$$dp = g'ydx + g\frac{\sum_{i=1}^{2K}\prod_{j\neq i}^{2K}(x - x_i)}{2y}dx = \frac{h(x)}{y}dx , \qquad (3.4.15)$$

where $\{x_i\}$ is the set of branch points. The asymptotic behavior near the $x = 0$ and $x = \infty$ restricts the meromorphic function h. From the definition of the quasi-momentum (3.4.12),

$$dp \sim \frac{1}{2x^2} + O(1) \quad (x \to 0). \qquad (3.4.16)$$

On the other hand, the asymptotic behavior around $x = \infty$ is specified from that of the resolvent $G(x)$

$$G(x) \sim \frac{\alpha}{x} + O(x^{-2}) \quad (x \to \infty), \quad \alpha = \int_C d\xi \rho(\xi) = \frac{M}{L} , \qquad (3.4.17)$$

where α is a filling fraction. With the definition for $p(x)$ (3.4.12), the asymptotics near the infinity is

$$dp \sim \left(\frac{1}{2} - \alpha\right)\frac{dx}{x^2} + O(x^{-3}) \quad (x \to \infty). \qquad (3.4.18)$$

Together with the (3.4.16), the meromorphic differential should be the form of

$$dp = \frac{1}{y}\sum_{i=-1}^{K-1} a_k x^{K-1} , \quad a_{-1} = \frac{\sqrt{r_{2K}}}{2} , \quad a_0 = \frac{r_{2K-1}}{4\sqrt{r_{2K}}} . \qquad (3.4.19)$$

a_{-1} and a_0 are determined from the asymptotics (3.4.16). Furthermore, the meromorphic differential dp is specified by the period integrals for the cycles

$$\int_{A_i} dp = 0 , \qquad (3.4.20)$$

$$\int_{B_i} dp = 2\pi n_i , \qquad (3.4.21)$$

where the first conditions follows from the single valuedness of the quasi-momentum on the physical sheet and the B-cycle conditions are derived from (3.4.10) by following the argument that we did to derive (3.3.65) conversely. The remaining $K - 1$ coefficients

a_i $(i = 1, \ldots, K - 1)$ can be determined from the A-cycle conditions and the B-cycle conditions and the moduli of the curve are related to the mode numbers for the cuts. We have still free parameters that are characterized by the filling fractions for the cuts

$$S_i = \frac{1}{2\pi i} \int_{A_i} p(x)dx = \int_{C_i} \rho(x)dx \quad (i = 1, \ldots, K - 1) \,, \tag{3.4.22}$$

$$S_K = \frac{1}{2\pi i} \int_{A_i} p(x)dx = \int_{C_i} \rho(x)dx = \alpha - \sum_{i=1}^{K-1} S_i \,. \tag{3.4.23}$$

The filling fraction S_i is the normalized occupation number for the Bethe roots with mode number n_i, thus the sum of the filling fractions must be the total number of excitations divided by the length of the spin chain $\alpha = M/L$. The total momentum P should be quantized as $2\pi\mathbb{Z}$ due to the cyclicity condition (3.4.2), hence, we have

$$P = -G(0) = \sum_{i=1}^{K} \int_{C_i} \frac{\rho(x)}{x} dx = 2\pi \sum_{i=1}^{K} n_i \int_{C_i} \rho(x)dx$$

$$= 2\pi \sum_{i=1}^{K} n_i S_i = 2\pi m \,, \quad m \in \mathbb{Z} \,. \tag{3.4.24}$$

Therefore, we can determine the quasi-momentum in terms of the free parameters $(S_i, n_i; \alpha, m)$ as we did in the case of string. From (3.4.7), (3.4.8) and the definition of $p(x)$, we find

$$p(x) = -\frac{1}{2x} - 2\pi m - \frac{8\pi^2 \gamma L}{\lambda} x + O(x^2) \,. \tag{3.4.25}$$

Therefore, we can read the anomalous dimension from the coefficient of the quasi-momentum. In [79], anomalous dimensions are compared with the energies of the classical string for various solutions and match at 2-loop level. Let us consider two simple examples.

BMN states

The BMN operators holographycally describe small fluctuations around the point-like string orbiting an equator of S^5 with a large angular momentum [18]. These operators are mapped to low energy excited states with a small number of magnons in the spin chain with large length L. Further, the momenta of magnons in such excited states are assumed to be scale as $p \sim 1/L$ (or equivalently $u_i \sim L$). Since the number of magnon excitations M is small, the first term in the r.h.s. of (3.4.4) is negligible[27] and we have

$$x_k \simeq \frac{1}{2\pi n_k} \Leftrightarrow u_k \simeq \frac{L}{2\pi n_k} \,. \tag{3.4.26}$$

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[27]A typical distance between the Bethe roots is assumed to be well separated $O(x_k - x_l) \sim 1$.

From a different point of view, the S-matrix of magnons becomes trivial in the scaling limit due to the large size of the system and the Bethe ansatz equation is reduced to the usual quantization condition for the momenta of magnons. Notice that the occupation number or equivalently the filling fraction S_n for the mode number n is always 1 or 0 since we are assuming that the Bethe roots are well separated and there is no degeneracy. With these rapidities, the anomalous dimension for a BMN operator is given by

$$\Delta - L = \gamma = \frac{\lambda}{8\pi^2} \sum_{k=1}^{M} \frac{1}{u_k^2 + 1/4} \simeq \frac{\lambda}{2L^2} \sum_k^{M} n_k^2 \qquad (3.4.27)$$

Here, mode numbers should satisfy $\sum_k^M n_k = 0$ due to the trace condition (3.4.2) or the level matching condition. On the other hand, the BMN frequency ω_n can be expanded in $\lambda' = \lambda/J^2$ as $\omega_n = 1 + \frac{\lambda'^2 n^2}{2}$ hence, we obtain the light-cone energy as follows

$$E - J = \sum_{k=1}^{M} N_{n_k} \omega_{n_k} \simeq \sum_{k=1}^{M} \left(N_{n_k} + \frac{\lambda'^2 n_k^2}{2} + \cdots \right), \qquad (3.4.28)$$

where N_{n_k} is the occupation number for the mode number n_k and they satisfy the level matching condition $P = \sum_n n N_n = 0$. Therefore, under the identification $J = L - M \simeq L$, $N_{n_k} = S_{n_k} = 1$, we see the precise matching.

1-cut solution

We next consider the so-called 1-cut solutions, whose quasi-momentum only has a single cut with mode number n. The momentum constraint (3.4.24) implies $\alpha = M/L$ should satisfy $n\alpha = m$ for an integer m. In this case, one can easily find the solution of the equation (3.4.10) as follows

$$G(x) = \frac{1}{2} \left(\frac{1}{x} - 2\pi n + \frac{1}{x}\sqrt{(2\pi nx - 1)^2 + 8\pi mx} \right). \qquad (3.4.29)$$

The coefficient inside the root are chosen so that they eliminate the pole at $x = 0$ and reproduce the asymptotic behavior $G(x) \sim \alpha/x + O(x)$ $(x \to \infty)$. From this resolvent and (3.4.8), the anomalous dimension is given by

$$\gamma = -\frac{\lambda}{8\pi^2 L} \frac{dG}{dx}\bigg|_{x=0} = \frac{\lambda m(n-m)}{2L}. \qquad (3.4.30)$$

On the other hand, the dual classical string solutions for such 1-cut solutions are known to be circular strings:

$$X_1 + iX_2 = \sqrt{\frac{\mathcal{J}_1}{w_1}} e^{iw_1\tau + im_1\sigma}, \quad X_3 + iX_4 = \sqrt{\frac{\mathcal{J}_2}{w_2}} e^{iw_2\tau + im_2\sigma}, \quad t = \kappa\tau, \qquad (3.4.31)$$

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where X_i's are the embedding coordinates of the unit sphere S^3, t is the global AdS time and (τ, σ) are the world-sheet coordinates in the static gauge. For this solution, the global charges (2.2.23) are given by

$$E = \sqrt{\lambda}\kappa, \ J_1 = \sqrt{\lambda}\mathcal{J}_1, \ J_2 = \sqrt{\lambda}\mathcal{J}_2. \tag{3.4.32}$$

The embedding $X_i X^i = 1$ constraints the parameters as $1 = \mathcal{J}_1/w_1 + \mathcal{J}_2/w_2$ and the level matching condition is given by $m_1 \mathcal{J}_1 + m_2 \mathcal{J}_2 = 0$. By introducing new parameters as $\mathcal{J} = \mathcal{J}_1 + \mathcal{J}_2$, $\alpha = \mathcal{J}_2/\mathcal{J}$, $m = m_1$ and $n = m_1 - m_2$, the embedding constraint and the level matching condition become $\mathcal{J}^{-1} = (1-\alpha)/w_1 + \alpha/w_2$ and $m = n\alpha$ respectively. Furthermore, these parameters are related to each other via the equations of motion and the Virasoro constrains as follows

$$w_i^2 = \nu^2 + m_i^2, \ \nu^2 \equiv \mathcal{J}_1 \frac{w_1^2 - m_1^2}{w_1} + \mathcal{J}_2 \frac{w_2^2 - m_2^2}{w_2}, \tag{3.4.33}$$

$$\kappa^2 = \mathcal{J}_1 \frac{w_1^2 + m_1^2}{w_1} + \mathcal{J}_2 \frac{w_2^2 + m_2^2}{w_2}. \tag{3.4.34}$$

These complicated constrains are simplified in the large $\mathcal{J}$ limit and solved by

$$\kappa \simeq \mathcal{J} + \frac{\alpha(1-\alpha)n^2}{2\mathcal{J}}, \ w_1 \simeq \mathcal{J} + \frac{\alpha(2\alpha - 1)n^2}{2\mathcal{J}}, w_2 \simeq \mathcal{J} + \frac{(\alpha - 1)(2\alpha - 1)n^2}{2\mathcal{J}}. \tag{3.4.35}$$

In particular, energy of this circular string is given by

$$E - J = \lambda \frac{\alpha(1-\alpha)n^2}{2J} + \cdots. \tag{3.4.36}$$

This form of the large J expansion for the classical energy resembles the semiclassical expansion for the anomalous dimension of a 1-cut solution in the weak coupling. Actually, once we identify the parameters as $J = L$, $J_2 = M$, we obtain the precise matching. This is a particular example of the Frolov-Tseytlin limit where $\lambda, J \to \infty$ with keeping $\lambda/J^2 \ll 1$.

Similarly, the other solutions such as folded strings[28] are discussed within the framework of the finite-gap solution as well [79]. The analysis of semiclassical spectrum based on algebraic curves was extended to other sectors including the $SO(6)$ sector, the noncompact $SL(2)$ sector [80–82], and later the whole $PSU(2, 2|4)$ sector [83,84].

It is well-known that there is a mismatch between the result of the gauge theory and the string theory at 3-loop level [23], which is known as the infamous *3-loop discrepancy*. To see this, let us consider the integral equation for the resolvent of the string theory that is defined from the quasi-momentum of string $p_{st}(x)$ as follows [29]

$$G_{st}(x) = p_{st}(x) + \frac{\pi\kappa}{x+1} + \frac{\pi\kappa}{x-1}. \tag{3.4.37}$$

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[28]They correspond to two-cut solutions.

[29]We denote the quantities of string theory with the subscript st to distinguish from the gauge theory quantities.

Here, we define the resolvent so that it is regular on the physical sheet from the asymptotics at $x = \pm 1$ (3.3.49). Since the quasi momentum satisfies (3.3.65) on the branch cuts for the spectral curve, the resolvent satisfies

$$G_{st}(x + i0) + G_{st}(x - i0) = 2\mathcal{P} \int_C dy \frac{\rho_{st}(y)}{x - y} = \frac{Ex}{x^2 - g^2} + 2\pi n_k, \quad x \in C_k, \qquad (3.4.38)$$

where we have used the relation $E = \sqrt{\lambda}\kappa$, $\sqrt{\lambda} = 4\pi g$ and rescaled the parameter as $x \to gx$. The asymptotics of the quasi-momentum allows us to re-express the conserved charges in terms of the resolvent or the density as follows[30]

$$-\oint \frac{dx}{2\pi i} G_{st}(x) = \int_C dx \rho(x) = J_2 + \frac{E - J}{2}, \qquad (3.4.39)$$

$$-\oint \frac{dx}{2\pi i} \frac{G_{st}(x)}{x} = \int_C dx \frac{\rho_{st}(x)}{x} = 2\pi m, \qquad (3.4.40)$$

$$-\oint \frac{dx}{2\pi i} \frac{G_{st}(x)}{x^2} = \int_C dx \frac{\rho_{st}(x)}{x^2} = \frac{E - J}{2g^2}. \qquad (3.4.41)$$

where $J = J_1 + J_2$ is the total spin. By substituting the last equation to (3.4.38), we obtain the following integral equation

$$2\mathcal{P} \int_C dy \frac{\rho_{st}(y)}{x - y} = \frac{Jx}{x^2 - g^2} + 2g^2 x \int_C dy \frac{\rho_{st}(y)}{y^2(x^2 - g^2)} + 2\pi n_k, \quad x \in C_k, \qquad (3.4.42)$$

This is the analogue of the scaling limit of the Bethe ansatz equation (3.4.10). It is obvious that the spectrum matches at 1-loop level.

In [28], the integrable spin chain model that describes the higher-loop dilatation operator for the $SU(2)$ sector, together with a conjectured all-loop Bethe ansatz equation was proposed. The model is sometimes called the BDS model named after Beisert, Dippel and Staudacher. The structure of the Bethe ansatz equation is the same as in the case of the Heisenberg model (3.2.46), however, the S-matrix and the magnon dispersion are modified. The scaling limit of this Bethe ansatz equation leads to the following integral equation

$$2\mathcal{P} \int_C dy \frac{\rho_{gauge}(y)}{x - y} = \frac{Jx}{x^2 - g^2} + 2g^2 x \int_C dy \frac{\rho_{gauge}(y)}{y^2 x^2 (1 - g^2/x^2 y^2)} + 2\pi n_k, \quad x \in C_k. \quad (3.4.43)$$

Therefore, we see that the only difference between the result of the gauge theory and the string theory is the second term in the right hand side and the discrepancy start at the 3-loop level. However, it does not immediately means that the 3-loop discrepancies are counterexamples of the AdS/CFT correspondence since the AdS/CFT is a strong/weak

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[30]C.f., the similar equations for gauge theory (3.4.7) and (3.4.8).

duality and thus there is a possibility that some non-trivial factor, which appears in perturbative regime of one theory but does not contribute in the perturbative regime of the other theory, interpolates the two regimes. Indeed, the result of the gauge theory side (3.4.43) assumes the BMN scaling hypothesis that the expansion with respect to λ/J^2 is sensible. If there is a factor that violates the BMN scaling, the problem can be explained in the following *order of limits* mechanism. Notice that the order of limit of the BMN scaling are different in the gauge theory and the string theory. In the gauge theory side, we first compute the dilatation operator perturbatively around $\lambda = 0$ and take the scaling limit $J \to \infty$, while we first take the large spin limit $J \to \infty$ to suppress the quantum corrections and expand the solution in the BMN coupling λ/J^2 later in the string theory side. To clarify the importance of the order of limit, let us consider the following toy model.

$$F(\lambda, J) = J \left[1 + a_1 \frac{\lambda}{J^2} + a_2 \left(\frac{\lambda}{J^2} \right)^2 + a_3 \left(\frac{\lambda}{J^2} \right)^3 + \cdots \right] + b \left(\frac{\lambda}{J^2} \right)^3 \frac{\lambda^{J-3}}{1 + \lambda^{J-3}} . \quad (3.4.44)$$

The first term respects the BMN scaling, however, the second term violates it. Then, if we take the limit as in the gauge theory, the coefficient of the $(\lambda/J^2)^3$ becomes simply a_3. On the other hand, if we take the limit as in the string theory, it becomes $a_3 + b$.

In fact, the BDS model is based on the BMN scaling hypothesis. Together with the assumption of the integrability and the field theoretic considerations, it determines the model uniquely and fixes the S-matrix of the magnons completely. However, it turned out that a non-trivial overall phase, which violates the BMN scaling, must be present so that the Bethe ansatz correctly reproduces the strong coupling results [35]. This phase is called dressing phase and this is the first evidence that the dressing phase is non-trivial.

We end this subsection with some remarks on the related topics. We have assumed so far that there are no condensate cuts, or "string" configurations of the Bethe roots, which are the distribution of the Bethe roots with a constant interval such as $u_k = u + ik$ in the scaling limit. Such condensates of the Bethe roots can be viewed as extra logarithmic cuts in the complex plane since the density is constant along such condensates $\rho(x) = -i$ and $p(x + i0) - p(x - i0) = -2\pi\rho(x) = 2\pi i$.[31] Condensate cuts typically have the tails with the square root cuts, thus they connect pairs of the cuts C_i. A string configuration of the Bethe roots represents bound states of magnons and the dual string states are known to be bound states of giant magnons [85–87]. The giant magnons and their bound states are realized as the finite-gap solutions by degenerating some of the cuts in the spectral curve [88, 89].

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[31]Note that $G(x + i0) - G(x - i0) = \frac{1}{L} \sum_{k=1}^{M} -2\pi\delta(x - x_k) = -2\pi\rho$.

Furthermore, it turns out that the 1-loop corrections to the energies of classical strings or anomalous dimensions of corresponding composite operators are efficiently calculated from the spectral curve [99, 100]. The 1-loop shifted energy is schematically given as follows

$$E = E_{cl} + E_0 + O(L^{-1}) , \tag{3.4.45}$$

where E_{cl} is the classical energy of order $O(L)$ and the second term is the ground state energy of order $O(1)$. The ground state energy is the summation over all (quadratic) fluctuations around the background configuration, thus it is given by

$$E_0 = \frac{1}{2} \sum_{A,n} (-1)^{F_A} \Omega_{A,n} , \tag{3.4.46}$$

where A labels the possible types of fluctuations, n is the mode number and $(-1)^F$ is ± 1 for a bosonic/fermionic excitation. This is indeed equivalent to the summation over the zero point energies for the Harmonic oscillators since we are considering the 1-loop fluctuations around the classical configuration. In [99, 100], this summation over fluctuations is translated into the language of the spectral curve. The main idea is as follows. The first step is to map the classical configuration to the algebraic curve or spectral curve. Then, the fluctuations perturb the background configuration by adding microscopic cuts to the original algebraic curve. The position of the microscopic cuts, which are indeed poles with finite residues, are determined from the requirement that the perturbed solution also satisfies (3.3.65), or (3.4.13), then their positions turn out to be the singular points. This can be understood from the fact that the singular points are shrunk cuts and they correspond to the unexcited modes. Moreover, for the case of string moving on $\mathbb{R} \times S^3$, the fluctuation frequency $\Omega_{A,n}$ is the value of the quasi-energy $q(x)$ defined in (3.3.50) at each singular point [97]. Hence, E_0 can be rewritten as

$$E_0 = \frac{1}{2} \oint_{C_{\text{sing}}} \frac{dx}{2\pi i} q(x) \frac{d}{dx} \ln \sin p(x) , \tag{3.4.47}$$

where the contour C_{sing} encircles all the singular points and we have rewritten the summation over singular points as the contour integral by using $p(x_k^*) = n_k \pi$. This expression suggests the importance of singular points and we will encounter a similar expression in the case of the three-point functions.

3.4.2 Spin chain from string

In the previous subsection, we discussed how the scaling limit of the Bethe ansatz equation reproduces the finite-gap solutions. Actually, we can see such a 1-loop agreement of the

spectrum even at the level of an effective action, as shown in [78]. The outline of the derivation is the following. The coherent state representation allows us to rewrite the Heisenberg Hamiltonian H_{XXX} as a path integral form with an effective action, then the continuum limit leads to the 1+1 dimensional sigma model which is so-called the Landau-Lifshitz model. The same sigma model is obtained from the string sigma model moving on $\mathbb{R} \times S^3$ by expanding the action around a classical solution with a large angular momentum and removing the "fast" variables.

The details for the Landau-Lifshitz model is summarized in the appendix 8.2. The action is given by

$$S_{LL} = \int d\tau \int_0^L d\sigma \left[\frac{1}{2} \frac{(\vec{n} \times \partial_\tau \vec{n}) \cdot \vec{n}_0}{1 + \vec{n} \cdot \vec{n}_0} - \frac{\lambda}{32\pi^2} \int_0^L d\sigma \partial_\sigma \vec{n} \cdot \partial_\sigma \vec{n} \right] \tag{3.4.48}$$

$$= -\frac{L}{2\pi} \int d\tau \int_0^{2\pi} d\sigma \left[\frac{1}{2} \cos\theta \partial_\tau \phi + \frac{\lambda}{16\pi^2 L^2} ((\partial_\sigma \theta)^2 + \sin^2\theta (\partial_\sigma \phi)^2) \right] . \tag{3.4.49}$$

The equation of motion for the polarization vector follows from the action

$$\partial_\tau \vec{n} = \frac{\lambda}{8\pi^2} \vec{n} \times \partial_\sigma^2 \vec{n} \tag{3.4.50}$$

This is called Landau-Lifshitz equation and a familiar equation that describes the time-evolution of the magnetization vector for a classical magnet. It is known that this equation is classically integrable, which means that the equation of motion can be recasted into the form of a flat connection depending on an arbitrary complex parameter, as with the case of the string on $\mathbb{R} \times S^3$. This is not surprising since the original Heisenberg model is a quantum integrable system.

We now derive the same action from the string sigma model. Recall that the metric of the $\mathbb{R} \times S^3$ in the polar coordinates is given by

$$ds^2 = -dt^2 + d\Omega_{[3]}^2 = -dt^2 + d\psi^2 + \cos^2\psi d\phi_1^2 + \sin^2\psi d\phi_2^2 . \tag{3.4.51}$$

Here, we introduce new coordinates $\varphi_1 = (\phi_1 + \phi_2)/2$ and $\varphi_2 = (\phi_1 - \phi_2)/2$. In this coordinate patch, the center of mass for the BMN-like solution moves in φ_1 direction with a large angular momentum. In other words, (t, φ_1) are "fast" variables and other variables are "slow" variables. Thus, it is natural to shift the coordinate as $\varphi_1 \to t + \tilde{\varphi}_1$. With these change of variables, the metric becomes

$$ds^2 = 2dt d\tilde{\varphi}_1 + d\psi^2 + d\tilde{\varphi}_1^2 + d\varphi_2^2 + 2\cos(2\psi)dt d\varphi_2 + 2\cos(2\psi)d\tilde{\varphi}_1 d\varphi_2 . \tag{3.4.52}$$

For simplicity, we will omit the tilde for $\tilde{\varphi}_1$ in the following. After taking the static gauge $t = \kappa\tau$, the action for string moving on this background is given by

$$S = \frac{\sqrt{\lambda}}{4\pi} \int d\tau d\sigma (2\kappa\dot{\varphi}_1 + \dot{\psi}^2 + \dot{\varphi}_1^2 + \dot{\varphi}_2^2 + 2\cos(2\psi)\kappa\dot{\varphi}_2 + 2\cos(2\psi)\dot{\varphi}_1\dot{\varphi}_2$$
$$-\psi'^2 - \varphi_1'^2 - \varphi_2'^2 - 2\cos(2\psi)\varphi_1'\varphi_2'), \qquad (3.4.53)$$

where we denote derivatives with respect to τ by dots and those with respect to σ by primes. The Virasoro constraints become

$$2\kappa\varphi_1' + \dot{\psi}\psi' + \dot{\varphi}_1\varphi_1' + \dot{\varphi}_2\varphi_2' + 2\cos(2\psi)(\kappa\varphi_2' + \dot{\varphi}_1\varphi_2' + \dot{\varphi}_2\varphi_1') = 0, \qquad (3.4.54)$$
$$2\kappa\dot{\varphi}_1 + \dot{\psi}^2 + \dot{\varphi}_1^2 + \dot{\varphi}_2^2 + \psi'^2 + \varphi_1'^2 + \varphi_2'^2 + 2\cos(2\psi)(\kappa\dot{\varphi}_2 + \dot{\varphi}_1\dot{\varphi}_2 + \varphi_1'\varphi_2'). \qquad (3.4.55)$$

We have just rewritten the action and not performed any approximations so far. Since we would like to write down the effective action for the "slow" variables, we wish to take an appropriate decoupling limit. This can be achieved by dropping time derivatives, assuming that the motion of string is mainly captured by the rotation. However, if we neglect all the time derivatives, we only have the BPS solutions or massless modes moving around the big circle. To obtain the solutions whose energy behaves as $E \sim J$ in the large spin limit, we need to take the following limit as in [78][32]

$$\kappa \to \infty, \quad \dot{X} \to 0, \quad \kappa\dot{X} : \text{fixed} \quad \text{for} \quad X = \psi, \varphi_1, \varphi_2 \qquad (3.4.56)$$

To obtain the action (3.4.49), we need to use the Virasoro constraints in the limit and remove the total derivative term. Moreover, we should identify the variables as $\phi = -2\varphi_2$, $\theta = \Psi$, and rescale the world-sheet variables as $\tilde{\tau} = \tau/\kappa$, $\tilde{\sigma} = \sqrt{\lambda}\kappa\sigma/J$. Finally, we obtain the effective action for the string sigma model

$$S_{st} = -\frac{J}{2\pi} \int d\tilde{\tau} \int_0^{2\pi} d\tilde{\sigma} \left[\frac{1}{2}\cos\theta\partial_\tau\phi + \frac{\lambda}{16\pi^2 J^2}((\partial_\sigma\theta)^2 + \sin^2\theta(\partial_\sigma\phi)^2) \right] \qquad (3.4.57)$$

This is indeed equivalent to the action (3.4.49) under the identifications $J \leftrightarrow L$, $\tilde{\sigma} \leftrightarrow \sigma$ and $\tilde{\tau} \leftrightarrow \tau$. The energy and the z-component of the spin are obtained from this action

$$E = \frac{\lambda}{16\pi^2 J} \int_0^{2\pi} d\sigma(\theta'^2 + \phi'^2 \sin^2\theta), \qquad (3.4.58)$$

$$S_z = \frac{J}{4\pi} \int_0^{2\pi} d\sigma \cos\theta, \qquad (3.4.59)$$

where S_z and the total spin J are related to the two spins of S^3 as $J = J_1 + J_2$, $S_z = J_1 - J_2$.

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[32]This is actually the same limit considered in [73], namely, the Frolov-Tseytlin (FT) limit that takes $\lambda, J \to \infty$ keeping $\lambda/J^2 \ll 1$.

We have seen that the Landau-Lifshitz model action arises in the scaling limit from the string sigma model. A natural question is that what is the exact relation between the all-loop spin chain model and the string sigma model, at least in the scaling limit. As we explained at the end of the section 3.2.1, the higher-loop dilatation operator of $\mathcal{N} = 4$ SYM is mapped to the spin chain Hamiltonian with long-range interactions. Since the long-range interactions lead to higher derivative terms in the continuum limit, assuming the BMN scaling hypothesis, the schematic form of the action in the continuum limit of such a spin chain model is expected to be

$$S = S_{WZ} + J \int d\tau d\sigma \left(\frac{\lambda}{L^2}(\partial_\sigma \vec{n})^2 + \cdots + c_n \frac{\lambda^n}{L^{2n}}(\partial_\sigma \vec{n})^{2n} + \cdots \right). \tag{3.4.60}$$

The coefficient in front of the Wess-Zumino term should not be renormalized due to the topological argument. Although the BMN scaling hypothesis breaks down at 3-loop level [23], this form of the action is suggestive. The above action is first order in time derivatives and infinite order with respect to spacial derivatives. In other words, it is non-relativistic and has non-local interactions. On the other hand, the string action is quadratic in world-sheet coordinates. It seems that the exact relation between them involves integrating out some degrees of freedom so that it reproduces non-local interactions in the spin chain.

3.5 Beyond perturbative analysis

In this section, we will explain the subsequent developments beyond the perturbative analysis. In ordinary quantum field theories, perturbative calculations at higher loop level are surely hopeless as the relevant Feynman diagrams exponentially increase. However, in the present situation, spectrum problem can be solved even at all-loop level due to the underlying integrable structure which governs the both weak and strong coupling regimes. Crucial steps are divided into three parts:

1. Consider a infinitely long spin chain.

2. Determine the asymptotic spectrum.

3. Include the finite size effect.

Let us explain these in turn.

Decompactification limit

We start form the two-point functions for BPS operators with infinite length:

$$\text{Tr}(\ldots ZZZ \ldots). \tag{3.5.1}$$

By considering such a decompactified configuration, we can neglect the troublesomeness coming from the finite size effect. To treat generic composite operators, we wish to add "excitations", namely, the other fields or covariant derivatives upon the "vacuum". Such excitations are classified according to the symmetry of the vacuum, which is given by the centrally extended $\mathfrak{su}(2|2)_L \oplus \mathfrak{su}(2|2)_R$ symmetry since the original $\mathfrak{psu}(2,2|4)$ symmetry is spontaneously broken down due to the choice of the vacuum. A generic single trace operator is schematically expressed as follows

$$|\ldots Z \ldots Z\chi Z \ldots Z\chi' Z \ldots\rangle \leftrightarrow \mathrm{Tr}(\ldots Z \ldots Z\chi Z \ldots Z\chi' Z \ldots)\,, \qquad (3.5.2)$$

$$\chi, \chi', \ldots, \in \{\Phi_{a\dot{a}}, \Psi_{\dot{a}\alpha}, \bar{\Psi}_{a\dot{\alpha}}, D_{\alpha\dot{\alpha}}Z\}\,, \ a, \dot{a} = 1, 2\,, \ \alpha, \dot{\alpha} = 3, 4\,. \qquad (3.5.3)$$

Here, the roman and greek indices represent fermonic and bosonic part of $\mathfrak{su}(2|2)$ respectively and the dot (undot) indices denote the label of $\mathfrak{su}(2|2)_R$ ($\mathfrak{su}(2|2)_L$). All the other fields can be expressed as a multiple excitations on a single site [33].

Asymptotic Bethe ansatz

Now let us see how this residual symmetry powerfully constrains the asymptotic spectrum. The essence is to exploit the Bethe ansatz technique introduced in section 3.2.2. The momenta of magnon excitations are quantized due to the periodicity, which leads to the Bethe ansatz equation. Recall that the scattering of magnons plays an important role. Since we are considering a infinitely long spin chain, it is possible to define asymptotic states of the excitations and S-matrices of them. Such asymptotic states are belong to the bi-fundamental representation of the centrally extended $\mathfrak{su}(2|2)_L \oplus \mathfrak{su}(2|2)_R$ symmetry, under which the S-matrix is invariant. It turned out that the same structure appears in the uniform light-cone gauge fixed string sigma model [32–34], after relaxing the level matching condition and taking decompactifying limit namely, infinite light-cone momentum. The S-matrix acts on the multi-magnon states and it is known to be factorized into two $\mathfrak{su}(2|2)$ parts

$$\mathbb{S} = \mathcal{S} \otimes \dot{\mathcal{S}}\,. \qquad (3.5.4)$$

To understand the role of symmetry, it is of use to introduce the Zamolodchikov-Faddeev (ZF) operators, which are creation operators of magnons

$$|\chi_{A_1\dot{A}_1}, \ldots, \chi_{A_n\dot{A}_n}\rangle = A^\dagger_{A_1\dot{A}_1}(p_1) \cdots A^\dagger_{A_n\dot{A}_n}(p_n)|0\rangle\,. \qquad (3.5.5)$$

Since the S-matrix is factorized, the ZF operator is also given by the tensor product form

$$A^\dagger_{A\dot{A}}(p) = A^\dagger_A(p) \otimes A^\dagger_{\dot{A}}(p)\,, \qquad (3.5.6)$$

---

[33]For instance, $D\Psi$ can be regarded as $Z \to \Psi \to D\Psi$ or equivalently, $Z \to DZ \to D\Psi$

and thus it is enough to consider the half of $\mathfrak{su}(2|2)$ part. The commutation relations between the ZF operator and $\mathfrak{su}(2|2)$ generators can be read from the representation of 1-magnon state with a definite momentum on a infinite spin chain:

$$|\chi\rangle = \sum_n e^{ipn} |\ldots Z \overset{n}{\chi} Z \ldots\rangle. \tag{3.5.7}$$

As for the bosonic generators, the ZF operator transforms as a fundamental representation, while the action of the fermonic generators are not trivial as they could change the length of spin chain. Furthermore, the coefficient of the action depends non-trivially on the momentum and the coupling constant, which are determined by imposing the closure of the algebra and central charge condition [30]. As a result, the action of generators is fixed and we can read off the dispersion relation since one of the central charges is essentially the dilatation operator or the light-cone Hamiltonian.

$$e^{ip} = \frac{x^+}{x^-}, \quad \mathcal{H} = -ig\left(x^+ - \frac{1}{x^+} - x^- + \frac{1}{x^-}\right) = \sqrt{1 + 16g^2 \sin^2 \frac{p}{2}}, \tag{3.5.8}$$

where $x^\pm$ are the familiar Zhukowsky variables defined as $x^\pm = x(u \pm i/2)$ with $u/g = x + 1/x$ and $g = \sqrt{\lambda}/4\pi$.

The ZF operator is a natural generalization of free field oscillator. It takes into account the effect of interaction by deforming the free exchanging algebra using the non-trivial scattering matrix and satisfies the following ZF algebra:

$$A^\dagger_{A_1}(p_1) A^\dagger_{A_2}(p_2) = S^{B_1 B_2}_{A_1 A_2}(p_1, p_2) A^\dagger_{B_2}(p_2) A^\dagger_{B_1}(p_1). \tag{3.5.9}$$

Starting from $A^\dagger_{A_1}(p_1) A^\dagger_{A_2}(p_2) A^\dagger_{A_3}(p_3)$, we can rewrite it into a linear combination of $A^\dagger_{B_1}(p_3) A^\dagger_{B_2}(p_2) A^\dagger_{B_3}(p_1)$ using (3.5.9) in two different way, which leads to the following consistency condition

$$S_{12}(p_1, p_2) S_{13}(p_1, p_3) S_{23}(p_2, p_3) = S_{23}(p_2, p_3) S_{13}(p_1, p_3) S_{12}(p_1, p_2). \tag{3.5.10}$$

This is nothing but the Yang-Baxter equation which ensures that the multi particle scattering is factorized into the product of two-body ones. Furthermore, the invariance under $\mathfrak{su}(2|2)$

$$[\mathbb{J}_{12}, \mathcal{S}_{12}] = 0, \tag{3.5.11}$$

gives the constrains for the S-matrix, which can be obtained by computing $\mathbb{J}_{12} A^\dagger_{A_1}(p_1) A^\dagger_{A_2}(p_2)|0\rangle$ and expanding in terms of $A^\dagger_{B_2}(p_2) A^\dagger_{B_1}(p_1)|0\rangle$ in two different ways. Namely, we first push the generator through the oscillators to act the vacuum and then use the ZF algebra. The

other way is to use the ZF algebra first and push the generator to the next of the vacuum. By comparing these two expressions, we can determine the S-matrix $S^{B_1 B_2}_{A_1 A_2}(p_1, p_2)$ up to a scalar phase $\sigma^2_{12}(p_1, p_2)$ [30, 33]. The resulting S-matrix automatically satisfies the Yang-Baxter equation. In order to fix the dressing factor, we further need to impose several conditions such as unitarity, crossing symmetry and the analytical properties concerning the bound state spectrum. The crossing symmetry is the condition that the scattering of a pair of particle and anti-particle with zero total charges through another physical particle should be inconsequential. (See, for example, [45]) To obtain the crossing equation, we need to explain the particle to anti-particle transformation. As an anti-particle has energy and momentum with the opposite sign $(-E(u), -p(u))$, it is given by an analytical continuation on the rapidity plane $2\gamma : u \to u^{2\gamma}$, which can be defined through crossing the cut of $x^+(u)$, $x^-(u)$ and then going back to the original value of u. As a result, the Zhukovski variables change as follows

$$2\gamma : \quad x^\pm \to \frac{1}{x^\pm} \, . \tag{3.5.12}$$

Therefore, it gives the particle to anti-particle map as seen from (3.5.8). Using this transformation, the crossing equation is given by

$$\sigma(u_1, u_2)\sigma(u_1^{2\gamma}, u_2) = \frac{1 - \frac{1}{x_1^+ x_2^+}}{1 - \frac{x_1^-}{x_2^-}} \frac{1 - \frac{x_1^-}{x_2^+}}{1 - \frac{1}{x_1^+ x_2^-}} \, . \tag{3.5.13}$$

However, there are infinitely many solutions for the crossing equation, hence, we need to resort to a physical argument of the analyticity to single out the correct solution. It is proven that the solution with minimal number singularities is actually the so-called BES/BHL phase, which was first guessed in [38, 39].

With the dispersion relation and the S-matrix, we can write down the asymptotic spectrum. The essence of the idea of the Bethe ansatz method was explained in section 3.2.2. The Bethe ansatz equation takes the form

$$e^{i p_k L} \overset{?}{=} \prod_{l \neq k} \mathcal{S}(p_k, p_l) \, . \tag{3.5.14}$$

However, unlike the SU(2) case, the S-matrix is not a phase but a matrix in the present case. Therefore, we first need to diagonalize the transfer matrix, which is the product of the S-matrix by introducing auxiliary rapidities. After diagonalization, it is possible to apply the usual Bethe anstaz equation with the eigenvalue of the transfer matrix. The resultant asymptotic Bethe ansatz equation or the Beisert-Staudacher equation was first
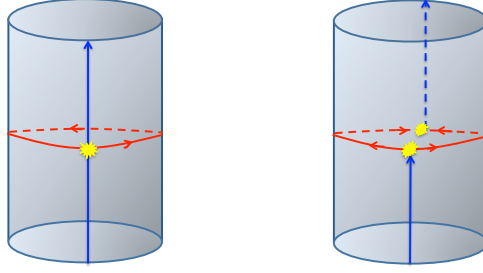
Figure 3.5.1: The one-particle Lüscher correction consists of the two terms, namely, the F-term and the μ-term.

given in [29], becomes as follows in the SU(2) sector:

$$\left(\frac{x^+(u_k)}{x^-(u_k)}\right)^L = \prod_{l\neq k} \sigma^2(u_k, u_l) \frac{u_k - u_l + i}{u_k - u_l - i}. \qquad (3.5.15)$$

The anomalous dimension reads from the dispersion as

$$\gamma = 2ig \sum_k \left(\frac{1}{x^+(u_k)} - \frac{1}{x^-(u_k)}\right). \qquad (3.5.16)$$

Notice that the above equations are reduced to the result obtained in section 3.2.2 at the weak coupling since the Zhukovski variable becomes $x \approx u$ and the dressing phase is trivial in the limit.

Finite size correction

Although the spectrum obtained from the asymptotic Bethe ansatz includes all the power like finite size corrections of the form $1/L$, it cannot capture the exponentially small corrections like e^{-cL}. Since anomalous dimensions of the $\mathcal{N} = 4$ SYM correspond to the energies of string on $AdS_5 \times S^5$, which are calculated by world-sheet energies after the light-cone gauge fixing, it is quite natural to apply the formula for the correction to the energy spectrum in massive QFT in finite volume. The formula derived by Lüscher [46] for the leading finite size correction to the mass spectrum in infinite volume is called Lüscher's formula or Lüscher's correction. It is an universal formula in the sense that it can be applied to any massive QFT in arbitrary space-time dimension and the correction is expressed in terms of the infinite volume S-matrix of the theory. However, this formula is the most useful in the case of integrable two-dimensional QFT as the S-matrix in such a theory is factorized into two-body ones and it is completely fixed by the symmetry. The

Lüscher's correction is composed of two parts

$$\delta m_F = -m \int_{-\infty}^{\infty} \frac{d\theta}{2\pi} e^{-mL\cosh\theta} \cosh\theta \sum_b \left(S_{ab}^{ab}(\theta + i/2) - 1 \right) , \qquad (3.5.17)$$

$$\delta m_\mu = -\frac{\sqrt{3}}{2} m \sum_{b,c} M_{abc}(-i) \mathrm{res}_{\theta=2\pi i/3} S_{ab}^{ab}(\theta) e^{-\frac{\sqrt{3}}{2}mL} , \qquad (3.5.18)$$

where the first and second term are called F-term and μ-term, respectively (figure 3.5.1). $S_{ab}^{ab}(\theta)$ is the S-matrix of the theory and M_{abc} is 1 if the particle label c is a bound state of a and b. The physical pictures behind these two terms are explained as follows. The F-term corresponds to the interaction between a physical particle and a virtual particle circulating around the circle, while the μ-term corresponds to the process in which a physical particle splits into two other particles and recombines after they circulate around the circle. These picture are interesting and give a useful intuitions to interpret the finite size correction.

In order to apply the Lüscher's formula in the case of our interest, namely, the light-cone gauge fixed string theory on $AdS_5 \times S^5$, we need to generalize the above formula as regard to the following two aspects. The first point is that the world-sheet theory is no more Lorentz invariant after the gauge fixing, hence, the dispersion relation for the magnon excitation is not relativistic. Furthermore, the S-matrix does depend on two rapidities θ_1, θ_2, while the S-matrix in a relativistic theory depend only on a relative rapidity $\theta = \theta_1 - \theta_2$. The second point is that we necessarily treat the multi-particle Lüscher's correction as the world-sheet momentum of a single particle state should be zero due to the level matching condition, or cyclicity of the single trace operator.

One of the most important method to compute the finite size effect in integrable QFT is the thermodynamic Bethe ansatz (TBA) approach, which was first proposed by Zamolodchikov [49]. Actually, the multi-particle Lüscher's correction can be derived from the TBA equation (see, for example, [48]). The idea of the TBA method is as follows. We first consider the Euclidean partition function by Wick rotating the time direction with large imaginary time R. In other words, we treat the Euclidean two-dimensional theory on the torus with radius (L, R). The ground state energy of the original theory can be read from the large R limit of the partition function:

$$\lim_{R\to\infty} Z(L, R) = \lim_{R\to\infty} \mathrm{Tr}(e^{-RH(L)}) = e^{-RE_0(L)} + \cdots . \qquad (3.5.19)$$

Second, we exploit the fact that exactly the same partition function can be computed from the double Wick rotated theory, which is so-called mirror theory (figure 3.5.2). By

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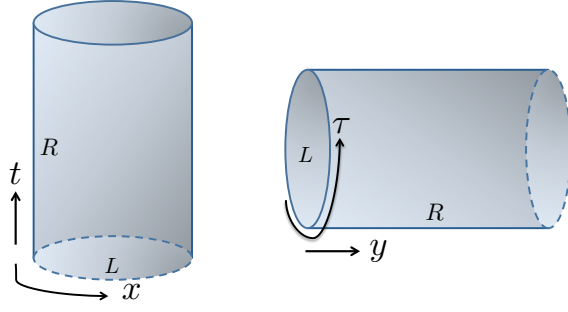
Figure 3.5.2: The same partition function can be computed in two different ways. The right figure shows the mirror theory which is obtained by the double Wick rotations: $(t, x) \to (\tau, y) = (-ix, it)$.

definition, the energy and momentum in the mirror theory $(\tilde{p}, \tilde{E})$ are given by[34]

$$\tilde{E} = ip, \quad \tilde{p} = iE, \tag{3.5.20}$$

$$\tilde{E}(\tilde{p}) = 2 \operatorname{arc\,sinh} \left(\frac{1}{2g} \sqrt{Q^2 + \tilde{p}^2} \right). \tag{3.5.21}$$

Then, the partition function is given by

$$\lim_{R \to \infty} Z(L, R) = \lim_{R \to \infty} \tilde{Z}(R, L) = \lim_{R \to \infty} \operatorname{Tr}(e^{-L\tilde{H}(R)}) = \sum_n e^{-L\tilde{E}_n}. \tag{3.5.22}$$

From this expression, it is easy to see that the exponentially small finite size corrections are coming from the summation over the contribution of the virtual particles or mirror particles. In summary, the computation of the ground state energy of the finite volume original theory can be converted into the computation of the double Wick rotated theory with infinite volume via the Euclidean partition function. In the latter theory, the spectrum is completely governed by the asymptotic Bethe ansatz method, as the theory is defined in an infinite volume system. In the large volume limit, it is of use to introduce the density for magnons in the rapidity plane $\rho_Q(u)$ and the asymptotic Bethe ansatz equation of the mirror theory becomes

$$\tilde{p}(u_k) + \sum_{Q'} \int du' (-i \log S_{Q_k Q'}) \rho_{Q'}(u') = \frac{2\pi n_k}{R}. \tag{3.5.23}$$

This can be rewritten as

$$\partial_u \tilde{p} - 2\pi(\rho_Q + \bar{\rho}_Q) = -\sum_{Q'} \int du' K_{QQ'}(u, u') \rho_{Q'}(u'), \tag{3.5.24}$$

$$K_{QQ'}(u, u') = -i\partial_u \log S_{QQ'}(u, u'), \tag{3.5.25}$$

---

[34]Q is the integer which labels the bound states of the magnons.

where $\bar{\rho}_Q$ denote the density of holes, which satisfy the same equation but not being excited, and $K_{QQ'}$ is the kernel. Using the density of particles and holes, the partition function is given by

$$Z(L, R) = \sum_Q \int \mathcal{D}\rho_Q e^{-L\tilde{E}[\rho_Q] + S[\rho_Q, \bar{\rho}_Q]}, \tag{3.5.26}$$

$$S[\rho_Q, \bar{\rho}_Q] = R \int du [(\rho_Q + \bar{\rho}_Q) \log(\rho_Q + \bar{\rho}_Q) - \rho_Q \log \rho_Q - \bar{\rho}_Q \log \bar{\rho}_Q], \tag{3.5.27}$$

where the entropy factor $S[\rho_Q, \bar{\rho}_Q]$ comes from the possibilities how solutions of the Bethe ansatz equation are distributed to particles and holes. In the large R limit, the above path integral is dominated by the saddle point. With the constraint on the variation of the density $\delta\rho_Q, \delta\bar{\rho}_Q$ coming from (3.5.24), the saddle point equation is

$$\epsilon_Q(u) - L\tilde{E}_Q(u) = -\sum_{Q'} (\log(1 + e^{-\epsilon_{Q'}}) * K_{Q'Q})(u), \tag{3.5.28}$$

where $\epsilon_Q = \log \frac{\bar{\rho}_Q}{\rho_Q}$ is so-called the pseudo energy and $*$ denotes the convolution integral. This is nothing but the TBA equation. With the solution of the TBA equation, it turns out that the ground state energy in the original theory is given by

$$E_0(L) = -\sum_Q \int \frac{du}{2\pi} (\partial_u \tilde{p}) \log(1 + e^{-\epsilon_Q}). \tag{3.5.29}$$

Furthermore, it is possible to describe the spectrum for the excited states by an analytical continuation. To explain this, we first rewrite the above expression into the following by integral by part

$$E = \int \frac{du}{2\pi} \tilde{p}(u) \partial_u \log(1 + e^{-\epsilon(u)}), \tag{3.5.30}$$

and deform the integration contour so that it pick up the residues on the poles satisfying $1 + e^{-\epsilon(u_k)} = 0$, by pushing the contour to the real axis. The resulting expression is

$$E = \sum_k E(u_k) - \int_{-\infty}^{\infty} \frac{du}{2\pi} \partial_u \tilde{p} \log(1 + e^{-\epsilon(u)}), \tag{3.5.31}$$

where we used $E(u) = i\tilde{p}(u)$. Similarly, the equation which the pseudo energy satisfies is also analytically continued as follows

$$\epsilon(u) = L\tilde{E}(u) + \sum_k \log S(u_k, u) - \int_{-\infty}^{\infty} \frac{dv}{2\pi} K(v, u) \log(1 + e^{-\epsilon(v)}). \tag{3.5.32}$$

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From these expression, we can actually derive the multi-particle Lüscher's correction by solving them iteratively. First, in the leading of large L, we can neglect the integral in (3.5.32), hence, by substituting the remaining part into (3.5.31), the energy becomes[35]

$$E = \sum_k E(p_k) - \int_{-\infty}^{\infty} \frac{dp}{2\pi} e^{-L\tilde{E}(p)} \prod_k S(p, p_k) \, . \tag{3.5.33}$$

Here, we have done the change of variables from the rapidity to momentum in the integral. For completeness, we need to fix the position of physical rapidities which satisfy $1 + e^{-\epsilon(u_k)} = 0$. This means $\epsilon(u_k) = (2n_k + 1)\pi i$ and we find that the asymptotic Bethe ansatz equation

$$(2n_k + 1)\pi i = ip_k L + i\pi - \sum_{j \neq k} \log S(p_k, p_j) \, , \tag{3.5.34}$$

where we have used $S(u, u) = -1$. In order to incorporate exponentially small correction to the asymptotic Bethe equation, we further consider the next to leading correction to the pseudo energy

$$\epsilon(u) = L\tilde{E}(u) + \sum_k \log S(u_k, u) - \int_{-\infty}^{\infty} \frac{dv}{2\pi} \frac{\partial_v S(v, u)}{S(v, u)} e^{-L\tilde{E}(p)} \prod_k S(v, u_k) \, , \tag{3.5.35}$$

and impose the quantization condition $\epsilon(u_k) = (2n_k + 1)\pi i$

$$2\pi n_k = BY_k + \delta\Phi_k \, , \tag{3.5.36}$$

$$BY_k = e^{ip_k L} \prod_{j \neq k} S(p_k, p_j) \, , \quad \delta\Phi_k = -\int_{-\infty}^{\infty} \frac{dp}{2\pi} e^{-L\tilde{E}(p)} \partial_p T(p) \, , \tag{3.5.37}$$

where $T(p) := \prod_j S(p, p_j)$ is the transfer matrix. Since the integral defining Φ is exponentially small, the corrections to magnon momenta δp_k are small as well and the equations for them are written as follows

$$\delta\Phi_k + \sum_j \frac{\partial BY_k}{\partial p_j} \delta p_j = 0 \, . \tag{3.5.38}$$

By solving the above equation and expressing δp_j in terms of $\delta\Phi_k$[36] and using (3.5.33), we find

$$E = \sum_k E(p_k) + \sum_k \frac{dE(p_k)}{dp_k} \delta p_k - \int_{-\infty}^{\infty} \frac{dp}{2\pi} e^{-L\tilde{E}(p)} T(p) \, . \tag{3.5.39}$$

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[35]In what follows, we will express the function depending on momenta of magnons through rapidities $f(u(p))$ as $f(p)$ for simplicity. For example, we denote $S(u(p_1), (p_2))$ as $S(p_1, p_2)$.

[36]$\delta p_j = -M_{jk}^{-1} \delta\Phi_k$ with the matrix $M_{kj} := \left(\frac{\partial BY_k}{\partial p_j} \right)$.
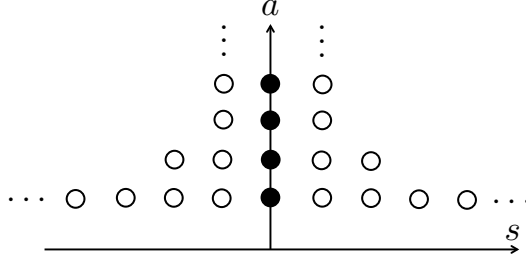
Figure 3.5.3: The T-shape hook for the Y-system or T-system is shown. Each circle corresponds to the allowed representations in the mirror theory and in particular, the black circles represents the physical momentum carrying excitations.

The first term is nothing but the asymptotic Bethe ansatz spectrum. The second term can be interpreted as the correction to the ABA due to the quantization condition including exponentially small corrections. The third term has a more interesting physical interpretation. It is the contribution coming from the scattering between physical magnons and a mirror particle circulating around the cylinder, namely, F-term. Indeed, it is divided into the phase factor $e^{ip_{ph}} = e^{-\tilde{E}L}$ and the scattering part $T(p) = \prod_k S(p, p_k)$. The μ-term would also appear through the analytic continuation if the deformation of the contour picks up poles of bound states.

In order to apply the TBA method to $\mathrm{AdS}_5/\mathrm{CFT}_4$, we need to pay attention to the following two points. The first main difference from the previous discussion is that the S-matrix is not diagonal and the auxiliary particles should be introduced in order to diagonalize the matrix structure. Even though the auxiliary particles do not contribute to the energy, however, they modify the allowed physical momenta through the scattering process. As a result, the TBA equations are complicatedly coupled. The second point is that the bound state spectrum and their spectrum in the mirror theory are necessary to derive the TBA equations. The S-matrix for fundamental magnons in the mirror theory and their TBA equation are discussed in [50] and the bound states spectrum is conjectured to be the so-called string hypothesis [52] [37]. The full TBA equations are derived in [56–58].

Refined formalisms

Before concluding this section, we should mention to recent progresses in the spectrum program. Although the TBA method gives a formalism in which the exact spectrum for planar $\mathrm{AdS}_5/\mathrm{CFT}_4$ is calculable in principle, however, it would be impossible, at least quite difficult, to solve the set of infinitely many coupled integral equations in a practical

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[37]The S-matrix for the bound states are discussed in [51, 53–55].

manner. To develop a more sophisticated formalism which is equivalent to the TBA, it is of particular importance to note that the TBA equations can be recast into a set of functional equations so-called the Y-system [56] (see also [61]). The fundamental constituents in the Y-system are the Y-functions $Y_{a,s}(u)$,[38] which are the functions of the spectral parameter u and labeled by the two integers (a, s). These two integers take value on the lattice so-called "T-hook" (figure 3.5.3) [39]. The functional equation is given by the following form[40]

$$\frac{Y_{a,s}^+ Y_{a,s}^-}{Y_{a+1,s} Y_{a,s+1}} = \frac{(1+Y_{a,s+1})(1+Y_{a,s-1})}{(1+Y_{a+1,s})(1+Y_{a-1,s})}, \tag{3.5.40}$$

equipped with the boundary conditions $Y_{0,s} = Y_{a,|s|>2} = \infty$, $Y_{a>2,\pm 2} = 0$ and some analytical properties[41]. The energy is expressed as

$$E = \sum_k \epsilon^{\mathrm{ph}}(u_k) + \sum_{a=1}^{\infty} \int_{-\infty}^{\infty} \frac{du}{2\pi i} \partial_u \epsilon^{\mathrm{mir}}(u) \log(1 + Y_{a,0}), \tag{3.5.41}$$

where the physical energy $\epsilon^{\mathrm{ph}}(u)$ and the mirror momentum $\epsilon^{\mathrm{mir}}(u)$ is defined through

$$\epsilon_a^{\mathrm{ph,mir}}(u) = a + \frac{2ig}{x_{\mathrm{ph,mir}}^{[a]}} - \frac{2ig}{x_{\mathrm{ph,mir}}^{[-a]}}, \tag{3.5.42}$$

$$x_{\mathrm{ph}}(u) = \frac{1}{2}\left(\frac{u}{g} + \sqrt{\frac{u}{g} - 2}\sqrt{\frac{u}{g} - 2}\right), \quad x_{\mathrm{mir}}(u) = \frac{1}{2}\left(\frac{u}{g} + i\sqrt{4 - \frac{u^2}{g^2}}\right), \tag{3.5.43}$$

$(x_{\mathrm{ph}}, x_{\mathrm{mir}})$ have the branch cuts at $(-2g, 2g)$ and $(-\infty, -2g) \cup (2g, \infty)$, respectively [42]. The energy formula is very reminiscent of the TBA formula. The physical Bethe roots u_k satisfy the following exact Bethe ansatz equation

$$Y_{1,0}^{\mathrm{ph}}(u_k) = -1, \tag{3.5.44}$$

where $Y_{1,0}^{\mathrm{ph}}(u)$ is defined in the physical region by the analytic continuation through the branch cut $(-\infty, -2g + i/2) \cup (2g + i/2, \infty)$.

There are other variants of the Y-system, such as T-system and a finite system of non linear integral equations (FiNLIE) [65]. However, the most elegant and elaborated

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[38]It is related to pseudo energy and density as $Y_{a,s}(u) = e^{\epsilon_{a,s}(u)} = \frac{\bar{\rho}_{a,s}}{\rho_{a,s}}$.

[39]They turn out to be the representation label for particles, including auxiliary particles and bound states, in the mirror theory.

[40]In what follows, we denote $f^{\pm} = f(x \pm i/2)$ and $f^{[a]} = f(u + ia/2)$

[41]Minimal analytical assumptions being consistent with ABA are as follows: (1) $Y_{1,\pm 1}, Y_{2,\pm 2}$ should have a cut on $(-\infty, -2g) \cup (2g, \infty)$ and they are related to each other by $Y_{2,\pm 2}(u + i\epsilon) = (Y_{1,\pm 1}(u - i\epsilon))^{-1}$ on the cut. (2) $Y_{1,s}$ should have no branch cuts inside the strip $-\frac{s-1}{2} < \mathrm{Im}u < \frac{s-1}{2}$. (3) $Y_{a,1}$ should have no branch cuts inside the strip $-\frac{a-1}{2} < \mathrm{Im}u < \frac{a-1}{2}$. (4) $Y_{a,0}$ should have no branch cuts inside the strip $-\frac{a}{2} < \mathrm{Im}u < \frac{a}{2}$. See [62]

[42]Notice that they are related to each other by analytic continuation through the cuts.

formalism recently proposed is the quantum spectral curve method (QSC) [66, 67]. The QSC formalism gives a simple finite set of non-linear Riemann-Hilbert equations, whose solutions are Q-functions. The Q-functions also appear in the context of T-system [63, 64] and they are conjectured to be the exact wave functions of the spin chain in the Skylyanin's separated variable basis. Under the WKB arroximation or semi-classical limit, the Q function Q_j is approximated as

$$Q_j \sim \exp\left(\int du p_j\right),\qquad(3.5.45)$$

where p_j is the corresponding quasi-momentum. Actually, the quasi-momenta defined in this way satisfy exactly the same constrains coming from the semi-classical limit of the ABA, which in turn means that the quasi momenta are defined on the classical spectral curve. This is the reason why the formalism is called quantum spectral curve as it is a quantum analogue of the classical one.

3.6 Lessons from spectrum problem

So far, we have discussed how the integrability based approaches work for the spectrum problem of AdS_5/CFT_4. Before moving to the argument of the three-point functions, it is of use to extract several lessons in order to emphasize the power of integrability and work out our strategy for the three-point function. There are two important lessons:

- Semi-classical analysis.

- Use of the (residual) symmetry.

Let us take a close look at the above points in turn. First, as we stress repeatedly, the semi-classical analysis is a necessary ingredient to avoid the difficulty such as strong-weak nature of the duality and quantization of string on a curved background. Furthermore, combined with the integrability analysis, semi-classical spectrum can be nicely expressed in terms of the data of algebraic curves so-called spectral curve. It should be noticed that the same mathematical structure appears and governs the spectrum after taking the semi-classical limit, even though its origin seems to be different in the weak and strong coupling regimes. Indeed, in the weak coupling, the distribution of Bethe roots are condensed into several cuts on the plane in the semi-classical limit, and the curve is defined so that the quasi-momentum, which is essentially defined through the density of Bethe roots, is a single valued function on the curve. On the other hand, in the strong coupling, the equation of motion can be converted into the form of a flat connection and

the spectral curve is defined by the characteristic equation of the monodromy matrix, which is given by the path ordered exponential of the integral of the flat connection. The structural similarity in the semi-classical limit would help to build a bridge between the two different regimes and the existence of the spectral curve and quasi-momentum can provide a tool to access other observables such as three-point functions. Actually, we will reveal the cognate structure in the semi-classical limit in the case of three-point function (chapter 8) and this would be a key step for the development of the three-point function.

The second point involves the development beyond the perturbative analysis. By considering a infinite length spin chain or light-coned gauge fixed string with infinite light-cone momentum, we can define asymptotic states and the S-matrix between of them. Due to the centrally extended $\mathfrak{su}(2|2)_L \oplus \mathfrak{su}(2|2)_R$ symmetry, the S-matrix can be uniquely fixed up to an overall scalar factor and the magnon dispersion relation is determined as well. It is quite surprising that the part of global symmetry gives such a sever constraint. We will see that the residual symmetry severely constrain three-point functions as well.

We should also comment on the importance of the role of the central charges. Actually, the dispersion relation is determined through the action of central charge, combined with the closure of the algebra. This is because the one of the central charge is light-cone Hamiltonian and the remaining central charges are related to the world-sheet momentum. The central charges appearing at the commutators of the same type of supercharges $P \sim \{Q, Q\}$, $K \sim \{S, S\}$ annihilate the physical state. This is because they generate the following gauge transformation $P : \chi \to \alpha[Z, \chi]$ and $K : \chi \to \beta[Z^-, \chi]$, which removes a field Z, hence, they are trivial on gauge invariant physical states. However, it should be noticed that their action do not kill a decompactified state as the level matching condition is relaxed, or equivalently, the periodicity is removed. It is interesting to note this fact that the action of gauge transformation of the original theory transforms into the central charges as if it is a part of global symmetry. This is a reminiscent of the asymptotic symmetry of the large gauge transformation.

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Chapter 4

Integrability and AdS/CFT
-Three-point function-

In this chapter, we will discuss the planar three-point functions of gauge invariant composite operators in $\mathcal{N} = 4$ SYM theory by using the underlying integrable structures of the theory. As explained in the section 3.2, the computations of two-point functions, or conformal dimensions of operators in $\mathcal{N} = 4$ SYM theory are greatly understood due to the integrability based approaches such as the Bethe ansatz techniques in spin chain models. We will mainly forcus at the weak coupling computation, however, the bootstrap method using the remnant symmetry successfully works in three-point functions as well.

The rest of this section is organized as follows. We first overview the ongoing development history briefly in section 4.1. In section 4.2, we will review the method of computing the three-point functions developed in [115]. This method is called "tailoring" since one first "cuts" spin chains and then sews each part of the spin chains to obtain the results in this procedure. The tailoring allows us to express the structure constants in terms of the scalar products of the relevant spin chains. We will compute three-point functions of operators in the $SU(2)$ sectors by applying the tailoring procedure in section 4.2.1. Then, we will consider the semiclassical limit of the three-point functions for the $SU(2)$ sector in section 4.2.2. We finally mention to the recent non-perturbative framework for the three-point functions, so-called the hexagon bootstrap program (section 4.3).

4.1   Overview

- **Tailoring**

   Seminal works [112–114] initiated studies of three-point functions in $\mathcal{N} = 4$ SYM at weak coupling based on the integrability. In particular, the physical picture of cutting and gluing spin chains was introduced in [112], and the usefulness of the

algebraic Bethe ansatz techniques in computing the overlaps of the wave functions for quantum spin chains, which are relevant to the calculations of three-point functions, was emphasized in [113]. More recently, based on these works, the method of computing the three-point functions has been elaborated [115–119], in such a way that the three-point functions, or structure constants are expressed in terms of the scalar products of the Bethe states of the spin chain Hamiltonian.

- **Beyond SU(2) sector**

  The tailoring method was first developed in the SU(2) sector and it was latter applied to other sectors such as SU(3) [140], SL(2) [141, 145] and SU(1|1) [142, 143]. As a result, structure constants are expressed in terms of the inner products of corresponding spin chains. However, for the higher rank sectors, no simple expression for the inner product between Bethe states is known and thus semi-classical analysis remains to be understood. One of the interesting expression for the structure constant is the Skylyanin's separation of variable (SoV) representation [93, 94]. Such SoV representation was discussed in the SL(2) sector [145] and SU(2) sector [146] (see also [135]). For the twist 2 operator, the three-point functions were studied in the so-called BFKL limit [144] and the obtained structure constant is exact in N_c.

- **Loop corrections**

  Although higher loop corrections for the three-point function have not been fully understood, it should be noted that the 1-loop correction in the SO(6) sector turned out to be understood as the insertion of the 1-loop spin chain Hamiltonian at the splitting point [112]. Furthermore, 1-loop correction in the SU(2) sector using integrability was discussed [118, 119]. The technique is called the Θ-morphism, which maps the tree-level structure constant to the 1-loop one. In this formalism, one introduces inhomogeneous parameters on each site of the spin chain, and act particular differential operators with respect to them. This operation play a role of mapping 1-loop eigenstates of the spin chain Hamiltonian, which are relevant to tree-level structure constants, to those of the long range spin chain Hamiltonian. Indeed, it is discussed that the Θ-morphism generates the transformation from the inhomogeneous XXX spin chain to the long-range spin chains [127].

- **Holographic calculations**

  Holographic calculations of the three-point function goes back to the "light-light-light" three-point function, namely, three BPS operators [102, 103]. Another important class of three-point functions is so-called "heavy-heavy-light" correlator

[104–106], which involves two non-BPS operators with large charges and one BPS operator. In this case, the three-point function is evaluated at the saddle point configuration of two heavy operators, which is essentially the two-point function or the trajectory connecting the heavy operators. As the back reaction to this saddle is negligible, the three-point function is given by evaluating the light operator on the saddle. In [107], the authors consider the holographic three-point functions for operators dual to short string states, namely, operators with dimensions $\Delta \sim \lambda^{\frac{1}{4}}$. Since such short operators are insensitive to the radius of AdS space, the authors assume that the string vertex operators are well approximated by the flat space ones with definite momenta. The most interesting and important class of three-point functions is the "heavy-heavy-heavy" correlator, whose operators all have large charges. First such an attempt was done in [108] and the contribution coming from AdS_2 part are calculated. Namely, they calculated the contribution of the action part for string states in $AdS_2 \times S^5$ whose rotation are constrained in $S^k \subset S^5$. Similarly, the computations of the action part for the GKP string [72] was done in [109] and the complete answer including the contribution of vertex operators was first given in [110]. Finally, heavy-heavy-heavy three-point functions for the so-called 1-cut solution in the SU(2) sector was performed in [111]. Although the result has a quite similar structure to the weak coupling result obtained in [124, 125], it does not match even after taking the Frolov-Tseytlin limit. The reason of this slight difference is discussed in chapter 8 and we will see that the weak and strong coupling computations actually match in the Frolov-Tseytlin limit.

- **Form factor approach**

  Toward a non-perturbative understanding for the three-point functions, an interesting approach is the form factor bootstrap program. Form factors are defined by the matrix elements of local operators in the basis of asymptotic in and out states. In particular, in two-dimensional integrable massive field theories, such form factors are severely constrained by the global symmetry, unitarity, crossing symmetry, accompanied with an analytic expression for the S-matrix. An axiomatic bootstrap approach for world-sheet form factor was first discussed in [147, 148], and latter diagonal form factors in finite volume are applied to the heavy-heavy-light three-point functions [149]. This conjecture was first proposed in the strong coupling but tested also in the weak coupling [150]. From a world-sheet form factor point of view, the axioms for the light-cone string field theory vertex was proposed and the connection to the OPE coefficients was discussed in [151, 152].

- **Hexagon bootstrap**

  The most elegant non-perturbative formulation for the three-point function recently proposed is the hexagon bootstrap approach [153]. In this formalism, the three-point function or structure constant can be decomposed into a more fundamental building block so-called the hexagon form factor. The idea behind this approach is that one can cut the world-sheet with three punctures, namely, a pair of pants diagram. In this way, two hexagons are obtained and each hexagon is identified with a form factor. It is possible to determine the form factor using the power of symmetry and integrability techniques uniquely, and several checks were done. At weak coupling, the finite size effect due to the mirror particle conjectured in the hexagon formalism was confirmed in [154, 155]. Furthermore, the hexagon form factor in the diagonal limit reproduces the structure of the form factors used in heavy-heavy-light correlators [156, 157], and the heavy-heavy-heavy three-point functions was studied in [158].

4.2 Tailoring of three-point functions

Our main interest in this section is the three-point functions of gauge invariant composite operators in $\mathcal{N} = 4$ SYM theory. Since the theory enjoys the conformal invariance even at quantum level, two- and three-point functions of local operators are uniquely fixed by the conformal symmetry. Once we choose a basis of local operators so that their two-point functions have the following normalized forms

$$\langle \mathcal{O}_i(x_i)\bar{\mathcal{O}}_j(x_j)\rangle = \frac{\delta_{ij}}{|x_i - x_j|^{2\Delta_i}} \,, \tag{4.2.1}$$

then, the three-point functions of such local operators are given as follows

$$\langle \mathcal{O}_i(x_i)\mathcal{O}_j(x_j)\mathcal{O}_k(x_k)\rangle = \frac{C_{ijk}}{|x_{ij}|^{\Delta_i+\Delta_j-\Delta_k}|x_{jk}|^{\Delta_j+\Delta_k-\Delta_i}|x_{ki}|^{\Delta_k+\Delta_i-\Delta_j}} \,, \tag{4.2.2}$$

where Δ_i is the conformal dimension of the operator $\mathcal{O}_i$ and C_{ijk} is the structure constant. The computations of conformal dimensions from integrability have been the main theme of many studies. In what follows, we will study the structure constants for single trace operators in the planar limit of $\mathcal{N} = 4$ SYM theory at the leading order in $1/N$. Since the expansion in $1/N$ for the structure constants involving three single trace operators starts from $O(1/N)$, we have the following weak coupling perturbation series

$$C_{ijk} = \frac{1}{N}(c_{ijk}^{(0)} + \lambda c_{ijk}^{(1)} + \lambda^2 c_{ijk}^{(2)} + \cdots) \,. \tag{4.2.3}$$

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We will concentrate on the tree-level structure constants. As with the usual quantum field theories, one can compute the tree-level correlation functions by summing up all the possible planar Wick contractions with free propagators in principle. For example, let us consider the three-point functions of scalar single trace operators with lengths L_i, L_j and L_k, which are the main subjects in the following discussions. Note that the total number of the lengths of the operators must be even since each propagator connects two fields. Furthermore, assuming $L_i + L_j - L_k$ does not vanish, the number of contractions between $\mathcal{O}_i$ and $\mathcal{O}_j$ is $l_{ij} = (L_i + L_j - L_k)/2$. The correlators with $L_i = L_j + L_k$ correspond to the extremal correlators, namely, the three-point functions of three half-BPS operators such as $\mathcal{O}_i = \text{Tr} Z^{L_i}$, $\mathcal{O}_j = \text{Tr} Z^{L_j}$ and $\mathcal{O}_k = \text{Tr} \bar{Z}^{L_i}$. It is known that such extremal correlators are protected due to the supersymmetry and the tree-level structure constants are exact even at finite coupling. With the bare conformal dimensions $\Delta_i^{(0)} = L_i$, the contractions with free propagators reproduce the correct space-time dependence $|x_{ij}|^{\Delta_i + \Delta_j - \Delta_k}$. Then, the problem is reduced to the combinatorics of the planar contractions.

Hence, the computations of the tree-level three-point functions or structure constants seem to be trivial since they only count the combinatorics of the possible Feynman diagrams. However, this is not the case in the following reason. At $\lambda = 0$, namely in the free theory, there are no operator mixings and all the anomalous dimensions of the operators vanish. Hence, a large number of operators have the same (bare) conformal dimension and are highly degenerated. This huge degeneracy is lifted at 1-loop level due to the quantum corrections which generate anomalous dimensions. Therefore, according to the standard degenerated perturbation theory in quantum mechanics, we need to take appropriate linear combinations of bare composite operators so that the operators $\mathcal{O}_i$ have definite 1-loop anomalous dimensions $\Delta_i = \Delta_0 + \lambda \gamma_i^{(1)}$ in order to obtain the tree-level structure constants correctly. In other words, we need to know the exact form of the 1-loop renormalized composite operators which diagonalize the action of the 1-loop dilatation operator. Generally, the problem becomes more and more difficult as the sizes of the operators become large since the number of operators involving the mixing grows exponentially and we have to diagonalize the mixing and then take the all the possible contractions of them.

Fortunately, the diagonalization of the mixing can be solved by the integrability and the operators are mapped to the eigenstates of the integrable spin chain Hamiltonian, as explained in section 3.2,

$$\mathcal{O}_i(x) \longleftrightarrow |\Psi_i\rangle. \tag{4.2.4}$$

Thus, the structure constants are expressed in terms of the spin chain language and

efficiently computed by the integrability techniques. Inspired by the picture that dual three closed strings interact via splitting/joining, the planar tree-level structure constants are constructed in the following manner.

1. Mapping each operator to the eigenstate of the spin chain Hamiltonian $|\Psi_i\rangle$, $|\Psi_j\rangle$, $|\Psi_k\rangle$ and choosing a cyclic order such as (i, j, k).

2. **Cutting**

   Breaking the i-th closed spin chain into left and right open subchains with length l_{ik} and l_{ij} and expressing $|\Psi_i\rangle$ as an entangled state in the tensor product of the left and right open subchain Hilbert spaces:

   $$|\Psi_i\rangle = \sum_a |\Psi_i^a\rangle_l \otimes |\Psi_i^a\rangle_r \,. \tag{4.2.5}$$

   Similar for the other two spin chains.

3. **Flipping**

   Converting the kets to bras with the flipping operations.

   $$|\Psi_i\rangle = \sum_a |\Psi_i^a\rangle_l \otimes |\Psi_i^a\rangle_r \longrightarrow \hat{\Psi}_i := |\Psi_i^a\rangle_l \otimes {}_r\langle\overleftarrow{\Psi_i^a}| \tag{4.2.6}$$

   The flipping operation (i) reverses the orientation of the spin chain, (ii) does not change the wave functions (no conjugations), (iii) preserves the charges. We will explain in detain when we consider examples of the $SU(2)$ sector in the next subsection.

4. **Sewing**

   Gluing each open subchain and taking scalar products of subchain to obtain the corresponding Wick contractions.

   $$\sum_{a,b,c} {}_r\langle\overleftarrow{\Psi_i^a}|\Psi_j^b\rangle_l \ {}_r\langle\overleftarrow{\Psi_j^b}|\Psi_k^c\rangle_l \ {}_r\langle\overleftarrow{\Psi_k^a}|\Psi_i^b\rangle_l = \mathrm{Tr}\,\hat{\Psi}_i \hat{\Psi}_j \hat{\Psi}_k \,. \tag{4.2.7}$$

The structure constants are obtained by this "tailoring" procedure and the results are given as follows

$$c_{ijk}^{(0)} = \frac{L_i L_j L_k \sum_{a,b,c} {}_r\langle\overleftarrow{\Psi_i^a}|\Psi_j^b\rangle_l \ {}_r\langle\overleftarrow{\Psi_j^b}|\Psi_k^c\rangle_l \ {}_r\langle\overleftarrow{\Psi_k^a}|\Psi_i^b\rangle_l}{\sqrt{L_i\langle\Psi_i|\Psi_i\rangle}\sqrt{L_j\langle\Psi_j|\Psi_j\rangle}\sqrt{L_k\langle\Psi_k|\Psi_k\rangle}} = \frac{\sqrt{L_i L_j L_k}\,\mathrm{Tr}\,\hat{\Psi}_i \hat{\Psi}_j \hat{\Psi}_k}{\sqrt{\mathrm{Tr}\,\hat{\Psi}_i}\sqrt{\mathrm{Tr}\,\hat{\Psi}_j}\sqrt{\mathrm{Tr}\,\hat{\Psi}_k}} \,, \tag{4.2.8}$$

where L_i, L_j and L_k come from the summation over all the cutting positions of the closed spin chains. The denominator factor appears so that external operators are properly normalized and give canonical two-point functions. In the following sections, we will see this tailoring of three-point functions are more efficient method than a brute force one, focusing on the calculation of the $SU(2)$ sector. In particular, it allows us to compute the structure constants of long operators.

4.2.1 Three-point functions in SU(2) sector

In the following section, we will focus on particular subsets of single trace operators, namely, the $SU(2)$ sectors. In the $SU(2)$ sector, single trace operators are made up of two kinds of complex scalar fields and they are mapped to the eigenstates for the Hamiltonian of spin 1/2 XXX Heisenberg model. We first clarify the set up of the tailoring of three-point functions for the $SU(2)$ sector. We consider three operators with lengths L_1, L_2 and L_3. The number of contractions between $\mathcal{O}_i$ and $\mathcal{O}_j$ is given by

$$l_{ij} = \frac{1}{2}(L_i + L_j - L_k),\tag{4.2.9}$$

where (i, j, k) is a permutation of $(1, 2, 3)$. Since we are considering operators from the $SU(2)$ sector, one type of scalars are identified with up-spin states (vacuum) while the other type of scalars are identified with the down-spins and interpreted as the excitations over the ferromagnetic vacuum, which is the state with all up-spins. The spin chain lengths are indeed the operator lengths L_i and the number of excitations are denoted as M_i. Let us consider the three operators of the following form

$$\mathcal{O}_1 = \text{Tr}[ZZ \ldots Y \ldots], \ \ \mathcal{O}_2 = \text{Tr}[\bar{Z}\bar{Z} \ldots \bar{Y} \ldots], \ \ \mathcal{O}_3 = \text{Tr}[ZZ \ldots \bar{Y} \ldots].\tag{4.2.10}$$

The notations for the three operators are summarized in the following table.

	Vacuum	Excitations	Number of fields
$\mathcal{O}_1$	Z	Y	$\#\{Z, Y\} = \{L_1 - M_1, M_1\}$
$\mathcal{O}_2$	$\bar{Z}$	$\bar{Y}$	$\#\{\bar{Z}, \bar{Y}\} = \{L_2 - M_2, M_2\}$
$\mathcal{O}_3$	Z	$\bar{Y}$	$\#\{Z, \bar{Y}\} = \{L_3 - M_3, M_3\}$

Notice that the extremal correlators can also be obtained by the limit $l_{ij} \to 0$ [103]. The limit of the non-extremal correlators is more advantageous than to consider the extremal correlators directly since the contributions from the mixing with double trace operators are suppressed in $1/N$ in the former, while they are not negligible in the latter. Notice that the minimal sector that contains $Z, Y, \bar{Z}, \bar{Y}$ is actually not $SU(2)$ but $SU(2)_L \times$
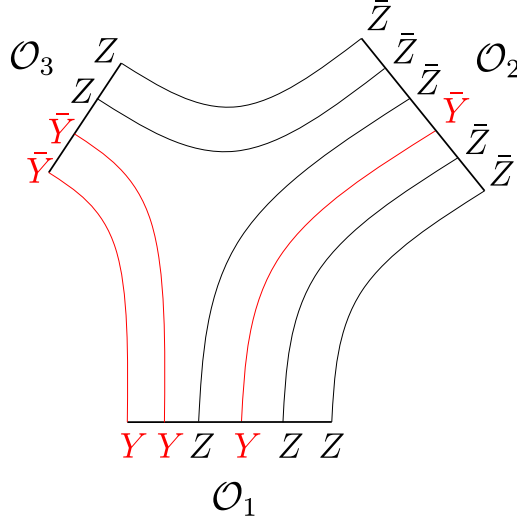
Figure 4.2.1: A planar tree-level Feynman diagram for the three-point function of $SU(2)$ sector. All contractions are taken so that the R-charges are conserved. In particular, all the $\bar{Y}$ fields of $\mathcal{O}_3$ are connected to $\mathcal{O}_1$ and all the Z fields in $\mathcal{O}_3$ are connected with $\mathcal{O}_2$. Note that the number of excitations for each spin chain satisfies $M_1 = M_2 + M_3$. Furthermore, the number of contractions between operator i and j are given by $l_{12} = L_1 - M_3$, $l_{13} = M_3$ and $l_{23} = L_3 - M_3$.

$SU(2)_R$ sector. However, the three-point function for the operators (4.2.10) is usually referred to simply $SU(2)$ sector since each external operator is belong to the single $SU(2)$ sector. The symmetry group $SU(2)_L \times SU(2)_R \cong SO(4)$ is indeed identified with the isometry of S^3 of the dual string moving on $\mathbb{R} \times S^3$ as stressed in [111].

Due to the R-charge conservation, the only non-vanishing way of contractions is connecting each field with its conjugate. Hence, from the table 4.2.1, one can easily see that all the Z fields (vacuum) in the operator $\mathcal{O}_3$ must be contracted with $\mathcal{O}_2$ and all the $\bar{Y}$ fields (excitations) are connected with $\mathcal{O}_1$. The number of propagators between $\mathcal{O}_3$ and the other two operators is

$$l_{13} = L_3 - M_3 , \quad l_{23} = M_3 \tag{4.2.11}$$

The contractions involving $\mathcal{O}_3$ are rather trivial due to the asymmetry of the transformation properties of three operators with respect to the symmetry group $SU(2)_L \times SU(2)_R$. To see the transformation properties, consider the following matrix composed of four complex scalars

$$\begin{pmatrix} Z & Y \\ -\bar{Y} & \bar{Z} \end{pmatrix} . \tag{4.2.12}$$

The action of $SU(2)_L$ is defined by the left multiplication and the action of $SU(2)_R$ is defined by the right multiplication. Then, it turns out that (Z, Y) and $(-\bar{Y}, \bar{Z})$, which are the constituents for $\mathcal{O}_1$ and $\mathcal{O}_2$ respectively, transform as the doublet under the action of $SU(2)_R$, while $(Z, -\bar{Y})$, which is the constituent of $\mathcal{O}_3$ transform as the doublet under the action of $SU(2)_L$. In other words, the definition of the $SU(2)$ subgroup for three operators are asymmetric in the present case. Therefore, the non-trivial contractions occur between $\mathcal{O}_1$ and $\mathcal{O}_2$. The number of contractions between them is

$$l_{12} = L_1 - M_3\,. \tag{4.2.13}$$

Moreover, the R-charge conservation constrains the total number of excitations as follows

$$M_1 = M_2 + M_3\,. \tag{4.2.14}$$

In what follows, we apply the tailoring method which was introduced in the previous section to the three-point function of $\mathcal{O}_1$, $\mathcal{O}_2$ and $\mathcal{O}_3$. We first map these operators to the corresponding eigenstates of the Hamiltonian of the spin chain, namely, the on-shell Bethe states of the XXX Heisenberg model

$$\mathcal{O}_1 \longrightarrow |\boldsymbol{u}\rangle\,, \;\; \mathcal{O}_2 \longrightarrow |\boldsymbol{v}\rangle\,, \;\; \mathcal{O}_3 \longrightarrow |\boldsymbol{w}\rangle\,, \tag{4.2.15}$$

where $\boldsymbol{u} = \{u_1, \ldots, u_{M_1}\}$ is a set of the Bethe roots that satisfy the Bethe ansatz equations for M_1 magnon excitations and the trace condition. The other two sets of the Bethe roots are the similar.

Cutting

We cut each spin chain into two open spin chains and express the Bethe states as an entangled state. Let us consider a spin chain of length L. We denote the first l spins starting from the left of the chain as the left subchain and the remaining $r = L - l$ spins as the right subchain. A Bethe state $|\boldsymbol{u}\rangle$ with M excitations in the spin chain can be represented as an entangled state of the tensor product of the left and right subchains as follows

$$|\boldsymbol{u}\rangle = \sum_{\alpha \cup \bar{\alpha}} H(\alpha, \bar{\alpha})|\alpha\rangle_l \otimes |\bar{\alpha}\rangle_r\,, \tag{4.2.16}$$

where the summation is taken over all the possible partitions of the Bethe roots $\{u_i\}$ into two subsets α, $\bar{\alpha}$ which satisfy $\alpha \cup \bar{\alpha} = \{u_i\}$. For example, in the case of $M = 2$, there are four possible partitions such as $(\alpha, \bar{\alpha}) = (\{\}, \{u_1, u_2\}), (\{u_1\}, \{u_2\}), (\{u_2\}, \{u_1\}), (\{u_1, u_2\}, \{\})$. The coefficients $H(\alpha, \bar{\alpha})$ will be determined soon later.[1]

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[1]The coefficients crucially depend of the normalization of the Bethe state. In particular, they take different forms in the coordinate Bethe ansatz framework and the algebraic Bethe ansatz framework. We will concentrate on the latter basis since the coefficients take more simple forms.

Note that this is a quite simplified form compared with the explicit spin basis such as $| \uparrow \ldots \downarrow \ldots \uparrow \rangle$. Indeed, if we expand a generic Bethe state in terms of the spin basis for the left and right chain such as $|n_1, \ldots, n_k\rangle_l \otimes |n_{k+1} - l, \ldots, n_M - l\rangle_r$, the number of partitions of excitations (down spins) is order of $L!(L - M)!/M!$, while the number of partitions for (4.2.16) is 2^M, which is much smaller if $L \gg M$. This simplification is due to the locality of the action of the Hamiltonian. Since the action of the Hamiltonian is the nearest neighbor, the form of the wave function (3.2.44) should be preserved locally. In other words, each magnon does not know whether the spin chain is closed or opened at least locally.

To show that Bethe states in the closed spin chain is actually written in the form (4.2.16) and determine the coefficients $H(\alpha, \bar{\alpha})$, let us divide the monodromy matrix for closed spin chain into the product of the left and right monodromy matrix as follows

$$\Omega(u) = \begin{pmatrix} A(u) & B(u) \\ C(u) & D(u) \end{pmatrix} = \begin{pmatrix} A_l(u) & B_l(u) \\ C_l(u) & D_l(u) \end{pmatrix} \begin{pmatrix} A_r(u) & B_r(u) \\ C_r(u) & D_r(u) \end{pmatrix}. \tag{4.2.17}$$

Notice that the components of the left and right monodromy matrix still have the same form of the exchanging relations as those of the total monodromy matrix. Furthermore, the components of the left monodromy matrix commute with the components of the right one since they are made up of local spin operators in the left and right spin chain respectively and local variables mutually commute. From the definition of the Bethe state, we find

$$|\boldsymbol{u}\rangle = \prod_{i=1}^{M} B(u_i)| \uparrow^L \rangle = \prod_{i=1}^{M} (A_l(u_i)B_r(u_i) + B_l(u_i)D_r(u_i))| \uparrow^l \rangle_l \otimes | \uparrow^r \rangle_r. \tag{4.2.18}$$

By expanding the parentheses, we obtain the 2^M terms and they are exactly the form of the summation over the partitions. To determine the coefficients, we have to commute A and D through the B's and act them on the vacuum using the exchanging relations for A, B and A, D. In this way, we have

$$H(\alpha, \bar{\alpha}) = \prod_{u_i \in \alpha, \; u_j \in \bar{\alpha}} \frac{u_i - u_j + i}{u_i - u_j} \prod_{u_k \in \alpha} \left(u_k - \frac{i}{2} \right)^r \prod_{u_l \in \bar{\alpha}} \left(u_l + \frac{i}{2} \right)^l. \tag{4.2.19}$$

Finally, the three Bethe states $|\boldsymbol{u}\rangle$, $|\boldsymbol{v}\rangle$ and $|\boldsymbol{w}\rangle$ corresponding to the single trace operators

$\mathcal{O}_1$, $\mathcal{O}_2$ and $\mathcal{O}_3$ are decomposed as follows

$$|\boldsymbol{u}\rangle = \sum_{\alpha \cup \bar{\alpha} = \{u\}} H^{\alpha \bar{\alpha}} |\alpha\rangle_l \otimes |\bar{\alpha}\rangle_r \,, \tag{4.2.20}$$

$$|\boldsymbol{v}\rangle = \prod_{k=1}^{M_2} \left(v_k - \frac{i}{2} \right)^{l_{12}} |\boldsymbol{v}\rangle_l \otimes |\uparrow^{l_{23}}\rangle_r + \cdots \,, \tag{4.2.21}$$

$$|\boldsymbol{w}\rangle = \prod_{l=1}^{M_3} \left(w_l + \frac{i}{2} \right)^{l_{13}} |\uparrow^{l_{23}}\rangle_l \otimes |\boldsymbol{w}\rangle_r + \cdots \,, \tag{4.2.22}$$

where $\cdots$ in the decomposition for $|\boldsymbol{v}\rangle$ and $|\boldsymbol{w}\rangle$ denotes the terms that are irrelevant for the computation of the structure constant, as seen from the figure 4.2.1.

Flipping

After cutting, which breaks the states into two pieces, we need to interpret the Wick contractions as an operation of the spin chain to obtain the structure constants. Since each Wick contraction maps a pair of two constituent fields of the different single trace operators into a number, the Wick contractions are represented as the scalar products involving two different subchains. Therefore, it is necessary to appropriately define a operation which converts one of the two kets of subchains into a bra (flipping) as $|\Psi\rangle_l \otimes \mathcal{F}|\Psi\rangle_r \to |\Psi\rangle_l \otimes {}_r\langle\overleftarrow{\Psi}|$, and take pairings of bras and kets for the neighboring subchains (sewing).

As already mentioned, the flipping operation $\mathcal{F}$ is not usual conjugation $\dagger$ since the conjugation flips the order of the constituents of the operator and their charges while the flipping does not change the operator. To see this, let us consider a example of the contraction between $\mathcal{O}_1$ and $\mathcal{O}_2$ which appears in the figure 4.2.1. Note that the state in the right subchain for $\mathcal{O}_1$ is not $|YY\rangle$ but $|ZZYZZYZ\rangle$. One can find the difference between the conjugation and the flipping from the following action on this state

$$\dagger \;:\; c|ZZYZZYZ\rangle \longmapsto \langle ZZYZZYZ|c^* \,, \tag{4.2.23}$$

$$\mathcal{F} \;:\; c|ZZYZZYZ\rangle \longmapsto \langle \bar{Z}\bar{Y}\bar{Z}\bar{Z}\bar{Y}\bar{Z}\bar{Z}|c \,, \tag{4.2.24}$$

Here, we have adopted the convention in which the fields (or spins) in the bras and kets are aligned from left to right. For example, $\langle Z\bar{Y}|Z\bar{Y}\rangle = 1$, while $\langle \bar{Y}Z|Z\bar{Y}\rangle = 0$ in this convention. One of the important property for the flipping is that it preserves the scalar product

$$\langle \overleftarrow{\Psi}_1 | \Psi_2 \rangle = \langle \overleftarrow{\Psi}_2 | \Psi_1 \rangle \,. \tag{4.2.25}$$

This can be explicitly checked in the spin basis such as $|ZZZY\ldots\rangle$. As a result, one can see that the contraction in the example of figure 4.2.1 gives 1. Then, we find that the action of the flipping on the coordinate Bethe ansatz basis is given by

$$\mathcal{F} : \psi(n_1,\ldots,n_M)|n_1,\ldots,n_M\rangle \mapsto \psi(n_1,\ldots,n_M)\langle L - n_M + 1,\ldots, L - n_1 + 1|\hat{C}\,, \tag{4.2.26}$$

where $\hat{C}$ is the charge conjugation, which exchange $Z \leftrightarrow \bar{Z}$ and $Y \leftrightarrow \bar{Y}$. Hence, we can compute the action of the flipping for any Bethe state in principle. For example, we find the action of the flipping on two magnon state in the coordinate basis as follows[2]

$$\mathcal{F}|u_1, u_2\rangle^{co} = \sum_{1 \le n_1 \le L} (e^{ip_1 n_1 + ip_2 n_2} + S(p_2, p_1)e^{ip_2 n_1 + ip_1 n_2})\langle L - n_2 + 1, L - n_1 + 1|\,, \tag{4.2.27}$$

By changing the labels as $m_1 = L - n_2 + 1$ and $m_2 = L - n_1 + 1$, we obtain

$$\mathcal{F}|u_1, u_2\rangle^{co} = e^{i(L+1)(p_1+p_2)}S(p_1, p_2)^{co}\langle u_1^*, u_2^*|\hat{C}\,. \tag{4.2.28}$$

Similarly, we can determine the action of the flipping on the generic Bethe states and it turns out $\mathcal{F}|\boldsymbol{u}\rangle \propto \langle \boldsymbol{u^*}|\hat{C}$. The proportional constants generally depend on the basis and rapidities. However, for the Bethe states in the algebraic Bethe ansatz basis, they become simply numerical constants which are independent of rapidities as follows [115]

$$\mathcal{F}|\boldsymbol{u}\rangle = (-1)^M \langle \boldsymbol{u^*}|\hat{C}\,. \tag{4.2.29}$$

By using the hermite conjugate property $B^\dagger(u) = -C(u^*)$, the flipping operation after cutting gives

$$\mathbf{1}_l \otimes \mathcal{F}_r|\boldsymbol{u}\rangle = \sum_{\alpha \cup \bar{\alpha}} H^{\alpha\bar{\alpha}} \prod_{u_i \in \alpha}^{M} B(u_i)|\uparrow^l\rangle_l \otimes {}_r\langle\uparrow^r| \prod_{u_j \in \bar{\alpha}} C(u_j)\hat{C}\,. \tag{4.2.30}$$

Sewing

The last procedure of the tailoring is the sewing, namely, taking the overlaps of the wave functions for the Bethe states in the neighboring subchains. From the form of the Bethe state after the cutting and flipping (4.2.30), we find that the basic ingredient in the sewing procedure is the following pairing of a bra and a ket.

$$\langle\uparrow^L| \prod_{i=1}^{M} C(v_i) \prod_{j=1}^{M} B(u_j)|\uparrow^L\rangle\,. \tag{4.2.31}$$

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[2]We denote the Bethe states in the coordinate basis as $|\ldots\rangle^{co}$ to distinguish them from the algebraic basis.

This quantity is related to the scalar product of the Bethe states in the following way

$$\langle \boldsymbol{v}|\boldsymbol{u}\rangle = (-1)^M \langle \uparrow^L | \prod_{i=1}^M C(v_i^*) \prod_{j=1}^M B(u_j)| \uparrow^L \rangle. \tag{4.2.32}$$

Here, we have used $B^\dagger(u) = -C(u^*)$. The natural quantity which appears in the tailoring procedure is rather (4.2.31) than the scalar product $\langle \boldsymbol{v}|\boldsymbol{u}\rangle$ itself. Therefore, sometimes (4.2.31) is denoted simply as $\langle \boldsymbol{v}|\boldsymbol{u}\rangle$. [3] In what follows, we follow the convention that the pairing of a Bethe state with a dual Bethe state (4.2.31) is denoted as $\langle \boldsymbol{v}|\boldsymbol{u}\rangle$ and we will refer it to as a scalar product of Bethe states. It is known that if rapidities $\{u\}$ or $\{v\}$ in (4.2.31) satisfy the Bethe ansatz equations, then the scalar product can be converted into a more simple form, which is a determinant of a matrix. We will briefly review determinant formulas when we consider the semiclassical limit of the three-point function in the next subsection.

From (4.2.20)-(4.2.22) and the general expression for the structure constant (4.2.8), we obtain

$$c_{123} = \sqrt{L_1 L_2 L_3} \frac{\sum_{\alpha \cup \bar{\alpha} = \{u\}} H^{\alpha \bar{\alpha}} d_{l_{12}}^{\boldsymbol{v}} a_{l_{13}}^{\boldsymbol{w}} \langle \overset{\leftarrow}{\bar{\alpha}} |\boldsymbol{v}\rangle_{l_{12}} \langle \overset{\leftarrow}{\boldsymbol{w}} |\alpha\rangle_{l_{13}}}{\sqrt{\langle \boldsymbol{u}|\boldsymbol{u}\rangle} \sqrt{\langle \boldsymbol{v}|\boldsymbol{v}\rangle} \sqrt{\langle \boldsymbol{w}|\boldsymbol{w}\rangle}}, \tag{4.2.33}$$

where $d_{l_{12}}^{\boldsymbol{v}} = \prod_{k=1}^{M_2}(v_k - i/2)^{l_{12}}$, $a_{l_{13}}^{\boldsymbol{w}} = \prod_{l=1}^{M_3}(w_l - i/2)^{l_{13}}$ and subscripts l_{ij} for the scalar products denote the lengths of the spin chains in which the Bethe states are defined. Furthermore, with the property of flipping $\langle \overset{\leftarrow}{\Psi}_1|\Psi_2\rangle = \langle \overset{\leftarrow}{\Psi}_2|\Psi_1\rangle$ and the action on the Bethe states (4.2.30), the above expression can be rewritten as follows

$$c_{123} = \sqrt{L_1 L_2 L_3} \frac{d_{l_{12}}^{\boldsymbol{v}} a_{l_{13}}^{\boldsymbol{w}} \langle \boldsymbol{v}|\boldsymbol{u}\rangle_{l_{12}} \langle \downarrow^{M_3} |\boldsymbol{w}\rangle_{l_{13}}}{\sqrt{\langle \boldsymbol{u}|\boldsymbol{u}\rangle} \sqrt{\langle \boldsymbol{v}|\boldsymbol{v}\rangle} \sqrt{\langle \boldsymbol{w}|\boldsymbol{w}\rangle}}, \tag{4.2.34}$$

where we have used the summation over the partitions for $|\boldsymbol{u}\rangle$ conversely. This is a quite simple form expressed in terms of the scalar products of the Bethe states. Note that the Bethe states appear in the above scalar products are *not* on-shell, since the lengths of the spin chain are changed and the periodicity conditions for each magnon are no more satisfied. We can compute the scalar product by using exchanging relations in principle. However, this brute force computation is not suitable for the analysis of the semiclassical limit, namely, the limit $L_i, M_i \to \infty$ keeping M_i/L_i finite as we will discuss in the next subsection. Fortunately, it is possible to rewrite the scalar product in (4.2.34) as a scalar

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[3]The dual Fock space is spanned by the states generated by the action of $C(v)$'s upon the dual pseudo vacuum $\langle \uparrow |$ satisfying $\langle \uparrow |B(u) = 0$ and $\langle \uparrow^L | \uparrow^L \rangle = 1$, such as $\langle \uparrow |C(u_1)\ldots C(u_M)$. Furthermore, if the rapidities $u_1, \ldots u_M$ satisfy the Bethe ansatz equation, then the state is a eigenstate of the transfer matrix $T(u)$ with eigenvalue $t_{\boldsymbol{u}}(u)$ given in (3.2.76).

product between a on-shell Bethe state and a off-shell Bethe state [123] and thus further simplifications occur.

