Construction of String Field Theory from Matrix Models

（行列模型からの弦の場の理論の構成）

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1

学 位 論 文

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Doctor Thesis

Construction of String Field Theory from Matrix Models

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1 Introduction

One of the most important problems in particle physics is to make a consistent quantum field theory of gravity. We have two ways for attacking it at present.

One is a non-perturbative formulation of quantum gravity such as a lattice gravity theory. Since the quantum gravity based on the Einstein action is perturbatively non-renormalizable in more than two space-time dimensions, it is impossible to define it based on the conventional perturbation theory.

The other is string theory. There is a spin-two massless particle in the closed string spectrum. Thus we can interprete it as a graviton [1]. However, at the first quantized level of the string theory, we have infinitely many classical solutions. If the total central charge (including the contribution of ghosts), i.e. an anomaly of the world sheet conformal symmetry is zero, any two-dimensional conformal field theory on a complex plane can be the classical solution (the perturbative vacuum of the string). If the string theory truly describes our world, it must have a mechanism selecting a true vacuum from the infinitely many perturbative vacua by its own non-perturbative effects. Or when considering a problem of space-time compactification, since it is a kind of phase transitions, we can not give a definite answer without examining the non-perturbative effects. Thus in the string theory the non-perturbative effects should be considered seriously. However, in the first quantized framework we can hardly know about them. In order to do so, it is natural to construct the second-quantized string theory (the string field theory) and to investigate its non-perturbative structure.

In these several years, much progress has been made about two-dimensional quantum gravity coupled with a matter which is a conformal field of the central charge $c \leq 1$. Now, the partition function and various correlation functions can be estimated, and we have some knowledge about an integrable structure of the system. Also, the two-dimensional quantum gravity is equivalent to a string theory such that the target space dimensionality is $c \leq 1$ (the $c \leq 1$ non-critical string). Thus the result is very stimulating to string theorists.

In particular, the development in the lattice approach based on the dynamical triangulation of the string world sheet by matrix models is very fascinating. The matrix models can be regarded as a constructive definition of a lattice regularization of the string field theory, and that by taking the continuum limit (the so-called double scaling limit) we can obtain some knowledge about the non-perturbative effects in the case of $c \leq 1$.

However, the matrix models are not in the conventional field theoretical form, and aspects of the string field theory do not appear manifestly. Thus it will be useful to reformulate the $c \leq 1$ matrix models in a field theoretical form for investigating the non-perturbative effects, based on analogy with conventional local field theories. Moreover it might be possible to get the unified understanding of lower-dimensional string field theories and by studying the symmetry of the resultant formalisms to find some guiding principle of constructing the generic (not restricted $c \leq 1$) string field theory.

Recently, Ishibashi and Kawai [2] proposed a simple string field theory which reproduces the results of the one-matrix model ($c = 0$). Its Hamiltonian has only three terms: a loop splitting vertex, a loop merging vertex and a tadpole term which represents the annihilation of a string. A string creation term and a kinetic term are absent. In this theory one can derive the correlation functions by solving the equation of motion obtained from the string field Hamiltonian. This is in contrast to the situation that in the one-matrix model the correlation functions can only be determined by considering an infinite number of components of the Schwinger-Dyson equations (the S-D eqs.). Thus the Hamiltonian has the same information as the infinitely many S-D eqs. in spite of its simple form. Further, Jevicki and Rodrigues [3] pointed out the relation of the theory to stochastic quantization of the one-matrix model. Although some ambiguous points are remaining in their argument about the continuum limit, it gives an interesting interpretation of the fictitious time in the stochastic quantization.

In respect to higher dimensional case ($c \leq 1$), Ishibashi and Kawai [4] proposed a string field theory of a similar type as before. They took the time differently from the $c = 0$ case, and considered a theory containing only string fields with the simplest spin configuration on the equal time loop. (All spins are aligned.) However, validity of their proposal has not been proven. Further, in the case that c takes the values $c = 1 - \frac{6}{m(m+1)}$ of the unitary minimal series (corresponding to the $(m-1)$-matrix model) Ikehara et al. [5] discussed a possibility of considering all spin configurations on the loop created by the string field. Most of their analysis is, however, devoted to the integrable structure of the *usual* S-D eqs. (the W_m constraint), and the Hamiltonian they gave is not in a definite form. The tadpole terms are not determined at all, and the inner product of the string states contains a divergence, which comes from that of the boundary states of boundary conformal field theory.

In view of this situation, we will present a more systematic derivation of string field Hamiltonians in the case of $c = 0$ and $1/2$ by directly constructing them from the matrix models and consider the nature of the resultant Hamiltonians.

This thesis is composed of four sections and four appendices. Section 2 is devoted to a brief review of the one- and two-matrix models for later convenience.

In Section 3 we construct the $c = 0$ string field theory from the one-matrix model for purposes of a warming up for the two-matrix case and of giving an improved derivation which does not contain the ambiguous points in the argument of Jevicki-Rodrigues. In Section 4 we construct the $c = 1/2$ string field theory directly from the two-matrix model without restricting the spin configurations of the string states. We show that the Hamiltonian has indeed no tadpole terms and no dependence of a cosmological constant. Because of these properties, the Hamiltonian alone can not given a sufficient set of equations to determine the correlation functions uniquely, without introducing further constraints. Then we discuss candidates for such constraints. Section 5 is devoted to a summary of our results and a discussion about the problems which are remained as subjects in future. Many of the calculations in Section 4 is straightforward but quite long. In appendices we show several examples of such calculations in detail. Appendix A is devoted to the estimations of disk amplitudes (genus zero one-point functons) of various spin configurations. Here we have extensively used the Mathematica package. The continuum limits of the amplitudes $W^{(4)}(\zeta_1, \sigma_1, \zeta_2, \sigma_2)$ and $W_1(\zeta_1; \zeta_2, \sigma_2)$ have not appeared in the literatures. Also, we derive the whole expressions of the disk amplitudes including the non-universal pieces. In Appendix B we obtain the spin-flip operator in the continuum limit using the results of Appendix A. It is necessary to represent the Hamiltonian in the definite form. Appendices C and D are devoted to the calculations which are necessary to taking the continuum limit of the Hamiltonian.

2 Brief Review of Matrix Models

2.1 One-Matrix Model

The partition function of the one-matrix model is a matrix integral of a $N \times N$ hermitian matrix M:

$$Z = \int d^{N^2}M \; e^{-N\mathrm{tr}V(M)} \tag{2.1}$$

where

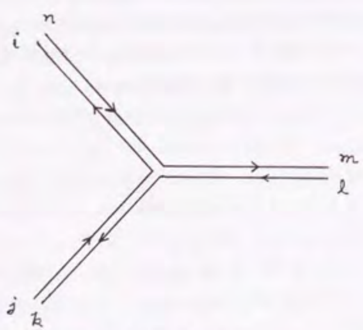
$$V(M) = \frac{1}{2}M^2 - \frac{g}{3}M^3. \tag{2.2}$$

After a perturbative expansion with respect to the coupling g, we obtain the following Feynman rules:

propagator

$$\langle M_{ij}M_{kl}\rangle = \frac{1}{N}\delta_{il}\delta_{jk} \quad =$$

vertex



$$= Ng\delta_{ij}\delta_{kl}\delta_{mn},$$

and N is assigned to each index loop.

Feynman graphs are constructed by connecting the above pieces with the direction of the arrows preserved. Thus the following weight is assigned to each Feynman graph

$$N^{-\sharp(\text{propagator})+\sharp(\text{loop})}(Ng)^{\sharp(\text{vertex})}.$$

a triangle and each propagator by a edge of the triangle (see Fig.2-1).

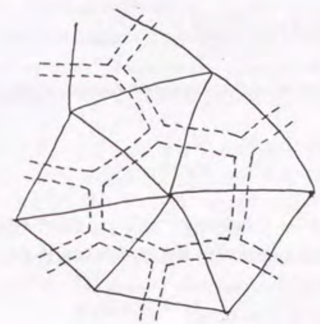


Fig.2-1

Then the above weight is rewritten in terms of the dual gragh G as

$$N^{-\sharp(\text{edge})+\sharp(\text{vertex})}(Ng)^{\sharp(\text{triangle})} = N^{\chi(G)}g^{A(G)},$$

where $\chi(G)$ and $A(G)$ represent the Euler number and the bare area of the graph G respectively.

Thus the logarithm of the partition function Z is shown to be

$$\begin{aligned}\ln Z &= \sum_{G:\text{connected graphs}} N^{\chi(G)}g^{A(G)} \\ &= \sum_{h=0}^{\infty} N^{2-2h}F_h(g) \\ &= N^2 \qquad + N^0 \qquad + N^{-2} \qquad + \cdots,\end{aligned}$$

where $F_h(g)$ is the partition function of a dynamically triangulated genus h surface or the h-loop string amplitude represented by the triangle lattice. Therefore Z can be interpreted as a lattice version of the string vacuum amplitude summed over the all string loop corrections.

In order to obtain a continuum theory we have to search the critical point of g where the graphs with sufficiently large area dominate. According to ref. [6] this point is known to be

$$g_* = \frac{3^{1/4}}{6}$$

and the behavior of $F_h(g)$ in the case of $g \sim g_*$ is given by

$$\frac{d^2}{dg^2}F_h(g) \sim \left(\frac{g_* - g}{g_*}\right)^{\frac{5}{2}(1-h)-2}.$$

Here we introduce a lattice spacing a, a renormalized cosmological constant t and a string coupling constant $g_{\rm st}$ as

$$g_* - g = g_* a^2 t, \quad N a^{5/2} = g_{\rm st}^{-1}.$$

By taking a continuum limit as $a \longrightarrow 0$ keeping t and $g_{\rm st}$ fixed (the double scaling limit) the partition function of the continuum theory is obtained

$$\frac{d^2}{dt^2}\ln Z = \sum_{h=0}^{\infty} \text{const.}\ g_{\rm st}^{2h-2}\ t^{\frac{5}{2}(1-h)-2}. \tag{2.3}$$

In the above it is remarkable that in the result after taking the continuum limit the contributions of the all topologies survive. Moreover according to refs. [7], [8], [9] some information about the non-perturbative sum (2.3) can be obtained. Namely, the double derivative of $\ln Z$ about t

$$f(t) = g_{\rm st}^2 \frac{d^2}{dt^2}\ln Z$$

obeys the Painlevé equation of the first kind:

$$f^2 + \frac{g_{\rm st}^2}{3}\frac{d^2}{dt^2}f = t.$$

If the above equation is solved, we are to determine the non-perturbative behavior of the string amplitude. But since it is a differential equation of the second degree, two initial conditions are needed. One of them is decided as the asymptotic expansion of f about $g_{\rm st} \sim 0$ which is coincident with eq.(2.3). However the other is unknown.

2.2 Two-Matrix Model

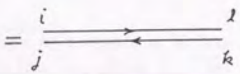
As seen in the previous subsection, in the one-matrix model there is no height degree on the random surface interpreted as a string world sheet. Here we consider the Ising spin (the $c = 1/2$ conformal matter) as a height on the random surface. This is represented by the following two-matrix model:

$$\begin{aligned} Z &= \int d^{N^2}A\, d^{N^2}B\ e^{-N\,{\rm tr}V(A,B)}, \\ V(A,B) &= \frac{1}{2}A^2 + \frac{1}{2}B^2 - \frac{g}{3}A^3 - \frac{g}{3}B^3 - cAB. \end{aligned} \tag{2.4}$$

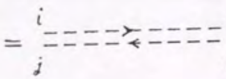
Here A and B are $N \times N$ hermitian matrices. Considering a perturbative expansion with respect to g, the Feynman rules are as follows:

propagators

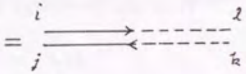
$$\langle A_{ij}A_{kl}\rangle = \frac{1}{N}\frac{1}{1-c^2}\delta_{il}\delta_{jk} \quad =$$



$$\langle B_{ij}B_{kl}\rangle = \frac{1}{N}\frac{1}{1-c^2}\delta_{il}\delta_{jk} \quad =$$

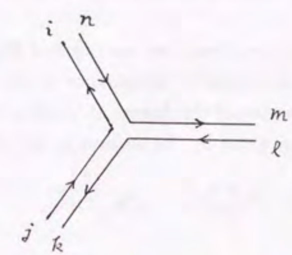


$$\langle A_{ij}B_{kl}\rangle = \frac{1}{N}\frac{c}{1-c^2}\delta_{il}\delta_{jk} \quad =$$



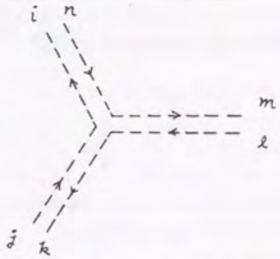
vertices

${\rm tr}A^3 \quad \leftrightarrow$



$= Ng\delta_{ij}\delta_{kl}\delta_{mn}$

$$\mathrm{tr} B^3 \quad \leftrightarrow$$



$$= N g \delta_{ij} \delta_{kl} \delta_{mn}$$

and N is assigned to each index loop. Considering the dual graph similarly as in the case of the one-matrix model, it can be seen that (2.4) represents a statistical model on the dynamically triangulated surface where the Ising spin is on the center of each triangle. (A and B are associated with up- and down-spins respectively.) In this case, the Boltzmann weights of the nearest neighbor spin-interaction is given by

$$\begin{aligned} e^{-\beta E_{\uparrow\uparrow}} = e^{-\beta E_{\downarrow\downarrow}} &= \frac{1}{1-c^2} \\ e^{-\beta E_{\uparrow\downarrow}} &= \frac{c}{1-c^2}. \end{aligned} \tag{2.5}$$

Thus the free energy $\ln Z$ is shown to be

$$\begin{aligned} \ln Z &= \sum_{G:\text{triangulated surface}} N^{\chi(G)} g^{A(G)} F(c, G) \\ &= \sum_{h=0}^{\infty} N^{2-2h} F_h(c, g) \end{aligned}$$

where $F(c, G)$ is the partition function of the Ising model on randomly trianglated lattice G.

In order to obtain a continuum theory we have to find the critical points of the both of g and c. The former is needed for dominance of the sufficiently large area surfaces, and the latter for survive of the degree of freedom of the Ising spin after the continuum limit. Those are given by the authors in ref.[10] as follows

$$c_* = \frac{-1 + 2\sqrt{7}}{27}, \quad g_* = \sqrt{10 c_*^3},$$

and when g approaches g_*, $F_h(c_*, g)$ behaves as

$$\frac{d^2}{dg^2} F_h(c_*, g) \sim \left(\frac{g_* - g}{g_*} \right)^{\frac{7}{3}(1-h)-2}.$$

Here we note that only when c is just tuned to c_*, there appears the degree of the freedom of the Ising spin.

Thus we can take the double scaling limit of $\ln Z$ by introducing the variables of the continuum theory t and g_{st} as

$$c = c_*, \quad g = g_*(1 - a^2 t), \quad N a^{7/3} = g_{\mathrm{st}}^{-1}$$

and tuning $a \to 0$. The result is

$$\frac{d^2}{dt^2} \ln Z = \sum_{h=0}^{\infty} \text{const.}\ g_{\mathrm{st}}^{2h-2}\ t^{\frac{7}{3}(1-h)-2}.$$

Moreover the authors of ref.[11] derived the non-linear differential equation satisfied by the above non-perturbative sum:

$$f^3 - f f'' - \frac{1}{2}(f')^2 + \frac{2}{27} f^{(4)} = t \tag{2.6}$$

where

$$f(t) = \frac{d^2}{dt^2} \ln Z$$

and for notational simplicity we set $g_{\mathrm{st}} = 1$ and rescaled f and t properly.

As in the one-matrix case there is a problem of the unknown initial conditions in eq.(2.6) also.

3 $c = 0$ Non-Critical String Theory from the One-Matrix Model

As seen in the previous section, the one-matrix model presents the constructive definition of the $c = 0$ non-critical string field theory. However, we can hardly see string field theoretic aspects directly from the one-matrix model.

In this section, introducing a string field operator we reformulate the one-matrix model to the string theoretic form. In Section 3.1 we mention a stochastic quantization method in case of a simple model. In Section 3.2 we apply it for the one-matrix model and derive a stochastic Hamiltonian, which has a remarkable property that it can uniquely determine the all correlation functions of the $c = 0$ non-critical string field at least perturbatively. And in Section 3.3 it is shown that the fictitious time of the stochastic quantization can be interpreted as the proper time on the string world sheet, and the Hamiltonian describes the evolution of strings along the proper time.

3.1 Stochastic Quantization

We will briefly explain a stochastic quantization method taking the system of the integrals of N-variables q_i's for example:

$$\begin{aligned} Z &= \int \prod_{i=1}^{N} dq_i \; e^{-V(q_1,\cdots,q_N)} \\ \langle \mathcal{O}(q) \rangle &= \frac{1}{Z} \int \prod_{i=1}^{N} dq_i \; \mathcal{O}(q) e^{-V(q_1,\cdots,q_N)}. \end{aligned} \tag{3.1}$$

We consider to reformulate this system 'classically' as follows. Firstly, introducing the fictitious time D and the random force (the white noise) $\eta_i(D)$, we define the 'time' evolution of q_i's by the Langevin equation:

$$\partial_D q_i(D) = -\partial_{q_i} V + \eta_i(D). \tag{3.2}$$

And since the force is random, the mean value under η_i's is defined by the Gaussian form:

$$\langle f(\eta) \rangle_\eta = \text{const.} \int \prod_{i,D} d\eta_i(D) \; f(\eta) e^{-\int dD \sum_i \eta_i(D)^2}, \tag{3.3}$$

where const. is determined so that $\langle 1 \rangle_\eta = 1$.

Then

$$\begin{aligned} \langle \eta_i(D) \rangle_\eta &= 0, \\ \langle \eta_i(D) \eta_j(D') \rangle_\eta &= 2\delta_{ij} \delta(D - D'), \\ &\cdots. \end{aligned}$$

Secondly, the probability distribution function $\Phi(q, D)$ is introduced by

$$\langle \mathcal{O}(q(D)) \rangle_\eta = \int \prod_{i=1}^{N} dq_i \; \mathcal{O}(q) \Phi(q, D). \tag{3.4}$$

Here we will derive the evolution of $\Phi(q, D)$. In order to make the process well-defined we use the discretized time $\{D_k\}$. Then eq. (3.2) changes to

$$q_i(D_{k+1}) - q_i(D_k) = -\partial_{q_i} V(q_i(D_k)) \Delta D + \eta_i(D_k) \Delta D.$$

After expanding the difference $\mathcal{O}(q(D_{k+1})) - \mathcal{O}(q(D_k))$ using the above and taking the mean value by the discretized version of (3.3), we obtain

$$\begin{aligned} &\langle \mathcal{O}(q(D_{k+1})) \rangle_\eta - \langle \mathcal{O}(q(D_k)) \rangle_\eta = \\ &\quad -\langle \sum_i \partial_{q_i} \mathcal{O}(q(D_k)) \partial_{q_i} V(q(D_k)) \rangle_\eta \Delta D + \langle \sum_i \partial_{q_i}^2 \mathcal{O}(q(D_k)) \rangle_\eta \Delta D + O((\Delta D)^2). \end{aligned}$$

Now we take the $\Delta D \to 0$ limit, and after a partial integral the result is

$$\frac{\partial}{\partial D}\langle \mathcal{O}(q(D))\rangle_\eta = \int \prod_{i=1}^{N} dq_i \; \mathcal{O}(q(D)) \sum_i \frac{\partial}{\partial q_i}\left(\frac{\partial}{\partial q_i} + \frac{\partial V}{\partial q_i}\right)\Phi(q, D).$$

Comparing with the r.h.s. of (3.4), we obtain the so-called Fokker-Planck equation

$$\frac{\partial}{\partial D}\Phi(q, D) = \sum_{i=1}^{N} \frac{\partial}{\partial q_i}\left(\frac{\partial}{\partial q_i} + \frac{\partial V}{\partial q_i}\right)\Phi(q, D). \tag{3.5}$$

We note that the time-independence of the normalization

$$\int \prod_{i=1}^{N} dq_i \Phi(q, D) = 1 \tag{3.6}$$

is consistent with (3.5).

Here the existence of the thermal equibrium state ($D \to \infty$ limit):

$$\lim_{D\to\infty} \frac{\partial}{\partial D}\Phi(q, D) = 0 \tag{3.7}$$

leads to

$$\Phi_{eq}(q) \equiv \lim_{D\to\infty} \Phi(q, D) = \text{const.} e^{-V(q)}, \tag{3.8}$$

where we used the fact that the kernel of the r.h.s. of (3.5) can be rewritten to the positive definite Laplacian

$$\sum_i \frac{\partial}{\partial q_i}\left(\frac{\partial}{\partial q_i} + \frac{\partial V}{\partial q_i}\right)\Phi(q, D) =$$
$$-e^{-\frac{1}{2}V}\left\{-\sum_i \left(\frac{\partial}{\partial q_i} - \frac{1}{2}\frac{\partial V}{\partial q_i}\right)\left(\frac{\partial}{\partial q_i} + \frac{1}{2}\frac{\partial V}{\partial q_i}\right)\right\}(e^{\frac{1}{2}V}\Phi(q, D)), \tag{3.9}$$

and the normalization constant is determined by

$$\int \prod_{i=1}^{N} dq_i \Phi_{eq}(q) = 1. \tag{3.10}$$

Thus we obtain

$$\langle \mathcal{O}(q)\rangle = \lim_{D\to\infty} \langle \mathcal{O}(q(D))\rangle_\eta.$$

Here it is noted that $\lim_{D\to\infty}\langle \mathcal{O}(q(D))\rangle_\eta$ can be determined by the 'classical' equation of motion (3.2) under the random force, independently of the initial condition. (The randomness of the force reflects the Gaussian contraction (3.3).) In this sense the correlator $\langle \mathcal{O}(q)\rangle$ can be calculated in the 'classical' framework.

Stochastic Hamiltonian and Schwinger-Dyson equation Next, we will mention about a stochastic Hamiltonian. Using the formal solution of eq. (3.5)

$$\Phi(q, D) = e^{D\sum_i \frac{\partial}{\partial q_i}(\frac{\partial}{\partial q_i} + \frac{\partial V}{\partial q_i})}\Phi_{\text{in}}(q),$$

we rewrite (3.1)

$$\langle \mathcal{O}(q)\rangle = \lim_{D\to\infty} \int \prod_{i=1}^{N} dq_i \; \Phi_{\text{in}}(q) e^{D\sum_i(\frac{\partial}{\partial q_i} - \frac{\partial V}{\partial q_i})\frac{\partial}{\partial q_i}}\mathcal{O}(q). \tag{3.11}$$

Here we represent it by an operator form. Applying the following realization

$$\begin{aligned}
[\frac{\partial}{\partial q_i},\; q_j] = \delta_{ij} \quad &\leftrightarrow \quad [\phi_i,\; \phi_j^\dagger] = \delta_{ij} \\
\mathcal{O}(q) \quad &\leftrightarrow \quad \mathcal{O}(\phi^\dagger)|0\rangle \\
\frac{\partial}{\partial q_i}\mathcal{O}(q) \quad &\leftrightarrow \quad \phi_i \mathcal{O}(\phi^\dagger)|0\rangle \quad (\phi_i|0\rangle = 0) \\
q_i\mathcal{O}(q) \quad &\leftrightarrow \quad \phi_i^\dagger \mathcal{O}(\phi^\dagger)|0\rangle \\
\mathcal{O}(0) \quad &\leftrightarrow \quad \langle 0|\mathcal{O}(\phi^\dagger)|0\rangle \quad (\langle 0|\phi_i^\dagger = 0)
\end{aligned}$$

and putting $\Phi_{\text{in}}(q) = \prod_i \delta(q_i)$ since $\Phi_{\text{in}}(q)$ can be chosen freely if normalized, we have

$$\langle \mathcal{O}(q)\rangle = \lim_{D\to\infty} \langle 0|e^{-DH}\mathcal{O}(\phi^\dagger)|0\rangle \tag{3.12}$$

where

$$H = \sum_{i=1}^{N}\left[-\phi_i^2 + \left(\frac{\partial V}{\partial q_i}\right)(\phi^\dagger)\phi_i\right]. \tag{3.13}$$

we will call this a stochastic Hamiltonian.

