博士論文

Entanglement Entropy in Algebraic Quantum Field Theory
(代数的場の量子論におけるエンタングルメント・エントロピー)

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Entanglement entropy in algebraic quantum field theory

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Abstract

We consider the problem of defining the entanglement entropy for chiral nets in the framework of algebraic quantum field theory. Considering a Möbius covariant local net with the split property, we give a sensible definition for the entropy H_I^E of a state restricted to a local algebra of an open connected non-dense interval $I \subset S^1$, with a given "conformal energy cutoff E". Considering the vacuum state restricted to any such interval, we prove that the latter is finite, and we give some upper bound estimates in terms of the dimensions of eigenspaces of the conformal Hamiltonian. This thesis is based on a joint work with Yoh Tanimoto.

1 Introduction

Entropy and entanglement entropy

Entropy is a concept as old as thermodynamics, a quantity that assigns to a macrostate our "ignorance" towards the randomness of its microscopic constituents' behavior, reflected in a certain coarse graining of observables. Entropy is also defined in quantum mechanics (see *e.g.* [OP04]). Recall that Hilbert spaces play a important part in those theories: given $\mathcal{H}$ the Hilbert space of the theory, observables correspond to self-adjoint operators in $\mathcal{B}(\mathcal{H})$, (normal) states are defined by density matrices ρ in $\mathcal{H}$ acting as positive linear functionals on $\mathcal{B}(\mathcal{H})$, and their von Neumann entropy $S(\rho)$ is defined by $S(\rho) := -k_B \operatorname{Tr}(\rho \log \rho)$. The concept agrees with our classical intuition, since it is zero for pure states (the least "random" a state can be), whereas, the "more mixed" the state is, the higher is its entropy. Entropy here is then a quantity associated to a state, but contrary to usual quantum mechanical observables, it is not associated to a quantum observable, in the sense that it is not represented by a self-adjoint linear operator. In the classical limit, in which the relative phases of the decomposition in pure states are purely random, the von Neumann formula recovers the expression equivalent to the familiar classical definition of entropy.

In quantum mechanics, however, something novel happens with respect to restrictions of a system in restricted subsystems. As a consequence of the superposition principle, the states of a composed system might present "non-classical correlations" with respect to measurements localized in those subsystems, giving rise to randomness even without no "lack of knowledge" when preparing the global state. This phenomenon is called *entanglement*, and states that present it are called *entangled*. Entanglement has then been pinpointed as a discerning characteristic of the quantum nature of a system, and has been investigated profoundly as a means of probing the very foundations of quantum mechanics (as in the EPR paradox and Bell's inequalities) as well as a resource for quantum information theory.

For illustrative purposes, consider a system with Hilbert space $\mathcal{H}$ composed by a subsystem A with Hilbert space $\mathcal{H}_A$ and its complement A' with Hilbert space $H_{A'}$, so that $\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_{A'}$. Localizations of states into the subsystem A are then restrictions of those to the subalgebra $\mathcal{B}(\mathcal{H}_A) \cong \mathcal{B}(\mathcal{H}_A) \otimes 1_{\mathcal{H}_{A'}}$. Algebraically, this corresponds to taking the global density matrix and "tracing out" the inaccessible degrees of freedom of A' with the partial trace $\operatorname{Tr}_{\mathcal{H}_{A'}}$, a process analogous to the coarse graining process. The von Neumann entropy of the resulting reduced density matrix is then called the *entanglement entropy* of the global state with respect to the subsystem A, *viz.*

$$S_A(\rho) := S_{\mathrm{vN}}(\rho_A) = S_{\mathrm{vN}}\big(\mathrm{Tr}_{\mathcal{H}_{A'}}(\rho)\big) \qquad (\rho \text{ density matrix on } \mathcal{H}_A \otimes \mathcal{H}_{A'}).$$

The following is a standard example. Consider a pure state ω_Ω induced by a unit vector $\Omega \in \mathcal{H}_A \otimes \mathcal{H}_{A'}$ with decomposition[1] $\Omega = \sum_k \lambda_k \Phi_k \otimes \Psi_k$, where $\{\Phi_k\}_k$ and $\{\Psi_k\}_k$ are orthonormal basis of $\mathcal{H}_A$ and $\mathcal{H}_{A'}$ respectively (and therefore $\sum |\lambda_k|^2 = 1$). The reduced state is described by the density matrix is $\rho_{A'} = \sum_k |\lambda_k|^2 |\Phi_k\rangle\langle\Phi_k|$. When the unit vector Ω is separable, that is, written solely as a unique product $\Psi \otimes \Phi$, the reduced state is also pure. But otherwise, the state is said to be entangled, and the resulting reduced state is mixed. Thus, the entanglement entropy is non vanishing. Indeed, for pure states in the global algebra, the entanglement entropy is often used as an operationally defined measure of entanglement [BBPS96] (although not the unique one, see *e.g.* [VP98, HHHH09]). It is indeed in the context of pure states, specifically for ground states, that many results involving entanglement entropy have been developed (although many other works have appeared dealing with finite temperature states and other scenarios, see [CC09, Section 7] and references within).