The first key observation is that the scalar products $\langle \boldsymbol{v}|\boldsymbol{u}\rangle_{l_{12}}$ and $\langle\downarrow^{M_3}|\boldsymbol{w}\rangle_{l_{13}}$ can be recast in the following form

$$d^{\boldsymbol{v}}_{l_{12}}\langle \boldsymbol{v}|\boldsymbol{u}\rangle_{l_{12}} \to \left(_{l_{13}}\langle\downarrow^{l_{13}}| \otimes {}_{l_{12}}\langle\boldsymbol{v}|\right)|\boldsymbol{u}\rangle_{L_1}\,, \ a^{\boldsymbol{w}}_{l_{13}}\langle\downarrow^{M_3}|\boldsymbol{w}\rangle_{l_{13}} \to \left(_{l_{23}}\langle\uparrow^{l_{23}}| \otimes {}_{l_{13}}\langle\downarrow^{M_3}|\right)|\boldsymbol{w}\rangle_{L_3}\,. \tag{4.2.35}$$

This can be easily seen from the figure 4.2.1. We find that the Bethe states $|\boldsymbol{u}\rangle_{L_1}$ and $|\boldsymbol{w}\rangle_{L_3}$ become on-shell again since they are defined in the spin chains before cutting with lengths L_1 and L_3 respectively. The bras in the above expressions do not seem to be Bethe states, however, it turns out that they are written as the forms of the Bethe states with specific rapidities.

To convert the bras into such forms, let us introduce inhomogeneity parameters $\boldsymbol{\theta}^{(i)} = (\theta_1,\ldots,\theta_{L_i})$ for each site of the i-th spin chain. The second key observation is that the action of $B(\theta_1 + i/2)$ on the pseudo-vacuum flips the spin of the first site of the spin chain

$$B(\theta_1 + i/2)|\uparrow^L\rangle \propto |\downarrow\uparrow\ldots\uparrow\rangle = s_1^-|\uparrow^L\rangle\,. \tag{4.2.36}$$

Actually, any local spin variables in the XXX model can be expressed in terms of the components of the monodromy matrix. This is known to be the quantum inverse scattering problem, whose solution was first given in [122]

$$s_n^j = i\prod_{k=1}^{n-1} T(\xi_k)\mathrm{Tr}(\sigma^j\Omega(\xi_n))\prod_{l=1}^{n} T^{-1}(\xi_l)\,, \tag{4.2.37}$$

where $T(u) = A(u) + D(u)$ is the transfer matrix, σ^j $(j = 1, 2, 3)$ are the Pauli matrices and $\xi_m = \theta_m + i/2$. Notice that the order of transfer matrices is irrelevant due to the commutativity of them. The proof of this formula is based on the fact that the Lax operator $L_n(u)$ becomes the permutation operator $P_{n,0}$ at $u = \xi_n$. See [122] for details. Then, using this master formula, We find

$$|\underbrace{\downarrow\ldots\downarrow}_{l}\overbrace{\uparrow\ldots\uparrow}^{L-l}\rangle = \left(iB(\xi_1)T^{-1}(\xi_1)\right)\cdots\left(i\prod_{i=1}^{l-1} T(\xi_i)B(\xi_l)\prod_{j=1}^{l} T^{-1}(\xi_j)\right)|\uparrow^L\rangle = \prod_{i=1}^{l} B(\xi_i)|\uparrow^L\rangle\,, \tag{4.2.38}$$

where we have used the action of the transfer matrix of the vacuum $T(\xi_i)|\uparrow^L\rangle = i|\uparrow^L\rangle$ (see (3.2.64)). Due to the commutativity of the actions of $B(u)$'s, we further obtain the following expression for the excited states on the "kink pseudo-vacuum"

$$|\downarrow^l\rangle \otimes |\boldsymbol{v}\rangle_{L-l} = \prod_{i=1}^{M} B(v_i)|\downarrow^l\uparrow^{L-l}\rangle = |\boldsymbol{v}\cup\boldsymbol{\xi}\rangle_L\,, \tag{4.2.39}$$

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where $\boldsymbol{\xi} = (\xi_1, \ldots, \xi_l)$. Therefore, by taking the conjugate and combining with the result (4.2.35), we finally obtain the following expression for the structure constant

$$c_{123}^{(0)} = \lim_{\boldsymbol{\theta}^{(i)} \to 0} \sqrt{L_1 L_2 L_3} \frac{\langle \boldsymbol{v} \cup \boldsymbol{\xi} | \boldsymbol{u} \rangle_{L_1} \langle \boldsymbol{\xi} | \boldsymbol{w} \rangle_{L_3}}{\sqrt{\langle \boldsymbol{u} | \boldsymbol{u} \rangle} \sqrt{\langle \boldsymbol{v} | \boldsymbol{v} \rangle} \sqrt{\langle \boldsymbol{w} | \boldsymbol{w} \rangle}} , \qquad (4.2.40)$$

with $\boldsymbol{\xi} = (\xi_1, \ldots, \xi_{l_{13}})$. This expression is written in terms of the scalar products for on-shell Bethe states and off-shell Bethe states and thus it will play a crucial role in the context of the semiclassical analysis.

Before discussing the semiclassical analysis, let us see some examples of the expression (4.2.40). The three-point function for three-BPS operators is the simplest cases. They are protected by supersymmetry and the tree-level results hold at any coupling [102]. The multiplets of BPS operators are obtained by the action of the global symmetry on the highest weight state. In the spin chain language, they correspond to the vacuum descendants in the $SU(2)$ sense, which are obtained by setting some of rapidities of magnons to infinity, as explained in the end of section 3.2.2. The structure constant for three BPS operators is obtained in this limit and it reduces to the simply combinatorial factor as follows

$$C_{123}^{\circ\circ\circ} = \frac{\sqrt{L_1 L_2 L_3} \begin{pmatrix} l_{12} \\ M_2 \end{pmatrix} \begin{pmatrix} l_{23} \\ M_3 \end{pmatrix}}{\sqrt{\prod_{i=1}^{3} \begin{pmatrix} L_i \\ M_i \end{pmatrix}}} . \qquad (4.2.41)$$

This is actually the result obtained in [102]. Here, we denote BPS operators as $\circ$. Similarly, we will denote non-BPS operators as $\bullet$ in the following.

The next-to-simplest three-point functions involve two BPS operators and one non-BPS operator, namely, $C_{123}^{\circ\bullet\circ}$. Here, we choose $\mathcal{O}_2$ as the non-BPS operator for definiteness, hence $\mathcal{O}_2$ are mapped to the Bethe state while the other two operators correspond to vacuum descendants. In this case, the structure constant is no more protected by the supersymmetry and it depends on the M_2 rapidities that characterize $\mathcal{O}_2 \cong |\boldsymbol{v}\rangle$. For example, the simplest case is the descendant of the Bethe state of two magnons with finite opposite momenta, say $p_1 = -p_2$ being acted by the action of the lowering operator twice $(S^-)^2$. In other words, the set of rapidities is given by $\boldsymbol{v} = (\frac{1}{2} \cot p, -\frac{1}{2} \cot p, \infty, \infty)$ where p is quantized as $p = 2\pi n/(L_2 - 1)$ due to the periodicity condition, namely, Bethe ansatz equations. Then, the result is

$$c_{123}^{\circ\bullet\circ} = e^{-ip/2} \sqrt{\frac{L_1 L_2 L_3}{2 \begin{pmatrix} L_1 \\ M_1 \end{pmatrix} \begin{pmatrix} L_2 \\ 2 \end{pmatrix} \begin{pmatrix} L_3 \\ M_3 \end{pmatrix}}} \frac{\sin\left(\frac{p l_{12}}{2}\right) \sin\left(\frac{p}{2}(l_{12} - 1)\right)}{\sin^2\left(\frac{p}{2}\right)} , \qquad (4.2.42)$$

with $M_3 = l_{13}$ and $M_1 = M_3 + 2$. The semiclassical limit of the $c_{123}^{\circ\bullet\bullet}$ for generic non-BPS operator $\mathcal{O}_2$ was studied in [117], which holographically corresponds to the classical tunneling of the dual string.

The most interesting case is the three-point function with three non-BPS operators. We will discuss the semiclassical limit of such three-point functions in the next subsection.

4.2.2 Determinant formula and semi-classical limit

The main theme of this section is the semiclassical limit of the three-point functions for the $SU(2)$ sector derived in the previous section, or the structure constant (4.2.40). As with the case of spectrum or two-point functions, semiclassical limit would be expected to establish a first bridge between weak coupling and strong couplings in the case of three-point functions. For the case of the $SU(2)$ sector, such computations of the structure constant are reduced to the evaluations of the scalar product of the form

$$\langle \boldsymbol{v} | \boldsymbol{u} \rangle = \langle \uparrow^L | \prod_{i=1}^{M} C(v_i) \prod_{j=1}^{M} B(u_j) | \uparrow^L \rangle \tag{4.2.43}$$

for the $SU(2)$ XXX Heisenberg spin chain. Traditionally, however, the computation of such a product has been pursued in the framework of the algebraic Bethe ansatz reviewed in the section 3.2.2. Although the computation is conceptually quite straightforward as one simply needs to move $C(v_i)$'s all the way through $B(u_j)$'s, using the exchange algebra, and act on the pseudo-vacuum, in practice this procedure produces a multitude of terms which grow exponentially in the number of magnons and becomes intractable. Fortunately, in the case of the product between an on-shell and an off-shell Bethe states, Slavnov discovered a much more concise expression in the form of a determinant, which was to be called Slavnov's determinant formula [121]. More recently, various other types of determinant formulas have been developed, which are intimately related to the Slavnov's determinant. Since the configuration for which the Slavnov's formula is valid is precisely the one needed for the computation of the three-point functions in the super Yang-Mills theory, which is our motivation, it is of interest to sketch in advance that how these different variants of determinant formulas are directly or indirectly related to each other.

As stated above, let us consider the case where either one of the set of rapidities $\boldsymbol{u}$ or $\boldsymbol{v}$ are on-shell. For definiteness, let us take $\boldsymbol{v}$ to be on-shell. Then the original Slavnov's

formula computes the scalar product $\langle \boldsymbol{v}|\boldsymbol{u}\rangle$ as a $M \times M$ determinant of the form

$$
\begin{aligned}
\langle \boldsymbol{v}|\boldsymbol{u}\rangle = &\frac{\prod_{i=1}^{M} Q_\theta^+(u_i) Q_\theta^-(v_i)}{\prod_{i<j}(u_i - u_j)(v_j - v_i)} \\
&\times \det \left(\frac{1}{u_m - v_n} \left(\prod_{k\neq n}^{M}(u_m - v_k - i) - \prod_{k\neq n}^{M}(u_m - v_k + i) \prod_{l=1}^{L} \frac{u_m - \theta_l - \frac{i}{2}}{u_m - \theta_l + \frac{i}{2}} \right) \right)_{1\leq m,n\leq M}.
\end{aligned}
$$

$$(4.2.44)$$

Very recently, Kostov and Matsuo [134] showed that this expression is equivalent to an alternative determinantal expression of the form

$$
\langle \boldsymbol{v}|\boldsymbol{u}\rangle = (-1)^M Z^{\mathrm{KM}}(\boldsymbol{z}|\boldsymbol{\theta}), \quad \boldsymbol{z} \equiv \boldsymbol{u} \cup \boldsymbol{v} \tag{4.2.45}
$$

where $Z^{\mathrm{KM}}(\boldsymbol{z}|\boldsymbol{\theta})$ is now a $2M \times 2M$ determinant given by

$$
Z^{\mathrm{KM}}(\boldsymbol{z}|\boldsymbol{\theta}) = \frac{\prod_{i=1}^{2M} Q_\theta^-(z_i)}{\prod_{i<j}(z_i - z_j)}\det \left(z_m^{n-1} - \prod_{l=1}^{L} \frac{z_m - \theta_l + i/2}{z_m - \theta_l - i/2}(z_m + i)^{n-1} \right)_{1\leq m,n\leq 2M}.
$$

$$(4.2.46)$$

They also pointed out that this equivalence is due essentially to the following equality valid when $\boldsymbol{u}$ or $\boldsymbol{v}$ are on-shell:

$$
\langle \uparrow^L | \prod_{i=1}^{M} C(v_i) \prod_{j=1}^{M} B(u_j) | \uparrow^L \rangle \propto \langle \downarrow^L | (S^-)^{L-2M} \prod_{i=1}^{M} B(v_i) \prod_{j=1}^{M} B(u_j) | \uparrow^L \rangle. \tag{4.2.47}
$$

Intuitively this can be understood in the following way. Suppose the set of rapidities $\boldsymbol{v}$ are on-shell. Then the Bethe state $\prod_{i=1}^{M} B(v_i) | \uparrow^L \rangle$ built on the up vacuum is the highest weight state of global $SU(2)$ with spin $\frac{L}{2} - M$. On the other hand, the state $\prod_{i=1}^{M} C(v_i) | \downarrow^L \rangle$ generated by the action of $C(v)$ on the down pseudo-vacuum has the same eigenvalue for the transfer matrix $T(u)$. Generally, an on-shell state corresponding to the same solution of the Bethe ansatz equations is expected to belong to the same $SU(2)$ multiplet. Since $\prod_{i=1}^{M} C(v_i) | \downarrow^L \rangle$ is a lowest weight state with spin $-\frac{L}{2} + M$, we can make it into the highest weight state with spin $\frac{L}{2} - M$ by the action of $(S^+)^{L-2M}$. Therefore we should have the equality

$$
\prod_{i=1}^{M} B(v_i) | \uparrow^L \rangle \propto (S^+)^{L-2M} \prod_{i=1}^{M} C(v_i) | \downarrow^L \rangle. \tag{4.2.48}
$$

Taking the conjugate of this relation, we obtain (4.2.47).

Now it turns out that there is another variant of the determinant formula, found by Foda and Wheeler [133]. They showed that the Kostov-Matsuo expression $Z^{\mathrm{KM}}(\boldsymbol{z})$

can be identified with the so-called partial domain wall partition function (pDWPF) $Z^{\mathrm{pDWPF}}(\boldsymbol{z}|\boldsymbol{\theta})$, which naturally arises in the context of the six vertex model:

$$
Z^{\mathrm{pDWPF}}(\boldsymbol{z}|\boldsymbol{\theta}) = \frac{\prod_{\alpha=1}^{2M} Q_\theta^+(z_\alpha) Q_\theta^-(z_\alpha)}{\prod_{\alpha<\beta}(z_\alpha - z_\beta) \prod_{i<j}(\theta_j - \theta_i)}
$$

$$
\times \det \begin{pmatrix}
\frac{i}{(z_1-\theta_1+i/2)(z_1-\theta_1-i/2)} & \cdots & \frac{i}{(z_1-\theta_L+i/2)(z_1-\theta_L-i/2)} \\
\vdots & & \vdots \\
\frac{i}{(z_{2M}-\theta_1+i/2)(z_{2M}-\theta_1-i/2)} & \cdots & \frac{i}{(z_{2M}-\theta_L+i/2)(z_{2M}-\theta_L-i/2)} \\
\theta_1^{L-2M-1} & \cdots & \theta_L^{L-2M-1} \\
\vdots & & \vdots \\
\theta_1^0 & \cdots & \theta_L^0
\end{pmatrix}.
$$

$$(4.2.49)$$

The forms of these determinant formulas are rather simple and they hold for any size of the spin chain L and the number of excitations M. Thus, the semiclassical analysis has been heavily dependent on such determinant formulas. However, the semiclassical analysis based on the determinant formula still involves technical calculations, hence we will briefly summarize the results of the semiclassical analysis without derivations and give physical interpretations for them in the rest of this subsection. The more comprehensive semiclassical analysis based on the physical picture should be worked out. As a first step toward such an understanding, a new expression for the scalar products of the Bethe states was derived in [136]. This is given as follows

$$
Z^{\mathrm{pDWPF}}(\boldsymbol{z}|\boldsymbol{\theta})
$$
$$
\propto \prod_{n=1}^{L-1} \oint_{\mathcal{C}_{\mathrm{all}}} \frac{dx_n}{2\pi i} \prod_{k<l}(x_k - x_l)(e^{2\pi x_k} - e^{2\pi x_l}) \prod_{m=1}^{L-1} \frac{Q_{\boldsymbol{u}}(x_m) Q_{\boldsymbol{v}}(x_m) e^{2\pi x_m}}{Q_{\boldsymbol{\theta}}^+(x_m) Q_{\boldsymbol{\theta}}^-(x_m)}, \qquad (4.2.50)
$$

where Q-functions are defined in section 3.2.2 and the integration contour contains all the poles coming from the denominator of integrand, namely, $\{\theta_k \pm i/2\}$.

The most interesting semiclassical limit for the three-point functions is those of heavy-heavy-heavy correlators involving three large non-BPS operators, which holographically describe the interaction of three classical strings. Such heavy-heavy-heavy three-point functions are first derived in [124, 125] and holographic three-point functions for three classical strings are studied in [108–111].

The derivation of the structure constants for three classical operators $c_{123}^{\bullet\bullet\bullet}$ in [124, 125] is based on (4.2.40) and the determinant formula for the scalar product of the Bethe states. The outline of the derivation is as follows. First, the determinant formula for the scalar product of an on-shell Bethe state and an off-shell Bethe is converted into a

correlator of free fermions on the complex plane. Then, it is bosonized and written in terms of a functional depending on product of two operators. In the semiclassical limit, the argument in the functional can be replaced with the classical values for operators and the result is obtained from the evaluation of such a functional. The main result is the following

$$\log\langle \boldsymbol{v}|\boldsymbol{u}\rangle \sim \oint_{A_{\boldsymbol{u}}+A_{\boldsymbol{v}}} \frac{dx}{2\pi} \mathrm{Li}_2(e^{iq(x)})\,, \tag{4.2.51}$$

$$q(x) := G_{\boldsymbol{u}}(x) + G_{\boldsymbol{v}}(x) - G_{\boldsymbol{\theta}}(x)\,, \tag{4.2.52}$$

where $\boldsymbol{u}$ or $\boldsymbol{u}$ satisfy the Bethe ansatz equations, $G_{\boldsymbol{u}}$, $G_{\boldsymbol{v}}$ and $G_{\boldsymbol{u}}$ are the resolvents for $\boldsymbol{u}$, $\boldsymbol{v}$ and $\boldsymbol{\theta}$ respectively.[4] With this formula of the scalar product, (4.2.40) and setting all inhomogeneity parameters to zero, the tree-level structure constant of three heavy operators is given as follows

$$\log c_{123}^{\bullet\bullet\bullet} \simeq \oint_{A_{\boldsymbol{u}}\cup A_{\boldsymbol{v}}} \frac{dx}{2\pi} \mathrm{Li}_2(e^{ip_{\boldsymbol{u}}(x)+ip_{\boldsymbol{v}}(x)+iL_3/2x}) + \oint_{A_{\boldsymbol{w}}} \frac{dx}{2\pi} \mathrm{Li}_2(e^{ip_{\boldsymbol{w}}(x)-i(L_2-L_1)/2x})$$

$$-\frac{1}{2}\oint_{A_{\boldsymbol{u}}} \frac{dx}{2\pi} \mathrm{Li}_2(e^{2ip_{\boldsymbol{u}}(x)}) - \frac{1}{2}\oint_{A_{\boldsymbol{v}}} \frac{dx}{2\pi} \mathrm{Li}_2(e^{2ip_{\boldsymbol{v}}(x)}) - \frac{1}{2}\oint_{A_{\boldsymbol{w}}} \frac{dx}{2\pi} \mathrm{Li}_2(e^{2ip_{\boldsymbol{w}}(x)})\,. \tag{4.2.53}$$

4.3 Recent non-perturbative development

In this section, we shall explain the recently developed non-perturbative approach for the three-point function so-called the hexagon bootstrap program [153]. The main idea is that the three-point function or structure constant can be calculated from new elementary constituents which geometrically resemble hexagonal patches. The new building block can be regarded as a form factor of the underlying integrable model and an integrable bootstrap approach combined with the residual symmetry for the three-point function uniquely determine the form factor. The rest of this section is organized as follows. We will first see that how the hexagons appear from the structure constant by cutting a pair of pants. Then, we consider the symmetry of such hexagon vertices, which will play a crucial role in the determination of it. In the subsequent subsection, it turns out that the hexagon can be completely fixed using the symmetry of the hexagon and some integrable bootstrap axioms. Finally, we will discuss several checks for the conjecture.

4.3.1 Basic idea

From a string theory point of view, the planar three-point function is identified as a pair of pants, namely, the sphere with three punctures. Here, it would be helpful to recall that

---

[4]For example, $G_{\boldsymbol{u}} = \sum_{i=1}^{M} \frac{1}{x-u_i}$, $G_{\boldsymbol{\theta}} = \sum_{i=1}^{L} \frac{1}{x-\theta_i}$. Note that the argument x is not rescaled. If we rescale as $x \to Lx$, they are reduced to the form defined in 3.4.1.
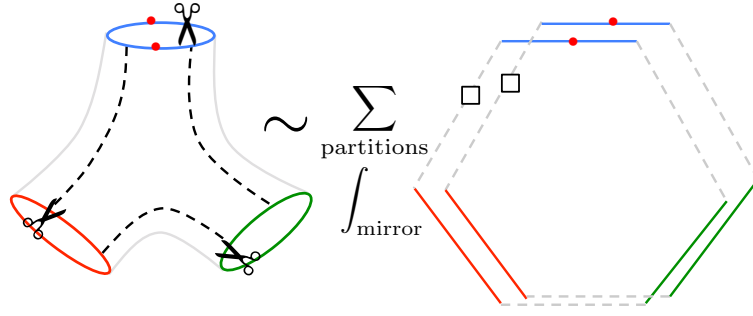
Figure 4.3.1: A pair of pants can be cut into two hexagons. There are magnon excitations on the physical edges, which are decorated by colors and there are also excitations on the mirror edges, which are represented by dashed lines. The resulting expression is a sum over all possible partitions for physical excitations and involves integrals of the momenta for mirror excitations.

a closed string amplitude is a square of open string amplitude under certain situations. With this idea in mind, it is quite natural to cut a pair of pants in such a way that the resulting two planes are hexagons (figure 4.3.1). In other words, a pair of pants is obtained by sewing two hexagons.

We shall elaborate the idea and see that this naive idea actually works in what follows. If we cut the pair of pants into two hexagons, three of the six edge of each hexagon correspond to open strings or the half of the original strings and the rest three are new segments, which we call the mirror edges. As for the physical edge, there are physical magnon excitations since the original closed string states have magnons whose rapidities satisfy the quantization condition. There are many possibilities that each magnon excitations is put on one hexagon or the other and we need to take the summation over these partitions. On the other hand, we find mirror particles exist on the mirror edge as with the case for the spectrum problem. For these mirror particles, we should integrate over the rapidities of them in addition to sum over all the possible types of excitations. This is a natural operation once we regard the decomposition as an insertion of the complete sets of the excitations of the underlying world-sheet theory on each segment of the cutting line. In summary, we introduce two hexagons by cutting a pair of pants and the resulting hexagons have physical particles and mirror particles on each edge. In other words, the structure constant can be reproduced by summing all the possible intermediate mirror particles and all the possible partitions of physical particles.

In order to proceed the analysis, it is necessary to define the hexagon in a more rigorous way. For this purpose, it is useful to first take a infinite length limit and consider

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three-point functions in the asymptotic regime, namely, three-point functions with infinite bridge lengths $l_{ij} = (L_i + L_j - L_k)/2 \to \infty$. Let us recall that the asymptotic spectrum is completely determined by the asymptotic Bethe ansatz due to the power of the residual symmetry. Furthermore, the finite size correction can be incorporated and understood by taking into account the interactions between physical particles and mirror particles. Since there are no mirror particles and the only non-vanishing contribution is the vacuum in the mirror edge, it would be expected that the asymptotic regime is a good starting point as well in the case of the three-point function. We shall soon comment on the finite size effect for the hexagon. In the asymptotic regime, it is possible to identify such a hexagon configuration with a form factor defined between asymptotic states on an infinite volume

$$\langle \mathcal{O}_1 | \hat{\mathfrak{h}}(|\mathcal{O}_2\rangle \otimes |\mathcal{O}_3\rangle) \,. \tag{4.3.1}$$

Actually, from a classical world-sheet point of view, the hexagon can be seen as a configuration with a defect operator creating a conical singularity with an excess angle π. Each asymptotic state is identified with a Bethe state whose magnons carry the bi-fundamental representation and rapidities satisfying the periodicity condition. To obtain a more manageable expression, it is of use to regard the above as a overlap between a hexagon vertex and the three asymptotic states:

$$\langle \mathfrak{h} | (|\mathcal{O}_1\rangle \otimes |\mathcal{O}_2\rangle \otimes |\mathcal{O}_3\rangle) \,. \tag{4.3.2}$$

One might afraid that it is impossible to calculate this quantity as we just replace the problem with the hexagon vertex and we do not know the precise definition of it. However, due to the fact that generic three-point functions with three half BPS operators possess the residual symmetry and generic non-BPS three-point functions are obtained by adding magnon excitations upon such a maximally symmetric correlator, it turns out that the hexagon vertex is invariant under the residual symmetry generators

$$\langle \mathfrak{h} | \mathcal{J} = 0 \,. \tag{4.3.3}$$

Combined with integrable bootstrap method, this symmetry condition severely constrains and uniquely determines the hexagon form factor, as with the case of the S-matrix in the spectrum problem. We will investigate what the residual symmetry is and solve the constraints in the next subsequent subsections.

We should comment on the finite size effect or the contributions from mirror particles. As already mentioned, we can neglect mirror particles in the asymptotic regime, $l_{ij} \to \infty$. It should be noticed that $L_i \to \infty$ is not sufficient as there are two types of the finite size effects. One is the usual wrapping effects for each operator, which can be identified with

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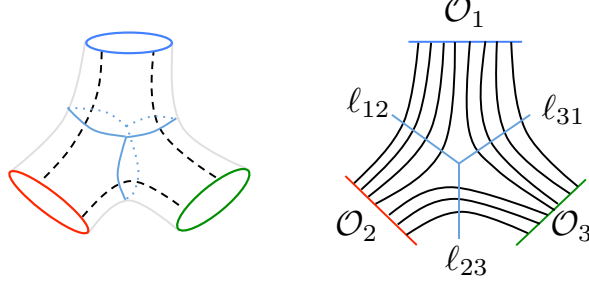
Figure 4.3.2: The new wrapping effects occur due to the propagations of mirror particles from a hexagon to another. The order of the new effects begin at $\mathcal{O}(g^{2\ell_{ij}})$, while usual wrappings for operators begin at $\mathcal{O}(g^{2L_i})$.

a contribution coming from particles circulating around the cylinder. On the other hand, the other finite size effect is peculiar to the three-point functions and it originates from the mirror particles which transit from one hexagon to the other with the propagation length l_{ij} (figure 4.3.2). Due to the presence of this new effect, there might arise the finite size corrections [5].

We shall next consider the form factor in a finite volume theory in order to include the finite size effects. To put it differently, it is necessary to deal the hexagons with mirror excitations. Such hexagons with mirror excitations are related with those without excitations on the mirror edges by the so-called mirror transformation. The mirror transformation is operatively defined by the analytical continuation for a rapidity so that the roles of momentum and energy are exchanged. This is achieved by continuing a rapidity across the branch cut of $x^+(u)$ and then going back to the original value of u. As a result, the Zhukovski variables change $x^+ \to 1/x^+$, $x^- \to x^-$ and thus the momentum and energy are exchanged $(p, E) \to (iE, ip)$. Since the mirror theory is originally defined by the double Wick rotation and the time and space are interchanged, excitations on the physical edge are mapped to those of the neighboring mirror direction under the mirror transformation (figure 4.3.3). We can also apply the mirror transformation several times. The transformation rules are summarized as follows

$$
\begin{aligned}
\gamma &: u \to u^{\gamma}, & x^+ &\to 1/x^+, & x^- &\to x^-, \\
2\gamma &: u \to u^{2\gamma}, & x^+ &\to 1/x^+, & x^- &\to 1/x^-, \\
3\gamma &: u \to u^{3\gamma}, & x^+ &\to 1/x^+, & x^- &\to x^-, \\
4\gamma &: u \to u^{4\gamma}, & x^+ &\to x^+, & x^- &\to x^-.
\end{aligned}
\tag{4.3.4}
$$

Twice the mirror transformation 2γ is nothing but the crossing transformation exchanging

---

[5]For example, $l_{ij} = 0$ even for large operators for the extremal correlators.
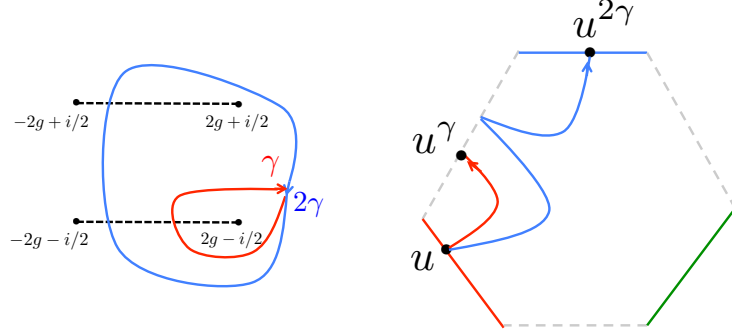
Figure 4.3.3: The mirror transformations are realized by the analytical continuation through the cuts between $(-2g \pm i/2, 2g \pm i/2)$. By applying the mirror transformation several times, an excitation on some edge can be moved to another edge.

particles and anti-particles. It should be noticed that 4γ is not trivial as it maps a excitation on one edge to another edge in the case of hexagon [6]. Successively applying the mirror transformations, it is possible to gather all the excitations including both physical and mirror ones into the top of a single physical edge. Hence, starting from the most general hexagon form factor, which has six sets of excitations carrying rapidities, we obtain a more simple hexagon whose excitations are collected on a single physical edge by using the mirror transformations repeatedly [7]. We call it as the fundamental hexagon form factor and defined as

$$\mathfrak{h}_{A_1\dot{A}_1,\ldots,A_N\dot{A}_N}(u_1,\ldots,u_N) = \langle\mathfrak{h}|(|\chi_{A_1\dot{A}_1}(u_1)\ldots\chi_{A_N\dot{A}_N}(u_N)\rangle_1 \otimes |0\rangle_2 \otimes |0\rangle_3) \,. \qquad (4.3.5)$$

In what follows, we denote the fundamental hexagon as $\langle\mathfrak{h}|\chi_{A_1\dot{A}_1}\ldots\chi_{A_N\dot{A}_N}\rangle$. The goal of the next two subsections is to explain how the symmetry and integrability work for the determination of the above building block. However, it would be useful to write down the explicit form of it here.

$$\mathfrak{h}_{A_1\dot{A}_1,\ldots,A_N\dot{A}_N} = (-1)^{\mathfrak{f}} \prod_{i<j} h_{ij} \langle\mathcal{X}^N_{\dot{A}_N}\ldots\mathcal{X}^1_{\dot{A}_1}|\mathcal{S}|\mathcal{X}^1_{A_1}\ldots\mathcal{X}^N_{A_N}\rangle \,, \qquad (4.3.6)$$

where $(-1)^{\mathfrak{f}} = (-1)^{\sum_{i<j} f_i \dot{f}_i}$ is a sign factor involving fermion gradings $(f_i, \dot{f}_i)$. The matrix structure for the indices are completely fixed by the multi-particle $SU(2|2)$ S-matrix and it is factorized into the two-body ones, which are explicitly given in the appendix C by

---

[6]Even for the dressing phase, it has a non-trivial monodromy under 4γ.

[7]The same idea is employed for the pentagon transition [163], which is a fundamental building block for the scattering amplitude or the null polygonal Wilson loop in $\mathcal{N} = 4$ SYM.

setting $S_{12}^0 = 1$. The overall scalar factor is

$$h_{12} = \frac{x_1^- - x_2^-}{x_1^- - x_2^+} \frac{1 - 1/(x_1^- x_2^+)}{1 - 1/(x_1^+ x_2^+)} \frac{1}{\sigma_{12}}\,, \tag{4.3.7}$$

with the BES dressing phase σ_{12}.

4.3.2 Symmetry of the hexagon factor

In this subsection, we will study the symmetry of the hexagon vertex. For this purpose, let us recall that we study the spectrum problem starting with the two-point function of the half BPS operators:

$$\langle \mathrm{Tr} Z^L(0) \mathrm{Tr} \bar{Z}^L(\infty) \rangle\,. \tag{4.3.8}$$

Once we choose the vacuum of this form, the $\mathfrak{psu}(2,2|4)$ symmetry is broken down to $\mathfrak{su}(2|2)_L \oplus \mathfrak{su}(2|2)_R$, whose generators are

$$\mathfrak{psu}(2|2)_L : \quad L_\alpha{}^\beta\,,\; R_a{}^b\,,\; Q_\alpha^a\,,\; S_a^\alpha\,, \tag{4.3.9}$$

$$\mathfrak{psu}(2|2)_R : \quad L_{\dot\alpha}{}^{\dot\beta}\,,\; R_{\dot a}{}^{\dot b}\,,\; Q_{\dot\alpha}^{\dot a}\,,\; S_{\dot a}^{\dot\alpha}\,, \tag{4.3.10}$$

$$\mathbb{R}^3 : \quad P\,,\; K\,,\; C\,. \tag{4.3.11}$$

The algebra is summarized in the appendix C. The starting point for the three-point functions involves the BPS operators of the

$$\mathcal{O}_i(x_i) = \mathrm{Tr}(\vec{Y}_i \cdot \Phi)^{L_i}(x_i)\,, \tag{4.3.12}$$

where the polarization vectors for the R-symmetry indices $\vec{Y}_i$ are the six components null vectors satisfying $\vec{Y}_i \cdot \vec{Y}_i = 0$. By using the superconformal transformation, it is helpful to set the positions of the operators along a line and adjust the R-symmetry polarizations. For example, let us take the following configurations:

$$\mathcal{O}_1(x_1) = \mathrm{Tr}(\Phi_1 + i\Phi_2)^{L_1}(x_1)|_{x_1=(0,0,0,0)}\,, \tag{4.3.13}$$

$$\mathcal{O}_2(x_2) = \mathrm{Tr}(\Phi_1 + i\Phi_3)^{L_2}(x_2)|_{x_2=(1,0,0,0)}\,, \tag{4.3.14}$$

$$\mathcal{O}_3(x_3) = \mathrm{Tr}(\Phi_1 - i\Phi_2)^{L_3}(x_3)|_{x_3=(\infty,0,0,0)}\,. \tag{4.3.15}$$

In this case, the Bosonic symmetry is manifest as it is given by $O(3)_{\mathrm{rotation}} \times O(3)_{\mathrm{R-sym}}$, whose first $O(3)$ is the rotation around the line and the second $O(3)$ is the rotation among the other scalar components (Φ_4, Φ_5, Φ_6). Actually, the full symmetry of the above configuration turns out be $PSU(2|2)_D$ symmetry, which is a natural supersymmetric extension

of $O(3)_{\text{rotation}} \times O(3)_{\text{R−sym}}$ and the diagonal subgroup of $PSU(2|2)_L \times PSU(2|2)_R$. To clarify the residual symmetry, it is useful to employ the "twisted-translation" defined by

$$\mathfrak{Z}(a) = e^{\mathcal{T}a}Z(0)e^{-\mathcal{T}a} = (Z + \kappa^2 a^2 \bar{Z} + \kappa a(Y - \bar{Y}))(0, a, 0, 0)\,, \tag{4.3.16}$$

$$\mathcal{T} = -i\epsilon_{\alpha\dot{\alpha}}P^{\alpha\dot{\alpha}} + \kappa\epsilon_{\dot{a}a}R^{a\dot{a}}\,, \tag{4.3.17}$$

where κ is a constant which counts the R-charge. The three-point function of the twisted-translated BPS operators is just $\kappa^{L_1+L_2+L_3}$. Among the generators of $\mathfrak{su}(2|2)_L \oplus \mathfrak{su}(2|2)_R$, the following generators commute with T and thus they preserve the three-point function

$$\mathcal{L}_\alpha{}^\beta = L_\alpha{}^\beta + \bar{L}_{\dot{\alpha}}{}^{\dot{\beta}}\,, \quad \mathcal{R}_a{}^b = R_a{}^b + \bar{R}_{\dot{a}}{}^{\dot{b}}\,, \tag{4.3.18}$$

$$\mathcal{Q}_\alpha^a = Q_\alpha^a + i\kappa\epsilon_{\alpha\dot{\beta}}\epsilon^{a\dot{b}}\bar{S}_{\dot{b}}^{\dot{\beta}}\,, \quad \mathcal{S}_a^\alpha = S_a^\alpha + \frac{i}{\kappa}\epsilon_{a\dot{b}}\epsilon^{\alpha\dot{\beta}}\bar{Q}_{\dot{\beta}}^{\dot{b}}\,, \tag{4.3.19}$$

$$\mathcal{P} = P - \kappa^2 K\,, \quad \mathcal{K} = K - \frac{1}{\kappa^2}P\,, \tag{4.3.20}$$

whose algebra is nothing but $\mathfrak{su}(2|2)$.

Non-BPS three-point functions can be obtained by adding magnon excitations on each twisted-translated BPS operator. Put it differently, we start with the non-BPS operators defined at $x = 0$ upon Z vacuum, and then perform the twisted-translation

$$\mathcal{O}_i(0) = \text{Tr}(ZZ \ldots \chi \ldots) \xrightarrow{\mathcal{T}} \mathfrak{O}_i(a_i) = e^{\mathcal{T}a_i}\mathcal{O}_i(0)e^{-\mathcal{T}a_i}\,. \tag{4.3.21}$$

Due to the highest weight condition, which is justified for on-shell Bethe states, the dependence of the twist parameter can be uniquely fixed by the Ward identities as follows

$$\langle \mathfrak{O}_1(a_1)\mathfrak{O}_2(a_2)\mathfrak{O}_3(a_3) \rangle = \frac{\kappa^{J_1+J_2+J_3}\mathbb{C}_{123}}{(a_1 - a_2)^{\delta_{12|3}}(a_2 - a_3)^{\delta_{23|1}}(a_3 - a_1)^{\delta_{31|2}}}\,. \tag{4.3.22}$$

Here, J_i is the R-charge and $\delta_{ij|k}$ is given by $\delta_{ij|k} = 2(C_i + C_j - C_k)$ with $C_i = \frac{1}{2}(\Delta_i - J_i)$. The hexagon formalism computes the "structure constant" $\mathbb{C}_{123}$. It should be noticed that $\mathbb{C}_{123}$ is completely independent of a_i's and it contains all the other tensor structures involving the space-time or R-symmetry polarizations.

4.3.3 Integrable bootstrap for the hexagon

In this section, we investigate how the symmetry and integrability techniques constrain the fundamental hexagon form factor (4.3.5). As a simple exercise, we first consider the 1-particle hexagon:

$$\mathfrak{h}_{A\dot{A}} = \langle \mathfrak{h}|\chi_{A\dot{A}} \rangle\,. \tag{4.3.23}$$
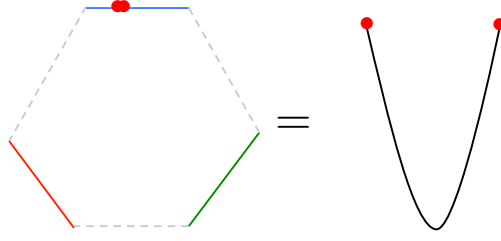
Figure 4.3.4: The 1-particle hexagon is proportional to the invariant tensor. The right figure shows the contraction of $SU(2|2)_L$ and $SU(2|2)_R$ indices.

Using the singlet condition of the form, $\langle \mathfrak{h}| \mathcal{L} = \langle \mathfrak{h}| \mathcal{R} = 0$, it turns out that it should be proportion to the $\mathfrak{su}(2|2)_D$ invariant tensor

$$\mathfrak{h}_{A\dot{A}} \propto \epsilon_{A\dot{A}}, \tag{4.3.24}$$

and the overall normalization factor can be fixed to unity. As a result, the non-vanishing 1-particle hexagon involves the so-called longitudinal magnons, namely, the two scalars $\Phi_{1\dot{2}} = Y, \Phi_{2\dot{1}} = \bar{Y}$, and the two derivatives $\mathcal{D}_{1\dot{2}} = D, \mathcal{D}_{2\dot{1}} = \bar{D}$, whose polarizations are along the three-point function. A convenient interpretation for the 1-particle result is to regard it as a $\mathfrak{su}(2|2)_D$ invariant scalar product (see also 4.3.4)

$$\langle \mathcal{X}_{\dot{A}} | \mathcal{X}_A \rangle = \mathfrak{h}_{A\dot{A}}, \tag{4.3.25}$$

which is consistent with (4.3.6).

Next, let us consider the 2-particle hexagon:

$$\langle \mathfrak{h}| \chi_{A\dot{A}} \chi_{B\dot{B}} \rangle. \tag{4.3.26}$$

As with the case of the 2-particle hexagon, the symmetry constraint of the bosonic generators $\langle \mathfrak{h}| \mathcal{J} = 0$ fixes the hexagon as follows

$$\langle \mathfrak{h}| \Phi^1_{a\dot{a}} \Phi^2_{b\dot{b}} \rangle = \mathcal{A}_{12} \epsilon_{ab} \epsilon_{b\dot{b}} + \frac{1}{2} (\mathcal{A}_{12} - \mathcal{B}_{12}) \epsilon_{ab} \epsilon_{\dot{a}\dot{b}}, \tag{4.3.27}$$

$$\langle \mathfrak{h}| \Phi^1_{a\dot{a}} \mathcal{D}^2_{\beta\dot{\beta}} \rangle = \mathcal{G}_{12} \epsilon_{a\dot{a}} \epsilon_{\beta\dot{\beta}}, \quad \langle \mathfrak{h}| \mathcal{D}^1_{\alpha\dot{\alpha}} \Phi^2_{b\dot{b}} \rangle = \mathcal{G}_{12} \epsilon_{\alpha\dot{\alpha}} \epsilon_{b\dot{b}}, \tag{4.3.28}$$

$$\langle \mathfrak{h}| \mathcal{D}^1_{\alpha\dot{\alpha}} \mathcal{D}^2_{\beta\dot{\beta}} \rangle = \mathcal{D}_{12} \epsilon_{\alpha\dot{\beta}} \epsilon_{\beta\dot{\alpha}} + \frac{1}{2} (\mathcal{D}_{12} - \mathcal{E}_{12}) \epsilon_{\alpha\beta} \epsilon_{\dot{\alpha}\dot{\beta}}, \tag{4.3.29}$$

$$\langle \mathfrak{h}| \Psi^1_{a\dot{\alpha}} \Psi^2_{b\dot{b}} \rangle = \frac{1}{2} \mathcal{C}_{12} \epsilon_{ab} \epsilon_{\dot{\alpha}\dot{\beta}}, \quad \langle \mathfrak{h}| \Psi^1_{a\dot{\alpha}} \Psi^2_{\beta\dot{b}} \rangle = \mathcal{H}_{12} \epsilon_{a\dot{b}} \epsilon_{\beta\dot{\alpha}}, \tag{4.3.30}$$

$$\langle \mathfrak{h}| \Psi^1_{\alpha\dot{a}} \Psi^2_{b\dot{\beta}} \rangle = \mathcal{K}_{12} \epsilon_{b\dot{a}} \epsilon_{\alpha\dot{\beta}}, \quad \langle \mathfrak{h}| \Psi^1_{\alpha\dot{a}} \Psi^2_{\beta\dot{b}} \rangle = \frac{1}{2} \mathcal{F}_{12} \epsilon_{\dot{a}\dot{b}} \epsilon_{\alpha\beta}, \tag{4.3.31}$$

Figure 4.3.5: As shown in the right figure, the 2-particle hexagon is essentially the Beisert's $\mathfrak{su}(2|2)$ invariant S-matrix, up to a scalar factor h_{12}.

where the coefficients $\mathcal{A}_{12}, \ldots, \mathcal{L}_{12}$ are determined using the invariance under the fermionic generators up to an overall factor[8]

$$\mathcal{A}_{12} = h_{12}A_{12}, \quad \mathcal{B}_{12} = h_{12}B_{12}, \tag{4.3.32}$$

$$\mathcal{G}_{12} = h_{12}G_{12}, \quad \mathcal{L}_{12} = h_{12}L_{12}, \tag{4.3.33}$$

$$\mathcal{D}_{12} = -h_{12}D_{12}, \quad \mathcal{E}_{12} = -h_{12}E_{12}, \tag{4.3.34}$$

$$\mathcal{C}_{12} = -h_{12}z^{-1}C_{12}, \quad \mathcal{F}_{12} = -zh_{12}F_{12}, \tag{4.3.35}$$

$$\mathcal{K}_{12} = h_{12}K_{12}, \quad \mathcal{H}_{12} = -h_{12}H_{12}. \tag{4.3.36}$$

Here, z satisfies $z^2 = -\frac{\kappa^2}{\zeta^2}e^{-ip}$ [9] with $p = p_1 + p_2$ and the coefficients $A_{12}, \ldots L_{12}$ are the S-matrix elements whose scalar factor is set to unity $S_{12}^0 = 1$ (see appendix C). Combined with the action of the S-matrix and the inner product (4.3.25), the results are simply summarized as follows (figure 4.3.5)

$$\mathfrak{h}_{A\dot{A},B\dot{B}} = (-1)^{\dot{f}_1 f_2} h_{12} \langle \mathcal{X}_{\dot{B}}^2 \mathcal{X}_{\dot{A}}^1 | \mathcal{S}_{12} | \mathcal{X}_A^1 \mathcal{X}_B^2 \rangle. \tag{4.3.37}$$

For example, two-scalar hexagon $\mathfrak{h}_{a\dot{a},b\dot{b}}$ is

$$\begin{aligned}
\mathfrak{h}_{a\dot{a},b\dot{b}}/h_{12} &= \langle \phi_{\dot{b}}^2 \phi_{\dot{a}}^1 | \mathcal{S}_{12} | \phi_a^1 \phi_b^2 \rangle \\
&= \langle \phi_{\dot{b}}^2 \phi_{\dot{a}}^1 | (A_{12}|\phi_{\{a}^2 \phi_{b\}}^1\rangle + B_{12}|\phi_{[a}^2 \phi_{b]}^1\rangle + \frac{1}{2}C_{12}\epsilon_{ab}\epsilon^{\alpha\beta}|Z^-\psi_\alpha^2\psi_\beta^1\rangle) \\
&= A_{12}\epsilon_{ab}\epsilon_{b\dot{a}} + \frac{1}{2}(A_{12} - B_{12})\epsilon_{ab}\epsilon_{\dot{a}\dot{b}}.
\end{aligned} \tag{4.3.38}$$

---

[8]For generic normalization $\mathfrak{h}_{a\dot{a}} = \epsilon_{a\dot{a}}, \mathfrak{h}_{\alpha\dot{\alpha}} = N\epsilon_{\alpha\dot{\alpha}}$, each component is multiplied by $N_i = \frac{\kappa(x^+-x^-)}{\eta\dot{\eta}}$ if $\mathfrak{psu}(2|2)_R$ index for $\chi_{A\dot{A}}^i$ carry a fermionic index.

[9]From the central charge condition $\langle \mathfrak{h}|\mathcal{P} = 0$, we find $0 = g\zeta(1-e^{ip})\langle\mathfrak{h}|Z^+\chi\rangle - g\frac{\kappa^2}{\zeta}\langle\mathfrak{h}|Z^-\chi\rangle$. Assuming $\langle\mathfrak{h}|Z^n\chi\rangle = z^n\langle\mathfrak{h}|\chi\rangle$, we find $z^2 = -\frac{\kappa^2}{\zeta^2}e^{-ip}$.
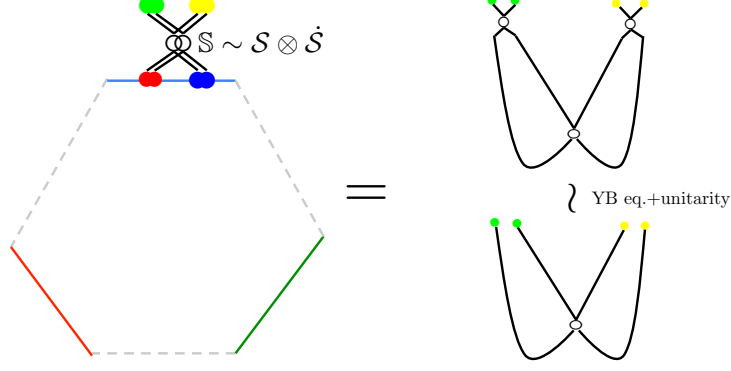
Figure 4.3.6: The multi-particle ansatz for the hexagon (4.3.6) actually satisfies the Watson equation. This can be shown by the unitarity and the Yang-Baxter equation for the S-matrix.

This precisely matches with the obtained result. The fact that the two-particle hexagon is expressed in terms of the Beisert's $\mathfrak{su}(2|2)$ S-matrix can be understood in the following way. Firstly, magnon excitations of $\mathfrak{psu}(2|2)_L$ part and those of $\mathfrak{psu}(2|2)_L$ part behave as if they are particle and anti-particle respectively, under the diagonal $\mathfrak{psu}(2|2)_D$. Hence, by the crossing symmetry for the magnons in the right part, the hexagon can be identified with the S-matrix, which is nothing but the transition amplitude from the $\mathfrak{psu}(2|2)_L$ in-state to the $\mathfrak{psu}(2|2)_R$ out-state, up to a scalar factor.

For the case of generic multi-particle hexagon, the symmetry alone cannot fix it, however, it is natural to expect that it could be described in terms of the S-matrix (4.3.6), as with the case of the two-particle hexagon. In order to justify the ansatz, the following two axioms play a quite important role

(i) Watson equation : $\quad \langle \mathfrak{h}|(\mathbb{S}_{ii+1} - 1)|\chi_{A_1 \dot{A}_1} \ldots \chi_{A_n \dot{A}_n} \rangle = 0 \,,$ (4.3.39)

(ii) Decoupling equation : $\quad \langle \mathfrak{h}|\chi_{A \dot{A}}(u)\chi_{B \dot{B}}(u^{2\gamma}) \prod_{\text{rest}} \chi_{A_j \dot{A}_j} \rangle \propto \langle \mathfrak{h}| \prod_{\text{rest}} \chi_{A_j \dot{A}_j} \rangle \,.$ (4.3.40)

with the $SU(2|2)_L \times SU(2|2)_R$ S-matrix $\mathbb{S} = S^0 (-1)^{|\dot{F}|} \mathcal{S} \dot{\mathcal{S}} (-1)^{|F|}$. Let us explain these conditions in turn and see (4.3.6), (4.3.7) actually satisfies them.

The first equation is so-called Watson equation (figure 4.3.6), which is a axiom often imposed for form factors in integrable massive quantum field theories. It is easy to see that the Watson equation is satisfied due to the Yang-Baxter equation and unitarity of the S-matrix, assuming the following condition for the scalar factor

$$h_{12}/h_{21} = S^0_{12} = \frac{x_1^+ - x_2^-}{x_1^- - x_2^+} \frac{1 - 1/(x_1^- x_2^+)}{1 - 1/(x_1^+ x_2^-)} \frac{1}{\sigma_{12}^2} \,.$$ (4.3.41)
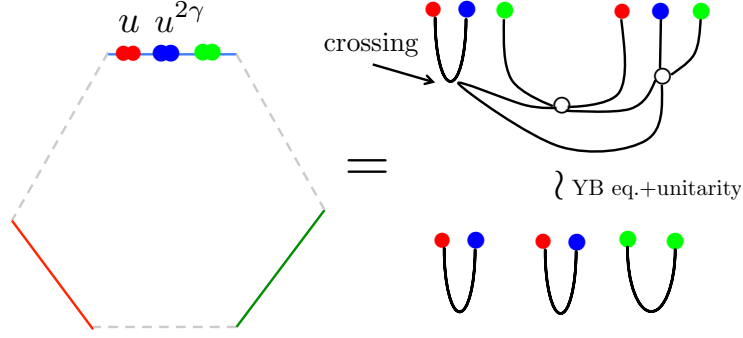
Figure 4.3.7: A pair of particle and anti-particle decouples from the rest. This can be shown by using the crossing, unitarity and the Yang-Baxter equation.

From a different point of view, the Watson equation requires the scalar factor h_{12} satisfies the unitarity condition.

The decoupling equation states that a pair of a particle and an anti-particle decouples from the rest. To put it precisely, such a particle-anti-particle pair develops a pole in the form factor and the residue of it is in proportion to the form factor whose asymptotic state is removed of the pair. From the ansatz (4.3.6), it turns out that the decoupling equation gives another condition for the scalar factor h_{12}. In fact, the S-matrix also has a pole on any particle-anti-particle pair and its residue produces the scattering of the signet with the rest (figure 4.3.7)

$$\mathrm{res}_{(12)}\mathcal{S}|\mathcal{X}_A^1\mathcal{X}_B^1\prod_{\mathrm{rest}}\mathcal{X}_j\rangle = \mathcal{S}_{\mathrm{rest}}\prod_{\mathrm{rest}}\mathcal{S}_{2j}\mathcal{S}_{1j}|\prod_{\mathrm{rest}}\mathcal{X}_j\times\mathbf{1}_{12}\rangle, \qquad (4.3.42)$$

where the singlet is consisted of the pair and given in [30]. Hence, the decoupling is ensured if the scattering between any excitations j and the pair is trivial. This is essentially equivalent to the crossing equation [37] for the S-matrix which is derived in the same way in [30]. In the present case, the decoupling equation gives the following crossing equation for the scalar factor h_{12}

$$h(u_1^{2\gamma}, u_2)h(u_1, u_2) = \frac{x_1^- - x_2^-}{x_1^- - x_2^+}\frac{1 - 1/(x_1^+ x_2^-)}{1 - 1/(x_1^+ x_2^+)}, \qquad (4.3.43)$$

whose simplest solution is (4.3.7) combined with (4.3.41).

We need to make some comments in order to provide necessary ingredients. As stated above, all the hexagon form factors can be converted into the fundament hexagon which has excitations only on a single edge by using mirror and crossing transformations several

times. Although the explicit form of such transformations depends on the frame in which we work, namely, the spin chain frame or the string frame, it is neatly implemented in the string frame. In particular, the crossing transformation in the string frame is given as follows [10]

$$|\chi(u^{2\gamma})\rangle_1 \otimes |0\rangle_2 \otimes |0\rangle_3 = |0\rangle_1 \otimes |-\tilde{\chi}(u)\rangle_2 \otimes |0\rangle_3 \,, \qquad (4.3.44)$$

$$\tilde{\chi}_{A\dot{B}} = \chi_{B\dot{A}} \,. \qquad (4.3.45)$$

Among the excitations, the crossing transformation for derivatives is given by the above form in either frame. Therefore, the hexagon with one derivative on the first physical edge and its conjugate on the second edge is

$$h_{D|\bar{D}}(u_1|u_2) = -h_{DD}(u_2^{2\gamma}, u_1) = -\frac{1 - 1/(x_1^- x_2^+)}{1 - 1/(x_1^+ x_2^-)}\sigma(u_2, u_1) \,. \qquad (4.3.46)$$

The second important point is that the hexagon transition $h_{X|X}(u|v)$ should have a pole at $u = v$, whose residue governs the normalization of the one-particle state and is related to the measure:

$$\text{res}_{v=u}h_{X|X}(u|v) = \frac{i}{\mu_X(u)} \,. \qquad (4.3.47)$$

A similar measure factor is introduced in the pentagon transition, which is a fundamental building block for the null polygonal Wilson loop [163].

4.3.4 Some checks

In this subsection, we shall give several checks for the hexagon formalism, including the both weak and strong couplings.

Weak coupling results

The simplest example for the check of hexagon formalism is the weak coupling computation. In order to clarify the role of the hexagon and concentrate on the asymptotic part, we first consider the three-point function involving two long non-BPS operators, each of which has only one magnon excitation. In other words, the first (top) operator $\mathcal{O}_1$ is build upon Z vacuum at $x = 0$ and has a single excitation χ_{top} with rapidity u, while the second (bottom) operator $\mathcal{O}_2$ is build upon $\bar{Z}$ vacuum at $x = \infty$ and has another excitation χ_{bot} with rapidity v. The other operator $\mathcal{O}_3$ is a BPS operator. Since we are now considering

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[10]In the spin chain frame, we need to multiply an extra momentum dependent factor whose explicit form depends on types of fields.

the asymptotic three-point function, we can relax the zero momentum condition or the level matching condition. Nevertheless, we impose the periodicity conditions for each magnon

$$e^{ip_1 L_1} = e^{ip_2 L_2} = 1 .$$

(4.3.48)

By cutting a pair of pants, the expected expression for the structure constant is

$$C_{123}^{\bullet\bullet\circ} \propto h_{\chi_{\text{top}}|\chi_{\text{bot}}}(u|v) + e^{ip\ell_{31}+iq\ell_{12}} h_{\chi_{\text{bot}}|\chi_{\text{top}}}(v|u)$$
$$- e^{iq\ell_{12}} h_{\chi_{\text{top}}}(u) h_{\chi_{\text{bot}}}(v) - e^{ip\ell_{31}} h_{\chi_{\text{top}}}(u) h_{\chi_{\text{bot}}}(v) .$$

(4.3.49)

Each term corresponds to a partition of rapidities into the two hexagons and the exponential factors in front of the hexagons originate from the propagation of magnons. Notice that extra minus signs are necessary when excitations are put on the second hexagon. All the above hexagons can be guessed from the finite coupling proposal. For instance, when χ's are the longitudinal scalars $Y, \bar{Y}$,

$$h_{Y|Y} = h_{\bar{Y}|\bar{Y}} = 1 - \frac{i}{u-v} , \; h_{Y|\bar{Y}} = h_{\bar{Y}|Y} = -1 , \; h_Y = -h_{\bar{Y}} = 1 ,$$

(4.3.50)

at the leading order of weak coupling.

We will compare this conjecture with the explicit weak coupling computation whose setup is realized in the following way. As already mentioned, the first and second operators are respectively of the form $\mathcal{O}_1 = \text{Tr}(Z \ldots \chi_{\text{top}} \ldots Z) + \cdots |_{x=0}$, $\mathcal{O}_2 = \text{Tr}(\bar{Z} \ldots \tilde{\chi}_{\text{bot}} \ldots \bar{Z}) + \cdots |_{x=\infty}$. The third operator is a BPS operator

$$\mathcal{O}_3 = \text{Tr}(Z + \bar{Z} + Y - \bar{Y})^{L_3} ,$$

(4.3.51)

inserted at $x = (0, 1, 0, 0)$. Since the longitudinal excitations on the first or second operator can be absorbed into the third operator, namely, Wick contracted with it, the setup is called reservoir picture. The remaining transverse excitations, including fermionic excitations and the other scalars cannot be directly contracted with the reservoir and a pair of transverse excitations on the first and second operators must be contracted to each other. For example, if we introduce a transverse scalar $\Phi_{1\dot{1}} = X$ on the top, the conjugate transverse scalar $\Phi_{2\dot{2}} = -\bar{X}$ should be put on the bottom, otherwise the correlator vanishes.

For the longitudinal excitations, say Y or $\bar{Y}$, there are two types of the contribution, which arise in different ways of the contractions (figure 4.3.8). If we contract an excitation on the top with one on the bottom, we find

$$C^{\text{direct}} = \sum_{n=1}^{\ell_{12}} e^{ip(L_1 - n + 1) + iqn} .$$

(4.3.52)

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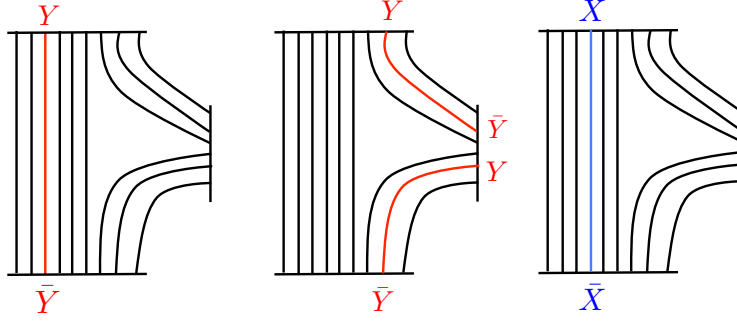
Figure 4.3.8: The figures shows the Wick contractions from a reservoir point of view. The longitudinal scalar excitations on the top and bottom operators can be Wick contracted with each other, and with the reservoir as well. On the other hand, the transverse scalars cannot be absorbed into the reservoir.

On the other hand, when the excitations on the top or bottom are contracted with the reservoir, their contribution is given by

$$C_{\text{top}}^{\text{reservoir}} = \sum_{n=1}^{\ell_{31}} e^{ipn}, \quad C_{\text{bottom}}^{\text{reservoir}} = \sum_{n=\ell_{12}+1}^{L_2} e^{iqn}. \tag{4.3.53}$$

The question now is how to integrate these contributions.

Let us first consider the case where there is only Y excitation on the top. In this case, the top reservoir term is the only non-vanishing term and can be summed up as

$$C_{\text{top}}^{\text{reservoir}} = N(p)(1 - e^{ip\ell_{31}}), \tag{4.3.54}$$

where the normalization is $N(p) = 1/(e^{-ip} - 1)$ and each term in the bracket corresponds to the one-point hexagon.

Then, we next study the case in which the transverse scalar Y and its conjugate $\bar{Y}$ are excited on the top and the bottom, respectively. In other words, $\chi_{\text{top}} = \chi_{\text{bottom}} = Y$ and the hexagon transitions $h_{Y|Y}$, h_Y involve. These excitations are directly connected to each other if they are on the half of the spin chain, otherwise they are absorbed into the reservoir. The contribution of the latter has a relative minus sign due to the sign in the vacuum constituent fields of the reservoir (4.3.51). As a result, we find

$$C_{123}^{\bullet\bullet\circ}|_{Y\to Y} \propto C^{\text{direct}} - C_{\text{top}}^{\text{reservoir}} \times C_{\text{bottom}}^{\text{reservoir}}, \tag{4.3.55}$$

which can be converted into a more simple expression once we use the rapidity representation and the Bethe ansatz equations $e^{ipL_1} = e^{iqL_2} = 1$

$$C_{123}^{\bullet\bullet\circ}|_{Y\to Y} \propto \left(1 - \frac{i}{u-v}\right) + e^{ip\ell_{31}+iq\ell_{12}} \times \left(1 - \frac{i}{v-u}\right) - e^{iq\ell_{12}} \times 1 - e^{ip\ell_{31}} \times 1. \tag{4.3.56}$$

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This result precisely matches the conjecture (4.3.49), (4.3.50).

Another interesting example is that the excitations on the top and bottom are both Y. As it is impossible to connect the excitations directly in the tree-level, there is no contribution of the form C^{direct}. Hence, the non-vanishing term only comes from the reservoir

$$C^{\bullet\bullet\circ}_{123}|_{Y\to\bar{Y}} \propto C^{\text{reservoir}}_{\text{top}} \times C^{\text{reservoir}}_{\text{bottom}}, \tag{4.3.57}$$

which is simply rewritten as follows

$$C^{\bullet\bullet\circ}_{123}|_{Y\to\bar{Y}} \propto (-1) + e^{ip\ell_{31}+iq\ell_{12}} \times (-1) - e^{iq\ell_{12}} \times (-1) - e^{ip\ell_{31}} \times (-1). \tag{4.3.58}$$

This result again reproduces the hexagon conjecture.

Finally, let us consider the case of the transverse excitations $X, \bar{X}$ (figure 4.3.8). As the transverse excitations should be contracted to each other, there is no term concerning the reservoir.

$$C^{\bullet\bullet\circ}_{123}|_{X\to\bar{X}} \propto C^{\text{direct}} \propto \frac{-i}{u-v} + e^{ip\ell_{31}+iq\ell_{12}} \times \frac{-i}{v-u} - e^{iq\ell_{12}} \times 0 - e^{ip\ell_{31}} \times 0. \tag{4.3.59}$$

The fact that the last two terms vanish reflects that the one-particle hexagon is zero for the transverse excitations. Furthermore, the hexagon transition for X can read off

$$h_{X|\bar{X}} = \frac{-i}{u-v}, \tag{4.3.60}$$

which agrees with the hexagon conjecture as well.

It is possible to incorporate the 1-loop computations into the hexagon formalism. For the SO(6) sector, the 1-loop result is just obtained from the tree-level result by inserting the 1-loop Hamiltonian at the splitting points [112, 114]. Such intentions of the Hamiltonian slightly modify the hexagon transitions as follows

$$h_{Y|Y}(u,v) = \frac{u-v-i}{u-v}\left[1 + \frac{g^2}{(u^2+1/4)(v^2+1/4)} + \mathcal{O}(g^4)\right], \tag{4.3.61}$$

$$h_{Y|\bar{Y}}(u,v) = -1 + \mathcal{O}(g^4), \tag{4.3.62}$$

$$h_{X|\bar{X}}(u,v) = h_{Y|Y}(u,v) + h_{Y|\bar{Y}}(u,v), \tag{4.3.63}$$

which are completely in agreement with the weak coupling expansion for the hexagon transitions. It should be noticed that the transition $h_{Y|Y}(u,v)$ is nothing but the function $f(u,v)$ in [118] and such a function is studied in the fermionic SU(1|1) sector [142, 143].

Asymptotic all-loop conjecture for rank 1 sector

With all the ingredients in our hands, it is possible to write down the conjectured expression for asymptotic all-loop structure constant with one non-BPS and two BPS operators in the rank-1 sectors, SU(2) and SL(2) sector. Both results can be read by replacing h with h_{DD} for SL(2), and with h_{YY} for SU(2). The result is given by

$$\left(\frac{C_{123}^{\bullet\circ\circ}}{C_{123}^{\circ\circ\circ}} \right)^2 = \frac{\prod_{k=1}^{M} \mu(u_k)}{\det\partial_{u_i}\phi_j \prod_{k<l} S(u_k, u_l)} \times \mathcal{A}^2 \,, \tag{4.3.64}$$

where M is the magnon number and $\mu(u)$ is the measure defined as the residue at the pole of the hexagon transition (4.3.47). In the denominator of the prefactor, the scattering phase ϕ_i in infinite volume appears and its definition is

$$e^{i\phi_i} = e^{ip_i L_1} \prod_{j\neq i} S(u_i, u_j) \,. \tag{4.3.65}$$

Since the denominator gives the so-called Gaudin norm, which is the norm of Bethe state in a infinite volume, the prefactor is actually the relative normalization between infinite and fiite volume theory. Lastly, the most important dynamical part is $\mathcal{A}$ which is provided by the summation over partitions of the rapidities of the hexagons

$$\mathcal{A} = \prod_{i<j} h(u_i, u_j) \sum_{\alpha\cup\bar\alpha=\boldsymbol{u}} (-1)^{|\bar\alpha|} \prod_{j\in\bar\alpha} e^{ip_j\ell} \prod_{i\in\alpha, j\in\bar\alpha} f(u_i, u_j) \,. \tag{4.3.66}$$

Here, the summation is taken over all possible partitions $\alpha\cup\bar\alpha = \boldsymbol{u}$ with $f(u,v) = 1/h(u,v)$ and the bridge length $\ell = \ell_{31}$.

Leading finite size correction

By taking into account the contributions of the mirror particles, it is possible to improve the asymptotic result (4.3.64). The leading finite size correction are composed of three terms and the corrected structure constant schematically becomes

$$\mathcal{A} \to \mathcal{A} + \delta\mathcal{A}_{12} + \delta\mathcal{A}_{23} + \delta\mathcal{A}_{31} \,, \tag{4.3.67}$$

where $\delta\mathcal{A}_{12}$, $\delta\mathcal{A}_{31}$ correspond to the neighboring bridge, while the rest one $\delta\mathcal{A}_{23}$ is coming from the opposite bridge. These contributions are seemingly different, however, the leading contribution turns out to have the following universal form regardless of which bridges we consider

$$\delta\mathcal{A} = \sum_{a\geq 1} \int \frac{du}{2\pi} \mu_a^\gamma(u) \times \left(\frac{1}{x^{[+a]}x^{[-a]}} \right)^\ell \times \text{int}_a(u|\{u_i\}) \,, \tag{4.3.68}$$

where ℓ is the corresponding bridge length and the integer a denotes the bound state label. The above term consists of three parts: the measure factor $\mu_a^\gamma(u) = \mu_a(u^\gamma)$, the propagation factor $e^{-E_{\mathrm{mir}}(u)\ell}$ and the scattering part $\mathrm{int}_a(u|\{u_i\})$. Let us explain these in turn.

For a bound state of a derivatives, the measure is

$$\mu_a(u^\gamma) = \frac{a(x^{[+a]}x^{[-a]})^2}{g^2(x^{[+a]}x^{[-a]} - 1)^2((x^{[+a]})^2 - 1)((x^{[-a]})^2 - 1)}. \qquad (4.3.69)$$

To put it more concretely, this measure can be read from the residue of the hexagon at the zero momentum pole, which is exactly parallel for the pentagon transition of the scattering amplitude [163].

The second part is coming from the "phase" of the propagation of mirror magnon $e^{ip(u^\gamma)} = e^{-E_{\mathrm{mir}}(u)}$. Indeed, the mirror energy is $E_a = \log x^{[+a]}x^{[-a]}$ and the factor gives the second contribution.

Lastly, but most importantly, the integrand $\mathrm{int}_a(u|\{u_i\})$ represents the interaction between physical magnons and a mirror particle. For the simplest case $a = 1$, it is given by

$$\mathrm{int}_a(u|\{u_i\}) = \sum_{\alpha \cup \bar\alpha = \{u_i\}} w(\alpha, \bar\alpha)(-1)^{|\alpha|} \sum_{\chi = D, \bar D, Y, \bar Y} (-1)^{f_\chi} h_{\chi D\dots D}(u^\gamma, \alpha) h_{D\dots D\bar\chi}(\bar\alpha, u^{-\gamma}),$$

$$(4.3.70)$$

$$w(\alpha, \bar\alpha) = \prod_{u_i \in \bar\alpha} (e^{ip(u_i)\ell} \prod_{u_j \in \alpha, j > i} S(u_i, u_j)). \qquad (4.3.71)$$

As apparent from the above expression, it is given as the hexagon form factors, in which a pair of mirror and anti-mirror particle appears together with the sum over partitions for physical magnon excitations. Notice that if χ in the summand would be replaced by the vacuum, the two hexagon are completely factorized and we find the asymptotic result is reproduced.

The leading finite size effect (4.3.68) is quite reminiscent of the Lüscher's correction (3.5.33) in the spectrum problem. Indeed, the scattering part $\mathrm{int}_a(u|\{u_i\})$ can be converted into the transfer matrix whose auxiliary space is the a-th anti-symmetric irreducible representation, up to an overall factor depending on rapidities (figure 4.3.9). By performing the explicit sum over partitions, it turns out that[11]

$$\mathrm{int}_a^{3\gamma}(u|\{u_i\}) = \mathcal{A} \times \frac{(-1)^a T_a(u^\gamma)}{\prod_i h_{D_a D}(u^\gamma, u_i)}, \qquad (4.3.72)$$

[11]In the case of the adjacent mirror edges, the scattering part is proportion to $T_a(u^{-\gamma})$.

Figure 4.3.9: For the case of one mirror particle contribution, the sum over mirror particle flavors can be nicely rewritten as the transfer matrix times the asymptotic result.

where $\mathcal{A}$ is the asymptotic part, T_a is the transfer matrix with the a-th anti-symmetric irrep auxiliary space and D_a denote the bound state of a derivatives.

Without doing any explicit calculations, it is possible to estimate the order of the finite size effect (4.3.68) at weak coupling. Since $x^{[\pm a]} \sim (u + ia/2)/g$ in the weak coupling, the integrand becomes $\mathcal{O}(g^{2\ell+2})$, assuming that $\text{int}_a(u|\{u_i\})$ is order $\mathcal{O}(1)$. Actually, for the mirror edge which is facing the excitations, the scattering part is proportion to the transfer matrix of the form (4.3.72) and becomes $\mathcal{O}(1)$, while $\text{int}_a^\gamma(u|\{u_i\}) \propto T_a(u^{-\gamma}|\{u_i\})$ and becomes $\mathcal{O}(g^2)$ for the adjacent mirror edges, due to the supersymmetry. From these order counting, the finite size effect being intrinsic to three-point functions starts from the $\ell + 1$-loop level, which in turn suggests two-loop is the lowest level to see the effect.

Combined with the asymptotic result and the leading finite size effect, non-trivial weak coupling checks are performed in [153–155]. The laboratory of their study is the three-point functions involving two BPS operators and one low-twist non-BPS operator in the SL(2) sector. Such a structure constant can be investigated by using the four-point functions of BPS operators since the OPE expansion of two BPS operators can produce non-BPS operators with lower twist and thus the structure constants of the form $C_{123}^{\bullet\circ\circ}$ naturally involve. Therefore, several invaluable data are available in this set up. In particular, an interesting example is the three-point functions of a twist-two operator and two short BPS operators. The results for the structure constant for twist-2 operator with lower spin S and the bridge length $\ell = 1, 2$ is summarized in the following table.

Notice that non-trivial zeta function appears at the 1-loop level in the case of $\ell = 1$, due to the presence of the new finite size effect. Furthermore, it is confirmed that the finite size effects are perfectly consistent with perturbative results even at three-loop level [154, 155].

Table 4.3.1: $\ell = 2$

S	$\left(\frac{C_{123}^{\bullet\circ\circ}}{C_{123}^{\circ\circ\circ}}\right)^2$ for twist 2, bridge length $\ell = 2$
2	$\frac{1}{6} - 2g^2 + 28g^4 + \cdots$
4	$\frac{1}{70} - \frac{205}{882}g^2 + \frac{36653}{9216}g^4 + \cdots$
6	$\frac{1}{924} - \frac{553}{27225}g^2 + \frac{826643623}{2156220000}g^4 + \cdots$
8	$\frac{1}{12870} - \frac{14380057}{9018009000}g^2 + \frac{2748342985341731}{85305405235050000}g^4 + \cdots$
10	$\frac{1}{184756} - \frac{3313402433}{27991929747600}g^2 + \frac{156422034186391633909}{62201169404983234080000}g^4 + \cdots$

Table 4.3.2: $\ell = 1$

S	$\left(\frac{C_{123}^{\bullet\circ\circ}}{C_{123}^{\circ\circ\circ}}\right)^2$ for twist 2, bridge length $\ell = 1$
2	$\frac{1}{6} - 2g^2 + (28 + 12\zeta(3))g^4 + \cdots$
4	$\frac{1}{70} - \frac{205}{882}g^2 + \left(\frac{76393}{18522} + \frac{10\zeta(3)}{7}\right)g^4 + \cdots$
6	$\frac{1}{924} - \frac{553}{27225}g^2 + \left(\frac{826643623}{2156220000} + \frac{7\zeta(3)}{55}\right)g^4 + \cdots$
8	$\frac{1}{12870} - \frac{14380057}{9018009000}g^2 + \left(\frac{2748342985341731}{85305405235050000} + \frac{761\zeta(3)}{75075}\right)g^4 + \cdots$
10	$\frac{1}{184756} - \frac{3313402433}{27991929747600}g^2 + \left(\frac{171050793565932326659}{62201169404983234080000} + \frac{671\zeta(3)}{881790}\right)g^4 + \cdots$

Strong coupling result

The strong coupling regime is a interesting regime in which both the asymptotic results and finite size corrections can be studied. Let us consider the structure constant with two BPS and one non-BPS operators in the asymptotic regime, based on (4.3.64). In order to discuss the connection with the strong coupling result [109–111, 139], we need to take the semi-classical limit, or the Frolov-Tseytlin limit, $u_i, M, L \sim \sqrt{\lambda}$. A summation over partitions appearing in (4.3.64) is converted into the form of "path integral" by introducing the "density" for each partition in the scaling limit. It is schematically represented as follows

$$\sum_{\alpha \cup \bar{\alpha} = \boldsymbol{u}} (\text{summand}) \longrightarrow \int \mathcal{D}\rho_\alpha \mathcal{D}\rho_{\bar{\alpha}} \delta(\rho_\alpha + \rho_{\bar{\alpha}} - \rho_{\boldsymbol{u}})\mu[\rho_\alpha, \rho_{\bar{\alpha}}](\text{integrand}), \qquad (4.3.73)$$

where ρ_α, $\rho_{\bar{\alpha}}$ are the densities corresponding to a partition $\alpha \cup \bar{\alpha} = \boldsymbol{u}$ satisfying the constraint $\rho_\alpha + \rho_{\bar{\alpha}} = \rho_{\boldsymbol{u}}$ with total density $\rho_{\boldsymbol{u}}$ and μ is the measure factor. Since the measure originates from the "density of states" for the microscopic partitions corresponding to a given macroscopic density, the logarithm of μ can be regarded as the "entropy" factor.

The resulting path integral can be well approximated by the saddle point method and it is possible to perform such a computation by applying the same technique which also

appears in the weak coupling [117] [12]. As a result, we find

$$\ln C_{123}^{\bullet\bullet\circ\circ}|_{\text{asymptotic}} = \oint_{C_1} \frac{du}{2\pi} \left[\text{Li}_2(e^{ip_1+ip_2-ip_3}) - \frac{1}{2}\text{Li}_2(e^{2ip_1}) \right], \qquad (4.3.74)$$

where the contour C_1 encircles all the cuts of the quasi-momentum p_1. The quasi-momentum p_1 can be interpreted as that of the sphere part $\tilde{p}_1$ for the SU(2) sector or that of the AdS part $\hat{p}_1$ for the SL(2) sector. Notice that the remaining quasi-momenta are given by $p_i = 2\pi x L_i/((x^2 - 1)\sqrt{\lambda})$, which are nothing but those for BMN strings.

We next discuss the leading finite size correction corresponding to a single mirror particle contribution. For example, in the SL(2) sector, we can easily take the strong coupling limit of (4.3.72), the result becomes

$$\int_{U^-} \frac{du}{2\pi} e^{ip_2+ip_3}(e^{-i\hat{p}_1(x)} + e^{i\hat{p}_1(1/x)} - 2e^{-i\tilde{p}_1(x)}), \qquad (4.3.75)$$

where p_1 is given by the that of the BMN string in the present case. The important point here is that the integration contour U^- is given as the unit circle on the lower half plane connecting $x = -1$ to $x = +1$. It should be noted that such a integration around the unit circle appears in the case of the finite size correction for the spectrum. Surprisingly, these results completely match with those obtained in [139].

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[12]This is possible due to the fact that the absolute value of the summand f, which is dominated by the singular behavior of the form $1/(u_i - u_j)$, is the same with the weak coupling, while the phase of the summand does change the result. In turn, the computation can be parallel once we replace the quasi-momentum by the strong coupling counter part.

Chapter 5

Interlude -other observables-

We have so far discussed the application of integrability techniques for the two-point functions and the three-point functions. With tremendous efforts, conjectured non-perturbative formulations have been proposed for these observables. However, the integrability techniques turn out be quite powerful to determine other observables such as Wilson loops and scattering amplitudes. Although the main theme of this thesis is the correlation functions, especially the three-point functions, it is of use to explain how the integrability is applied to other observables and see the power of it. We will first discuss the scattering amplitude or null polygonal Wilson loop based on an idea of the OPE decomposition, and introduce the fundamental object so-called pentagon transition. Furthermore, we will cover the topic of half-BPS Wilson line with a cusp or quark-anti-quark potential. This is closely related to the spectrum problem of open spin chain and analogous to the two-point functions of single trace operators.

5.1 Scattering amplitude and null polygonal Wilson loop

The null polygonal Wilson loops in $\mathcal{N} = 4$ SYM are known to be dual to the scattering amplitudes of massless particles by the T-duality. A seminal paper by Alday and Maldacena [159] initiated the studies of the scattering amplitudes or null Wilson loops using integrability techniques. In that paper, the strong coupling limit of the amplitude or null Wilson loop is evaluated at the saddle of string action in AdS space, which is equivalent to the area of a minimal surface. It turned out that such an evaluation of the minimal area is efficiently performed with the helps of the fact that the equation of motion for the string in AdS space is classically integrable. See, for example, [160].

A non-perturbative approach for the scattering amplitudes relies on the idea of the OPE like decomposition [161]. It is well known that correlation functions in confor-
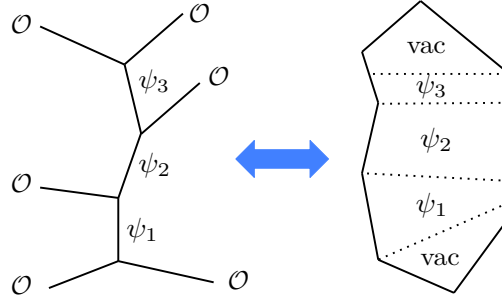
Figure 5.1.1: Correlation functions in conformal field theories can be decomposed into three- and two-point functions by using the operator product expansion. Similarly, the OPE like decompositions hold for a null polygonal Wilson loop. As a result, $(n-3)$ null squares appear and neighboring null squares form $n-4$ pentagons for a n-edged null polygon.

mal field theories are decomposed into lower-point functions, using the operator product expansions. The resulting expression is a multiple sum over states corresponding to operators, which arise in the product of the local operators. A similar OPE like decomposition can be applied to the vacuum expectation value of the null polygonal Wilson loop [161]. The basis for the decomposition consists of two building blocks, the square and the pentagon, which are quite analogous to the two-point function and the three-point function respectively in the OPE decomposition for the correlation function.

As depicted in the figure 5.1.1, a null polygon is decomposed into a sequence of squares and these adjacent squares form pentagons. Since the Wilson line can be regarded as a trajectory of a flux tube, it is natural that a flux tube state is defined on each square. In particular, middle squares are interpreted as intermediate states and the top and bottom squares are identified as the vacua. Notice that there are $n-5$ middle squares for an n-edged Wilson loop.

Let us explain the flux tube states in detail. The most convenient way to describe the flux tube states is to regard them as the dual of the (excited) GKP strings. The GKP string is a long string with a large spin, which ends on the null square at the boundary of AdS and it is dual to the single trace operator with a large number of derivatives along a null direction. It is the vacuum state of the flux tube and excitations on the flux tube are dual to fluctuations of the GKP string. For example, the gluonic excitations are dual to bumps in folded strings and the scalar excitations correspond to fluctuations in the sphere S^5. As with the case of spectrum problem for excitations on the BMN vacuum, the Bethe ansatz techniques work in the present case as well and the energies for the excitations on the GKP vacuum and the S-matrix among them at finite coupling can be determined [162].

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As a result, the flux tube states are described in terms of Bethe states, which are labelled by flavors and rapidities for excitations: $\boldsymbol{a} = (a_1, \ldots, a_N)$, $\boldsymbol{u} = (u_1, \ldots, u_N)$.

For the kinematical dependence of the null polygon, we need to pay attention to the middle squares since each square corresponds to the GKP string with excitations and it has a local time τ, space σ and angle for the rotation being transverse to the square. All conformally inequivalent polygons are parametrized by these variables by acting the symmetry generators conjugate to them for each square. As a result, we find $3(n-5)$ kinematical variables $\{\tau_i, \sigma_i, \phi_i\}_{i=1}^{n-5}$ parametrizing a null polygon, which turn out to be associated with the cross ratios.

Let us move to the dynamics of the null polygonal Wilson loop. From a GKP string point of view, such a null polygonal Wilson loop can be regarded as a transition process in which a GKP vacuum state evolves and kicked by the cusps to excited states and then goes back to another GKP vacuum again. Schematically, it is represented as follows

$$\text{vacuum} \rightarrow \psi_1 \rightarrow \cdots \rightarrow \psi_{n-5} \rightarrow \text{vacuum} \,. \tag{5.1.1}$$

Transitions from a state to another are occurred at the cusps of the polygon and they can be regarded as insertions of defect operators. Hence, we reach the following expression.

$$\mathcal{W} = \langle \text{vac} | \hat{\mathcal{P}} e^{-\tau_{n-5}\hat{H} + i\sigma_{n-5}\hat{P} + i\phi_{n-5}\hat{J}} \hat{\mathcal{P}} \ldots \hat{\mathcal{P}} e^{-\tau_1 \hat{H} + i\sigma_1 \hat{P} + i\phi_1 \hat{J}} \hat{\mathcal{P}} | \text{vac} \rangle \,, \tag{5.1.2}$$

where $\hat{\mathcal{P}}$ is the so-called pentagon operator and the propagation factor is given by $e^{-\tau_j \hat{H} + i\sigma_j \hat{P} + i\phi_j \hat{J}}$ with the three conformal symmetry generators of the square $(\hat{H}, \hat{P}, \hat{J})$. Alternatively, by inserting the completeness relations $\mathbf{1} = \sum_{\psi_i} |\psi_i\rangle\langle\psi_i|$ into the above expression,

$$\mathcal{W} = \sum_{\psi_i} e^{\sum_j (-E_j \tau_j + iP_j \sigma_j + im_i \phi_j)} \mathcal{P}(\text{vac}|\psi_1)\mathcal{P}(\psi_1|\psi_2)\ldots\mathcal{P}(\psi_{n-6}|\psi_{n-5})\mathcal{P}(\psi_{n-5}|\text{vac}) \,. \tag{5.1.3}$$

Here, the pentagon transition is defined as $\mathcal{P}(\psi_j|\psi_k) = \langle\psi_j|\hat{\mathcal{P}}|\psi_k\rangle$ and the state ψ_i has definite energy E_i, momentum p_i and $U(1)$ angular momentum m_i. As already stated, these eigenstates can be identified with Bethe states whose excitations on the GKP vacuum are magnons with flavors and rapidities $\boldsymbol{a} = (a_1, \ldots, a_N)$, $\boldsymbol{u} = (u_1, \ldots, u_N)$ and the eigenvalues (E_i, p_i, m_i) are sum of the charges of individual magnons. Therefore, sum over states in the intermediate processes and the pentagon transitions can be read as

$$\sum_{\psi_i} \rightarrow \sum_{\boldsymbol{a}^{(i)}} \int \prod_{m=1}^{N} \mu_{a_m^{(i)}}(u_m) du_m \,, \tag{5.1.4}$$

$$\mathcal{P}(\psi_j|\psi_k) \rightarrow P_{\boldsymbol{a}^{(j)}\boldsymbol{a}^{(k)}}(\boldsymbol{u}^{(j)}|\boldsymbol{u}^{(k)}) \,. \tag{5.1.5}$$

The remaining task is to determine the measure factor and the pantaloon transitions. Actually, the measure factor is not independent object and it is related to the pentagon transition in the following way

$$\text{Res}_{v=u} P_{aa}(u|v) = \frac{i}{\mu_a(u)} \, . \tag{5.1.6}$$

Therefore, what remains to be done is to determine the paragon transitions.

In [163, 164], the pentagon transitions involving gluonic and scalar excitations are discussed based on several axioms as with the case of the hexagon form factor. For example, the one-particle pentagon transitions $P(u|v) := P_{FF}(u|v)$, $\bar{P}(u|v) := P_{F\bar{F}}(u|v)$[1] are required to satisfy the three axioms

$$(i): \quad P(-u|-v) = P(v|u) \, , \tag{5.1.7}$$

$$(ii): \quad P(u|v) = S(u,v)P(v,u) \, , \tag{5.1.8}$$

$$(iii): \quad P(u^{-\gamma}|v) = \bar{P}(v|u) \, . \tag{5.1.9}$$

The first axiom is a consequence of the reflection symmetry of the pentagon. The second axiom is the so-called fundamental relation and $S(u,v)$ is the GKP S-matrix for the scattering of gluonic excitations. The third axiom reflects the invariance of the pentagon transition under the mirror transformation. Combined these axioms, immediately follows the relation $\bar{P}(u^{2\gamma}|v) = P(u^{-3\gamma}|v) = S(v,u)$ and we have a clear physical interpretation for it. In fact, the excitation with rapidity u can be brought on the same edge of the excitation with rapidity v by using mirror transformation $u \to u^{2\gamma}$ or $u \to u^{-3\gamma}$. However, the transformed excitation is on the left or right of the excitation with v depending on which mirror transformations we apply. Therefore, we need to exchange the two excitation in order to compare the result by using the fundamental relation. As a result, we reach the relation mentioned above. This is a interesting consistency check for the axioms.

The finite coupling solution for the axioms is simply given as follows

$$P(u|v)^2 = \left[\frac{f(u,v)}{g^2(u-v)(u-v-i)} \right]^\eta \frac{S(u,v)}{S(u^\gamma, v)} \, , \tag{5.1.10}$$

with $\eta = 1$ and the function $f(u_1, u_2)$ defined by

$$f(u_1, u_2) = x_1^+ x_1^- x_2^+ x_2^- (1 - g^2/(x_1^+ x_2^-))(1 - g^2/(x_1^- x_2^+))(1 - g^2/(x_1^+ x_2^+))(1 - g^2/(x_1^- x_2^-)) \, . \tag{5.1.11}$$

For the transition $\bar{P}(u|v)$, the solution is given by the above expression with $\eta = -1$. Surprisingly, it is possible to write down the conjectured multi-particle pentagon transition

$$P(\boldsymbol{u}|\boldsymbol{v}) = \frac{\prod_{i,j} P(u_i, u_j)}{\prod_{i>j} P(u_i|u_j) \prod_{i<j} P(v_i|v_j)} \, . \tag{5.1.12}$$

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[1]Here, $F = F_{z-}$ and $\bar{F} = F_{\bar{z}-}$.
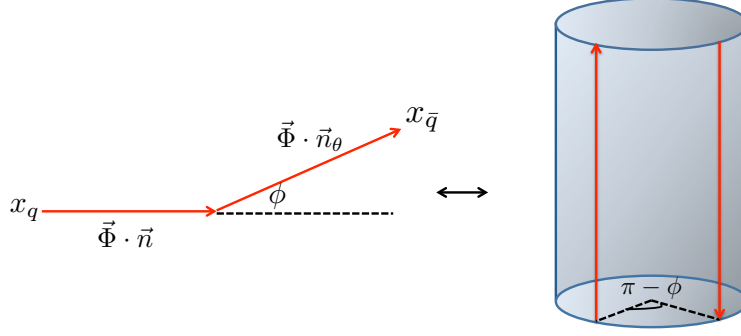
Figure 5.2.1: It is possible to introduce two cusps for a locally supersymmetric Wilson loop. One is an usual cusp ϕ in space-time and the other is a cusp θ in the R-symmetry space. The latter is defined by the polarization vectors $\vec{n}, \vec{n}_\theta$ of the scalars coupled to each segment of the line as $\vec{n} \cdot \vec{n} = \cos\theta$. Using the conformal map from the plane to the cylinder, the Wilson loop is mapped to a pair of quark and anti-quark trajectories stretching along the time direction.

It should be noted that the transition is expressed in terms of the single transitions, which is a manifestation of the underlying integrability.

5.2 Half-BPS Wilson loop with a cusp and $q\bar{q}$ potential

Another important example for the power of the integrability is the generalized cusp anomalous dimension of the locally supersymmetric Wilson loop. The Wilson loop with a cusp is known to have divergence of the form

$$\langle W \rangle \sim e^{-\Gamma_{\text{cusp}} \log \frac{\Lambda_{\text{UV}}}{\Lambda_{\text{IR}}}} , \tag{5.2.1}$$

where $\Lambda_{\text{UV,IR}}$ are the UV or IR cut-off. For the locally supersymmetric Wilson loop, it is possible to introduce another "cusp" in the R-symmetry space which changes the scalar coupled to the line

$$W = P \exp \int_{-\infty}^{0} dt \left[iA \cdot \dot{x}_q + \vec{\Phi} \cdot \vec{n} |\dot{x}_q| \right] \times P \exp \int_{0}^{\infty} dt \left[iA \cdot \dot{x}_{\bar{q}} + \vec{\Phi} \cdot \vec{n}_\theta |\dot{x}_{\bar{q}}| \right] . \tag{5.2.2}$$

Here, $x_q(t), x_{\bar{q}}(t)$ are straight lines which form an angle ϕ and they respectively correspond to the trajectory of a heavy quark and an anti-quark. Furthermore, the polarization vectors $\vec{n}, \vec{n}_\theta \in S^5$ form an angle θ and satisfy $\vec{n} \cdot \vec{n}_\theta = \cos\theta$. As a result, the generalized cusp anomalous dimension Γ_{cusp} depends on the three parameters (λ, ϕ, θ).

Before going into the detail descriptions, let us emphasize the relation with other observables.

- **Quark-anti-quark potential**

  Under the conformal map from the plane to cylinder, the two lines are mapped to a pair of anti-parallel lines, which stretch along the time direction of $\mathbb{R} \times S^3$, being separated by an angle $\pi - \phi$ in the sphere. As the dilatation operator is mapped to the Hamiltonian under this transformation, the cusp anomalous dimension is identified with the static energy of a heavy quark-anti-quark pair on S^3 with the separation angle $\pi - \phi$ (figure 5.2.1):

  $$\Gamma_{\text{cusp}}(\lambda, \phi, \theta) = V(\lambda, \phi, \theta) \,. \tag{5.2.3}$$

  Of particular interest, we find the quark-anti-quark potential in the flat space by considering the small $r = \pi - \phi$ limit, as we can neglect the curvature in this limit

  $$\Gamma_{\text{cusp}}(\lambda, \theta) \to -\frac{\Omega(\lambda, \theta)}{r}, \quad (r \to 0) \,, \tag{5.2.4}$$

  where the function $\Omega(\lambda, \theta)$ is the analogue of the Coulomb charge. Weak coupling computations were performed in [167] at three-loop level, and strong cooling computations were done at 1-loop level [168].

- **Bremsstrahlung function**

  In the limit $\phi \sim \theta$, the cusp anomalous dimension vanishes due to the supersymmetry. Furthermore, in the small angle limit, it is known that Γ_{cusp} behaves as follows

  $$\Gamma_{\text{cusp}} \sim -(\phi^2 - \theta^2)B(\lambda), \quad (\phi, \theta \ll 1) \,, \tag{5.2.5}$$

  where the coefficient function $B(\lambda)$ defined here is the so-called Bremsstrahlung function and can be computed exactly by using the localization [169]. The energy emitted from a moving quark in the small velocity limit is given by

  $$\Delta E = B \int dt (\dot{v})^2 \,, \tag{5.2.6}$$

  and thus B is called the Bremsstrahlung function.

- **IR behavior of scattering amplitudes**

  The cusp anomalous dimension also characterizes various IR behaviors. Actually, the IR divergences for the scattering of massive particles in any conformal gauge

theory are governed by $\Gamma_{\text{cusp}}(\lambda, \varphi, \theta)$, where $\varphi = -\frac{p_1 \cdot p_2}{\sqrt{p_1^2 p_2^2}}$ is the Lorentz boost angle defined through the momenta of massive particles [170]. In particular, the massive particles in $\mathcal{N} = 4$ SYM arise when we give some Higgs vevs for $\vec{\Phi}$. The angle θ can be interpreted as the angle defined between the Higgs vevs for massive W-bosons.

Furthermore, the IR divergences of massless particles are characterized by $\Gamma_{\text{cusp}}^{\infty}$, which is obtained by the infinite boost limit: $\Gamma_{\text{cusp}} \sim \varphi \Gamma_{\text{cusp}}^{\infty}$ ($\varphi \to \infty$) [2]. The computation of $\Gamma_{\text{cusp}}^{\infty}$ was performed in [39] using integrability techniques.

In what follows, we will briefly review how to compute the cusp anomalous dimension in the framework of the integrability. The basic idea is essentially the same as the spectrum problem for the closed spin chain, except that we need to take into account the effect of the boundaries as we soon explain. The strategy is summarized as three steps: (1) We first consider the cusped Wilson line with an operator insertion, (2) then, we apply the asymptotic Bethe ansatz with boundaries, (3) finally, we include finite size effects which are summed up into boundary TBA equations [171, 172]. Let us see in detail.

The first step starts from the spectrum problem of local operators inserted on a Wilson line. In particular, we consider a BPS operator with a large length of the form

$$\underbrace{P \exp \int_{-\infty}^{0} dt \left[iA \cdot \dot{x}_q + \vec{\Phi} \cdot \vec{n} |\dot{x}_q| \right]}_{B_l} \times Z^L(0) \times \underbrace{P \exp \int_{0}^{\infty} dt \left[iA \cdot \dot{x}_{\bar{q}} + \vec{\Phi} \cdot \vec{n}_\theta |\dot{x}_{\bar{q}}| \right]}_{B_r(\phi, \theta)}, \quad (5.2.7)$$

where $Z = \Phi_1 + i\Phi_2$ and $L \gg 1$. From a local operator point of view, it is reasonable to regard $B_l Z^L B_r$ as open spin chain with two boundaries. Hence, the cusp anomalous dimension can be obtained by determining the energy spectrum problem of the spin chain with boundaries and analytically continuing $L \to 0$. In the large L limit, the effect of the two Wilson lines or boundaries is negligible, as the operator is infinitely long and decompactified. This situation is exactly the same for the bulk excitations on the BMN vacuum, hence, they are classified according to $PSU(2|2)_L \times PSU(2|2)_R$ and the S-matrix between them can be determined uniquely. However, in order to impose the periodicity condition for each magnon, we should be careful at the boundaries since the magnons are reflected and scattered at the boundary. In other words, it is necessary to fix the form of the reflection matrix of the boundary.