Further, we show that the stochastic Hamiltonian is derived from a (symmetrized) Schwinger-Dyson equation (S-D eq.) for the generating function

$$Z(J) = \frac{1}{Z}\int \prod_{i=1}^{N} dq_i \; e^{-V(q)}e^{J\cdot q} \quad (J \cdot q = \sum_i J_i q_i).$$

That S-D eq. is written as

$$0 = -\mathcal{H}Z(J), \quad \mathcal{H} = -\sum_i J_i\left(J_i - \left(\frac{\partial V}{\partial q_i}\right)\left(\frac{\partial}{\partial J}\right)\right), \tag{3.14}$$

which is derived from the identity

$$0 = \frac{1}{Z}\int \prod_{i=1}^{N} dq_i \; \sum_i \frac{\partial}{\partial q_i}\left(e^{-V(q)}\frac{\partial}{\partial q_i}e^{J\cdot q}\right). \tag{3.15}$$

From eq.(3.13) and the operator representation of $Z(J)$:

$$Z(J) = \lim_{D\to\infty} \langle 0|e^{-DH}e^{J\cdot\phi^\dagger}|0\rangle,$$

it can be seen that $\mathcal{H}$ is nothing but the J-representation of (3.13).

Moreover we remark that the solution of (3.14) is uniquely determined as follows. For pure imaginary J's, the Fourier transformed version of (3.14) is

$$0 = \sum_i \frac{\partial}{\partial Q_i}\left(\frac{\partial}{\partial Q_i} + \left(\frac{\partial V}{\partial q_i}\right)(Q)\right)\tilde{Z}(Q) \tag{3.16}$$

where

$$Z(J) = \int \prod_i dQ_i \, e^{J\cdot Q}\tilde{Z}(Q).$$

Repeating the same discussion as in (3.9) and (3.10), we find out the unique solution

$$\tilde{Z}(Q) = \text{const.}e^{-V(Q)},$$

where the multiplicative constant is determined as $Z(J=0)=1$.

3.2 Stochastic Hamiltonian of the One-Matrix Model

Here we derive the stochastic Hamiltonian from the one-matrix model using a method in the previous subsection. We follow a suggestion made by Jevicki and Rodrigues [3], however in their collective field formalism the non-trivial Jacobian factor is induced and the evaluation of that factor is very difficult in the scaling limit. Besides, there is an ambiguous point in their argument when determining the tadpole term in the Hamiltonian. From this reason we will present a derivation in such a way that the Jacobian does not appear.

We define a generating functional of the one-matrix model as

$$\begin{aligned} Z[J] &= \frac{1}{Z}\int d^{N^2}M \; e^{-N\mathrm{tr}V(M)}e^{J\cdot\Phi}, \\ Z &= \int d^{N^2}M \; e^{-N\mathrm{tr}V(M)}, \quad V(M) = \frac{1}{2}M^2 - \frac{g}{3}M^3, \\ J\cdot\Phi &= \oint \frac{d\zeta}{2\pi i}J(\zeta)\Phi(\zeta), \end{aligned}$$

where

$$\Phi(\zeta) = \frac{1}{N}\mathrm{tr}\frac{1}{\zeta - M} = \frac{1}{N}\sum_{n=0}^{\infty}\zeta^{-n-1}\mathrm{tr}M^n \tag{3.17}$$

is a loop operator and a contour of ζ-integral is taken as the convergence circle of a series in r.h.s. of (3.17) as depicted in Fig.3-1[2]. We take as the source $J(\zeta)$ a regular function inside the contour .

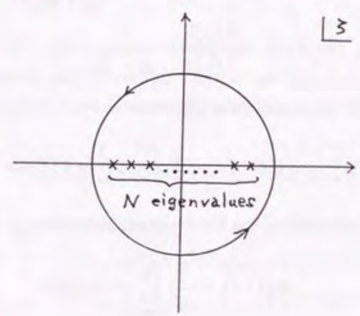


Fig.3-1

[2]In this theory with finite N, the $\Phi(\zeta)$ in the correlation functions will have no convergence circle since the region of M-integral is non-compact. However we are interested in $N \to \infty$ limit of the theory, where it is known that the master field configrations with the finite eigenvalues of M are dominant in the M-integral[6]. Thus we can regard that $\Phi(\zeta)$ has the finite convergence circle.

We will start the analogue of (3.15):

$$0 = \frac{1}{Z}\int d^{N^2}M \sum_{\alpha=1}^{N^2} \frac{\partial}{\partial M_\alpha}\left(e^{-N\mathrm{tr}V(M)}\frac{\partial}{\partial M_\alpha}e^{J\cdot\Phi}\right) \tag{3.18}$$

where M is expanded by the basis of $N \times N$ hermitian matrices $\{t^\alpha\}$:

$$M = \sum_{\alpha=1}^{N^2} M_\alpha\, t^\alpha.$$

Using the identities

$$\begin{aligned} \sum_\alpha \mathrm{tr}(At^\alpha Bt^\alpha) &= \mathrm{tr}A\mathrm{tr}B, \\ \sum_\alpha \mathrm{tr}(At^\alpha)\mathrm{tr}(Bt^\alpha) &= \mathrm{tr}AB, \end{aligned}$$

eq.(3.18) is rewritten as follows

$$0 = -\mathcal{H}Z[J] \tag{3.19}$$

$$\begin{aligned} \mathcal{H} = &-\oint\frac{d\zeta}{2\pi i}\left[\partial_\zeta J(\zeta)\left\{\left(\frac{\delta}{\delta J(\zeta)} - \frac{1}{2}(\zeta - g\zeta^2)\right)^2 - \frac{1}{4}(\zeta - g\zeta^2)^2 - g\zeta\right\}\right. \\ &\left.+\frac{1}{N^2}(\partial_\zeta J(\zeta))^2\frac{\delta}{\delta J(\zeta)}\right]. \end{aligned} \tag{3.20}$$

Here the functional derivative $\frac{\delta}{\delta J(\zeta)}$ is defined for ζ outside the contour as

$$\frac{\delta J(\zeta')}{\delta J(\zeta)} = \frac{1}{\zeta - \zeta'}. \tag{3.21}$$

Then eq.(3.21) shows the δ-function like nature

$$\oint\frac{d\zeta'}{2\pi i}\frac{\delta J(\zeta')}{\delta J(\zeta)}\Phi(\zeta') = \oint\frac{d\zeta'}{2\pi i}\frac{1}{\zeta - \zeta'}\Phi(\zeta') = \Phi(\zeta),$$

where for the above calculation we used a transformation $\zeta' = 1/z'$ (see Fig.3-2) and the fact that the

$$\Phi(1/z') = \frac{1}{N}\sum_{n=0}^{\infty} z'^{n+1}\mathrm{tr}M^n$$

is regular inside the contour in Fig.3-2.

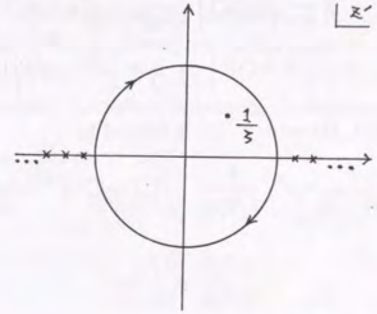


Fig.3-2

Continuum limit of $\mathcal{H}$ Introducing a lattice spacing a, the continuum limit (the so-called double scaling limit) is taken by $a \to 0$ with

$$\zeta = \zeta_*(1 + ay), \quad g = g_*(1 - a^2t), \quad \frac{1}{N} = a^{5/2}g_{\mathrm{st}}$$

where from ref.[6]

$$\zeta_* = (\sqrt{3} + 1)\cdot 3^{1/4}, \quad g_* = \frac{3^{1/4}}{6}.$$

Here in order to obtain the correct continuum limit, we must subtract a non-universal cut-off dependent part from the correlation functions. It appears only in the planar one-point function (the disk amplitude). Namely, the connected K-point function

$$W(\zeta_1, \cdots, \zeta_K) = \langle\Phi(\zeta_1)\cdots\Phi(\zeta_K)\rangle_c$$

is written as

$$\begin{aligned} W(\zeta) &= \frac{1}{2}(\zeta - g\zeta^2) + a^{3/2}w(y) + O(a^2) \\ W(\zeta_1, \cdots, \zeta_K) &= a^{3K/2}w(y_1, \cdots, y_K) + O(a^{(3K+1)/2}) \quad (K \geq 2) \end{aligned}$$

where $\frac{1}{2}(\zeta - g\zeta^2)$ is the non-universal part of the disk amplitude and w is a universal piece giving the correct continuum limit. This subtraction means the following shift of the J-derivative in $\mathcal{H}$:

$$\frac{\delta}{\delta J(\zeta)} \to \frac{\delta}{\delta J(\zeta)} + \frac{1}{2}(\zeta - g\zeta^2).$$

After the shift, $\mathcal{H}$ is in the form:

$$\mathcal{H} = -\oint \frac{d\zeta}{2\pi i}\left[\partial_\zeta J(\zeta)\left\{\frac{\delta^2}{\delta J(\zeta)^2} - \frac{1}{4}(\zeta - g\zeta^2)^2 - g\zeta\right\} + \frac{1}{N^2}(\partial_\zeta J(\zeta))^2\left(\frac{\delta}{\delta J(\zeta)} + \frac{1}{2}(\zeta - g\zeta^2)\right)\right].$$

In the continuum limit, the source $\tilde{J}(y)$ is defined by

$$\frac{\delta}{\delta J(\zeta)} = a^{3/2}\frac{\delta}{\delta \tilde{J}(y)}, \quad J(\zeta) = \zeta_*^{-1}a^{-5/2}\tilde{J}(y),$$

so that

$$\frac{\delta \tilde{J}(y')}{\delta \tilde{J}(y)} = \frac{1}{y - y'}.$$

Here we note that the inside (outside) of the contour in ζ-plane is mapped to the left (right) half plane in y-space, thus $\tilde{J}(y)$ is regular in the left half plane.

The above $\mathcal{H}$ becomes

$$\mathcal{H} = -a^{1/2}\zeta_*^{-1}\int_{-i\infty}^{i\infty}\frac{dy}{2\pi i}\left[\partial_y\tilde{J}(y)\left(\frac{\delta^2}{\delta\tilde{J}(y)^2} - C(y^3 - \frac{3}{4}\tau y) + O(a^1)\right) + g_{\mathrm{st}}^2\zeta_*^{-2}(\partial_y\tilde{J}(y))^2\left(\frac{\delta}{\delta\tilde{J}(y)} + \frac{3^{3/4}}{6}a^{-3/2} - \frac{1+\sqrt{3}}{2\cdot 3^{1/4}}a^{-1/2}y + O(a^{1/2})\right)\right],$$

where

$$\tau = \frac{16}{3(1+\sqrt{3})^2}t, \quad C = \frac{\sqrt{3}}{12}(1+\sqrt{3})^3.$$

At a glance there seems to exist undesirable terms: $\int_{-i\infty}^{i\infty}\frac{dy}{2\pi i}(\partial_y\tilde{J}(y))^2$ and $\int_{-i\infty}^{i\infty}\frac{dy}{2\pi i}(\partial_y\tilde{J}(y))^2 y$, however in fact it turns out that they vanish. We note that since $\int_{-i\infty}^{i\infty}\frac{dy}{2\pi i}(\partial_y\tilde{J}(y))^2 y^n$ is a quadratic functional of $\tilde{J}$ it vanishes if

$$I_n \equiv \frac{\delta}{\delta\tilde{J}(y_1)}\frac{\delta}{\delta\tilde{J}(y_2)}\int_{-i\infty}^{i\infty}\frac{dy}{2\pi i}(\partial_y\tilde{J}(y))^2 y^n = 0. \tag{3.22}$$

It is easy to see that I_n vanishes for $n = 0, 1, 2$ when $\mathrm{Re}y_1, \mathrm{Re}y_2 > 0$.

Thus after the rescaling as

$$\tilde{J}(y) \to \tilde{J}(y)C^{-1/2}, \quad g_{\mathrm{st}}^2\zeta_*^{-2}C^{-1} \to g_{\mathrm{st}}^2,$$

we have the stochastic Hamiltonian in the continuum theory

$$\mathcal{H} = -\int_{-i\infty}^{i\infty}\frac{dy}{2\pi i}\left[\partial_y\tilde{J}(y)\frac{\delta^2}{\delta\tilde{J}(y)^2} + g_{\mathrm{st}}^2(\partial_y\tilde{J}(y))^2\frac{\delta}{\delta\tilde{J}(y)} - \partial_y\tilde{J}(y)(y^3 - \frac{3}{4}\tau y)\right], \tag{3.23}$$

where the overall factor $a^{1/2}\zeta_*^{-1}C^{1/2}$ was absorbed by a redefinition of the fictitious time.

From this factor, we can see that the dimension of the fictitious time is half of that of the loop length, which is related to a fractal structure of the random surface [13].

Turning to the operator formalism, we obtain the following string field theory

$$\begin{aligned} Z[J] &= \lim_{D\to\infty}\langle 0|e^{-DH}e^{J\cdot\Psi^\dagger}|0\rangle, \\ H &= -\int_{-i\infty}^{i\infty}\frac{dy}{2\pi i}[\tilde{\Psi}^\dagger(y)^2\partial_y\tilde{\Psi}(y) + g_{\mathrm{st}}^2\tilde{\Psi}^\dagger(y)(\partial_y\tilde{\Psi}(y))^2 \\ &\quad -(y^3 - \frac{3}{4}\tau y)\partial_y\tilde{\Psi}(y)], \end{aligned} \tag{3.24}$$

$$\begin{aligned} [\tilde{\Psi}(y),\ \tilde{\Psi}^\dagger(y')] &= \frac{-1}{y - y'}, \quad \tilde{\Psi}(y)|0\rangle = \langle 0|\tilde{\Psi}^\dagger(y) = 0 \\ J\cdot\Psi^\dagger &= \int_{-i\infty}^{i\infty}\frac{dy}{2\pi i}\tilde{J}(y)\tilde{\Psi}^\dagger(y). \end{aligned}$$

3.3 Physical Meaning of the String Field Theory

In order to see a physical meaning of the theory it is convenient to change the argument y of the string field into a loop length l by making the Laplace tranformation:

$$\begin{aligned}
\Psi^\dagger(l) &= \int_{-i\infty}^{i\infty} \frac{dy}{2\pi i} e^{ly} \tilde{\Psi}^\dagger(y), \\
\Psi(l) &= \int_{-i\infty}^{i\infty} \frac{dy}{2\pi i} e^{-ly} \tilde{\Psi}(y), \\
J(l) &= \int_{-i\infty}^{i\infty} \frac{dy}{2\pi i} e^{-ly} \tilde{J}(y).
\end{aligned}$$

Then

$$\begin{aligned}
[\Psi(l),\ \Psi^\dagger(l')] &= \delta(l-l'), \quad \Psi(l)|0\rangle = \langle 0|\Psi(l) = 0, \\
J\cdot\Psi^\dagger &= \int_0^\infty dl J(l)\Psi^\dagger(l),
\end{aligned}$$

$$\begin{aligned}
H &= -\int_0^\infty dl_1 \int_0^\infty dl_2 \Psi^\dagger(l_1)\Psi^\dagger(l_2)(l_1+l_2)\Psi(l_1+l_2) \\
&\quad -g_{\mathrm{st}}^2 \int_0^\infty dl_1 \int_0^\infty dl_2 \Psi^\dagger(l_1+l_2) l_1 \Psi(l_1) l_2 \Psi(l_2) \\
&\quad -\int_0^\infty dl \rho(l)\Psi(l), \qquad (3.25) \\
\rho(l) &= 3\delta''(l) - \frac{3}{4}\tau\delta(l).
\end{aligned}$$

This is coincident with the form of the $c=0$ non-critical string field theory firstly proposed by Ishibashi and Kawai [2][3] and constructed using the dynamical triangulation method by Watabiki [12]. It is known that the theory (3.25) reproduces the correlation functions and the relations satisfied by them (the Virasoro constraint) calculated from the one-matrix model.

Indeed, from the existence of the $D \to \infty$ limit in the partition function the following equation of motion is obtained:

$$0 = \mathcal{H}Z[J] \qquad (3.26)$$

By expanding each order of J in (3.26), we have infinitely many relations to give the correlation functions. For example, the first component is in the form:

$$-\partial_y(w(y,y) + w(y)^2) + 3y^2 - \frac{3}{4}\tau = 0,$$

[3]Strictly speaking, the Hamiltonian (3.25) has the reciprocal sign to that of Ishibashi-Kawai. Their sign is misleading. Our Hamiltonian has the right sign, due to the direct construction from the representation of the positive definite Laplacian as (3.18)-(3.20).

in which expanding w's perturbatively with respect to g_{st}, the disk amplitude can be determined from the tadpole term by imposing an appropriate boundary condition (the analyticity in the right half plane of the y-space.) Also, we can show that the other generic correlation functions can be uniquely determined by imposing a suitable boundary condition.

Thus it is remarkable that the only one equation of motion (3.26) uniquely determines the system in the $c=0$ non-critical string. So to speak, the Hamiltonian by itself plays the same role as the infinitely many S-D eqs. required for giving the all correlation functions unambiguously.

In the Hamiltonian (3.25) it can be interpreted that the string field $\Psi(l)$ is an annihilation operator of a length l string with no marked point, and $\Psi^\dagger(l)$ is a creation operator of that with a marked point. The Hamiltonian H describes 'time'-evolution of a string, i.e. the first (second) term of (3.25) shows a splitting (merging) process and the third term an annihilation of a string. It is noted that the kinetic term representing a string propagation like $\int_0^\infty dl_1 \int_0^\infty dl_2 K(l_1,l_2)\Psi^\dagger(l_1)\Psi(l_2)$ and a string creation term like $\int_0^\infty dl C(l)\Psi^\dagger(l)$ do not appear.

The absence of the creation term is due to a definition of the 'time'. Here the 'time' is the proper time (geodesic distance) on the world sheet, which is defined unambiguously in the lattice theory [13]. The 'time' of a point P on a surface with boundary loops is determined as follows. Let us denote the geodesic distance from P to each loop (C_i) by d_i. Then the 'time' of P is defined by the minimum of d_i's (see Fig.3-3)

$$(\text{'time' of } P) = \min_i(\{d_i\}).$$

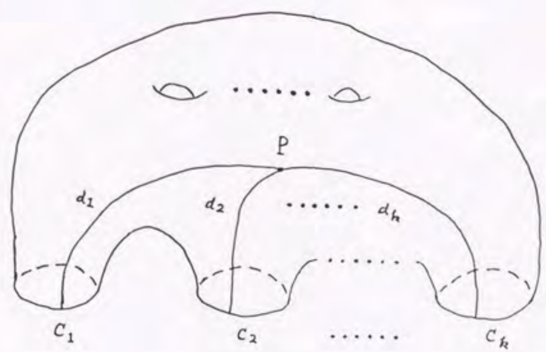


Fig.3-3

In this definition, the 'time' of the point not on the boundary loops can not take the local minimum value. Thus the string creation can not occur.

Further we note that the proper time appears in the stochastic quantization of the matrix model as the fictitious time. In the lattice definition, the stochastic Hamiltonian is equivalent to a superposition of the symmetrized versions of *usual* S-D eqs., which describes the one-step (one-lattice unit) deformation of incident loops on a dynamically triangulated surface. It is nothing but the one-unit evolution of the proper time [13]. Thus it could be naturally understood that the fictitious time is identified with the proper time.

Also the absence of the kinetic term will be understood from the dimensions of the 'time' D and the loop length l. Because $l \sim D^2 \gg D$ for the large D, loops would be densely packed in the surface. Then even the infinitesimal 'time' evolution would cause the loops to touch each other or themselves, and the string could not propagate freely. Thus the propagation by the kinetic term will not be needed in order to represent the dynamics of the $c = 0$ non-critical strings.

4 $c = 1/2$ Non-Critical String Field Theory from the Two-Matrix Model

In this section we construct the $c = 1/2$ non-critical string field theory from the two-matrix model without restricting the spin configuration of a string field, as the first step of considering the string field theory in realistic dimensions. It is done by the similar procedure to the one-matrix case. However we must introduce an infinite number of components as a string field because the configuration space of the Ising spins on the loop is infinitely large.

We will derive the string field Hamiltonian in the continuum theory under a few reasonable assumptions whose validity is confirmed in several simple cases. We will, however, point out that without imposing further constraints, it can not determine the partition function uniquely by itself contrary to the one-matrix case.

4.1 Stochastic Hamiltonian of the Two-Matrix Model

Here we start with the following generating functional of the two-matrix model:

$$\begin{aligned} Z[J] &= \frac{1}{Z}\int d^{N^2}A\, d^{N^2}B\; e^{-S} e^{J\cdot\Phi}, \\ Z &= \int d^{N^2}A\, d^{N^2}B\; e^{-S}, \\ S &= N\mathrm{tr}(V(A)+V(B)-cAB), \quad V(A)=\frac{1}{2}A^2-\frac{g}{3}A^3, \end{aligned}$$

$$\begin{aligned} J\cdot\Phi = &\oint\frac{d\zeta}{2\pi i}J_A(\zeta)\Phi_A(\zeta)+\oint\frac{d\sigma}{2\pi i}J_B(\sigma)\Phi_B(\sigma) \\ &+\sum_{n=1}^{\infty}\oint\prod_{i=1}^{n}\frac{d\zeta_i}{2\pi i}\frac{d\sigma_i}{2\pi i}J_n(\zeta_1,\sigma_1,\cdots,\zeta_n,\sigma_n)\Phi_n(\zeta_1,\sigma_1,\cdots,\zeta_n,\sigma_n), \end{aligned} \tag{4.1}$$

where Φ is a loop operator of the infinitely multi-components:

$$\begin{aligned} \Phi_A(\zeta) &= \frac{1}{N}\mathrm{tr}\frac{1}{\zeta-A}, \\ \Phi_B(\sigma) &= \frac{1}{N}\mathrm{tr}\frac{1}{\sigma-B}, \\ \Phi_n(\zeta_1,\sigma_1,\cdots,\zeta_n,\sigma_n) &= \frac{1}{N}\mathrm{tr}\left(\frac{1}{\zeta_1-A}\frac{1}{\sigma_1-B}\cdots\frac{1}{\zeta_n-A}\frac{1}{\sigma_n-B}\right) \quad (n=1,2,\cdots), \end{aligned}$$

and integration contours and source functions are taken similarly as in the one-matrix case (see Section 3.2).