Area Law for ground states of quantum many-body systems

Entanglement entropy has been also employed on more complex systems, as in the case of quantum many-body systems. There, the entanglement entropy of the ground state probes the geometric nature of the microscopic interactions. Unlike in thermodynamics, where the entropy is an extensive quantity (scaling proportionally to the volume), it was found that, for many cases, the entanglement entropy of the ground state scales proportionally to the area of the boundary (maybe added by a small, often logarithmic correction). In such cases, the entanglement entropy is said to fulfill an *Area Law* (see *e.g.* [ECP10]). As an heuristic reasoning for such, we might say, considering the interactions to be of short-range, that the quantum correlations between a region and its exterior are established on the boundary between the two.

This remarkable fact has consequences that go as deep as the computational complexity of quantum many-body systems. In one dimensional systems, the scaling of entanglement entropy relates to how well the ground state can be approximated by a matrix-product state, which translates to a good performance of the density matrix renormalization group (DMFG) for determining the ground state. This is in contrast to bare bone methods, which deal with an exponential growth of complexity with respect to the size of the lattice [Sch05]. Also in one dimension, another potential use of the characterization of the scaling of entanglement entropy is to distinguish phases of quantum matter. It is also expected that entanglement entropy might play an important role towards the study of exotic phases in condensed matter, which are more closely related to entanglement that to symmetry breaking (see *e.g.* [ECP10] and references within).

Entanglement entropy for the ground state of a lattice QFT

Many results have also been devoted to the continuum limit of lattices, specially regarding QFT (quantum field theory). In fact, one of the first results on the subject was as early as in 1986, due to Bombelli, Koul, Lee and Sorkin [BKLS86]. Before the concept acquired the name of entanglement entropy, the authors used the term "geometric entropy" and derived its formula for the vacuum of a free scalar massive field theory in 1+3 dimensional Minkowski spacetime restricted to the outside of a ball of finite radius R. The methods used involved basically the computation of the reduced density matrix in terms of correlators. The lattice separation parameter a approaching zero in the continuum limit was then shown to produce an *ultraviolet divergence* in the entanglement entropy. Indeed, this should happen since even the vacuum has too many high energy fluctuations which account for an infinite number

[1] There is always a choice of basis such that this decomposition, called Schmidt decomposition, is possible.

of degrees of freedom. Then, taking the parameter a as an *ultraviolet cutoff* (approaching zero), they determined the entanglement entropy as proportional to the boundary area $S_R = (\text{const})(\text{Area})/(2\pi a^2)$, *i.e.* establishing the *area law* for the vacuum of the field theory. The quantity was suggested to be connected to the Bekenstein-Hawking entropy of a black hole (see *e.g.* [Wal94]), which is also abiding to an area law. In fact, the work was itself based on the premise of such link, where the ball behaves as an black hole, and the restriction of the state to the region outside the ball was due to the impossibility of retrieving information from observables inside it. Similar results were then obtained in [Sre93] for a massless free scalar field in the same geometrical setting, and in [VLRK03] for the fermionic case.

While the later results deal with a "real time" calculation, alternative methods were developed, involving imaginary "Euclidean" time methods. Here, the space is described by a discrete lattice of separation parameter $a > 0$, and the time is "Wick-rotated" to a imaginary continuous parameter τ. Then, one can define the localized density matrix ρ_A by path integrals on the Euclidean spacetime, tracing out the field components outside A. Analogously, for positive integers n, integration on certain n-sheeted Riemannian surfaces allows the calculation of $\operatorname{Tr} \rho_A^n$. This, in turn, allows the computation of n-entropies, defined by $S_n(\rho) = (1-n)^{-1} \log \operatorname{Tr} \rho_A^n$. Finally, given some conditions, one can analytically extend the latter to $n \searrow 1$ and obtain the entanglement entropy. This was used for calculating the vacuum entanglement entropy for lattices whose critical points corresponded to 1+1 dimensional conformal field theories in [HLW94], arriving at the famous formula

$$S_I(\omega) = (c/3) \log(l/a) + c'.$$

Here, I interval of length l, and again, an a is an ultraviolet cutoff parameter a. This ultraviolet divergence can be attributed to the infinite amount of localized excitations near the boundary of I, which in turns correlate the subsystem with its exterior. But most notable is the presence of the central charge c, an important parameter of the underlying theory. The result was later refined in [CC04], and appeared constantly in many other works (see *e.g.* [CC09, CH09] and references within).

