博士論文

On the quantitative shadowing property of topological dynamical systems

(位相的力学系の量的擬軌道追跡性について)

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ON THE QUANTITATIVE SHADOWING PROPERTY OF TOPOLOGICAL DYNAMICAL SYSTEMS

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1. Introduction

1.1. **General Introduction.**

The notion of the shadowing property was initially introduced in the study of hyperbolic differentiable dynamics. It originated in the influential early works of Anosov and Bowen [5, 11] and was later captured in the framework of topological dynamics. Since then it has been shown that the notion has many interesting consequences in qualitative study of dynamical systems, and it is still a developing branch of modern theory of dynamical systems [6, 33].

The shadowing property is defined based on the intuitive idea of the shadowing of rough orbits by true orbits. The formal definition is given as follows. A *topological dynamical system* is a pair (X, f) of a compact metric space X with a metric d and a continuous map f from X to itself. An infinite sequence $(x_i)_{i=0}^{\infty}$ of points in X is a *δ-pseudo orbit* of f if $d(f(x_i), x_{i+1}) \leq \delta$ for all $i \geq 0$. For given $\epsilon > 0$, a δ-pseudo orbit $(x_i)_{i=0}^{\infty}$ is said to be *ϵ-shadowed* by $x \in X$ if $d(x_i, f^i(x)) \leq \epsilon$ for all $i \geq 0$. We say that f has the *shadowing property* if for every $\epsilon > 0$, there is $\delta > 0$ such that every δ-pseudo orbit of f is ϵ-shadowed by some point of X. When f is a homeomorphism, a *δ-pseudo orbit* of f is defined to be a bi-infinite sequence of points $(x_i)_{i\in\mathbb{Z}}$ with $d(f(x_i), x_{i+1}) \leq \delta$ for all $i \in \mathbb{Z}$. We say that a point $x \in X$ *ϵ-shadows* $(x_i)_{i\in\mathbb{Z}}$ if $d(x_i, f^i(x)) \leq \epsilon$ for all $i \in \mathbb{Z}$. The corresponding *shadowing property* for homeomorphisms is defined similarly as above

Pseudo orbits appear naturally in the following two situations. One is the perturbation of the system and another is the numerical experiments. In the former case, an orbit of the perturbed system is considered as a pseudo orbit of the original system, and the shadowing property ensures the existence of corresponding orbit of the original system near it. In this respect, the shadowing property is closely related to the notion of *stability*. In the latter case, the numerical orbits are thought of as pseudo orbits because of their errors introduced by discretization methods and by finite-precision calculation. Then, the shadowing property of the system guarantees the existence of true orbits near them, and so it can be considered as a basis for the effectiveness of numerical methods.

A classical shadowing lemma states that a diffeomorphism on a smooth closed manifold satisfies the shadowing property around its hyperbolic invariant set. In such a context, the shadowing property can be considered as a topological expression of *hyperbolicity*, which is also closely related to the notion of stability. For instance, the class of structurally stable diffeomorphisms on a smooth closed manifold can be characterized in terms of the shadowing property [35, 37], and also about the topological dynamical systems, there are some results linking the topological stability and the shadowing property [40].

For topological dynamical systems, the shadowing property is common in the sense that C^0-generic homeomorphisms on a smooth closed manifold satisfy the shadowing property [34]. In the classical topological theory of hyperbolic dynamics, some kind of *expansiveness*, which is also an topological expression of hyperbolicity, is often assumed with the shadowing property. However, in recent years, several attempts have been made to explore the implications of the sole shadowing property in a purely topological setting. Such studies should contribute to understand more general dynamical systems exhibiting a certain non-hyperbolic behavior. It is worth mentioning that there exists a C^1-open set of diffeomorphisms on a certain manifold which do not satisfy the shadowing property [10]. Further research on the shadowing property is also expected to understand such phenomena.

As mentioned above, the shadowing property is important from the point of view of applications as well. It enables us to extract meaningful information of the system from coarse knowledge of the orbit structure. The method of shadowing is also used to rigorously prove the existence of periodic orbits and chaotic behavior [32]. There have been extensive studies on the shadowability of chaotic dynamical systems, and the idea of pointwise shadowing can be seen there [18].

From the general perspective as above, we study the shadowing property of topological dynamical systems. The main subject of this work is to quantify the arguments on the shadowing property. While giving the exact definition of quantitative shadowing properties in subsection 1.3, we pursue the consequences of quantitative shadowing properties and attempt to give the results on the shadowing property as their corollaries. Such an attempt will broaden the possibility for the applications of our results and also lead to a deeper understanding of the shadowing property.

This thesis consists of three main sections. The contents of Sections 2, 3, and 4 are based on [20], [21], and [22], respectively. In Section 2, we prove that a quantitative shadowing property combined with sensitivity implies the abundance of entropy points. As a corollary, for instance, it is proved that if a sensitive continuous map $f : X \to X$ has the shadowing property, then every point of X is an entropy point. The passage from the result on a quantitative shadowing property to the result on the shadowing property is a prototype of the arguments in later sections. Our results complement the previous results by Moothathu *et al.* [27, 29, 30]. In Section 3, we quantify the notion of shadowable points recently introduced by Morales [31], and define quantitative shadowable points. By giving basic properties of quantitative shadowable points and examples related to them, we provide a basis for studying the shadowing property from a local and quantitative viewpoint. Then, we prove a quantitative version of a Morales' theorem in [31] to understand it as a continuous limit. Moreover, we answer two questions asked in [31]. In Section 4, we extend the study on the (quantitative) shadowable points in relation to chaos and equicontinuity. The results in the first half of this section are on the chaotic side. We examine the implications of quantitative pointwise shadowability in connection with various chaotic properties. In the second half of this section, through the extension of a recent result by Li and Oprocha [28] and a method of chain decomposition, we give a detailed description of local features of interior points in the set of shadowable points, under the assumption of chain recurrence. Such points are characterized as chaotic or non-chaotic points in terms of two canonical dynamics, the full shift and the odometers.

In the following subsections, we describe the background of each main section and formulate the results therein.

1.2. **Section 2: Shadowing, sensitivity, and entropy points.**

In Section 2, we prove a sufficient condition that a point can be approximated by an entropy point in terms of the sensitivity and the shadowing property. More precisely, we prove that for a continuous map $f : X \to X$ and a closed f-invariant subset $S \subset X$, if eventually sensitive points of $f|_S$ are dense in S, then any point of S can be approximated by an entropy point with an accuracy corresponding to that of the shadowing.

Sensitivity (which is also known as *Sensitive dependence on initial conditions*) is one of the features of chaotic dynamical systems. It means, intuitively, that an extremely small difference of initial conditions at every point is amplified to be a significant difference in a later state, and so one may expect that such dynamical systems tend to have a positive topological entropy. However, as seen from a simple example (Example 2.2.1), some additional condition is necessary to ensure that a system has a positive entropy besides the sensitivity. Recently, several results in this direction have appeared [27, 29, 30], which proved sufficient conditions for a point to be an entropy point in terms of sensitivity at the point, recurrence near the point, and the shadowing property. Our results are related to them but different, especially in the point that we do not assume the recurrence.

Let us give some basic definitions and notations. Let (X, d) be a compact metric space and let $f : X \to X$ be a continuous map. For $S \subset X$ and $b > 0$, we say that f has a *b-shadowing property around S* if there is $\delta > 0$ such that every δ-pseudo orbit contained in S is b-shadowed by some $x \in X$. We say that f has the *shadowing property around S* if f has a b-shadowing property around S for every $b > 0$.

For $e > 0$, a point $x \in X$ is said to be an *e-sensitive point* of f if for any neighborhood U of x, there exist $y, z \in U$ and $n \in \mathbb{N}$ such that $d(f^n(y), f^n(z)) > e$. We denote by $Sen_e(f)$ the set of e-sensitive points of f and define $Sen(f) = \bigcup_{e>0} Sen_e(f)$. A point of $Sen(f)$ is called a *sensitive point* of f. We say that f is *sensitive* if $X = Sen_e(f)$ for some $e > 0$, and such $e > 0$ is called a *sensitive constant* for f (see [16] for an in-depth look at the concept of sensitivity).

Given a continuous map $f : X \to X$ and $n \geq 1$, define a metric d_n on X by $d_n(x, y) = \max_{0 \leq j \leq n-1} d(f^j(x), f^j(y))$. For $n \geq 1$ and $\epsilon > 0$, a subset $E \subset X$ is called *(n, ϵ)-separated* if $x \neq y$ $(x, y \in E)$ implies $d_n(x, y) > \epsilon$. For $A \subset X$, let $S(A, n, \epsilon)$ denote the maximal cardinality of an (n, ϵ)-separated set contained in A and consider

$$h(f, A, \epsilon) = \limsup_{n\to\infty} \frac{1}{n} \log S(A, n, \epsilon).$$

Note that $\epsilon_2 < \epsilon_1$ implies $h(f, A, \epsilon_1) \leq h(f, A, \epsilon_2)$, which guarantees the existence of $\lim_{\epsilon\to 0} h(f, A, \epsilon) \in [0, \infty]$. The *topological entropy of f on A*, denoted by $h(f, A)$, is $h(f, A) = \lim_{\epsilon\to 0} h(f, A, \epsilon)$. Then, the *topological entropy* of f, denoted by $h_{top}(f)$, is defined by $h_{top}(f) = h(f, X)$. In [43], Ye and Zhang introduced the notion of entropy points. A point $x \in X$ is said to be an *entropy point* of f if $h(f, \overline{U}) > 0$ for any neighborhood U of x. Let $Ent(f)$ denote the set of entropy points of f. It is known that $Ent(f)$ is a closed f-invariant subset of X, and $h_{top}(f) > 0$ iff $Ent(f) \neq \emptyset$ (see [41] and [43]).

Now let us state our first theorem.

Theorem 1.2.1. *Let $f : X \to X$ be a continuous map and let $S \subset X$ be a closed f-invariant subset. If there is $e > 0$ for which $\{x \in S : \omega(x) \cap Sen_e(f|_S) \neq \emptyset\}$ is dense in S and f has a b-shadowing property around S with $0 < 2b < e$, then for any $x \in S$, there exists $y \in Ent(f)$ such that $d(x, y) \leq b$. In particular, if f has the shadowing property around S, then $S \subset Ent(f)$.*

A point $x \in X$ is said to be a *recurrent point* of f if $x \in \omega(x)$, where $\omega(x)$ denotes the ω-limit set of x under f, and the set of recurrent points of f is denoted by $R(f)$. In [29], Moothathu proved that for a continuous map $f : X \to X$ with the shadowing property around a closed f-invariant subset $S \subset X$, letting $g = f|_S$, then for every $z \in S$, we have $z \in Ent(f)$ if the following two conditions are satisfied:

(M1) $z \in Sen(g)$;
(M2) $(z, z) \in \mathrm{Int}[\overline{R(g \times g)}]$ (where the closure and the interior are taken in $S \times S$).

In Theorem 1.2.1, instead of the simultaneous recurrence condition (M2) for a certain sensitive point, we assume the density of eventually sensitive points and show that all points of S are entropy points especially when f has the shadowing property around S. A finite sequence $(x_i)_{i=a}^{b}$ of points in X, where $0 \leq a < b < \infty$, is a *δ-chain* of f if $d(f(x_i), x_{i+1}) \leq \delta$ for all $a \leq i < b$. The proof of Theorem 1.2.1 is based on the construction of a tree-like chain structure. We apply the shortcut lemma (Lemma 2.1.2) to each edge of the tree, and then by shadowing them, prove that the root point is approximated by an entropy point.

It is immediate to obtain the following corollary from Theorem 1.2.1.

Corollary 1.2.1. *Let $f : X \to X$ be a continuous map. If f is sensitive and has the shadowing property, then $X = Ent(f)$.*

When f is a homeomorphism, we define $Sen_e^*(f) = Sen_e(f) \cup Sen_e(f^{-1})$. Then, we say that a homeomorphism f is *weakly sensitive* if $X = Sen_e^*(f)$ for some $e > 0$, and such $e > 0$ is called a *weakly sensitive constant* for f. Here we should mention that some authors call such a homeomorphism a *sensitive* homeomorphism (see [6] for instance). The weak sensitivity can be understood as an extended notion of *expansiveness*. As for the relation between the positivity of topological entropy and the expansiveness, there is a classical result by Fathi [15], proving that if a homeomorphism $f : X \to X$ is expansive and $\dim X > 0$, then $h_{top}(f) > 0$. Later, Kato [19] generalized the result for continuum-wise expansive homeomorphisms. However, the weak sensitivity in itself does not necessarily yield a positive topological entropy as shown Example 2.2.1.

The following theorem is a version of Theorem 1.2.1 for homeomorphisms, which shows that the weak sensitivity together with a shadowing property ensures a positive topological entropy.

Theorem 1.2.2. *Let $f : X \to X$ be a homeomorphism and let $S \subset X$ be a closed f-invariant subset. If $f|_S$ is weakly sensitive with a weakly sensitive constant $e > 0$ and f has a b-shadowing property around S with $0 < 2b < e$, then for any $x \in S$, there exists*

$y \in Ent(f) \cup Ent(f^{-1})$ such that $d(x, y) \le b$. In particular, if f has the shadowing property around S, then $S \subset Ent(f) \cup Ent(f^{-1})$.

As a corollary, we obtain the following.

Corollary 1.2.2. *Let $f : X \to X$ be a homeomorphism If f is weakly sensitive and has the shadowing property, then $X = Ent(f) \cup Ent(f^{-1})$, and especially $h_{top}(f) > 0$.*

Here, it is worth mentioning that the hypothesis of Theorem 1.2.2 cannot be replaced by a weaker condition that

$$\{x \in S : \omega(x) \cap Sen_e(f|_S) \neq \emptyset\} \cup \{x \in S : \alpha(x) \cap Sen_e(f|_S^{-1}) \neq \emptyset\}$$

is dense in S, which will be shown in Example 2.2.2.

Finally, we give two more corollaries of Theorem 1.2.1. For a map $f : X \to X$ and a point $x \in X$, we denote by $O_f(x)$ the orbit of x under f; i.e., $O_f(x) = \{f^n(x) : n \ge 0\}$. The following corollary is an application of Theorem 1.2.1 under the existence of a dense orbit.

Corollary 1.2.3. *Let $f : X \to X$ be a continuous map and let $S \subset X$ be a closed f-invariant subset. If f has a b-shadowing property around S, $S = \overline{O_f(x)}$ for some $x \in S$ and $Sen_e(f|_S) \neq \emptyset$ for some $e > 2b$, then for any $y \in S$, there exists $z \in Ent(f)$ such that $d(y, z) \le b$. In particular, if f has the shadowing property around S, then $S \subset Ent(f)$.*

The next corollary gives a sufficient condition for a point to be an entropy point under the hypothesis of the shadowing property.

Corollary 1.2.4. *Let $f : X \to X$ be a continuous map and let $S \subset X$ be a closed f-invariant subset. If f has the shadowing property around S, then we have $\{x \in S : \omega(x)$ is non-minimal $\} \subset Ent(f)$.*

It should be mentioned that Corollary 1.2.4 also follows from Moothathu's result ([29, Theorem 3]) with an additional argument, while this corollary will be proved as a direct consequence of Corollary 1.2.3.

1.3. **Section 3: Quantitative shadowable points.**

Recently, Morales introduced the notion of shadowable points for homeomorphisms by splitting the shadowing property into pointwise shadowings [31]. A shadowable point of a homeomorphism is defined to be a point such that the shadowing lemma holds for pseudo orbits passing through the point. Such an idea has been also seen [2] or [44] for instance, but it has not been explicitly formulated until Morales did in [31]. It prompts us to reconsider the theory of shadowing from a local point of view. In this section, we extend the notion of shadowable points for continuous maps and study quantitative shadowable points with a given shadowing accuracy. A quantitative version of Morales' theorem in [31] is proved. In addition, we prove a dichotomy on the set of quantitative shadowable points for chain transitive or transitive homeomorphisms, and also give a characterization of homeomorphisms with a quantitative shadowing property as those having full measure of quantitative shadowable points for all ergodic measures.

Let us define a quantitative shadowing property. Let (X, d) be a compact metric space and let $f : X \to X$ be a continuous map. For $b > 0$, we say that f has a *b-shadowing* *property* if there exists $\delta > 0$ such that every δ-pseudo orbit of f is b-shadowed by some $x \in X$. Then, for $c \geq 0$, we say that f has a *$c+$-shadowing property* if f has the b-shadowing property for every $b > c$. Note that the $0+$-shadowing property corresponds with the usual shadowing property.

Then, we define quantitative shadowable points. For $b > 0$, a *b-shadowable point* is a point $x \in X$ such that there exists $\delta > 0$ for which every δ-pseudo orbit $(x_i)_{i=0}^{\infty}$ with $x_0 = x$ is b-shadowed by some point of X. We denote by $Sh_b^+(f)$ the set of b-shadowable points of f. Then, for $c \geq 0$, we define

$$Sh_{c+}^+(f) = \bigcap_{b>c} Sh_b^+(f),$$

and a point of $Sh_{c+}^+(f)$ is called a *$c+$-shadowable point* of f. A $0+$-shadowable point is also simply called a *shadowable point*, and $Sh_{0+}^+(f)$ is also denoted by $Sh^+(f)$.

The corresponding *b-shadowing property* and *$c+$-shadowing property* for homeomorphisms are defined similarly as above for $b > 0$ and $c \geq 0$, and so do *b-shadowable points* and *$c+$-shadowable points*. Given a homeomorphism $f : X \to X$, we denote by $Sh_b(f)$ the set of b-shadowable points of f and by $Sh_{c+}(f)$ the set of $c+$-shadowable points of f. Then, we have

$$Sh_{c+}(f) = \bigcap_{b>c} Sh_b(f).$$

A point of $Sh_{0+}(f)$(or a $0+$-shadowable point) is also simply called a *shadowable point*, and $Sh_{0+}(f)$ is also denoted by $Sh(f)$. This terminology is consistent with that of [31]. When we need to distinguish pseudo orbits, shadowing property, and shadowable points in the map sense from those in the homeomorphism sense, the former will be called with "forward" (e.g. forward shadowing property).

With the notion of quantitative shadowable points, we can measure the shadowability of a map at a point quantitatively even if the point is not a shadowable point in the strict sense. In numerical experiments, numerical orbits are thought of as pseudo orbits, and the shadowing property of the system guarantees the existence of true orbits near them. Then, it is natural to ask what we can know about the system under the situation that a finite shadowing accuracy is ensured. Given a statement for the systems with the shadowing property, it is of theoretical interest to ask what still holds true if we weaken the assumption of the shadowing property to a quantitative shadowing property, and in particular, ask whether we can recover the original statement as a continuous limit or not. These facts motivate us to introduce the notion of quantitative shadowable points and quantitative shadowing property. It will be shown that for every homeomorphism $f : X \to X$ and every $c \geq 0$, $Sh_{c+}(f)$ is an f-invariant Borel set in X (Corollary 3.1.2 and Lemma 3.4.1), and f has the $c+$-shadowing property iff $Sh_{c+}(f) = X$ (Corollary 3.1.1). Hence, the $c+$-shadowable points (resp. $c+$-shadowing property) seems to be a natural extension of the shadowable points (resp. shadowing property).

The notion of shadowable points is closely related to the *chain continuity*. We recall the definition of the chain continuity from [2]. Let $f : X \to X$ be a continuous map. A point $x \in X$ is said to be a *chain continuity point* for f if for every $\epsilon > 0$, there is $\delta > 0$ such that every δ-pseudo orbit $(x_i)_{i=0}^{\infty}$ of f with $x_0 = x$ is ϵ-shadowed by x itself. Recall that a point $x \in X$ is an *equicontinuity point* for f if for every $\epsilon > 0$, there is $\delta > 0$

such that $d(x, y) \leq \delta$ implies $\sup_{n\geq 0} d(f^n(x), f^n(y)) \leq \epsilon$ for all $y \in X$. It is easy to see that $x \in X$ is a chain continuity point for f iff $x \in Sh^+(f)$ and x is an equicontinuity point for f. Therefore, when $x \in X$ is an isolated point of X, x is a chain continuity point for f iff $x \in Sh^+(f)$. When f is a homeomorphism, it is clear that if $x \in X$ is a chain continuity point for both f and f^{-1}, then $x \in Sh(f)$, and the converse holds if x is an isolated point of X. In general, chain continuity is a much stronger property than shadowability. For example, when X is a compact smooth boundaryless manifold whose dimension is at least 2, a C^0-generic homeomorphism satisfies the shadowing property [34]. On the other hand, while a C^0-generic homeomorphism has a dense G_δ-subset of chain continuity points, it simultaneously contains many non-chain continuity points [4]. The full shift on a finite alphabet satisfies the shadowing property but has no chain continuity point.

The first result in this section concerns a question in [31]. For the statement, we need a few definitions. A homeomorphism $f : X \to X$ is said to be *transitive* if for every non-empty open subsets U and V in X, there is an $n \in \mathbb{Z}$ such that $f^n(U) \cap V \neq \emptyset$. Since X is a compact metric space, f is transitive iff there exists $x \in X$ such that the orbit $O_f(x) = \{f^n(x) : n \in \mathbb{Z}\}$ is dense in X (see [41]). In [31, Remark 1.4], Morales asked if there is a transitive homeomorphism f for which $Sh(f)$ is a non-empty and non-compact subset. We first consider the problem in a slightly different way. Given a continuous map $f : X \to X$, a finite sequence of points $(x_i)_{i=0}^k$ in X (where k is a positive integer) is called a *δ-chain of f* if $d(f(x_i), x_{i+1}) \leq \delta$ for every $0 \leq i \leq k-1$. Then, we say that f is *chain transitive* if for any $x, y \in X$ and $\delta > 0$, there is a δ-chain $(x_i)_{i=0}^k$of f such that $x_0 = x$ and $x_k = y$. A continuous map $f : X \to X$ is *transitive in the strong sense* if for every non-empty open subsets U and V in X, there is an $n > 0$ such that $f^n(U) \cap V \neq \emptyset$. It is obvious that if f is transitive in the strong sense, then f is chain transitive, and the converse holds if f has the shadowing property. When f is a transitive homeomorphism, if X is perfect, i.e., X contains no isolated point, then f is transitive in the strong sense, and if X contains an isolated point, then the set of isolated points of X coincides with a single dense orbit (see [3] for details). We prove the following theorem.

Theorem 1.3.1. *If $f : X \to X$ is a chain transitive homeomorphism, then for every $c \geq 0$, either $Sh_{c+}(f) = \emptyset$ or $Sh_{c+}(f) = X$.*

Then, the following theorem answers the question above.

Theorem 1.3.2. *If $f : X \to X$ is a transitive homeomorphism, then for every $c \geq 0$, either $Sh_{c+}(f) = \emptyset$ or $Sh_{c+}(f) = X$.*

By this theorem, we see that if a homeomorphism f is transitive, then $Sh(f)$ cannot be non-empty and non-compact simultaneously. Note that the chain transitivity and the transitivity are independent of each other.

We say that a continuous map $f : X \to X$ is *pointwise recurrent* if $R(f) = X$. A continuous map $f : X \to X$ is said to be *equicontinuous* if for every $\epsilon > 0$, there is $\delta > 0$ such that $d(x, y) \leq \delta$ implies $\sup_{n\geq 0} d(f^n(x), f^n(y)) \leq \epsilon$ for all $x, y \in X$. Since X is compact, $f : X \to X$ is equicontinuous iff every point of X is an equicontinuity point for f. We remark that if f is surjective and equicontinuous, then f is a homeomorphism and f^{-1} is also equicontinuous. For $x \in X$, the connected component of X containing x is denoted by $C(x)$ and let $X^{deg} = \{x \in X : C(x) = \{x\}\}$.