$\Phi_A(\zeta)$ ($\Phi_B(\sigma)$) represents a configuration on the loop consisting of only one domain of A-spin (B-spin) and $\Phi_n(\zeta_1,\sigma_1,\cdots,\zeta_n,\sigma_n)$ represents $2n$ domains of A- and B-spins (see Fig.4-1).

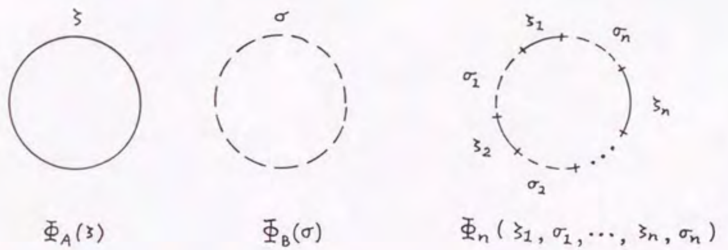


Fig.4-1

In order to derive the stochastic Hamiltonian, we shall consider the following identity

$$0=\int d^{N^2}A\,d^{N^2}B\sum_{a=1}^{N^2}\left[\frac{\partial}{\partial A_a}e^{-S}\frac{\partial}{\partial A_a}+\frac{\partial}{\partial B_a}e^{-S}\frac{\partial}{\partial B_a}\right]e^{J\cdot\Phi}. \tag{4.2}$$

As a result of the similar calculation as in the one-matrix case, the Hamiltonian constraint can be arranged in the following form:

$$\begin{aligned} 0 &= -\mathcal{H}Z[J] \\ \mathcal{H} &= -J\cdot K\frac{\delta}{\delta J}-J\cdot\left(\frac{\delta}{\delta J}\vee\frac{\delta}{\delta J}\right)-\frac{1}{N^2}(J\wedge J)\cdot\frac{\delta}{\delta J}-J\cdot T. \end{aligned} \tag{4.3}$$

Here we note that due to the cyclic symmetry of pairs (ζ_i,σ_i), a functional derivative of J_n is defined as

$$\begin{aligned} &\frac{\delta J_n(\zeta_1',\sigma_1',\cdots,\zeta_n',\sigma_n')}{\delta J_n(\zeta_1,\sigma_1,\cdots,\zeta_n,\sigma_n)} \\ &\quad=\frac{1}{n}\sum_{c:\text{cyclic permutation}}\frac{1}{\zeta_1-\zeta_{c(1)}'}\frac{1}{\sigma_1-\sigma_{c(1)}'}\cdots\frac{1}{\zeta_n-\zeta_{c(n)}'}\frac{1}{\sigma_n-\sigma_{c(n)}'}. \end{aligned}$$

Of course the derivative of J_n by the other components is zero.

Explanations of the notations in $\mathcal{H}$ are in order.

i) The first term (a kinetic term): The first few components of $K\frac{\delta}{\delta J}$ are

$$\begin{aligned} \left(K\frac{\delta}{\delta J}\right)_A(\zeta) &= \vec{\partial_\zeta}(\zeta-g\zeta^2)\frac{\delta}{\delta J_A(\zeta)}-c\partial_\zeta\oint\frac{d\sigma}{2\pi i}\sigma\frac{\delta}{\delta J_1(\zeta,\sigma)}, \\ \left(K\frac{\delta}{\delta J}\right)_B(\sigma) &= \vec{\partial_\sigma}(\sigma-g\sigma^2)\frac{\delta}{\delta J_B(\sigma)}-c\partial_\sigma\oint\frac{d\zeta}{2\pi i}\zeta\frac{\delta}{\delta J_1(\zeta,\sigma)}, \\ \left(K\frac{\delta}{\delta J}\right)_1(\zeta_1,\sigma_1) &= (\vec{\partial_{\zeta_1}}(\zeta_1-g\zeta_1^2)+\vec{\partial_{\sigma_1}}(\sigma_1-g\sigma_1^2))\frac{\delta}{\delta J_1(\zeta_1,\sigma_1)} \\ &\quad+c\oint\frac{d\sigma}{2\pi i}\sigma\frac{\delta}{\delta J_2(\zeta_1,\sigma,\zeta_1,\sigma_1)}+c\oint\frac{d\zeta}{2\pi i}\zeta\frac{\delta}{\delta J_2(\zeta_1,\sigma_1,\zeta,\sigma_1)} \\ &\quad+g\left(\frac{\delta}{\delta J_A(\zeta_1)}+\frac{\delta}{\delta J_B(\sigma_1)}\right), \end{aligned}$$

$$\begin{aligned} &\left(K\frac{\delta}{\delta J}\right)_2(\zeta_1,\sigma_1,\zeta_2,\sigma_2)= \\ &\quad\sum_{j=1}^{2}\{\vec{\partial_{\zeta_j}}(\zeta_j-g\zeta_j^2)+\vec{\partial_{\sigma_j}}(\sigma_j-g\sigma_j^2)\}\frac{\delta}{\delta J_2(\zeta_1,\sigma_1,\zeta_2,\sigma_2)} \\ &\quad+c\oint\frac{d\sigma}{2\pi i}\sigma\left(\frac{\delta}{\delta J_3(\zeta_1,\sigma,\zeta_1,\sigma_1,\zeta_2,\sigma_2)}+\frac{\delta}{\delta J_3(\zeta_1,\sigma_1,\zeta_2,\sigma,\zeta_2,\sigma_2)}\right) \end{aligned}$$

$$
\begin{aligned}
&+c\oint \frac{d\zeta}{2\pi i}\zeta\left(\frac{\delta}{\delta J_3(\zeta_1,\sigma_1,\zeta,\sigma_1,\zeta_2,\sigma_2)}+\frac{\delta}{\delta J_3(\zeta_1,\sigma_1,\zeta_2,\sigma_2,\zeta,\sigma_2)}\right)\\
&-gD_\sigma(\sigma_1,\sigma_2)\left(\frac{\delta}{\delta J_1(\zeta_1,\sigma)}+\frac{\delta}{\delta J_1(\zeta_2,\sigma)}\right)\\
&-gD_\zeta(\zeta_1,\zeta_2)\left(\frac{\delta}{\delta J_1(\zeta,\sigma_1)}+\frac{\delta}{\delta J_1(\zeta,\sigma_2)}\right),\\
&\cdots,
\end{aligned}
\tag{4.4}
$$

where an arrow over ∂ stands for that the derivative acts the whole functions following it and the 'combinatorial derivative' $D_P(P_1,P_2)$ is defined as

$$
D_P(P_1,P_2)f(P)=\frac{f(P_1)-f(P_2)}{P_1-P_2}.
$$

The structure of the generic component can be guessed from these expressions as explained below.

Each component represents two kinds of the processes. One is a string propagation preserving a spin configuration on a loop with the loop length changed by one- or two-lattice units. And the other is the process with the only one spin flipped but keeping the loop length. For example, let us see $\left(K\frac{\delta}{\delta J}\right)_1(\zeta_1,\sigma_1)$. The first and last columns show the former process. It is noted that as a special case if the A-(or B-)spin domain consists of only one- or two-lattice units, the process annihilates the domain. The last column represents this. And the middle shows the latter process. The operator $\oint\frac{d\sigma}{2\pi i}\sigma\frac{\delta}{\delta J_2(\zeta_1,\sigma,\zeta_1,\sigma_1)}$ corresponds to

$$
\oint\frac{d\sigma}{2\pi i}\sigma\Phi_2(\zeta_1,\sigma,\zeta_1,\sigma_1)=\frac{1}{N}\mathrm{tr}\left(\frac{1}{\zeta_1-A}B\frac{1}{\zeta_1-A}\frac{1}{\sigma_1-B}\right)
$$

which is obtained by flipping a A-spin in the ζ_1-domain of $\Phi_1(\zeta_1,\sigma_1)$. Also we note that $-D_\sigma(\sigma_1,\sigma_2)\frac{\delta}{\delta J_1(\zeta_1,\sigma)}$ in $\left(K\frac{\delta}{\delta J}\right)_2(\zeta_1,\sigma_1,\zeta_2,\sigma_2)$ corresponds to

$$
-D_\sigma(\sigma_1,\sigma_2)\Phi_1(\zeta_1,\sigma)=\frac{1}{N}\mathrm{tr}\left(\frac{1}{\zeta_1-A}\frac{1}{\sigma_1-B}\frac{1}{\sigma_2-B}\right)
$$

which is a result that the ζ_2-domain has disappeared in $\Phi_2(\zeta_1,\sigma_1,\zeta_2,\sigma_2)$.

ii) The second term represents a process where a string splits into two:

$$
\begin{aligned}
\left(\frac{\delta}{\delta J}\vee\frac{\delta}{\delta J}\right)_A(\zeta) &= -\partial_\zeta\frac{\delta^2}{\delta J_A(\zeta)^2},\\
\left(\frac{\delta}{\delta J}\vee\frac{\delta}{\delta J}\right)_B(\sigma) &= -\partial_\sigma\frac{\delta^2}{\delta J_B(\sigma)^2},\\
\left(\frac{\delta}{\delta J}\vee\frac{\delta}{\delta J}\right)_1(\zeta_1,\sigma_1) &= -2\left(\frac{\delta}{\delta J_A(\zeta_1)}\partial_{\zeta_1}+\frac{\delta}{\delta J_B(\sigma_1)}\partial_{\sigma_1}\right)\frac{\delta}{J_1(\zeta_1,\sigma_1)},
\end{aligned}
$$

$$
\begin{aligned}
\left(\frac{\delta}{\delta J}\vee\frac{\delta}{\delta J}\right)_2(\zeta_1,\sigma_1,\zeta_2,\sigma_2) &= -2\sum_{j=1}^{2}\left(\frac{\delta}{\delta J_A(\zeta_j)}\partial_{\zeta_j}+\frac{\delta}{\delta J_B(\sigma_j)}\partial_{\sigma_j}\right)\frac{\delta}{\delta J_2(\zeta_1,\sigma_1,\zeta_2,\sigma_2)}\\
&\quad +2D_\zeta(\zeta_1,\zeta_2)\frac{\delta}{\delta J_1(\zeta,\sigma_1)}D_\zeta(\zeta_1,\zeta_2)\frac{\delta}{\delta J_1(\zeta,\sigma_2)}\\
&\quad +2D_\sigma(\sigma_1,\sigma_2)\frac{\delta}{\delta J_1(\zeta_1,\sigma)}D_\sigma(\sigma_1,\sigma_2)\frac{\delta}{\delta J_1(\zeta_2,\sigma)},\\
&\quad \cdots.
\end{aligned}
\tag{4.5}
$$

iii) The third term corresponds to a merging process of two strings:

$$
\begin{aligned}
(J\wedge J)_A(\zeta) &= (\partial_\zeta J_A(\zeta))^2,\\
(J\wedge J)_B(\sigma) &= (\partial_\sigma J_B(\sigma))^2,\\
(J\wedge J)_1(\zeta_1,\sigma_1) &= 2((\partial_{\zeta_1}J_A(\zeta_1))\partial_{\zeta_1}+(\partial_{\sigma_1}J_B(\sigma_1))\partial_{\sigma_1})J_1(\zeta_1,\sigma_1),
\end{aligned}
$$

$$
\begin{aligned}
&(J\wedge J)_2(\zeta_1,\sigma_1,\zeta_2,\sigma_2)=\\
&\quad 2\sum_{j=1}^{2}\{(\partial_{\zeta_j}J_A(\zeta_j))\partial_{\zeta_j}+(\partial_{\sigma_j}J_B(\sigma_j))\partial_{\sigma_j}\}J_2(\zeta_1,\sigma_1\zeta_2,\sigma_2)\\
&\quad +J_1(\zeta_1,\sigma_1)J_1(\zeta_2,\sigma_2)(D_{\zeta_1}(z_1,z_2)D_{\zeta_2}(z_1,z_2)+D_{\sigma_1}(s_1,s_2)D_{\sigma_2}(s_1,s_2))\Big|_{\substack{z_i=\zeta_i,\\ s_i=\sigma_i}},\\
&\quad \cdots,
\end{aligned}
\tag{4.6}
$$

where $\Big|_{\substack{z_i=\zeta_i,\\ s_i=\sigma_i}}$ means that this substitution is performed after the combinatorial derivatives have acted.

iv) The last term (a tadpole term) shows an annihilation of a string:

$$
J\cdot T=\oint\frac{d\zeta}{2\pi i}J_A(\zeta)g+\oint\frac{d\sigma}{2\pi i}J_B(\sigma)g.
$$

Here it is noted that these processes in general occur in local with respect to the domains. Namely, more than two domains never create and annihilate at the same time and in the splitting and merging processes the recombinations between the only one pair of the domains occur . This property will be important in considering the continuum limit of $\mathcal{H}$ afterwards.

4.2 The Hamiltonian in the Continuum Theory

In this subsection we will take the continuum limit of the Hamiltonian (4.3). In order to do so, we have to subtract the non-universal pieces of disk amplitudes as in the one-matrix case. As we can see in Appendix A, a new feature of the two-matrix model that does not appear in the one-matrix is that the non-universal part of a disk amplitude contains the universal pieces of disk amplitudes with simpler spin configurations as well as the non-universal c-number function.

When we write a connected k-point correlator of the $J = 0$ background as

$$\begin{aligned} G^{(k)}_{I_1,\cdots,I_k} &= \langle \Phi_{I_1} \cdots \Phi_{I_k} \rangle_c, \\ & \qquad I_1, \cdots, I_k = A, B, 1, 2, \cdots, \end{aligned} \tag{4.7}$$

the generating functional $Z[J]$ is

$$Z[J] = \exp\left[J \cdot G^{(1)} + \frac{1}{2!} J \cdot (J \cdot G^{(2)}) + \frac{1}{3!} J \cdot (J \cdot (J \cdot G^{(3)})) + \cdots \right] \tag{4.8}$$

where the multiple inner product

$$\underbrace{J \cdot (J \cdots (J}_{k} \cdot G^{(k)}) \cdots)$$

is successively performed w.r.t. the each index of $G^{(k)}_{I_1,\cdots,I_k}$ as (4.1).

From the investigation of disk amplitudes we can see that for the universal part $\hat{\Phi}_I$ of the operator Φ_I is obtained by the linear transformation:

$$\Phi_I = \sum_J \mathcal{M}_{IJ} \hat{\Phi}_J + \phi_I, \tag{4.9}$$

where $\mathcal{M}_{IJ}$ is a mixing matrix of the universal part and ϕ_I is the non-universal c-number function. A few components of (4.9) is given by

$$\Phi_A(\zeta) = \hat{\Phi}_A(\zeta) + \phi_A(\zeta), \quad \phi_A(\zeta) = -\frac{c}{3g} + \frac{2}{3}(\zeta - g\zeta^2), \tag{4.10}$$

$$\Phi_B(\sigma) = \hat{\Phi}_B(\sigma) + \phi_B(\sigma), \quad \phi_B(\sigma) = -\frac{c}{3g} + \frac{2}{3}(\sigma - g\sigma^2), \tag{4.11}$$

$$\begin{aligned} \Phi_1(\zeta,\sigma) &= \sqrt{\frac{10}{27}} r (\hat{\Phi}_A(\zeta) + \hat{\Phi}_B(\sigma)) + \hat{\Phi}_1(\zeta,\sigma) + \phi_1(\zeta,\sigma), \\ \phi_1(\zeta,\sigma) &= \frac{r}{27}\left(1 - \frac{s}{P_-}(\zeta + \sigma - 2P_-)\right). \end{aligned} \tag{4.12}$$

$$\begin{aligned} \Phi_2(\zeta_1,\sigma_1,\zeta_2,\sigma_2) &= -\frac{10}{27} r (D_\zeta(\zeta_1,\zeta_2)\hat{\Phi}_A(\zeta) + D_\sigma(\sigma_1,\sigma_2)\hat{\Phi}_B(\sigma)) \\ &\quad -\sqrt{\frac{10}{27}} r [D_\zeta(\zeta_1,\zeta_2)(\hat{\Phi}_1(\zeta,\sigma_1) + \hat{\Phi}_1(\zeta,\sigma_2)) \\ &\quad + D_\sigma(\sigma_1,\sigma_2)(\hat{\Phi}_1(\zeta_1,\sigma) + \hat{\Phi}_1(\zeta_2,\sigma))] \\ &\quad + \hat{\Phi}_2(\zeta_1,\sigma_1,\zeta_2,\sigma_2) + \phi_2(\zeta_1,\sigma_1,\zeta_2,\sigma_2), \\ \phi_2(\zeta_1,\sigma_1,\zeta_2,\sigma_2) &= 10\left(\frac{r}{27}\right)^2, \end{aligned} \tag{4.13}$$

where r and s are two irrational numbers: $r = -1 + 2\sqrt{7}, \quad s = 2 + \sqrt{7}$.

Then the connected correlators are transformed as

$$\begin{aligned} G^{(1)}_I &= \sum_I \mathcal{M}_{IJ} \hat{G}^{(1)}_J + \phi_I, \\ G^{(k)}_{I_1,\cdots,I_k} &= \sum_{J_1,\cdots,J_k} \mathcal{M}_{I_1J_1} \cdots \mathcal{M}_{I_kJ_k} \hat{G}^{(k)}_{J_1,\cdots,J_k} \quad (k = 2,3,\cdots), \end{aligned}$$

where $\hat{G}^{(k)}_{I_1,\cdots,I_k}$ stands for the universal piece of $G^{(k)}_{I_1,\cdots,I_k}$.

Thus introducing a transformed source

$$J_I = \sum_K \hat{J}_K (\mathcal{M}^{-1})_{KI} \tag{4.14}$$

the universal partition function $\hat{Z}[\hat{J}]$ is written as

$$\begin{aligned} Z[J] &= e^{J\cdot\phi} \hat{Z}[\hat{J}], \\ \hat{Z}[\hat{J}] &= \exp\left[\hat{J} \cdot \hat{G}^{(1)} + \frac{1}{2!}\hat{J} \cdot (\hat{J} \cdot \hat{G}^{(2)}) + \frac{1}{3!}\hat{J} \cdot (\hat{J} \cdot (\hat{J} \cdot \hat{G}^{(3)})) + \cdots\right]. \end{aligned}$$

Further the Hamiltonian constraint (4.3) is in the form of

$$0 = -\mathcal{H}\hat{Z}[\hat{J}], \tag{4.15}$$

$$\begin{aligned} \mathcal{H} &= -(\hat{J}\mathcal{M}^{-1}) \cdot K\left(\mathcal{M}\frac{\delta}{\delta \hat{J}} + \phi\right) \\ &\quad -(\hat{J}\mathcal{M}^{-1}) \cdot \left(\left(\mathcal{M}\frac{\delta}{\delta \hat{J}} + \phi\right) \vee \left(\mathcal{M}\frac{\delta}{\delta \hat{J}} + \phi\right)\right) \\ &\quad -\frac{1}{N^2}((\hat{J}\mathcal{M}^{-1}) \wedge (\hat{J}\mathcal{M}^{-1})) \cdot \left(\mathcal{M}\frac{\delta}{\delta \hat{J}} + \phi\right) - (\hat{J}\mathcal{M}^{-1}) \cdot T. \end{aligned} \tag{4.16}$$

Here a few components of (4.14) are given by

$$\begin{aligned} J_A(\zeta) &= \hat{J}_A(\zeta) - \sqrt{\frac{10}{27}} r \oint \frac{d\sigma}{2\pi i} \hat{J}_1(\zeta,\sigma) \\ &\quad + \frac{10}{27} r \oint \frac{d\zeta_1}{2\pi i} \prod_{i=1}^{2} \frac{d\sigma_i}{2\pi i} \frac{1}{\zeta_1 - \zeta} \hat{J}_2(\zeta_1,\sigma_1,\zeta,\sigma_2) \\ &\quad + \frac{10}{27} r \oint \frac{d\zeta_2}{2\pi i} \prod_{i=1}^{2} \frac{d\sigma_i}{2\pi i} \frac{1}{\zeta_2 - \zeta} \hat{J}_2(\zeta,\sigma_1,\zeta_2,\sigma_2) + \cdots, \end{aligned} \tag{4.17}$$

$$J_B(\sigma) = \hat{J}_B(\sigma) - \sqrt{\frac{10}{27}} r \oint \frac{d\zeta}{2\pi i} \hat{J}_1(\zeta,\sigma)$$

$$
\begin{aligned}
&+\frac{10}{27}r\oint\frac{d\sigma_1}{2\pi i}\prod_{i=1}^{2}\frac{d\zeta_i}{2\pi i}\frac{1}{\sigma_1-\sigma}\hat{J}_2(\zeta_1,\sigma_1,\zeta_2,\sigma)\\
&+\frac{10}{27}r\oint\frac{d\sigma_2}{2\pi i}\prod_{i=1}^{2}\frac{d\zeta_i}{2\pi i}\frac{1}{\sigma_2-\sigma}\hat{J}_2(\zeta_1,\sigma,\zeta_2,\sigma_2)+\cdots, \qquad (4.18)
\end{aligned}
$$

$$
\begin{aligned}
J_1(\zeta,\sigma) &= \hat{J}_1(\zeta,\sigma)\\
&-\sqrt{\frac{10}{27}r}\oint\frac{d\zeta_1}{2\pi i}\frac{d\sigma_1}{2\pi i}\left(\frac{1}{\zeta_1-\zeta}+\frac{1}{\sigma_1-\sigma}\right)\hat{J}_2(\zeta_1,\sigma_1,\zeta,\sigma)\\
&-\sqrt{\frac{10}{27}r}\oint\frac{d\zeta_2}{2\pi i}\frac{d\sigma_2}{2\pi i}\left(\frac{1}{\zeta_2-\zeta}+\frac{1}{\sigma_2-\sigma}\right)\hat{J}_2(\zeta,\sigma,\zeta_2,\sigma_2)\\
&-\sqrt{\frac{10}{27}r}\oint\frac{d\zeta_1}{2\pi i}\frac{d\sigma_2}{2\pi i}\left(\frac{1}{\zeta_1-\zeta}+\frac{1}{\sigma_2-\sigma}\right)\hat{J}_2(\zeta_1,\sigma,\zeta,\sigma_2)\\
&-\sqrt{\frac{10}{27}r}\oint\frac{d\zeta_2}{2\pi i}\frac{d\sigma_1}{2\pi i}\left(\frac{1}{\zeta_2-\zeta}+\frac{1}{\sigma_1-\sigma}\right)\hat{J}_2(\zeta,\sigma_1,\zeta_2,\sigma)\\
&+\cdots, \qquad (4.19)\\
J_2(\zeta_1,\sigma_1,\zeta_2,\sigma_2) &= \hat{J}_2(\zeta_1,\sigma_1,\zeta_2,\sigma_2)+\cdots, \qquad (4.20)
\end{aligned}
$$

where $\cdots$ represents the terms which consists of $\hat{J}_I \quad (I=3,4,\cdots)$.

