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２次元近傍の量子重力理論におけるくりこみ群と
コンフォーマルモードの動力学

平成６年１２月博士（理学）申請
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Doctor Thesis

Abstract

We study quantum gravity near two dimensions ($(2+\epsilon)$-dimensional quantum gravity) from the viewpoint of renormalization group. We formulate the theory in such a way to separate the conformal mode of the metric explicitly. Special care is taken of an oversubtraction problem in the conformal mode dynamics. As an application to two-dimensional quantum gravity, we calculate the scaling dimensions of manifestly generally covariant operators. To obtain a deeper insight into the formalism, we study R^2 gravity in $(2+\epsilon)$-dimensional quantum gravity, which also serves as a success in treating R^2 gravity other than Liouville approach. We next perform the two-loop renormalization of the theory, which is constructed consistently at one-loop level. As the first step toward full calculations, we concentrate on the part proportional to the number of matter fields. The results suggest that we can construct a consistent theory of quantum gravity by ϵ expansion around two dimensions.

Contents

1 Introduction

Constructing a quantum theory of gravitation is one of the most important and challenging problems which remain in theoretical physics. Among various approaches to the problem, quantum gravity near two dimensions ($(2+\epsilon)$-dimensional quantum gravity) has been studied for several years mainly in order to understand qualitative features of four-dimensional quantum gravity from the viewpoint of renormalization group. [1, 2, 3, 4, 5, 6, 7]. It corresponds to ϵ expansion for quantum gravity. ϵ expansion has been used together with renormalization group in quantum field theory and statistical mechanics and was found to be a powerful tool to analyze nonperturbative properties, as was seen in the study of $O(N)$ linear sigma model (ϕ^4 theory with nontrivial interaction) in $4-\epsilon$ dimensions [8], which belongs to the same universality class as three-dimensional $O(N)$ nonlinear sigma model. Though quantum gravity is not renormalizable perturbatively in four dimensions, it may be possible that there is a nontrivial ultraviolet fixed point of renormalization group and that one can construct a quantum field theory of gravitation by taking the continuum limit near the fixed point. In the case of $O(N)$ nonlinear sigma model, it is renormalizable with asymptotic freedom in two dimensions and has a well-defined $(2+\epsilon)$-dimensional expansion with a nontrivial fixed point [9]. Such an expansion gives us information about phase transitions and scaling properties of higher dimensional $O(N)$ nonlinear sigma models, to which perturbation theory is not accessible. We expect due to the existence of two-dimensional theory of gravitation that we can construct consistent quantum gravity in $2+\epsilon$ dimensions and derive from it information of phase structure and scaling properties in four-dimensional quantum gravity.

As in the nonperturbative analysis of ordinary quantum field theories, numerical simulation is another powerful tool for investigating a field theoretical approach to quantum gravity. We hope that we can compare the results obtained in the both methods in near future .

In fact, one may require more than local field theory such as string theory [10] to construct consistent quantum theory of gravitation. We should not, however, forget this simpler possibility. Also, it is not inconceivable that apparently different approaches eventually come to the same final goal. At least we can learn lessons of quantum gravity and acquire insights into the common problems of it in a simpler setting in $2+\epsilon$ dimensions.

As another motivation to investigating $(2+\epsilon)$-dimensional quantum gravity, we have its application to two-dimensional quantum gravity [11, 12]. Two-dimensional quantum gravity [13] has been studied intensively through Liouville theory [14] and the matrix model [15] in these several years, and its progress has provided us with much insight into quantum gravity and string theory. Taking the $\epsilon \to 0$ limit, we expect that we can adopt $(2+\epsilon)$-dimensional quantum gravity as another regularization scheme of two-dimensional gravity and that we can resolve problems which are difficult to treat in the matrix model or in Liouville theory.

In this paper, we analyze renormalization group in terms of the renormalization point. Although we may introduce, for an example, a cut-off in the momentum space, such a procedure makes our analysis difficult since the cut-off break the general covariance explicitly. Another technically essential point in our strategy is that in the calculations of effective actions we adopt a background field method. It has an advantage that the gauge invariance with respect to the background field is preserved manifestly, which enables us to determine counter terms easily.

Recently Kawai, Kitazawa and Ninomiya formulated quantum gravity near two-dimensions in such a way that the conformal mode is explicitly separated [2, 3]. Such a formulation seems natural since the conformal mode plays an important role as the dynamical degree of freedom in two-dimensional quantum gravity and the conformal mode sets the length scale, which is crucial for renormalization group. They pointed out the oversubtraction problem in the conformal mode dynamics at one loop level and presented a prescription to answer the problem. They took $\epsilon \to 0$ limit carefully in the strong coupling regime and succeeded in computing the scaling dimensions of the gravitationally dressed primary fields in two dimensions, which agree with the ones calculated in the matrix model or in Liouville theory. However this prescription does not hold true for the regime near the ultraviolet fixed point, in which we are interested for higher dimensional quantum gravity. Subsequently the authors of ref.[6] gave a formalism applicable to the regime near the fixed point, avoiding the oversubtraction for the conformal mode. They started with the action having generalized dependence on the conformal mode and invariant only under the explicit volume preserving diffeomorphism, where they simplified the problem by considering the conformal matters. They showed, at one-loop level, (1) the existence of an ultraviolet fixed point, (2) the existence of an renormalization group flow from it to the infrared fixed point in which the theory coincided with the Einstein gravity and (3) the restoration of the general covariance along

the renormalization group flow. In particular, the fixed point possessed the Z_2 symmetry with respect to the conformal mode in their parametrization, which is interesting since it may be possible to interpret this as vanishing of the expectation value of the metric there.

It must be noted that the oversubtraction problem is independent of the parametrization of metric. In fact, it has been shown in ref. [16] that Einstein gravity near two dimensions which is formulated in the generally covariant way without the separation of conformal mode is not renormalizable beyond one-loop level. This phenomenon reflects the oversubtraction at one-loop level. One can clarify the problem by separating the conformal mode.

In this paper, we investigate $(2+\epsilon)$-dimensional quantum gravity further based on three works [11, 4, 7]. The paper consists of two parts. In the first part, we treat two concrete problems. One is, as an application of quantum gravity near two dimensions to two-dimensional quantum gravity, to calculate the scaling dimensions of manifestly generally covariant operators such as $\int \sqrt{g}R^n \mathrm{d}^2x$, where $n = 0, 1, 2, \cdots$, utilizing the prescription mentioned above which is valid and powerful for two-dimensions limit [11]. These operators have clear physical meanings, but are difficult to study in the conventional approaches. We compare the properties of these operators with those of the scaling operators appearing in the matrix model or in Liouville theory. The other is to treat R^2 gravity in $(2+\epsilon)$-dimensional quantum gravity. Quantum R^2 gravity in two-dimensions [17] is worth investigating as a new type of universality class of two-dimensional quantum gravity and as an example of models overcoming the so-called $c = 1$ barrier though not having unitarity. It has been studied recently in Liouville approach, yet in contrast to ordinary gravity it is difficult to treat it in the matrix model and we do not have so far any other results which can be compared with the ones in Liouville approach [18]. It seems, therefore, desirable to investigate it in other approaches. And, in view of a subtlety of the conformal mode dynamics in $(2+\epsilon)$-dimensional quantum gravity, we feel that it is worthwhile acquiring a deeper insight into the formalism by applying it to other theories, such as R^2 gravity, than the ordinary Einstein gravity.

In the second part of the present paper, we study the two-loop renormalization of the theory mentioned previously which possesses the generalized dependance on the conformal mode and has been constructed consistently at one-loop level [7]. It is important to perform the two-loop renormalization of the theory since there is no proof that the procedure gives consistent results to higher orders. We can establish the validity of ϵ expansion around two dimensions of quantum gravity by showing that the higher order corrections can be computed systematically and that the theory renormalized up to higher orders level satisfies the requirement from general covariance. Thus two-loop calculation serves a check of ϵ expansion approach to quantum gravity. However the two-loop calculations in quantum gravity is a formidable task due to the proliferation of diagrams and tensor indices. Therefore we have decided to calculate the two-loop counter terms which is proportional to the number of matter fields (the central charge) first. Since the number of scalar fields we couple to gravity is a free parameter, the counter terms must be of the renormalizable form. We also simplify the task by imposing Z_2 symmetry to pay attention to the ultraviolet fixed point. Eventually the number of diagrams we have to calculate reduces to one hundred twenty three.

The organization of the paper is as follows. The next section is devoted to reviewing the recent works on $(2+\epsilon)$-dimensional quantum gravity. In section 3, we calculate the scaling dimensions of manifestly generally covariant operators using the formalism developed in the previous section and interpret the results. In section 4, we treat R^2 gravity in the framework of $(2+\epsilon)$-dimensional quantum gravity and obtain the result consistent with the one derived in Liouville approach. In section 5, we perform the calculations of the two-loop counter terms and see consistency of the theory. The last section is devoted to summary and outlook. In appendix A, we explain the background field method which we use in this paper and present an interpretation of renormalization point. In appendixes B $\sim$ D, some of cumbersome calculations are collected.

2 Formalism of Quantum Gravity near Two Dimensions

In this section, we review quantum gravity in $2+\epsilon$ dimensions formulated in such a way that the conformal mode of the metric is explicitly separated, following ref. [2, 3, 6]. Section 2.1 is devoted to an explanation for an oversubtraction problem in the conformal mode dynamics appearing when the theory is renormalized in a manifestly generally covariant way [2]. In section 2.2, a prescription to resolve the problem is presented, which is, however, not reliable near the ultraviolet fixed point. $\epsilon \to 0$ limit is taken in the strong coupling regime and the exact results in two dimensions is reproduced [2]. In section 2.3, we explain generalization of the conformal mode dependence of the action in order to overcome the oversubtraction and treat the regime near the ultraviolet fixed point [6]. We see that how the general covariance is restored by considering the one-loop effects.

2.1 Dynamics of Conformal Mode in $(2+\epsilon)$–Dimensional Quantum Gravity

As is seen in Liouville theory, the conformal mode of the metric plays an important role as the dynamical degree of freedom in two-dimensional quantum gravity. Therefore it is natural to adopt a parametrization and a gauge which single out the conformal mode. Let us write the metric as

$$g_{\mu\nu} = \hat{g}_{\mu\rho}(e^h)^{\rho}{}_{\nu} e^{-\phi}, \tag{2.1}$$

where $h^{\mu}{}_{\nu}$ is a traceless hermitian tensor, ϕ is a conformal mode, and $\hat{g}_{\mu\nu}$ is a background metric. A more detailed explanation about this parametrization is presented in appendix A. We start from the Einstein action and the action for c species of massless scalar field in $D = 2+\epsilon$ dimensions,

$$S = \frac{\mu^{\epsilon}}{G}\int \mathrm{d}^D x\sqrt{g}R + \sum_{i=1}^{c}\int \mathrm{d}^D x\sqrt{g}g^{\mu\nu}\partial_{\mu}\varphi_i\partial_{\nu}\varphi_i, \tag{2.2}$$

where G is a dimensionless gravitational constant and μ is a renormalization point (renormalization scale). We also present an interpretation of the renormalization point in appendix A. Using the background field method, we calculate the one-loop divergence. The gauge fixing term is chosen as

$$\frac{\mu^\epsilon}{G}\int d^D x\sqrt{\hat{g}}\frac{1}{2}(\hat{\nabla}_\nu h^\nu{}_\mu + \frac{\epsilon}{2}\partial_\mu\phi)(\hat{\nabla}_\rho h^{\rho\mu} + \frac{\epsilon}{2}\partial^\mu\phi), \tag{2.3}$$

which makes the kinetic term of $h^\mu{}_\nu$ canonical and removes the mixing of $h^\mu{}_\nu$ and ϕ in the quadratic terms. By adding this term to the action (2.2), we obtain the total quadratic action,

$$\begin{aligned}\frac{\mu^\epsilon}{G}\int d^D x\sqrt{\hat{g}} \quad \{ \quad & \frac{1}{4}\hat{\nabla}_\nu h^\rho_\mu \hat{\nabla}^\nu h^\mu_\rho + \frac{1}{2}\hat{R}^\sigma_{\mu\nu\rho} h^\rho{}_\sigma h^{\mu\nu} \\ - \quad & \frac{\epsilon}{8}D\hat{g}^{\mu\nu}\partial_\mu\phi\partial_\nu\phi + \frac{\epsilon}{2}\phi h^\mu{}_\nu \hat{R}^\nu_\mu + \frac{\epsilon^2}{8}\phi^2\hat{R}\} + (\text{matters}). \end{aligned} \tag{2.4}$$

Note that there is an $\frac{1}{\epsilon}$ pole in the propagator of ϕ due to the ϵ factor in the kinetic term. The ghost action is derived from the gauge fixing term (2.3) in the standard way,

$$\frac{\mu^\epsilon}{G}\int d^D x\sqrt{\hat{g}}\{\hat{\nabla}_\nu\bar{\eta}^\mu\hat{\nabla}^\nu\eta_\mu - \bar{\eta}^\mu\hat{R}^\nu{}_\mu\eta_\nu + \cdots\}. \tag{2.5}$$

Exploiting the 't Hooft-Veltman formalism [19], we can evaluate the one–loop divergences from (2.4) and (2.5) as

$$\frac{25-c}{24\pi}\frac{\mu^\epsilon}{\epsilon}\int \mathrm{d}^D x\sqrt{\hat{g}}\hat{R}, \tag{2.6}$$

which forces us to choose the following one–loop counter term

$$S_{\text{c.t.}} = -\frac{25-c}{24\pi}\frac{\mu^\epsilon}{\epsilon}\int \mathrm{d}^D x\sqrt{g}R. \tag{2.7}$$

It follows from (2.2) and (2.7) that the bare coupling constant G_0 is related to the renormalized coupling constant G through

$$\frac{1}{G_0} = \mu^\epsilon\left(\frac{1}{G} - \frac{25-c}{24\pi}\frac{1}{\epsilon}\right). \tag{2.8}$$

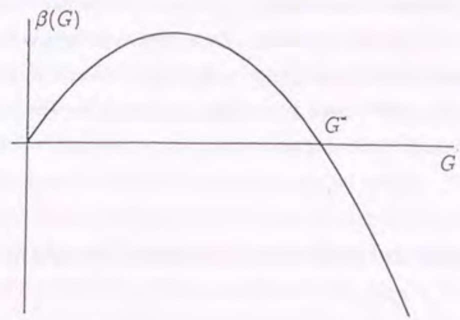
Using this relation, one can calculate the β-function as

$$\begin{aligned}\beta(G) &= \mu\frac{\partial G}{\partial\mu} \\ &= \epsilon G - \frac{25-c}{24\pi}G^2,\end{aligned} \tag{2.9}$$

which means there is an ultraviolet fixed point

$$G^* = \frac{24\pi}{25-c}\epsilon, \tag{2.10}$$

as long as $c < 25$. As is expected, this fixed point separates weak and strong phases. The weak phase should contain massless gravitons and resemble our universe. The continuum limit must be taken by approaching the ultraviolet fixed point from the weak coupling regime.



We find, however, a subtle problem in this situation before taking the next step. The counter term (2.7), which preserves general covariance, causes an oversubtraction problem for the conformal mode. A detailed explanation of this problem is as follows. Noting that

$$\begin{aligned}\int \mathrm{d}^D x\sqrt{g}R &= \int \mathrm{d}^D x\sqrt{\tilde{g}}e^{-\frac{\epsilon\phi}{2}}\tilde{R} \\ &\quad -\int \mathrm{d}^D x\sqrt{\tilde{g}}e^{-\frac{\epsilon\phi}{2}}\frac{1}{4}\epsilon(D-1)\tilde{g}^{\mu\nu}\partial_\mu\phi\partial_\nu\phi,\end{aligned} \tag{2.11}$$

where $\tilde{g}_{\mu\nu} = \hat{g}_{\mu\rho}(e^h)^\rho{}_\nu$, one can see that the kinetic term of the conformal mode in the tree action is an $O(\epsilon)$ quantity while that in the one–loop counter term is an $O(1)$ quantity. Here we decompose the background metric further as $\hat{g}_{\mu\nu} = \bar{g}_{\mu\nu}e^{-\bar{\phi}}$. In Fig. 1 the diagrams which give the two–point functions of $\bar{\phi}$ are listed up. The diagrams (1) $\sim$ (5) give $O(\epsilon)$ quantities due to the ϵ factors in the vertices. One can also easily verify that the $O(1)$ contributions from the diagrams (6) and (7) cancel out. As a result, there remains the $O(\epsilon)$ 'divergence' for $\partial_\mu\bar{\phi}\partial_\mu\bar{\phi}$. This puzzle is resolved by considering the full effective action. In fact, we must obtain the finite non–local Liouville term

$$-\frac{25-c}{96\pi}\int \mathrm{d}^D x\sqrt{\hat{g}}\hat{R}\frac{1}{\hat{\Delta}}\hat{R} \tag{2.12}$$

together with the divergent term (2.6). Its $\bar{\phi}$ dependence is evaluated as follows,

$$-\frac{25-c}{96\pi}\int \mathrm{d}^D x\sqrt{\hat{g}}\bar{R}\frac{1}{\bar{\Delta}}\bar{R} + \frac{25-c}{96\pi}\int \mathrm{d}^D x\sqrt{\bar{g}}(\bar{g}^{\mu\nu}\partial_\mu\bar{\phi}\partial_\nu\bar{\phi} + 2\bar{R}\bar{\phi}) + O(\epsilon), \tag{2.13}$$

which cancels out the $O(1)$ dependence on $\bar{\phi}$ in (2.6). It follows that we subtract $O(1)$ quantity from $O(\epsilon)$ quantity for the conformal mode when we add the general covariant one–loop counter term (2.7). That is the oversubtraction problem for the conformal mode dynamics. It is impossible to renormalize the theory above one–loop level due to this oversubtraction. The ordinary renormalization procedure breaks down unless the general covariance of the procedure is discarded, which leads to treating a model in section 2.3 which has generalized dependence on the conformal mode.

2.2 An Answer to the Oversubtraction Problem and Two–Dimensions Limit

Before the conformal mode dependence of the action is generalized for the purpose of resolving the oversubtraction, an answer to the problem is presented here. On the grounds that the oversubtraction problem is nothing but the counter term dominance for the kinetic term of the conformal mode, it is natural to redefine the conformal mode propagator by summing up conformal mode propagators with arbitrary times of insertion of the counter term $\frac{25-c}{24\pi}\frac{1}{4}\partial_\mu\phi\partial_\mu\phi$. The conformal mode propagator after this resummation becomes

$$\begin{aligned}
& \left(-\frac{2G}{\epsilon}\frac{1}{p^2}\right)\sum_{n=0}^{\infty}\left[\left(-\frac{25-c}{24\pi}\frac{1}{2}p^2\right)\left(-\frac{2G}{\epsilon}\frac{1}{p^2}\right)\right]^n \\
=& \left(-\frac{2G}{\epsilon}\frac{1}{p^2}\right)\frac{1}{1-\frac{25-c}{24\pi}\frac{G}{\epsilon}} \\
=& -\frac{2G_0\mu^\epsilon}{\epsilon}\frac{1}{p^2}. \qquad (2.14)
\end{aligned}$$

This leads us to use $G_0\mu^\epsilon$ as an expansion parameter instead of G. The effective coupling becomes then large near the ultraviolet fixed point $G \sim G^*$. Therefore the dynamics near the fixed point is quite nontrivial and cannot be treated well in this way. Here the authors of ref.[2] propose that the $\epsilon \to 0$ limit should be taken in the strong coupling regime ($G \gg G^* = O(\epsilon)$). In this regime, the effective coupling becomes an $O(\epsilon)$ quantity,

$$G_0\mu^\epsilon \longrightarrow -\frac{24\pi}{25-c}\epsilon, \qquad (2.15)$$

and the loop expansion works well. Note that this negative coupling make the kinetic term of the conformal mode positive and hence the problem of the conformal mode instability is resolved as in Liouville theory. In the weak coupling regime ($G \ll G^*$), the expansion parameter is G itself and the conformal mode instability remains. One can verify correspondence with the ordinary formalisms of two–dimensional quantum gravity by renormalizing

$\int\sqrt{g}^{1-\Delta_0}\Phi_{\Delta_0}d^2x$ type operators, where Φ_{Δ_0} is a spinless primary field with conformal dimension Δ_0, and calculating their scaling dimensions. In doing so, a technically important point in the formalism is that the dynamics is completely determined by the conformal mode in the sense that the other fields, the $h_{\mu\nu}$, ghost and matter fields, can be dropped from the beginning, as is checked explicitly up to the two-loop level. It is natural since the conformal mode governs dynamics of two-dimensional quantum gravity. After this simplification the theory can be reduced to a free field theory, which makes it possible to perform a full order calculation of the scaling dimensions. The calculation reproduces the exact result of refs. [14]. In section 3.2 we extend this calculation to the case of the manifestly generally covariant operators. We refer the reader to section 3.2 in which essences of this calculation are included.

2.3 Conformal Gravity near Two Dimensions

As is mentioned in section 2.1, treating quantum gravity near two–dimensions in such a way that manifestly general covariance is maintained leads to the oversubtraction for the conformal mode. In this subsection, we review the study of a model which possesses generalized dependence on the conformal mode and is invariant only under the volume preserving diffeomorphism [6], and in particular see how the general covariance is guaranteed in the last stage. The system considered here is a $(2+\epsilon)$–dimensional quantum gravity coupled to c copies of scalar fields in the conformally invariant way with the following action.

$$\frac{\mu^\epsilon}{G}\int d^Dx\sqrt{\tilde{g}}\{R(1-\frac{\epsilon}{8(D-1)}\varphi_i^2)+\frac{1}{2}g^{\mu\nu}\partial_\mu\varphi_i\partial_\nu\varphi_i\} \qquad (2.16)$$

where i runs from 1 to c. The fact that the conformal matter decouples the conformal mode makes the analysis easier. It is hard to treat the matter fields not conformally invariant in the following formulation though it must be possible in principle [3]. This action can be rewritten as

$$\frac{\mu^\epsilon}{G}\int d^Dx\sqrt{\tilde{g}}\{\tilde{R}((1+\frac{1}{2}\sqrt{\frac{\epsilon}{2(D-1)}}\psi)^2-\frac{\epsilon}{8(D-1)}\varphi_i^2)-\frac{1}{2}\tilde{g}^{\mu\nu}\partial_\mu\psi\partial_\nu\psi+\frac{1}{2}\tilde{g}^{\mu\nu}\partial_\mu\varphi_i\partial_\nu\varphi_i\} \qquad (2.17)$$

where the conformal mode is parameterized as $e^{-\frac{\epsilon\phi}{4}} = 1+\frac{1}{2}\sqrt{\frac{\epsilon}{2(D-1)}}\psi$. In this way the kinetic term of ψ becomes canonical. Note that the conformal mode ψ can be viewed as another conformally coupled scalar field in this parametrization. Therefore one can quantize the theory treating the conformal mode as a matter field coupled in the conformally invariant way. In such a quantization procedure it is important to keep the conformal invariance. Since it is well known that the conformal anomaly arises in quantum field theory, we need to modify the tree action to cancel the quantum conformal anomaly. In other words, the general covariance is broken when the theory is renormalized without the oversubtraction. Therefore we have to break it in the beginning in order to restore it at last.