In [CH09] it is argued that the above also satisfies the area law, given certain conditions, being multiplied by the number of disjoint segments composing the subset I. The log-divergence also makes frequent apparitions in formulas of entanglement entropy. For instance, for free fields in $1+d$ dimensions, we expect the area law to hold in the form of [CH09]:

$$S_V(\omega) = g_{d-1}[\partial V]\epsilon^{-(d-1)} + \ldots + g_1[\partial V]\epsilon^{-(1)} + g_0[\partial V]\log(\epsilon) + S_{\text{fin},V}(\omega)$$

where all g_k are "local and extensive functions of the boundary ∂V, homogeneous of degree k". Whereas g_k depend on factors such as the specific regularization prescriptions, the term g_0 accompanying the log is expected to be universal, independent of the particular discrete model used to obtain the QFT in the continuum limit (there are other universal terms in the finite part the entropy , see *e.g.* [CH04]).

Before further discussing entanglement entropy in QFT, we make two small remarks on related topics. First, we digress a little on the (supposed) connection between entanglement entropy and black hole entropy via the area law. This law also relates to the "holographic principle", a conjecture which states that the information contained in a volume of space is completely encoded in a theory that lives in the boundary of the region. One of the prominent representatives of such principles is the $\text{AdS}_{d+2}/\text{CFT}_{d+1}$ correspondence, in which a quantum field theory in an asymptotically anti-de Sitter spacetime (AdS) is expected to be dual to a quantum conformal field theory in the latter's conformal infinity (CFT). In [RT06b, RT06a, NRT09], the authors pinpoint the entanglement entropy as an important concept which, given its universal features, may help to elucidate further geometrical properties of holography. They then introduce the entanglement entropy (for the vacuum) in an holographic manner, tautologically defined to follow the area law by $S_A := \text{const.} \times \text{Area}(\gamma A)$, where γA is the d dimensional minimal surface living in AdS_{d+2} whose boundary coincides with the boundary of A. For $d = 1$, it was shown that the holographic entanglement entropy coincides with the entanglement entropy of CFT_1. Secondly, we comment on experimental measurements of entanglement entropy. Those are also under investigation. Take for instance the work [KL09], in which the entanglement entropy relates to measurable electric noise (see also [CC09, Section 8]). Also related is [IMP+15], where experimental measures of Rènyi entropies and mutual information using quantum many-body interference are realized.

Algebraic QFT

As mentioned, the entanglement entropy of the vacuum in critical QFT limits has shown to obey certain characteristic features, such as an ultraviolet divergence that, when "regularized", displays itself in interesting area laws, where certain "universal terms" appear, and might "distinguish different theories apart". Nevertheless, the apparition of terms that are not universal makes an interpretation of the results quite difficult. Therefore, one would also like to probe results directly on the continuum, without the need of any lattice regularization.

When dealing with formulations of QFTs[2], the algebraic approach seems like a good choice, since the tools regarding quantum entropy can be formulated very precisely in terms of operator algebras. In AQFT, (algebraic quantum field theory, also called "local quantum physics", see *e.g.* [Haa96, Ara99]), the main ingredient of the theory is a net of operator algebras $\{\mathfrak{A}(\mathcal{O})\}$ indexed by (bounded) regions of the spacetime, in which each element $\mathfrak{A}(\mathcal{O}) \subset \mathcal{B}(\mathcal{H})$ is an operator algebra corresponding to measurements that can be done in the corresponding region $\mathcal{O} \subset \mathbf{R}^{1+d}$. Those nets are usually called Haag-Kastler nets, and each one "constitutes the intrinsic mathematical description of the theory" [Haa96, p.105]. One major drawback of this framework is the lack of nontrivial examples: indeed, no interacting models have been constructed in 4 dimensional spacetimes. However, this general approach was not only interesting as a mathematical object, but also gave insight into many general features expected in quantum field theories, such as the theory of thermal states and superselection theory. It is in such setting that we base our work.