4.2.1 Comtinuum Limit of Kinetic, Splitting and Tadpole Terms

In order to show the existence of the continuum limit there are several points to be checked. Here we consider about the kinetic, splitting and tadpole terms in (4.16). Firstly, it is needed to rewrite the operators creating the locally spin-flipped loops

$$
\begin{aligned}
&-\partial_\zeta\oint\frac{d\sigma}{2\pi i}\sigma\left[\left(\mathcal{M}\frac{\delta}{\delta\hat{J}}\right)_1(\zeta,\sigma)+\phi_1(\zeta,\sigma)\right] \quad \text{in} \quad \left(K\left(\mathcal{M}\frac{\delta}{\delta\hat{J}}+\phi\right)\right)_A(\zeta), \qquad (4.21)\\
&-\partial_\sigma\oint\frac{d\zeta}{2\pi i}\zeta\left[\left(\mathcal{M}\frac{\delta}{\delta\hat{J}}\right)_1(\zeta,\sigma)+\phi_1(\zeta,\sigma)\right] \quad \text{in} \quad \left(K\left(\mathcal{M}\frac{\delta}{\delta\hat{J}}+\phi\right)\right)_B(\sigma), \qquad (4.22)\\
&\oint\frac{d\sigma}{2\pi i}\sigma\left[\left(\mathcal{M}\frac{\delta}{\delta\hat{J}}\right)_2(\zeta_1,\sigma,\zeta_1,\sigma_1)+\phi_2\right] \quad \text{and} \qquad (4.23)\\
&\oint\frac{d\zeta}{2\pi i}\zeta\left[\left(\mathcal{M}\frac{\delta}{\delta\hat{J}}\right)_2(\zeta_1,\sigma_1,\zeta,\sigma_1)+\phi_2\right] \quad \text{in} \quad \left(K\left(\mathcal{M}\frac{\delta}{\delta\hat{J}}+\phi\right)\right)_1(\zeta_1,\sigma_1) \qquad (4.24)\\
&\cdots
\end{aligned}
$$

to forms appropriate for taking the continuum limit. As we can see in Appendix A and B, for the disk amplitude (the genus zero one-point function) of (4.21), the universal part is

$$
-\partial_\zeta\hat{W}_1(\zeta)=-s^{-1}\oint\frac{d\sigma}{2\pi i}\sigma\partial_\zeta\hat{W}^{(2)}(\zeta,\sigma) \qquad (4.25)
$$

and the non-universal one is nothing but $W_1^{\text{non}}(\zeta)$ in (A.24).

Also for the disk amplitude of (4.23) the universal and non-universal pieces are

$$
\hat{W}_1(\zeta_1;\zeta_1,\sigma_1)=s^{-1}\oint\frac{d\sigma}{2\pi i}\sigma\hat{W}^{(4)}(\zeta_1,\sigma,\zeta_1,\sigma_1) \qquad (4.26)
$$

and $W_1^{\text{non}}(\zeta_1;\zeta_1,\sigma_1)$ in (A.31) respectively. The integral symbol $\oint\frac{d\sigma}{2\pi i}\sigma$ in (4.25), (4.26) is used in the sense of

$$
\oint\frac{d\sigma}{2\pi i}\sigma=P_*^2 a\int_C\frac{dx}{2\pi i}
$$

in the continuum limit, where the contour C is around the negative real axis and the singularities in the left half plane. This symbol can be regarded as the continuum spin-flip operator.

Since the spin-flip operation occurs in local with respect to domains, it could be considered that the result of (4.25), (4.26) holds even in the surface with non-zero genus and the loop with generic spin configuration. Thus the eqs.(4.21)–(4.24) can be rewritten as

$$
\begin{aligned}
&-\partial_\zeta\oint\frac{d\sigma}{2\pi i}\sigma\left[\left(\mathcal{M}\frac{\delta}{\delta\hat{J}}\right)_1(\zeta,\sigma)+\phi_1(\zeta,\sigma)\right]=\\
&-s^{-1}\oint\frac{d\sigma}{2\pi i}\sigma\partial_\zeta\frac{\delta}{\delta\hat{J}_1(\zeta,\sigma)}-\partial_\zeta\left(\left(\frac{2}{3g}-\frac{1}{3c}(\zeta-g\zeta^2)\right)\frac{\delta}{\delta\hat{J}_A(\zeta)}\right)\\
&-\frac{1}{c}\partial_\zeta[g\zeta+\frac{c}{9g}(\zeta-g\zeta^2)+\frac{2}{9}(\zeta-g\zeta^2)^2].
\end{aligned}
$$

$$-\partial_\sigma \oint \frac{d\zeta}{2\pi i}\zeta \left[\left(\mathcal{M}\frac{\delta}{\delta \hat{J}} \right)_1 (\zeta,\sigma) + \phi_1(\zeta,\sigma) \right] = (\zeta \leftrightarrow \sigma, \quad A \leftrightarrow B \text{ in the above}),$$

$$\begin{aligned}
\oint \frac{d\sigma}{2\pi i}\sigma \left[\left(\mathcal{M}\frac{\delta}{\delta \hat{J}} \right)_2 (\zeta_1,\sigma,\zeta_1,\sigma_1) + \phi_2 \right] = {} & s^{-1} \oint \frac{d\sigma}{2\pi i}\sigma \frac{\delta}{\delta \hat{J}_2(\zeta_1,\sigma,\zeta_1,\sigma_1)} \\
& + \frac{1}{c}\left(-\frac{2rs}{27P_*} - \sqrt{\frac{10}{27}}r(1-2g\zeta_1) \right) \frac{\delta}{\delta \hat{J}_A(\zeta_1)} \\
& + \frac{1}{c}\left(-\sqrt{\frac{10}{27}}r(1-2g\zeta_1) - g \right) \frac{\delta}{\delta \hat{J}_B(\sigma_1)} \\
& + \sqrt{\frac{10}{27}}r \left(-\frac{2}{3g} + \frac{1}{3c}(\zeta_1 - g\zeta_1^2) \right) \partial_{\zeta_1} \frac{\delta}{\delta \hat{J}_A(\zeta_1)} \\
& - \sqrt{\frac{10}{27}}rs^{-1}\partial_{\zeta_1} \oint \frac{d\sigma}{2\pi i}\sigma \frac{\delta}{\delta \hat{J}_1(\zeta_1,\sigma)} \\
& - \frac{1}{c}(1-2g\zeta_1)\frac{\delta}{\delta \hat{J}_1(\zeta_1,\sigma_1)} \\
& + \left(-\frac{2}{3g} + \frac{1}{3c}(\zeta_1 - g\zeta_1^2) \right) \partial_{\zeta_1} \frac{\delta}{\delta \hat{J}_1(\zeta_1,\sigma_1)} \\
& + \left(-\frac{rs}{27P_*}\left(-\frac{2}{3g} + \frac{1}{3c}(\zeta_1 - g\zeta_1^2) \right) - (1-2g\zeta_1)(1 - \frac{s}{P_*}(\zeta_1+\sigma_1-2P_*)) \right. \\
& \left. + \frac{1}{3} - \frac{2g}{3c}(\sigma_1 - g\sigma_1^2) \right),
\end{aligned}$$

$$\oint \frac{d\zeta}{2\pi i}\zeta \left[\left(\mathcal{M}\frac{\delta}{\delta \hat{J}} \right)_2 (\zeta_1,\sigma_1,\zeta,\sigma_1) + \phi_2 \right] = (\zeta \leftrightarrow \sigma, \quad A \leftrightarrow B \text{ in the above}),$$
$$\cdots.$$

Plugging altogether, after a straightforward calculation we have

$$\begin{aligned}
& -(\hat{J}\mathcal{M}^{-1}) \cdot K\left(\mathcal{M}\frac{\delta}{\delta \hat{J}} + \phi \right) - (\hat{J}\mathcal{M}^{-1}) \cdot \left(\left(\mathcal{M}\frac{\delta}{\delta \hat{J}} + \phi \right) \vee \left(\mathcal{M}\frac{\delta}{\delta \hat{J}} + \phi \right) \right) - (\hat{J}\mathcal{M}^{-1}) \cdot T \\
= {} & -(\hat{J}\mathcal{M}^{-1}) \cdot \left(\left(\mathcal{M}\frac{\delta}{\delta \hat{J}} \right) \vee \left(\mathcal{M}\frac{\delta}{\delta \hat{J}} \right) \right) \\
& - \oint \frac{d\zeta}{2\pi i}\hat{J}_A(\zeta)cs^{-1}(-\partial_\zeta) \oint \frac{d\sigma}{2\pi i}\sigma \frac{\delta}{\delta \hat{J}_1(\zeta,\sigma)} \\
& - \oint \frac{d\sigma}{2\pi i}\hat{J}_B(\sigma)cs^{-1}(-\partial_\sigma) \oint \frac{d\zeta}{2\pi i}\zeta \frac{\delta}{\delta \hat{J}_1(\zeta,\sigma)} \\
& - \oint \frac{d\zeta_1}{2\pi i}\frac{d\sigma_1}{2\pi i}\hat{J}_1(\zeta_1,\sigma_1)cs^{-1} \left[\oint \frac{d\sigma}{2\pi i}\sigma \frac{\delta}{\delta \hat{J}_2(\zeta_1,\sigma,\zeta_1,\sigma_1)} + \oint \frac{d\zeta}{2\pi i}\zeta \frac{\delta}{\delta \hat{J}_2(\zeta_1,\sigma_1,\zeta,\sigma_1)} \right] \\
& + \cdots,
\end{aligned} \tag{4.27}$$

where $\cdots$ stands for the terms containing $\hat{J}_n$ $(n \geq 2)$. It is noted that the tadpole term is cancelled with contribution of the kinetic term.

Secondly, we have to rewrite the splitting term (the first term in the above). After some algebras presented in Appendix C, it can be shown that

$$\left(\left(\mathcal{M}\frac{\delta}{\delta \hat{J}} \right) \vee \left(\mathcal{M}\frac{\delta}{\delta \hat{J}} \right) \right)_I = \left(\mathcal{M}\left(\frac{\delta}{\delta \hat{J}} \vee \frac{\delta}{\delta \hat{J}} \right) \right)_I \tag{4.28}$$

in the case of $I = A, B, 1, 2$. When acting to the loop operator $\frac{\delta}{\delta \hat{J}_I}$, $\mathcal{M}$ generates loops with simpler spin configurations by collapsing some domains of $\frac{\delta}{\delta \hat{J}_I}$. Thus eq.(4.28) represents that the two operations: loop splitting and domain collapse are commutative. Since the splitting process preserves a spin configuration, the commutativity is obvious when the domains which are related to splitting do not overlap the domains which are collapsed. Also even when there is such an overlap, since the derivatives or the combinatorial derivatives always act to the variables of the domains where the recombination occurs as seen in (4.5), there is no contribution to eq. (4.28). (We note that the loop operators have no dependence on the variables of the collapsed domains.) Thus we have good reasons for expecting that it holds for the general components. At present, however, we have no rigorous proof because of technical complexity.

Under the assumption that eq.(4.28) holds in general, the splitting term can be written as

$$-(\hat{J}\mathcal{M}^{-1}) \cdot \left(\left(\mathcal{M}\frac{\delta}{\delta \hat{J}} \right) \vee \left(\mathcal{M}\frac{\delta}{\delta \hat{J}} \right) \right) = -\hat{J} \cdot \left(\frac{\delta}{\delta \hat{J}} \vee \frac{\delta}{\delta \hat{J}} \right). \tag{4.29}$$

Now we are ready to take the continuum limit except the merging term. From the scaling behaviors of the disk amplitudes seen in Appendix A we should take the scaling of the various variables as follows:

$$\begin{aligned}
g &= g_*(1 - a^2\frac{s^2}{20}T), \quad \zeta = P_*(1+ay), \quad \sigma = P_*(1+ax), \\
N &= a^{-7/3}g_{\mathrm{st}}^{-1}, \\
\frac{\delta}{\delta \hat{J}_A(\zeta)} &= a^{4/3}P_*^{-1}\frac{\delta}{\delta \tilde{J}_A(y)}, \quad \hat{J}_A(\zeta) = a^{-7/3}\tilde{J}_A(y), \\
\frac{\delta}{\delta \hat{J}_B(\sigma)} &= a^{4/3}P_*^{-1}\frac{\delta}{\delta \tilde{J}_B(x)}, \quad \hat{J}_B(\sigma) = a^{-7/3}\tilde{J}_B(x), \\
\frac{\delta}{\delta \hat{J}_1(\zeta,\sigma)} &= a^{5/3}P_*^{-2}\frac{\delta}{\delta \tilde{J}_1(y,x)}, \quad \hat{J}_1(\zeta,\sigma) = a^{-11/3}\tilde{J}_1(y,x), \\
\frac{\delta}{\delta \hat{J}_2(\zeta_1,\sigma_1,\zeta_2,\sigma_2)} &= a^1 P_*^{-4}\frac{\delta}{\delta \tilde{J}_2(y_1,x_1,y_2,x_2)}, \\
& \quad \hat{J}_2(\zeta_1,\sigma_1,\zeta_2,\sigma_2) = a^{-5}\tilde{J}_2(y_1,x_1,y_2,x_2), \\
& \quad \cdots.
\end{aligned} \tag{4.30}$$

In fact, it turns out that in the continuum limit all the universal contributions in (4.27) starts with $O(a^{1/3})$, and thus we are convinced that the correct continuum limit can be taken by (4.30).

4.2.2 Continuum Limit of Merging Term

Next we consider about the merging process

$$-\frac{1}{N^2}((\hat{J}\mathcal{M}^{-1})\wedge(\hat{J}\mathcal{M}^{-1}))\cdot\left(\mathcal{M}\frac{\delta}{\delta\hat{J}}\right)-\frac{1}{N^2}((\hat{J}\mathcal{M}^{-1})\wedge(\hat{J}\mathcal{M}^{-1}))\cdot\phi.$$

By expanding the second term w.r.t. the components, we can see that the terms of the following types vanish:

$$\begin{array}{ll}(\hat{J}_A\hat{J}_I\phi_A) & (I=A,1,2),\\ (\hat{J}_B\hat{J}_I\phi_B) & (I=B,1,2),\\ (\hat{J}_1\hat{J}_1\phi_1), & \end{array}$$

since it can be shown that in the case of $n=0,1,2$

$$\begin{aligned}\int_{-i\infty}^{i\infty}\frac{dy}{2\pi i}(\partial_y\tilde{J}_A(y))^2y^n &= 0,\\ \int_{-i\infty}^{i\infty}\frac{dy}{2\pi i}\partial_y\tilde{J}_A(y)\partial_y\tilde{J}_1(y,x)y^n &= 0,\\ \int_{-i\infty}^{i\infty}\frac{dy}{2\pi i}\partial_y\tilde{J}_A(y)\int_{-i\infty}^{i\infty}\frac{dy_1}{2\pi i}\partial_y\left(\frac{1}{y-y_1}\tilde{J}_2(y_1,x_1,y,x)\right)y^n &= 0,\end{aligned}$$

by using the same logic as in the one-matrix model (3.22).

Also for the first term similarly, by using

$$\int_{-i\infty}^{i\infty}\frac{dy}{2\pi i}(\partial_y\tilde{J}_1(y,x))^2y^n=0,\quad(n=0,1,2),$$

$$\begin{aligned}&\int_{-i\infty}^{i\infty}\prod_{i=1}^{2}\frac{dy_i}{2\pi i}\frac{dx_i}{2\pi i}\tilde{J}_1(y_1,x_1)\tilde{J}_1(y_2,x_2)(-\partial_{y_1}\partial_{y_2})D_y(y_1,y_2)\frac{\delta}{\delta\tilde{J}_A(y)}\\ &=\int_{-i\infty}^{i\infty}\frac{dy}{2\pi i}\prod_{i=1}^{2}\frac{dx_i}{2\pi i}\partial_y\tilde{J}_1(y,x_1)\partial_y\tilde{J}_1(y,x_2)\frac{\delta}{\delta\tilde{J}_A(y)},\end{aligned}$$

after a rather long calculation which is given in Appendix D, it can be shown that for the first few or several components $(I=A,B,1,2)$

$$((\hat{J}\mathcal{M}^{-1})\wedge(\hat{J}\mathcal{M}^{-1}))\cdot\left(\mathcal{M}\frac{\delta}{\delta\hat{J}}\right)=(\hat{J}\wedge\hat{J})\cdot\frac{\delta}{\delta\hat{J}}. \tag{4.31}$$

Although we can not give general proofs for

$$((\hat{J}\mathcal{M}^{-1})\wedge(\hat{J}\mathcal{M}^{-1}))\cdot\phi=0 \tag{4.32}$$

and (4.31), it is natural to expect that the results confirmed in the first few or several components hold in general, because of the local property of the interaction with respect to the domains.

In the following we shall assume that (4.31) and (4.32) hold in general. Then the merging term can be rewritten as

$$-\frac{1}{N^2}(\hat{J} \wedge \hat{J}) \cdot \frac{\delta}{\delta \hat{J}},$$

which is the appropriate form for taking the continuum limit (4.30).

4.2.3 Continuum Limit of $\mathcal{H}$

At this stage we can obtain the Hamiltonian in the continuum theory immediately. After rescaling as

$$\tilde{J}_I \to P_*^{-2} \tilde{J}_I, \quad \frac{\delta}{\delta \tilde{J}_I} \to P_*^2 \frac{\delta}{\delta \tilde{J}_I}, \quad g_{\mathrm{st}}^2 \to P_*^4 g_{\mathrm{st}}^2,$$

and absorbing the overall factor $a^{1/3}$ into the fictitious time, the Hamiltonian $\mathcal{H}$ in the continuum theory can be written as

$$\mathcal{H} = -\tilde{J} \cdot \tilde{K} \cdot \frac{\delta}{\delta \tilde{J}} - \tilde{J} \cdot \left(\frac{\delta}{\delta \tilde{J}} \vee \frac{\delta}{\delta \tilde{J}} \right) - g_{\mathrm{st}}^2 (\tilde{J} \wedge \tilde{J}) \cdot \frac{\delta}{\delta \tilde{J}}, \tag{4.33}$$

where the forms of the splitting $\frac{\delta}{\delta \tilde{J}} \vee \frac{\delta}{\delta \tilde{J}}$ and the merging $\tilde{J} \wedge \tilde{J}$ are the same as in the lattice theory (4.5) and (4.6) respectively, while in the kinetic term only the flipping-spin process survives:

$$\begin{aligned}
\left(\tilde{K} \cdot \frac{\delta}{\delta \tilde{J}}\right)_A (y) &= cs^{-1}(-\partial_y) \int_C \frac{dx}{2\pi i} \frac{\delta}{\delta \tilde{J}_1(y,x)}, \\
\left(\tilde{K} \cdot \frac{\delta}{\delta \tilde{J}}\right)_B (x) &= cs^{-1}(-\partial_x) \int_C \frac{dy}{2\pi i} \frac{\delta}{\delta \tilde{J}_1(y,x)}, \\
\left(\tilde{K} \cdot \frac{\delta}{\delta \tilde{J}}\right)_1 (y_1,x_1) &= cs^{-1} \left[\int_C \frac{dx}{2\pi i} \frac{\delta}{\delta \tilde{J}_2(y_1,x,y_1,x_1)} + \int_C \frac{dy}{2\pi i} \frac{\delta}{\delta \tilde{J}_2(y_1,x_1,y,x_1)} \right],
\end{aligned}$$

$$\begin{aligned}
&\left(\tilde{K} \cdot \frac{\delta}{\delta \tilde{J}}\right)_2 (y_1,x_1,y_2,x_2) = \\
&cs^{-1} \left[\int_C \frac{dx}{2\pi i} \left(\frac{\delta}{\delta \tilde{J}_3(y_1,x,y_1,x_1,y_2,x_2)} + \frac{\delta}{\delta \tilde{J}_3(y_1,x_1,y_2,x,y_2,x_2)} \right) \right. \\
&\left. + \int_C \frac{dy}{2\pi i} \left(\frac{\delta}{\delta \tilde{J}_3(y_1,x_1,y,x_1,y_2,x_2)} + \frac{\delta}{\delta \tilde{J}_3(y_1,x_1,y_2,x_2,y,x_2)} \right) \right], \\
&\cdots.
\end{aligned} \tag{4.34}$$

Contrary to the one-matrix case, this Hamiltonian can not uniquely determine the partition function (the generating functional). It leads to the set of S-D eqs. which only relate the amplitude with a certain spin configuration to the one with a flipped-spin added, and thus we can get no closed equation of the amplitude from the Hamiltonian alone. Also it is noted that the Hamiltonian has no dependence on the cosmological constant T.

4.2.4 Detailed Observation of the Hamiltonian

In order to make the detailed physical observation of the Hamiltonian, it is convenient to rewrite it in the operator formalism and in the Laplace transformed variables (loop length) of y and x. In the operator formalism, $\tilde{J}$ and $\frac{\delta}{\delta \tilde{J}}$ are replaced by the loop operators $\tilde{\Psi}$ and $\tilde{\Psi}^\dagger$, and the Hamitonian becomes

$$H = -(\tilde{K}\tilde{\Psi}^\dagger) \cdot \tilde{\Psi} - (\tilde{\Psi}^\dagger \vee \tilde{\Psi}^\dagger) \cdot \tilde{\Psi} - g_{\mathrm{st}}^2 \tilde{\Psi}^\dagger \cdot (\tilde{\Psi} \wedge \tilde{\Psi}),$$

where

$$[\tilde{\Psi}_A(y), \tilde{\Psi}_A^\dagger(y')] = \frac{-1}{y - y'},$$

$$[\tilde{\Psi}_B(x), \tilde{\Psi}_B^\dagger(x')] = \frac{-1}{x - x'},$$

$$\begin{aligned}
&[\tilde{\Psi}_n(y_1, x_1, \cdots, y_n, x_n), \tilde{\Psi}_n^\dagger(y_1', x_1', \cdots, y_n', x_n')] \\
&= \frac{1}{n} \sum_{c:\text{cyclic perm.}} \frac{-1}{y_1 - y'_{c(1)}} \frac{-1}{x_1 - x'_{c(1)}} \cdots \frac{-1}{y_n - y'_{c(n)}} \frac{-1}{x_n - x'_{c(n)}}, \qquad (n = 1, 2, \cdots)
\end{aligned}$$

the other commutators vanish.