Thus the action is generalized in the following form which possesses the manifest volume preserving diffeomorphism invariance.

$$\frac{\mu^\epsilon}{G}\int d^Dx\sqrt{\hat{g}}\{\tilde{R}L(\psi,\varphi_i)-\frac{1}{2}\tilde{g}^{\mu\nu}\partial_\mu\psi\partial_\nu\psi+\frac{1}{2}\tilde{g}^{\mu\nu}\partial_\mu\varphi_i\partial_\nu\varphi_i\} \tag{2.18}$$

where $L=1+a\psi+b\psi^2-d\varphi_i^2$. Note that in the case of the conformally coupled Einstein gravity (2.17) $a=\sqrt{\frac{\epsilon}{2(D-1)}}$ and $b=d=\frac{\epsilon}{8(D-1)}$. As the natural extension of this case, we assume in the general cases that a is an $O(\sqrt{\epsilon})$ quantity and b and d are $O(\epsilon)$ quantities. Here we consider the following local gauge transformation of the fields,

$$\begin{aligned}
\delta\tilde{g}_{\mu\nu} &= \epsilon^\rho\partial_\rho\tilde{g}_{\mu\nu}+\partial_\mu\epsilon^\rho\tilde{g}_{\rho\nu}+\tilde{g}_{\mu\rho}\partial_\nu\epsilon^\rho-\frac{2}{D}\partial_\rho\epsilon^\rho\tilde{g}_{\mu\nu},\\
\delta\psi &= \epsilon^\rho\partial_\rho\psi+(D-1)\frac{\partial L}{\partial\psi}\frac{2}{D}\partial_\rho\epsilon^\rho,\\
\delta\varphi_i &= \epsilon^\rho\partial_\rho\varphi_i-(D-1)\frac{\partial L}{\partial\varphi_i}\frac{2}{D}\partial_\rho\epsilon^\rho.
\end{aligned} \tag{2.19}$$

In the Einstein gravity, this transformation is nothing but the general coordinate transformation and indeed the action (2.17) is invariant under it. The change of the generalized Einstein action (2.18) under the transformation (2.19) is evaluated as

$$\begin{aligned}
-\frac{\mu^\epsilon}{G}\int d^Dx\sqrt{\hat{g}}\;\{\; &\frac{1}{2}\{\epsilon L-2(D-1)((\frac{\partial L}{\partial\psi})^2-(\frac{\partial L}{\partial\varphi_i})^2)\}\hat{R}\\
&-\frac{1}{4}\{\epsilon-4(D-1)\frac{\partial^2 L}{\partial\psi^2}\}\partial_\mu\psi\partial^\mu\psi\\
&+\frac{1}{4}\{\epsilon+4(D-1)\frac{\partial^2 L}{\partial\varphi_i^2}\}\partial_\mu\varphi_i\partial^\mu\varphi^i\}\delta\rho,
\end{aligned} \tag{2.20}$$

where $\delta\rho$ is $\frac{2}{D}\partial_\rho\epsilon^\rho$. This vanishes when $\delta\rho$ is equal to zero, which is expected because the action (2.18) always possesses the volume preserving diffeomorphism invariance. One can say that the action has general covariance if it is invariant under the following conformal transformation

$$\delta\tilde{g}_{\mu\nu} = -\tilde{g}_{\mu\nu}\delta\rho\;\;(\delta\hat{g}_{\mu\nu}=-\hat{g}_{\mu\nu}\delta\rho),$$

$$\begin{aligned}
\delta\psi &= (D-1)\frac{\partial L}{\partial\psi}\delta\rho,\\
\delta\varphi_i &= -(D-1)\frac{\partial L}{\partial\varphi_i}\delta\rho.
\end{aligned} \tag{2.21}$$

In other words, the general covariance can be recovered by demanding the conformal invariance further. The strategy is that we demand the action only possessing the volume preserving diffeomorphism at tree level to gain the conformal invariance at quantum level. The general covariance should be established at the bare action level. To calculate the one–loop counter term, we decompose the fields into the backgrounds and the quantum fields as $\varphi_i\to\hat{\varphi}_i+\varphi_i,\psi\to\hat{\psi}+\psi$ and $\tilde{g}_{\mu\nu}=\hat{g}_{\mu\rho}(e^h)^\rho{}_\nu$. Here $h^\mu{}_\nu$ is a traceless tensor as before. The effective action can be computed by summing the one particle irreducible diagrams with respect to the quantum fields φ_i, ψ and $h^\mu{}_\nu$. In addition to expand the action around the background fields, we need to fix the gauge invariance (2.19) in order to perform the functional integration. We adopt the following background gauge,

$$\frac{1}{2}L(\hat{\nabla}^\mu h_{\mu\nu}-\frac{\partial_\nu L}{L})(\hat{\nabla}_\rho h^{\rho\nu}-\frac{\partial^\nu L}{L}), \tag{2.22}$$

from which the ghost terms is derived as

$$\hat{\nabla}_\mu\bar{\eta}_\nu\hat{\nabla}^\mu\eta^\nu+\hat{R}^\mu_\nu\bar{\eta}_\mu\eta^\nu\frac{\partial_\nu L}{L}(\hat{\nabla}^\mu\bar{\eta}_\mu)\eta^\nu+\cdots \tag{2.23}$$

The one loop counter term of this theory is evaluated to be

$$-\frac{A}{\epsilon}\int d^Dx\sqrt{\hat{g}}\hat{R}, \tag{2.24}$$

where $A=\frac{25-c}{24\pi}$. By adding (2.24) to (2.18), we obtain the one–loop bare action, from which the relations between the bare quantities and the renormalized ones are read off as follows,

$$\begin{aligned}
\frac{1}{G_0} &= \mu^\epsilon(\frac{1}{G}-\frac{A}{\epsilon})\\
a_0 &= a(1+\frac{AG}{2\epsilon})\\
b_0 &= b\\
d_0 &= d.
\end{aligned} \tag{2.25}$$

β functions are obtained by demanding that the bare quantities do not depend on the renormalization scale μ as

$$\begin{aligned}
\beta_G &= \epsilon G-AG^2,\\
\beta_a &= -\frac{GA}{2}a,\\
\beta_b &= \beta_d=0.
\end{aligned} \tag{2.26}$$

The Einstein action is the infrared fixed point with $G=0, a=\sqrt{\frac{\epsilon}{2(D-1)}}$ and $b=d=\frac{\epsilon}{8(D-1)}$. The theory possesses the ultraviolet fixed point with $G=\frac{\epsilon}{A}, a=0$ and $b=d=\frac{\epsilon}{8(D-1)}$. There is a renormalization group flow from ultraviolet fixed point to the infrared fixed point. Note that at the ultraviolet fixed point the action possesses the Z_2 invariance ($\psi \to -\psi$ and $\varphi_i \to -\varphi_i$). The enhancement of this symmetry may be due to the fact that the theory is expanded around the symmetric vacuum in which the expectation value of the metric vanishes. In fact, Z_2 symmetry appears in the Einstein action (2.16) when the metric is expanded around zero. The variation of the conformal trasformation is calculated as before in the following.

$$\begin{aligned}-\frac{\mu^\epsilon}{G}\int d^Dx\sqrt{\hat{g}} \quad \{ \quad & \frac{1}{2}\{\epsilon L - AG - 2(D-1)((\frac{\partial L}{\partial \psi})^2-(\frac{\partial L}{\partial \varphi_i})^2)\}\hat{R} \\ - \quad & \frac{1}{4}\{\epsilon - 4(D-1)\frac{\partial^2 L}{\partial \psi^2}\}\partial_\mu\psi\partial^\mu\psi \\ + \quad & \frac{1}{4}\{\epsilon + 4(D-1)\frac{\partial^2 L}{\partial \psi^2}\}\partial_\mu\varphi_i\partial^\mu\varphi^i\}\delta\rho. \end{aligned} \tag{2.27}$$

By substituting the values of b and d on the renormalization group flow into the above, we obtain

$$-\frac{\mu^\epsilon}{G}\int d^Dx\sqrt{\hat{g}}\frac{1}{2}\{\epsilon - GA - 2(D-1)a^2\}\hat{R}\delta\rho. \tag{2.28}$$

This quantity vanishes both on the ultraviolet fixed point and on the infrared fixed point. One can also show that it vanishes along the renormalization group trajectory, using the relation

$$\mu\frac{\partial}{\partial\mu}\{\frac{\mu^\epsilon}{G}(\epsilon - GA - 2(D-1)a^2)\} = 0, \tag{2.29}$$

which is derived from the β functions (2.27). Note that the conformal invariance is crucial to restore the general covariance in the action which possesses only the volume preserving diffeomorphism invariance. Therefore the general covariance is maintained along the renormalization group trajectory also.

3 Scaling Dimensions of Manifestly Generally Covariant Operators in Two–Dimensional Quantum Gravity

In this section we consider scaling operators in two-dimensional quantum gravity. Section 3.1 is devoted to summarizing the properties of the scaling operators appearing in Liouville theory or in the matrix model. In section 3.2 we calculate the scaling dimensions of manifestly generally covariant operators using the formalism developed in section 2.2. This is the first success in treating such operators in a consistent way. In section 3.3 we interpret the results. We compare the scaling dimensions with those of the operators in the matrix model and in Liouville theory. Our spectrum includes all the scaling dimensions of the scaling operators in the matrix model except the boundary operators. However there are also many others which do not appear in the matrix model. Though there is a possibility that the scaling operators in the matrix model corresponds to the operators considered here, we conjecture that the partial agreement of the scaling dimensions should be considered as accidental and that the operators considered give a new series of operators in two-dimensional quantum gravity. This section is based on the work [11].

3.1 Scaling Operators in Two–Dimensional Quantum Gravity

Although the equivalence of the two approaches, Liouville theory and the matrix model, is almost confirmed based on the agreement of the correlation functions of the operators [20, 21, 22], the notion of operators comes out in each approach in quite a different way. In the matrix model, the scaling operators appear when a macroscopic loop on the surface is shrunk. They form a complete set in the sense that their correlators satisfy closed recursive relations [23]. We must say, however, that they come out in such a geometrical way that it is not clear how they can be written in terms of the metric and the matter fields. In Liouville theory, on the other hand, one can carry out the BRST cohomological analysis [24] to obtain the physical operators, whose scaling dimensions have the same spectrum as that appearing in the matrix model except for those operators in the matrix model known as the boundary operators [25] or the redundant operators [25, 26]. Here the operators with zero ghost number can be understood as primary fields with gravitational dressing, while the operators with nonzero ghost number do not allow such a clear interpretation. Alternatively, without taking Felder's resolution [27], one can construct the gravitationally dressed primary fields inside and outside the minimal Kac table, which have a one-to-one correspondence to the scaling operators in the matrix model up to the correlation function level [21, 22]. The *inside* ones are nothing but the operators with zero ghost number in the BRST analysis, while the *outside* ones include the operators with nonzero ghost number in the BRST analysis and the boundary operators. Here the physical meaning of the dressed primary fields *outside* the minimal Kac table is quite obscure.

In these circumstances, we think it is worthwhile studying manifestly generally covariant operators, whose physical meaning is clear. Specifically, we consider in this paper manifestly generally covariant operators written as a volume integral of a local scalar density composed of the metric and the matter fields. For example, in the case of pure gravity, the operators we consider are $\int \sqrt{g}R^n \mathrm{d}^2x$, where $n = 0, 1, 2, \cdots$. In spite of the clarity of their physical meaning, such operators are difficult to study in the conventional approaches. In Liouville theory, there is no unambiguous way to define such composite operators, while in the matrix model, or in dynamical triangulation in general, one may consider their formal counterparts by identifying the scalar curvature with the deficit angle per volume, but it is not clear whether they really correspond to the desired operators in the continuum limit. The formalism of $(2+\epsilon)$–dimensional quantum gravity, however, seems most suitable for our purpose. Here we would like to generalize the calculation in section 2.2 to the scaling dimensions of the manifestly generally covariant operators explained above.

3.2 Calculations of Scaling Dimensions of Manifestly Generally Covariant Operators

Using the formalism described in section 2.2, we first calculate the scaling dimensions of $\int \sqrt{g}R^n \mathrm{d}^2x$ type operators in pure gravity. Dropping the h-field, the Einstein action can be written in terms of the conformal mode as

$$\int \sqrt{g} R \mathrm{d}^D x = \int \mathrm{d}^D x \left[\sqrt{\hat{g}} \hat{R} \mathrm{e}^{-\frac{\epsilon}{2}\phi} - \frac{\epsilon(D-1)}{4} \sqrt{\hat{g}} \mathrm{e}^{-\frac{\epsilon}{2}\phi} \hat{g}^{\mu\nu} \partial_\mu \phi \partial_\nu \phi \right]. \tag{3.1}$$

By introducing a new variable ψ through

$$\mathrm{e}^{-\frac{\epsilon}{4}\phi} = 1 + \frac{\epsilon}{4}\psi \tag{3.2}$$

the action can be written in terms of ψ as

$$S = \frac{1}{G_0} \int \mathrm{d}^D x \left[\frac{\epsilon^2}{16} \sqrt{\hat{g}} \hat{R} \psi^2 - \frac{D-1}{4} \epsilon \sqrt{\hat{g}} \hat{g}^{\mu\nu} \partial_\mu \psi \partial_\nu \psi \right], \tag{3.3}$$

where we drop the linear term, following the usual prescription of the background field method. As is seen in section 2.2, the expansion parameter is $G_0 \mu^\epsilon$, which is equal to $-\frac{24\pi}{25-c}\epsilon$ in this case. We make use of the general covariance of the theory to proceed further; namely, instead of keeping the full background field dependence, we expand the background field around the flat metric as

$$\hat{g}_{\mu\nu} = \delta_{\mu\nu} + \hat{h}_{\mu\nu} \tag{3.4}$$

and, after calculating the one-point function of an operator up to sufficient order in $\hat{h}_{\mu\nu}$, we read off the corresponding generally covariant form to reproduce the full result. Defining H and $G_{\mu\nu}$ through

$$\begin{aligned} \sqrt{\hat{g}} &= 1 + H & (3.5) \\ \sqrt{\hat{g}} \hat{g}^{\mu\nu} &= \delta_{\mu\nu} + G_{\mu\nu}, & (3.6) \end{aligned}$$

the action reads

$$S = \frac{1}{G_0} \int \mathrm{d}^D x \left[\frac{\epsilon^2}{16} \hat{R}(1+H)\psi^2 - \frac{D-1}{4} \epsilon (\partial_\mu \psi \partial_\mu \psi + G_{\mu\nu} \partial_\mu \psi \partial_\nu \psi) \right]. \tag{3.7}$$

The terms with $\hat{R}$ and $G_{\mu\nu}$ will be treated perturbatively. After we integrate ψ field in the calculation of the one-point function of $\int \sqrt{g}R^n \mathrm{d}^2x$, we obtain, in general, terms such as $\int \sqrt{g}R^n \mathrm{d}^2x$, $\int \sqrt{g}R^{n-2} \triangle R \mathrm{d}^2x$, $\int \sqrt{g}R^n \frac{1}{\triangle} R \mathrm{d}^2x$ and so on, which are $O(\hat{h}^n)$, $O(\hat{h}^{n-1})$, $O(\hat{h}^{n+1})$ quantities and so on respectively. We can verify that we may ignore the term such as the third one at last because the power of its divergence is decreased due to its nonlocality. We, therefore, keep terms up to $O(\hat{h}^n)$ here. Special care should be taken for the $n = 1$ case, which will be treated later. $\int \sqrt{g}R^n \mathrm{d}^D x$ can be expressed in terms of ψ as

$$\begin{aligned} \int \sqrt{g} R^n \mathrm{d}^D x &= \int \mathrm{d}^D x \sqrt{\hat{g}} \mathrm{e}^{(-\frac{D}{2}+n)\phi} \left\{ \hat{R} - (D-1)\hat{g}^{\mu\nu} \hat{\nabla}_\mu \partial_\nu \phi + \frac{1}{4}\epsilon(D-1)\hat{g}^{\mu\nu} \partial_\mu \phi \partial_\nu \phi \right\}^n \\ &= \int \mathrm{d}^D x \sqrt{\hat{g}} \mathrm{e}^{-\frac{4}{\epsilon}(-\frac{D}{2}+n)\log(1+\frac{\epsilon}{4}\psi)} \left\{ \hat{R} + (D-1)\frac{1}{1+\frac{\epsilon}{4}\psi} \hat{g}^{\mu\nu} \hat{\nabla}_\mu \partial_\nu \psi \right\}^n. \quad (3.8) \end{aligned}$$

In the following, we set $H = 0$ and $G_{\mu\nu} = 0$ in the action (3.7) and replace $\hat{g}^{\mu\nu} \hat{\nabla}_\mu \partial_\nu$ in the expression (3.8) with ∂^2. That this does not affect the result is shown in appendix B. The expectation value of the expression (3.8) can be written down for $n = 2$, for example, as

$$\begin{aligned}\langle\int\sqrt{g}R^2\mathrm{d}^Dx\rangle &= \int\mathrm{d}^Dx\sqrt{\hat{g}}\langle\mathrm{e}^{-\frac{4}{\epsilon}(-\frac{D}{2}+2)\log(1+\frac{\epsilon}{4}\psi)}\rangle\hat{R}^2\\ &+2(D-1)\int\mathrm{d}^Dx\sqrt{\hat{g}}\langle\mathrm{e}^{-\frac{4}{\epsilon}(-\frac{D}{2}+2+\frac{\epsilon}{4})\log(1+\frac{\epsilon}{4}\psi)}\partial^2\psi\rangle\hat{R}\\ &+(D-1)^2\int\mathrm{d}^Dx\sqrt{\hat{g}}\langle\mathrm{e}^{-\frac{4}{\epsilon}(-\frac{D}{2}+2+\frac{\epsilon}{2})\log(1+\frac{\epsilon}{4}\psi)}(\partial^2\psi)^2\rangle.\end{aligned} \tag{3.9}$$

Since we are dealing with free field theory, the expectation value within each term can be calculated to full order. For the details of the calculation, we refer the reader to appendix C, where we show that the relevant $\frac{1}{\epsilon}$ divergence comes from the $\mathrm{e}^{-\frac{4}{\epsilon}(-\frac{D}{2}+2)\log(1+\frac{\epsilon}{4}\psi)}$ in each term. The same argument holds for arbitrary n, and the wave function renormalization of the operator $\int\sqrt{g}R^n\mathrm{d}^2x$ is given by

$$Z_n = \exp\{-\frac{4}{\epsilon}(1-n)\log(1+\rho_n) + \frac{8\pi}{G_0\mu^\epsilon}\rho_n^2\}, \tag{3.10}$$

where ρ_n is

$$\frac{1}{2}\{-1+\sqrt{1+\frac{G_0\mu^\epsilon}{\pi\epsilon}(1-n)}\}. \tag{3.11}$$

From this, the gravitational anomalous dimension is evaluated such as

$$\gamma_n = \mu\frac{\partial}{\partial\mu}\log Z_n = -\epsilon\frac{8\pi}{G_0\mu^\epsilon}\rho_n^2. \tag{3.12}$$

We relate the gravitational scaling dimensions Δ_n of the operator $\int\sqrt{g}R^n\mathrm{d}^2x$ with the gravitational anomalous dimension through

$$1-\Delta_n = \frac{2(1-n)+\gamma_n}{2+\gamma_0}, \tag{3.13}$$

where γ_0 is the gravitational anomalous dimension of the cosmological term. This relation is derived through the following scaling argument together with the renormalization group analysis. Let us consider the one-point function

$$\langle Z_n\int\sqrt{g}R^n\mathrm{d}^2x\rangle|_\mu. \tag{3.14}$$

Here we scale the background metric $\hat{g}_{\mu\nu}\to\lambda\hat{g}_{\mu\nu}$. The one-point function change as

$$\lambda^{1-n}\langle Z_n\int\sqrt{g}R^n\mathrm{d}^2x\rangle|_{\mu\lambda^{\frac{1}{2}}}, \tag{3.15}$$

where the renormalization scale grows $\mu\lambda^{\frac{1}{2}}$. By the renormalization group this is equal to

$$\lambda^{1-n+\frac{\gamma_n}{2}}\langle Z_n\int\sqrt{g}R^n\mathrm{d}^2x\rangle|_\mu, \tag{3.16}$$

where the running of the coupling constant can be ignored as long as $G \gg \epsilon$. If we choose the cosmological term as the standard scale, we obtain the relation (3.13). From (3.12) and (3.13), we can evaluate the scaling dimension of $\int\sqrt{g}R^n\mathrm{d}^2x$ as

$$\Delta_n = \frac{\sqrt{1-c+24n}-\sqrt{1-c+24\Delta_0^{(0)}}}{\sqrt{25-c}-\sqrt{1-c+24\Delta_0^{(0)}}}, \tag{3.17}$$

where $\Delta_0^{(0)}$ is the half of the canonical dimension of the cosmological term. Since $c = 0$ and $\Delta^{(0)} = 0$ for pure gravity, we obtain

$$\Delta_n = \frac{\sqrt{1+24n}-1}{4}. \tag{3.18}$$

By the way, the scaling dimension of the operator $\int\sqrt{g}^{1-\Delta_0}\Phi_{\Delta_0}\mathrm{d}^2x$ is obtained obviously by replacing n in (3.17) with Δ_0. The result agrees with that of ref. [15], as is mentioned in section 2.2.

For $n = 1$, since the $O(\hat{h})$ contribution to $\int\sqrt{g}R\mathrm{d}^2x$ is a total derivative, we have to look at the $O(\hat{h}^2)$ contributions instead of the $O(\hat{h})$ contributions. In this case, however, the exponent of $\mathrm{e}^{-\frac{4}{\epsilon}(-\frac{D}{2}+n)\log(1+\frac{\epsilon}{4}\psi)}$ in (3.8) gets an extra $O(\epsilon)$ factor and therefore we do not have any $\frac{1}{\epsilon}$ divergence, which means the scaling dimension is unity and the expression (3.18) holds for $n = 1$ as well.

Let us extend the above result to two-dimensional quantum gravity coupled to (p, q) minimal conformal matter. Recall that p and q are coprime integers and satisfy $p < q$. The central charge of the (p, q) minimal model is

$$c = 1 - \frac{6(p-q)^2}{pq}, \tag{3.19}$$

and the conformal weight of the (r, s) primary field $\Phi_{r,s}$ is given by the Kac table as

$$h_{r,s} = \frac{(qr-ps)^2-(p-q)^2}{4pq}, \tag{3.20}$$

where r and s are positive integers which satisfy

$$ps < qr, \qquad r < p, \quad \text{and} \quad s < q. \tag{3.21}$$

$\Phi_{1,1}$ corresponds to the identity operator, whose conformal weight is 0.

Since $\sqrt{g}^{1-h_{r,s}}\Phi_{r,s}$ is a scalar density, we can define a set of manifestly generally covariant operators by

$$\int\sqrt{g}^{1-h_{r,s}}\Phi_{r,s}R^n\mathrm{d}^2x \qquad (n = 0, 1, 2, \cdots). \tag{3.22}$$

The cosmological term, which we take as a standard scale to define the scaling dimensions, is identified, as in Liouville theory, with the operator

$$\int \sqrt{g}^{1-h_{\rm min}}\Phi_{\rm min}\mathrm{d}^2x, \tag{3.23}$$

where $\Phi_{\rm min}$ is the primary field with the least conformal weight $h_{\rm min}$ given by

$$h_{\rm min} = \frac{1-(p-q)^2}{4pq}. \tag{3.24}$$

For unitary models ($q = p+1$), $h_{\rm min} = 0$ and $\Phi_{\rm min} = \Phi_{1,1}$ (the identity operator), and therefore (3.23) reduces to the naive cosmological term $\int\sqrt{g}\mathrm{d}^2x$. The scaling dimension $\Delta^{\rm MGC}_{r,s;n}$ of the operator $\int\sqrt{g}^{1-h_{r,s}}\Phi_{r,s}R^n\mathrm{d}^2x$ can be obtained by setting $n \to n + h_{r,s}$ and $\Delta_0^{(0)} = h_{\rm min}$ in the expression (3.17), which gives

$$\begin{aligned}\Delta^{\rm MGC}_{r,s;n} &= \frac{\sqrt{1-c+24(h_{r,s}+n)}-\sqrt{1-c+24h_{\rm min}}}{\sqrt{25-c}-\sqrt{1-c+24h_{\rm min}}} \\ &= \frac{\sqrt{(qr-ps)^2+4pqn}-1}{p+q-1},\end{aligned} \tag{3.25}$$

where (3.19), (3.20) and (3.24) are used in the last equality. One can see that the scaling dimension of $\int\sqrt{g}R\mathrm{d}^2x$ is 1, which is to be expected since $\int\sqrt{g}R\mathrm{d}^2x$ is topological in the sense that it is a constant for a fixed topology.

We comment here that there are also such generally covariant operators as $\int\sqrt{g}R\triangle R\mathrm{d}^2x$, which we do not consider in this paper. The only difficulty in dealing with such operators is that the argument made in appendix B does not work in this case. Consequently even the renormalizability of such operators is not obvious. We can say, however, that if they are renormalizable at all, they form eigenvectors with a $\int\sqrt{g}R^n\mathrm{d}^2x$ type operator having the same canonical dimension and have the same scaling dimension as it.