The presented notions of vacuum entanglement entropy are *not* straightforwardly adaptable to the context of algebraic QFT. For a given spatial region O which generates the subregion $\mathcal{O}$[3], one has to calculate the entropy of the state vacuum ω restricted to the local algebra $\mathfrak{A}(\mathcal{O})$. However, contrary to the lattice case in which a local algebra is a type I factor of form $\mathcal{B}(\mathcal{H}_{\mathcal{O}})$, the local algebras $\mathfrak{A}(\mathcal{O})$ of Haag-Kastler nets are commonly von Neumann algebras of type III_1 [BDF87, BV95]. Those algebras have no normal pure states, which reflects the fact that the vacuum, when considered locally, is a mixed state. There is also no canonical trace, which makes an extension of the von Neumann entropy formula non-trivial. In fact, using the concept of relative entropy, one can generalize the entropy to von Neumann algebras, resulting in a divergent quantity for algebras of type II and III [OP04].

Perhaps because of those difficulties, the literature regarding entropy in AQFT is unfortunately still very scarce. It isn't however, void. In [Nar94], the author introduces a slightly altered definition of localized entropy of the vacuum ω restricted to an local algebra $\mathfrak{A}(\mathcal{O})$, using an auxiliary quantity $\delta > 0$ as a regularizing spatial parameter (in the sense that, when $\delta = 0$, the author's definition would recover the standard divergent definition). Furthermore, it was shown that if the local net satisfies the nuclearity condition of Buchholz and Wichmann (in a stronger version), then one could take the region $\mathcal{O}$ growing to the whole Minkowski space (with δ growing also appropriately) in a way such that the local entropy would tend to zero, as expected from the vacuum state. The author's intentions were only focused on the determination of a global entropy density, but its relation to entanglement entropy is evident. It is unfortunate, however, that there are still no results concerning adaptations of those ideas, the reason being that the analytical methods used could not straightforwardly be adapted to a regularization scheme leaving finite results when δ goes to zero (See also [Nar02] for a different perspective, regarding *relative* entropy of entanglement in AQFT).

[2] Now, whereas Lagrangian QFT has been an epitome in modern science, it is plagued with problems of mathematical rigor. This argument is commonly circumvented by the view that QFT is merely an effective theory describing phenomena on the realm of an energy limit (see *e.g.* [Zee10, VIII.3]). From another perspective, during the late 1950s and 1960s much effort was done in establishing quantum field theory on more mathematically sound grounds.

[3] Typically, O will be a "spatial" ball in $t = 0$, and $\mathcal{O}$ will be the double cone with O as its base.

AQFT in 2 dimensions and chiral components

One could argue that it is not only the entanglement entropy that poses problems for AQFT, but the theory itself is too "heavy". In fact, even after more then 50 years since its formulation, no interacting models have been constructed in the 4 dimensional Minkowski spacetime. The situation is better, however, on the 2 dimensional scenario. As a start, a variety of models can be constructed [GJ81, Lec08, Tan14]. In addition, the 2 dimensional geometry presents cases where a theory is a product of two theories of different "chiralities". This arises from the fact that the lightlike cone of a point splits into two lines of different chiralities (one going to the right, another to the left). There is a resemblance to the D'Alambert solution of the Klein-Gordon wave equation in 1+1 dimensions, where any solution f of $(\partial_t^2 - \partial_x^2)\, f(t,x) = 0$ decomposes into the chiral terms f_R and f_L, such that $f(t,x) = f_R(x-t) + f_L(x+t)$. Here, with the change of coordinates $x_\pm = 2^{-1/2}(x \pm t)$, any double cone is $\mathcal{O} \subset \mathbb{R}^{1+1}$ is then a product $I_+ \times I_-$ of intervals in those lightlike lines. Then, a local net $\mathfrak{A}$ is chiral when it decomposes as $\mathfrak{A}(I_+ \times I_-) = \mathfrak{A}_+(I_+) \otimes \mathfrak{A}_-(I_-)$. This happens for nets of massless QFT in 2 dimensions, but no similar factorization happens on higher dimensions.

Those are good justifications for the study of algebraic quantum field theory in two dimensional spaces: The theory is simpler but richer at the same time. Moreover, besides providing a rich playground for physical theories, this framework has also deep mathematical connections to the modular theory of operator algebras, theory of subfactors, and tensor categories.