As a result of the Laplace tranformation

$$\begin{aligned}
\Psi_A^\dagger(l) &= \int_{-i\infty}^{i\infty} \frac{dy}{2\pi i} e^{ly} \tilde{\Psi}_A^\dagger(y), \\
\Psi_A(l) &= \int_{-i\infty}^{i\infty} \frac{dy}{2\pi i} e^{-ly} \tilde{\Psi}_A(y), \\
\Psi_B^\dagger(k) &= \int_{-i\infty}^{i\infty} \frac{dx}{2\pi i} e^{kx} \tilde{\Psi}_B^\dagger(x), \\
\Psi_B(k) &= \int_{-i\infty}^{i\infty} \frac{dx}{2\pi i} e^{-kx} \tilde{\Psi}_B(x), \\
\Psi_1^\dagger(l, k) &= \int_{-i\infty}^{i\infty} \frac{dy}{2\pi i} \frac{dx}{2\pi i} e^{ly + kx} \tilde{\Psi}_1^\dagger(y, x), \\
\Psi_1(l, k) &= \int_{-i\infty}^{i\infty} \frac{dy}{2\pi i} \frac{dx}{2\pi i} e^{-ly - kx} \tilde{\Psi}_1(y, x), \\
&\cdots,
\end{aligned}$$

the non-vanising commutators become

$$[\Psi_A(l), \Psi_A^\dagger(l')] = \delta(l - l'),$$

$$[\Psi_B(k), \Psi_B^\dagger(k')] = \delta(k - k'),$$

$$\begin{aligned}
&[\Psi_n(l_1, k_1, \cdots, l_n, k_n), \Psi_n^\dagger(l_1', k_1', \cdots, l_n', k_n')] \\
&= \frac{1}{n} \sum_{c:\text{cyclic perm.}} \delta(l_1 - l'_{c(1)}) \delta(k_1 - k'_{c(1)}) \cdots \delta(l_n - l'_{c(n)}) \delta(k_n - k'_{c(n)}) \\
&(n = 1, 2, \cdots).
\end{aligned}$$

And the Hamiltonian H is

$$H = -(K\Psi^\dagger) \cdot \Psi - (\Psi^\dagger \vee \Psi^\dagger) \cdot \Psi - g_{\mathrm{st}}^2 \Psi^\dagger \cdot (\Psi \wedge \Psi),$$

where the inner product is given by

$$\begin{aligned}
f \cdot g = &\int_0^\infty dl\, f_A(l) g_A(l) + \int_0^\infty dk\, f_B(k) g_B(k) \\
&+ \sum_{n=1}^\infty \int_0^\infty \prod_{i=1}^n dl_i dk_i f_n(l_1, k_1, \cdots, l_n, k_n) g_n(l_1, k_1, \cdots, l_n, k_n).
\end{aligned}$$

Each term of H is written in the loop length representation as follows.

The first term (the kinetic term) is

$$\begin{aligned}
(K\Psi^\dagger)_A(l) &= cl(F\Psi_1^\dagger)(l;), \\
(K\Psi^\dagger)_B(k) &= ck(F\Psi_1^\dagger)(k;), \\
(K\Psi^\dagger)_1(l, k) &= c\int_0^{l_1} dl (F\Psi_2^\dagger)(l; l_1 - l, k_1) \\
&\quad + c\int_0^{k_1} dk (F\Psi_2^\dagger)(l_1, k; k_1 - k), \\
(K\Psi^\dagger)_2(l_1, k_1, l_2, k_2) &= c\int_0^{l_1} dl (F\Psi_3^\dagger)(l; l_1 - l, k_1, l_2, k_2) \\
&\quad + c\int_0^{l_2} dl (F\Psi_3^\dagger)(l_1, k_1, l; l_2 - l, k_2) \\
&\quad + c\int_0^{k_1} dk (F\Psi_3^\dagger)(l_1, k; k_1 - k, l_2, k_2) \\
&\quad + c\int_0^{k_2} dk (F\Psi_3^\dagger)(l_1, k_1, l_2, k; k_2 - k), \\
&\cdots,
\end{aligned}$$

where the operator F collapses the one domain at the position of ";" into the infinitesimal one which consists of the only one flipped spin. The precise definition of F is hardly given in the terms of the loop length, but in the Laplace conjugate variables it is given as the contour integral in (4.34). From the above we can see that the kinetic term represents the process of flipping one spin without changing the loop length. The factor $l(k)$ or the integral $\int dl$ ($\int dk$) shows the sum up of the position of the spin to be flipped.

The second term (the splitting term) is

$$\begin{aligned}
(\Psi^\dagger \vee \Psi^\dagger)_A(l) &= l\int_0^l dl' \Psi_A^\dagger(l') \Psi_A^\dagger(l - l'), \\
(\Psi^\dagger \vee \Psi^\dagger)_B(k) &= k\int_0^k dk' \Psi_B^\dagger(k') \Psi_B^\dagger(k - k'), \\
(\Psi^\dagger \vee \Psi^\dagger)_1(l_1, k_1) &= 2\int_0^{l_1} dl \Psi_A^\dagger(l_1 - l) \Psi_1^\dagger(l, k_1) l \\
&\quad + 2\int_0^{k_1} dk \Psi_B^\dagger(k_1 - k) \Psi_1^\dagger(l_1, k) k,
\end{aligned}$$

$$
\begin{aligned}
&(\Psi^\dagger \vee \Psi^\dagger)_2(l_1,k_1,l_2,k_2)\\
&= 2\{\int_0^{l_1} dl \Psi_A^\dagger(l_1-l)\Psi_2^\dagger(l,k_1,l_2,k_2)l\\
&\quad +\int_0^{l_2} dl \Psi_A^\dagger(l_2-l)\Psi_2^\dagger(l_1,k_1,l,k_2)l\\
&\quad +\int_0^{k_1} dk \Psi_B^\dagger(k_1-k)\Psi_2^\dagger(l_1,k,l_2,k_2)k\\
&\quad +\int_0^{k_2} dk \Psi_B^\dagger(k_2-k)\Psi_2^\dagger(l_1,k_1,l_2,k)k\}\\
&\quad +2\{\int_0^{l_1+l_2} dl \Psi_1^\dagger(l,k_1)\Psi_1^\dagger(l_1+l_2-l,k_2)(l_1+l_2-l)\\
&\quad -\int_0^{l_1} dl \Psi_1^\dagger(l,k_1)\Psi_1^\dagger(l_1+l_2-l,k_2)(l_1-l)\\
&\quad -\int_0^{l_2} dl \Psi_1^\dagger(l,k_1)\Psi_1^\dagger(l_1+l_2-l,k_2)(l_2-l)\}\\
&\quad +2\{\int_0^{k_1+k_2} dk \Psi_1^\dagger(l_1,k)\Psi_1^\dagger(l_2,k_1+k_2-k)(k_1+k_2-k)\\
&\quad -\int_0^{k_1} dk \Psi_1^\dagger(l_1,k)\Psi_1^\dagger(l_2,k_1+k_2-k)(k_1-k)\\
&\quad -\int_0^{k_2} dk \Psi_1^\dagger(l_1,k)\Psi_1^\dagger(l_2,k_1+k_2-k)(k_2-k)\},\\
&\cdots.
\end{aligned}
$$

The A-(B-) component has the same structure as in the $c=0$ case. In the component 1 or the first brace of the component 2, the factor two can be regarded as a symmetric factor of the vertex. The second (third) brace of the component 2 is interpreted as follows. Precisely speaking, it is interpreted that the string has a marked point. When it approaches to some other point, the string splitts.[4] Let us consider the string with the spin configuration (l_1,k_1,l_2,k_2) and the marked point in the domain l_1. In the case of $l_1 < l_2$, by considering the three cases about the length l of the string which splitts off, the splitting process is written as

$$
\begin{aligned}
&\theta(l_2-l_1)\{\int_0^{l_1} dl \Psi_1^\dagger(l,k_1)\Psi_1^\dagger(l_1+l_2-l,k_2)l\\
&\quad +\int_{l_1}^{l_2} dl \Psi_1^\dagger(l,k_1)\Psi_1^\dagger(l_1+l_2-l,k_2)l_1\\
&\quad +\int_{l_2}^{l_1+l_2} dl \Psi_1^\dagger(l,k_1)\Psi_1^\dagger(l_1+l_2-l,k_2)(l_1+l_2-l)\}
\end{aligned}
$$

[4]The origin of the splitting is, say,

$$
\sum_\alpha \frac{\partial^2}{\partial A_\alpha^2}\mathrm{tr}\left(\frac{1}{\zeta_1-A}\frac{1}{\sigma_1-B}\frac{1}{\zeta_2-A}\frac{1}{\sigma_2-B}\right)
$$

in the S-D eq. (4.2). It can be interpreted the first operation of the derivative makes the marked point and the second makes the some other point.

$$
= \theta(l_2-l_1)\{\text{inside the second brace of } (\Psi^\dagger \vee \Psi^\dagger)_2\}.
$$

Since the inside of the brace in the r.h.s. is symmetric w.r.t. $l_1 \leftrightarrow l_2$, adding the result of the case $l_1 > l_2$ we have

$$
\{\text{inside the second brace of } (\Psi^\dagger \vee \Psi^\dagger)_2\}.
$$

Further, a factor two is brought by considering the case that the marked point is in the domain l_2. Also the third brace can be explained by the same observation when the point in the domain k_1 or k_2 is marked.

The last term (the merging term) is given by

$$
\begin{aligned}
(\Psi\wedge\Psi)_A(l) &= \int_0^l dl' \Psi_A(l')\Psi_A(l-l')l'(l-l'),\\
(\Psi\wedge\Psi)_B(k) &= \int_0^k dk' \Psi_B(k')\Psi_B(k-k')k'(k-k'),\\
(\Psi\wedge\Psi)_1(l_1,k_1) &= 2\int_0^{l_1} dl \Psi_A(l)\Psi_1(l_1-l,k_1)l(l_1-l)\\
&\quad +2\int_0^{k_1} dk \Psi_B(k)\Psi_1(l_1,k_1-k)k(k_1-k),
\end{aligned}
$$

$$
\begin{aligned}
&(\Psi\wedge\Psi)_2(l_1,k_1,l_2,k_2)\\
&= 2\{\int_0^{l_1} dl \Psi_A(l)\Psi_2(l_1-l,k_1,l_2,k_2)l(l_1-l)\\
&\quad +\int_0^{l_2} dl \Psi_A(l)\Psi_2(l,k_1,l_2-l,k_2)l(l_2-l)\\
&\quad +\int_0^{k_1} dk \Psi_B(k)\Psi_2(l_1,k_1-k,l_2,k_2)k(k_1-k)\\
&\quad +\int_0^{k_2} dl \Psi_B(k)\Psi_2(l_1,k_1,l_2,k_2-k)k(k_2-k)\}\\
&\quad +\{\int_0^{l_1+l_2} dl \Psi_1(l,k_1)\Psi_1(l_1+l_2-l,k_2)(l_1+l_2-l)\\
&\quad -\int_0^{l_1} dl \Psi_1(l,k_1)\Psi_1(l_1+l_2-l,k_2)(l_1-l)\\
&\quad -\int_0^{l_2} dl \Psi_1(l,k_1)\Psi_1(l_1+l_2-l,k_2)(l_2-l)\}\\
&\quad +\{\int_0^{k_1+k_2} dk \Psi_1(l_1,k)\Psi_1(l_2,k_1+k_2-k)(k_1+k_2-k)\\
&\quad -\int_0^{k_1} dk \Psi_1(l_1,k)\Psi_1(l_2,k_1+k_2-k)(k_1-k)\\
&\quad -\int_0^{k_2} dk \Psi_1(l_1,k)\Psi_1(l_2,k_1+k_2-k)(k_2-k)\},\\
&\cdots.
\end{aligned}
$$

As before the component $A(B)$ has the same structure as in the $c=0$ case, and the factor two can be regarded as a symmetric factor. The second or third brace of the component 2 can be understood by considering the merging as the inverse process of the splitting. Then we note that two loops (l, k) and (l', k') are able to merge only when they are marked in the domains l and l' or k and k', which explains that the factor two does not appear differently from the case of the splitting.

Now we also note that the dimension of the fictitious time (identified with the proper time on the world sheet) is one-third of that of the loop length. Namely, the loop length is a cube of the typical size in the time direction: $L \sim D^3 \gg D$, which indicates that the fractal nature of the loop is stronger than the $c = 0$ case.

4.3 Candidates for the Constraints

Here we expand some speculative discussions about candidates for the constraints and its relationship to the definiteness of the partition function.

Remembering the discussion of the stochastic quantization (Section 3.1), the uniqueness of the solution of the Hamiltonian constraint (3.14) is a natural result from the fact that the Fourier transformation of the operator $\mathcal{H}$ is nothing but the positive definite Laplacian. Thus we might naively guess that if we start with the positive definite Laplacian in the matrix models and construct the string field Hamiltonian, it could determine uniquely the generating functional (the normalized partition function with source terms).

In fact, in the one-matrix model we consider the following generating function

$$Z(j) = \frac{1}{Z} \int d^{N^2}M \; e^{-S} e^{\sum_\alpha j_\alpha M_\alpha} \tag{4.35}$$

as a substitute for

$$Z[J] = \frac{1}{Z} \int d^{N^2}M \; e^{-S} e^{J \cdot \Phi}.$$

The correlation function of $\Phi(\zeta)$ can be written as

$$\begin{aligned} \langle \Phi(\zeta_1) \cdots \Phi(\zeta_n) \rangle &= \varphi(\zeta_1) \cdots \varphi(\zeta_n) Z(j)|_{j=0}, \\ \varphi(\zeta) &= \frac{1}{N} \sum_{n=0}^{\infty} \zeta^{-n-1} \mathrm{tr} \left(\sum_\alpha t^\alpha \frac{\partial}{\partial j_\alpha} \right)^n . \end{aligned}$$

Then from the argument of Section 3.1 we can see that the equation of motion which is derived from

$$0 = \frac{1}{Z} \int d^{N^2}M \; \sum_\alpha \frac{\partial}{\partial M_\alpha} \left(e^{-S} \frac{\partial}{\partial M_\alpha} e^{\sum_\alpha j_\alpha M_\alpha} \right) \tag{4.36}$$

has an unique solution. Further, when we set $j = 0$ after acting $\varphi(\zeta_1) \cdots \varphi(\zeta_n)$ to (4.36), we have the S-D eqs. equivalent to those obtained from the expansion of (3.20) w.r.t. J. Thus it can be concluded that the equation of motion from

$$0 = \frac{1}{Z} \int d^{N^2}M \; \sum_\alpha \frac{\partial}{\partial M_\alpha} \left(e^{-S} \frac{\partial}{\partial M_\alpha} e^{J \cdot \Phi} \right)$$

has an unique solution.

However, the result we have obtained in the continuum limit indicates that it works for the one-matrix model but not for the two-matrix model. At present, we do not have appropriate understanding why this must so. However we could think the following two possibilities as the reason.

For one thing, we must remark that when we expand the Hamiltonian constraint (the equation of motion) of the matrix models with respect to the lattice spacing a, infinitely many constraint equations are obtained: the leading order of a, the subleading order, the subsubleading order and so on. Our discussion until now has been restricted to the leading order. There is no guarantee that all the necessary information is contained only in the leading order results.

In principle, in the two-matrix model we could estimate the contributions of the non-leading orders in the Hamiltonian constraint, and might find out some new constraint equations which determine the generating functional uniquely together with (the leading order of) the Hamiltonian. It may, however, be a formidable task to go beyond the calculation done in this work.

For the other thing, since the potentials in the matrix models are unbounded below, we must remark that the arguments of the stochastic quantization are formal. The theory might change corresponding to taking a different initial probability density $\Phi_{\rm in}$. Then we have to impose some constraints for the proper choice of the $\Phi_{\rm in}$.

As seen in Appendix A, we indeed can determine the various disk amplitudes starting with the *usual* S-D eqs. in the two-matrix model. Thus it will be useful to consider the solvable structure for a search of the constraints. We notice that there are two kinds of the S-D eqs. in the two-matrix model. One is derived from the deformation accompanied with flipping only one spin (e.g. (A.1), (A.2)), and the other is from the deformation with flipping two neighboring spins at the same time (e.g. (A.3)). Indeed, our Hamiltonian contains only the former deformation. The both are needed in order to determine the disk amplitude $W(\zeta)$ uniquely. Then, firstly we can consider the constraint containing the latter deformation which comes from the formula:

$$0 = \int d^{N^2}Ad^{N^2}B \sum_\alpha \left(\frac{\partial}{\partial A_\alpha} e^{-S} \frac{\partial}{\partial B_\alpha} + \frac{\partial}{\partial B_\alpha} e^{-S} \frac{\partial}{\partial A_\alpha} \right) e^{J\cdot\Phi}. \tag{4.37}$$

And it becomes the following form

$$0 = \mathcal{C}_1 Z[J],$$

$$\begin{aligned}
\mathcal{C}_1 \;=\; & \oint \frac{d\zeta}{2\pi i} J_A(\zeta) \left[\partial_\zeta \oint \frac{d\sigma}{2\pi i} \sigma(\sigma - g\sigma^2) \frac{\delta}{\delta J_1(\zeta,\sigma)} - c\partial_\zeta \left(\zeta \frac{\delta}{\delta J_A(\zeta)} \right) \right] \\
& + \oint \frac{d\sigma}{2\pi i} J_B(\sigma) \left[\partial_\sigma \oint \frac{d\zeta}{2\pi i} \zeta(\zeta - g\zeta^2) \frac{\delta}{\delta J_1(\zeta,\sigma)} - c\partial_\sigma \left(\sigma \frac{\delta}{\delta J_B(\sigma)} \right) \right] \\
& + \oint \frac{d\zeta_1}{2\pi i} \frac{d\sigma_1}{2\pi i} J_1(\zeta_1,\sigma_1) \left[- \oint \frac{d\sigma}{2\pi i} (\sigma - g\sigma^2) \frac{\delta}{\delta J_2(\zeta_1,\sigma,\zeta_1,\sigma_1)} \right.
\end{aligned}$$

$$\begin{aligned}
& - \oint \frac{d\zeta}{2\pi i} (\zeta - g\zeta^2) \frac{\delta}{\delta J_2(\zeta_1,\sigma_1,\zeta,\sigma_1)} \\
& \left. - c(\vec{\partial_{\zeta_1}} \zeta_1 + \vec{\partial_{\sigma_1}} \sigma_1) \frac{\delta}{\delta J_1(\zeta_1,\sigma_1)} \right] \\
& + \frac{2}{N^2} \oint \frac{d\zeta_1}{2\pi i} \frac{d\sigma_1}{2\pi i} J_A(\zeta_1) J_B(\sigma_1) \partial_{\zeta_1} \partial_{\sigma_1} \frac{\delta}{\delta J_1(\zeta_1,\sigma_1)} \\
& + \oint \frac{d\zeta_1}{2\pi i} \frac{d\sigma_1}{2\pi i} J_1(\zeta_1,\sigma_1) 2 \frac{\delta^2}{\delta J_1(\zeta_1,\sigma_1)^2} \\
& + \cdots,
\end{aligned} \tag{4.38}$$

where $\cdots$ stands for the terms containing J_1^2 and $J_n \quad (n \geq 2)$.

Though we have not accomplished taking the continuum limit of (4.38), comparing the first or second column of (4.38) with the continuum limit of the disk amplitude $W_2(\zeta)$:

$$\hat{W}_2(\zeta) - a^4 \frac{9r^{-1/2}s^4}{160\sqrt{30}} (16y^4 - 16Ty^2 + 2T^2) = 0,$$

we could expect that in the continuum limit the constraint $\mathcal{C}_1$ has a tadpole term which gives a dependence on the cosmological constant.

Further, since from the Hamiltonian we can not make the amplitude where two-neighboring spins are flipped, in order to obtain a closed equation of the amplitude another relation $\mathcal{C}_2$ analogous to (A.2) would be needed.

Here if we write naively the generating functional as

$$Z[J] = \lim_{D\to\infty} \langle 0| e^{-DH} e^{J\cdot\Psi^\dagger} |0\rangle, \tag{4.39}$$

with the vacuum such that $\Psi|0\rangle = \langle 0|\Psi^\dagger = 0$, it might look as if the generating functional is determined uniquely. However, eq.(4.39) gives a trivial result $Z[J] = 1$ since $\langle 0|H = 0$ from the absence of tadpole terms in H. In order to make (4.39) meaningful, we would have to consider the constraints with the tadpole terms. We guess that the proper definition of $Z[J]$ would be

$$Z[J] = \lim_{D\to\infty} \langle * | e^{-DH} e^{J\cdot\Psi^\dagger} |0\rangle, \tag{4.40}$$

where the state $\langle *|$ is taken as

$$0 = \lim_{D\to\infty} \langle * | e^{-DH} \mathcal{C}_i$$

satisfies for the operator form of the constraints $\mathcal{C}_i$. If there are the tadpole terms in the constraints, (4.40) is expected to give a nontrivial result.[5]

[5] Ishibashi and Kawai have proposed a Hamiltonian which has no tadpole term and no depen-

5 Discussions

We have derived the string field Hamiltonians which describe the evolution of the loops along the proper time on the world sheet in the $c = 0$ and $1/2$ case from the one- and two-matrix models directly. The common features in the $c = 0$ and $1/2$ theory are the following two points.

1) The loop splitting or merging process occurs without any changes of the loop length (and the spin configuration).

2) The Hamilonian has no kinetic terms corresponding to free propagation of strings.

For the $c = 0$ case it coincides with the results obtained by the authors in [2],[3],[12], and by the Hamiltonian itself all the correlation functions can be determined uniquely in the form of the perturbative expansion.

For the $c = 1/2$ case the Hamiltonian is written in a rather simple form, though the string field has infinitely many components. However, the Hamiltonian has no dependence on the cosmological constant, and the equation of motion can not determine the amplitudes. Thus we need to introduce some constraints in order to fix the partition function uniquely.

Also the other interesting results of this work are in order.

1) The fractal structure of the boundary loops.

Comparing the dimensions of the loop length L and the proper time D, we have the following result:

$$L \sim D^2 \quad (c = 0), \qquad L \sim D^3 \quad (c = 1/2).$$

From the result, we will tend to guess for the unitary minimal matter

$$L \sim D^m \quad (c = 1 - \frac{6}{m(m+1)}).$$

It indicates that the fractal structure of the loops becomes stronger as m increases, and the limit of $c \to 1$ is very singular. This would suggest the existence of the $c = 1$ barrier which is considered as a phase transition point of the two-dimensional quantum gravity.

2) The simple form of the Hamiltonian.

dence of the cosmological constant in the $c \leq 1$ string field theory [4]. From the above argument, their generating functional seems to be trivial. Also they obtain the disk amplitude $W(\zeta)$ from a S-D eq., but we might think that this S-D eq. could not have an unique solution since in the definition of the partition function (the generating functional) the cosmological constant does not appear. Thus we doubt that their definition is proper or not. It would be necessary that some constraints are considered other than the Hamiltonian in their theory too.

We have derived the Hamiltonian in the definite form. In the $c = 1/2$ case in spite of an infinite number of components of string fields, it takes quite a simple form. As seen in Section 4.2.4 all terms in the Hamiltonian can be explained by the combinatorial arguments. It will be easy to generalize the result in the case of higher dimensional string theories.

Finally we point out the remaining subjects, which are the following two points.

1) Finding out the constraints in the continuum theory.

It is important for the correct representation of the partition function in the present framework of string field theory. It will be interesting that we consider the algebra formed by the Hamiltonian and the constraints, relating to the symmetry and the integrable structure in the theory. It may be useful for constructing the string field theories in the $c > 1/2$ case and for understanding the string field theories in unified manner.

2) Studying the non-perturbative effects in the string field theory.

For example, it will be interesting and important to understand how to derive the Painlevé equation and the tunnelling effects in the one-matrix model, starting from the string field theory obtained here. If it is possible, we might find the way to derive the correlation functions non-perturbatively.