3.3 Interpretation of the Results

We compare the spectrum of the scaling dimensions obtained in the above with that appearing in the matrix model. Let us begin with the case of pure gravity. In the matrix model, we have a set of scaling operators $\mathcal{O}_k$ ($k = 1, 3, 5\cdots$) whose scaling dimension is $\frac{k-1}{4}$ [15]. Our result (3.18) agrees with this scaling dimension when

$$\begin{aligned} n &= \frac{k^2-1}{24} \\ &= \frac{(k+1)(k-1)}{24}. \end{aligned} \tag{3.26}$$

Since k is a positive odd integer, the righthand side of the above expression becomes integer except when $k = 0$ mod 3. Thus we have confirmed that in the case of pure gravity our spectrum includes all the scaling dimensions of the scaling operators in the matrix model except $\mathcal{O}_k$ ($k = 0$ mod 3), which are called the boundary operators due to the fact that $\mathcal{O}_3$ can be interpreted as a 'cosmological term' for the boundary of the surface [25].

Let us next examine the case in which (p,q) minimal conformal matter is coupled. In the matrix model, we have a set of scaling operators $\mathcal{O}_k$ ($k > 0$, $k \neq 0$ mod p) whose scaling dimension is given by [28]

$$\Delta_k^{\rm MM} = \frac{k-1}{p+q-1}. \tag{3.27}$$

We can check explicitly that when

$$n = \begin{cases} pqt^2 + (qr+ps)t + rs \\ pqt^2 + (qr-ps)t \end{cases} \tag{3.28}$$

with $t \in \boldsymbol{Z}$, our result (3.25) reduces to

$$\frac{|2pqt + qr \pm ps| - 1}{p+q-1}, \tag{3.29}$$

which agrees with the spectrum obtained in the BRST analysis of the Liouville theory [24]. Note that the righthand side of (3.28) is a non-negative integer for any $t \in \boldsymbol{Z}$. This means that, just as in pure gravity, our spectrum includes all the scaling dimensions of the scaling operators in the matrix model except the boundary operators $\mathcal{O}_k$ ($k = 0$ mod q).

To illustrate our result, we show, in Tables 1,2 and 3, our spectrum as well as that appearing in the matrix model for three typical cases : pure gravity ($p = 2$, $q = 3$), the $k = 3$ case of Kazakov's k-series ($p = 2$, $q = 5$), and quantum gravity coupled to the critical Ising model ($p = 3$, $q = 4$).

As is mentioned in section 3.1, although the scaling operators in the matrix model form a complete set, in the sense that their correlators satisfy closed recursive relations, their physical picture is not clear except for the ones which can be understood as primary fields with gravitational dressing. Our result might suggest the interesting possibility that the rest of the scaling operators correspond to $\int\sqrt{g}^{1-\Delta_{r,s}}\Phi_{r,s}R^n\mathrm{d}^2x$ ($n = 1, 2, 3, \cdots$) except for the boundary operators. Moreover, one might expect that the indices r and s in the spectrum (3.29) obtained in the BRST analysis are nothing but those of the (r,s) primary field $\Phi_{r,s}$ and that the ghost number $-(2t+1)$ or $2t$ respectively for the plus/minus sign in the expression (3.29) is related to the n of R^n through the expression (3.28), though the correspondence at the correlation function level between the physical operators in the BRST analysis and the scaling operators in the matrix model has not been proved yet for the operators with nonzero ghost number.

One should note, however, that in our spectrum there are also many generically irrational scaling dimensions which do not appear in the matrix model. This may be a clue that the operators considered in this paper, except for the ones with $n=0$, are completely different from those appearing in the matrix model. Indeed one can argue as follows[29]. Take, for example, the operators in pure gravity, $\mathcal{O}_7$ and $\int\sqrt{g}R^2\mathrm{d}^2x$, which have been shown to have the same scaling dimension 3/2. Recently the theory with $\int\sqrt{g}R^2\mathrm{d}^2x$ in the action has been investigated and the partition function is shown to behave as a function of the area as [18]

$$f(A)\sim A^{\gamma_{\rm str}-3}\mathrm{e}^{-\frac{\rm const.}{m^2A}}, \tag{3.30}$$

for $m^2A\ll 1$, where $1/m^2$ is the coefficient of the R^2 term in the action. This formula is derived also in section 4 using the formalism of $(2+\epsilon)$-dimensional quantum gravity [4]. On the other hand, the theory with the action $S=t\mathcal{O}_1+\mathcal{O}_5+x_7\mathcal{O}_7$ in the matrix model gives the string equation

$$t+f^2+x_7f^3=0, \tag{3.31}$$

which means that the area dependence of the partition function for this case gives a power behavior, which is obviously different from that in the R^2 gravity. We also see in section 4 that all of manifestly generally covariant operators including the cosmological term in R^2 gravity has no gravitationally anomalous dimension in $m^2\to 0$ limit. Therefore their scaling dimensions are the same as the canonical ones and do not agree with those of the scaling operators in $k=3$ theory, which is defined in (3.31) in the limit of x_7 dominance. We conjecture, therefore, that $\mathcal{O}_7$ and $\int\sqrt{g}R^2\mathrm{d}^2x$ cannot be identified, in spite of the agreement of the scaling dimensions and that the operators $\int\sqrt{g}^{1-\Delta_{r,s}}\Phi_{r,s}R^n\mathrm{d}^2x$ $(n=1,2,3,\cdots)$ give a new series of operators in two-dimensional quantum gravity. To obtain the definite conclusion, we need information for multi-point functions of the manifestly generally covariant operators, which seem hard to treat in our formalism

scaling operator	scaling dimension	generally covariant operator	scaling dimension
$\mathcal{O}_1$	0	$\int\sqrt{g}\mathrm{d}^2x$	0
$\mathcal{O}_3$	1/2		
$\mathcal{O}_5$	1	$\int\sqrt{g}R\mathrm{d}^2x$	1
$\mathcal{O}_7$	3/2	$\int\sqrt{g}R^2\mathrm{d}^2x$	3/2
		$\int\sqrt{g}R^3\mathrm{d}^2x$	$(\sqrt{73}-1)/4$
		$\int\sqrt{g}R^4\mathrm{d}^2x$	$(\sqrt{97}-1)/4$
$\mathcal{O}_9$	2		
$\mathcal{O}_{11}$	5/2	$\int\sqrt{g}R^5\mathrm{d}^2x$	5/2
		$\int\sqrt{g}R^6\mathrm{d}^2x$	$(\sqrt{145}-1)/4$
$\mathcal{O}_{13}$	3	$\int\sqrt{g}R^7\mathrm{d}^2x$	3
		$\int\sqrt{g}R^8\mathrm{d}^2x$	$(\sqrt{193}-1)/4$
		$\int\sqrt{g}R^9\mathrm{d}^2x$	$(\sqrt{217}-1)/4$
		$\int\sqrt{g}R^{10}\mathrm{d}^2x$	$(\sqrt{241}-1)/4$
		$\int\sqrt{g}R^{11}\mathrm{d}^2x$	$(\sqrt{265}-1)/4$
$\mathcal{O}_{15}$	7/2		
$\mathcal{O}_{17}$	4	$\int\sqrt{g}R^{12}\mathrm{d}^2x$	4
		$\int\sqrt{g}R^{13}\mathrm{d}^2x$	$(\sqrt{313}-1)/4$
		$\int\sqrt{g}R^{14}\mathrm{d}^2x$	$(\sqrt{337}-1)/4$
$\mathcal{O}_{19}$	9/2	$\int\sqrt{g}R^{15}\mathrm{d}^2x$	9/2
$\vdots$	$\vdots$	$\vdots$	$\vdots$

Table 1: Comparison of the scaling dimensions in pure gravity.

scaling operator	scaling dimension	generally covariant operator	scaling dimension
$\mathcal{O}_1$	0	$\int \sqrt{g}^{1-h_{1,2}}\Phi_{1,2}\mathrm{d}^2x$	0
$\mathcal{O}_3$	1/3	$\int \sqrt{g}\mathrm{d}^2x$	1/3
$\mathcal{O}_5$	2/3		
		$\int \sqrt{g}^{1-h_{1,2}}\Phi_{1,2}R\mathrm{d}^2x$	$(\sqrt{41}-1)/6$
$\mathcal{O}_7$	1	$\int \sqrt{g}R\mathrm{d}^2x$	1
$\mathcal{O}_9$	4/3	$\int \sqrt{g}^{1-h_{1,2}}\Phi_{1,2}R^2\mathrm{d}^2x$	4/3
		$\int \sqrt{g}R^2\mathrm{d}^2x$	$(\sqrt{89}-1)/6$
$\mathcal{O}_{11}$	5/3	$\int \sqrt{g}^{1-h_{1,2}}\Phi_{1,2}R^3\mathrm{d}^2x$	5/3
		$\int \sqrt{g}R^3\mathrm{d}^2x$	$(\sqrt{129}-1)/6$
		$\int \sqrt{g}^{1-h_{1,2}}\Phi_{1,2}R^4\mathrm{d}^2x$	$(\sqrt{161}-1)/6$
$\mathcal{O}_{13}$	2	$\int \sqrt{g}R^4\mathrm{d}^2x$	2
		$\int \sqrt{g}^{1-h_{1,2}}\Phi_{1,2}R^5\mathrm{d}^2x$	$(\sqrt{201}-1)/6$
		$\int \sqrt{g}R^5\mathrm{d}^2x$	$(\sqrt{209}-1)/6$
$\mathcal{O}_{15}$	7/3		
		$\int \sqrt{g}^{1-h_{1,2}}\Phi_{1,2}R^6\mathrm{d}^2x$	$(\sqrt{241}-1)/6$
		$\int \sqrt{g}R^6\mathrm{d}^2x$	$(\sqrt{249}-1)/6$
		$\int \sqrt{g}^{1-h_{1,2}}\Phi_{1,2}R^7\mathrm{d}^2x$	$(\sqrt{281}-1)/6$
$\mathcal{O}_{17}$	8/3	$\int \sqrt{g}R^7\mathrm{d}^2x$	8/3
		$\int \sqrt{g}^{1-h_{1,2}}\Phi_{1,2}R^8\mathrm{d}^2x$	$(\sqrt{321}-1)/6$
		$\int \sqrt{g}R^8\mathrm{d}^2x$	$(\sqrt{329}-1)/6$
$\mathcal{O}_{19}$	3	$\int \sqrt{g}^{1-h_{1,2}}\Phi_{1,2}R^9\mathrm{d}^2x$	3
$\vdots$	$\vdots$	$\vdots$	$\vdots$

Table 2: Comparison of the scaling dimensions in the $k = 3$ case of Kazakov's k-series ($p = 2$, $q = 5$). Note that the $(2, 5)$ minimal model has two primary fields, namely the identity operator and $\Phi_{1,2}$ which has a negative conformal weight ($h_{1,2} = -\frac{1}{5}$).

scaling operator	scaling dimension	generally covariant operator	scaling dimension
$\mathcal{O}_1$	0	$\int \sqrt{g}\mathrm{d}^2x$	0
$\mathcal{O}_2$	1/6	$\int \sqrt{g}^{1-h_{2,2}}\Phi_{2,2}\mathrm{d}^2x$	1/6
$\mathcal{O}_4$	1/2		
$\mathcal{O}_5$	2/3	$\int \sqrt{g}^{1-h_{2,1}}\Phi_{2,1}\mathrm{d}^2x$	2/3
$\mathcal{O}_7$	1	$\int \sqrt{g}R\mathrm{d}^2x$	1
		$\int \sqrt{g}^{1-h_{2,2}}\Phi_{2,2}R\mathrm{d}^2x$	$(2\sqrt{13}-1)/6$
$\mathcal{O}_8$	7/6		
		$\int \sqrt{g}^{1-h_{2,1}}\Phi_{2,1}R\mathrm{d}^2x$	$(\sqrt{73}-1)/6$
		$\int \sqrt{g}R^2\mathrm{d}^2x$	$(\sqrt{97}-1)/6$
$\mathcal{O}_{10}$	3/2	$\int \sqrt{g}^{1-h_{2,2}}\Phi_{2,2}R^2\mathrm{d}^2x$	3/2
$\mathcal{O}_{11}$	5/3	$\int \sqrt{g}^{1-h_{2,1}}\Phi_{2,1}R^2\mathrm{d}^2x$	5/3
		$\int \sqrt{g}R^3\mathrm{d}^2x$	$(\sqrt{145}-1)/6$
		$\int \sqrt{g}^{1-h_{2,2}}\Phi_{2,2}R^3\mathrm{d}^2x$	$(2\sqrt{37}-1)/6$
$\mathcal{O}_{13}$	2	$\int \sqrt{g}^{1-h_{2,1}}\Phi_{2,1}R^3\mathrm{d}^2x$	2
		$\int \sqrt{g}R^4\mathrm{d}^2x$	$(\sqrt{193}-1)/6$
$\mathcal{O}_{14}$	13/6	$\int \sqrt{g}^{1-h_{2,2}}\Phi_{2,2}R^4\mathrm{d}^2x$	13/6
$\vdots$	$\vdots$	$\vdots$	$\vdots$

Table 3: Comparison of the scaling dimensions in two-dimensional quantum gravity coupled to the critical Ising model ($p = 3$, $q = 4$). Note that the $(3, 4)$ minimal model has three primary fields, namely the identity operator, the energy density operator $\Phi_{2,1}$ ($h_{2,1} = \frac{1}{2}$) and the local spin operator $\Phi_{2,2}$ ($h_{2,2} = \frac{1}{16}$).

4 R^2 Gravity in $(2+\epsilon)$–Dimensional Quantum Gravity

In this section, we treat R^2 gravity in $(2+\epsilon)$-dimensional quantum gravity. In section 4.1, we calculate one-loop counter terms and see that the oversubtraction for the conformal mode occurs as in the ordinary cases. We present a prescription similar to the one in section 2.2. In the next subsection, taking $\epsilon \to 0$ limit in the strong coupling regime, we calculate the string susceptibility of the system and compare the result with that obtained through Liouville approach. This section is based on the ref. [4].

4.1 Calculation of One–loop Counter Terms

We define a $(2+\epsilon)$-dimensional system corresponding to R^2 gravity by the following action,

$$\begin{aligned} S = \frac{\mu^\epsilon}{G}\int \mathrm{d}^D x\sqrt{g}R \; &+ \; \frac{\mu^\epsilon}{4m^2}\int \mathrm{d}^D x\sqrt{g}R^2 + \Lambda\mu^\epsilon\int \mathrm{d}^D x\sqrt{g} \\ &+ \; \mu^\epsilon\sum_{i=1}^{c}\int \mathrm{d}^D x\sqrt{g}\frac{1}{2}g^{\mu\nu}\partial_\mu\varphi_i\partial_\nu\varphi_i, \end{aligned} \tag{4.1}$$

where G is the gravitational constant, Λ is the cosmological constant, and φ_i is the matter field. m is a parameter with mass dimension, which corresponds to the inverse of the range controlled by the R^2 term. Since the above action contains higher derivatives, which is difficult to deal with, we introduce an auxiliary field χ and replace the R^2 term with

$$\mu^\epsilon\int \mathrm{d}^D x\sqrt{g}(-iR\chi + m^2\chi^2). \tag{4.2}$$

In appendix D, we calculate the one-loop counter terms for the generalized action

$$S = \mu^\epsilon\int \mathrm{d}^D x\sqrt{g}\left(\frac{1}{2}K(\chi)g^{\mu\nu}\partial_\mu\chi\partial_\nu\chi + L(\chi)R + M(\chi)\right) + (\text{matters}), \tag{4.3}$$

which reduces to the action considered by setting $K(\chi) = 0$, $L(\chi) = \frac{1}{G} - i\chi$ and $M(\chi) = \Lambda + m^2\chi^2$. As can be seen in (D.21) and (D.22), the counter term for the χ-kinetic term can be set to 0 by choosing appropriately the function f, which comes from the freedom of gauge fixing (D.15). Note also that the renormalization of $M(\chi)$ is self-contained, which enables us to treat it separately as an inserted operator. Thus the action including the one-loop counter term reads

$$S + S_{\text{c.t.}} = \mu^\epsilon\int \mathrm{d}^D x\sqrt{g}\left(\frac{1}{G} - i\chi - \frac{1}{2\pi\epsilon}\frac{24-c}{12}\right)R + (\text{matters}). \tag{4.4}$$

Special care should be taken for the counter term

$$-\frac{24-c}{24\pi\epsilon}\mu^\epsilon\int \mathrm{d}^D x\sqrt{g}R. \tag{4.5}$$

As in the case of section 2.2, this counter term (4.5) is an oversubtraction for the conformal mode, which forces us to incorporate this counter term in the tree-level action and redo the perturbative expansion with the effective action

$$S_{\text{eff}} = \mu^\epsilon\int \mathrm{d}^D x\sqrt{g}\left(\frac{1}{G} - i\chi - \frac{1}{2\pi\epsilon}\frac{24-c}{12}\right)R + (\text{matters}). \tag{4.6}$$

This amounts to redefining the $L(\chi)$ as $L(\chi) = \frac{1}{G} - i\chi - \frac{24-c}{24\pi\epsilon}$.

We show in the following that one can obtain results consistent with ref. [18] in the $\epsilon \to 0$ limit in the strong coupling regime, *i.e.* $G \gg \epsilon$. This is to be expected, since in the infrared limit R^2 gravity reduces to ordinary gravity without R^2 term, which was reproduced in ref. [2, 3] also in the strong coupling regime.

4.2 Derivation of String Susceptibility

Let us consider the renormalization of the operators $\int \mathrm{d}^D x\sqrt{g}$ and $\int \mathrm{d}^D x\sqrt{g}\chi^2$. We first show, up to two-loop level, that the divergent parts coming from the diagrams with $h_{\mu\nu}$ line cancel as a whole and therefore do not contribute to the renormalization of the operators considered.

In order to diagonalize the kinetic terms in the action after gauge fixing, we introduce the new quantum fields Φ, X and $\Omega^\mu{}_\nu$ through

$$\begin{aligned} \phi &= \tilde{F}\Phi + \frac{2L'}{\epsilon L}\tilde{I}X \\ \chi &= \tilde{I}X \\ h^\mu{}_\nu &= L^{-1/2}\Omega^\mu{}_\nu, \end{aligned} \tag{4.7}$$

where $\tilde{I}$ and $\tilde{F}$ are given through

$$\begin{aligned} \frac{1}{\tilde{I}^2} &= \frac{L'^2}{L}\left(1 + \frac{D}{\epsilon}\right) \\ \tilde{F}^2 &= -\frac{4}{\epsilon DL}. \end{aligned}$$

After this field redefinition, the kinetic term reduces to the following standard form

$$\int \mathrm{d}^D x\sqrt{\hat{g}}\left\{\frac{1}{4}\Omega^\mu{}_{\nu,\rho}\Omega^\nu{}_{\mu,}{}^\rho + \frac{1}{2}\hat{g}^{\mu\nu}\partial_\mu\Phi\partial_\nu\Phi + \frac{1}{2}\hat{g}^{\mu\nu}\partial_\mu X\partial_\nu X\right\}. \tag{4.8}$$

The interaction vertices including $\Omega_{\mu\nu}$ are

$$
\begin{aligned}
\int \mathrm{d}^D x\sqrt{\hat{g}}\Big\{&\frac{\epsilon}{4}(D-1)L^{1/2}\tilde{F}^2\Omega^{\mu\nu}\partial_\mu\Phi\partial_\nu\Phi - \frac{1}{\epsilon}(D-1)L^{-3/2}L'^2\tilde{I}^2\Omega^{\mu\nu}\partial_\mu X\partial_\nu X \\
&-\frac{\epsilon}{8}\tilde{F}^2\Omega^{\mu\rho}\Omega_\rho{}^\nu\partial_\mu\Phi\partial_\nu\Phi + \frac{1}{2\epsilon}(D-1)L^{-2}L'^2\tilde{I}^2\Omega^{\mu\rho}\Omega_\rho{}^\nu\partial_\mu X\partial_\nu X + \cdots\Big\}
\end{aligned}
\tag{4.9}
$$

The operators can be written in terms of the new quantum fields as

$$
\begin{aligned}
\int \mathrm{d}^D x\sqrt{g} &= \int \mathrm{d}^D x\sqrt{\hat{g}}\mathrm{e}^{-\frac{D}{2}(\tilde{F}\Phi+\frac{2L'}{\epsilon L}\tilde{I}X)} \\
\int \mathrm{d}^D x\sqrt{g}\chi^2 &= \int \mathrm{d}^D x\sqrt{\hat{g}}\mathrm{e}^{-\frac{D}{2}(\tilde{F}\Phi+\frac{2L'}{\epsilon L}\tilde{I}X)}(\hat{\chi}+\tilde{I}X)^2.
\end{aligned}
$$

The Fig. 3 shows the list of the diagrams with $\Omega_{\mu\nu}$ line we have to consider when we evaluate the one-point functions of the above operators up to two-loop level. (a) and (b) correspond to $\langle\Phi^2\rangle$, while (c) and (d) correspond to $\langle X^2\rangle$. Although each diagram has $O(\frac{1}{\epsilon})$ divergence (Note that $L\sim O(\frac{1}{\epsilon})$.), an explicit calculation shows that the divergent parts of (a) and (b), as well as (c) and (d), cancel each other. One can also check that the contribution of the ghosts and the matters is finite, due to the suppression factors of ϵ and $L^{-1}\sim O(\epsilon)$ in the action. Thus we have shown that the diagrams containing $h_{\mu\nu}$, ghosts or matters do not affect the renormalization of the operators at least up to two-loop level. We expect that this holds true to all orders of the loop expansion and that two-dimensional R^2 gravity is completely governed by the dynamics of the conformal mode ϕ and the auxiliary field χ.

Dropping the $h_{\mu\nu}$ field, the ghosts and the matters, the effective action reads

$$
\begin{aligned}
&\int \mathrm{d}^D x\sqrt{g}\left(\frac{1}{G}-i\chi-\frac{24-c}{24\pi\epsilon}\right)R \\
\sim\ &\int \mathrm{d}^D x\sqrt{\hat{g}}\Big\{-i\mathrm{e}^{-\frac{\epsilon}{2}\phi}\left(\hat{R}-(D-1)\hat{g}^{\mu\nu}\nabla_\mu\partial_\nu\phi+\frac{1}{4}\epsilon(D-1)\hat{g}^{\mu\nu}\partial_\mu\phi\partial_\nu\phi\right)(\chi+\hat{\chi}) \\
&\qquad+\left(\frac{1}{G}-\frac{24-c}{24\pi\epsilon}\right)\mathrm{e}^{-\frac{\epsilon}{2}\phi}\left(\hat{R}-\frac{1}{4}\epsilon(D-1)\hat{g}^{\mu\nu}\partial_\mu\phi\partial_\nu\phi\right)\Big\}.
\end{aligned}
\tag{4.10}
$$

Introducing new variables ψ and ξ through

$$
\begin{aligned}
\mathrm{e}^{-\frac{\epsilon}{4}\phi}\chi &= \xi \\
\mathrm{e}^{-\frac{\epsilon}{4}\phi} &= 1+\frac{\epsilon}{4}\psi,
\end{aligned}
\tag{4.11}
$$

the terms relevant to the renormalization of the operators considered are

$$
\sim\int \mathrm{d}^D x\sqrt{g}\left\{i(D-1)\hat{g}^{\mu\nu}\partial_\mu\psi\partial_\nu\xi+\frac{24-c}{96\pi}(D-1)\hat{g}^{\mu\nu}\partial_\mu\psi\partial_\nu\psi\right\},
\tag{4.12}
$$

which means that the problem is reduced to a free field theory with the propagators

$$
\begin{aligned}
\langle\psi(p)\psi(-p)\rangle &= 0 \\
\langle\psi(p)\xi(-p)\rangle &= \frac{-i}{D-1}\frac{1}{p^2} \\
\langle\xi(p)\xi(-p)\rangle &= \frac{24-c}{48\pi}\frac{1}{D-1}\frac{1}{p^2}.
\end{aligned}
\tag{4.13}
$$

Let us evaluate the divergence of the one-point functions of the operators. As for the cosmological term, one gets

$$
\begin{aligned}
\left\langle\int \mathrm{d}^D x\sqrt{g}\right\rangle &= \int \mathrm{d}^D x\sqrt{\hat{g}}\left\langle\mathrm{e}^{\frac{2D}{\epsilon}\log(1+\frac{\epsilon}{4}\psi)}\right\rangle \\
&= \int \mathrm{d}^D x\sqrt{\hat{g}},
\end{aligned}
$$

due to $\langle\psi(p)\psi(-p)\rangle = 0$. Thus one finds that the cosmological term is not renormalized. As for the mass term, one gets

$$
\begin{aligned}
&\left\langle\int \mathrm{d}^D x\sqrt{g}\chi^2\right\rangle \\
=\ &\int \mathrm{d}^D x\sqrt{\hat{g}}\left\langle\left(\hat{\chi}^2+2\hat{\chi}\chi+\chi^2\right)\mathrm{e}^{-\frac{D}{2}\phi}\right\rangle \\
=\ &\int \mathrm{d}^D x\sqrt{\hat{g}}\left\{\hat{\chi}^2+2i\frac{1}{2\pi\epsilon}\hat{\chi}-\frac{18-c}{48\pi}\frac{1}{2\pi\epsilon}-\left(\frac{1}{2\pi\epsilon}\right)^2\right\} \\
=\ &\int \mathrm{d}^D x\sqrt{\hat{g}}\left\{\left(\hat{\chi}+\frac{i}{2\pi\epsilon}\right)^2-\frac{18-c}{48\pi}\frac{1}{2\pi\epsilon}\right\}.
\end{aligned}
$$

Strictly speaking, one should have taken care of the $O(1)$ contributions to the term proportional to $\hat{\chi}$ in the last step of the equality. One can check, however, that starting from the action with χ-linear term and adopting the minimal subtraction scheme is equivalent to the above manipulation.