Morales proved the following two results (which are Theorems 1.2 and 1.3 in [31]).

(M1) If $f : X \to X$ is a pointwise recurrent homeomorphism, then $Sh(f) \subset X^{deg}$.
(M2) If $f : X \to X$ is an equicontinuous homeomorphism, then $Sh(f) = X^{deg}$.

We prove a quantitative version of (M1) for continuous maps as follows.

Theorem 1.3.3. *Let $f : X \to X$ be a pointwise recurrent continuous map. For every $x \in X$ and every $c \geq 0$, if $x \in Sh^+_{c+}(f)$, then* $\operatorname{diam} C(x) \leq 2c$.

By putting $c = 0$ in Theorem 1.3.3, we obtain (M1) again. We prove Theorem 1.3.3 following Morales' argument in the proof of (M1), but his argument itself falls short of proving Theorem 1.3.3.

Remark 1.3.1. Let $X = [0, 1]$ be the unit interval and let $Id : X \to X$ be the identity map. Then, Id is obviously pointwise recurrent. It is also obvious that $0 \in Sh^+_{1/2}(Id)$, while $\operatorname{diam} C(0) = \operatorname{diam} [0, 1] = 1$. This shows that the bound given in Theorem 1.3.3 is sharp.

It is immediate to obtain the following corollary from Theorem 1.3.3.

Corollary 1.3.1. *Let $f : X \to X$ be a pointwise recurrent continuous map. If X is connected, then for any $0 \leq c < \operatorname{diam} X/2$, we have $Sh^+_{c+}(f) = \emptyset$.*

It is well known that every equicontinuous homeomorphism $f : X \to X$ is pointwise recurrent, therefore we have $Sh(f) \subset X^{deg}$ by (M1). The Morales' second result (M2) claims that the converse $X^{deg} \subset Sh(f)$ also holds for any equicontinuous homeomorphism. We remark that there is a pointwise recurrent homeomorphism for which $X^{deg} \subset Sh(f)$ does not hold. Such an example is given in Example 3.5.1 (see [31, Remark 1.3]).

Example 3.5.2 shows that (M2) does not necessarily hold for an equicontinuous map in its own form. It is worth mentioning that, according to Theorem 3.3 in [2], we have a version of (M2) for equicontinuous maps as follows. For a continuous map $f : X \to X$, let $C^+_f(x)$ denote the set of points $y \in X$ such that for every $\delta > 0$, there is a δ-chain $(x_i)_{i=0}^k$ with $x_0 = x$ and $x_k = y$. Then, for any equicontinuous map $f : X \to X$, we have $Sh^+(f) = \{x \in X : \dim C^+_f(x) = 0\}$.

A map $f : X \to X$ is said to be an *isometry* if $d(f(x), f(y)) = d(x, y)$ for every $x, y \in X$. Note that an isometry on a compact metric space must be surjective. It is well known that for any equicontinuous homeomorphism $f : X \to X$, there is a metric D on X equivalent to d for which f is an isometry with respect to D. In fact, $D : X \times X \to [0, \infty)$ defined by $D(x, y) = \sup_{n \in \mathbb{Z}} d(f^n(x), f^n(y))$ is such a metric. If f is an isometry, we can prove a simple quantitative statement by the same argument as in the proof of [31, Theorem 1.3].

Proposition 1.3.1. *Let $f : X \to X$ be an isometry and let $c \geq 0$. If* $\operatorname{diam} C(x) \leq c$ *for $x \in X$, then $x \in Sh_{c+}(f)$.*

If $f : X \to X$ is an equicontinuous homeomorphism, then by choosing the compatible metric D above and putting $c = 0$ in Proposition 1.3.1, we can recover (M2). Example

3.5.3 shows that the assertion"$x \in Sh_{c+}(f)$" in Proposition 1.3.1 cannot be replaced by "$x \in Sh_c(f)$". It also shows that $Sh_{c+}(f) = Sh_c(f)$ does not always hold.

The following two theorems deal with a shadowing property for homeomorphisms from an ergodic theoretical viewpoint. Let $f : X \to X$ be a homeomorphism and let $\mathcal{M}_f^{erg}(X)$ be the set of ergodic f-invariant Borel measures on X. As shown in Corollary 3.1.2, $Sh_{c+}(f)$ is f-invariant for every $c \geq 0$, and hence we have $\mu(Sh_{c+}(f)) = 0$ or $\mu(Sh_{c+}(f)) = 1$ for each $\mu \in \mathcal{M}_f^{erg}(X)$. For $S \subset X$ and $b > 0$, we say that f has a *b-shadowing property around S* if there is $\delta > 0$ such that every δ-pseudo orbit contained in S is b-shadowed by some $x \in X$. Moreover, we say that f has the *$c+$-shadowing property around S* if f has a b-shadowing property around S for every $b > c \geq 0$. The 0+-shadowing property around S is also simply called *shadowing property around S*. Denote by $\mathrm{supp}(\mu)$ the support of $\mu \in \mathcal{M}_f^{erg}(X)$.

Theorem 1.3.4. *Let $f : X \to X$ be a homeomorphism. If $\mu(Sh_{c+}(f)) = 1$ with $c \geq 0$ and $\mu \in \mathcal{M}_f^{erg}(X)$, then f has the $c+$-shadowing property around* $\mathrm{supp}(\mu)$. *In particular, if $\mu(Sh(f)) = 1$ with $\mu \in \mathcal{M}_f^{erg}(X)$, then f has the shadowing property around* $\mathrm{supp}(\mu)$.

Remark 1.3.2. The fact that $Sh_{c+}(f)$, $c \geq 0$, are Borel sets in X has not been proved yet, which will be given as Lemma 3.4.1 (see [31, Remark 1.2]).

Theorem 1.3.5. *Let $f : X \to X$ be a homeomorphism and let $c \geq 0$. If $\mu(Sh_{c+}(f)) = 1$ for every $\mu \in \mathcal{M}_f^{erg}(X)$, then f has the $c+$-shadowing property. In particular, if $\mu(Sh(f)) = 1$ for every $\mu \in \mathcal{M}_f^{erg}(X)$, then f has the shadowing property.*

1.4. **Section 4: Properties of shadowable points: chaos and equicontinuity.**

Some basic properties and several results of (quantitative) shadowable points are obtained in Section 3. In this section, we extend the study on (quantitative) shadowable points. The main idea is to localize and quantify the arguments on the shadowing property in connection with chaos and equicontinuity. The chaos includes the positive entropy, sensitivity, and Li-Yorke chaos, and corresponds to the full-shift, while the equicontinuity corresponds to the odometers (or adding machines).

The first main result in this section gives sufficient conditions for a quantitative shadowable point to be approximated by an entropy point, which concern the notions of sensitivity and Li-Yorke pairs (Theorem 1.4.1). As a corollary, we obtain relatively simple sufficient conditions for a shadowable point to be an entropy point (Corollary 1.4.1). By this corollary, owing to the notion of shadowable points, we can concisely specify entropy points in connection with other chaotic properties of dynamical systems. As a consequence, we establish the equivalence of two definitions of chaos, Li-Yorke chaos and the positive topological entropy, under the shadowing property (Corollary 1.4.2). Moreover, we give a lower estimate of the positive topological entropy under the presence of a Li-Yorke pair and a quantitative shadowing property (Theorem 1.4.2).

The second main result in this section provides a dichotomy on interior points in the set of shadowable points under the assumption of chain recurrence (Theorem 1.4.3). It tells us that being an interior point in the set of shadowable points (with chain recurrence) enables us to characterize the point as a chaotic point or a non-chaotic point in comparison with two canonical dynamics, i.e., the full shift and odometers (see properties (S2) and (E2) in Theorem 1.4.3). It also depicts how chaotic points and non-chaotic points, or full shift extensions and odometers are mixed in the chain recurrent set. According to [4], the mixture of full-shift extensions and odometers is a C^0-generic property of homeomorphisms on a smooth closed manifold, so our results complement such a picture. In the classical topological theory of hyperbolic dynamics, some kind of expansiveness, which is also a topological expression of hyperbolicity, is often assumed with the shadowing property, where the possibility of the presence of non-trivial equicontinuous subsystems is excluded. Therefore, our results seem to give an insight into a certain non-hyperbolic behavior.

Here, we give some basic definitions used in this section. Given a continuous map $f : X \to X$, a subset $S \subset X$ is *f-invariant* if $f(S) \subset S$. A *subsystem* of (X, f) is a pair of a closed f-invariant subset $S \subset X$ and $f|_S$. We say that (X, f) (or f) is *minimal* if X does not contain any non-empty, proper, and closed f-invariant subset. For dynamical systems (X, f) and (Y, g), a *factor map* is a continuous surjection $\pi : X \to Y$ with $\pi \circ f = g \circ \pi$. When there is a factor map $\pi : X \to Y$, then we say that (Y, g) is a *factor* of (X, f), and (X, f) is an *extension* of (Y, g). When a factor map $\pi : X \to Y$ is 1-1, π is said to be a *conjugacy*, and we say that (X, f) is *conjugate* to (Y, g). Factor maps and subsystems are basic notions for describing the properties of dynamical systems.

Before stating our first result in this section, we need to define *Li-Yorke pairs* and *Li-Yorke chaos*. For a dynamical system (X, f), a pair of points $\{x, y\} \subset X$ is said to be a *Li-Yorke pair* (with modulus $e > 0$) if one has simultaneously,

$$\liminf_{n\to\infty} d(f^n(x), f^n(y)) = 0 \quad \text{and} \quad \limsup_{n\to\infty} d(f^n(x), f^n(y)) > e > 0.$$

A subset $S \subset X$ is called *scrambled* if any pair of distinct points $x, y \in S$ is a Li-Yorke pair. Then, a system (X, f) is called *Li-Yorke chaotic* if X contains an uncountable scrambled set.

Throughout this section, for a continuous map $g : Y \to Y$, we denote by $CR(g)$ the chain recurrent set of g and by $EC(g)$ the set of equicontinuity points for g. Now let us state our first result in this section.

Theorem 1.4.1. *Let $f : X \to X$ be a continuous map. Given $x \in Sh^+_{c+}(f)$ with $c \geq 0$, if $e > 2c$ and one of the following conditions is satisfied, then there exists $w \in Ent(f)$ such that $d(x, w) \leq c$.*

(1) *There is a closed f-invariant subset $S \subset X$ such that $CR(f|_S) = S$ and $\omega(x, f) \cap Sen_e(f|_S) \neq \emptyset$.*
(2) *There is $y \in X$ such that $\{x, y\} \subset X$ is a Li-Yorke pair with modulus e.*
(3) *There is a closed f-invariant subset $S \subset \omega(x, f)$ such that $\omega(x, f) \setminus B_e(S) \neq \emptyset$, where $B_e(S) = \{y \in X : d(y, S) \leq e\}$.*

Theorem 1.4.1 gives three sufficient conditions for a quantitative shadowable point to be approximated by an entropy point. Roughly speaking, our proof of Theorem 1.4.1 is based on the observation that if one of the conditions (1)-(3) is satisfied, then x limits to a point such that there are sufficiently "separated" pairs of two cycles through the point. By constructing pseudo orbits beginning at x and eventually turning around the cycles, and then by shadowing them, we prove that x is approximated by an entropy point (Lemma 4.1.2). Indeed, it has been observed so far that the existence of such a "separated" pair of two cycles near a point together with the shadowing property enables us to obtain a factor map onto the full shift from a subsystem of some power of the map (see, for example, [24, 27, 30]). As far as the author knows, such an idea goes back to 80's [23]. In this section, we explicitly define "e-separated pairs of two δ-cycles at a point" (Definition 4.1.1) and provide three sufficient conditions for the existence of such objects, each of which corresponds to one of the conditions in Theorem 1.4.1 (Lemma 4.1.1). By using them, we prove Theorem 1.4.1. The method to obtain a factor map onto the full shift is described in Lemma 4.1.3.

Applying Theorem 1.4.1 with $c = 0$, it is immediate to obtain the following corollary, which provides sufficient conditions for a shadowable point to be an entropy point.

Corollary 1.4.1. *Let $f : X \to X$ be a continuous map. Given $x \in Sh^+(f)$, if one of the following conditions is satisfied, then $x \in Ent(f)$.*

(1) *There is a closed f-invariant subset $S \subset X$ such that $CR(f|_S) = S$ and $\omega(x, f) \cap Sen(f|_S) \neq \emptyset$.*
(2) *There is $y \in X$ such that $\{x, y\} \subset X$ is a Li-Yorke pair.*
(3) *$\omega(x, f)$ is non-minimal for f.*

By Theorem 1.4.1 and Lemma 4.1.3 together with the result of [7], we obtain the following corollary.

Corollary 1.4.2. *Let $f : X \to X$ be a continuous map with the shadowing property. Then, the following properties are equivalent.*

(1) *$h_{top}(f) > 0$.*
(2) *(X, f) has a Li-Yorke pair.*
(3) *(X, f) is Li-Yorke chaotic.*
(4) *There exists $x \in X$ such that $\omega(x, f)$ is non-minimal for f.*

Positive topological entropy is a characteristic feature of chaos. It is well-known that positive topological entropy implies Li-Yorke chaos for any surjective continuous map on a compact metric space ([7, Corollary 2.4]). Corollary 1.4.2 claims that when the shadowing property is assumed, the presence of a Li-Yorke pair implies positive topological entropy, and so does Li-Yorke chaos by the fact above. As a consequence, two definitions of chaos coincide under the shadowing property. We remark here that for interval maps, the presence of a Li-Yorke pair implies Li-Yorke chaos, but there are Li-Yorke chaotic interval maps with zero topological entropy [25, 39, 42].

As the next step, we give a lower estimate of the topological entropy under the presence of a Li-Yorke pair and a quantitative shadowing property. Let d_2 denote the metric on $X^2 = X \times X$ defined by $d_2((a, b), (a', b')) = \max\{d(a, a'), d(b, b')\}$.

Theorem 1.4.2. *Let $f : X \to X$ be a continuous map and suppose that the following three conditions hold:*

(1) *$e > 2b > 0$;*
(2) *$x \in Sh_b^+(f)$ and every δ-pseudo orbit $(x_i)_{i=0}^{\infty}$ of f with $x_0 = x$ is b-shadowed by some point of X; and*
(3) *There is $y \in X$ such that $\{x, y\} \subset X$ is a Li-Yorke pair with modulus e.*

Then, we have

$$h_{top}(f) \geq \frac{1}{2N_2(\delta)} \log 2,$$

where $N_2(\delta)$ denotes the minimum cardinality of an open cover of (X^2, d_2) whose mesh is $\leq \delta$.

Then, we proceed to a study on the presence of regularly recurrent points near a chain recurrent point in the interior of the set of shadowable points. The following proposition claims that there is a periodic point or a point whose orbit closure is conjugate to an odometer in any neighborhood of such a point. It is a slight extension of a recent result by Li and Oprocha [28, Corollary 3.3], and we also give an alternative proof of it through the construction of a factor map (Lemma 4.3.1).

Let us briefly review the definition of odometers. Given a continuous map $f : X \to X$, a point $x \in X$ is said to be *regularly recurrent* if for every neighborhood U of x, there is $k \in \mathbb{N}$ such that $f^{kn}(x) \in U$ for all $n \geq 0$, and *minimal* (or *almost periodic*) if the restriction of f to the orbit closure $\overline{O_f(x)} = \overline{\{f^n(x) : n \geq 0\}}$ is minimal. We denote by $RR(f)$ (resp. $M(f)$) the set of regularly recurrent (resp. minimal) points of f. Note that $RR(f) \subset M(f)$. It holds that $M(f) = M(f^m)$ for every $m \in \mathbb{N}$ (see, for example, [29]). An *odometer* (also called an *adding machine*) is defined as follows. Given a strictly increasing sequence $m = (m_k)_{k=1}^{\infty}$ of positive integers such that $m_1 \geq 2$ and m_k divides m_{k+1} for each $k = 1, 2, \dots$, we define

- $X(k) = \{0, 1, \dots, m_k - 1\}$ (with the discrete topology);
- $X_m = \{x = (x_k)_{k=1}^{\infty} \in \prod_{k=1}^{\infty} X(k) : x_k \equiv x_{k+1} \pmod{m_k}\}$;
- $g(x)_k = x_k + 1 \pmod{m_k}$ for $x \in X_m$.

The resulting dynamical system (X_m, g) is called an odometer with the periodic structure m. An odometer is characterized as a minimal equicontinuous system on Cantor space (see [26]). Any infinite minimal system with the shadowing property is conjugate to an odometer. It is also known that for every continuous map $f : X \to X$ and $x \in RR(f) \setminus Per(f)$, a dynamical system $(\overline{O_f(x)}, f)$ is an almost 1-1 extension of an odometer. Moreover, if $\overline{O_f(x)} \subset RR(f)$, then $(\overline{O_f(x)}, f)$ is conjugate to an odometer (see [9, 14]).

Proposition 1.4.1. *Let $f : X \to X$ be a continuous map and let $p \in \operatorname{Int} Sh^+(f) \cap CR(f)$. Then, for every $\epsilon > 0$, there exists $q \in X$ with $d(p, q) \leq \epsilon$ such that $q \in Per(f)$ or $(\overline{O_f(q)}, f)$ is conjugate to an odometer.*

Remark 1.4.1. If a continuous map $f : X \to X$ satisfies the shadowing property, then $Sh^+(f) = X$. In this case, as seen from Proposition 1.4.1, $RR(f)$ is dense in the non-wandering set of f. Therefore, one may expect that if $f : X \to X$ has the b-shadowing property with $b > 0$, then for every $x \in \Omega(f)$, there exists $y \in RR(f)$ with $d(x, y) \leq b$, but this is not the case as shown in the following example. Let $\sigma : \{0,1\}^{\mathbb{Z}} \to \{0,1\}^{\mathbb{Z}}$ be the full shift and let $g_b : Y_b \to Y_b$ be a minimal rigid rotation on a circle Y_b with radius $b > 0$. Then, since σ has the shadowing property, $\sigma \times g_b : \{0,1\}^{\mathbb{Z}} \times Y_b \to \{0,1\}^{\mathbb{Z}} \times Y_b$ has the b-shadowing property. However, $RR(\sigma \times g_b) = RR(\sigma) \times RR(g_b) = \emptyset$ because $RR(g_b) = \emptyset$.

The next theorem describes local features of interior points in the set of shadowable points, under the assumption that $\operatorname{Int} Sh^+(f)$ is contained in a chain recurrent subset. A key idea of the proof is *Bowen type decomposition* of chain recurrent subsets.

It has been observed so far that if a continuous map $f : X \to X$ is chain recurrent, then X admits a canonical decomposition into finitely many chain components. Such an idea goes back to Smale's spectral decomposition theorem on Axiom A diffeomorphisms. It states that the non-wandering set of an Axiom A diffeomorphism is decomposed into finitely many clopen transitive components [38]. Then, Bowen decomposed each of the components into cyclically alternating clopen components for which the power of the diffeomorphism restricted to each component is topologically mixing, and used it to develop the ergodic theory of Axiom A diffeomorphisms [12]. A topological version of Smale and Bowen decomposition is presented in [6] for instance.

Relatively recently, such a type of decomposition is generalized for chain transitive maps. An idea leading to the generalization was already presented in [1]. It was used in [36] to give a structure theorem of chain transitive maps, and used in [13] to prove a certain kind of shadowing property for chain transitive maps. We consider such a type of decomposition of chain recurrent subsets by chain equivalence relations without assuming the chain transitivity, and use it to prove Theorem 1.4.3.

Theorem 1.4.3. *Let $f : X \to X$ be a continuous map and let $\sigma : \{0,1\}^{\mathbb{N}} \to \{0,1\}^{\mathbb{N}}$ be the full shift. Suppose that there is a closed f-invariant subset $S \subset X$ such that $CR(f|_S) = S$ and $\mathrm{Int}\, Sh^+(f) \subset S$. Then, for any $x \in \mathrm{Int}\, Sh^+(f)$, each of the following two families of properties (S1)-(S5) and (E1)-(E4) consists of equivalent properties, and either (S1) or (E1) holds.*

- (S1) $x \in \overline{Sen(f)}$.
- (S2) *For every $\epsilon > 0$, there are $m \in \mathbb{N}$ and a closed f^m-invariant subset $Y \subset B_\epsilon(x)$ for which we have a factor map $\pi : (Y, f^m) \to (\{0,1\}^{\mathbb{N}}, \sigma)$, and there exists $y \in X$ with $d(x,y) \le \epsilon$ such that $y \in Per(f)$ or $(\overline{O_f(y)}, f)$ is conjugate to an odometer.*
- (S3) *For every $\epsilon > 0$, there exists $y \in X$ with $d(x,y) \le \epsilon$ such that $(\overline{O_f(y)}, f)$ is a minimal sensitive subsystem.*
- (S4) $x \in Ent(f)$.
- (S5) $x \notin \mathrm{Int}\, RR(f)$.
- (E1) $x \in \mathrm{Int}\, EC(f)$.
- (E2) *There is a neighborhood U of x such that for every $y \in U$, $y \in Per(f)$ or $(\overline{O_f(y)}, f)$ is conjugate to an odometer.*
- (E3) $x \notin Ent(f)$.
- (E4) $x \in \mathrm{Int}\, RR(f)$.

Moreover, if $x \in EC(f)$, then $x \in Per(f)$ or $(\overline{O_f(x)}, f)$ is conjugate to an odometer.

Remark 1.4.2. If $CR(f) = X$, then the hypothesis of Theorem 1.4.3 is satisfied for $S = X$. When $f : X \to X$ is a homeomorphism or an open map and $\mathrm{Int}\, Sh^+(f) \subset CR(f)$, putting $S = \overline{\mathrm{Int}\, Sh^+(f)}$, we have $f(S) \subset S$ and $\Omega(f|_S) = S$, implying $CR(f|_S) = S$. Then, the hypothesis of Theorem 1.4.3 is satisfied. Note that if a continuous map $f : X \to X$ satisfies the shadowing property and $S = \Omega(f)$, then we have $Sh^+(f|_S) = S$ and $CR(f|_S) = S$. Hence, Theorem 1.4.3 applies to any dynamical system with the shadowing property restricted to its non-wandering set.

2. Shadowing, sensitivity, and entropy points

Throughout this section, (X, d) is a compact metric space, and $f : X \to X$ is a continuous map.

2.1. **Proof of Theorems 1.2.1 and 1.2.2.**

In this subsection, we prove Theorems 1.2.1 and 1.2.2. Let $\mathcal{K}(X) = \{A \subset X : A \text{ is non-empty and closed}\}$ and let d_H denote the Hausdorff metric on $\mathcal{K}(X)$, i.e., $d_H(A, B) = \inf\{\epsilon > 0 : A \subset B_\epsilon(B) \text{ and } B \subset B_\epsilon(A)\}$ $(A, B \in \mathcal{K}(X))$, where $B_\epsilon(Y) = \{x \in X : d(x, Y) \le \epsilon\}$ for any subset $Y \subset X$ and $\epsilon > 0$.

We need a few lemmas to prove Theorem 1.2.1. The first lemma is almost obvious, but we give a proof for the sake of completeness.