Outline of this work

Motivated by those factors, this work focuses on studying the entanglement entropy on AQFT by considering the simplest case: the vacuum state on a chiral net living in a one dimensional light ray spacetime. The simpler geometry (indeed, the simplest) and the presence of a "conformal Hamiltonian" L_0 (even though diffeomorphism covariance is not required) provides us with better analytical tools to tackle the problem. In fact, it should be said that our work is not so different from the lattice approach, and we explain what we mean by this. If the net satisfies a natural assumption called *split property* (or *funnel property*), even though an local algebra $\mathfrak{A}(I)$ is of type III_1, we can "approximate it from the outside" by factor of type I. More concretely, we take a spacing parameter $\delta > 0$ and a interval I_δ which is apart from I by an amount of δ. The split property then assures the existence of a factor $\mathfrak{R}_\delta$ of type I between $\mathfrak{A}(I)$ and $\mathfrak{A}(S^1 \setminus I_\delta)$. On those algebras $\mathfrak{R}_\delta$, the von Neumann entropy has then the possibility of not being infinite! We, however, have no means to prove its finiteness (at least up to our current research). What we do is then a tweak on the definition: in the algebra $\mathfrak{R}_\delta$, we consider or entropy $H_{I,\delta}$ to be the infimum (with some scaling parameter) of the von Neumann entropies of all states which approximate ω well enough, *except on the vicinity of the boundary* (corresponding to the two δ-sized intervals adjacent to I). Indeed, since the divergent contribution to the entropy is credited precisely to high energy fluctuations around this boundary, it is no surprise that our definition provides a finite result (technically, in this step we require from the net one further extra prerequisite).

One could call this result a regularized entropy with an UV cutoff parameter δ, but we argue that this geometrical cutoff might be better understood when replaced by one on the energy space. In fact, what we do next is to consider, while δ is still present, a new cutoff parameter E. This cutoff parameter is then used to regularize our considered states by cutting off the contributions of "conformal energy higher than E". Since the conformal Hamiltonian L_0 has a discrete spectrum, many of the calculations can be simplified, and we acquire an upper bound for the entropy that is independent of δ. This result presents, in our view, a better understanding of the phenomena behind the entanglement entropy in chiral nets, since the "geometrical ignorance" is then substituted by an energetic cutoff of fluctuations. Moreover, the technical prerequisites for the finiteness of the former are shown to be less restrictive than for its geometric counterpart: it is only required that the chiral net satisfies the *split property*. Our main result is then the finiteness of the regularized entropy

H_I^E.

This work is divided as follows. In Section 2, we present the mathematical tools to be used, including a brief review of the basics of von Neumann entropy and the basics of Möbius covariant nets. In particular, we define those nets with its basic assumptions (Definition 2.11) and list important extra assumptions (Definition 2.19) used later in this work. In Section 3, we make our definition of regularized entanglement entropy for a Möbius covariant local net satisfying the split property, and prove its finiteness (see Theorem 3.9) given that the net satisfies the extra assumption 2.18(e) of conformal nuclearity. We finish with our conclusions in Section 4.

2 Mathematical preliminaries

This section sets up all the mathematical background to be used later in this work. As a starting point, we only assume the reader has a basic knowledge on functional analysis and operator algebras. In 2.1, we review the essentials of von Neumann entropy, and show how it is problematic for algebras of type III, which are just the case of local algebras in AQFT. In 2.2, we present the basic geometrical structure of the Möbius group, and use it in 2.3 to define Möbius covariant local nets (Definition 2.11), our main objects of interest. We present the basic notions regarding nuclear maps in 2.5, to then use those in defining the notion of conformal nuclearity condition, as well as other related extra assumptions (Definition 2.18).

2.1 Von Neumann entropy

Here we make a brief review on von Neumann entropy to set the notation and basic properties used later in this work. We mainly follow [OP04] (including conventions and many symbols), which we recommend for a good exposition on the subject. Let $\mathcal{H}$ be separable Hilbert space, and $\mathcal{B}(\mathcal{H})$ the algebra of bounded linear operators on $\mathcal{H}$. Denote the set of states of a von Neumann subalgebra $\mathfrak{M} \subset \mathcal{B}(\mathcal{H})$ by $\mathfrak{S}(\mathfrak{M})$. This is a convex subset of the space $\mathfrak{M}_+^*$ of positive linear functionals. We shall often deal with convex decompositions of states, which we define in the following.