The bare operators, therefore, can be written as

$$
\begin{aligned}
&{m_0}^2\int \mathrm{d}^D x\sqrt{g}{\chi_0}^2+\Lambda_0\int \mathrm{d}^D x\sqrt{g} \\
=\ &m^2\mu^\epsilon\int \mathrm{d}^D x\sqrt{g}\left\{\left(\chi-i\frac{1}{2\pi\epsilon}\right)^2+\frac{18-c}{48\pi}\frac{1}{2\pi\epsilon}\right\}+\Lambda\mu^\epsilon\int \mathrm{d}^D x\sqrt{g},
\end{aligned}
\tag{4.14}
$$

from which one can read off the relations between the bare parameters and the renormalized ones as

$$
\begin{aligned}
\chi_0 &= \chi-i\frac{1}{2\pi\epsilon} \\
{m_0}^2 &= m^2\mu^\epsilon \\
\Lambda_0 &= \Lambda\mu^\epsilon+\frac{18-c}{48\pi}\frac{1}{2\pi\epsilon}m^2\mu^\epsilon.
\end{aligned}
\tag{4.15}
$$

Using the above relations, one can evaluate the area dependence of the partition function in the $\epsilon \to 0$ limit as follows [3].

$$
\begin{aligned}
Z(A) = \int \mathcal{D}g_{\mu\nu}\mathcal{D}\chi_0 \exp\left[-\mu^\epsilon \int \mathrm{d}^D x\sqrt{g}\left(\frac{1}{G} - i\chi - \frac{24-c}{24\pi\epsilon}\right) R - {m_0}^2 \int \mathrm{d}^D x\sqrt{g}{\chi_0}^2 - \Lambda_0 \int \mathrm{d}^D x\sqrt{g}\right] \\
\cdot\, \delta\left(\mu^\epsilon \int \mathrm{d}^D x\sqrt{g}\bigg|_\mu - A\right).
\end{aligned}
$$

Rescaling the metric as $g_{\mu\nu} \to \lambda g_{\mu\nu}$,

$$
\begin{aligned}
Z(A) &= \int \mathcal{D}g_{\mu\nu}\mathcal{D}\chi_0 \exp\left[-\lambda^{\epsilon/2}\mu^\epsilon \int \mathrm{d}^D x\sqrt{g}\left(\frac{1}{G} - i\chi_0 - \frac{12-c}{24\pi\epsilon}\right) R - \lambda^{D/2} m^2 \mu^\epsilon \int \mathrm{d}^D x\sqrt{g}{\chi_0}^2\right. \\
&\qquad \left. - \lambda^{D/2}\left(\Lambda\mu^\epsilon + \frac{18-c}{48\pi}\frac{1}{2\pi\epsilon}m^2\mu^\epsilon\right)\int \mathrm{d}^D x\sqrt{g}\right] \cdot \delta\left(\lambda^{D/2}\,\mu^\epsilon \int \mathrm{d}^D x\sqrt{g}\bigg|_{\lambda^{1/2}\mu} - A\right) \\
&= \int \mathcal{D}g_{\mu\nu} \exp\left[\frac{\epsilon}{2}\log\lambda\frac{12-c}{24\pi\epsilon}\mu^\epsilon \int \mathrm{d}^D x\sqrt{g}R - \frac{\epsilon}{2}\log\lambda\frac{18-c}{48\pi}\frac{1}{2\pi\epsilon}m^2\lambda\mu^\epsilon \int \mathrm{d}^D x\sqrt{g}\right] \\
&\qquad \cdot \exp\left[-\mu^\epsilon \int \mathrm{d}^D x\sqrt{g}\left(\frac{1}{G}R - \frac{12-c}{24\pi\epsilon}R + \frac{1}{4m^2\lambda}R^2\right) - \Lambda_0\lambda \int \mathrm{d}^D x\sqrt{g}\right] \\
&\qquad \cdot\, \delta\left(\lambda^{D/2}\,\mu^\epsilon \int \mathrm{d}^D x\sqrt{g}\bigg|_\mu - A\right).
\end{aligned}
$$

Setting $\lambda = A$,

$$
\begin{aligned}
Z(A) &= A^{\gamma_{\rm str}-3}\mathrm{e}^{-\Lambda_0 A} \int \mathcal{D}g_{\mu\nu} \exp\left[-\mu^\epsilon \int \mathrm{d}^D x\sqrt{g}\left(\frac{1}{G} - \frac{12-c}{24\pi\epsilon}R + \frac{1}{4m^2 A}R^2\right)\right] \\
&\qquad \cdot\, \delta\left(\mu^\epsilon \int \mathrm{d}^D x\sqrt{g}\bigg|_\mu - 1\right),
\end{aligned}
$$

where $\gamma_{\rm str}$ is the string susceptibility given by

$$
\gamma_{\rm str} = 2 + \frac{c-12}{6}(1-h) + \frac{c-18}{192\pi^2}m^2 A. \tag{4.16}
$$

Here h is the number of handle on the surface. For $m^2 A \ll 1$, the classical solution dominates in the path integral. One gets, after taking account of the fluctuation around the classical solution, the area dependence of the partition function as,

$$
Z(A) \sim A^{\gamma_{\rm str}-3}\mathrm{e}^{-\Lambda A} \exp\left(-\frac{16\pi^2(1-h)^2}{m^2 A}\right). \tag{4.17}
$$

One should note here that our m^2 corresponds to 8π times the m^2 of ref. [18]. Comparing our result with that of ref. [18], the only discrepancy is the c-independent coefficient of $m^2 A$ in eq.(4.16), which is subtraction scheme dependent. We can, therefore, conclude that the two results are consistent. In our calculation, the peculiar $(c-12)$ factor comes from the shift of the χ-field and the A-dependent term comes from the fact that the χ^2 operator generates a cosmological term after renormalization.

Finally, we comment on the scaling dimensions of manifestly generally covariant operators in R^2 gravity. From the first equation of (4.14), one can easily see that these operators are not renormalized in all order. Therefore, their gravitational scaling dimensions are the same as the canonical dimensions Δ_0 in the $m^2 \to 0$ limit; they have no gravitational dressings. It is natural because the space-time is expected to be locally flat in this limit.

5 Two–loop Renormalization in Quantum Gravity near Two Dimensions

In this section we perform the two–loop renormalization of the theory developed in section 2.3. We concentrate on the part proportional to the number of matter fields and the ultraviolet fixed point. Section 5.1 is devoted to the explanation of the strategy for our two–loop calculations. In section 5.2 we present the results for two–loop divergences and verify that nonlocal divergences cancel out among the diagrams as well as infrared divergences mixed with ultraviolet divergences and only the local divergences remain at last. In section 5.3, we see that the theory is renormalized multiplicatively. Conformal invariance is established at the ultraviolet fixed point. Hence the general covariance is also guaranteed. Our result is that the theory is constructed consistently at least in this part up to two–loop level. This section is based on the work [7].

5.1 Calculation of Two–loop Counter Terms

As is seen in section 2.3, the one–loop bare action of the theory we consider is written as

$$\frac{\mu^{\epsilon}}{G}\int d^D x\sqrt{\hat{g}}\{\tilde{R}L(X)+\frac{1}{2}\eta^{ij}\tilde{g}^{\mu\nu}\partial_\mu X_i\partial_\nu X_j-\frac{AG}{\epsilon}\tilde{R}\}, \tag{5.1}$$

where $X_i=(\psi,\varphi_i)$ and $\eta^{ij}=\mathrm{diag}(-1,1,\cdots,1)$. Paying our attention to the ultraviolet fixed point, we set

$$L(X)=1-\frac{1}{2}\epsilon b\eta^{ij}X_iX_j, \tag{5.2}$$

where we have replaced b of section 2.3 with $\epsilon b/2$ to keep ϵ factor explicitly. The Z_2 symmetry of the fixed point action is preserved also in two–loop calculations. As the first step to the complete two–loop renormalization, we evaluate only the counter terms proportional to the number of matter fields in this paper.

As in the one–loop calculation, we expand the fields around the backgrounds as $\tilde{g}_{\mu\nu}=\hat{g}_{\mu\rho}(e^h)^\rho{}_\nu$, $X_i\to\hat{X}_i+X_i$ and employ a background gauge. We adopt the same gauge fixing term (2.22) as the one–loop case, which is not renormalized at one–loop level. The ghost action (2.23) is also used in this case. In two–loop calculations, we must in general expand the action up to the fourth order of the quantum fields ($h_{\mu\nu}$, X_i and ghosts). However we

need only the three and four point vertices which are quadratic with respect to the X_i fields, since we compute the counter terms proportional to the number of matter fields.

We expand the one–loop bare action (5.1), the gauge fixing term (2.22) and the ghost action (2.23) around the background fields in a sufficient order as is explained in the above. Here we exploit the formula

$$\begin{aligned}\tilde{R} \;=\;& \hat{R}-h^\mu{}_\nu\hat{R}^\nu{}_\mu-\hat{\nabla}_\mu\hat{\nabla}_\nu h^{\mu\nu}+\frac{1}{4}\hat{\nabla}_\rho h^\mu{}_\nu\hat{\nabla}^\rho h^\nu{}_\mu\\ &+\frac{1}{2}\hat{R}^\sigma_{\mu\nu\rho}h^\rho{}_\sigma h^{\mu\nu}-\frac{1}{2}\hat{\nabla}_\nu h^\nu{}_\mu\hat{\nabla}_\rho h^{\rho\mu}+\hat{\nabla}_\mu(h^\mu{}_\nu\hat{\nabla}^\rho h^\nu{}_\rho)+O(h^3).\end{aligned} \tag{5.3}$$

The background metric is expanded around the flat one as

$$\hat{g}_{\mu\nu}=\delta_{\mu\nu}+\hat{h}_{\mu\nu}, \tag{5.4}$$

where $\hat{h}_{\mu\mu}=0$ can be assumed, for simplicity, without any loss of generality, and the propagators of quantum fields is defined on the flat metric. In this way, the term $L(\hat{X})\hat{\nabla}_\rho h^\mu{}_\nu\hat{\nabla}^\rho h^\nu{}_\mu/4$ is obtained as the kinetic term for the $h_{\mu\nu}$ field. To make this kinetic term a canonical one and define a propagator on the flat metric, we introduce a symmetric traceless matrix $H_{\mu\nu}$ which satisfies $H_{\mu\mu}=0$. Namely we express $h_{\mu\nu}$ as

$$h^{\mu\nu}=\frac{1}{\sqrt{L(\hat{X})}}T^{\mu\nu\lambda\rho}H_{\lambda\rho}, \tag{5.5}$$

where $T^{\mu\nu\lambda\rho}$ is defined as

$$T^{\mu\nu\lambda\rho}=\frac{1}{2}(\hat{g}^{\mu\lambda}\hat{g}^{\nu\rho}+\hat{g}^{\mu\rho}\hat{g}^{\nu\lambda}-\frac{2}{D}\hat{g}^{\mu\nu}\hat{g}^{\lambda\rho}). \tag{5.6}$$

After this prescription, we obtain the propagators and the vertices for the $H_{\mu\nu}$, X_i and ghost fields which are required in our calculation. They are as follows.

<u>propagators</u>

$$\langle H_{\mu\nu}(p)H_{\lambda\rho}(-p)\rangle=\frac{1}{p^2}P_{\mu\nu\lambda\rho} \tag{5.7}$$

$$\langle X_i(p)X_j(-p)\rangle=\frac{\eta_{ij}}{p^2} \tag{5.8}$$

$$\langle \eta_\mu(p)\bar{\eta}_\nu(-p)\rangle=\frac{\delta_{\mu\nu}}{p^2} \tag{5.9}$$

Here $P_{\mu\nu\lambda\rho}$ is defined as

$$P_{\mu\nu\lambda\rho}=\delta_{\mu\lambda}\delta_{\nu\rho}+\delta_{\mu\rho}\delta_{\nu\lambda}-\frac{2}{D}\delta_{\mu\nu}\delta_{\lambda\rho}. \tag{5.10}$$

two–point vertices

$$
\begin{aligned}
K_1^{\mu\nu\lambda\rho\alpha\beta}\partial_\alpha H_{\mu\nu}\partial_\beta H_{\lambda\rho} &: \\
K_1^{\mu\nu\lambda\rho\alpha\beta} &= \frac{1}{4}\sqrt{\hat{g}}T^{\mu\nu\lambda\rho}\hat{g}^{\alpha\beta}-\frac{1}{8}P^{\mu\nu\lambda\rho}\delta^{\alpha\beta} \qquad (5.11)\\
K_2^{\mu\nu\lambda\rho\alpha}H_{\mu\nu}\partial_\alpha H_{\lambda\rho} &: \\
K_2^{\mu\nu\lambda\rho\alpha} &= -\frac{i}{4L}\epsilon b\sqrt{\hat{g}}\hat{X}\partial_\beta\hat{X}T^{\mu\nu\lambda\rho}\hat{g}^{\alpha\beta}+i\sqrt{\hat{g}}\hat{g}^{\alpha\beta}T^{\gamma\nu\lambda\rho}\hat{\Gamma}^{\mu}_{\beta\gamma}\\
&\quad -i\frac{1}{L}\epsilon b\sqrt{\hat{g}}\hat{X}\partial_\gamma\hat{X}T_\beta{}^{\gamma\mu\nu}T^{\beta\alpha\lambda\rho} \qquad (5.12)\\
K_3^{\mu\nu\lambda\rho}H_{\mu\nu}H_{\lambda\rho} &: \\
K_3^{\mu\nu\lambda\rho} &= \frac{1}{4L}\epsilon b\sqrt{\hat{g}}\hat{X}\partial_\gamma\hat{X}T^{\mu\nu\alpha\rho}\hat{g}^{\beta\gamma}\hat{\Gamma}^{\lambda}_{\alpha\beta}+\frac{1}{L}\epsilon b\sqrt{\hat{g}}\hat{X}\partial_\delta\hat{X}T_\gamma{}^{\delta\mu\nu}T^{\beta\gamma\alpha\rho}\hat{\Gamma}^{\lambda}_{\alpha\beta}\\
&\quad +\frac{1}{4L}\epsilon b\sqrt{\hat{g}}\hat{X}\partial_\gamma\hat{X}T^{\lambda\rho\alpha\nu}\hat{g}^{\beta\gamma}\hat{\Gamma}^{\mu}_{\alpha\beta}+\frac{1}{L}\epsilon b\sqrt{\hat{g}}\hat{X}\partial_\delta\hat{X}T_\gamma{}^{\delta\lambda\rho}T^{\beta\gamma\alpha\nu}\hat{\Gamma}^{\mu}_{\alpha\beta}\\
&\quad -\sqrt{\hat{g}}T^{\alpha\nu\delta\rho}\hat{g}^{\beta\gamma}\hat{\Gamma}^{\mu}_{\alpha\beta}\hat{\Gamma}^{\lambda}_{\gamma\delta}-\frac{1}{2}\sqrt{\hat{g}}\hat{R}^{\alpha}_{\beta\gamma\delta}T_\alpha{}^{\delta\mu\nu}T^{\beta\gamma\lambda\rho}\\
&\quad -\frac{1}{4L}\sqrt{\hat{g}}\partial_\alpha\hat{X}\partial_\beta\hat{X}T_\gamma{}^{\alpha\mu\nu}T^{\beta\gamma\lambda\rho} \qquad (5.13)\\
K_4^{\mu\nu\alpha}H_{\mu\nu}\partial_\alpha X &: \\
K_4^{\mu\nu\alpha} &= i\frac{1}{\sqrt{L}}\sqrt{\hat{g}}\partial_\beta\hat{X}T^{\alpha\beta\mu\nu} \qquad (5.14)\\
K_5^{\mu\nu}H_{\mu\nu}X &: \\
K_5^{\mu\nu} &= -\frac{1}{\sqrt{L}}\epsilon b\sqrt{\hat{g}}\hat{X}\hat{R}_{\alpha\beta}T^{\alpha\beta\mu\nu} \qquad (5.15)\\
K_6^{\alpha\beta}\partial_\alpha X\partial_\beta X &: \\
K_6^{\alpha\beta} &= \frac{1}{2}(\sqrt{\hat{g}}\hat{g}^{\alpha\beta}-\delta^{\alpha\beta}) \qquad (5.16)\\
K_7X^2 &: \\
K_7 &= \frac{1}{2}\epsilon b\sqrt{\hat{g}}\hat{R} \qquad (5.17)\\
\tilde{K}_1^{\mu\nu\lambda\rho\alpha\beta}\partial_\alpha H_{\mu\nu}\partial_\beta H_{\lambda\rho} &: \\
\tilde{K}_1^{\mu\nu\lambda\rho\alpha\beta} &= -\frac{AG}{\epsilon}\left(\frac{1}{4L}\sqrt{\hat{g}}T^{\mu\nu\lambda\rho}\hat{g}^{\alpha\beta}-\frac{1}{2L}\sqrt{\hat{g}}T_\gamma{}^{\alpha\mu\nu}T^{\gamma\beta\lambda\rho}\right) \qquad (5.18)\\
\tilde{K}_2^{\mu\nu\lambda\rho\alpha}H_{\mu\nu}\partial_\alpha H_{\lambda\rho} &: \\
\tilde{K}_2^{\mu\nu\lambda\rho\alpha} &= -\frac{AG}{\epsilon}\left(-\frac{i}{4L^2}\epsilon b\sqrt{\hat{g}}\hat{X}\partial_\beta\hat{X}T^{\mu\nu\lambda\rho}\hat{g}^{\alpha\beta}+i\frac{1}{L}\sqrt{\hat{g}}T^{\gamma\nu\lambda\rho}\hat{g}^{\alpha\beta}\hat{\Gamma}^{\mu}_{\beta\gamma}\right.\\
&\quad \left.-i\frac{2}{L}\sqrt{\hat{g}}T^{\gamma\beta\delta\nu}T_\beta{}^{\alpha\lambda\rho}\hat{\Gamma}^{\mu}_{\gamma\delta}+\frac{i}{2L^2}\epsilon b\sqrt{\hat{g}}\hat{X}\partial_\beta\hat{X}T_\gamma{}^{\beta\mu\nu}T^{\alpha\gamma\lambda\rho}\right) \qquad (5.19)\\
\tilde{K}_3^{\mu\nu\lambda\rho}H_{\mu\nu}H_{\lambda\rho} &: \\
\tilde{K}_3^{\mu\nu\lambda\rho} &= -\frac{AG}{\epsilon}\left(\frac{1}{4L^2}\epsilon b\sqrt{\hat{g}}\hat{X}\partial_\gamma\hat{X}T^{\mu\nu\alpha\rho}\hat{g}^{\beta\gamma}\hat{\Gamma}^{\lambda}_{\alpha\beta}-\frac{1}{2L^2}\epsilon b\sqrt{\hat{g}}\hat{X}\partial_\delta\hat{X}T_\gamma{}^{\delta\mu\nu}T^{\gamma\beta\alpha\rho}\hat{\Gamma}^{\lambda}_{\alpha\beta}\right.\\
&\quad +\frac{1}{4L^2}\epsilon b\sqrt{\hat{g}}\hat{X}\partial_\gamma\hat{X}T^{\lambda\rho\alpha\nu}\hat{g}^{\beta\gamma}\hat{\Gamma}^{\mu}_{\alpha\beta}-\frac{1}{2L^2}\epsilon b\sqrt{\hat{g}}\hat{X}\partial_\delta\hat{X}T_\gamma{}^{\delta\lambda\rho}T^{\gamma\beta\alpha\nu}\hat{\Gamma}^{\mu}_{\alpha\beta}
\end{aligned}
$$

$$
\begin{aligned}
&\quad -\frac{1}{L}\sqrt{\hat{g}}T^{\alpha\nu\delta\rho}\hat{g}^{\beta\gamma}\hat{\Gamma}^{\mu}_{\alpha\beta}\hat{\Gamma}^{\lambda}_{\gamma\delta}-\frac{1}{2L}\sqrt{\hat{g}}\hat{R}^{\alpha}_{\beta\gamma\delta}T_\alpha{}^{\delta\mu\nu}T^{\beta\gamma\lambda\rho}\\
&\quad \left.+\frac{2}{L}\sqrt{\hat{g}}T_\alpha{}^{\beta\gamma\nu}T^{\delta\alpha\eta\rho}\hat{\Gamma}^{\mu}_{\beta\gamma}\hat{\Gamma}^{\lambda}_{\delta\eta}\right) \qquad (5.20)\\
\bar{K}_1^{\mu\nu\alpha\beta}\partial_\alpha\bar{\eta}_\mu\partial_\beta\eta_\nu &: \\
\bar{K}_1^{\mu\nu\alpha\beta} &= \sqrt{\hat{g}}\hat{g}^{\mu\nu}\hat{g}^{\alpha\beta}-\delta^{\mu\nu}\delta^{\alpha\beta} \qquad (5.21)\\
\bar{K}_2^{\mu\nu\alpha}\partial_\alpha\bar{\eta}_\mu\eta_\nu &: \\
\bar{K}_2^{\mu\nu\alpha} &= i\sqrt{\hat{g}}\hat{g}^{\alpha\beta}\hat{g}^{\mu\gamma}\hat{\Gamma}^{\nu}_{\beta\gamma} \qquad (5.22)\\
\bar{K}_3^{\mu\nu\alpha}\bar{\eta}_\mu\partial_\alpha\eta_\nu &: \\
\bar{K}_3^{\mu\nu\alpha} &= i\sqrt{\hat{g}}\hat{g}^{\alpha\beta}\hat{g}^{\nu\gamma}\hat{\Gamma}^{\mu}_{\beta\gamma} \qquad (5.23)\\
\bar{K}_4^{\mu\nu}\bar{\eta}_\mu\eta_\nu &: \\
\bar{K}_4^{\mu\nu} &= -\sqrt{\hat{g}}\hat{g}^{\alpha\beta}\hat{g}^{\gamma\delta}\hat{\Gamma}^{\mu}_{\alpha\gamma}\hat{\Gamma}^{\nu}_{\beta\delta}-\sqrt{\hat{g}}\hat{R}^{\mu\nu} \qquad (5.24)
\end{aligned}
$$

three–point vertices

$$
\begin{aligned}
V_1^{\mu\nu\alpha\beta}H_{\mu\nu}\partial_\alpha X\partial_\beta X &: \\
V_1^{\mu\nu\alpha\beta} &= -\frac{1}{2\sqrt{L}}\sqrt{\hat{g}}T^{\mu\nu\alpha\beta} \qquad (5.25)\\
V_2^{\mu\nu}H_{\mu\nu}X^2 &: \\
V_2^{\mu\nu} &= -\frac{1}{2\sqrt{L}}\epsilon b\sqrt{\hat{g}}\hat{R}_{\alpha\beta}T^{\mu\nu\alpha\beta} \qquad (5.26)
\end{aligned}
$$