For this purpose, we first concentrate on the supersymmetric case $\phi = \theta = 0$, namely, a straight Wilson line without any insertions and cusps. The bosonic symmetry of this Wilson line easily turns out to be $SL(2) \times SO(3) \times SO(5)$. The first $SL(2)$ represents the conformal transformation along the time direction and the second $SO(3)$ are the spatial

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[2]Notice that $\Gamma_{\text{cusp}}^{\infty}$ is also called cusp anomalous dimension as well.
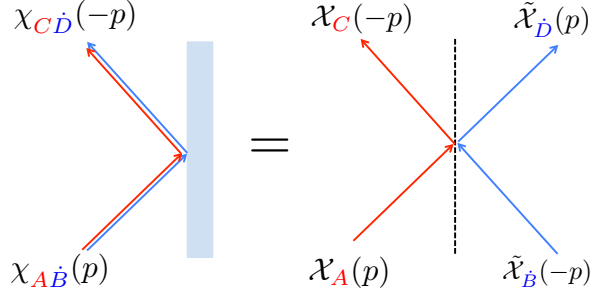
Figure 5.2.2: The reflection matrix $R_{A\dot{B}}^{C\dot{D}}(p)$ turns out to be related to the $\mathfrak{su}(2|2)_D$ invariant S-matrix $S_{A\dot{B}}^{C\dot{D}}(p, -p)$ using the unfolding trick.

rotations around the line. The last $SO(5)$ corresponds to the R-symmetry rotations which leaves the scalar coupled to the line invariant. The supersymmetric extension is known to be $OSP(4^*|4)$[3]. Next, let us insert the BPS operator Z^L in the middle of the Wilson line. Recall that the symmetry group preserving the BMN vacuum is $PSU(2|2)_L \times PSU(2|2)_R$. As a result, the symmetry is broken down to $PSU(2|2)_D$, which is the maximal subgroup of $PSU(2|2)^2 \cap OSP(4^*|4)$. The bosonic subgroup of $PSU(2|2)_D$ consists of two $SO(3)$, which are the rotations around the line and the rotations among scalars $(\Phi_1, \Phi_2, \vec{\Phi} \cdot \vec{n})$ However, notice that $PSU(2|2)_D$ in the present case is not coincident with that symmetry of the hexagon. Actually, the fermionic generators are slightly different and they are schematically $Q_D = Q + \bar{Q}$, $S_D = S + \bar{S}$, while $Q_D = Q + \bar{S}$, $S_D = S + \bar{Q}$ in the case of the hexagon.

With the symmetry of the boundary, unfolding trick depicted in figure 5.2.2 and the crossing equation, we can completely fix the reflection matrix. The boundary reflection matrix satisfies the (boundary) Yang-Baxter equation and it suggests that the existence of the boundaries do not spoil the integrability of the system.

The next goal is to derive the formula for the finite size effects since we already know that the asymptotic energy for $B_l Z^L B_r$ is 0 [4], up to the exponentially suppressed terms e^{-cL}. Therefore, we shall discuss the partition function for the system with boundaries and then treat the mirror theory by exchanging the roles of space and time. (See figure 5.2.3.)

$$Z_{B_l,B_r}^{\text{open}} = \text{Tr}[e^{-TH_{B_l,B_r}^{\text{open}}}] = \langle B_l | e^{-LH^{\text{closed}}} | B_r \rangle, \qquad (5.2.8)$$

where $H_{B_l,B_r}^{\text{open}}$ is the Hamiltonian of the open chanel and the trace is taken over the open

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[3]The star means that the real form of $SO(4)$ is taken so that $SO(4^*) \equiv SL(2) \times SU(2)$.

[4]This operator is no longer BPS for generic (ϕ, θ), however, the energy is approximately 0 in the large L regime.
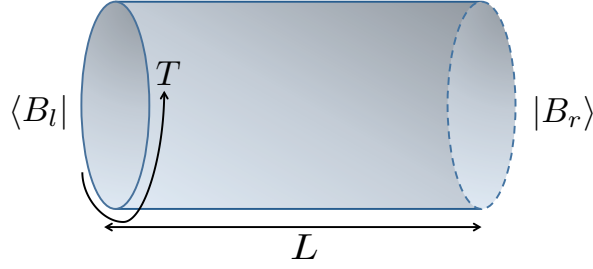
Figure 5.2.3: The boundary TBA technique is of reminiscent of the open-closed duality. From a open string point of view, the 1-loop partition function can be computed by taking the trace over open string Hilbert space. On the other hand, the same partition function is obtained from the overlap between two boundary states, with a propagation of a closed string with the Hamiltonian H^{closed}.

chain Hilbert space. On the other hand, the same partition function can be described as an overlap between two boundary states with the propagation time L and the closed chain Hamiltonian H^{closed}. This is quite analogous to the open-string duality. The boundary states are expressed in terms of the the reflection matrix in the mirror chanel, which is obtained by an analytical continuation via mirror transformation. The boundary TBA equations can be derived by computing the overlap introducing the densities, which are given by

$$\log Y_A = \log(\kappa_A^l \kappa_A^r) - 2L\tilde{E}_{m,A} + K_{AB} * \log(1 + Y_B). \tag{5.2.9}$$

The equation is almost the same as the case of the closed string, except for the factors $\kappa_A^{l,r}$, which come from the reflection matrices and depend on the boundary states. Finally, the cusp anomalous dimension is schematically represented as follows

$$\Gamma_{\text{cusp}} = -\frac{1}{2\pi} \sum_A \int_0^\infty du \partial_u \tilde{p}_A \log(1 + Y_A), \tag{5.2.10}$$

with the energies $\tilde{E}_{m,A}$ and the momenta $\tilde{p}_A$ of the excitations in the mirror theory.

Part II

Main result

Chapter 6

Novel construction and the monodromy relation -SU(2) sector-

For the weak coupling perturbative computation[1], a systematic procedure called "tailoring" has been developed in 4.2 [115–119], and with a useful technical improvement [123], a special class of three-point functions for non-BPS operators have been expressed explicitly in terms of Slavnov determinants [121]. Furthermore, the semi-classical limit of such three point functions with large charges were successfully evaluated in a remarkably compact form (4.2.53) [117, 124, 125, 127].

On the other hand, the strong coupling computation was performed using the string theory in $AdS_3 \times S^3$ spacetime [111], with the vertex operators possessing the same global quantum numbers as the operators in the "SU(2) sector" considered at weak coupling. Since the canonical quantization of the string in such a curved space is not available at present, the saddle point approximation was used, which is valid for the case of vertex operators carrying large charges. Although the precise form of the vertex operators nor the exact saddle point configuration were not known, the judicious use of classical integrability, with a certain natural assumption, was powerful enough to produce explicit answers for the desired three-point functions. Surprisingly, even before taking any limits, the results exhibited structures rather similar to those at weak coupling. On the other hand, upon taking the so-called Frolov-Tseytlin limit, in which the strong and the weak coupling results were expected to agree, small discrepancies were observed, the understanding of which is left as a future problem.

Evidently, besides making the comparison of the results, the principal goal of these investigations is to uncover common concepts and structures threading the both sides of the duality and understand how they are realized to make the duality work. For this purpose, it is desirable to be able to treat the both sides in as much the same way as

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[1]For earlier pioneering investigations, see [112–114].

possible and try to extract the key principle. In this chapter, we shall present two new significant results in the weak coupling analysis for such a purpose, which are actually hinted by the strong coupling investigation of [108–111]. Let us now briefly describe them one by one.

The first result concerns the computation of the three-point functions much more general than those treated so far in the existing literature. As is customary, let Φ_i ($i = 1, 2, 3, 4$) be the four of the six adjoint scalar fields forming the SU(2) sector and denote their complex combinations as

$$
\begin{aligned}
Z &= \Phi_1 + i\Phi_2\,, & \bar{Z} &= \Phi_1 - i\Phi_2\,, \\
Y &= \Phi_3 + i\Phi_4\,, & \bar{Y} &= \Phi_3 - i\Phi_4\,.
\end{aligned}
\tag{6.0.1}
$$

In the systematic investigation initiated in [115], two of the three operators interpretable as $\mathrm{XXX}_{1/2}$ spin chains were taken to be built upon the pseudo-vacuum $\mathrm{Tr}\,(Z^\ell)$, and the remaining one was built upon $\mathrm{Tr}\,(\bar{Z}^\ell)$. As long as one identifies Z and $\bar{Z}$ as "ground state" up-spins and Y and $\bar{Y}$ as down-spins representing the excitations, such a choice of operators were essentially unique in order to produce non-extremal correlators.

In the work of [111], however, a detailed analysis has been made of the operators built upon more general "vacuum" states where an arbitrary linear combination of Φ_i is regarded as the "up-spin". This study revealed that the natural way to characterize the general operators so constructed is by a pair of two-component vectors n and $\tilde{n}$, termed "polarization spinors", associated to each of the SU(2) factors of the global symmetry group SO(4) $\cong$ SU(2)$_L$×SU(2)$_R$. By applying this characterization to the string vertex operators, three-point functions of operators carrying general polarization spinors were computed at strong coupling.

Since the analysis of the general operators mentioned above was inspired in the spin-chain picture of the operators, one would expect that similar generalization can and should be done at weak coupling. Clearly this would be important in the comparison with the strong coupling results. Unfortunately, however, there are apparent problems to overcome. One is that when the three operators are built on different "rotated vacua", it is non-trivial to perform the Wick contractions keeping the spin-chain interpretation intact. Another difficulty is that, for the general configurations under consideration, $\langle$off-shell|off-shell$\rangle$ inner products produced through the usual tailoring procedure cannot in general be converted into $\langle$on-shell|off-shell$\rangle$ form by the known trick [123]. This hampers the expressions in terms of tractable determinants.

As will be explained fully in sections 6.1 and 6.2, these problems will be neatly solved by (i) the "double spin-chain" formulation of the conventional spin-chain and (ii) the novel interpretation of the Wick contraction as skew-symmetric singlet paring acting on

the double spin-chain Hilbert space. These ideas allow us to characterize the general operators by a pair of polarization spinors and moreover naturally factorize the three-point functions into the product of $SU(2)_L$ and $SU(2)_R$ factors, just as it happened for the wave function part of the strong coupling computation [111]. The most important advantage, however, is the fact that under the new singlet pairing interpretation, the Wick contraction procedure produces only the matrix elements of the $B(u_i)$ components of the monodromy matrix, without the appearance of $C(u_i)$ components. Therefore the building blocks of the three-point functions take the form of the so-called partial domain wall partition function (pDWPF) [124, 125, 129, 130, 132, 134] and immediately possess determinant expressions. In particular, for certain class of correlators the expression in terms of the sum of pDWPF's collapses into a single term and yields a remarkably simple result.

Now let us move on to the second new result, which again is motivated by the structure of the strong coupling computations [108–111]. One of the crucial difficulties in the strong coupling computation is that one does not know the exact three-pronged saddle point solution with which to evaluate the three-point function. In the framework of the classical integrable system, the most important available information is the form of the solution of the auxiliary linear problem (ALP) in the vicinity of the vertex operator insertion point z_i, which can be approximated[2] by the saddle point configuration for the two-point solution. Differently put, the local monodromy operator Ω_i and its linearly independent eigenfunctions $i_\pm$ of ALP around z_i are the only available secure yet local data. It is clear that in addition one definitely needs some global information to capture the properties of the three-point function. As was demonstrated in the previous works [108–111], such a global information was provided by the triviality of the total monodromy, namely $\Omega_1 \Omega_2 \Omega_3 = 1$. This seemingly weak constraint turned out to be surprisingly powerful and played a key role in computing the Wronskians of the eigenfunctions $\langle i_+, j_+ \rangle$, etc. with which the three-point functions are constructed.

This experience strongly suggests that one should formulate a similar monodromy relations for the three-point functions at weak coupling as well. The corresponding quantities are the three-point functions with three local monodromy operators inserted. As will be explained in section 6.4, non-trivial relations, which contain certain constant shifts of the spectral parameter, can be obtained through the use of the so-called "unitarity" and "crossing" relations for the Lax operator. Generically such monodromy identity relates three-point functions composed of different operators and hence may be regarded as a kind

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[2] Actually, as far as the evaluation of the wave function for the three-point function is concerned, the slight deviation from the two-point function near the puncture contains a crucial information [111].

of Schwinger-Dyson equation. As a simple application, one can obtain the counterpart of the total trivial monodromy relation $\Omega_1 \Omega_2 \Omega_3 = 1$ in the semi-classical limit of the large spectral parameter, where the constant shifts can be ignored. Just as in the case of strong coupling, such a relation provides vital information in the computation of the three-point functions, the details of which will be fully described in chapter 8 [139]. In any event, this structure may provide a key to the understanding of the notion of "integrability" beyond the spectral level, especially if it can be generalized to higher loop correlators.

The organization of the rest of this chapter is as follows: In section 6.1, we will begin by explaining the double spin-chain formalism for the SU(2) sector and introduce the general rotated vacua and construct the non-BPS operators built upon such vacua. Then in section 6.2, we will formulate the new group-theoretical view of the Wick contractions of constituent fields and the composite operators made out of them, which is natural for the double spin-chain formulation. With theses preparations, we will describe in section 6.3 how one can compute the three-point functions which are much more general than the ones considered so far in the literature. The advantage of our new formalism becomes apparent in this computation in that the correlators factorize into the $SU(2)_L$ - and the $SU(2)_R$- pieces and will be naturally expressed in terms of the determinants which describe the partial domain wall partition functions. The new global monodromy relations for the three-point functions will be derived in section 6.4. In the double spin-chain formalism, this relation will also enjoy the factorized properties. Finally in section 6.5, we will discuss future directions and briefly comment on a direct computation of the semi-classical three-point functions without the use of the determinant formulas, being prepared as a separate treatise [139]. Two short appendices are provided to explain the kinematical dependence of the three-point functions and the general form of the monodromy relation.

6.1   Double spin-chain formalism for the SU(2) sector

As described in the introduction, one of the two major aims of this paper is to develop a scheme in which the three-point functions of a more general class in the SU(2) sector can be computed systematically. This is of value since such a computation has already been done in the strong coupling regime [111] and it is important to be able to make a comparison of their general structures. In this section, we shall explain the basic idea of this formalism, to be called the "double spin-chain formalism".

6.1.1 Scalar fields as tensor products of two spins

Let us begin with the description of a new way of mapping each of the basic fields $Z, \bar{Z}, X, \bar{X}$ of the "SU(2)" sector to a tensor product of two spin-chain states. In the previous approach [115], one makes the identifications of the basic up- and down- spin pair as $(Z, Y) \mapsto (|\uparrow\rangle, |\downarrow\rangle)$, $(Z, \bar{Y}) \mapsto (|\uparrow\rangle, |\downarrow\rangle)$ and $(\bar{Z}, \bar{Y}) \mapsto (|\uparrow\rangle, |\downarrow\rangle)$[3]. Thus, although the content of these three pairs are obviously different, the spin chains composed of them are regarded as the *same type* of SU(2) spin chain. This somewhat redundant characterization of the constituents of the spin chains makes it difficult to construct the correlators of three operators forming spin chains where their relevant SU(2) groups are embedded in more general ways in the total symmetry group SO(4).

A natural and simple solution to this problem is to make use of the fact that the basic fields $(Z, \bar{Z}, Y, \bar{Y})$ carry distinct charges with respect to $\mathrm{SU}(2)_L \times \mathrm{SU}(2)_R$ ($\cong \mathrm{SO}(4)$). This is best expressed by assembling them into the 2×2 matrix

$$\Phi_{a\tilde{a}} \equiv \begin{pmatrix} Z & Y \\ -\bar{Y} & \bar{Z} \end{pmatrix}_{a\tilde{a}}, \tag{6.1.1}$$

which transforms as

$$\Phi \to U_L \Phi U_R, \tag{6.1.2}$$

where $U_L \in \mathrm{SU}(2)_L$ and $U_R \in \mathrm{SU}(2)_R$. This means that these fields carry left and the right charges (L, R) of the form

$$\begin{aligned} Z &: (+1/2, +1/2), & Y &: (+1/2, -1/2), \\ \bar{Z} &: (-1/2, -1/2), & -\bar{Y} &: (-1/2, +1/2). \end{aligned} \tag{6.1.3}$$

Thus, from the representation-theoretic point of view, it is natural to map each of these fields to a tensor product of two spin-states in the following way:

$$\begin{aligned} Z &\mapsto |\uparrow\rangle_L \otimes |\uparrow\rangle_R, & Y &\mapsto |\uparrow\rangle_L \otimes |\downarrow\rangle_R, \\ \bar{Z} &\mapsto |\downarrow\rangle_L \otimes |\downarrow\rangle_R, & -\bar{Y} &\mapsto |\downarrow\rangle_L \otimes |\uparrow\rangle_R, \end{aligned} \tag{6.1.4}$$

This evidently leads to the double spin-chain formalism, which will be much more versatile than the conventional single spin-chain treatment. As an example, consider a general linear combination of the four fields, which can be written as

$$\mathrm{P} \cdot \Phi \equiv \sum_{a,\tilde{a}} \mathrm{P}^{a\tilde{a}} \Phi_{a\tilde{a}}, \tag{6.1.5}$$

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[3]In the "tailoring" formulation [115], the pair $(\bar{Z}, Y)$ is not needed for the construction of three distinct spin-chains making up the three-point functions.

where $\mathrm{P}^{a\tilde{a}}$ is a 2×2 matrix. Then, clearly this quantity maps to the double spin-chain state as

$$\mathrm{P} \cdot \Phi \mapsto \mathrm{P}^{1\tilde{1}}|\uparrow\rangle_L \otimes |\uparrow\rangle_R + \mathrm{P}^{1\tilde{2}}|\uparrow\rangle_L \otimes |\downarrow\rangle_R + \mathrm{P}^{2\tilde{1}}|\downarrow\rangle_L \otimes |\uparrow\rangle_R + \mathrm{P}^{2\tilde{2}}|\downarrow\rangle_L \otimes |\downarrow\rangle_R. \quad (6.1.6)$$

6.1.2 General rotated vacua

Next let us turn to the construction and the description of the general spin-chains. To do this, we must first prepare a general vacuum state upon which the SU(2) magnon excitations are created. The most transparent way to construct such a general vacuum state is to make an arbitrary $\mathrm{SU}(2)_L \times \mathrm{SU}(2)_R$ transformation to the conventional BPS vacuum state $\mathrm{Tr}\,(Z^\ell)$, where ℓ is the length of the spin-chain. Under the transformation (6.1.2), Z itself turns into

$$Z = (\Phi)_{11} \longrightarrow (U_L \Phi U_R)_{11} = (U_L)_1{}^a \Phi_{a\tilde{a}} (U_R)^{\tilde{a}}{}_1 \quad (6.1.7)$$

Comparing this with the general linear combination $\mathrm{P}^{a\tilde{a}}\Phi_{a\tilde{a}}$, we learn that $\mathrm{P}^{a\tilde{a}}$ can be written as a product

$$\mathrm{P}^{a\tilde{a}} = \mathfrak{n}^a \tilde{\mathfrak{n}}^{\tilde{a}}, \quad (6.1.8)$$
$$\mathfrak{n}^a = (U_L)_1{}^a, \qquad \tilde{\mathfrak{n}}^{\tilde{a}} = (U_R)^{\tilde{a}}{}_1$$

Hereafter, we use the notations where the indices a and $\tilde{a}$ are lowered and raised by the ϵ tensors $\epsilon_{ab}, \epsilon_{\tilde{a}\tilde{b}}, \epsilon^{a\tilde{a}}, \epsilon^{a\tilde{a}}$, with the convention $\epsilon_{12} = 1, \epsilon^{12} = 1$. This means $\epsilon_{ab}\epsilon^{bc} = -\delta_a^c$ and $\epsilon_{ab}\epsilon^{ab} = 2$, etc. For instance, $\mathrm{P}_{a\tilde{a}}$ is defined as $\mathrm{P}_{a\tilde{a}} = \epsilon_{ab}\epsilon_{\tilde{a}\tilde{b}}\mathrm{P}^{b\tilde{b}}$. Then it is easy to see that $\mathrm{P}^{a\tilde{a}}$ is nilpotent in the sense that $\mathrm{P}^{a\tilde{a}}\mathrm{P}_{a\tilde{a}} = \mathfrak{n}^a \tilde{\mathfrak{n}}^{\tilde{a}} \epsilon_{ab}\epsilon_{\tilde{a}\tilde{b}} \mathfrak{n}^b \tilde{\mathfrak{n}}^{\tilde{b}} = 0$.

Now because of the structure (6.1.8), the combination $\mathrm{P} \cdot \Phi$ is mapped to the spin state

$$\mathrm{P} \cdot \Phi \mapsto |\mathfrak{n}\rangle_L \otimes |\tilde{\mathfrak{n}}\rangle_R, \quad (6.1.9)$$

where

$$|\mathfrak{n}\rangle_L \equiv \mathfrak{n}^1|\uparrow\rangle_L + \mathfrak{n}^2|\downarrow\rangle_L, \qquad |\tilde{\mathfrak{n}}\rangle_R \equiv \tilde{\mathfrak{n}}^1|\uparrow\rangle_R + \tilde{\mathfrak{n}}^2|\downarrow\rangle_R. \quad (6.1.10)$$

This makes it clear that the two dimensional vectors $\mathfrak{n}^a$ and $\tilde{\mathfrak{n}}^{\tilde{a}}$ characterize the scalar fields completely. Such vectors were introduced in [111] and were termed "polarization spinors"[4].

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[4]Note that the notation for the polarizations is slightly different from the one in [111]: In [111], we denoted the $\mathrm{SU}(2)_L$ polarization spinor by $\tilde{n}$ and $\mathrm{SU}(2)_R$ polarization spinor by n.

It is now easy to see that the rotated BPS vacuum

$$\text{Tr}\left((\mathrm{P}\cdot\Phi)^{\ell}\right) \tag{6.1.11}$$

is mapped to the spin-chain state of the form

$$\text{Tr}\left((\mathrm{P}\cdot\Phi)^{\ell}\right) \mapsto |\mathfrak{n}^{\ell}\rangle_L \otimes |\tilde{\mathfrak{n}}^{\ell}\rangle_R , \tag{6.1.12}$$

where $|\mathfrak{n}^{\ell}\rangle_L$ and $|\tilde{\mathfrak{n}}^{\ell}\rangle_R$ are given by

$$|\mathfrak{n}^{\ell}\rangle_L = \underbrace{|\mathfrak{n}\rangle_L \otimes \cdots \otimes |\mathfrak{n}\rangle_L}_{\ell}, \quad |\tilde{\mathfrak{n}}^{\ell}\rangle_R = \underbrace{|\tilde{\mathfrak{n}}\rangle_R \otimes \cdots \otimes |\tilde{\mathfrak{n}}\rangle_R}_{\ell} . \tag{6.1.13}$$

For later convenience, we impose the following normalization conditions on the polarization spinors:

$$\mathfrak{n}^a \overline{\mathfrak{n}}_a = 1 , \quad \tilde{\mathfrak{n}}^{\tilde{a}} \overline{\tilde{\mathfrak{n}}}_{\tilde{a}} = 1 , \tag{6.1.14}$$

where the "conjugate spinors" $\overline{\mathfrak{n}}$ and $\overline{\tilde{\mathfrak{n}}}$ are defined by

$$\overline{\mathfrak{n}}_a \equiv (\mathfrak{n}^a)^* , \quad \overline{\tilde{\mathfrak{n}}}_{\tilde{a}} \equiv (\tilde{\mathfrak{n}}^{\tilde{a}})^* , \tag{6.1.15}$$

The condition (8.2.17) determines the normalization of the operator (H.9) up to a phase. The phases of the operators only affect the overall phase of the structure constant, which we will not discuss in this chapter.

6.1.3 Non-BPS operators as excitations on rotated vacua

We will now express non-BPS operators as excited states on the general rotated vacua constructed in the previous subsection.

The strategy is straightforward. We will first consider the excited states built upon the conventional vacuum $|\uparrow^{\ell}\rangle$ in both the left and the right sectors by the algebraic Bethe ansatz procedure. Explicitly, the states obtained are

$$|\boldsymbol{u};\uparrow^{\ell}\rangle_L = B(u_1)\cdots B(u_M)|\uparrow^{\ell}\rangle_L , \qquad |\tilde{\boldsymbol{u}};\uparrow^{\ell}\rangle_R = B(\tilde{u}_1)\cdots B(\tilde{u}_{\tilde{M}})|\uparrow^{\ell}\rangle_R , \tag{6.1.16}$$

where the sets of rapidities $\boldsymbol{u}$ and $\tilde{\boldsymbol{u}}$ are assumed to satisfy the Bethe equation. As is customary, the magnon creation operator $B(u)$ is defined through the monodromy matrix as

$$\Omega(u) \equiv \mathrm{L}_1(u-\theta_1)\mathrm{L}_2(u-\theta_2)\cdots\mathrm{L}_{\ell}(u-\theta_{\ell}) = \begin{pmatrix} A(u) & B(u) \\ C(u) & D(u) \end{pmatrix} ,$$
$$\mathrm{L}_k(u) = \begin{pmatrix} u+iS_3^k & iS_-^k \\ iS_+^k & u-iS_3^k \end{pmatrix} , \tag{6.1.17}$$

where S_*^k denotes the SU(2) spin operator acting on the k-th site of the spin chain and $L_k(u)$ is the Lax operator associated to site k. The extra parameters θ's introduced here are called the inhomogeneities. To compute the tree-level correlation functions, we do not need such parameters and they should be simply set to zero. However, as discussed in [?, ?, 119, 127], the inhomogeneities are known to be useful for discussing the loop corrections to the three-point functions. Therefore, we will keep them in the following discussions.

Now in order to obtain the state which can be interpreted as an SU(2) spin-chain, we may excite either the left sector or the right sector, but not both. If we excite both, such a state cannot be obtained by any embedding of SU(2) in SO(4). Therefore, we have the following two types of excited states, which we call type I and type II:

$$\text{Type I}: |\boldsymbol{u};\uparrow^\ell\rangle_L \otimes |\uparrow^\ell\rangle_R\,, \qquad \text{Type II}: |\uparrow^\ell\rangle_L \otimes |\tilde{\boldsymbol{u}};\uparrow^\ell\rangle_R\,. \tag{6.1.18}$$

It is important to note that they cannot be related by an SO(4) rotation since there is no transformation within SO(4) which interchanges $\text{SU}(2)_L$ and $\text{SU}(2)_R$.

Once we have these basic states, we can now rotate them by an arbitrary $\text{SU}(2)_L \times \text{SU}(2)_R$ transformation to produce general excited states. A very useful way to parametrize the $\text{SU}(2)_L$ and $\text{SU}(2)_R$ transformations is as follows. As shown previously the polarization spinors characterize the rotated fields precisely. Therefore one can specify, for example, an element $\mathfrak{g}_\mathfrak{n} \in \text{SU}(2)_L$ by the equation

$$\mathfrak{g}_\mathfrak{n}|\uparrow\rangle_L = |\mathfrak{n}\rangle_L\,, \tag{6.1.19}$$

up to a phase coming from the U(1) rotation h which leaves $|\uparrow\rangle$ invariant. Since we shall ignore such a phase in this work, what is relevant is actually the parametrization of the coset $\text{SU}(2)/\text{U}(1)$, the element of which will be denoted by $g_\mathfrak{n}$, where

$$\mathfrak{g}_\mathfrak{n} = g_\mathfrak{n} h\,, \qquad \mathfrak{g}_\mathfrak{n} \in \text{SU}(2)\,, \quad g_\mathfrak{n} \in \text{SU}(2)/\text{U}(1)\,, \quad h \in \text{U}(1)\,. \tag{6.1.20}$$

Among the various parametrizations of $\text{SU}(2)/\text{U}(1)$, the one which will be most useful is the so-called the coherent state parametrization. In the spin $1/2$ highest weight representation we are adopting, the useful expression for the coset element $g_\mathfrak{n}$ is obtained by the SU(2) Baker-Campbell-Hausdorff formula in the form [175]

$$g_\mathfrak{n} = e^{-\bar{\zeta}S_+ + \zeta S_-} = e^{zS_-}e^{-\ln(1+|z|^2)S_3}e^{-\bar{z}S_+}\,, \tag{6.1.21}$$

$$= e^{-\bar{z}S_+}e^{\ln(1+|z|^2)S_3}e^{zS_-} \tag{6.1.22}$$

where $z = (\zeta/|\zeta|)\tan|\zeta|$ and S_i's are the generators of the global SU(2), with the convention $S_\pm \equiv S_1 \pm iS_2$. Since $|\uparrow\rangle_L$ corresponds to $\mathfrak{n}^a = (1,0)^t$, applying (6.1.21) we

get

$$\mathfrak{n}^a = g_{\mathfrak{n}} \begin{pmatrix} 1 \\ 0 \end{pmatrix}^a = \frac{1}{\sqrt{1+|z|^2}} \begin{pmatrix} 1 \\ z \end{pmatrix}^a . \tag{6.1.23}$$

Similarly, coset elements corresponding to $|\uparrow\rangle_R, |\downarrow\rangle_L, |\downarrow\rangle_R$ are characterized by

$$g_{\mathfrak{n}} |\downarrow\rangle_L = |\bar{\mathfrak{n}}\rangle_R , \qquad \tilde{g}_{\tilde{\mathfrak{n}}} |\uparrow\rangle_L = |\tilde{\mathfrak{n}}\rangle_R , \qquad \tilde{g}_{\tilde{\mathfrak{n}}} |\downarrow\rangle_R = |\bar{\tilde{\mathfrak{n}}}\rangle_R , \tag{6.1.24}$$

and the corresponding polarization spinors can be computed similarly, using (6.1.21) or (6.1.22) where appropriate, as[5]

$$\mathfrak{n}^a = \frac{1}{\sqrt{1+|z|^2}} \begin{pmatrix} 1 \\ z \end{pmatrix}^a , \qquad \bar{\mathfrak{n}}^a = \frac{1}{\sqrt{1+|z|^2}} \begin{pmatrix} -\bar{z} \\ 1 \end{pmatrix}^a , \tag{6.1.25}$$

$$\tilde{\mathfrak{n}}^{\tilde{a}} = \frac{1}{\sqrt{1+|\tilde{z}|^2}} \begin{pmatrix} 1 \\ \tilde{z} \end{pmatrix}^a , \qquad \bar{\tilde{\mathfrak{n}}}^{\tilde{a}} = \frac{1}{\sqrt{1+|\tilde{z}|^2}} \begin{pmatrix} -\bar{\tilde{z}} \\ 1 \end{pmatrix}^{\tilde{a}} . \tag{6.1.26}$$

With this preparation, it is now straightforward to write down the general excited states of type I and II built upon the rotated vacuum $|\mathfrak{n}^\ell\rangle \otimes |\tilde{\mathfrak{n}}^\ell\rangle$ as

$$\text{Type I} : |\boldsymbol{u}; \mathfrak{n}^\ell\rangle_L \otimes |\tilde{\mathfrak{n}}^\ell\rangle_R , \qquad \text{Type II} : |\mathfrak{n}^\ell\rangle_L \otimes |\tilde{\boldsymbol{u}}; \tilde{\mathfrak{n}}^\ell\rangle_R , \tag{6.1.27}$$

where $|\boldsymbol{u}; \mathfrak{n}^\ell\rangle_L$ and $|\tilde{\boldsymbol{u}}; \tilde{\mathfrak{n}}^\ell\rangle_R$ are obtained by the SU(2)$_L$ and SU(2)$_R$ rotations discussed above:

$$|\boldsymbol{u}; \mathfrak{n}^\ell\rangle_L \equiv g_{\mathfrak{n}} |\boldsymbol{u}; \uparrow^\ell\rangle_L , \quad |\tilde{\boldsymbol{u}}; \tilde{\mathfrak{n}}^\ell\rangle_R \equiv \tilde{g}_{\tilde{\mathfrak{n}}} |\tilde{\boldsymbol{u}}; \uparrow^\ell\rangle_R . \tag{6.1.28}$$

Now it is well-known that, when the rapidities $\boldsymbol{u}$ and $\tilde{\boldsymbol{u}}$ are all finite, the on-shell Bethe states constructed upon the up-spin vacuum (6.1.16) satisfy the highest weight condition

$$S_+ |\boldsymbol{u}; \uparrow^\ell\rangle_L = 0 , \qquad S_+ |\tilde{\boldsymbol{u}}; \uparrow^\ell\rangle_R = 0 . \tag{6.1.29}$$

Upon such states, the actions of $g_{\mathfrak{n}}$ and $\tilde{g}_{\tilde{\mathfrak{n}}}$ simplify because the last factor in (6.1.21) becomes unity. As a result, we obtain the following expressions[6]:

$$\begin{aligned}
|\boldsymbol{u}; \mathfrak{n}^\ell\rangle_L &= \left(\frac{1}{1+|z|^2} \right)^{\ell/2-M} e^{zS_-} |\boldsymbol{u}; \uparrow^\ell\rangle_L , \\
|\tilde{\boldsymbol{u}}; \tilde{\mathfrak{n}}^\ell\rangle_R &= \left(\frac{1}{1+|\tilde{z}|^2} \right)^{\ell/2-\tilde{M}} e^{\tilde{z}S_-} |\tilde{\boldsymbol{u}}; \uparrow^\ell\rangle_R .
\end{aligned} \tag{6.1.30}$$

We shall see that the representations (6.1.30) will be quite useful when we evaluate the three-point functions in section 6.3.

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[5]We redisplay the result for $\mathfrak{n}^a$ as well for convenience.
[6]The idea to characterize the rotated state in a similar way was proposed previously in [126].

6.2 Wick contraction as skew-symmetric singlet pairing

Having prepared the operators interpretable as general classes of spin-chains built upon rotated vacua, we now discuss how to perform the Wick contractions of such objects in an efficient manner based on a group-theoretical point of view.

6.2.1 Wick contraction for general constituent fields

To begin, let us discuss the Wick contraction of the constituent fields. At the tree level, the contraction rules for the basic complex scalar fields are given by

$$\underset{\textstyle\smile}{Z\ Z} = 0\,, \quad \underset{\textstyle\smile}{Z\ Y} = 0\,, \quad \underset{\textstyle\smile}{Z\ \bar{Y}} = 0\,, \quad \underset{\textstyle\smile}{Z\ \bar{Z}} = 1\,, \quad \text{etc}\,. \tag{6.2.1}$$

It will be most useful to regard these rules as those for the elements of the matrix $\Phi_{a\tilde{a}}$ given in (H.5). It is easy to check that the above rules are neatly summarized as

$$\underset{\textstyle\smile}{\Phi_{a\tilde{a}}\ \Phi_{b\tilde{b}}} = \epsilon_{ab}\epsilon_{\tilde{a}\tilde{b}}\,. \tag{6.2.2}$$

Now recall that the general linear combination of these fields can be written as

$$\mathrm{P}^{a\tilde{a}}\Phi_{a\tilde{a}} = \mathfrak{n}^a\tilde{\mathfrak{n}}^{\tilde{a}}\Phi_{a\tilde{a}}\,, \tag{6.2.3}$$

where we used the factorized expression of $\mathrm{P}^{a\tilde{a}}$ in terms of the polarization spinors (6.1.8). Then, using (6.2.2) and (6.2.3), the contraction of two general combinations denoted as $F_1 = \mathrm{P}_1 \cdot \Phi$ and $F_2 = \mathrm{P}_2 \cdot \Phi$ can be immediately computed as

$$\underset{\textstyle\smile}{F_1\ F_2} = \left(\mathfrak{n}_1^a\mathfrak{n}_{2a}\right)\left(\tilde{\mathfrak{n}}_1^{\tilde{a}}\tilde{\mathfrak{n}}_{2\tilde{a}}\right)\,. \tag{6.2.4}$$

This formula reveals that in terms of the polarization spinors the Wick contraction is nothing but the operation of forming singlets in both the $\mathrm{SU}(2)_L$ and the $\mathrm{SU}(2)_R$ sectors.

We now would like to transplant this structure in the spin-chain language. For this purpose, it is convenient to write the up and the down spin state collectively as $|a\rangle$ with the definition[7]

$$\begin{aligned} |a\rangle : \quad |1\rangle &\equiv |\uparrow\rangle\,, \quad |2\rangle \equiv |\downarrow\rangle\,, \\ \langle a|b\rangle &= \delta_{ab}\,. \end{aligned} \tag{6.2.5}$$

Then, from the definition of $|\mathfrak{n}\rangle$ given in (8.1.5) we have

$$|\mathfrak{n}\rangle = \mathfrak{n}^a|a\rangle\,, \qquad \mathfrak{n}^a = \langle a|\mathfrak{n}\rangle\,. \tag{6.2.6}$$

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[7]Of course we do this for both the left and the right sectors. Here for simplicity we suppress the subscripts L and R, as the structure is common.
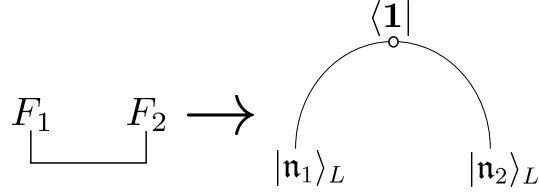
Figure 6.2.1: The wick contraction and the singlet pairing (8.1.36). The white blob denotes the singlet state $\langle \mathbf{1} |$. Here we only depicted $\mathrm{SU}(2)_L$ sector.

Let us now introduce the singlet projection operator $\langle \mathbf{1} |$ in the following way:

$$\langle \mathbf{1} | \equiv \epsilon_{ab} \langle a | \otimes \langle b | . \tag{6.2.7}$$

When acted on the state of the form $| \mathfrak{n}_1 \rangle \otimes | \mathfrak{n}_2 \rangle$, it projects out the singlet in the manner

$$\langle \mathbf{1} | \left(| \mathfrak{n}_1 \rangle \otimes | \mathfrak{n}_2 \rangle \right) = \epsilon_{ab} \langle a | \mathfrak{n}_1 \rangle \langle b | \mathfrak{n}_2 \rangle = \epsilon_{ab} \mathfrak{n}_1^a \mathfrak{n}_2^b = \mathfrak{n}_1^a \mathfrak{n}_{2a} . \tag{6.2.8}$$

Therefore the contraction $F_1\,F_2$ given in (6.2.4) is reproduced as

$$F_1\,F_2 = \langle \mathbf{1} | \left(| \mathfrak{n}_1 \rangle_L \otimes | \mathfrak{n}_2 \rangle_L \right) \langle \mathbf{1} | \left(| \tilde{\mathfrak{n}}_1 \rangle_R \otimes | \tilde{\mathfrak{n}}_2 \rangle_R \right) \tag{6.2.9}$$

This relation is expressed pictorially in Figure 6.2.1. Note that each factor on the right hand side of (8.1.36) is anti-symmetric under the interchange of two spin states, unlike the ordinary inner product used in the previous works [115–117, 119, 123, 127].

It should be remarked that the appearance of the singlet state in the expression (8.1.36) is quite natural from a physical point of view: Every Feynman diagram, including the ones with vertices, can be viewed, from an appropriate direction, as a virtual process in which the fields annihilate into the vacuum. Since the vacuum is not charged under any symmetry, it belongs to the singlet representation for all the symmetry groups. Thus, different Feynman diagrams account for different ways of producing the singlet representation starting from a given field-configuration. The simplest way to achieve this is to take a pair of fields and project it to the singlet representation, which is exactly what (8.1.36) does. This argument suggests that the singlet state will play an important role also in other sectors[8] and at higher-loop order, although the expression will certainly be more complicated than (8.1.36).

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[8]Although our motivation was to provide a new interpretation for the Wick contraction of the fields forming a spin chain, a very similar idea of invariant pairing was introduced in a different context, namely the mapping from CFT_4 to TFT_2 in [176]. Thieir description is likely to be quite useful for the construction of three-point functions for the non-compact sectors of the PSU(2,2|4) spin chain.

$$\left\langle |\mathcal{O}_1\rangle_L , |\mathcal{O}_2\rangle_L \right\rangle \;=\;$$

$$|\mathcal{O}_1\rangle_L \qquad\qquad |\mathcal{O}_2\rangle_L$$

Figure 6.2.2: A pictorial definition of the skew-symmetric inner product for two spin-chain states: We first compute an overlap between the singlet state $\langle \mathbf{1}|$ and a tensor product of two spins connected to a single blob ($\circ$), and then, take a product of such overlaps. Here we only described the $\mathrm{SU}(2)_L$ chain. The definition for the $\mathrm{SU}(2)_R$ chain is basically the same.

6.2.2   Wick contraction for two composite operators

Let us next express the Wick contraction between two composite operators $\mathcal{O}_1$ and $\mathcal{O}_2$ using the skew-symmetric inner product defined above. In what follows, we denote the spin-chain states corresponding to the operators $\mathcal{O}_1$ and $\mathcal{O}_2$ abstractly as[9]

$$\mathcal{O}_1 \mapsto |\mathcal{O}_1\rangle_L \otimes |\tilde{\mathcal{O}}_1\rangle_R, \qquad \mathcal{O}_2 \mapsto |\mathcal{O}_2\rangle_L \otimes |\tilde{\mathcal{O}}_2\rangle_R. \tag{6.2.10}$$

As we are working in the large N_c limit, the unsuppressed Wick contractions between two composite operators are of a special type, an example of which is given by

$$\mathrm{tr}\big(\cdots Y Z\big) \qquad \mathrm{tr}\big(\bar{Z}\bar{Y}\cdots\big). \tag{6.2.11}$$

The structure should be clear: the allowed contractions are between the rightmost field in $\mathcal{O}_1$ with the leftmost field in $\mathcal{O}_2$ and so on, as indicated. Obviously the two spin chains must be of the same length to be non-vanishing under the contractions.

This type of contraction rule is expressed in the spin-chain language by using the following skew-symmetric inner product between two states of the same length, $|\Psi_1\rangle$ and $|\Psi_2\rangle$:

$$\left\langle |\Psi_1\rangle , |\Psi_2\rangle \right\rangle \equiv \left(\prod_{k=1}^{\ell} \langle \mathbf{1}_{k;\ell+1-k}| \right) |\Psi_1\rangle \otimes |\Psi_2\rangle. \tag{6.2.12}$$

Here ℓ is the length of the spin chain and $\langle \mathbf{1}_{k;\ell+1-k}|$ is the state which projects out the singlet part made out of the spin state at k-th site of $|\Psi_1\rangle$ and the one at the $(\ell+1-k)$-th site of $|\Psi_2\rangle$. The operation should be quite clear from Figure 6.2.2. In terms of the bracket

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[9]$\mathcal{O}_1$ and $\mathcal{O}_2$ can be either of type I or type II in (8.1.13).

defined in (6.2.12), the contraction between $\mathcal{O}_1$ and $\mathcal{O}_2$ can be expressed as

$$\underline{\mathcal{O}_1\,\mathcal{O}_2} = \big\langle |\mathcal{O}_1\rangle_L\,, |\mathcal{O}_2\rangle_L \big\rangle \big\langle |\tilde{\mathcal{O}}_1\rangle_R\,, |\tilde{\mathcal{O}}_2\rangle_R \big\rangle\,. \tag{6.2.13}$$

As is manifest in (8.3.9), the Wick contraction between two operators factorizes into the part coming from the $SU(2)_L$ chain and the part coming from the $SU(2)_R$ chain. This factorization property continues to hold for the tree-level three-point functions, since they are computed through the contractions between the composite operators in the manner described above.

6.3 Construction and evaluation of three-point functions

Up to this point, we have developed a new way of performing the Wick contractions between the composite operators in the spin-chain language suitable for dealing with a certain general class of operators in the $SU(2)$ sector. We now use this technology to assemble the three-point functions and show that they will possess determinant expressions.

6.3.1 Three-point function as factorized spin-chain products

To perform the actual calculations, let us first clarify the basic structure of the three-point functions, in particular their characteristic feature of the factorization into the left and the right sector.

As explained in [115], the three-point function can be computed by first mapping the operators to the spin-chain states, then splitting each spin chain into the left and the right sub-chains (the *cutting* procedure) and finally computing the Wick contractions between the right sub-chain of $\mathcal{O}_1$ and the left sub-chain of $\mathcal{O}_2$ etc., using a suitably-chosen inner product for the spin chains (the *sewing* procedure).

In our formulation, the situation might at first sight appear more involved, since each operator $\mathcal{O}_i$ is expressed as a tensor product of two spin-chain states, $|\mathcal{O}_i\rangle_L$ and $|\tilde{\mathcal{O}}_i\rangle_R$, and then we need to split each of them into two sub-chains. However, it is actually more transparent since, as already emphasized, the contributions from the $SU(2)_L$- and $SU(2)_R$-chains completely factorize and hence the $SU(2)_L$- and $SU(2)_R$-chains can be discussed separately. Thus, below let us first focus only on the $SU(2)_L$-chain.

After the cutting, each spin-chain state is expressed as an entangled state of two states

defined on the sub-chains in the following manner:

$$|\mathcal{O}_1\rangle_L = \sum_a |\mathcal{O}_{1_a}\rangle^l \otimes |\mathcal{O}_{1_a}\rangle^r,$$

$$|\mathcal{O}_2\rangle_L = \sum_b |\mathcal{O}_{2_b}\rangle^l \otimes |\mathcal{O}_{2_b}\rangle^r, \tag{6.3.1}$$

$$|\mathcal{O}_3\rangle_L = \sum_c |\mathcal{O}_{3_c}\rangle^l \otimes |\mathcal{O}_{3_c}\rangle^r.$$

Here the superscripts l and r denote the left and the right sub-chain. The length of each sub-chain is determined from the Wick contraction rule and is given by

$$\text{Length of } |\mathcal{O}_{1_a}\rangle^r \text{ and } |\mathcal{O}_{2_b}\rangle^l : \frac{\ell_1 + \ell_2 - \ell_3}{2} \equiv \ell_{12},$$

$$\text{Length of } |\mathcal{O}_{2_b}\rangle^r \text{ and } |\mathcal{O}_{3_c}\rangle^l : \frac{\ell_2 + \ell_3 - \ell_1}{2} \equiv \ell_{23}, \tag{6.3.2}$$

$$\text{Length of } |\mathcal{O}_{3_c}\rangle^r \text{ and } |\mathcal{O}_{1_a}\rangle^l : \frac{\ell_3 + \ell_1 - \ell_2}{2} \equiv \ell_{31},$$

where ℓ_i is the length of the spin chain $|\mathcal{O}_i\rangle_L$.

Once the cutting is performed, the rest is to compute the Wick contractions between various sub-chains using the inner product (6.2.12). As a result, we get the "three-spin-chain product" defined in the following way (see also Figure 6.3.1):

$$\langle |\mathcal{O}_1\rangle_L, |\mathcal{O}_2\rangle_L, |\mathcal{O}_3\rangle_L \rangle \equiv \sum_{a,b,c} \langle |\mathcal{O}_{1_a}\rangle^r, |\mathcal{O}_{2_b}\rangle^l \rangle \langle |\mathcal{O}_{2_b}\rangle^r, |\mathcal{O}_{3_c}\rangle^l \rangle \langle |\mathcal{O}_{3_c}\rangle^r . |\mathcal{O}_{1_a}\rangle^l \rangle. \tag{6.3.3}$$

Multiplying the contribution from the $SU(2)_R$ sector, which is entirely similar to (6.3.3), the final formal expression for the structure constant is given by

$$C_{123} = \frac{\sqrt{\ell_1 \ell_2 \ell_3}}{N_c \sqrt{\mathcal{N}_1 \mathcal{N}_2 \mathcal{N}_3}} \langle |\mathcal{O}_1\rangle_L, |\mathcal{O}_2\rangle_L, |\mathcal{O}_3\rangle_L \rangle \langle |\tilde{\mathcal{O}}_1\rangle_R, |\tilde{\mathcal{O}}_2\rangle_R, |\tilde{\mathcal{O}}_3\rangle_R \rangle, \tag{6.3.4}$$

where $\mathcal{N}_k$ denotes a factor coming from the normalization of the operator $\mathcal{O}_k$. As advertised several times already, the expression (6.3.4) of the structure constant completely factorizes into the contributions from the $SU(2)_L$ and the $SU(2)_R$ parts. This phenomenon was already observed in [115,123] for a restricted class of three-point functions but (6.3.4) tells us that it is a much more general property as long as three $SU(2)$-operators can be embedded in a single $SO(4)$. In any case, the expression of the structure constant above is as yet formal, and in the rest of this section we shall perform the cutting and sewing explicitly in our new formalism and that will naturally lead to the determinantal formula for the three-point functions.

$$\langle |\mathcal{O}_1\rangle_L, |\mathcal{O}_2\rangle_L, |\mathcal{O}_3\rangle_L\rangle =$$

$$|\mathcal{O}_1\rangle_L \qquad |\mathcal{O}_2\rangle_L \qquad |\mathcal{O}_3\rangle_L$$

Figure 6.3.1: A pictorial definition of the three-spin-chain product. As in Figure 6.2.2, at each white blob, we compute the overlap with the singlet state $\langle \mathbf{1}|$. The number of curves connecting the state $|\mathcal{O}_i\rangle_L$ and $|\mathcal{O}_j\rangle_L$ is determined solely by the length of the operators to be $(\ell_i + \ell_j - \ell_k)/2$.

6.3.2 "Cutting and sewing" in the new formulation

Let us begin with the explanation of the cutting procedure in our formalism. Due to the factorization property we only need to focus on the $\mathrm{SU}(2)_L$ part. Below we only consider the operators satisfying the highest weight conditions. As shown in (6.1.30), such operators can be expressed as Bethe states multiplied by the operator e^{zS_-}. The cutting procedure of Bethe states is already studied in [115] using the method called "generalized two-component model" and the result in our notation takes the form

$$|\boldsymbol{u};\uparrow^\ell\rangle_L = \sum_{\boldsymbol{\alpha}_l \cup \boldsymbol{\alpha}_r = \boldsymbol{u}} H_\ell(\boldsymbol{\alpha}_l, \boldsymbol{\alpha}_r|\boldsymbol{\theta}) |\boldsymbol{\alpha}_l;\uparrow^{\ell_l}\rangle \otimes |\boldsymbol{\alpha}_r;\uparrow^{\ell_r}\rangle. \tag{6.3.5}$$

The sum is over all possible ways of splitting the rapidities $\boldsymbol{u}$ into two groups $\boldsymbol{\alpha}_l$ and $\boldsymbol{\alpha}_r$, the symbols ℓ_l and ℓ_r denote respectively the length of the left and the right sub-chains[10] and the coefficient function $H_\ell(\boldsymbol{\alpha}_l, \boldsymbol{\alpha}_r|\boldsymbol{\theta})$ reads[11]

$$H_\ell(\boldsymbol{\alpha}_l, \boldsymbol{\alpha}_r|\boldsymbol{\theta}) \equiv \prod_{u\in\boldsymbol{\alpha}_l} \prod_{v\in\boldsymbol{\alpha}_r} \prod_{a=\ell_l+1}^{\ell} \prod_{b=1}^{\ell_l} \left(\frac{u-v+i}{u-v}\right)\left(u-\theta_a-\frac{i}{2}\right)\left(v-\theta_b+\frac{i}{2}\right). \tag{6.3.6}$$

On the other hand, the splitting of the prefactor e^{zS_-} is simple since the global $\mathrm{SU}(2)$ generator S_- for the full chain is just a sum of the generators for the sub-chains: $S_- = S_-^l \otimes \mathbf{1} + \mathbf{1} \otimes S_-^r$. Therefore, after the cutting procedure, the rotated excited state (6.1.30) is expressed as

$$e^{zS_-}|\boldsymbol{u};\uparrow^\ell\rangle_L = \sum_{\boldsymbol{\alpha}_l \cup \boldsymbol{\alpha}_r = \boldsymbol{u}} H_\ell(\boldsymbol{\alpha}_l, \boldsymbol{\alpha}_r)\left(e^{zS_-^l}|\boldsymbol{\alpha}_l;\uparrow^{\ell_l}\rangle\right) \otimes \left(e^{zS_-^r}|\boldsymbol{\alpha}_r;\uparrow^{\ell_r}\rangle\right). \tag{6.3.7}$$

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[10]Note that ℓ_l and ℓ_r satisfy $\ell = \ell_l + \ell_r$.

[11]Just as in [115], this coefficient is obtained by re-expressing $B(u)$ of the original chain in terms the elements of the monodromy matrices Ω_l and Ω_r of the left and the right sub-chains through the relation $B(u) = \Omega(u)_{12} = (\Omega_l(u)\Omega_r(u))_{12}$ and then pushing the operators A_l and D_r to the right using the Yang-Baxter algebra.

Although the cutting procedure described above is quite similar to the one developed in [115], except for the $SU(2)_L$-$SU(2)_R$ factorization property, the sewing procedure in our formalism is substantially different, with a definite advantage. To describe this, we use an important property, which we call the "crossing" relation, of the Lax operator

$$L(u) = \begin{pmatrix} u + iS_3 & iS_- \\ iS_+ & u - iS_3 \end{pmatrix}. \tag{6.3.8}$$

Let $|s_1\rangle$ and $|s_2\rangle$ be two arbitrary spin $1/2$ states and consider the overlap with the singlet state with the Lax-operator insertion: $\langle \mathbf{1}| (L(u - \theta)|s_1\rangle \otimes |s_2\rangle)$. Using the definition of the singlet state, one can show the following relation by direct computation:

$$\langle \mathbf{1}| (L(u - \theta)|s_1\rangle \otimes |s_2\rangle) = \langle \mathbf{1}| (|s_1\rangle \otimes \mathcal{C} \circ L(u - \theta)|s_2\rangle), \tag{6.3.9}$$

where $\mathcal{C} \circ L(u)$ is the "crossed Lax operator", which is given by

$$\mathcal{C} \circ L(u) \equiv \sigma_2 L^t(u) \sigma_2 = \begin{pmatrix} u - iS_3 & -iS_- \\ -iS_+ & u + iS_3 \end{pmatrix}. \tag{6.3.10}$$

In (6.3.10), σ_2 acts on the auxiliary space and the superscript t denotes the transposition in the auxiliary space. This relation, if we regard σ_2 as the charge conjugation matrix, can be viewed as a sort of the crossing relation of the factorized S-matrices[12] and we therefore call (6.3.10) the "crossing" relation.

The relation (6.3.10) leads to a useful nontrivial identity of the monodromy matrix. Let $|\psi_1\rangle$ and $|\psi_2\rangle$ to be arbitrary spin-chain states of the same length. They can be either on-shell Bethe states describing each operator or off-shell Bethe states which appear after the cutting procedure. Then, from the fundamental relation (6.3.10), the following important relation can be obtained, as we shall prove shortly:

$$\langle \Omega_1(u)|\psi_1\rangle, |\psi_2\rangle \rangle = \langle |\psi_1\rangle, \sigma_2 \Omega_2^t(u) \sigma_2|\psi_2\rangle \rangle. \tag{6.3.11}$$

Here, again t and σ_2 act on the auxiliary space and Ω_n is the monodromy matrix acting on $|\psi_n\rangle$ defined by

$$\Omega_1(u) = L_1(u - \theta_1^{(1)}) \cdots L_\ell(u - \theta_\ell^{(1)}), \quad \Omega_2(u) = L_1(u - \theta_1^{(2)}) \cdots L_\ell(u - \theta_\ell^{(2)}). \tag{6.3.12}$$

The parameters $\theta^{(n)}$'s are the inhomogeneities for $|\psi_n\rangle$. In order for (6.3.11) to be satisfied, we need to make the following identification between the inhomogeneities (see Figure 6.3.2):

$$\theta_k^{(1)} = \theta_{\ell-k+1}^{(2)}. \tag{6.3.13}$$

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[12]For the relation between the crossing symmetry and the scattering with the singlet state, see, for example, [31, 45]. See also the footnote 20.
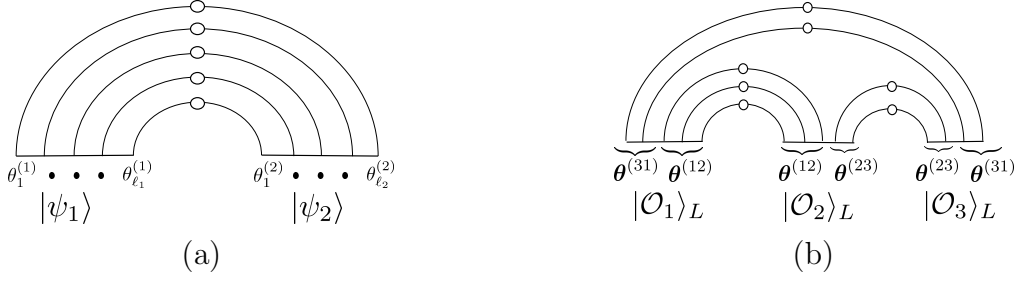
Figure 6.3.2: The identification of the inhomogeneities for the two-point functions and the three-point functions. In both cases, we identify the inhomogeneities connected by a propagator (connected to the same blob). (a) For the two-point functions, the identification is given by (6.3.13). (b) For the three-point functions, the sets of the inhomogeneities are related as (6.3.27).

In terms of the Wick contraction in the gauge theory, this amounts to assigning the same inhomogeneity parameter to each two spin sites contracted by a propagator. This is precisely the identification we need when we study the one-loop correction using the inhomogeneities [119, 124, 125, 127] and we impose such relation throughout this paper.

Let us now prove the relation (6.3.11). From the definition of Ω (6.3.12), we can express the LHS of (6.3.11) as

$$\left\langle \left(L_1(u - \theta_1^{(1)})\right)_{i_1 i_2} \left(L_2(u - \theta_2^{(1)})\right)_{i_2 i_3} \cdots \left(L_\ell(u - \theta_\ell^{(1)})\right)_{i_\ell i_{\ell+1}} |\psi_1\rangle, |\psi_2\rangle \right\rangle, \qquad (6.3.14)$$

where, for definiteness, we wrote down the indices for the auxiliary space. Since the k-th site of $|\psi_1\rangle$ is contracted with the $(\ell - k + 1)$-th site of $|\psi_2\rangle$ as shown in Figure 6.3.2 and the inhomogeneities are identified as (6.3.13), the Lax operator transforms as follows under the application of the crossing relation (6.3.10):

$$\left(L_k(u - \theta_k^{(1)})\right)_{i_k i_{k+1}} \quad \rightarrow \quad \left(\sigma_2 L_{\ell-k+1}^t(u - \theta_{\ell-k+1}^{(2)})\sigma_2\right)_{i_k i_{k+1}}. \qquad (6.3.15)$$

Then, moving the Lax operators one by one, we obtain

$$\left\langle |\psi_1\rangle, \left(\sigma_2 L_\ell^t(u - \theta_\ell^{(2)})\sigma_2\right)_{i_1 i_2} \left(\sigma_2 L_{\ell-1}^t(u - \theta_{\ell-1}^{(2)})\sigma_2\right)_{i_2 i_3} \cdots \left(\sigma_2 L_1^t(u - \theta_1^{(2)})\sigma_2\right)_{i_\ell i_{\ell+1}} |\psi_2\rangle \right\rangle$$

$$= \left\langle |\psi_1\rangle, \left(\sigma_2 (L_1(u - \theta_1^{(2)}) \cdots L_\ell(u - \theta_\ell^{(2)}))^t \sigma_2\right)_{i_1 i_{\ell+1}} |\psi_2\rangle \right\rangle$$

$$= \left\langle |\psi_1\rangle, \left(\sigma_2 \Omega_2^t(u)\sigma_2\right)_{i_1 i_{\ell+1}} |\psi_2\rangle \right\rangle. \qquad (6.3.16)$$

In terms of components, $\sigma_2 \Omega_2^t(u)\sigma_2$ is given by

$$\sigma_2 \Omega_2^t(u)\sigma_2 = \sigma_2 \begin{pmatrix} A^{(2)}(u) & B^{(2)}(u) \\ C^{(2)}(u) & D^{(2)}(u) \end{pmatrix}^t \sigma_2 = \begin{pmatrix} D^{(2)}(u) & -B^{(2)}(u) \\ -C^{(2)}(u) & A^{(2)}(u) \end{pmatrix}. \qquad (6.3.17)$$

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Here and throughout this subsection we put superscripts (1) or (2) in order to distinguish the components of Ω_1 from those of Ω_2. The formula (6.3.11) in particular contains the crucial relation

$$\left\langle B^{(1)}(u)|\psi_1\rangle, |\psi_2\rangle\right\rangle = -\left\langle |\psi_1\rangle, B^{(2)}(u)|\psi_2\rangle\right\rangle, \tag{6.3.18}$$

which only involves $B(u)$ operators. In the $u \to \infty$ limit, the relation (6.3.18) produces

$$\left\langle S_-^{(1)}|\psi_1\rangle, |\psi_2\rangle\right\rangle = -\left\langle |\psi_1\rangle, S_-^{(2)}|\psi_2\rangle\right\rangle. \tag{6.3.19}$$

Then, by a repeated use of (6.3.18) and (6.3.19), we can collect all $B(u)$'s and S_-'s on one side and transform the skew-symmetric inner product which appear in the sewing procedure, such as

$$\left\langle e^{xS_-^{(1)}}B^{(1)}(u_1)\cdots B^{(1)}(u_{M_1})|\!\uparrow^\ell\rangle, e^{yS_-^{(2)}}B^{(2)}(v_1)\cdots B^{(2)}(v_{M_2})|\!\uparrow^\ell\rangle\right\rangle, \tag{6.3.20}$$

into the following expression:

$$(-1)^{M_1}\left\langle |\!\uparrow^\ell\rangle, e^{(y-x)S_-^{(2)}}B^{(2)}(u_1)\cdots B^{(2)}(u_{M_1})B^{(2)}(v_1)\cdots B^{(2)}(v_{M_2})|\!\uparrow^\ell\rangle\right\rangle. \tag{6.3.21}$$

From the definition of the skew-symmetric inner product (6.2.12), this expression can be readily evaluated[13] as a matrix element in the spin-chain Hilbert space as follows:

$$\begin{aligned}
&(-1)^{M_1}\left\langle |\!\uparrow^\ell\rangle, e^{(y-x)S_-^{(2)}}B^{(2)}(u_1)\cdots B^{(2)}(u_{M_1})B^{(2)}(v_1)\cdots B^{(2)}(v_{M_2})|\!\uparrow^\ell\rangle\right\rangle \\
&= (-1)^{M_1}\langle\downarrow^\ell|e^{(y-x)S_-^{(2)}}B^{(2)}(u_1)\cdots B^{(2)}(u_{M_1})B^{(2)}(v_1)\cdots B^{(2)}(v_{M_2})|\!\uparrow^\ell\rangle.
\end{aligned} \tag{6.3.22}$$

It is important to recognize that a matrix element of the form (6.3.22) can be identified with the so-called partial domain wall partition function. More precisely, we can show

$$\langle\downarrow^\ell|e^{zS_-}B(x_1)\cdots B(x_M)|\!\uparrow^\ell\rangle = z^{\ell-M}Z_p(\boldsymbol{x}|\boldsymbol{\theta}), \tag{6.3.23}$$

where $Z_p(\boldsymbol{x}|\boldsymbol{\theta})$ is the partial domain wall partition function (pDWPF), which is given by [124, 125]:

$$\begin{aligned}
Z_p(\boldsymbol{x}|\boldsymbol{\theta}) &\equiv \frac{1}{(\ell-M)!}\langle\downarrow^\ell|S_-^{\ell-M}B(x_1)\cdots B(x_M)|\!\uparrow^\ell\rangle \\
&= \frac{\prod_{i=1}^M\prod_{j=1}^\ell(x_i-\theta_j-i/2)}{\prod_{i<j}(x_i-x_j)}\det\left(x_b^{a-1} - \prod_{c=1}^\ell\frac{x_b-\theta_c+i/2}{x_b-\theta_c-i/2}(x_b-i)^{a-1}\right)_{a,b}.
\end{aligned} \tag{6.3.24}$$

---

[13]Essentially, due to the skew-symmetry, each time the singlet projector acts on a pair of spins, the up-spin is converted to the down-spin and this produces $\langle\downarrow^\ell|$.

The indices a and b run from 1 to M and θ's are the inhomogeneity parameters for the chain. To understand (6.3.23), we just need to expand the exponential e^{zS_-} on the LHS of (6.3.23). Upon doing so, (6.3.23) yields infinitely many terms, each of which has a different number of S_-'s. However, among such terms, only one term

$$\frac{z^{\ell-M}}{(\ell-M)!}\langle\downarrow^\ell|\,(S_-)^{\ell-M}\,B(x_1)\cdots B(x_M)|\uparrow^\ell\rangle \tag{6.3.25}$$

is non-vanishing because of the conservation of the SU(2) spin and it can be readily identified with the pDWPF.

Let us stress that the discussion above is valid both for the on-shell and the off-shell Bethe states. In [134], it was shown that the scalar product between the on-shell Bethe state and the off-shell Bethe state can be transformed into the pDWPF[14]. However, such an argument cannot be applied to the scalar products between two off-shell Bethe states and this was considered to be the main obstacle in studying more general SU(2) three-point functions. In this respect, the argument above clearly shows the advantage of our formulation based on the skew-symmetric inner product as it allows us to use the determinant expression irrespective of whether the Bethe states are on-shell or not.

6.3.3 Representation in terms of the partial domain wall partition function

Let us now combine the results in the previous subsections to write down an explicit expression for general three-point functions. As in the previous subsections, we focus on the contribution from the SU(2)$_L$ sector $\langle|\mathcal{O}_1\rangle_L,|\mathcal{O}_2\rangle_L,|\mathcal{O}_3\rangle_L\rangle$.

First, using the coset parametrization (6.1.30), each spin-chain state can be expressed as

$$|\mathcal{O}_1\rangle_L = \left(\frac{1}{1+|z_1|^2}\right)^{\ell_1/2-M_1} e^{z_1 S_-}|\boldsymbol{u}^{(1)};\uparrow^{\ell_1}\rangle,$$

$$|\mathcal{O}_2\rangle_L = \left(\frac{1}{1+|z_2|^2}\right)^{\ell_2/2-M_2} e^{z_2 S_-}|\boldsymbol{u}^{(2)};\uparrow^{\ell_2}\rangle, \tag{6.3.26}$$

$$|\mathcal{O}_3\rangle_L = \left(\frac{1}{1+|z_3|^2}\right)^{\ell_3/2-M_3} e^{z_3 S_-}|\boldsymbol{u}^{(3)};\uparrow^{\ell_3}\rangle,$$

where $\boldsymbol{u}^{(k)}$ denotes the set of rapidities for the operator $\mathcal{O}_k$ and its number of elements is denoted by M_k. Then, we can apply the formula (6.3.5) to split each chain into two and compute the skew-symmetric inner product using (6.3.23). When computing the inner product, it is important that we assign the same inhomogeneity parameter to any two spin

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[14]In fact, using our formulation, one can prove the equivalence between the pDWPF and the on-shell-off-shell scalar product (the so-called Kostov-Matsuo trick) by a simple calculation.

sites contracted by a propagator as discussed in the previous subsection. In the current setup, this leads to the following relation among the sets of inhomogeneities (see Figure 6.3.2):

$$\boldsymbol{\theta}^{(1)} = \boldsymbol{\theta}^{(31)} \cup \boldsymbol{\theta}^{(12)}, \quad \boldsymbol{\theta}^{(2)} = \boldsymbol{\theta}^{(12)} \cup \boldsymbol{\theta}^{(23)}, \quad \boldsymbol{\theta}^{(3)} = \boldsymbol{\theta}^{(23)} \cup \boldsymbol{\theta}^{(31)}, \quad (6.3.27)$$

where $\boldsymbol{\theta}^{(n)}$ is the set of inhomogeneities for $|\mathcal{O}_n\rangle_L$ and $\boldsymbol{\theta}^{(nm)}$ denote the set of the inhomogeneities common to $|\mathcal{O}_n\rangle_L$ and $|\mathcal{O}_m\rangle_L$. As a result of these operations, we obtain the following final form expressed in terms of the sum-over-partitions

$$
\begin{aligned}
&\langle |\mathcal{O}_1\rangle_L, |\mathcal{O}_2\rangle_L, |\mathcal{O}_3\rangle_L \rangle \\
&= \left(\frac{1}{1+|z_1|^2}\right)^{\ell_1/2 - M_1} \left(\frac{1}{1+|z_2|^2}\right)^{\ell_2/2 - M_2} \left(\frac{1}{1+|z_3|^2}\right)^{\ell_3/2 - M_3} \\
&\quad \times \sum_{\boldsymbol{\alpha}_l^{(k)} \cup \boldsymbol{\alpha}_r^{(k)} = \boldsymbol{u}^{(k)}} z_{21}^{\ell_{12} - |\boldsymbol{\alpha}_r^{(1)}| - |\boldsymbol{\alpha}_l^{(2)}|} z_{32}^{\ell_{23} - |\boldsymbol{\alpha}_r^{(2)}| - |\boldsymbol{\alpha}_l^{(3)}|} z_{13}^{\ell_{31} - |\boldsymbol{\alpha}_r^{(3)}| - |\boldsymbol{\alpha}_l^{(1)}|} \mathcal{D}_{\{\boldsymbol{\alpha}_{l,r}^{(1)}, \boldsymbol{\alpha}_{l,r}^{(2)} \boldsymbol{\alpha}_{l,r}^{(3)}\}},
\end{aligned}
\quad (6.3.28)
$$

In this expression, $|\boldsymbol{\alpha}_{l,r}^{(k)}|$ stands for the number of elements of $\boldsymbol{\alpha}_{l,r}^{(k)}$ and z_{nm} denotes the difference $z_n - z_m$. The last factor $\mathcal{D}_{\{\boldsymbol{\alpha}_{l,r}^{(1)}, \boldsymbol{\alpha}_{l,r}^{(2)} \boldsymbol{\alpha}_{l,r}^{(3)}\}}$, which is independent of the polarizations, is given in terms of the pDWPF as

$$
\begin{aligned}
\mathcal{D}_{\{\boldsymbol{\alpha}_{l,r}^{(1)}, \boldsymbol{\alpha}_{l,r}^{(2)} \boldsymbol{\alpha}_{l,r}^{(3)}\}} &\equiv (-1)^{|\boldsymbol{\alpha}_r^{(1)}| + |\boldsymbol{\alpha}_r^{(2)}| + |\boldsymbol{\alpha}_r^{(3)}|} \prod_{k=1}^{3} H_{\ell_k}(\boldsymbol{\alpha}_l^{(k)}, \boldsymbol{\alpha}_r^{(k)} | \boldsymbol{\theta}^{(k)}) \\
&\quad \times Z_p\left(\boldsymbol{\alpha}_r^{(1)} \cup \boldsymbol{\alpha}_l^{(2)} | \boldsymbol{\theta}^{(12)}\right) Z_p\left(\boldsymbol{\alpha}_r^{(2)} \cup \boldsymbol{\alpha}_l^{(3)} | \boldsymbol{\theta}^{(23)}\right) Z_p\left(\boldsymbol{\alpha}_r^{(3)} \cup \boldsymbol{\alpha}_l^{(1)} | \boldsymbol{\theta}^{(31)}\right).
\end{aligned}
\quad (6.3.29)
$$

Let us emphasize that our final expression (6.3.28) has a number of advantages. Firstly, the result is valid for the three-point functions built upon more general spin-chain vacua than the ones studied in the literature. Secondly, the result already demonstrates certain separation into the kinematical factor and the dynamical factor. Thirdly, the dynamical factor $\mathcal{D}_{\{\boldsymbol{\alpha}_{l,r}^{(1)}, \boldsymbol{\alpha}_{l,r}^{(2)} \boldsymbol{\alpha}_{l,r}^{(3)}\}}$ is given essentially by a product of the pDWPF, each of which possesses determinant representation. One apparently unsatisfactory feature of (6.3.28) is that it still involves the sums over partitions, which become quite nontrivial especially when the number of magnons is large. As we shall show in the next subsection, however, for certain class of correlators the sum can be reduced to just a single term, by exploiting the SU(2) symmetry[15]. This leads to a remarkably simple expression for which the semi-classical limit can be easily taken.

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[15] A similar idea was utilized to simplify the three-point functions in the SL(2) sector in [141].

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6.3.4 Determinant expressions for a large class of three-point functions

Let us now show that for certain correlators the expression (6.3.28) can be drastically simplified. The correlators we consider are those for which two of the operators belong to one type (type I or type II) and the third to the other type. We call such three-point functions "mixed correlators". In what follows, we study the case in which $\mathcal{O}_1$ and $\mathcal{O}_2$ are of type I and $\mathcal{O}_3$ is of type II since the generalization to other cases is simply a matter of renaming.

The crucial observation for the simplification is the fact that the dependence on the parameters z_i characterizing the operators forming the three-point functions is completely dictated by the SU(2) symmetry. This is shown in Appendix D and is quite analogous to the determination of the position dependence for the three-point functions in two-dimensional conformal field theory. Now as $\mathcal{O}_3$ is of type II and therefore $|\mathcal{O}_3\rangle_L$ contains no magnons in the present case, we can set M_3 in the formula (D.7) to zero and obtain

$$
\langle |\mathcal{O}_1\rangle_L, |\mathcal{O}_2\rangle_L, |\mathcal{O}_3\rangle_L \rangle = \left(\frac{1}{1+|z_1|^2} \right)^{\frac{\ell_1}{2}-M_1} \left(\frac{1}{1+|z_2|^2} \right)^{\frac{\ell_2}{2}-M_2} \left(\frac{1}{1+|z_3|^2} \right)^{\frac{\ell_3}{2}}
$$
$$
\times z_{21}^{\ell_{12}-M_1-M_2} z_{32}^{\ell_{23}-M_2+M_1} z_{13}^{\ell_{31}-M_1+M_2} \mathcal{G},
\qquad (6.3.30)
$$

where the factor $\mathcal{G}$ stands for the term independent of z_i's. As can be easily seen, the first line of (6.3.30) coincides with the second line of (6.3.28). On the other hand, the structure given in the second line of (6.3.30) is not visible in the sum-over-partition expression (6.3.28). In order to compare them more closely, let us expand both sides in powers of z_3. Upon this expansion, the second line of (6.3.30) yields the following term as the highest-order term:

$$
(-1)^{\ell_{31}-M_1+M_2} z_3^{\ell_3} \left(z_{21}^{\ell_{12}-M_1-M_2} \mathcal{G} \right).
\qquad (6.3.31)
$$

On the other hand, if we expand each term in the sum in (6.3.28), we obtain the following expression as the highest-order term[16]:

$$
(-1)^{\ell_{31}-|\boldsymbol{\alpha}_l^{(1)}|} z_3^{\ell_3-|\boldsymbol{\alpha}_l^{(1)}|-|\boldsymbol{\alpha}_r^{(2)}|} \left(z_{21}^{\ell_{12}-|\boldsymbol{\alpha}_r^{(1)}|-|\boldsymbol{\alpha}_l^{(2)}|} \right) \mathcal{D}_{\{\boldsymbol{\alpha}_{l,r}^{(1)}, \boldsymbol{\alpha}_{l,r}^{(2)}, \varnothing\}}.
\qquad (6.3.32)
$$

This shows that only a single term in the sum, for which $|\boldsymbol{\alpha}_l^{(1)}| = |\boldsymbol{\alpha}_r^{(2)}| = 0$ holds, can produce the highest power $z_3^{\ell_3}$. Therefore, comparing the coefficients in front of $z_3^{\ell_3}$, we

---

[16]Note that, since $|\mathcal{O}_3\rangle_L$ does not have any magnons, there is no sum over the partitions coming from $\mathcal{O}_3$.

can determine $\mathcal{G}$ to be of the form

$$
\begin{aligned}
\mathcal{G} &= (-1)^{-M_1+M_2} \mathcal{D}_{\{\boldsymbol{\alpha}_{l,r}^{(1)},\boldsymbol{\alpha}_{l,r}^{(2)},\varnothing\}} \Big|_{\boldsymbol{\alpha}_l^{(1)}=\boldsymbol{\alpha}_r^{(2)}=\varnothing} \\
&= (-1)^{M_2} \prod_{a=1}^{M_1} Q_{\boldsymbol{\theta}^{(31)}}^+(u_a^{(1)}) \prod_{b=1}^{M_2} Q_{\boldsymbol{\theta}^{(23)}}^-(u_b^{(2)}) Z_p\left(\boldsymbol{u}^{(1)} \cup \boldsymbol{u}^{(2)}|\boldsymbol{\theta}^{(12)}\right),
\end{aligned}
\tag{6.3.33}
$$

where the function $Q_{\boldsymbol{\theta}}(x)$ is defined by

$$
Q_{\boldsymbol{\theta}}(x) \equiv \prod_{\theta \in \boldsymbol{\theta}}(x - \theta),
\tag{6.3.34}
$$

and the superscripts $\pm$ denote the shift of the argument by $\pm i/2$.

Let us now study the semi-classical limit of our three-point function. For this purpose, it is more convenient to introduce the "rescaled" partial domain wall partition function[17] defined by

$$
\mathcal{Z}_p\left(\boldsymbol{u}^{(1)} \cup \boldsymbol{u}^{(2)}|\boldsymbol{\theta}^{(12)}\right) \equiv \frac{Z_p\left(\boldsymbol{u}^{(1)} \cup \boldsymbol{u}^{(2)}|\boldsymbol{\theta}^{(12)}\right)}{\prod_{x \in \boldsymbol{u}^{(1)}} Q_{\boldsymbol{\theta}^{(12)}}^+(x) \prod_{y \in \boldsymbol{u}^{(2)}} Q_{\boldsymbol{\theta}^{(12)}}^-(y)}.
\tag{6.3.35}
$$

Then, $\langle |\mathcal{O}_1\rangle_L, |\mathcal{O}_2\rangle_L, |\mathcal{O}_3\rangle_L \rangle$ takes the form

$$
\begin{aligned}
\langle |\mathcal{O}_1\rangle_L, |\mathcal{O}_2\rangle_L, |\mathcal{O}_3\rangle_L \rangle &= \left(\frac{1}{1+|z_1|^2}\right)^{\frac{\ell_1}{2}-M_1} \left(\frac{1}{1+|z_2|^2}\right)^{\frac{\ell_2}{2}-M_2} \left(\frac{1}{1+|z_3|^2}\right)^{\frac{\ell_3}{2}} \\
&\quad \times (z_1 - z_2)^{\ell_{12}-M_1-M_2}(z_2 - z_3)^{\ell_{23}-M_2+M_1}(z_3 - z_1)^{\ell_{31}-M_1+M_2} \\
&\quad \times \left(\prod_{a=1}^{M_1} Q_{\boldsymbol{\theta}^{(1)}}^+(u_a^{(1)}) \prod_{b=1}^{M_2} Q_{\boldsymbol{\theta}^{(2)}}^-(u_b^{(2)}) \mathcal{Z}_p\left(\boldsymbol{u}^{(1)} \cup \boldsymbol{u}^{(2)}|\boldsymbol{\theta}^{(12)}\right)\right),
\end{aligned}
\tag{6.3.36}
$$

where we have neglected the factor $(-1)^{M_2}$ as it only changes the overall sign. Performing a similar analysis, we can also determine the contribution from the $\mathrm{SU}(2)_R$ spin chain and the result is given by

$$
\begin{aligned}
\langle |\tilde{\mathcal{O}}_1\rangle_R, |\tilde{\mathcal{O}}_2\rangle_R, |\tilde{\mathcal{O}}_3\rangle_R \rangle &= \left(\frac{1}{1+|\tilde{z}_1|^2}\right)^{\frac{\ell_1}{2}} \left(\frac{1}{1+|\tilde{z}_2|^2}\right)^{\frac{\ell_2}{2}} \left(\frac{1}{1+|\tilde{z}_3|^2}\right)^{\frac{\ell_3}{2}-\tilde{M}_3} \\
&\quad \times (\tilde{z}_1 - \tilde{z}_2)^{\ell_{12}+\tilde{M}_3}(\tilde{z}_2 - \tilde{z}_3)^{\ell_{23}-\tilde{M}_3}(\tilde{z}_3 - \tilde{z}_1)^{\ell_{31}-\tilde{M}_3} \\
&\quad \times \left(\prod_{a=1}^{M_3} Q_{\tilde{\boldsymbol{\theta}}^{(3)}}^+(\tilde{u}_a^{(3)}) \mathcal{Z}_p\left(\tilde{\boldsymbol{u}}^{(3)} \cup \varnothing|\tilde{\boldsymbol{\theta}}^{(31)}\right)\right).
\end{aligned}
\tag{6.3.37}
$$

---

[17]Note that it is the rescaled partial domain wall partition function, which has a simple semi-classical limit. In [117], it is called $\mathcal{A}$-functional.

Note that, since the result is completely factorized into the $SU(2)_L$ and the $SU(2)_R$ parts, we can introduce independent sets of the inhomogeneities for the $SU(2)_R$ sector denoted by $\tilde{\theta}$'s. The tree-level structure constant can then be obtained by setting θ's and $\tilde{\theta}$'s to zero. Now the semi-classical limit of the three-point coupling constant can also be easily studied using the results of [117, 124, 125] and we obtain, up to a phase,

$$
C_{123} = \frac{\sqrt{\ell_1 \ell_2 \ell_3}}{N_c} k_L \, k_R \, c_{123} \,,
$$

$$
\log c_{123} \sim \oint_{\mathcal{C}_{\boldsymbol{u}^{(1)}} \cup \mathcal{C}_{\boldsymbol{u}^{(2)}}} \frac{du}{2\pi i} \mathrm{Li}_2 \left(e^{ip_1 + ip_2 + i\ell_3/2u} \right) + \oint_{\mathcal{C}_{\tilde{\boldsymbol{u}}^{(3)}}} \frac{du}{2\pi i} \mathrm{Li}_2 \left(e^{ip_3 + i(\ell_2 - \ell_1)/2u} \right) \qquad (6.3.38)
$$

$$
- \frac{1}{2} \oint_{\mathcal{C}_{\boldsymbol{u}^{(1)}}} \frac{du}{2\pi} \mathrm{Li}_2 \left(e^{2ip_1} \right) - \frac{1}{2} \oint_{\mathcal{C}_{\boldsymbol{u}^{(2)}}} \frac{du}{2\pi} \mathrm{Li}_2 \left(e^{2ip_2} \right) - \frac{1}{2} \oint_{\mathcal{C}_{\tilde{\boldsymbol{u}}^{(3)}}} \frac{du}{2\pi} \mathrm{Li}_2 \left(e^{2i\tilde{p}_3} \right) \,.
$$

Here k_L and k_R are kinematical factors given by the first two lines on the left hand side of (6.3.36) and (6.3.37) respectively, $p_n(u)$ and $\tilde{p}_n(u)$ are the quasi-momenta given by

$$
p_n(u) = \sum_{v \in \boldsymbol{u}^{(n)}} \frac{1}{u - v} - \frac{\ell_n}{2u} \,, \qquad \tilde{p}_n(u) = \sum_{v \in \tilde{\boldsymbol{u}}^{(n)}} \frac{1}{u - v} - \frac{\ell_n}{2u} \,, \qquad (6.3.39)
$$

and the integration contours $\mathcal{C}_{\boldsymbol{u}^{(n)}}$ and $\mathcal{C}_{\tilde{\boldsymbol{u}}^{(n)}}$ encircle[18] the Bethe roots $\boldsymbol{u}^{(n)}$ and $\tilde{\boldsymbol{u}}^{(n)}$ respectively.

So far, we have seen that the mixed correlators have simple expressions, which allow us to study the semi-classical limit with ease. The remaining class of three-point functions are the ones for which all the three operators are of the same type. We call such three-point functions "unmixed". It turns out that, in the case of the unmixed correlators, several different terms in the sum in (6.3.28) contribute to the highest power of z_i's, and therefore the result cannot be simplified by the straightforward application of the aforementioned logic. In addition, the prediction from the semi-classical computation based on the coherent states (to be reported in [139]) does not take a form which can be readily obtained from the pDWPF. These two observations indicate that the unmixed correlators are much more complicated objects. Nevertheless, studying such three-point functions is important for the following reason: The pDWPF is the quantity which describes the skew-symmetric product of two spin-chain state. Therefore, the fact that the mixed correlators can be reduced to the pDWPF suggests that such three-point functions are characterized essentially by the integrability governing the two-point function, which is already fairly well-understood. This in turn means that, in order to reveal the genuine "integrability for the three-point functions", we do need to study the unmixed correlators, which cannot be simplified into the pDWPF.

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[18]As briefly discussed in [125], the contours are in general complicated and the case-by-case analysis is necessary.

6.4 Monodromy relation

Based on the framework developed so far, we now derive the second main result of this paper, namely the nontrivial identities, to be called the monodromy relations, satisfied by the two-point and the three-point functions with the monodromy operators inserted. This identity is a direct consequence of the two fundamental properties of the Lax operator, *i.e.* the "unitarity" and the "crossing", and might provide a hint for the essence of the integrability of the correlation functions that we are eager to capture.

6.4.1 Monodromy relation for two-point functions

First, let us derive the monodromy relation for the two-point functions using the aforementioned two basic properties of the Lax operator.

The first is the "unitarity" relation[19]. From the definition of the Lax operator (6.3.8), one can straightforwardly check the following identity:

$$\mathrm{L}(\theta - u + i/2)\mathrm{L}(u - \theta + i/2) = -f(u) \cdot \mathbf{1}\,. \tag{6.4.1}$$

Here the symbol $\mathbf{1}$ denotes the identity operator both for the spin and the auxiliary spaces and $f(u)$ is given by

$$f(u) \equiv (u - \theta)^2 + 1\,. \tag{6.4.2}$$

The relation (6.4.1) is an analogue of the unitarity condition for the factorized S-matrices and can be understood pictorially as shown in the upper figure of Figure 6.4.1.

The second property is the "crossing" relation (6.3.9). What is important for the following discussions is that the crossed Lax operator $\mathcal{C} \circ \mathrm{L}(u)$ can be written alternatively as[20]

$$\mathcal{C} \circ \mathrm{L}(u) = -\mathrm{L}(-u)\,. \tag{6.4.5}$$

---

[19]It is also called the inversion identity.

[20]Written in terms of the R-matrix

$$(R(u))_{i_1 j_1}^{i_2 j_2} \equiv u\delta_{i_1}^{i_2}\delta_{j_1}^{j_2} + i\delta_{i_1}^{j_2}\delta_{i_2}^{j_1} \qquad (i_1, i_2, j_1, j_2 = 1, 2)\,, \tag{6.4.3}$$

which is related to the Lax operator by $R(u) = \mathrm{L}(u + i/2)$, the equation (6.4.5) takes the form of the crossing relation for the factorizable S-matrices,

$$\sum_{i_1', i_2'} -(\sigma_2)_{i_1 i_2'} (R(u))_{i_1' j_1}^{i_2' j_2} (\sigma_2)^{i_1' i_2} = (R(i - u))_{i_1 j_1}^{i_2 j_2}\,. \tag{6.4.4}$$

This is the reason why we call $\mathcal{C} \circ \mathrm{L}(u)$ the crossed Lax operator.

"Unitarity" relation.
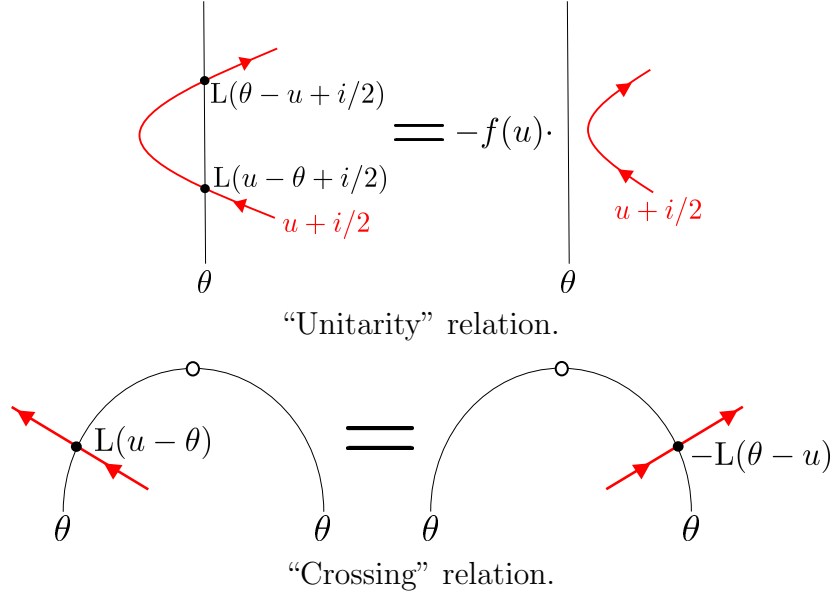
"Crossing" relation.

Figure 6.4.1: The "unitarity" and the "crossing" relation of the Lax operator for the XXX spin chain. In both figures, the black line refers to the spin space and the red line to the auxiliary space. Upper figure: A product of two Lax operators acting on the same spin space equals to the identity as shown in (6.4.1). Lower figure: A skew-symmetric product with a Lax operator insertion on one side is equivalent to the skew-symmetric product with a crossed Lax operator insertion on the other side as shown in (6.4.6).

With this relation, the crossing relation (6.3.9) takes the following form:

$$\langle \mathbf{1} | \left(\mathrm{L}(u - \theta)|s_1\rangle \otimes |s_2\rangle \right) = -\langle \mathbf{1} | \left(|s_1\rangle \otimes \mathrm{L}(\theta - u)|s_2\rangle \right) . \tag{6.4.6}$$

A pictorial representation of this relation is given in the lower figure of Figure 6.4.1.

Making use of these two properties[21], let us now derive the monodromy relation for the two-point functions . First, consider the following quantity, which is depicted in the figure (a) of Figure 6.4.2:

$$\langle |\mathcal{O}_1\rangle_L , \left(\overleftarrow{\Omega}_2(-u + i/2) \right)_{ij} \left(\Omega_2(u + i/2) \right)_{jk} |\mathcal{O}_2\rangle_L \rangle , \tag{6.4.7}$$

where i, j and k are the indices for the auxiliary space, and Ω_n and $\overleftarrow{\Omega}_n$ are the monodromy and the "reverse-ordered" monodromy[22] for the operator $\mathcal{O}_n$, defined by

$$\Omega_n(u + i/2) \equiv \mathrm{L}_1^{(n)}(u - \theta_1^{(n)} + i/2) \cdots \mathrm{L}_{\ell_n}^{(n)}(u - \theta_{\ell_n}^{(n)} + i/2) , \tag{6.4.8}$$

$$\overleftarrow{\Omega}_n(-u + i/2) \equiv \mathrm{L}_{\ell_n}^{(n)}(\theta_{\ell_n}^{(n)} - u + i/2) \cdots \mathrm{L}_1^{(n)}(\theta_1^{(n)} - u + i/2) . \tag{6.4.9}$$

---

[21]For the moment, we only consider the $\mathrm{SU}(2)_L$ sector since the generalization to the $\mathrm{SU}(2)_R$ sector is straightforward.

[22]Note that, owing to the relation (6.4.5), the reverse-ordered monodromy is equivalent to the monodromy which appeared in (6.3.11): $\overleftarrow{\Omega}_n(-u) = (-1)^{\ell_n} \mathcal{C} \circ \Omega_n(u)$.

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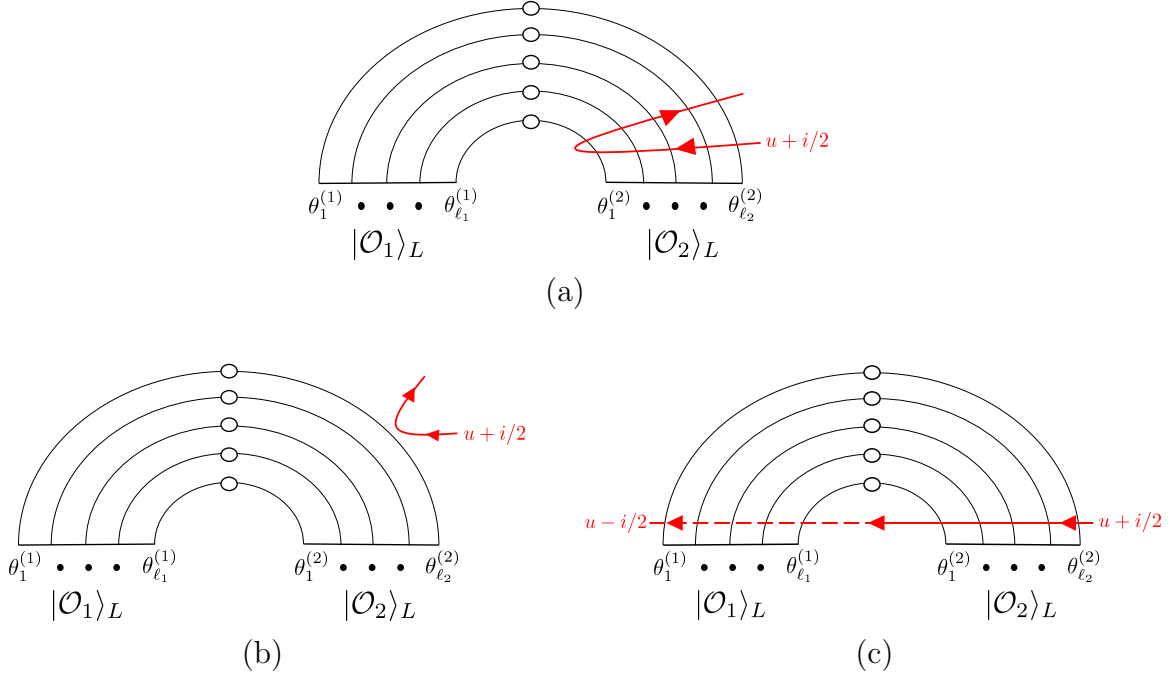
Figure 6.4.2: The derivation of the monodromy relation for the two-point function. The figure (a) describes the skew-symmetric product with the usual monodromy and the reverse-ordered monodromy, given in (6.4.7). By applying the unitarity relations, one can show that it is proportional to the skew-symmetric product without any monodromy insertions, which is given in (6.4.10) and depicted in the figure (b). On the other hand, if we apply the crossing relations repeatedly to (6.4.7), we reach the right hand side of (6.4.13), which is shown in the figure (c). In the figure (c), the solid red line denotes the monodromy matrix whose argument is shifted by $+i/2$ whereas the dashed red line denotes the monodromy matrix whose argument is shifted by $-i/2$. The equivalence between the figures (b) and (c) is the monodromy relation for the two-point function given in (6.4.14).

Here $\mathrm{L}_k^{(n)}$ and $\theta_k^{(n)}$ respectively denote the Lax operator and the inhomogeneity parameter for the k-th site of the spin-chain state $|\mathcal{O}_n\rangle_L$, and ℓ_n is the length of the operator $\mathcal{O}_n$. Here again the inhomogeneities are identified as $\theta_k^{(1)} = \theta_{\ell-k+1}^{(2)}$, as discussed already in section 6.3.2. Using the unitarity relation (6.4.1) repeatedly, we can show that (6.4.7) is proportional to the skew-symmetric product without monodromy insertions, which is depicted in the figure (b) of Figure 6.4.2:

$$(6.4.7) = \delta_{ik}(-1)^\ell f_{12}(u)\big\langle|\mathcal{O}_1\rangle_L, |\mathcal{O}_2\rangle_L\big\rangle, \tag{6.4.10}$$

where the prefactor $f_{12}(u)$ is given by

$$f_{12}(u) \equiv \prod_{k=1}^{\ell}\left((u-\theta_k^{(1)})^2+1\right) = \prod_{k=1}^{\ell}\left((u-\theta_k^{(2)})^2+1\right). \tag{6.4.11}$$

Let us next apply the crossing relation to each Lax operator constituting $\overleftarrow{\Omega}_2$ in (6.4.7). Since the k-th site of the operator $\mathcal{O}_2$ is contracted with the $(\ell - k + 1)$-th site of the operator $\mathcal{O}_1$, the Lax operator transforms under the application of the crossing relation as

$$\mathrm{L}_k^{(2)}(-u + \theta_k^{(2)}) \rightarrow -\mathrm{L}_{\ell-k+1}^{(1)}(u - \theta_{\ell-k+1}^{(1)}) , \tag{6.4.12}$$

where we used the identifications of the inhomogeneity parameters (6.3.13). Thus, after the successive application of the crossing relation, we arrive at the following expression, which is depicted in the figure of Figure 6.4.2:

$$(6.4.7) = (-1)^\ell \langle \left(\Omega_1^-(u) \right)_{ij} |\mathcal{O}_1\rangle_L , \left(\Omega_2^+(u) \right)_{jk} |\mathcal{O}_2\rangle_L \rangle . \tag{6.4.13}$$

The superscripts $\pm$ on the monodromy operator denotes the shift of the argument $\Omega^\pm(u) \equiv \Omega(u \pm i/2)$.