Definition 2.1 (Convex decomposition of a state). Let $\phi \in \mathfrak{S}(\mathfrak{M})$. We say that $\phi = \sum_{k\in\mathbb{N}} \lambda_k \phi_k$ is a *convex decomposition* of ϕ if $\{\phi_k \in \mathfrak{S}(\mathfrak{M})\}_{k\in\mathbb{N}}$ is a countable collection of states and $\{\lambda_k \in [0,1]\}_{k\in\mathbb{N}}$ is a summable sequence of positive indices, such that $\sum_{k\in\mathbb{N}} \lambda_k = 1$ and $\phi = \sum_{k\in\mathbb{N}} \lambda_k \phi_k$ in the weak* topology.

Remark. If the state ϕ is normal, a convex decomposition $\phi = \sum_k \lambda_k \phi_k$ is comprised only of normal states ϕ_k. We shall only be interested in this scenario.

Definition 2.2 (Function η). Define the η function as $\eta : \mathbb{R}_{\geq 0} \to \mathbb{R}$ as $\eta(x) := -x\log(x)$ for $x > 0$. The function is continuously extended to zero by $\eta(0) := 0$.

Proposition 2.3. For a parameter p with $0 < p < 1$, there is a constant $c_p > 0$ such that

$$\eta(x) \leq c_p x^p \qquad (x \geq 0).$$

Moreover, the optimal value is $c_p = \frac{1}{(1-p)e}$, where e is the Euler number.

Proof. Define the function $f : x \in \mathbb{R}_{\geq 0} \mapsto -x^{(1-p)}\log(x) \in \mathbb{R}$. By elementary calculus, the differentiable function f attains its maximum at $x_0 = e^{-1/\epsilon} \in (0,1)$, with value $c_p := f(x_0) = ((1-p)e)^{-1}$. Multiplying the inequality $f(x) \leq c_p$ by x^p concludes the proof. □

Recall that any normal positive functional ϕ on $\mathcal{B}(\mathcal{H})$ has an associated positive trace-class operator $\rho_\phi \in L^1(\mathcal{H})$, such that $\rho_\phi = \mathrm{Tr}(\rho_\phi\,\cdot)$, where Tr is the (non-normalized) trace functional. When ϕ is a normal state, we call ρ_ϕ its density matrix.

Definition 2.4 (Von Neumann entropy). Let ϕ be a normal state on $\mathcal{B}(\mathcal{H})$, and ρ_ϕ its associated density matrix. Then its von Neumann entropy is defined as

$$S_{\mathrm{vN}}(\phi) := \mathrm{Tr}(\eta(\rho_\phi)) = \mathrm{Tr}(\,-\rho_\phi \log(\rho_\phi)\,).$$

Proposition 2.5. The von Neumann entropy has the following properties:

1. *Positivity.* $S_{\mathrm{vN}}(\phi) \in [0, \dim \mathcal{H}]$, vanishing iff ϕ is a pure state.
2. *Invariance.* If σ is a $*$-automorphism of $\mathcal{B}(\mathcal{H})$, then $S_{\mathrm{vN}}(\phi \circ \sigma) = S_{\mathrm{vN}}(\phi)$.
3. *Weak* lower semicontinuity* of $\phi \in \mathfrak{S}(\mathcal{B}(\mathcal{H})) \mapsto S_{\mathrm{vN}}(\phi) \in [0, +\infty]$.
4. *Concavity.* If $\phi = \sum_k \lambda_k \phi_k$ is a convex decomposition of ϕ, then it holds that

$$\sum_{k\in\mathbb{N}} \lambda_k S_{\mathrm{vN}}(\phi_k) \leq S_{\mathrm{vN}}(\phi) \leq \sum_{k\in\mathbb{N}} \lambda_k S_{\mathrm{vN}}(\phi_k) + \sum_{k\in\mathbb{N}} \eta(\lambda_k).$$

Proof (Sketch). Positivity and invariance are straight forward from the definition. For a proof of concavity, see [OP04, Proposition 1.6 and 6.2]. Lower semicontinuity follows from Kosaki's formula for the relative entropy, see [OP04, Theorem 5.11 and Formula (6.9)]. □

We rephrase the concavity property in the following corollary, which will be used in Subsection 3.4.