four–point vertices

$$
\begin{aligned}
W_1^{\mu\nu\lambda\rho\alpha\beta}\partial_\alpha H_{\mu\nu}\partial_\beta H_{\lambda\rho}X^2 &: \\
W_1^{\mu\nu\lambda\rho\alpha\beta} &= -\frac{1}{8L}\epsilon b\sqrt{\hat{g}}T^{\mu\nu\lambda\rho}\hat{g}^{\alpha\beta} \qquad (5.27)\\
W_2^{\mu\nu\lambda\rho\alpha\beta}H_{\mu\nu}\partial_\alpha H_{\lambda\rho}X\partial_\beta X &: \\
W_2^{\mu\nu\lambda\rho\alpha\beta} &= \frac{1}{L}\epsilon b\sqrt{\hat{g}}T_\gamma{}^{\beta\mu\nu}T^{\gamma\alpha\lambda\rho} \qquad (5.28)\\
W_3^{\mu\nu\lambda\rho\alpha\beta}H_{\mu\nu}H_{\lambda\rho}\partial_\alpha X\partial_\beta X &: \\
W_3^{\mu\nu\lambda\rho\alpha\beta} &= \frac{1}{8L}\sqrt{\hat{g}}(T_\gamma{}^{\alpha\mu\nu}T^{\gamma\beta\lambda\rho}+T_\gamma{}^{\beta\mu\nu}T^{\gamma\alpha\lambda\rho}) \qquad (5.29)\\
W_4^{\mu\nu\lambda\rho\alpha}H_{\mu\nu}\partial_\alpha H_{\lambda\rho}X^2 &: \\
W_4^{\mu\nu\lambda\rho\alpha} &= -i\frac{1}{2L}\epsilon b\sqrt{\hat{g}}T^{\beta\nu\lambda\rho}\hat{g}^{\alpha\gamma}\hat{\Gamma}^{\mu}_{\beta\gamma} \qquad (5.30)\\
W_5^{\mu\nu\lambda\rho\alpha}H_{\mu\nu}H_{\lambda\rho}X\partial_\alpha X &: \\
W_5^{\mu\nu\lambda\rho\alpha} &= i\frac{1}{L}\epsilon b\sqrt{\hat{g}}T_\delta{}^{\alpha\mu\nu}T^{\delta\beta\gamma\rho}\hat{\Gamma}^{\lambda}_{\beta\gamma}\\
&\quad +i\frac{1}{L}\epsilon b\sqrt{\hat{g}}T_\delta{}^{\alpha\lambda\rho}T^{\delta\beta\gamma\nu}\hat{\Gamma}^{\mu}_{\beta\gamma} \qquad (5.31)
\end{aligned}
$$

$$
\begin{aligned}
W_6^{\mu\nu\lambda\rho} H_{\mu\nu} H_{\lambda\rho} X^2 &: \\
W_6^{\mu\nu\lambda\rho} &= \frac{1}{2L}\epsilon b\sqrt{\hat{g}}T^{\alpha\nu\delta\rho}\hat{g}^{\beta\gamma}\hat{\Gamma}^{\mu}_{\alpha\beta}\hat{\Gamma}^{\lambda}_{\gamma\delta} \\
&\quad + \frac{1}{4L}\epsilon b\sqrt{\hat{g}}\hat{R}^{\alpha}_{\beta\gamma\delta}T_{\alpha}{}^{\delta\mu\nu}T^{\beta\gamma\lambda\rho} \qquad (5.32) \\
\bar{W}_1^{\mu\nu\alpha\beta}\partial_\alpha\bar{\eta}_\mu\eta_\nu X\partial_\beta X &: \\
\bar{W}_1^{\mu\nu\alpha\beta} &= \epsilon b\sqrt{\hat{g}}\hat{g}^{\mu\alpha}\hat{g}^{\nu\beta} \qquad (5.33) \\
\bar{W}_2^{\mu\nu\alpha}\bar{\eta}_\mu\eta_\nu X\partial_\alpha X &: \\
\bar{W}_2^{\mu\nu\alpha} &= -\epsilon b\sqrt{\hat{g}}\hat{g}^{\beta\gamma}\hat{g}^{\nu\alpha}\hat{\Gamma}^{\mu}_{\beta\gamma} \qquad (5.34)
\end{aligned}
$$

In these expressions, L is equal to $1 - \frac{1}{2}\epsilon b\hat{X}_i^2$ and the suffices for the $\hat{X}_i$ are omitted.

To determine the counter terms, we exploit the manifest general covariance with respect to the background metric $\hat{g}_{\mu\nu}$. We keep the appropriate order of $\hat{h}_{\mu\nu}$ in the calculation of the diagrams and read off the general covariant forms from the results.

Our strategy of calculation is as follows. First, we set $\hat{h}_{\mu\nu} = 0$ and evaluate divergences proportional to $\partial_\mu\hat{X}\partial_\mu\hat{X}/2$ to determine the counter term proportional to $\tilde{g}^{\mu\nu}\partial_\mu X\partial_\nu X/2$. The coefficient can, in general, depend on $\hat{X}_i$(or X_i). Next, we calculate the diagrams which are the first order of $\hat{h}_{\mu\nu}$ and subtract from the results the $O(\hat{h})$ contributions coming from the term proportional to $\hat{g}^{\mu\nu}\partial_\mu\hat{X}\partial_\nu\hat{X}/2$ derived in the first step. Exploiting $\sqrt{\hat{g}}\hat{R} = -\partial_\mu\partial_\nu\hat{h}_{\mu\nu} + O(\hat{h}^2)$, we obtain the counter term for the X–dependent part of $\tilde{R}L(X)$. Finally, we compute the diagrams which are the second order of $\hat{h}_{\mu\nu}$ setting $\hat{X} = 0$. By making use of the relation

$$
\int d^Dx\sqrt{\hat{g}}\hat{R} = \int d^Dx(\frac{1}{4}\partial_\mu\hat{h}_{\lambda\rho}\partial_\mu\hat{h}_{\lambda\rho} - \frac{1}{2}\partial_\mu\hat{h}_{\mu\nu}\partial_\rho\hat{h}_{\rho\nu}) + O(\hat{h}^3), \qquad (5.35)
$$

we fix the counter term for $\tilde{R}$.

In order to renormalize the theory up to the two–loop level, we have to make the two–loop divergences local. It is possible to do so only if we subtract the sub-divergences of the one–loop sub-diagrams from the two–loop diagrams properly. The one–loop renormalization of quantum fields shows that the only one–loop counter term proportional to c is $\frac{c}{24\pi\epsilon}\tilde{R}$. This means that the sub-divergences should arise only from the matter sub–loops connected to the quantum $H_{\mu\nu}$ or the background $\hat{h}_{\mu\nu}$-lines. Keeping this point in mind, we can classify all the diagrams into groups within which local divergences are obtained.

There is a subtlety in computing the short distance divergences of the two–loop diagrams:

The subdiagrams containing the $H_{\mu\nu}$ propagators, in general, cause infrared divergences. In order to regularize them, we introduce a mass term in the $H_{\mu\nu}$ propagator; $\frac{1}{p^2} \to \frac{1}{p^2+m^2}$. We take $m \to 0$ limit after extracting $\frac{1}{\epsilon}\log(\frac{m^2}{k^2})$ type divergences. It is seen later that such divergences are canceled out among the diagrams and do not appear in the final results. Therefore the short distance divergences are separated from the infrared divergences and there are no mixed divergences.

5.2 Results for Two–loop Counter Terms

In this subsection, we calculate (the minus signs of) two–loop divergences in the effective action following the strategy described in the previous section and show the results in detail.[1] The two–loop counter terms are readily obtained by performing the replacements, $\hat{X}_i \to X_i$ and $\hat{g}_{\mu\nu} \to \tilde{g}_{\mu\nu}$ in the two–loop divergences.

5.2.1 Divergences for $\hat{g}^{\mu\nu}\partial_\mu\hat{X}_i\partial_\nu\hat{X}^i$

To evaluate the divergences for the kinetic terms of $\hat{X}_i$, we must consider the six diagrams (Fig. 3), where $\hat{h}_{\mu\nu}$ is set to be equal to zero. The divergence of each diagram is obtained as

$$
\frac{cG}{(4\pi)^2}\alpha\int d^Dx\frac{1}{2}\partial_\mu\hat{X}_i\partial_\mu\hat{X}^i, \qquad (5.36)
$$

where α is independent of $\hat{X}_i$ and contains the pole. We summarize the results in Table 4, from which we can see that each of the diagrams gives a local and single–pole divergence, and has no infrared divergence. The final result is written in the covariant form as

$$
\frac{cG}{(4\pi)^2}\left(-\frac{cG}{8\epsilon}\right)\int d^Dx\sqrt{\hat{g}}\hat{g}^{\mu\nu}\frac{1}{2}\partial_\mu\hat{X}_i\partial_\nu\hat{X}^i. \qquad (5.37)
$$

5.2.2 Divergences for $\hat{R}\ \hat{X}_i^2$

We write down all of the diagrams which are the first order of $\hat{h}_{\mu\nu}$ and subtract from them the contributions of $\hat{g}^{\mu\nu}\partial_\mu\hat{X}_i\partial_\nu\hat{X}_i$. Thus we obtain thirty three diagrams, which indeed give the divergences for $\hat{R}\ \hat{X}_i^2$. As all of them include the vertices proportional to $\epsilon\ b$, there is in principle neither nonlocal nor infrared divergence. After a lengthy calculation, we find that

[1] Tremendous amount of tensor calculations involved in this subsection has been performed with the aid of MathTensor. It is our pleasure to acknowledge S. Christensen of MathSolution Inc. for kind advice concerning the usage of this powerful tool.

only the five diagrams provide nontrivial contributions, which are canceled out among them. As a result, we find no divergences for $\hat{R}\,\hat{X}_i^2$.

5.2.3 Divergences for $\hat{R}$

We set $\hat{X}_i = 0$ and evaluate two-point functions of $\hat{h}_{\mu\nu}$. The diagrams we have to calculate are classified into two categories. The one category consists of forty one diagrams, which contain the vertices proportional to ϵb and give no nonlocal or infrared divergences. Among them, there are four diagrams which include ghost loops and each of them is found to give no contribution. As for the remaining thirty seven diagrams which have in general single pole divergences, our calculation shows that such divergences cancel among the diagrams.

The other category is a set of forty three diagrams which possess no overall ϵb factor and are able to give nonlocal and infrared divergences. We classify them into thirteen groups (Fig. 4-Fig.16), such that the contribution from each of them become local.

The Group 1, 2, 3 resemble the others in topology of the diagrams, while they differ in the vertices connected to the external $\hat{h}_{\mu\nu}$ line. The diagrams of the Group 4 share the diagram $4-5$ with the one-loop counter term insertion. Each of the Groups $5-12$ is the combination of two diagrams, where the one-loop counter term insertion cancels the sub-divergence from a matter sub-loop. The two diagrams of the Group 13 have the sub-diagrams, which are two-point functions of matters at one-loop level. As is seen in the one-loop calculation, the divergent contributions from these sub-diagrams cancel each other, which implies that the Group 13 does not need one-loop counter term insertions and gives a local divergence.

The calculations of diagrams such as $3-1$ and so on are performed by the method presented in [30]. We summarize the results for each of the Groups in Tables 5-17. We obtain generally the divergences in the momentum space such as

$$\begin{aligned} \frac{cG}{(4\pi)^2}\int\frac{d^Dk}{(2\pi)^D} \quad \{ & \quad Ak_\mu k_\nu \hat{h}(k)_{\mu\lambda}\hat{h}(-k)_{\nu\lambda} + Bk^2\hat{h}(k)_{\mu\nu}\hat{h}(-k)_{\mu\nu} \\ & + \quad C\frac{1}{k^2}k_\mu k_\nu \hat{h}(k)_{\mu\nu}k_\lambda k_\rho \hat{h}(-k)_{\lambda\rho}\}. \end{aligned} \tag{5.38}$$

The coefficients A, B and C are shown in Tables. Here ρ and σ are defined as follows,

$$\rho = \log(\frac{k^2}{4\pi}),$$
$$\sigma = \log(\frac{m^2}{k^2}).$$

We can see from Tables that in the total of each of the Groups ρ and σ do not appear and C is equal to zero . This means that the nonlocal and infrared divergences in the $\frac{1}{\epsilon}$ poles cancel among the diagrams in each of the Groups. We also collect the total results of the Groups in Table 18 and sum them up in the Total. Remarkably, the double-pole singularity does vanish in the final result although it remains in each Group. From the relation between A and B in the final result and the formula (5.35), we can verify the preservation of the general covariance with respect to the background, which serves as a check of our calculation. The final result for the divergent contribution is found to be

$$\frac{cG}{(4\pi)^2}\frac{5}{24\epsilon}\int d^Dx\sqrt{\hat{g}}\hat{R}. \tag{5.39}$$

Diagram	a	b	c	d	e	f	Total
α	0	$\frac{1}{8\epsilon}$	$\frac{2b}{\epsilon}$	$-\frac{2b}{\epsilon}$	$-\frac{1}{12\epsilon}$	$-\frac{1}{6\epsilon}$	$-\frac{1}{8\epsilon}$

Table 4: The divergences for $\partial_\mu \hat{X}_i \partial_\mu \hat{X}^i$

Diagram	A	B	C
$1-1$	$-\frac{1}{4\epsilon^2}+(\frac{13}{48}-\frac{\gamma}{4}-\frac{\varrho}{4}-\sigma)\frac{1}{\epsilon}$	$\frac{1}{8\epsilon^2}+(-\frac{13}{96}+\frac{\gamma}{8}+\frac{\varrho}{8}+\frac{\sigma}{2})\frac{1}{\epsilon}$	0
$1-2$	$\frac{1}{2\epsilon^2}+(-\frac{4}{3}+\frac{\gamma}{4}+\frac{\varrho}{4})\frac{1}{\epsilon}$	$-\frac{1}{4\epsilon^2}+(\frac{2}{3}-\frac{\gamma}{8}-\frac{\varrho}{8})\frac{1}{\epsilon}$	0
$1-3$	$(\frac{13}{12}+\frac{\sigma}{3})\frac{1}{\epsilon}$	$(-\frac{13}{24}-\frac{\sigma}{6})\frac{1}{\epsilon}$	0
$1-4$	$\frac{\sigma}{2\epsilon}$	$-\frac{\sigma}{4\epsilon}$	0
$1-5$	$-\frac{\sigma}{2\epsilon}$	$\frac{\sigma}{4\epsilon}$	$\frac{\sigma}{3\epsilon}$
$1-6$	$\frac{2\sigma}{3\epsilon}$	$-\frac{\sigma}{3\epsilon}$	$-\frac{\sigma}{3\epsilon}$
Total	$\frac{1}{4\epsilon^2}+\frac{1}{48\epsilon}$	$-\frac{1}{8\epsilon^2}-\frac{1}{96\epsilon}$	0

Table 5: The results for Group 1

Diagram	A	B	C
$2-1$	$-\frac{1}{4\epsilon^2}+(\frac{5}{16}-\frac{\gamma}{4}-\frac{\varrho}{4})\frac{1}{\epsilon}$	$\frac{1}{8\epsilon^2}+(-\frac{7}{32}+\frac{\gamma}{8}+\frac{\varrho}{8})\frac{1}{\epsilon}$	0
$2-2$	$-\frac{5}{6\epsilon^2}+(\frac{5}{12}-\frac{5\gamma}{12}-\frac{5\varrho}{12})\frac{1}{\epsilon}$	$\frac{5}{12\epsilon^2}+(-\frac{1}{12}+\frac{5\gamma}{24}+\frac{5\varrho}{24})\frac{1}{\epsilon}$	0
$2-3$	$\frac{1}{4\epsilon^2}+(-\frac{7}{48}+\frac{\gamma}{4}+\frac{\varrho}{4}+\frac{\sigma}{2})\frac{1}{\epsilon}$	$-\frac{1}{8\epsilon^2}+(\frac{13}{96}-\frac{\gamma}{8}-\frac{\varrho}{8}-\frac{\sigma}{4})\frac{1}{\epsilon}$	0
$2-4$	$-\frac{1}{4\epsilon^2}+(\frac{31}{48}-\frac{\gamma}{4}-\frac{\varrho}{4})\frac{1}{\epsilon}$	$\frac{1}{8\epsilon^2}+(-\frac{37}{96}+\frac{\gamma}{8}+\frac{\varrho}{8})\frac{1}{\epsilon}$	0
$2-5$	0	0	0
$2-6$	$\frac{2}{3\epsilon^2}+(-\frac{1}{6}+\frac{\gamma}{3}+\frac{\varrho}{3})\frac{1}{\epsilon}$	$-\frac{1}{3\epsilon^2}+(\frac{1}{8}-\frac{\gamma}{6}-\frac{\varrho}{6})\frac{1}{\epsilon}$	0
$2-7$	$\frac{2}{3\epsilon^2}+(-\frac{5}{6}+\frac{\gamma}{3}+\frac{\varrho}{3})\frac{1}{\epsilon}$	$-\frac{1}{3\epsilon^2}+(\frac{11}{24}-\frac{\gamma}{6}-\frac{\varrho}{6})\frac{1}{\epsilon}$	0
$2-8$	$-\frac{\sigma}{2\epsilon}$	$\frac{\sigma}{4\epsilon}$	0
Total	$\frac{1}{4\epsilon^2}+\frac{11}{48\epsilon}$	$-\frac{1}{8\epsilon^2}+\frac{1}{32\epsilon}$	0

Table 6: The results for Group 2

Diagram	A	B	C
$3-1$	$\frac{19}{24\epsilon}$	$-\frac{11}{24\epsilon}$	0
$3-2$	$\frac{11}{36\epsilon}$	$-\frac{5}{18\epsilon}$	0
$3-3$	$\frac{1}{2\epsilon^2}+(-\frac{4}{3}+\frac{\gamma}{2}+\frac{\varrho}{2})\frac{1}{\epsilon}$	$-\frac{1}{4\epsilon^2}+(\frac{19}{24}-\frac{\gamma}{4}-\frac{\varrho}{4})\frac{1}{\epsilon}$	0
$3-4$	$-\frac{1}{2\epsilon}$	$\frac{1}{4\epsilon}$	0
$3-5$	$\frac{2}{3\epsilon}$	$-\frac{1}{3\epsilon}$	$\frac{1}{6\epsilon}$
$3-6$	$-\frac{1}{\epsilon^2}+(\frac{1}{3}-\frac{\gamma}{2}-\frac{\varrho}{2})\frac{1}{\epsilon}$	$\frac{1}{2\epsilon^2}+(-\frac{1}{6}+\frac{\gamma}{4}+\frac{\varrho}{4})\frac{1}{\epsilon}$	$-\frac{1}{6\epsilon}$
Total	$-\frac{1}{2\epsilon^2}+\frac{19}{72\epsilon}$	$\frac{1}{4\epsilon^2}-\frac{7}{36\epsilon}$	0

Table 7: The results for Group 3

Diagram	A	B	C
$4-1$	$-\frac{1}{2\epsilon^2}+(\frac{13}{24}-\frac{\gamma}{2}-\frac{\varrho}{2}-\frac{\sigma}{2})\frac{1}{\epsilon}$	$\frac{1}{8\epsilon^2}+(-\frac{19}{48}+\frac{\gamma}{8}+\frac{\varrho}{8}+\frac{\sigma}{8})\frac{1}{\epsilon}$	0
$4-2$	$\frac{1}{4\epsilon^2}+(-\frac{1}{16}+\frac{\gamma}{4}+\frac{\varrho}{4}+\frac{\sigma}{4})\frac{1}{\epsilon}$	$-\frac{1}{4\epsilon^2}+(\frac{1}{32}-\frac{\gamma}{4}-\frac{\varrho}{4}-\frac{\sigma}{4})\frac{1}{\epsilon}$	0
$4-3$	$\frac{1}{3\epsilon^2}+(\frac{11}{36}+\frac{\gamma}{3}+\frac{\varrho}{3}+\frac{\sigma}{6})\frac{1}{\epsilon}$	$-\frac{1}{6\epsilon^2}+(-\frac{11}{72}-\frac{\gamma}{6}-\frac{\varrho}{6}-\frac{\sigma}{12})\frac{1}{\epsilon}$	$-\frac{1}{6\epsilon}$
$4-4$	$-\frac{1}{3\epsilon^2}+(-\frac{13}{72}-\frac{\gamma}{3}-\frac{\varrho}{3}-\frac{\sigma}{6})\frac{1}{\epsilon}$	$\frac{5}{12\epsilon^2}+(\frac{31}{144}+\frac{5\gamma}{12}+\frac{5\varrho}{12}+\frac{\sigma}{3})\frac{1}{\epsilon}$	$\frac{1}{6\epsilon}$
$4-5$	$\frac{1}{2\epsilon^2}+(-\frac{7}{8}+\frac{\gamma}{4}+\frac{\varrho}{4}+\frac{\sigma}{4})\frac{1}{\epsilon}$	$-\frac{1}{4\epsilon^2}+(\frac{19}{48}-\frac{\gamma}{8}-\frac{\varrho}{8}-\frac{\sigma}{8})\frac{1}{\epsilon}$	0
Total	$\frac{1}{4\epsilon^2}-\frac{13}{48\epsilon}$	$-\frac{1}{8\epsilon^2}+\frac{3}{32\epsilon}$	0

Table 8: The results for Group 4

Diagram	A	B	C
$5-1$	$-\frac{1}{8\epsilon^2}+(\frac{13}{32}-\frac{\gamma}{8}-\frac{\varrho}{8})\frac{1}{\epsilon}$	$\frac{1}{16\epsilon^2}+(-\frac{19}{64}+\frac{\gamma}{16}+\frac{\rho}{16})\frac{1}{\epsilon}$	0
$5-2$	$\frac{1}{4\epsilon^2}+(-\frac{17}{36}+\frac{\gamma}{8}+\frac{\varrho}{8})\frac{1}{\epsilon}$	$-\frac{1}{8\epsilon^2}+(\frac{61}{144}-\frac{\gamma}{16}-\frac{\rho}{16})\frac{1}{\epsilon}$	0
Total	$\frac{1}{8\epsilon^2}-\frac{19}{288\epsilon}$	$-\frac{1}{16\epsilon^2}+\frac{73}{576\epsilon}$	0

Table 9: The results for Group 5

Diagram	A	B	C
$6-1$	$(-\frac{13}{192}-\frac{\sigma}{8})\frac{1}{\epsilon}$	$(\frac{7}{384}+\frac{\sigma}{16})\frac{1}{\epsilon}$	0
$6-2$	$(\frac{1}{48}+\frac{\sigma}{8})\frac{1}{\epsilon}$	$-\frac{\sigma}{16}$	0
Total	$-\frac{3}{64\epsilon}$	$\frac{7}{384\epsilon}$	0

Table 10: The results for Group 6

Diagram	A	B	C
$7-1$	$\frac{11}{192\epsilon}$	$-\frac{29}{384\epsilon}$	0
$7-2$	$-\frac{5}{48\epsilon}$	$\frac{1}{8\epsilon}$	0
Total	$-\frac{3}{64\epsilon}$	$\frac{19}{384\epsilon}$	0

Table 11: The results for Group 7

Diagram	A	B	C
$8-1$	$\frac{1}{8\epsilon^2}+(-\frac{13}{48}+\frac{\gamma}{8}+\frac{\varrho}{8})\frac{1}{\epsilon}$	$-\frac{1}{16\epsilon^2}+(\frac{5}{48}-\frac{\gamma}{16}-\frac{\rho}{16})\frac{1}{\epsilon}$	0
$8-2$	$-\frac{1}{4\epsilon^2}+(\frac{5}{24}-\frac{\gamma}{8}-\frac{\varrho}{8})\frac{1}{\epsilon}$	$\frac{1}{8\epsilon^2}+(-\frac{1}{12}+\frac{\gamma}{16}+\frac{\rho}{16})\frac{1}{\epsilon}$	0
Total	$-\frac{1}{8\epsilon^2}-\frac{1}{16\epsilon}$	$\frac{1}{16\epsilon^2}+\frac{1}{48\epsilon}$	0