Then, by equating the right hand sides of (6.4.10) and (6.4.13), we obtain the monodromy relation for the two-point function:

$$\langle \left(\Omega_1^-(u) \right)_{ij} |\mathcal{O}_1\rangle_L , \left(\Omega_2^+(u) \right)_{jk} |\mathcal{O}_2\rangle_L \rangle = \delta_{ik} f_{12}(u) \langle |\mathcal{O}_1\rangle_L , |\mathcal{O}_2\rangle_L \rangle . \tag{6.4.14}$$

One can write down a similar relation also for the $SU(2)_R$ chain as

$$\langle \left(\tilde{\Omega}_1^-(u) \right)_{ij} |\tilde{\mathcal{O}}_1\rangle_R , \left(\tilde{\Omega}_2^+(u) \right)_{jk} |\tilde{\mathcal{O}}_2\rangle_R \rangle = \delta_{ik} \tilde{f}_{12}(u) \langle |\tilde{\mathcal{O}}_1\rangle_R , |\tilde{\mathcal{O}}_2\rangle_R \rangle , \tag{6.4.15}$$

where $\tilde{\Omega}_n(u)$ are the monodromy matrices for the $SU(2)_R$ chain and $\tilde{f}_{12}(u)$ is given in terms of the inhomogeneity for the $SU(2)_R$ chain $\tilde{\theta}_k^{(n)}$ by

$$\tilde{f}_{12}(u) \equiv \prod_{k=1}^\ell \left((u - \tilde{\theta}_k^{(1)})^2 + 1 \right) = \prod_{k=1}^\ell \left((u - \tilde{\theta}_k^{(2)})^2 + 1 \right) . \tag{6.4.16}$$

The monodromy relations (6.4.14) and (6.4.15) are the embodiment of the integrability for the two-point function. As the two-point function is determined by the spectrum of the operators, they should be essentially equivalent to the integrable structures already known in the spectral problem. However, it might be interesting to clarify the relation with the conventional formalism and ask if these new formalism helps to deepen the understanding of the spectral problem.

6.4.2 Monodromy relation for three-point functions

Let us now turn to the three-point functions. As explained in section 6.3.1, the three-point functions are given by a product of two factors coming from the $SU(2)_L$ and the

$SU(2)_R$ respectively and each factor is expressed in terms of the skew-symmetric products between sub-chains. Therefore, one can apply the unitarity (6.4.1) and the crossing relation (6.4.6) to each sub-chain and derive a nontrivial monodromy relation for the three-point functions. Although the essence of the derivation is entirely similar to the one for the two-point function, for the three-point function there is a certain freedom in the form of the monodromy relation which comes from the choice of the shifts of the spectral parameter for the three monodromy matrices. To give an intuitive picture of the monodromy relation, however, below we shall exhibit a specific example which can be easily understood from a figure Figure 6.4.3 and relegate the discussion of how the more general forms of the relation arise to Appendix E.

Now for the $SU(2)_L$ sector a simple monodromy relation can be given in the form

$$
\begin{aligned}
&\left\langle \left(\Omega_1^-(u)\right)_{ij}|\mathcal{O}_1\rangle_L , \left(\Omega_2^{+|-}(u)\right)_{jk}|\mathcal{O}_2\rangle_L , \left(\Omega_3^+(u)\right)_{kl}|\mathcal{O}_3\rangle_L\right\rangle \\
&= \delta_{il} f_{123}(u)\langle |\mathcal{O}_1\rangle_L , |\mathcal{O}_2\rangle_L , |\mathcal{O}_3\rangle_L\rangle ,
\end{aligned}
\tag{6.4.17}
$$

where $f_{123}(u)$ is defined by[23]

$$
f_{123}(u) \equiv \prod_{i=1}^{\ell_{31}} \left((u - \theta_i^{(1)})^2 + 1\right) \prod_{j=1}^{\ell_{12}} \left((u - \theta_j^{(2)})^2 + 1\right) \prod_{k=1}^{\ell_{23}} \left((u - \theta_k^{(3)})^2 + 1\right) ,
\tag{6.4.18}
$$

and $\Omega_2^{+|-}(u)$ denotes a product of the monodromy matrices on the left and the right sub-chains of $\mathcal{O}_2$ whose arguments are shifted by $+i/2$ and $-i/2$ respectively. More specifically, the relevant monodromy matrices are given by

$$
\begin{aligned}
\Omega_1^-(u) &= L_1^-(u - \theta_1^{(1)})\cdots L_{\ell_{31}}^-(u - \theta_{\ell_{31}}^{(1)})L_{\ell_{31}+1}^-(u - \theta_{\ell_{31}+1}^{(1)})\cdots L_{\ell_1}^-(u - \theta_{\ell_1}^{(1)}) , \\
\Omega_2^{+|-}(u) &= L_1^+(u - \theta_1^{(2)})\cdots L_{\ell_{12}}^+(u - \theta_{\ell_{12}}^{(2)})L_{\ell_{12}+1}^-(u - \theta_{\ell_{12}+1}^{(2)})\cdots L_{\ell_2}^-(u - \theta_{\ell_2}^{(2)}) , \\
\Omega_3^+(u) &= L_1^+(u - \theta_1^{(3)})\cdots L_{\ell_{23}}^+(u - \theta_{\ell_{23}}^{(3)})L_{\ell_{23}+1}^+(u - \theta_{\ell_{23}+1}^{(3)})\cdots L_{\ell_3}^+(u - \theta_{\ell_3}^{(3)}) .
\end{aligned}
\tag{6.4.19}
$$

For the $SU(2)_R$ chain, the corresponding form of the monodromy relation can be written as

$$
\begin{aligned}
&\left\langle \left(\tilde{\Omega}_1^-(u)\right)_{ij}|\tilde{\mathcal{O}}_1\rangle_R , \left(\tilde{\Omega}_2^{+|-}(u)\right)_{jk}|\tilde{\mathcal{O}}_2\rangle_R , \left(\tilde{\Omega}_3^+(u)\right)_{kl}|\tilde{\mathcal{O}}_3\rangle_R\right\rangle \\
&= \delta_{il} \tilde{f}_{123}(u)\langle |\tilde{\mathcal{O}}_1\rangle_R , |\tilde{\mathcal{O}}_2\rangle_R , |\tilde{\mathcal{O}}_3\rangle_R\rangle ,
\end{aligned}
\tag{6.4.20}
$$

where $\tilde{f}_{123}(u)$ is defined by

$$
\tilde{f}_{123}(u) \equiv \prod_{i=1}^{\ell_{31}} \left((u - \tilde{\theta}_i^{(1)})^2 + 1\right) \prod_{j=1}^{\ell_{12}} \left((u - \tilde{\theta}_j^{(2)})^2 + 1\right) \prod_{k=1}^{\ell_{23}} \left((u - \tilde{\theta}_k^{(3)})^2 + 1\right) .
\tag{6.4.21}
$$

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[23]For a definition of ℓ_{ij}, see (6.3.2).

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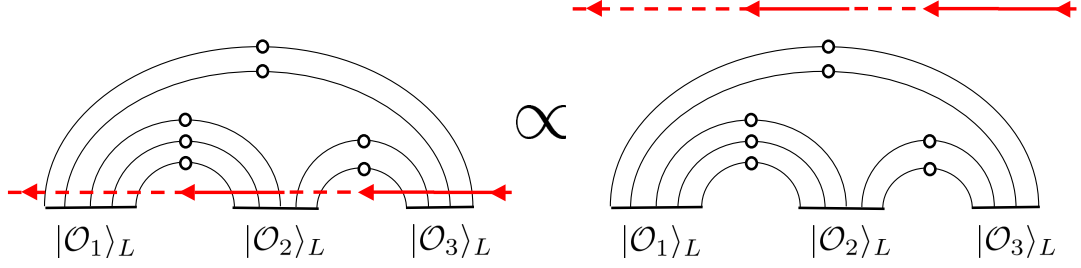
Figure 6.4.3: The monodromy relation for the three-point function (6.4.17). The thick red line denotes a part of the monodromy matrix with a $+i/2$ shift of the spectral parameter and the dashed red line denotes a part of the monodromy matrix with a $-i/2$ shift of the spectral parameter.

As in the $SU(2)_L$ sector, $\tilde{\Omega}_2^{+|-}(u)$ denotes a product of the monodromy matrices on the left and the right sub-chains whose arguments are shifted by $+i/2$ and $-i/2$ respectively.

Let us now discuss the implications of the typical monodromy relations of the form (6.4.17) and (6.4.20). Firstly, the monodromy relations in general relate three-point functions of different spin-chain states and therefore can be regarded as a kind of Schwinger-Dyson equation. It would be extremely interesting if we could compute the three-point functions by directly solving these relations. Secondly, the relations imply the existence of infinite number of conserved charges together with the existence of the associated Ward identities. For instance, by expanding (6.4.17) around $u = \infty$, at the leading order we obtain the usual Ward identities for the global $SU(2)_L$-symmetry of the form

$$\left\langle S_*|\mathcal{O}_1\rangle_L\,,|\mathcal{O}_2\rangle_L\,,|\mathcal{O}_3\rangle_L\right\rangle + \left\langle|\mathcal{O}_1\rangle_L\,,S_*|\mathcal{O}_2\rangle_L\,,|\mathcal{O}_3\rangle_L\right\rangle + \left\langle|\mathcal{O}_1\rangle_L\,,|\mathcal{O}_2\rangle_L\,,S_*|\mathcal{O}_3\rangle_L\right\rangle = 0\,,$$

$$(6.4.22)$$

where S_* are the global $SU(2)$ generators and $*$ stands for 1, 2 or 3. These global Ward identities are quite useful in fixing the kinematical dependence of the three-point functions, as described in the Appendix D. Naturally it would be quite interesting and important to study the non-trivial relations obtained at the sub-leading levels and see if we can exploit them to understand the structure of the three-point functions[24].

As for the importance of the monodromy relation, we already have a supporting evidence from the strong coupling computation performed in [111]. In that analysis the three-point function in the $SU(2)$ sector was determined from the following relation for

---

[24]At the sub-leading order, (6.4.17) produces a set of non-trivial identities involving operators which act non-locally on the spin chains. These identities can be regarded as a sort of Yangian invariance for the three-point functions. Similar relations are discussed in the context of the scattering amplitudes in [182–189] and it would be interesting to clarify the connection.

the monodromy matrices defined on the classical string world-sheet:

$$\Omega_1(x)\Omega_2(x)\Omega_3(x) = \mathbf{1}\,. \tag{6.4.23}$$

This relation, which is a direct consequence of the classical integrability of the string sigma model, is a clear manifestation of the integrability for the three-point function at strong coupling and was indeed an essential ingriedient in the computation of the three-point functions. The relations we derived here, (6.4.17) and (6.4.20), can be regarded as the weak coupling counter-part of (6.4.23) and its generalization. The similarity becomes more apparent if we take the so-called semi-classical limit of the spin chain, in which the length of the chain and the number of the magnons are both large. To study the low energy excitation in this limit, we need to use the rescaled spectral parameter $u = \ell u'$, and send ℓ to ∞ keeping u' finite. In terms of this rescaled parameter, the shifts of the spectral parameter in $\Omega_n^\pm$, $\Omega_2^{+|-}$ and so on become negligible. Furthermore, in this limit, the three-point function will be well-approximated by coherent states. Then the monodromy matrices, which are originally quantum operators acting on the spin chains, become classical. Therefore, in such a limit the relations (6.4.17) and (6.4.20) exactly take the same form as (6.4.23). As will be discussed in the forthcoming publication [139], we can use (6.4.17) and (6.4.20) to directly study the semi-classical behavior of the three-point functions at weak coupling without relying on the explicit determinantal expressions for the scalar products of the XXX spin chain.

6.5 Summary of this chapter

In this chapter, we proposed a novel way of understanding the tree-level three-point functions in the SU(2) sector. In the previous approaches, each operator was mapped to a single spin-chain state and the Wick contraction was interpreted as the scalar product of the spin chain. However, in order to study more general three-point functions, it is much more advantageous to associate a tensor product of two spin-chain states to each operator and express the Wick contraction as the overlap with the singlet state. Using this new formalism, we showed that a broader class of three-point functions, which we call mixed correlators, have simple determinant representation. Moreover, we derived nontrivial identities satisfied by the three-point functions with monodromy operators inserted. The identities can be regarded as the weak-coupling counterpart of the relation $\Omega_1\Omega_2\Omega_3 = 1$, which played an important role in the computation at strong coupling.

Chapter 7

PSU(2,2|4) singlet projector and monodromy relation

In the previous chapter 6, we have developed two novel viewpoints and applied them explicitly to a class of three point functions in the SU(2) sector which are much more general than had been treated by the tailoring procedure. Let us summarize these two ideas as (I) and (II) below:

(I) One is the group theoretic reinterpretation of the Wick contraction of basic fields as a singlet projection of a tensor product of two fields.

(Ia) In the case of the SU(2) sector, one can apply this idea with respect to the more refined $SU(2)_L \times SU(2)_R$ structure present in that sector. This feature can be succinctly referred to as "double spin-chain" and it leads to the factorization of the left and the right sector and simplifies various formulas. This formalism allowed us to study a class of three-point functions of operators built upon more general spin-chain vacua than the special configuration discussed so far in the literature. This formulation makes the correspondence with the strong coupling computation done in [111] quite apparent.

(Ib) Another conspicuous advantage of the new interpretation of the Wick contraction is that one can avoid the scalar products of off-shell states which appear in the tailoring prescription. Emergence of such an object required the trick [123] to turn one of the off-shell states into an on-shell state in order to write it in terms of the Slavnov determinant. In contrast, in our formulation one can directly obtain the expression in terms of the partial domain wall partition functions, and for a certain class of correlators it can be readily expressed as a determinant even for two off-shell states.

(II) The second new idea formulated explicitly in [138] is so-called the monodromy relation, which can be obtained by inserting the monodromy matrices Ω's inside the two-point or the three-point functions and using the unitarity and the crossing relations. This produces a relation between correlation functions of different operators and hence acts like the Schwinger-Dyson equation. In particular, in the special limit where the spectral parameter u goes to ∞, it reduces to the Ward identity for $\mathrm{SU}(2)_{\mathrm{L}}$ and $\mathrm{SU}(2)_{\mathrm{R}}$. Moreover, for the three-point function in the semiclassical limit, where the operators carry large quantum numbers, it takes the classical relation $\Omega_1\Omega_2\Omega_3 = 1$, which is precisely of the form of the monodromy relation that follows from the integrability of the string theory, which played such a crucial role in the computation at strong coupling [108, 111]. Thus, its super Yang-Mills counterpart should also be considered as a major part of the concept of "integrability" beyond the spectral level.

Now the main purpose of the present chapter is to extend these two main ideas explicitly to the full $\mathfrak{psu}(2,2|4)$ sector and discuss how various general formulas are modified and simplified when we reduce them to the various subsectors. As for the monodromy relation (II), for the sake of clarity of presentation, we shall mainly concentrate on the case of the two-point functions. However, the extension to the three-point functions of our interest is straightforward, as was demonstrated in the case of SU(2) sector in [138], and the form of the result will also be briefly presented.

As far as the basic ideas (I) and (II) sketched above are concerned, similar ideas on the Wick contraction and the monodromy relations have also been discussed independently by [137][1]. Their work was based largely on the work by [114], which observed and utilized certain similarity of the Wick-contracting operator to the string field theory (SFT) vertex in the spirit of the string bit formulation. In this fashion, the work of [137] discussed already the full $\mathfrak{psu}(2,2|4)$ sector making use of the similar vertex, as well as the same oscillator representations and some associated basic formulas, as [114].

However, as far as the result (I) for the $\mathfrak{psu}(2,2|4)$ is concerned, the exponential form of the vertex written down by [114] was guessed by an analogy with the delta-function overlap in SFT and unfortunately was not $\mathfrak{psu}(2,2|4)$ singlet. The work of [137], which was based on [114], modified certain parts of the exponent and checked that it is a singlet projector a posteriori. However, there are two points that one wishes to improve on.

One is the understanding of why the singlet projector is of a simple exponential form, which was assumed in the work of [114] and hence in [137]. Such a form may be natural

---

[1]They did not discuss, however, the explicit advantage described in (Ia) and (Ib) gained by the new interpretation of the Wick contraction.

as the oscillator description of the δ-function overlap familiar in SFT context is indeed exponential. However, the analogy should be taken with care. For one thing the discrete indices of the oscillators for the string case are the Fourier mode numbers, whereas the similar indices in the super Yang-Mills case designate the location along the spin chain. Furthermore, in the case of the string the parts to be identified are rather homogeneous and hence it is natural to employ the (oscillator representation of) a delta-function to connect them. On the other hand, in the case of the $\mathfrak{psu}(2,2|4)$ spin chain the adjacent "string bits" can be quite different and the analogy to the delta function overlap is not intuitively obvious. In fact it is a simple exercise to construct the singlet state in the case of the spin j representation of SU(2) and confirm that it does *not* take an exponential form.

Therefore the surest way to obtain the desired vertex which effects the Wick contraction is to construct the most general singlet projector systematically in the space of tensor product of two spin chains. We shall show that the singlet projector exists for $\mathfrak{su}(2,2|4)$ (as well as its restriction $\mathfrak{psu}(2,2|4)$) but not for $\mathfrak{su}(2,2|4)$ and, strictly speaking, for each sector of the representation of the $\mathfrak{su}(2,2|4)$ with a definite central charge C, the singlet vertex is not of a simple exponential form. However, provided that one is interested only in a sector with one definite value of the central charge[2], one is allowed to use the simple exponential form, which is much more tractable. (We shall further elaborate on this later.) In this connection, we shall also explain in an appendix how the simple non-exponential singlet projector constructed for the SU(2) subsector in our previous work can be obtained from the general exponential projector for $\mathfrak{psu}(2,2|4)$.

The second point is that one wishes to improve the situation that the singlet projector of [137] is not manifestly conformally invariant, which is not useful for the treatment of the computation of the correlation functions of the local composite operators. In the present work we shall construct the version of the singlet projector which is manifestly conformally invariant and hence much simpler to use.

To explain what we mean by this, it is instructive to recall the following basic facts. In constructing the representations of the superalgebra $\mathfrak{u}(2,2|4)$, there are basically two different schemes, depending on which maximal bosonic subgroups of the supergroup U(2,2|4) to make use of:

$$(E): \quad \mathrm{U}(2,2|4) \supset \mathrm{U}(1)_E \times \mathrm{SU}(2)_L \times \mathrm{SU}(2)_R \times \mathrm{SU}(4)\,, \qquad (7.0.1)$$

$$(D): \quad \mathrm{U}(2,2|4) \supset \mathrm{U}(1)_D \times \mathrm{SL}(2,\mathbb{C}) \times \overline{\mathrm{SL}(2,\mathbb{C})} \times \mathrm{SU}(4) \qquad (7.0.2)$$

---

[2]This is the case for the $\mathcal{N} = 4$ super Yang-Mills, since the basic fields all carry $C = 0$.

Their difference resides in the choice of the subgroups in the SO(4,2) part. The one, which we shall call E-scheme, makes use of the compact subgroups with the AdS energy E being diagonal. On the other hand, in the scheme to be called D-scheme, the dilatation generator D is diagonal and the rest of the subgroup chosen in SO(4,2) is the non-compact Lorentz group $\mathrm{SL}(2,\mathbb{C}) \times \overline{\mathrm{SL}(2,\mathbb{C})}$. Therefore, the D-scheme is manifestly conformal covariant. It is well-known and fully discussed in [177] that these two schemes are connected by a non-unitary similarity transformation generated by the operator $U = e^{(\pi/4)(P_0 - K_0)}$ such that $U^{-1}DU = iE$.

In the treatment of [114], and hence [137], the oscillators appropriate for the E-scheme are used as basic building blocks for the generators of $\mathfrak{u}(2,2|4)$ and the relevant vertex operators. Since the D-scheme is more natural for the main purpose of computing the correlation functions of the basic super Yang-Mills fields, they transformed various quantities to that scheme by the similarity transformation using the operator U. However, since U does not map an individual component group, such as $\mathrm{SU}(2)_\mathrm{L}$, in the E-scheme to a definite component group, such as $\mathrm{SL}(2,\mathbb{C})$, in the D-scheme[3], the mapping does not make the description manifestly conformally covariant.

In our construction, to be described fully in section 7.1, we will stick to the D-scheme throughout, by using the oscillators which transform covariantly under the maximal subgroups shown in (7.0.2). This will make the entire description quite transparent without the need of the operator U.

Let us next turn to the $\mathfrak{psu}(2,2|4)$ version of the monodromy relation (II). There are two natural types of monodromy matrices depending on the choice of the auxiliary space. One is the simpler and the fundamental one, for which the auxiliary space is taken to be $\mathbb{C}^{4|4}$. For this case, the derivation of the monodromy relation is a straightforward extension of the one for the SU(2) sector given in our previous work [138] and agrees with the description given in [137]. Another type is the monodromy relation associated with the so-called harmonic R-matrix, for which the structure of the auxiliary space is the same as that of the physical quantum space [178–181]. This case may be useful for obtaining local conserved quantities as well as for the study of scattering amplitudes [182–189]. In the present work, we shall derive the monodromy relation for this more complicated case as well, which was not discussed in [137]. As was demonstrated in [138], the monodromy relation for the three-point functions can be straightforwardly derived once that for the two-point functions is established, in this article we shall concentrate on the case of two-point functions.

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[3]Obviously, any similarity transformation, unitary or non-unitary, does not change the structure of the group.

Now the monodromy relations for the entire $\mathfrak{psu}(2,2|4)$ sector is practically too complicated to analyze at present. In this sense, it is of interest to look first at such relations for simpler subsectors. This has already been done for the SU(2) sector in our previous work. In the present work, we shall first sketch how this result can be rederived by the reduction of the relations for the $\mathfrak{psu}(2,2|4)$ sector and then apply similar techniques to the non-compact SL(2) subsector with much more detailed expositions. Under such reductions we shall see that certain non-trivial shifts in the spectral parameters are produced. It should be stated that all the discussions in this chapter are at the tree level. It will be an important future task to extend some of the basic concepts to the loop level.

Having explained the essence of the new findings of the present work, let us briefly summarize the organization of the rest of this article.

In section 7.1, we start with a review, where we present the representation of the generators of $\mathfrak{u}(2,2|4)$ in terms of the oscillators, which transform covariantly under the maximal subgroups $U(1)_D \times SL(2,\mathbb{C}) \times \overline{SL(2,\mathbb{C})}$ of SO(4,2) (the D-scheme choice discussed above) and $U(1)_J \times SU(2)_L \times SU(2)_R$ of SO(6) respectively. With this set-up, we solve the conditions for the most general singlet state in the tensor product of two Hilbert spaces. This gives a state the form of which is not quite an exponential in the tensor product of oscillators. We shall then explain that nevertheless for the application to the super Yang-Mills fields with $C = 0$, one can promote it to a simple exponential form. As a check, we compute the relevant two-point functions of basic super Yang-Mills fields using this singlet state. (The demonstration that it reduces to a simple non-exponential form for the SU(2) subsector obtained in our previous work will be given in Appendix F.)

In section 7.2, we derive explicitly the formulas for the monodromy relations for the correlation functions in the $\mathfrak{psu}(2,2|4)$ spin chain systems, first in the case of the fundamental R-matrix and then in the case of the harmonic R-matrix, which is more involved (some of the details are relegated to Appendix G.).

In section 7.3, we explain how the monodromy relations for the $\mathfrak{psu}(2,2|4)$ can be reduced to the ones for the subsectors. In particular, we study the case of the compact SU(2) subsector and the non-compact SL(2) subsector and see that the reduction produces certain shifts in the spectral parameters.

In the final discussion section (section 7.4), we shall summarize the essential ideas and methods employed to obtain the new results in this work and discuss how they should be utilized to try to capture the principles through which to relate the super Yang-Mills theory and the string theory in AdS spaces.

As already indicated, three appendices (including Appendix A where we list all the generators of $\mathfrak{u}(2,2|4)$ in the D-scheme notation for convenience) are provided to supple-

ment the discussions given in the main text.

7.1 Oscillator description of $\mathfrak{psu}(2,2|4)$ and the singlet projector

We begin by constructing the singlet projector for the full $\mathfrak{psu}(2,2|4)$ sector from the first principle, with which one can efficiently perform the Wick contraction between the basic fields of the $\mathcal{N} = 4$ super Yang-Mills fields. As already emphasized in our previous work [138], the use of this object is quite natural and versatile in computing fairly general class of correlation functions of gauge-invariant composite operators made out of SYM fields, at least at the tree level and possibly at the higher loop levels.

7.1.1 Oscillator representation of the generators of $\mathfrak{u}(2,2|4)$ in the D-scheme

In the case of the SU(2) subsector discussed in our previous paper, the construction of the singlet projector was nothing but the elementary problem of forming a singlet state out of two spin 1/2 particles, once we regard the SU(2) spin chain as a double-chain associated with the two distinct SU(2) groups[4], $\mathrm{SU(2)_L}$ and $\mathrm{SU(2)_R}$, acting on the chain. In the case of the full $\mathfrak{psu}(2,2|4)$ spin chain, however, the structure of the algebra and its representation are sufficiently involved to render the general construction non-trivial. Luckily, as our aim is to be able to perform the Wick contraction of only the basic SYM fields, we may restrict ourselves to the singleton representation, which can be realized by a minimal set of oscillators [190, 191]. However, before introducing the oscillators, we must recall that there are basically two different bases for the representations of the $\mathfrak{u}(2,2|4)$ algebra, depending on which maximal subgroups of the conformal group SO(2,4) are used and the properties of the oscillators depend on such bases. Let us describe and compare them in some detail below following [114].

E-scheme and the D-scheme

As already mentioned in the introduction, we shall call these two schemes E-scheme and D-scheme, where "E" and "D" stand for the energy and the dilatation respectively, for

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[4]It should be clear that these two groups, belonging to SU(4), are quite different from the $\mathrm{SU(2)_L} \times \mathrm{SU(2)_R}$ groups which will appear as a part of the maximal subgroups of SO(2,4) in the E-scheme described below in (7.1.1).

which the subgroups taken are shown below:

$$(E): \quad SO(2,4) \supset SO(2)_E \times SU(2)_L \times SU(2)_R \,, \tag{7.1.1}$$

$$(D): \quad SO(2,4) \supset SO(1,1)_D \times SL(2,\mathbb{C}) \times \overline{SL(2,\mathbb{C})} \,. \tag{7.1.2}$$

For the E-scheme, the maximal subgroups are all compact, including the $SO(2)_E$ factor, the eigenvalue of which is identified with the AdS energy[5]. Thus, in the context of AdS/CFT, this scheme is useful in describing the states and their spectra on the gravity/string side. On the other hand, for the D-scheme, the maximal subgroups are all non-compact, consisting of the dilatation and the Lorentz groups. As the interpretation of SO(2,4) as the conformal group in four dimensions is manifest in this scheme, D-scheme is more natural in discussing the correlation functions in the SYM theory. Accordingly, the set of oscillators used in these two schemes are different, each set transforming covariantly under the respective maximal subgroups.

Before introducing them and discussing their difference, it is useful to first recall how the SO(2,4) algebra and its representations are described according to these two schemes. From the point of view of the conformal algebra in four dimensions, the commutation relations of the generators of SO(2,4) are given by

$$[M_{\mu\nu}, M_{\rho\sigma}] = -i(\eta_{\mu\rho}M_{\nu\sigma} - \eta_{\nu\rho}M_{\mu\sigma} + \eta_{\mu\sigma}M_{\rho\nu} - \eta_{\nu\sigma}M_{\rho\mu}) \,, \tag{7.1.3}$$

$$[M_{\mu\nu}, P_{\rho}] = -i(\eta_{\mu\rho}P_{\nu} - \eta_{\nu\rho}P_{\mu}) \,, \tag{7.1.4}$$

$$[M_{\mu\nu}, K_{\rho}] = -i(\eta_{\mu\rho}K_{\nu} - \eta_{\nu\rho}K_{\mu}) \,, \tag{7.1.5}$$

$$[D, M_{\mu\nu}] = [P_{\mu}, P_{\nu}] = [K_{\mu}, K_{\nu}] = 0 \,, \tag{7.1.6}$$

$$[-iD, P_{\mu}] = P_{\mu} \,, \quad [-iD, K_{\mu}] = -K_{\mu} \,, \tag{7.1.7}$$

$$[P_{\mu}, K_{\nu}] = 2i(\eta_{\mu\nu}D + M_{\mu\nu}) \,, \tag{7.1.8}$$

where $\mu, \nu = 0, 1, 2, 3$ and the metric signature is taken to be $\eta_{\mu\nu} = \mathrm{diag}(-1,1,1,1)$. $M_{\mu\nu}$, P_{μ}, K_{μ} and D are, respectively, the Lorentz, the momentum, the special conformal and the dilatation generators. This set of commutation relations can be expressed more compactly as

$$[J_{KL}, J_{MN}] = -i(\eta_{KM}J_{LN} - \eta_{LM}J_{KN} + \eta_{KN}J_{ML} - \eta_{LN}J_{MK}) \,, \tag{7.1.9}$$

$$J_{\mu\nu} := M_{\mu\nu} \,, \quad J_{\mu-1} := \frac{1}{2}(P_{\mu} + K_{\mu}) \,, \quad J_{\mu4} := \frac{1}{2}(P_{\mu} - K_{\mu}) \,, \quad J_{-14} := D \,, \tag{7.1.10}$$

for which the structure of SO(2,4) is manifest. In this representation, the range of six-dimensional indices and the metric are taken to be $M, N = -1, 0, 1 \cdots, 3, 4$ and $\eta_{MN} := \mathrm{diag}(-1, -1, 1, 1, 1, 1)$.

---

[5]Strictly speaking, one considers its universal cover.

Now consider this algebra from the point of view of the E-scheme. It is easy to find that the generators of the compact maximal subgroups $U(1)_E$, $SU(2)_L$ and $SU(2)_R$ are given respectively by

$$E := J_{0-1} = \frac{1}{2}(P_0 + K_0),\tag{7.1.11}$$

$$L_m := \frac{1}{2}\left(\frac{1}{2}\epsilon_{mnl}M_{nl} + M_{m4}\right),\tag{7.1.12}$$

$$R_m := \frac{1}{2}\left(\frac{1}{2}\epsilon_{mnl}M_{nl} - M_{m4}\right).\tag{7.1.13}$$

where $m, n, l = 1, 2, 3$. Obviously, L_m and R_m commute with E and hence carry zero energy. The rest of the generators of $SO(2,4)$ carry either positive or negative energy and thus the generators of the entire algebra are decomposed in the following fashion:

$$\mathfrak{so}(2,4) = \mathcal{E}^+ \oplus \mathcal{E}^0 \oplus \mathcal{E}^-,$$
$$[E, \mathcal{E}^\pm] = \pm\mathcal{E}^\pm, \quad [E, \mathcal{E}^0] = 0, \quad [\mathcal{E}^0, \mathcal{E}^\pm] \subset \pm\mathcal{E}^\pm, \quad [\mathcal{E}^+, \mathcal{E}^-] \subset \mathcal{E}^0.\tag{7.1.14}$$

Let $|e, j_L, j_R\rangle$ be the simultaneous eigenstate of the energy E and the third components L_3 and R_3 of $SU(2)_L$ and $SU(2)_R$ respectively with the eigenvalues denoted by e, j_L and j_R. Namely

$$E|e, j_L, j_R\rangle = e|e, j_L, j_R\rangle, \quad L_3|e, j_L, j_R\rangle = j_L|e, j_L, j_R\rangle, \quad R_3|e, j_L, j_R\rangle = j_R|e, j_L, j_R\rangle.\tag{7.1.15}$$

The physically relevant unitary positive energy representations are built upon the lowest weight state among the set $\{|e, j_L, j_R\rangle\}$, which is annihilated by all the energy-lowering generators belonging to $\mathcal{E}^-$. We denote them by L^{ij}, where the i and j are actually the spinor indices of $SU(2)_L$ and $SU(2)_R$ respectively and run from 1 to 2. Therefore we have four annihilation operators in total and the lowest weight state is characterized by

$$L^{ij}|e, j_L, j_R\rangle = 0.\tag{7.1.16}$$

By acting onto this vacuum the four raising operators belonging to $\mathcal{E}^+$, which we denote by L_{ij}, one obtains unitary representations in the E-scheme.

Next consider the algebra $\mathfrak{so}(2,4)$ from the D-scheme point of view. In this scheme, the generators of the maximal subgroups are given by

$$D = J_{-14},\tag{7.1.17}$$

$$\mathcal{M}_m := \frac{1}{2}\left(\frac{1}{2}\epsilon_{mnl}M_{nl} + iM_{0m}\right),\tag{7.1.18}$$

$$\mathcal{N}_m := \frac{1}{2}\left(\frac{1}{2}\epsilon_{mnl}M_{nl} - iM_{0m}\right).\tag{7.1.19}$$

Here, $\mathcal{M}_m$ and $\mathcal{N}_m$ denote the generators of the Lorentz group $\mathrm{SL}(2,\mathbb{C})\times\overline{\mathrm{SL}(2,\mathbb{C})}$. In this scheme, as is apparent from the commutation relations (7.1.7), P_μ and K_μ are, respectively the raising or lowering operators. Hence, the decomposition of the conformal algebra $\mathfrak{so}(2,4)$ is of the structure

$$\mathfrak{so}(2,4) = \mathcal{D}^+ \oplus \mathcal{D}^0 \oplus \mathcal{D}^- \,,$$
$$[-iD, \mathcal{D}^\pm] = \pm\mathcal{D}^\pm \,, \ \ [-iD, \mathcal{D}^0] = 0 \,, \ \ [\mathcal{D}^0, \mathcal{D}^\pm] \subset \pm\mathcal{D}^\pm \,, \ \ [\mathcal{D}^+, \mathcal{D}^-] \subset \mathcal{D}^0 \,, \tag{7.1.20}$$

where $P_\mu \in \mathcal{D}^+$, $K_\mu \in \mathcal{D}^-$ and $D, \mathcal{M}_m, \mathcal{N}_m \in \mathcal{D}^0$.

From the point of view of CFT in four dimensions, which is directly expressed in the D-scheme, the multiplets of operators are built upon the conformal primaries placed at the origin $x^\mu = 0$. They carry definite dilatation charges, belong to the definite Lorentz representations, and are annihilated by the lowering operators K_μ. Using the state-operator correspondence, such a primary state, denoted by $|\Delta, j_\mathcal{M}, \bar{j}_\mathcal{N}\rangle$ with Δ and $(j_\mathcal{M}, \bar{j}_\mathcal{N})$ being the dilatation charge and the Lorentz spins, is characterized by

$$- iD|\Delta, j_\mathcal{M}, j_\mathcal{N}\rangle = \Delta|\Delta, j_\mathcal{M}, j_\mathcal{N}\rangle \,, \ \ K_\mu|\Delta, j_\mathcal{M}, j_\mathcal{N}\rangle = 0 \,,$$
$$\mathcal{M}_3|\Delta, j_\mathcal{M}, j_\mathcal{N}\rangle = j_\mathcal{M}|\Delta, j_\mathcal{M}, j_\mathcal{N}\rangle \,, \ \ \mathcal{N}_3|\Delta, j_\mathcal{M}, j_\mathcal{N}\rangle = j_\mathcal{N}|\Delta, j_\mathcal{M}, j_\mathcal{N}\rangle \,. \tag{7.1.21}$$

Then the module is built up by the descendants generated by the multiplicative actions of the raising operators P_μ. It should be emphasized that such a representation relevant for discussing the correlation functions is non-unitary, since the anti-hermitian operator $-iD$ has real eigenvalues.

Now let us give a brief description of the relation between the E-scheme used in [114, 137] and the D-scheme to be employed exclusively in this work. It is well-known by the work of [177] that there exists a non-unitary similarity transformation between the generators of these two schemes. The correspondence between $E = J_{-1,0}$ and $-iD = -iJ_{-1,4}$ indicates that such a transformation should rotate the non-compact 0-th direction into the compact 4-th direction and indeed it is effected by the operator

$$U = \exp\left(\frac{\pi}{2}M_{04}\right) = \exp\left(\frac{\pi}{4}(P_0 - K_0)\right) \,. \tag{7.1.22}$$

Explicit transformations are given by

$$U^{-1}(-iD)U = E \,, \ \ \ U^{-1}L_m U = \mathcal{M}_m \,, \ \ \ U^{-1}R_m U = \mathcal{N}_m \,, \tag{7.1.23}$$

$$U^{-1}P_\mu U \in \mathcal{E}^+ \,, \ \ \ U^{-1}K_\mu U \in \mathcal{E}^- \,, \tag{7.1.24}$$

$$|\Delta, j_\mathcal{M}, j_\mathcal{N}\rangle = U|e, j_L, j_R\rangle \,, \ \ \ \text{with } \Delta = e \,, \ j_\mathcal{M} = j_L \,, \ j_\mathcal{N} = j_R \,. \tag{7.1.25}$$

As already mentioned, for the purpose of discussing the CFT correlation functions, the D-scheme is much more natural and if one starts from the E-scheme description as in

[114, 137], one must necessarily manipulate with the operator U in the intermediate step. Also, in the oscillator representations of the generators, to be elaborated below, the D-scheme oscillators always keep the conformal covariance manifest as opposed to those in the E-scheme. We shall make the comparison more explicit later.

Oscillator representation in the D-scheme

Having argued the advantage of the D-scheme for our purpose, let us now introduce appropriate oscillators for this scheme and express the generators as their quadratic combinations.

For this purpose, it is useful to rewrite first the generators of the SO(2,4) algebra using the dotted and the undotted spinor indices of the Lorentz group. We will adopt the following conventions for the conversions of vectors and the tensors:

$$P_{\alpha\dot\beta} := -\frac{1}{2}(\sigma^\mu)_{\alpha\dot\beta}P_\mu\,, \quad K^{\dot\alpha\beta} := +\frac{1}{2}(\bar\sigma^\mu)^{\dot\alpha\beta}K_\mu\,, \tag{7.1.26}$$

$$M_\alpha{}^\beta := \frac{i}{2}(\sigma^{\mu\nu})_\alpha{}^\beta M_{\mu\nu}\,, \quad \bar M^{\dot\alpha}{}_{\dot\beta} := \frac{i}{2}(\bar\sigma^{\mu\nu})^{\dot\alpha}{}_{\dot\beta}M_{\mu\nu}\,, \tag{7.1.27}$$

where the Lorentz sigma matrices are defined in terms of the Paul matrices in the following way

$$(\sigma^\mu)_{\alpha\dot\beta} = (-1, \sigma^i)_{\alpha\dot\beta}\,, \quad (\bar\sigma^\mu)^{\dot\alpha\beta} = \epsilon^{\dot\alpha\dot\gamma}\epsilon^{\beta\delta}(\sigma^\mu)_{\delta\dot\gamma} = (-1, -\sigma^i)^{\dot\alpha\beta}\,, \tag{7.1.28}$$

$$(\sigma^{\mu\nu})_\alpha{}^\beta = (\sigma^{[\mu}\bar\sigma^{\nu]})_\alpha{}^\beta\,, \quad (\bar\sigma^{\mu\nu})^{\dot\alpha}{}_{\dot\beta} = (\bar\sigma^{[\mu}\sigma^{\nu]})^{\dot\alpha}{}_{\dot\beta}\,. \tag{7.1.29}$$

Our convention for the epsilon tensor, with which the indices are raised and lowered, will be $\epsilon_{12} = \epsilon_{\dot1\dot2} = -\epsilon^{12} = -\epsilon^{\dot1\dot2} = 1$.

In this notation, the SO(2,4) commutation relations take the form

$$[M_\alpha{}^\beta, J_\gamma] = \delta_\gamma^\beta J_\alpha - \frac{1}{2}\delta_\alpha{}^\beta J_\gamma\,, \quad [M_\alpha{}^\beta, J^\gamma] = -\delta_\alpha^\gamma J^\beta + \frac{1}{2}\delta_\alpha{}^\beta J^\gamma\,, \tag{7.1.30}$$

$$[\bar M^{\dot\alpha}{}_{\dot\beta}, J_{\dot\gamma}] = -\delta_{\dot\gamma}^{\dot\alpha} J_{\dot\beta} + \frac{1}{2}\delta^{\dot\alpha}{}_{\dot\beta} J^{\dot\gamma}\,, \quad [\bar M^{\dot\alpha}{}_{\dot\beta}, J_{\dot\gamma}] = \delta_{\dot\beta}^{\dot\gamma} J^{\dot\alpha} - \frac{1}{2}\delta^{\dot\alpha}{}_{\dot\beta} J_{\dot\gamma}\,, \tag{7.1.31}$$

$$[D, P_{\alpha\dot\beta}] = iP_{\alpha\dot\beta}\,, \quad [D, K^{\dot\alpha\beta}] = -iK^{\dot\alpha\beta}\,, \quad [D, M_\alpha{}^\beta] = [D, \bar M^{\dot\alpha}{}_{\dot\beta}] = 0 \tag{7.1.32}$$

$$[P_{\alpha\dot\beta}, K^{\dot\gamma\delta}] = \delta_\alpha^\delta \bar M^{\dot\gamma}{}_{\dot\beta} - \delta_{\dot\beta}^{\dot\gamma}M_\alpha{}^\delta + i\delta_\alpha^\delta\delta_{\dot\beta}^{\dot\gamma}D\,, \tag{7.1.33}$$

where J^γ and $J_{\dot\gamma}$ generically stand for quantities with undotted and dotted spinor indices.

With this preparation, it is now quite natural to introduce two sets of bosonic oscillators, undotted and dotted, which transform under $\mathrm{SL}(2,\mathbb{C})$ and $\overline{\mathrm{SL}(2,\mathbb{C})}$ respectively

$$[\mu^\alpha, \lambda_\beta] = \delta_\beta^\alpha, \quad (\alpha, \beta = 1, 2)\,, \quad [\tilde\mu^{\dot\alpha}, \tilde\lambda_{\dot\beta}] = \delta_{\dot\beta}^{\dot\alpha} \quad (\dot\alpha, \dot\beta = \dot1, \dot2)\,. \tag{7.1.34}$$

In terms of these oscillators, the conformal generators can be expressed rather simply as
[6]

$$M_\alpha{}^\beta = \lambda_\alpha \mu^\beta - \frac{1}{2}\delta_\alpha^\beta \lambda_\gamma \mu^\gamma \,, \quad \bar{M}^{\dot\alpha}{}_{\dot\beta} = -\tilde\lambda_{\dot\beta}\tilde\mu^{\dot\alpha} + \frac{1}{2}\delta^{\dot\alpha}_{\dot\beta}\tilde\lambda_{\dot\gamma}\tilde\mu^{\dot\gamma} \,, \tag{7.1.35}$$

$$P_{\alpha\dot\beta} = \lambda_\alpha\tilde\lambda_{\dot\beta} \,, \quad K^{\dot\alpha\beta} = \tilde\mu^{\dot\alpha}\mu^\beta \,, \quad D = \frac{i}{2}(\lambda_\alpha\mu^\alpha + \tilde\lambda_{\dot\alpha}\tilde\mu^{\dot\alpha} + 2) \,. \tag{7.1.36}$$

Essentially the same oscillator representation was given in [21, 192]. We shall follow [192] with slight changes of signs and conventions.

Now to construct the SU(4) R-symmetry generators, we introduce four sets of fermionic oscillators satisfying the anti-commutation relations

$$\{\xi^a, \bar\xi_b\} = \delta_b^a \,, \quad (a, b = 1, 2, 3, 4) \,. \tag{7.1.37}$$

Then the SU(4) generators can be constructed as

$$R_a{}^b = \bar\xi_a\xi^b - \frac{1}{4}\delta_a^b\bar\xi_c\xi^c \,, \tag{7.1.38}$$

which indeed satisfy the correct commutation relations $[R_a{}^b, R_c{}^d] = \delta_b{}^c R_a{}^d - \delta_a{}^d R_c{}^b$. It is easy to check that under $R_a{}^b$ the oscillators $\bar\xi_b$ and ξ^a transform under the fundamental and anti-fundamental representations. As the SU(4) indices of any generator will be carried by these fundamental oscillators, this guarantees that a generator J_c (J^c) having a lower (upper) index transforms as a fundamental (anti-fundamental), *i.e.*

$$[R_a{}^b, J_c] = \delta_c^b R_a{}^b - \frac{1}{4}\delta_a{}^b J_c \,, \quad [R_a{}^b, J^c] = -\delta_a^c J^b + \frac{1}{4}\delta_a{}^b J^c \,. \tag{7.1.39}$$

The remaining generators, namely the fermionic supersymmetry and superconformal generators, are expressed in a very simple way where the transformation properties are directly expressed by those of the constituent oscillators:

$$Q_\alpha^a := \lambda_\alpha\xi^a \,, \quad \bar{Q}_{\dot\alpha a} = \tilde\lambda_{\dot\alpha}\bar\xi_a \,, \tag{7.1.40}$$

$$S_a^\alpha = \mu^\alpha\bar\xi_a \,, \quad \bar{S}^{\dot\alpha a} = \tilde\mu^{\dot\alpha}\xi^a \,. \tag{7.1.41}$$

Central charge and hyper charge

The 30 bosonic and 32 fermionic generators constructed in terms of the oscillators above constitute the generators of the $\mathfrak{psu}(2, 2|4)$. Actually, they do not close under (anti-) commutation. The closure requires the operator called the central charge given by

$$C = \frac{1}{2}(\lambda_\alpha\mu^\alpha - \tilde\lambda_{\dot\alpha}\tilde\mu^{\dot\alpha} + \bar\xi_a\xi^a) - 1 \,. \tag{7.1.42}$$

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[6]Recall that $-iD$ has positive real eigenvalues in our convention.

As the name indicates, C commutes with all the generators of $\mathfrak{psu}(2,2|4)$ and hence it takes a constant value for an irreducible representation. In particular, as we shall describe shortly, for the basic fields of the $\mathcal{N} = 4$ SYM (*i.e.* for the field strength multiplet) of our interest, C vanishes. Thus in this sector, we can neglect this operator. Another additional operator of interest is the so-called the hypercharge operator[7] given by

$$B = \frac{1}{2}\bar{\xi}_a \xi^a \,. \tag{7.1.43}$$

This is essentially the fermion number operator. One notices that B does not appear in all the (anti-) commutation relations of $\mathfrak{psu}(2,2|4)$ and thus it can be regarded as an outer automorphism of $\mathfrak{psu}(2,2|4)$. By adding B and C to $\mathfrak{psu}(2,2|4)$, we obtain the closed algebra called $\mathfrak{u}(2,2|4)$.

The generators of $\mathfrak{u}(2,2|4)$ can be expressed succinctly in terms of the oscillators as

$$J^A_{\ B} = \bar{\zeta}^A \zeta_B \,, \quad \bar{\zeta}^A = \begin{pmatrix} \lambda_\alpha \\ i\tilde{\mu}^{\dot\alpha} \\ \bar{\xi}_a \end{pmatrix}^A \,, \quad \zeta_A = \begin{pmatrix} \mu^\alpha \\ i\tilde{\lambda}_{\dot\alpha} \\ \xi^a \end{pmatrix}_A \,. \tag{7.1.44}$$

One can check that ζ's satisfy the graded commutator of the form

$$[\zeta_A, \bar{\zeta}^B] = \zeta_A \bar{\zeta}^B - (-1)^{|A||B|}\bar{\zeta}^B \zeta^A = \delta_A^{\ B} \,, \tag{7.1.45}$$

where $|A|$ is 1 for fermions and 0 for bosons. Hereafter, for simplicity, all the commutators should be interpreted as graded commutator as above. Then, the graded commutators between the generators of $\mathfrak{u}(2,2|4)$ are neatly summarized in the following form:

$$[J^A_{\ B}, J^C_{\ D}] = \delta^C_{\ B} J^A_{\ D} - (-1)^{(|A|+|B|)(|C|+|D|)}\delta^A_{\ D} J^C_{\ D} \,. \tag{7.1.46}$$

It is useful to write down the elements $J^A_{\ B}$ of $\mathfrak{u}(2,2|4)$ in a matrix form in the following way:

$$J^A_{\ B} := \left(\begin{array}{cc|c} J_\alpha^{\ \beta} & J_{\alpha\dot\beta} & J_\alpha^{\ b} \\ J^{\dot\alpha\beta} & J^{\dot\alpha}_{\ \dot\beta} & J^{\dot\alpha b} \\ \hline J_a^{\ \beta} & J_{a\dot\beta} & J_a^{\ b} \end{array} \right)_{AB} = \left(\begin{array}{cc|c} Y_\alpha^{\ \beta} & iP_{\alpha\dot\beta} & Q_\alpha^b \\ iK^{\dot\alpha\beta} & Y^{\dot\alpha}_{\ \dot\beta} & i\bar{S}^{\dot\alpha b} \\ \hline S_a^{\ \beta} & iQ_{\dot\beta a} & W_a^{\ b} \end{array} \right)_{AB} \,, \tag{7.1.47}$$

$$Y_\alpha^{\ \beta} = \lambda_\alpha \mu^\beta = M_\alpha^{\ \beta} + \frac{1}{2}\delta_\alpha^{\ \beta}(-iD + C - B) \,, \tag{7.1.48}$$

$$Y^{\dot\alpha}_{\ \dot\beta} = -\tilde{\mu}^{\dot\alpha}\tilde{\lambda}_{\dot\beta} = \bar{M}^{\dot\alpha}_{\ \dot\beta} + \frac{1}{2}\delta^{\dot\alpha}_{\ \dot\beta}(iD + C - B) \,, \tag{7.1.49}$$

$$W_a^{\ b} = \bar{\xi}_a \xi^b = R_a^{\ b} + \frac{1}{2}\delta_a^{\ b}B \,. \tag{7.1.50}$$

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[7]The definition of the hypercharge is ambiguous in the sense that we can add the central charge to it. For example, in the literature [21], the hypercharge is defined by $Z := \frac{1}{2}(\lambda_\alpha \mu^\alpha - \tilde{\lambda}_{\dot\alpha}\tilde{\mu}^{\dot\alpha})$ and it plays a role of the chirality operator. The relation to our definition is $Z = C - B + 1$.

From this one can see that the central charge and the hypercharge are related to the trace and supertrace in the following way

$$\text{tr}J := \sum_A J^A_{\ A} = 2C\,, \qquad \text{str}J := \sum_A (-1)^A J^A_{\ A} = 2C - 4B\,. \tag{7.1.51}$$

As we shall see latter, the singlet projector we shall construct will be valid for the $\mathfrak{su}(2,2|4)$ algebra as well as for $\mathfrak{psu}(2,2|4)$, where the generators $\hat{J}^A_{\ B}$ of the former is obtained from $\mathfrak{u}(2,2|4)$ by imposing the supertraceless condition as

$$\hat{J}^A_{\ B} := J^A_{\ B} - \frac{\text{str}J}{8}(-1)^{|A|}\delta^A_{\ B}\,. \tag{7.1.52}$$

In particular this gives $\sum_A (-1)^{|A|}\hat{J}^A_{\ A} = 0$, which tells us that the hypercharge B is completely removed from $\mathfrak{su}(2,2|4)$.

Oscillator vacuum and the representations of the fundamental SYM fields

We now move on to the oscillator representation for the fundamental fields which appear in $\mathcal{N}=4$ SYM. For this purpose, we define the Fock vacuum $|0\rangle$ to be the state annihilated by all the annihilation operators:

$$\mu^\alpha|0\rangle = \tilde{\mu}^{\dot{\alpha}}|0\rangle = \xi^a|0\rangle = 0\,. \tag{7.1.53}$$

To be more precise, $|0\rangle$ is a tensor product of two vacua, one for the bosonic oscillators and the other for the fermionic ones. Namely,

$$|0\rangle = |0\rangle_B \otimes |0\rangle_F\,, \tag{7.1.54}$$

Then the Fock space is built upon this vacuum by acting by the creation operators $\lambda_\alpha, \tilde{\lambda}_{\dot{\alpha}}, \bar{\xi}_a$. However, not all the states produced this way correspond to the fields of $\mathcal{N}=4$ SYM. The relevant ones are only those carrying zero central charge. This can be explicitly checked by the expressions of the basic $\mathcal{N}=4$ SYM fields in terms of the oscillators given by [21]

$$F_{\alpha\beta}(0) \leftrightarrow \lambda_\alpha\lambda_\beta|0\rangle\,, \tag{7.1.55}$$

$$\psi_{\alpha a}(0) \leftrightarrow \lambda_\alpha\bar{\xi}_a|0\rangle\,, \tag{7.1.56}$$

$$\phi_{ab}(0) \leftrightarrow \bar{\xi}_a\bar{\xi}_b|0\rangle\,, \tag{7.1.57}$$

$$\bar{\psi}^a_{\dot{\alpha}}(0) \leftrightarrow \frac{1}{3!}\epsilon^{abcd}\tilde{\lambda}_{\dot{\alpha}}\bar{\xi}_b\bar{\xi}_c\bar{\xi}_d|0\rangle\,, \tag{7.1.58}$$

$$\bar{F}_{\dot{\alpha}\dot{\beta}}(0) \leftrightarrow \frac{1}{4!}\epsilon^{abcd}\tilde{\lambda}_{\dot{\alpha}}\tilde{\lambda}_{\dot{\beta}}\bar{\xi}_a\bar{\xi}_b\bar{\xi}_c\bar{\xi}_d|0\rangle\,. \tag{7.1.59}$$

From the form of C given in (7.1.42) it is clear that they all carry $C = 0$. Also, it is easy to check that these oscillator expressions of the fields carry the correct Lorentz and R-symmetry quantum numbers.

In addition to these fundamental fields, we need to express their derivatives. The field at the general position x is obtained by the action of the translation operator $e^{iP \cdot x}$ as

$$|\mathcal{O}(0)\rangle \rightarrow |\mathcal{O}(x)\rangle := e^{iP \cdot x}|\mathcal{O}(0)\rangle. \tag{7.1.60}$$

From the oscillator representation $P_{\alpha\dot\beta} = \lambda_\alpha \tilde\lambda_{\dot\beta}$, we see that the derivatives of a field can be expressed as

$$\partial_{(\alpha(\dot\beta} \cdots \partial_{\gamma)\dot\delta)} \mathcal{O}(x) \cong (i\lambda_\alpha \tilde\lambda_{\dot\beta}) \dots (i\lambda_\gamma \tilde\lambda_{\dot\delta})|\mathcal{O}(x)\rangle, \tag{7.1.61}$$

where $\partial_{\alpha\dot\beta} = \partial/\partial x^{\dot\beta\alpha}$ and $x^{\dot\alpha\beta} := x^\mu (\bar\sigma_\mu)^{\dot\alpha\beta}$ and we have used $P \cdot x = P_{\alpha\dot\beta} x^{\dot\beta\alpha}$ and $\partial_{\alpha\dot\beta} e^{iP \cdot x} = iP_{\alpha\dot\beta} e^{iP \cdot x}$. Notice that the spinor indices $(\alpha, \gamma, \dots)$ and $(\dot\beta, \dot\delta, \dots)$ are symmetrized as the bosonic oscillators λ mutually commute. Also note that we can replace some combinations of the (covariant) derivatives by appropriate fields without derivatives using the equations of motion and the Bianchi identities. For example, we can set $\partial_{\alpha\dot\beta}\partial^{\dot\beta\alpha}\phi \propto \square\phi$ and $\epsilon^{\alpha\beta}\partial_{\alpha\dot\alpha}\psi^a_\beta$ to zero due to the free equations of motion[8]. As a result, we can express fields with derivatives by expressions where all the spinor indices are totally symmetrized. Therefore, the independent fields with derivatives are simply generated by acting $P_{\alpha\dot\beta} = \lambda_\alpha \tilde\lambda_{\dot\beta}$ on the oscillator representations for the fundamental fields (7.1.55)-(7.1.59). Since $P_{\alpha\dot\beta}$ commutes with the central charge, these states with derivatives are still within the subspace with vanishing central charge.

Various "vacua" and their relations

It is an elementary exercise in quantum mechanics to construct the singlet state from two spin 1/2 states by forming a suitable combination of the highest and the lowest states. It is a slightly more involved exercise to extend this to the case of the general spin j, but the structure is similar: One combines the states built upon the lowest weight states and those built upon the highest weight states with simple weights. Indeed, up to an overall constant, the singlet state is given by

$$|\mathbf{1}_j\rangle = \sum_{l=0}^{2j} (-1)^l |-(j-l)\rangle \otimes |j-l\rangle \tag{7.1.62}$$

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[8]In the interacting case, it is possible to replace the combinations of covariant derivatives such as $\mathcal{D}_{\alpha\dot\beta}\mathcal{D}^{\dot\beta\alpha}\phi$ and $\epsilon^{\alpha\beta}\mathcal{D}_{\alpha\dot\alpha}\psi^a_\beta$ by the fields without derivatives using the equations of motion as well.

This indicates that for the construction of the singlet state for much more complicated case of $\mathfrak{psu}(2,2|4)$, the basic idea should be the same and one would combine the Fock states built upon the lowest weight oscillator vacuum $|0\rangle$, already introduced, with the states built upon the highest weight oscillator vacuum $|\bar{0}\rangle$, which should be defined to be annihilated by the creation operators as

$$\lambda_\alpha|\bar{0}\rangle = \tilde{\lambda}_{\dot{\alpha}}|\bar{0}\rangle = \bar{\xi}_a|\bar{0}\rangle = 0 \,. \tag{7.1.63}$$

Just as for $|0\rangle$ given in (7.1.54), the more precise definition of $|\bar{0}\rangle$ is

$$|\bar{0}\rangle \equiv |\bar{0}\rangle_B \otimes |\bar{0}\rangle_F \,. \tag{7.1.64}$$

From (7.1.63) it immediately follows that $|\bar{0}\rangle$ is annihilated by $P_{\alpha\dot{\beta}} = \lambda_\alpha \tilde{\lambda}_{\dot{\beta}}$ and thus the Fock space built on $|\bar{0}\rangle$ is a highest weight module as opposed to the lowest weight module built on $|0\rangle$.

There is an essential difference between the bosonic sector and the fermionic sector. For the bosonic sector, $|0\rangle_B$ and $|\bar{0}\rangle_B$ cannot be related by the action of a finite number of oscillators[9], but for the fermionic sector one can readily identify $|\bar{0}\rangle_F = \bar{\xi}_1\bar{\xi}_2\bar{\xi}_3\bar{\xi}_4|0\rangle_F$.

It will turn out, however, that as for the fermionic oscillator Fock space describing the R-symmetry quantum numbers, "vacua" slightly different from $|0\rangle$ and $|\bar{0}\rangle$ will be more useful and more physical. To introduce them, we rename the fermionic oscillators in the following way so that half of the creation (annihilation) operators are switched to annihilation (creation) operators[10]:

$$c^i = \xi^i \ (i = 1, 2) \,, \quad d^i = \bar{\xi}_{i+2} \ (i = 1, 2) \,, \tag{7.1.65}$$

$$\bar{c}_i = \bar{\xi}_i \ (i = 1, 2) \,, \quad \bar{d}_i = \xi^{i+2} \ (i = 1, 2) \,. \tag{7.1.66}$$

We define the state $|Z\rangle$ as annihilated by the new annihilation operators c^i and d^i, while $|\bar{Z}\rangle$ is defined to be annihilated by the new creation operators $\bar{c}_i$ and $\bar{d}_i$.

$$c^i|Z\rangle = d^i|Z\rangle = 0 \,, \tag{7.1.67}$$

$$\bar{c}_i|\bar{Z}\rangle = \bar{d}_i|\bar{Z}\rangle = 0 \,. \tag{7.1.68}$$

As states built on the original vacuum $|0\rangle$, these new "vacua" can be written as

$$|Z\rangle = d^1 d^2|0\rangle = \bar{\xi}_3\bar{\xi}_4|0\rangle \,, \tag{7.1.69}$$

$$|\bar{Z}\rangle = \bar{c}_1\bar{c}_2|0\rangle = \bar{\xi}_1\bar{\xi}_2|0\rangle \tag{7.1.70}$$

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[9]Actually, by using the operator $U^2 = \exp[\frac{\pi}{2}(P_0 - K_0)]$ one can map $|0\rangle$ to $|\bar{0}\rangle$ and exchange the role of the annihilation and the creation operators.

[10]Such a transformation is sometimes called a particle-hole transformation.

and they are related as $|\bar{Z}\rangle = -\bar{c}_1\bar{c}_2\bar{d}_1\bar{d}_2|Z\rangle$ or $|Z\rangle = -c^1c^2d^1d^2|\bar{Z}\rangle$. Now if we recall that the SO(6) scalars are represented by $\bar{\xi}_a\bar{\xi}_b|0\rangle$ as shown in (7.1.57), $|Z\rangle$ and $|\bar{Z}\rangle$ correspond to some physical scalars. To be definite let us identify them as states carrying $\mathrm{SU}(2)_L \times \mathrm{SU}(2)_R$ quantum numbers of the SU(2) sector, where the generators are given by[11]

$$J_+^L = c^1 d^1\,, \quad J_-^L = \bar{d}_1\bar{c}_1\,, \quad J_3^L = \frac{1}{2}(d^1\bar{d}_1 - \bar{c}_1 c^1)\,, \tag{7.1.71}$$

$$J_+^R = c^2 d^2\,, \quad J_-^R = \bar{d}_2\bar{c}_2\,, \quad J_3^R = \frac{1}{2}(d^2\bar{d}_2 - \bar{c}_2 c^2)\,. \tag{7.1.72}$$

Then, it is easy to see that $|Z\rangle$ and $|\bar{Z}\rangle$ carry the quantum numbers $(\frac{1}{2}, \frac{1}{2})$ and $(-\frac{1}{2}, -\frac{1}{2})$ respectively and hence can be identified with, say, $\Phi_1 + i\Phi_2$ and its complex conjugate. The (de-)excitations of these vacua in the SU(2) sector with the quantum numbers $(\frac{1}{2}, -\frac{1}{2})$ and $(-\frac{1}{2}, \frac{1}{2})$ respectively, which are often denoted by $|Y\rangle$ and $|-\bar{Y}\rangle$, are given by

$$|Y\rangle = J_-^R|Z\rangle = J_+^L|\bar{Z}\rangle = \bar{c}_2 d^1|0\rangle\,, \tag{7.1.73}$$

$$|-\bar{Y}\rangle = J_-^L|Z\rangle = J_+^R|\bar{Z}\rangle = d^2\bar{c}_1|0\rangle\,. \tag{7.1.74}$$

Now in order to construct the singlet projector in the next section, it will turn out to be convenient to define the scalar states similar to the above, except that their bosonic part of the vacuum is switched from $|0\rangle_B$ to $|\bar{0}\rangle_B$. We will place a line over the kets (or the corresponding bra) to denote such scalar states. For example,

$$\overline{|Z\rangle} \equiv |\bar{0}\rangle_B \otimes d^1 d^2|0\rangle_F\,, \tag{7.1.75}$$

$$\overline{|\bar{Z}\rangle} \equiv |\bar{0}\rangle_B \otimes \bar{c}^1\bar{c}^2|0\rangle_F\,. \tag{7.1.76}$$

In this more precise notation, the previously defined $|Z\rangle$ and $|\bar{Z}\rangle$ are written as

$$|Z\rangle = |0\rangle_B \otimes d^1 d^2|0\rangle_F\,, \tag{7.1.77}$$

$$|\bar{Z}\rangle = |0\rangle_B \otimes \bar{c}^1\bar{c}^2|0\rangle_F\,. \tag{7.1.78}$$

Since overlined scalar states differ only in the bosonic sector, the properties of such states under the action of the fermionic oscillators are exactly the same as the un-overlined ones. For example, $c^i\overline{|Z\rangle} = 0$, etc., just as in (7.1.67) and (7.1.68).

$$\bar{c}_i\overline{|Z\rangle} = \bar{d}_i\overline{|Z\rangle} = 0\,, \quad \overline{|Z\rangle} := \xi^1\xi^2|\bar{0}\rangle\,, \tag{7.1.79}$$

$$c^i\overline{|\bar{Z}\rangle} = d^i\overline{|\bar{Z}\rangle} = 0\,, \quad \overline{|\bar{Z}\rangle} := \xi^3\xi^4|\bar{0}\rangle\,. \tag{7.1.80}$$

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[11]Of course the choice of $\mathrm{SU}(2)_L \times \mathrm{SU}(2)_R$ in SU(4) is not unique. We are simply taking a convenient one. Incidentally, our normalization for $J_\pm$ is $J_\pm = J_1 \pm i J_2$.

As we will need them later, it should be convenient to list the properties of the bra (or dual) vacua, which evidently follow from those of the ket vacua. $\langle 0|$ and $\langle \bar{0}|$ have the properties

$$\langle 0|\lambda_\alpha = \langle 0|\tilde{\lambda}_{\dot{\alpha}} = \langle 0|\bar{\xi}_a = 0\,, \quad \langle 0|0\rangle = 1\,, \tag{7.1.81}$$

$$\langle \bar{0}|\mu^\alpha = \langle \bar{0}|\tilde{\mu}^{\dot{\alpha}} = \langle \bar{0}|\xi^a = 0\,, \quad \langle \bar{0}|\bar{0}\rangle = 1\,. \tag{7.1.82}$$

This means that the dual Fock space is generated either by the action of $(\mu^\alpha, \tilde{\mu}^{\dot{\alpha}}, \xi_a)$ on $\langle 0|$ or by the action of $(\lambda_\alpha, \tilde{\lambda}_{\dot{\alpha}}, \bar{\xi}^a)$ on $\langle \bar{0}|$. As for the properties of the scalar bra vacua under the action of the fermionic oscillators, they satisfy

$$\langle Z|\bar{c}_i = \langle Z|\bar{d}_i = 0\,, \quad \langle \bar{Z}|c^i = \langle \bar{Z}|d^i = 0\,, \tag{7.1.83}$$

$$\langle Z|Z\rangle = \langle \bar{Z}|\bar{Z}\rangle = 1\,, \tag{7.1.84}$$

and exactly the same equations hold for the overlined bra states $\overline{\langle Z|}$ and $\overline{\langle \bar{Z}|}$.

Comparison with the E-scheme formulation

Before we start the systematic construction of the singlet projector using these oscillator representations, let us end this subsection with some comments on the difference between the oscillator representation we use and the one employed in [114, 137]. They basically work in the E-scheme, where the oscillators are covariant under the compact subgroup shown in (7.1.1). This in turn means that the representations corresponding to local composite operators are obtained indirectly by the use of the complicated operator $U = \exp[\frac{\pi}{4}(P_0 - K_0)]$. To be a little more specific, let us display the E-scheme oscillators and the Fock vacuum used in [114, 137]. The difference from ours is in the bosonic oscillators, which are given by

$$[a_i, \bar{a}^j] = \delta_i^j\,, (i, j = 1, 2)\ [b_s, \bar{b}^t] = \delta_s^t\,, \ (s, t = 1, 2) \tag{7.1.85}$$

$$a_i|0\rangle_E = b_s|0\rangle_E = 0\,. \tag{7.1.86}$$

Then the SO(2,4) generators are expressed as bi-linears of these oscillators as

$$L_j^i = \bar{a}^i a_j - \frac{1}{2}\delta_j^i(\bar{a}^k a_k)\,, \ R_t^s = \bar{b}^s b_t - \frac{1}{2}\delta_t^s(\bar{b}^u b_u)\,, \tag{7.1.87}$$

$$E = \frac{1}{2}(\bar{a}^i a_i + \bar{b}^s b_s) + 1\,, \ L_{is} = a_i b_s\,, \ L^{is} = \bar{a}^i \bar{b}^s\,, \tag{7.1.88}$$

where L_j^i, R_t^s are SU(2)$_L \times$ SU(2)$_R$ generators, E is the AdS energy and L_{is}, L^{is} are the elements of $\mathcal{E}^-$, $\mathcal{E}^+$ respectively. Hence, the bosonic oscillators $a_i, \bar{a}^i$ transform covariantly under SU(2)$_L$ as a doublet and $b_s, \bar{b}^s$ are doublets of SU(2)$_R$. To convert them to the D-scheme oscillators, one needs to employ the similarity transformation using the operator U.

However, as we mentioned in the introduction, any similarity transformation preserves the structure of the algebra and hence, for example, $SU(2)_L$ does not become a Lorentz group $SL(2, \mathbb{C})$. This is reflected in the transformation of the oscillators themselves. By using the explicit oscillator representation of U, we easily find, for example, $U^{-1}\bar{a}^i U = \frac{1}{\sqrt{2}}(\bar{a}^i + b_i)$ etc., which is not informative as far as the useful re-interpretation to the D-scheme is concerned. Thus although the E- and the D- schemes are connected by a similarity transformation U, the conformal covariance cannot be made manifest by just such a transformation. Hence for the purpose of dealing with the local composite operators, the use of D-scheme is much more transparent and indeed in what follows we shall never need the operator U.

7.1.2 Construction of the singlet projector for $\mathfrak{psu}(2,2|4)$

Singlet condition and its solution

We shall now give a detailed construction of the singlet projector $_{\text{psu}}\langle\mathbf{1}_{12}|$ for the states in the product of a pair of Hilbert spaces $\mathcal{H}_1 \otimes \mathcal{H}_2$, which satisfies the defining equation for the singlet projector

$$_{\text{psu}}\langle\mathbf{1}_{12}|(J^A{}_B \otimes \mathbf{1} + \mathbf{1} \otimes J^A{}_B) = 0, \qquad J^A{}_B \in \mathfrak{psu}(2,2|4). \tag{7.1.89}$$

In order to find the most general singlet projector, we must proceed systematically. As it will become clear below, actually the desired singlet projector satisfying the relation above can be constructed for $\mathfrak{su}(2,2|4)$ as well as for $\mathfrak{psu}(2,2|4)$, but not for $\mathfrak{u}(2,2|4)$. Recall that the generators $J^A{}_B$ of $\mathfrak{su}(2,2|4)$ are obtained from the generator of $\mathfrak{u}(2,2|4)$, to be tentatively denoted by $\hat{J}^A{}_B$, by making them supertraceless, $i.e.$

$$J^A{}_B = \hat{J}^A{}_B - \frac{1}{8}\delta^A_B(-1)^{|A|}(\text{Str}\hat{J}). \tag{7.1.90}$$

Because of this condition, when we interchange the order of the two conjugate oscillators making up any diagonal generator $J^A{}_A$, constant terms produced from the (anti-)commutation relations precisely cancel. This property will be of crucial importance for the construction of the true singlet projector.

First, let us begin by identifying the building block for the sector involving the oscillator pair λ_α and μ^α. Since the generators $J^A{}_B$ are quadratic in the oscillators, the building block which would realize the relation (7.1.89) in the above sector should be of the form

$$_{\lambda\mu}\langle\mathbf{1}_{12}| \propto (\langle Z| \otimes \overline{\langle\bar{Z}|})(\mu^\alpha)^{n_\mu} \otimes (\lambda_\beta)^{n_\lambda}) \tag{7.1.91}$$

Now consider a useful combination of generators $\mathcal{J}^{(1)} \equiv \lambda_1\mu^1 - \lambda_2\mu^2 = \mu^1\lambda_1 - \mu^2\lambda_2$, which belongs to $\mathfrak{su}(2,2|4)$ and hence the interchange of the order of λ_α and μ^α does not produce

any constant. When we apply $\mathcal{J}^{(1)} \otimes \mathbf{1}$, we should use the form $\mathcal{J}^{(1)} \equiv \lambda_1 \mu^1 - \lambda_2 \mu^2$ since λ_α annihilates $\langle Z|$. Then we easily obtain

$$(\langle Z| \otimes \overline{\langle \bar{Z}|})(\mu^\alpha)^{n_\mu} \otimes (\lambda_\beta)^{n_\lambda})(\mathcal{J}^{(1)} \otimes \mathbf{1}) = n_\mu(\delta_1^\alpha - \delta_2^\alpha)(\langle Z| \otimes \overline{\langle \bar{Z}|})(\mu^\alpha)^{n_\mu} \otimes (\lambda_\beta)^{n_\lambda}). \tag{7.1.92}$$

On the other hand, when we apply $\mathbf{1} \otimes \mathcal{J}^{(1)}$, since $\overline{\langle \bar{Z}|}$ is annihilated by μ^α, we should use the form $\mathcal{J}^{(1)} = \mu^1 \lambda_1 - \mu^2 \lambda_2$. Then, we get

$$(\langle Z| \otimes \overline{\langle \bar{Z}|})(\mu^\alpha)^{n_\mu} \otimes (\lambda_\beta)^{n_\lambda})(\mathbf{1} \otimes \mathcal{J}^{(1)}) = -n_\lambda(\delta_\beta^1 - \delta_\beta^2)(\langle Z| \otimes \overline{\langle \bar{Z}|})(\mu^\alpha)^{n_\mu} \otimes (\lambda_\beta)^{n_\lambda}). \tag{7.1.93}$$

In order for the sum of (7.1.92) and (7.1.93) to vanish, we must have $n_\mu = n_\lambda$ and $\alpha = \beta$. Hence, the form of the oscillator factor should actually be the combination $(\mu^\alpha \otimes \lambda_\alpha)^{n_{\mu\alpha}}$

We can apply the same logic to the sectors consisting of other conjugate pairs, namely $(\tilde{\mu}^{\dot{\alpha}}, \tilde{\lambda}_{\dot{\alpha}}), (\bar{c}_i, c^i)$ and $(\bar{d}_j, d^j)$ and find similar conditions. In this way, we find that the necessary form for the singlet projector for $\mathfrak{su}(2, 2|4)$ can be written as

$$_{\mathrm{su}}\langle \mathbf{1}_{12}| = \sum_{\mathbf{n}} f(\mathbf{n}) \langle \mathbf{n}| \tag{7.1.94}$$

$$\langle \mathbf{n}| \equiv \langle Z| \otimes \overline{\langle \bar{Z}|} \prod_{\alpha, \dot{\beta}, i, j} \frac{(\mu^\alpha \otimes \lambda_\alpha)^{n_{\lambda_\alpha}}}{n_{\mu_\alpha}!} \frac{(\tilde{\mu}^{\dot{\alpha}} \otimes \tilde{\lambda}_{\dot{\alpha}})^{n_{\tilde{\lambda}_{\dot{\alpha}}}}}{n_{\tilde{\mu}_{\dot{\alpha}}}!} \frac{(c^i \otimes \bar{c}_i)^{n_{c_i}}}{n_{c_i}!} \frac{(d^j \otimes \bar{d}_j)^{n_{d_j}}}{n_{d_j}!}, \tag{7.1.95}$$

$$f(\mathbf{n}) = f(n_{\lambda_1}, n_{\lambda_2}, n_{\tilde{\lambda}_{\dot{1}}}, n_{\tilde{\lambda}_{\dot{2}}}, \ldots), \tag{7.1.96}$$

where $f(\mathbf{n})$ at this stage is an arbitrary function and is to be determined by the requirement of the singlet condition. As for the sum over the powers n_{μ_α} etc, we shall allow them to be arbitrary non-negative integers.

To see what conditions should be satisfied by the function $f(\mathbf{n})$, let us focus first on a simple generator in the (μ, λ) sector of the form $J_\alpha{}^\beta = \lambda_\alpha \mu^\beta$, where $\alpha \neq \beta$. Since λ_α is the annihilation operator for the bra state $\langle Z|$, just as before, we easily get

$$\sum_{\mathbf{n}} f(\mathbf{n}) \langle \mathbf{n}|(\lambda_\alpha \mu^\beta \otimes 1) = \sum_{\mathbf{n}} f(\mathbf{n}) \langle \mathbf{n}|(\lambda_\alpha \otimes 1)(\mu^\beta \otimes 1)$$

$$= \sum_{\mathbf{n}} f(\mathbf{n}) \langle \mathbf{n}| \frac{n_{\lambda_\alpha}}{n_{\lambda_\alpha}!}(\mu^\alpha \otimes \lambda_\alpha)^{n_{\lambda_\alpha}-1}(1 \otimes \lambda_\alpha)(\mu^\beta \otimes 1) \cdots$$

$$= \sum_{\mathbf{n}} f(n_{\lambda_\alpha} + 1, \ldots) \langle \mathbf{n}|(\mu^\beta \otimes 1)(1 \otimes \lambda_\alpha) + \cdots. \tag{7.1.97}$$

In the third line we have shifted $\mathfrak{n}_{\lambda_\alpha}$ by 1 and interchanged the order of the factors $1 \otimes \lambda_\alpha$ and $\mu^\beta \otimes 1$. Now the action of $\mu^\beta \otimes 1$ on $\langle \mathbf{n}|$ is easily seen to produce the structure $1 \otimes \mu^\beta$ with an overall minus sign, together with a shift of $\mathfrak{n}_{\lambda_\beta}$ by minus one unit in $f(\mathbf{n})$ under

the sum. As for the structure of the operator part, combined with the factor $(1 \otimes \lambda_\alpha)$ already produced, we get

$$(1 \otimes \mu^\beta)(1 \otimes \lambda_\alpha) = (1 \otimes \mu^\beta \lambda_\alpha) = (1 \otimes \lambda_\alpha \mu^\beta) = 1 \otimes J_\alpha{}^\beta, \qquad (7.1.98)$$

where we have interchanged the order of μ^β and λ_α to get back $J_\alpha{}^\beta$ without producing any constant since we are considering the case with $\alpha \neq \beta$. Altogether we obtain the formula

$$\sum_{\mathbf{n}} f(n_{\lambda_\alpha}, n_{\lambda_\beta}, \ldots)\langle\mathbf{n}|(J_\alpha{}^\beta \otimes 1) = -\sum_{\mathbf{n}} f(n_{\lambda_\alpha} + 1, n_{\lambda_\beta} - 1, \ldots)\langle\mathbf{n}|(1 \otimes J_\alpha{}^\beta) \qquad (7.1.99)$$

Thus the singlet condition demands

$$f(n_{\lambda_\alpha}, n_{\lambda_\beta}, \ldots) = f(n_{\lambda_\alpha} + 1, n_{\lambda_\beta} - 1, \ldots). \qquad (7.1.100)$$

The general solution of this equation is

$$f(n_{\lambda_1}, n_{\lambda_2}, \ldots) = g(n_{\lambda_1} + n_{\lambda_2}, \ldots), \qquad (7.1.101)$$

where g is an arbitrary function except that n_{λ_α}'s must appear as the sum $n_{\lambda_1} + n_{\lambda_2}$.

Repeating similar analyses for all the off-diagonal[12] generators of $\mathfrak{su}(2,2|4)$, one obtains the following list of singlet conditions.

For the bosonic generators, we get

(b1)	$\lambda_\alpha \mu^\beta$	$f(n_{\lambda_\alpha}, n_{\lambda_\beta}, \ldots) = f(n_{\lambda_\alpha} + 1, n_{\lambda_\beta} - 1, \ldots)$
(b2)	$\lambda_\alpha \tilde{\lambda}_{\dot\beta}$	$f(n_{\lambda_\alpha}, n_{\tilde{\lambda}_{\dot\beta}}, \ldots) = -f(n_{\lambda_\alpha} + 1, n_{\tilde{\lambda}_{\dot\beta}} + 1, \ldots)$
(b3)	$\tilde{\mu}^{\dot\alpha} \tilde{\lambda}_{\dot\beta}$	$f(n_{\tilde{\lambda}_{\dot\alpha}}, n_{\tilde{\lambda}_{\dot\beta}}, \ldots) = f(n_{\tilde{\lambda}_{\dot\alpha}} - 1, n_{\tilde{\lambda}_{\dot\beta}} + 1, \ldots)$
(b4)	$\tilde{\mu}^{\dot\alpha} \mu^\alpha$	$f(n_{\tilde{\lambda}_{\dot\alpha}}, n_{\lambda_\alpha}, \ldots) = -f(n_{\tilde{\lambda}_{\dot\alpha}} - 1, n_{\lambda_\alpha} - 1, \ldots)$
(b5)	$\bar{c}_i \bar{d}_j$	$f(n_{c_i}, n_{d_j}, \ldots) = -f(n_{c_i} + 1, n_{d_j} + 1, \ldots)$
(b6)	$\bar{c}_i c^j$	$f(n_{c_i}, n_{c_j}, \ldots) = f(n_{c_i} + 1, n_{c_j} - 1, \ldots)$
(b7)	$d^j c^i$	$f(n_{d_j}, n_{c_i}, \ldots) = -f(n_{d_j} - 1, n_{c_i} - 1, \ldots)$
(b8)	$d^j \bar{d}_k$	$f(n_{d_j}, n_{d_k}, \ldots) = f(n_{d_j} - 1, n_{d_k} + 1, \ldots)$

For the fermionic generators, the conditions are

(f1)	$\lambda_\alpha c^i$	$f(n_{\lambda_\alpha}, n_{c_i}, \ldots) = -f(n_{\lambda_\alpha} + 1, n_{c_i} - 1, \ldots)$
(f2)	$\lambda_\alpha \bar{d}_j$	$f(n_{\lambda_\alpha}, n_{d_j}, \ldots) = f(n_{\lambda_\alpha} + 1, n_{d_j} + 1, \ldots)$
(f3)	$\tilde{\mu}^{\dot\alpha} c^i$	$f(n_{\tilde{\lambda}_{\dot\alpha}}, n_{c_i}, \ldots) = f(n_{\tilde{\lambda}_{\dot\alpha}} - 1, n_{c_i} - 1, \ldots)$
(f4)	$\tilde{\mu}^{\dot\alpha} \bar{d}_j$	$f(n_{\tilde{\lambda}_{\dot\alpha}}, n_{d_j}, \ldots) = -f(n_{\tilde{\lambda}_{\dot\alpha}} - 1, n_{d_j} + 1, \ldots)$
(f5)	$\bar{c}_i \mu^\alpha$	$f(n_{c_i}, n_{\lambda_\alpha}, \ldots) = -f(n_{c_i} + 1, n_{\lambda_\alpha} - 1, \ldots)$
(f6)	$\bar{c}_i \tilde{\lambda}_{\dot\alpha}$	$f(n_{c_i}, n_{\tilde{\lambda}_{\dot\alpha}}, \ldots) = f(n_{c_i} + 1, n_{\tilde{\lambda}_{\dot\alpha}} + 1, \ldots)$
(f7)	$d^j \mu^\alpha$	$f(n_{d_j}, n_{\lambda_\alpha}, \ldots) = f(n_{d_j} - 1, n_{\lambda_\alpha} + 1, \ldots)$
(f8)	$d^j \tilde{\lambda}_{\dot\alpha}$	$f(n_{d_j}, n_{\tilde{\lambda}_{\dot\alpha}}, \ldots) = -f(n_{d_j} - 1, n_{\tilde{\lambda}_{\dot\alpha}} + 1, \ldots)$

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[12] "Off-diagonal" here means the generators like $\lambda_\alpha \mu^\beta$ with $\alpha \neq \beta$, etc. so that their (anti)commutators vanish. For them there is no difference between $\mathfrak{u}(2,2|4)$ and $\mathfrak{su}(2,2|4)$.

With the hint from the analysis of the bosonic (μ, λ) sector, it is actually easy to write down the most general solution satisfying these equations. The answer is

$$f(n_{\lambda_1}, n_{\lambda_2}, \ldots) = (-1)^{n_{\tilde{\lambda}_{\dot{1}}} + n_{\tilde{\lambda}_{\dot{2}}} + n_{c_1} + n_{c_2}} h(C) \tag{7.1.102}$$

$$2C = (n_{\lambda_1} + n_{\lambda_2}) - (n_{\tilde{\lambda}_{\dot{1}}} + n_{\tilde{\lambda}_{\dot{2}}}) + (n_{c_1} + n_{c_2}) - (n_{d_1} + n_{d_2}) \tag{7.1.103}$$

where $h(x)$ is an arbitrary function of one argument. It is important to note that C is precisely the central charge of $\mathfrak{u}(2, 2|4)$. As such it can be set to a number in an irreducible representation. In particular, the fundamental SYM fields of our interest belong to the sector where $C = 0$ and $h(0)$ is just an overall constant, which we shall set to unity for simplicity.

Now we must examine the diagonal generators, such as $\lambda_\alpha \mu^\alpha$ and $\bar{c}_i c^i$, etc. Because an extra constant is produced upon interchanging the order of the oscillators, for example like $\lambda_\alpha \mu^\alpha = \mu^\alpha \lambda_\alpha - 2$, etc. in the process of the manipulation as in (7.1.98), in general the singlet condition is not satisfied. However, as we already stressed, for the diagonal generators which belong to $\mathfrak{su}(2, 2|4)$ and $\mathfrak{psu}(2, 2|4)$ such constants cancel. Therefore, the conditions we obtained for the function $f(\mathbf{n})$ do not change and the singlet projector for the physical SYM states is obtained as[13]

$$_{\text{psu}}\langle \mathbf{1}_{12}| = \langle Z| \otimes \overline{\langle \bar{Z}|} \sum_{\mathbf{n} \geq 0, C = 0} (-1)^{n_{\tilde{\lambda}_{\dot{1}}} + n_{\tilde{\lambda}_{\dot{2}}} + n_{c_1} + n_{c_2}}$$

$$\times \prod_{\alpha, \dot{\beta}, i, j} \frac{(\mu^\alpha \otimes \lambda_\alpha)^{n_{\lambda_\alpha}}}{n_{\lambda_\alpha}!} \frac{(\tilde{\mu}^{\dot{\alpha}} \otimes \tilde{\lambda}_{\dot{\alpha}})^{n_{\tilde{\lambda}_{\dot{\alpha}}}}}{n_{\tilde{\lambda}_{\dot{\alpha}}}!} \frac{(c^i \otimes \bar{c}_i)^{n_{c_i}}}{n_{c_i}!} \frac{(d^j \otimes \bar{d}_j)^{n_{d_j}}}{n_{d_j}!} \tag{7.1.104}$$

Because of the restriction $C = 0$ in the sum over the n_*'s, this expression *does not* quite take the form of an exponential. However, we can remove the restriction $C = 0$ in the sum when applying $\langle \mathbf{1}_{12}|$ to the physical SYM states, since the extra states with $C \neq 0$ produced are orthogonal to $C = 0$ states and do not contribute to the inner product with the physical states. Thus, with the $C = 0$ restriction removed, the singlet state above *can be* written as a simple exponential given by

$$_{\text{psu}}\langle \mathbf{1}_{12}| = \langle Z| \otimes \overline{\langle \bar{Z}|} \exp \left(\lambda_\alpha \otimes \mu^\alpha - \tilde{\lambda}_{\dot{\alpha}} \otimes \tilde{\mu}^{\dot{\alpha}} + \bar{c}_i \otimes c^i - \bar{d}_j \otimes d^j \right), \tag{7.1.105}$$

where in the exponent the sum is implied for the repeated indices. If one wishes to perform the Wick contraction in a manifestly symmetric fashion, one can use the form

$$\frac{1}{2} \left(_{\text{psu}}\langle \mathbf{1}_{12}| + _{\text{psu}}\langle \tilde{\mathbf{1}}_{12}| \right), \tag{7.1.106}$$

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[13]If one takes a different value of C one obtains a singlet projector for that sector. Here we focus on the physical SYM fields for which $C = 0$.

where

$$_{\text{psu}}\langle \tilde{\mathbf{1}}_{12}| = \overline{\langle \bar{Z}|} \otimes \langle Z| \exp\left(\mu^\alpha \otimes \lambda_\alpha - \tilde{\mu}^{\dot\alpha} \otimes \tilde{\lambda}_{\dot\alpha} + c^i \otimes \bar{c}_i - d^j \otimes \bar{d}_j \right). \qquad (7.1.107)$$

Hereafter, we shall suppress for simplicity the subscript psu and write $\langle \mathbf{1}_{12}|$ for $_{\text{psu}}\langle \mathbf{1}_{12}|$.

Crossing relations for the oscillators

Before ending this subsection, let us make an important remark on the property of the singlet projector (7.1.105). Although we have constructed this state by demanding that it be singlet under the generators of $\mathfrak{psu}(2,2|4)$ satisfying (7.1.89), it is easy to see from the process of construction above that actually the singlet projector (7.1.105) effects the following "crossing relations" for the *individual oscillators*:

$$\langle \mathbf{1}_{12}|(\bar\zeta^A \otimes 1) = \langle \mathbf{1}_{12}|(1 \otimes \bar\zeta^A), \qquad (7.1.108)$$

$$\langle \mathbf{1}_{12}|(\zeta_A \otimes 1) = -\langle \mathbf{1}_{12}|(1 \otimes \zeta_A). \qquad (7.1.109)$$

Clearly these relations themselves have no group theoretical meaning and appear to be stronger than the singlet condition. It is remarkable that yet they follow from the requirement of the singlet condition and will be quite useful in the computation of the correlation function, as we shall see in the next subsection.

7.1.3 Wick contraction of the basic fields using the singlet projector

Let us now show that the Wick contraction of the basic fields of the $\mathcal{N}=4$ super Yang-Mills theory can be computed quite easily by using the singlet projector constructed in the previous subsection. This can be identified as the method of Ward identity already introduced in [114]. However, as we use the D-scheme from the outset, our method is much more direct and simpler, without the need of rather complicated conversion operator U.

Consider first the scalar field $\phi_{ab}(x)$ belonging to the 6-dimensional anti-symmetric representation of SU(4), which corresponds to the state $\bar\xi_{[a}\bar\xi_{b]}e^{iP\cdot x}|0\rangle$. Then the Wick contraction of two such fields $\overbrace{\phi_{ab}(x)\phi_{cd}}(y)$ can be computed as $\langle \mathbf{1}_{12}|(\bar\xi_{[a}\bar\xi_{b]}e^{iP\cdot x}|0\rangle \otimes \bar\xi_{[c}\bar\xi_{d]}e^{iP\cdot y}|0\rangle)$. Since the singlet structure for the SU(4) part gets extracted as the unique factor ϵ_{abcd}, we obtain

$$\overbrace{\phi_{ab}(x)\phi_{cd}}(y) \propto \epsilon_{abcd}I(x,y), \qquad (7.1.110)$$

where

$$I(x,y) \equiv \langle \mathbf{1}_{12}|(e^{iP\cdot x}|0\rangle \otimes e^{iP\cdot y}|0\rangle). \qquad (7.1.111)$$

The function $I(x, y)$ will be seen below to be the basic building block for the contractions of all the super Yang-Mills fields and can be easily fixed by the singlet conditions[14] with $J^A{}_B$ taken to be translation and the dilatation generators in the following way. First, applying the singlet condition (7.1.89) taking $J^A{}_B$ to be the translation generator, we have

$$0 = \langle \mathbf{1}_{12} | (iP_\mu e^{iP \cdot x} |0\rangle \otimes e^{iP \cdot y} |0\rangle) + \langle \mathbf{1}_{12} | (e^{iP \cdot x} |0\rangle \otimes iP_\mu e^{iP \cdot y} |0\rangle)$$
$$= \left(\frac{\partial}{\partial x^\mu} + \frac{\partial}{\partial y^\mu} \right) I(x, y) \tag{7.1.112}$$

This gives $I(x, y) = I(x - y)$. Next, we use the dilatation operator given by $D = (i/2)(\lambda_\alpha \mu^\alpha + \tilde{\lambda}_{\dot{\alpha}} \tilde{\mu}^{\dot{\alpha}} + 2)$. Since $P \cdot x$ can be written as $\lambda_\alpha \tilde{\lambda}_{\dot{\alpha}} x^{\dot{\alpha}\alpha}$, the action of $\lambda_\alpha \mu^\alpha$ in D on $e^{iP \cdot x} |0\rangle$ gives

$$\lambda_\alpha \mu^\alpha e^{iP \cdot x} |0\rangle = i\lambda_\alpha \tilde{\lambda}_{\dot{\alpha}} x^{\dot{\alpha}\alpha} e^{i\lambda_\alpha \tilde{\lambda}_{\dot{\alpha}} x^{\dot{\alpha}\alpha}} |0\rangle = iP \cdot x e^{iP \cdot x} |0\rangle = x^\mu \frac{\partial}{\partial x^\mu} e^{iP \cdot x} |0\rangle. \tag{7.1.113}$$

Evidently, the action of $\tilde{\lambda}_{\dot{\alpha}} \tilde{\mu}^{\dot{\alpha}}$ on $e^{iP \cdot x} |0\rangle$ gives exactly the same contribution. In a similar manner, the contribution from the D acting on $e^{iP \cdot y} |0\rangle$ in the singlet condition relation produces the same result with x^μ replaced by y^μ. Altogether, the singlet condition with $J^A{}_B = D$ yields

$$\left(x^\mu \frac{\partial}{\partial x^\mu} + y^\mu \frac{\partial}{\partial y^\mu} + 2 \right) I(x - y) = 0. \tag{7.1.114}$$

The solution is obviously

$$I(x - y) \propto \frac{1}{(x - y)^2}. \tag{7.1.115}$$

Let us now describe how the contraction of the fundamental fermions, i.e. $\overparen{\psi_{\alpha a}(x) \bar{\psi}^b_{\dot{\alpha}}(y)}$ can be done using the singlet projector. The singlet part for the R-symmetry obviously gives δ^b_a and hence we have

$$\overparen{\psi_{\alpha a}(x) \bar{\psi}^b_{\dot{\alpha}}(y)} \propto \delta_a{}^b \langle \mathbf{1}_{12} | (e^{iP \cdot x} \lambda_\alpha |0\rangle_D \otimes e^{iP \cdot y} \tilde{\lambda}_{\dot{\alpha}} |0\rangle_D). \tag{7.1.116}$$

In this case, we may use the crossing relation (7.1.109) for the oscillators to rewrite the RHS as

$$\langle \mathbf{1}_{12} | (e^{iP \cdot x} \lambda_\alpha |0\rangle_D \otimes e^{iP \cdot y} \tilde{\lambda}_{\dot{\alpha}} |0\rangle_D) = \langle \mathbf{1}_{12} | (e^{iP \cdot x} |0\rangle_D \otimes e^{iP \cdot y} \lambda_\alpha \tilde{\lambda}_{\dot{\alpha}} |0\rangle_D) = -i \frac{\partial}{\partial y^{\dot{\alpha}\alpha}} I(x - y) \tag{7.1.117}$$

---

[14]As we shall see below, the singlet conditions produce Ward identities.

Therefore, up to an overall normalization, we obtain

$$\overbrace{\psi_{\alpha a}(x)\bar{\psi}^b_{\dot\alpha}(y)} \propto 2i\delta^b_a \frac{(x-y)_{\alpha\dot\alpha}}{|x-y|^4} \tag{7.1.118}$$

Likewise, the Wick contraction for the self-dual field strength can be computed, again using the crossing relations for the oscillators, as

$$
\begin{aligned}
\overbrace{F_{\alpha\beta}(x)\bar{F}_{\dot\alpha\dot\beta}(y)} &\propto \langle \mathbf{1}_{12}|(e^{iP\cdot x}\lambda_\alpha\lambda_\beta|0\rangle_D \otimes e^{iP\cdot y}\tilde\lambda_{\dot\alpha}\tilde\lambda_{\dot\beta}|0\rangle_D) \\
&= -\frac{1}{2}\Big(\langle \mathbf{1}_{12}|(e^{iPx}\lambda_\alpha\tilde\lambda_{\dot\alpha}|0\rangle_D \otimes e^{iPy}\lambda_\beta\tilde\lambda_{\dot\beta}|0\rangle_D) \\
&\quad + \langle \mathbf{1}_{12}|(e^{iPx}\lambda_\alpha\tilde\lambda_{\dot\beta}|0\rangle_D \otimes e^{iPy}\lambda_\beta\tilde\lambda_{\dot\alpha}|0\rangle_D)\Big) \\
&= \frac{1}{2}\left(\frac{\partial}{\partial x^{\dot\alpha\alpha}}\frac{\partial}{\partial y^{\dot\beta\beta}} + \frac{\partial}{\partial x^{\dot\beta\alpha}}\frac{\partial}{\partial y^{\dot\alpha\beta}} \right) I(x-y) \\
&= -6\frac{(x-y)_{(\alpha\dot\alpha}(x-y)_{\beta)\dot\beta}}{|x-y|^6}\,.
\end{aligned}
\tag{7.1.119}
$$

Normalizations of these two point functions depend of course on the choice of the normalization of the individual fields but once we fix one of them, then the rest can be determined by supersymmetry.

With the demonstrations above, we wish to emphasize that our method of using the conformally covariant D-scheme is quite simple and useful in that the properties of the singlet projector can be directly and effectively utilized.

7.2 Monodromy relations for correlation functions in $\mathfrak{psu}(2,2|4)$ spin chain system

Having constructed the singlet projector in the conformally covariant basis, we shall now generalize the so-called monodromy relations for the correlation functions developed in our previous work [110] for the SU(2) sector to the full $\mathfrak{psu}(2,2|4)$ sector. Here one must first note the following new features. In the case of the SU(2) sector, the structure of the auxiliary Hilbert space is unequivocally chosen to be identical to that of the quantum Hilbert space, both two dimensional, describing the up and down "spin" states. On the other hand, for $\mathfrak{psu}(2,2|4)$ there are two appropriate choices for the auxiliary space. To see this, we should recall the properties of the general R-matrix, to be denoted by $\mathbb{R}_{ij}(u)$, from which the monodromy matrix is constructed. It is a linear map acting on the tensor product of two vector spaces $V_i \otimes V_j$, i.e. $\mathbb{R}_{ij} \in \mathrm{End}(V_i \otimes V_j)$, and satisfying the following Yang-Baxter equation:

$$\mathbb{R}_{12}(u_1-u_2)\mathbb{R}_{13}(u_1)\mathbb{R}_{23}(u_2) = \mathbb{R}_{23}(u_2)\mathbb{R}_{13}(u_1)\mathbb{R}_{12}(u_1-u_2)\,, \tag{7.2.1}$$

where complex parameters u_i are the spectral parameters. From such $\mathbb{R}_{ij}(u)$ matrices, one constructs the monodromy matrix $\Omega(u) = \mathbb{R}_{a1}(u) \cdots \mathbb{R}_{a\ell}(u)$, where a here is the label for the auxiliary space V_a and the numbers 1 through ℓ denote the location of the site at which $\mathfrak{psu}(2,2|4)$ spin state resides to make up a spin chain[15] . Then, out of the monodromy matrix, one defines the transfer matrix $T(u)$ by taking the trace over the auxiliary space, namely $T(u) := \text{Tr}_a \Omega(u)$. The prime importance of the Yang-Baxter equation (7.2.1) is that it ensures the commutativity of the transfer matrices at different spectral parameters, i.e. $[T(u), T(v)] = 0$, which in turn implies that the quantities obtained as the coefficients of the power expansion in the spectral parameter all commute. In particular, as one of such quantities can be identified with the Hamiltonian of the spin chain, all the coefficients can be interpreted as conserved charges. This is at the heart of the integrability.