Lemma 2.1.1. *Let $A, B \in \mathcal{K}(X)$. For any $A' \in \mathcal{K}(X)$ with $A' \subset A$, there is $B' \in \mathcal{K}(X)$ such that $B' \subset B$ and $d_H(A', B') \le d_H(A, B)$.*

Proof. Put $d_H(A, B) = \delta$ and $B' = B_\delta(A') \cap B$. Then, $B' \in \mathcal{K}(X)$, $B' \subset B$, and $B' \subset B_\delta(A')$. Suppose that $x \in A'$ to see $A' \subset B_\delta(B')$. Since $x \in A' \subset A$ and $d_H(A, B) = \delta$, there is $y \in B$ such that $d(x, y) \le \delta$. Then, we have $y \in B_\delta(A') \cap B = B'$, implying $x \in B_\delta(B')$. Therefore, we have $A' \subset B_\delta(B')$, and hence $d_H(A', B') \le \delta = d_H(A, B)$. □

The next lemma is a shortcut lemma, which is a key ingredient of our proof of Theorem 1.2.1. Since $(\mathcal{K}(X), d_H)$ is a compact metric space, for any positive constant $\delta_1 > 0$, there is an open cover $\mathcal{U} = \{U_1, \dots, U_K\}$ such that $\text{mesh}\,\mathcal{U} = \max_{1 \le i \le K} \text{diam}\, U_i \le \delta_1$.

Lemma 2.1.2. *Let $\delta_1 > 0$ and let $\mathcal{U} = \{U_1, \dots, U_K\}$ be an open cover of $(\mathcal{K}(X), d_H)$ with $\text{mesh}\,\mathcal{U} \le \delta_1$. Given $A \in \mathcal{K}(X)$ and $N \ge 1$, there is a number $k = k(A, N) \in \{1, \dots, K\}$ satisfying the following condition. For any $B \in \mathcal{K}(X)$ with $B \subset f^N(A)$, there is a δ_1-pseudo orbit $(x_i)_{i=0}^k$ of f such that $x_0 \in A$ and $x_k \in B$. If f is a homeomorphism, a similar statement also holds for f^{-1}.*

Proof. Firstly, let us show the following claim.

<u>Claim 1</u>: There exist a number $k \in \{1, \dots, K\}$ and a sequence of numbers $0 = j_0 < j_1 < \dots < j_k = N$ such that $d_H(f^{j_{l-1}+1}(A), f^{j_l}(A)) \le \delta_1$ for all $1 \le l \le k$.

For each $1 \le j \le N$, we take $1 \le i_j \le K$ so that $f^j(A) \in U_{i_j}$. Put $j_1 = \max\{1 \le j \le N : f^j(A) \in U_{i_1}\}$. Since $f(A), f^{j_1}(A) \in U_{i_1}$, we have $d_H(f(A), f^{j_1}(A)) \le \delta_1$. If $j_1 < N$, then put $j_2 = \max\{j_1 + 1 \le j \le N : f^j(A) \in U_{i_{j_1+1}}\}$. Since $f^{j_1+1}(A) \notin U_{i_1}$ and $f^{j_1+1}(A) \in U_{i_{j_1+1}}$, we have $U_{i_{j_1+1}} \ne U_{i_1}$. Similarly to the above, $f^{j_1+1}(A), f^{j_2}(A) \in U_{i_{j_1+1}}$ gives $d_H(f^{j_1+1}(A), f^{j_2}(A)) \le \delta_1$. If $j_2 < N$, we repeat the process, and so on. Inductively, we obtain a sequence of integers $0 = j_0 < j_1 < j_2 < \cdots$. If $j_K < N$, then $U_{i_1}, U_{i_{j_1+1}}, \dots, U_{i_{j_K+1}}$ would be $K + 1$ distinct elements of $\mathcal{U}$, which is absurd. Therefore, we have $j_k = N$ for some $k \in \{1, \dots, K\}$, and $d_H(f^{j_{l-1}+1}(A), f^{j_l}(A)) \le \delta_1$ for all $1 \le l \le k$.

Next, using Claim 1, we show the following claim.

<u>Claim 2</u>: Given $B \subset f^N(A)$ with $B \in \mathcal{K}(X)$, put $B_k = B$. Then, we have a sequence $B_0, B_1, \dots, B_k \in \mathcal{K}(X)$ such that $B_l \subset f^{j_l}(A)$ and $d_H(f(B_{l-1}), B_l) \le \delta_1$ for all $1 \le l \le k$.

Suppose that B_j has been chosen for all $l \leq j \leq k$. Since $B_l \subset f^{j_l}(A)$ and

$$d_H(f^{j_{l-1}+1}(A), f^{j_l}(A)) \leq \delta_1,$$

using Lemma 2.1.1, we can take $C_l \in \mathcal{K}(X)$ such that $C_l \subset f^{j_{l-1}+1}(A)$ and $d_H(C_l, B_l) \leq \delta_1$. Set $B_{l-1} = f^{-1}(C_l) \cap f^{j_{l-1}}(A)$. Then, $B_{l-1} \in \mathcal{K}(X)$, $B_{l-1} \subset f^{j_{l-1}}(A)$ and $d_H(f(B_{l-1}), B_l) = d_H(C_l, B_l) \leq \delta_1$. Thus, we get $B_0, B_1, \ldots, B_k$ inductively.

To finish the proof, fix $x_0 \in B_0$. Since $f(x_0) \in f(B_0)$ and $d_H(f(B_0), B_1) \leq \delta_1$, there is $x_1 \in B_1$ such that $d(f(x_0), x_1) \leq \delta_1$. Then, since $f(x_1) \in f(B_1)$ and $d_H(f(B_1), B_2) \leq \delta_1$, there is $x_2 \in B_2$ such that $d(f(x_1), x_2) \leq \delta_1$. Repeating this process, we obtain $x_0, x_1, \ldots, x_k$ such that $x_l \in B_l$ and $d(f(x_{l-1}), x_l) \leq \delta_1$ for all $1 \leq l \leq k$. The statement for f^{-1} is proved similarly. □

We need the following lemma given in [43] by Ye and Zhang.

Lemma 2.1.3 ([43, Proposition 2.5]). *If $h(f, A) > 0$ for a closed subset $A \subset X$, then $A \cap \mathrm{Ent}(f) \neq \emptyset$.*

This lemma is obtained by the fact that for any choice of $K_1, \ldots, K_m \in \mathcal{K}(X)$, we have $h(f, \bigcup_{i=1}^m K_i) = \max\{h(f, K_i) : 1 \leq i \leq m\}$ and a simple concentration argument.

Before giving a proof of Theorem 1.2.1, to simplify the notation, we put $S' = \{x \in S : \omega(x) \cap Sen_e(f|_S) \neq \emptyset\}$ and denote by $\{0,1\}^*$ the set of all binary words of finite length including the empty word λ (i.e. the unique word of length zero). Note that $\lambda s = s$ for any element $s \in \{0,1\}^*$ and regard $i_1 \cdots i_{k-1}$ as λ when $k = 1$.

Proof of Theorem 1.2.1. Since $Ent(f) \subset X$ is a closed subset, it suffices to show that for any $x \in S$ and any $b' > b$, there exists $u \in Ent(f)$ such that $d(x, u) < b'$. Given $x \in S$ and $r > 0$, by assumption, there are $v \in S'$, $w \in \omega(v) \cap Sen_e(f|_S)$, and $M > 0$ such that $d(x, v) < r$ and $d(f^M(v), w) < r/2$. Then, since $w \in Sen_e(f|_S)$, we can choose $y, z \in S$, and $N > 0$ such that $\max\{d(w, y), d(w, z)\} < r/2$ and $d(f^N(y), f^N(z)) > e$. We fix positive constants δ, δ_1, and r satisfying the following conditions:

(1) Every δ-pseudo orbit of f contained in S is b-shadowed by some point in X;
(2) $0 < 2\delta_1 \leq \delta$, $b + r < b'$, and $d(s, t) \leq r$ implies $d(f(s), f(t)) \leq \delta_1$ for all $s, t \in X$.

Starting from any $x = x_\lambda \in S$ and repeating the above choice of $\{v, w, y, z\}$, we can construct inductively "branching" sequences of points in S, $x_{i_1\cdots i_k}$, $y_{i_1\cdots i_k}$, $z_{i_1\cdots i_k}$, $v_{i_1\cdots i_k}$, $w_{i_1\cdots i_k} \in S$, $i_1 \cdots i_k \in \{0,1\}^*$, with the following properties:

(3) $d(x_{i_1\cdots i_{k-1}}, v_{i_1\cdots i_{k-1}}) < r$;
(4) $d\big(f^{M_{i_1\cdots i_{k-1}}}(v_{i_1\cdots i_{k-1}}), w_{i_1\cdots i_{k-1}}\big) < r/2$ for some $M_{i_1\cdots i_{k-1}} > 0$;
(5) $\max\{d(w_{i_1\cdots i_{k-1}}, y_{i_1\cdots i_{k-1}}), d(w_{i_1\cdots i_{k-1}}, z_{i_1\cdots i_{k-1}})\} < r/2$;
(6) $f^{N_{i_1\cdots i_{k-1}}}(y_{i_1\cdots i_{k-1}}) = x_{i_1\cdots i_{k-1}0}$, $f^{N_{i_1\cdots i_{k-1}}}(z_{i_1\cdots i_{k-1}}) = x_{i_1\cdots i_{k-1}1}$ and $d(x_{i_1\cdots i_{k-1}0}, x_{i_1\cdots i_{k-1}1}) > e$ for some $N_{i_1\cdots i_{k-1}} > 0$.

Put $A_{i_1\cdots i_{k-1}} = \{y_{i_1\cdots i_{k-1}}, z_{i_1\cdots i_{k-1}}\}$. Then, by (6) and the shortcut lemma (Lemma 2.1.2), for every $k \geq 1$ and every $i_1 \cdots i_{k-1} \in \{0,1\}^*$, there exist integers $L_{i_1\cdots i_{k-1}}$, $K_{i_1\cdots i_{k-1}}$, and δ_1-chains $(q_j^{i_1\cdots i_{k-1}})_{j=0}^{L_{i_1\cdots i_{k-1}}}$, $(p_j^{i_1\cdots i_{k-1}i})_{j=0}^{K_{i_1\cdots i_{k-1}}}$, $i \in \{0,1\}$, of f satisfying the following properties:

(7) $1 \le L_{i_1\cdots i_{k-1}},\, K_{i_1\cdots i_{k-1}} \le K$;
(8) $q_0^{i_1\cdots i_{k-1}} = v_{i_1\cdots i_{k-1}}$ and $q_{L_{i_1\cdots i_{k-1}}}^{i_1\cdots i_{k-1}} = f^{M_{i_1\cdots i_{k-1}}}(v_{i_1\cdots i_{k-1}})$;
(9) $p_0^{i_1\cdots i_{k-1}i} \in A_{i_1\cdots i_{k-1}}$ and $p_{K_{i_1\cdots i_{k-1}}}^{i_1\cdots i_{k-1}i} = x_{i_1\cdots i_{k-1}i}$ for $i \in \{0,1\}$.

Given $k \ge 2$ and $i_1 \cdots i_k \in \{0,1\}^*$, we consider the following chain:

$$(q_0^\lambda, \dots, q_{L_\lambda}^\lambda, p_1^{i_1}, \dots, p_{K_\lambda}^{i_1}, q_1^{i_1}, \dots, q_{L_{i_1}}^{i_1}, p_1^{i_1i_2}, \dots, p_{K_{i_1}}^{i_1i_2}, \dots$$
$$\dots, q_1^{i_1\cdots i_{k-1}}, \dots, q_{L_{i_1\cdots i_{k-1}}}^{i_1\cdots i_{k-1}}, p_1^{i_1\cdots i_k}, \dots, p_{K_{i_1\cdots i_{k-1}}}^{i_1\cdots i_k}, q_1^{i_1\cdots i_k}).$$

Then, for every $1 \le l \le k$, using (2), (3), (4), (5), (8), and (9), we have

$$d\big(f(q_{L_{i_1\cdots i_{l-1}}}^{i_1\cdots i_{l-1}}),\, p_1^{i_1\cdots i_l}\big) = d\big(f\big(f^{M_{i_1\cdots i_{l-1}}}(v_{i_1\cdots i_{l-1}})\big),\, p_1^{i_1\cdots i_l}\big)$$
$$\le d\big(f\big(f^{M_{i_1\cdots i_{l-1}}}(v_{i_1\cdots i_{l-1}})\big),\, f(p_0^{i_1\cdots i_l})\big) + d\big(f(p_0^{i_1\cdots i_l}),\, p_1^{i_1\cdots i_l}\big) \le \delta_1 + \delta_1 \le \delta$$

and

$$d\big(f(p_{K_{i_1\cdots i_{l-1}}}^{i_1\cdots i_l}), q_1^{i_1\cdots i_l}\big) = d\big(f(x_{i_1\cdots i_l}),\, q_1^{i_1\cdots i_l}\big)$$
$$\le d\big(f(x_{i_1\cdots i_l}),\, f(q_0^{i_1\cdots i_l})\big) + d\big(f(q_0^{i_1\cdots i_l}),\, q_1^{i_1\cdots i_l}\big) \le \delta_1 + \delta_1 \le \delta.$$

Hence, the above chain is a δ-chain of f, and then b-shadowed by a point $p_{i_1\cdots i_k} \in X$ by (1). By (3) and (8), we have

$$d(x, p_{i_1\cdots i_k}) \le d(x_\lambda, v_\lambda) + d(v_\lambda, p_{i_1\cdots i_k}) = d(x_\lambda, v_\lambda) + d(q_0^\lambda, p_{i_1\cdots i_k}) \le r + b,$$

that is $p_{i_1\cdots i_k} \in B_{b+r}(x)$. For all $k \ge 2$, let $E_k = \{p_{i_1\cdots i_k} : i_1 \cdots i_k \in \{0,1\}^k\}$. Let us claim that E_k is a $(2kK+1, e-2b)$-separated set. To prove the claim, put $s = p_{i_1\cdots i_k}$ and $t = p_{i_1'\cdots i_k'}$ with $i_1 \cdots i_k \ne i_1' \cdots i_k'$. Take l with $1 \le l \le k$ such that $i_l \ne i_l'$ and $i_j = i_j'$ if $j < l$. We can assume that $i_l = 0$ and $i_l' = 1$. Set $J = L_\lambda + K_\lambda + L_{i_1} + K_{i_1} + \cdots + L_{i_1\cdots i_{l-1}} + K_{i_1\cdots i_{l-1}}$. Then, using (6) and (9), we have

$$d(f^J(s), f^J(t)) \ge d(x_{i_1\cdots i_l}, x_{i_1'\cdots i_l'}) - d(x_{i_1\cdots i_l}, f^J(s)) - d(x_{i_1'\cdots i_l'}, f^J(t))$$
$$= d(x_{i_1\cdots i_{l-1}0}, x_{i_1\cdots i_{l-1}1}) - d\big(p_{K_{i_1\cdots i_{l-1}}}^{i_1\cdots i_l}, f^J(s)\big) - d\big(p_{K_{i_1'\cdots i_{l-1}'}}^{i_1'\cdots i_l'}, f^J(t)\big)$$
$$> e - 2b.$$

Since $J \le 2lK \le 2kK$ by (7), we have $d_{2kK+1}(s,t) > e - 2b$. Therefore, E_k is a $(2kK+1, e-2b)$-separated set. Note that $E_k \subset B_{b+r}(x)$ and the cardinality of E_k is 2^k. Hence, we have $S(B_{b+r}(x), 2kK+1, e-2b) \ge 2^k$, and then

$$\begin{aligned}
h(f, B_{b+r}(x)) &\ge h(f, B_{b+r}(x), e-2b) = \limsup_{n\to\infty} \frac{1}{n} \log S(B_{b+r}(x), n, e-2b) \\
&\ge \limsup_{k\to\infty} \frac{1}{2kK+1} \log S(B_{b+r}(x), 2kK+1, e-2b) \\
&\ge \limsup_{k\to\infty} \frac{1}{2kK+1} \log 2^k \\
&= \frac{1}{2K} \log 2 > 0.
\end{aligned}$$

Thus, by Lemma 2.1.3, there exists $u \in Ent(f)$ such that $d(x,u) \le b + r < b'$. Finally, this proof also shows that if f has the shadowing property around S, then $S \subset \overline{Ent(f)} = Ent(f)$. □

For the proof of Theorem 1.2.2, we put $S' = \mathrm{Int}\,[Sen_e(f|_S)]$ (where the interior is taken in S) and $S'' = S \setminus Sen_e(f|_S)$. Note that $Sen_e(f|_S)$ is a closed f-invariant subset of S, and hence S' and S'' are both open f-invariant subsets of S. It is easy to see that $S = \overline{S' \cup S''} = \overline{S'} \cup \overline{S''}$ holds. Note also that if a homeomorphism f has a b-shadowing property around S, then so does f^{-1}.

Proof of Theorem 1.2.2. Firstly, let us prove that for any $x \in \overline{S'}$, there exists $y \in Ent(f)$ such that $d(x,y) \le b$. Since $S' \subset Sen_e(f|_S)$ and S' is an open f-invariant subset of S, we have $S' = Sen_e(f|_{S'})$. Then, $\overline{S'}$ is closed f-invariant subset of S, and $\overline{S'} = Sen_e(f|_{\overline{S'}})$. Hence, Theorem 1.2.1 applies to $f|_{\overline{S'}}$, proving the claim.

Secondly, let us prove that for any $x \in \overline{S''}$, there exists $y \in Ent(f^{-1})$ such that $d(x,y) \le b$. Note that $S'' \subset Sen_e(f|_S^{-1})$, since f is weakly sensitive with the weakly sensitive constant e. Then, similarly to the above, we have $\overline{S''} = Sen_e(f|_{\overline{S''}}^{-1})$. Hence, Theorem 1.2.1 applies again to $f|_{\overline{S''}}^{-1}$, proving the second claim.

Now let us finish the proof. Since $S = \overline{S'} \cup \overline{S''}$, for any $x \in S$, there exists $y \in Ent(f) \cup Ent(f^{-1})$ such that $d(x,y) \le b$. If f has the shadowing property around S, then we have $S \subset \overline{Ent(f) \cup Ent(f^{-1})} = Ent(f) \cup Ent(f^{-1})$. □

2.2. **Examples and Proof of the Corollaries.**

In this subsection, we present some examples complementing the results of this section, and prove Corollaries 1.2.3 and 1.2.4. The first example shows that the sensitivity does not always imply positive topological entropy.

Example 2.2.1. Let $\mathbb{T}^2 = (\mathbb{R}/\mathbb{Z})^2$ and $f : \mathbb{T}^2 \to \mathbb{T}^2$ be a homeomorphism defined by $f(x,y) = (x+y, y)$. We define metrics on $\mathbb{R}/\mathbb{Z}$ and $\mathbb{T}^2$ by $\mathrm{dist}(x,x') = \min_{m\in\mathbb{Z}} |x-x'+m|$ and $d((x,y),(x',y')) = \max\{\mathrm{dist}(x,x'), \mathrm{dist}(y,y')\}$, respectively. For each $n \in \mathbb{Z}$, we have $f^n(x,y) = (x+ny, y)$. Let us claim that $\mathbb{T}^2 = Sen_e(f)$ for every $0 < e < 1/2$. Suppose that $0 < \epsilon < 1$ and $0 < |y - y'| < \epsilon$. Then, we have

$$d(f^n(x,y), f^n(x,y')) = d((x+ny, y), (x+ny', y')) \ge \mathrm{dist}(n(y-y'), 0) > (1-\epsilon)/2$$

for some $n > 0$. It follows that $\mathbb{T}^2 = Sen_{(1-\epsilon)/2}(f)$ for every $0 < \epsilon < 1$, proving the claim. On the other hand, we have $h_{top}(f) = 0$. In fact,

$$d(f^n(x,y), f^n(x',y')) = d((x+ny, y), (x'+ny', y')) \le (n+1)d((x,y),(x',y')),$$

and hence $S(\mathbb{T}^2, n, \epsilon) = O(n^2)$ for any fixed $\epsilon > 0$, leading to $h_{top}(f) = 0$. This follows also from the fact that f is distal and every distal homeomorphism has zero topological entropy (see [17]). Let us show that f does not have the b-shadowing property for any $0 < b < 1/2$. Consider the pseudo orbit $(x_i)_{i=0}^{\infty}$ defined by $x_i = (i(i-1)\delta/2, i\delta)$, $i \ge 0$, for any $0 < \delta < 1$. Then, we have

$$d(f(x_i), x_{i+1}) = d((i(i+1)\delta/2, i\delta), (i(i+1)\delta/2, (i+1)\delta)) \le \delta$$

for every $i \geq 0$, and hence it is a δ-pseudo orbit of f. However, for any $(x, y) \in \mathbb{T}^2$, we have

$$d(f^i(x, y), x_i) = d((x + iy, y), (i(i-1)\delta/2, i\delta)) \geq \mathrm{dist}(y, i\delta) \geq (1-\delta)/2$$

for some $i \geq 0$. Therefore, if $b < (1-\delta)/2 < 1/2$, then $(x_i)_{i=0}^{\infty}$ has no b-shadowing point, proving the claim.

As mentioned in subsection 1.2, the next example shows that the hypothesis of Theorem 1.2.2 cannot be replaced by a weaker condition that

$$\{x \in S : \omega(x) \cap Sen_e(f|_S) \neq \emptyset\} \cup \{x \in S : \alpha(x) \cap Sen_e(f|_S^{-1}) \neq \emptyset\}$$

is dense in S.

Example 2.2.2. Let $\sigma : \{1, 2, 3\}^{\mathbb{Z}} \to \{1, 2, 3\}^{\mathbb{Z}}$ be the shift map. We define a metric d on $\{1, 2, 3\}^{\mathbb{Z}}$ by $d(x, y) = \sum_{i \in \mathbb{Z}} 2^{-|i|}\delta(x_i, y_i)$ for $x = (x_i)_{i \in \mathbb{Z}}$, $y = (y_i)_{i \in \mathbb{Z}} \in \{1, 2, 3\}^{\mathbb{Z}}$, where $\delta(a, b) = 1$ if $a \neq b$ and $\delta(a, b) = 0$ otherwise. Consider two points $p, q \in \{1, 2, 3\}^{\mathbb{Z}}$ defined as follows:

- $p_i = 1$ for $i \leq 0$ and $p_i = 2$ for $i > 0$;
- $q_i = 2$ for $i \leq 0$ and $q_i = 3$ for $i > 0$.

Set $\gamma_1 = \{\sigma^n(p) : n \in \mathbb{Z}\}$ and $\gamma_2 = \{\sigma^n(q) : n \in \mathbb{Z}\}$. For $a \in \{1, 2, 3\}$, we denote by $\overline{a}$ the point $(\dots, a, a, a, \dots) \in \{1, 2, 3\}^{\mathbb{Z}}$. Now let $S = \{\overline{1}, \overline{2}, \overline{3}\} \cup \gamma_1 \cup \gamma_2$. Note that S is a closed σ-invariant subset of $\{1, 2, 3\}^{\mathbb{Z}}$. Then, for any fixed $0 < e < 3$, we have $\overline{2} \in Sen_e(\sigma|_S) \cap Sen_e(\sigma|_S^{-1})$. For any $x \in \gamma_1$ and $y \in \gamma_2$, we have $\omega(x) = \alpha(y) = \{\overline{2}\}$. Since $S = \overline{\gamma_1 \cup \gamma_2}$, we see that

$$\{x \in S : \omega(x) \cap Sen_e(\sigma|_S) \neq \emptyset\} \cup \{x \in S : \alpha(x) \cap Sen_e(\sigma|_S^{-1}) \neq \emptyset\}$$

is dense in S. Let $X \subset \{1, 2, 3\}^{\mathbb{Z}}$ be a subshift of finite type defined by the transition matrix

$$A = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix},$$

i.e. $X = \{x \in \{1, 2, 3\}^{\mathbb{Z}} : A_{x_i x_{i+1}} = 1 (\forall i \in \mathbb{Z})\}$. Let $f = \sigma|_X$. Then, f has the shadowing property and S is a closed f-invariant subset of X. On the other hand, it is easy to see that $h_{top}(f) = 0$. Hence, this gives the required example.