Corollary 2.6. Let $\{\phi_k\}_{k\in\mathbb{N}}$ be a family of pure states on $\mathcal{B}(\mathcal{H})$, and $\{\lambda_k \geq 0\}_{k\in\mathbb{N}} \in l^1_+(\mathbb{N})$ be a summable sequence of positive parameters. Define the positive functional ϕ acting on $\mathcal{B}(\mathcal{H})$ by $\phi := \sum_{k\in\mathbb{N}} \lambda_k \phi_k$. Clearly, its norm is given by $\|\phi\| = \sum_{k\in\mathbb{N}} \lambda_k$. Then, the entropy of the state $\phi/\|\phi\|$ satisfies the following inequality

$$S_{\mathrm{vN}}\left(\frac{\phi}{\|\phi\|}\right) \leq \log(\|\phi\|) + \frac{1}{\|\phi\|} \sum_{k\in\mathbb{N}} \eta(\lambda_k).$$

The definition of the von Neumann entropy relies on the underlying Hilbert space, where normal positive functionals can be associated with density matrices. However, by the properties of Proposition 2.5, one can easily define the von Neumann entropy for states on an abstract type I factor by the following.

Definition 2.7 (Von Neumann entropy for algebras of type I). Let $\mathfrak{N}$ be a factor of type I. By definition, there is Hilbert space $\mathcal{K}$ such that a $*$-isomorphism $\sigma : \mathcal{B}(\mathcal{K}) \to \mathfrak{N}$ exists. Then, for any normal positive functional ϕ on $\mathfrak{N}$, its von Neumann entropy $S_{\mathfrak{N}}(\phi)$ is defined by

$$S_{\mathfrak{N}}(\phi) := S_{\mathrm{vN}}(\phi \circ \sigma).$$

If $\mathfrak{N} = \oplus_k \mathfrak{N}_k$ is a countable sum of factors of type I, then any normal state ϕ on $\mathfrak{N}$ is a sum $\phi = \oplus_k \lambda_k \phi_k$ such that each ϕ_k a normal state on $\mathfrak{N}_k$. The von Neumann entropy of ϕ is then defined as

$$S_{\mathfrak{N}}(\phi) := \sum_k \lambda_k S_{\mathfrak{N}_k}(\phi_k) + \sum_k \eta(\lambda_k).$$

Remark. Since the invariance property in Proposition 2.5 holds, the above definition is independent on the choice of σ, and thus is well-defined. This will be used in our definitions in Section 3.

For general von Neumann algebras (such as local algebras, which are in many cases factors of type III_1), there might not be corresponding density matrices nor traces, so the usual definition of von Neumann entropy does not make sense. An alternative definition by means of relative entropy is explained in Appendix A.1. The definitions in Section 3 will however exploit the split property and depend only on entropy of algebras of type I.

To close this section, we review some aspects of *entanglement entropy*.

We consider a quantum system with Hilbert space $\mathcal{H}$ and observable algebra $\mathfrak{A}$ over a finite lattice L: to each vertex $k \in L$, there is an associated Hilbert space $\mathcal{H}_k$, such that the Hilbert space of the system L is $\mathcal{H} = \otimes_{k\in L}\mathcal{H}_k$, and the algebra of observables is $\mathfrak{A} = \mathcal{B}(\mathcal{H})$.

Subsystems of L are described by subsets of vertices $A \subset L$, to which there is an associated Hilbert space $\mathcal{H}_A := \otimes_{k\in A}\mathcal{H}_k$, and an associated algebra of observables $\mathfrak{A}(A) = \mathcal{B}(\mathcal{H}_A)$. Also, for each subsystem $A \in L$, its complement subsystem is denoted by $A' = L \setminus A$. We thus have the decomposition $\mathcal{H} \cong \mathcal{H}_A \otimes \mathcal{H}_{A'}$, and $\mathfrak{A} \cong \mathfrak{A}(A) \otimes \mathfrak{A}(A')$.

For a subsystem $A \subset L$, the Hilbert space $\mathcal{H}_A$ has the canonical trace $\mathrm{Tr}_{\mathcal{H}_A} : L^1(\mathcal{H}_A) \to \mathbb{C}$. The partial trace $\mathrm{Tr}_A : L^1(\mathcal{H}) \to L^1(\mathcal{H}_{A'})$ "traces out" the degrees of freedom associated to A, and is defined by the linear extension of the operator

$$\mathrm{Tr}_A(a \otimes b) \mapsto \big(\mathrm{Tr}_{\mathcal{H}_A}(a)\big)\, b \in L^1(\mathcal{H}_{A'}) \qquad (a \in L^1(\mathcal{H}_A), b \in L^1(\mathcal{H}_{A'})).$$

The partial trace is the total trace when $A = L$, but in general it is not tracial (in the sense that $\mathrm{Tr}_A(xy) \neq \mathrm{Tr}_A(yx)$ in general, which is clear from the formula).