Now in the case of the $\mathfrak{u}(2,2|4)$ spin chain, while the quantum Hilbert space is taken to be the Fock space $\mathcal{V}$ constructed by the oscillators introduced in the previous section, there are two natural choices for the auxiliary space V_a, which should form a representation of $\mathfrak{u}(2,2|4)$ or its complexified version $\mathfrak{gl}(4|4)$[16]. One is the fundamental representation of $\mathfrak{gl}(4|4)$, i.e. $V_a = \mathbb{C}^{4|4}$ and the other is the choice $V_a = \mathcal{V}$, i.e. the auxiliary space being the same as the infinite dimensional quantum space in structure. We shall call the corresponding R-matrix as "fundamental" for the former case and "harmonic" for the latter choice.

For the former case, the monodromy matrix is finite dimensional and its components are operators acting on the quantum space. These components satisfy the exchange relations (or Yang-Baxter algebra) coming from the Yang-Baxter equations, and are quite powerful in diagonalizing the transfer matrix in the context of algebraic Bethe ansatz. It should be noted that a similar finite dimensional monodromy matrix can be defined classically in the strong coupling regime using the flat connections of the string sigma model and can be used to determine the semi-classical spectrum [84]. Further, beyond the spectral problem, the monodromy relation of this type has its counterpart in the computation of the three-point functions in the strong coupling regime [108], [109–111] as the triviality of the total monodromy of the form $\Omega_1 \Omega_2 \Omega_3 = 1$, where Ω_i is the local monodromy produced around the i-th vertex operator in the so-called auxiliary linear problem. As explained in [108,111], this seemingly weak relation is disguisingly powerful, as it captures the important global information governing the three-point functions.

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[15]For the discussion of concepts requiring the Yang-Baxter equation, we must consider $\mathfrak{u}(2,2|4)$, but not $\mathfrak{psu}(2,2|4)$, as it is the R-matrix associated with the former which satisfies the Yang-Baxter equation. We shall give more detailed discussion on this point later.

[16]The most of the discussion to follow is insensitive to whether we consider $\mathfrak{u}(2,2|4)$ or its complexified version $\mathfrak{gl}(4|4)$. Thus, when the description is easier with the complexified version, we shall use $\mathfrak{gl}(4|4)$ in place of $\mathfrak{u}(2,2|4)$.

Such monodromy relations for the fundamental R-matrix for $\mathfrak{gl}(4|4)$ can be derived through a procedure similar to the one for the SU(2) case worked out in detail in our previous work [110] and has been discussed in [137]. Besides the purpose of completeness, we shall re-derive these relations below since we shall use the definition of the Lax operator, slightly different from the one used in [137], which is more natural in connection with the strong coupling counterpart.

Next let us briefly describe the characteristics of the monodromy relations we shall derive for the harmonic R-matrix, which are completely new. In this case, the monodromy matrix is no longer finite dimensional since the auxiliary space V_a is the same as the infinite dimensional quantum spin-chain Hilbert space $\mathcal{V}$. One of the virtues of considering such a harmonic R-matrix is that, just as in the case of the SU(2) Heisenberg spin chain, the construction of the conserved charges including the Hamiltonian is much easier, since due to the identical structure of V_a and $\mathcal{V}$ the R-matrix at specific value of the spectral parameter serves as the permutation operator P_{an}. Such an operator is known to be extremely useful in extracting the Hamiltonian (*i.e.* the dilatation operator). Because of this and other features, the harmonic R-matrix and the related quantities have already found an interesting applications in the computation of the scattering amplitudes from the point of view of integrability [182–189] and are expected to be useful in the realm of the correlation functions as well.

In any case, since the monodromy matrix, constructed out of either "fundamental" or "harmonic" R-matrices, is a generating function of an infinite number of conserved charges, the monodromy relations can be regarded as a collection of "Ward identities" associated with such higher charges, which should characterize the important properties of the correlation functions.

Remarks on the relevance of of $\mathfrak{u}(2,2|4)$ for the monodromy relation
 and $\mathfrak{psu}(2,2|4)$ for the singlet projector

Before we begin the discussion of the monodromy relations, let us give some important clarifying remarks on the relevance of the different super algebras for the two topics we discuss in this work and the role of their oscillator representation.

- The monodromy relation, to be discussed below, is deeply rooted in the integrability of the theory and hence it is crucial that the relevant R-matrix and the Lax matrix must satisfy the Yang-Baxter equations and the RLL=LLR equations. A method has long been known [178–181] that one can construct such an R-matrix and a Lax matrix from a suitable Lie super algebra. In the present case, one can do so for $\mathfrak{u}(2,2|4)$ algebra but not for $\mathfrak{su}(2,2|4)$ or $\mathfrak{psu}(2,2|4)$. This is a general mathematical statement and has nothing to do with a particular oscillator representation nor

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with the super Yang-Mills theory. However, when one makes use of the singleton oscillator representation, then one can easily construct the states which form the fundamental Field strength multiplet of SYM theory and the R-matrix and the monodromy matrix can be constructed in terms of the generators bilinear in the oscillators. Although the basic SYM fields carry a special value of the central charge $C = 0$ and the global symmetry of the $\mathcal{N} = 4$ SYM theory is $\mathfrak{psu}(2, 2|4)$, still when we discuss the monodromy relations for the correlation functions for the composite operators made up of these SYM fields, the generators and the related quantities to be used must be those of $\mathfrak{u}(2, 2|4)$.

- On the other hand, when we use the singlet projector to perform the Wick contractions efficiently in the computation of the correlation functions, the projector is a singlet for $\mathfrak{su}(2, 2|4)$ and $\mathfrak{psu}(2, 2|4)$. This notion has nothing to do with the integrability. In fact a singlet projector for $\mathfrak{u}(2, 2|4)$ does not exist at least in the oscillator representation utilized and this point gives a subtle effect in the crossing relation, to be discussed in the next subsection.

Thus, in the monodromy relations for the correlation functions, two different superalgebras are playing their respective role. The monodromy matrices to be inserted are associated with $\mathfrak{u}(2, 2|4)$, while the singlet projector which works as an elegant device in forming the correlation function for the physical SYM fields is valid for $\mathfrak{psu}(2, 2|4)$.

7.2.1 Basic monodromy relation in the case of fundamental R-matrix

Let us begin with the case of the monodromy relations with the use of the fundamental R-matrix. We shall first give the definitions and conventions for the fundamental R-matrix and the associated Lax matrix, which are slightly different from the ones used in [137], and then discuss the two important relations, namely the crossing relations and the inversion relations, which will lead immediately to the monodromy relations of interest.

Fundamental R-matrix and Lax operator

Consider the fundamental R-matrix, for which the quantum space is $\mathcal{V}$ and the auxiliary space is taken to be $\mathbb{C}^{4|4}$. This kind of R-matrix is often called the Lax operator and will be denoted by $L_{an}(u)$, where a and n refer, respectively, to the auxiliary space and the position on the spin chain. It satisfies the important relation called RLL=LLR relation

$$R_{12}(u_1 - u_2)L_{a_1n}(u_1)L_{a_2n}(u_2) = L_{a_2n}(u_2)L_{a_1n}(u_1)R_{12}(u_1 - u_2), \qquad (7.2.2)$$

which follows from the basic Yang-Baxter equation (7.2.1) by setting $V_1 = V_2 = \mathbb{C}^{4|4}$ and $V_3 = \mathcal{V}_n$, where n is the position of the spin. The R-matrix $R_{12}(u)$ appearing in this equation acts on the tensor product of two copies of the auxiliary space $V_1 \otimes V_2$ and, besides the RLL=LLR equation, it also satisfies the original Yang-Baxter equation denoted as RRR=RRR equation:

$$R_{12}(u_1 - u_2)R_{13}(u_1)R_{23}(u_2) = R_{23}(u_2)R_{13}(u_1)R_{12}(u_1 - u_2)\,. \tag{7.2.3}$$

The solution of the above RRR relation turns out to be of the form

$$R_{ij}(u) = u + \eta(-1)^{|B|}E_{i\ B}^A \otimes E_{j\ A}^B\,, \qquad (E_{\ B}^A)_D^C \equiv \delta_C^A \delta_D^B\,, \tag{7.2.4}$$

where η is an arbitrary complex parameter[17] and $E_{i\ B}^A$ is the fundamental representation of $\mathfrak{gl}(4|4)$ acting non-trivially on $V_i \cong \mathbb{C}^{4|4}$. To check that the R-matrix above actually satisfies the Yang-Baxter it is useful to note that the operator $\Pi_{ij} := (-1)^{|B|}E_{i\ B}^A \otimes E_{j\ A}^B$ serves as the graded permutation operator[18]. For example, $\Pi_{12}(a \otimes b \otimes c) = (-1)^{|a||b|}(b \otimes a \otimes c)$, $\Pi_{13}(a \otimes b \otimes c) = (-1)^{|a|(|b|+|c|)+|b||c|}(c \otimes b \otimes a)$ and so on. Then, the Lax operator satisfying (7.2.3) is given by

$$L_{a_i n}(u) = u + \eta(-1)^{|B|}E_{i\ B}^A \otimes J_{n\ B}^A\,, \tag{7.2.7}$$

where $J_{n\ B}^A$'s are the generators of $\mathfrak{gl}(4|4)$ defined on the n-th site of the spin chain. It is tedious but straightforward to show that the Lax operator indeed satisfies the RLL=LLR relation, by explicitly computing the both sides. In performing this calculation, one should remember that there are no grading relations between the auxiliary space and quantum spaces, which are two independent spaces. Explicitly, this means

$$(E_{\ B}^A \otimes J_{\ B}^A)(E_{\ D}^C \otimes J_{\ D}^C) = (E_{\ B}^A E_{\ D}^C) \otimes (J_{\ B}^A J_{\ D}^C) = E_{\ D}^A \otimes (J_{\ B}^A J_{\ D}^B)\,, \tag{7.2.8}$$

where we have used $E_{\ B}^A E_{\ D}^C = \delta_C^B E_{\ D}^A$. Differently put, the definition of the product of the Lax operators is not as supermatrices but as usual matrices. Although the choice for the mutual grading between these two spaces is a matter of convention[19], our choice is a

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[17] Although the Yang-Baxter equation holds for arbitrary η, we will later set $\eta = \pm i$ for our interest.

[18] To prove this, we should pay attention to the non-trivial gradings between two auxiliary spaces

$$(E_{\ B}^A \otimes E_{\ D}^C)(a \otimes b) = (-1)^{a(|C|+|D|)}(E_{\ B}^A a) \otimes (E_{\ D}^C b) \quad a \otimes b \in \mathbb{C}^{4|4} \otimes \mathbb{C}^{4|4}\,, \tag{7.2.5}$$

$$(E_{\ B}^A \otimes E_{\ D}^C)(E_{\ F}^E \otimes E_{\ H}^G) = (-1)^{(|C|+|D|)(|E|+|F|)}(E_{\ B}^A E_{\ F}^E) \otimes (E_{\ D}^C E_{\ H}^G)\,. \tag{7.2.6}$$

.

[19] In [137], the authors adopt the convention where non-trivial gradings between the auxiliary space and the quantum space exist. Namely, $(E_{\ B}^A \otimes J_{\ B}^A)(E_{\ D}^C \otimes J_{\ D}^C) = (-1)^{(|A|+|B|)(|C|+|D|)}(E_{\ B}^A E_{\ D}^C) \otimes (J_{\ B}^A J_{\ D}^C)$. Because of this, the definition of the Lax operator they use, i.e. $L(u) := u - i/2 - i(-1)^{|A|}E_{\ B}^A \otimes J_{\ A}^B$, is slightly different from ours.

natural one from the point of view of connecting to the strong coupling regime. This is simply because the monodromy matrix at strong coupling is defined by the path ordered exponential of the integral of the flat connection and the multiplication rule for such matrices is the ordinary one. With this convention, the explicit form of the Lax operator is given in terms of the superconformal generators by

$$(L(u))^A{}_B = u\delta^A{}_B + \eta(-1)^{|B|}J^A{}_B = \begin{pmatrix} u + \eta Y_\alpha{}^\beta & i\eta P_{\alpha\dot\beta} & -\eta Q^b_\alpha \\ i\eta K^{\dot\alpha\beta} & u + \eta \bar{Y}^{\dot\alpha}{}_{\dot\beta} & -i\eta \bar{S}^{\dot\alpha b} \\ \hline \eta S^\beta_a & i\eta \bar{Q}_{\dot\beta a} & u - \eta W^b_a \end{pmatrix}_{AB} . \qquad (7.2.9)$$

As usual the monodromy matrix is defined as the product of the Lax operators on each site going around the spin chain of length ℓ:

$$\Omega_a(u) := L_{a1}(u)\cdots L_{a\ell}(u) . \qquad (7.2.10)$$

The monodromy matrix so defined satisfies the following relation, since each Lax operator satisfies the RLL=LLR relation:

$$R_{12}(u_1 - u_2)\Omega_{a_1}(u_1)\Omega_{a_2}(u_2) = \Omega_{a_2}(u_2)\Omega_{a_1}(u_1)R_{12}(u_1 - u_2) . \qquad (7.2.11)$$

If we write out the above equation for each component, we obtain the so-called Yang-Baxter exchange algebra. In the rest of this subsection, when there is no confusion we drop the indices for the auxiliary space for simplicity.

Monodromy relation

Let us now derive the generic monodromy relation. This can be achieved by proving the following two important relations for the Lax operators, called the crossing relation and the inversion relation. They are respectively of the form

$$(C): \quad \langle \mathbf{1}_{12}|L^{(1)}_n(u) = -\langle \mathbf{1}_{12}|L^{(2)}_{\ell-n+1}(\eta - u) , \qquad (7.2.12)$$

$$(I): \quad L^{(i)}_n(u)L^{(i)}_n(v) = u(\eta - u) , \quad (u + v = \eta) , \qquad (7.2.13)$$

where the superscript (i) on $L^{(i)}_n$ denotes the i-th spin chain. The crossing relation (C) connects the Lax operator defined on the n-th site of a spin chain called 1 to that defined on the $\ell - n + 1$-th site of another spin chain called 2. To get the feeling for the crossing relation, it suffices to recall that the singlet projector $\langle \mathbf{1}_{12}|$ effects the Wick contraction between a field at the n-th site of one spin chain and a field at the $\ell - n + 1$-th site of another chain. Actually, it is easy to prove it more precisely from the defining property of the singlet $\langle \mathbf{1}_{12}|$. As it was already emphasized in section 2.2, the operator $\langle \mathbf{1}_{12}|$ is a singlet projector for $\mathfrak{su}(2,2|4)$ or $\mathfrak{psu}(2,2|4)$ but *not* for $\mathfrak{u}(2,2|4)$ which is of our concern

here. So the operator $\langle \mathbf{1}_{12}|$ transforms the generator $J^A{}_B$ of $\mathfrak{u}(2,2|4)$ acting on the first spin chain into the operator $-J^A{}_B - (-1)^{|A|}\delta^A{}_B$ acting on the second spin chain, where the constant piece $-(-1)^{|A|}\delta^A{}_B$ comes from the (anti-)commutator term. Applying this to the Lax operator $(L(u))^A{}_B = u\delta^A{}_B + \eta(-1)^{|B|}J^A{}_B$, one sees that the constant term shifts the spectral parameter by η and we get the crossing relation as shown in (7.2.12).

The proof of the inversion relation (I), which says that the Lax operator can be inverted for a specific value of the spectral parameter, is slightly more involved. The product of two Lax operators gives

$$(L(u)L(v))^A{}_B = uv\delta^A{}_B + \eta(u+v)(-1)^{|B|}J^A{}_B + \eta^2(-1)^{|B|+|C|}J^A{}_C J^C{}_B. \tag{7.2.14}$$

First look at the last term quadratic in the generators. In general this cannot be simplified further. However, as we are specializing in the oscillator representation, we can write $J^A{}_B = \bar\zeta^A \zeta_B$ using the oscillators satisfying $[\zeta_A, \bar\zeta^B] = \delta_A{}^B$, as described in (7.1.44). Therefore we can reduce the product of the generator in the following way:

$$(-1)^{|C|}J^A{}_C J^C{}_B = \bar\zeta^A(\bar\zeta^C \zeta_C)\zeta_B = (2C-1)J^A{}_B. \tag{7.2.15}$$

For the first equality, we have used the fact that the constant term $(-1)^{|C|}\delta^C{}_C$, which appears from the commutation relation, vanishes. Further, since $[\bar\zeta^C \zeta_C, \bar\zeta^A] = 1$ and the central charge is given by $2C = \mathrm{tr}J = \bar\zeta^C \zeta_C$, we obtain the result above. Hence the product of the Lax operators (7.2.14) is simplified to

$$(L(u)L(v))^A{}_B = uv\delta^A{}_B + \eta(u+v+\eta(2C-1))(-1)^{|B|}J^A{}_B. \tag{7.2.16}$$

Now as we repeatedly emphasized, for the Yang-Mills fields of our interest we can set $C = 0$ and hence the coefficient in front of $(-1)^{|B|}J^A{}_B$ becomes $\eta(u+v-\eta)$. Therefore when $u+v = \eta$, the RHS of (7.2.16) becomes $uv\delta^A{}_B = u(\eta-u)\delta^A{}_B$, which is precisely the RHS of the inversion equation (7.2.13).

We are now ready to present the generic form of the monodromy relation, which takes the form

$$\langle \mathbf{1}_{12}|\Omega^{(1)}(u)\Omega^{(2)}(u) = \langle \mathbf{1}_{12}|F_\ell(u), \tag{7.2.17}$$

where $F_\ell(u)$ is some function of u, to be given shortly. Both sides of this relation should be understood as acting on a tensor product of states on two Hilbert spaces of the form $|\mathcal{O}_1\rangle \otimes |\mathcal{O}_2\rangle$. To show (7.2.17), we first prove the following relation with the use of the crossing relation (7.2.12):

$$\langle \mathbf{1}_{12}|\Omega^{(1)}(u) = (-1)^\ell \langle \mathbf{1}_{12}|\overleftarrow{\Omega}^{(2)}(\eta-u), \tag{7.2.18}$$

$$\overleftarrow{\Omega}^{(2)}(u) := L_\ell^{(2)}(u)\cdots L_1^{(2)}(u). \tag{7.2.19}$$

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Focus first on the LHS of (7.2.18) and consider moving the Lax operator $L_n^{(1)}(u)$ at the n-th site in $\Omega^{(1)}$ to the left towards $\langle \mathbf{1}_{12}|$. Since the components of the Lax operators on different sites commute in the graded sense, namely

$$(L_n(u))_{AB}(L_m(v))_{CD} = (-1)^{(|A|+|B|)(|C|+|D|)}(L_m(v))_{CD}(L_n(u))_{AB}, \qquad (7.2.20)$$

we can move $L_n^{(1)}(u)$ all the way to the left and hit $\langle \mathbf{1}_{12}|$ like

$$\langle \mathbf{1}_{12}| \ldots (L_n^{(1)}(u))_{AB} \cdots = (-1)^{(|A|+|B|)(\cdots)} \langle \mathbf{1}_{12}|(L_n^{(1)}(u))_{AB} \ldots \cdots . \qquad (7.2.21)$$

We can now use the crossing relation (C) to replace the Lax operator $(L_n^{(1)}(u))_{AB}$ with $(L_{\ell-n+1}^{(2)}(\eta - u))_{AB}$ and move it back again to the original position. In this process, the sign factors which appear through the exchange of operators exactly cancel with those produced in the previous process and we get

$$\langle \mathbf{1}_{12}| \cdots L_n^{(1)}(u) \cdots = -\langle \mathbf{1}_{12}| \cdots L_{\ell-n+1}^{(2)}(\eta - u) \cdots . \qquad (7.2.22)$$

Repeating this to all the Lax operators making up the monodromy matrices, we immediately get (7.2.18). Now apply $\Omega^{(2)}(u)$ to the both sides of (7.2.18) and use the relation $\overleftarrow{\Omega}^{(2)}(\eta - u)\Omega^{(2)}(u) \propto 1$, with the overall factor which is readily computable using the inversion relation (I). In this way we obtain (7.2.17) with the function $F_\ell(u)$ given by $F_\ell(u) = (u(u-\eta))^\ell$. Now if we apply (7.2.17) explicitly to the state $|\psi_1\rangle \otimes |\psi_2\rangle$, we obtain the more explicit monodromy relation for the two-point function

$$(-1)^{(|C|+|B|)|\psi_1|} \left\langle (\Omega^{(1)}(u))_{AB}|\psi_1\rangle, (\Omega^{(2)}(u))_{BC}|\psi_2\rangle \right\rangle = F_\ell(u)\delta_{AC} \left\langle |\psi_1\rangle, |\psi_2\rangle \right\rangle, \qquad (7.2.23)$$

$$F_\ell(u) = (u(u-\eta))^\ell. \qquad (7.2.24)$$

Here $\langle , \rangle$ denotes the pairing with the singlet, which gives the Wick contraction between two operators. The sign in front arises when we pass the monodromy through the first state $|\psi_1\rangle$.

At this point, it is of importance to remark that we obtain the usual Ward identities at the leading order in the expansion of the above equation around $u = \infty$. This is a direct consequence of the $\mathfrak{su}(2,2|4)$ invariance of the singlet projector.

Once the monodromy relation is obtained for the two-point functions, the one for the three-point functions can be obtained easily, just as was shown explicitly for the SU(2) sector in [138]. The only differences from that case are the form of the prefactor function $F_{123}(u)$ and some sign factors due to the superalgebra nature of $\mathfrak{psu}(2,2|4)$. Thus the monodromy relation for the three-point function takes the form

$$\left\langle (\Omega(u))_{AB}|\psi_1\rangle, (-1)^{|\psi_1|(|B|+|C|)}(\Omega(u))_{BC}|\psi_2\rangle, (-1)^{(|\psi_1|+|\psi_2|)(|C|+|D|)}(\Omega(u))_{CD}|\psi_3\rangle \right\rangle$$
$$= F_{123}(u)\delta_{AD} \left\langle |\psi_1\rangle, |\psi_2\rangle, |\psi_3\rangle \right\rangle, \qquad F_{123}(u) = (u(u-\eta))^{\ell_1+\ell_2+\ell_3}.$$
$$(7.2.25)$$

7.2.2 Basic monodromy relation in the case of harmonic R-matrix

We shall now discuss another important version of the R-matrix, called the harmonic R-matrix, to be denoted by the bold letter $\mathbf{R}$. We shall derive the inversion and the crossing relations for it and finally prove the relations for the correlation functions obtained with the insertion of monodromy matrices constructed out of the harmonic R-matrices.

The term "harmonic" stems from the form of the Hamiltonian (or dilatation) density first derived in [21], which can be expressed as[20] $H_{12} = h(J_{12})$, where the function $h(j) = \sum_{k=1}^{j} 1/k$ is the so-called harmonic number and J_{12}^2 is the quadratic Casimir operator. This Hamiltonian is intimately related to the one in the SL(2) subsector and in that context was derived also as the logarithmic derivative of the R-matrix, just as in the case of the SU(2) Heisenberg spin chain.

The harmonic R-matrix is recently applied in the context of the scattering amplitudes for the $\mathcal{N} = 4$ SYM theory, as a tool to construct the building blocks for the deformed Grassmannian formulas characterized as Yangian invariants [182–189]. The spectral parameter can be naturally introduced in the deformed formulas and turned out to serve as a regulator for the IR divergences.

Since this type of R-matrix is less well-known, we shall first give a brief review of the basic facts on the harmonic R-matrix following [183] and then using these properties derive the crossing and inversion relations, which are essential, as in the case of the fundamental R-matrix, in obtaining the monodromy relations we seek.

Review of the harmonic R-matrix

The harmonic R-matrix $\mathbf{R}_{12}$ acting on the tensor product of two copies of the Fock space $\mathcal{V}_1 \otimes \mathcal{V}_2$ should satisfy the following RLL=LLR relation

$$\mathbf{R}_{12}(u_1 - u_2)L_1(u_1)L_2(u_2) = L_2(u_2)L_1(u_1)\mathbf{R}_{12}(u_1 - u_2). \qquad (7.2.26)$$

This is obtained from the general formula (7.2.1) by setting $V_1 = \mathcal{V}_1$, $V_2 = \mathcal{V}_2$, $V_3 = \mathbb{C}^{4|4}$ and replacing $\mathbb{R}_{i3}(u_i)$ with the Lax operator $L_i(u_i) = u_i + \eta(-1)^{|B|}E^A_{\ B} \otimes J^A_{i\ B}$. Renaming $u = u_1 - u_2$, $u_2 = \tilde{u}$ and expanding the above equation in powers of $\tilde{u}$, one can show that the harmonic R-matrix satisfies the following two types of equations:

(i) $[\mathbf{R}_{12}(u), J^A_{1\ B} + J^A_{2\ B}] = 0\,,$ $\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ (7.2.27)

(ii) $(-1)^{|C|}\eta(\mathbf{R}_{12}(u)J^A_{1\ C}J^C_{2\ B} - J^A_{2\ C}J^C_{1\ B}\mathbf{R}_{12}(u)) - u(J^A_{2\ B}\mathbf{R}_{12}(u) - \mathbf{R}_{12}(u)J^A_{2\ B}) = 0\,.$
$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ (7.2.28)

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[20]The subscript 12 signifies that the Hamiltonian is restricted to two fields 1 and 2.

The first expresses the invariance of the harmonic R-matrix under $\mathfrak{gl}(4|4)$, while the second implies the invariance under the level 1 generators of the Yangian algebra. They together ensure the full Yangian invariance of the harmonic R-matrix. As it can be explicitly verified after constructing $\mathbf{R}_{12}$ explicitly, the product $\mathbf{R}_{12}(u)\mathbf{R}_{12}(-u)$ is proportional to the unit operator $\mathbf{1}_{12}$ but the overall normalization can be arbitrary, since the equations (i) and (ii) above are both linear in $\mathbf{R}$. Therefore one can impose the following unitarity condition, or inversion relation, to fix the overall scale:

$$\mathbf{R}_{12}(u)\mathbf{R}_{12}(-u) = \mathbf{1}_{12}. \tag{7.2.29}$$

Conversely, the solution satisfying (7.2.27)-(7.2.29) is unique, as we demonstrate later.

To actually find the form of $\mathbf{R}_{12}(u)$, we will first solve the equation (7.2.27). For this purpose, it is convenient to introduce the following notations for sets of oscillators:

$$\bar{\alpha}^{\mathsf{A}} = \begin{pmatrix} \lambda_\alpha \\ \bar{c}_i \end{pmatrix}, \quad \alpha_{\mathsf{A}} = \begin{pmatrix} \mu^\alpha \\ c^i \end{pmatrix}, \tag{7.2.30}$$

$$\bar{\beta}^{\dot{\mathsf{A}}} = \begin{pmatrix} \tilde{\lambda}_{\dot\alpha} \\ \bar{d}_i \end{pmatrix}, \quad \beta_{\dot{\mathsf{A}}} = \begin{pmatrix} \tilde{\mu}^{\dot\alpha} \\ d^i \end{pmatrix}. \tag{7.2.31}$$

Notice that $\alpha_{\mathsf{A}}|Z\rangle = \beta_{\dot{\mathsf{A}}}|Z\rangle = 0$. These oscillators transform covariantly under the subalgebras $\mathfrak{gl}(2|2) \oplus \mathfrak{gl}(2|2) \subset \mathfrak{gl}(4|4)$, whose bosonic parts are given by the Lorentz and $\mathfrak{su}(2)_L \oplus \mathfrak{su}(2)_R$ R-symmetry subalgebras. In other words, the indices $\mathsf{A}, \mathsf{B}, \dots$ are associated with the (anti-) fundamental representation of one $\mathfrak{gl}(2|2)$ and the indices $\dot{\mathsf{A}}, \dot{\mathsf{B}}, \dots$ describe the (anti-) fundamental representation of the other $\mathfrak{gl}(2|2)$. Accordingly, the $\mathfrak{gl}(4|4)$ generators are decomposed into diagonal parts and the off-diagonal parts with respect to these two $\mathfrak{gl}(2|2)$ subalgebras in the following form:

$$J^A_B \longrightarrow \begin{pmatrix} J^{\mathsf{A}}_{\mathsf{B}} & J^{\mathsf{A}}_{\dot{\mathsf{B}}} \\ J^{\dot{\mathsf{A}}}_{\mathsf{B}} & J^{\dot{\mathsf{A}}}_{\dot{\mathsf{B}}} \end{pmatrix}. \tag{7.2.32}$$

The explicit form of these generators in terms of α, β-oscillators are given in [183], but we shall not write them down here.

Now using the oscillators above, one introduces the following basis of linear operators acting on $\mathcal{V}_1 \otimes \mathcal{V}_2$, which will be useful for solving the condition (7.2.27):

$$\begin{aligned}
\mathbf{Hop}^{(12)}_{k,l,m,n} &= \; : \frac{(\bar\alpha_2\alpha^1)^k}{k!} \frac{(\bar\beta^2\beta_1)^l}{l!} \frac{(\bar\alpha_1\alpha^2)^m}{m!} \frac{(\bar\beta^1\beta_2)^n}{n!} : \\
&= \frac{1}{k!l!m!n!} \bar\alpha_2^{\mathsf{A}_1} \cdots \bar\alpha_2^{\mathsf{A}_k} \bar\beta^2_{\dot{\mathsf{A}}_1} \cdots \bar\beta^2_{\dot{\mathsf{A}}_l} \bar\alpha_1^{\mathsf{B}_1} \cdots \bar\alpha_1^{\mathsf{B}_m} \bar\beta^1_{\dot{\mathsf{B}}_1} \cdots \bar\beta^1_{\dot{\mathsf{B}}_n} \\
&\quad \cdot \beta_2^{\dot{\mathsf{B}}_n} \cdots \beta_2^{\dot{\mathsf{B}}_1} \alpha_{\mathsf{B}_m}^2 \cdots \alpha_{\mathsf{B}_1}^2 \beta_1^{\dot{\mathsf{A}}_l} \cdots \beta_1^{\dot{\mathsf{A}}_1} \alpha_{\mathsf{A}_k}^1 \cdots \alpha_{\mathsf{A}_1}^1.
\end{aligned} \tag{7.2.33}$$

Here the symbol $: * :$ in the first line denotes the normal ordering of the oscillators. The name **Hop** stems from the following properties of this operator. Its action transforms $k + l$ oscillators with label 1 to those with label 2 and $m + n$ oscillators with label 2 to those with label 1. Thus it effects a kind of hopping operation. Note that these operators are manifestly invariant under the diagonal part of (7.2.32), namely, $J^{\mathsf{A}}_{\mathsf{B}}$ and $J^{\dot{\mathsf{A}}}_{\dot{\mathsf{B}}}$, as all the relevant indices in (7.2.33) are contracted. Therefore, the general solution $\mathbf{R}_{12}(u)$ of (7.2.27) should be obtained as a linear combination of $\mathbf{Hop}^{(12)}_{k,l,m,n}$ of the form

$$\mathbf{R}_{12}(u) = \sum_{k,l,m,n} \mathcal{A}^{(\mathbf{N})}_{k,l,m,n}(u)\mathbf{Hop}^{(12)}_{k,l,m,n}, \tag{7.2.34}$$

where $\mathbf{N}$ stands for the total number operator defined by

$$\mathbf{N} = \mathbf{N}^{(1)} + \mathbf{N}^{(2)}, \quad \mathbf{N}^{(i)} = \mathbf{N}^{(i)}_{\alpha} + \mathbf{N}^{(i)}_{\beta} = \bar{\alpha}^{\mathsf{A}}_i \alpha^i_{\mathsf{A}} + \bar{\beta}^i_{\dot{\mathsf{A}}} \beta^{\dot{\mathsf{A}}}_i. \tag{7.2.35}$$

Note that the coefficients $\mathcal{A}^{(\mathbf{N})}_{k,l,m,n}(u)$ can depend in general on the spectral parameter u, the total number operator $\mathbf{N}$ and the central charge[21], since they all commute with the diagonal generators. As explained in detail in [183], the invariance under the remaining off-diagonal generators $J^{\mathsf{A}}_{\mathsf{B}}$, $J^{\dot{\mathsf{A}}}_{\dot{\mathsf{B}}}$ together with the invariance under the level-one Yangian generators (7.2.28) uniquely fix the coefficients up to an overall coefficient $\rho(u)$ as

$$\mathcal{A}^{(\mathbf{N})}_{k,l,m,n} = a_{k,l,m,n}\mathcal{A}^{(\mathbf{N})}_I, \quad a_{k,l,m,n} := \delta_{k+n,l+m}(-1)^{(k+l)(m+n)},$$
$$\mathcal{A}^{(\mathbf{N})}_I(u) = \rho(u)(-1)^{I+\frac{\mathbf{N}}{2}}\mathcal{B}(I, u + \mathbf{N}/2), \quad \mathcal{B}(x,y) := \frac{\Gamma(x+1)}{\Gamma(x-y+1)\Gamma(y+1)}. \tag{7.2.36}$$

Here $I := \frac{k+l+m+n}{2}$ is an integer since $k+n = l+m$, and $\mathcal{B}(x,y)$ is a natural generalization of the binomial coefficient, whose arguments can be complex. Notice that it satisfies $\mathcal{B}(x,y) = \mathcal{B}(x, x-y)$ by definition. As already mentioned before, the overall coefficient $\rho(u)$ is determined so that the unitarity condition (7.2.29) is satisfied. Explicit calculation gives

$$\rho(u) = \Gamma(u+1)\Gamma(1-u). \tag{7.2.37}$$

An important characteristic of the harmonic R-matrix, for which the quantum and the auxiliary spaces are identical just as for the SU(2) Heisenberg spin chain, is that the R-matrix at $u = 0$ yields precisely the permutation operator. Because of this fact, through the well-known manipulation, the Hamiltonian can be extracted from the R-matrix simply

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[21] Notice that the central charge is given by $C^{(i)} = \mathbf{N}^{(i)}_{\alpha} - \mathbf{N}^{(i)}_{\beta}$. Since we are interested in the representation in which the central charge vanishes, we neglect dependence on this combination.

as a logarithmic derivative. This is summarized as[22]

$$\mathbf{P}_{12} = \mathbf{R}_{12}(0) \,, \tag{7.2.38}$$

$$\mathbf{H}_{12} = \frac{d}{du} \ln \mathbf{R}_{12}(u)|_{u=0} \,. \tag{7.2.39}$$

Monodromy relation

Having reviewed the basic facts on the harmonic R-matrix and displayed its explicit form in terms of the oscillators, we now discuss the monodromy relations involving such R-matrices. As in the case of the fundamental R-matrix, the basic ingredients for the derivation is (i) the inversion relation and (ii) the crossing relation.

The inversion relation is already given in (7.2.29), together with the computation of the factor $\rho(u)$, shown in (7.2.37), needed for the normalization.

As for the crossing relation, its form for the harmonic R-matrix turned out to be similar to (but not identical with) the one for the Lax matrix for the case of the fundamental R-matrix shown in (7.2.12) and is given by

$$\langle \mathbf{1}_{12}|\mathbf{R}_{an}^{(1)}(u) = \langle \mathbf{1}_{12}|\mathbf{R}_{a\ell-n+1}^{(2)}(-u) \,. \tag{7.2.40}$$

Once this is verified, the crossing relation for the product of harmonic R-matrices is easily given by

$$\langle \mathbf{1}_{12}|\mathbf{R}_{a1}^{(1)}(u) \cdots \mathbf{R}_{a\ell}^{(1)}(u) = \langle \mathbf{1}_{12}|\mathbf{R}_{a\ell}^{(2)}(-u) \cdots \mathbf{R}_{a1}^{(2)}(-u) \,. \tag{7.2.41}$$

Now define the monodromy matrix as

$$\Omega_{\boldsymbol{mn}}^{(i)}(u) := \langle \boldsymbol{m}|\mathbf{R}_{a1}^{(i)}(u) \cdots \mathbf{R}_{a\ell}^{(i)}(u)|\boldsymbol{n}\rangle_a \,, \tag{7.2.42}$$

where $\{|\boldsymbol{n}\rangle\}$ is a complete set of states in the auxiliary space satisfying $1 = \sum_{\boldsymbol{n}} |\boldsymbol{n}\rangle\langle\boldsymbol{n}|$. Note that the components of the monodromy matrix take values in operators acting on the quantum space. The crossing relation for them follow immediately from (7.2.41) by taking the matrix element between the states $\langle \boldsymbol{m}|$ and $|\boldsymbol{n}\rangle$ and is expressed as

$$\langle \mathbf{1}_{12}|\Omega_{\boldsymbol{mn}}^{(1)}(u) = \langle \mathbf{1}_{12}|\overleftarrow{\Omega}_{\boldsymbol{mn}}^{(2)}(-u) \,. \tag{7.2.43}$$

Now contract this relation with $\overleftarrow{\Omega}_{\boldsymbol{nl}}^{(2)}(-u)$ and sum over $\boldsymbol{n}$. Then, since the completeness of the states $\{|\boldsymbol{n}\rangle\}$ in the auxiliary space implies $\overleftarrow{\Omega}_{\boldsymbol{kl}}^{(2)}(-u)\Omega_{\boldsymbol{lm}}^{(2)}(u) = \delta_{\boldsymbol{km}}$, we obtain the basic monodromy relation

$$\sum_n \langle \mathbf{1}_{12}|\Omega_{\boldsymbol{mn}}^{(1)}(u)\Omega_{\boldsymbol{nl}}^{(2)}(u) = \langle \mathbf{1}_{12}|\delta_{\boldsymbol{ml}} \,. \tag{7.2.44}$$

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[22] Equivalently, the expansion of the R-matirx around $u = 0$ is of the form $\mathbf{R}_{12}(u) = \mathbf{P}_{12}(1+u\mathbf{H}_{12}+\cdots)$.

As an example for the use of this relation, contract both sides with $|\mathcal{O}_1\rangle \otimes |\mathcal{O}_2\rangle$. We then obtain the monodromy relation for a two-point function of the form

$$\sum_l \left\langle \Omega_{kl}^{(1)}(u)|\mathcal{O}_1\rangle, \Omega_{lm}^{(2)}(u)|\mathcal{O}_2\rangle \right\rangle = \delta_{km} \left\langle |\mathcal{O}_1\rangle, |\mathcal{O}_2\rangle \right\rangle, \qquad (7.2.45)$$

where $\langle\,,\,\rangle$ denotes the Wick contraction via the singlet projector.

Just as in the case of the fundamental R-matrix, the derivation of the monodromy relation for the three-point functions is straightforward. In fact, the prefactor function in this case is trivial and the result takes the simple form:

$$\sum_{l,m} \left\langle \Omega_{kl}^{(1)}(u)|\mathcal{O}_1\rangle, \Omega_{lm}^{(2)}(u)|\mathcal{O}_2\rangle, \Omega_{mn}^{(3)}(u)|\mathcal{O}_3\rangle \right\rangle = \delta_{kn} \left\langle |\mathcal{O}_1\rangle, |\mathcal{O}_2\rangle, |\mathcal{O}_3\rangle \right\rangle. \qquad (7.2.46)$$

The reason for the absence of the sign factors in contrast to the case of the fundamental R-matrix (7.2.25) is because all the components of the harmonic R-matrix $\langle \boldsymbol{m}|\mathbf{R}(u)|\boldsymbol{n}\rangle$ are bosonic: They are composed of even number of oscillators, as seen from the definitions (7.2.33)-(7.2.36).

Let us now describe how one can prove the basic crossing relation (7.2.40). As shown in (7.2.34), the harmonic R-matrix is made up of the hopping operators (7.2.33) and the coefficients (7.2.36). Therefore we need to derive the crossing relations for these two quantities. Since the manipulations are somewhat involved, we relegate the details to Appendix C and only sketch the procedures and some relevant intermediate results below.

First, using the crossing relations for the oscillators, it is easy to derive the crossing relation for an arbitrary function of the number operators. The result reads

$$\langle \mathbf{1}_{12}|f(\mathbf{N}^{(a)} + \mathbf{N}^{(1)}) = \langle \mathbf{1}_{12}|f(\mathbf{N}^{(a)} - \mathbf{N}^{(2)}). \qquad (7.2.47)$$

In particular, the coefficient $\mathcal{A}_I^{(\mathbf{N}^{(a)}+\mathbf{N}^{(1)})}$ becomes $\mathcal{A}_I^{(\mathbf{N}^{(a)}-\mathbf{N}^{(2)})}$ under such crossing.

Next one can show that the crossing relation for the hopping operator takes the form

$$\langle \mathbf{1}_{12}|\mathbf{Hop}_{k,l,m,n}^{(a1)} = \langle \mathbf{1}_{12}|\mathcal{C} \circ \mathbf{Hop}_{k,l,m,n}^{(a1)}$$

$$:= (-1)^{l+m} \langle \mathbf{1}_{12}| \sum_{p=0}^{\min(k,m)} \sum_{q=0}^{\min(l,n)} \mathcal{B}(\mathbf{N}_\alpha^{(a)} - m + p, p)\mathcal{B}(\mathbf{N}_\beta^{(a)} - n + q, q)\mathbf{Hop}_{k-p,l-q,m-p,n-q}^{(a2)},$$

$$\qquad (7.2.48)$$

where $\mathcal{C} \circ \mathbf{Hop}_{k,l,m,n}^{(a1)}$ denotes the crossed $\mathbf{Hop}$ operator and the function $\mathcal{B}(x,y)$ was defined in (7.2.36).

Combining these crossing operations, we find that

$$\langle \mathbf{1}_{12}|\mathbf{R}^{(1)}(u) = \langle \mathbf{1}_{12}| \sum_{k,l,m,n} \sum_{p,q} a_{k,l,m,n} \mathcal{A}_I^{(\mathbf{N}^{(a)}-\mathbf{N}^{(2)}+k+l-m-n)} \mathcal{C} \circ \mathbf{Hop}_{k,l,m,n}^{(a1)} \tag{7.2.49}$$

$$= \langle \mathbf{1}_{12}| \sum_{k,l,m,n} a_{k,l,m,n} \tilde{\mathcal{A}}_I^{(\mathbf{N})} \mathbf{Hop}_{k,l,m,n}^{(a2)} \,, \tag{7.2.50}$$

where

$$\tilde{\mathcal{A}}_I^{(\mathbf{N})}(u) = \sum_{p,q}^{\infty} (-1)^I \mathcal{B}(\mathbf{N}_\alpha^{(a)}-m,p)\mathcal{B}(\mathbf{N}_\beta^{(a)}-n,q)\mathcal{A}_{I+p+q}^{(2I-\mathbf{N}+2\mathbf{M})} \,. \tag{7.2.51}$$

In the above expression of $\tilde{\mathcal{A}}_I^{(\mathbf{N})}(u)$, we have renamed $(k-p, l-q, m-p, n-q) \to (k,l,m,n)$ and defined $\mathbf{M} := \mathbf{N}^{(a)} - m - n$. Note that, under this change of labels, $a_{k,l,m,n} = \delta_{k+n,l+m}(-1)^{(k+l)(m+n)}$ becomes $a_{k,l,m,n}(-1)^{p+q}$ and $I = \frac{k+l+m+n}{2}$ changes into $I+p+q$. Using the binomial identity $\mathcal{B}(\alpha+\beta, k) = \sum_{j=0}^k \mathcal{B}(\alpha, k-j)\mathcal{B}(\beta, j)$[23], the summation over p, q in $\tilde{\mathcal{A}}_I^{(\mathbf{N})}$ can be converted into a simpler expression

$$\begin{aligned}
\tilde{\mathcal{A}}_I^{(\mathbf{N})}(u) &= \sum_{r=0}^{\infty} (-1)^I \left(\sum_{p=0}^{r} \mathcal{B}(\mathbf{N}_\alpha^{(a)}-m,p)\mathcal{B}(\mathbf{N}_\beta^{(a)}-n,r-p) \right) \mathcal{A}_{I+r}^{(2I-\mathbf{N}+2\mathbf{M})}(u) \\
&= \sum_{r=0}^{\infty} (-1)^I \mathcal{B}(\mathbf{M},r)\mathcal{A}_{I+r}^{(2I-\mathbf{N}+2\mathbf{M})}(u) \,.
\end{aligned} \tag{7.2.52}$$

In Appendix C, we will show that this sum giving $\tilde{\mathcal{A}}_I^{(\mathbf{N})}(u)$ can be evaluated and leads to the desired equality

$$\tilde{\mathcal{A}}_I^{(\mathbf{N})}(u) = \sum_{r=0}^{\infty} (-1)^I \mathcal{B}(\mathbf{M},r)\mathcal{A}_{I+r}^{(2I-\mathbf{N}+2\mathbf{M})} = \mathcal{A}_I^{(\mathbf{N})}(-u) \,. \tag{7.2.53}$$

Putting this result into (7.2.50) and summing over k, l, m, n, we find that the RHS of (7.2.50) becomes $\langle \mathbf{1}_{12}|\mathbf{R}^{(2)}(-u)$ and this proves the crossing relation (7.2.40).

7.3 Reduction of monodromy relation to subsectors

In the previous section, we have derived the monodromy relation for the full $\mathfrak{psu}(2,2|4)$ sector both in the case of the fundamental R-matrix and of the harmonic R-matrix. The former can be obtained by a rather straightforward generalization of the SU(2) case discussed in our previous work [138] and we tried to give a slightly more detailed exposition compared with the result already given in [137]. On the other hand, the case for the

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[23]This is a direct consequence from the binomial theorem, namely, $(1+x)^\alpha = \sum_{k=0}^{\infty} \mathcal{B}(\alpha,k)x^k$. The formula readily follows by considering $(1+x)^{\alpha+\beta} = (1+x)^\alpha \cdot (1+x)^\beta$.

harmonic R-matrix is new. Although its derivation turned out to be substantially more involved than the case of the fundamental R-matrix, the symmetric set up for which the quantum and the auxiliary spaces carry identical representation of $\mathfrak{psu}(2,2|4)$ can be of particular value, as was already indicated in the application to the scattering amplitude. Also the fact that the Hamiltonian can be obtained simply as the logarithmic derivative of the harmonic R-matrix should find useful applications.

The most important original purpose for formulating the monodromy relations, however, is their possible use as the set of powerful equations which govern, from the integrability perspective, the structures of the correlation functions. Although this idea has not yet been studied explicitly, to perform such an analysis it is natural to begin with the simplest set-ups, namely the cases of important tractable subsectors of the full theory. In what follows, we shall consider the compact SU(2) sector and the non-compact SL(2) sector as typical examples, and derive the monodromy relations for them from the point of view of a systematic reduction of the general $\mathfrak{psu}(2,2|4)$ case.

7.3.1 Reduction to the SU(2) subsector

Embedding of $\mathfrak{su}(2)_L \oplus \mathfrak{su}(2)_R$ in $\mathfrak{u}(2,2|4)$

In the case of the SU(2) sector, the monodromy relation was already obtained in our previous work [110]. Therefore the purpose here is to re-derive it through the reduction of the $\mathfrak{psu}(2,2|4)$ case. To do this, the first step is to identify the generators of $\mathrm{SU}(2)_\mathrm{L}$ and $\mathrm{SU}(2)_\mathrm{R}$, given in (7.1.71) and (7.1.72) in terms of those of $\mathfrak{u}(2,2|4)$ shown in (C.6). This is simple since $\mathrm{SU}(2)_\mathrm{L} \times \mathrm{SU}(2)_\mathrm{R}$ is contained entirely in the R-symmetry group SU(4), and hence the relevant $\mathfrak{u}(2,2|4)$ generators are $W_a{}^b = \bar{\xi}_a \xi^b = R_a{}^b + \frac{1}{2}\delta_a^b B$ given in (7.1.50). Explicitly, we have

$$J_3^L = \frac{1}{2}(W_3{}^3 - W_1{}^1), \quad J_+^L = -W_3{}^1, \quad J_-^L = -W_1{}^3, \tag{7.3.1}$$

$$J_3^R = \frac{1}{2}(W_4{}^4 - W_2{}^2), \quad J_+^R = -W_4{}^2, \quad J_-^R = -W_2{}^4. \tag{7.3.2}$$

It will be important to recognize that the following combinations, B_L and B_R, of diagonal $W_a{}^a$ generators act as central charges for the group $\mathrm{SU}(2)_\mathrm{L} \times \mathrm{SU}(2)_\mathrm{R}$:

$$B_L = \frac{1}{2}(W_3{}^3 + W_1{}^1), \qquad B_R = \frac{1}{2}(W_4{}^4 + W_2{}^2). \tag{7.3.3}$$

For example, B_L together with J_i^L form $\mathrm{U}(2)_\mathrm{L} = \mathrm{U}(1) \times \mathrm{SU}(2)_\mathrm{L}$ of which B_L is the U(1) part. Thus, $[B_L, J_i^L] = 0$. Obviously B_L commutes with $\mathrm{SU}(2)_\mathrm{R}$. The argument for B_R is entirely similar. Being the central charges, they take definite values in an irreducible

representation, which in our case of interest is the spin $(\frac{1}{2})_L \times (\frac{1}{2})_R$ representation. Evaluating B_L and B_R on any of the states in this representation, say $|Z\rangle = \bar{\xi}_3\bar{\xi}_4|0\rangle$, it is easy to obtain $B_L = B_R = \frac{1}{2}$.

With this in mind, let us write down the $\mathrm{SU}(2)_\mathrm{L}$ Lax operators as embedded in that of $\mathfrak{gl}(4|4)$, given in (7.2.9). We get

$$L_{\mathrm{SU}(2)_L}(u) = \begin{pmatrix} u + \frac{i}{2}(W_3{}^3 - W_1{}^1) & -iW_1{}^3 \\ -iW_3{}^1 & u - \frac{i}{2}(W_3{}^3 - W_1{}^1) \end{pmatrix} \tag{7.3.4}$$

Using the relation $B_L = \frac{1}{2} = \frac{1}{2}(W_3{}^3 + W_1{}^1)$, i.e. $W_3{}^3 + W_1{}^1 = 1$, this can be re-written as

$$L_{\mathrm{SU}(2)_L}(u) = \begin{pmatrix} u + \frac{i}{2} - iW_1{}^1 & -iW_1{}^3 \\ -iW_3{}^1 & u + \frac{i}{2} - iW_3{}^3 \end{pmatrix} \tag{7.3.5}$$

Let us now recall the general form of the Lax operator for $\mathfrak{gl}(4|4)$, which is $L(u)^A{}_B = u\delta^A_B + \eta(-1)^{|B|}J^A{}_B$, and its more refined form in terms of the superconformal generators given in (7.2.9). The part relevant for the $\mathrm{SU}(2)$ sector is in the lower diagonal corner given by $u\delta^b_a - \eta W_a{}^b$. For the $\mathrm{SU}(2)_\mathrm{L}$, we can identify the indices for a and b to be 1 and 3. In the entirely similar manner, the Lax operator for the $\mathrm{SU}(2)_\mathrm{R}$ sector is obtained from the one for the $\mathrm{SU}(2)_\mathrm{L}$ sector by substitution of the indices $1 \to 2$ and $3 \to 4$. Then, it is easy to see that the Lax operators for the $\mathrm{SU}(2)_\mathrm{L}$ and $\mathrm{SU}(2)_\mathrm{R}$ sectors are obtained from the $\mathfrak{gl}(4|4)$ Lax operator by taking into account the shift $u \to u + i/2$ in the form

$$L_{\mathrm{SU}(2)_L}(u)_{ab} = L(u + i/2)_{ab}, \qquad \text{with } \eta = i, \quad \{a, b\} = \{1, 3\}, \tag{7.3.6}$$

$$L_{\mathrm{SU}(2)_R}(u)_{\bar{a}\bar{b}} = L(u + i/2)_{\bar{a}\bar{b}}, \qquad \text{with } \eta = i, \quad \{\bar{a}, \bar{b}\} = \{2, 4\}. \tag{7.3.7}$$

It is important to note that the occurrence of the shift of the spectral parameter is due to the emergence of the extra central charges when a group is restricted to its subgroup and hence is a rather general phenomenon.

Inversion relation

Let us now derive the inversion relation. As in the previous discussion, we shall concentrate on the $\mathrm{SU}(2)_\mathrm{L}$ part. The object to consider is the product of two $\mathfrak{u}(2, 2|4)$ Lax operators given in (7.2.14), with the indices A, B taken to be the $\mathrm{SU}(2)$ indices[24] a, b. This gives

$$(L(u)L(v))_{\mathrm{ab}} = (L(u))_{\mathrm{ac}}(L(v))_{\mathrm{cb}} + \eta^2(-1)^\gamma(-1)^{|\mathrm{b}|}J^\mathrm{a}{}_\gamma J^\gamma{}_\mathrm{b}, \tag{7.3.8}$$

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[24]Here and *until* the end of subsection 4.1, we shall use roman letters a, b, etc. to denote the genuine $\mathrm{SU}(2)$ indices, which take the values 1 and 2, in order to distinguish them from the italic letters a, b, etc., which are $\mathrm{SU}(4)$ indices taking values from 1 to 4.

where γ runs over the indices of the auxiliary space other than the SU(2) indices a, b, ... Using the oscillator representation, the second terms can be simplified as

$$(-1)^\gamma J^{\mathrm{a}}{}_\gamma J^\gamma{}_{\mathrm{b}} = \bar\zeta^{\mathrm{a}}(\mu^\alpha \lambda_\alpha - \tilde\lambda_{\dot\alpha}\tilde\mu^{\dot\alpha} - \xi^{\bar d}\bar\xi_{\bar d})\zeta_{\mathrm{b}}$$
$$= J^{\mathrm{a}}{}_{\mathrm{b}}(2Z_L + 2Z_R + 2B_R)\,, \tag{7.3.9}$$

where

$$Z_L := \frac{1}{2}(\lambda_1 \mu^1 - \tilde\lambda_{\mathrm{i}}\tilde\mu^{\mathrm{i}}) = \frac{1}{2}(J_1{}^1 + J_{\mathrm{i}}{}^{\mathrm{i}})\,, \tag{7.3.10}$$

$$Z_R := \frac{1}{2}(\lambda_2 \mu^2 - \tilde\lambda_{\dot 2}\tilde\mu^{\dot 2}) = \frac{1}{2}(J_2{}^2 + J_{\dot 2}{}^{\dot 2})\,, \tag{7.3.11}$$

and B_R was already defined in (7.3.3). In the SU(2) sector the quantities Z_L and Z_R vanish[25] and $B_R = 1/2$, as was already explained. Therefore the factor $2Z_L + 2Z_R + 2B_R$ is simply unity and we simply obtain

$$(-1)^\gamma J^{\mathrm{a}}{}_\gamma J^\gamma{}_{\mathrm{b}} = J^{\mathrm{a}}{}_{\mathrm{b}}\,. \tag{7.3.12}$$

Substituting this back into (7.3.9) and rewriting the spectral parameters in order to express the result in terms of the Lax operator for SU(2)$_{\mathrm{L}}$, we obtain the relation

$$(L(u)L(v))_{\mathrm{ab}} = (L(u))_{\mathrm{ac}}(L(v))_{\mathrm{cb}} - \eta^2 J^{\mathrm{a}}{}_{\mathrm{b}}$$
$$= (L(u+\eta/2))_{\mathrm{ac}}(L(v+\eta/2))_{\mathrm{cb}} - ((u+\eta/2)(v+\eta/2) - uv)\delta_{\mathrm{ab}} \tag{7.3.13}$$
$$= (L_{\mathrm{SU}(2)}(u))_{\mathrm{ac}}(L_{\mathrm{SU}(2)}(v))_{\mathrm{cb}} - ((u+i/2)(v+i/2) - uv)\delta_{\mathrm{ab}}\,.$$

Now from the inversion relation for the Lax operator for $\mathfrak{u}(2,2|4)$ given in (7.2.13), when $u+v = \eta = i$, the left hand side of the above equation becomes $uv\delta_{ab}$, and thus we obtain the inversion relation for the SU(2)$_{\mathrm{L}}$ Lax operator to be of the form

$$(L_{\mathrm{SU}(2)}(u)L_{\mathrm{SU}(2)}(v))_{\mathrm{ab}} = f_{\mathrm{SU}}(u,v)\delta_{\mathrm{ab}}\,, \quad u+v = i\,, \tag{7.3.14}$$

$$f_{\mathrm{SU}}(u,v) = uv - \frac{3}{4} \tag{7.3.15}$$

To compare with the result of [138], we must substitute $u \to -u+\frac{i}{2}$ and $v \to u+\frac{i}{2}$. Then the relation above takes the form

$$(L_{\mathrm{SU}(2)}(-u+(i/2))L_{\mathrm{SU}(2)}(u+(i/2)))_{\mathrm{ab}} = -(u^2+1)\delta_{\mathrm{ab}}\,, \tag{7.3.16}$$

which agrees with the equation (5.1) of [138] (with the inhomogeneity parameter θ set to zero.

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[25]Actually, as it will be explained in the next subsection, the quantities Z_L and Z_R are central charges for SL(2)$_{\mathrm{L}} \times$ SL(2)$_{\mathrm{R}}$.

From this inversion relation, one can easily obtain the monodromy relation, as already described in [110]. So we shall omit this derivation for the SU(2) sector. Instead, in the next subsection, we shall present a derivation of the monodromy relation for the SL(2) sector *directly* from that for the $\mathfrak{psu}(2,2|4)$ sector. The result is new and the method can of course be applied to the SU(2) case as well to provide an alternative derivation of a known result given in [138].

7.3.2 Reduction to the SL(2) subsector

Having been warmed up with the reduction to the simplest SU(2) sector, we now perform the reduction to the SL(2) subsector to derive the explicit form of its monodromy relation, which is new.

Embedding of $\mathfrak{sl}(2)_L \oplus \mathfrak{sl}(2)_R$ in $\mathfrak{u}(2,2|4)$ and a derivation of monodromy relation

Let us begin by clarifying how the generators of $SL(2)_L \times SL(2)_R$ are embedded in $\mathfrak{u}(2,2|4)$. First consider the simple "light-cone" combinations of operators given by

$$p_+ = \frac{1}{2}(P_0 - P_3)\,, \quad k^+ = -\frac{1}{2}(K_0 + K_3)\,, \quad d_+ = \frac{i}{2}(M_{03} - D)\,, \tag{7.3.17}$$

$$p_- = \frac{1}{2}(P_0 + P_3)\,, \quad k^- = \frac{1}{2}(-K_0 + K_3)\,, \quad d_- = -\frac{i}{2}(M_{03} + D)\,. \tag{7.3.18}$$

They satisfy the following simple set of commutation relations:

$$[d_\pm, p_\pm] = p_\pm\,, \quad [d_\pm, k^\pm] = -k^\pm\,, \quad [k^\pm, p_\pm] = 2d_\pm\,. \tag{7.3.19}$$

This shows that the generators of the $SL(2)_L \times SL(2)_R$ can be taken as

$$SL(2)_L: \quad S_- = -ip_+\,, \quad S_+ = -ik^+\,, \quad S_0 = -d_+\,, \tag{7.3.20}$$

$$SL(2)_R: \quad \tilde{S}_- = -ip_-\,, \quad \tilde{S}_+ = -ik^-\,, \quad \tilde{S}_0 = -d_-\,. \tag{7.3.21}$$

In this notation, the commutation relations are

$$[S_0, S_\pm] = \pm S_\pm\,, \qquad [S_+, S_-] = 2S_0\,, \tag{7.3.22}$$

$$\left[\tilde{S}_0, \tilde{S}_\pm\right] = \pm \tilde{S}_\pm\,, \qquad \left[\tilde{S}_+, \tilde{S}_-\right] = 2\tilde{S}_0\,. \tag{7.3.23}$$

Now from the definition of spinor notations (7.1.26), (7.1.27) and the form of the $\mathfrak{u}(2,2|4)$ generators J^A_B given in (C.6), one finds that, for example, the generators $\{S_0, S_\pm\}$ of

$SL(2)_L$ are embedded in $\mathfrak{u}(2,2|4)$ as[26]

$$J_1{}^1 = M_1{}^1 - \frac{i}{2}D - \frac{1}{2}B = \frac{i}{2}(M_{03} - iM_{12}) - \frac{i}{2}(D - iB) = -S_0 + \frac{1}{2}M_{12} - \frac{1}{2}B \quad (7.3.24)$$

$$J^{\dot{1}}{}_{\dot{1}} = M^{\dot{1}}{}_{\dot{1}} + \frac{i}{2}D - \frac{1}{2}B = \frac{i}{2}(-M_{03} - iM_{12}) + \frac{i}{2}(D + iB) = S_0 + \frac{1}{2}M_{12} - \frac{1}{2}B, \quad (7.3.25)$$

$$J_{1\dot{1}} = iP_{1\dot{1}} = ip_+ = -S_-, \quad J^{\dot{1}1} = iK^{\dot{1}1} = ik^+ = -S_+. \quad (7.3.26)$$

At this point we note that, just as the extra central charges B_L and B_R appeared for the subgroup $SU(2)_L \times SU(2)_R$, the quantities Z_L and Z_R previously defined in (7.3.10) and (7.3.11) behave as central charges for $SL(2)_L \times SL(2)_R$. This will be important below.

Now in what follows, let us concentrate on the $SL(2)_L$ subsector, where the composite operators are constructed by multiple actions of the covariant derivatives along the light-cone direction as

$$\mathcal{O}(x) = \frac{1}{n_1! n_2! \dots} \mathrm{Tr}(\mathcal{D}_{1\dot{1}}^{n_1} Z \mathcal{D}_{1\dot{1}}^{n_2} Z \cdots)$$

$$\mapsto \exp(i(x^+ p_+ + x^- p_-)) \frac{(\lambda_1 \tilde{\lambda}_{\dot{1}})^{n_1}}{n_1!} |Z\rangle \otimes \frac{(\lambda_1 \tilde{\lambda}_{\dot{1}})^{n_2}}{n_2!} |Z\rangle \otimes \cdots. \quad (7.3.27)$$

Here, $|Z\rangle$ is the scalar state $\bar{\xi}_3 \bar{\xi}_4 |0\rangle$ and each factor in the total tensor product signifies the operator at different positions of the spin chain. Note that on this type of states the rotation operator M_{12} vanishes and the hypercharge operator $B = \frac{1}{2}\bar{\xi}_a \xi^a$ takes the definite value 1. Furthermore, one can easily check that $Z_L = Z_R = 0$ by acting them on such a state. All these relations is consistent with the vanishing of the central charge $C = Z_L + Z_R + B - 1$ for the physical SYM operators.

With these properties, the relations (7.3.24) and (7.3.25) simplify and we can easily embed the $SL(2)_L$ Lax operator into that of $\mathfrak{u}(2,2|4)$ in the following fashion:

$$L_{SL(2)}(u) := \begin{pmatrix} u + iS_0 & iS_- \\ iS_+ & u - iS_0 \end{pmatrix} = \begin{pmatrix} u - \frac{i}{2} - iJ_1{}^1 & -iJ_{1\dot{1}} \\ -iJ^{\dot{1}1} & u - \frac{i}{2} + iJ^{\dot{1}}{}_{\dot{1}} \end{pmatrix}. \quad (7.3.28)$$

Here again the shift of $u \to u - \frac{i}{2}$ is due to the effect of the central charge, as in the case of $SU(2)$. Now comparing this with the form of the , we readily find that the $SL(2)_L$ Lax operator is embedded in the $\mathfrak{u}(2,2|4)$ Lax operator as[27]

$$(L_{SL(2)}(u))_{ab} = (L(u - i/2))_{ab}, \quad (a, b = 1, \dot{1}), \quad \eta = -i. \quad (7.3.29)$$

(Just as we did in the discussion of the $SU(2)$ sector in the previous subsection, we shall hereafter use the roman letters a, b, etc. to denote the Lorentz spinor indices $\{a, b\} =$

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[26] 1 and $\dot{1}$ are the indices for $SL(2, \mathbb{C}) \times \overline{SL(2, \mathbb{C})}$.

[27] Similarly, the Lax operator for $SL(2)_R$ part is embedded as $(L_{\widetilde{SL}(2)}(u))_{ab} = (L(u - i/2))_{ab}$ (a, b = 2, $\dot{2}$) with $\eta = -i$. In this equation, a and b refer to the indices of 2×2 matrix, which are part of the $SL(2, \mathbb{C}) \times \overline{SL(2, \mathbb{C})}$ indices.

$\{1, \dot{1}\}$ for the $\mathrm{SL}(2)_\mathrm{L}$ sector, in order to distinguish them from the $\mathrm{SU}(4)$ indices a, b, etc, which run over 1 to 4.)

To derive the monodromy relation, we first need to find the inversion relation for the Lax operator. Just as in the $\mathrm{SU}(2)$ case, the product of the $\mathrm{SL}(2)_\mathrm{L}$ part of the $\mathfrak{u}(2, 2|4)$ Lax operators give

$$(L(u)L(v))_{\mathrm{ab}} = (L(u))_{\mathrm{ac}}(L(v))_{\mathrm{cb}} + \eta^2 (-1)^{|\gamma|} J^{\mathrm{a}}{}_\gamma J^\gamma{}_{\mathrm{b}} \tag{7.3.30}$$

where γ runs over the all indices of the auxiliary except for those of $\mathrm{SL}(2)$, namely $\mathrm{a}, \mathrm{b} = 1, \dot{1}$. Note at this stage the sign of the second term is opposite to the one for $\mathrm{SU}(2)$ case given in (7.3.8). Now as before we can simplify this term quadratic in the generators in the following manner:

$$\begin{aligned}
(-1)^{|\gamma|} J^{\mathrm{a}}{}_\gamma J^\gamma{}_{\mathrm{b}} &= \begin{pmatrix} \lambda_1 \\ \tilde{\mu}^{\dot{1}} \end{pmatrix}^{\mathrm{a}} (\mu^2 \lambda_2 - \tilde{\lambda}_{\dot{2}} \tilde{\mu}^{\dot{2}} - \xi^i \bar{\xi}_i)(\mu^1 - \tilde{\lambda}_{\dot{1}})_{\mathrm{b}} \\
&= J^{\mathrm{a}}{}_{\mathrm{b}}(2Z_R + 2B - 3) \\
&= -J^{\mathrm{a}}{}_{\mathrm{b}},
\end{aligned} \tag{7.3.31}$$

where we used $Z_R = 0, B = 1$. Thus we get an extra minus sign from this manipulation and hence obtains the same sign for the second term as for the $\mathrm{SU}(2)$ case. Now we make an appropriate shift of the spectral parameter and get

$$(L(u)L(v))_{\mathrm{ab}} = (L(u))_{\mathrm{ac}}(L(v))_{\mathrm{cb}} - \eta^2 J^{\mathrm{a}}{}_{\mathrm{b}} \tag{7.3.32}$$

$$= (L(u - \eta/2))_{\mathrm{ac}}(L(v - \eta/2))_{\mathrm{cb}} + \frac{\eta}{2}(u + v - \eta)\delta_{\mathrm{ab}} + \frac{\eta^2}{4}\delta_{\mathrm{ab}} \tag{7.3.33}$$

$$= (L_{\mathrm{SL}(2)})(u))_{\mathrm{ac}}(L_{\mathrm{SL}(2)}(v))_{\mathrm{cb}} + \frac{\eta}{2}(u + v - \eta)\delta_{\mathrm{ab}} + \frac{\eta^2}{4}\delta_{\mathrm{ab}}, . \tag{7.3.34}$$

Thus if we set $u + v = \eta = -i$, then since the left hand side of the above equation becomes $uv\delta_{\mathrm{ab}}$, we get

$$(L_{\mathrm{SL}(2)})(u))_{\mathrm{ac}}(L_{\mathrm{SL}(2)}(v))_{\mathrm{cb}} = f_{\mathrm{SL}}(u, v)\delta_{\mathrm{ab}}, \qquad u + v = -i \tag{7.3.35}$$

$$f_{\mathrm{SL}}(u, v) = uv + \frac{1}{4}. \tag{7.3.36}$$

Note that if we compare (7.3.36) with (7.3.15), we can recognize that the function $f(u, v)$ for the $\mathrm{SU}(2)$ and the $\mathrm{SL}(2)$ cases can be written in a unified manner as

$$f(u, v) = uv - \mathbb{S}^2 = \begin{cases} uv - s(s + 1) & \text{for } \mathrm{SU}(2) \\ uv - s(s - 1) & \text{for } \mathrm{SL}(2) \end{cases}, \qquad s = \frac{1}{2}, \tag{7.3.37}$$

where $\mathbb{S}^2$ is the Casimir operator for the respective group.

Method of direct reduction from $\mathfrak{psu}(2,2|4)$ monodromy relation

We now wish to demonstrate that we can derive the monodromy relation for the two-point function in the SL(2) sector *more directly* from that for $\mathfrak{psu}(2,2|4)$, with the judicious use of the product relation (7.3.34) for the Lax operators.

The monodromy relation for $\mathfrak{psu}(2,2|4)$ is given in (7.2.23). To reduce it to the SL(2) sector, we set the indices A and C to be those of SL(2), say a and c which take values in $\{1,\dot{1}\}$, and for convenience make a shift of the spectral parameter $u \to u - i/2$. Hereafter we shall employ the notation $f^-(u) \equiv f(u - i/2)$ for such a shift for any function $f(u)$. Then from (7.2.23) we get

$$\langle \Omega_\ell^-(u)_{aB}|\psi_1\rangle, \Omega_\ell^-(u)_{Bc}|\psi_2\rangle\rangle = F_\ell^-(u)\delta_{ac}\langle|\psi_1\rangle, |\psi_2\rangle\rangle\,, \tag{7.3.38}$$

or more succinctly, before taking the inner product with $|\psi_1\rangle \otimes |\psi_2\rangle$,

$$\langle \mathbf{1}_{12}|\Omega_\ell^{(1)-}(u)_{aB}\Omega_\ell^{(2)-}(u)_{Bc} = F_\ell^-(u)\delta_{ac}\langle \mathbf{1}_{12}|\,. \tag{7.3.39}$$

What we wish to derive from it is the relation involving the SL(2) monodromy matrices with only the SL(2) indices $\{1,\dot{1}\}$. However, obviously this reduction is non-trivial since the general $\mathfrak{gl}(4|4)$ indices A, B which occur for a neighboring product of two Lax operators $(LL)_{AB}$ in Ω_{aB} may take all possible $\mathfrak{gl}(4|4)$ values.

This difficulty can be overcome by noting that the formula for the crossing relation for the individual $\mathfrak{u}(2,2|4)$ Lax matrices is quite simple and that a certain product of the $\mathfrak{u}(2,2|4)$ generators which appear in the product of two Lax matrices can be reduced to a single generator, as was demonstrated already in (7.3.31). To make use of these properties, we first focus on the leftmost and the rightmost Lax operators forming the two monodromy matrices and make the split

$$\Omega_\ell^{(1)-}(u)_{aB} = L_1^{(1)-}(u)_{aD}\Omega_{\ell-1}^{(1)-}(u)_{DB}\,, \tag{7.3.40}$$

$$\Omega_\ell^{(2)-}(u)_{Bc} = \Omega_{\ell-1}^{(2)-}(u)_{BE}L_\ell^{(2)-}(u)_{Ec}\,, \tag{7.3.41}$$

Then, the LHS of (7.3.39) becomes

$$\langle \mathbf{1}_{12}|\Omega^{(1)-}{}_\ell(u)_{aB}\Omega^{(2)-}{}_\ell(u)_{Bc} = \langle \mathbf{1}_{12}|L_1^{(1)-}(u)_{aD}\Omega^{(1)-}{}_{\ell-1}(u)_{DB}\Omega^{(2)-}{}_{\ell-1}(u)_{BE}L_\ell^{(2)-}(u)_{Ec}\,. \tag{7.3.42}$$

As the operators at different positions all commute, we can move $\Omega^{(1)-}{}_{\ell-1}\Omega^{(2)-}{}_{\ell-1}$ in the middle to the left until they hit $\langle \mathbf{1}_{12}|$. Then we can use the monodromy relation (7.3.38) for the case of length $\ell-1$ and write (7.3.42) as

$$\langle \mathbf{1}_{12}|\Omega^{(1)-}{}_\ell(u)_{aB}\Omega^{(2)-}{}_\ell(u)_{Bc} = F_{\ell-1}^-\langle \mathbf{1}_{12}|L_1^{(1)-}(u)_{aB}L_\ell^{(2)-}(u)_{Bc}\,. \tag{7.3.43}$$

Now substitute the definition $L(u)^A{}_B = u\delta^A_B + \eta(-1)^{|B|}J^A{}_B$ for the Lax operators on the RHS and expand. This gives

$$\langle \mathbf{1}_{12}|L_1^{(1)-}(u)_{aB}L_1^{(2)-}(u)_{Bc} = \langle \mathbf{1}_{12}|L_1^{(1)-}(u)_{ab}L_\ell^{(2)-}(u)_{bc} + \langle \mathbf{1}_{12}|L_1^{(1)-}(u)_{a\beta}L_\ell^{(2)-}(u)_{\beta c} \,,$$
$$(7.3.44)$$

where β stands for indices other than those of SL(2). Further the second term on the RHS can be written out explicitly as

$$L_1^{(1)-}(u)_{a\beta}L_\ell^{(2)-}(u)_{\beta c} = \eta^2(-1)^{|\beta|}J_1^{(1)a}{}_\beta J_\ell^{(2)\beta}{}_c \,,$$
$$(7.3.45)$$

where we used $(-1)^{|c|} = 1$ since c is a bosonic index. Apply both sides now to the singlet projector $\langle \mathbf{1}_{12}|$ and use the crossing relation for the generator $\langle \mathbf{1}_{12}|J_1^{(1)a}{}_\beta = -\langle \mathbf{1}_{12}|J_\ell^{(2)a}{}_\beta$, which is valid since a $\neq \beta$. Then using the identity (7.3.31) to the RHS, (7.3.45) becomes $\langle \mathbf{1}_{12}|L_1^{(1)-}(u)_{a\beta}L_\ell^{(2)-}(u)_{\beta c} = -\eta^2\langle \mathbf{1}_{12}|J_\ell^{(2)a}{}_c$ and the equation (7.3.43) can be simplified to

$$\langle \mathbf{1}_{12}|\Omega^{(1)-}{}_\ell(u)_{aB}\Omega^{(2)-}{}_\ell(u)_{Bc} = F_{\ell-1}^-\langle \mathbf{1}_{12}| \left(L_1^{(1)-}(u)_{ab}L_\ell^{(2)-}(u)_{bc} - \eta^2 J_\ell^{(2)a}{}_c\right) \,.$$
$$(7.3.46)$$

Note that on the RHS all the indices have turned into SL(2) indices.

Now we make a slight trick to split the last term of the RHS into identical halves[28] as $J_\ell^{(2)a}{}_c = \frac{1}{2}J_\ell^{(2)a}{}_c + \frac{1}{2}J_\ell^{(2)a}{}_c$ and then hit just one half to $\langle \mathbf{1}_{12}|$ to change it into $J_\ell^{(1)a}{}_c$ acting on the spin chain 1. Since the generator here is that of $\mathfrak{u}(2,2|4)$, in doing so there appears an extra constant term $\propto \delta^a_c$ coming from the commutator of the oscillators forming this generator. Then we get

$$\langle \mathbf{1}_{12}|J_\ell^{(2)a}{}_c = \frac{1}{2}\langle \mathbf{1}_{12}|J_\ell^{(2)a}{}_c - \frac{1}{2}\langle \mathbf{1}_{12}| \left(J_1^{(1)a}{}_c + \delta^a_c\right)$$
$$(7.3.47)$$

We shall now show that the terms linear in the generators appearing on the RHS can be absorbed by a judicious shifts of the spectral parameters of the expression $L_1^{(1)-}(u)_{ab}L_\ell^{(2)-}(u)_{bc}$, which is the first term on the RHS of (7.3.44). In fact, one can easily check that (7.3.44) can be re-expressed as

$$\langle \mathbf{1}_{12}|L_1^-(u + \eta/2)_{ab}L_\ell^-(u - \eta/2)_{bc} + f_1^-(\eta)\delta_{ac}\langle \mathbf{1}_{12}| \,,$$
$$(7.3.48)$$

where the factor $f_1(\eta)$ is given by $f_1(\eta) = -\eta^2/4$. Thus combining with (7.3.43), the LHS of the original $\mathfrak{psu}(2,2|4)$ monodromy relation (7.3.39), namely $\langle \mathbf{1}_{12}|\Omega_\ell^{(1)-}(u)_{aB}\Omega_\ell^{(2)-}(u)_{Bc}$, becomes

$$\langle \mathbf{1}_{12}| \left(F_{\ell-1}^-\delta_{bd}L_1^-(u + \eta/2)_{ab}L_\ell^-(u - \eta/2)_{dc} + f_1(u)F_{\ell-1}^-\delta_{ac}\right) \,.$$
$$(7.3.49)$$

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[28]If one wishes, one can actually use a more general split with coefficients α and β satisfying $\alpha + \beta = 1$ and follow the same logic to be described below for the $\frac{1}{2} + \frac{1}{2}$ split. This will lead to more general forms of the SL(2) monodromy relations. Below we shall only describe the simplest split for the sake of clarity.

As the final step, we now rewrite the quantity $\langle \mathbf{1}_{12}|F^-_{\ell-1}\delta_{\mathrm{bd}}$ in the first term by using the fundamental monodromy relation (7.3.39) with ℓ replaced by $\ell - 1$. This process is a reversal of the splitting procedure we started out with and inserts the products of the monodromy matrices defined on the RHS of the splitting equations (7.3.40) and (7.3.41). Then, the equation above becomes

$$\langle \mathbf{1}_{12}|L^-_1(u + \eta/2)_{ab}(\Omega^{(1)-}_{\ell-1})_{\mathrm{b}C}(\Omega^{(2)-}_{\ell-1})_{C\mathrm{d}}L^-_\ell(u - \eta/2)_{\mathrm{dc}}$$
$$+ \langle \mathbf{1}_{12}|f_1(u)F^-_{\ell-1}\delta_{\mathrm{ac}}. \tag{7.3.50}$$

On the other hand, each Lax operator on the RHS of the relation (7.3.43) can be interpreted as a special monodromy matrix of length one and hence we can use the monodromy relation to write the RHS as

$$F^-_{\ell-1}\langle \mathbf{1}_{12}|L^{(1)-}_1(u)_{\mathrm{a}B}L^{(2)-}_\ell(u)_{B\mathrm{c}} = F^-_{\ell-1}F^-_1\delta_{\mathrm{ac}}\langle \mathbf{1}_{12}| \tag{7.3.51}$$

Thus equating (7.3.50) and (7.3.51) and rearranging, we obtain an important relation

$$\langle \mathbf{1}_{12}|L^-_1(u + \eta/2)_{\mathrm{ab}}(\Omega^{(1)-}_{\ell-1})_{\mathrm{b}C}(\Omega^{(2)-}_{\ell-1})_{C\mathrm{d}}L^-_\ell(u - \eta/2)_{\mathrm{dc}} = g_1(u)F^-_{\ell-1}\langle\delta|_{\mathrm{ac}}\mathbf{1}_{12}, \tag{7.3.52}$$

where the function $g_1(u)$ is given by

$$g_1(u) = F^-_1(u) - f^-_1(u) = u^2. \tag{7.3.53}$$

The point to be noted here is that the part containing the unrestricted indices in the above relation is $(\Omega^{(1)-}_{\ell-1})_{\mathrm{b}C}(\Omega^{(2)-}_{\ell-1})_{C\mathrm{d}}$, namely the monodromy matrices of length $\ell - 1$, shorter by one unit from the original ℓ.

It should now be clear that we can perform this reduction process repeatedly until all the indices become those of SL(2) only. Then, taking η to be $-i$ and identifying the SL(2) Lax operator as

$$L_{\mathrm{SL}(2)}(u)_{\mathrm{ab}} \equiv L^-(u)_{\mathrm{ab}}, \tag{7.3.54}$$

upon acting on the state $|\psi_1\rangle \otimes |\psi_2\rangle$ we obtain the monodromy relation for the genuine SL(2) monodromy matrices inserted as

$$\langle(\Omega_{\mathrm{SL}(2)}(u - i/2))_{\mathrm{ab}}|\psi_1\rangle, (\Omega_{\mathrm{SL}(2)}(u + i/2))_{\mathrm{bc}}|\psi_2\rangle\rangle = u^{2\ell}\delta_{\mathrm{ac}}\langle|\psi_1\rangle, |\psi_2\rangle\rangle. \tag{7.3.55}$$

This completes the direct derivation of the SL(2) monodromy relation from that of $\mathfrak{psu}(2, 2|4)$ relations.

7.4 Summary of this chapter

In this chaper, we studied the tree-level three-point functions in the entire $\mathfrak{psu}(2,2|4)$ sector of $\mathcal{N}=4$ super Yang-Mills theory from a group theoretic and integrability-based point of view. We in particular developed the manifestly conformally invariant construction of the singlet-projection operator and used it to express the Wick contraction. Unlike the preceding works [114, 137], our construction doesn't necessitate the "U-operator" which intertwines two schemes of representations of the superconformal algebra. This property greatly simplifies the analysis and allowed us to derive the monodromy relation for the harmonic R-matrix, as well as for the usual fundamental R-matrix.

It would also be interesting to study the loop correction in our formulation. For this purpose, a more detailed analysis of the harmonic R-matrix may be useful since the harmonic R-matrix is intimately related to the local conserved charges including the one-loop Hamiltonian. Another avenue of research is to explore the relation with the scattering amplitudes [182–189]. Also in that case, the harmonic R-matrix and the monodromy relation played an important role. It would be interesting if one could make a more direct connection.

Chapter 8

Cognate structure for semi-classical three-point functions

As explained in the section 4.3, a non-perturbative framework capable of studying the three-point functions at finite coupling was put forward in [153]. The basic idea of this approach is to decompose the three-point functions into more fundamental building blocks called the hexagon form factors and determine them using assumed all-loop integrability[1]. Although quite powerful, as this method refers only to the magnon and its mirror excitations without referring to their specific origins, it is difficult to see how the gauge theory and the string theory are related. In this sense, our present work connecting the weak and the strong coupling representations based on the known integrability properties should be considered as complementary to such a universal approach.

Now concerning both the weak-coupling and the hexagon form factor methods, the three-point functions are expressed in terms of the sums over partitions of rapidities[2], which become increasingly more complicated as the number of magnons increases. However, it turned out that, in the semiclassical limit, where both the number of magnons and the length of the spin chain become very large, the result at weak coupling can be written in a surprisingly concise form, namely a simple integral on the spectral curve, whose integrand is expressed solely in terms of the so-called pseudo-momenta [117, 124, 125]. Now it is important to recall at this point that also at strong coupling in the semiclassical approximation the form of the three-point functions exhibits the same simple structure. A natural question then is whether there is an underlying physical mechanism by which one can produce such a simple expression more directly.

In the case of the two-point functions, a similar question was addressed in [78, 79]. In

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[1]An attempt in a similar spirit using the assumed all-loop integrability to determine the string field theory vertex was made in [151].

[2] One can sometimes further simplify the expression into a determinant form [123]. However, such an expression is known at the moment only for certain rank 1 sectors at weak coupling.

the semiclassical limit, the collective dynamics of magnons is described by the so-called Landau-Lifshitz model, a classically integrable nonrelativistic sigma model which can be obtained as a continuum limit of the Heisenberg spin chain. This formulation allows one to compute the semi-classical two-point function directly using the classical integrability and moreover makes it possible to describe the weak and strong coupling computations in a similar manner.

So the main purpose of the present work is to develop a formulation for the computation of the three-point functions at weak coupling, which in the semiclassical limit produces in a direct way the compact integral expressions similar to those in the strong coupling and to understand its basic mechanism. This will not only be quite useful from the point of view of the computation of the semiclassical limit, which is often physically most interesting, but the understanding of its mechanism would also reveal an aspect of integrability common to apparently disparate regimes. We will indeed see that a formulation extremely similar to that of the strong coupling analysis performed in [111] is possible and it will not only reproduce existing results in the literature but also make predictions for a class of three-point functions whose semi-classical limit have not yet been computed.

Let us now describe the idea and the structure of our formulation more explicitly. The basic starting point is the result of our previous paper [136] where the tree-level three-point function in the SU(2) sector can be expressed as the overlap between the singlet state and the three spin-chain states. By preparing a coherent state basis, we can then express such an overlap as a product of integrals over the coherent state variables. Now for the semiclassical situation of our interest, each spin chain reduces to a Landau-Lifshitz string and, more importantly, the overlap can be evaluated by the saddle point method. The situation is quite similar to the one at strong coupling, and just as in that case the determination of the saddle point configuration is quite difficult. However, the similarity to the strong coupling case goes further in the semiclassical situation. We also have the monodromy relation identical in form, derived in [136, 137] for the weak coupling, which was one of the crucial ingredients in the strong coupling case in determining the three-point function without the knowledge of the saddle point configuration. This relation is natural and powerful as it is a direct consequence of the classical integrability of the string sigma model and encodes infinitely many conservation laws.

Now, with such a monodromy relation at hand, most of the crucial ingredients for the strong coupling computation can be transplanted, with some modifications, to the present weak coupling case. More precisely, what this means is the following:

- The semi-classical three-point functions can again be expressed in terms of the "Wronskians" between the eigenvectors of the monodromy matrices.

- The monodromy relations, which are identical in form to the strong coupling case, determine the product of the Wronskians in terms of the quasimomenta.

- The individual Wronskian can be projected out by solving the Riemann-Hilbert problem using the analyticity property concerning the positions of the zeros and the poles.

It should be noted, however, that there is an important difference from the strong coupling case, concerning the determination of the analyticity property of the Wronskians. For the strong coupling case, the analyticity was determined by assuming the smoothness of the worldsheet for the saddle-point configuration connecting the three strings. In the present case, however, the three spin chains are glued together directly by the singlet projector, which is nothing but a convenient way of performing the Wick contractions dictated by the super Yang-Mills dynamics. There is no concept of worldsheet and hence the smoothness argument above does not apply.

Therefore, in this chapter we developed a new different argument, which is more powerful and universal. The basic idea is to study the response of certain fundamental quantities to an addition of a small number of Bethe roots. In the semiclassical context, such an addition corresponds to the continuous variation of the filling fraction of the Bethe roots and when applied to the (log of) the structure constant $\ln C_{123}$, it reveals that $\ln C_{123}$ plays the role of the generating function of the angle variables and provides the key equation for obtaining C_{123}. On the other hand, as it will be elaborated fully in section 8.3, we can also apply such a variation to the norm of the exact spin-chain eigenstate. When the original and the deformed states are both on-shell Bethe eigenstates, they must be orthogonal and we demand that this exact quantum property must be smoothly connected to the semiclassical structure for consistency. This requirement will turn out to be powerful enough to determine the configuration of the zeros and the poles on the spectral curve. The Wronskians determined through this logic not only leads to the known semiclassical results for the three-point functions in the literature but also allow us to compute more general SU(2) correlators, which have not been computed before.

It is then extremely interesting to apply this new orthogonality argument to the strong coupling case and see what happens. It turned out that this more universal argument lead to the modification of the integration contours obtained in the previous investigation, and the results with the modified contours are consistent with the hexagon form factor approach of [153] and exactly match the Frolov-Tseytlin limit [73] in the weak coupling regime. This indicates that, as already suspected and discussed in [111], the apparently natural requirement of smoothness of the saddle-point worldsheet configuration in the

strong coupling case is not quite correct and our new logic for determining the analyticity in the semiclassical spectral curve is more reliable.

The rest of the chapter is structured as follows. In section 8.1, after reviewing the formulation of the tree-level structure constant in terms of the overlap with the singlet projector, we derive a path-integral representation for such an overlap using the coherent state basis, which is subsequently evaluated by its saddle point in the semi-classical limit. We then show that the variation of the semi-classical structure constant with respect to a conserved charge of the spin chain states produces the angle variable which is canonically conjugate to that charge. In section 8.2, we construct the angle variables for the Landau-Lifshitz model using its classical integrability. Based on the results in the previous sections, we express, in section 8.3, the semi-classical structure constant in terms of the Wronskians of the eigenvectors of the monodromy matrices. In section 8.4, we evaluate such Wronskians, making use of the monodromy relation and the orthogonality of two on-shell states. Putting together all the results in the preceding sections, we finally derive the explicit expression for semi-classical structure constants at weak coupling in section 8.5. In section 8.6, we describe how the argument developed in the present paper applied to the strong coupling computation modifies the results obtained previously. We give a summary and conclude this chapter in section 8.7.

8.1 Semi-classical structure constant and the monodromy relation

8.1.1 Wick contraction represented as the singlet projection

We begin with a brief review of the two devices introduced in our previous work [138], namely the double spin-chain representation for the SU(2) sector and the interpretation of the Wick contraction as the group singlet projection, which greatly facilitate the construction of the correlation functions.

The four scalar fields ϕ_i $(i = 1, 2, 3, 4)$ forming the so-called the SU(2) sector of the super Yang-Mills theory can be assembled into a 2×2 matrix $\Phi_{\tilde{a}a}$ given by

$$\Phi_{\tilde{a}a} \equiv \left(\begin{array}{cc} Z & Y \\ -\bar{Y} & \bar{Z} \end{array} \right)_{\tilde{a}a}, \tag{8.1.1}$$

where $Z \equiv \phi_1 + i\phi_2$, $Y \equiv \phi_3 + i\phi_4$ and $\bar{Z}$ and $\bar{Y}$ are their hermitian conjugates respectively. Evidently, the symmetry of the SU(2) sector is actually SO(4)=SU(2)$_L \times$ SU(2)$_R$ and the matrix Φ transforms under these two SU(2) factors as $\Phi \rightarrow U_L \Phi U_R$, where U_L (U_R) belongs to SU(2)$_L$ (SU(2)$_R$). This suggests that it is natural to consider the spin-chain

consisting of these basic fields as forming a tensor product of two spin-chains, which we called the double spin-chain. Consider first the individual spin states $|\uparrow\rangle$ and $|\downarrow\rangle$ and denote them by $|\uparrow\rangle = |1\rangle$ and $|\downarrow\rangle = |2\rangle$ for convenience. Then, from the transformation property above, the basic fields correspond to the tensor product states as $Z \mapsto |\uparrow\rangle_L \otimes |\uparrow\rangle_R = |1\rangle \otimes |1\rangle$, etc. It is easy to see that this mapping is succinctly summarized as

$$\Phi_{\tilde{a}a} \mapsto |\tilde{a}\rangle \otimes |a\rangle\,, \quad \tilde{a}, a = 1, 2\,. \tag{8.1.2}$$

To construct the correlation functions at the tree level, we need to Wick contract these fields. For the Wick contraction of $\Phi_{\tilde{a}a}\,\Phi_{\tilde{b}b}$, the only non-vanishing ones are (suppressing the coordinate dependence) $Z\,\bar{Z} = 1$ and $X\,\bar{X} = 1$. This gives the simple formula

$$\Phi_{\tilde{a}a}\,\Phi_{\tilde{b}b} = \epsilon_{\tilde{a}\tilde{b}}\epsilon_{ab}\,. \tag{8.1.3}$$

In terms of the corresponding spin states, this rule is equivalent to

$$|\tilde{a}\rangle\,|\tilde{b}\rangle = \epsilon_{\tilde{a}\tilde{b}}\,, \qquad |a\rangle\,|b\rangle = \epsilon_{ab}\,. \tag{8.1.4}$$

Now consider the general linear combination of states $F = |\tilde{f}\rangle \otimes |f\rangle$, with

$$|\tilde{f}\rangle = \tilde{f}^1|\uparrow\rangle + \tilde{f}^2|\downarrow\rangle = \tilde{f}^{\tilde{a}}|\tilde{a}\rangle\,, \tag{8.1.5}$$

$$|f\rangle = f^1|\uparrow\rangle + f^2|\downarrow\rangle = f^a|a\rangle\,. \tag{8.1.6}$$

From the above rules, the Wick contraction between F_1 and F_2, where $F_i = |\tilde{f}_i\rangle \otimes |f_i\rangle$, can be easily computed as $F_1\,F_2 = (\tilde{f}_1^{\tilde{a}}\epsilon_{\tilde{a}\tilde{b}}\tilde{f}_2^{\tilde{b}})(f_1^a\epsilon_{ab}f_2^b)$. This form shows that one can perform the Wick contraction by taking the inner product with the singlet projection operator

$$\langle\mathbf{1}| = \epsilon_{ab}\langle a| \otimes \langle b|, \qquad \text{for both SU(2)}_L \text{ and SU(2)}_R\,, \tag{8.1.7}$$

namely

$$F_1\,F_2 = \langle\mathbf{1}|\Big(|\tilde{f}_1\rangle \otimes |\tilde{f}_2\rangle\Big)\langle\mathbf{1}|\Big(|f_1\rangle \otimes |f_2\rangle\Big)\,. \tag{8.1.8}$$

This representation allows us to perform the Wick contractions for any complicated operators easily and systematically[3].

Now let us apply this scheme to the single-trace operators. The contractions which survive in the large N limit are the ones which connect the $(L + 1 - i)$-th field in the operator $\mathcal{O}_1$ with the i-th field in the operator $\mathcal{O}_2$, where L is the length common to both operators. Explicitly, an example of this structure looks like

$$\mathcal{O}_1 : \mathrm{tr}\big(\cdots XZ\big) \qquad \mathcal{O}_2 : \mathrm{tr}\big(\bar{Z}\bar{X}\cdots\big)\,. \tag{8.1.9}$$

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[3] For the full PSU(2, 2|4) sector, the singlet projection operator has been constructed in [137, 138].
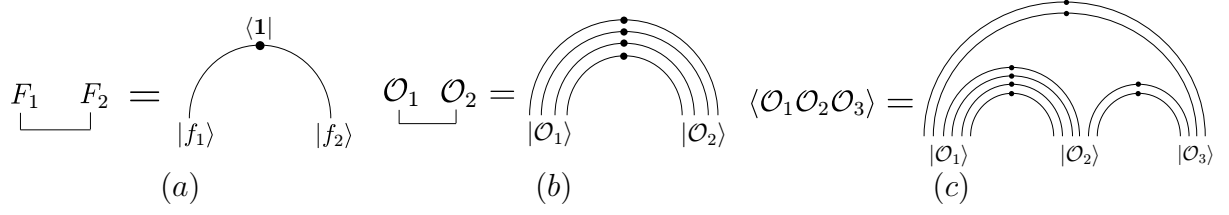
Figure 8.1.1: The Wick contractions represented as the overlap with the singlet state. Here we only depict the SU(2)$_R$ part. (a) The Wick contraction between two fields can be expressed as the overlap between two spin states and the singlet, which is denoted by the black dot. (b) The pictorial explanation of the formula for the two-point function (8.1.11). Here again each dot denotes the overlap with the singlet state. (c) The three-point function expressed as the overlap with the singlet state (8.1.12).

This structure motivates us to consider the following tensor product of singlet states,

$$\langle \mathbf{1}_{12}| = \prod_{i=1}^{L} \left(\epsilon_{ab} \langle a|_{L+1-i}^{(1)} \otimes \langle b|_i^{(2)} \right), \tag{8.1.10}$$

where $\langle *|_i^{(k)}$ denotes the single-spin state living on the i-the site of the spin chain corresponding to the operator $\mathcal{O}_k$. Then, the contractions between the operators can be reproduced by taking the inner product

$$\mathcal{O}_1 \, \mathcal{O}_2 = \langle \mathbf{1}_{12}| \left(|\tilde{\mathcal{O}}_1\rangle \otimes |\tilde{\mathcal{O}}_2\rangle \right) \langle \mathbf{1}_{12}| \left(|\mathcal{O}_1\rangle \otimes |\mathcal{O}_2\rangle \right). \tag{8.1.11}$$

Here $|\tilde{\mathcal{O}}_k\rangle \otimes |\mathcal{O}_k\rangle$ and $|\tilde{\mathcal{O}}_k\rangle \otimes |\mathcal{O}_k\rangle$ are the spin-chain states corresponding to the operators $\mathcal{O}_k$. For a pictorial explanation, see figure 8.1.1-(b).