Now let us prove Corollaries 1.2.3 and 1.2.4

Proof of Corollary 1.2.3. Since $S = \overline{O_f(x)}$ and a sensitive point cannot be an isolated point, we have $Sen_e(f|_S) \subset \omega(x)$. Hence, for any $n \geq 0$, it holds that $\omega(f^n(x)) \cap Sen_e(f|_S) = \omega(x) \cap Sen_e(f|_S) \neq \emptyset$. Thus, the hypothesis of Theorem 1.2.1 is satisfied, and then this corollary follows from the theorem. □

Proof of Corollary 1.2.4. By Corollary 1.2.3, the proof of this corollary is reduced to proving the following claim:

Claim: Let $f : X \to X$ be a continuous map. For a point $x \in X$, if $\omega(x)$ is non-minimal, then $f|_{\overline{O_f(x)}}$ has a sensitive point.

Since $\omega(x)$ is non-minimal, there are an orbit $\gamma \subset \omega(x)$ and $y \in \omega(x)$ such that $y \notin \overline{\gamma}$. We put $e = d(y, \overline{\gamma})(> 0)$. Fix $z \in \overline{\gamma}$ and take any neighborhood U of z. Since $y, z \in \omega(x)$, there are positive integers $M > N > 0$ such that $f^N(x) \in U$ and $d(y, f^M(x)) < e/2$. Then, we have

$$\begin{aligned} d(f^{M-N}(f^N(x)), f^{M-N}(z)) &= d(f^M(x), f^{M-N}(z)) \\ &\geq d(y, f^{M-N}(z)) - d(y, f^M(x)) > e/2 \end{aligned}$$

since $f^{M-N}(z) \in \overline{\gamma}$. Thus, z is a $e/2$-sensitive point of $f|_{\overline{O_f(x)}}$, proving the claim and finishing the proof of Corollary 1.2.4. □

3. Quantitative shadowable points

3.1. **Quantitative Lemmas.**

In this subsection, we prove quantitative lemmas needed for the proof of the theorems stated in subsection 1.3.

The first lemma claims that $x \in Sh_b^+(f)$ is equivalent to $y \in Sh_b^+(f)$ for every $x, y \in X$ when f is chain transitive. It is an analog of [2, Corollary 2.3] for quantitative shadowable points. Recall that the definition of $C_f^+(x)$, $x \in X$, for a continuous map $f : X \to X$ was given as follows. For any $y \in X$, $y \in C_f^+(x)$ iff for every $\delta > 0$, there is a δ-chain $(x_i)_{i=0}^k$ with $x_0 = x$ and $x_k = y$.

Lemma 3.1.1. *Let $f : X \to X$ be a continuous map. Given $b > 0$, if $x \in Sh_b^+(f)$, then $y \in Sh_b^+(f)$ for every $y \in C_f^+(x)$. In particular, if f is chain transitive, then $Sh_b^+(f) = \emptyset$ or $Sh_b^+(f) = X$.*

Proof. Suppose $x \in Sh_b^+(f)$. Then, there exists $\delta > 0$ such that every δ-pseudo orbit $(x_i)_{i=0}^\infty$ with $x_0 = x$ is b-shadowed by some point of X. Fix $y \in C_f^+(x)$ and take a δ-chain $(x_i)_{i=0}^k$ of f with $x_0 = x$ and $x_k = y$. For any δ-pseudo orbit $(y_i)_{i=0}^\infty$ with $y_0 = y$, define $z_i = x_i$ for $0 \leq i \leq k$ and $z_i = y_{i-k}$ for $i \geq k$. Then, $(z_i)_{i=0}^\infty$ is a δ-pseudo orbit with $z_0 = x$, and hence b-shadowed by some $p \in X$. Then, $f^k(p)$ is a b-shadowing point of $(y_i)_{i=0}^\infty$. Therefore, we have $y \in Sh_b^+(f)$. If f is chain transitive, then $C_f^+(x) = X$ for every $x \in X$. Hence, $Sh_b^+(f) \neq \emptyset$ implies $Sh_b^+(f) = X$. □

The following lemma is an essential part of the proof of Lemma 3.1.3, which is a quantitative and pointwise version of [31, Lemma 2.1] for continuous maps.

Lemma 3.1.2. *Let $f : X \to X$ be a continuous map. If $x \in Sh_{c+}^+(f)$ with $c \geq 0$, then for every $b > c$, there exists $\delta = \delta(x, b) > 0$ such that every δ-pseudo orbit $(x_i)_{i=0}^\infty$ with $d(x, x_0) < \delta$ is b-shadowed by some point of X.*

Proof. Take $\epsilon > 0$ with $c + 2\epsilon < b$. Since $x \in Sh_{c+}^+(f) \subset Sh_{c+\epsilon}^+(f)$, there exists $\delta_1 > 0$ such that every δ_1-pseudo orbit $(z_i)_{i=0}^\infty$ with $z_0 = x$ is $(c+\epsilon)$-shadowed by some point of X. Take a constant $0 < \delta < \min\{\delta_1/2, \epsilon\}$ such that $d(u, v) < \delta$ implies $d(f(u), f(v)) < \delta_1/2$ for every $u, v \in X$. Given a δ-pseudo orbit $(x_i)_{i=0}^\infty$ with $d(x, x_0) < \delta$, define $(y_i)_{i=0}^\infty$

by $y_0 = x$ and $y_i = x_i$ for $i \geq 1$. Then, we have

$$d(f(y_0), y_1) = d(f(x), x_1) \leq d(f(x), f(x_0)) + d(f(x_0), x_1) < \frac{\delta_1}{2} + \delta < \delta_1,$$

and

$$d(f(y_i), y_{i+1}) = d(f(x_i), x_{i+1}) \leq \delta < \delta_1/2$$

for all $i \geq 1$. Therefore, $(y_i)_{i=0}^{\infty}$ is δ_1-pseudo orbit with $y_0 = x$, and hence $(c + \epsilon)$-shadowed by some $p \in X$. Then, we have

$$d(x_0, p) \leq d(x_0, x) + d(x, p) = d(x_0, x) + d(y_0, p) < \delta + c + \epsilon < b,$$

and

$$d(x_i, f^i(p)) = d(y_i, f^i(p)) \leq c + \epsilon < b$$

for all $i \geq 1$. Hence, p is a b-shadowing point of $(x_i)_{i=0}^{\infty}$, proving the lemma. □

Similarly, we can prove the following statement for homeomorphisms.

Proposition 3.1.1. *Let $f : X \to X$ be a homeomorphism. If $x \in Sh_{c+}(f)$ with $c \geq 0$, then for every $b > c$, there exists $\delta = \delta(x, b) > 0$ such that every δ-pseudo orbit $(x_i)_{i\in\mathbb{Z}}$ with $d(x, x_0) < \delta$ is b-shadowed by some point of X.*

Lemma 3.1.3. *Let $f : X \to X$ be a continuous map and let $K \subset X$ be a compact subset. Given $c \geq 0$, if $K \subset Sh^+_{c+}(f)$, then for any $b > c$, there exist a neighborhood U of K and $\delta_0 > 0$ such that every δ_0-pseudo orbit $(x_i)_{i=0}^{\infty}$ with $x_0 \in U$ is b-shadowed by some point of X. In particular, if $Sh^+_{c+}(f) = X$, then f has the forward $c+$-shadowing property.*

Proof. Take $\delta(x, b) > 0$ as in Lemma 3.1.2 for every $x \in K$. Since K is compact, we can take a finite subset $S \subset K$ for which $K \subset \bigcup_{x\in S} V_{\delta(x,b)}(x)$, where $V_{\delta(x,b)}(x) = \{z \in X : d(z, x) < \delta(x, b)\}$. Then, $U = \bigcup_{x\in S} V_{\delta(x,b)}(x)$ and $\delta_0 = \min\{\delta(x, b) : x \in S\} > 0$ provide the desired neighborhood and constant. □

The next statement for homeomorphisms can be also proved similarly.

Proposition 3.1.2. *Let $f : X \to X$ be a homeomorphism and let $K \subset X$ be a compact subset. Given $c \geq 0$, if $K \subset Sh_{c+}(f)$, then for any $b > c$, there exist a neighborhood U of K and $\delta_0 > 0$ such that every δ_0-pseudo orbit $(x_i)_{i\in\mathbb{Z}}$ with $x_0 \in U$ is b-shadowed by some point of X. In particular, if $Sh_{c+}(f) = X$, then f has the $c+$-shadowing property.*

The following lemma enables us to pass from the forward shadowing property to the full shadowing property.

Lemma 3.1.4. *Let $f : X \to X$ be a homeomorphism and let $S \subset X$. Given $b, \delta > 0$, if every forward δ-pseudo orbit $(x_i)_{i=0}^{\infty}$ contained in S is b-shadowed by some $x \in X$, then every δ-pseudo orbit $(x_i)_{i\in\mathbb{Z}}$ contained in S is also b-shadowed by some $x \in X$. In particular, if $f : X \to X$ is a homeomorphism and has the forward b-shadowing property, then f has the b-shadowing property.*

Proof. Suppose that $(x_i)_{i\in\mathbb{Z}}$ is a δ-pseudo orbit contained in S. For every integer $n > 0$, define $x_i^{(n)} = x_{i-n}$ for $i \geq 0$. Then, $(x_i^{(n)})_{i=0}^{\infty}$ is a forward δ-pseudo orbit contained in S, and hence b-shadowed by some $y^{(n)} \in X$. Put $z_n = f^n(y^{(n)})$. Then, we have

$$d(x_i, f^i(z_n)) = d(x_{i+n}^{(n)}, f^{i+n}(y^{(n)})) \leq b$$

for every $-n \le i \le n$. Take a subsequence $(n_j)_{j=1}^{\infty}$ so that $0 < n_1 < n_2 < \cdots$ and $\lim_{j\to\infty} z_{n_j} = z$ for some $z \in X$. Then, for any $i \in \mathbb{Z}$, we have $d(x_i, f^i(z)) = \lim_{j\to\infty} d(x_i, f^i(z_{n_j})) \le b$, showing that z is a b-shadowing point of $(x_i)_{i\in\mathbb{Z}}$. □

Lemma 3.1.3 combined with Lemma 3.1.4 yields the following corollary.

Corollary 3.1.1. *Let $f : X \to X$ be a homeomorphism and let $c \ge 0$. Then, the following properties are equivalent.*

1. $Sh^{+}_{c+}(f) = X$.
2. $Sh_{c+}(f) = X$.
3. *f has the forward $c+$-shadowing property.*
4. *f has the $c+$-shadowing property.*

The next lemma translates a b-shadowing property of f^{-1} into a b-shadowing property of f.

Lemma 3.1.5. *Let $f : X \to X$ be a homeomorphism and let $S \subset X$. Given $b, \delta_1 > 0$, if every δ_1-pseudo orbit $(y_i)_{i\in\mathbb{Z}}$ of f^{-1} contained in S is b-shadowed by some point of X with respect to f^{-1}, then there exists $\delta_2 > 0$ such that every δ_2-pseudo orbit $(x_i)_{i\in\mathbb{Z}}$ of f contained in S is b-shadowed by some point of X with respect to f.*

Proof. Take $\delta_2 > 0$ such that $d(u, v) \le \delta_2$ implies $d(f^{-1}(u), f^{-1}(v)) \le \delta_1$ for all $u, v \in X$. Given any δ_2-pseudo orbit $(x_i)_{i\in\mathbb{Z}}$ of f contained in S, put $y_i = x_{-i}$ for every $i \in \mathbb{Z}$. Then, since $d(f(x_{-i-1}), x_{-i}) \le \delta_2$, we have

$$d(f^{-1}(y_i), y_{i+1}) = d(f^{-1}(x_{-i}), x_{-i-1}) = d\big(f^{-1}(x_{-i}), f^{-1}(f(x_{-i-1}))\big) \le \delta_1$$

for each $i \in \mathbb{Z}$. Hence, $(y_i)_{i\in\mathbb{Z}}$ is a δ_1-pseudo orbit of f^{-1} contained in S, and so there is $z \in X$ such that $d(y_i, f^{-i}(z)) \le b$ for every $i \in \mathbb{Z}$. From this we see that $d(x_i, f^i(z)) \le b$ for every $i \in \mathbb{Z}$, proving the lemma. □

One more lemma will be needed for the proof of Theorem 1.3.2.

Lemma 3.1.6. *Let $f : X \to X$ be a homeomorphism. Given $x \in X$ and $c \ge 0$, if $\overline{O_f(x)} \cap Sh_{c+}(f) \ne \emptyset$, then $x \in Sh_{c+}(f)$. In particular, given $x \in X$, if $\overline{O_f(x)} \cap Sh(f) \ne \emptyset$, then $x \in Sh(f)$.*

Proof. Suppose that $\overline{O_f(x)} \cap Sh_{c+}(f) \ne \emptyset$ and $b > c$. Then, by Proposition 3.1.1, we can take $y \in \overline{O_f(x)} \cap Sh_{c+}(f)$ and $\delta > 0$ so that every δ-pseudo orbit $(y_i)_{i\in\mathbb{Z}}$ with $d(y, y_0) < \delta$ is b-shadowed by some point of X. Fix an integer $M \in \mathbb{Z}$ with $d(y, f^M(x)) < \delta/2$. By the continuity of f, there is $0 < \delta' < \delta$ such that for every δ'-pseudo orbit $(x_i)_{i\in\mathbb{Z}}$ with $x_0 = x$, it holds that $d(f^M(x), x_M) < \delta/2$, and hence $d(y, x_M) < \delta$. Then, it is easy to see that all such δ'-pseudo orbits have b-shadowing points. Thus, we have $x \in Sh_b(f)$, and since $b > c$ is arbitrary, we have obtained $x \in Sh_{c+}(f)$. The claim for $Sh(f)$ follows by putting $c = 0$. □

As a corollary of Lemma 3.1.6, we obtain the following.

Corollary 3.1.2. *For every homeomorphism $f : X \to X$ and every $c \ge 0$, $Sh_{c+}(f)$ is f-invariant, i.e., $f(Sh_{c+}(f)) = Sh_{c+}(f)$.*

3.2. **Proof of Theorems 1.3.1 and 1.3.2.**

In this subsection, we prove Theorems 1.3.1 and 1.3.2 using lemmas proved in the last subsection. As for the proof of Theorem 1.3.2, we deal with the general $c \geq 0$ case and the $c = 0$ case separately. This is because, while the first proof is general, we can further clarify the matter with an another proof that is only effective for the $c = 0$ case. A related example will be given in subsection 3.5 (Example 3.5.6).

3.2.1. *The general $c \geq 0$ case.* We first prove Theorem 1.3.1.

Proof of Theorem 1.3.1. Let $c \geq 0$ and suppose that $Sh_{c+}(f) \neq \emptyset$. Then, since $\emptyset \neq Sh_{c+}(f) \subset Sh^+_{c+}(f)$ and f is chain transitive, we have $Sh^+_{c+}(f) = X$ by Lemma 3.1.1. From Corollary 3.1.1, it follows that $Sh_{c+}(f) = X$. □

Then, we give a proof of Theorem 1.3.2.

Proof of Theorem 1.3.2. Let us assume $Sh_{c+}(f) \neq \emptyset$ and then prove $Sh_{c+}(f) = X$. Take $x \in X$ such that $O_f(x) = \{f^n(x) : n \in \mathbb{Z}\}$ is dense in X. Note that $X = O_f(x) \cup \omega(x) \cup \alpha(x)$. Since both $f|_{\omega(x)}$ and $f|_{\alpha(x)}$ are chain transitive, if $\omega(x) \cap \alpha(x) \neq \emptyset$, then f is chain transitive, and so by applying Theorem 1.3.1, we obtain $Sh_{c+}(f) = X$. In what follows, we assume that $\omega(x) \cap \alpha(x) = \emptyset$. Applying Lemma 3.1.6, we have $x \in Sh_{c+}(f)$, and so $O_f(x) \subset Sh_{c+}(f)$ by the f-invariance of $Sh_{c+}(f)$ (Corollary 3.1.2). From Lemma 3.1.1, it follows that $\omega(x) \subset Sh^+_{c+}(f)$ and $\alpha(x) \subset Sh^+_{c+}(f^{-1})$. By Corollary 3.1.1, $Sh^+_{c+}(f) = X$ implies $Sh_{c+}(f) = X$. Hence, it only remains to show that $\alpha(x) \subset Sh^+_{c+}(f)$. Applying Lemma 3.1.3 with f^{-1} and $K = \alpha(x)$, for any $b > c$, there is $\delta_1 = \delta_1(b) > 0$ and a neighborhood $U = U(b)$ of $\alpha(x)$ such that every forward δ_1-pseudo orbit $(y_i)_{i=0}^\infty$ of f^{-1} with $y_0 \in U$ is b-shadowed by some point of X with respect to f^{-1}, and then using Lemma 3.1.4, we can replace such $(y_i)_{i=0}^\infty$ by any δ_1-pseudo orbit $(y_i)_{i\in\mathbb{Z}}$ of f^{-1} contained in U. Then, by Lemma 3.1.5, there exists $\delta = \delta(b) > 0$ such that every δ-pseudo orbit of f contained in U is b-shadowed by some point of X with respect to f. Put $\alpha(x) = A$ and $\omega(x) = B$. Given $b > c$, choose $\delta > 0$ and a neighborhood U of A as above, and take a neighborhood V of B. Then, there is an integer $N > 0$ such that $|n| > N$ implies $f^n(x) \in U \cup V$. For such $N > 0$, since $O_f(x) \subset Sh_{c+}(f)$, we can find $0 < \delta' < \delta$ such that every δ'-pseudo orbit $(x_i)_{i\in\mathbb{Z}}$ of f with $x_0 = f^n(x)$ for some $|n| \leq N$ is b-shadowed by some point of X. Suppose that $(z_i)_{i\in\mathbb{Z}}$ is a δ'-pseudo orbit of f with $z_0 \in A$. From the facts that A and B are f-invariant, and $X = U \cup V \cup \{f^n(x) : |n| \leq N\}$, we see that if U, V, and then $\delta' > 0$ are taken sufficiently small, either $(z_i)_{i\in\mathbb{Z}}$ is contained in U or there is $i \in \mathbb{Z}$ such that $z_i = f^n(x)$ for some $|n| \leq N$. In both cases, $(z_i)_{i\in\mathbb{Z}}$ has a b-shadowing point. Thus, we have $A \subset Sh_b(f)$, and since $b > c$ is arbitrary, we have obtained $A \subset Sh_{c+}(f)$, proving the theorem. □

3.2.2. *The $c = 0$ case.* We give an another proof of Theorem 1.3.2 in the case where $c = 0$.

Proof. Suppose that $Sh(f) \neq \emptyset$. If X is perfect, then f is transitive in the strong sense, and hence $Sh(f) = X$ follows from Theorem 1.3.1. If X contains an isolated point, then X contains a single dense orbit denoted by $O_f(x)$. Note that x is especially an isolated point of X. By Lemma 3.1.6, we have $x \in Sh(f)$. There are two cases to consider: (1)

The case where $\omega(x) \cap \alpha(x) \neq \emptyset$. In this case, f is chain transitive, and so by Theorem 1.3.1, we have $Sh(f) = X$. This implies that f has the shadowing property, and hence f is transitive in the strong sense. Since x is an isolated point of X, x should be a periodic point of f, and consequently X coincides with a single periodic orbit. The same conclusion can be drawn by constructing a periodic pseudo orbit through x. (2) The case where $\omega(x) \cap \alpha(x) = \emptyset$. Note that x is a chain continuity point for both f and f^{-1}, and so by Theorems 3.3 and 3.5 in [2], each $f|_{\omega(x)}$ and $f|_{\alpha(x)}$ is a periodic orbit or conjugate to an odometer. It also holds that every $y \in O_f(x) \cup \omega(x)$ is a chain continuity point for f, and hence $O_f(x) \cup \omega(x) \subset Sh^+(f)$. It only remains to prove that $\alpha(x) \subset Sh^+(f)$. Since $f|_{\alpha(x)}$ has the shadowing property, we can use the argument in the proof of Theorem 1.3.2 above to prove $\alpha(x) \subset Sh^+(f)$. □

3.3. **Proof of Theorem 1.3.3.**

For the proof of Theorem 1.3.3, we need a few definitions. A δ-chain $(x_i)_{i=0}^k$ of a continuous map f is said to be a *δ-cycle* of f if $x_0 = x_k$. A point $x \in X$ is a *chain recurrent point* of f if for any $\delta > 0$, there is a δ-cycle $(x_i)_{i=0}^k$ of f with $x_0 = x_k = x$. We denote by $CR(f)$ the set of chain recurrent points of f, and then f is said to be *chain recurrent* if $CR(f) = X$. The following lemma claims that if $f : X \to X$ is a chain recurrent continuous map, then for all $m \in \mathbb{N}$, any pair of points in a connected subset of X are chainable with an arbitrarily precise chain of f whose length is a multiple of m.

Lemma 3.3.1. *Let $f : X \to X$ be a chain recurrent continuous map. Then, for every connected subset $C \subset X$, $x, y \in C$, $\delta > 0$, and $m \in \mathbb{N}$, there is a δ-chain $(x_i)_{i=0}^k$ of f with $x_0 = x$, $x_k = y$ and $m|k$.*

Proof. It is enough to show that there is a δ-chain $(y_i)_{i=0}^l$ of f^m with $y_0 = x$, $y_l = y$. Take a sequence $(z_j)_{j=0}^n$ of points in C with $z_0 = x$, $z_n = y$ and $d(z_j, z_{j+1}) \leq \delta/2$ for all $0 \leq j \leq n-1$. By Proposition 44 in Chapter V of [8], we have $CR(f^m) = CR(f) = X$. Therefore, for each $0 \leq j \leq n-1$, there is a $\delta/2$-cycle $c_j = (x_i^{(j)})_{i=0}^{k_j}$ of f^m with $x_0^{(j)} = x_{k_j}^{(j)} = z_j$. Define $c'_j = (y_i^{(j)})_{i=0}^{k_j}$ by $y_i^{(j)} = x_i^{(j)}$ for $0 \leq i \leq k_j - 1$ and $y_{k_j}^{(j)} = z_{j+1}$. Then, c'_j is a δ-chain of f^m from z_j to z_{j+1}. Thus, the chain formed by $c'_0, \ldots, c'_{n-1}$ gives the desired δ-chain of f^m. □

Now let us prove Theorem 1.3.3.