Table 12: The results for Group 8

Diagram	A	B	C
$9-1$	$\frac{1}{4\epsilon^2}+(-\frac{25}{16}+\frac{\gamma}{4}+\frac{\varrho}{4})\frac{1}{\epsilon}$	$-\frac{1}{8\epsilon^2}+(\frac{25}{32}-\frac{\gamma}{8}-\frac{\varrho}{8})\frac{1}{\epsilon}$	0
$9-2$	$-\frac{1}{2\epsilon^2}+(\frac{5}{3}-\frac{\gamma}{4}-\frac{\varrho}{4})\frac{1}{\epsilon}$	$\frac{1}{4\epsilon^2}+(-\frac{5}{6}+\frac{\gamma}{8}+\frac{\varrho}{8})\frac{1}{\epsilon}$	0
Total	$-\frac{1}{4\epsilon^2}+\frac{5}{48\epsilon}$	$\frac{1}{8\epsilon^2}-\frac{5}{96\epsilon}$	0

Table 13: The results for Group 9

Diagram	A	B	C
$10-1$	$-\frac{1}{8\epsilon^2}+(\frac{13}{32}-\frac{\gamma}{8}-\frac{\varrho}{8})\frac{1}{\epsilon}$	$\frac{1}{16\epsilon^2}+(-\frac{11}{64}+\frac{\gamma}{16}+\frac{\rho}{16})\frac{1}{\epsilon}$	0
$10-2$	$\frac{1}{4\epsilon^2}+(-\frac{11}{24}+\frac{\gamma}{8}+\frac{\varrho}{8})\frac{1}{\epsilon}$	$-\frac{1}{8\epsilon^2}+(\frac{1}{6}-\frac{\gamma}{16}-\frac{\rho}{16})\frac{1}{\epsilon}$	0
Total	$\frac{1}{8\epsilon^2}-\frac{5}{96\epsilon}$	$-\frac{1}{16\epsilon^2}-\frac{1}{192\epsilon}$	0

Table 14: The results for Group 10

Diagram	A	B	C
$11-1$	$\frac{1}{8\epsilon^2}+(-\frac{13}{32}+\frac{\gamma}{8}+\frac{\rho}{8})\frac{1}{\epsilon}$	$-\frac{1}{16\epsilon^2}+(\frac{19}{64}-\frac{\gamma}{16}-\frac{\rho}{16})\frac{1}{\epsilon}$	0
$11-2$	$-\frac{1}{4\epsilon^2}+(\frac{11}{24}-\frac{\gamma}{8}-\frac{\rho}{8})\frac{1}{\epsilon}$	$\frac{1}{8\epsilon^2}+(-\frac{5}{12}+\frac{\gamma}{16}+\frac{\rho}{16})\frac{1}{\epsilon}$	0
Total	$-\frac{1}{8\epsilon^2}+\frac{5}{96\epsilon}$	$\frac{1}{16\epsilon^2}-\frac{23}{192\epsilon}$	0

Table 15: The results for Group 11

Diagram	A	B	C
$12-1$	$\frac{5}{16\epsilon}$	$-\frac{3}{32\epsilon}$	0
$12-2$	$-\frac{5}{12\epsilon}$	$\frac{1}{8\epsilon}$	0
Total	$-\frac{5}{48\epsilon}$	$\frac{1}{32\epsilon}$	0

Table 16: The results for Group 12

Diagram	A	B	C
$13-1$	$\frac{1}{3\epsilon^2}+(-\frac{23}{72}+\frac{\gamma}{3}+\frac{\rho}{3}+\frac{\sigma}{6})\frac{1}{\epsilon}$	$-\frac{1}{6\epsilon^2}+(\frac{23}{144}-\frac{\gamma}{6}-\frac{\rho}{6}-\frac{\sigma}{12})\frac{1}{\epsilon}$	$-\frac{1}{6\epsilon}$
$13-2$	$-\frac{1}{3\epsilon^2}+(\frac{7}{36}-\frac{\gamma}{3}-\frac{\rho}{3}-\frac{\sigma}{6})\frac{1}{\epsilon}$	$\frac{1}{6\epsilon^2}+(-\frac{7}{72}+\frac{\gamma}{6}+\frac{\rho}{6}+\frac{\sigma}{12})\frac{1}{\epsilon}$	$\frac{1}{6\epsilon}$
Total	$-\frac{1}{8\epsilon}$	$\frac{1}{16\epsilon}$	0

Table 17: The results for Group 13

Group	A	B
1	$\frac{1}{4\epsilon^2}+\frac{1}{48\epsilon}$	$-\frac{1}{8\epsilon^2}-\frac{1}{96\epsilon}$
2	$\frac{1}{4\epsilon^2}+\frac{11}{48\epsilon}$	$-\frac{1}{8\epsilon^2}+\frac{1}{32\epsilon}$
3	$-\frac{1}{2\epsilon^2}+\frac{19}{72\epsilon}$	$\frac{1}{4\epsilon^2}-\frac{7}{36\epsilon}$
4	$\frac{1}{4\epsilon^2}-\frac{13}{48\epsilon}$	$-\frac{1}{8\epsilon^2}+\frac{3}{32\epsilon}$
5	$\frac{1}{8\epsilon^2}-\frac{19}{288\epsilon}$	$-\frac{1}{16\epsilon^2}+\frac{73}{576\epsilon}$
6	$-\frac{3}{64\epsilon}$	$\frac{7}{384\epsilon}$
7	$-\frac{3}{64\epsilon}$	$\frac{19}{384\epsilon}$
8	$-\frac{1}{8\epsilon^2}-\frac{1}{16\epsilon}$	$\frac{1}{16\epsilon^2}+\frac{1}{48\epsilon}$
9	$-\frac{1}{4\epsilon^2}+\frac{5}{48\epsilon}$	$\frac{1}{8\epsilon^2}-\frac{5}{96\epsilon}$
10	$\frac{1}{8\epsilon^2}-\frac{5}{96\epsilon}$	$-\frac{1}{16\epsilon^2}-\frac{1}{192\epsilon}$
11	$-\frac{1}{8\epsilon^2}+\frac{5}{96\epsilon}$	$\frac{1}{16\epsilon^2}-\frac{23}{192\epsilon}$
12	$-\frac{5}{48\epsilon}$	$\frac{1}{32\epsilon}$
13	$-\frac{1}{8\epsilon}$	$\frac{1}{16\epsilon}$
Total	$-\frac{5}{48\epsilon}$	$\frac{5}{96\epsilon}$

Table 18: Divergences for $\hat{R}$

5.3 Conformal Invariance and Ultraviolet Fixed Point

In the previous subsection, we have calculated the divergences in the effective action and determined the counterterms by the background field method. The bare action can be written as

$$\begin{aligned} S_0 &= \int \mathrm{d}^D x \sqrt{\hat{g}} \frac{\mu^\epsilon}{G} \Big\{ \left(1 - \frac{25-c}{6\epsilon}\hat{G} + \frac{5c}{24\epsilon}\hat{G}^2 \right) \tilde{R} \\ &\qquad -\frac{1}{2}(\epsilon b - w\hat{G}^2 b) X_i X^i \tilde{R} \\ &\qquad +\frac{1}{2}\left(1 - \frac{c}{8\epsilon}\hat{G}^2\right) \tilde{g}^{\mu\nu}\partial_\mu X_i \partial_\nu X^i \Big\}, \end{aligned} \tag{5.40}$$

where $\hat{G}$ is equal to $G/4\pi$. We have introduced a parameter w , which corresponds to a finite renormalization of the coupling $X_i X^i \tilde{R}$. Although it can be taken arbitrary as far as the divergence of the theory is concerned, we have to keep it since the corresponding tree–level coupling constant is $O(\epsilon)$. From this expression, we see that the theory can be renormalized multiplicatively at least to this extent, which is nontrivial since the X_i field is dimensionless in two dimensions.

We parametrize the bare action as

$$S_0 = \frac{1}{G_0} \int \mathrm{d}^D x \sqrt{\hat{g}} \left\{ \left(1 - \frac{1}{2}\epsilon b_0 X_{0i} X_0{}^i \right) \tilde{R} + \frac{1}{2}\tilde{g}^{\mu\nu}\partial_\mu X_{0i} \partial_\nu X_0{}^i \right\}, \tag{5.41}$$

which gives the following relations between the bare quantities and the renormalized quantities.

$$\frac{1}{G_0} = \frac{\mu^\epsilon}{G}\left(1 - \frac{25-c}{6\epsilon}\hat{G} + \frac{5c}{24\epsilon}\hat{G}^2 \right), \tag{5.42}$$

$$\epsilon b_0 = \epsilon b + \left(\frac{c}{8} - w \right) \hat{G}^2 b. \tag{5.43}$$

Using these relations, the β functions can be obtained as

$$\beta_G = G \left(\epsilon - \frac{25-c}{6}\hat{G} + \frac{5c}{12}\hat{G}^2 \right), \tag{5.44}$$

$$\beta_b = \left(2w - \frac{c}{4} \right) \hat{G}^2 b. \tag{5.45}$$

As is seen in the expression for β_b, the free parameter w is relevant to the physics of the system. We will fix this ambiguity by imposing general covariance on the bare action.

Since we have maintained only the volume–preserving diffeomorphism, we have to impose the conformal invariance on the bare action so that the theory is generally covariant. We

consider the conformal transformation

$$\delta\tilde{g}_{\mu\nu} = \tilde{g}_{\mu\nu}\delta\rho, \tag{5.46}$$

$$\delta X_{0i} = (D-1)\frac{\partial L_0}{\partial X_0{}^i}\delta\rho, \tag{5.47}$$

where $L_0 = 1 - \frac{1}{2}\epsilon b_0 X_{0i} X_0{}^i$. Under this transformation, the bare action transforms as

$$\begin{aligned} \delta S_0 &= \int \mathrm{d}^D x \sqrt{\hat{g}} \frac{\mu^\epsilon}{2G} \Big[\left\{ \epsilon - \frac{25-c}{6}\hat{G} + \frac{5c}{24}\hat{G}^2 \right\} \tilde{R} \\ &\qquad -\frac{1}{2}\left\{ \epsilon(\epsilon b - w\hat{G}^2 b) - 4(D-1)(\epsilon b - w\hat{G}^2 b)\left(\epsilon - w\hat{G}^2 + \frac{c}{8}\hat{G}^2 \right) b \right\} X_i X^i \tilde{R} \\ &\qquad + \frac{1}{2}\left\{ \epsilon\left(1 - \frac{c}{8\epsilon}\hat{G}^2 \right) - 4(D-1)(\epsilon b - w\hat{G}^2 b) \right\} \partial_\mu X_i \partial^\mu X^i \Big] \delta\rho. \end{aligned} \tag{5.48}$$

When we consider symmetry at the quantum level within the counterterm formalism, we have to replace the operators in (5.48) with the corresponding renormalized operators [32, 33, 34].

In order to define the renormalized operator for $X_i X^i \tilde{R}$, we differentiate the bare action with respect to the finite parameter b

$$\int \mathrm{d}^D x \sqrt{\hat{g}} (X_i X^i \tilde{R})_r \equiv -\frac{2G}{\epsilon\mu^\epsilon}\frac{\partial S_0}{\partial b} = \left(1 - \frac{w\hat{G}^2}{\epsilon} \right) \int \mathrm{d}^D x \sqrt{\hat{g}} X_i X^i \tilde{R}. \tag{5.49}$$

We need to translate this relation into the local one, where, in general, one may have some total derivative terms. We note, however, that a complete set of operators can be written without total derivative terms by making use of the equations of motion [34]. We can, therefore, define the renormalized operator $(X_i X^i \tilde{R})_r$ as

$$(X_i X^i \tilde{R})_r = \left(1 - \frac{w\hat{G}^2}{\epsilon} \right) X_i X^i \tilde{R}. \tag{5.50}$$

The same reasoning holds in the case of the other operators, which we omit to mention in the following.

The renormalized operator for $\partial_\mu X_i \partial^\mu X^i$ can be obtained by introducing a parameter "f" in front of the tree–level kinetic term of X_i and keep track of the parameter in the divergent diagrams. The propagator of X_i is multiplied by a factor $\frac{1}{f}$ and the vertices which originate from the kinetic term of X_i are multiplied by a factor f. One finds that all the diagrams corresponding to the renormalization of the kinetic term are multiplied by a factor f and the other diagrams remain the same. This implies that the renormalized operator for $\partial_\mu X_i \partial^\mu X^i$ can be defined through

$$\partial_\mu X_i \partial^\mu X^i = \left(1 + \frac{c\hat{G}^2}{8\epsilon} \right) (\partial_\mu X_i \partial^\mu X^i)_r. \tag{5.51}$$

Finally the renormalized operator for $\tilde{R}$ can be obtained as follows. We renormalize X_i so that the only G dependence comes from the coefficient of $\tilde{R}^{\,2}$. Also we have to perform the wave function renormalization in order to avoid picking up unphysical contributions from the kinetic term of X_i. Thus we define

$$Y_i = \sqrt{\frac{1}{G}\left(1 - \frac{c}{8\epsilon}\hat{G}^2\right)} X_i, \tag{5.52}$$

and rewrite the action in terms of Y_i as

$$\begin{aligned} S_0(G, Y_i) &= \int \mathrm{d}^D x \sqrt{\hat{g}} \mu^\epsilon \left[\frac{1}{G}\left(1 - \frac{25-c}{6\epsilon}\hat{G} + \frac{5c}{24\epsilon}\hat{G}^2\right)\tilde{R} \right. \\ &\quad - \frac{1}{2}\left(\epsilon b - w\hat{G}^2 b + \frac{c}{8}\hat{G}^2 b\right) Y_i Y^i \tilde{R} \\ &\quad \left. + \frac{1}{2}\partial_\mu Y_i \partial^\mu Y^i \right]. \end{aligned} \tag{5.53}$$

By differentiating the above bare action with respect to $1/G$, we can obtain the renormalized operator for $\tilde{R}$ as

$$(\tilde{R})_r = \left(1 - \frac{5c}{24\epsilon}\hat{G}^2\right)\tilde{R} - \left(w\hat{G}^2 b - \frac{c}{8}\hat{G}^2 b\right) X_i X^i \tilde{R}. \tag{5.54}$$

Using eqs. (5.50), (5.51) and (5.54), the conformal anomaly (5.48) can be written in terms of the renormalized operators as

$$\begin{aligned} &\int \mathrm{d}^D x \sqrt{\hat{g}} \frac{\mu^\epsilon}{2G} \left[\left\{ \epsilon - \frac{25-c}{6}\hat{G} + \frac{5c}{12}\hat{G}^2 \right\} (\tilde{R})_r \right. \\ &\quad - \frac{1}{2}\epsilon b \left\{ \epsilon - 4(D-1)\left(\epsilon b - w\hat{G}^2 b + \frac{c}{8}\hat{G}^2 b\right) - 2\left(w\hat{G}^2 - \frac{c}{8}\hat{G}^2\right) \right\} (X_i X^i \tilde{R})_r \\ &\quad \left. - \frac{1}{2}\left\{ \epsilon - 4(D-1)\left(\epsilon b - w\hat{G}^2 b + \frac{c}{8}\hat{G}^2 b\right) \right\} (\partial_\mu X_i \partial^\mu X^i)_r \right] \delta\rho. \end{aligned} \tag{5.55}$$

This result is reasonable since each term includes the expression for the β function. At the ultraviolet fixed point, where the β functions vanish, the conformal anomaly vanishes if and only if the fixed-point value of b is given by

$$b = b^* = \frac{1}{4(D-1)}. \tag{5.56}$$

This is the same as the one-loop result. In order that this nonvanishing fixed-point value of b may be realized, the coefficient $(2w - c/4)$ in the β function of b should vanish. Thus

[2]Here we assume that the G dependence in the gauge fixing term does not affect the physical conclusion. An explicit check of this assumption by performing the operator renormalization of $\tilde{R}$ requires as much work as has been done in this study.

the free parameter w should be chosen to be $\frac{c}{8}$ to the leading order of c. Note also that the fixed-point value of b coincides with the value of b that corresponds to the classical Einstein gravity. This is consistent with the one-loop result where it has been shown that the β function of b remains zero throughout the renormalization group trajectory from the ultraviolet fixed point to the infrared fixed point which corresponds to Einstein gravity.

6 Summary and Outlook

In this paper, we have studied quantum gravity near two dimensions from the viewpoint of renormalization group. In our formulation the conformal mode of the metric is separated explicitly and treated carefully in view of subtleties of the oversubtraction problem.

In the first part of this paper, we have treated two problems in the framework of $(2+\epsilon)$-dimensional quantum gravity. Firstly, using the prescription which holds for two-dimensions limit, we computed the scaling dimensions of manifestly generally covariant operators, whose physical and geometrical interpretations are clear and which are not easily tractable in conventional approaches. We succeeded in treating them in a consistent way for the first time. Our conjecture is that we obtained a new series of scaling operators in two-dimensional quantum gravity. We calculated only one point functions of manifestly generally covariant operators. It is desirable to calculate their multi-point functions and compare them with those of the scaling operators in the matrix model to verify our conjecture. However it is not an easy task. It might be possible that there appears a new type of closed algebras in their correlators similar to Virasoro or W algebras as is well known in the matrix model and in Liouville theory.

Secondly, we have studied R^2 gravity in the formalism of $(2+\epsilon)$-dimensional quantum gravity. We presented a prescription similar to the one given in the ordinary Einstein gravity in order to take into account the oversubtraction which occurs at one loop level also in this case. Dropping the h-field, the ghosts and the matters, the theory reduces to a free field theory, which enables a full order calculation of the string susceptibility in the $\epsilon \to 0$ limit. The result is consistent with that of ref [18]. In addition, we have seen that the scaling dimensions of manifestly generally covariant operators in this theory are the canonical ones in the $m^2 \to 0$ limit, which supports our conjecture mentioned above. Through this study, we shed light on the oversubtraction problem of the conformal mode dynamics in $(2+\epsilon)$-dimensional quantum gravity and showed its usefulness as a regularization scheme of two-dimensional quantum gravity.

As for an original motivation to studying $(2+\epsilon)$-dimensional quantum gravity, we should investigate further the theory developed in ref. [6]. In the second part of this paper, we have studied two-loop renormalization of the theory to check the consistency of ϵ expansion of quantum gravity. We concentrated on the part proportional to the number of matter fields and imposed Z_2 symmetry on the system. We examined how the nonlocal term as well as the infrared divergence mixed with the ultraviolet divergence cancels among the diagram. We succeeded in dealing with the infrared divergence in a consistent way and showed that the theory is multiplicatively renomalizable at least to this extent. It has been shown that the conformal invariance is restored at the ultraviolet fixed point when we choose the finite renormalization properly. This ensures the existence of the ultraviolet fixed point which possesses general covariance up to two-loop level. In this way, we have shown that the theory can be constructed at two-loop level to this extent. We can say that we have established the systematic method of two-loop calculation through this work and given a prototype of two-loop calculation.

In order to establish $(2+\epsilon)$-dimensional quantum gravity, we have to perform full calculation of two-loop renormalization. We examine whether the theory can satisfy the requirement from the general covariance and higer order corrections can be evaluated systematically. In particular, the renormalization of operators such as the cosmological term is highly nontrivial. If it is possible, we hope that we can eventually calculate physical quantities such as critical exponents and compare the results with the ones which may be obtained in numerical simulations of three or four dimensions. At last it might be possible that we identify an order parameter in $(2+\epsilon)$-dimensional quantum gravity, which provides us with a new and deeper understanding of quantum gravity. We will continue calculations of the two-loop renormalization further.

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Appendix A

In this appendix, we present explanations of some concepts essential in the formalism of $(2+\epsilon)$–dimensional quantum gravity. First, we explain in detail the background field method, which we utilize to compute the effective action [31]. The generating functional for the connected Green's functions in the field theory is

$$e^{-W[J]} = \int D\varphi exp(-S - J\cdot\varphi) \tag{A.1}$$

where S is the action and $J\cdot\varphi = \int d^Dx J(x)\varphi(x)$. φ denotes a collection of fields in the theory. The effective action is obtained by the Legendre transform

$$\Gamma[<\varphi>] = W[J] - J\cdot <\varphi>, \tag{A.2}$$

where $<\varphi(x)>= \frac{\delta W[J]}{\delta J(x)}$. Therefore the effective action is

$$e^{-\Gamma[<\varphi>]} = \int D\delta\varphi exp(-S[<\varphi> + \delta\varphi] + \frac{\delta\Gamma[<\varphi>]}{\delta <\varphi>}\cdot\delta\varphi) \tag{A.3}$$

where $\delta\varphi = \varphi - <\varphi>$ since $J = -\frac{\delta\Gamma}{\delta<\varphi>}$. The effective action can be expanded in terms of $\hbar$.

$$\Gamma = S + \hbar\Gamma^{(1)} + \hbar^2\Gamma^{(2)} + \cdots \tag{A.4}$$

Hence we can compute the effective action by expanding the action S around the background $<\varphi>$ and dropping the linear terms in $\delta\varphi$. Namely the effective action is the sum of the one particle irreducible diagrams with respect to $\delta\varphi$.

In this paper, we parameterize the metric as follows.

$$g_{\mu\nu} = \tilde{g}_{\mu\nu}e^{-\phi} = (e^h)_{\mu\nu}e^{-\phi}, \tag{A.5}$$

where ϕ is the conformal mode of the metric, $\tilde{g}_{\mu\nu}$ satisfies $\det(g_{\mu\nu}) = 1$ and hence $h_{\mu\mu} = 0$. Since $h_{\mu\nu}$ is dimensionless, $h_{\mu\nu}$ and $\tilde{g}_{\mu\nu}$ are equally valid fields to functionally integrate. We expand the fields around the backgrounds as $\phi = \hat{\phi} + \delta\phi$ and $h_{\mu\nu} = \hat{h}_{\mu\nu} + \delta\tilde{h}_{\mu\nu}$. However it is more convenient to expand $\tilde{g}_{\mu\nu} = \hat{g}_{\mu\rho}(e^{\delta h})^\rho{}_\nu$ where $\delta h^\rho{}_\rho = 0$ and the tensor indices are raised and lowered by the background metric $\hat{g}_{\mu\nu}$. We utilize this parametrization in this paper and compute the one particle irreducible diagrams with respect to $\delta h_{\mu\nu}$. Although $\hat{g}_{\mu\nu} = (e^{\hat{h}})_{\mu\nu}$ classically, the wave function renormalization is involved at quantum level in general since $\delta h_{\mu\nu}$ and $\delta\tilde{h}_{\mu\nu}$ are related nonlinearly. However physical consequences are not altered by this simplification. We call $\delta\varphi_i, \delta\psi$ and $\delta h_{\mu\nu}$ as quantum fields and δ in front of them will be dropped throughout this paper.