Since the tree-level three-point function is essentially given by a product of Wick contractions, one can also map the computation of the three point function to that in the spin-chain Hilbert space:

$$\langle \mathcal{O}_1 \mathcal{O}_2 \mathcal{O}_3 \rangle = \langle \mathbf{1}_{123}| \left(|\tilde{\mathcal{O}}_1\rangle \otimes |\tilde{\mathcal{O}}_2\rangle \otimes |\tilde{\mathcal{O}}_3\rangle \right) \langle \mathbf{1}_{123}| \left(|\mathcal{O}_1\rangle \otimes |\mathcal{O}_2\rangle \otimes |\mathcal{O}_3\rangle \right). \tag{8.1.12}$$

As in the previous case, the structure of the singlet state $\langle \mathbf{1}_{123}|$ is determined by the structure of the Wick contraction, which is depicted in figure 8.1.1-(c). Explicitly, it is given by

$$\langle \mathbf{1}_{123}| = \left(\prod_{i=1}^{L_{12}} \epsilon_{ab} \langle a|_{L_1+1-i}^{(1)} \otimes \langle b|_i^{(2)} \right) \otimes \left(\prod_{i=1}^{L_{23}} \epsilon_{ab} \langle a|_{L_2+1-i}^{(2)} \otimes \langle b|_i^{(3)} \right) \otimes \left(\prod_{i=1}^{L_{31}} \epsilon_{ab} \langle a|_{L_3+1-i}^{(3)} \otimes \langle b|_i^{(1)} \right).$$

Here L_k is the length of the operator $\mathcal{O}_k$ and $L_{ij} = (L_i + L_j - L_k)/2$ is the number of Wick contractions connecting $\mathcal{O}_i$ and $\mathcal{O}_j$.
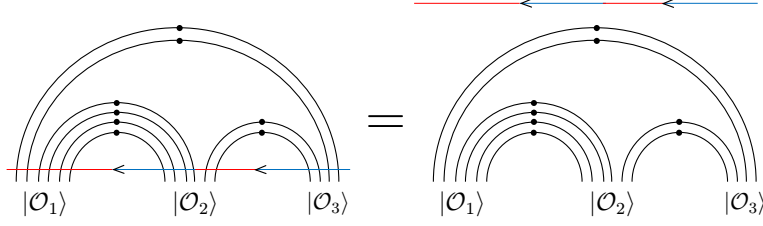
252

Figure 8.1.2: The monodromy relation given in (8.1.14). The three-point function with and without the monodromy matrix are equal up to the prefactor, which we omit writing here. The red lines denote parts of the monodromy matrix with a $-i/2$ shift of the spectral parameter whereas the blue lines denote parts of the monodromy matrix with a $+i/2$ shift of the spectral parameter.

Now taking into account the normalization factors correctly, we arrive at the following basic formula for the structure constant[4]:

$$C_{123} = \frac{\sqrt{L_1 L_2 L_3}}{N_c} \langle \mathbf{1}_{123} | \Big(|\tilde{\mathcal{O}}_1\rangle \otimes |\tilde{\mathcal{O}}_2\rangle \otimes |\tilde{\mathcal{O}}_3\rangle \Big) \langle \mathbf{1}_{123} | \Big(|\mathcal{O}_1\rangle \otimes |\mathcal{O}_2\rangle \otimes |\mathcal{O}_3\rangle \Big). \tag{8.1.13}$$

In the above, N_c denotes the rank of the gauge group.

An important consequence of this formalism is the so-called monodromy relation, which is an identity connecting the structure constant with and without the insertion of the monodromy matrix. It was derived in [136, 137] and, for the $SU(2)_R$ part, it reads[5]

$$\langle \mathbf{1}_{123} | \Big(\big(\Omega_1^-(u) \big)_{ij} |\mathcal{O}_1\rangle \otimes \big(\Omega_2^{+|-}(u) \big)_{jk} |\mathcal{O}_2\rangle \otimes \big(\Omega_3^+(u) \big)_{kl} |\mathcal{O}_3\rangle \Big)$$
$$= f_{123}(u) \delta_{il} \langle \mathbf{1}_{123} | \Big(|\mathcal{O}_1\rangle \otimes |\mathcal{O}_2\rangle \otimes |\mathcal{O}_3\rangle \Big). \tag{8.1.14}$$

Here $\Omega(u)$ is the monodromy matrix constructed from the Lax operator

$$\Omega(u) \equiv L_1(u) L_2(u) \cdots L_L(u),$$
$$L_k(u) \equiv \begin{pmatrix} 1 + i S_3^k/u & i S_-^k/u \\ i S_+^k/u & 1 - i S_3^k/u \end{pmatrix}, \tag{8.1.15}$$

and the superscripts $\Omega^{\pm,+|-}(u)$ indicates the shift of the argument by $\pm i/2$ (for a precise definition, see figure 8.1.2). The constant factor $f_{123}(u)$ is given by

$$f_{123}(u) = \left(1 + \frac{1}{u^2} \right)^{(L_1+L_2+L_3)/2} \tag{8.1.16}$$

The identity (8.1.14) encodes infinitely many conservation laws for the structure constant. As we will see in section 8.1.3, the semi-classical limit of (8.1.14) takes a form identical to the one at strong coupling and will play a key role in the subsequent analysis.

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[4]See [115] for the origin of the prefactor in (8.1.13).

[5]Here we adopt the normalization of $L_k(u)$ to be such that $L_k(\infty) = \mathbf{1}$, which is slightly different from the one used in [136, 137]. The monodromy matrix in the present normalization can be naturally identified with the monodromy matrix in the Landau-Lifshiz sigma model in the semi-classical limit.

8.1.2 On-shell Bethe states and polarization vectors

Before discussing the semi-classical limit, let us briefly explain how to characterize the general SU(2) state in the double spin-chain representation.

In the Bethe-ansatz approach, we construct the general eigenstates of the Hamiltonian by first considering the vacuum, which is typically taken to be $\mathrm{tr}\left(Z^L\right)$, and then introducing the magnons (X or $\bar{X}$) with a set of rapidities satisfying the Bethe ansatz equation. An important property of such states, to be called the on-shell Bethe states, is that they are the highest weight state [70] if the rapidities are all finite. In the double spin-chain representation, this translates to

$$\tilde{S}_+|\tilde{\mathcal{O}}\rangle = 0\,, \quad S_+|\mathcal{O}\rangle = 0\,, \tag{8.1.17}$$

where $\tilde{S}_+$ (S_+) is the raising operator for the total spin in $\mathrm{SU}(2)_L$ ($\mathrm{SU}(2)_R$).

In the case of three-point functions, we cannot take all the states to be the ones constructed upon $\mathrm{tr}\left(Z^L\right)$ since such three-point functions vanish owing to the charge conservation. To study nonvanishing three-point functions, we have to consider the states constructed upon more general vacua, which can be obtained from $\mathrm{tr}\left(Z^L\right)$ by the $\mathrm{SU}(2)_L \times \mathrm{SU}(2)_R$ rotations. As shown in [136], such vacua can be characterized in terms of the *polarization vectors*[6] n and $\tilde{n}$, in the following way[7]:

$$\mathrm{tr}\left((\tilde{n}^{\tilde{a}}n^a\Phi_{\tilde{a}a})^L\right) \qquad (\tilde{a}, a = 1, 2)\,. \tag{8.1.19}$$

The highest weight condition satisifed by the on-shell Bethe states constructed upon this rotated vacuum reads

$$\tilde{S}'_+|\tilde{\mathcal{O}}\rangle = 0\,, \quad S'_+|\mathcal{O}\rangle = 0\,, \tag{8.1.20}$$

where S'_+ and $\tilde{S}'_+$ are rotated generators given by

$$\begin{aligned}
\begin{pmatrix} \tilde{S}'_3 & \tilde{S}'_- \\ \tilde{S}'_+ & -\tilde{S}'_3 \end{pmatrix} &= \tilde{N}^{-1} \begin{pmatrix} \tilde{S}_3 & \tilde{S}_- \\ \tilde{S}_+ & -\tilde{S}_3 \end{pmatrix} \tilde{N}\,, \\
\begin{pmatrix} S'_3 & S'_- \\ S'_+ & -S'_3 \end{pmatrix} &= N^{-1} \begin{pmatrix} S_3 & S_- \\ S_+ & -S_3 \end{pmatrix} N\,,
\end{aligned} \tag{8.1.21}$$

with

$$N = \begin{pmatrix} n^1 & -n^2 \\ n^2 & n^1 \end{pmatrix}\,, \quad \tilde{N} = \begin{pmatrix} \tilde{n}^1 & -\tilde{n}^2 \\ \tilde{n}^2 & \tilde{n}^1 \end{pmatrix}\,. \tag{8.1.22}$$

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[6]In the previous paper [111], they were called polarization spinors.
[7]More explicitly, (8.1.19) reads

$$\mathrm{tr}\left((n_1\tilde{n}_1 Z + n_2\tilde{n}_1 Y - n_1\tilde{n}_2\bar{Y} + n_1\tilde{n}_2\bar{Z})^L\right)\,. \tag{8.1.18}$$

The highest weight condition (8.1.20) will play an important role when deriving the expression for the semi-classical structure constant in section 8.3.3.

8.1.3 Coherent-state representation and the semi-classical limit of C_{123}

We will now study the semi-classical limit of the expression (8.1.13) for C_{123}. Unlike the previous methods [115, 123, 124], where one first evaluates this quantity exactly and then take the semi-classical limit, we shall take the semi-classical limit at the outset by deriving a path integral representation of the structure constant and applying the saddle-point method. This scheme will be seen to be valuable as a novel computational method universally applicable for a large class of SU(2) three-point functions, including the cases which previously could not be treated easily. Actually the more important aspect of this method is that it reveals a cognate structure between the weak coupling computation under consideration and the strong coupling counterpart performed in [111], as we shall see.

The semi-classical limit of our interest is a sort of the continuum limit of the Heisenberg spin chain. More precisely, it is the following scaling limit,

$$L \to \infty, \quad M \to \infty, \quad Lp, L/M : \text{ fixed}. \tag{8.1.23}$$

Here L, M and p are, respectively, the length of the spin chain, the number of magnons and the momentum of each magnon. As such it is efficiently described by some continuous field along the chain, which should provide a representation of SU(2). The so-called coherent state representation is ideal for such a purpose. It is a representation realized on the coset space SU(2)/U(1), which is isomorphic to a unit sphere S^2. As briefly reviewed in Appedix A, a coherent state representation for a single spin 1/2 state can be taken to be

$$|\mathbf{n}\rangle = \exp\left(i\theta \frac{\mathbf{n}_0 \times \mathbf{n}}{|\mathbf{n}_0 \times \mathbf{n}|} \cdot \vec{S}\right) |\uparrow\rangle = \cos\frac{\theta}{2}|\uparrow\rangle - e^{i\phi}\sin\frac{\theta}{2}|\downarrow\rangle, \tag{8.1.24}$$

where $\mathbf{n}_0 = (0, 0, 1)$ is a unit vector in the z direction and $\mathbf{n} = (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta)$ is a unit vector in a general direction. To express C_{123} in this basis, we just need to insert the completeness relation

$$1 = \int \mathcal{D}\mathbf{n} \, |\mathbf{n}\rangle\langle\mathbf{n}| \qquad \left(\mathcal{D}\mathbf{n} \equiv d^3\mathbf{n}\,\delta(\mathbf{n}^2 - 1)\right), \tag{8.1.25}$$

to each inner product in (8.1.13). As a result, we obtain the following path-integral

expression:

$$C_{123} = \frac{\sqrt{L_1 L_2 L_3}}{N_c} \texttt{Left} \times \texttt{Right} \, ,$$

$$\texttt{Left} = \int \mathcal{D}\vec{\tilde{\mathfrak{n}}}_1 \mathcal{D}\vec{\tilde{\mathfrak{n}}}_2 \mathcal{D}\vec{\tilde{\mathfrak{n}}}_3 e^{-S[\vec{\tilde{\mathfrak{n}}}_1, \vec{\tilde{\mathfrak{n}}}_2, \vec{\tilde{\mathfrak{n}}}_3]} \tilde{\Psi}_1[\vec{\tilde{\mathfrak{n}}}_1] \tilde{\Psi}_2[\vec{\tilde{\mathfrak{n}}}_2] \tilde{\Psi}_3[\vec{\tilde{\mathfrak{n}}}_3] \, , \qquad (8.1.26)$$

$$\texttt{Right} = \int \mathcal{D}\vec{\mathfrak{n}}_1 \mathcal{D}\vec{\mathfrak{n}}_2 \mathcal{D}\vec{\mathfrak{n}}_3 e^{-S[\vec{\mathfrak{n}}_1, \vec{\mathfrak{n}}_2, \vec{\mathfrak{n}}_3]} \Psi_1[\vec{\mathfrak{n}}_1] \Psi_2[\vec{\mathfrak{n}}_2] \Psi_3[\vec{\mathfrak{n}}_3] \, .$$

Here $\vec{\tilde{\mathfrak{n}}}$ and $\vec{\mathfrak{n}}$ denote a chain of coherent states

$$|\vec{\mathfrak{n}}\rangle \equiv |\mathfrak{n}\rangle_1 \otimes |\mathfrak{n}\rangle_2 \otimes \cdots \otimes |\mathfrak{n}\rangle_L \, . \qquad (8.1.27)$$

$e^{-S[\vec{\mathfrak{n}}, \vec{\mathfrak{m}}, \vec{\mathfrak{l}}]}$ is the overlap between the singlet and the coherent states

$$e^{-S[\vec{\mathfrak{n}}, \vec{\mathfrak{m}}, \vec{\mathfrak{l}}]} \equiv \langle \mathbf{1}_{123}| \left(|\vec{\mathfrak{n}}\rangle \otimes |\vec{\mathfrak{m}}\rangle \otimes |\vec{\mathfrak{l}}\rangle \right) \, , \qquad (8.1.28)$$

while the wave functions $\tilde{\Psi}$ and Ψ are defined by

$$\tilde{\Psi}_k[\vec{\tilde{\mathfrak{n}}}_k] = \langle \vec{\tilde{\mathfrak{n}}}_k | \tilde{\mathcal{O}}_k \rangle \, , \qquad \Psi_k[\vec{\mathfrak{n}}_k] = \langle \vec{\mathfrak{n}}_k | \mathcal{O}_k \rangle \, . \qquad (8.1.29)$$

Now in the semi-classical limit, this expression can be well-approximated by the saddle-point of the integrand, which gives

$$C_{123} = \frac{\sqrt{L_1 L_2 L_3}}{N_c} \left(e^{-S[\vec{\tilde{\mathfrak{n}}}_1^*, \vec{\tilde{\mathfrak{n}}}_2^*, \vec{\tilde{\mathfrak{n}}}_3^*]} \tilde{\Psi}_1[\vec{\tilde{\mathfrak{n}}}_1^*] \tilde{\Psi}_2[\vec{\tilde{\mathfrak{n}}}_2^*] \tilde{\Psi}_3[\vec{\tilde{\mathfrak{n}}}_3^*] \right) \left(e^{-S[\vec{\mathfrak{n}}_1^*, \vec{\mathfrak{n}}_2^*, \vec{\mathfrak{n}}_3^*]} \Psi_1[\vec{\mathfrak{n}}_1^*] \Psi_2[\vec{\mathfrak{n}}_2^*] \Psi_3[\vec{\mathfrak{n}}_3^*] \right) \, ,$$
$$(8.1.30)$$

where $\vec{\mathfrak{n}}_k^*$ ($\vec{\tilde{\mathfrak{n}}}_k^*$) represents the saddle point of the $\mathcal{D}\vec{\mathfrak{n}}_k^*$ ($\mathcal{D}\vec{\tilde{\mathfrak{n}}}_k^*$) integral. Evidently, the result (8.1.30) factorizes into the $\mathrm{SU}(2)_L$ part and the $\mathrm{SU}(2)_R$ part. In the discussions in the following sections, we mainly focus on the $\mathrm{SU}(2)_R$ part since the computation in the $\mathrm{SU}(2)_L$ part is similar.

Let us now study the semi-classical limit of the monodromy relation. Since the monodromy matrix is an $O(1)$ quantity, the insertion of the monodromy matrix does not affect the saddle point. Thus, in the semi-classical limit, we can replace the monodromy matrix, which is originally the quantum operator acting on the spin chain, with the classical value evaluated on the saddle point given in (8.1.30). Furthermore, since we scale the spectral parameter as $u \sim L$ in the semi-classical limit, the shifts of the arguments in $\Omega^{\pm}$ etc. become negligible and the factor f_{123} can be approximated by unity. Therefore we arrive at the relation

$$\Omega_1(u)\Omega_2(u)\Omega_3(u)|_{\text{saddle}} = \mathbf{1} \, . \qquad (8.1.31)$$

Importantly, (8.1.31) has exactly the same form as the monodromy relation in the string sigma model. This allows us to transplant most of the crucial ingredients for the strong coupling computation as we shall see in the next section.

8.1.4 $\ln C_{123}$ as the "generating function" of the angle variable

Once we choose the operators $\mathcal{O}_i$ of definite conformal dimensions for which to compute the three-point function, each part of the expression in (8.1.30) can be explicitly computed in principle with the judicious use of the integrability. This is indeed the approach taken in the previous study at strong coupling [111]. However, in this brute-force method, we shall encoutner extremely complicated intermediate expressions, most of which should cancel in the final result. Therefore, below we shall devise an entirely different method, which at the same time reveals the important meaning of $\ln C_{123}$ as a whole. This approach also enables us to study the semi-classical states with arbitrary number of cuts in the spectral curve, unlike the method in [111], which was restricted to the so-called one-cut solutions.

The basic idea is to see how $\ln C_{123}$ changes as we introduce a small number of additional Bethe roots. In the semi-classical context, this means to study the variation of the structure constant with respect to the variation of the filling fraction[8] $S_i^{(m)}$ given by

$$\frac{\partial \ln C_{123}}{\partial S_i^{(m)}} \, , \tag{8.1.32}$$

where the subscript i labels the filling fraction for the different cut belonging to the same operator, while the superscript (m) labels the three different operators. By "integrating" this quantity, one can determine the ratio between the structure constant involving non-BPS operators and the one for three BPS operators, for which all the filling fractions vanish.

Specifically, the change of the filling fraction produces the following two effects: (i) A slight change of the saddle point configuration $\vec{n}^*$ and (ii) the direct small change of the wave functions $\Psi[S_i, \vec{n}^*]$ due to δS_i. Actually, the contribution from (i) takes the form,

$$\frac{\partial \vec{n}_m^*}{\partial S_i^{(m)}} \frac{\delta \ln C_{123}|_{\mathrm{saddle}}}{\delta \vec{n}_m^*} \, , \tag{8.1.33}$$

and hence it vanishes owing to the saddle-point equation $\delta \, C_{123}|_{\mathrm{saddle}} / \delta \vec{n}_m^* = 0$.

Now from the general theory, the wave function in the semi-classical limit is given by the following WKB form

$$\ln \Psi \sim i \sum_k \int P_k dQ_k \, , \tag{8.1.34}$$

where in the present case Q_k's correspond to the coherent-state variables, $\vec{n}$, and P_k's to their canonical conjugates. The right hand side of (8.1.34) can be regarded as the

---

[8]The precise definition of the filling fraction will be given in section 8.2.2.

generating function of the canonical transformation. Therefore, by differentiating with respect to the filling fraction, which is known to be the conserved action variables, we obtain

$$\frac{\partial}{\partial S_i^{(n)}} \ln \Psi = i \frac{\partial}{\partial S_i^{(n)}} \sum_k \int P_k dQ_k = i\phi_i^{(n)} , \qquad (8.1.35)$$

where $\phi_i^{(n)}$ are the angle variables conjugate to $S_i^{(n)}$. Putting altogether, we find that $\ln C_{123}$ plays the role of the "generating function" giving the angle variable under the variation of the filling fraction and we get the simple formula

$$\frac{\partial \ln C_{123}}{\partial S_i^{(n)}} = i\phi_i^{(n)} . \qquad (8.1.36)$$

Concerning this formula, two comments are in order. First, as we have already indicated by the use of quotation marks, the quantity $\ln C_{123}$ is not the generating function of the action variables in the usual sense. The precise meaning is that at the saddle point it behavs as if it were a generating function of the value of the angle variable under the variation of $S_i^{(m)}$.

The second comment concerns the normalization of the structure constant C_{123} or rather the normalization of the operators making the three-point function. As it will be discussed in the next section, in the general integral expressing the angle variable ϕ_i in (8.2.21), we will not specify the initial point of integration. Therefore the expression (8.1.36) is actually ambiguous as it stands. To fix this ambiguity, we require that the operators we use produce the normalized two-point functions correctly. This can be achieved in practice by replacing the right hand side of (8.1.36) by the difference between the angle variable for the three-point function and the one for the two-point function in the following way:

$$\frac{\partial \ln C_{123}}{\partial S_i^{(n)}} = i \left(\phi_i^{(n)} - \phi_{i,2\text{pt}}^{(n)} \right) \equiv i\varphi_i^{(n)} . \qquad (8.1.37)$$

Unlike (8.1.36), the expression (8.1.37) is entirely unambiguous and we will adopt thhis form in the rest of this article.

8.2 Classical integrability of the Landau-Lifshitz model

We shall now apply the general formalism developed in the previous section more explicitly to the semi-classical limit of the Heisenberg spin chain. It is well-known that such a limit gives rise to so-called the Landau-Lifshitz model, a classically integrable field theory in $1 + 1$ dimensions.

8.2.1 Landau-Lifshitz model, its Lax pair and the monodromy matrix

Let us briefly summarize the basic properties of the Landau-Lifshitz model and its integrable nature. In the semi-classical limit, the coherent state variable $\vec{n}(m, \tau)$, where m is an integer specifying the position along the spin chain, becomes a continuous field $\vec{n}(\sigma, \tau)$. It is convenient to take the range of σ to be $0 \leq \sigma \leq L$, where L is the length of the spin chain. The action is given by[9]

$$S_{\mathrm{LL}} = \frac{1}{2} \int d\tau d\sigma \int_0^1 ds\, \vec{n} \cdot (\partial_\tau \vec{n} \times \partial_s \vec{n}) - \frac{g^2}{2} \int d\tau d\sigma\, \partial_\sigma \vec{n} \cdot \partial_\sigma \vec{n}, \qquad (8.2.1)$$

where $g = \sqrt{\lambda}/4\pi$ is the 't Hooft coupling constant. The first term in (8.2.1) is the Wess-Zumino term and the s-dependence of $\vec{n}$ is defined such that $\vec{n}(s = 1) = (0, 0, 1)$ and $\vec{n}(s = 0) = \vec{n}$. The equation of motion obtained by varying the above action reads

$$\partial_\tau \vec{n} = 2g^2 \vec{n} \times \partial_\sigma^2 \vec{n}. \qquad (8.2.2)$$

One of the important features of this model is its classically integrability, whose clearest manifestation is the existence of the following Lax pair structure

$$[\partial_\sigma - J_\sigma, \partial_\tau - J_\tau] = 0,$$
$$J_\sigma = \frac{i}{2u} \vec{n}\vec{\sigma} = \frac{i}{2u} \begin{pmatrix} \mathrm{n}_3 & \mathrm{n}_1 - i\mathrm{n}_2 \\ \mathrm{n}_1 + i\mathrm{n}_2 & -\mathrm{n}_3 \end{pmatrix}, \qquad J_\tau = \frac{2ig^2}{u^2} \vec{n}\vec{\sigma} + \frac{2ig^2}{u} (\vec{n} \times \partial_\sigma \vec{n})\, \vec{\sigma}, \qquad (8.2.3)$$

where u is the spectral parameter. From the above Lax pair, one can construct the monodromy matrix in the usual way[10]:

$$\Omega(u) \equiv \mathrm{P} \exp\left(\int_0^L d\sigma\, J_\sigma\right). \qquad (8.2.4)$$

As in the case of the integrable string sigma model, one defines the quasi-momentum $p(x)$ as the logarithm of the eigenvalue of the monodromy matrix:

$$\Omega(u) \sim \begin{pmatrix} e^{ip(u)} & 0 \\ 0 & e^{-ip(u)} \end{pmatrix}. \qquad (8.2.5)$$

The asymptotic properties of the quasi-momentum at $u = 0$ and $u = \infty$ can be easily obtained from the above definitions and contain useful information: Its residue at $u = 0$ is related to the length of the spin chain [79] as

$$p(u) = -\frac{L}{2u} + O(1), \qquad (8.2.6)$$

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[9]A review of the derivation is provided in Appendix H.

[10]The monodromy matrix defined here can be identified with the semi-classical limit of the monodromy matrix the Heisenberg spin chain (8.1.15).
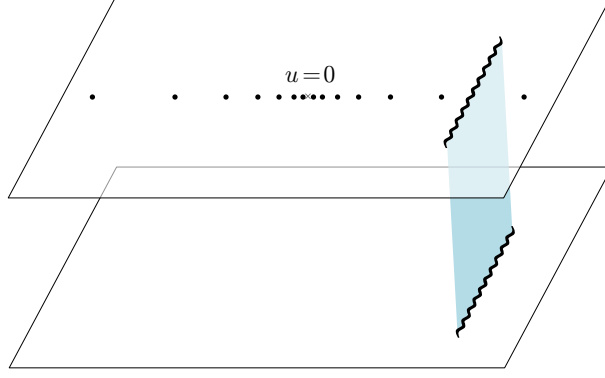
Figure 8.2.1: The structure of the spectral curve at weak coupling. In general, it has several branch cuts and infinitely many singular points, denoted by black dots, which accumulate to $u = 0$. The singular points can be regarded as degenerate branch points.

while the leading behavior at infinity provides the information of the number M of magnon excitations of the system:

$$p(u) = \frac{2M - L}{2u} + O(u^{-2}).$$

(8.2.7)

The spectral curve is defined from the monodromy matrix as

$$\det\,(y - \Omega(u)) = (y - e^{ip(u)})(y - e^{-ip(u)}) = 0.$$

(8.2.8)

Owing to the singular behavior of the quasi-momentum (8.2.6), the spectral curve contains an infinite number of points satisfying $e^{2ip(u^*)} = 1$. Such points are called the singular points and can be regarded as the infinitesimal branch cut [95,96] (see also figure 8.2.1).

Now, it is well-known that the information contained in the non-linear Lax equation can be recovered by the simultaneous solution of the auxiliary linear problem given by

$$(\partial_\sigma - J_\sigma)\psi = 0, \qquad (\partial_\tau - J_\tau)\psi = 0.$$

(8.2.9)

In particular, for the i-th spin chain, the solutions which are at the same time the eigenfunctions of the monodromy matrix Ω_i with eigenvalues $e^{\pm ip_i(u)}$ will be denoted by $i_\pm$, i.e.

$$\Omega_i i_\pm = e^{\pm ip_i} i_\pm,$$

(8.2.10)

and they will be of great importance in what follows.

8.2.2 Action-angle variables

As was already indicated in section 8.1.4, the concept of action-angle variables plays an essential role in the computation of the structure constant. Therefore in this subsection,

with the use of the method of Sklyanin [93] we shall construct the action-angle variables for the Landau-Lifshitz model.

First, we must compute the Poisson bracket between the elements of the monodromy matrix, which is characterized by the classical r-matrix in the form

$$\{\Omega(u) \overset{\otimes}{,} \Omega(v)\} = [\Omega(u) \otimes \Omega(v) , \mathbf{r}(u - v)] \,. \tag{8.2.11}$$

In the case of the Landau-Lifshitz sigma model, since the quantum R-matrix $R(x)$ for the XXX spin chain is well-known, a quick way[11] to obtain the classical r-matrix is to take the classical limit of $R(x)$. Explicitly, we obtain

$$
\begin{aligned}
R(u) &= \mathbb{I} + i\frac{\mathbb{P}}{u} \longmapsto \mathbb{I} + i\mathbf{r}(u) \,, \\
\Rightarrow \quad \mathbf{r}(u) &= \frac{\mathbb{P}}{u} \,,
\end{aligned}
\tag{8.2.12}
$$

where $\mathbb{I}$ is the identity operator and $\mathbb{P}$ is the permutation operator. Using the form (8.2.12) in (8.2.11), one can obtain the explicit form of the Poisson bracket between the individual components of the monodromy matrix, which are denoted as usual in the form

$$\Omega(u) \equiv \begin{pmatrix} \mathcal{A}(u) & \mathcal{B}(u) \\ \mathcal{C}(u) & \mathcal{D}(u) \end{pmatrix} \,. \tag{8.2.13}$$

The resulting Poisson brackets obtained in this way are displayed in Appedix C.

We now describe the Sklyanin's approach [93] for the construction of the action-angle variables. Consider the eigenfunctions $\psi_\pm(u)$ of the monodromy matrix $\Omega(u)$, with eigenvalues $e^{\pm ip(u)}$, *i.e.* defined by

$$\Omega(u)\psi_\pm(u;\tau) = e^{\pm ip(u)}\psi_\pm(u;\tau) \,. \tag{8.2.14}$$

Such eigenstates are called Baker-Akhiezer vectors. Now an important information is encoded in the *normalized* Baker-Akhiezer vector $h(u;\tau)$ defined to be proportional to $\psi_+(u;\tau)$ and satisfying the normalization condition

$$\langle n \,, h \rangle \equiv \epsilon_{ab} n^a h^b = 1 \,, \qquad h = \frac{1}{\langle n \,, \psi_+ \rangle}\psi_+ \,, \tag{8.2.15}$$

Here $n = (n^1, n^2)^t$ is a constant vector with unit norm. In the original formalism by Sklyanin, n can be arbitrary as long as it is independent of the spectral parameter. However, in the present context, we must choose it to be equal to the polarization vector diuscussed in section 2.2 in order to guarantee the highest weight property of the semi-classical wave function (see Appendix I for a detailed explanation).

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[11] For the first-principle derivation of the classical r-matrix, see Appendix H.3.

For general solutions, there are infinitely many poles in $h(u, \tau)$, the position of which are denoted by γ_i, $i = 1, 2, \ldots$. Sklyanin observed that to each such pole γ_i, which becomes a dynamical variable through the relations (8.2.14) and (8.2.15), a canonical pair of variables are associated. Relegating the details of the derivation to Appendix H.4, the result is the following set of commutation relations

$$\{\gamma_i, \gamma_j\} = \{p(\gamma_i), p(\gamma_j)\} = 0, \quad -i\{\gamma_i, p(\gamma_j)\} = \delta_{ij}. \tag{8.2.16}$$

where $p(\gamma_i)$ is the quasi-momentum $p(u)$ with the substitution $u = \gamma_i$. This shows that $(\gamma_i, -ip(\gamma_i))$'s are canonical pairs of variables.

Once the canonical pairs are obtained, one can easily construct the action variables, which should be identified with the conserved filling fractions S_i, as

$$S_i \equiv \frac{1}{2\pi i} \oint_{\mathcal{C}_i} p(u) du. \tag{8.2.17}$$

Here $\mathcal{C}_i$ denotes the i-th branch cut.

Now to construct the angle variables ϕ_i conjugate to S_i, we need to to find the generating function of the type $F(\gamma_i, S_i)$, which effects the canonical transformation from $(\gamma_i, -ip(\gamma_i))$ to the action-angle variables. Such a function is defined by the properties

$$\frac{\partial F}{\partial \gamma_i} = -ip(\gamma_i), \qquad \frac{\partial F}{\partial S_i} = \phi_i. \tag{8.2.18}$$

In the present context, the first equation should be viewed as defining the function F, while the second equation should be regarded as the definition of ϕ_i. Therefore, to determine F, we need to integrate the first equation with S_i fixed. As the filling fractions are given by the integral of $p(u)$ on the spectral curve, fixing all S_i's is equivalent to fixing the functional form of $p(u)$. Therefore, F can be determined as

$$F = -i \sum_i \int^{\gamma_i} p(u) du. \tag{8.2.19}$$

Next we compute $\phi_i = \partial F / \partial S_i$. This requires changing S_i with all the other filling fractions fixed. In the Heisenberg spin chain we started with, this corresponds to adding a small number of Bethe roots to the branch cut $\mathcal{C}_i$. As is clear from (8.2.5), this addition of magnon inevitably changes the asymptotic behavior of $p(u)$ at $u = \infty$. Therefore, changing S_i is precisely equivalent to adding to $p(u)du$ a one-form whose period integral is non-vanishing only for the cycle around $\mathcal{C}_i$ and the cycle at infinity. Such a one-form should be proportional to a holomorphic differential ω_i satisfying the following properties:

$$\oint_{\mathcal{C}_j} \omega_i = \delta_{ij}, \quad \int_{\infty^+} \omega_i = -1 = \oint_{\infty^-} (-\omega_i). \tag{8.2.20}$$

Here ∞^+ (∞^-) denotes the infinity on the first (second) sheet. Using such ω_i, the partial derivative $\partial F/\partial S_i$ is expressed as[12]

$$\phi_i = 2\pi \sum_j \int^{\gamma_j} \omega_i \,. \tag{8.2.21}$$

8.3 Angle variables and the Wronskians

In this section, we shall show that the angle variables constructed in the previous section can be expressed in terms of the skew-symmetric product, to be called the Wronskians, of the solutions of the auxiliary linear problem corresponding to the Lax pair and of the polarization vectors. In what follows the Wronskian of any two-component vectors χ^a and ϕ^a is defined as

$$\langle \chi, \phi \rangle \equiv \chi^a \epsilon_{ab} \phi^b \,. \tag{8.3.1}$$

8.3.1 Normalization of the solutions to the auxiliary linear problems

By using the Wronskian, we shall conveniently normalize the solutions $k_\pm$ of the auxiliary linear problem for the k-th spin chain as

$$\langle k_+, k_- \rangle = 1 \,. \tag{8.3.2}$$

In addition to this condition, it is consistent to require that the two solutions $k_\pm$ are related across the cut by

$$k_+|_{\text{2nd-sheet}} = -i \, k_-|_{\text{1st-sheet}} \,, \quad k_-|_{\text{2nd-sheet}} = -i \, k_+|_{\text{1st-sheet}} \,. \tag{8.3.3}$$

Then from (8.3.2) and (8.3.3), one can show that $k_\pm$ develop the following singularity at the branch points of the spectral curve

$$k_\pm \propto \frac{1}{\sqrt{u - u_b}} \quad (u \to u_b) \,. \tag{8.3.4}$$

Let us briefly explain how this comes about. As the eigenvectors of the monodromy matrix are determined only up to an overall factor, we may first choose an eigenvector k_+^0 which remains non-singular even at the position of the branch points. Then the other eigenvector k_-^0 can be obtained by the smooth analytic continuation of k_+^0 to the second sheet, since upon this operation the quasimomentum $p_k(u)$ flips sign and hence the eigenvalue changes from $e^{ip_k(u)}$ to $e^{-ip_k(u)}$. By this definition, $k_\pm^0$ clearly satisfy the following relations:

$$k_+^0\big|_{\text{2nd-sheet}} = k_-^0\big|_{\text{1st-sheet}} \,, \quad k_-^0\big|_{\text{2nd-sheet}} = k_+^0\big|_{\text{1st-sheet}} \,. \tag{8.3.5}$$

---

[12]This expression is a generalization of the so-called Abel map known in the theory of Riemann surfaces.

Now let us normalize these two eigenvectors so that they satisfy the normalization condition (8.3.2). This can be achieved by the rescaling

$$k_+ \equiv \frac{1}{\sqrt{\langle k_+^0 , k_-^0 \rangle}} k_+^0 \,, \quad k_- \equiv \frac{1}{\sqrt{\langle k_+^0 , k_-^0 \rangle}} k_-^0 \,. \tag{8.3.6}$$

Since two eigenvectors $k_\pm^0$ become degenerate at the branch points, $\langle k_+^0 , k_-^0 \rangle$ has a simple zero at such points. This yields the singularity structure given in (8.3.4).

Note that the aforementioned conditions do not completely fix the normalization, since we can always "renormalize" the eigenvectors as

$$k_+ \to c(u) k_+ \,, \qquad k_- \to k_-/c(u) \,, \tag{8.3.7}$$

without violating the conditions (8.3.2) and (8.3.3), if the function $c(u)$ satisfies

$$c(u)\big|_{\text{1st-sheet}} = \frac{1}{c(u)}\bigg|_{\text{2nd-sheet}} \,. \tag{8.3.8}$$

In section 8.3.3, we will utilize this freedom to express the angle variable in terms of the Wronskians.

8.3.2 Separated variables for two-point functions and orthogonality

In order to obtain the formula for the difference of the angle variables appearing in (8.1.37) in terms of appropriate Wronskians, we must first clarify the structure of the separated variables on the two-sheeted spectral curve. Similar information was crucial also in the case of the strong coupling, treated by the string theory representation. In that case, certain assumptions on the analyticity as a function on the string worldsheet helped determine some important structure. However, in the present case there is no worldsheet and we must devise a different logic to get a handle on the structure of the separated variables.

Before delving into the discussion of the case of the three-point function, it is necessary to understand in detail the separated variables for the two-point functions. It will turn out that the logic that we shall employ is of such a general validity that it can also be applied for strong coupling, as well as for the weak coupling that we are analyzing.

Let us consider the norm of a physical spin-chain state $\langle \Psi | \Psi \rangle$ (or equivalently a two-point function) and perturb one of the states by adding a small number of Bethe roots to produce the inner product $\langle \Psi | \Psi + \delta\Psi \rangle$. Clearly $\langle \Psi | \Psi + \delta\Psi \rangle$ should be non-vanishing for a general perturbation. However, when the perturbed state is such that it becomes on-shell again, $\langle \Psi | \Psi + \delta\Psi \rangle$ must vanish because of the orthogonality of different eigenstates of the

spin Hamiltonian. Therefore we have

$$\langle\Psi|\Psi+\delta\Psi\rangle = 0\,, \quad \text{if } |\Psi\rangle \text{ and } |\Psi+\delta\Psi\rangle \text{ are on-shell}\,. \tag{8.3.9}$$

It should be emphasized that this is an exact quantum statement.

Now we perform the same type of perturbation in the semi-classical regime. Specifically, consider the norm $\langle\Psi|\Psi\rangle$ of a semiclassical on-shell state and perturb only the ket state $|\Psi\rangle$ by adding a small cut at the position of one of the singular points[13], u_*, which corresponds to adding a small number of Bethe roots. When the added cut is small enough, the log of this quantity (normalized by the original norm) can be expressed as

$$\ln\left(\frac{\langle\Psi|\Psi+\delta\Psi\rangle}{\langle\Psi|\Psi\rangle}\right) \simeq \left.\frac{\partial\ln\langle\Psi|\Psi'\rangle}{\partial S_{u_*}}\right|_{\Psi'=\Psi} \delta S_{u_s}\,, \tag{8.3.10}$$

where the derivative with respect to S_{u_*} acts only on $|\Psi'\rangle$. We have denoted the action variable associated with the degenerate cut at the singular point u_* by S_{u_*} and δS_{u_*} denotes the filling fraction corresponding to the small cut added. Since the state $|\Psi\rangle$ is semiclassical, we can evaluate the quantity $\partial\langle\Psi|\Psi'\rangle/\partial S_{u_*}$ using the saddle point approximation. This operation is exactly the same as the one performed on $\ln C_{123}$ previously, and taking into account the saddle point equation itself the contribution that remains is

$$\left.\frac{\partial\ln\langle\Psi|\Psi'\rangle}{\partial S_{u_*}}\right|_{\Psi'=\Psi} = i\phi_{u_*}\,, \tag{8.3.11}$$

where ϕ_{u_*} is the angle variable evaluated on the unperturbed state. As the small cut added in this regime is actually made of some number m of on-shell Bethe roots, with the positive integer m being of $\mathcal{O}(1)$, we can identify δS_{u_*} as m and hence (8.3.10) together with (8.3.11) can be written as[14]

$$\ln\left(\frac{\langle\Psi|\Psi+\delta\Psi\rangle}{\langle\Psi|\Psi\rangle}\right) \simeq im\phi_{u_*}\,. \tag{8.3.12}$$

This means that when the perturbed state $|\Psi+\delta\Psi\rangle$ is again on-shell, according to the exact quantum property (8.3.9), which must hold in the semi-classical regime as well, we must have

$$\frac{\langle\Psi|\Psi+\delta\Psi\rangle}{\langle\Psi|\Psi\rangle} \simeq e^{im\phi_{u_*}} \to 0\,. \tag{8.3.13}$$

---

[13]As discussed in [99,100], the on-shell perturbation of the classical solution corresponds to the insertion of an infinitesimal cut at singular points.

[14]Note that in the semi-classical limit, anything which does not scale as the length of the chain L can be regarded as small numbers.

To examine this, let us compute ϕ_{u_*} using the formula (8.2.21) applied to this case. We have

$$\phi_{u_*} = 2\pi \sum_j \int^{\gamma_j} \omega_{u_*} \,, \tag{8.3.14}$$

where γ_j are the separated variables and ω_{u_*} is the holomorphic differential which satisfies the following properties on the first and the second sheet.

$$
\begin{aligned}
\text{1st sheet:} \quad & \oint_{u_*} \omega_{u_*} = 1 \,, \quad \oint_\infty \omega_{u_*} = -1 \,, \quad \oint_{\mathcal{C}_i} \omega_{u_*} = 0 \,, \\
& \omega_{u_*} \sim \frac{1}{2\pi i} \frac{1}{u - u_*} \quad (u \to u_*) \,, \\
\text{2nd sheet:} \quad & \oint_{u_*} \omega_{u_*} = -1 \,, \quad \oint_\infty \omega_{u_*} = 1 \,, \quad \oint_{\mathcal{C}_i} \omega_{u_*} = 0 \,, \\
& \omega_{u_*} \sim -\frac{1}{2\pi i} \frac{1}{u - u_*} \quad (u \to u_*) \,.
\end{aligned} \tag{8.3.15}
$$

This means that when one of the γ_j's is at $u = u_*$ on the first sheet, ϕ_{u_*} behaves like

$$\phi_{u_*} \sim \frac{1}{i} \ln(u - u_*) \quad (u \to u_*) \,, \tag{8.3.16}$$

while if such a situation occurs on the second sheet, we have

$$\phi_{u_*} \sim -\frac{1}{i} \ln(u - u_*) \quad (u \to u_*) \,, \tag{8.3.17}$$

Thus in order for $e^{im\phi_{u_*}}$ to vanish as dictated by (8.3.13), there must be a separated variable at each singular point on the first sheet. This information will be of prime importance in section 4.3 and 5.2: In section 4.3, it will provide the information of the zeros and poles of the important quantity $\langle n, \psi_+^{\rm 3pt} \rangle$. Such analyticity properties will in turn be imperative in determining those of the Wronskians, in terms of which the correlation functions will be expressed.

We once again stress that the preceding argument only uses the exact quantum property and its validity for the semi-classical regime as a special case, it is applicable regardless of the strength of the coupling constant.

8.3.3 Angle variables expressed in terms of the Wronskians

Using the properties discussed above, we now rewrite the angle variables in terms of the Wronskians.

As described in section 8.2.2, to construct the angle variables, we need to know the separated variables, which are associated to the poles of the normalized eigenvector of the

monodromy matrix given in (8.2.15). Clearly some of the zeros of $\langle n, \psi_+ \rangle$, where ψ_+ is the *unnormalized* eigenvector, correspond to such poles. However, $\langle n, \psi_+ \rangle$ may contain additional zeros, which do not appear in the normalized eigenvector $\psi_+ / \langle n, \psi_+ \rangle$ since ψ_+ itself becomes a zero-vector at such points and the ratio becomes finite. In addition to zeros, $\langle n, \psi_+ \rangle$ in general has poles where ψ_+ itself diverges. Likewise, these poles do not appear in the normalized eigenvector as they cancel between the numerator and the denominator. In what follows, we call these zeros and poles *spurious zeros and poles*. It is important to note that the positions of the spurious zeros and poles may move if we change the normalization of the eigenvector as (8.3.7) whereas those of the separated variables do not.

With these properties in mind, let us study the analytic structure of the factor $\langle n, \psi_+^{2\text{pt}} \rangle$ for the two-point function. When the spectral curve contains m-cuts, there are m "dynamical" separated variables, which evolve in the worldsheet time [95]. In addition to those, there are infinitely many non-dynamical separated variables which are trapped at the singular points on the first sheet of the spectral curve as discussed in the previous subsection. Both of them must correspond to zeros of $\langle n, \psi_+^{2\text{pt}} \rangle$. However, if we construct the solution explicitly using the finite gap method [95], we do not find infinitely many zeros corresponding to the nondynamical separated variables. This is because $\langle n, \psi_+^{2\text{pt}} \rangle$ has spurious poles, which cancel the zeros associated with those separated variables. Furthermore, it has a square-root singularity at the branch points as shown in (8.3.4). Thus the divisor of $\langle n, \psi_+^{2\text{pt}} \rangle$ is given by[15]:

$$\left(\langle n, \psi_+^{2\text{pt}} \rangle \right) = \sum_k \gamma_k^{2\text{pt}} - \sum_l s_l - \frac{1}{2} \sum_m b_m \,. \tag{8.3.18}$$

Here $\gamma_k^{2\text{pt}}$'s correspond to the separated variables (both dynamical and nondynamical), s_l's denote the singular points on the first sheet, and b_m's denote the branch points.

We now turn to the corresponding quantity for the three-point function $\langle n, \psi_+^{3\text{pt}} \rangle$. To compute the normalized three-point functions, it is convenient to use the same normalization for the eigenvectors $\psi_+^{2\text{pt}}$ and $\psi_+^{3\text{pt}}$. More precisely, we require $\psi_+^{3\text{pt}}$ (and therefore $\langle n, \psi_+^{3\text{pt}} \rangle$) to have the same spurious zeros and poles as $\psi_+^{2\text{pt}}$. This can always be achieved by multiplying by an appropriate function of the spectral parameter as (8.3.7). However, in that process, we often need to introduce extra spurious zeros and poles to $\langle n, \psi_+^{3\text{pt}} \rangle$ in order to make $c(u)$ in (8.3.7) to be single-valued on the spectral curve. Therefore, the

---

[15]The solution for the two-point function has moduli, and for generic values of the moduli $\langle n, \psi_+^{2\text{pt}} \rangle$ can have other spurious zeros and poles. However, on physical grounds, we expect that it is possible to choose a solution for which such poles and zeros are absent (although the argument is not completely rigorous). For a more detailed discussion on this point, see Appendix J.

general structure of the divisor takes the following form:

$$\left(\langle n\,, \psi_+^{\mathrm{3pt}} \rangle \right) = \sum_k \gamma_k^{\mathrm{3pt}} - \sum_l s_l - \frac{1}{2} \sum_m b_m + \sum_n (\eta_n - \delta_n)\,. \tag{8.3.19}$$

Here γ_k^{3pt} are the separated variables for the three-point functions whereas the last term (η_n and δ_n) correspond to the extra spurious zeros and poles alluded to above.

Let us make two important remarks regarding (8.3.19). First, owing to the normalization condition $\langle \psi_+^{\mathrm{3pt}}\,, \psi_-^{\mathrm{3pt}} \rangle = 1$, ψ_-^{3pt} has zeros at δ_n and poles at η_n. Since $\psi_\pm^{\mathrm{3pt}}$ are related to each other by (8.3.3), η_n and δ_n must satisfy

$$\eta_n = \hat{\sigma}\delta_n\,, \tag{8.3.20}$$

where $\hat{\sigma}$ is the holomorphic involution, which exchanges two sheets of the Riemann surface. Second, as (8.3.19) shows, ψ_+^{3pt} becomes singular at the singular points on the first sheet. This property plays a key role in the determination of the analyticity structure in section 8.4.2.

Taking into account the analyticity structure discussed above, we now compute the right hand side of (8.1.37), which is the shift of the angle variables for the three-point function relative to those of the two-point function. (In what follows, we suppress the indices distinguishing operators until the very end when we write down the final expression (8.3.35).) For this purpose, it is useful to introduce a one-form df defined by

$$df = d\ln \frac{\langle n\,, \psi_+^{\mathrm{3pt}} \rangle}{\langle n\,, \psi_+^{\mathrm{2pt}} \rangle}\,, \tag{8.3.21}$$

the divisor of which is given by

$$(df) = \sum_k \gamma_k^{\mathrm{3pt}} - \gamma_k^{\mathrm{2pt}} + \sum_n \eta_n - \delta_n\,. \tag{8.3.22}$$

Now, using the formula (8.2.21), we can express the shift φ_k as

$$\varphi_k = 2\pi \sum_{j=1}^{\infty} \int_{\gamma_j^{\mathrm{2pt}}}^{\gamma_j^{\mathrm{3pt}}} \omega_k\,. \tag{8.3.23}$$

where ω_k satisfies

$$\oint_{\mathcal{C}_j} \omega_k = \delta_{jk}\,, \quad \int_{\infty^+} \omega_k = -1 = \oint_{\infty^-} (-\omega_k)\,. \tag{8.3.24}$$

This expression can be simplified further using the generalized Riemann bilinear identity[16], which reads

$$\int_Q^P \tilde{\omega}_{RS;k} = \int_S^R \tilde{\omega}_{PQ;k}\,. \tag{8.3.25}$$

---

[16]This form is given in [97] and was used in a similar manner as below in [110, 111].

Here $\tilde{\omega}_{PQ;k}$ and $\tilde{\omega}_{RS;k}$ are normalized Abelian differentials satisfying[17]

$$\oint_P \tilde{\omega}_{PQ;k} = 1 \,, \quad \oint_Q \tilde{\omega}_{PQ;k} = -1 \,, \quad \oint_{\mathcal{C}_j} \tilde{\omega}_{PQ;k} = -\delta_{jk} \,. \tag{8.3.26}$$

Since ω_k can be identified with $-\tilde{\omega}_{\infty^+\infty^-;k}$, we can use the Riemann bilinear identity to rewrite φ_k as

$$\varphi_k = 2\pi \sum_{j=1}^{\infty} \int_{\gamma_j^{2\mathrm{pt}}}^{\gamma_j^{3\mathrm{pt}}} \omega_k = -2\pi \int_{\infty^-}^{\infty^+} \sum_{j=1}^{\infty} \tilde{\omega}_{\gamma_j^{3\mathrm{pt}}\gamma_j^{2\mathrm{pt}};k} = i \int_{\infty^-}^{\infty^+} (df - e_k) \tag{8.3.27}$$

where the integration contour starts from ∞^-, goes through the cut $\mathcal{C}_k$ and ends at ∞^+, and e_k is the one-form uniquely characterized by the following conditions:

$$(e_k) = \sum_n (\eta_n - \delta_n) \,, \qquad e_k(\hat{\sigma}u) = -e_k(u) \,, \qquad \oint_{\mathcal{C}_j} e_k = 0 \quad \text{for } j \neq k \,. \tag{8.3.28}$$

Here the second property follows from (8.3.20). Finally, substituting the definition of df into (8.3.27), we obtain[18]

$$\varphi_k = i \ln \frac{\langle n \,, \psi_+^{3\mathrm{pt}} \rangle \langle n \,, \psi_-^{2\mathrm{pt}} \rangle}{\langle n \,, \psi_-^{3\mathrm{pt}} \rangle \langle n \,, \psi_+^{2\mathrm{pt}} \rangle} \Bigg|_{x=\infty^+} - i \int_{\infty^-}^{\infty^+} e_k \,. \tag{8.3.29}$$

This expression can be further rewritten in terms of the Wronskians by judicious use of the highest weight condition. To see this, recall that the on-shell states constructed upon the rotated vacuum with the polarization vector n is annihilated by the operator S_+', given in (8.1.21). Such global charges can be read off from the asymptotic behavior of the monodromy matrix as

$$\begin{aligned}
\Omega(u) &= \mathbf{1} + \frac{i}{u} \begin{pmatrix} S_3 & S_- \\ S_+ & -S_3 \end{pmatrix} + O(u^{-2}) \\
&= \mathbf{1} + \frac{i}{u} N \begin{pmatrix} S_3' & S_-' \\ S_+' & -S_3' \end{pmatrix} N^{-1} + O(u^{-2}) \,.
\end{aligned} \tag{8.3.30}$$

where the second equality follows from (8.1.21). This leads to the following asymptotic form of the monodromy matrix in the semi-classical limit:

$$\Omega(u)|_{\mathrm{saddle}} = \mathbf{1} + \frac{i}{2u} N \begin{pmatrix} L - 2M & * \\ 0 & -(L-2M) \end{pmatrix} N^{-1} + O(u^{-2}) \qquad (u \to \infty^+) \,. \tag{8.3.31}$$

---

[17]To make connection with the formulas in [97], we need to choose the basis of the a-cycle as the cycles around the cuts except $\mathcal{C}_k$. Then (8.3.26) coincides with the definition of the normalized Abelian differential of the third kind.

[18]Here we used the relation that $\psi_+(\infty^-) = \psi_-(\infty^+)/i$ which follows from (8.3.3).

Note that this is true both for two-point functions and multi-point functions. From (8.3.31) and the asymptotic form of the quasi-momentum (8.2.7), we can determine the asymptotic behavior of the eigenvectors $\psi_\pm$ to be of the form

$$\psi_-(\infty^+) = a\,n\,, \qquad \psi_+(\infty^+) = -a^{-1}(i\sigma_2 n) + b\,n\,, \tag{8.3.32}$$

where a, b are arbitrary and we have imposed the normalization condition $\langle \psi_+\,, \psi_-\rangle = 1$. In the special case of two-point functions, by the explicit construction based on the finite-gap method, we can check[19] that $a_{2\text{pt}} = 1$. Thus the ratio appearing in (8.3.29) can be evaluated as

$$\left.\frac{\langle n\,, \psi_+^{3\text{pt}}\rangle\langle n\,, \psi_-^{2\text{pt}}\rangle}{\langle n\,, \psi_-^{3\text{pt}}\rangle\langle n\,, \psi_+^{2\text{pt}}\rangle}\right|_{x=\infty^+} = a_{3\text{pt}}^{-2}\,. \tag{8.3.33}$$

For three-point functions, the same quantity can be extracted from different combinations of the Wronskians. For instance, it is easy to verify, using (8.3.32), that the following combination gives the $a_{3\text{pt}}^{-2}$ for the operator $\mathcal{O}_i$:

$$a_{3\text{pt}}^{-2}\big|_{\mathcal{O}_i} = \left.\frac{\langle n_i\,, n_j\rangle\langle n_k\,, n_i\rangle}{\langle n_j\,, n_k\rangle}\frac{\langle j_-\,, k_-\rangle}{\langle i_-\,, j_-\rangle\langle k_-\,, i_-\rangle}\right|_{x=\infty^+}. \tag{8.3.34}$$

Thus, restoring the indices for the operators, we arrive at the final expression,

$$\varphi_{k_i}^{(i)} = i\ln\left(\left.\frac{\langle n_i\,, n_j\rangle\langle n_k\,, n_i\rangle}{\langle n_j\,, n_k\rangle}\frac{\langle j_-\,, k_-\rangle}{\langle i_-\,, j_-\rangle\langle k_-\,, i_-\rangle}\right|_{x=\infty^+}\right) - i\int_{\infty^-}^{\infty^+} e_{k_i}^{(i)}\,. \tag{8.3.35}$$

Here $e_{k_i}^{(i)}$ is a one form defined on the spectral curve of the i-th operator $\mathcal{O}_i$, which satisfies

$$\left(e_{k_i}^{(i)}\right) = \sum_n \eta^{(i)} - \delta_n^{(i)}\,, \qquad e_{k_i}^{(i)}(\hat{\sigma}_i u) = e_{k_i}^{(i)}(u)\,, \qquad \oint_{\mathcal{C}_s^{(i)}} e_{k_i}^{(i)} = 0 \quad \text{for } s \neq k_i\,, \tag{8.3.36}$$

where $\hat{\sigma}_i$ and $\mathcal{C}_s^{(i)}$ denote the holomorphic involution and the branch cuts respectively, for the spectral curve of $\mathcal{O}_i$.

Let us make a remark on the nature of the angle variables for the general multi-cut solutions that we are capable of dealing with in this chapter, in comparison to the previous work [111], where only the one-cut solution could be studied. In that work, the only allowed perturbation is to vary the filling fraction associated with the unique cut and at the same time the one at infinity, *i.e.* S_∞, in the opposite direction for consistency. This is why the angle variable conjugate to the global charge S_∞ showed up in the previous work. However, in the more general case of multi-cut solutions, we have to specify the cut to be perturbed among many and the corresponding angle variable is not necessarily the one conjugate to the global charge but the one associated with the more general filling fraction.

---

[19]See also Appendix J.

8.4 Evaluation of the Wronskians

Now that we have expressed the structure constant in terms of Wronskians, our final task is to evaluate these Wronskians.

8.4.1 Products of Wronskians from monodromy relation

Let us recall that at strong coupling, the monodromy relation was of crucial importance and it allowed us to express certain products of Wronskians in terms of quasi-momenta [108, 109, 111]. Since the relation derived in (8.1.31) is identical in form to the one in that analysis, one can apply the same argument also to the present case.

First, take the basis in which Ω_1 is diagonal of the form $\mathrm{diag}(e^{ip_1}, e^{-ip_1})$. Since the eigenvectors of different monodromy matrices are related with each other as

$$i_\pm = \langle i_\pm, j_- \rangle j_+ - \langle i_\pm, j_+ \rangle j_- \,, \tag{8.4.1}$$

Ω_2 in this basis is given by

$$\Omega_2 = \mathrm{M}_{12} \, \mathrm{diag}(e^{ip_2}, e^{-ip_2}) \, \mathrm{M}_{21} \,, \tag{8.4.2}$$

where M_{ij} is the basis-transformation matrix defined by

$$\mathrm{M}_{ij} = \begin{pmatrix} -\langle i_-, j_+ \rangle & \langle i_-, j_- \rangle \\ \langle i_+, j_+ \rangle & \langle i_+, j_- \rangle \end{pmatrix} \,. \tag{8.4.3}$$

Now, using the relation $\Omega_1 \Omega_2 \Omega_3 = 1$ (8.1.31), we can express the trace of the monodromy matrix Ω_3 as

$$\mathrm{tr}\, \Omega_3 \, (= 2\cos p_3) = \mathrm{tr}\Omega_2^{-1}\Omega_1^{-1} \,. \tag{8.4.4}$$

Substituting the explicit expressions of Ω_1 and Ω_2 above to the right hand side of (8.4.4), we get

$$\cos p_3 = \cos(p_1 - p_2)\langle 1_+, 2_+ \rangle \langle 1_-, 2_- \rangle - \cos(p_1 + p_2)\langle 1_+, 2_- \rangle \langle 1_-, 2_+ \rangle \,. \tag{8.4.5}$$

Combining this equation with the Schouten identity,

$$1 = \langle 1_+, 2_+ \rangle \langle 1_-, 2_- \rangle - \langle 1_+, 2_- \rangle \langle 1_-, 2_+ \rangle \,, \tag{8.4.6}$$

we can determine the products $\langle 1_+, 2_+ \rangle \langle 1_-, 2_- \rangle$ and $\langle 1_+, 2_- \rangle \langle 1_-, 2_+ \rangle$. Products of other Wronskians can be computed in a similar manner and the results can be summarized as

$$
\begin{aligned}
\langle i_+, j_+ \rangle \langle i_-, j_- \rangle &= \frac{\sin \dfrac{p_i + p_j + p_k}{2} \sin \dfrac{p_i + p_j - p_k}{2}}{\sin p_i \sin p_j} \,, \\[2mm]
\langle i_+, j_- \rangle \langle i_-, j_+ \rangle &= \frac{\sin \dfrac{p_i - p_j + p_k}{2} \sin \dfrac{p_i - p_j - p_k}{2}}{\sin p_i \sin p_j} \,,
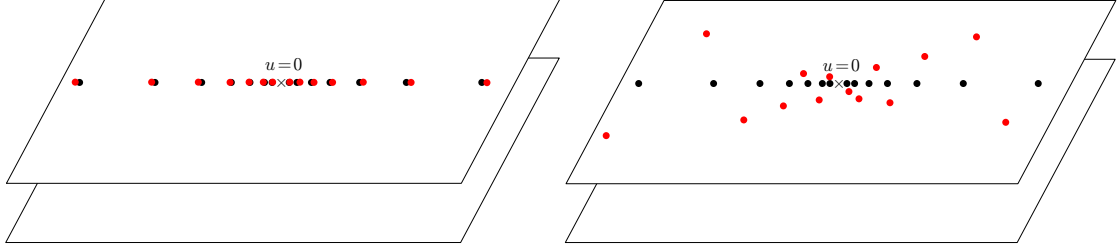\end{aligned}
\tag{8.4.7}
$$

Figure 8.4.1: The analytic structure of $\langle i_+, j_+ \rangle$ on the $[u, u, u]$-sheet when all the operators are BPS. Left: In the limit $p_i \to p_j$ and $p_k \to 0$, the poles (denoted by black dots) and the zeros (denoted by red dots) are on top of each other, and the Wronskians are free of singularities. Right: Since there are no branch cuts, even after p_k becomes nonzero and p_i starts to differ from p_j, the zeros and the poles cannot move away from the $[u, u, u]$-sheet.

where the cyclic permutation of $(1, 2, 3)$ should be applied to (i, j, k).

8.4.2 Analytic properties of the Wronskians

Now to compute three-point functions, we need to know the individual Wronskians, not just their products (8.4.7). For this purpose, below we need to determine the analytic properties of the Wronskians as a function of the spectral parameter. This knowledge will later be indispensable in decomposing the products into individual Wronskians.

Before proceeding, let us make one general remark: Since each set of eigenvectors $i_\pm$ live on a two-sheeted Riemann surface, Wronskians generally live on a 2^3-sheeted Riemann surface. Each of these eight-fold sheets will be denoted as $[*, *, *]$-sheet, where the n-th entry $*$ is written as "u" for the upper sheet of $p_n(u)$ and "l" when it refers to the lower sheet of $p_n(u)$. The determination of the analyticity on this eight-sheeted Riemann surface may at first sight seem a formidable task. However, as the eigenvectors on different sheets are related with each other by (8.3.3), once we know the analyticity of all the Wronskians on the $[u, u, u]$-sheet, the analyticity on the other sheets can be automatically deduced. For instance, the analyticity of $\langle 1_+, 2_+ \rangle$ on the $[l, u, u]$-sheet are the same as the analyticity properties of $\langle 1_-, 2_+ \rangle$ on the $[u, u, u]$-sheet. Thus, in what follows, it suffices to determine the analyticity on the $[u, u, u]$-sheet.

BPS correlators

We first study the simplest possible case, namely the three-point function of BPS operators. A distinct feature of such correlators is that the quasi-momenta do not contain any branch cuts. This simplifies the determination of the analyticity of Wronskians drastically, as we see below.

Let us first consider the Wronskians with the same signs, $\langle i_+, j_+ \rangle$ and $\langle i_-, j_- \rangle$. As

(8.4.7) shows, their products contain poles at $\sin p_i = 0$ and $\sin p_j = 0$, which are at the singular points of the spectral curve. In our normalization of the eigenvectors, the plus solutions i_+ and j_+ become singular at the singular points on the first sheet of the spectral curve (see the discussion below (8.3.19)). This means that the poles on the $[u, u, u]$-sheet belong to $\langle i_+, j_+ \rangle$ (and not to $\langle i_-, j_- \rangle$). In addition to poles, the right hand side of (8.4.7) has zeros at $\sin(p_i + p_j + p_k)/2 = 0$ and $\sin(p_i + p_j - p_k)/2 = 0$. To determine which Wronskian contains these zeros, we first consider the limit where $p_k \to 0$ and $p_i \to p_j$. In this limit, the classical solution for the three-point function approaches to the one for the two-point function. As mentioned in section 8.3.3, for the two-point function, the eigenvectors are nonsingular even at $\sin p_i = 0$ and so are the Wronskians. This means that in this limit the zeros of the Wronskians must cancel the pole singularities. In order for such cancellations to occur, all the zeros on the $[u, u, u]$-sheet must belong to $\langle i_+, j_+ \rangle$ when p_k is very small. Now, since all the operators are BPS and there are no branch cuts connecting different sheets, those zeros cannot leave the $[u, u, u]$-sheet even if we increase the value of p_k (see figure 8.4.1). We therefore conclude that all the zeros on the $[u, u, u]$-sheet must always belong to $\langle i_+, j_+ \rangle$, not to $\langle i_-, j_- \rangle$, when the three operators are BPS.

Next we consider the Wronskians with opposite signs $\langle i_+, j_- \rangle$ and $\langle i_-, j_+ \rangle$. Also in this case, the determination of the poles are straightforward since they are associated with the eigenvectors themselves. By the same reasoning as above, we conclude that the poles at $\sin p_i = 0$ belong to $\langle i_+, j_- \rangle$ whereas the poles at $\sin p_j$ belong to $\langle i_-, j_+ \rangle$. On the other hand, the determination of zeros is more complicated and we need to resort to the argument given in [111]. As reviewed in Appendix L, it leads to the following general rules:

1. *When a factor $\sin \left(\sum_i \epsilon_i\, p_i/2 \right)$ vanishes, the Wronskians which vanish are the ones among $\{1_{\epsilon_1}, 2_{\epsilon_2}, 3_{\epsilon_3}\}$ or the ones among $\{1_{-\epsilon_1}, 2_{-\epsilon_2}, 3_{-\epsilon_3}\}$. (Here ϵ_i takes $+$ or $-$.)*

2. *On the $[u, u, u]$-sheet, the Wronskians from the group which contains more than one $+$ eigenvectors all vanish whereas the Wronskians from the other group do not.*

Applying these rules, we can determine the analyticity on the $[u, u, u]$-sheet as given in tables 8.4.1 and 8.4.2. As already mentioned, the analyticity on other sheets can be deduced from the relations (8.3.3).

Extension to non-BPS

Now we turn to non-BPS correlators. Their analytic properties can be inferred from those of the BPS correlators if we make the following physically reasonable assumption:

	$1/\sin p_i$	$1/\sin p_j$	$\sin\dfrac{p_i+p_j+p_k}{2}$	$\sin\dfrac{p_i+p_j-p_k}{2}$
$\langle i_+, j_+\rangle$ $\langle i_-, j_-\rangle$	✓	✓	✓	✓

Table 8.4.1: The analytic properties of $\langle i_+, j_+\rangle$ and $\langle i_-, j_-\rangle$ on the $[u, u, u]$-sheet. The checked entries indicate existence of poles/zeros.

	$1/\sin p_i$	$1/\sin p_j$	$\sin\dfrac{p_i-p_j+p_k}{2}$	$\sin\dfrac{-p_i+p_j+p_k}{2}$
$\langle i_+, j_-\rangle$	✓		✓	
$\langle i_-, j_+\rangle$		✓		✓

Table 8.4.2: The analytic properties of $\langle i_+, j_-\rangle$ and $\langle i_-, j_+\rangle$ on the $[u, u, u]$-sheet.

Continuity Assumption:

When all the branch cuts of the quasi-momenta $p_i(x)$ shrink to zero size, the classical solution for the non-BPS correlator smoothly goes over to those for the BPS correlators.

This assumption implies in particular that the Wronskians for the BPS and the non-BPS cases are also smoothly connected. Now, let us consider the three BPS correlator discussed above and insert a very small cut to make it non-BPS. Because of the continuity assumption, the zeros and the poles of the Wronskians cannot move to a different sheet as long as the cut is sufficiently small and therefore the analyticity of Wronskians for such non-BPS correlators must be the same as the one for the BPS correlators. If we gradually increase the sizes of the cuts, at some point the zeros and the poles start crossing the branch cuts and move over to a different sheet, leading to a change in the analytic property of the Wronskians. A simple way to take such effects into account is to first compute the correlators with small cuts and then analytically continue the final results with respect to the sizes of the cuts (see figure 8.4.2). This analytic continuation to larger cuts will be discussed in section 8.5.2, and we will comment on how it affects the integration contours[20]. Thus, until then, we will restrict ourselves to the spectral curves with small cuts.

---

[20]A similar phenomenon was observed in the context of one-loop corrections to the classical energy both at weak [101] and strong coupling [99,100]. In both cases, as the sizes of the cuts become bigger, some of the physical excitations cross the cuts. At weak coupling, this leads to the change in the distribution of the Bethe roots. Nevertheless, the final result turns out to be a smooth function of the sizes of the cuts, and we expect that this is also the case for three-point functions.
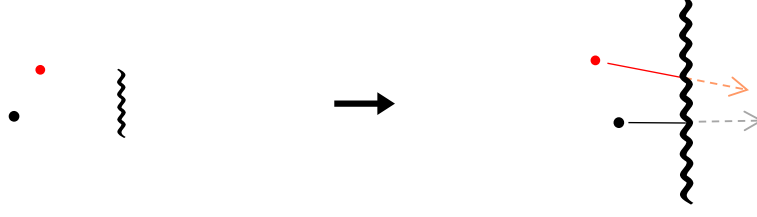
Figure 8.4.2: The analyticity of the Wronskians for the non-BPS correlators. Under the continuity assumption, the analyticity remains the same as the one for the BPS correlators as long as the cuts are sufficiently small (left figure). As we increase the size of the cut, the zeros and the poles start to move and, at some point, they cross the cut and cause the change in the analyticity (right figure). Such effects affect the integration contours of the final result as we shall see in section 8.5.2.

Spurious zeros and poles

In addition to the structures we discussed so far, there are extra spurious zeros and poles of the eigenvectors (η_n and δ_n in (8.3.19)). Owing to the normalization condition $\langle i_+, i_- \rangle = 1$, whenever i_+ has such a zero i_- has a pole, and vice versa. Thus, these extra zeros and poles cancel out if we consider the products of the Wronskians appearing on the left hand side of (8.4.7). Nevertheless we should keep in mind that such poles and zeros are present as we solve for the individual Wronskians below. In the next section, we first subtract such zeros and poles from the Wronskians and study the Riemann-Hilbert problem for these subtracted quantities.

8.4.3 Solving the Riemann-Hilbert problem

Let us now use the analytic properties to set up and solve the Riemann-Hilbert problem to determine the Wronskians. Here we will only discuss $\langle i_+, j_+ \rangle$ and $\langle i_-, j_- \rangle$ since these are the Wronskians relevant for the computation of the structure constant. (The argument below can be straightforwardly generalized to other Wronskians but we will not elaborate on it here.)

In what follows, we analyze the logarithmic derivative of the relation (8.4.7), namely

$$
\begin{aligned}
&\partial_u \ln \langle i_+, j_+ \rangle + \partial_u \ln \langle i_-, j_- \rangle \\
&= \partial_u \ln \sin \frac{p_i + p_j - p_k}{2} + \partial_u \ln \sin \frac{p_i + p_j + p_k}{2} - \partial_u \ln \sin p_i - \partial_x \ln \sin p_j \, .
\end{aligned}
\tag{8.4.8}
$$

Since the Wronskians contain extra zeros and poles which are absent on the right hand side of (8.4.8) as mentioned above, it is convenient to consider the following quantities

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from which extra zeros or poles are subtracted:

$$
\begin{aligned}
W_{++}^{ij} &= \partial_u \ln\langle i_+, j_+\rangle - e_{k_i}^{(i)} - e_{k_j}^{(j)}, \\
W_{--}^{ij} &= \partial_u \ln\langle i_-, j_-\rangle + e_{k_i}^{(i)} + e_{k_j}^{(j)}.
\end{aligned}
\tag{8.4.9}
$$

Here $e_{k_i}^{(i)}$ is a one-form given by (8.3.36) in section 8.3.3. As explained there, it depends on the choice of the cut $\mathcal{C}_{k_i}^{(i)}$, which we are perturbing. In terms of $W_{\pm\pm}^{ij}$, (8.4.8) can be written as

$$
\begin{aligned}
&W_{++}^{ij} + W_{--}^{ij} \\
&= \partial_u \ln\sin\frac{p_i + p_j - p_k}{2} + \partial_u \ln\sin\frac{p_i + p_j + p_k}{2} - \partial_u \ln\sin p_i - \partial_x \ln\sin p_j.
\end{aligned}
\tag{8.4.10}
$$

This is the Riemann-Hilbert problem we need to solve.

To uniquely characterize the solution to the Riemann-Hilbert problem, we have to specify its period integrals in addition to its analyticity. In the case at hand, the period integrals of $W_{\pm\pm}^{ij}$ are given by

$$
\oint_{\mathcal{C}_s^{(i)}} W_{\pm\pm}^{ij} = 0 \quad (s \neq k_i), \qquad \oint_{\mathcal{C}_s^{(j)}} W_{\pm\pm}^{ij} = 0 \quad (s \neq k_j).
\tag{8.4.11}
$$

These properties can be shown as follows: Since $\langle i_\pm, j_\pm\rangle$ is a single-valued function on the Riemann surface, the integral of $\partial_u \ln\langle i_\pm, j_\pm\rangle$ along any closed curve gives $n\pi$ where n is an integer. As we are considering the spectral curves with small cuts, which are continuously connected to the curves without cuts, n must be zero in such a case. Together with the property of $e_{k_i}^{(i)}$ given in (8.3.36), this leads to (8.4.11). This property will be later utilized in checking the correctness of the expressions of $W_{\pm\pm}^{ij}$ we shall construct.

Wiener-Hopf method

Before solving (8.4.10), let us briefly review the standard Wiener-Hopf method, which decomposes a function into the part regular on the upper-half plane and the part regular on the lower-half plane. Suppose $f(x)$ is a function which decreases sufficiently fast at infinity and does not have a pole on the real axis. Then $f(x)$ can be decomposed as $f(x) = f_+(x) + f_-(x)$ where f_+ and f_- are defined on the half planes by

$$
\begin{aligned}
f_+(x) &= \int_{-\infty}^{\infty} \frac{dx'}{2\pi i} \frac{1}{x' - x} f(x') \quad (\operatorname{Im} x > 0), \\
f_-(x) &= -\int_{-\infty}^{\infty} \frac{dx'}{2\pi i} \frac{1}{x' - x} f(x') \quad (\operatorname{Im} x < 0).
\end{aligned}
\tag{8.4.12}
$$

Using (8.4.12), it is easy to verify that f_+ is regular on the upper-half plane and f_- is regular on the lower-half plane. When x is not in the region specified in (8.4.12), we
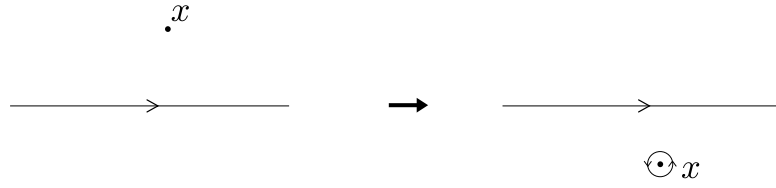
Figure 8.4.3: Analytic continuation of the Wiener-Hopf integral. As a result of the analytic continuation, the integral picks up a pole at $x' = x$. This yields the first term in (8.4.13).

need to analytically-continue these formulas. This leads, for instance, to the following expression for $f_+(x)$ on the lower-half plane ($\text{Im}\, x < 0$)

$$f_+(x) = f(x) + \int_{-\infty}^{\infty} \frac{dx'}{2\pi i} \frac{1}{x' - x} f(x') = f(x) - f_-(x)\,. \tag{8.4.13}$$

Here the first term $f(x)$ is produced by the integral along a small circle around $x' = x$ as shown in figure 8.4.3. An important feature of this method is that the contour of integration separates domains with different analyticity structures. This is true also in a more complicated situation where functions are defined on a multi-sheeted Riemann surface as in (8.4.10).

For later convenience, let us also present the version of (8.4.14) obtained by integration by parts:

$$
\begin{aligned}
f_+(x) &= -\int_{-\infty}^{\infty} \frac{dx'}{2\pi i} \frac{1}{(x' - x)^2} F(x') \quad (\text{Im}\, x > 0)\,, \\
f_-(x) &= \int_{-\infty}^{\infty} \frac{dx'}{2\pi i} \frac{1}{(x' - x)^2} F(x') \quad (\text{Im}\, x < 0)\,.
\end{aligned}
\tag{8.4.14}
$$

Here $F(x)$ is the integral of $f(x)$, *i.e.*, $F(x) = \int^x f(x')$. It is this form of the Wiener-Hopf decomposition that will be generalized below in order to deal with the multi-sheeted Riemann surface on which $p_i(x)$'s are defined.

Decomposition of the poles

We first study the factors $\partial_u \ln \sin p_i$ and $\partial_u \ln \sin p_j$, which give rise to the poles of the Wronskians. Below we focus on $\partial_u \ln \sin p_i$ since the case for $\partial_u \ln \sin p_j$ is completely similar.

As in the standard Wiener-Hopf decomposition, we should be able to decompose it by considering a convolution integral whose contour separates the domains with different analyticity. As summarized in table 8.4.1, the poles of $1/\sin p_i$ belong to $\langle i_+, j_+ \rangle$ when the rapidity is on the first sheet of p_i while they belong to $\langle i_-, j_- \rangle$ when it is on the second sheet of p_i. Obviously, these two regions are separated by the branch cuts of p_i and so the

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contour should be taken to go around the cuts. Now what we must properly deal with is the choice of the convolution kernel, as we have a two-sheeted Riemann surface instead of a simple complex plane. The natural generalization of the kernel (8.4.14) in the present case is given by the bidifferential characterized by the properties listed below, which is often called the *Bergman kernel* in physics literature[21] . To define the Bergman kernel, we must first pick a basis of cycles. Then, the Bergman kernel $B(p, q)$ is a differential in both p and q and is uniquely specified by the following properties;

1. *Symmetry:*

$$B(p, q) = B(q, p) \, . \tag{8.4.15}$$

2. *Normalization:*

$$\oint_{p \in a_j} B(p, q) = 0 \, , \tag{8.4.16}$$

   where $\{a_j\}$ is the basis of a-cycles.

3. *Analyticity: $B(p, q)$ is meromorphic in p with only a double pole at $p = q$ with the following structure:*

$$B(p, q) \sim \frac{1}{2\pi i (\zeta(p) - \zeta(q))^2} d\zeta(p) d\zeta(q) \, . \tag{8.4.17}$$

   Here ζ is a local coordinate around $p \simeq q$.

In addition to these properties, when the curve is hyperelliptic, it satisfies

4. *Involution property:*

$$B(\hat{\sigma} p, \hat{\sigma} q) = B(p, q) \, . \tag{8.4.18}$$

This is because the kernel $B(\hat{\sigma} p, \hat{\sigma} q)$ satisfies all three properties listed above, which specify the Bergman kernel uniquely. In the present case, we can define the Bergman kernel for each of the three spectral curves and we denote them by

$$B_{k_i}^{(i)}(p, q) \qquad i = 1, 2, 3 \, . \tag{8.4.19}$$

---

[21] This quantity is introduced by J. Fay [193] as the bidifferential made from the prime form and is called "the normalized bidifferential of the second kind" (see also [194]). Although in mathematics the Bergman kernel, strictly speaking, refers to slightly broader notion, we shall follow the physics nomenclature. We thank M. Jimbo and A. Nakayashiki for useful information on these matters.
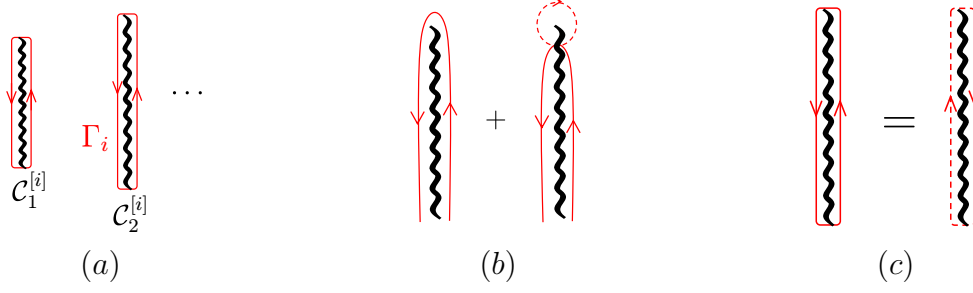
Figure 8.4.4: The integration contours relevant to the decomposition of poles. (a) The contour Γ_i goes along the branch cuts $\mathcal{C}_s^{(i)}$ on the first sheet of p_i counterclockwise. (b) Near the branch point, one must average over all possible ways to avoid the branch point as shown in the figure. The dashed curve denotes the contour on the second sheet. (c) By the continuous deformation, one can move the contour to the second sheet of p_i. The contour on the second sheet has the opposite orientation from the one on the first sheet. This leads to the minus sign on the right hand side of (8.4.24)

Here the subscript k_i designates our choice of the basis of the cycles; namely we choose the a-cycles as the cycles that surround each cut except $\mathcal{C}_{k_i}^{(i)22}$.

Now, using these kernels, one can decompose $\partial_u \ln \sin p_i$ as follows:

$$
\begin{aligned}
W_{++}^{ij} du &= \oint_{u' \in \Gamma_i} B_{k_i}^{(i)}(u, u') \ln \sin p_i(u') + \texttt{rest} && (u \in \text{2nd sheet of } p_i)\,, \\
W_{--}^{ij} du &= -\oint_{u' \in \Gamma_i} B_{k_i}^{(i)}(u, u') \ln \sin p_i(u') + \texttt{rest} && (u \in \text{1st sheet of } p_i)\,.
\end{aligned}
\tag{8.4.20}
$$

Here $\texttt{rest}$ represents the terms coming from decomposition of the rest of terms on the right hand side of (8.4.10). The integration contour Γ_i is on the 1st sheet of p_i and goes along the cuts $\mathcal{C}_s^{(i)}$ as depicted in figure 8.4.4-(a). Let us now make a remark on the contour: Unlike other poles, the poles at the branch points are shared equally by $\partial_u \ln\langle i_+, j_+\rangle$ and $\partial_u \ln\langle i_-, j_-\rangle$, since each eigenvector has a square-root singularity as shown in (8.3.4). To realize this structure, one must average over different ways of avoiding the branch points as shown in figure 8.4.4-(b). Apart from this small subtlety, these formulas are natural generalization of (8.4.14) and more importantly they are consistent with the property of $W_{\pm\pm}^{ij}$ (8.4.11), thanks to the normalization of the Bergman kernel (8.4.16). As in the standard Wiener-Hopf method, the expressions in the other regions can be obtained by analytic continuation.

Before proceeding, let us rewrite (8.4.20) in a form where the action of the holomorphic involution is more clearly seen. For this purpose, we first make a change of the integration variable from u' to $\hat{\sigma}_i u'$, with $\hat{\sigma}_i$ being the holomorphic involution for the spectral curve

---

[22]This means that the integration of the Bergman kernel around $\mathcal{C}_{k_i}^{(i)}$ does not vanish, $\oint_{p \in \mathcal{C}_{k_i}^{(i)}} B_{k_i}^{(i)} \neq 0$.

of p_i. This, of course, leaves the value of the integral intact, but its form gets slightly modified. For instance, the integrand is transformed as

$$\ln \sin p_i(\hat{\sigma}_i u') = \ln\left(-\sin p_i(u')\right), \qquad (8.4.21)$$

$$B_{k_i}^{(i)}(u, \hat{\sigma}_i u') = \check{B}_{k_i}^{(i)}(u, u'). \qquad (8.4.22)$$

Here, the new kernel $\check{B}_{k_i}^{(i)}(p, q)$ is defined by (8.4.22) and has a pole when $p = \hat{\sigma}_i q$,

$$\check{B}_{k_i}^{(i)}(p, q) \sim \frac{1}{2\pi i(\zeta(p) - \zeta(\hat{\sigma}_i q))} d\zeta(p) d\zeta(\hat{\sigma}_i q), \qquad (8.4.23)$$

where ζ is a local coordinate around p and $\hat{\sigma}_i q$. Similarly, the integration contour is modified as follows (see figure 8.4.4-(c) for more explanation):

$$\oint_{\Gamma_i} d\left(\hat{\sigma}_i u'\right) = -\oint_{\Gamma_i} du' \qquad (8.4.24)$$

From these transformation rules, we can rewrite the integral appearing in (8.4.20) as

$$\oint_{u' \in \Gamma_i} B_{k_i}^{(i)}(u, u') \ln \sin p_i(u') = -\oint_{u' \in \Gamma_i} \check{B}_{k_i}^{(i)}(u, u') \ln \sin p_i(u'). \qquad (8.4.25)$$

Here and below we neglect the term coming from $\ln(-1)$ since it changes the structure constant only by an overall phase. Now, by averaging two sides of (8.4.25), we arrive at the following expressions for $W_{\pm\pm}^{ij}$:

$$
\begin{aligned}
W_{++}^{ij} du &= \oint_{\Gamma_i} A_{k_i}^{(i)} * \ln \sin p_i + \texttt{rest} \qquad (u \in \text{2nd sheet of } p_i)\,, \\
W_{--}^{ij} du &= -\oint_{\Gamma_i} A_{k_i}^{(i)} * \ln \sin p_i + \texttt{rest} \quad (u \in \text{1st sheet of } p_i)\,.
\end{aligned}
\qquad (8.4.26)
$$

Here $A_{k_i}^{(i)}$ is the "anti-symmetrized" kernel defined by

$$A_{k_i}^{(i)}(p, q) \equiv \frac{1}{2}\left(B_{k_i}^{(i)}(p, q) - \check{B}_{k_i}^{(i)}(p, q)\right) = \frac{1}{2}\left(B_{k_i}^{(i)}(p, q) - B_{k_i}^{(i)}(p, \hat{\sigma}_i q)\right), \qquad (8.4.27)$$

and the notation $\oint_{\mathcal{C}} F * f$ denotes the convolution integral

$$\oint_{\mathcal{C}} F * f = \oint_{u' \in \mathcal{C}} F(u, u') f(u'). \qquad (8.4.28)$$

The kernel $A_{k_i}^{(i)}$ is odd under the holomorphic involution of each of the arguments:

$$A_{k_i}^{(i)}(p, \hat{\sigma}_i q) = -A_{k_i}^{(i)}(p, q)\,, \qquad A_{k_i}^{(i)}(\hat{\sigma}_i p, q) = -A_{k_i}^{(i)}(p, q)\,. \qquad (8.4.29)$$

The first equality follows immediately from the definition whereas the second equality can be derived using the property of the Bergman kernel (8.4.18). This property is used in section 8.5 when we write down the expression for the semi-classical structure constant.
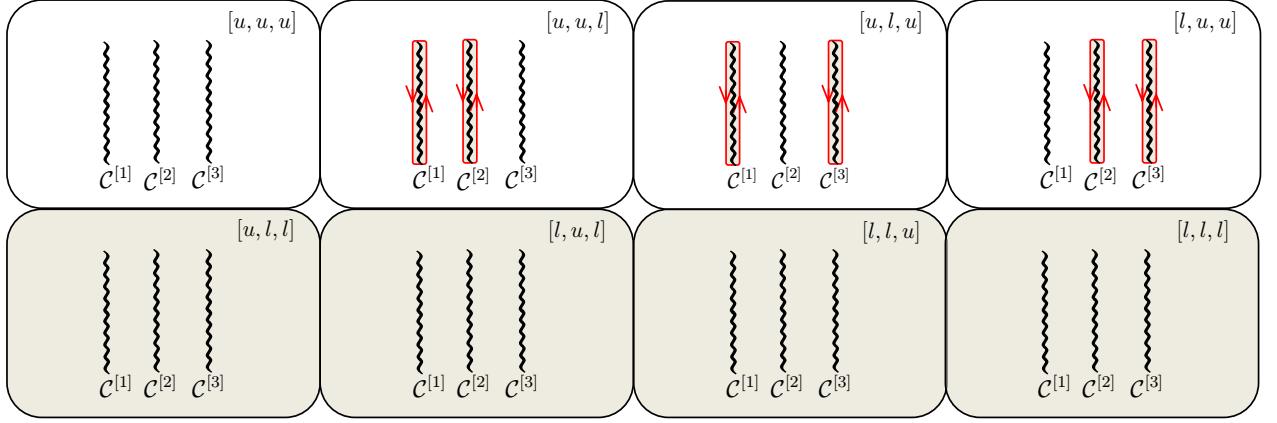
Figure 8.4.5: The analyticity structure and the integration contour for the decomposition of $\sin(p_i + p_j + p_k)/2$. In the region depicted in white, the Wronskians among $\{1_+, 2_+, 3_+\}$ have zeros, whereas in the region depicted in gray, the Wronskians among $\{1_-, 2_-, 3_-\}$ have zeros. The integration contour Γ_{+++}, denoted in red, separates the white region from the gray region.

Decomposition of the zeros

We now decompose $\partial_u \ln \sin(p_i + p_j + p_k)/2$ and $\partial_u \ln \sin(p_i + p_j - p_k)/2$, which are responsible for the zeros of the Wronskians. Since these quantities are defined on the 2^3-sheeted Riemann surface, both the integration contour and the convolution kernel must also be defined on the same eight-sheeted surface.

Let us first specify the integration contour. As in the previous case, the contour should be taken such that it separates the domains with different analyticity. As stated in the rules in section 8.4.2, when $\sin\left(\sum_i \epsilon_i p_i\right)$ vanishes, the Wronskians that vanish are the ones among $\{1_{\epsilon_1}, 2_{\epsilon_2}, 3_{\epsilon_3}\}$ or the ones among $\{1_{-\epsilon_1}, 2_{-\epsilon_2}, 3_{-\epsilon_3}\}$. Depending on which of the two groups contain vanishing Wronskians, the eight-sheeted Riemann surface is divided into two regions. Then the integration contour, denoted by $\Gamma_{\epsilon_1\epsilon_2\epsilon_3}$, will be placed at the boundary of the two regions. For instance, for the case of $\sin(p_1 + p_2 + p_3)/2$, the two regions and the contour are depicted in figure 8.4.5. To find the analyticity structure and the contour of other factors, one just needs to exchange the sheets appropriately, thanks to the property (8.3.3). For example, the analyticity structure and the contour of $\sin(p_1 + p_2 - p_3)/2$ are given by those in figure 8.4.5 with $[*, *, u]$-sheets and $[*, *, l]$-sheets swapped.

We next determine the convolution kernel. To carry out the desired decomposition, we use the kernel $B_{\text{all}}(p, q)$, which satisfies the first and the third properties, (8.4.15) and (8.4.17) respectively, of the Bergman kernel and the following slightly different normalization condition:

Normalization:

$$\oint_{p \in \mathcal{C}_s^{(i)}} B_{\text{all}}(p, q) = 0, \qquad s \neq k_i, \quad i = 1, 2, 3. \tag{8.4.30}$$

Using this kernel, we can, for instance, decompose $\partial_u \ln(p_i + p_j + p_k)/2$ in the following way:

$$\begin{aligned}
W_{++}^{ij} du &= -\oint_{u' \in \Gamma_{+++}} B_{\text{all}}(u, u') \ln \sin \frac{p_i + p_j + p_k}{2}(u') + \texttt{rest} \\
&\quad (u \in \text{gray region in figure } 8.4.5), \\
W_{--}^{ij} du &= \oint_{u' \in \Gamma_{+++}} B_{\text{all}}(u, u') \ln \sin \frac{p_i + p_j + p_k}{2}(u') + \texttt{rest} \\
&\quad (u \in \text{white region in figure } 8.4.5).
\end{aligned} \tag{8.4.31}$$

Again, in virtue of the normalization condition (8.4.30), this is consistent with the property of $W_{\pm\pm}^{ij}$ (8.4.11). The decomposition of $\partial_u \ln \sin(p_i + p_j - p_k)$ can be performed in a similar manner.

Let us make a clarifying remark. Although the existence of the kernel B_{all} with the properties listed above has not been explicitly proven, the convolution integral (8.4.31) can be rewritten entirely in terms of the standard Bergman kernel, the existence of which is firmly established. To show this, we again make use of the holomorphic involution. To illustrate the idea, let us consider the following terms that appear in the expression for W_{++}^{12} and W_{--}^{12}:

$$\oint_{u' \in \Gamma_{+++}} B_{\text{all}}(u, u') \ln \sin \frac{p_1 + p_2 + p_3}{2}(u') + \oint_{u' \in \Gamma_{++-}} B_{\text{all}}(u, u') \ln \sin \frac{p_1 + p_2 - p_3}{2}(u'). \tag{8.4.32}$$

If we change the integration variables from u' to $\hat{\sigma}_3 u$, the integrand and the contour of (8.4.32) transform as

$$\begin{aligned}
\ln \sin \frac{p_1 + p_2 \pm p_3}{2}(\hat{\sigma}_3 u') &= \ln \sin \frac{p_1 + p_2 \mp p_3}{2}(u'), \\
B_{\text{all}}(u, \hat{\sigma}_3 u') &= B_{\text{all}}^{(3)}(u, u'), \\
\oint_{\Gamma_{++\pm}} d(\hat{\sigma}_3 u') &= \oint_{\Gamma_{++\mp}} du',
\end{aligned} \tag{8.4.33}$$

where the new kernel, $B_{\text{all}}^{(3)}(p, q)$, has a pole at $p = \hat{\sigma}_3 q$. Using this transformation rule, we can re-express the integral (8.4.32) as

$$\oint_{u' \in \Gamma_{+++}} B_{\text{all}}^{(3)}(u, u') \ln \sin \frac{p_1 + p_2 + p_3}{2}(u') + \oint_{u' \in \Gamma_{++-}} B_{\text{all}}^{(3)}(u, u') \ln \sin \frac{p_1 + p_2 - p_3}{2}(u'). \tag{8.4.34}$$

	q	$\hat{\sigma}_1 q$	$\hat{\sigma}_2 q$	$\hat{\sigma}_3 q$	$\hat{\sigma}_1\hat{\sigma}_2 q$	$\hat{\sigma}_1\hat{\sigma}_3 q$	$\hat{\sigma}_2\hat{\sigma}_3 q$	$\hat{\sigma}_1\hat{\sigma}_2\hat{\sigma}_3 q$
$K(p,q)$	$+1$			$+1$	-1			-1
$A^{(1)}_{k_1}(p,q)$	$+1/2$	$-1/2$	$+1/2$	$+1/2$	$-1/2$	$-1/2$	$+1/2$	$-1/2$
$A^{(2)}_{k_2}(p,q)$	$+1/2$	$+1/2$	$-1/2$	$+1/2$	$-1/2$	$+1/2$	$-1/2$	$-1/2$

Table 8.4.3: The structure of the poles of the kernels K, $A^{(1)}_{k_1}$ and $A^{(2)}_{k_2}$. The first row designates the position of the double pole as a function of p and the numbers within the table are the coefficient of each pole. One can easily see that K and $A^{(1)}_{k_1} + A^{(2)}_{k_2}$ have the same pole structure.

Considering all possible combinations of holomorphic involutions, we obtain 2^3 different expressions for (8.4.32). Then averaging over these 2^3 expressions, we get

$$\frac{1}{8} \oint_{u' \in \Gamma_{+++}} K(u, u') \ln \sin \frac{p_1 + p_2 + p_3}{2}(u') + (4 \text{ other terms}), \tag{8.4.35}$$

with

$$K(p, q) \equiv \left(B_{\text{all}} + B^{(3)}_{\text{all}} - B^{(12)}_{\text{all}} - B^{(123)}_{\text{all}} \right)(p, q),$$
$$B^{(12)}_{\text{all}}(u, u') \equiv B_{\text{all}}(u, \hat{\sigma}_1\hat{\sigma}_2 u'), \quad B^{(123)}_{\text{all}}(u, u') \equiv B_{\text{all}}(u, \hat{\sigma}_1\hat{\sigma}_2\hat{\sigma}_3 u'). \tag{8.4.36}$$

Now the kernel $K(p,q)$ has four double poles as shown in table 8.4.3. As is clear from table 8.4.3, the analytic properties of $K(p,q)$ are identical to those of $A^{(1)}_{k_1} + A^{(2)}_{k_2}$, which are expressed in terms of the usual Bergman kernels. Thus we can replace $K(p,q)$ in (8.4.35) with $A^{(1)}_{k_1} + A^{(2)}_{k_2}$. Performing similar analysis to other 4 terms, we obtain the following expression for W^{12}_{++}:

$$W^{12}_{++} du = \texttt{rest}$$
$$- \frac{1}{8} \left(\oint_{\Gamma_{+++}} (A^{(1)}_{k_1} + A^{(2)}_{k_2}) * \ln \sin \frac{p_1 + p_2 + p_3}{2} + \oint_{\Gamma_{++-}} (A^{(1)}_{k_1} + A^{(2)}_{k_2}) * \ln \sin \frac{p_1 + p_2 - p_3}{2} \right.$$
$$\left. + \oint_{\Gamma_{+-+}} (A^{(1)}_{k_1} - A^{(2)}_{k_2}) * \ln \sin \frac{p_1 - p_2 + p_3}{2} + \oint_{\Gamma_{-++}} (-A^{(1)}_{k_1} + A^{(2)}_{k_2}) * \ln \sin \frac{-p_1 + p_2 + p_3}{2} \right).$$
$$\tag{8.4.37}$$

As in the standard Wiener-Hopf decomposition, this integral expression is valid in the region where W^{12}_{++} does not have any poles, which in this case correspond to the $[l, l, *]$-sheets[23].