Proof of Theorem 1.3.3. Assume $\operatorname{diam} C(x) > 2c$ to exhibit a contradiction. Choose $y, z \in C(x)$, and $\epsilon > 0$ with $d(y, z) > 2c + 3\epsilon$. Fix $\delta > 0$ such that every δ-pseudo orbit starting from x is $(c+\epsilon)$-shadowed by some point of X. Note that f is chain recurrent since it is pointwise recurrent. Applying Lemma 3.3.1, we can take a δ-chain $(x_i)_{i=0}^l$ and a δ-cycle $(z_i)_{i=0}^m$ with $x_0 = x$, $x_l = y$, and $z_0 = z_m = z$. By Lemma 3.3.1 again, there is an integer $a > 0$ and a δ-chain $(y_i)_{i=0}^{ma}$ such that $y_0 = y$ and $y_{ma} = z$. Consider the following forward δ-pseudo orbit starting from x:

$$(x_0, x_1, \ldots, x_l, y_1, \ldots, y_{ma}, z_1, \ldots, z_m, z_1, \ldots, z_m, z_1, \ldots, z_m, \ldots),$$

which is $(c+\epsilon)$-shadowed by some $p \in X$. Put $q = f^l(p)$. Then, we have $d(y, q) = d(x_l, f^l(p)) \leq c + \epsilon$ and $d(f^{m(a+n)}(q), z) = d(f^{l+m(a+n)}(p), z_m) \leq c + \epsilon$ for every integer

$n > 0$. Since $q \in R(f) = R(f^m)$ by Lemma 25 in Chapter IV of [8], there exists an integer $n_0 > 0$ such that $d(q, f^{m(a+n_0)}(q)) \le \epsilon$. Then,

$$d(y, z) \le d(y, q) + d(q, f^{m(a+n_0)}(q)) + d(f^{m(a+n_0)}(q), z) \le 2c + 3\epsilon,$$

which contradicts the choice of $y, z \in C(x)$ with $d(y, z) > 2c + 3\epsilon$. Thus, we have obtained $\operatorname{diam} C(x) \le 2c$. □

3.4. **Proof of Theorems 1.3.4 and 1.3.5.**

In this subsection, we show that the set of shadowable points of any homeomorphism is Borel measurable and prove Theorems 1.3.4 and 1.3.5.

Lemma 3.4.1. *Let $f : X \to X$ be a homeomorphism. Then, $Sh_b(f)$ is a Borel set in X for every $b > 0$. Hence, $Sh_{c+}(f)$ is a Borel set in X for every $c \ge 0$, and in particular, $Sh(f)$ is a Borel set in X.*

Proof. We first prove that $Sh_b(f)$ is a Borel set in X for any given $b > 0$. For $\delta > 0$, let $S_{\delta,b}(f)$ be the set of points $z \in X$ such that every pseudo orbit $(x_i)_{i\in\mathbb{Z}}$ with $d(f(x_i), x_{i+1}) < \delta$ ($\forall i \in \mathbb{Z}$) and $x_0 = z$ is b-shadowed by some $y \in X$. Then, it is easy to see that

$$Sh_b(f) = \bigcup_{m\in\mathbb{N}} S_{1/m,b}(f).$$

Hence, it suffices to show that $S_{\delta,b}(f)$ is a closed subset of X for all $\delta > 0$. Let $(z_n)_{n\in\mathbb{N}}$ be a sequence of points in $S_{\delta,b}(f)$ such that $\lim_{n\to\infty} z_n = z$ for some $z \in X$. Given any pseudo orbit $(x_i)_{i\in\mathbb{Z}}$ with $d(f(x_i), x_{i+1}) < \delta$ ($\forall i \in \mathbb{Z}$) and $x_0 = z$, we define a sequence of pseudo orbits $(x_i^{(n)})_{i\in\mathbb{Z}}$, $n \in \mathbb{N}$, by $x_0^{(n)} = z_n$ and $x_i^{(n)} = x_i$ for $i \ne 0$. Then, for sufficiently large $n \in \mathbb{N}$, we have $d(f(x_i^{(n)}), x_{i+1}^{(n)}) < \delta$ for all $i \in \mathbb{Z}$. For such $n \in \mathbb{N}$, the pseudo orbit $(x_i^{(n)})_{i\in\mathbb{Z}}$ is b-shadowed by some $y_n \in X$ because $x_0^{(n)} = z_n \in S_{\delta,b}(f)$. Take a subsequence $(y_{n_j})_{j\in\mathbb{N}}$ such that $\lim_{j\to\infty} y_{n_j} = \bar{y}$ for some $\bar{y} \in X$. Then, we have

$$d(x_0, \bar{y}) = d(z, \bar{y}) = \lim_{j\to\infty} d(z_{n_j}, y_{n_j}) = \lim_{j\to\infty} d(x_0^{(n_j)}, y_{n_j}) \le b$$

and

$$d(x_i, f^i(\bar{y})) = \lim_{j\to\infty} d(x_i^{(n_j)}, f^i(y_{n_j})) \le b$$

for $i \ne 0$. Thus, $\bar{y}$ is a b-shadowing point of $(x_i)_{i\in\mathbb{Z}}$, proving that $z \in S_{\delta,b}(f)$. From this we see that $S_{\delta,b}(f)$ is a closed subset of X, and therefore $Sh_b(f)$ is a Borel set in X. Given $c \ge 0$, we have

$$Sh_{c+}(f) = \bigcap_{n\in\mathbb{N}} Sh_{c+1/n}(f),$$

and every $Sh_{c+1/n}(f)$, $n \in \mathbb{N}$, is a Borel set in X. Hence, $Sh_{c+}(f)$ is a Borel set in X. The claim for $Sh(f)$ follows by putting $c = 0$. □

Remark 3.4.1. By the same argument as in the proof above, we can prove that $Sh_b^+(f)$, $b > 0$, $Sh_{c+}^+(f)$, $c \ge 0$, and $Sh^+(f)$ are Borel sets in X for every continuous map $f : X \to X$.

Remark 3.4.2. By Lemma 3.1.2, we see that for any continuous map $f: X \to X$ and any $c \geq 0$, $Sh^+_{c+}(f) \subset \bigcap_{b>c} \operatorname{Int} Sh^+_b(f)$. Since

$$\bigcap_{b>c} \operatorname{Int} Sh^+_b(f) \subset \bigcap_{b>c} Sh^+_b(f) = Sh^+_{c+}(f),$$

we have $Sh^+_{c+}(f) = \bigcap_{b>c} \operatorname{Int} Sh^+_b(f)$, and hence $Sh^+_{c+}(f)$ is a G_δ-set in X. Similarly, by Proposition 3.1.1, for any homeomorphism $f: X \to X$ and any $c \geq 0$, we have $Sh_{c+}(f) = \bigcap_{b>c} \operatorname{Int} Sh_b(f)$, and hence $Sh^+_{c+}(f)$ is a G_δ-set in X.

Here we give a proof of Theorems 1.3.4 and 1.3.5.

Proof of Theorem 1.3.4. Since $\mu(\{x \in \operatorname{supp}(\mu) : \overline{O_f(x)} = \operatorname{supp}(\mu)\}) = 1$ by [41, Theorem 5.15] and $\mu(R(f)) = 1$ by Poincaré recurrence theorem, we have $\mu(\{x \in \operatorname{supp}(\mu) : \omega(x) = \operatorname{supp}(\mu)\}) = 1$. Since $\mu(Sh_{c+}(f)) = 1$, there exists $x \in Sh_{c+}(f) \cap \operatorname{supp}(\mu)$ such that $\omega(x) = \operatorname{supp}(\mu)$. Then, we have $\operatorname{supp}(\mu) \subset Sh^+_{c+}(f)$ by Lemma 3.1.1. Let $b > c$. Then, by Lemmas 3.1.3 and 3.1.4, there exist $\delta > 0$ and a neighborhood U of $\operatorname{supp}(\mu)$ such that every δ-pseudo orbit $(x_i)_{i\in\mathbb{Z}}$ contained in U is b-shadowed by some $y \in X$. In particular, every δ-pseudo orbit contained in $\operatorname{supp}(\mu)$ is b-shadowed by some point of X. Thus, f has the b-shadowing property around $\operatorname{supp}(\mu)$, and since $b > c$ is arbitrary, f has the $c+$-shadowing property around $\operatorname{supp}(\mu)$. The claim for $Sh(f)$ follows by putting $c = 0$. □

Proof of Theorem 1.3.5. Note that for every minimal subset $Y \subset X$, there exists $\mu \in \mathcal{M}^{erg}_f(X)$ such that $\operatorname{supp}(\mu) = Y$. Since $\mu(Sh_{c+}(f)) = 1$ for such μ, we have $Y \cap Sh_{c+}(f) \neq \emptyset$. Now, for any given $x \in X$, there is a minimal subset $Y_0 \subset \omega(x)$. Then, since $Y_0 \cap Sh_{c+}(f) \neq \emptyset$, we have $\omega(x) \cap Sh_{c+}(f) \neq \emptyset$. Applying Lemma 3.1.6, we obtain $x \in Sh_{c+}(f)$. Hence, $Sh_{c+}(f) = X$, and thus f has the $c+$-shadowing property by Corollary 3.1.1. The claim for $Sh(f)$ follows by putting $c = 0$. □

3.5. **Miscellaneous Examples.**

In this subsection, we present some examples which complement the results of this section. The first example shows that there is a pointwise recurrent homeomorphism for which $X^{deg} \subset Sh(f)$ does not hold.

Example 3.5.1. Let $\sigma : \{0,1\}^{\mathbb{Z}} \to \{0,1\}^{\mathbb{Z}}$ be the shift map and let $X \subset \{0,1\}^{\mathbb{Z}}$ be an infinite minimal subshift. Note that $f \equiv \sigma|_X : X \to X$ is pointwise recurrent and expansive. It is easy to show that for any expansive homeomorphism $g : Y \to Y$ on a compact metric space Y, we have $CR(g) \cap Sh(g) \subset \overline{Per(g)}$. On the other hand, we have $CR(f) = X$ and $Per(f) = \emptyset$, since X is infinite and minimal. Hence, $Sh(f) = \emptyset$, but $X^{deg} = X \neq \emptyset$.

As mentioned in subsection 1.3, the next example shows that Morales' result (M2) ([31, Theorem 1.3]) does not necessarily hold for an equicontinuous map in its own form.

Example 3.5.2. Let $x_n = (0, \frac{1}{2n-1}) \in \mathbb{R}^2$, $I_n = [0,1] \times \{\frac{1}{2n}\} \subset \mathbb{R}^2$ for $n \in \mathbb{N}$ and $I_0 = [0,1] \times \{0\} \subset \mathbb{R}^2$. Set $X = \{x_n : n \in \mathbb{N}\} \cup I_0 \cup \bigcup_{n\in\mathbb{N}} I_n$ and define $f : X \to X$ by $f(x_n) = x_{n+1}$, $f(t, \frac{1}{2n}) = (t, \frac{1}{2n+2})$ ($n \in \mathbb{N}$, $t \in [0,1]$) and $f(x) = x$ on I_0. Then, f is open, injective, and equicontinuous, but $X^{deg} = \{x_n : n \in \mathbb{N}\}$ is not contained in $Sh^+(f)$, because $Sh^+(f) = \emptyset$.

As mentioned in subsection 1.3, the next example shows that we cannot replace the assertion"$x \in Sh_{c+}(f)$" in Proposition 1.3.1 by "$x \in Sh_c(f)$". This example also shows that $Sh_{c+}(f) = Sh_c(f)$ does not always hold.

Example 3.5.3. Let $C_r = \{(x,y) \in \mathbb{R}^2 : x^2 + y^2 = r^2\}$ for $r > 0$ and define $X = C_1 \cup \bigcup_{n\in\mathbb{N}} C_{1+1/n}$. Then, the identity map Id on X is trivially an isometry. Since $\operatorname{diam} C((1,0)) = \operatorname{diam} C_1 = 2$, we have $(1,0) \in Sh_{2+}(Id)$ by Proposition 1.3.1. Now, let us claim that $(1,0) \notin Sh_2(Id)$. To prove the claim, given $\delta > 0$, we take an integer $n_0 > 0$ with $1/n_0 < \delta$. Then, there is a δ-pseudo orbit $(x_i)_{i\in\mathbb{Z}}$ of Id with $x_0 = (1,0)$ such that the closure of $\{x_i : i \in \mathbb{Z}\}$ contains $X_0 = C_1 \cup \bigcup_{n\geq n_0} C_{1+1/n}$. Note that there is no $x \in X$ such that $X_0 \subset B_2(x) = \{y \in X : d(x,y) \leq 2\}$, and hence $(x_i)_{i\in\mathbb{Z}}$ has no 2-shadowing point. Thus, we have $(1,0) \notin Sh_2(Id)$. As a consequence, $Sh_{2+}(Id) \neq Sh_2(Id)$.

Then, we give an example showing that for a homeomorphism $f : X \to X$ and $b > 0$, $Sh_b(f)$ is not always f-invariant. Note that $Sh_{c+}(f)$ is f-invariant for every $c \geq 0$ (Corollary 3.1.2). It indicates that despite of its slightly complicated definition, $c+$-shadowable points are more suitable for theoretical study compared to b-shadowable points.

Example 3.5.4. Let $\mathbb{C}$ be the complex plane and let $B_1(0)$ denote the closed unit disk centered at the origin 0. Set $\theta_n = \tan^{-1}(n)/2$ and $r_n = 1 - 1/(1 + |n|)$ for $n \in \mathbb{Z}$, and define $I_0 = \{r \in \mathbb{R} : 1 \leq r \leq 2\} \subset \mathbb{C}$ and $I_n = \{re^{\sqrt{-1}\theta_n} \in \mathbb{C} : r_n \leq r \leq 1\}$ for $n \in \mathbb{Z} \setminus \{0\}$. Note that I_n is inside $B_1(0)$ for $n \in \mathbb{Z} \setminus \{0\}$ and I_0 is outside $B_1(0)$. Set $J = \{e^{\sqrt{-1}\theta} \in \mathbb{C} : \pi/4 \leq |\theta| \leq \pi/2\}$ and define

$$X = \{0\} \cup J \cup \bigcup_{n\in\mathbb{Z}} I_n \subset \mathbb{C}.$$

Then, X is a compact subset of $\mathbb{C}$. Put $x_n = e^{\sqrt{-1}\theta_n} \in I_n$ for $n \in \mathbb{Z}$ and take a homeomorphism $f_n : I_n \to I_{n+1}$ such that $f_n(x_n) = x_{n+1}$ for each $n \in \mathbb{Z}$. Then, a homeomorphism $f : X \to X$ can be defined by

$$f(x) = \begin{cases} x & \text{if } x \in \{0\} \cup J \\ f_n(x) & \text{if } x \in I_n,\ n \in \mathbb{Z} \end{cases}.$$

Note that we have $f(x_n) = x_{n+1}$ for all $n \in \mathbb{Z}$. Let us claim that $x_0 \in Sh_1(f)$ and $x_{\pm 1} \notin Sh_1(f)$, which implies that $Sh_1(f)$ is not f-invariant. In fact, it is easy to see that if $\delta > 0$ is sufficiently small, then every δ-pseudo orbit $(y_i)_{i\in\mathbb{Z}}$ of f with $y_0 = x_0 = 1$ is contained in $B_1(0)$, and hence 1-shadowed by $0 \in X$. On the other hand, for any $\delta > 0$, since $f(x_{-1}) = x_0 = 1$, $f(1) = f(x_0) = x_1$, and $f^n(x_0) = x_n \to e^{\pm\sqrt{-1}\pi/4}$ as $n \to \pm\infty$, there is a δ-pseudo orbit $(z_i)_{i\in\mathbb{Z}}$ of f such that $z_0 = x_1$, $z_{-1} = 1 + \epsilon$ for some $\epsilon > 0$, and $z_{\pm m} = \pm\sqrt{-1}$ for some large $m \in \mathbb{N}$. Note that no point of X other than 0 is at a distance of no more than 1 from both $\pm\sqrt{-1}$. However, we have $|z_{-1}| = 1 + \epsilon > 1$. Hence, $(z_i)_{i\in\mathbb{Z}}$ has no 1-shadowing point, and thus $x_1 \notin Sh_1(f)$. We can also prove that $x_{-1} \notin Sh_1(f)$ similarly.

Next, we give an example of a homeomorphism $f : X \to X$ satisfying the following properties.

(P1) There is $x \in Sh(f)$ such that $\omega(x) \cup \alpha(x) \subset X \setminus Sh(f)$.
(P2) There is $x \in Sh^+(f) \cap Sh^+(f^{-1})$ such that $x \notin Sh(f)$.
(P3) There is $x \in X$ such that $\omega(x) \cap Sh^+(f) \neq \emptyset$ but $x \notin Sh^+(f)$.

(P1) is a property relevant to Lemma 3.1.6. It shows that $x \in Sh(f)$ does not always imply $\omega(x) \cap Sh(f) \neq \emptyset$ or $\alpha(x) \cap Sh(f) \neq \emptyset$. It is obvious that if $x \in Sh(f)$, then $x \in Sh^+(f) \cap Sh^+(f^{-1})$. However, by (P2), we see that the converse does not always hold. (P3) shows that Lemma 3.1.6 does not work for $Sh^+(f)$.

Example 3.5.5. Let $\sigma : \{0,1,2\}^{\mathbb{Z}} \to \{0,1,2\}^{\mathbb{Z}}$ be the shift map. Define a metric d on $\{0,1,2\}^{\mathbb{Z}}$ by $d(x,y) = \sum_{i\in\mathbb{Z}} 2^{-|i|}\delta(x_i, y_i)$ for $x = (x_i)_{i\in\mathbb{Z}}$, $y = (y_i)_{i\in\mathbb{Z}} \in \{0,1,2\}^{\mathbb{Z}}$, where $\delta(a,b) = 1$ if $a \neq b$ and $\delta(a,b) = 0$ otherwise. Put $\overline{a} = (\dots, a, a, a, \dots) \in \{0,1,2\}^{\mathbb{Z}}$ for $a \in \{0,1,2\}$ and consider the following three points $p, q, r \in \{0,1,2\}^{\mathbb{Z}}$:

$$p = (\dots, 0, 0, 0, \overset{0}{\check{1}}, 1, 1, \dots),\ q = (\dots, 1, 1, 1, \overset{0}{\check{2}}, 2, 2, \dots),\ r = (\dots, 0, 0, 0, \overset{0}{\check{2}}, 2, 2, \dots).$$

Note that

$$d(\sigma^n(p), \overline{1}) = d(\sigma^n(q), \overline{2}) = d(\sigma^n(r), \overline{2}) \searrow 0 \quad (n \to +\infty)$$

and

$$d(\sigma^n(p), \overline{0}) = d(\sigma^n(q), \overline{1}) = d(\sigma^n(r), \overline{0}) \searrow 0 \quad (n \to -\infty).$$

Set $A = \{\overline{0}, \overline{1}\} \cup \{\sigma^n(p) : n \in \mathbb{Z}\}$, $B = \{\overline{1}, \overline{2}\} \cup \{\sigma^n(q) : n \in \mathbb{Z}\}$, $C = \{\sigma^n(r) : n \in \mathbb{Z}\}$, and then let $X = A \cup B \cup C$. Finally, define $f = \sigma|_X$. Let us claim that f has the following properties.

(1) $C \subset Sh(f)$.
(2) $B \subset Sh^+(f)$ and $A \subset Sh^+(f^{-1})$.
(3) $A \setminus \{\overline{1}\} \subset X \setminus Sh^+(f)$ and $B \setminus \{\overline{1}\} \subset X \setminus Sh^+(f^{-1})$.
(4) $\overline{1} \notin Sh(f)$.

(1) Let $x \in C$. Then, we have:

- $d(f^n(x), \overline{2}) \to 0$ as $n \to +\infty$ and $d(f^n(x), \overline{0}) \to 0$ as $n \to -\infty$; and
- $\overline{2}$ is a sink of f and $\overline{0}$ is a source of f.

From these facts, we see that for any given $\epsilon > 0$, there is $\delta > 0$ such that every δ-pseudo orbit $(z_n)_{n\in\mathbb{Z}}$ of f with $z_0 = x$ is ϵ-shadowed by x itself, and hence $x \in Sh(f)$.
(2) Since

- B is an attractor of f; and
- $f|_B$ has the shadowing property because it is a subshift of finite type,

it holds that $B \subset Sh^+(f)$. Similarly, we have $A \subset Sh^+(f^{-1})$.
(3) Let $x \in A\setminus\{\overline{1}\}$ and fix any b with $0 < b < d(x, B) \le 3$. Let us show that $x \notin Sh_b^+(f)$. It is easy to see that for any given $\delta > 0$, there is a forward δ-pseudo orbit $(z_i)_{i=0}^{\infty}$ of f with $z_0 = x$ such that $z_k = \overline{1}$ and $z_l = \overline{2}$ for some $0 < k < l$. Suppose that $y \in X$ is a b-shadowing point of such $(z_i)_{i=0}^{\infty}$. Then, $d(\overline{1}, f^k(y)) = d(z_k, f^k(y)) \le b < 3$ and $d(\overline{2}, f^l(y)) = d(z_l, f^l(y)) \le b < 3$. Since $d(\overline{1}, w) = 3$ for all $w \in C$, we have $f^k(y) \notin C$, implying $f^k(y) \in A \cup B$ and then $y \in A \cup B$. Similarly, since $d(\overline{2}, w) = 3$ for all $w \in A$, we have $f^l(y) \notin A$, implying $f^l(y) \in B \cup C$ and then $y \in B \cup C$. From $A \cap C = \emptyset$, it follows that $y \in B$, but this implies $d(z_0, y) = d(x, y) \ge d(x, B) > b$, which contradicts that y is a b-shadowing point of $(z_i)_{i=0}^{\infty}$. Hence, $(z_i)_{i=0}^{\infty}$ has no b-shadowing point. Since

$\delta > 0$ can be taken arbitrarily small, we conclude that $x \notin Sh_b^+(f)$. Thus, we have $x \notin Sh^+(f)$ for all $x \in A \setminus \{\overline{1}\}$. Similarly, we can prove that $B \setminus \{\overline{1}\} \subset X \setminus Sh^+(f^{-1})$.
(4) Fix any $0 < b < 3$. It is easy to see that for any given $\delta > 0$, there is a δ-pseudo orbit $(z_i)_{i\in\mathbb{Z}}$ of f with $z_0 = \overline{1}$ such that $z_k = \overline{0}$ and $z_l = \overline{2}$ for some $k < 0$ and $l > 0$. Then, by the same argument as above, we see that such $(z_i)_{i\in\mathbb{Z}}$ has no b-shadowing point. Hence, $\overline{1} \notin Sh_b(f)$, and in particular, $\overline{1} \notin Sh(f)$.

Let us check that f satisfies the three properties (P1), (P2), and (P3). Firstly, by (1) and (3), we see that $r \in C \subset Sh(f)$ and $\omega(r) \cup \alpha(r) = \{\overline{0}, \overline{2}\} \subset X \setminus Sh(f)$. Secondly, by (2) and (4), we have $\overline{1} \in A \cap B \subset Sh^+(f) \cap Sh^+(f^{-1})$ and $\overline{1} \notin Sh(f)$. Finally, from (2) and (3), it follows that $\overline{1} \in \omega(p) \cap B \subset \omega(p) \cap Sh^+(f)$ and $p \in A \setminus \{\overline{1}\} \subset X \setminus Sh^+(f)$.

If $f : X \to X$ is a chain transitive homeomorphism, $Sh^+(f) \neq \emptyset$ implies $Sh^+(f) = X$ (Lemma 3.1.1), and so of course this dichotomy holds if f is transitive in the strong sense. However, as the following example shows, this is not the case when f is merely a transitive homeomorphism. It is also an example of a homeomorphism $f : X \to X$ for which there is $x \in X$ such that $\omega(x) \cap Sh^+(f) \neq \emptyset$ but $x \notin Sh^+(f)$, showing that Lemma 3.1.6 does not work for $Sh^+(f)$ (see also Example 3.5.5).