Next, we give an interpretation of the meanings of renormalization group and renormalization point (renormalization scale) in $(2+\epsilon)$-dimensional quantum gravity intensively. Special care should be taken of it because the spacetime itself fluctuates in quantum gravity. Let us consider the Einstein action in $D = 2+\epsilon$ dimensions:

$$\int d^D x \frac{1}{G_0}\sqrt{g}R, \tag{A.6}$$

where G_0 is the gravitational coupling constant. We parameterize the metric $g_{\mu\nu}$ as in (A.5). A dimensionless coupling constant G is further introduced together with the renormalization group scale μ through $1/G_0 = \mu^\epsilon/G$. In this parametrization the action (A.6) becomes

$$\int d^D x \frac{\mu^\epsilon}{G} e^{-\frac{\epsilon}{2}\phi}(\tilde{R} - \frac{\epsilon(D-1)}{4}\tilde{g}^{\mu\nu}\partial_\mu\phi\partial_\nu\phi), \tag{A.7}$$

where $\tilde{R}$ is the scalar curvature made out of $\tilde{g}_{\mu\nu}$. We note the similarity of this action to the nonlinear sigma model:

$$\int d^D x \frac{\mu^\epsilon}{\omega} e^{-\frac{\epsilon}{2}\phi}\partial_\mu\vec{n}\cdot\partial^\mu\vec{n} \tag{A.8}$$

Since $\tilde{R}$ involves two derivatives, it is analogous to the kinetic term of the nonlinear sigma model. ω is the coupling constant for $\vec{n}$ field and G is that for $h_{\mu\nu}$ field. The physical length scale is set by the line element $ds^2 = e^{-\phi}dx_\mu dx^\mu$. If we scale the length as $ds^2 \to \lambda^2 ds^2$, the coupling constant changes as $1/\omega \to \lambda^\epsilon/\omega$ and $1/G \to \lambda^\epsilon/G$ respectively. In this sense the coupling constant grows canonically at short distance in both theories. However we can choose the renormalization scale μ such that $\mu\lambda = 1$ and consider the running coupling constants. If the running coupling constants possess the short distance fixed points, we can control the theory well. The crucial point in quantum gravity is that the zero mode of ϕ field sets the scale of the metric and hence the scale of the length. In fact the zero mode is determined by the classical solution of the theory. Since the definite combination $\mu^\epsilon e^{-\frac{\epsilon}{2}\phi}$ appears in the action, it is most advantageous to choose the renormalization scale μ to compensate the scale factor of the metric (or constant mode of ϕ). It is analogous to choose the renormalization scale to match the momentum scale of the relevant scattering in the conventional field theory problem. In this way the renormalization scale of the dimensionless gravitational coupling constant $G(\mu)$ is related to the scale factor of the metric. In particular, large renormalization scale is relevant at short distances. We need to consider all possible values of the constant mode of ϕ for the whole theory since we are integrating over it. It

follows that we consider the whole renormalization group trajectory as the whole quantum theory of gravitation. Such an idea is consistent with the independence of the theory from the scale factor of a particular metric. Furthermore, considering that the scale factor of the metric expand with time and can be identified with time itself, it may be possible to say that the renormalization scale is identified with time and the renormalization group evolution is hence naturally related to the time evolution in quantum gravity

Appendix B

In this appendix, we check that the result is not affected by the simplification we have made concerning the background field dependence in section 3.2. We expand the background field around the flat metric as

$$\hat{g}_{\mu\nu} = \delta_{\mu\nu} + \hat{h}_{\mu\nu}. \tag{B.9}$$

Then the action and the operator considered are written as

$$S = \frac{1}{G_0}\int \mathrm{d}^D x \left[\frac{\epsilon^2}{16}\hat{R}(1+H)\psi^2 - \frac{D-1}{4}\epsilon(\partial_\mu\psi\partial_\mu\psi + G_{\mu\nu}\partial_\mu\psi\partial_\nu\psi)\right], \tag{B.10}$$

$$\begin{aligned}\int \mathrm{d}^D x\sqrt{g}R^n &= \int \mathrm{d}^D x\sqrt{\hat{g}}\mathrm{e}^{-\frac{4}{\epsilon}(-\frac{D}{2}+n)\log(1+\frac{\epsilon}{4}\psi)} \\ &\quad\cdot\{\hat{R} + \partial^2\psi - \hat{\Gamma}^\nu{}_{\mu\mu}\partial_\nu\psi + I_{\mu\nu}(\partial_\mu\partial_\nu\psi - \hat{\Gamma}^\lambda{}_{\mu\nu}\partial_\lambda\psi)\}^n, \end{aligned} \tag{B.11}$$

respectively, where H, $G_{\mu\nu}$ and $I_{\mu\nu}$ are $O(\hat{h})$ quantities defined through

$$\sqrt{\hat{g}} = 1 + H \tag{B.12}$$

$$\sqrt{\hat{g}}\hat{g}^{\mu\nu} = \delta_{\mu\nu} + G_{\mu\nu} \tag{B.13}$$

$$\hat{g}^{\mu\nu} = \delta_{\mu\nu} + I_{\mu\nu}. \tag{B.14}$$

We have set $H = 0$, $G_{\mu\nu} = 0$, $I_{\mu\nu} = 0$ and $\hat{\Gamma}^\lambda{}_{\mu\nu} = 0$ at the beginning of our calculation. In order to justify this simplification, we have to check that there is no extra $\frac{1}{\epsilon}$ divergence coming from the terms with the above $O(\hat{h})$ coefficients.

For each use of $G_{\mu\nu}\partial_\mu\psi\partial_\nu\psi$ in the action, we have to use $\partial^2\psi$ in the operator in order to keep $O(\hat{h}^n)$. The diagrams we have to consider are listed in Fig.17, where the dot represents a derivative. The first one, for example, gives

$$\int \frac{\mathrm{d}^D p}{(2\pi)^D}\frac{(p+k)^2(p+k)_\mu p_\nu}{(p+k)^2 p^2}. \tag{B.15}$$

In order to have a logarithmic divergence, we have to factor out k^2 from the integrand, which is not possible due to the fact that $(p+k)^2$ in the numerator coming from the $\partial^2\psi$ in the operator cancels the propagator $\frac{1}{(p+k)^2}$. This occurs for each of the diagrams in Fig. 17 and one can also check that the above situation is not altered even if one takes into account the terms $\hat{\Gamma}^\nu{}_{\mu\mu}\partial_\nu\psi$, $I_{\mu\nu}\partial_\mu\partial_\nu\psi$ and $I_{\mu\nu}\hat{\Gamma}^\lambda{}_{\mu\nu}\partial_\lambda\psi$ in the operator and the $\frac{\epsilon^2}{16}\hat{R}H\psi^2$ term in the action.

Appendix C

In this appendix, we explain how to evaluate the expectation values appearing in eq. (3.9), namely,

$$\begin{aligned}\langle\int\sqrt{g}R^2\mathrm{d}^D x\rangle &= \int \mathrm{d}^D x\sqrt{\hat{g}}\langle\mathrm{e}^{-\frac{4}{\epsilon}(-\frac{D}{2}+2)\log(1+\frac{\epsilon}{4}\psi)}\rangle\hat{R}^2 \\ &\quad+2(D-1)\int \mathrm{d}^D x\sqrt{\hat{g}}\langle\mathrm{e}^{-\frac{4}{\epsilon}(-\frac{D}{2}+2+\frac{\epsilon}{4})\log(1+\frac{\epsilon}{4}\psi)}\partial^2\psi\rangle\hat{R} \\ &\quad+(D-1)^2\int \mathrm{d}^D x\sqrt{\hat{g}}\langle\mathrm{e}^{-\frac{4}{\epsilon}(-\frac{D}{2}+2+\frac{\epsilon}{2})\log(1+\frac{\epsilon}{4}\psi)}(\partial^2\psi)^2\rangle. \end{aligned} \tag{C.1}$$

The diagrams which appear in calculating the expectation value in each term can be drawn generally as (a),(b) and (c) respectively in Fig. 18, where the dot represents a derivative and the cross represents a mass insertion. Note that the expectation value in the first term is just the one we encounter in the case of $\int\sqrt{g}^{1-\Delta_0}\Phi_{\Delta_0}\mathrm{d}^2 x$ type operators. Since each plain loop contributes a factor

$$-\frac{2G_0}{\epsilon}\int\frac{\mathrm{d}^D p}{(2\pi)^D}\frac{1}{p^2} = -\frac{2G_0}{\epsilon}\left(-\frac{\mu^\epsilon}{2\pi\epsilon}\right) = \frac{G_0\mu^\epsilon}{\pi\epsilon^2}, \tag{C.2}$$

we can calculate, for example, the diagram (a) by introducing a zero-dimensional field theory whose action is $S(X) = \frac{1}{2}\frac{\pi\epsilon^2}{G_0\mu^\epsilon}X^2$, and considering the expectation value of

$$\mathrm{e}^{-\frac{4}{\epsilon}(-\frac{D}{2}+2)\log(1+\frac{\epsilon}{4}X)}. \tag{C.3}$$

Thus we have

$$\langle\mathrm{e}^{-\frac{4}{\epsilon}(-\frac{D}{2}+2)\log(1+\frac{\epsilon}{4}\psi)}\rangle = \frac{1}{Z}\int_{-\infty}^{\infty}\mathrm{d}X\mathrm{e}^{-\frac{4}{\epsilon}(-\frac{D}{2}+2)\log(1+\frac{\epsilon}{4}X)}\mathrm{e}^{-\frac{1}{2}\frac{\pi\epsilon^2}{G_0\mu^\epsilon}X^2}, \tag{C.4}$$

where

$$Z = \int_{-\infty}^{\infty}\mathrm{d}X\mathrm{e}^{-\frac{1}{2}\frac{\pi\epsilon^2}{G_0\mu^\epsilon}X^2} = \frac{\sqrt{2G_0\mu^\epsilon}}{\epsilon}. \tag{C.5}$$

Introducing a new variable $Y = \frac{1}{4}\epsilon X$, the integral becomes

$$\frac{\epsilon}{\sqrt{2G_0\mu^\epsilon}}\frac{4}{\epsilon}\int_{-\infty}^{\infty}\mathrm{d}Y\mathrm{e}^{-\frac{1}{\epsilon}\left[4(-\frac{D}{2}+2)\log(1+Y)+\frac{8\pi\epsilon}{G_0\mu^\epsilon}Y^2\right]}, \tag{C.6}$$

whose asymptotic behavior for $\epsilon\to 0$ can be readily evaluated by means of the saddle-point method. The saddle point $Y = \rho$ is given through

$$\frac{\mathrm{d}}{\mathrm{d}Y}\left[4\left(-\frac{D}{2}+2\right)\log(1+Y) + \frac{8\pi\epsilon}{G_0\mu^\epsilon}Y^2\right]_{Y=\rho} = 0, \tag{C.7}$$

namely,

$$4\left(-\frac{D}{2}+2\right)\frac{1}{1+\rho}+\frac{16\pi\epsilon}{G_0\mu^\epsilon}\rho=0, \tag{C.8}$$

from which we obtain

$$\rho=\frac{1}{2}\left\{-1\pm\sqrt{1-\frac{G_0\mu^\epsilon}{\pi\epsilon}}\right\}. \tag{C.9}$$

Thus we obtain the asymptotic behavior of the expectation value up to a factor of $O(1)$ as

$$\sim \exp\left[-\frac{4}{\epsilon}\log(1+\rho)-\frac{8\pi}{G_0\mu^\epsilon}\rho^2\right]. \tag{C.10}$$

We have to choose '+' for the double sign in the expression (C.9) so that we may reproduce the correct perturbative expansion. Let us now turn to the second term in eq. (C.1). The expectation value in this term can be evaluated with the diagram (b) as,

$$\begin{aligned}&\frac{1}{Z}\int_{-\infty}^{\infty}\mathrm{d}X\left\{\frac{\mathrm{d}}{\mathrm{d}X}\mathrm{e}^{-\frac{4}{\epsilon}\left(-\frac{D}{2}+2+\frac{\epsilon}{4}\right)\log\left(1+\frac{\epsilon}{4}X\right)}\right\}\mathrm{e}^{-\frac{1}{2}\frac{\pi\epsilon^2}{G_0\mu^\epsilon}X^2}\\ &\qquad\cdot\left(-\frac{2G_0}{\epsilon}\right)^2\cdot\left(-\frac{1}{G_0}\frac{\epsilon^2}{8}\right)\int\frac{\mathrm{d}^Dp}{(2\pi)^D}\frac{-p^2}{(p^2)^2}.\end{aligned}$$

Since the expression in the curly bracket gives

$$\begin{aligned}&-\frac{4}{\epsilon}\left(-\frac{D}{2}+2+\frac{\epsilon}{4}\right)\frac{1}{1+\frac{\epsilon}{4}X}\frac{\epsilon}{4}\mathrm{e}^{-\frac{4}{\epsilon}\left(-\frac{D}{2}+2+\frac{\epsilon}{4}\right)\log\left(1+\frac{\epsilon}{4}X\right)} &\text{(C.11)}\\ =\;&-\left(-\frac{D}{2}+2+\frac{\epsilon}{4}\right)\mathrm{e}^{-\frac{4}{\epsilon}\left(-\frac{D}{2}+2+\frac{\epsilon}{2}\right)\log\left(1+\frac{\epsilon}{4}X\right)}, &\text{(C.12)}\end{aligned}$$

the result for the asymptotic behavior is the same as (C.10) up to a factor of $O(1)$. As for the third term, there are two diagrams we have to consider, as is shown in Fig. (18-c). The left one can be evaluated as

$$\begin{aligned}&\frac{1}{Z}\int_{-\infty}^{\infty}\mathrm{d}X\left\{\frac{\mathrm{d}^2}{\mathrm{d}X^2}\mathrm{e}^{-\frac{4}{\epsilon}\left(-\frac{D}{2}+2+\frac{\epsilon}{4}\right)\log\left(1+\frac{\epsilon}{4}X\right)}\right\}\mathrm{e}^{-\frac{1}{2}\frac{\pi\epsilon^2}{G_0\mu^\epsilon}X^2}\\ &\qquad\cdot\left[\left(-\frac{2G_0}{\epsilon}\right)^2\cdot\left(-\frac{1}{G_0}\frac{\epsilon^2}{8}\right)\int\frac{\mathrm{d}^Dp}{(2\pi)^D}\frac{-p^2}{(p^2)^2}\right]^2,\end{aligned}$$

whose asymptotic behavior is also the same as (C.10) up to a factor of $O(1)$, while the right one can be evaluated as

$$\begin{aligned}&\frac{1}{Z}\int_{-\infty}^{\infty}\mathrm{d}\mathrm{e}^{-\frac{4}{\epsilon}\left(-\frac{D}{2}+2\pm\frac{\epsilon}{4}\right)\log\left(1+\frac{\epsilon}{4}X\right)}\mathrm{e}^{-\frac{1}{2}\frac{\pi\epsilon^2}{G_0\mu^\epsilon}X^2}\\ &\cdot\left(-\frac{2G_0}{\epsilon}\right)^3\cdot\left(-\frac{1}{G_0}\frac{\epsilon^2}{8}\right)^2\int\frac{\mathrm{d}^Dp}{(2\pi)^D}\frac{(-p^2)^2}{(p^2)^3},\end{aligned}$$

which has the asymptotic behavior of (C.10) multiplied by an $O(\epsilon)$ factor. Altogether, we get

$$\langle\int\sqrt{g}R^2\mathrm{d}^Dx\rangle\sim\exp\left[-\frac{4}{\epsilon}\log(1+\rho)-\frac{8\pi}{G_0\mu^\epsilon}\rho^2\right]\int\sqrt{\hat{g}}\hat{R}^2\mathrm{d}^Dx, \tag{C.13}$$

which means that the relevant $\frac{1}{\epsilon}$ divergence in calculating the scaling dimension comes from the $\mathrm{e}^{-\frac{4}{\epsilon}\left(-\frac{D}{2}+2\right)\log\left(1+\frac{\epsilon}{4}\psi\right)}$ in each term of (C.1).

Appendix D

In this appendix, we calculate the one-loop counter terms for the most general renormalizable action with a scalar field χ and c species of conformal matter,

$$S = \mu^\epsilon \int \mathrm{d}^D x \sqrt{g} \left(\frac{1}{2} K(\chi) g^{\mu\nu} \partial_\mu \chi \partial_\nu \chi + L(\chi) R + M(\chi) \right) + (\text{matters}). \tag{D.14}$$

Adopting the background field method, we replace χ with $\hat{\chi} + \chi$, where $\hat{\chi}$ is the background field, and parameterize $g_{\mu\nu}$ as in (2.1). We expand the action up to the second order of χ, ϕ and h, and drop the first order terms following the prescription of the background field method. We can choose the gauge fixing term so that the mixing terms between h and the other fields may be cancelled,

$$S_{\text{g.f.}} = \mu^\epsilon \int \mathrm{d}^D x \sqrt{\hat{g}} \frac{1}{2} L(\hat{\chi}) \left(\hat{\nabla}_\nu h^\nu{}_\mu + \frac{\epsilon}{2} \partial_\mu \phi - \frac{L'(\hat{\chi})}{L(\hat{\chi})} \partial_\mu \chi - f(\hat{\chi}) \partial_\mu \hat{\chi} \chi \right)^2 . \tag{D.15}$$

Note that the function f can be taken arbitrary. The ghost action can be determined from the gauge fixing term as

$$\begin{aligned} S_{\text{ghost}} &= \mu^\epsilon \int \mathrm{d}^D x \sqrt{\hat{g}} \left\{ \hat{\nabla}_\nu \bar{\eta}^\mu \hat{\nabla}^\nu \eta_\mu + \bar{\eta}^\mu \hat{R}_\mu{}^\nu \eta_\nu - \hat{\nabla}_\mu \bar{\eta}^\mu \frac{L'(\hat{\chi})}{L(\hat{\chi})} \partial_\nu \hat{\chi} \eta^\nu \right. \\ & \left. \qquad - \left(\frac{L''(\hat{\chi}) L(\hat{\chi}) - L'(\hat{\chi})^2}{L(\hat{\chi})^2} - f(\hat{\chi}) \right) \bar{\eta}^\mu \partial_\mu \hat{\chi} \partial_\nu \hat{\chi} \eta^\nu \right\} . \end{aligned} \tag{D.16}$$

The kinetic terms of χ, h and ϕ, including those from the gauge fixing term, thus read

$$\begin{aligned} &\mu^\epsilon \int \mathrm{d}^D x \sqrt{\hat{g}} \left\{ \frac{1}{2} \left(K(\hat{\chi}) + \frac{L'(\hat{\chi})^2}{L(\hat{\chi})} \right) \hat{g}^{\mu\nu} \partial_\mu \chi \partial_\nu \chi + \frac{1}{4} L(\hat{\chi}) \hat{\nabla}_\rho h^\mu{}_\nu \hat{\nabla}^\rho h^\nu{}_\mu \right. \\ & \left. \qquad + \frac{D}{2} L'(\hat{\chi}) \hat{g}^{\mu\nu} \partial_\mu \chi \partial_\nu \phi - \frac{\epsilon D}{8} \hat{g}^{\mu\nu} \partial_\mu \phi \partial_\nu \phi \right\} . \end{aligned} \tag{D.17}$$

Note that the kinematical pole which appears in the case of ordinary gravity does not show up here due to the ϕ-χ coupling. We introduce the new quantum fields X, Φ and $\Omega^\mu{}_\nu$ through

$$\begin{aligned} \chi &= F(\hat{\chi}) X - \frac{D}{2} F(\hat{\chi})^2 L'(\hat{\chi}) I(\hat{\chi}) \Phi \\ \phi &= I(\hat{\chi}) \Phi \\ h^\mu{}_\nu &= L(\hat{\chi})^{-1/2} \Omega^\mu{}_\nu , \end{aligned} \tag{D.18}$$

where $F(\hat{\chi})$ and $I(\hat{\chi})$ are defined through

$$\begin{aligned} \frac{1}{F(\hat{\chi})^2} &= K(\hat{\chi}) + \frac{L'(\hat{\chi})^2}{L(\hat{\chi})} \\ \frac{1}{I(\hat{\chi})^2} &= \frac{1}{4} \left(\epsilon D L(\hat{\chi}) + D^2 F(\hat{\chi})^2 L'(\hat{\chi})^2 \right) . \end{aligned} \tag{D.19}$$

After this field redefinition, the kinetic term reduces to the standard form

$$\mu^\epsilon \int \mathrm{d}^D x \sqrt{\hat{g}} \left\{ \frac{1}{4} \Omega^\mu{}_{\nu,\rho} \Omega^\nu{}_{\mu,}{}^\rho - \frac{1}{2} \hat{g}^{\mu\nu} \partial_\mu \Phi \partial_\nu \Phi + \frac{1}{2} \hat{g}^{\mu\nu} \partial_\mu X \partial_\nu X \right\} . \tag{D.20}$$

Taking account of the other terms coming from S, $S_{\text{g.f.}}$ and S_{ghost}, and calculating the one-loop counter terms through the 't Hooft-Veltman formalism [19], one obtains the final result for the action with the counter terms as

$$\begin{aligned} S + S_{\text{c.t.}} &= \mu^\epsilon \int \mathrm{d}^D x \sqrt{g} \left[\frac{1}{2} \left\{ K(\chi) - \frac{1}{2\pi\epsilon} P(\chi) \right\} g^{\mu\nu} \partial_\mu \chi \partial_\nu \chi \right. \\ & \qquad + \left\{ L(\chi) - \frac{1}{2\pi\epsilon} \frac{24 - c}{12} \right\} R \\ & \left. \qquad + \left\{ M(\chi) - \frac{1}{2\pi\epsilon} \left(\frac{1}{2} M(\chi) I(\chi)^2 + M'(\chi) \frac{1}{L'(\chi)} \right) \right\} \right] \\ & \qquad + (\text{matters}), \end{aligned} \tag{D.21}$$

where $P(\chi)$ is given by

$$\begin{aligned} P &= -2L'^{-2} L'' + 2L^{-1} L'' + 2F^2 L'^{-1} L'' (K' + 2L^{-1} L' L'' - L^{-2} L'^3) \\ & \quad - \frac{1}{2} F^4 (K' + 2L^{-1} L' L'' - L^{-2} L'^3)^2 \\ & \quad + 3L^{-1} K - 2L^{-1} F^{-2} + \frac{9}{2} L^{-2} L'^2 - f . \end{aligned} \tag{D.22}$$

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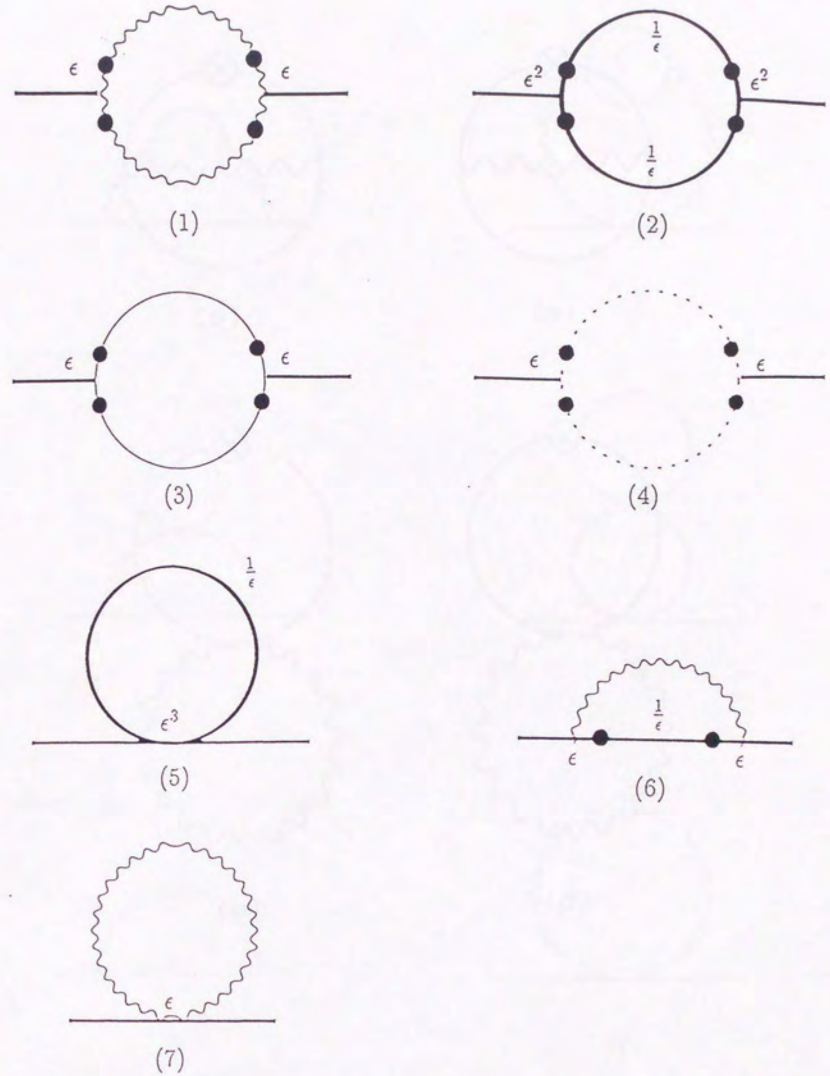


Fig. 1 Diagrams for $\partial_\mu\bar{\phi}\partial_\mu\bar{\phi}$

The wavy line, the solid line and the dashed line represent the propagators of $h_{\mu\nu}$, matter and ghost respectively. The thick line represents the φ propagator or the external $\bar{\varphi}$ line. The dot denotes a derivative. ϵ factors in the verticies and $\frac{1}{\epsilon}$ poles in the φ propagator are written down explicitly.