Finally, let us discuss the simplification of the integration contours. The contours of (8.4.37) are defined on the eight-sheeted Riemann surface. However, for comparison with the results in the literature, it is more convenient to convert them into integrals

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[23]As discussed in section 8.4.2, when the spectral parameter is on these sheets, $\langle 1_+, 2_+ \rangle$ does not have any poles or zeros except for extra poles and zeros which are now subtracted. See tables 8.4.1 and 8.4.2.

defined purely on the $[u, u, u]$-sheet. This can be achieved again by making use of the holomorphic involution: For instance, take the integral along Γ_{+++} in (8.4.37) and consider the portion of the integral on the $[u, l, u]$-sheet. If we perform the holomorphic involution $\hat{\sigma}_2$ to the integrated variable u', this contribution becomes identical to the third term in (8.4.37) evaluated on the $[u, u, u]$-sheet. Repeating the same analysis for the other relevant integrals, we arrive at the expression

$$
W_{++}^{12} du = \mathtt{rest}
$$
$$
- \frac{1}{2} \left(\oint_{\gamma_{+++}} (A_{k_1}^{(1)} + A_{k_2}^{(2)}) * \ln \sin \frac{p_1 + p_2 + p_3}{2} + \oint_{\gamma_{++-}} (A_{k_1}^{(1)} + A_{k_2}^{(2)}) * \ln \sin \frac{p_1 + p_2 - p_3}{2} \right.
$$
$$
\left. + \oint_{\gamma_{+-+}} (A_{k_1}^{(1)} - A_{k_2}^{(2)}) * \ln \sin \frac{p_1 - p_2 + p_3}{2} + \oint_{\gamma_{-++}} (-A_{k_1}^{(1)} + A_{k_2}^{(2)}) * \ln \sin \frac{-p_1 + p_2 + p_3}{2} \right),
$$
$$
(8.4.38)
$$

where $\gamma_{\epsilon_1 \epsilon_2 \epsilon_3}$ is a portion of $\Gamma_{\epsilon_1 \epsilon_2 \epsilon_3}$ on the $[u, u, u]$-sheet. It is clear from figure 8.4.5 that Γ_{+++} does not have any portion on the $[u, u, u]$-sheet, and thus $\gamma_{+++} = \varnothing$. The other contours are along the cuts of some of the quasi-momenta as shown below:

$$
\gamma_{++-} = \Gamma_1 \cup \Gamma_2, \qquad \gamma_{+-+} = \Gamma_1 \cup \Gamma_3, \qquad \gamma_{-++} = \Gamma_2 \cup \Gamma_3. \qquad (8.4.39)
$$

Here Γ_i's are the contours given in figure 8.4.4-(a). Substituting (8.4.39) into (8.4.38) and restoring the terms coming from the decomposition of poles, we finally obtain

$$
W_{++}^{12} du =
$$
$$
\oint_{\Gamma_1} A_{k_1}^{(1)} * \ln \sin p_2 + \oint_{\Gamma_2} A_{k_2}^{(2)} * \ln \sin p_1 - \frac{1}{2} \left(\oint_{\Gamma_1 \cup \Gamma_2} (A_{k_1}^{(1)} + A_{k_2}^{(2)}) * \ln \sin \frac{p_1 + p_2 - p_3}{2} \right.
$$
$$
\left. + \oint_{\Gamma_1 \cup \Gamma_3} (A_{k_1}^{(1)} - A_{k_2}^{(2)}) * \ln \sin \frac{p_1 - p_2 + p_3}{2} + \oint_{\Gamma_2 \cup \Gamma_3} (-A_{k_1}^{(1)} + A_{k_2}^{(2)}) * \ln \sin \frac{-p_1 + p_2 + p_3}{2} \right).
$$
$$
(8.4.40)
$$

Similarly, we can write down the expression for W_{--}^{12}, which is valid when the spectral parameter is on the $[u, u, *]$-sheets:

$$
W_{--}^{12} du =
$$
$$
- \oint_{\Gamma_1} A_{k_1}^{(1)} * \ln \sin p_1 - \oint_{\Gamma_2} A_{k_2}^{(2)} * \ln \sin p_2 + \frac{1}{2} \left(\oint_{\Gamma_1 \cup \Gamma_2} (A_{k_1}^{(1)} + A_{k_2}^{(2)}) * \ln \sin \frac{p_1 + p_2 - p_3}{2} \right.
$$
$$
\left. + \oint_{\Gamma_1 \cup \Gamma_3} (A_{k_1}^{(1)} - A_{k_2}^{(2)}) * \ln \sin \frac{p_1 - p_2 + p_3}{2} + \oint_{\Gamma_2 \cup \Gamma_3} (-A_{k_1}^{(1)} + A_{k_2}^{(2)}) * \ln \sin \frac{-p_1 + p_2 + p_3}{2} \right).
$$
$$
(8.4.41)
$$

The expressions for the other W_{++}^{ij}'s and W_{--}^{ij}'s can be obtained from (8.4.40) and (8.4.41) by the permutation of the indices.

8.5 Results at weak coupling

Now we combine the results of sections 8.1, 8.3 and 8.4 and write down the explicit integral expression for the structure constant.

8.5.1 Integral expression for the semi-classical structure constant

As shown in (8.3.35), the variation of the semi-classical structure constant is given in terms of the Wronskians. To compute those Wronskians, we integrate the results obtained in the previous section (8.4.40) and (8.4.41). The net effect of integration is to replace the kernels $A^{(i)}_{k_i}(u, u')$ by their integrals $\int_{v=v_0}^{v=u} A^{(i)}_{k_i}(v, u')$. This is, however, still ambiguous since the initial point of the u-integration v_0 is not fixed. To determine v_0, we impose the following condition which comes from the normalization of the eigenvectors (8.3.3):

$$\langle i_-, j_- \rangle (\hat{\sigma}_i \hat{\sigma}_j u) = -\langle i_+, j_+ \rangle (u) \,. \tag{8.5.1}$$

In terms of the logarithm of the Wronskians, this reads

$$\ln \langle i_-, j_- \rangle (\hat{\sigma}_i \hat{\sigma}_j u) = \ln \langle i_+, j_+ \rangle (u) \,. \tag{8.5.2}$$

As in the previous analyses, we have neglected the minus sign in (8.5.1), which only affects the overall phase of the final result. We shall now show that (8.5.2) is satisified if we choose v_0 to be the branch point of $\mathcal{C}^{(i)}_{k_i}$, which we denote by b_{k_i} (see figure 8.5.1). Under this choice, the Wronskians are given by

$$\ln \langle i_+, j_+ \rangle = E^{(i)}_{k_i} + E^{(j)}_{k_j} + \oint_{\Gamma_i} \alpha^{(i)}_{k_i} * \ln \sin p_i + \oint_{\Gamma_j} \alpha^{(j)}_{k_j} * \ln \sin p_j$$

$$- \left(\oint_{\Gamma_i \cup \Gamma_j} (\alpha^{(i)}_{k_i} + \alpha^{(j)}_{k_j}) * \ln \sin \frac{p_i + p_j - p_k}{2} + \oint_{\Gamma_i \cup \Gamma_k} (\alpha^{(i)}_{k_i} - \alpha^{(j)}_{k_j}) * \ln \sin \frac{p_i - p_j + p_k}{2} \right.$$

$$\left. + \oint_{\Gamma_j \cup \Gamma_k} (-\alpha^{(i)}_{k_i} + \alpha^{(j)}_{k_j}) * \ln \sin \frac{-p_i + p_j + p_k}{2} \right) , \tag{8.5.3}$$

$$\ln \langle i_-, j_- \rangle = -E^{(i)}_{k_i} - E^{(j)}_{k_j} - \oint_{\Gamma_i} \alpha^{(i)}_{k_i} * \ln \sin p_i - \oint_{\Gamma_j} \alpha^{(j)}_{k_j} * \ln \sin p_j$$

$$+ \oint_{\Gamma_i \cup \Gamma_j} (\alpha^{(i)}_{k_i} + \alpha^{(j)}_{k_j}) * \ln \sin \frac{p_i + p_j - p_k}{2} + \oint_{\Gamma_i \cup \Gamma_k} (\alpha^{(i)}_{k_i} - \alpha^{(j)}_{k_j}) * \ln \sin \frac{p_i - p_j + p_k}{2}$$

$$+ \oint_{\Gamma_j \cup \Gamma_k} (-\alpha^{(i)}_{k_i} + \alpha^{(j)}_{k_j}) * \ln \sin \frac{-p_i + p_j + p_k}{2} , \tag{8.5.4}$$
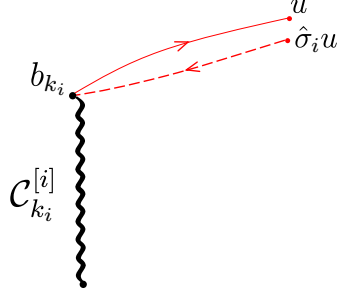
Figure 8.5.1: The branch point b_{k_i} and the integration contour in (8.5.7).

with $E_{k_i}^{(i)}$ and $\alpha_{k_i}^{(i)}$ given by

$$E_{k_i}^{(i)} \equiv \int_{v=b_{k_i}}^{v=u} e_{k_i}^{(i)}(v)\,, \qquad \alpha_{k_i}^{(i)}(u, u') \equiv \int_{v=b_{k_i}}^{v=u} A_{k_i}^{(i)}(v, u')\,. \tag{8.5.5}$$

As with the expressions in the previous section, (8.5.3) and (8.5.4) are valid on the $[l, l, l]$-sheet and on the $[u, u, u]$-sheet respectively. To see that (8.5.3) and (8.5.4) indeed satisfy the condition (8.5.1), we just need to use the fact that $e_{k_i}^{(i)}$ and $A_{k_i}^{(i)}$ are odd while the branch point b_{k_i} is invariant under the holomorphic involution (see (8.3.36) and (8.4.29)). Using these properties, we can express $E_{k_i}^{(i)}$ and $\alpha_{k_i}^{(i)}$ in a more symmetric way as follows[24]:

$$E_{k_i}^{(i)} = \frac{1}{2} \int_{v=\hat{\sigma}_i u}^{v=u} e_{k_i}^{(i)}(v)\,, \qquad \alpha_{k_i}^{(i)}(u, u') = \frac{1}{2} \int_{v=\hat{\sigma}_i u}^{v=u} B_{k_i}^{(i)}(v, u')\,. \tag{8.5.7}$$

Here, the precise integration contour is the one given in figure 8.5.1.

Having determined the Wronskians, we can now compute the angle variable by evaluating (8.5.4) at $u = \infty$ and substituting them into (8.3.35). It turns out that the terms $E_{k_i}^{(i)}(u = \infty)$ precisely cancel the last term in (8.3.35). Thus, as anticipated, the contribution from extra zeros and poles do not appear in the final expression, which takes the form

$$\varphi_{k_i}^{(i)} = i \left(\ln \frac{\langle n_i, n_j \rangle \langle n_k, n_i \rangle}{\langle n_j, n_k \rangle} + 2 \oint_{\Gamma_i} \bar{\alpha}_{k_i}^{(i)} \ln \sin p_i - \oint_{\Gamma_i \cup \Gamma_j} \bar{\alpha}_{k_i}^{(i)} \ln \sin \frac{p_i + p_j - p_k}{2} \right.$$
$$\left. - \oint_{\Gamma_i \cup \Gamma_k} \bar{\alpha}_{k_i}^{(i)} \ln \sin \frac{p_i - p_j + p_k}{2} + \oint_{\Gamma_j \cup \Gamma_k} \bar{\alpha}_{k_i}^{(i)} \ln \sin \frac{-p_i + p_j + p_k}{2} \right)\,. \tag{8.5.8}$$

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[24]For instance, the expression for $E_{k_i}^{(i)}$ can be derived as follows:

$$E_{k_i}^{(i)} = \int_{v=b_{k_i}}^{v=u} e_{k_i}^{(i)}(v) = \frac{1}{2} \left(\int_{v=b_{k_i}}^{v=u} e_{k_i}^{(i)}(v) + \int_{\hat{\sigma}_i v=b_{k_i}}^{\hat{\sigma}_i v=u} e_{k_i}^{(i)}(\hat{\sigma}_i v) \right)$$
$$= \frac{1}{2} \left(\int_{v=b_{k_i}}^{v=u} e_{k_i}^{(i)}(v) + \int_{v=b_{k_i}}^{v=\hat{\sigma}_i u} e_{k_i}^{(i)}(\hat{\sigma}_i v) \right) = \frac{1}{2} \int_{v=\hat{\sigma}_i u}^{v=u} e_{k_i}^{(i)}(v)\,. \tag{8.5.6}$$

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Here the one form $\bar{\alpha}_{k_i}^{(i)}$ is defined by

$$\bar{\alpha}_{k_i}^{(i)}(u') \equiv \alpha_{k_i}^{(i)}(\infty, u') = \frac{1}{2} \int_{v=\hat{\sigma}_i\infty}^{v=\infty} B_{k_i}^{(i)}(v, u'). \tag{8.5.9}$$

From the properties of the Bergman kernel, one can show that $\bar{\alpha}_{k_i}^{(i)}$ has the following analytic properties:

$$\operatorname*{Res}_{u=\infty} \bar{\alpha}_{k_i}^{(i)} = -\frac{1}{2}, \qquad \operatorname*{Res}_{u=\hat{\sigma}_i\infty} \bar{\alpha}_{k_i}^{(i)} = +\frac{1}{2},$$

$$\oint_{\mathcal{C}_s^{(i)}} \bar{\alpha}_{k_i}^{(i)} = 0 \quad (s \neq k_i), \qquad \oint_{\mathcal{C}_{k_i}^{(i)}} \bar{\alpha}_{k_i}^{(i)} = +\frac{1}{2}. \tag{8.5.10}$$

Now, it is easy to check that the one form[25]

$$\frac{\partial (p_i du/4\pi i)}{\partial S_{k_i}^{(i)}} \tag{8.5.11}$$

also satisfies the same analytic properties. Since (8.5.10) specifies the one form uniquely, this means that $\bar{\alpha}_{k_i}^{(i)}$ is identical to (8.5.11). Using this fact and the identity,

$$\int_0^x dx' \ln \sin x' = \frac{i}{2}\left(\operatorname{Li}_2(e^{2ix}) - \frac{\pi^2}{6}\right) + \ln(i/2)x - \frac{i}{2}x^2, \tag{8.5.12}$$

we can integrate the relation $\partial \ln C_{123}/\partial S_{k_i}^{(i)} = i\delta\phi_{k_i}^{(i)}$ to get the following integral expression:

$$\ln\left(\frac{C_{123}}{C_{123}^{\text{BPS}}}\right)\bigg|_{\text{SU(2)}_R} = \sum_{\{i,j,k\}\in\text{cperm}\{1,2,3\}} \left[(M_k - M_i - M_j)\ln\langle n_i, n_j\rangle + \frac{1}{2}\oint_{\Gamma_i\cup\Gamma_j} \frac{du}{2\pi}\operatorname{Li}_2\left(e^{ip_i+ip_j-ip_k}\right)\right]$$

$$- \frac{1}{2}\sum_{k=1}^3 \operatorname{Li}_2(e^{2ip_k}). \tag{8.5.13}$$

Here the summation in the first line denotes the sum over the cyclic permutation (abbreviated as "cperm") of $\{1, 2, 3\}$ and M_i is the total number of magnons in p_i. Note that (8.5.13) is the contribution from the SU(2)$_R$ sector only. For the complete result for the structure function for the distinct types of three-point functions, to be analyzed in the next section, it must be combined with the contribution from the SU(2)$_L$ sector as well.

8.5.2 Results and comparison with the literature

The operators forming the three-point functions we are studying transform under a single group SO(4) = SU(2)$_L \times$ SU(2)$_R$. For such correlators, there are two distinct classes, as discussed in [111].

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[25]One can show (8.5.11) using the argument similar to the one given in section 8.2.2: To perturb $S_{k_i}^{(i)}$, one needs to add to $p_i du$ a one form whose period integral does not vanish only along the cycle at infinity and the cycle around $\mathcal{C}_{k_i}^{(i)}$. By comparing the residues carefully, we arrive at (8.5.11).
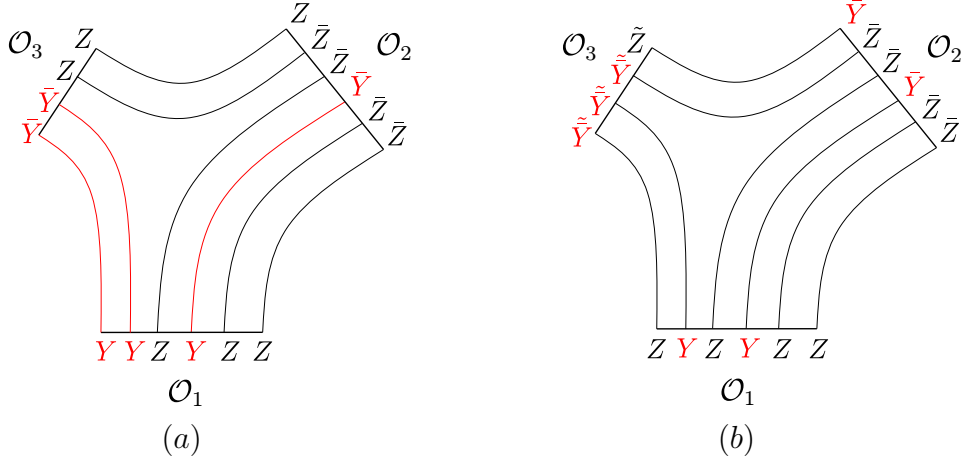
Figure 8.5.2: Two examples of type I-I-II three-point functions. In both figures, the fields denoted by black letters correspond to the vacuum and the fields denoted by red letters correspond to the magnons. (*a*) The configuration studied in most of the literature (see e. g. [115]). It amounts to choosing the polarization vectors as $n_1 = n_3 = \tilde{n}_1 = \tilde{n}_3 = (1,0)^t$ and $n_2 = \tilde{n}_2 = (0,1)^t$ (*b*) The configuration used in [153]. $\tilde{Z}$ and $\tilde{\bar{Y}}$ are given by $\tilde{Z} = Z + \bar{Z} + Y - \bar{Y}$ and $\tilde{\bar{Y}} = \bar{Y} - \bar{Z}$ respectively. The polarization vectors in this case are given by $n_1 = \tilde{n}_1 = (1,0)^t$, $n_2 = \tilde{n}_2 = (0,1)^t$ and $n_3 = \tilde{n}_3 = (1,1)^t$.

Type I-I-II three-point function

The first class of such three-point functions is called *type I-I-II*. These are the ones for which two of the operators have magnon excitations in the same SU(2), whereas the magnons for the third operator are in the other SU(2). Examples of such configurations are depicted in figure 8.5.2. This class of three-point functions were studied extensively in the literature and it was shown in [123, 134] that they can be expressed as a product of two Izergin-Korepin determinants [130, 131]. From such exact expressions, the semi-classical limit was extracted in [117, 124, 125]. In what follows, we shall reproduce it from our result (8.5.13).

Let us, for simplicity, consider the case where $\mathcal{O}_1$ and $\mathcal{O}_2$ belong to SU(2)$_R$ and $\mathcal{O}_3$ belongs to SU(2)$_L$. The structure constant factorizes into the left and the right parts as explained in section 8.1 and each part can be expressed in terms of integrals of the type given in (8.5.13). To get an explicit expression for C_{123} from (8.5.13), we also need to know the BPS three-point functions C_{123}^{BPS}. This can be easily computed as they are just a simple product of Wick contractions. The result is

$$\ln C_{123}^{\text{BPS}} = \sum_{\{i,j,k\}\in \text{cperm}\{1,2,3\}} \frac{L_i + L_j - L_k}{2} \left(\ln\langle n_i, n_j\rangle + \ln\langle \tilde{n}_i, \tilde{n}_j\rangle\right) . \tag{8.5.14}$$

Using this expression, we can write down the result for the type I-I-II three-point function

as

$$\ln C_{123} = \mathcal{K} + \mathcal{L} + \mathcal{R} + \mathcal{N}, \tag{8.5.15}$$

where each part is given by

$$\mathcal{K} = \sum_{\{i,j,k\}\in\mathrm{cperm}\{1,2,3\}} (Q_i + Q_j - Q_k)\ln\langle n_i, n_j\rangle + (\tilde{Q}_i + \tilde{Q}_j - \tilde{Q}_k)\ln\langle\tilde{n}_i, \tilde{n}_j\rangle, \tag{8.5.16}$$

$$\mathcal{L} = \frac{1}{2}\left(\oint_{\Gamma_3}\frac{du}{2\pi}\mathrm{Li}_2\left(e^{ip_3+(L_1-L_2)/2u}\right) + \oint_{\Gamma_3}\frac{du}{2\pi}\mathrm{Li}_2\left(e^{ip_3+(L_2-L_1)/2u}\right)\right), \tag{8.5.17}$$

$$\mathcal{R} = \frac{1}{2}\left(\oint_{\Gamma_1\cup\Gamma_2}\frac{du}{2\pi}\mathrm{Li}_2\left(e^{ip_1+ip_2-iL_3/2u}\right) + \oint_{\Gamma_1}\frac{du}{2\pi}\mathrm{Li}_2\left(e^{ip_1-ip_2+iL_3/2u}\right)\right.$$
$$\left. + \oint_{\Gamma_2}\frac{du}{2\pi}\mathrm{Li}_2\left(e^{-ip_1+ip_2+iL_3/2u}\right)\right), \tag{8.5.18}$$

$$\mathcal{N} = -\frac{1}{2}\sum_k\oint_{\Gamma_k}\frac{du}{2\pi}\mathrm{Li}_2\left(e^{2ip_k}\right). \tag{8.5.19}$$

Here $\mathcal{K}$ denotes the contribution determined purely by kinematics and the $\mathrm{SU}(2)_{L,R}$ global charges l_i and r_i are given by

$$\begin{aligned}\tilde{Q}_1 &= \frac{L_1}{2}, & \tilde{Q}_2 &= \frac{L_2}{2}, & \tilde{Q}_3 &= \frac{L_3}{2} - M_3,\\ Q_1 &= \frac{L_1}{2} - M_1, & Q_2 &= \frac{L_2}{2} - M_2, & Q_3 &= \frac{L_3}{2}.\end{aligned} \tag{8.5.20}$$

The second and the third terms $\mathcal{L}$ and $\mathcal{R}$ contain the dynamical information of the three-point functions and come from $\mathrm{SU}(2)_L$ and $\mathrm{SU}(2)_R$ respectively. The last term $\mathcal{N}$ is the part corresponding to the norms of the Bethe states in the exact quantum expression (see for instance [115]). To make a direct connection with the results in [125], we now rewrite the second and the third terms in $\mathcal{R}$ by pushing the contour onto the second sheet as we did in figure 8.4.4-(c). Then the two terms read

$$-\oint_{\Gamma_1}\frac{du}{2\pi}\mathrm{Li}_2\left(e^{-(ip_1-ip_2+iL_3/2u)}\right) - \oint_{\Gamma_2}\frac{du}{2\pi}\mathrm{Li}_2\left(e^{-(-ip_1+ip_2+iL_3/2u)}\right). \tag{8.5.21}$$

Now using the dilogarithm identity,

$$\mathrm{Li}_2\left(\frac{1}{x}\right) = -\mathrm{Li}_2(x) - \frac{\pi^2}{6} - \frac{1}{2}\ln^2(-x), \tag{8.5.22}$$

we can show[26] that (8.5.21) is identical to the first term in (8.5.18). By performing similar manipulation, we can also show that the first and the second terms in (8.5.17) are equivalent. In this way, we can obtain the following alternative expression for $\mathcal{L} + \mathcal{R}$:

$$\mathcal{L} + \mathcal{R} = \oint_{\Gamma_1\cup\Gamma_2}\frac{du}{2\pi}\mathrm{Li}_2\left(e^{ip_1+ip_2-iL_3/2u}\right) + \oint_{\Gamma_3}\frac{du}{2\pi}\mathrm{Li}_2\left(e^{ip_3+(L_2-L_1)/2u}\right). \tag{8.5.23}$$

Together with the norm part $\mathcal{N}$, this perfectly agrees with the result in [125].

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[26]The terms coming from the second and the third terms in the identity (8.5.22) vanish.
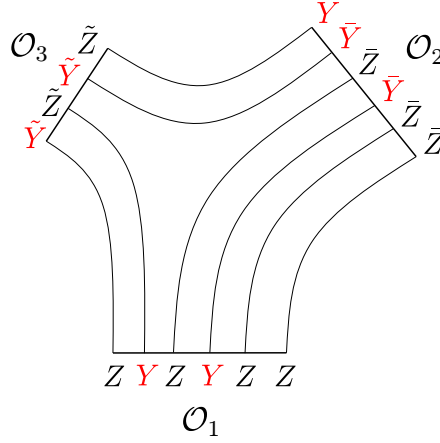
Figure 8.5.3: An example of the type I-I-I three-point functions studied in the [153]. $\tilde{Y}$ in the figure represents $Y + \bar{Z}$. The polarization vectors are given by $n_1 = \tilde{n}_1 = (1,0)^t$, $n_2 = \tilde{n}_2 = (0,1)^t$ and $n_3 = \tilde{n}_3 = (1,1)^t$

Type I-I-I three-point function

Let us now turn to the case where all the three operators have magnons in the same SU(2)-sector. They are called *type I-I-I* in [136]. An example of this class of correlators is given in figure 8.5.3. As compared to the type I-I-II correlators, they have much more complicated structure and the exact results known at weak coupling are given either in terms of the sum of the triple product of determinants [136] or in terms of the multiple-integral expression based on the separation of variables [146]. Both of these expressions are hard to deal with and their semi-classical limit has not been computed. Despite such complications for the exact result, the semiclassical result we derive below turned out to take a remarkably simple form. It would certainly be a challenging future problem to reproduce it from the expressions given in [136] and [146].

For definiteness, let us consider the case where all the operators belong to $\mathrm{SU}(2)_R$. In this case, there is no dynamical contribution from $\mathrm{SU}(2)_L$ and we can write down the full expression using (8.5.13) as

$$\ln C_{123} = \mathcal{K} + \mathcal{R} + \mathcal{N}\,, \tag{8.5.24}$$

with each part given by

$$\mathcal{K} = \sum_{\{i,j,k\}\in\mathrm{cperm}\{1,2,3\}} (Q_i + Q_j - Q_k)\ln\langle n_i, n_j\rangle + (\tilde{Q}_i + \tilde{Q}_j - \tilde{Q}_k)\ln\langle\tilde{n}_i, \tilde{n}_j\rangle\,, \tag{8.5.25}$$

$$\mathcal{R} = \frac{1}{2}\sum_{\{i,j,k\}\in\mathrm{cperm}\{1,2,3\}} \left(\oint_{\Gamma_i\cup\Gamma_j} \frac{du}{2\pi}\mathrm{Li}_2\left(e^{ip_i+ip_j-ip_k}\right)\right)\,, \tag{8.5.26}$$

$$\mathcal{N} = -\frac{1}{2}\sum_k \oint_{\Gamma_k} \frac{du}{2\pi}\mathrm{Li}_2\left(e^{2ip_k}\right)\,. \tag{8.5.27}$$
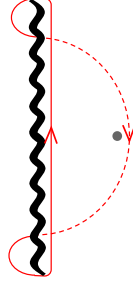
Figure 8.5.4: The deformation of the contour due to the branch-point singularities. When the branch point (denoted by a black dot in the figure) crosses the cut, the contour around it must be deformed such that it avoids the point.

Here the definitions of l_i and r_i are modified from (8.5.20) in the following manner:

$$
\begin{aligned}
\tilde{Q}_1 &= \frac{L_1}{2}\,, & \tilde{Q}_2 &= \frac{L_2}{2}\,, & \tilde{Q}_3 &= \frac{L_3}{2}\,, \\
Q_1 &= \frac{L_1}{2} - M_1\,, & Q_2 &= \frac{L_2}{2} - M_2\,, & Q_3 &= \frac{L_3}{2} - M_3\,.
\end{aligned}
\tag{8.5.28}
$$

As advertized, the expression above for the structure constant is as simple as the one for the I-I-II type.

Remark on the integration contour

So far, we have been assuming that the cuts in p_i are sufficiently small. In particular, we used this assumption when we derive the analyticity of the Wronskians. Let us briefly explain what we expect when we gradually increase the sizes of the cuts in the integral expression (8.5.13).

Since the dilogarithm $\mathrm{Li}_2(x)$ has a branch cut emanating from $x = 1$, the integrands of (8.5.13) contain infinitely many branch-point sigularities at $e^{i(p_i+p_j-p_k)} = 1$ and $e^{2ip_i} = 1$. These correspond to the zeros and the poles of the Wronskians respectively. As we increase the size of the cut, at some point, they start crossing the cut. When this happens, we need to deform the contour as depicted in figure 8.5.4 in order to keep the final result continuous with respect to the size of the cut. Thus, if we consider the operators with large cuts, the integration contours will be substantially deformed and will no longer be given by the ones around the cuts. This would explain the observation made in [125] that one must deform the contours appropriately in order to reproduce the value obtained by numerics. It would be important to perform detailed numerical computation and confirm the claim we made here.

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8.6 Application to the strong coupling

One of the important findings of the present work is that, as far as the semi-classical behaviors are concerned, the same structure and the logic underlie the three point functions both at weak and strong couplings. In this section we shall apply the machineries developed so far to the computation at strong coupling.

8.6.1 Classical integrability of string sigma model on S^3

Let us first give a brief review[27] of the classical integrability of the string sigma model on S^3 emphasizing the similarity to and the difference from the Landau-Lifshitz model discussed in section 8.2.

For the string sigma model on S^3, we can define two sets of Lax pairs as

$$\left[\partial + \frac{j_z}{1-x}\,,\ \bar{\partial} + \frac{j_{\bar{z}}}{1+x}\right] = 0\,, \qquad \left[\partial + \frac{x\tilde{j}_z}{1-x}\,,\ \bar{\partial} - \frac{x\tilde{j}_{\bar{z}}}{1+x}\right] = 0\,. \tag{8.6.1}$$

Here x is the spectral parameter and the currents j and $\tilde{j}$ are defined using the embedding coordinate Y_i $(i = 1, \ldots 4)$ as

$$j = G^{-1}dG\,, \qquad \tilde{j} = dGG^{-1}\,, \qquad G \equiv \begin{pmatrix} Y_1 + iY_2 & Y_3 + iY_4 \\ -Y_3 + iY_4 & Y_1 - iY_2 \end{pmatrix}. \tag{8.6.2}$$

For each Lax pair, we have an auxiliary linear problem and a monodromy matrix:

$$\left(\partial + \frac{j_z}{1-x}\right)\psi = 0\,, \quad \left(\bar{\partial} + \frac{j_{\bar{z}}}{1+x}\right)\psi = 0\,, \quad \Omega(x) \equiv \mathrm{Pexp}\left[-\oint\left(\frac{j_z dz}{1-x} + \frac{j_{\bar{z}} d\bar{z}}{1+x}\right)\right], \tag{8.6.3}$$

$$\left(\partial + \frac{x\tilde{j}_z}{1-x}\right)\tilde{\psi} = 0\,, \quad \left(\bar{\partial} + \frac{xj_{\bar{z}}}{1+x}\right)\tilde{\psi} = 0\,, \quad \tilde{\Omega}(x) \equiv \mathrm{Pexp}\left[-\oint\left(\frac{x\tilde{j}_z dz}{1-x} - \frac{x\tilde{j}_{\bar{z}} d\bar{z}}{1+x}\right)\right]. \tag{8.6.4}$$

Note that the two sets of quantities defined above are related with each other by $\tilde{j} = GjG^{-1}$, $\tilde{\psi} = G\psi$ and $\tilde{\Omega} = G\Omega G^{-1}$. As with the Landau-Lifshitz model, the quasi-momentum $p(x)$ is given by the logarithm of the eigenvalue of the monodromy matrix $\Omega \sim \tilde{\Omega} \sim \mathrm{diag}(e^{ip}, e^{-ip})$. The spectral curve is defined also in a similar way as

$$\det\left(y - \Omega(x)\right) = \det\left(y - \tilde{\Omega}(x)\right) = (y - e^{ip})(y - e^{-ip}) = 0\,. \tag{8.6.5}$$

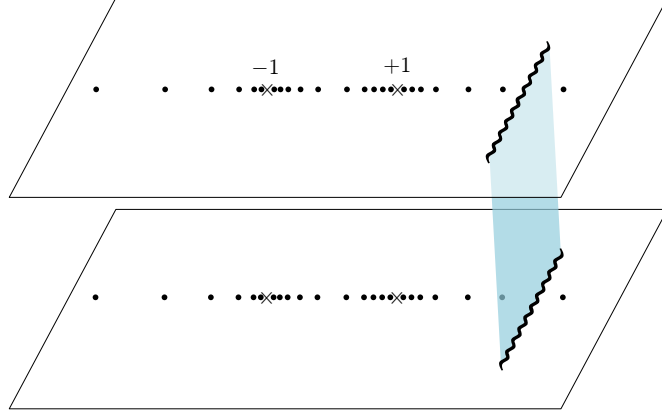
[27]For a more detailed account, see [79, 95–97, 111].

Figure 8.6.1: The structure of the spectral curve at strong coupling. In addition to the branch cuts, it has infinitely many singular points, denoted by black dots, which accumulate to $x = \pm 1$. Those singular points should be regarded as degenerate branch points.

The asymptotic behavior of the quansi-momentum around 0 and ∞ encodes the information of the global charges[28] as

$$p(x) \sim -\frac{Q}{g}\frac{1}{x} \quad (x \to \infty), \qquad p(x) \sim \frac{\tilde{Q}}{g}x \quad (x \to 0), \tag{8.6.6}$$

where Q and $\tilde{Q}$ are the charges of the $SU(2)_R$ and $SU(2)_L$ respectively. We should note that, unlike the Landau-Lifshitz model, the quasi-momentum does not have a pole at $x = 0$. Instead, it has poles at $x = \pm 1$ with residues given by the worldsheet[29] energy $\mathcal{E}$ and momentum $\mathcal{P}$:

$$p(x) \sim -\frac{\sqrt{(\mathcal{E} \pm \mathcal{P})/2g}}{x \mp 1} \qquad (x \to \pm 1). \tag{8.6.7}$$

Owing to this pole structure, the singular points of the spectral curve accumulate to $x = \pm 1$ as shown in figure 8.6.1.

As in the Landau-Lifshitz sigma model, the filling fractions are given by contour integrals on the spectral curve. However, their explicit forms are slightly modified:

$$S_k \equiv \frac{1}{2\pi i} \oint_{\mathcal{C}_k} p(x) du(x). \tag{8.6.8}$$

Here $u(x)$ is the *rapidity* variable, given by

$$u(x) = g\left(x + \frac{1}{x}\right), \tag{8.6.9}$$

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[28]In the most general situation, the quasi-momentum around $x = 0$ behaves as $p(x) \sim 2\pi m + x\tilde{Q}/g + \cdots$, where m is an integer called the winding number. Here we are considering the $m = 0$ case for simplicity.

[29]$\mathcal{E}$ and $\mathcal{P}$ defined here are the energy and the momentum of the S^3 sigma model in the conformal gauge. They therefore do not have definite physical meaning. In particular $\mathcal{E}$ is in general different from the lightcone energy of the string sigma model in $AdS_5 \times S^5$.

and the integration contour goes around the k-th branch cut[30] $\mathcal{C}_k$ counterclockwise on the first sheet.

8.6.2 $\mathrm{SU}(2)_L$ and $\mathrm{SU}(2)_R$ excitations at strong coupling

One of the conspicuous differences from the Landau-Lifshitz model is that the filling fractions given by (8.6.8) can be negative at strong coupling, and it turns out that the signs of the filling fractions are tied to whether the state has excitations in the $\mathrm{SU}(2)_L$ sector or in the $\mathrm{SU}(2)_R$ sector.

To understand this point, let us consider the perturbation around the BMN vacuum. It was shown in [99, 100] that the quasi-momentum receives the following correction when an infinitesimal cut is inserted at $x = x_*$:

$$\delta p(x) = n \frac{dx}{du} \frac{1}{x - x_*} = n \frac{x^2}{g(x^2 - 1)} \frac{1}{x - x_*} . \tag{8.6.10}$$

Here n is the filling fraction inserted at $x = x_*$ and the factor dx/du in (8.6.10) is necessary due to the definition of S_k in (8.6.8). We can also compute the energy shift using the results[31] in [99, 100] as

$$\delta \Delta = \frac{2n}{x_*^2 - 1} . \tag{8.6.11}$$

In (8.6.11), the quantity $1/(x_*^2 - 1)$ is positive when $|x_*| > 1$, while it is negative when $|x_*| < 1$. Since all the physical excitations around the BMN vacuum must have the positive energy shift[32], this means that n must be positive if $|x_*| > 1$ whereas it must be negative when $|x_*| < 1$. This is in marked contrast with the situation in the Landau-Lifshitz model, where we always needed to take n to be positive to describe the physical states. Physically, this is because the Bethe roots in the region $|x_*| < 1$ correspond to anti-particles: In order to construct a physical state from the anti-particles, we need to insert "holes" just as in the Dirac's fermi sea.

We can show more generally that the filling fraction defined by (8.6.8) must be positive whenever the cut is outside the unit circle whereas they must be negative whenever the cut is inside the unit circle. Now, to understand the physical meaning of these two types

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[30] As in the Landau Lifshitz sigma model, we should consider the (infinitely many) singular points satisfying $e^{2ip(x)} = 1$ also as (degenerate) branch cuts.

[31] The argument roughly goes as follows: As is clear from (8.6.10), the perturbation modifies the behavior around $x = \pm 1$. Owing to the Virasoro constraint, the AdS quasi-momentum $\hat{p}$ around $x = \pm 1$ must also be deformed in the same way. Once we understand how $\hat{p}$ is modified, we can then read off the energy shift from its asymptotic behavior at $x = \infty$.

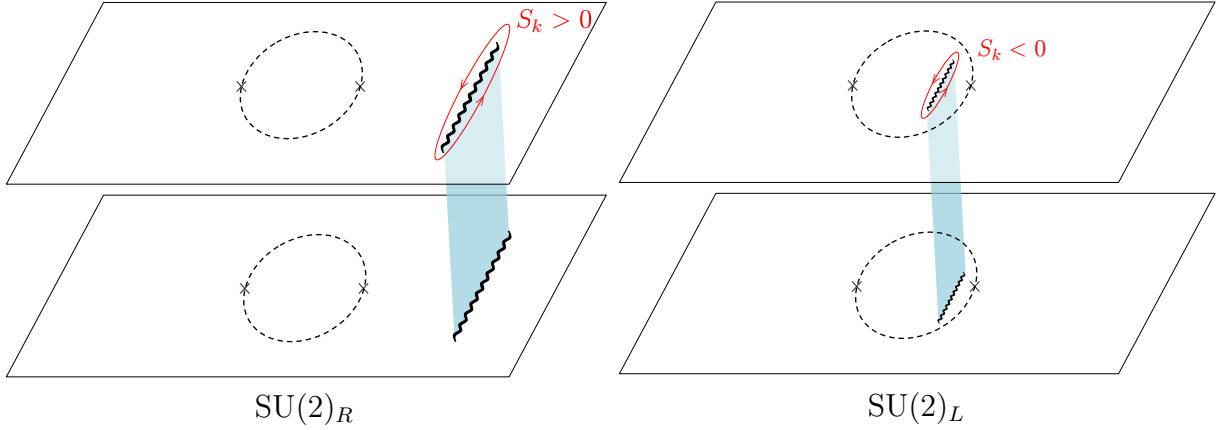[32] In other words, one cannot lower the energy starting from the BMN vacuum.

Figure 8.6.2: The spectral curves for $SU(2)_R$- and $SU(2)_L$-sectors. The curve for $SU(2)_R$ (left figure) contains branch cuts outside the unit circle and the filling fractions are positive. On the other hand, the curve $SU(2)_L$ (right figure) has branch cuts inside the unit circle and the filling fractions are negative.

of cuts, let us consider the relation[33] between the global charges and the filling fractions:

$$Q - \tilde{Q} + \sum_k S_k = 0 \,. \tag{8.6.13}$$

For the BMN vacuum, all the filling fractions are zero and Q and $\tilde{Q}$ are equal. Now, if we insert cuts outside the unit circle, which have positive filling fractions, we must either decrease Q or increase $\tilde{Q}$ in order to satisfy (8.6.13). However, since the BMN vacuum has the maximal[34] Q and $\tilde{Q}$, the only way to achieve this is to decrease Q. This clearly tells us that those states correspond to the ones with excitations in $SU(2)_R$. By a similar argument, we can show that the states with cuts inside the unit circle correspond to the states with $SU(2)_L$ excitations. For a summary, see figure 8.6.2. In Appendix K, we provide an interpretation of the $SU(2)_L$- and $SU(2)_R$-sectors from the point of view of the full spectral curve of the $AdS_5 \times S^5$ sigma model.

8.6.3 Angle variables, $\ln C_{123}$ and Wronskians at strong coupling

With this knowledge, we now construct the angle variables which compute the derivative of $\ln C_{123}$, and express them in terms of the Wronskians. Below we shall treat the $SU(2)_R$-sector and the $SU(2)_L$-sector separately.

---

[33](8.6.13) follows from the fact that r and l can be expressed as

$$Q = \frac{1}{2\pi i} \oint_{x=\infty} p(x)du(x) \,, \qquad \tilde{Q} = -\frac{1}{2\pi i} \oint_{x=0} p(x)du(x) \,. \tag{8.6.12}$$

[34]This is clear in particular at weak coupling. Whenever we excite magnons on the spin chain, the total global charge must always decrease as shown in (8.2.7).

$SU(2)_R$-**sector**

Let us first discuss the states with $SU(2)_R$ excitations. To construct the angle variables, we should study the normalized eigenvectors of the monodromy matrix as in the Landau-Lifshitz model. One important difference in the present situation is that we now have two sets of linear problems and monodromy matrices. For the $SU(2)_R$, the appropriate one to use is (8.6.3). This is because (8.6.4) is invariant under the $SU(2)_R$ transformations and is therefore insensible to the $SU(2)_R$ excitations.

As in section 8.2.2, the separated variables can be constructed from the poles γ_i of the normalized eigenvector $h(x)$,

$$h(x) \equiv \frac{1}{\langle n\,,\psi_+\rangle}\psi_+\,. \tag{8.6.14}$$

Here $\psi_\pm$ are the solutions to the auxiliary linear problem (8.6.3) satisfying

$$\Omega(x)\psi_\pm(x) = e^{\pm ip(x)}\psi_\pm(x)\,. \tag{8.6.15}$$

As shown in [96], a pair of canonically conjugate variables at strong coupling is given not by $(\gamma_i, -ip(\gamma_i))$ but by $(u(\gamma_i), -ip(\gamma_i))$, where $u(x)$ is the rapidity defined by (8.6.9). This explains the form of the filling fraction given in (8.6.8).

Now, to construct the angle variables, we need to consider the generating function of the canonical transformation (8.2.19) and then differentiate it with respect to S_k. As explained in the previous subsection, we should simultaneously decrease the global charge Q when we vary S_k. This amounts to adding to $p(x)du(x)$ a one form whose integral does not vanish only for the cycle around $\mathcal{C}_k$ and the cycle at infinity. As a result, we get

$$\phi_k = 2\pi \sum_j \int_{\gamma_j^{2\mathrm{pt}}}^{\gamma_j^{3\mathrm{pt}}} \omega_k\,, \tag{8.6.16}$$

where ω_k is the one form satisfying

$$\oint_{\mathcal{C}_j} \omega_k = \delta_{kj}\,, \qquad \oint_0 \omega_k = 0\,, \qquad \oint_\infty \omega_k = -1\,. \tag{8.6.17}$$

Let us next express the derivative of $\ln C_{123}$ in terms of the angle variables. The arguments leading to (8.1.37) are by and large applicable also to the present case, except for one important point. At strong coupling, in addition to the contribution from the S^3 part of the sigma model, we should also include the contribution from the AdS part. In particular, whenever we perturb the filling fraction in the S^3 part, we inevitably change the conformal dimension Δ_i, which is a global charge in AdS. This leads to the following

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modification of (8.1.37):

$$\frac{\partial \ln C_{123}}{\partial S_{k_i}^{(i)}} = i\phi_{k_i}^{(i)} + i\frac{\partial \Delta_i}{\partial S_{k_i}^{(i)}}\phi_\Delta^{(i)} \,. \tag{8.6.18}$$

Here $\phi_\Delta^{(i)}$ is the angle variable conjugate to Δ_i, whose definition is given in Appendix M.

Now, following the argument in section 8.3.3, we can express the angle variable $\phi_{k_i}^{(i)}$ in terms of the Wronskians and the result takes the same form as (8.3.35). We can perform similar analysis also to the AdS part (see Appendix M for details) to get the following expression of the angle variable $\phi_\Delta^{(i)}$:

$$\phi_\Delta^{(i)} = \frac{i}{2}\ln\left(\frac{|x_i - x_j|^2|x_k - x_i|^2}{|x_j - x_k|^2}\frac{\langle j_-, k_-\rangle}{\langle i_-, j_-\rangle\langle k_-, i_-\rangle}\bigg|_{x=\infty}\frac{\langle j_+, k_+\rangle}{\langle i_+, j_+\rangle\langle k_+, i_+\rangle}\bigg|_{x=0}\right)\,. \tag{8.6.19}$$

Here x_i denotes the position of the operator $\mathcal{O}_i$ and the eigenvectors i_+'s and $\tilde{i}_+$'s are the solutions to the auxiliary linear problems of the AdS_3 sigma model.

SU(2)$_L$-sector

For the SU(2)$_L$-sector, the linear problem we should consider is (8.6.4), as it is the one that transforms nontrivially under the SU(2)$_L$ transformation.

In this case, the separated variables in the SU(2)$_L$ sector can be constructed from the poles $\tilde{\gamma}_i$ of the normalized eigenvector $\tilde{h}(x)$,

$$\tilde{h}(x) \equiv \frac{1}{\langle \tilde{n}, \tilde{\psi}_+\rangle}\tilde{\psi}_+\,. \tag{8.6.20}$$

Here $\tilde{\psi}_+$ is the solution to the auxiliary linear problem (8.6.4) satisfying

$$\tilde{\Omega}(x)\tilde{\psi}_\pm(x) = e^{\pm ip(x)}\tilde{\psi}_\pm(x)\,. \tag{8.6.21}$$

Then the separated variables can be constructed from the poles at $\tilde{\gamma}_i$ as $(u(\tilde{\gamma}_i), -ip(\tilde{\gamma}_i))$.

From the separated variables, we can construct the angle variables in the same manner as for the SU(2)$_L$-sector. The only modification in the present case is that, when we change the filling fraction S_k, we need to change l but not r as discussed in section 8.6.2. This can be achieved by adding to $p(x)du(x)$ a one form whose integral does not vanish only for the cycle around $\mathcal{C}_k$ and the cycle around $x = 0$. Then we get the expression,

$$\tilde{\phi}_k = 2\pi\sum_j\int_{\gamma_j^{\mathrm{2pt}}}^{\gamma_j^{\mathrm{3pt}}}\omega_k\,, \tag{8.6.22}$$

where ω_k is the one form satisfying[35]

$$\oint_{\mathcal{C}_j} \omega_k = \delta_{jk}, \qquad \oint_0 \omega_k = -1, \qquad \oint_\infty \omega_k = 0. \tag{8.6.23}$$

Using these angle variables, we can express the derivative of $\ln C_{123}$ as[36]

$$\frac{\partial \ln C_{123}}{\partial S_{k_i}^{(i)}} = i\tilde{\phi}_{k_i}^{(i)} + i\frac{\partial \Delta_i}{\partial S_{k_i}^{(i)}}\phi_\Delta^{(i)}. \tag{8.6.24}$$

Here, as in the previous relation (8.6.18), $\phi_\Delta^{(i)}$ is the AdS angle variable (8.6.19).

Let us next express the angle variables in terms of the Wronskians. Although the basic logic in section 8.3.3 applies also to the present case, we have to modify (8.3.29) and (8.3.35) appropriately as follows:

$$\begin{aligned}
\tilde{\phi}_k &= 2\pi \sum_j \int_{\gamma_j^{2\mathrm{pt}}}^{\gamma_j^{3\mathrm{pt}}} \omega_k = -2\pi \int_{0^-}^{0^+} \sum_j \tilde{\omega}_{\gamma_j^{3\mathrm{pt}} \gamma_j^{2\mathrm{pt}};k} = i\int_{0^-}^{0^+} d\ln\frac{\langle \tilde{n}, \tilde{\psi}_+^{3\mathrm{pt}}\rangle}{\langle \tilde{n}, \tilde{\psi}_+^{2\mathrm{pt}}\rangle} - e_k \\
&= i\ln\left(\frac{\langle \tilde{n}, \tilde{\psi}_+^{3\mathrm{pt}}\rangle\langle \tilde{n}, \tilde{\psi}_-^{2\mathrm{pt}}\rangle}{\langle \tilde{n}, \tilde{\psi}_-^{3\mathrm{pt}}\rangle\langle \tilde{n}, \tilde{\psi}_+^{2\mathrm{pt}}\rangle}\right)\Bigg|_{x=0} - i\int_{0^-}^{0^+} e_k.
\end{aligned} \tag{8.6.25}$$

Here the one forms $\tilde{\omega}_{PQ;k}$ and e_k are defined by (8.3.26) and (8.3.28) respectively.

To express (8.6.25) in terms of Wronskians, we use the highest weight condition again. In this case, we should study the behavior of the monodromy matrix $\tilde{\Omega}(x)$ around $x = 0$ on the first sheet,

$$\tilde{\Omega}(x) = \mathbf{1} + ix\begin{pmatrix} \tilde{S}_3 & \tilde{S}_- \\ \tilde{S}_+ & -\tilde{S}_3 \end{pmatrix} + \cdots. \tag{8.6.26}$$

Applying the argument similar to the one in section 8.3.3, we arrive at the following form of the eigenvectors at $x = 0$ (on the first sheet):

$$\tilde{\psi}_+(0) = a\tilde{n}, \qquad \tilde{\psi}_-(0) = a^{-1}i\sigma_2\tilde{n} + b\tilde{n}. \tag{8.6.27}$$

Using (8.6.25) and (8.6.27), we finally get the expression for the angle variables in terms of the Wronskians:

$$\tilde{\phi}_{k_i}^{(i)} = i\ln\left(\frac{\langle \tilde{n}_i, \tilde{n}_j\rangle\langle \tilde{n}_k, \tilde{n}_i\rangle}{\langle \tilde{n}_j, \tilde{n}_k\rangle}\frac{\langle j_+, k_+\rangle}{\langle i_+, j_+\rangle\langle k_+, i_+\rangle}\Bigg|_{x=0}\right) - i\int_{0^-}^{0^+} e_{k_i}^{(i)}. \tag{8.6.28}$$

Here the Wronskians are evaluated on the first sheet and $\bar{\phi}_{k_i}^{(i)}$ denotes the angle variable of the operator $\mathcal{O}_i$ associated with the k_i-th cut whereas $\tilde{n}_i$ is the SU(2)$_L$ polarization vector for $\mathcal{O}_i$. To derive (8.6.28), we used the fact that the Wronskians among i_+'s are equivalent to the Wronskians among $\tilde{i}_+$'s, $\langle i_+, j_+\rangle = \langle \tilde{i}_+, \tilde{j}_+\rangle$. This is because two sets of eigenvectors are related with each other by the similarity transformation, $\tilde{i}_+ = G\,i_+$, and the Wronskians are invariant under such transformation.

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[35]Here the contour for the second integral goes around $x = 0$ on the first sheet counterclockwise.

[36]We shall not present the derivation here since it closely parallels the one for the SU(2)$_R$.
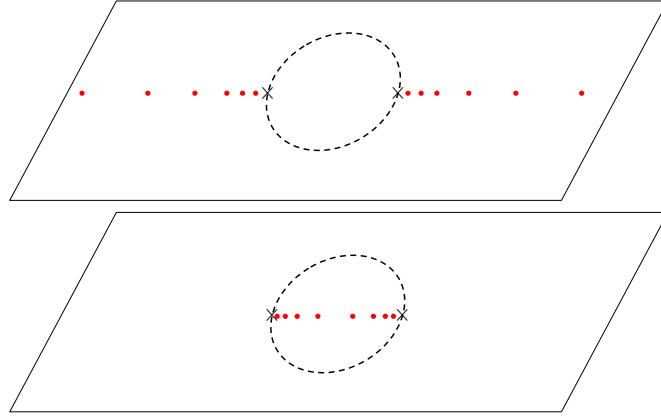
Figure 8.6.3: The positions of the separated variables for two-point functions at strong coupling. From the orthogonality of on-shell states (8.6.29), we conclude that the separated variables are either at the singular points outside the unit circle on the first sheet, or at the singular points inside the unit circle on the second sheet.

8.6.4 Semi-classical orthogonality of on-shell states at strong coupling

The key ideas for determining the analyticity of the Wronskians in the Landau-Lifshitz model were the requirement of the semi-classical orthogonality between two different on-shell states and the assumption of the continuity between the BPS correlators and the non-BPS correlators. Here we apply these two ideas to the analysis at strong coupling.

Just as for weak coupling, one can construct, at strong coupling, a different on-shell state by introducing an infinitesimal cut at the position of the singular point. We should, however, be careful about whether the perturbation is physical or not: As explained in section 8.6.2, in order to obtain a physical state, we should insert a positive filling fraction when the singular point is outside the unit circle, whereas we should insert a negative filling fraction when the singular point is inside the unit circle. With this in mind, we now impose the orthogonality condition

$$\langle \psi | \psi + \delta\psi \rangle = 0 \,. \tag{8.6.29}$$

Here $\delta\psi$ must correspond to a physical perturbation in the sense explained above. Now it is not so hard to verify that the argument in section 8.3.2 applied to the present case leads to the conclusion that the separated variables are at the singular points outside the unit circle on the first sheet, or at the singular points inside the unit circle on the second sheet (see figure 8.6.3). Then, repeating the argument[37] given in section 8.4.2, with the above modification taken into account, we can determine the poles and the zeros of the Wronskians. The results are summarized in table 8.6.1.

---

[37]Since the monodromy relation at strong coupling takes exactly the same form as (8.1.31), the equation (8.4.7) holds without modification.

Now, using these analyticity properties, we can solve the Riemann-Hilbert problem and determine the individual Wronskians as described in section 8.4.3. The main difference in the present case is that the Wronskians change the analyticity when they cross $|x| = 1$. This leads to extra integration contours around the unit circle. Once the Wronskians are determined, we can compute the angle variable and then determine the structure constants using (8.6.18) and (8.6.24). The results will be given explicitly in the next subsection.

		$1/\sin p_i$	$1/\sin p_j$	$\sin \dfrac{p_i + p_j + p_k}{2}$	$\sin \dfrac{p_i + p_j - p_k}{2}$
$\|x\| > 1$	$\langle i_+, j_+\rangle$	✓	✓	✓	✓
	$\langle i_-, j_-\rangle$				
$\|x\| < 1$	$\langle i_+, j_+\rangle$				
	$\langle i_-, j_-\rangle$	✓	✓	✓	✓

		$1/\sin p_i$	$1/\sin p_j$	$\sin \dfrac{p_i - p_j + p_k}{2}$	$\sin \dfrac{-p_i + p_j + p_k}{2}$
$\|x\| > 1$	$\langle i_+, j_-\rangle$	✓		✓	
	$\langle i_-, j_+\rangle$		✓		✓
$\|x\| < 1$	$\langle i_+, j_-\rangle$		✓		✓
	$\langle i_-, j_+\rangle$	✓		✓	

Table 8.6.1: The analytic properties of $\langle i_\pm, j_\pm\rangle$ on the $[u, u, u]$-sheet at strong coupling.

8.6.5 Results and discussions

We now write down the results for the three-point functions at strong coupling explicitly and compare them with the results in [111].

Type I-I-II three-point functions

Let us first consider the type I-I-II three-point functions. Below we assume that $\mathcal{O}_1$ and $\mathcal{O}_2$ belong to $\mathrm{SU}(2)_R$ while $\mathcal{O}_3$ belongs to $\mathrm{SU}(2)_L$. For such a three-point function, the result has the following structure:

$$\ln C_{123} = \mathcal{K} + \mathcal{D}_{\mathrm{S}} - \mathcal{D}_{\mathrm{AdS}} \tag{8.6.30}$$

Here $\mathcal{K}$ is the kinematical part given by

$$\mathcal{K} = \sum_{\{i,j,k\}\in\mathrm{cperm}\{1,2,3\}} (Q_i + Q_j - Q_k)\ln\langle n_i, n_j\rangle + (\tilde{Q}_i + \tilde{Q}_j - \tilde{Q}_k)\ln\langle \tilde{n}_i, \tilde{n}_j\rangle$$
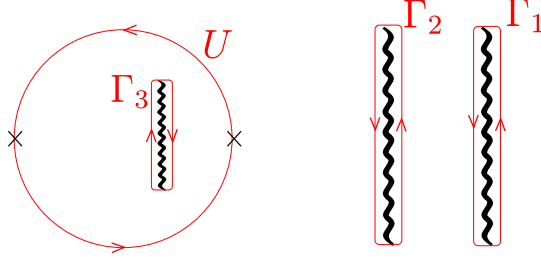$$- (\Delta_i + \Delta_j - \Delta_k)\ln|x_i - x_j|, \tag{8.6.31}$$

Figure 8.6.4: Integration contours for the type I-I-II three-point functions. Γ_1 and Γ_2 encircle counterclockwise the branch cuts of p_1 and p_2 respectively, whereas Γ_3 goes around the branch cuts of p_3 clockwise. U is the contour which goes counterclockwise around the unit circle.

where Q_i and $\tilde{Q}_i$ are the S^3 global charges of the operator $\mathcal{O}_i$, and the term in the second line comes form the AdS part. $\mathcal{D}_\mathrm{S}$ and $\mathcal{D}_\mathrm{AdS}$ denote the dynamical parts coming from the S^3 part and AdS_3 part respectively. Both $\mathcal{D}_\mathrm{S}$ and $\mathcal{D}_\mathrm{AdS}$ consist of several factors as

$$\mathcal{D}_\mathrm{S} = (\mathcal{L} + \mathcal{R})_\mathrm{S} + \mathcal{N}_\mathrm{S}, \qquad \mathcal{D}_\mathrm{AdS} = (\mathcal{L} + \mathcal{R})_\mathrm{AdS} + \mathcal{N}_\mathrm{AdS}, \tag{8.6.32}$$

and each factor is given as follows:

$$
\begin{aligned}
(\mathcal{L} + \mathcal{R})_\mathrm{S} =& \frac{1}{2} \left(\oint_{2U} \frac{du}{2\pi} \mathrm{Li}_2 \left(e^{ip_1 + ip_2 + ip_3} \right) + \oint_{\Gamma_1 \cup \Gamma_2 \cup 2U} \frac{du}{2\pi} \mathrm{Li}_2 \left(e^{ip_1 + ip_2 - ip_3} \right) \right. \\
& \left. + \oint_{\Gamma_1 \cup \Gamma_3 \cup 2U} \frac{du}{2\pi} \mathrm{Li}_2 \left(e^{ip_1 - ip_2 + ip_3} \right) + \oint_{\Gamma_2 \cup \Gamma_3 \cup 2U} \frac{du}{2\pi} \mathrm{Li}_2 \left(e^{-ip_1 + ip_2 + ip_3} \right) \right), \\
\mathcal{N}_\mathrm{S} =& -\frac{1}{2} \sum_k \oint_{\Gamma_k \cup 2U} \frac{du}{2\pi} \mathrm{Li}_2 \left(e^{2ip_k} \right),
\end{aligned}
$$

$$\tag{8.6.33}$$

$$
\begin{aligned}
(\mathcal{L} + \mathcal{R})_\mathrm{AdS} =& \sum_{\{i,j,k\} \in \mathrm{cperm}\{1,2,3\}} \oint_U \frac{du}{2\pi} \mathrm{Li}_2 \left(e^{i\hat{p}_i + i\hat{p}_j - i\hat{p}_k} \right), \\
\mathcal{N}_\mathrm{AdS} =& -\sum_k \oint_U \frac{du}{2\pi} \mathrm{Li}_2 \left(e^{2i\hat{p}_k} \right).
\end{aligned}
$$

$$\tag{8.6.34}$$

The contours of integration are depicted in figure 8.6.4 and $\hat{p}_i$ is the AdS quasi-momentum given by

$$\hat{p}_i = \frac{\Delta_i x}{2g(x^2 - 1)}. \tag{8.6.35}$$

A few remarks are in order. Firstly, as shown in (8.6.33), the integrals along the unit circle are multiplied by the extra factor of 2 (denoted by $2U$) as compared to the integrals

along the cuts. This factor can be deduced by carefully applying the argument given in section 8.5 to the strong coupling analysis. Roughly speaking, this is because the integrals along the unit circle exist on every sheet of the eight-sheeted Riemann surface whereas the integrals along the cuts exist only on some (roughly the half) of the sheets (see figure 8.4.5). Secondly, each integral along U is actually divergent owing to the poles in p_i at $x = \pm 1$. However, such divergences cancel out when we combine all the terms in (7.30). To illustrate this point, let us consider the integral

$$\int_U \frac{du}{2\pi} \mathrm{Li}_2\left(e^{i(p_1+p_2+p_3)}\right) . \tag{8.6.36}$$

Since we are interested in the behavior around $x = \pm 1$, where the integrand develops singularities, we can approximate the quasi-momenta by their asymptotic form just as in (7.35), namely $p_i(x) \sim \Delta_i x/(2g(x^2 - 1))$. To see the behavior in the vicinity of $x = \pm 1$ on the unit circle, we parametrize the Zhukowsky variable as $x = e^{i\theta}$ near $x = +1$ and as $x = -e^{-i\theta}$ near $x = -1$ and expand the expression for $p_i(x)$ above with respect to θ. In both cases, the result reads

$$p_i(x) \sim -\frac{i}{2\theta} + O(\theta) . \tag{8.6.37}$$

Plugging this expression into the dilogarithm, we obtain

$$\mathrm{Li}_2\left(e^{i(p_1+p_2+p_3)}\right) \sim \mathrm{Li}_2\left(e^{(\Delta_1+\Delta_2+\Delta_3)/(4g\theta)}\right) . \tag{8.6.38}$$

Since $\mathrm{Li}_2(0)$ is finite, there will be no singularity when θ approaches zero from below (*i.e.* when the integration variable is on the lower semi-circle). On the other hand, when θ approaches zero from above, the argument of the dilogarithm diverges and we need to use its asymptotic expression

$$\mathrm{Li}_2(z) \propto -\frac{1}{2}\log^2(-z) - \frac{\pi^2}{6} + O(z^{-1}) \qquad (|z| \to \infty) , \tag{8.6.39}$$

to obtain

$$\mathrm{Li}_2\left(e^{i(p_1+p_2+p_3)}\right) \sim -\frac{1}{2}\left(\frac{\Delta_1 + \Delta_2 + \Delta_3}{4g\theta} \pm \pi i\right)^2 . \tag{8.6.40}$$

Here the sign in front of πi depends on the choice of the branch of the logarithm. However, as we see below, the final result does not depend on the choice of this sign. As can be seen clearly, this expression contains a double pole and a single pole with respect to θ.

Now if we combine all the terms contained in (7.30), we get

$$-\frac{1}{2}\left[\left(\frac{\Delta_1 + \Delta_2 + \Delta_3}{4g\theta} \pm \pi i\right)^2 + \left(\frac{\Delta_1 + \Delta_2 - \Delta_3}{4g\theta} \pm \pi i\right)^2 \right.$$
$$+ \left(\frac{\Delta_1 - \Delta_2 + \Delta_3}{4g\theta} \pm \pi i\right)^2 + \left(\frac{-\Delta_1 + \Delta_2 + \Delta_3}{4g\theta} \pm \pi i\right)^2 \tag{8.6.41}$$
$$\left. - \left(\frac{2\Delta_1}{4g\theta} \pm \pi i\right)^2 - \left(\frac{2\Delta_2}{4g\theta} \pm \pi i\right)^2 - \left(\frac{2\Delta_3}{4g\theta} \pm \pi i\right)^2 \right],$$

which add up to the finite result $-(\pi^2)/2$. This confirms the absence of the singularities in the full expression (7.30). Thirdly, as in the weak coupling, the integrals along the cuts can be re-expressed by pushing some of the contours onto the second sheet:

$$(\mathcal{L} + \mathcal{R})_{\mathrm{S}}|_{\text{along }\Gamma_i} = \oint_{\Gamma_1 \cup \Gamma_2} \frac{du}{2\pi} \mathrm{Li}_2\left(e^{ip_1 + ip_2 - ip_3}\right) + \oint_{\Gamma_3} \frac{du}{2\pi} \mathrm{Li}_2\left(e^{ip_3 + ip_1 - ip_2}\right). \tag{8.6.42}$$

Here the first term can be interpreted as the contribution from the $\mathrm{SU}(2)_R$ whereas the second term can be regarded as coming from the $\mathrm{SU}(2)_L$. However, such factorization is not complete at strong coupling since the integrals along the unit circles cannot be rewritten in a similar manner.

Relation with the hexagon form factor

Let us make a comment on the relation with the hexagon form factor approach. As given in [153], the result from the hexagon form factor consists of two parts: One is the *asymptotic part*, which is given by the sum over partitions of the physical rapidities, and the other is the *wrapping correction*, which is the contribution from the mirror particles. In [153], they showed in simple cases that the integration along the branch cuts arises from the asymptotic part whereas the integration along the unit circle contains the first leading wrapping correction. More recently, it was demonstrated in [158] that, by partially resumming the mirror particle contributions, one could get an integral of the dilogarithm along the unit circle and correctly reproduce a part of our results (8.6.30). It would be an very interesting future problem to try to resum all the hexagon form factors at strong coupling and reproduce our full result, which would account for various more complicated processes involving the mirror particles.

BPS limit and Frolov-Tseytlin limit

We now study several limits of the result (8.6.30) and perform the consistency checks. Let us first consider the three-point functions of the BMN vacuum. As the quasi-momentum

for the BMN vacuum does not have any branch cuts, we only have integrals around the unit circle in that case. Furthermore, since the quasi-momenta in S^3 and AdS_3 coincide for the BMN vacuum, the two dynamical factors become identical, *i.e.* $\mathcal{D}_\mathrm{S} = \mathcal{D}_\mathrm{AdS}$, and cancel out in (8.6.30). Therefore we only have a contribution from the kinematical part in the final answer. This is consistent with the fact that the BPS three-point function does not receive quantum corrections.

Let us next study the Frolov-Tseytlin limit [73] by taking the charges r and l to be much larger than the coupling constant g while keeping the mode numbers of the cuts to be finite. In terms of the spectral curve, this amounts to pushing the branch cuts far away from the unit circle. More precisely, the branch cuts for p_1 and p_2 are pushed out into the region $|x| \gg 1$ whereas the branch cuts of p_3 are confined to the region $|x| \ll 1$. In such a limit, we can approximate the quasi-momenta on the unit circle by the quasi-momenta of the BMN vacuum. As explained above, for the BMN vacuum, integrals along the unit circle cancel out between S^3 and AdS_3. Thus, in the Frolov-Tseytlin limit, the integrals along the unit circle become negligible.

To study the remaining contributions, it is convenient to express the result (8.6.30) in terms of $\bar{p}_3$ defined by

$$\bar{p}_3(x) \equiv -p_3(1/x) \, . \tag{8.6.43}$$

As explained in Appendix K, $\bar{p}$ can be interpreted as the quasi-momentum defined on a different sheet in the full eight-sheeted spectral curve and the relation (8.6.43) is nothing but the $\mathbb{Z}_4$ automorphism of the string sigma model in $AdS_5 \times S^5$. It is $\bar{p}_3$ that is connected to the quasi-momentum for the $SU(2)_L$-sector at weak coupling. Now, to write down the expression in the Frolov-Tseytlin limit, we need to know the limiting forms of the quasi-momenta and the rapidity variable. In the region $|x| \gg 1$, $p_1(x)$ and $p_2(x)$ become the quasi-momenta in the Landau-Lifshitz model, whereas if $|x| \ll 1$ they approach their asymptotic forms around $x = 0$,

$$p_{1,2} \sim \frac{\tilde{Q}_{1,2}}{g} x \, . \tag{8.6.44}$$

Similarly, $\bar{p}_3(x)$ becomes the quasi-momentum of the Landau-Lifshitz model if $|x| \gg 1$ whereas it approaches

$$\bar{p}_3 \sim \frac{Q_3}{g} x \, , \tag{8.6.45}$$

in the region $|x| \ll 1$. As for the rapidity variable, it takes the following asymptotic form:

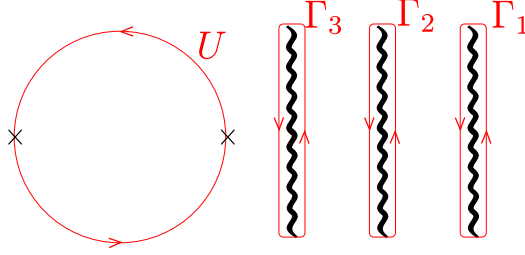$$u(x) \sim \begin{cases} gx & |x| \gg 1 \\ g/x & |x| \ll 1 \end{cases} . \tag{8.6.46}$$

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Figure 8.6.5: Integration contours for the type I-I-I correlators. Γ_i encircles the branch cuts of p_i counterclockwise. Here again U is the contour that goes counterclockwise around the unit circle.

Using these asymptotic forms and replacing the global charges Q_i and $\tilde{Q}_i$ with the spin-chain variables as given in (8.5.20), we can verify that (8.6.30) in the Frolov-Tseytlin limit coincides with the result at weak coupling (8.5.15).

One can also study the next-leading order correction to the Frolov-Tseytlin limit and compare it with the results in [127]. In [127], based on the previous results at strong coupling [111], they concluded that the next-leading order correction to the Frolov-Tseytlin limit agrees with the one-loop structure constant at weak coupling except for integration contours. Since we now have contours[38] which coincide with the weak coupling ones in the Frolov-Tseytlin limit, the results match also at this order. For details, see section 6 of [127].

Type I-I-I three-point functions

Next we consider the Type I-I-I three-point functions. As in section 8.5.2, we consider the case where all the operators belong to $\mathrm{SU}(2)_R$. Also in this case, the result can be expressed as

$$\ln C_{123} = \mathcal{K} + \mathcal{D}_\mathrm{S} - \mathcal{D}_\mathrm{AdS} . \tag{8.6.47}$$

Here $\mathcal{K}$ and $\mathcal{D}_\mathrm{AdS}$ are given by the same expressions as before, namely (8.6.31), (8.6.32) and (8.6.34). On the other hand, $\mathcal{D}_\mathrm{S}$ for the Type I-I-I three-point function is given by

$$\mathcal{D}_\mathrm{S} = (\mathcal{L} + \mathcal{R})_\mathrm{S} + \mathcal{N}_\mathrm{S} , \tag{8.6.48}$$

with

$$(\mathcal{L} + \mathcal{R})_\mathrm{S} = \frac{1}{2} \sum_{\{i,j,k\} \in \mathrm{cperm}\{1,2,3\}} \left(\oint_{\Gamma_i \cup \Gamma_j \cup 2U} \frac{du}{2\pi} \mathrm{Li}_2 \left(e^{ip_i + ip_j - ip_k} \right) \right) ,$$
$$\mathcal{N}_\mathrm{S} = -\frac{1}{2} \sum_k \oint_{\Gamma_k \cup 2U} \frac{du}{2\pi} \mathrm{Li}_2 \left(e^{2ip_k} \right) . \tag{8.6.49}$$

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[38]The relation between the results in this chapter and the results in [111] will be briefly discussed later.

The integration contours in (8.6.49) are depicted in figure 8.6.5.

We can study the Frolov-Tseytlin limit also in this case and the result again matches with the result at weak coupling (8.5.24).

Comparison with the result in [111]

Before ending this section, let us comment on the relation with the previous results for the three-point functions at strong coupling [111].

In [111], we determined the analyticity of the Wronskians assuming that the saddle-point configuration of the worldsheet is smooth except at the positions of the vertex operators. The integration contours obtained under this assumption are more complicated than what we have found in this chapter and the result in the Frolov-Tseytlin limit did not quite agree with the result at weak coupling. This implies that the assumption of smoothness is not quite correct and the saddle-point configuration has extra singularities. Although counterintuitive it may seem at first thought, such extra singularities are not so uncommon as already pointed out in [111]. For instance, consider the finite gap solution for the two-point function whose spectral curve contains more than one cuts. Such a solution is given in terms of the ratio of the theta functions defined on the higher-genus Riemann surface. Although those ratios are free of singularities on the Lorentzian worldsheet, they have infinitely many poles[39] on the Euclidean worldsheet, which is more appropriate for studying the correlation functions. Such extra poles, if present, can affect the argument in [111] and change the integration contours. By contrast, the logic presented in this chapter is based on the orthogonality of the on-shell states, which is the *exact* quantum property of the system, and therefore would be more universal and reliable.

8.7 Summary of this chapter

In this chapter, we studied the semi-classical three-point function in the SU(2)-sector of $\mathcal{N} = 4$ super Yang-Mills theory in four dimensions at weak coupling. The key idea was to express it as a saddle-point value of the coherent-state path integral and utilize the classical integrability of the Landau-Lifshitz model. This revealed the nature of the semi-classical structure constant as a generating function of the angle variables. For the computation of the angle variables, many of the machineries developed for the strong coupling analysis could be transplanted, the most important among which are the expression of the angle variables in terms of the Wronskians and the functional equation for the Wronskians. To

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[39]Such poles do not correspond to the insertion of vertex operators and do not affect the monodromy relation.

solve the functional equation, we developed a new logic to determine the analyticity, which is based on the orthogonality of two different on-shell states. The final results agree with the results in the literature and also make predictions for as-yet-unknown semi-classical structure constants for certain types of three-point functions.

We then re-examined the strong coupling analysis based on our new logic. It led to a modification of the integration contours of the result obtained in [111] and rendered the result in the Frolov-Tseytlin limit to be in agreement with the weak coupling one. In addition, the new result is consistent with the recent hexagon form factor approach [153].

Part III

Conclusion

Chapter 9

Finale -summary and prospects-

In this thesis, we have discussed the power of integrability techniques to investigate various observables in $\mathrm{AdS}_5/\mathrm{CFT}_4$, particularly focusing on the three-point functions. In what follows, we conclude this thesis with a short summary of the contents and some prospects on the future directions.

9.1 Summary

We first gave a brief review on the basic facts about the $\mathrm{AdS}_5/\mathrm{CFT}_4$ correspondence in section 2.

The spectrum problem was reviewed in chapter 3. The 1-loop dilatation operator of $\mathcal{N} = 4$ SYM is identified with the integrable spin chain Hamiltonian which is solved by the Bethe ansatz method (section 3.2). On the other hand, the classical equation of motion for the string on $AdS_5 \times S^5$ background can be rewritten as a flat connection and the solutions are described in terms of the algebraic curve language using the finite-gap method (section 3.3). Although the integrability makes the problem simplified and solved separately on each side, the matching of the spectrum is not clear at this point. To compare the spectrum of the both side, we need to take the semi-classical limit which is necessary to avoid the strong-weak nature of the duality. As a result, the spectrum of the both side precisely match and share exactly the same structure, namely, the spectral curve and the quasi-momentum (section 3.4). The developments beyond the perturbative analysis is briefly reviewed in 3.5, which gives an intuition on the integrability technique to build a bridge between the weak and strong coupling. In section 3.6, we extract several lessons from the studies on the spectrum problem and emphasize the importance of the semi-classical limit and the residual centrally extended $\mathfrak{su}(2|2)_L \oplus \mathfrak{su}(2|2)_R$ symmetry

The chapter 4 is devoted to the review of the three-point functions. The tree-level structure constant is obtained by taking all possible planar Wick contractions. Such com-

putations are efficiently performed by the so-called tailoring, which maps the problem to the computations of the overlap between two Bethe states (section 4.2). With the determinant formula for the scalar product of the Bethe states, the semi-classical structure constant can be obtained. We also explained the recent non-perturbative bootstrap approach for the three-point function in section 4.3. in that program, the structure constant is decomposed into a more fundamental object so-called the hexagon form factor, which is bootstrapped using the residual symmetry and some integrability axioms.

The main result of the thesis is composed of three chapters. In chapter 6, we developed and formulated two novel viewpoints and properties concerning the three-point functions at weak coupling in the SU(2) sector. One is a double spin-chain formulation of the spin-chain and the associated new interpretation of the operation of the Wick contraction. It was regarded as a signet projector which acts as a projection onto a singlet in the entire SO(4) sector, instead of an scalar product in the spin-chain Hilbert space. This formalism allows us to study a class of three-point functions of operators built upon more general spin-chain vacua than the special configuration discussed so far in the literature. Furthermore, this new viewpoint has the significant advantage over the conventional method: In the usual "tailoring" operation, the Wick contraction produces inner products between off-shell Bethe states, which cannot be in general converted into simple expressions. In contrast, our procedure directly produces the so-called partial domain wall partition functions, which can be expressed as determinants. Using this property, we derive simple determinantal representation for a broader class of three-point functions. The second new property uncovered in chapter 6 is the non-trivial identity satisfied by the three-point functions with monodromy operators inserted. Generically this relation connects three-point functions of different operators and can be regarded as a kind of Schwinger-Dyson equation. We also stated that it is a collection of the Ward identities reflecting the underlying symmetry.

In chapter 7, we generalize the SU(2) sector result to the full sector exploiting the oscillator representation for $\mathfrak{psu}(2,2|4)$. Furthermore, we derived the monodromy relation for the case of the so-called "harmonic R-matrix", as well as for the usual fundamental R-matrtix.

Finally, in chapter 8, we developed a new method of computing three-point functions in the SU(2) sector in the semi-classical regime at weak coupling, which closely parallels the strong coupling analysis. The structure threading two disparate regimes is the monodromy relation. We showed that this relation can be put to use directly for the semi-classical regime, where the dynamics is governed by the classical Landau-Lifshitz sigma model. Specifically, it reduces the problem to a set of functional equations, which can be solved

with the knowledge of the analyticity. To determine the analyticity, we developed a new universal logic applicable at both weak and strong couplings. As a result, compact semi-classical formulas are obtained for a general class of three-point functions at weak coupling including the ones whose semi-classical behaviors were not known before. In addition, the new analyticity argument applied to the strong coupling analysis leads to a modification of the integration contour, producing the results consistent with the recent hexagon bootstrap approach. It turned out that the modification also makes the Frolov-Tseytlin limit perfectly agree with the weak coupling form.

9.2   Future directions

There are several future directions worth exploring. Apart from challenging and far-reaching questions, there are numerous future directions that could be explored with the results and the techniques developed in our result. Below we briefly address some of them.

- **Loop corrections**

  To understand the loop corrections in our formalism is an interesting question. It was shown in [119, 127] that the loop corrections in the SU(2) sector can be neatly accounted for by the ingenious use of the inhomogeneities. It would be interesting if we can combine our formalism with the method in [119,127], and simplify and extend the computation at loop level. Since our formalism is based on the construction of the singlet projector, it is of particular importance to impose the singlet condition under the 1-loop corrected generators and find the vertex with which we can compute the 1-loop three-point functions.

- **Separation of variables (SoV) representation**

  The construction of separated variables is essential in classical integrable systems and also in our computations. Recently, the quantum counterpart of the SoV in XXX spin chain with $SU(N)$ symmetry is discussed [202]. Furthermore, the Q-functions in the quantum spectral curve formalism are expected to be the exact wave functions in the SoV basis. Hence, it would be worth to explore the formulation for the three-point functions in which the SoV plays a central role in order to build a bridge to the spectral problem.

- **Semi-classical limit of type I-I-I three-point functions**

  In chapter 8, we made predictions for the semi-classical limit of type I-I-I three-point functions at weak coupling (8.5.24). It would be an interesting problem to reproduce

it from the exact quantum expression given in [136, 146]. Since the result in [136, 146] has a more complicated structure than the type I-I-II three-point function, we probably need to develop new tools for studying it.

- **Resummation of the hexagon form factor at strong coupling**

  Another interesting direction of research is to analyze the strong-coupling semiclassical limit using the hexagon form factor formalism. It was shown in [153, 158] that a part of our result can be reproduced from the resummation of the hexagon form factor at strong coupling. It is important to further push this line of research and try to obtain the full strong coupling result from the hexagon form factor. This would be a litmus test for the hexagon form factor approach.

- **One-loop corrections at strong coupling**

  In the spectral problem, the power of the classical integrability and the associated spectral curve was not limited to the leading strong coupling limit. It also provided an efficient framework to study one-loop corrections around the classical solution such as (3.4.47) [99, 100]. The main idea there was to describe fluctuations as infinitesimal cuts inserted in the classical spectral curve. In chapter 8, we employed a very similar idea to determine the analytic properties of the Wronskians. It would then be extremely interesting, by extending our argument, to try to include the one-loop corrections . As a first step, it may be simpler to first analyze the weak coupling limit since the next-leading correction in the semi-classical limit was already computed by other means [128].

- **Application to other quantities**

  It would also be interesting to apply the method discussed here to other quantities in $\mathcal{N} = 4$ SYM. Of particular interest among them is the four-point function. The four-point function at the tree-level was studied in the paper [195] using integrability. However, even at that level, the resultant expression is rather involved owing to the complicated combinatorics of Wick contractions. In order to uncover a hidden structure, it might be helpful to study their semi-classical limit using our formalism. Such a structure is already known at strong coupling where it was shown that the four-point function of semi-classical operators can be described by the functional equation called χ-system [196]. It would be interesting to try to construct the weak-coupling counterpart of the χ-system. In addition, it might also be possible to use our framework to study non-planar observables such as the non-planar dilatation operator.

- **Entanglement entropy in integrable spin chains and field theories**

  Another interesting possibility is to apply the ideas and the techniques of chapter 8 to the computation of entanglement entropy in general integrable spin chains and field theories. To compute the entanglement entropy, one must first construct a reduced density matrix. In the case of spin chains, this can be achieved by preparing two identical states, cutting them into two halves and gluing the left (or the right) halves. This procedure is similar to the tailoring method for the three-point function [115]. Thus, it may be possible to study the entanglement entropy of the semi-classical state, which contains a large number of long wave-length excitations, using the formalism developed in chapter 8. This would be of particular interest since the entanglement entropy for such a highly excited state is difficult to compute by other methods.

More on monodromy relation

As for the prospects, of paramount importance is to further explore the implication of the cognate integrability structure at weak and strong coupling, which we discussed in chapter 8. Given the importance of the monodromy relation at the tree level and at strong coupling, a natural next step is to study it at higher loops. This may lead to a first-principle derivation of the integrable structure for three-point functions at finite coupling.

Another important structure worth mentioning in this regard is the striking similarity between our functional equations (8.4.7), which are the direct consequence of the monodromy relation, and the relations[1] constraining the light-cone string vertex in the pp-wave background [151]. It would be interesting to figure out the reason for this similarity. More generally, clarifying the integrable structure threading gauge and string theories would be a cornerstone for deeper understanding of the AdS/CFT correspondence. It may also yield practical merit if it leads to a new formulation of integrability for structure constants, which is more powerful than the existing approaches.

As a step toward this direction, the higher spin/vector model duality [204] is a nice laboratory since the operators have no anomalous dimensions in the large N limit and receive no renormalizations. Although the spectrum problem is trivial, the three-point functions for the higher spin operators depend non-trivially on the 'tHooft coupling λ, whose structure is severely constrained by the higher spin symmetry [6, 7]. To put it differently, we can study the non-trivial three-point functions without concerning the spectrum problem in a concrete set up. In [205], the three-point functions for higher spin

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[1]See (5.26) and (5.27) in [151].

currents are obtained by solving the tree-level monodromy relation using a momentum-twistor like representation. If one turns on the Chern-Simons coupling, it is known that the parity violating structures appear [7]. Accordingly, the monodromy relation is deformed in the finite coupling, however, it is expected that such deformation should be integrable in some sense as the three-point functions are completely fixed by the higher spin symmetry. Hence, it would be interesting to explore the relation between the monodromy relation which is a collection of the non-trivial Ward identities and the higher spin symmetry. Furthermore, in [206], it is pointed out that the integrable deformation parameter for the scattering amplitude in the ABJM theory, which is a supersymmetric Chern-Simon vector model, is naturally identified with the central charge of the $\mathfrak{osp}(6|4)$ algebra of the form $\exp(i\pi\tilde{\lambda})$, where $\tilde{\lambda}$ is a deformation parameter or the spectral parameter. In the case of $\tilde{\lambda} = 0$ or $\tilde{\lambda} = 1$, it is just a fermion number and an element of the three-dimensional little group $\mathbb{Z}_2$. This is quite suggestive since the statistics of particles are continuously modified and anyonic objects seem to appear, which often happens in the Chern-Simon theory and recently discussed in the context of the three-dimensional bosonization [6,7,207]. To sum up, we are led to the following bold conjecture:

$$\lambda = \text{spectral (defomation) parameter} = \text{anyonic phase}\,. \qquad (9.2.1)$$

It would be quite interesting to find a clue for the relation.

Mirror theory and quantum auxiliary problem

The hexagon program gives a non-perturbative approach for the three-point functions. In the asymptotic regime, excitations on the mirror edges do not contribute at all but we should incorporate them in general. Although any hexagon form factors with mirror particles can be transformed into the fundamental hexagons, it is necessary to perform the mirror transformation several times to obtain them. In anyway, we must understand the physic in the mirror edges to reveal a non-trivial connection between the hexagon and three-point functions.

For this purpose, the three-point function for "defect changing operators" on a half-BPS Wilson line is one of the simplest testing ground. This is analogous to the situation in section 5.2 as defect changing operators are actually the cusps which only change the scalars coupled to the segments of the line. Since the defect changing operator is a length zero operator, the contribution of mirror particles is crucial. It is possible to resum the Feynman diagrams using the Schwinger-Dyson equation in the ladder limit and confirm the match with perturbative computations at 2-loop level [208].

As repeatedly stressed in the chapter 8, the classical integrability allows us to express

the structure constant in terms of the fundamental building blocks, namely, the Wronskians of the eigenvectors of the monodromy matrix. Therefore, it is quite natural to ask whether the quantum analogues of such objects exist. In [209], it is pointed out that the solutions of the auxiliary linear problem in some integrable systems are the classical counterpart of the Zamolodchikov-Faddev operators, which are the creation operators for magnon excitations satisfying (3.5.9). Hence, it would be significant to explore the relation between the solution of auxiliary linear problem and the hexagon formalism to deepen our understanding of the mirror physics.

New formalisms

It would be important to understand more conceptual aspects of our new formalism. As can be seen from the figures 6.2.2 and 6.3.1, the way we computed the three-point functions is analogous to the description of the interaction in the string field theory. If our formalism proves to be powerful also at the loop level, it may provide a useful framework to understand how the string field theory in the AdS background emerges from perturbative gauge theory[2]. To reveal the mechanism in which the world-sheet emerges from a perturbative gauge theory is an important issue from an integrability point of view. In fact, the integrable structure of the world-sheet theory would persist and be robust regardless to the number of insertions of the vertex operators. Therefore, it suggests the world-sheet theory reflects the universal integrable structure, at least in the classical level.

Recently, the four-point functions are discussed in the context of the hexagon program [203]. The four-point functions are decomposed into four hexagons using the triangulation of the four-punctured Riemann sphere. Unlike the OPE expansion, the decomposition include not only the contributions of the single-trace operators but also those of the multi-trace operators. One of the interesting point is that the dependence of the cross ratios are encoded as the weight factor for the mirror particles on the gluing edge. It would be interesting to explore the general correlation functions using the hexagonization.

Taking these facts into account, it would be interesting to pursue a string theory-like formulation as with the case of scattering amplitudes for massless particles [210]. Similarly, it would be of use to develop a formulation based on the Grassmannian representation for the scattering amplitudes [211] since the relation with the integrability has been discussed in several papers [182–189].

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[2]Regarding this direction, there are several quite interesting works [114, 176], which discuss the connection between the perturbative computation in the field theory and the string-field-theoretic formalism from a slightly different point of view.

We hope to revisit some of these questions in the future and uncover the underlying mechanism of the unprecedented duality.

Appendix A

Commutation relations for $\mathfrak{u}(2,2|4)$

In this appendix, all the explicit forms of the commutation relations for the superconformal generators are listed in the D-scheme basis. First, the algebra for the bosonic generators, namely, SO(2,4) and SU(4) generators are given by

$$[M_\alpha{}^\beta, M_\gamma{}^\delta] = \delta_\gamma{}^\beta M_\alpha{}^\delta - \delta_\alpha{}^\delta M_\gamma{}^\beta, \quad [\bar{M}^{\dot\alpha}_{\dot\beta}, \bar{M}^{\dot\gamma}_{\dot\delta}] = \delta^{\dot\gamma}_{\dot\beta} \bar{M}^{\dot\alpha}_{\dot\delta} - \delta^{\dot\alpha}_{\dot\delta} \bar{M}^{\dot\gamma}_{\dot\beta}, \tag{A.1}$$

$$[M_\alpha{}^\beta, P_{\gamma\dot\delta}] = \delta^\beta_\gamma P_{\alpha\dot\delta} - \frac{1}{2}\delta_\alpha{}^\beta P_{\gamma\dot\delta}, \quad [\bar{M}^{\dot\alpha}_{\dot\beta}, P_{\gamma\dot\delta}] = -\delta^{\dot\alpha}_{\dot\delta} P_{\gamma\dot\beta} + \frac{1}{2}\delta^{\dot\alpha}_{\dot\beta} P_{\gamma\dot\delta}, \tag{A.2}$$

$$[M_\alpha{}^\beta, K^{\dot\gamma\delta}] = -\delta^\delta_\alpha K^{\dot\gamma\beta} + \frac{1}{2}\delta_\alpha{}^\beta K^{\dot\gamma\delta}, \quad [\bar{M}^{\dot\alpha}_{\dot\beta}, K^{\dot\gamma\delta}] = \delta^{\dot\gamma}_{\dot\beta} K^{\dot\alpha\delta} - \frac{1}{2}\delta^{\dot\alpha}_{\dot\beta} K^{\dot\gamma\delta}, \tag{A.3}$$

$$[D, P_{\alpha\dot\beta}] = iP_{\alpha\dot\beta}, \; [D, K^{\dot\alpha\beta}] = -iK^{\dot\alpha\beta}, \; [D, M_\alpha{}^\beta] = [D, \bar{M}^{\dot\alpha}_{\dot\beta}] = 0, \tag{A.4}$$

$$[P_{\alpha\dot\beta}, K^{\dot\gamma\delta}] = \delta^\delta_\alpha \bar{M}^{\dot\gamma}_{\dot\beta} - \delta^{\dot\gamma}_{\dot\beta} M_\alpha{}^\delta + i\delta^\delta_\alpha \delta^{\dot\gamma}_{\dot\beta} D, \tag{A.5}$$

$$[R_a{}^b, R_c{}^d] = \delta_c{}^b R_a{}^d - \delta_a{}^d R_c{}^b. \tag{A.6}$$

The commutators between the fermionic generators and the conformal generators D, P, K are

$$[D, Q_\alpha^a] = \frac{i}{2}Q_\alpha^a, \; [D, Q_{\dot\alpha a}] = \frac{i}{2}Q_{\dot\alpha a}, \; [D, S_a^\alpha] = -\frac{i}{2}S_a^\alpha, \; [D, \bar{S}^{\dot\alpha a}] = -\frac{i}{2}\bar{S}^{\dot\alpha a}, \tag{A.7}$$

$$[P_{\alpha\dot\beta}, S_a^\gamma] = -\delta^\gamma_\alpha \bar{Q}_{\dot\beta a}, \quad [P_{\alpha\dot\beta}, \bar{S}^{\dot\gamma a}] = -\delta^{\dot\gamma}_{\dot\beta} Q_\alpha^a, \tag{A.8}$$

$$[K^{\dot\alpha\beta}, Q_\gamma^a] = \delta^\beta_\gamma \bar{S}^{\dot\alpha a}, \quad [K^{\dot\alpha\beta}, \bar{Q}_{\dot\gamma a}] = \delta^{\dot\alpha}_{\dot\gamma} S_a^\beta. \tag{A.9}$$

Under the action of the Lorentz generators and the R-symmetry generators, the fermionic generators transform as follows

$$[M_\alpha{}^\beta, Q_\gamma^a] = \delta^\beta_\gamma Q_\alpha^a - \frac{1}{2}\delta_\alpha{}^\beta Q_\gamma^a, \quad [\bar{M}^{\dot\alpha}_{\dot\beta}, \bar{Q}_{\dot\gamma a}] = -\delta^{\dot\alpha}_{\dot\gamma} \bar{Q}_{\dot\beta a} + \frac{1}{2}\delta^{\dot\alpha}_{\dot\beta} \bar{Q}_{\dot\gamma a}, \tag{A.10}$$

$$[M_\alpha{}^\beta, S^{\gamma a}] = -\delta^\gamma_\alpha S^{\beta a} + \frac{1}{2}\delta_\alpha{}^\beta S^{\gamma a}, \quad [\bar{M}^{\dot\alpha}_{\dot\beta}, \bar{S}^{\dot\gamma a}] = \delta^{\dot\gamma}_{\dot\beta} \bar{S}^{\dot\alpha a} - \frac{1}{2}\delta^{\dot\alpha}_{\dot\beta} \bar{S}^{\dot\gamma a}, \tag{A.11}$$

$$[R_a{}^b, Q_\alpha^c] = -\delta_a{}^c Q_\alpha^b + \frac{1}{4}\delta_a{}^b Q_\alpha^a, \quad [R_a{}^b, \bar{Q}_{\dot\alpha c}] = \delta_c{}^b \bar{Q}_{\dot\alpha a} - \frac{1}{4}\delta_a{}^b \bar{Q}_{\dot\alpha c}, \tag{A.12}$$

$$[R_a{}^b, S_c^\alpha] = \delta_c{}^b S_a^\alpha - \frac{1}{4}\delta_a{}^b S_c^\alpha, \quad [R_a{}^b, \bar{S}^{\dot\alpha c}] = -\delta_a{}^c \bar{S}^{\dot\alpha b} + \frac{1}{4}\delta_a{}^b \bar{S}^{\dot\alpha c}. \tag{A.13}$$

The anti-commutators for the fermionic generators are

$$\{Q^a_\alpha, \bar{Q}_{\dot{\beta}b}\} = \delta^a_b P_{\alpha\dot{\beta}}\,, \quad \{\bar{S}^{\dot{\alpha}a}, S^\beta_b\} = \delta^a_b K^{\dot{\alpha}\beta}\,, \tag{A.14}$$

$$\{Q^a_\alpha, S^\beta_b\} = \delta^a_b M_\alpha{}^\beta - \frac{i}{2}\delta^a_b \delta_\alpha{}^\beta (D + iC) + \delta^\beta_\alpha R_b{}^a\,, \tag{A.15}$$

$$\{\bar{S}^{\dot{\alpha}a}, \bar{Q}_{\dot{\beta}b}\} = -\delta^a_b \bar{M}^{\dot{\alpha}}{}_{\dot{\beta}} - \frac{i}{2}\delta^a_b \delta^{\dot{\alpha}}{}_{\dot{\beta}} (D - iC) - \delta^{\dot{\alpha}}_{\dot{\beta}} R_b{}^a\,. \tag{A.16}$$

Notice that the central charge C appears in the anti-commutators of supercharges and superconformal charges. When we impose the condition of supertracelessness for the generators, we obtain $\mathfrak{su}(2,2|4)$. If we further drop the central charges on the RHS of the anti-commutators, we get $\mathfrak{psu}(2,2|4)$. Of course the central charge commutes with all the generators and the hypercharge essentially counts the fermion number $F(J)$ of the generator J:

$$[C, J] = 0\,, \quad [B, J] = \frac{1}{2}F(J)J\,. \tag{A.17}$$

The only generators carrying non-vanishing fermion numbers are the supercharges and the superconformal charges. Their fermion numbers are $F(\bar{Q}_{\dot{\alpha}a}) = F(S^\alpha_a) = 1$ and $F(Q^a_\alpha) = F(\bar{S}^{\dot{\alpha}a}) = -1$.

Appendix B

Details of algebraic Bethe ansatz

In this appendix we will explain the details of the algebraic Bethe ansatz which are omitted in 3.2.2. First, we will derive the Yang-Baxter algebra for the entries of the monodromy matrix by using a R-matrix. R-matrix is defined as a solution of the Yang-Baxter equation which ensures the existence of mutually commuting operators or integrability of the system. Then, we will show that the Hamiltonian of the Heisenberg model is actually obtained from the transfer matrix as (3.2.58).

Recall that the structure of the total Hilbert space $\mathcal{H}$ is the tensor product of the Hilbert spaces h_n defined on each site

$$\mathcal{H} = \bigotimes_{n=1}^{L} h_k \,, \tag{B.1}$$

where each Hilbert space h_n is spanned by the up-spin state and the down-spin state. Lax operator $L_{n,a}$ acts on the tensor product space of h_n and auxiliary space $V_a \cong \mathbb{C}^2$ and it is defined as (3.2.53). Here the subscript a of Lax operator denotes the auxiliary space V_a introduced for clarity. The Lax operator can be re-expressed in terms of the permutation operator $P_{n,a}$ as follows[1]:

$$L_{n,a}(u) = \left(u - \frac{i}{2} \right) \mathbf{1} + i P_{n,a} \,. \tag{B.2}$$

Although our goal is to derive the Yang-Baxter algebra, which is the set of exchange relations for the components of the monodromy matrix, it is convenient to first consider the exchanging relation for the Lax operators. For this purpose, let us introduce R-matrix that acts on the tensor product of the two auxiliary space $V_{a_1} \otimes V_{a_2}$

$$R_{a_1,a_2}(u) := u\mathbf{1} + i P_{a_1,a_2} \,. \tag{B.3}$$

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[1]For simplicity, we will consider the case of homogeneous limit, *i.e.* all θ_n's are set to zero. We can easily generalize to the inhomogeneous case.