Example 3.5.6. Let $Y = \{0,1\}^{\mathbb{Z}}$ and let $g : Y \to Y$ be the shift map. Take $y \in Y$ so that $Y = \omega(y) = \alpha(y)$ and let $y_i = g^i(y) \in Y$ for every $i \in \mathbb{Z}$. Define $x_i \in Y \times [-1,1]$, $i \in \mathbb{Z}$, as follows:

$$x_i = \begin{cases} \big(y_i, 1 - 1/(i+1)\big) & \text{if } i > 0 \\ (y, 0) & \text{if } i = 0 \\ \big(y_i, -1 - 1/(i-1)\big) & \text{if } i < 0 \end{cases}.$$

Let $X = Y \times \{-1,1\} \cup \{x_i : i \in \mathbb{Z}\}$ and define $f : X \to X$ by $f(y,a) = (g(y), a)$ for $y \in Y$, $a \in \{-1,1\}$, and $f(x_i) = x_{i+1}$ for $i \in \mathbb{Z}$.

Since $X = \overline{O_f(x_0)}$, $f : X \to X$ is a transitive homeomorphism. From the facts that $Y \times \{1\}$ is an attractor of f, and $f|_{Y\times\{1\}} \simeq g$ has the shadowing property, we see that $Y \times \{1\} \subset Sh^+(f)$. Note that for every $i \in \mathbb{Z}$, x_i is an isolated point of X and $\omega(x_i) = Y \times \{1\}$. If $x_i \in Sh^+(f)$ for some $i \in \mathbb{Z}$, then x_i is a chain continuity point for f, and so $f|_{\omega(x_i)} = f|_{Y\times\{1\}} \simeq g$ should be a periodic orbit or conjugate to an odometer (see the proof of Theorem 1.3.2 in subsection 3.2.2), but this is not the case. Hence, we have $\{x_i : i \in \mathbb{Z}\} \subset X \setminus Sh^+(f)$. Then, it follows from Lemma 3.1.1 that $Y \times \{-1\} \subset X \setminus Sh^+(f)$. Thus, $Sh^+(f) = Y \times \{1\}$, a non-empty proper subset of X. It also holds that $\omega(x_0) \subset Sh^+(f)$ but $x_0 \notin Sh^+(f)$.

4. Properties of shadowable points: chaos and equicontinuity

4.1. **Proof of Theorem 1.4.1 and Corollary 1.4.2.**

In this subsection, we prove Theorem 1.4.1 and Corollary 1.4.2. We first give the definition of "e-separated pairs of two δ-cycles at a point" mentioned in subsection 1.4. Let $f : X \to X$ be a continuous map on a compact metric space (X, d).

Definition 4.1.1. *For $x \in X$, a δ-chain $(x_i)_{i=0}^k$ of f is said to be a δ-cycle of f at x if $x_0 = x_k = x$. For $e > 0$, we say that a pair $((z_i^{(0)})_{i=0}^m, (z_i^{(1)})_{i=0}^m)$ of two δ-cycles of f at x is e-separated if $d(z_i^{(0)}, z_i^{(1)}) > e$ for some $0 < i < m$.*

Note that when we say that a pair of δ-cycles of f is e-separated, the two δ-cycles have the same length, which will be called the *period* of the pair. In what follows, δ-cycles mean δ-cycles of f unless otherwise specified.

Remark 4.1.1. Let $((z_i^{(0)})_{i=0}^k, (z_i^{(1)})_{i=0}^l)$ be a pair of δ-cycles at x with $d(z_j^{(0)}, z_j^{(1)}) > e$ for some $0 < j < \min\{k, l\}$. Then, the pair of the following δ-cycles:

$$(z_0^{(0)}, z_1^{(0)}, \dots, z_j^{(0)}, \dots, z_{k-1}^{(0)}, z_0^{(1)}, z_1^{(1)}, \dots, z_{l-1}^{(1)}, z_0^{(1)}),$$
$$(z_0^{(1)}, z_1^{(1)}, \dots, z_j^{(1)}, \dots, z_{l-1}^{(1)}, z_0^{(0)}, z_1^{(0)}, \dots, z_{k-1}^{(0)}, z_0^{(0)})$$

is an e-separated pair of δ-cycles at x with the period $k + l$.

Lemma 4.1.1. *Let $f : X \to X$ be a continuous map. Given $e > 0$ and $z \in X$, if one of the following conditions is satisfied, then for any $\delta > 0$, X contains an e-separated pair of two δ-cycles of f at z.*

(1) *There is a closed f-invariant subset $S \subset X$ such that $CR(f|_S) = S$ and $z \in Sen_e(f|_S)$.*
(2) *There are a Li-Yorke pair $\{x, y\} \subset X$ with modulus e and a sequence of integers $0 < n_1 < n_2 < \cdots$ such that*
$$\lim_{j\to\infty} d(f^{n_j}(x), f^{n_j}(y)) = 0 \quad \textit{and} \quad \lim_{j\to\infty} f^{n_j}(x) = z.$$
(3) *There are $x \in X$ and a closed f-invariant subset $S \subset X$ such that $z \in S \subset \omega(x, f)$ and $\omega(x, f) \setminus B_e(S) \neq \emptyset$.*

Proof.
(1): This proof is a modification of that of [24, Theorem 2]. Given $\delta > 0$, fix $0 < \delta_0 < \delta/2$ and take $0 < \delta_1 < \delta/2$ so that $d(a, b) < \delta_1$ implies $d(f(a), f(b)) < \delta_0$ for all $a, b \in X$. Then, since $z \in Sen_e(f|_S)$, there are $z_0^{(0)}, z_0^{(1)} \in S$ and $N \in \mathbb{N}$ such that $\max\{d(z, z_0^{(0)}), d(z, z_0^{(1)})\} < \delta_1$ and $d(f^N(z_0^{(0)}), f^N(z_0^{(1)})) > e$. Choose $\epsilon > 0$ with $d(f^N(z_0^{(0)}), f^N(z_0^{(1)})) > e + 2\epsilon$ and take $0 < \delta_2 < \delta/2$ such that for every δ_2-chain $(x_0, x_1, \dots, x_N)$ of f, we have $d(f^N(x_0), x_N) < \epsilon$. Since $z_0^{(0)}, z_0^{(1)} \in S = CR(f|_S)$, there exists a pair of δ_2-cycles in S

$$((z_0^{(0)}, z_1^{(0)}, \dots, z_{k-1}^{(0)}, z_0^{(0)}), (z_0^{(1)}, z_1^{(1)}, \dots, z_{l-1}^{(1)}, z_0^{(1)}))$$

with $\min\{k, l\} > N$. By the choice of δ_2, we have

$$\begin{aligned} d(z_N^{(0)}, z_N^{(1)}) &\geq d(f^N(z_0^{(0)}), f^N(z_0^{(1)})) - d(f^N(z_0^{(0)}), z_N^{(0)}) - d(f^N(z_0^{(1)}), z_N^{(1)}) \\ &> e + 2\epsilon - 2\epsilon = e. \end{aligned}$$

From

$$d(f(z), z_1^{(0)}) \leq d(f(z), f(z_0^{(0)})) + d(f(z_0^{(0)}), z_1^{(0)}) < \delta_0 + \delta_2 < \delta$$

and

$$d(f(z_{k-1}^{(0)}), z) \leq d(f(z_{k-1}^{(0)}), z_0^{(0)}) + d(z_0^{(0)}, z) < \delta_2 + \delta_1 < \delta,$$

it follows that $(z, z_1^{(0)}, \dots, z_{k-1}^{(0)}, z)$ is a δ-cycle at z. Similarly, $(z, z_1^{(1)}, \dots, z_{l-1}^{(1)}, z)$ is also a δ-cycle at z. Hence, as in Remark 4.1.1, S contains an e-separated pair of δ-cycles at z with period $k + l$.

(2): Given $\delta > 0$, take $0 < \eta = \eta(\delta) < \delta$ such that $d(a, b) \le \eta$ implies $d(f(a), f(b)) \le \delta$ for all $a, b \in X$. Then, there are $1 \le N_1 < N_2 < N_3$ with $N_2 - N_1$ and $N_3 - N_2$ arbitrarily large such that

$$\{f^{N_1}(x), f^{N_1}(y), f^{N_3}(x), f^{N_3}(y)\} \subset B_\eta(z) = \{u \in X : d(z, u) \le \eta\}.$$

and $d(f^{N_2}(x), f^{N_2}(y)) > e$. Then, the pair of the following

$$(z, f^{N_1+1}(x), \dots, f^{N_2-1}(x), f^{N_2}(x), f^{N_2+1}(x), \dots, f^{N_3-1}(x), z),$$
$$(z, f^{N_1+1}(y), \dots, f^{N_2-1}(y), f^{N_2}(y), f^{N_2+1}(y), \dots, f^{N_3-1}(y), z)$$

is an e-separated pair of δ-cycles at z.

(3): Fix $p \in \omega(x, f)$ with $d(p, S) > e$. Given $\delta > 0$, since $f|_{\omega(x,f)}$ is chain transitive, there is a δ-chain $(x_i^{(1)})_{i=0}^a$ of $f|_{\omega(x,f)}$ such that $x_0^{(1)} = z$ and $x_a^{(1)} = p$. Note that $f^a(z) \in S$, and hence $d(f^a(z), p) \ge d(p, S) > e$. By the chain transitivity of $f|_{\omega(x,f)}$ again, there is a pair $((y_i^{(0)})_{i=0}^b, (y_i^{(1)})_{i=0}^c)$ of δ-chains of $f|_{\omega(x,f)}$ such that $(y_0^{(0)}, y_0^{(1)}) = (f^a(z), p)$ and $(y_b^{(0)}, y_c^{(1)}) = (z, z)$. Consider the following pair of δ-cycles of f:

$$((z, f(z), \dots, f^{a-1}(z), y_0^{(0)}, y_1^{(0)}, \dots, y_{b-1}^{(0)}, z), (z, x_1^{(1)}, \dots, x_{a-1}^{(1)}, y_0^{(1)}, y_1^{(1)}, \dots, y_{c-1}^{(1)}, z)).$$

Since $d(y_0^{(0)}, y_0^{(1)}) = d(f^a(z), p) > e$, as in Remark 4.1.1, there is an e-separated pair of δ-cycles at z with period $2a + b + c$ contained in $\omega(x, f)$. □

Remark 4.1.2. Under the assumption of (1) (resp. (3)), the e-separated pairs of δ-cycles of f at z can be taken in S (resp. $\omega(x, f)$).

The next lemma is essential in the proof of Theorem 1.4.1.

Lemma 4.1.2. *Let $f : X \to X$ be a continuous map and let $x \in Sh_b^+(f)$ with $b > 0$. Given $e, \delta > 0$, and $z \in \omega(x, f)$, suppose that the following conditions are satisfied.*

- *$e > 2b$.*
- *Every δ-pseudo orbit $(x_i)_{i=0}^\infty$ of f with $x_0 = x$ is b-shadowed by some point of X.*
- *There is an e-separated pair $((z_i^{(0)})_{i=0}^m, (z_i^{(1)})_{i=0}^m)$ of δ-cycles of f at z with period m.*

Then, $h_{top}(f) \ge (\log 2)/m$, and there exists $w \in Ent(f)$ such that $d(x, w) \le b$.

Proof. Fix $0 < j < m$ with $d(z_j^{(0)}, z_j^{(1)}) > e$ and take $k > 0$ with $d(f^k(x), z) \le \delta$. By the hypothesis, given $n \in \mathbb{N}$, for each $s = (s_1, \dots, s_n) \in \{0, 1\}^n$, we can consider the following δ-chain of f:

$$(x, f(x), \dots, f^{k-1}(x), z_0^{(s_1)}, z_1^{(s_1)}, \dots, z_{m-1}^{(s_1)}, \dots, z_0^{(s_n)}, z_1^{(s_n)}, \dots, z_{m-1}^{(s_n)}),$$

which is b-shadowed by $y(s) \in B_b(x)$. Put $E_n = \{y(s) \in X : s \in \{0, 1\}^n\}$ and let us claim that E_n is a $(k + mn, e - 2b)$-separated set. In fact, for any $s, t \in \{0, 1\}^n$, if $s \ne t$, then $s_a \ne t_a$ for some $1 \le a \le n$, and letting $K = k + (a-1)m + j$, we have $K < k + mn$, and

$$d\big(f^K(y(s)), f^K(y(t))\big) \ge d(z_j^{(s_a)}, z_j^{(t_a)}) - d(f^K(y(s)), z_j^{(s_a)}) - d(f^K(y(t)), z_j^{(t_a)}) > e - 2b.$$

Note that $E_n \subset B_b(x)$ and the cardinality of E_n is 2^n. Hence, we have $S(B_b(x), k + mn, e - 2b) \geq 2^n$ for every $n \in \mathbb{N}$, and then

$$\begin{aligned} h(f, B_b(x)) &\geq h(f, B_b(x), e-2b) = \limsup_{n\to\infty} \frac{1}{n} \log S(B_b(x), n, e-2b) \\ &\geq \limsup_{n\to\infty} \frac{1}{k+mn} \log S(B_b(x), k+mn, e-2b) \\ &\geq \limsup_{n\to\infty} \frac{1}{k+mn} \log 2^n \\ &= \frac{1}{m} \log 2 > 0. \end{aligned}$$

Thus, we obtain $h_{top}(f) \geq h(f, B_b(x)) \geq (\log 2)/m$, and from Lemma 2.1.3 in subsection 2.1, it follows that $B_b(x) \cap Ent(f) \neq \emptyset$. □

Now let us prove Theorem 1.4.1.

Proof of Theorem 1.4.1. Take $b > c$ with $e > 2b > 2c$ and choose $\delta > 0$ such that every δ-pseudo orbit $(x_i)_{i=0}^{\infty}$ of f with $x_0 = x$ is b-shadowed by some point of X. For such δ, if one of the conditions (1)-(3) in Theorem 1.4.1 (corresponding to those in Lemma 4.1.1) is satisfied, then there exist $z \in \omega(x, f)$ and an e-separated pair of δ-cycles of f at z by Lemma 4.1.1. Hence, using Lemma 4.1.2, we see that there exists $w \in Ent(f)$ such that $d(x, w) \leq b$. Since $b > c$ can be taken arbitrarily close to c, and $Ent(f)$ is a closed subset of X, there exists $w \in Ent(f)$ such that $d(x, w) \leq c$, proving the theorem. □

Let $\sigma : \{0,1\}^{\mathbb{N}} \to \{0,1\}^{\mathbb{N}}$ be the full shift. The following lemma is a restatement (with modification) of Proposition 2 in Section 2 of [23]. It describes how we obtain from an e-separated pair of δ-cycles at x together with the shadowing property, a subsystem of some power of f which is an extension of the full shift.

Lemma 4.1.3. *Let $e \geq 2b > 0$ and let $((z_i^{(0)})_{i=0}^m, (z_i^{(1)})_{i=0}^m)$ be an e-separated pair of δ-cycles at $x \in X$. For each $s = (s_1, s_2, \dots) \in \{0,1\}^{\mathbb{N}}$, define a δ-pseudo orbit $\gamma(s)$ as follows:*

$$\gamma(s) = (z_0^{(s_1)}, z_1^{(s_1)}, \dots, z_{m-1}^{(s_1)}, z_0^{(s_2)}, z_1^{(s_2)}, \dots, z_{m-1}^{(s_2)}, z_0^{(s_3)}, z_1^{(s_3)}, \dots, z_{m-1}^{(s_3)}, \dots).$$

If every $\gamma(s)$, $s \in \{0,1\}^{\mathbb{N}}$, is b-shadowed by some point of X, then there exist a closed f^m-invariant subset $Y \subset B_b(x)$ and a factor map $\pi : (Y, f^m) \to (\{0,1\}^{\mathbb{N}}, \sigma)$.

Proof. Let

$$Y = \{y \in X : \ y \text{ is a } b\text{-shadowing point of } \gamma(s) \text{ for some } s \in \{0,1\}^{\mathbb{N}}\},$$

and define a map $\pi : Y \to \{0,1\}^{\mathbb{N}}$ so that y is a b-shadowing point of $\gamma(\pi(y))$. Then, it is easy to see that the following properties hold.

(1) Y is a closed subset of X;
(2) $f^m(Y) \subset Y$;
(3) π is well-defined;
(4) π is surjective;
(5) π is continuous; and
(6) $\pi \circ f^m = \sigma \circ \pi$.

Hence, $\pi : (Y, f^m) \to (\{0,1\}^{\mathbb{N}}, \sigma)$ is a factor map, and $Y \subset B_b(x)$ is obvious. $\square$

Remark 4.1.3. Let $Id : [0,1] \to [0,1]$ be the identity map on the unit interval. Then, for any $\delta > 0$, if $m \geq 1$ is large enough, we can take a δ-cycle $(z_i^{(0)})_{i=0}^m$ of Id at 0 with $z_j^{(0)} = 1$ for some $0 < j < m$. Consider the δ-cycle $(z_i^{(1)})_{i=0}^m$ of Id at 0 defined by $z_i^{(1)} = 0$ for all $0 \leq i \leq m$. Then, we have $d(z_j^{(0)}, z_j^{(1)}) = 1$, and every δ-pseudo orbit $\gamma(s)$, $s \in \{0,1\}^{\mathbb{N}}$, defined as in Lemma 4.1.3 is 1/2-shadowed by 1/2. But, it is obvious that there is no subsystem of (powers of) Id admitting a factor map to the full shift. Note that $((z_i^{(0)})_{i=0}^m, (z_i^{(1)})_{i=0}^m)$ is not a 1-separated pair of δ-cycles at 0 by the definition. This example shows that the assumption of the separation $> e$ cannot be replaced by $\geq e$ in order that Lemma 4.1.3 holds.

Remark 4.1.4. There is a sensitive continuous map $f : X \to X$ with the shadowing property such that (X, f) admits $(\{0,1\}^{\mathbb{N}}, \sigma)$ as a factor, but any subsystem of powers of f is not conjugate to $(\{0,1\}^{\mathbb{N}}, \sigma)$. In fact, $(\{0,1\}^{\mathbb{N}} \times X_m, \sigma \times g)$ with an odometer (X_m, g) gives such an example. The natural projection onto $(\{0,1\}^{\mathbb{N}}, \sigma)$ is a factor map. Note that $Per(\sigma \times g) = \emptyset$ because $Per(g) = \emptyset$, but if some subsystem of some power of $\sigma \times g$ were conjugate to $(\{0,1\}^{\mathbb{N}}, \sigma)$, then $Per(\sigma \times g)$ would be non-empty.

As a corollary of Lemma 4.1.1 and Lemma 4.1.3, we obtain the following lemma.

Lemma 4.1.4. *Let $f : X \to X$ be a continuous map and let $S \subset X$ be a closed f-invariant subset such that $CR(f|_S) = S$. If $x \in Sen(f|_S) \cap Sh^+(f)$, then for every $\epsilon > 0$, there are $m \in \mathbb{N}$ and a closed f^m-invariant subset $Y \subset B_\epsilon(x)$ for which we have a factor map $\pi : (Y, f^m) \to (\{0,1\}^{\mathbb{N}}, \sigma)$.*

Proof. Take positive constants e, ϵ, and $\delta > 0$ with the following properties.

- $x \in Sen_e(f|_S)$.
- $e > 2\epsilon$.
- Every δ-pseudo orbit $(x_i)_{i=0}^\infty$ of f with $x_0 = x$ is ϵ-shadowed by some point of X.

Then, the condition (1) of Lemma 4.1.1 is satisfied, and therefore S contains an e-separated pair of δ-cycles of f at x by Lemma 4.1.1. Hence, we can use Lemma 4.1.3 to obtain the conclusion. $\square$

For the proof of Corollary 1.4.2, we need the following lemma, which is also used in the proof of Theorem 1.4.3.

Lemma 4.1.5. *Let $f : X \to X$ be a continuous map. If $Y \subset X$ is a closed f^m-invariant subset with $m \in \mathbb{N}$, and $\pi : (Y, f^m) \to (\{0,1\}^{\mathbb{N}}, \sigma)$ is a factor map, then we have the following properties.*

1. *There is $y \in Y$ such that $\omega(y, f)$ is non-minimal for f.*
2. *There is $y \in Y$ such that $(\overline{O_f(y)}, f)$ is a minimal sensitive subsystem.*

Proof. (1): Take $s \in \{0,1\}^{\mathbb{N}}$ with $\omega(s, \sigma) = \{0,1\}^{\mathbb{N}}$ and $y \in \pi^{-1}(s)$. Putting $Z = \omega(y, f^m)$, we have $f^m(Z) \subset Z$ and $\pi(Z) = \{0,1\}^{\mathbb{N}}$, which implies that $\pi : (Z, f^m) \to (\{0,1\}^{\mathbb{N}}, \sigma)$ is a factor map. Then, defining $W = Z \cup f(Z) \cup \cdots \cup f^{m-1}(Z)$, we have $W = \omega(y, f)$. To show that W is non-minimal for f by contradiction, assume that W

is minimal for f. Then, we have $Z \subset W \subset M(f) = M(f^m)$, which contradicts that $M(\sigma) \neq \{0,1\}^{\mathbb{N}}$, because the π-image of any minimal point for f^m is also a minimal point for σ. Thus, W is non-minimal for f.
(2): It suffices to show that there exists $y \in Y$ such that $(\overline{O_{f^m}(y)}, f^m)$ is a minimal sensitive subsystem of (Y, f^m). Put $g = f^m$ and take an infinite minimal subshift $\Sigma \subset \{0,1\}^{\mathbb{N}}$. Since $Z = \pi^{-1}(\Sigma)$ is g-invariant, there is a minimal g-invariant subset $W \subset Z$. Then, since $\pi(W) \subset \Sigma$ is σ-invariant and Σ is minimal, we have $\pi(W) = \Sigma$. Let us claim that $g|_W$ is sensitive. Fix any $w \in W$. Note that $\sigma|_\Sigma$ is positively expansive, and hence if $\pi(U) \neq \{\pi(w)\}$ for every neighborhood U of w in W, then w is a sensitive point of $g|_W$. Assume the contrary, i.e, there is a neighborhood U of w in W such that $\pi(U) = \{\pi(w)\}$ to exhibit a contradiction. Since W is minimal for g, there is $n > 0$ such that $g^n(w) \in U$. Then, $\pi(w) \in \Sigma$ and $\sigma^n(\pi(w)) = \pi(g^n(w)) = \pi(w)$, which contradicts that Σ is infinite and minimal. Thus, for every $w \in W$, $(\overline{O_g(w)}, g)$=(W, g) is a minimal sensitive subsystem of (Y, g), proving the lemma. □

As the final proof of this subsection, we give a proof of Corollary 1.4.2.