Appendix

A Disk Amplitudes in the Two-Matrix Model

Here we obtain various disk amplitudes (genus zero one-point functions) in the two-matrix model by taking the continuum limit of the Schwinger-Dyson equations (S-D eqs.) which give the relations among them. The S-D eqs. have been discussed partially by Staudacher [14]. Here we will give detailed forms of various disk amplitudes in the continuum limit. The results (including the non-universal parts) have not appeared in the literatures.

We introduce the following notations for the disk amplitudes (some of which are borrowed from [14]):

$$
\begin{aligned}
W_n &= \left\langle \frac{1}{N}\mathrm{tr}A^n \right\rangle_0 , \\
W^{(2)}_{n,m} &= \left\langle \frac{1}{N}\mathrm{tr}A^n B^m \right\rangle_0 , \\
W(\zeta) &= \left\langle \frac{1}{N}\mathrm{tr}\frac{1}{\zeta - A} \right\rangle_0 , \\
W(\sigma) &= \left\langle \frac{1}{N}\mathrm{tr}\frac{1}{\sigma - B} \right\rangle_0 , \\
W_j(\zeta) &= \left\langle \frac{1}{N}\mathrm{tr}\frac{1}{\zeta - A} B^j \right\rangle_0 , \\
W^{(2)}(\zeta,\sigma) &= \left\langle \frac{1}{N}\mathrm{tr}\frac{1}{\zeta - A}\frac{1}{\sigma - B} \right\rangle_0 , \\
W_j(\zeta_1;\zeta_2,\sigma_2) &= \left\langle \frac{1}{N}\mathrm{tr}\frac{1}{\zeta_1 - A} B^j \frac{1}{\zeta_2 - A}\frac{1}{\sigma_2 - B} \right\rangle_0 , \\
W^{(2k)}(\zeta_1,\sigma_1,\cdots,\zeta_n,\sigma_n) &= \left\langle \frac{1}{N}\mathrm{tr}\frac{1}{\zeta_1 - A}\frac{1}{\sigma_1 - B}\cdots\frac{1}{\zeta_n - A}\frac{1}{\sigma_n - B} \right\rangle_0 \quad (k = 1,2,\cdots).
\end{aligned}
$$

A.1 $W(\zeta)$

Firstly we derive the disk amplitude with the most simple spin configuration on the loop. (The spins on the loop are all A-.) It is obtained by combining the following three S-D eqs.:

$$\begin{aligned}(\zeta - g\zeta^2)W(\zeta) &= cW_1(\zeta) + W(\zeta)^2 + 1 - g(\zeta + W_1), & \text{(A.1)}\\ (\zeta - g\zeta^2)W_1(\zeta) &= cW_2(\zeta) + W(\zeta)W_1(\zeta) + W_1 - g(\zeta W_1 + W_{1,1}^{(2)}), & \text{(A.2)}\\ W_1(\zeta) - gW_2(\zeta) &= c\zeta W(\zeta) - c, & \text{(A.3)}\end{aligned}$$

which come respectively from the identities at large-N limit

$$\begin{aligned}0 &= \int d^{N^2}Ad^{N^2}B \sum_{\alpha=1}^{N^2} \frac{\partial}{\partial A_\alpha}\left(\mathrm{tr}\left(\frac{1}{\zeta - A}t^\alpha\right)e^{-S}\right), & \text{(A.4)}\\ 0 &= \int d^{N^2}Ad^{N^2}B \sum_{\alpha=1}^{N^2} \frac{\partial}{\partial A_\alpha}\left(\mathrm{tr}\left(\frac{1}{\zeta - A}Bt^\alpha\right)e^{-S}\right), &\\ 0 &= \int d^{N^2}Ad^{N^2}B \sum_{\alpha=1}^{N^2} \frac{\partial}{\partial B_\alpha}\left(\mathrm{tr}\left(\frac{1}{\zeta - A}t^\alpha\right)e^{-S}\right). & \text{(A.5)}\end{aligned}$$

From (A.1)–(A.3) we can delete $W_1(\zeta)$ and $W_2(\zeta)$, and have a cubic equation of $W(\zeta)$:

$$W(\zeta)^3 + a_1W(\zeta)^2 + a_2W(\zeta) + a_3 = 0, \qquad \text{(A.6)}$$

$$\begin{aligned}a_1 &= \frac{c}{g} - 2(\zeta - g\zeta^2),\\ a_2 &= (\zeta - g\zeta^2)^2 - \frac{c}{g}(\zeta - g\zeta^2) + (\frac{c^3}{g} - g)\zeta + 1 - gW_1,\\ a_3 &= (-1 + gW_1 + g\zeta)(\zeta - g\zeta^2) + (1 - 3c + cg\zeta)W_1\\ &\quad -g^2W_3 - g + \frac{c}{g}(1 - c^2) - c\zeta,\end{aligned}$$

where in order to delete $W_2, W_{1,1}^{(2)}$ we used the S-D eqs.

$$W_1 - gW_2 = cW_1, \quad W_2 - gW_3 = cW_{1,1}^{(2)} + 1$$

which are obtained by considering the coefficients of ζ^{-1}, ζ^{-2} in (A.4).

In (A.6) we need to evaluate W_1 and W_3 in order to obtain $W(\zeta)$.

A.1.1 W_1, W_3

W_1 and W_3 are evaluated by the orthogonal polynomial method [15][10]. They are given implicitly as follows:

$$W_1 = \frac{1}{64g^3}[3\rho^4 - 6c\rho^3 - 2(1-2c)\rho^2 - 2c(1-2c)^2\rho^{-1}$$

$$\begin{aligned}&\quad +32g^2 - (1 - 2c + 4c^2)(1 - 2c)], & \text{(A.7)}\\ W_3 &= \frac{1}{16 \cdot 64g^5}[-16(\rho^6 - 1) + 90c(\rho^5 - 1) &\\ &\quad + \left(80(1-2c) - \frac{531}{4}c^2\right)(\rho^4 - 1) &\\ &\quad +(-64 - 94c + 380c^2 + 60c^3)(\rho^3 - 1) &\\ &\quad +(-48 + 336c - 333c^2 - 150c^3 - 54c^4)(\rho^2 - 1) &\\ &\quad +2(1-2c)(32 - 41c - 14c^2 - 66c^3)(\rho - 1) &\\ &\quad -6c(1-2c)^2(7 - 14c - 2c^2)(\rho^{-1} - 1) &\\ &\quad +c(1-2c)^2(16 - 21c - 6c^2 - 6c^3)(\rho^{-2} - 1) &\\ &\quad -4c^3(1-2c)^3(\rho^{-3} - 1) - \frac{3}{4}c^2(1-2c)^4(\rho^{-4} - 1)], & \text{(A.8)}\end{aligned}$$

where ρ is implicitly determined by

$$\begin{aligned}g^2 &= -\frac{1}{32}[4\rho^3 - 9c\rho^2 - 4(1-2c)\rho + 2c(1 - 2c + 2c^2)\\ &\quad -c(1-2c)^2\rho^{-2}]. \qquad \text{(A.9)}\end{aligned}$$

We can not solve (A.9) w.r.t. ρ analytically and thus can not write W_1 and W_3 as a function of g explicitly. However in the continuum limit expanding g and ρ about the critical points

$$g_* = \sqrt{10c_*}, \quad \rho_* = 3c_* \quad (c_* = \frac{-1 + 2\sqrt{7}}{27})$$

eq.(A.9) can be solved iteratively:

$$\begin{aligned}\rho &= \rho_* + a^{2/3}\frac{2}{3}\rho_*(5t)^{1/3} + a^{4/3}\frac{5}{36}\rho_*(5t)^{2/3}\\ &\quad -a^2\frac{35}{288}\rho_*t - a^{8/3}\frac{8557}{311040}\rho_*(5t)^{4/3} - a^{10/3}\frac{3523}{746496}\rho_*(5t)^{5/3}\\ &\quad +a^4\frac{21205}{442368}\rho_*t^2 + O(a^{14/3}) \qquad \text{(A.10)}\end{aligned}$$

where g is expanded as $g = g_*(1 - a^2t)$.

Substituting this into eqs.(A.7),(A.8) we have W_1 and W_3 in the expanded form:

$$\begin{aligned}W_1 &= W_1^{\mathrm{non}} + \hat{W}_1, \qquad \text{(A.11)}\\ W_1^{\mathrm{non}} &= \frac{-8\rho^4 + 3(2g_*)^2}{3(2g_*)^3} + a^2\frac{-136\rho^4 + (2g_*)^2}{27(2g_*)^3}t,\\ \hat{W}_1 &= a^{8/3}\frac{8\rho_*^4}{27(2g_*)^3}(5t)^{4/3} + a^{10/3}\frac{4\rho_*^4}{81(2g_*)^3}(5t)^{5/3}\\ &\quad +a^4\frac{-8527\rho_*^4 + (2g_*)^2}{972(2g_*)^3}t^2 + O(a^{14/3}),\end{aligned}$$

$$
\begin{aligned}
W_3 &= W_3^{\text{non}} + \hat{W}_3, &\text{(A.12)}\\
W_3^{\text{non}} &= \frac{32(420-839\rho_*)\rho_*^5}{729(2g_*)^5} + a^2\frac{160(252-611\rho_*)\rho_*^5}{729(2g_*)^5}t,\\
\hat{W}_3 &= a^{8/3}\frac{320\rho_*^6}{81(2g_*)^5}(5t)^{4/3} + a^{10/3}\frac{160\rho_*^6}{243(2g_*)^5}(5t)^{5/3}\\
&\quad + a^4\frac{70(1152-3593\rho_*)\rho_*^5}{729(2g_*)^5}t^2 + O(a^{14/3}),
\end{aligned}
$$

where we denoted the non-universal pieces by $W_1^{\text{non}}, W_3^{\text{non}}$ and the universal ones which give the continuum limit by $\hat{W}_1, \hat{W}_3$.

A.1.2 Evaluation of $W(\zeta)$

Shifting $W(\zeta)$ as

$$W(\zeta) = -\frac{a_1}{3} + Y, \tag{A.13}$$

eq.(A.6) becomes

$$Y^3 - \frac{1}{3}A_2 Y - \frac{1}{27}A_1 = 0 \tag{A.14}$$

where

$$A_1 = 9a_1a_2 - 2a_1^3 - 27a_3, \quad A_2 = a_1^2 - 3a_2.$$

Then the critical point of ζ denoted by P_* is determined by

$$A_1|_* = A_2|_* = 0, \tag{A.15}$$

where $|_*$ means that g, W_1 and W_3 are set to the critical values. It turns out that eq.(A.15) gives a cubic equation of P_* and the solution is $P_* = \frac{1+3c_*}{2g_*}$ which is triple folded.

Replacing ζ to the expansion $\zeta = P_*(1+ay)$ and arranging (A.14) w.r.t. the power of a, we can see that the contribution of Y starts with $O(a^{4/3})$ and it is expanded as

$$Y = a^{4/3}Y_4 + a^{5/3}Y_5 + a^2 Y_6 + \cdots.$$

The leading order of (A.14) gives

$$Y_4^3 - \frac{rs^{8/3}}{360\cdot 2^{2/3}}T^{4/3}Y_4^2 - \frac{r^{3/2}s^4}{12960\sqrt{30}}(16y^4 - 16Ty^2 + 2T^2) = 0, \tag{A.16}$$

where r and s are the irrational numbers: $r = -1 + 2\sqrt{7}$, $s = 2 + \sqrt{7}$ and the rescaled variable $t = \frac{s^2}{20}T$ is introduced.

The solution of (A.16) is

$$Y_4 = \frac{r^{1/2}s^{4/3}}{3\sqrt{30}\cdot 2^{4/3}}[(y + \sqrt{y^2 - T})^{4/3} + (y - \sqrt{y^2 - T})^{4/3}], \tag{A.17}$$

which gives the disk amplitude in the continuum theory.

For a later convenience, we divide $W(\zeta)$ into the two pieces: the non-universal part $W^{\text{non}}(\zeta)$ and the universal part $\hat{W}(\zeta)$, which are written as

$$
\begin{aligned}
W(\zeta) &= W^{\text{non}}(\zeta) + \hat{W}(\zeta), &\text{(A.18)}\\
W^{\text{non}}(\zeta) &= -\frac{a_1}{3} = -\frac{c}{3g} + \frac{2}{3}(\zeta - g\zeta^2), &\text{(A.19)}\\
\hat{W}(\zeta) &= a^{4/3}Y_4 + O(a^{5/3}). &\text{(A.20)}
\end{aligned}
$$

A.2 $W_1(\zeta)$, $W_2(\zeta)$

The amplitude $W_1(\zeta)$ ($W_2(\zeta)$) represents the configuration that the spins on the loop all align A- except a small B- domain which consists of only one spin (two spins).

In order to obtain the universal part of $W_1(\zeta)$ in the continuum limit we have to identify the non-universal part in eq.(A.1). From (A.1),

$$\begin{aligned} W_1(\zeta) &= \frac{1}{c}[(\zeta - g\zeta^2)W^{\text{non}}(\zeta) - W^{\text{non}}(\zeta)^2 - 1 + g(\zeta + W_1^{\text{non}})] \\ &\quad + \frac{1}{c}[\zeta - g\zeta^2 - 2W^{\text{non}}(\zeta)]\hat{W}(\zeta) \\ &\quad + \frac{1}{c}[-\hat{W}(\zeta)^2 + g\hat{W}_1]. \end{aligned} \tag{A.21}$$

In the above we identify the non-universal part as follows. Firstly if there are polynomials of y or T, they are the non-universal part (the first column of (A.21)). Secondly if there are amplitudes with the simpler spin configuration than $W_1(\zeta)$ multiplicated by polynomials of y and T, they are also (the middle column of (A.21))[6]. Using this rule the universal part is

$$\begin{aligned} \hat{W}_1(\zeta) &= \frac{1}{c}[-\hat{W}(\zeta)^2 + g\hat{W}_1] \\ &= a^{8/3}\frac{s^{8/3}}{40 \cdot 2^{2/3}}[-(y + \sqrt{y^2 - T})^{8/3} - (y - \sqrt{y^2 - T})^{8/3} + T^{4/3}] + O(a^3) \\ &\equiv a^{8/3}\frac{s^{8/3}}{40 \cdot 2^{2/3}} w_1(y) + O(a^3), \end{aligned} \tag{A.22}$$

where

$$W_1(\zeta) = W_1^{\text{non}}(\zeta) + \hat{W}_1(\zeta), \tag{A.23}$$

$$\begin{aligned} W_1^{\text{non}} &= \frac{1}{c}[(\zeta - g\zeta^2)W^{\text{non}}(\zeta) - W^{\text{non}}(\zeta)^2 - 1 + g(\zeta + W_1^{\text{non}})] \\ &\quad + \frac{1}{c}[\zeta - g\zeta^2 - 2W^{\text{non}}(\zeta)]\hat{W}(\zeta). \end{aligned} \tag{A.24}$$

When we do similarly for eq.(A.2), after using

$$\hat{W}(\zeta)^2 = -c\hat{W}_1(\zeta) + g\hat{W}_1$$

we have

$$W_2(\zeta) = W_2^{\text{non}}(\zeta) + \hat{W}_2(\zeta),$$

[6] $\hat{W}_1$ corresponds to the simpler spin configuration too. However since in this case it behaves the same scaling ($O(a^{8/3})$) as the truly universal piece $\hat{W}(\zeta)^2$ accidentally, it is ambiguous whether it should be subtracted or not and how much should be subtracted even if it should be. Here we will take the definition where no subtraction is performed.

$$\begin{aligned} W_2^{\text{non}}(\zeta) &= \left(\frac{1}{3g} + \frac{1}{3c}(\zeta - g\zeta^2)\right)\left[-\frac{1}{c} + \frac{g}{c}\zeta - \frac{c}{9g^2} + \frac{1}{9g}(\zeta - g\zeta^2)\right. \\ &\quad \left. + \frac{2}{9c}(\zeta - g\zeta^2)^2 + \frac{g}{c}W_1^{\text{non}}\right] \\ &\quad - \frac{g}{c}\left(\frac{2}{3g} - \frac{1}{3c}(\zeta - g\zeta^2)\right)\hat{W}_1 - \frac{1}{c}(2 - \frac{1}{c} - g\zeta)W_1 - \frac{g^2}{c^2}W_3 - \frac{g}{c^2} \\ &\quad + \left[\frac{1}{3g^2} - \frac{1}{3c^2}(\zeta - g\zeta^2)^2 + \frac{1}{c^2}(1 - g\zeta) - \frac{g}{c^2}W_1^{\text{non}}\right]\hat{W}(\zeta) \\ &\quad + \frac{1}{g}\hat{W}_1(\zeta), \\ \hat{W}_2(\zeta) &= -\frac{1}{c}\hat{W}(\zeta)\hat{W}_1(\zeta) \\ &= a^4\frac{9r^{-1/2}s^4}{160\sqrt{30}}(16y^4 - 16Ty^2 + 2T^2) + O(a^{13/3}). \end{aligned}$$

At a glance $W_2^{\text{non}}(\zeta)$ seems to have the terms of a fractional power of T in the third and fourth columns. However after the detailed calculation we can see that $O(a^{8/3}), O(a^{10/3})$ and $O(a^{11/3})$ terms of them which more dominantly contribute than $\hat{W}_2(\zeta)$ are cancelled out.

A.3 $W^{(2)}(\zeta,\sigma)$

From the identity

$$0=\int d^{N^2}Ad^{N^2}B\sum_{\alpha=1}^{N^2}\frac{\partial}{\partial A_\alpha}\left(\mathrm{tr}\left(t^\alpha\frac{1}{\zeta-A}\frac{1}{\sigma-B}\right)e^{-S}\right),$$

$W^{(2)}(\zeta,\sigma)$ is written as

$$W^{(2)}(\zeta,\sigma)=\frac{(1-g\zeta)W(\sigma)-cW(\zeta)-gW_1(\sigma)}{\zeta-g\zeta^2-c\sigma-W(\zeta)}. \tag{A.25}$$

Putting $\zeta=P_*(1+ay),\ \ \sigma=P_*(1+ax)$ and expanding with respect to a we obtain

$$\begin{aligned}
W^{(2)}(\zeta,\sigma) &= W^{(2)\,\mathrm{non}}(\zeta,\sigma)+\hat{W}^{(2)}(\zeta,\sigma), && \text{(A.26)}\\
W^{(2)\,\mathrm{non}}(\zeta,\sigma) &= \frac{r}{27}(1-as(y+x))+\sqrt{\frac{10}{27}}r(\hat{W}(\zeta)+\hat{W}(\sigma)), && \text{(A.27)}\\
\hat{W}^{(2)}(\zeta,\sigma) &= a^{5/3}\frac{rs^{5/3}}{108\cdot 2^{2/3}}w^{(2)}(y,x)\\
&\quad +a^2\frac{30^{3/2}}{r^{1/2}s^2}\frac{\hat{W}(\zeta)^2\hat{W}(\sigma)+\hat{W}(\zeta)\hat{W}(\sigma)^2}{(y+x)^2}\\
&\quad +a^2\frac{rs^2}{27\cdot 120}\frac{1}{(y+x)^2}[160(y^4+x^4)+80(yx^3+y^3x)-40y^2x^2\\
&\quad +12(s-10)T(y^2+x^2)+24(s-5)Tyx-120T(y^2+x^2+yx)\\
&\quad +12sT(y+x)^2+15T^2]\\
&\quad +O(a^{7/3}), && \text{(A.28)}
\end{aligned}$$

where $w(y)$ is the leading part of $\hat{W}(\zeta)$ divided by the overall constant

$$w(y)=(y+\sqrt{y^2-T})^{4/3}+(y-\sqrt{y^2-T})^{4/3},$$

and

$$w^{(2)}(y,x)=\frac{-w(y)^2-w(y)w(x)-w(x)^2+3T^{4/3}}{y+x}.$$

Here we calculate $\hat{W}^{(2)}(\zeta,\sigma)$ up to the next leading order since it will be necessary to have $W^{(4)}(\zeta_1,\sigma_1,\zeta_2,\sigma_2)$ after.

It is noted that eq.(A.25) does not seem to possess the symmetry of $\zeta\leftrightarrow\sigma$, which manifestly appears after the expansion w.r.t. a.