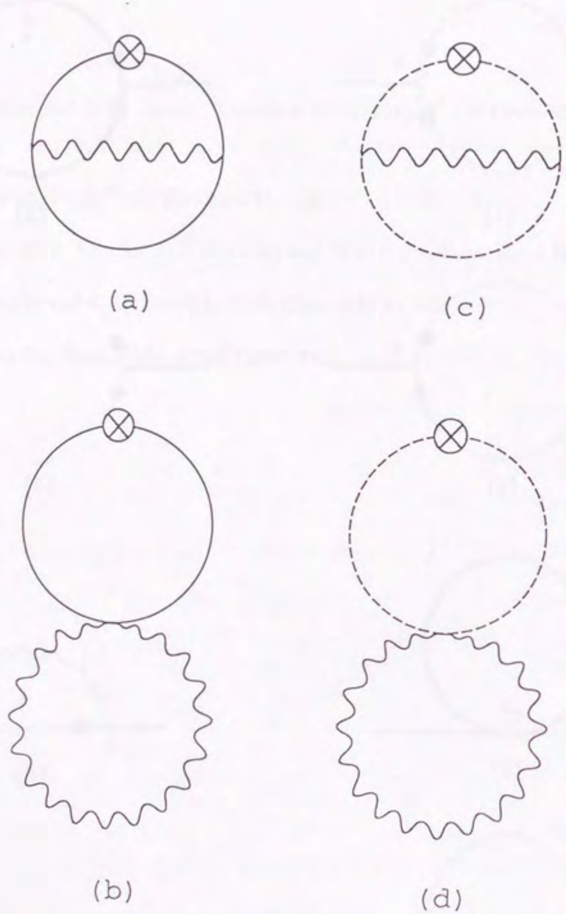


Fig. 2
The diagrams with $h_{\mu\nu}$ line we have to consider at two-loop level. The solid line, the dash line and the wavy line represent the propagators of the Φ-field, the X-field and the Ω-field respectively. (a) and (b) correspond to $\langle\Phi^2\rangle$, while (c) and (d) correspond to $\langle X^2\rangle$.

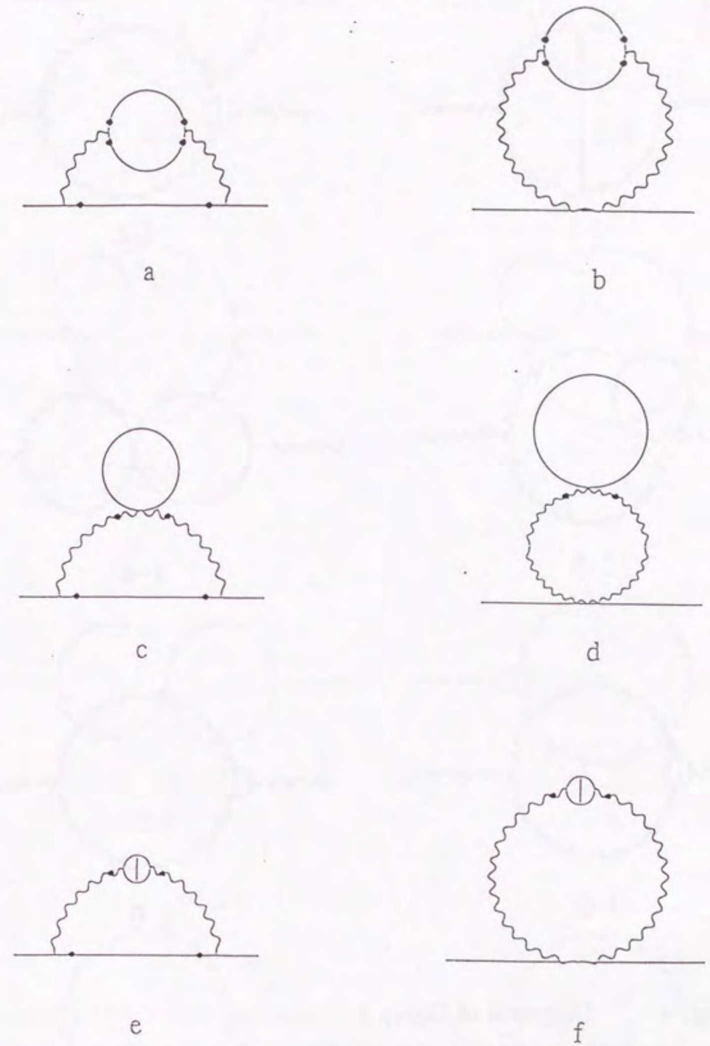


Fig. 3 Diagrams for $\partial_\mu \hat{X}_i \partial_\mu \hat{X}^i$
The wavy line represents the propagator of $H_{\mu\nu}$, and the solid line represents the propagator of X_i or the external X_i line. The dot denots a derivative and the symbol in the vertex represents the counter term insertion.

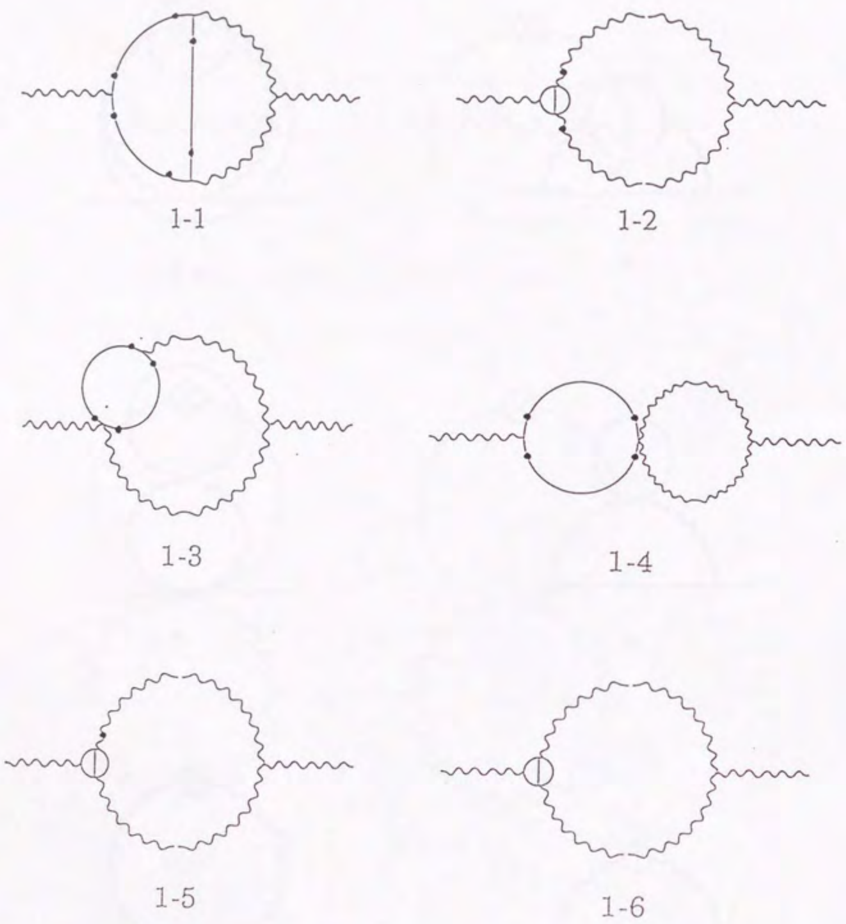


Fig. 4 Diagrams of Group 1

The wavy line represents the propagator of $H_{\mu\nu}$ or the external $\hat{h}_{\mu\nu}$ line, and the solid line represents the propagator of X_i. The dot denots a derivative and the symbol in the vertex represents the counter term insertion.

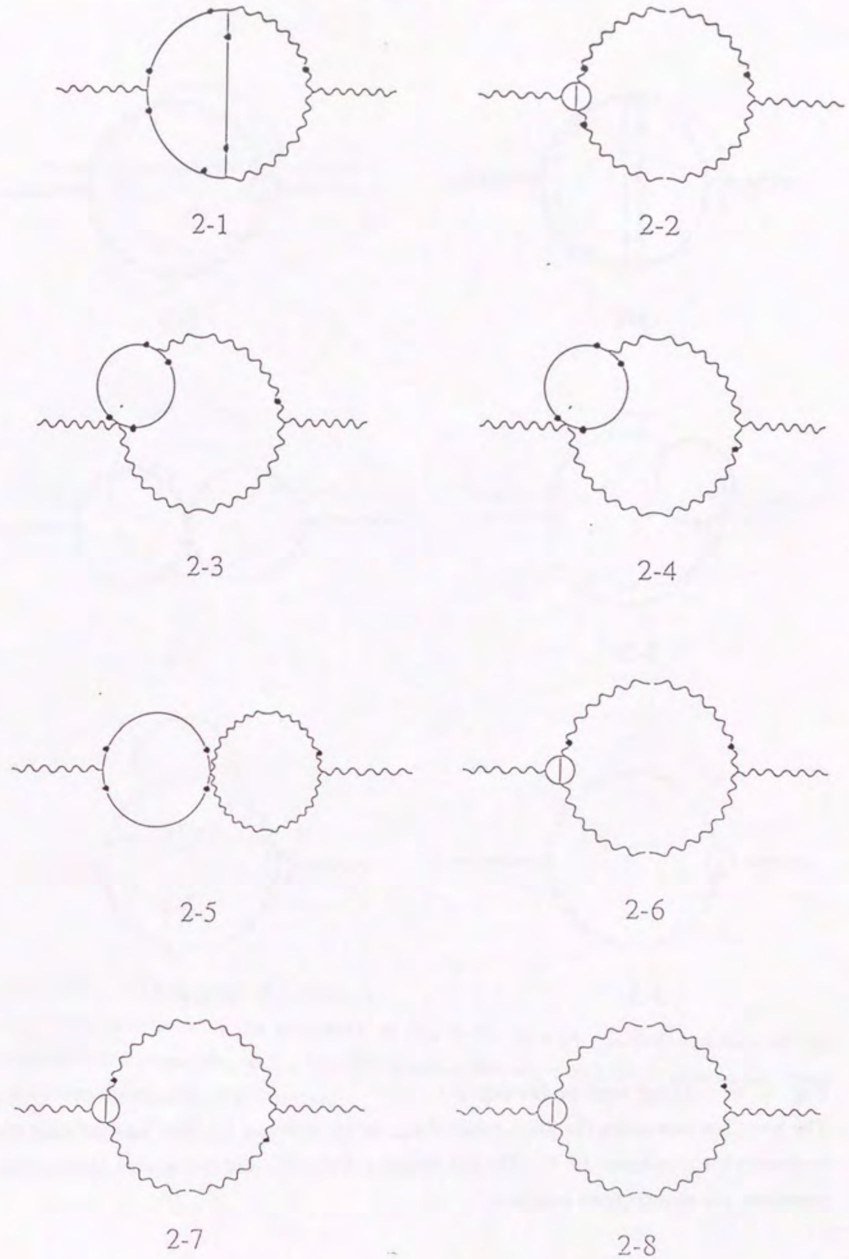


Fig. 5 Diagrams of Group 2

The wavy line represents the propagator of $H_{\mu\nu}$ or the external $\hat{h}_{\mu\nu}$ line, and the solid line represents the propagator of X_i. The dot denots a derivative and the symbol in the vertex represents the counter term insertion.

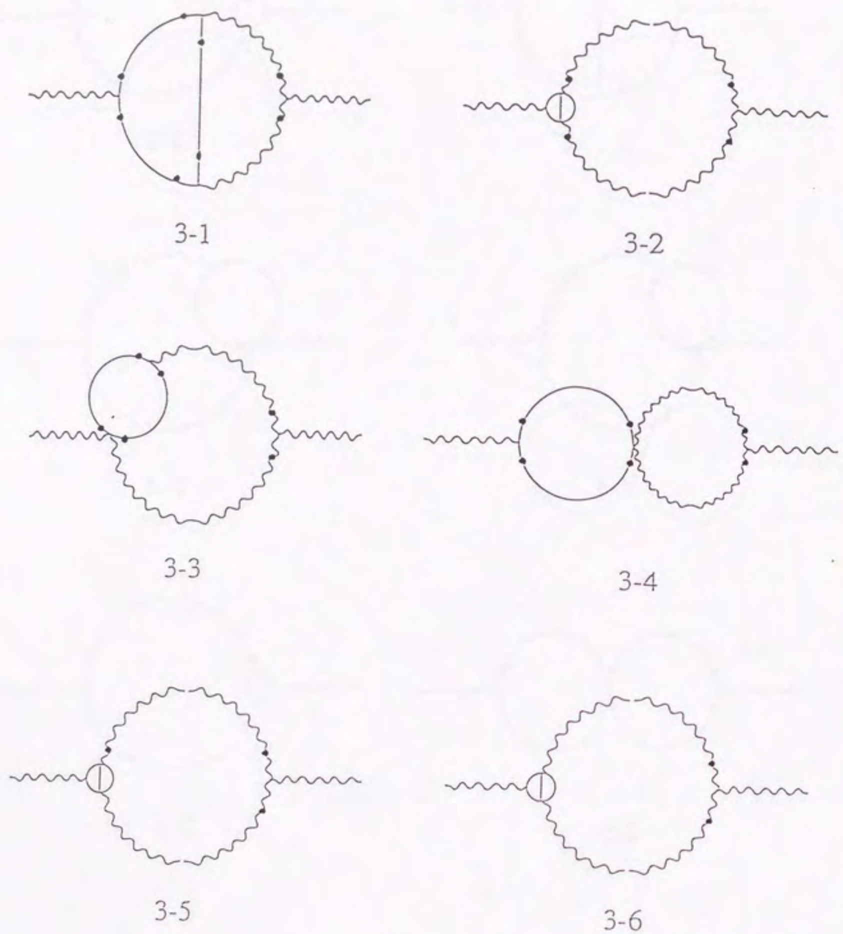


Fig. 6 Diagrams of Group 3

The wavy line represents the propagator of $H_{\mu\nu}$ or the external $\hat{h}_{\mu\nu}$ line, and the solid line represents the propagator of X_i. The dot denotes a derivative and the symbol in the vertex represents the counter term insertion.

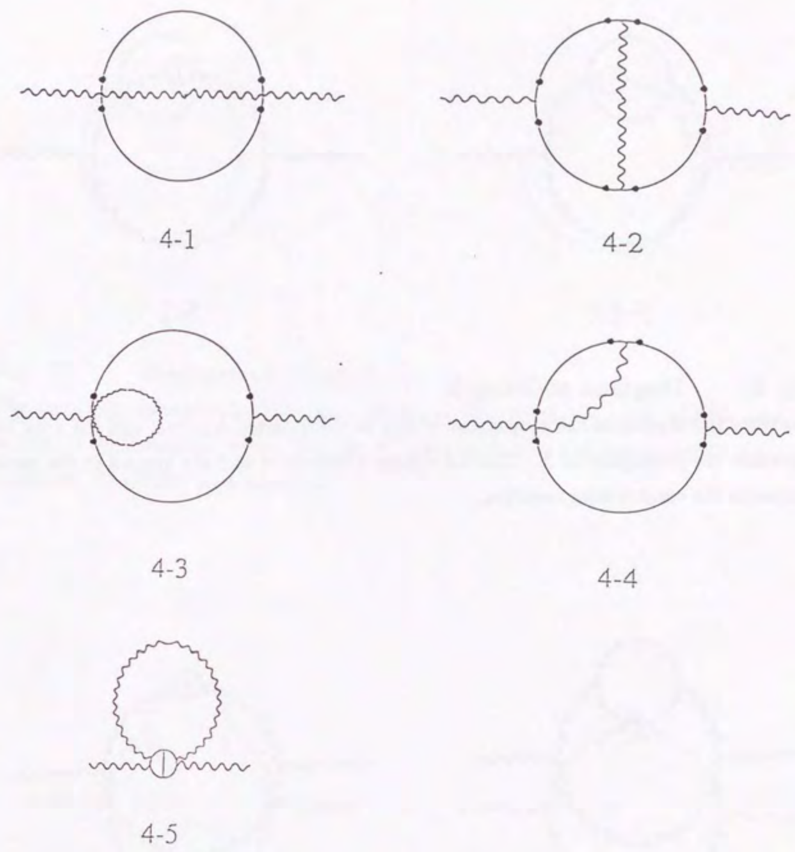


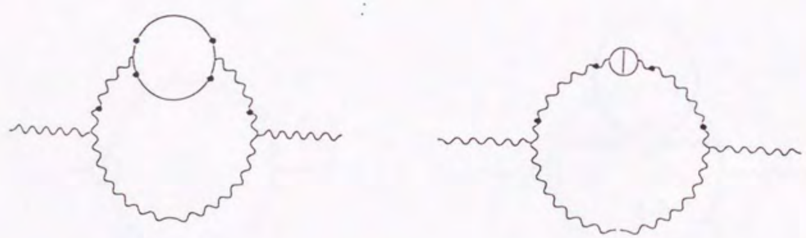
Fig. 7 Diagrams of Group 4

The wavy line represents the propagator of $H_{\mu\nu}$ or the external $\hat{h}_{\mu\nu}$ line, and the solid line represents the propagator of X_i. The dot denotes a derivative and the symbol in the vertex represents the counter term insertion.

5-1 5-2

Fig. 8 Diagrams of Group 5

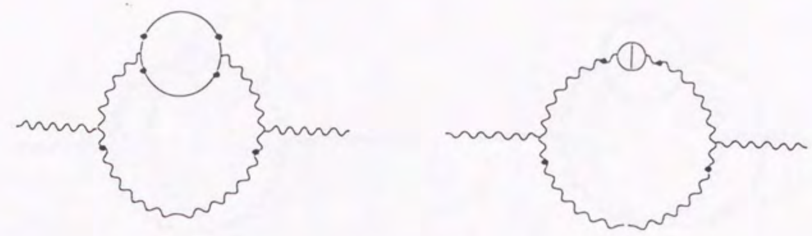
The wavy line represents the propagator of $H_{\mu\nu}$ or the external $\hat{h}_{\mu\nu}$ line, and the solid line represents the propagator of X_i. The dot denotes a derivative and the symbol in the vertex represents the counter term insertion.



6-1 6-2

Fig. 9 Diagrams of Group 6

The wavy line represents the propagator of $H_{\mu\nu}$ or the external $\hat{h}_{\mu\nu}$ line, and the solid line represents the propagator of X_i. The dot denotes a derivative and the symbol in the vertex represents the counter term insertion.



7-1 7-2

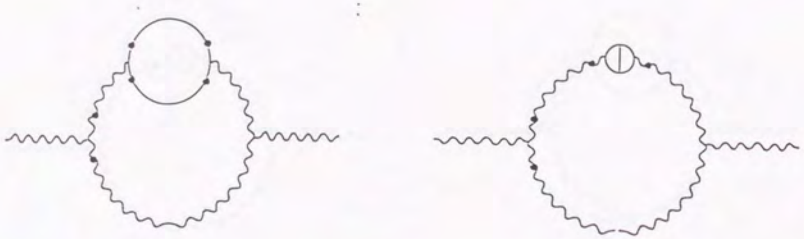
Fig. 10 Diagrams of Group 7

The wavy line represents the propagator of $H_{\mu\nu}$ or the external $\hat{h}_{\mu\nu}$ line, and the solid line represents the propagator of X_i. The dot denotes a derivative and the symbol in the vertex represents the counter term insertion.

8-1 8-2

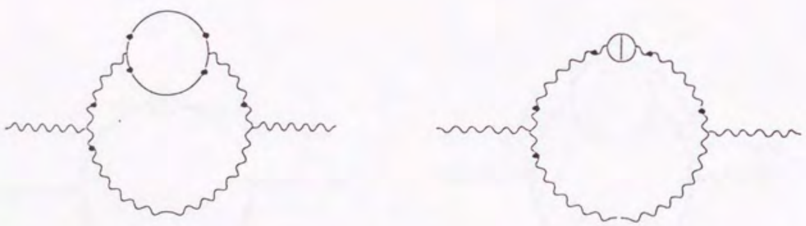
Fig. 11 Diagrams of Group 8

The wavy line represents the propagator of $H_{\mu\nu}$ or the external $\hat{h}_{\mu\nu}$ line, and the solid line represents the propagator of X_i. The dot denotes a derivative and the symbol in the vertex represents the counter term insertion.



9-1 9-2

Fig. 12 Diagrams of Group 9
The wavy line represents the propagator of $H_{\mu\nu}$ or the external $\hat{h}_{\mu\nu}$ line, and the solid line represents the propagator of X_i. The dot denotes a derivative and the symbol in the vertex represents the counter term insertion.



10-1 10-2

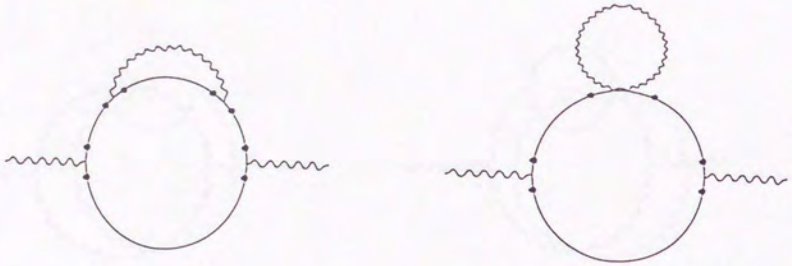
Fig. 13 Diagrams of Group 10
The wavy line represents the propagator of $H_{\mu\nu}$ or the external $\hat{h}_{\mu\nu}$ line, and the solid line represents the propagator of X_i. The dot denots a derivative and the symbol in the vertex represents the counter term insertion.

11-1 11-2

Fig. 14 Diagrams of Group 11
The wavy line represents the propagator of $H_{\mu\nu}$ or the external $\hat{h}_{\mu\nu}$ line, and the solid line represents the propagator of X_i. The dot denotes a derivative and the symbol in the vertex represents the counter term insertion.

12-1 12-2

Fig. 15 Diagrams of Group 12
The wavy line represents the propagator of $H_{\mu\nu}$ or the external $\hat{h}_{\mu\nu}$ line, and the solid line represents the propagator of X_i. The dot denotes a derivative and the symbol in the vertex represents the counter term insertion.



13-1 13-2

Fig. 16 Diagrams of Group 13

The wavy line represents the propagator of $H_{\mu\nu}$ or the external $\hat{h}_{\mu\nu}$ line, and the solid line represents the propagator of X_i. The dot denots a derivative and the symbol in the vertex represents the counter term insertion.

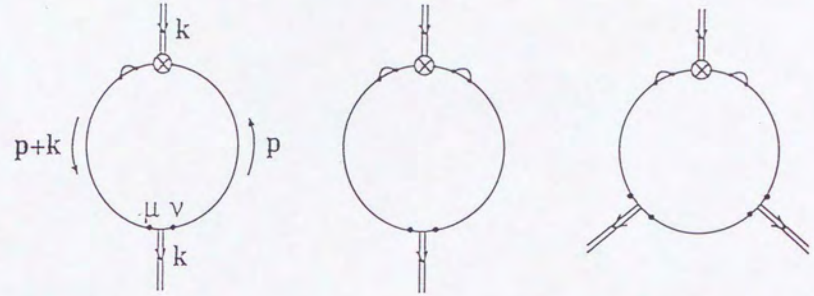
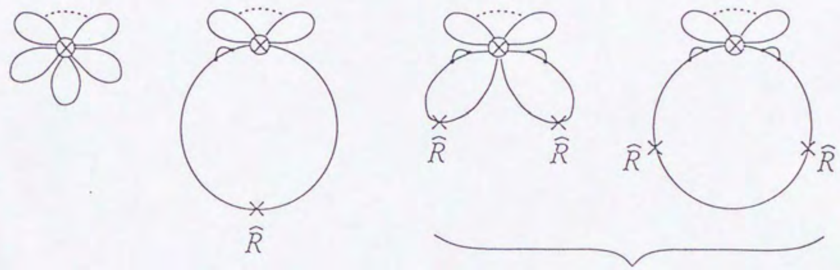


Fig. 17

The diagrams we have to evaluate in order to justify the simplification $G_{\mu\nu} = 0$. The dot represents a derivative and the arc connecting two dots implies a contraction.



(a) (b) (c)

Fig. 18

The diagrams which appear in calculating the expectation value of each term in (C.1). (a),(b) and (c) correspond to the first, second and third terms respectively. The dot represents a derivative and the arc connecting two dots implies a contraction, as in Figure 1. The cross represents a mass insertion using the $\frac{\epsilon^2}{16}\hat{R}c^{2}$ term in the action.