Proof of Corollary 1.4.2.
(1) ⇒ (3): $h_{top}(f) > 0$ implies that $h_{top}(f|_{\Omega(f)}) = h_{top}(f) > 0$. By the shadowing property of f, we see that $f|_{\Omega(f)}$ is surjective. Hence, from [7, Corollary 2.4], it follows that $f|_{\Omega(f)}$ is Li-Yorke chaotic, and so is f.
(3) ⇒ (2): This is obvious by the definition.
(2) ⇒ (1): Let $\{x, y\} \subset X$ be a Li-Yorke pair with modulus e. Note that $x \in Sh^+(f)$ since $Sh^+(f) = X$. Applying Theorem 1.4.1 (2) with $c = 0$, we have $x \in Ent(f)$, implying that $Ent(f) \neq \emptyset$, and thus $h_{top}(f) > 0$.
(1) ⇒ (4): If $Sen(f|_{\Omega(f)}) = \emptyset$, then $h_{top}(f) = h_{top}(f|_{\Omega(f)}) = 0$. Therefore, when $h_{top}(f) > 0$, we have $Sen(f|_{\Omega(f)}) \neq \emptyset$. The shadowing property of f implies that $\Omega(f) = \Omega(f|_{\Omega(f)}) \subset CR(f|_{\Omega(f)}) \subset \Omega(f)$, so $CR(f|_{\Omega(f)}) = \Omega(f)$. Since $Sh^+(f) = X$, we can apply Lemma 4.1.4 with $S = \Omega(f)$ to have $m \in \mathbb{N}$ and a closed f^m-invariant subset $Y \subset X$ for which we have a factor map $\pi : (Y, f^m) \to (\{0,1\}^{\mathbb{N}}, \sigma)$. Thus, by Lemma 4.1.5 (1), there is $x \in X$ such that $\omega(x, f)$ is non-minimal for f.
(4) ⇒ (1): Let $x \in X$ be a point such that $\omega(x, f)$ is non-minimal for f. Then, applying Theorem 1.4.1 (3) with $c = 0$, we have $x \in Ent(f)$, which implies $h_{top}(f) > 0$. □

4.2. **Proof of Theorem 1.4.2.**

To prove Theorem 1.4.2, we need the following technical lemma, which is a version of the shortcut lemma proved in subsection 2.1 (Lemma 2.1.2). Intuitively, the open cover $\mathcal{U}$ of (X^2, d_2) in the lemma works as a "scale", and any pair of chains of f with sufficiently small gaps and an arbitrary length can be replaced by a pair of chains with the same beginning and end points, whose gaps and length are bounded by the mesh and the cardinality of $\mathcal{U}$, respectively.

Lemma 4.2.1. *Let $\delta > 0$ and let $\mathcal{U} = \{U_1, \dots, U_K\}$ be an open cover of (X^2, d_2) with* $\mathrm{mesh}\,\mathcal{U} = \max_{1 \le i \le K} \mathrm{diam}\, U_i \le \delta$. *Suppose that $\beta > 0$ is a Lebesgue number of $\mathcal{U}$. Then, for every pair $((x_i^{(0)})_{i=0}^k, (x_i^{(1)})_{i=0}^k)$ of β-chains of f, there is a pair $((y_i^{(0)})_{i=0}^l, (y_i^{(1)})_{i=0}^l)$ of δ-chains of f such that $(y_0^{(0)}, y_0^{(1)}) = (x_0^{(0)}, x_0^{(1)})$ and $(y_l^{(0)}, y_l^{(1)}) = (x_k^{(0)}, x_k^{(1)})$ with $1 \le l \le K$.*

Proof. Put $g = f \times f$ and $z_i = (x_i^{(0)}, x_i^{(1)}) \in X^2$ for each $0 \le i \le k$. Then, we have $d_2(g(z_i), z_{i+1}) \le \beta$ for every $0 \le i < k$. Since $d_2(g(z_0), z_1)) \le \beta$, there is $1 \le i_0 \le K$ such that $\{g(z_0), z_1\} \subset U_{i_0}$. Put $j_1 = \max\{1 \le j \le k : z_j \in U_{i_0}\}$. Since $\{g(z_0), z_{j_1}\} \subset U_{i_0}$ and $\operatorname{diam} U_{i_0} \le \delta$, we have $d_2(g(z_0), z_{j_1}) \le \delta$. If $j_1 < k$, then since $d_2(g(z_{j_1}), z_{j_1+1}) \le \beta$, there is $1 \le i_1 \le K$ such that $\{g(z_{j_1}), z_{j_1+1}\} \subset U_{i_1}$. Put $j_2 = \max\{j_1 + 1 \le j \le k : z_j \in U_{i_1}\}$. Since $\{g(z_{j_1}), z_{j_2}\} \subset U_{i_1}$ and $\operatorname{diam} U_{i_1} \le \delta$, we have $d_2(g(z_{j_1}), z_{j_2}) \le \delta$. Note that $z_{j_1+1} \in U_{i_1} \setminus U_{i_0}$, and so $U_{i_1} \ne U_{i_0}$. If $j_2 < k$, we repeat the process, and so on. Inductively, we obtain a sequence of integers $0 = j_0 < j_1 < j_2 < \cdots$. If $j_K < k$, then $U_{i_0}, U_{i_1}, \dots, U_{i_K}$ would be $K + 1$ distinct elements of $\mathcal{U}$, which is absurd. Therefore, we have $j_l = k$ for some $1 \le l \le K$ and $d_2(g(z_{j_\alpha}), z_{j_{\alpha+1}}) \le \delta$ for every $0 \le \alpha < l$. Put $z_{j_\alpha} = (x_{j_\alpha}^{(0)}, x_{j_\alpha}^{(1)}) = (y_\alpha^{(0)}, y_\alpha^{(1)})$ for each $0 \le \alpha \le l$. Then, we have $(y_0^{(0)}, y_0^{(1)}) = (x_0^{(0)}, x_0^{(1)})$, $(y_l^{(0)}, y_l^{(1)}) = (x_k^{(0)}, x_k^{(1)})$, and

$$\begin{aligned}\max\{d(f(y_\alpha^{(0)}), y_{\alpha+1}^{(0)}), d(f(y_\alpha^{(1)}), y_{\alpha+1}^{(1)})\} &= d_2\big((f(y_\alpha^{(0)}), f(y_\alpha^{(1)})), (y_{\alpha+1}^{(0)}, y_{\alpha+1}^{(1)})\big)\\ &= d_2(g(z_{j_\alpha}), z_{j_{\alpha+1}}) \le \delta\end{aligned}$$

for all $0 \le \alpha < l$. Hence, $((y_\alpha^{(0)})_{\alpha=0}^l, (y_\alpha^{(1)})_{\alpha=0}^l)$ is a pair of δ-chains of f satisfying the required property. □

By the virtue of Lemma 4.2.1, we can reduce the period of a separated pair of cycles at a point in some case.

Lemma 4.2.2. *Under the same hypothesis as in Lemma 4.2.1, for every e-separated pair $((z_i^{(0)})_{i=0}^m, (z_i^{(1)})_{i=0}^m)$ of β-cycles at $x \in X$, there is an e-separated pair $((w_i^{(0)})_{i=0}^n, (w_i^{(1)})_{i=0}^n)$ of δ-cycles at x with period $n \le 2K$.*

Proof. Fix $0 < j < m$ with $d(z_j^{(0)}, z_j^{(1)}) > e$. We split the pair $((z_i^{(0)})_{i=0}^m, (z_i^{(1)})_{i=0}^m)$ into two parts corresponding to $0 \le i \le j$ and $j \le i \le m$ respectively, and apply the shortcut lemma (Lemma 4.2.1) to each part. Then, by joining them, we obtain a separated pair with a shortened period. Precisely, by Lemma 4.2.1, there exist two pairs $((x_i^{(0)})_{i=0}^k, (x_i^{(1)})_{i=0}^k)$ and $((y_i^{(0)})_{i=0}^l, (y_i^{(1)})_{i=0}^l)$ of δ-chains of f such that

- $(x_0^{(0)}, x_0^{(1)}) = (z_0^{(0)}, z_0^{(1)}) = (x, x)$ and $(x_k^{(0)}, x_k^{(1)}) = (z_j^{(0)}, z_j^{(1)})$;
- $(y_0^{(0)}, y_0^{(1)}) = (z_j^{(0)}, z_j^{(1)})$ and $(y_l^{(0)}, y_l^{(1)}) = (z_m^{(0)}, z_m^{(1)}) = (x, x)$; and
- $\max\{k, l\} \le K$.

Then, the pair of the following δ-cycles:

$$(x_0^{(0)}, x_1^{(0)}, \dots, x_{k-1}^{(0)}, y_0^{(0)}, y_1^{(0)}, \dots, y_{l-1}^{(0)}, y_l^{(0)}),$$
$$(x_0^{(1)}, x_1^{(1)}, \dots, x_{k-1}^{(1)}, y_0^{(1)}, y_1^{(1)}, \dots, y_{l-1}^{(1)}, y_l^{(1)})$$

is an e-separated pair of δ-cycles at x with period $n = k + l \le 2K$. □

Finally, using Lemma 4.2.2, we prove Theorem 1.4.2.

Proof of Theorem 1.4.2. Take an open cover $\mathcal{U}$ of (X^2, d_2) such that $\operatorname{mesh}\mathcal{U} \le \delta$ and $card\,\mathcal{U} = N_2(\delta)$. Let $\beta > 0$ be a Lebesgue number of $\mathcal{U}$. Then, by assumption (3) and Lemma 4.1.1, there is $z \in \omega(x, f)$ such that X contains an e-separated pair of β-cycles at z. Applying Lemma 4.2.2 to this pair, we obtain an e-separated pair of δ-cycles at

z whose period n is $\le 2N_2(\delta)$. Then, by assumptions (1) and (2), we can use Lemma 4.1.2 to conclude that

$$h_{top}(f) \ge \frac{1}{n}\log 2 \ge \frac{1}{2N_2(\delta)}\log 2.$$

□

4.3. **Proof of Proposition 1.4.1.**

In this subsection, we prove Proposition 1.4.1. We first prove the following lemma.

Lemma 4.3.1. *Let $f : X \to X$ be a continuous map. If $p \in \operatorname{Int} Sh^+(f)$ is a chain recurrent point of f, then for every $\epsilon > 0$, there exist a closed f-invariant subset $Y \subset X$ with $Y \cap B_\epsilon(p) \neq \emptyset$ and a factor map $\pi : (X_m, g) \to (Y, f)$, where $B_\epsilon(p) = \{x \in X : d(p, x) \le \epsilon\}$ and (X_m, g) is an odometer.*

Given $p \in \operatorname{Int} Sh^+(f) \cap CR(f)$ and $\epsilon > 0$, take $\epsilon_k > 0$, $k \in \mathbb{N}$, with $\sum_{k\in\mathbb{N}} \epsilon_k \le \epsilon$. For any subset $S \subset X$ and any $\delta > 0$, let $B_\delta(S) = \{x \in X : d(x, S) \le \delta\}$. We may suppose that $B_\epsilon(p) \subset Sh^+(f)$.

The next two lemmas are needed to prove Lemma 4.3.1. The first lemma is similar to [30, Lemma 3.1], but we extend it to a sequence of finite collections of subsets of X.

Lemma 4.3.2. *There exist a strictly increasing sequence $(m_k)_{k\in\mathbb{N}}$ of positive integers and a sequence $(\{A_j^{(k)} : 0 \le j < m_k\})_{k\in\mathbb{N}}$ of finite collections of compact subsets of X such that the following properties are satisfied for each $k \in \mathbb{N}$.*

1. $A_0^{(k)} \subset B_{\sum_{i=1}^k \epsilon_i}(p)$.
2. *m_k divides m_{k+1}.*
3. $f^{m_k}(A_0^{(k)}) = A_0^{(k)}$ *and* $A_0^{(k)}$ *is minimal for* f^{m_k}.
4. $f^j(A_0^{(k)}) = A_j^{(k)}$ *for all* $0 \le j < m_k$.
5. $\operatorname{diam} A_j^{(k)} \le 2\epsilon_k$ *for all* $0 \le j < m_k$.
6. *For any $0 \le j < m_{k+1}$, if $j = qm_k + r$ with $0 \le r < m_k$, then $A_j^{(k+1)} \subset B_{\epsilon_{k+1}}(A_r^{(k)})$.*

Proof. Let us prove the claim by induction on k. When $k = 1$, since $p \in Sh^+(f)$, there is $\delta_1 > 0$ such that every δ_1-pseudo orbit $(z_i)_{i=0}^\infty$ with $z_0 = p$ is ϵ_1-shadowed by some point of X. Then, since $p \in CR(f)$, there is a δ_1-cycle $(x_i^{(1)})_{i=0}^{m_1}$ with $x_0^{(1)} = x_{m_1}^{(1)} = p$. Consider the following m_1-periodic δ_1-pseudo orbit

$$(x_0^{(1)}, x_1^{(1)}, \dots, x_{m_1-1}^{(1)}, x_0^{(1)}, x_1^{(1)}, \dots, x_{m_1-1}^{(1)}, \dots),$$

which is ϵ_1-shadowed by $y_1 \in X$. Then, for every $n \ge 0$, $f^{m_1 n}(y_1)$ is also an ϵ_1-shadowing point, and hence every $y \in \omega(y_1, f^{m_1})$ is an ϵ_1-shadowing point of the above pseudo orbit. Since $\omega(y_1, f^{m_1})$ is f^{m_1}-invariant, there is a minimal subset $Y_1 \subset \omega(y_1, f^{m_1})$ for f^{m_1}. Given $0 \le j < m_1$, we have $d(f^j(y), x_j^{(1)}) \le \epsilon_1$ for every $y \in Y_1$, and therefore $f^j(Y_1) \subset B_{\epsilon_1}(x_j^{(1)})$. For each $0 \le j < m_1$, put $A_j^{(1)} = f^j(Y_1)$. Then, $A_0^{(1)} = Y_1 \subset B_{\epsilon_1}(x_0^{(1)}) = B_{\epsilon_1}(p)$, and properties (3), (4), and (5) are satisfied for $k = 1$.

Now given $k \in \mathbb{N}$, assume that m_k and $\{A_j^{(k)} : 0 \le j < m_k\}$ satisfying (1), (3), (4), and (5) are chosen. Fix $x_0^{(k+1)} \in A_0^{(k)}$. Then, since $A_0^{(k)} \subset B_{\sum_{i=1}^{k} \epsilon_i}(p) \subset B_\epsilon(p) \subset Sh^+(f)$, there is $\delta_{k+1} > 0$ such that every δ_{k+1}-pseudo orbit $(z_i)_{i=0}^\infty$ with $z_0 = x_0^{(k+1)}$ is ϵ_{k+1}-shadowed by some point of X. Since $A_0^{(k)}$ is minimal for f^{m_k}, there is $a_k \ge 2$ such that $d(x_0^{(k+1)}, f^{a_k m_k}(x_0^{(k+1)})) \le \delta_{k+1}$. Put $m_{k+1} = a_k m_k$ and consider the following m_{k+1}-periodic δ_{k+1}-pseudo orbit

$$(x_0^{(k+1)}, f(x_0^{(k+1)}), \dots, f^{m_{k+1}-1}(x_0^{(k+1)}), x_0^{(k+1)}, f(x_0^{(k+1)}), \dots, f^{m_{k+1}-1}(x_0^{(k+1)}), \dots),$$

which is ϵ_{k+1}-shadowed by some $y_{k+1} \in X$. Similarly to the above, we take a minimal subset $Y_{k+1} \subset \omega(y_{k+1}, f^{m_{k+1}})$ for $f^{m_{k+1}}$. Note that every $y \in Y_{k+1}$ is an ϵ_{k+1}-shadowing point of the pseudo orbit above. Given $0 \le j < m_{k+1}$, we have $d(f^j(y), f^j(x_0^{(k+1)})) \le \epsilon_{k+1}$ for every $y \in Y_{k+1}$, and therefore $f^j(Y_{k+1}) \subset B_{\epsilon_{k+1}}(f^j(x_0^{(k+1)}))$. Put $A_j^{(k+1)} = f^j(Y_{k+1})$ for every $0 \le j < m_{k+1}$. Then,

$$A_0^{(k+1)} = Y_{k+1} \subset B_{\epsilon_{k+1}}(x_0^{(k+1)}) \subset B_{\epsilon_{k+1}}(A_0^{(k)}) \subset B_{\epsilon_{k+1}}(B_{\sum_{i=1}^{k} \epsilon_i}(p)) \subset B_{\sum_{i=1}^{k+1} \epsilon_i}(p),$$

(2) is satisfied for k, and (3), (4), and (5) are satisfied for $k+1$. Suppose that $0 \le j < m_{k+1}$ is written as $j = qm_k + r$ with $0 \le r < m_k$. Then, $f^j(x_0^{(k+1)}) \in f^j(A_0^{(k)}) = A_r^{(k)}$, which implies $A_j^{(k+1)} = f^j(Y_{k+1}) \subset B_{\epsilon_{k+1}}(A_r^{(k)})$. Hence, (6) is also satisfied for k, and thus the lemma has been proved. □

Recall that the definition of the odometer (X_m, g) with the periodic structure $m = (m_k)_{k \in \mathbb{N}}$ was given in subsection 1.4. For $m = (m_k)_{k \in \mathbb{N}}$ and $(\{A_j^{(k)} : 0 \le j < m_k\})_{k \in \mathbb{N}}$ constructed in Lemma 4.3.2, we have the following property.

Lemma 4.3.3. *Let $r = (r_l)_{l \in \mathbb{N}} \in X_m$ and $k \in \mathbb{N}$. Then, we have*

$$A_{r_{k+N}}^{(k+N)} \subset B_{\epsilon_{k+1} + \dots + \epsilon_{k+N}}(A_{r_k}^{(k)})$$

for every $N \in \mathbb{N}$.

Proof. We prove this lemma by induction on N. When $N = 1$, since $r_{k+1} \equiv r_k \pmod{m_k}$, by substituting r_{k+1} and r_k for j and r in Lemma 4.3.2 (6), we have $A_{r_{k+1}}^{(k+1)} \subset B_{\epsilon_{k+1}}(A_{r_k}^{(k)})$. Let us assume that the claim holds for some $N \in \mathbb{N}$ and prove it for $N+1$. Since $r_{k+N+1} \equiv r_{k+N} \pmod{m_{k+N}}$, we have $A_{r_{k+N+1}}^{(k+N+1)} \subset B_{\epsilon_{k+N+1}}(A_{r_{k+N}}^{(k+N)})$ by Lemma 4.3.2 (6). On the other hand, we have $A_{r_{k+N}}^{(k+N)} \subset B_{\epsilon_{k+1} + \dots + \epsilon_{k+N}}(A_{r_k}^{(k)})$ by the induction hypothesis. Hence,

$$A_{r_{k+N+1}}^{(k+N+1)} \subset B_{\epsilon_{k+N+1}}(B_{\epsilon_{k+1} + \dots + \epsilon_{k+N}}(A_{r_k}^{(k)})) \subset B_{\epsilon_{k+1} + \dots + \epsilon_{k+N} + \epsilon_{k+N+1}}(A_{r_k}^{(k)}),$$

which completes the induction. □

Now let us prove Lemma 4.3.1.

Proof of Lemma 4.3.1. Given $r = (r_l)_{l \in \mathbb{N}} \in X_m$, using Lemma 4.3.3, for every $k \in \mathbb{N}$ and every $N \in \mathbb{N}$, we have

$$A_{r_{k+N}}^{(k+N)} \subset B_{\epsilon_{k+1} + \dots + \epsilon_{k+N}}(A_{r_k}^{(k)}) \subset B_{\sum_{i=k+1}^{\infty} \epsilon_i}(A_{r_k}^{(k)}).$$

Since $\operatorname{diam} A^{(k)}_{r_k} \le 2\epsilon_k$ by Lemma 4.3.2 (5), we see that $d_H(A^{(k)}_{r_k}, A^{(k+N)}_{r_{k+N}}) \le \sum_{i=k}^{\infty} 2\epsilon_i \to 0$ as $k \to \infty$, where d_H denotes the Hausdorff distance. In other words, the sequence $(A^{(l)}_{r_l})_{l\in\mathbb{N}}$ is a Cauchy sequence with respect to d_H, and so $\lim_{l\to\infty} d_H(A^{(l)}_{r_l}, C) = 0$ for some closed subset $C \subset X$. Since $\operatorname{diam} A^{(l)}_{r_l} \le 2\epsilon_l \to 0$ as $l \to \infty$ by Lemma 4.3.2 (5) again, we have $C = \{x\}$ for some $x \in X$. Then, define a map $\pi : X_m \to X$ by putting $\pi(r) = x$, which implies

$$\lim_{l\to\infty} d_H(A^{(l)}_{r_l}, \{\pi(r)\}) = 0$$

for every $r = (r_l)_{l\in\mathbb{N}} \in X_m$. We need two claims concerning the map π.

Claim 1: $\pi : X_m \to X$ is continuous.

Given $r = (r_l)_{l\in\mathbb{N}}$ and $s = (s_l)_{l\in\mathbb{N}} \in X_m$, suppose $r_l = s_l$ for every $1 \le l \le K$. Then,

$$A^{(K+N)}_{r_{K+N}} \subset B_{\sum_{i=K+1}^{\infty}\epsilon_i}(A^{(K)}_{r_K})$$

for all $N \in \mathbb{N}$ as above. Taking the limit as $N \to \infty$, we obtain

$$\pi(r) \in B_{\sum_{i=K+1}^{\infty}\epsilon_i}(A^{(K)}_{r_K}).$$

Similarly,

$$\pi(s) \in B_{\sum_{i=K+1}^{\infty}\epsilon_i}(A^{(K)}_{s_K}) = B_{\sum_{i=K+1}^{\infty}\epsilon_i}(A^{(K)}_{r_K}).$$

By Lemma 4.3.2 (5), we have $\operatorname{diam} A^{(K)}_{r_K} \le 2\epsilon_K$, and therefore $d(\pi(r), \pi(s)) \le \sum_{i=K}^{\infty} 2\epsilon_i \to 0$ as $K \to \infty$. Thus, $\pi : X_m \to X$ is continuous.

Claim 2: $\pi \circ g = f \circ \pi$.

Given $r \in X_m$, put $g(r) = s$. Then, for each $l \in \mathbb{N}$, we have $s_l = r_l + 1 \pmod{m_l}$ by the definition of g, and hence $A^{(l)}_{s_l} = f(A^{(l)}_{r_l})$ by Lemma 4.3.2 (4). Taking the limit as $l \to \infty$, we obtain $\pi(s) = f(\pi(r))$, that is, $\pi(g(r)) = f(\pi(r))$. Since $r \in X_m$ is arbitrary, this claim has been proved.

Putting $\pi(X_m) = Y$, from Claims 1 and 2, we see that $Y \subset X$ is a closed f-invariant subset, and $\pi : (X_m, g) \to (Y, f)$ is a factor map. Hence, it only remains to prove that there exists $q \in Y$ such that $q \in B_\epsilon(p)$. Put $q = \pi(\mathbf{0}) \in Y$, where $\mathbf{0} = (0, 0, 0, \dots) \in X_m$. By Lemma 4.3.2 (1), we have

$$A^{(k)}_0 \subset B_{\sum_{i=1}^{k}\epsilon_i}(p) \subset B_{\sum_{i=1}^{\infty}\epsilon_i}(p) \subset B_\epsilon(p)$$

for every $k \in \mathbb{N}$. Taking the limit as $k \to \infty$, we obtain $q \in B_\epsilon(p)$, proving the theorem. □

Using Lemma 4.3.1, we prove Proposition 1.4.1.

Proof of Proposition 1.4.1. By Lemma 4.3.1, for any given $\epsilon > 0$, there are a closed f-invariant subset $Y \subset X$ with $Y \cap B_\epsilon(p) \ne \emptyset$ and a factor map $\pi : (X_m, g) \to (Y, f)$, where (X_m, g) is an odometer. Then, (Y, f) is minimal, and it holds that $Y \subset RR(f)$ because $X_m = RR(g)$. By [9, Corollary 2.5], we see that Y is a periodic orbit or (Y, f) is conjugate to an odometer. Thus, taking $q \in Y \cap B_\epsilon(p)$, we have $q \in Per(f)$ or $(\overline{O_f(q)}, f)$ is conjugate to an odometer. □

4.4. **Bowen type decomposition of chain recurrent subsets.**

In this subsection, we give Bowen type decomposition of chain recurrent subsets and present some consequences.