A.4 $W_1(\zeta_1;\zeta_2,\sigma_2)$

Due to the S-D eq. obtained from

$$0=\int d^{N^2}Ad^{N^2}B\sum_{\alpha=1}^{N^2}\frac{\partial}{\partial A_\alpha}\left(\mathrm{tr}\left(\frac{1}{\zeta_1-A}t^\alpha\frac{1}{\zeta_2-A}\frac{1}{\sigma_2-B}\right)e^{-S}\right),$$

we have

$$\begin{aligned}
W_1(\zeta_1;\zeta_2,\sigma_2) &= -\frac{1}{c}[D_\zeta(\zeta_1,\zeta_2)(\zeta-g\zeta^2-W(\zeta_1)-W(\zeta_2))W^{(2)}(\zeta,\sigma_2)\\
&\quad +gW(\sigma_2)]. && \text{(A.29)}
\end{aligned}$$

After the similar calculation as before, we find

$$W_1(\zeta_1;\zeta_2,\sigma_2)=W_1^{\mathrm{non}}(\zeta_1;\zeta_2,\sigma_2)+\hat{W}_1(\zeta_1;\zeta_2,\sigma_2), \tag{A.30}$$

$$\begin{aligned}
W^{\mathrm{non}}(\zeta_1;\zeta_2,\sigma_2) &= -D_\zeta(\zeta_1,\zeta_2)(\zeta-g\zeta^2)(1-\frac{s}{P_*}(\zeta+\sigma_2-2P_*))\\
&\quad +\frac{s}{P_*}\left(\frac{2c}{3g}-\frac{2}{3}(\zeta_1-g\zeta_1^2)-\frac{2}{3}(\zeta_2-g\zeta_2^2)\right)+g\left(\frac{1}{3g}-\frac{2}{3c}(\sigma_2-g\sigma_2^2)^2\right)\\
&\quad -\frac{s}{P_*}(\hat{W}(\zeta_1)+\hat{W}(\zeta_2))-\frac{g}{c}\hat{W}(\sigma_2)\\
&\quad -\frac{1}{c}\sqrt{\frac{10}{27}}r\left(\frac{2c}{3g}-\frac{2}{3}(\zeta_1-g\zeta_1^2)-\frac{2}{3}(\zeta_2-g\zeta_2^2)\right)D_\zeta(\zeta_1,\zeta_2)\hat{W}(\zeta)\\
&\quad -\frac{1}{c}\sqrt{\frac{10}{27}}rD_\zeta(\zeta_1,\zeta_2)(((\zeta-g\zeta^2)(\hat{W}(\zeta)+\hat{W}(\sigma_2)))\\
&\quad -\sqrt{\frac{10}{27}}rD_\zeta(\zeta_1,\zeta_2)\hat{W}_1(\zeta)-\frac{1}{c}D_\zeta(\zeta_1,\zeta_2)(((\zeta-g\zeta^2)\hat{W}^{(2)}(\zeta,\sigma_2))\\
&\quad -\frac{1}{c}\left(\frac{2c}{3g}-\frac{2}{3}(\zeta_1-g\zeta_1^2)-\frac{2}{3}(\zeta_2-g\zeta_2^2)\right)D_\zeta(\zeta_1,\zeta_2)\hat{W}^{(2)}(\zeta,\sigma_2), && \text{(A.31)}
\end{aligned}$$

$$\begin{aligned}
\hat{W}_1(\zeta_1;\zeta_2,\sigma_2) &= \frac{1}{c}(\hat{W}(\zeta_1)+\hat{W}(\zeta_2))D_\zeta(\zeta_1,\zeta_2)\hat{W}^{(2)}(\zeta,\sigma_2)\\
&= a^2\frac{rs^2}{27\cdot 16}(w(y_1)+w(y_2))D_y(y_1,y_2)w^{(2)}(y,x_2)\\
&\quad +O(a^{7/3}). && \text{(A.32)}
\end{aligned}$$

A.5 $W^{(4)}(\zeta_1, \sigma_1, \zeta_2, \sigma_2)$

By combining the two S-D eqs. which come from

$$
\begin{aligned}
0 &= \int d^{N^2}A d^{N^2}B \sum_{\alpha=1}^{N^2} \frac{\partial}{\partial A_\alpha}\left(\operatorname{tr}\left(\frac{1}{\zeta_1 - A}\frac{1}{\sigma_1 - B}\frac{1}{\zeta_2 - A}t^\alpha \frac{1}{\sigma_2 - B}\right)e^{-S}\right), \\
0 &= \int d^{N^2}A d^{N^2}B \sum_{\alpha=1}^{N^2} \frac{\partial}{\partial B_\alpha}\left(\operatorname{tr}\left(\frac{1}{\zeta_1 - A}\frac{1}{\sigma_1 - B}t^\alpha \frac{1}{\sigma_2 - B}\right)e^{-S}\right),
\end{aligned}
$$

the formula for $W^{(4)}(\zeta_1, \sigma_1, \zeta_2, \sigma_2)$ is obtained as

$$
\begin{aligned}
W^{(4)}(\zeta_1, \sigma_1, \zeta_2, \sigma_2) = &\frac{1}{\zeta_2 - g\zeta_2^2 - c\sigma_2 - W(\zeta_2)} \\
&\times\{(c - W^{(2)}(\zeta_1, \sigma_2))D_\zeta(\zeta_1, \zeta_2)W^{(2)}(\zeta, \sigma_1) \\
&+ D_\sigma(\sigma_1, \sigma_2)[\frac{g}{c}(\sigma - g\sigma^2 - W(\sigma_1) - W(\sigma_2)) - 1 + g\zeta_2]W^{(2)}(\zeta_1, \sigma) \\
&+ \frac{g^2}{c}W(\zeta_1)\}.
\end{aligned} \tag{A.33}
$$

Putting $\zeta_i = P_*(1 + ay_i)$, $\sigma_i = P_*(1 + ax_i)$ we expand the r.h.s. of the above and arrange the terms using (A.28). After a straightforward but quite long calculation we arrive at the result:

$$
W^{(4)}(\zeta_1, \sigma_1, \zeta_2, \sigma_2) = W^{(4)\ \mathrm{non}}(\zeta_1, \sigma_1, \zeta_2, \sigma_2) + \hat{W}^{(4)}(\zeta_1, \sigma_1, \zeta_2, \sigma_2), \tag{A.34}
$$

$$
\begin{aligned}
W^{(4)\ \mathrm{non}}(\zeta_1, \sigma_1, \zeta_2, \sigma_2) &= 10\left(\frac{r}{27}\right)^2 - \frac{10}{27}r[D_\zeta(\zeta_1, \zeta_2)\hat{W}(\zeta) + D_\sigma(\sigma_1, \sigma_2)\hat{W}(\sigma)] \\
&- \sqrt{\frac{10}{27}}r[D_\zeta(\zeta_1, \zeta_2)(\hat{W}^{(2)}(\zeta, \sigma_1) + \hat{W}^{(2)}(\zeta, \sigma_2)) \\
&+ D_\sigma(\sigma_1, \sigma_2)(\hat{W}^{(2)}(\zeta_1, \sigma) + \hat{W}^{(2)}(\zeta_2, \sigma))],
\end{aligned} \tag{A.35}
$$

$$
\begin{aligned}
\hat{W}^{(4)}&(\zeta_1, \sigma_1, \zeta_2, \sigma_2) = a\frac{5r^2 s}{27^2 \cdot 8}w^{(4)}(y_1, x_1, y_2, y_2) + O(a^{4/3}) \\
&= a\frac{5r^2 s}{27^2 \cdot 8}\frac{1}{(y_1 - y_2)(x_1 - x_2)}\Big[-\frac{8}{3}(y_1 - y_2)(x_1 - x_2)(y_1 + y_2 + x_1 + x_2) \\
&- \frac{1}{2}(w(y_1) + w(x_1) + 2w(y_2) + 2w(x_2))w^{(2)}(y_1, x_1) \\
&+ \frac{1}{2}(w(y_1) + w(x_2) + 2w(y_2) + 2w(x_1))w^{(2)}(y_1, x_2) \\
&+ \frac{1}{2}(w(y_2) + w(x_1) + 2w(y_1) + 2w(x_2))w^{(2)}(y_2, x_1) \\
&- \frac{1}{2}(w(y_2) + w(x_2) + 2w(y_1) + 2w(x_1))w^{(2)}(y_2, x_2)\Big] \\
&+ O(a^{4/3}).
\end{aligned} \tag{A.36}
$$

Here we note that the above results have the symmetry under the cyclic permutation of variables

$$
\zeta_1 \to \sigma_2, \quad \sigma_2 \to \zeta_2, \quad \zeta_2 \to \sigma_1, \quad \sigma_1 \to \zeta_1,
$$

which are expected by the definition of the amplitude but we can not see in eq.(A.33).

From the results until now, for the amplitude $W^{(2k)}(\zeta_1, \sigma_1, \cdots, \zeta_k, \sigma_k)$ the following scaling is expected:

$$
\hat{W}^{(2k)}(\zeta_1, \sigma_1, \cdots, \zeta_k, \sigma_k) = a^{\frac{7}{3} - \frac{2}{3}k}w^{(2k)}(y_1, x_1, \cdots, y_k, x_k). \tag{A.37}
$$

Appendix

B The Operation Making the Infinitesimal Domain

In the lattice theory the reduction of a domain to one-lattice unit is given by the formula

$$\frac{1}{N}\mathrm{tr}\left(\frac{1}{\zeta - A}B\cdots\right) = \oint\frac{d\sigma}{2\pi i}\sigma\frac{1}{N}\mathrm{tr}\left(\frac{1}{\zeta - A}\frac{1}{\sigma - B}\cdots\right). \tag{B.1}$$

Here we want to construct the continuum version of this in the disk amplitudes. However when naively considering the continuum limit of (B.1) it would be very difficult to evaluate the above integral. Rather we will find the relation between only the universal pieces by comparing their forms.

Firstly let us consider $\hat{W}^{(2)}(\zeta,\sigma)$ and $\hat{W}_1(\zeta)$. It is useful to observe the following formulae:

$$\int_{C_1}\frac{dx}{2\pi i}\frac{x^\alpha}{x+y} = 0 \quad (\alpha \notin Z), \tag{B.2}$$

$$\int_{C_1}\frac{dx}{2\pi i}\frac{1}{x+y} = 1, \tag{B.3}$$

where the contour C_1 is depicted as Fig.B-1 and we used the Beta-function regularization

$$\int_0^\infty dr\frac{r^\alpha}{1+r} = B(\alpha+1,-\alpha) = -\frac{\pi}{\sin\pi\alpha}.$$

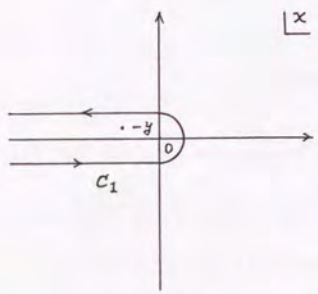


Fig.B-1

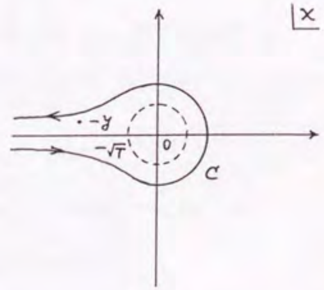


Fig.B-2

Then we consider the integral along the contour C in Fig.B-2

$$\int_C\frac{dx}{2\pi i}w^{(2)}(y,x) = \int_C\frac{dx}{2\pi i}\frac{-w(y)^2 - w(y)w(x) - w(x)^2 + 3T^{4/3}}{y+x}.$$

We note that when expanded in the case of $|x| > \sqrt{T}$, the numerator of the integrand contains only the terms with the fractional powers of x except $-w(y)^2 + T^{4/3}$. Thus using (B.2) and (B.3) we have

$$\int_C\frac{dx}{2\pi i}w^{(2)}(y,x) = -w(y)^2 + T^{4/3} = w_1(y) - 2T^{4/3}. \tag{B.4}$$

Including the overall factors it is written as

$$s^{-1}\oint\frac{d\sigma}{2\pi i}\sigma\hat{W}^{(2)}(\zeta,\sigma) = \hat{W}_1(\zeta) - a^{8/3}\frac{s^{8/3}}{40\cdot 2^{2/3}}2T^{4/3} + O(a^3) \tag{B.5}$$

where the integral symbol $\oint\frac{d\sigma}{2\pi i}\sigma$ is used in the sense of

$$\oint\frac{d\sigma}{2\pi i}\sigma = P_* a^2\int_C\frac{dx}{2\pi i}.$$

Or taking the ζ-derivative,

$$s^{-1}\oint\frac{d\sigma}{2\pi i}\sigma\partial_\zeta\hat{W}^{(2)}(\zeta,\sigma) = \partial_\zeta\hat{W}_1(\zeta) + O(a^2). \tag{B.6}$$

Secondly for $\hat{W}^{(4)}(\zeta_1,\sigma_1,\zeta_2,\sigma_2)$ and $\hat{W}_1(\zeta_1;\zeta_2,\sigma_2)$, we start with the formulae:

$$\int_{C_2}\frac{dx_1}{2\pi i}\frac{1}{x_1-x_2}\frac{x_1^\alpha}{x_1+y_1} = -\frac{x_2^\alpha}{y_1+x_2} \quad (\alpha\notin Z), \tag{B.7}$$

$$\int_{C_2}\frac{dx_1}{2\pi i}\frac{1}{x_1-x_2}\frac{x_1^n}{x_1+y_1} = -\frac{y_1^n}{y_1+x_2} \quad (n = 0,1,2,\cdots), \tag{B.8}$$

$$\int_{C_3}\frac{dx_1}{2\pi i}\frac{x_1^\alpha}{x_1-x_2} = -x_2^\alpha \quad (\alpha\notin Z), \tag{B.9}$$

where the contours C_2, C_3 are depicted as Fig.B-3,B-4.

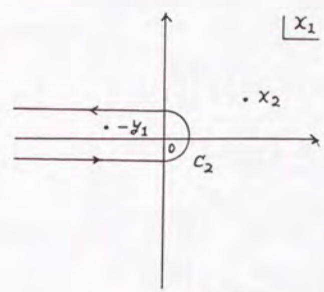


Fig.B-3

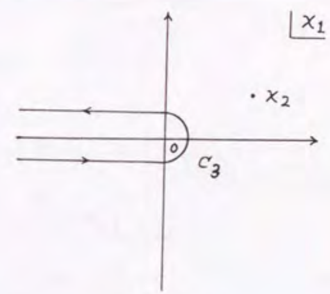


Fig.B-4

Using (B.7)–(B.9) as well as (B.2) we can calculate the integral

$$\int_C\frac{dx_1}{2\pi i}w^{(4)}(y_1,x_1,y_2,x_2)$$

as before. We take the variables y_1, y_2, x_2 in the right half plane and the contour C around the negative real axis and the poles in the left half plane. The result is as follows

$$\int_C \frac{dx_1}{2\pi i} w^{(4)}(y_1, x_1, y_2, x_2) = (w(y_1) + w(y_2)) D_y(y_1, y_2) w^{(2)}(y, x_2), \tag{B.10}$$

thus we can see that

$$s^{-1} \oint \frac{d\sigma_1}{2\pi i} \sigma_1 \hat{W}(\zeta_1, \sigma_1, \zeta_2, \sigma_2) = \hat{W}_1(\zeta_1; \zeta_2, \sigma_2) + O(a^{7/3}) \tag{B.11}$$

holds.

Reminding us of the ambiguity of a constant shift in the definition of $\hat{W}_1(\zeta)$ in Appendix A.2, now we can consider that eqs.(B.6), (B.11) have a universal meaning and that the constant shift of eq. (B.5) is non-universal.

Appendix

C Continuum Limit of Splitting Term for the several components

Here we will present the calculations of eq.(4.28):

$$\left(\left(\mathcal{M}\frac{\delta}{\delta\hat{J}}\right) \vee \left(\mathcal{M}\frac{\delta}{\delta\hat{J}}\right)\right)_I = \left(\mathcal{M}\left(\frac{\delta}{\delta\hat{J}} \vee \frac{\delta}{\delta\hat{J}}\right)\right)_I \tag{C.1}$$

for the first several components $I = A, B, 1, 2$.

Firstly for $I = A, B$, it is trivial from

$$\begin{aligned}
\left(\mathcal{M}\frac{\delta}{\delta\hat{J}}\right)_A &= \frac{\delta}{\delta\hat{J}_A(\zeta)}, \\
\left(\mathcal{M}\frac{\delta}{\delta\hat{J}}\right)_B &= \frac{\delta}{\delta\hat{J}_B(\sigma)}, \\
\left(\frac{\delta}{\delta\hat{J}} \vee \frac{\delta}{\delta\hat{J}}\right)_A &= -\partial_\zeta \left(\frac{\delta^2}{\delta\hat{J}_A(\zeta)^2}\right), \qquad (C.2) \\
\left(\frac{\delta}{\delta\hat{J}} \vee \frac{\delta}{\delta\hat{J}}\right)_B &= -\partial_\sigma \left(\frac{\delta^2}{\delta\hat{J}_B(\sigma)^2}\right). \qquad (C.3)
\end{aligned}$$

Secondly for $I = 1$, using

$$\begin{aligned}
\left(\mathcal{M}\frac{\delta}{\delta\hat{J}}\right)_1 &= \sqrt{\frac{10}{27}} r \left(\frac{\delta}{\delta\hat{J}_A(\zeta)} + \frac{\delta}{\delta\hat{J}_B(\sigma)}\right) + \frac{\delta}{\delta\hat{J}_1(\zeta, \sigma)}, \qquad (C.4) \\
\left(\frac{\delta}{\delta J} \vee \frac{\delta}{\delta J}\right)_1 &= -2\left(\frac{\delta}{\delta J_A(\zeta)}\partial_\zeta \frac{\delta}{\delta J_1(\zeta, \sigma)} + \frac{\delta}{\delta J_B(\sigma)}\partial_\sigma \frac{\delta}{\delta J_1(\zeta, \sigma)}\right) \qquad (C.5)
\end{aligned}$$

we have

$$\begin{aligned}
&\left(\left(\mathcal{M}\frac{\delta}{\delta\hat{J}}\right) \vee \left(\mathcal{M}\frac{\delta}{\delta\hat{J}}\right)\right)_1 \\
&= -2\left(\left(\mathcal{M}\frac{\delta}{\delta\hat{J}}\right)_A \partial_\zeta \left(\mathcal{M}\frac{\delta}{\delta\hat{J}}\right)_1 + \left(\mathcal{M}\frac{\delta}{\delta\hat{J}}\right)_B \partial_\sigma \left(\mathcal{M}\frac{\delta}{\delta\hat{J}}\right)_1\right) \\
&= -2\sqrt{\frac{10}{27}} r \left(\frac{dl}{\delta\hat{J}_A(\zeta)}\partial_\zeta \frac{\delta}{\delta\hat{J}_A(\zeta)} + \frac{\delta}{\delta\hat{J}_B(\sigma)}\partial_\sigma \frac{\delta}{\delta\hat{J}_B(\sigma)}\right) \\
&\quad -2\left(\frac{\delta}{\delta\ Jh_A(\zeta)}\partial_\zeta \frac{\delta}{\delta\hat{J}_1(\zeta, \sigma)} + \frac{\delta}{\delta\hat{J}_B(\sigma)}\partial_\sigma \frac{\delta}{\delta\hat{J}_1(\zeta, \sigma)}\right). \qquad (C.6)
\end{aligned}$$

On the other hand,

$$\left(\mathcal{M}\left(\frac{\delta}{\delta\hat{J}} \vee \frac{\delta}{\delta\hat{J}}\right)\right)_1 = \sqrt{\frac{10}{27}} r \left(\left(\frac{\delta}{\delta\hat{J}} \vee \frac{\delta}{\delta\hat{J}}\right)_A + \left(\frac{\delta}{\delta\hat{J}} \vee \frac{\delta}{\delta\hat{J}}\right)_B\right) + \left(\frac{\delta}{\delta\hat{J}} \vee \frac{\delta}{\delta\hat{J}}\right)_1,$$

which is nothing but the r.h.s. of (C.6). We note that in (C.6) the terms where the splitting domain coincides to the collapsed domain: $\frac{\delta}{\delta \hat{J}_A(\zeta)}\partial_\zeta \frac{\delta}{\delta \hat{J}_B(\sigma)}$, $\frac{\delta}{\delta \hat{J}_B(\sigma)}\partial_\sigma \frac{\delta}{\delta \hat{J}_A(\zeta)}$ vanish owing to the derivatives ∂_ζ, ∂_σ.

Next for $I = 2$, if we notice the following identities

$$\begin{aligned}
&\sum_{j=1}^{2} \frac{\delta}{\hat{J}_A(\zeta_j)}\partial_{\zeta_j} D_\zeta(\zeta_1,\zeta_2)\frac{\delta}{\delta \hat{J}_1(\zeta,\sigma_1)} \\
&= D_\zeta(\zeta_1,\zeta_2)\frac{\delta}{\hat{J}_A(\zeta)}\partial_\zeta \frac{\delta}{\delta \hat{J}_1(\zeta,\sigma_1)} - D_\zeta(\zeta_1,\zeta_2)\frac{\delta}{\delta J_A(\zeta)} D_\zeta(\zeta_1,\zeta_2)\frac{\delta}{\delta J_1(\zeta,\sigma_1)},
\end{aligned}$$

$$\sum_{j=1}^{2} \frac{\delta}{\hat{J}_A(\zeta_j)}\partial_{\zeta_j} D_\zeta(\zeta_1,\zeta_2)\frac{\delta}{\delta \hat{J}_A(\zeta)} + \left(D_\zeta(\zeta_1,\zeta_2)\frac{\delta}{\delta \hat{J}_A(\zeta)}\right)^2 = D_\zeta(\zeta_1,\zeta_2)\frac{\delta}{\delta \hat{J}_A(\zeta)}\partial_\zeta \frac{\delta}{\delta \hat{J}_A(\zeta)}, \tag{C.7}$$

then from

$$\begin{aligned}
\left(\mathcal{M}\frac{\delta}{\delta \hat{J}}\right)_2 &= \frac{10}{27}r\left[-D_\zeta(\zeta_1,\zeta_2)\frac{\delta}{\delta \hat{J}_A(\zeta)} - D_\sigma(\sigma_1,\sigma_2)\frac{\delta}{\delta \hat{J}_B(\sigma)}\right] \\
&\quad +\sqrt{\frac{10}{27}}r\left[-D_\zeta(\zeta_1,\zeta_2)\left(\frac{\delta}{\delta \hat{J}_1(\zeta,\sigma_1)} + \frac{\delta}{\delta \hat{J}_1(\zeta,\sigma_2)}\right)\right. \\
&\quad \left. -D_\sigma(\sigma_1,\sigma_2)\left(\frac{\delta}{\delta \hat{J}_1(\zeta_1,\sigma)} + \frac{\delta}{\delta \hat{J}_1(\zeta_2,\sigma)}\right)\right] \\
&\quad +\frac{\delta}{\delta \hat{J}_2(\zeta_1,\sigma_1,\zeta_2,\sigma_2)}
\end{aligned}$$

and

$$\begin{aligned}
\left(\frac{\delta}{\delta J}\vee\frac{\delta}{\delta J}\right)_2 &= -2\sum_{j=1}^{2}\left(\frac{\delta}{\delta J_A(\zeta_j)}\partial_{\zeta_j} + \frac{\delta}{\delta J_B(\sigma_j)}\partial_{\sigma_j}\right)\frac{\delta}{\delta J_2(\zeta_1,\sigma_1,\zeta_2,\sigma_2)} \\
&\quad +2D_\zeta(\zeta_1,\zeta_2)\frac{\delta}{\delta J_1(\zeta,\sigma_1)}D_\zeta(\zeta_1,\zeta_2)\frac{\delta}{\delta J_1(\zeta,\sigma_2)} \\
&\quad +2D_\sigma(\sigma_1,\sigma_2)\frac{\delta}{\delta J_1(\zeta_1,\sigma)}D_\sigma(\sigma_1,\sigma_2)\frac{\delta}{\delta J_1(\zeta_2,\sigma)},
\end{aligned}$$

we can show that (C.6) holds for $I = 2$.

Appendix

D Continuum Limit of Merging Term for the several components

Here we will give the calculation needed to show that eq.(4.31)

$$((\hat{J}\mathcal{M}^{-1})\wedge(\hat{J}\mathcal{M}^{-1}))\cdot\left(\mathcal{M}\frac{\delta}{\delta \hat{J}}\right) = (\hat{J}\wedge\hat{J})\cdot\frac{\delta}{\delta \hat{J}} \tag{D.1}$$

holds for the first several components.