It can be explicitly expressed by a matrix form as follows:

$$R_{a_1,a_2}(u) = \begin{pmatrix} u+i & 0 & 0 & 0 \\ 0 & u & i & 0 \\ 0 & i & u & 0 \\ 0 & 0 & 0 & u+i \end{pmatrix}. \tag{B.4}$$

The R-matrix is a solution of the following equation:

$$R_{a_1,a_2}(u-v)R_{a_1,a_3}(u)R_{a_2,a_3}(v) = R_{a_2,a_3}(v)R_{a_1,a_3}(u)R_{a_1,a_2}(u-v), \tag{B.5}$$

which is known to be the Yang-Baxter equation and it is graphically expressed in 3.2.4. We can easily confirm that the R-matrix (B.3) indeed satisfies the Yang-Baxter equation by explicitly checking the relation for a basis of $V_{a_1} \otimes V_{a_2} \otimes V_{a_3}$. Conversely, if we assume the form of R-matrix as $R_{a_1,a_2}(u) := f(u)\mathbf{1} + g(u)P_{a_1,a_2}$,[2] and plug this into (B.5), we obtain

$$\frac{f(u-v)}{g(u-v)} = \frac{f(u)}{g(u)} - \frac{f(v)}{g(v)}. \tag{B.6}$$

This means that we can set $f(u)/g(u) = u/i$ for rational class of the solutions and we get (B.3).[3]

Since the Lax operator (B.2) is essentially the form of the R-matrix (B.3), we find

$$R_{a_1,a_2}(u-v)L_{n,a_1}(u)L_{n,a_2}(v) = L_{n,a_2}(v)L_{n,a_1}(u)R_{a_1,a_2}(u-v). \tag{B.7}$$

This relation is sometimes called Yang-Baxter equation as well, however, we will call it $RLL = LLR$ relation. The similar relation holds for the monodromy matrices $\Omega_{a_1}(u) = L_{1,a_1}(u)\cdots L_{L,a_1}(u)$ and $\Omega_{a_2}(v) = L_{1,a_2}(v)\cdots L_{L,a_2}(v)$:

$$R_{a_1,a_2}(u-v)\Omega_{a_1}(u)\Omega_{a_2}(v) = \Omega_{a_2}(v)\Omega_{a_1}(u)R_{a_1,a_2}(u-v). \tag{B.8}$$

To prove this relation, notice that the left hand side of the equation (B.8) can be rearranged as follows:

$$R_{a_1,a_2}(u-v)\Omega_{a_1}(u)\Omega_{a_2}(v) = R_{a_1,a_2}(u-v)(L_{1,a_1}(u)L_{1,a_2}(v))\cdots(L_{L,a_1}(u)L_{L,a_2}(v)), \tag{B.9}$$

where we have used $[L_{a_1,n}(u), L_{m,a_2}(v)] = 0$ for $n \neq m$. One can prove (B.8) by successively using $RLL = LLR$ relation.

$$R_{a_1,a_2}(u-v)L_{1,a_1}(u)L_{1,a_2}(v)\cdots(L_{L,a_1}(u)L_{L,a_2}(v))$$
$$=L_{1,a_2}(v)L_{1,a_1}(u)R_{a_1,a_2}(u-v)\cdots(L_{L,a_1}(u)L_{L,a_2}(v))$$
$$\vdots$$
$$=(L_{1,a_2}(v)L_{1,a_1}(u))\cdots L_{L,a_2}(v)L_{L,a_1}(u)R_{a_1,a_2}(u-v) = (r.h.s). \tag{B.10}$$

---

[2]This is the most general assumption since $\mathbf{1}$ and P_{a_1,a_2} are the only $SU(2)$ invariant tensors.

[3]Of course, this does not uniquely fix the R-matrix since there is a choice of the normalization.

Thus, we have proved the relation (B.8) and the Yang-Baxter algebra immediately follows from it by expressing the relation in terms of the components explicitly. Note that $\Omega_{a_1}(u)$ and $\Omega_{a_2}(v)$ are the "matrix" of the $V_{a_1} \otimes V_{a_2}$ whose components are the operators acting on the Hilbert space $\mathcal{H}$:

$$\Omega_{a_1}(u) = \begin{pmatrix} A(u) & 0 & B(u) & 0 \\ 0 & A(u) & 0 & B(u) \\ C(u) & 0 & D(u) & 0 \\ 0 & C(u) & 0 & D(u) \end{pmatrix}, \quad \Omega_{a_2}(v) = \begin{pmatrix} A(v) & B(v) & 0 & 0 \\ C(v) & D(v) & 0 & 0 \\ 0 & 0 & A(v) & B(v) \\ 0 & 0 & C(v) & D(v) \end{pmatrix}.$$
(B.11)

Therefore we obtain the following set of exchanging relations

$$[A(u), A(v)] = 0 \tag{B.12}$$
$$[B(u), B(v)] = 0 \tag{B.13}$$
$$[C(u), C(v)] = 0 \tag{B.14}$$
$$[D(u), D(v)] = 0 \tag{B.15}$$
$$B(u)A(v) = b(u,v)A(v)B(u) + c(u,v)B(v)A(u) \tag{B.16}$$
$$B(v)D(u) = b(u,v)D(u)B(v) + c(u,v)B(u)D(v) \tag{B.17}$$
$$C(u)A(v) = b(v,u)A(v)C(u) + c(v,u)C(v)A(u) \tag{B.18}$$
$$C(v)D(u) = b(v,u)D(u)C(v) + c(v,u)C(u)D(v) \tag{B.19}$$
$$b(u,v)[C(u), B(v)] = -c(u,v)(A(u)D(v) - A(v)D(u)) \tag{B.20}$$
$$b(u,v)[A(u), D(v)] = -c(u,v)(C(u)B(v) - C(v)B(u)) \tag{B.21}$$

where $b(u,v) = u-v/u-v+i$, $c(u,v) = i/u-v+i$. Further, we can read the global $SU(2)$ transformation properties of $A(u) \dots D(u)$ from Yang-Baxter algebra by taking one of the rapidities to infinity. The complete set of commutation relations are given by

$$[S^3, A(u)] = [S^-, B(u)] = [S^+, C(u)] = [S^3, D(u)] = 0, \tag{B.22}$$
$$[S^3, C(u)] = +C(u), \quad [S^3, B(u)] = -B(u), \tag{B.23}$$
$$[S^+, A(u)] = -C(u), \quad [S^+, D(u)] = +C(u), \quad [S^+, B(u)] = A(u) - D(u), \tag{B.24}$$
$$[S^-, A(u)] = B(u), \quad [S^-, D(u)] = -B(u), \quad [S^-, C(u)] = -(A(u) - D(u)). \tag{B.25}$$

These relations are summarized as

$$\left[\Omega(u), \frac{\sigma^i}{2} + S^i \right] = 0, \tag{B.26}$$

This is a manifestation of the underlying $SU(2)$ symmetry since it simply states that the monodromy matrix is a "singlet" or invariant under the global $SU(2)$.

We will now show that the Hamiltonian is actually the logarithmic derivative of the transfer matrix as (3.2.58). Let us first confirm that the transfer matrices mutually commute: $[T(u), T(v)] = 0$. By multiplying $R^{-1}_{a_1,a_2}(u-v)$ to the both sides of the equation (B.8) from the left, we obtain

$$\Omega_{a_1}(u)\Omega_{a_2}(v) = R^{-1}_{a_1,a_2}(u-v)\Omega_{a_2}(v)\Omega_{a_1}(u)R_{a_1,a_2}(u-v). \tag{B.27}$$

Then, we take the trace on the space $V_{a_1} \otimes V_{a_2}$

$$\text{Tr}_{a_1}\Omega_{a_1}(u)\text{Tr}_{a_2}\Omega_{a_2}(v) = \text{Tr}_{a_2}\Omega_{a_2}(v)\text{Tr}_{a_1}\Omega_{a_1}(u), \tag{B.28}$$

where we have used the cyclicity of the trace and this is indeed $[T(u), T(v)] = 0$. To prove (3.2.58), we will exploit the fact that the Lax operator and its derivative at $u = i/2$ are simply the permutation operator and the identity operator respectively:

$$L_{n,a}(i/2) = iP_{a,n}, \quad \frac{dL_{n,a}}{du} = \mathbf{1}. \tag{B.29}$$

Therefore, the action of the monodromy matrix at $u = i/2$ becomes also simple:

$$\Omega(i/2) = i^L P_{1,a} \cdots P_{L,a}. \tag{B.30}$$

The product of the permutation operators can be rearranged as $P_{L,a}P_{L-1,L} \cdots P_{1,2}$, hence we obtain

$$T(u)|_{u=i/2} = \text{Tr}_a\Omega(u)|_{u=i/2} = i^L P_{L-1,L} \cdots P_{1,2}, \tag{B.31}$$

where we have used $\text{Tr}_a P_{L,a} = \mathbf{1}$. Similar calculation holds for the derivative of the transfer matrix thus we have

$$\frac{d}{du}T(u)|_{u=i/2} = i^{L-1}\sum_{n=1}^{L} P_{L-1,L} \cdots P_{n-1,n+1} \cdots P_{1,2}. \tag{B.32}$$

Multiplying $T(u)^{-1}|_{u=i/2}$ to the both sides of the above equation,

$$T^{-1}\frac{d}{du}T(u)|_{u=i/2} = -i\sum_{n=1}^{L} P_{n,n+1}. \tag{B.33}$$

This is essentially the form of the Hamiltonian and we have derived (3.2.58). Furthermore, $U = T(u)|_{u=i/2}/i^L$ is actually the shift operator. In fact, for any local operator X_n acting on n th site, we can show $UX_nU^{-1} = X_{n-1}$. To see this, note that $P_{m,n}X_nP_{m,n} = X_m$ and

the permutation operator $P_{n,m}$ commutes for any local operator X_l if l is neither equal to n nor m.

$$
\begin{aligned}
U X_n &= P_{L-1,L} \cdots P_{n-1,n} X_n P_{n-1,n-2} \cdots P_{1,2} \\
&= P_{L-1,L} \cdots \underbrace{P_{n-1,n} X_n P_{n-1,n}}_{X_{n-1}} P_{n-1,n} P_{n-1,n-2} \cdots P_{1,2} \\
&= X_{n-1} P_{L-1,L} \cdots P_{1,2} = X_{n-1} U .
\end{aligned}
\tag{B.34}
$$

At the second line, we have used $P_{n-1,n}^2 = \mathbf{1}$. Therefore, U is the shift operator and its logarithm is the momentum operator.

$$
e^{iP} = U = \frac{1}{i^L} T(u)\big|_{u=i/2} .
\tag{B.35}
$$

Appendix C

Centrally extended $\mathfrak{su}(2|2)_L \oplus \mathfrak{su}(2|2)_R$ symmetry and S-matrix

In this appendix, we summarize the basic facts on the centrally extended $\mathfrak{su}(2|2)_L \oplus \mathfrak{su}(2|2)_R$ symmetry and its bi-fundamental representation accompanied with the explicit form of the S-matrix, which is a necessary ingredient in the asymptotic Bethe ansatz.

C.1 Symmetry algebra

Once we choose a BMN vacuum of the spin chain, the original $\mathfrak{psu}(2,2|4)$ symmetry breaks down to the centrally extended $\mathfrak{su}(2|2)_L \oplus \mathfrak{su}(2|2)_R$ symmetry. From a string theory point of view, the uniform light-cone gauge fixed theory with the level matching condition relaxed enjoys the same symmetry as such generators commute with the Hamiltonian. An important things to note is that the symmetry is enhanced with three central charges, one of which is nothing but the Hamiltonian, due to the fact that the theory is decompactified and a part of the gauge transformation is mutated to the central charges. The centrally extended $\mathfrak{su}(2|2)$ algebra is given as follows

$$[\mathbb{R}_a{}^b, \mathbb{J}_c] = \delta_c^b \mathbb{J}_a - \frac{1}{2}\delta_a^b \mathbb{J}_c\,, \quad [\mathbb{R}_a{}^b, \mathbb{J}^c] = -\delta_a^c \mathbb{J}^b + \frac{1}{2}\delta_b^a \mathbb{J}^c\,, \tag{C.1}$$

$$[\mathbb{L}_\alpha{}^\beta, \mathbb{J}_\gamma] = \delta_\gamma^\beta \mathbb{J}_\alpha - \frac{1}{2}\delta_\alpha^\beta \mathbb{J}_\gamma\,, \quad [\mathbb{L}_\alpha{}^\beta, \mathbb{J}^\gamma] = -\delta_\alpha^\gamma \mathbb{J}^\alpha + \frac{1}{2}\delta_\alpha^\beta \mathbb{J}_c\,, \tag{C.2}$$

$$\{\mathbb{Q}_\alpha^a, \mathbb{Q}_\beta^b\} = \epsilon_{\alpha\beta}\epsilon^{ab}\mathbb{P}\,, \quad \{\mathbb{S}_a^\alpha, \mathbb{S}_b^\beta\} = \epsilon^{\alpha\beta}\epsilon_{ab}\mathbb{K}\,, \tag{C.3}$$

$$\{\mathbb{Q}_\alpha^a, \mathbb{S}_b^\beta\} = \delta_b^a \mathbb{L}_\alpha{}^\beta + \delta_\alpha^\beta \mathbb{R}_b{}^a + \delta_b^a \delta_\alpha^\beta \mathbb{C}\,. \tag{C.4}$$

Here, J^A (J_A) denotes any generators with upper (lower) indices and $(\mathbb{C}, \mathbb{P}, \mathbb{K})$ are the central charges. In order to see how the symmetry is embedded to $\mathfrak{psu}(2,2|4)$, it is of use

to use the oscillator representation. For definiteness, we write it down again[1]:

$$J^A_{\ B} = \bar\zeta^A \zeta_B\,, \quad \bar\zeta^A = \begin{pmatrix} \bar a_\alpha \\ b^{\dot\alpha} \\ \bar c_a \\ d^{\check a} \end{pmatrix}^A\,, \quad \zeta_A = \begin{pmatrix} a^\alpha \\ -\bar b_{\dot\alpha} \\ c^a \\ \bar d_{\check a} \end{pmatrix}_A\,. \tag{C.5}$$

Since the vacuum is annihilated by (a, b, c, d), the full $\mathfrak{u}(2,2|4)$ generators (7.1.44) and (C.6) are decomposed into the following way

$$J^A_{\ B} = \left(\begin{array}{cc|cc} Y_\alpha^{\ \beta} & -P_{\alpha\dot\beta} & Q_\alpha^b & \hat Q_{\alpha\dot b} \\ K^{\dot\alpha\beta} & Y^{\dot\alpha}_{\ \dot\beta} & \check S^{\dot\alpha b} & \bar S^{\dot\alpha}_{\ \dot b} \\ \hline S_a^\beta & -\bar Q_{\dot\beta a} & W_a^{\ b} & R_{a\dot b} \\ \hat S^{\dot a\beta} & -\bar Q^{\dot a}_{\ \dot\beta} & R^{\dot a b} & -W^{\dot b}_{\ \dot a} \end{array} \right)\,, \tag{C.6}$$

$$Y_\alpha^{\ \beta} = L_\alpha^{\ \beta} + \frac{1}{2}\delta_\alpha^{\ \beta}(D + C - B)\,, \quad Y^{\dot\alpha}_{\ \dot\beta} = -\bar L^{\dot\alpha}_{\ \dot\beta} + \frac{1}{2}\delta^{\dot\alpha}_{\ \dot\beta}(-D + C - B)\,, \tag{C.7}$$

$$W_a^{\ b} = R_a^{\ b} + \frac{1}{2}\delta_a^{\ b}(B - J)\,, \quad W^{\dot b}_{\ \dot a} = \bar R^{\dot b}_{\ \dot a} + \frac{1}{2}\delta^{\dot b}_{\ \dot a}(B + J)\,. \tag{C.8}$$

Here, C is the central charge of $\mathfrak{u}(2,2|4)$, B is the hypercharge, and J is related to the fermion number operator: $J = 1 - \frac{1}{2}(\bar c_a c^a + \bar d_{\dot a} d^{\dot a})$. Notice that the generators of the diagonal elements in each block do not mix the dotted and undotted indices. Hence, it turns out that $(L_\alpha^{\ \beta}, R_a^{\ b}, Q_\alpha^a, S_a^\alpha)$ form $\mathfrak{su}(2|2)_L$ and $(\bar L^{\dot\beta}_{\ \dot\alpha}, \bar R^{\dot b}_{\ \dot a}, \bar Q^{\dot a}_{\ \dot\alpha}, \bar S^{\dot\alpha}_{\ \dot a})$ form $\mathfrak{su}(2|2)_R$. Furthermore, from the anti-commutator between Q, S, we find one of the central charge $\mathbb{C}$ of $\mathfrak{su}(2|2)$ is given by $\frac{1}{2}(D - J)$. This is quite reasonable as $D - J$ is actually the total number operator for the oscillators and it commute with all the $\mathfrak{su}(2|2)_{L,R}$ generators above mentioned.

C.2   Representation

We shall start with the vacuum state, upon which we can add the excitations χ being the bi-fundamental representation of $\mathfrak{psu}(2|2)_L \oplus \mathfrak{psu}(2|2)_R$:

$$|0\rangle = |\ldots ZZZ\ldots\rangle\,. \tag{C.1}$$

$$\chi_{A\dot A} \in \{\Phi_{a\dot a}, \Psi_{\dot a\alpha}, \bar\Psi_{a\dot\alpha}, D_{\alpha\dot\alpha}Z\}\,, \quad a, \dot a = 1, 2\,, \quad \alpha, \dot\alpha = 3, 4\,. \tag{C.2}$$

Since the two $\mathfrak{su}(2|2)$ are completely factorized, it is of use to consider the tensor product of two $\mathfrak{su}(2|2)$ spin chain.

$$|\chi_{A\dot A}\rangle = |\mathcal{X}_A\rangle_L \otimes |\mathcal{X}_{\dot A}\rangle_R\,, \tag{C.3}$$

$$\mathcal{X}_A \in \{\phi_1, \phi_2 | \psi_1, \psi_2\}\,, \quad \mathcal{X}_{\dot A} \in \{\phi_{\dot 1}, \phi_{\dot 2} | \psi_{\dot 1}, \psi_{\dot 2}\}\,. \tag{C.4}$$

---

[1]Here, we simply denote $(\lambda_\alpha, \mu^\alpha)$ as $(\bar a_\alpha, a^\alpha)$ and $(\tilde\lambda_{\dot\alpha}, \tilde\mu^{\dot\alpha})$ as $(\bar b_{\dot\alpha}, b^{\dot\alpha})$.

Here, ϕ_a ($\phi_{\dot{a}}$) represents bosonic degrees of freedom and ψ_α ($\psi_{\dot{\alpha}}$) represents fermionic degrees of freedom for the $\mathfrak{su}(2|2)_L$ ($\mathfrak{su}(2|2)_R$). In what follows, we only concentrate on the half of the two $\mathfrak{su}(2|2)$ for simplicity. The asymptotic states are defined in the following way:

$$|\mathcal{X}_{A_1}(p_1)\dots\mathcal{X}_{A_n}(p_n)\rangle = \sum_{x_1 \ll \dots \ll x_n} e^{ip_k x_k} |\dots Z \overbrace{\mathcal{X}_{A_1}}^{x_1} Z \dots Z \overbrace{\mathcal{X}_{A_n}}^{x_n} Z \dots\rangle, \quad (\text{C.5})$$

Eigenstates of the Hamiltonian is an appropriate linear combination of the above asymptotic states in such a way that the state is periodic and invariant under the permutation. Due to the underlying integrable property, such an eigenstate is constructed from the Bethe ansatz and the-S-matrix is a crucial ingredient. Surprisingly, the residual symmetry uniquely determines the structure of the S-matrix up to a scalar phase. To see this, we first consider the symmetry representation on 1-particle state, which is summarized as follows:

$$R_a{}^b|\phi_c\rangle = \delta_c^b|\phi_a\rangle - \frac{1}{2}\delta_a^b|\phi_c\rangle\,, \quad L_\alpha{}^\beta|\psi_\gamma\rangle = \delta_\gamma^\beta|\psi_\alpha\rangle - \frac{1}{2}\delta_\alpha^\beta|\psi_\gamma\rangle\,, \quad (\text{C.6})$$

$$Q_\alpha^a|\phi_b\rangle = a\delta_b^a|\psi_\alpha\rangle\,, \quad Q_\alpha^a|\psi_\beta\rangle = b\epsilon_{\alpha\beta}\epsilon^{ab}|Z^+\phi_b\rangle\,, \quad (\text{C.7})$$

$$S_a^\alpha|\phi_b\rangle = c\epsilon_{ab}\epsilon^{\alpha\beta}|Z^-\psi_\beta\rangle\,, \quad S_a^\alpha|\psi_\beta\rangle = d\delta_\beta^\alpha|\phi_a\rangle\,, \quad (\text{C.8})$$

$$P|\mathcal{X}_A\rangle = ab|Z^+\mathcal{X}_A\rangle\,, \quad K|\mathcal{X}_A\rangle = cd|Z^-\mathcal{X}_A\rangle\,, \quad (\text{C.9})$$

$$C|\mathcal{X}_A\rangle = \frac{1}{2}(ad + bc)|\mathcal{X}_A\rangle\,, \quad |\mathcal{X}_A Z^\pm\rangle = e^{\pm ip}|Z^\pm\mathcal{X}_A\rangle\,. \quad (\text{C.10})$$

Here, $Z^\pm$ denotes the insertion or removal for the vacuum constituent field Z. From the action of the central charges P, K, it turns out that

$$P|\mathcal{X}_A\rangle = g\zeta(1 - e^{ip})|\mathcal{X}_A\rangle\,, \quad K|\mathcal{X}_A\rangle = \frac{g}{\zeta}(1 - e^{-ip})|\mathcal{X}_A\rangle\,. \quad (\text{C.11})$$

Therefore, $ab = g\zeta(1 - e^{ip})$ and $cd = \frac{g}{\zeta}(1 - e^{-ip})$. It is of use to introduce another parametrization for the coefficients of the supercharges

$$a = \sqrt{g}\eta\,, \quad b = \sqrt{g}\frac{\zeta}{\eta}\left(1 - \frac{x^+}{x^-}\right)\,, \quad c = \sqrt{g}\frac{i\eta}{\zeta x^+}\,, \quad d = \sqrt{g}\frac{x^+}{i\eta}\left(1 - \frac{x^-}{x^+}\right)\,, \quad (\text{C.12})$$

where $e^{ip} = x^+/x^-$. The closure of the algebra and the unitary representation further require the following constraints[2]

$$x^+ + \frac{1}{x^+} - x^- - \frac{1}{x^-} = \frac{i}{g}\,, \quad |g||\eta|^2 = -igx^+ + igx^-\,, \quad \frac{|\zeta|^2}{|x^-|^2} = \frac{1}{x^+x^-}\,. \quad (\text{C.13})$$

---

[2]The closure requires $ad - bc = 1$ and the unitarity leads $a = d^*, c = b^*$.

As the constant g is identified with the coupling constant and strictly positive, the above constraints can be solved as

$$x^\pm = x(u \pm i/2), \quad x(u) + \frac{1}{x(u)} = \frac{u}{g}, \quad \eta = \sqrt{-ix^+ + ix^-}, \quad \zeta = 1. \tag{C.14}$$

It should be noted that the above solution corresponds to the so-called "spin chain frame" [33] and η is given by $\eta = e^{ip/4}\sqrt{-ix^+ + ix^-}$ in the "string frame". At last, we find the following dispersion relation.

$$e^{ip} = \frac{x^+}{x^-}, \quad C = -ig\left(x^+ - x^- + \frac{1}{x^+} - \frac{1}{x^-}\right) = \sqrt{1 + 16g^2 \sin^2 \frac{p}{2}}. \tag{C.15}$$

C.3   S-matrix

The full action of the S-matrix is summarized as follows

$$\mathcal{S}_{12}|\phi_a^1 \phi_b^2\rangle = A_{12}|\phi_{\{a}^2 \phi_{b\}}^1\rangle + B_{12}|\phi_{[a}^2 \phi_{b]}^1\rangle + \frac{1}{2}C_{12}\epsilon_{ab}\epsilon^{\alpha\beta}|Z^- \psi_\alpha^2 \psi_\beta^1\rangle, \tag{C.1}$$

$$\mathcal{S}_{12}|\psi_\alpha^1 \psi_\beta^2\rangle = D_{12}|\psi_{\{\alpha}^2 \psi_{\beta\}}^1\rangle + E_{12}|\psi_{[\alpha}^2 \psi_{\beta]}^1\rangle + \frac{1}{2}F_{12}\epsilon_{\alpha\beta}\epsilon^{ab}|Z^+ \phi_a^2 \phi_b^1\rangle, \tag{C.2}$$

$$\mathcal{S}_{12}|\phi_a^1 \psi_\beta^2\rangle = G_{12}|\psi_\beta^2 \phi_a^1\rangle + H_{12}|\phi_a^2 \psi_\beta^1\rangle, \tag{C.3}$$

$$\mathcal{S}_{12}|\psi_\alpha^1 \psi_b^2\rangle = K_{12}|\psi_\alpha^2 \phi_b^1\rangle + L_{12}|\phi_b^2 \psi_\alpha^1\rangle, \tag{C.4}$$

where we employ the simplified notation $|\mathcal{X}_A(p_i)\mathcal{X}_B(p_j)\rangle = |\mathcal{X}_A^i\mathcal{X}_B^j\rangle$ and each component is given by

$$A_{12} = S_{12}^0 \frac{x_2^+ - x_1^-}{x_2^- - x_1^+} \frac{\eta_1\tilde{\eta}_2}{\tilde{\eta}_1\eta_2}\,, \tag{C.5}$$

$$B_{12} = S_{12}^0 \frac{x_2^+ - x_1-}{x_2^- - x_1^+}\left(1 - 2\frac{1 - 1/(x_2^- x_1^+)}{1 - 1/(x_2^+ x_1^+)}\frac{x_2^- - x_1^-}{x_2^+ - x_1^-}\right)\frac{\eta_1\tilde{\eta}_2}{\tilde{\eta}_1\eta_2}\,, \tag{C.6}$$

$$C_{12} = S_{12}^0 \frac{2\eta_1\tilde{\eta}_2}{x_1^+ x_2^+}\frac{1}{1 - 1/(x_1^+ x_2^+)}\frac{x_2^- - x_1^-}{x_2^- - x_1^+}\,, \tag{C.7}$$

$$D_{12} = -S_{12}^0\,, \tag{C.8}$$

$$E_{12} = -S_{12}^0\left(1 - 2\frac{1 - 1/(x_2^+ x_1^+)}{1 - 1/(x_2^- x_1^-)}\frac{x_2^+ - x_1^+}{x_2^- - x_1^+}\right)\,, \tag{C.9}$$

$$F_{12} = -S_{12}^0 \frac{2(x_1^+ - x_1^-)(x_2^+ - x_2^-)}{\tilde{\eta}_1\eta_2 x_1^- x_2^-}\frac{1}{1 - 1/(x_1^- x_2^-)}\frac{x_2^+ - x_1^+}{x_2^- - x_1^+}\,, \tag{C.10}$$

$$G_{12} = S_{12}^0 \frac{x_2^+ - x_1^+}{x_2^- - x_1^+}\frac{\eta_1}{\tilde{\eta}_1}\,, \tag{C.11}$$

$$H_{12} = S_{12}^0 \frac{\eta_1}{\eta_2}\frac{x_2^+ - x_2^-}{x_2^- - x_1^+}\,, \tag{C.12}$$

$$K_{12} = S_{12}^0 \frac{\tilde{\eta}_2}{\tilde{\eta}_1}\frac{x_1^+ - x_1^-}{x_2^- - x_1^+}\,, \tag{C.13}$$

$$L_{12} = S_{12}^0 \frac{\tilde{\eta}_2}{\eta_2}\frac{x_2^- - x_1^-}{x_2^- - x_1^+}\,. \tag{C.14}$$

Here, $\eta_k, \tilde{\eta}_k$ are given by $\eta_k = \tilde{\eta}_k = \gamma_k = \sqrt{-ix_k^+ + ix_k^-}$ in the spin chain frame, while $\eta_k = e^{ip_k/4}\gamma_k$, $\tilde{\eta}_1 = e^{ip_1/4+ip_2/2}\gamma_1$, $\tilde{\eta}_2 = e^{ip_2/2+ip_2/4}\gamma_2$ in the string frame. The scalar factor is determined by imposing the unitarity and crossing symmetry [45] and it is given by

$$S_{12}^0 = \frac{x_1^- - x_2^+}{x_1^+ - x_2^-}\frac{1 - 1/(x_1^+ x_2^-)}{1 - 1/(x_1^- x_2^+)}\sigma_{12}^2\,, \tag{C.15}$$

where σ_{12} is the BHL/BES dressing phase, which has the following integral expression [40, 45]:

$$\sigma_{12} = e^{i\theta}\,, \quad \theta = \chi(x_1^+, x_2^-) - \chi(x_1^-, x_2^-) - \chi(x_1^+, x_2^+) + \chi(x_1^-, x_2^+)\,, \tag{C.16}$$

$$\chi(x, y) = -i\oint_{|z|=1}\frac{dz}{2\pi i}\oint_{|z'|=1}\frac{dz'}{2\pi i}\frac{1}{x - z}\frac{1}{y - z'}\log\frac{\Gamma(1 + ig(z + 1/z - z' - 1/z'))}{\Gamma(1 + ig(z - 1/z - z' - 1/z'))}\,. \tag{C.17}$$

Appendix D

Kinematical dependence of the three-point function

In this appendix, we show that the kinematical dependence (*i.e.* the dependence on the parameters z_i) of the three-point functions can be completely determined by the invariance of the correlator under the symmetry group $\mathrm{SO}(4) \cong \mathrm{SU}(2)_L \times \mathrm{SU}(2)_R$ and the highest weight condition for the operators. This knowledge significantly simplifies the calculation, as elaborated in subsection 6.3.4.

As usual, we concentrate on the $\mathrm{SU}(2)_L$ sector. The key is the Ward identity (6.4.22) expressed using the coherent state parametrization (6.3.26). It is convenient to remove the trivial overall factor from the states in (6.1.30) and consider

$$|\hat{\mathcal{O}}_i\rangle_L := (1 + |z_i|^2)^{L_i} |\mathcal{O}_i\rangle_L = e^{z_i S_-} |\boldsymbol{u}^{(i)}; \uparrow^{\ell_i}\rangle_L \,, \tag{D.1}$$

where $L_i = \ell_i/2 - M_i$. Such a redefinition does not affect the Ward identity since $|\hat{\mathcal{O}}_i\rangle_L$ is related to $|\mathcal{O}_i\rangle_L$ simply by a multiplication of the scalar factor. It is important to note that the state $|\hat{\mathcal{O}}_i\rangle_L$ is independent of $\bar{z}_i$ as we have already implemented the highest weight condition. Hence, the remaining task is to determine the dependence on z_i.

As is rather well-known, on such a coherent state representation, the $\mathrm{SU}(2)_L$ generators have representations as differential operators. We can easily show

$$S_* |\hat{\mathcal{O}}_i\rangle_L = \rho_{z_i}(S_*) |\hat{\mathcal{O}}_i\rangle_L \,, \tag{D.2}$$

$$\rho_{z_i}(S_3) = L_i - z_i \frac{d}{dz_i} \,, \quad \rho_{z_i}(S_+) = L_i z_i - \frac{z_i^2}{2} \frac{d}{dz_i} \,, \quad \rho_{z_i}(S_-) = \frac{d}{dz_i} \,. \tag{D.3}$$

For instance, the action on S_3 on the state $|\hat{\mathcal{O}}_i\rangle_L$ can be computed as

$$
\begin{aligned}
S_3 |\hat{\mathcal{O}}_i\rangle_L &= S_3 e^{z_i S_-} |\boldsymbol{u}^{(i)}; \uparrow^{\ell_i}\rangle_L = e^{z_i S_-} (e^{-z_i S_-} S_3 e^{z_i S_-}) |\boldsymbol{u}^{(i)}; \uparrow^{\ell_i}\rangle_L \\
&= e^{z_i S_-} (S_3 - z_i S_-) |\boldsymbol{u}^{(i)}; \uparrow^{\ell_i}\rangle_L = \left(L_i - z_i \frac{d}{dz_i}\right) |\hat{\mathcal{O}}_i\rangle_L \,,
\end{aligned}
\tag{D.4}
$$

where L_i is the eigenvalue of S_3 on $|\boldsymbol{u}^{(i)}; \uparrow^{\ell_i}\rangle_L$.

Using this representation, Ward identity (6.4.22) can be expressed as

$$\sum_{i=1}^{3} \rho_{z_i}(S_*)\langle |\hat{\mathcal{O}}_1\rangle_L, |\hat{\mathcal{O}}_2\rangle_L, |\hat{\mathcal{O}}_3\rangle_L\rangle = 0 \,. \tag{D.5}$$

It is evident that this has exactly the same form as the global conformal Ward identity for three-point functions in 2d CFT if we identify $-L_i$ with the conformal dimensions. Thus, the z_i dependence can be uniquely fixed [197] as

$$\langle |\hat{\mathcal{O}}_1\rangle_L, |\hat{\mathcal{O}}_2\rangle_L, |\hat{\mathcal{O}}_3\rangle_L\rangle \propto z_{21}^{L_{12}} z_{32}^{L_{23}} z_{13}^{L_{31}} \,, \tag{D.6}$$

where $z_{ij} \equiv z_i - z_j$ and $L_{ij} \equiv L_i + L_j - L_k$. Therefore, the kinematical dependence of the three point function for the $\mathrm{SU}(2)_L$ sector is given by the simple form

$$\langle |\mathcal{O}_1\rangle_L, |\mathcal{O}_2\rangle_L, |\mathcal{O}_3\rangle_L\rangle \propto \left(\frac{1}{1+|z_1|^2}\right)^{L_1} \left(\frac{1}{1+|z_2|^2}\right)^{L_2} \left(\frac{1}{1+|z_3|^2}\right)^{L_3} \\ \times z_{21}^{L_{12}} z_{32}^{L_{23}} z_{13}^{L_{31}} \,. \tag{D.7}$$

Similarly, for the $\mathrm{SU}(2)_R$ sector the result is

$$\langle |\tilde{\mathcal{O}}_1\rangle_R, |\tilde{\mathcal{O}}_2\rangle_R, |\tilde{\mathcal{O}}_3\rangle_R\rangle \propto \left(\frac{1}{1+|\tilde{z}_1|^2}\right)^{R_1} \left(\frac{1}{1+|\tilde{z}_2|^2}\right)^{R_2} \left(\frac{1}{1+|\tilde{z}_3|^2}\right)^{R_3} \\ \times \tilde{z}_{21}^{R_{12}} \tilde{z}_{32}^{R_{23}} \tilde{z}_{13}^{R_{31}} \,, \tag{D.8}$$

where R_i is given by $\ell_i/2 - \tilde{M}_i$. It is important to note that the relations (D.7) and (D.8) take the following form in terms of the polarization spinors,

$$\begin{aligned} \langle |\mathcal{O}_1\rangle_L, |\mathcal{O}_2\rangle_L, |\mathcal{O}_3\rangle_L\rangle &\propto \langle \mathfrak{n}_1, \mathfrak{n}_2\rangle^{L_{12}} \langle \mathfrak{n}_2, \mathfrak{n}_3\rangle^{L_{23}} \langle \mathfrak{n}_3, \mathfrak{n}_1\rangle^{L_{31}} \,, \\ \langle |\tilde{\mathcal{O}}_1\rangle_R, |\tilde{\mathcal{O}}_2\rangle_R, |\tilde{\mathcal{O}}_3\rangle_R\rangle &\propto \langle \tilde{\mathfrak{n}}_1, \tilde{\mathfrak{n}}_2\rangle^{R_{12}} \langle \tilde{\mathfrak{n}}_2, \tilde{\mathfrak{n}}_3\rangle^{R_{23}} \langle \tilde{\mathfrak{n}}_3, \tilde{\mathfrak{n}}_1\rangle^{R_{31}} \,, \end{aligned} \tag{D.9}$$

where $\langle \mathfrak{n}, \mathfrak{m}\rangle \equiv \det(\mathfrak{n}, \mathfrak{m})$. This is precisely the structures observed in the computation at strong coupling [111].

It should be useful to make a small remark on the uniqueness of the kinematical dependence as determined by the symmetry argument. Although the results (D.7) and (D.8) above for the "SU(2) sector" are unique, this is not true in the case of higher rank sectors. For instance, in the SO(6) sector, the symmetry argument alone cannot fix the dependence completely and there exist several possible R-symmetry tensorial structures. In such cases, the three-point function is given by a linear combination of such allowed structures, whose coefficients depend on dynamics, for instance on 't Hooft coupling. Indeed, for the SO(2,4) sector, the existence of a large number of tensorial structures was found in [198].

Appendix E

General form of the monodromy relation for three-point functions

In this appendix we briefly discuss how more general forms of the monodromy relations can be constructed. Below, for simplicity we shall suppress the inhomogeneity parameters and consider the $\mathrm{SU}(2)_L$ sector.

The freedom in the form of the monodromy relation stems from the simple fact that by making an arbitrary shift of u the fundamental unitarity relation (6.4.1) can be rewritten as

$$\mathrm{L}(-u+a)\mathrm{L}(u+b) = -f(u+(b-a)/2) \cdot \mathbf{1} \,, \tag{E.1}$$

$$a+b = i \,. \tag{E.2}$$

Now suppose we split each monodromy operator into left and the right parts, similarly to the case of $\Omega_2^{+|-}$ in (6.4.19), in the form

$$\widehat{\Omega}_n(u) \equiv \Omega_n^l(u+a_n^l)\Omega_n^r(u+a_n^r) \,. \tag{E.3}$$

Then, by computing the three-point function $\langle \left(\widehat{\Omega}_1(u)\right)_{ij} |\mathcal{O}_1\rangle_L, \left(\widehat{\Omega}_2(u)\right)_{jk} |\mathcal{O}_2\rangle_L, \left(\widehat{\Omega}_3(u)\right)_{kl} |\mathcal{O}_3\rangle_L\rangle$, using the crossing relations and Wick contractions, we easily find that the conditions for the coefficients a_n^l and a_n^r for which the unitarity relation (E.1) works to yield the result proportional to $\delta_{il}\langle|\mathcal{O}_1\rangle_L, |\mathcal{O}_1\rangle_L, |\mathcal{O}_1\rangle_L\rangle$ are given by

$$a_2^l - a_1^r = a_3^l - a_2^r = a_3^r - a_1^l = i \,. \tag{E.4}$$

For the simple example we discussed in section 6.4.2, these relations are satisfied with $a_1^l = a_1^r = -i/2, a_2^l = i/2, a_2^r = -i/2, a_3^l = a_3^r = i/2$. In general, disregarding a common shift for all the $a_n^{r,l}$, there exist different monodromy relations which can be parametrized by two complex numbers. At the moment, the meaning of this freedom is unclear to us. It might be a special feature of the tree-level relation. In any case, deeper understanding of the monodromy relation is an important future problem.

Appendix F

Comment on the relation to the singlet state for the SU(2) sector

The exponential form of the singlet projector for $\mathfrak{u}(2,2|4)$ constructed in section 2.2 looks rather different from the simple non-exponential form given in our previous work [138] for the SU(2) subsector. If we write it explicitly in terms of the scalar states in this subsector, it is given by[1]

$$|\mathbf{1}_{12}\rangle_{\text{SO}(4)} = |Z\rangle \otimes |\bar{Z}\rangle - |X\rangle \otimes |(-\bar{X})\rangle - |(-\bar{X})\rangle \otimes |X\rangle + |\bar{Z}\rangle \otimes |Z\rangle. \qquad \text{(F.1)}$$

In this appendix, we briefly explain how this form is indeed obtained from the exponential form.

The SU(2) sector is only a part of the large Hilbert space in which the exponential state belongs. Also, in our previous paper, we were only considering the spin $1/2$ representation for $\text{SU}(2)_L$ and $\text{SU}(2)_R$. Thus to get our singlet formula (F.1) from the exponential form, we must project out such a sector from the full exponential projector.

It turns out that to do this appropriately, we must first write out the exponential state for the full $\text{SU}(4) \simeq \text{SO}(6)$ sector which are generated by the fermionic oscillators only. This is given by

$$|\mathbf{1}_{12}\rangle_{\text{SO}(6)} = e^A|Z\rangle \otimes |\bar{Z}\rangle \qquad \text{(F.2)}$$

$$A = \bar{c}_1 \otimes c^1 + \bar{c}_2 \otimes c^2 - \bar{d}_1 \otimes d^1 - \bar{d}_2 \otimes d^2 \qquad \text{(F.3)}$$

Since, for each Hilbert space component, A consists of four different fermionic oscillators, the expansion of e^A stops at order A^4. The terms coming from the odd powers of A, i.e. A and A^3, are fermionic. When we take inner product with scalar states, they do not contribute. Thus we only need to look at terms of order 1, A^2 and A^4.

---

[1]Here, since we are only concerned with the SU(2) sector where only the SU(4) oscillators are relevant, we shall denote $|\bar{\bar{Z}}\rangle$ by $|\bar{Z}\rangle$ for simplicity .

(i) At order 1, we simply get $|Z\rangle \otimes |\bar{Z}\rangle$. (ii) The next simplest contribution comes from A^4. Writing this out explicitly, we get

$$\frac{1}{4!}A^4|Z\rangle \otimes |\bar{Z}\rangle = (\bar{c}_1 \bar{d}_1)(\bar{c}_2 \bar{d}_2)|Z\rangle \otimes (d^1 c^1)(d^2 c^2)|\bar{Z}\rangle . \tag{F.4}$$

To see the meaning of this expression clearly, it is instructive to write down the generators of $\mathrm{SU}(2)_L \times \mathrm{SU}(2)_R$ in terms of these fermionic oscillators. They are given by

$$J_3^L = \frac{1}{2}(d^1 \bar{d}_1 - \bar{c}_1 c^1), \qquad J_+^L = d_1 c_1, \qquad J_-^L = \bar{c}_1 \bar{d}_1, \tag{F.5}$$

$$J_3^R = \frac{1}{2}(d^2 \bar{d}_2 - \bar{c}_2 c^2), \qquad J_+^R = d_2 c_2, \qquad J_-^R = \bar{c}_2 \bar{d}_2. \tag{F.6}$$

From this we see that the RHS of (F.4) can be written as $J_-^L J_-^R |Z\rangle \otimes J_+^L J_+^R |\bar{Z}\rangle$. The action of these lowering and raising operators turn $|Z\rangle$ into $|\bar{Z}\rangle$ and $|\bar{Z}\rangle$ into $|Z\rangle$, so that we get the simple result

$$\frac{1}{4!}A^4|Z\rangle \otimes |\bar{Z}\rangle = |\bar{Z}\rangle \otimes |Z\rangle . \tag{F.7}$$

(iii) Finally consider the A^2 terms. This produces 6 terms of various structures. To see which terms are relevant to the SO(4) sector, it is useful to look at the $\mathrm{SU}(2)_L \times \mathrm{SU}(2)_R$ quantum numbers of the oscillators:

$$c^1 = (\tfrac{1}{2}, 0), \quad \bar{c}_1 = (-\tfrac{1}{2}, 0), \quad c^2 = (0, \tfrac{1}{2}), \quad \bar{c}_2 = (0, -\tfrac{1}{2}) \tag{F.8}$$

$$d^1 = (\tfrac{1}{2}, 0), \quad \bar{d}_1 = (-\tfrac{1}{2}, 0), \quad d^2 = (0, \tfrac{1}{2}), \quad \bar{d}_2 = (0, -\tfrac{1}{2}) \tag{F.9}$$

Then, we can classify the 6 terms produced at order A^2 by their quantum numbers as follows:

$$\bar{c}_1 \bar{c}_2 |Z\rangle \otimes c^1 c^2 |\bar{Z}\rangle : \quad (0,0) \otimes (0,0) \tag{F.10}$$

$$-\bar{c}_1 \bar{d}_1 |Z\rangle \otimes c^1 d^1 |\bar{Z}\rangle : \quad -(-\tfrac{1}{2}, \tfrac{1}{2}) \otimes (\tfrac{1}{2}, -\tfrac{1}{2}) \simeq -|-\bar{X}\rangle \otimes |X\rangle \tag{F.11}$$

$$-\bar{c}_1 \bar{d}_2 |Z\rangle \otimes c^1 d^2 |\bar{Z}\rangle : \quad -(0,0) \otimes (0,0) \tag{F.12}$$

$$-\bar{c}_2 \bar{d}_1 |Z\rangle \otimes c^2 d^1 |\bar{Z}\rangle : \quad -(0,0) \otimes (0,0) \tag{F.13}$$

$$-\bar{c}_2 \bar{d}_2 |Z\rangle \otimes c^2 d^2 |\bar{Z}\rangle : \quad -(\tfrac{1}{2}, -\tfrac{1}{2}) \otimes (-\tfrac{1}{2}, \tfrac{1}{2}) \simeq -|X\rangle \otimes |-\bar{X}\rangle \tag{F.14}$$

$$\bar{d}_1 \bar{d}_2 |Z\rangle \otimes d^1 d^2 |\bar{Z}\rangle : \quad (0,0) \otimes (0,0) \tag{F.15}$$

The four terms with the quantum numbers $(0,0) \otimes (0,0)$ are orthogonal to the SO(4) scalar states of our interest and hence can be ignored in the singlet projector for the SU(2) sector. Thus, collecting the relevant states, we find

$$|\mathbf{1}_{12}\rangle_{\mathrm{SO}(6)} = e^A |Z\rangle \otimes |\bar{Z}\rangle$$

$$\ni |\mathbf{1}_{12}\rangle_{\mathrm{SO}(4)} = |Z\rangle \otimes |\bar{Z}\rangle - |X\rangle \otimes |(-\bar{X})\rangle - |(-\bar{X})\rangle \otimes |X\rangle + |\bar{Z}\rangle \otimes |Z\rangle, \tag{F.16}$$

which is precisely the singlet state (F.1) constructed in our previous work.

Appendix G

Details for the crossing relation of the harmonic R-matrix

In this appendix, we provide some details of the derivation of the intermediate formulas which are needed for the proof of the crossing relation for the harmonic R-matrix.

Proof of the formula (7.2.47)

To prove the formula (7.2.47) for the crossing of the number operators for the quantum and the auxiliary spaces, we should first recall the crossing property of the oscillators given in (7.2.30), (7.2.31):

$$\langle \mathbf{1}_{12} | \bar{\alpha}_{(1)} = \langle \mathbf{1}_{12} | \bar{\alpha}_{(2)} , \quad \langle \mathbf{1}_{12} | \alpha^{(1)} = -\langle \mathbf{1}_{12} | \alpha^{(2)} , \tag{G.1}$$

$$\langle \mathbf{1}_{12} | \bar{\beta}_{(1)} = -\langle \mathbf{1}_{12} | \bar{\beta}_{(2)} , \quad \langle \mathbf{1}_{12} | \beta^{(1)} = \langle \mathbf{1}_{12} | \beta^{(2)} . \tag{G.2}$$

Here, the subscripts (1),(2) label the two different spin chains corresponding to two operators. For simplicity we have suppressed the indices for the $\mathfrak{gl}(2|2) \oplus \mathfrak{gl}(2|2)$ and the labels for the different sites in the spin chain. Now from these relations, we immediately see that, under crossing, the number operator for the quantum space $\mathbf{N}^{(1)}$ transforms as $\mathbf{N}^{(1)} \to -\mathbf{N}^{(2)}$ [1], while the number operator for the auxiliary space $\mathbf{N}^{(a)}$ does not change.

Proof of the formula (7.2.48)

This formula can be understood in the following way. For simplicity, we concentrate on the oscillators $\bar{\alpha}, \alpha$. We first transform the creation operators $\bar{\alpha}_1^{\mathsf{A}}$ by crossing and get

$$\langle \mathbf{1}_{12} | \frac{1}{k!l!m!n!} \bar{\alpha}_2^{\mathsf{A}_1} \cdots \bar{\alpha}_2^{\mathsf{A}_k} \cdots \bar{\alpha}_a^{\mathsf{B}_1} \cdots \bar{\alpha}_a^{\mathsf{B}_m} \cdots \alpha_{\mathsf{B}_m}^1 \cdots \alpha_{\mathsf{B}_1}^1 \cdots \alpha_{\mathsf{A}_k}^a \cdots \alpha_{\mathsf{A}_1}^a . \tag{G.3}$$

Then, we wish to move the annihilation operators $\alpha_{\mathsf{A}_1}^1$ next to the singlet projector in order to use the crossing formula for them. This can be easily done, but after the crossing, we

---

[1]To be precise, $\mathbf{N}_\alpha^{(1)}$ transforms as $\mathbf{N}_\alpha^{(1)} \to -\mathbf{N}_\alpha^{(2)} - (-1)^{|\mathsf{A}|} \delta_{\mathsf{A}}^{\mathsf{A}}$. However, the constant term vanishes as the signs are opposite for the bosonic and fermionic oscillators and hence they exactly cancel with each other in the present case. This is also true for $\mathbf{N}_\beta^{(1)}$.

need to move them back to the original position, which in turn generates extra terms since α_A^2's do not commute with the creation operators $\bar{\alpha}_2^A$. As a result, the Kronecker delta δ_B^A appears, which contracts the indices for the oscillators of the auxiliary space $\bar{\alpha}_a^B, \alpha_A^a$. In this way we find that the number operator of the auxiliary space $\mathbf{N}_\alpha^{(a)}$ is inserted in the middle of the oscillators. Now when we move the number operator to the left most position, the number operator is shifted by a constant due to the presence of the creation operators on the way. This gives the expression of the form

$$
\begin{aligned}
&(\mathbf{N}_\alpha^{(a)} - m + 1) \cdots (\mathbf{N}_\alpha^{(a)} - m + p) \\
&\times \bar{\alpha}_2^{A_1} \cdots \bar{\alpha}_2^{A_{k-p}} \cdots \bar{\alpha}_a^{B_1} \cdots \bar{\alpha}_a^{B_{m-p}} \cdots \alpha_{B_{m-p}}^2 \cdots \alpha_{B_1}^2 \cdots \alpha_{A_{k-p}}^a \cdots \alpha_{A_1}^a .
\end{aligned} \tag{G.4}
$$

By carefully treating the numerical coefficients and performing the same calculation for the oscillators $\bar{\beta}, \beta$, we find that the crossing formula for the hopping operator is given by (7.2.48).

Explanation of (7.2.49)

Let us make cautionary remarks for using the crossing relations for the coefficients and the hopping operator already obtained to derive the crossing relation for the harmonic R-matrix given in (7.2.49). This has to do with the effects due to the order of crossing. Although the hopping operators preserve the total number of oscillators of the quantum space and as well as of the auxiliary space and commute with the number operator $\mathbf{N}$, the expression $\mathbf{Hop}_{k,l,m,n}^{(a2)}$ which appears *after* the crossing no longer commutes with the total number operator of the form $\mathbf{N}^{(a)} + \mathbf{N}^{(1)}$. In fact, since the hopping operator $\mathbf{Hop}_{k,l,m,n}^{(a2)}$ moves $k + l$ oscillators from the auxiliary space to the quantum space and moves $m + n$ oscillators from the quantum space to the auxiliary space, the following exchange relation holds:

$$
\mathbf{Hop}_{k,l,m,n}^{(a2)} f(\mathbf{N}^{(a)} + \mathbf{N}^{(1)}) = f(\mathbf{N}^{(a)} + \mathbf{N}^{(1)} + k + l - m - n)\mathbf{Hop}_{k,l,m,n}^{(a2)} . \tag{G.5}
$$

This effect has to be duly taken into account. More specifically, we first move the hopping operator $\mathbf{Hop}_{k,l,m,n}^{(a1)}$ to the left all the way until it hits the singlet projector. This operation does not shift the number operator as the labels for the quantum space are different and they commute with each other. Now, upon hitting the singlet state we use the crossing relation to convert it to $\mathbf{Hop}_{k,l,m,n}^{(a2)}$ and then we try to move it back to the original position. In this process we come across the shift for the number operator as in (G.5)[2]. After this procedure we make the crossing of the coefficients as $\mathcal{A}_I^{(\mathbf{N})} \to \mathcal{A}_I^{(\mathbf{N}^{(a)} - \mathbf{N}^{(2)})}$. In this way, we obtain the relation (7.2.49).

---

[2]Actually, we need to exchange $\mathbf{Hop}_{k-p,l-q,m-p,n-q}^{(a2)}$ through the coefficient $\mathcal{A}_I^{(\mathbf{N})}$. But this produces the same shift as in (G.5).

Proof of the relation (7.2.53)

Finally, we shall provide a proof of the relation (7.2.53). Let us recall that the definition for the coefficient $\mathcal{A}_I^{(\mathbf{N})}$ is given in terms of the generalized binomial (7.2.36). Hence, we have

$$\tilde{\mathcal{A}}_I^{(\mathbf{N})} = \rho(u)(-1)^{I+\frac{\mathbf{N}}{2}+\mathbf{M}} \sum_{r=0}^{\infty} (-1)^r \mathcal{B}(\mathbf{M}, r)\mathcal{B}(I+r, r-\mathbf{M}-u+\frac{\mathbf{N}}{2}), \qquad \text{(G.6)}$$

where we have used the identity $\mathcal{B}(x, y) = \mathcal{B}(x, x-y)$. From the above expression, it turns out that the proof for the relation (7.2.53) is equivalent to verify the relation

$$\sum_{r=0}^{\infty} (-1)^r \mathcal{B}(\mathbf{M}, r)\mathcal{B}(I+r, r-\mathbf{M}+-u+\frac{\mathbf{N}}{2}) = (-1)^{\mathbf{M}}\mathcal{B}(I, -u+\frac{\mathbf{N}}{2}). \qquad \text{(G.7)}$$

For this purpose, we will consider the following more general formula

$$\sum_{r=0}^{\infty} (-1)^r \mathcal{B}(\gamma, r)\mathcal{B}(\alpha+r, r-\gamma+\beta) = \frac{\sin \pi(\beta-\gamma)}{\sin \pi\beta}\mathcal{B}(\alpha, \beta), \qquad \text{(G.8)}$$

where α, γ are arbitrary complex numbers and we assume β to be generally a non-integer complex number. Once we can justify this relation, we easily obtain the relation we need by setting $\alpha = I$, $\beta = -u + \frac{\mathbf{N}}{2}$ and $\gamma = \mathbf{M}$. Using the definition for the generalized binomial and the well-known identity for the gamma function $\Gamma(x)\Gamma(1-x) = \frac{1}{\sin \pi x}$, the left hand side becomes

$$\begin{aligned}
(\text{L.H.S}) &= \sum_{r=0}^{\infty} (-1)^r \frac{\Gamma(\gamma+1)}{\Gamma(r+1)\Gamma(\gamma-r+1)} \frac{\Gamma(\alpha+r+1)}{\Gamma(r-\gamma+\beta+1)\Gamma(\alpha+\gamma-\beta+1)} \\
&= \frac{\Gamma(\gamma+1)\Gamma(\alpha+1)}{\Gamma(\alpha+\gamma-\beta+1)} \sum_{r=0}^{\infty} (-1)^r \frac{\Gamma(\alpha+r+1)}{\Gamma(r+1)\Gamma(\alpha+1)} \frac{\Gamma(\gamma-r-\beta)\sin\pi(r-\gamma+\beta+1)}{\Gamma(\gamma-r+1)} \qquad \text{(G.9)} \\
&= \frac{\sin\pi(\beta-\gamma)}{\sin\pi\beta} \frac{\Gamma(\gamma+1)\Gamma(\alpha+1)}{\Gamma(\alpha+\gamma-\beta+1)\Gamma(\beta+1)} \sum_{r=0}^{\infty} \mathcal{B}(\alpha+r, r)\mathcal{B}(\gamma-r-\beta-1, \gamma-r).
\end{aligned}$$

The summation over the products of binomials turns out to be equal to $\mathcal{B}(\alpha-\beta+\gamma, \gamma)$ since the following identity holds for arbitrary complex numbers a, b, c

$$\mathcal{B}(a+b+c-1, c) = \sum_{k=0}^{\infty} \mathcal{B}(a+k-1, k)\mathcal{B}(b+c-k-1, c-k). \qquad \text{(G.10)}$$

When c is any positive integer, the above relation immediately follows from

$$\frac{1}{(1+x)^a} = \sum_{k=0}^{\infty} (-1)^k \mathcal{B}(a+k-1, k)x^k, \quad \frac{1}{(1+x)^{a+b}} = \frac{1}{(1+x)^a} \cdot \frac{1}{(1+x)^b}. \qquad \text{(G.11)}$$

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As the both sides of (G.10) are analytic functions of c, it turns out that the relation holds for arbitrary complex c by analytic continuation. By using this identity with $a = \alpha + 1, b = -\beta, c = \gamma$, we find

$$\text{(L.H.S)} = \frac{\sin \pi(\beta - \gamma)}{\sin \pi \beta} \frac{\Gamma(\gamma + 1)\Gamma(\alpha + 1)}{\Gamma(\alpha + \gamma - \beta + 1)\Gamma(\beta + 1)} \mathcal{B}(\alpha - \beta + \gamma, \gamma) = \frac{\sin \pi(\beta - \gamma)}{\sin \pi \beta} \mathcal{B}(\alpha, \beta).$$
$$\text{(G.12)}$$

Therefore, we have shown (G.8), which completes the proof.

Appendix H

Details of the Landau-Lifshitz model

In this appendix, we shall give a brief description of how to obtain the Landau-Lifshitz model from the Heisenberg spin chain in the semi-classical limit.

H.1   Coherent state representation of SU(2)

To pave our way, we shall quickly review the coherent state representation of a Heisenberg spin chain (see [199] for the description relevant to the present context) and comment on its physical meaning. As mentioned in the main text, it is a representation of SU(2) on the functions on the coset space SU(2)/U(1), which is isomorphic to a unit sphere.

In this subsection we shall focus on a single spin $1/2$ state. Let $|\uparrow\rangle$ be the eigenstate of S_3 with the eigenvalue $1/2$. Then, a U(1) operator $h = e^{i\alpha S_3}$ around this direction only produces a phase and an arbitrary SU(2) element g can be decomposed as $g = \Omega h$, where Ω belongs to the coset SU(2)/U(1). Thus, $g|\uparrow\rangle = \Omega|\uparrow\rangle e^{i\alpha/2}$. On general grounds, Ω can be parametrized using the remaining generators $S_\pm = S_1 \pm iS_2$ as $\Omega(\eta) = \exp(\eta S_+ - \bar{\eta}S_-)$, where η is a complex parameter. For the Landau-Lifshitz model, one usually adopts the representation where the target space is easily seen to be a unit sphere. This is achieved by the choice of the parametrization[1] $\eta = -(\theta/2)e^{-i\phi}$. Then

$$\Omega(\eta)|\uparrow\rangle = \exp\left(-i\theta(S_2\cos\phi - S_1\sin\phi)\right)|\uparrow\rangle. \tag{H.1}$$

Now let $\mathbf{n}_0 = (0, 0, 1)$ be a unit vector in the z direction and $\mathbf{n} = (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta)$ be a unit vector in a general direction. Then, it is easy to see that $|\mathbf{n}_0 \times \mathbf{n}| = \sin\theta$ and

$$\frac{\mathbf{n}_0 \times \mathbf{n}}{|\mathbf{n}_0 \times \mathbf{n}|} = (-\sin\phi, \cos\phi, 0). \tag{H.2}$$

---

[1]The minus sign in front in η is a convention to conform to the one in [200].

Comparing with (H.1) we find

$$|\mathbf{n}\rangle \equiv \Omega(\eta)|\uparrow\rangle = \exp\left(-i\theta \frac{\mathbf{n}_0 \times \mathbf{n}}{|\mathbf{n}_0 \times \mathbf{n}|} \cdot \vec{S}\right)|\uparrow\rangle = \cos\frac{\theta}{2}|\uparrow\rangle + e^{i\phi}\sin\frac{\theta}{2}|\downarrow\rangle. \qquad \text{(H.3)}$$

At this point an alert reader may have noticed that the pair of coefficients $(\cos(\theta/2), e^{i\phi}\sin(\theta/2))$ coincide with the components of the so-called monopole harmonics, introduced in [200,201] as $Y_{q,l,m}$ defined on a unit sphere, in the case where $q = eg = \frac{1}{2}$, with e and g, respectively, being the electric charge of a particle on the sphere and the magnetic charge of a monopole situated at the origin. Actually, as it is a section of a non-trivial U(1) bundle and it has to be defined in two overlapping open sets, like those around the northern and the southern hemispheres, separately in such a way that in the overlap its expressions are connected by a non-trivial gauge transformation. What is happening is that in order to produce a spin 1/2 representation out of a vector $\mathbf{n}$, which obviously carries spin 1, it must be combined with an extra spin of magnitude 1/2, which can be interpreted as provided by a "minimum" charge-monopole system.

The monopole harmonics associated with the vector $\mathbf{n}$ as above corresponds to $Y_{\frac{1}{2},\frac{1}{2},m}(\mathbf{n})$, $(m = \pm\frac{1}{2})$. As described in [200], an important property of the monopole harmonics is that under the action of a rotation matrix $D(\mathbf{n}')_{m'm}$ around the direction $\mathbf{n}'$, the monopole harmonics does not simply rotate into a linear combination of monopole harmonics. This is because, under the rotation, while the open sets with respect to which the monopole harmonics is defined get rotated into different regions, the gauge connection $A_\mu(x)$ is not changed. Therefore in order to recover the same relative configuration of the open sets and the form of the connection one must make a suitable gauge transformation of $A_\mu(x)$. This produces an extra U(1) phase factor of the form $\exp(i\Phi(\mathbf{n}, \mathbf{n}')q)$, where $\Phi(\mathbf{n}, \mathbf{n}')$ is the area of the triangle on the unit sphere the vertices of which are defined by $\mathbf{n}, \mathbf{n}'$ and the vector $\mathbf{n}_0$. It is clear from the preceding discussions that this phase, to be called *the Wess-Zumino phase*, is an essential ingredient in realizing the spin $\frac{1}{2}$ representation in terms of the coherent states $|\mathbf{n}\rangle$.

An important quantity in which this phase appears is the inner product of the coherent states. One can show by direct calculation that

$$\begin{aligned}
\langle \mathbf{n}'|\mathbf{n}\rangle &= \cos\frac{\theta}{2}\cos\frac{\theta'}{2} + e^{i(\phi-\phi')}\sin\frac{\theta}{2}\sin\frac{\theta'}{2} \\
&= \exp\left(i\frac{\Phi(\mathbf{n}',\mathbf{n})}{2}\right)\sqrt{1 - \frac{(\mathbf{n} - \mathbf{n}')^2}{4}},
\end{aligned} \qquad \text{(H.4)}$$

where

$$\tan\frac{\Phi(\mathbf{n}',\mathbf{n})}{2} = \frac{(\mathbf{n}' \times \mathbf{n}) \cdot \mathbf{n}_0}{1 + \mathbf{n}_0 \cdot \mathbf{n} + \mathbf{n}_0 \cdot \mathbf{n}' + \mathbf{n} \cdot \mathbf{n}'}. \qquad \text{(H.5)}$$

More intuitive expression of the Wess-Zumino phase will also be given shortly.

Before leaving this subsection, let us record two basic relations we will use. One is the (over)completeness relation which reads

$$\mathbf{1} = \frac{1}{2\pi} \int d^3\mathbf{n} \, \delta(\mathbf{n}^2 - 1) |\mathbf{n}\rangle\langle\mathbf{n}| \, . \tag{H.6}$$

This can be readily verified by substituting the explicit form of $|\mathbf{n}\rangle$ given in (H.3) and performing the integration. One then obtains that the RHS is indeed equal to $|\uparrow\rangle\langle\uparrow| + |\downarrow\rangle\langle\downarrow| = 1$. Another basic relation of use is $\langle\mathbf{n}|\vec{S}|\mathbf{n}\rangle = \frac{1}{2}\mathbf{n}$, which can also be checked with ease.

H.2   Brief derivation of the Landau-Lifshitz model

Making use of the coherent state representation of the SU(2) spin 1/2 state explained above, we now briefly describe how the Landau-Lifshitz model arises from the Heisenberg spin chain in the semiclassical limit.

Let us denote by $|\vec{n}\rangle = |\mathbf{n}_1\rangle \otimes \cdots \otimes |\mathbf{n}_L\rangle$ a coherent state of the spin chain and consider the transition amplitude $\langle\vec{n}_{\text{final}}|e^{-iHt}|\vec{n}_{\text{initial}}\rangle$ from the initial state to the final state, through the Hamiltonian of the Heisenberg spin chain given (up to a convenient constant) by

$$H = 4g^2 \sum_{i=1}^{L} \left(\frac{1}{4} - \vec{S}_i \vec{S}_{i+1} \right) \, . \tag{H.1}$$

By the standard procedure, namely by performing the time evolution in infinitesimal steps with the insertions of the completeness relation (H.6) at each step, one obtains the coherent state path-integral representation

$$\langle\vec{n}_{\text{final}}|e^{-iHt}|\vec{n}_{\text{initial}}\rangle = \int \mathcal{D}\vec{n}(t) e^{iS} \, , \tag{H.2}$$

with the action S given by

$$S = \sum_{i=1}^{\ell} \int dt \left[\frac{(\mathbf{n}_i \times \partial_t \mathbf{n}_i) \cdot \mathbf{n}_0}{2(1 + \mathbf{n}_i \cdot \mathbf{n}_0)} - \frac{g^2}{2}(\mathbf{n}_i - \mathbf{n}_{i-1})^2 \right] \, . \tag{H.3}$$

By taking the continuum limit of this expression, we obtain the well-known action of the Landau-Lifshitz model.

$$S = \int dt \int_0^\ell d\sigma \left[\frac{(\mathbf{n} \times \partial_t \mathbf{n}) \cdot \mathbf{n}_0}{2(1 + \mathbf{n} \cdot \mathbf{n}_0)} - \frac{g^2}{2} \partial_\sigma \mathbf{n} \cdot \partial_\sigma \mathbf{n} \right] \, . \tag{H.4}$$

The first term on the RHS represents the Wess-Zumino phase produced through the inner product as given in (H.4) and (H.5).

Just as in the Wess-Zumino-Novikov-Witten model, for example, such a Wess-Zumino term has a representation in terms of an integral one dimension higher (in this case as a three dimensional integral) of the form

$$\frac{1}{2} \int_0^1 ds \int dt \int_0^L d\sigma \, \mathbf{n} \cdot (\partial_t \mathbf{n} \times \partial_s \mathbf{n}) \,, \tag{H.5}$$

where s-dependence of $\mathbf{n}$ is defined such that $\mathbf{n}(s = 1) = (0, 0, 1)$ and $\mathbf{n}(s = 0) = \mathbf{n}$. The expression (H.5) has a rather intuitive meaning. Since $\mathbf{n}$ is a unit vector, $\partial_t \mathbf{n}$ and $\partial_s \mathbf{n}$ are perpendicular to $\mathbf{n}$. Therefore the exterior product $\partial_t \mathbf{n} \times \partial_s \mathbf{n}$ is in the direction of $\mathbf{n}$ and $\mathbf{n} \cdot (\partial_t \mathbf{n} \times \partial_s \mathbf{n}) \, dt ds$ is nothing but the infinitesimal area element. Hence the integration gives the area and together with the factor of $1/2$, which is the value of $q = eg$ discussed in the previous subsection, we get the exponent of the Wess-Zumino phase factor.

H.3 Poisson brackets and the r-matrix for the Landau-Lifshitz model

As described in section 8.2.2, the classical r-matrix for the Landau-Lifshitz model can be obtained quickly as the classical limit of the well-known form of the quantum R-matrix of the Heisenberg spin chain.

However, it would be of interest to supply the first principle derivation of the r-matrix from the computation of the Poisson brackets among the coherent state variables $n_i(\sigma, \tau)$. Below we give a sketch of such a derivation.

Poisson brackets

First we derive the Poisson (Dirac) bracket structure of the Landau-Lifshitz model. The most straightforward way is to start from the action (8.2.1), regard $\vec{n}$ as the fundamental variable and derive the Dirac brackets, taking into account the constraints $\vec{n}^2 = 1$. However, in practice it turned out to be much easier to first parametrize the 2-sphere by θ and ϕ and then compute the Dirac brackets. In terms of these angle variables, the action of the Landau-Lifshitz sigma model takes the form

$$S = - \int d\tau d\sigma \left[\frac{1}{4} (\cos \theta \partial_\tau \phi + \phi \sin \theta \partial_\tau \theta) + \frac{g^2}{2} \left(\partial_\sigma \theta \partial_\sigma \theta + \sin^2 \theta \partial_\sigma \phi \partial_\sigma \phi \right) \right]. \tag{H.1}$$

From this action, the conjugate momenta can be determined as

$$\Pi_\phi = -\frac{1}{4} \cos \theta \,, \qquad \Pi_\theta = -\frac{1}{4} \phi \sin \theta \,. \tag{H.2}$$

Evidently, these two equations should be regarded as the constraints. The commutation relation of these two constraints is given by

$$\{\Pi_\phi + \frac{1}{4}\cos\theta\big|_\sigma, \Pi_\theta + \frac{1}{4}\phi\sin\theta\big|_{\sigma'}\} = -\frac{\sin\theta}{2}\delta(\sigma - \sigma').$$

(H.3)

Thus, the Dirac bracket for any dynamical variables A and B for this system is given by

$$\{A, B\}_D = \{A, B\}$$
$$+ \int d\sigma \frac{2}{\sin\theta}\left(\{A, \Pi_\phi + \frac{1}{4}\cos\theta\}\{\Pi_\theta + \frac{1}{4}\phi\sin\theta, B\} - \{A, \Pi_\theta + \frac{1}{4}\phi\sin\theta\}\{\Pi_\phi + \frac{1}{4}\cos\theta, B\}\right).$$

(H.4)

Applying this formula to the variables $\mathbf{n}_i(\sigma)$ and $\mathbf{n}_j(\sigma)$ at equal time yields

$$\{\mathbf{n}_i(\sigma), \mathbf{n}_j(\sigma')\}_D = 2\epsilon_{ijk}\mathbf{n}_k\delta(\sigma - \sigma'),$$

(H.5)

which is nothing but the classical commutation relations for the spin variables. (In what follows, we omit writing the subscript D.)

Classical r-matrix

Having derived the commutation relations for the variables $\vec{n}$, we can now derive the Poisson bracket between the Lax matrices and determine the classical r-matrix. The Poisson bracket between J_σ given in (8.2.3) can be calculated as

$$\{J_\sigma(\sigma|u) \overset{\otimes}{,} J_\sigma(\sigma'|v)\} = -\frac{1}{16\pi^2 uv}\{\mathbf{n}(\sigma)\cdot\vec{\sigma} \overset{\otimes}{,} \mathbf{n}(\sigma')\cdot\vec{\sigma}\}$$
$$= -\delta(\sigma - \sigma')\frac{1}{8\pi^2 uv}\epsilon_{ijk}\mathbf{n}_k(\sigma)\sigma_i \otimes \sigma_j.$$

(H.6)

One can simplify this expression by using the Fiertz identity

$$(\sigma_a)_{ij}(\sigma_b)_{kl} = \sum_{c,d}\frac{\text{tr}(\sigma_c\sigma_a\sigma_d\sigma_b)}{4}(\sigma_c)_{il}(\sigma_d)_{kj},$$

(H.7)

where the indices c and d run from 0 to 3 and σ_0 is defined to be equal to $\mathbf{1}$. Applying this identity, the factor $\epsilon_{ijk}\sigma_i \otimes \sigma_j$ can be re-expressed as

$$\epsilon_{ijk}(\sigma_i)_{\alpha\beta}(\sigma_j)_{\gamma\delta} = \frac{i}{2}\left((\sigma_k)_{\alpha\delta}\delta_{\beta\gamma} - (\sigma_k)_{\beta\gamma}\delta_{\alpha\delta}\right).$$

(H.8)

Utilizing such formulas, we can arrive at the following expression[2] for the Poisson bracket:

$$\{J_\sigma(\sigma|u) \overset{\otimes}{,} J_\sigma(\sigma'|v)\} = \delta(\sigma - \sigma')\left[\mathbf{r}(u - v), -(J_\sigma(u) \otimes \mathbf{1} + \mathbf{1} \otimes J_\sigma(v))\right].$$

(H.9)

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[2]To arrive at the expression (H.9), we use $(uv)^{-1} = (v^{-1} - u^{-1})(u - v)^{-1}$.

In this expression, $\mathbf{r}(u)$ is the so-called classical r-matrix, which in this case is given by

$$\mathbf{r}(u) = \frac{\mathbb{P}}{u} \,. \tag{H.10}$$

The symbol $\mathbb{P}$ denotes the operator which permutes the two spaces in the tensor product: $V_1 \otimes V_2 \longmapsto V_2 \otimes V_1$. It is well-known[3] that when the Poisson bracket between the Lax matrices can be expressed in terms of the classical r-matrix as in (H.9), the Poisson bracket between the monodromy matrices can also be expressed by the classical r-matrix as

$$\{\Omega(u) \overset{\otimes}{,} \Omega(v)\} = [\Omega(u) \otimes \Omega(v) \,, \, \mathbf{r}(u - v)] \,. \tag{H.11}$$

H.4 Construction of the separated variables

In this appendix, we will describe how the separated variables are obtained for the Landau-Lifshitz model.

Expressions of the Poisson brackets obtained from the r-matrix

First, let us give a list of Poisson bracket relations between the components of the monodromy matrix written as

$$\Omega(u) \equiv \begin{pmatrix} \mathcal{A}(u) & \mathcal{B}(u) \\ \mathcal{C}(u) & \mathcal{D}(u) \end{pmatrix} \,. \tag{H.1}$$

With the form of the r-matrix given in (8.2.12) and the basic Poisson bracket formula (8.2.11) involving the r-matrix, the Poisson bracket relations between the components of $\Omega(u)$ can be easily computed as

$$\{\mathcal{A}(u), \mathcal{B}(v)\} = \frac{-1}{u-v} \left(\mathcal{A}(u)\mathcal{B}(v) - \mathcal{A}(v)\mathcal{B}(u) \right), \quad \{\mathcal{A}(u), \mathcal{C}(v)\} = \frac{1}{u-v} \left(\mathcal{A}(u)\mathcal{C}(v) - \mathcal{A}(v)\mathcal{C}(u) \right),$$

$$\{\mathcal{A}(u), \mathcal{D}(v)\} = \frac{1}{u-v} \left(\mathcal{B}(u)\mathcal{C}(v) - \mathcal{B}(v)\mathcal{C}(u) \right), \quad \{\mathcal{B}(u), \mathcal{C}(v)\} = \frac{1}{u-v} \left(\mathcal{A}(u)\mathcal{D}(v) - \mathcal{A}(v)\mathcal{D}(u) \right),$$

$$\{\mathcal{B}(u), \mathcal{D}(v)\} = \frac{1}{u-v} \left(\mathcal{B}(u)\mathcal{D}(v) - \mathcal{B}(v)\mathcal{D}(u) \right), \quad \{\mathcal{C}(u), \mathcal{D}(v)\} = \frac{-1}{u-v} \left(\mathcal{C}(u)\mathcal{D}(v) - \mathcal{C}(v)\mathcal{D}(u) \right),$$

$$\{\mathcal{A}(u), \mathcal{A}(v)\} = \{\mathcal{B}(u), \mathcal{B}(v)\} = \{\mathcal{C}(u), \mathcal{C}(v)\} = \{\mathcal{D}(u), \mathcal{D}(v)\} = 0 \,. \tag{H.2}$$

These basic relaitions will be utilized in what follows.

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[3]A proof of (H.11) below can be found in page 106-107 of [120].

Separated variables à la Sklyanin

Having displayed the explicit expression for the Poisson brackets, we now construct the separated canonically conjugate variables by the so-called Sklyanin's magic recipe [93]. In this method, such variables are obtained as associated to the poles of the normalized eigenvector h of the monodromy matrix, defined in the following way[4]:

$$\Omega(u)h(u) = e^{ip(u)}h(u)\,, \quad \langle n\,, h \rangle = 1\,. \tag{H.3}$$

Here $n = (n^1, n^2)^t$ is the polarization vector. To simplify the construction it turns out to be convenient to first transform the monodromy matrix $\tilde{\Omega}(x)$ by a similarity transformation into the form

$$\tilde{\Omega}(x) \equiv \begin{pmatrix} n^2 & n^1 \\ -n^1 & n^2 \end{pmatrix} \Omega(x) \begin{pmatrix} n^2 & -n^1 \\ n^1 & n^2 \end{pmatrix} \equiv \begin{pmatrix} \tilde{\mathcal{A}}(x) & \tilde{\mathcal{B}}(x) \\ \tilde{\mathcal{C}}(x) & \tilde{\mathcal{D}}(x) \end{pmatrix}\,. \tag{H.4}$$

As the Lax pair equations are invariant under such a transformation, the components of $\tilde{\Omega}$ satisfy the same Poisson-bracket relation as those of the components of Ω displayed in (H.2).

Let us denote the poles of h by γ_i. Then the components of $\tilde{\Omega}$ satisfy the following relation[5].

$$\tilde{B}(\gamma_i) = 0\,, \quad \tilde{\mathcal{D}}(\gamma_i) = \tilde{\mathcal{A}}(\gamma_i)^{-1} = e^{ip(\gamma_i)}\,. \tag{H.5}$$

In what follows, we make use of these relations to derive the commutation relations between γ_i's and $p(\gamma_i)$'s.

We start from the analysis of $\{\tilde{\mathcal{B}}(u)\,, \tilde{\mathcal{B}}(v)\} = 0$. Since $\tilde{B}$ has zeros at γ_i and γ_j ($i \neq j$), it can be expressed in the form $\tilde{\mathcal{B}}(u) = (u - \gamma_i)\mathcal{B}'(u)$ or $\tilde{\mathcal{B}}(u) = (u - \gamma_j)\mathcal{B}''(u)$. The functions $\mathcal{B}'(u)$ and $\mathcal{B}''(u)$ are not known but what is important is that they have the properties $\mathcal{B}'(\gamma_i) \neq 0$ and $\mathcal{B}''(\gamma_j) \neq 0$. Then the commutation relation between $\tilde{\mathcal{B}}(u)$ and $\tilde{\mathcal{B}}(v)$ can be rewritten as

$$(u - \gamma_i)(v - \gamma_j)\{\mathcal{B}'(u)\,, \mathcal{B}''(v)\} - (v - \gamma_j)\mathcal{B}'(u)\{\gamma_i\,, \mathcal{B}''(v)\}$$
$$- (u - \gamma_i)\mathcal{B}''(v)\{\mathcal{B}'(u)\,, \gamma_j\} + \mathcal{B}'(u)\mathcal{B}''(v)\{\gamma_i\,, \gamma_j\} = 0\,. \tag{H.6}$$

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[4]In Sklyanin's original formulation, the normalization condition is expressed in terms of the ordinary inner product as $n' \cdot h = 1$. Here we are instead using the skew-symmetric inner product in order to make connection with the Wronskian. It is equivalent to the original formulation under the identification of n' with $i\sigma_2 n$.

[5]To see this, it is helpful to consider the relation between the *normalized* eigenvector h and the *unnormalized* eigenvector ψ_+. The normalized eigenvector can be constructed from the unnormalized eigenvector by $h = \psi_+/\langle n\,, \psi_+\rangle$. Therefore the poles of the normalized eigenvector arise when the unnormalized eigenvector satisfy $\langle n\,, \psi_+\rangle = 0$. Thus, at the poles of the normalized eigenvector, the vector parallel to n becomes the eigenvector of the monodromy matrix. Then, it is easy to see that (H.5) follows.

Now at this stage, we can safely take the limit $u \to \gamma_i$ and $v \to \gamma_j$. Then the first three terms vanish the last term gives the relation

$$\{\gamma_i, \gamma_j\} = 0. \tag{H.7}$$

Next consider the commutation relation between $\tilde{\mathcal{A}}(u)$ and $\tilde{\mathcal{B}}(v)$. Here again, we should substitute the expansion $\tilde{\mathcal{A}}(u) = \tilde{\mathcal{A}}(\gamma_i) + (u - \gamma_i)\mathcal{A}'(u)$ as well as the ones for $\mathcal{B}'$ and $\mathcal{B}''$. Then similarly to the previous case, the limit $u \to \gamma_i$ and $v \to \gamma_j$ can be easily taken and, making use of the relation (H.7), we can deduce the important relation

$$\{\tilde{\mathcal{A}}(\gamma_i), \gamma_j\} = \tilde{\mathcal{A}}(\gamma_i)\delta_{ij}. \tag{H.8}$$

As the last step, a similar calculation for $\{\tilde{\mathcal{A}}(x), \tilde{\mathcal{A}}(x')\} = 0$ leads to

$$\{\tilde{\mathcal{A}}(\gamma_i), \tilde{\mathcal{A}}(\gamma_j)\} = 0. \tag{H.9}$$

Using the expression of $\tilde{\mathcal{A}}(\gamma_i)$ and $p(\gamma_i)$ given in (H.5) and the equations (H.7)-(H.9), we can obtain the commutation relations among γ_i's and $p(\gamma_j)$'s as

$$\{\gamma_i, \gamma_j\} = \{p(\gamma_i), p(\gamma_j)\} = 0, \quad -i\{\gamma_i, p(\gamma_j)\} = \delta_{ij}. \tag{H.10}$$

This shows that $(\gamma_i, -ip(\gamma_i))$'s are the separated canonical pairs of variables associated to the poles of the normalized eigenvector.

Appendix I

Highest weight condition on the semi-classical wave function

Here we study constraints from the highest weight condition on the semi-classical wave function and show that the constant vector n appearing in the normalization condition (8.2.15) must be equal to the polarization vector.

As explained in section 8.1.2, the states constructed on the rotated vacuum characterized by the polarization vector $n = (n^1, n^2)^t$ satisfy the highest weight condition

$$S'_+ |\Psi\rangle = 0 \,, \tag{I.1}$$

with S'_+ given in (8.1.21). To understand the consequence of this condition in the semi-classical limit, let us recall the form of the semi-classical wave function (in the action-angle basis),

$$\Psi = \exp\left(i \sum_k S_k \phi_k \right) \,. \tag{I.2}$$

As explained in section 8.2.2, the angle variables ϕ_k can be constructed from the poles of the factor $\langle n', \psi_+ \rangle$, where n' is a constant vector which defines the normalization condition[1]. Thus, in order to gurantee the highest weight property of the semi-classical wave function, we need to choose n' such that $\langle n', \psi_+ \rangle$ is invariant under the transformation generated by S'_+.

For this purpose, let us first go back to the Heisenberg spin chain. In the Heisenberg spin chain, the Lax operator is given by

$$L(u) = \begin{pmatrix} 1 + iS_3/u & iS_-/u \\ iS_+/u & 1 - iS_3/u \end{pmatrix} \,. \tag{I.3}$$

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[1] Thus in literature this vector is usually referred to as the normalization vector.

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By the straightforward computation, one can show that it transforms under $e^{aS'_+}$ as

$$e^{aS'_+}L(u)e^{-aS'_+} = AL(u)A^{-1}, \tag{I.4}$$

where the matrix A is given by

$$A = N \begin{pmatrix} 1 & -a \\ 0 & 1 \end{pmatrix} N^{-1}. \tag{I.5}$$

Now, since the Landau-Lifshitz sigma model is obtained by taking the continuum limit of the Heisenberg spin chain, (I.4) implies the following transformation rule of the Lax matrix of the Landau-Lifshitz sigma model:

$$e^{aS'_+}\left(J_\sigma\right) = AJ_\sigma A^{-1}. \tag{I.6}$$

This means that a solution ψ_+ to the auxiliary linear problem transforms as $\psi_+ \to A\psi_+$ in order to compensate for the transformation (I.6). Thus the Wronskian $\langle n', \psi_+ \rangle$ gets transformed as

$$\langle n', \psi_+ \rangle \mapsto \langle n', A\psi_+ \rangle = \langle A^{-1}n', \psi_+ \rangle, \tag{I.7}$$

where the equality follows from the SL(2) invariance of the skew-symmetric product. It is then easy to see that the invariance under the transformation requires $n' = A^{-1}n'$ and this leads to the identification $n' = n$.

Appendix J

Baker-Akhiezer vectors for the two-point functions

In the case of two-point functions, the explicit solutions can be constructed by the finite gap method [95]. For the general spectral curve with genus g, the solutions to the auxiliary linear problem evaluated at $(\tau, \sigma) = (0, 0)$ reads[1]

$$\psi_+^0(u) = \left(\begin{array}{c} k_-(u) \\ k_+(u) \end{array} \right), \quad \psi_-^0(u) = \left(\begin{array}{c} k_-(\hat{\sigma}u) \\ k_+(\hat{\sigma}u) \end{array} \right). \tag{J.1}$$

where $\hat{\sigma}$ is the holomorphic involution and the functions $k_-(u)$ and $k_+(u)$ are characterized uniquely by their divisors and the normalization at infinity:

$$\begin{aligned} (k_+) &= \infty^+ + \sum_{i=1}^{g} \gamma_i' - \sum_{j=1}^{g+1} \hat{\gamma}_j, \qquad k_+(\infty^-) = 1, \\ (k_-) &= \infty^- + \sum_{i=1}^{g} \gamma_i - \sum_{j=1}^{g+1} \hat{\gamma}_j, \qquad k_-(\infty^+) = 1. \end{aligned} \tag{J.2}$$

Here γ_i' are the initial values of the separated variables parametrizing the moduli for two-point functions, and γ_i and $\hat{\gamma}_i$ are the divisors satisfying[2]

$$\{\hat{\gamma}_j\} \sim \{\infty^-, \gamma_i\} \sim \{\infty^+, \gamma_i'\}. \tag{J.3}$$

As noted in [95], the solutions (J.1) describe the highest weight eigenstate of S_3. This means that the corresponding polarization vector is $n = (1, 0)^t$. The solutions for more general rotated vacua can be obtained by the global rotation.

The solutions (J.1) do not satisfy the normalization conditions $\langle \psi_+^0, \psi_-^0 \rangle = 1$. To normalize the solutions, we need to divide them by $\sqrt{\langle \psi_+^0, \psi_-^0 \rangle}$ as in (8.3.6). After the

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[1]See (4.13) in [95].

[2]The symbol $a \sim b$ means that there is a single-valued function on the Riemann surface which has poles at a and zeros at b.

division, we obtain

$$\psi_+(u) = C(u) \begin{pmatrix} k_-(u) \\ k_+(u) \end{pmatrix}, \quad \psi_-(u) = C(u) \begin{pmatrix} k_-(\hat{\sigma}u) \\ k_+(\hat{\sigma}u) \end{pmatrix}, \tag{J.4}$$

with $C(u)$ given by

$$C(u) = \frac{1}{\sqrt{\langle \psi_+^0, \psi_-^0 \rangle}} = \frac{1}{\sqrt{k_-(u)k_+(\hat{\sigma}u) - k_+(u)k_-(\hat{\sigma}u)}}. \tag{J.5}$$

Now, owing to (J.2), $C(u)$ contains $2(g+1)$ square-root zeros at $\hat{\gamma}_i$ and $\hat{\sigma}\hat{\gamma}_i$. In addition, as argued in section 8.3.1, it must contain the square-root singularity at the positions of the branch points b_k. Thus the divisor of $C(u)$ is given by

$$(C) = \frac{1}{2}\sum_{j=1}^{g+1} \hat{\gamma}_j + \frac{1}{2}\sum_{j=1}^{g+1} \hat{\sigma}\hat{\gamma}_j - \frac{1}{2}\sum_{k=1}^{2(g+1)} b_k. \tag{J.6}$$

Combined with (J.2), this determines the divisor of the factor $\langle n, \psi_+ \rangle$ to be

$$(\langle n, \psi_+ \rangle) = \infty^+ + \sum_{i=1}^{g} \gamma_i' + \frac{1}{2}\sum_{j=1}^{g+1}(\hat{\sigma}\hat{\gamma}_j - \hat{\gamma}_j) - \frac{1}{2}\sum_{k=1}^{2(g+1)} b_k. \tag{J.7}$$

This shows that $\langle n, \psi_+ \rangle$ has spurious zeros and poles at $\hat{\gamma}_j$ and $\hat{\sigma}\hat{\gamma}_j$ unless we choose $\hat{\gamma}_j$ to be invariant under the holomorphic involution.

For the genus 0 solutions including the ones corresponding to the BPS operators, we can confirm that it is always possible to choose $\hat{\gamma}_j$ to be invariant under $\hat{\sigma}$ by analyzing the explicit form of the solution. On the other hand, the situation for the higher genus solutions is less obvious since it is in general not clear if we can choose $\hat{\gamma}_j$ to be invarint under $\hat{\sigma}$ without violating the relation (J.3). However, when the cuts are sufficiently small, the solution would be very close to the BPS one, and, therefore from the continuity argument similar to the one used in section 8.4, we expect that it is possible to choose $\hat{\gamma}_j$ to be invariant under the involution (at least for some appropriate choices[3] of γ_j'.)

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[3]Different choices of γ_j' in the moduli of two-point functions only change the overall phase of the structure constant. Thus, for the computation of the three-point functions, we can choose a convenient one.

Appendix K

Quasi-momentum in the full spectral curve

In this appednix, we shall clarify the relation (8.6.43).

For this purpose, consider the monodromy matrix for the full $AdS_5 \times S^5$ is a $(4|4) \times (4|4)$ matrix given by

$$\Omega_{AdS_5 \times S^5}(x) \sim \text{diag}\left(e^{i\tilde{p}_1}, e^{i\tilde{p}_2}, e^{i\tilde{p}_3}, e^{i\tilde{p}_4} | e^{i\hat{p}_1}, e^{i\hat{p}_2}, e^{i\hat{p}_3}, e^{i\hat{p}_4}\right). \tag{K.1}$$

Here $\tilde{p}_i$ and $\hat{p}_i$ denote the quasi-momenta for the S^5 part and for the AdS_5 part respectively.

The quasi-momenta in the $SU(2)_L \times SU(2)_R$ sector, which we studied in chapter 8, are identified with those in the full $AdS_5 \times S^5$ as follows (See figure K.0.1 above.):

$$p(x)|_{SU(2)_R} = \tilde{p}_2 - \tilde{p}_3, \qquad p(x)|_{SU(2)_L} = \tilde{p}_1 - \tilde{p}_4. \tag{K.2}$$

As explained in [84], owing to the $\mathbb{Z}_2$ automorphism of the coset, the quasi-momenta obey the following involution relation,

$$\tilde{p}_{1,2}(1/x) = -\tilde{p}_{2,1}(x), \qquad \tilde{p}_{3,4}(1/x) = -\tilde{p}_{4,3}(x). \tag{K.3}$$

In terms of the $SU(2)_L \times SU(2)_R$ quasi-momenta, this reads

$$p(1/x)|_{SU(2)_R} = -p(x)|_{SU(2)_L}, \qquad p(1/x)|_{SU(2)_L} = -p(x)|_{SU(2)_R}. \tag{K.4}$$

The quasi-momentum $p(x)$ used in the strong-coupling analysis in section 8.6 is the $SU(2)_R$ quasi-momentum. On the other hand, at weak coupling, the result factorizes into the $SU(2)_R$ and the $SU(2)_L$ sectors and the contribution from the $SU(2)_R$ $(SU(2)_L)$ sector is expressed purely in terms of $SU(2)_R$ $(SU(2)_L)$ quasi-momenta. Thus in order to make direct comparison between the weak-coupling and the strong-coupling results in the Frolov-Tseytlin limit, we need to rewrite a part of the strong-coupling result in terms of the $SU(2)_L$ quasi-momentum. This is precisely what we did in (8.6.43) and $\bar{p}$ defined there corresponds to the $SU(2)_L$ quasi-momentum.
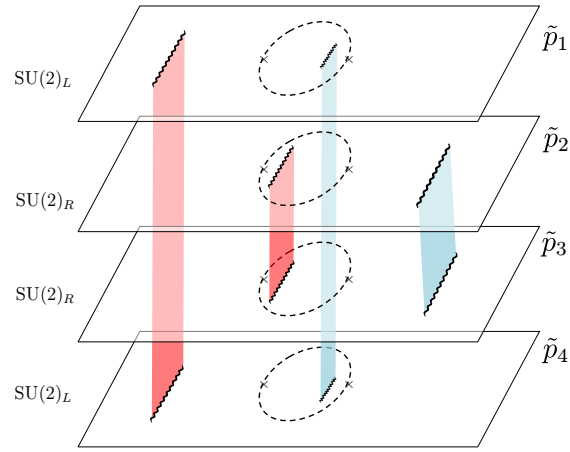
Figure K.0.1: The S^5 part of the full eight-sheeted spectral curve. The cuts denoted in the same color are related with each other by the $\mathbb{Z}_2$ automorphism. The $\mathrm{SU}(2)_L$ and $\mathrm{SU}(2)_R$ sectors discussed in chapter 8 correspond to the first and the fourth, and the second and the third sheets respectively.

Appendix L

Zeros of $\langle i_+, j_- \rangle$

Here we explain how to determine the zeros of the Wronskian for eigenfunctions with opposite sign eigenvalues, namely $\langle i_+, j_- \rangle$, by applying the argument given in [111].

As shown in (8.4.7), the product of $\langle i_+, j_- \rangle$ and $\langle i_-, j_+ \rangle$ contains zeros at $\sin(p_i - p_j + p_k)/2 = 0$ and $\sin(-p_i + p_j + p_k)/2 = 0$. For definiteness, we focus on zeros at $\sin(p_i - p_j + p_k)/2 = 0$ in what follows since the generalization to the zeros associated with $\sin(-p_i + p_j + p_k)/2 = 0$ is straightforward. When $\sin(p_i - p_j + p_k)/2 = 0$, all possible products of Wronskians which vanish are

$$\langle i_+, j_- \rangle \langle i_-, j_+ \rangle, \quad \langle j_-, k_+ \rangle \langle j_+, k_- \rangle, \quad \langle i_+, k_+ \rangle \langle i_-, k_- \rangle. \tag{L.1}$$

An important feature of (L.1) is that all the Wronskians that appear are the ones between the eigenstates in the same group, $\mathcal{S}_1 = \{i_+, j_-, k_+\}$ or $\mathcal{S}_2 = \{i_-, j_+, k_-\}$. Now, let us first note that the following lemma holds:

> **Lemma**: In each product of two Wronskians in (L.1), only one of the Wronskians can vanish.

This is because, if both of them vanish simultaneously, the product will have a double zero, and contradicts the fact that $\sin(p_i - p_k + p_k)/2$ only has simple zeros. Now, using this lemma, we will prove the following main theorem:

> **Theorem**: There are only two distinct possibilities concerning the zeros of the Wronskians in (L.1): Either (a) all the Wronnskians among the members of $\mathcal{S}_1$ are zero and those among $\mathcal{S}_2$ are nonzero, or (b) all the Wronskians among $\mathcal{S}_2$ are zero and those among $\mathcal{S}_1$ are nonzero.

A proof goes as follows. As stated in the Lemma, there are three distinct Wronskians which vanish at $\sin(p_i - p_j + p_k)/2$. This means that at least two of such Wronskians will be between the members of the same set, which can be $\mathcal{S}_1$ or $\mathcal{S}_2$. When the Wronskians

vanish, the two eigenvectors in the Wronskian become parallel to each other. Since each set contains only three vectors, if two different Wronskians among the same set vanish, all three eigenvectors in that set become parallel simultaneously. Then, the third Wronskian in that set must also vanish. This argument shows that all the Wronskians among one of two sets, $\mathcal{S}_1$ or $\mathcal{S}_2$, vanish simultaneously. Now, using the Lemma, we can conclude that the Wronskians among the other set must not vanish. This proves the theorem.

Since we already know the analyticity of the Wronskians of the same sign type, *i.e.* $\langle i_+, k_+ \rangle$ and $\langle i_-, k_- \rangle$, it is now straightforward to determine the zeros of the Wronskians with opposite signs using the Theorem above. This leads to the rule given in section 8.4.2.

Appendix M

Angle variable for the AdS part

In this appendix, we sketch the construction and the evaluation of the angle variable for the AdS part given in (8.6.19)(see also section 6.2 of [111]).

Since we are studying the solutions with no nontrivial motion in AdS, the quasi-momentum for the AdS part does not have any cut:

$$\hat{p}_i = \frac{\Delta_i x}{2g(x^2 - 1)}\,. \tag{M.1}$$

However, for the analysis of the angle variables, it turns out to be convenient to first consider the one-cut solution and then shrink the cut to get the result for (M.1). For one-cut solutions, there are two independent action variables,

$$S_\infty = \frac{1}{2\pi i} \int_\infty p(x)du(x)\,, \quad S_0 = \frac{1}{2\pi i} \int_0 p(x)du(x)\,. \tag{M.2}$$

Since the conformal dimension Δ is given by $S_0 - S_\infty$, the angle variable conjugate to Δ is given by $(\phi_0 - \phi_\infty)/2$, where ϕ_0 and ϕ_∞ are the variable conjugate to S_0 and S_∞ respectively.

Each angle variable ϕ_0 and ϕ_∞ can be constructed and evaluated in the similar manner as for the S^3 part. As a result, we obtain

$$
\begin{aligned}
\phi_0^{(i)} &= i \ln \left(\frac{\langle \tilde{n}_i\,, \tilde{n}_j \rangle \langle \tilde{n}_k\,, \tilde{n}_i \rangle}{\langle \tilde{n}_j\,, \tilde{n}_k \rangle} \frac{\langle j_+\,, k_+ \rangle}{\langle i_+\,, j_+ \rangle \langle k_+\,, i_+ \rangle} \Big|_{x=0^+} \right), \\
\phi_\infty^{(i)} &= i \ln \left(\frac{\langle n_i\,, n_j \rangle \langle n_k\,, n_i \rangle}{\langle n_j\,, n_k \rangle} \frac{\langle j_-\,, k_- \rangle}{\langle i_-\,, j_- \rangle \langle k_-\,, i_- \rangle} \Big|_{x=\infty^+} \right).
\end{aligned} \tag{M.3}
$$

As discussed in section 6.2 of [111], the polarization vectors in the AdS part are identified with the insertion points of the operators as

$$n_i = \begin{pmatrix} 1 \\ x_i \end{pmatrix}, \qquad \tilde{n}_i = \begin{pmatrix} 1 \\ \bar{x}_i \end{pmatrix}. \tag{M.4}$$

Substituting (M.4) to (M.3) and computing $\phi_\Delta = (\phi_0 - \phi_\infty)/2$, we arrive at the expression (8.6.19).

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