Let $g : S \to S$ be a chain recurrent continuous map on a compact metric space S. For $\delta > 0$, we define a relation $\sim_\delta$ on S as follows. For $x, y \in S$, $x \sim_\delta y$ iff there are a δ-chain $(x_i)_{i=0}^k$ of g with $x_0 = x$ and $x_k = y$, and a δ-chain $(y_i)_{i=0}^l$ of g with $y_0 = y$ and $y_l = x$. By the chain recurrence of g, we can show that $x \sim_\delta g(x)$ for every $x \in S$, and $x \sim_\delta y$ for all $x, y \in S$ with $d(x, y) < \delta$. Hence, every equivalence class C with respect to $\sim_\delta$ is clopen in S and g-invariant, i.e., $g(C) \subset C$. Then, each equivalence class is called a *δ-chain component* of S (with respect to g), and so S is decomposed into finitely many δ-chain components. Such a decomposition is called a *δ-chain decomposition* of S (with respect to g). Now, fix a δ-chain component C. Note that for any δ-cycle $c = (x_i)_{i=0}^n$ of g, if $x_i \in C$ for some $0 \le i \le n$, then $x_i \in C$ for all $0 \le i \le n$. In such a case, we write $c \subset C$. Set $l(c) = n$ for any δ-cycle $c = (x_i)_{i=0}^n$. Define

$$\mathcal{N} = \{n \in \mathbb{N} : \exists\ \delta\text{-cycle } c \text{ of } g \text{ with } c \subset C \text{ and } l(c) = n\},$$

and put

$$m = \gcd \mathcal{N} = \max\{j \in \mathbb{N} : j|n \text{ for every } n \in \mathcal{N}\}.$$

Then, we define a relation $\sim_{\delta,m}$ on C as follows. For any $x, y \in C$, $x \sim_{\delta,m} y$ iff there is a δ-chain $(x_i)_{i=0}^k$ of g with $x_0 = x$, $x_k = y$ and $m|k$. By the definition of m, we see that $\sim_{\delta,m}$ is an equivalence relation on C, and by the chain recurrence of g, for all $x, y \in C$ with $d(x, y) < \delta$, we have $x \sim_{\delta,m} y$. Hence, every equivalence class D with respect to $\sim_{\delta,m}$ is clopen in S. Take $p \in C$ and consider m points $p, g(p), \dots, g^{m-1}(p)$. Then, it is easy to see that $C = \bigsqcup_{i=0}^{m-1} [g^i(p)]$ is the partition of C into equivalence classes with respect to $\sim_{\delta,m}$, where $[g^i(p)]$ denotes the equivalence class containing $g^i(p)$. Put $D_i = [g^i(p)]$ for $0 \le i \le m-1$ and $D_m = D_0$. Then, we have

- (D1) $C = \bigsqcup_{i=0}^{m-1} D_i$ and every D_i, $0 \le i \le m-1$, is clopen in S;
- (D2) $g(D_i) \subset D_{i+1}$ for every $0 \le i \le m-1$ (**Lemma 4.4.1**);
- (D3) Given $x, y \in D_i$ with $0 \le i \le m-1$, there exists $M \in \mathbb{N}$ such that for any integer $N \ge M$, there is a δ-chain $c = (x_i)_{i=0}^k$ of g in C with $x_0 = x$, $x_k = y$, and $l(c) = k = mN$.

(D3) is proved in [13, Lemma 2.3]. The proof is based on the fact that for every positive integers $n_1, n_2, \dots, n_l \in \mathbb{N}$ with $\gcd\{n_1, n_2, \dots, n_l\} = m$, there exists $L \in \mathbb{N}$ such that for every integer $N \ge L$, we have $n_1a_1 + n_2a_2 + \dots + n_la_l = mN$ for some integers $a_1, a_2, \dots, a_l \ge 0$. We call each D_i, $0 \le i \le m-1$, a *δ-cyclic component* of C, and $C = \bigsqcup_{i=0}^{m-1} D_i$ is called a *δ-cyclic decomposition* of C.

Proof of Lemma 4.4.1. It is obvious from the definition that $x \sim_{\delta,m} g^m(x)$ for every $x \in C$, and hence $g^{3m+i}(p) \in D_i$ for every $0 \le i \le m-1$. Fix $0 \le i \le m-1$ and $x \in D_i$. Since both $g^{3m+i}(p)$ and x are in D_i, there are $N \in \mathbb{N}$ and a δ-chain $(x_i)_{i=0}^{mN}$ of g such that $x_0 = g^{3m+i}(p)$ and $x_{mN} = x$. Then, the following

$$(g^{i+1}(p), g^{i+2}(p), \dots, g^{i+3m}(p), x_1, \dots, x_{mN-1}, x, g(x))$$

is a δ-chain of g of length $m(N+3)$, which implies $g^{i+1}(p) \sim_{\delta,m} g(x)$. Thus, we have $g(x) \in D_{i+1}$, and since $x \in D_i$ is arbitrary, $g(D_i) \subset D_{i+1}$ has been proved. □

In what follows, for $x \in S$, we denote by $C(x,\delta,g)$ the δ-chain component containing x. For the given δ-cyclic decomposition $C(x,\delta,g) = \bigsqcup_{i=0}^{m-1} D_i$ with $x \in D_0$, we define

- $D(x,\delta,g) = D_0$;
- $r(x,\delta,g) = \max\{\operatorname{diam} D_i : 0 \le i \le m-1\}$; and
- $m(x,\delta,g) = m$.

Note that for any $0 < \delta_2 < \delta_1$, we have

- $C(x,\delta_2,g) \subset C(x,\delta_1,g)$;
- $D(x,\delta_2,g) \subset D(x,\delta_1,g)$;
- $r(x,\delta_2,g) \le r(x,\delta_1,g)$; and
- $m(x,\delta_1,g) | m(x,\delta_2,g)$.

Then, we present some consequences of the Bowen type decomposition. The following lemma characterizes the dynamics of a point $x \in S$ satisfying $\lim_{\delta\to 0} r(x,\delta,g) = 0$.

Lemma 4.4.2. *Let $g : S \to S$ be a chain recurrent continuous map. Suppose that $\lim_{\delta\to 0} r(x,\delta,g) = 0$. Then, we have $\overline{O_g(x)} \subset EC(g) \cap RR(g)$ and $\dim \overline{O_g(x)} = 0$. Moreover,*

(1) *If $\lim_{\delta\to 0} m(x,\delta,g) = \infty$, then $(\overline{O_g(x)}, g)$ is conjugate to an odometer.*
(2) *If $\lim_{\delta\to 0} m(x,\delta,g) < \infty$, then $x \in Per(g)$.*

Proof. It is obvious from the definition of the δ-cyclic decomposition that $\overline{O_g(x)} \subset EC(g) \cap RR(g)$ and $\dim \overline{O_g(x)} = 0$. Note that $\overline{O_g(x)}$ is minimal and $\overline{O_g(x)} \subset RR(f)$. If $\overline{O_g(x)}$ is a finite set, then $g^n(x) = x$ for some $n \in \mathbb{N}$, and hence $m(x,\delta,g) \le n$ for every $\delta > 0$. Therefore, if $\lim_{\delta\to 0} m(x,\delta,g) = \infty$, then $\overline{O_g(x)}$ is infinite, and thus by [9, Corollary 2.5], $(\overline{O_g(x)}, g)$ is conjugate to an odometer. If $\lim_{\delta\to 0} m(x,\delta,g) = n < \infty$, then it is easy to see that $g^n(x) = x$. □

The next lemma gives a quantitative relation between the Bowen type decomposition and the presence of separated pairs of cycles at a point, whose definition was given in Definition 4.1.1.

Lemma 4.4.3. *Let $g : S \to S$ be a chain recurrent continuous map. For every $x \in S$ and every $e, \delta > 0$, S contains an e-separated pair of δ-cycles of g at x iff $r(x,\delta,g) > e$.*

Proof. Put $m = m(x,\delta,g)$ and let $C(x,\delta,g) = \bigsqcup_{i=0}^{m-1} D_i$ be the δ-cyclic decomposition of $C(x,\delta,g)$ with $x \in D_0$.

Assume that $r(x,\delta,g) \le e$. Then, by the definition of $r(x,\delta,g)$, we have $\operatorname{diam} D_i \le e$ for every $0 \le i \le m-1$. Let $((z_j^{(0)})_{j=0}^n, (z_j^{(1)})_{j=0}^n)$ be a pair of δ-cycles of g such that $z_0^{(0)} = z_0^{(1)} = z_n^{(0)} = z_n^{(1)} = x$. Then, both $(z_j^{(0)})_{j=0}^n$ and $(z_j^{(1)})_{j=0}^n$ are contained in $C(x,\delta,g)$, and $m|n$. Moreover, for given $0 \le j \le n$ and $0 \le i \le m-1$, if $j \equiv i \pmod m$, then we have $\{z_j^{(0)}, z_j^{(1)}\} \subset D_i$. Hence, $d(z_j^{(0)}, z_j^{(1)}) \le e$ for all $0 \le j \le n$, and thus $((z_j^{(0)})_{j=0}^n, (z_j^{(1)})_{j=0}^n)$ is not e-separated.

Conversely, assume that $r(x,\delta,g)>e$ and take $0\le i\le m-1$ such that $\operatorname{diam} D_i>e$. Choose $y_0,y_1\in D_i$ with $d(y_0,y_1)>e$. Then, by (D3), there are $N_1\in\mathbb{N}$ and a pair of δ-chains $((x_j^{(0)})_{j=0}^{mN_1},(x_j^{(1)})_{j=0}^{mN_1})$ of g with $x_0^{(0)}=x_0^{(1)}=f^i(x)$ and $(x_{mN_1}^{(0)},x_{mN_1}^{(1)})=(y_0,y_1)$. Since $x\in D_0$ and $f^{m-i}(y_0),f^{m-i}(y_1)\in D_0$, using (D3) again, we have $N_2\in\mathbb{N}$ and a pair of δ-chains $((y_j^{(0)})_{j=0}^{mN_2},(y_j^{(1)})_{j=0}^{mN_2})$ of g with $(y_0^{(0)},y_0^{(1)})=(f^{m-i}(y_0),f^{m-i}(y_1))$ and $y_{mN_2}^{(0)}=y_{mN_2}^{(1)}=x$. Then, the pair of the following δ-cycles

$$(x,\dots,f^{i-1}(x),x_0^{(0)},x_1^{(0)},\dots,x_{mN_1-1}^{(0)},y_0,\dots,f^{m-i-1}(y_0),y_0^{(0)},y_1^{(0)},\dots,y_{mN_2-1}^{(0)},x),$$
$$(x,\dots,f^{i-1}(x),x_0^{(1)},x_1^{(1)},\dots,x_{mN_1-1}^{(1)},y_1,\dots,f^{m-i-1}(y_1),y_0^{(1)},y_1^{(1)},\dots,y_{mN_2-1}^{(1)},x)$$

is an e-separated pair of δ-cycles of g at x with period $m(N_1+N_2+1)$, proving the lemma. □

By Lemma 4.4.3 and Lemma 4.1.3, we obtain the following lemma.

Lemma 4.4.4. *Let $f:X\to X$ be a continuous map and let $S\subset X$ be a closed f-invariant subset such that $CR(f|_S)=S$. Given $x\in S$, suppose that the following conditions are satisfied.*

- *Every δ-pseudo orbit $(x_i)_{i=0}^\infty$ contained in S with $x_0=x$ is b-shadowed by some point of X.*
- *$r(x,\delta,f|_S)>2b$.*

Then, there exist $m\in\mathbb{N}$ and a closed f^m-invariant subset $Y\subset B_b(x)$ for which we have a factor map $\pi:(Y,f^m)\to(\{0,1\}^{\mathbb{N}},\sigma)$.

From Lemma 4.4.4, we obtain the following corollary, which is a quantitative localized version of [29, Corollary 6]. For $b>0$ and $S\subset X$, we say that a continuous map $f:X\to X$ has the *b-shadowing property around S* if there is $\delta>0$ such that every δ-pseudo orbit of f contained in S is b-shadowed by some point of X.

Corollary 4.4.1. *Let $f:X\to X$ be a continuous map with the b-shadowing property around a closed f-invariant subset S. If $CR(f|_S)=S$ and $h_{top}(f)=0$, then for every $x\in S$, there is a clopen subset D of S such that $x\in D$ and $\operatorname{diam} f^n(D)\le 2b$ for all $n\ge 0$.*

Proof. Choose $\delta>0$ such that every δ-pseudo orbit contained in S is b-shadowed by some point of X. Given $x\in S$, by Lemma 4.4.4, we have $r(x,\delta,f|_S)\le 2b$. Put $D=D(x,\delta,f|_S)$. Then, D is a clopen subset of S containing x, and we have $\operatorname{diam} f^n(D)\le 2b$ for all $n\ge 0$ by (D2) and the definition of $r(x,\delta,f|_S)$. □

By Corollary 4.4.1, we can recover [29, Corollary 6] as the continuous limit when $b\to 0$.

Corollary 4.4.2 ([29, Corollary 6])**.** *Let $f:X\to X$ be a continuous map with the shadowing property. If $h_{top}(f)=0$, then $\dim\Omega(f)=0$, and $f|_{\Omega(f)}$ is equicontinuous.*

Proof. The shadowing property of f implies that $\Omega(f)=\Omega(f|_{\Omega(f)})\subset CR(f|_{\Omega(f)})\subset\Omega(f)$, so $CR(f|_{\Omega(f)})=\Omega(f)$. Note that for every $b>0$, f has the b-shadowing property around $\Omega(f)$, and hence Corollary 4.4.1 applies to $S=\Omega(f)$. By taking the limit as $b\to 0$, we obtain $\dim\Omega(f)=0$, and $f|_{\Omega(f)}$ is equicontinuous. □

The next lemma gives a quantitative relation between the Bowen type decomposition and the distribution of sensitive points under quantitative pointwise shadowability.

Lemma 4.4.5. *Let $f : X \to X$ be a continuous map and let $S \subset X$ be a closed f-invariant subset such that $CR(f|_S) = S$. Given $x \in S$, suppose that every δ-pseudo orbit $(x_i)_{i=0}^{\infty}$ contained in S with $x_0 = x$ is b-shadowed by some point of X. Then, we have the following properties.*

(1) *If $r(x, \delta, f|_S) - 2b > e > 0$, then there exists $y \in Sen_e(f)$ such that $d(x, y) \leq b$.*
(2) *If $r(x, \delta, f|_S) \leq e$, then $x \notin Sen_e(f|_S)$.*

Proof. We first prove (1). Since $r(x, \delta, f|_S) > e + 2b$, by Lemma 4.4.3, S contains an $(e+2b)$-separated pair of δ-cycles at x. By Lemma 4.1.3, there exist $m \in \mathbb{N}$, a closed f^m-invariant subset $Y \subset B_b(x)$, and a factor map $\pi : (Y, f^m) \to (\{0, 1\}^{\mathbb{N}}, \sigma)$. Moreover, the construction of the factor map π in the proof of Lemma 4.1.3 implies that if $\pi(a) \neq \pi(b)$ for $a, b \in Y$, then $d(f^n(a), f^n(b)) > e$ for some $n \geq 0$. Now, since Y is compact and π is surjective, there exists $y \in Y$ such that for every neighborhood U of y in Y, there is $z \in U$ with $\pi(z) \neq \pi(y)$. Then, we have $y \in Sen_e(f)$ and $d(x, y) \leq b$, proving (1). Suppose that $r(x, \delta, f|_S) \leq e$. Then, putting $D = D(x, \delta, f|_S)$, we have $\operatorname{diam} f^n(D) \leq e$ for every $n \geq 0$ by (D2) and the definition of $r(x, \delta, f|_S)$. Hence, we have $x \notin Sen_e(f|_S)$, proving (2). □

4.5. **Proof of Theorem 1.4.3.**

In this subsection, we prove Theorem 1.4.3. Let $f : X \to X$ be a continuous map and let $S \subset X$ be a closed f-invariant subset such that $CR(f|_S) = S$ and $\operatorname{Int} Sh^+(f) \subset S$. Then, S admits Bowen type decomposition with respect to $f|_S$.

The following lemma claims that for any $x \in \operatorname{Int} Sh^+(f)$, we have a dichotomy, $\lim_{\delta\to0} r(x, \delta, f|_S) > 0$ with $x \in Sen(f)$ or $\lim_{\delta\to0} r(x, \delta, f|_S) = 0$ with $x \in EC(f)$.

Lemma 4.5.1. *For any $x \in \operatorname{Int} Sh^+(f)$, we have the following properties.*

(1) *If $\lim_{\delta\to0} r(x, \delta, f|_S) > 0$, then $x \in Sen(f)$.*
(2) *If $\lim_{\delta\to0} r(x, \delta, f|_S) = 0$, then $x \in EC(f) \cap RR(f)$.*

Proof. Let us suppose that $\lim_{\delta\to0} r(x, \delta, f|_S) > e > 0$ and prove that $x \in Sen_e(f)$. Take $\epsilon > 0$ and $\delta_0 > 0$ such that $\lim_{\delta\to0} r(x, \delta, f|_S) > e + 2\epsilon > e$ and every δ_0-pseudo orbit $(x_i)_{i=0}^{\infty}$ of f with $x_0 = x$ is ϵ-shadowed by some point of X. Then, since $r(x, \delta_0, f|_S) - 2\epsilon > e > 0$, by Lemma 4.4.5, there exists $y \in Sen_e(f)$ such that $d(x, y) \leq \epsilon$. Since $\epsilon > 0$ can be taken arbitrarily small, we obtain $x \in \overline{Sen_e(f)} = Sen_e(f)$. As for (2), if $\lim_{\delta\to0} r(x, \delta, f|_S) = 0$, then from Lemma 4.4.2, it follows that $x \in EC(f|_S) \cap RR(f|_S)$. Since $x \in \operatorname{Int} Sh^+(f) \subset \operatorname{Int} S$, we have $x \in EC(f)$, and obviously $x \in RR(f)$. □

By Lemmas 4.4.2 and 4.5.1, we obtain the following corollary.

Corollary 4.5.1. *For every $x \in \operatorname{Int} Sh^+(f)$, if $x \in EC(f)$, then $x \in Per(f)$ or $(\overline{O_f(x)}, f)$ is conjugate to an odometer.*

Now let us prove Theorem 1.4.3.

Proof of Theorem 1.4.3. $(S1) \Rightarrow (S2)$: Let $x \in \overline{Sen(f)}$. By Proposition 1.4.1, for every $\epsilon > 0$, there exists $y \in X$ with $d(x, y) \leq \epsilon$ such that $y \in Per(f)$ or $(\overline{O_f(y)}, f)$ is conjugate

to an odometer. On the other hand, given $\epsilon > 0$, take $y \in Sen(f) \cap \operatorname{Int} Sh^+(f)$ with $d(x,y) \le \epsilon/2$. Note that $y \in \operatorname{Int} Sh^+(f) \subset \operatorname{Int} S$, and so $y \in Sen(f|_S) \cap Sh^+(f)$. By Lemma 4.1.4, there are $m \in \mathbb{N}$ and a closed f^m-invariant subset $Y \subset B_{\epsilon/2}(y)$ for which we have a factor map $\pi : (Y, f^m) \to (\{0,1\}^{\mathbb{N}}, \sigma)$. Since $B_{\epsilon/2}(y) \subset B_\epsilon(x)$, $(S1) \Rightarrow (S2)$ has been proved.
$(S2) \Rightarrow (S3)$: This follows from Lemma 4.1.5 (2).
$(S2) \Rightarrow (S4)$: Let U be a neighborhood of x in X. Then, there are $m \in \mathbb{N}$, a closed f^m-invariant subset $Y \subset U$, and a factor map $\pi : (Y, f^m) \to (\{0,1\}^{\mathbb{N}}, \sigma)$. We have

$$h(f, \overline{U}) \ge \frac{1}{m} h(f^m, \overline{U}) \ge \frac{1}{m} h(f^m, Y) \ge \frac{1}{m} \log 2 > 0.$$

Since U is arbitrary, we have $x \in Ent(f)$.
$(S2) \Rightarrow (S5)$: Take $s \notin RR(\sigma)$ and $y \in \pi^{-1}(s)$. Then, we have $y \notin RR(f^m) = RR(f)$. Hence, there exists $y \in B_\epsilon(x)$ with $y \notin RR(f)$ for every $\epsilon > 0$, which implies (S5).
$(S3) \Rightarrow (S1)$: This is obvious.
$(S4) \Rightarrow (S1)$: Suppose that $x \notin \overline{Sen(f)}$. Then, $x \in \operatorname{Int} EC(f)$. Take a neighborhood U of x such that $\overline{U} \subset EC(f)$. Then, we have $h(f, \overline{U}) = 0$, and hence $x \notin Ent(f)$.
$(S5) \Rightarrow (S1)$: Suppose that $x \notin \overline{Sen(f)}$. Then, $x \in \operatorname{Int} EC(f)$. Take a neighborhood U of x such that $U \subset EC(f) \cap \operatorname{Int} Sh^+(f)$. Then, using Lemma 4.5.1, we have $\lim_{\delta \to 0} r(y, \delta, f|_S) = 0$ for every $y \in U$, and hence $y \in RR(f)$ by Lemma 4.4.2. Thus, we have $x \in \operatorname{Int} RR(f)$.
$(E1) \iff (E3) \iff (E4)$ has been already proved.
$(E1) \Rightarrow (E2)$: Take a neighborhood U of x such that $U \subset EC(f) \cap \operatorname{Int} Sh^+(f)$. Then, using Lemma 4.5.1, we have $\lim_{\delta \to 0} r(y, \delta, f|_S) = 0$ for every $y \in U$, and hence by Lemma 4.4.2, $y \in Per(f)$ or $(\overline{O_f(y)}, f)$ is conjugate to an odometer.
$(E2) \Rightarrow (E4)$: This is obvious.

The last claim has been already proved as Corollary 4.5.1. □

Finally, we give an example in which Theorem 1.4.3 holds.

Example 4.5.1. Let $C \subset [0,1]$ be the Cantor ternary set. Take a homeomorphism $g : C \to C$ which is conjugate to the full shift $\sigma : \{0,1\}^{\mathbb{Z}} \to \{0,1\}^{\mathbb{Z}}$. Then, g has the shadowing property. Set $x_n = 1/n$, $C_n = \{y/n : y \in C\}$ for each $n \in \mathbb{N}$, and let

$$X = \{(0,0)\} \cup \bigcup_{n \in \mathbb{N}} \{x_n\} \times C_n \subset \mathbb{R}^2.$$

Then, X is a compact subset of $\mathbb{R}^2$. Define a homeomorphism $f : X \to X$ by $f((0,0)) = (0,0)$, and $f(x_n, y/n) = (x_n, g(y)/n)$ for $y \in C_n$, $n \in \mathbb{N}$. Then, it is easy to see that $Sh^+(f) = X$, and so f has the shadowing property. It is also obvious that f is non-wandering. Note that $(0,0) \in EC(f)$, but $(0,0) \notin \operatorname{Int} EC(f)$. Then, we see that (S1)-(S5) are satisfied for $x = (0,0)$.

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