Using the definitions of the $\wedge$-operation and the transformed source $\hat{J}$: (4.6) and (4.17)–(4.20), we can write the first several components of the r.h.s. of (D.1) as

$$\begin{aligned}
&\oint \frac{d\zeta}{2\pi i}((\hat{J}\mathcal{M}^{-1})\wedge(\hat{J}\mathcal{M}^{-1}))_A(\zeta)\left(\mathcal{M}\frac{\delta}{\delta \hat{J}}\right)_A(\zeta) \\
&= \oint \frac{d\zeta}{2\pi i}(\hat{J}\wedge\hat{J})_A(\zeta)\frac{\delta}{\delta \hat{J}_A(\zeta)} + \mathcal{R}_A,
\end{aligned}$$

$$\begin{aligned}
&\oint \frac{d\sigma}{2\pi i}((\hat{J}\mathcal{M}^{-1})\wedge(\hat{J}\mathcal{M}^{-1}))_B(\zeta)\left(\mathcal{M}\frac{\delta}{\delta \hat{J}}\right)_B(\sigma) \\
&= \oint \frac{d\sigma}{2\pi i}(\hat{J}\wedge\hat{J})_B(\sigma)\frac{\delta}{\delta \hat{J}_B(\sigma)} + \mathcal{R}_B,
\end{aligned}$$

$$\begin{aligned}
&\oint \frac{d\zeta}{2\pi i}\frac{d\sigma}{2\pi i}((\hat{J}\mathcal{M}^{-1})\wedge(\hat{J}\mathcal{M}^{-1}))_1(\zeta,\sigma)\left(\mathcal{M}\frac{\delta}{\delta \hat{J}}\right)_1(\zeta,\sigma) \\
&= \oint \frac{d\zeta}{2\pi i}\frac{d\sigma}{2\pi i}(\hat{J}\wedge\hat{J})_1(\zeta,\sigma)\frac{\delta}{\delta \hat{J}_1(\zeta,\sigma)} + \mathcal{R}_1,
\end{aligned}$$

$$\begin{aligned}
&\oint \prod_{i=1}^{2}\frac{d\zeta_i}{2\pi i}\frac{d\sigma_i}{2\pi i}((\hat{J}\mathcal{M}^{-1})\wedge(\hat{J}\mathcal{M}^{-1}))_2(\zeta_1,\sigma_1,\zeta_2,\sigma_2)\left(\mathcal{M}\frac{\delta}{\delta \hat{J}}\right)_2(\zeta_1,\sigma_1,\zeta_2,\sigma_2) \\
&= \oint \prod_{i=1}^{2}\frac{d\zeta_i}{2\pi i}\frac{d\sigma_i}{2\pi i}(\hat{J}\wedge\hat{J})_2(\zeta_1,\sigma_1,\zeta_2,\sigma_2)\frac{\delta}{\delta \hat{J}_2(\zeta_1,\sigma_1,\zeta_2,\sigma_2)} + \mathcal{R}_2.
\end{aligned}$$

Here the sum of $\mathcal{R}_I$'s should vanish for the validity of (D.1). The form of $\mathcal{R}_I$'s is as follows:

$$\begin{aligned}
\mathcal{R}_A &= -2\sqrt{\frac{10}{27}}r\oint \frac{d\zeta}{2\pi i}\frac{d\sigma}{2\pi i}\partial_\zeta \hat{J}_A(\zeta)\partial_\zeta \hat{J}_1(\zeta,\sigma)\frac{\delta}{\delta \hat{J}_A(\zeta)} \\
&\quad +\frac{10}{27}r\oint \frac{d\zeta}{2\pi i}\prod_{i=1}^{2}\frac{d\sigma_i}{2\pi i}\partial_\zeta \hat{J}_1(\zeta,\sigma_1)\partial_\zeta \hat{J}_1(\zeta,\sigma_2)\frac{\delta}{\delta \hat{J}_A(\zeta)}
\end{aligned}$$

$$
\begin{aligned}
&+2\frac{10}{27}r\oint\prod_{i=1}^{2}\frac{d\zeta_i}{2\pi i}\frac{d\sigma_i}{2\pi i}\left[\partial_{\zeta_2}\hat{J}_A(\zeta_2)\partial_{\zeta_2}\left(\frac{1}{\zeta_1-\zeta_2}\hat{j}_2(\zeta_1,\sigma_1,\zeta_2,\sigma_2)\right)\frac{\delta}{\delta\hat{J}_A(\zeta_2)}\right.\\
&\left.-\partial_{\zeta_1}\hat{J}_A(\zeta_1)\partial_{\zeta_1}\left(\frac{1}{\zeta_1-\zeta_2}\hat{j}_2(\zeta_1,\sigma_1,\zeta_2,\sigma_2)\right)\frac{\delta}{\delta\hat{J}_A(\zeta_1)}\right]\\
&+\cdots,
\end{aligned}
\tag{D.2}
$$

$$
\begin{aligned}
\mathcal{R}_B \;=\;& -2\sqrt{\frac{10}{27}}r\oint\frac{d\zeta}{2\pi i}\frac{d\sigma}{2\pi i}\partial_\sigma\hat{J}_B(\sigma)\partial_\sigma\hat{J}_1(\zeta,\sigma)\frac{\delta}{\delta\hat{J}_B(\sigma)}\\
&+\frac{10}{27}r\oint\frac{d\sigma}{2\pi i}\prod_{i=1}^{2}\frac{d\zeta_i}{2\pi i}\partial_\sigma\hat{J}_1(\zeta_1,\sigma)\partial_\sigma\hat{J}_1(\zeta_1,\sigma)\frac{\delta}{\delta\hat{J}_B(\sigma)}\\
&+2\frac{10}{27}r\oint\prod_{i=1}^{2}\frac{d\zeta_i}{2\pi i}\frac{d\sigma_i}{2\pi i}\left[\partial_{\sigma_2}\hat{J}_B(\sigma_2)\partial_{\sigma_2}\left(\frac{1}{\sigma_1-\sigma_2}\hat{j}_2(\zeta_1,\sigma_1,\zeta_2,\sigma_2)\right)\frac{\delta}{\delta\hat{J}_B(\sigma_2)}\right.\\
&\left.-\partial_{\sigma_1}\hat{J}_B(\sigma_1)\partial_{\sigma_1}\left(\frac{1}{\sigma_1-\sigma_2}\hat{j}_2(\zeta_1,\sigma_1,\zeta_2,\sigma_2)\right)\frac{\delta}{\delta\hat{J}_B(\sigma_1)}\right]\\
&+\cdots,
\end{aligned}
\tag{D.3}
$$

$$
\begin{aligned}
\mathcal{R}_1 =\;& 2\sqrt{\frac{10}{27}}r\oint\frac{d\zeta}{2\pi i}\frac{d\sigma}{2\pi i}(\partial_\zeta\hat{J}_A(\zeta)\partial_\zeta\hat{J}_1(\zeta,\sigma)+\partial_\sigma\hat{J}_B(\sigma)\partial_\sigma\hat{J}_1(\zeta,\sigma))\\
&\times\left(\frac{\delta}{\hat{J}_A(\zeta)}+\frac{\delta}{\hat{J}_B(\sigma)}\right)\\
&-2\frac{10}{27}r\oint\frac{d\zeta}{2\pi i}\prod_{i=1}^{2}\frac{d\sigma_i}{2\pi i}\partial_\zeta\hat{J}_1(\zeta,\sigma_1)\partial_\zeta\hat{J}_1(\zeta,\sigma_2)\left(\frac{\delta}{\hat{J}_A(\zeta)}+\frac{\delta}{\hat{J}_B(\sigma_2)}\right)\\
&-2\frac{10}{27}r\oint\frac{d\sigma}{2\pi i}\prod_{i=1}^{2}\frac{d\zeta_i}{2\pi i}\partial_\sigma\hat{J}_1(\zeta_1,\sigma)\partial_\sigma\hat{J}_1(\zeta_2,\sigma)\left(\frac{\delta}{\hat{J}_A(\zeta_2)}+\frac{\delta}{\hat{J}_B(\sigma)}\right)\\
&-2\sqrt{\frac{10}{27}}r\oint\frac{d\zeta}{2\pi i}\prod_{i=1}^{2}\frac{d\sigma_i}{2\pi i}\partial_\zeta\hat{J}_1(\zeta,\sigma_1)\partial_\zeta\hat{J}_1(\zeta,\sigma_2)\frac{\delta}{\delta\hat{J}_1(\zeta,\sigma_2)}\\
&-2\sqrt{\frac{10}{27}}r\oint\frac{d\sigma}{2\pi i}\prod_{i=1}^{2}\frac{d\zeta_i}{2\pi i}\partial_\sigma\hat{J}_1(\zeta_1,\sigma)\partial_\sigma\hat{J}_1(\zeta_2,\sigma)\frac{\delta}{\delta\hat{J}_1(\zeta_2,\sigma)}\\
&+\left\{2\frac{10}{27}r\oint\prod_{i=1}^{2}\frac{d\zeta_i}{2\pi i}\frac{d\sigma_i}{2\pi i}\partial_{\zeta_1}\hat{J}_A(\zeta_1)\partial_{\zeta_1}\left(\left(\frac{1}{\zeta_1-\zeta_2}+\frac{1}{\sigma_1-\sigma_2}\right)\hat{J}_2(\zeta_1,\sigma_1,\zeta_2,\sigma_2)\right)\right.\\
&\times\left(\frac{\delta}{\hat{J}_A(\zeta_1)}+\frac{\delta}{\hat{J}_B(\sigma_1)}\right)\\
&+2\frac{10}{27}r\oint\prod_{i=1}^{2}\frac{d\zeta_i}{2\pi i}\frac{d\sigma_i}{2\pi i}\partial_{\sigma_1}\hat{J}_B(\sigma_1)\partial_{\sigma_1}\left(\left(\frac{1}{\zeta_1-\zeta_2}+\frac{1}{\sigma_1-\sigma_2}\right)\hat{J}_2(\zeta_1,\sigma_1,\zeta_2,\sigma_2)\right)
\end{aligned}
$$

$$
\begin{aligned}
&\times\left(\frac{\delta}{\hat{J}_A(\zeta_1)}+\frac{\delta}{\hat{J}_B(\sigma_1)}\right)\\
&+\left.\begin{array}{l}\text{(six terms by a replacement } \zeta_1\leftrightarrow\zeta_2 \text{ or } \sigma_1\leftrightarrow\sigma_2\\ \text{without changing } \hat{J}_2)\end{array}\right\}\\
&+\left\{\frac{10}{27}r\left(\frac{\delta}{\hat{J}_A(\zeta)}+\frac{\delta}{\hat{J}_B(\sigma)}\right)\leftrightarrow\sqrt{\frac{10}{27}}r\frac{\delta}{\delta\hat{J}_1(\zeta,\sigma)}\text{in the above brace}\right\}\\
+&\;\cdots,
\end{aligned}
\tag{D.4}
$$

$$
\begin{aligned}
\mathcal{R}_2 =\;& 2\frac{10}{27}r\oint\prod_{i=1}^{2}\frac{d\zeta_i}{2\pi i}\frac{d\sigma_i}{2\pi i}\sum_{j=1}^{2}[\partial_{\zeta_j}\hat{J}_A(\zeta_j)\partial_{\zeta_j}+\partial_{\sigma_j}\hat{J}_B(\sigma_j)\partial_{\sigma_j}]\hat{J}_2(\zeta_1,\sigma_1,\zeta_2,\sigma_2)\\
&\times\left(-D_\zeta(\zeta_1,\zeta_2)\frac{\delta}{\delta\hat{J}_A(\zeta)}-D_\sigma(\sigma_1,\sigma_2)\frac{\delta}{\delta\hat{J}_B(\sigma)}\right)\\
&+\frac{10}{27}r\oint\prod_{i=1}^{2}\frac{d\zeta_i}{2\pi i}\frac{d\sigma_i}{2\pi i}\hat{J}_1(\zeta_1,\sigma_1)\hat{J}_1(\zeta_2,\sigma_2)\\
&\times\left[-D_{\zeta_1}(z_1,z_2)D_{\zeta_2}(z_1,z_2)D_\zeta(\zeta_1,\zeta_2)\frac{\delta}{\delta\hat{J}_A(\zeta)}\right.\\
&\left.-D_{\sigma_1}(s_1,s_2)D_{\sigma_2}(s_1,s_2)D_\sigma(\sigma_1,\sigma_2)\frac{\delta}{\delta\hat{J}_B(\sigma)}\right]_{\substack{z_i=\zeta_i\\ s_i=\sigma_i}}\\
&+2\sqrt{\frac{10}{27}}r\oint\prod_{i=1}^{2}\frac{d\zeta_i}{2\pi i}\frac{d\sigma_i}{2\pi i}\sum_{j=1}^{2}[\partial_{\zeta_j}\hat{J}_A(\zeta_j)\partial_{\zeta_j}+\partial_{\sigma_j}\hat{J}_B(\sigma_j)\partial_{\sigma_j}]\hat{J}_2(\zeta_1,\sigma_1,\zeta_2,\sigma_2)\\
&\times\left[-D_\zeta(\zeta_1,\zeta_2)\left(\frac{\delta}{\delta\hat{J}_1(\zeta,\sigma_1)}+\frac{\delta}{\delta\hat{J}_1(\zeta,\sigma_2)}\right)\right.\\
&\left.-D_\sigma(\sigma_1,\sigma_2)\left(\frac{\delta}{\delta\hat{J}_1(\zeta_1,\sigma)}+\frac{\delta}{\delta\hat{J}_1(\zeta_2,\sigma)}\right)\right]\\
&+\frac{10}{27}r\oint\prod_{i=1}^{2}\frac{d\zeta_i}{2\pi i}\frac{d\sigma_i}{2\pi i}\hat{J}_1(\zeta_1,\sigma_1)\hat{J}_1(\zeta_2,\sigma_2)\\
&\times\left[-D_{\zeta_1}(z_1,z_2)D_{\zeta_2}(z_1,z_2)D_\zeta(\zeta_1,\zeta_2)\left(\frac{\delta}{\delta\hat{J}_1(\zeta,\sigma_1)}+\frac{\delta}{\delta\hat{J}_1(\zeta,\sigma_2)}\right)\right.\\
&\left.-D_{\sigma_1}(s_1,s_2)D_{\sigma_2}(s_1,s_2)D_\sigma(\sigma_1,\sigma_2)\left(\frac{\delta}{\delta\hat{J}_1(\zeta_1,\sigma)}+\frac{\delta}{\delta\hat{J}_1(\zeta_2,\sigma)}\right)\right]_{\substack{z_i=\zeta_i\\ s_i=\sigma_i}}\\
+&\;\cdots,
\end{aligned}
\tag{D.5}
$$

where $\cdots$ stands for the terms which contain $\hat{J}_A\hat{J}_I,\ \hat{J}_B\hat{J}_I\ \ (I\geq 3)$ or $\hat{J}_K\hat{J}_L\ \ (K+L\geq 3)$.

Firstly, we shall consider the terms in the form of

$$\hat{J}_A\hat{J}_1\left(\frac{\delta}{\delta\hat{J}_A}+\frac{\delta}{\delta\hat{J}_B}\right),\hat{J}_B\hat{J}_1\left(\frac{\delta}{\delta\hat{J}_A}+\frac{\delta}{\delta\hat{J}_B}\right) \tag{D.6}$$

which appear in the first term of $\mathcal{R}_A, \mathcal{R}_B$ and $\mathcal{R}_1$. Their contributions are partially cancelled each other. So the remaining terms are

$$2\sqrt{\frac{10}{27}}r\oint\frac{d\zeta}{2\pi i}\frac{d\sigma}{2\pi i}\left(\partial_\zeta\hat{J}_A(\zeta)\partial_\zeta\hat{J}_1(\zeta,\sigma)\frac{\delta}{\delta\hat{J}_B(\sigma)}+\partial_\sigma\hat{J}_B(\sigma)\partial_\sigma\hat{J}_1(\zeta,\sigma)\frac{\delta}{\delta\hat{J}_A(\zeta)}\right). \tag{D.7}$$

We can see that eq.(D.7) indeed does not contribute from the following argument. Since it is a linear functional of $\hat{J}_A$, $\hat{J}_1$ or $\hat{J}_B$, $\hat{J}_1$, (D.7) vanishes if

$$\oint\frac{d\zeta}{2\pi i}\partial_\zeta\frac{\delta\hat{J}_A(\zeta)}{\delta\hat{J}_A(\zeta')}\partial_\zeta\frac{\delta\hat{J}_1(\zeta,\sigma)}{\delta\hat{J}_1(\zeta'',\sigma')}=0 \tag{D.8}$$

and

$$\oint\frac{d\sigma}{2\pi i}\partial_\sigma\frac{\delta\hat{J}_B(\sigma)}{\delta\hat{J}_B(\sigma')}\partial_\sigma\frac{\delta\hat{J}_1(\zeta,\sigma)}{\delta\hat{J}_1(\zeta',\sigma'')}=0 \tag{D.9}$$

hold. In the continuum limit, the integral of the l.h.s. of (D.8) is

$$\int_{-i\infty}^{i\infty}\frac{dy}{2\pi i}\left(\partial_y\frac{1}{y-y'}\right)^2 \qquad (\mathrm{Re}y'>0),$$

which obviously vanishes. The similar argument holds for (D.9) also. Thus we can see that (D.6) vanishes.

Secondly, we consider $\hat{J}_1\hat{J}_1\frac{\delta}{\delta\hat{J}_A}$. After a trivial cancellation, the following terms remain

$$\begin{aligned}
&-\int_{-i\infty}^{i\infty}\frac{dy}{2\pi i}\prod_{i=1}^{2}\frac{dx_i}{2\pi i}\partial_y\tilde{J}_1(y,x_1)\partial_y\tilde{J}_1(y,x_2)\frac{\delta}{\delta\tilde{J}_A(y)}\\
&-2\int_{-i\infty}^{i\infty}\frac{dx}{2\pi i}\prod_{i=1}^{2}\frac{dy_i}{2\pi i}\partial_x\tilde{J}_1(y_1,x)\partial_x\tilde{J}_1(y_2,x)\frac{\delta}{\delta\tilde{J}_A(y_2)}\\
&+\int_{-i\infty}^{i\infty}\prod_{i=1}^{2}\frac{dy_i}{2\pi i}\frac{dx_i}{2\pi i}\tilde{J}_1(y_1,x_1)\tilde{J}_1(y_2,x_2)(-\partial_{y_1}\partial_{y_2})D_y(y_1,y_2)\frac{\delta}{\delta\tilde{J}_A(y)}
\end{aligned} \tag{D.10}$$

in the continuum limit. Here using the same logic as before, we can see that

$$\int_{-i\infty}^{i\infty}\frac{dx}{2\pi i}\partial_x\tilde{J}_1(y_1,x)\partial_x\tilde{J}_1(y_2,x)=0$$

and

$$\begin{aligned}
&\int_{-i\infty}^{i\infty}\frac{dy}{2\pi i}\prod_{i=1}^{2}\frac{dx_i}{2\pi i}\partial_y\tilde{J}_1(y,x_1)\partial_y\tilde{J}_1(y,x_2)\frac{\delta}{\delta\tilde{J}_A(y)}\\
&=\int_{-i\infty}^{i\infty}\prod_{i=1}^{2}\frac{dy_i}{2\pi i}\frac{dx_i}{2\pi i}\tilde{J}_1(y_1,x_1)\tilde{J}_1(y_2,x_2)(-\partial_{y_1}\partial_{y_2})D_y(y_1,y_2)\frac{\delta}{\delta\tilde{J}_A(y)}
\end{aligned}$$

hold. Thus eq.(D.10) vanishes. Also, the similar argument holds for $\hat{J}_1\hat{J}_1\frac{\delta}{\delta\hat{J}_B}$ because of the symmetry w.r.t. $A\leftrightarrow B$.

Doing the same analysis, we find that $\hat{J}_1\hat{J}_1\frac{\delta}{\delta\hat{J}_1}$ cancels due to the following idetity

$$\begin{aligned}
&\int_{-i\infty}^{i\infty}\frac{dy}{2\pi i}\prod_{i=1}^{2}\frac{dx_i}{2\pi i}\partial_y\tilde{J}_1(y,x_1)\partial_y\tilde{J}_1(y,x_2)\frac{\delta}{\delta\tilde{J}_1(y,x_1)}\\
&=\int_{-i\infty}^{i\infty}\prod_{i=1}^{2}\frac{dy_i}{2\pi i}\frac{dx_i}{2\pi i}\tilde{J}_1(y_1,x_1)\tilde{J}_1(y_2,x_2)(-\partial_{y_1}\partial_{y_2})D_y(y_1,y_2)\frac{\delta}{\delta\tilde{J}_1(y,x_1)}.
\end{aligned}$$

Next, for the terms of $\hat{J}_A\hat{J}_2\frac{\delta}{\delta\hat{J}_A}$, $\hat{J}_B\hat{J}_2\frac{\delta}{\delta\hat{J}_B}$, the remaining terms can be written in the form

$$\begin{aligned}
&\int_{-i\infty}^{i\infty}\prod_{i=1}^{2}\frac{dy_i}{2\pi i}\frac{dx_i}{2\pi i}D_y(y_1,y_2)\partial_y\tilde{J}_A(y)\partial_{y_1}\tilde{J}_2(y_1,x_1,y_2,x_2)\\
&\times\left(\frac{\delta}{\delta\tilde{J}_A(y_2)}+\frac{\delta}{\delta\tilde{J}_B(x_1)}+\frac{\delta}{\delta\tilde{J}_B(x_2)}\right)\\
&+\int_{-i\infty}^{i\infty}\prod_{i=1}^{2}\frac{dy_i}{2\pi i}\frac{dx_i}{2\pi i}D_y(y_1,y_2)\partial_y\tilde{J}_A(y)\partial_{y_2}\tilde{J}_2(y_1,x_1,y_2,x_2)\\
&\times\left(\frac{\delta}{\delta\tilde{J}_A(y_1)}+\frac{\delta}{\delta\tilde{J}_B(x_1)}+\frac{\delta}{\delta\tilde{J}_B(x_2)}\right).
\end{aligned}$$

It vanishes because we can show that the following identity holds

$$\int_{-i\infty}^{i\infty}\frac{dy_1}{2\pi i}D_y(y_1,y_2)\partial_y\tilde{J}_A(y)\partial_{y_1}\tilde{J}_2(y_1,x_1y_2,x_2)=0.$$

By repeating the same argument, we can also prove that

$$\hat{J}_B\hat{J}_2\frac{\delta}{\delta\hat{J}_1}=0 \quad\text{and}\quad \hat{J}_A\hat{J}_2\frac{\delta}{\delta\hat{J}_1}=0.$$

Finally we mention about $\hat{J}_A\hat{J}_2\frac{\delta}{\delta\hat{J}_1}$. The following terms remain:

$$\begin{aligned}
&\int_{-i\infty}^{i\infty}\prod_{i=1}^{2}\frac{dy_i}{2\pi i}\frac{dx_i}{2\pi i}\\
&\times\Bigg\{-\partial_{y_1}\tilde{J}_A(y_1)\partial_{y_1}\tilde{J}_2(y_1,x_1,y_2,x_2)D_x(x_1,x_2)\frac{\delta}{\delta\tilde{J}_1(y_2,x)}\\
&\quad-\partial_{y_2}\tilde{J}_A(y_2)\partial_{y_2}\tilde{J}_2(y_1,x_1,y_2,x_2)D_x(x_1,x_2)\frac{\delta}{\delta\tilde{J}_1(y_1,x)}\\
&\quad+D_y(y_1,y_2)\partial_y\tilde{J}_A(y)\partial_{y_1}\tilde{J}_2(y_1,x_1y_2,x_2)\left(\frac{\delta}{\delta\tilde{J}_1(y_2,x_1)}+\frac{\delta}{\delta\tilde{J}_1(y_2,x_2)}\right)\\
&\quad+D_y(y_1,y_2)\partial_y\tilde{J}_A(y)\partial_{y_2}\tilde{J}_2(y_1,x_1y_2,x_2)\left(\frac{\delta}{\delta\tilde{J}_1(y_1,x_1)}+\frac{\delta}{\delta\tilde{J}_1(y_1,x_2)}\right)\Bigg\}.
\end{aligned}$$

It turns out that the first and third terms vanish by considering y_1- integral after taking the functional derivative $\frac{\delta}{\delta J_A}\frac{\delta}{\delta J_2}$ and that the second and fourth terms vanish also by considering y_2-integral.

From the symmetric property, we can see that $\hat{J}_B\hat{J}_2\frac{\delta}{\delta J_1}$ vanish also.

We note that the contributions from more generic components $(I = 3, 4, \cdots)$ in the l.h.s. of (D.1) give no influence to the results obtained here.

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