A normal state ϕ in L is described by a density matrix $\rho_\phi \in L^1(\mathcal{H})$. The restriction of ϕ to the subalgebra $\mathfrak{A}(A)$ is described by a density matrix $\rho_{\phi,A} \in L^1(\mathcal{H}_A)$. This reduced density matrix is precisely acquired by the action of the partial trace $\mathrm{Tr}_{A'}$ on the global density matrix ρ_ϕ, *viz.* $\rho_{\phi,A} = \mathrm{Tr}_{A'}(\rho_\phi)$.

Definition 2.8 (Entanglement entropy for lattice systems). Let $(\mathcal{H}, \mathfrak{A})$ be a quantum system over a finite lattice L, and ϕ a normal state in it described by the density matrix ρ. The entanglement entropy of ϕ with respect to the subsystem $A \subset L$ is defined as

$$S_A(\phi) := S_{\mathrm{vN}}\big(\,\phi|_{\mathfrak{A}(A)}\big) = \mathrm{Tr}_{\mathcal{H}_A}\big(\eta(\rho_A)\big),$$

where $\rho_A = \mathrm{Tr}_{A'}(\rho) \in L^1(\mathcal{H}_A)$ is the reduced density matrix of ρ to the subsystem A.

The following properties of entanglement entropy are well known.

Proposition 2.9. Let ϕ be a normal state on a quantum system $(\mathcal{H}, \mathfrak{A})$ over a finite lattice L. Then, the entanglement entropy satisfies the following properties:

1. *Symmetry.* If ϕ is a pure state on $\mathfrak{A}$, then $S_A(\phi) = S_{A'}(\phi)$.
2. *Strong subadditivity.* For A, B, C three subsystems that do not intersect, it holds that $S_{A\cup B\cup C}(\phi) + S_B(\phi) \leq S_{A\cup B}(\phi) + S_{B\cup C}(\phi)$.
3. *Triangular inequality and subadditivity.* For A, B two subsystems that do not intersect, it holds that $|S_A(\phi) - S_B(\phi)| \leq S_{A\cup B}(\phi) \leq S_A(\phi) + S_B(\phi)$.

Proof (Sketch). The symmetry property follows from the Schmidt decomposition of vectors in a tensor product of two Hilbert spaces [OP04, Lemma 6.4]. Strong subadditivity follows from properties of relative entropy, see [OP04, Proposition 1.9 and 6.3]. Subadditivity follows from strong subadditivity with $B = \emptyset$ (to which corresponds $S_\emptyset(\phi) = 0$). The triangular inequality can be derived by a purification process and a combination of the symmetry and strong subadditivity properties [OP04, Proposition 6.5]. □

The concept of entanglement entropy for lattices rely heavily on the assumption that states can be lifted to density matrices, which result as the action of partial traces on the global density matrix. This tool is not present in algebraic quantum field theory, and we shall need other methods for defining and regularizing the entanglement entropy.

2.2 The one-dimensional spacetime and the Möbius group

The main objects of this work are theories living in the one dimensional spacetime lightray $\mathbb{R}$ which are covariant for relativistic symmetries of the Poincaré group. Recall that the Poincaré group restricted to such spacetime consists of transformations $x \mapsto e^s x + a$, where s corresponds to a dilation (Lorentz boosts of rapidity s) and a corresponds to translations.

Those theories can often be seen as restrictions of ones defined in the one-point compactification $\overline{\mathbb{R}} \cong S^1$ of the lightray, where the symmetry group is the Möbius group Möb, an important extension of the Poincaré group.

In this subsection, we shall discuss the one dimensional spacetime $\overline{\mathbb{R}} = S^1$, the Möbius symmetry group Möb, and its representation theory.

The **one-dimensional spacetime** considered will be $\overline{\mathbb{R}} := \mathbb{R} \cup \{\infty\}$ (viewed as the one point compactification of the light-ray spacetime $\mathbb{R}$). The spacetime $\overline{\mathbb{R}}$ is identified with $S^1 = \{z \in \mathbb{C}\,,\, |z| = 1\}$ with the infinity point corresponding to $z = -1$. Such identification is given by the Cayley map as follows:

$$\mathbb{R} \cup \{\infty\} \ni x = -i\frac{z-1}{z+1} = \tan\left(\frac{\arg(z)}{2}\right) \leftrightarrow z = \frac{i-x}{i+x} = \exp\left(2i\tan^{-1}(x)\right) \in S^1 \qquad (1)$$

Intervals. We denote as $\mathcal{J}$ the **collection of intervals**, or the **collection of connected spacetime regions**, *i.e.* the collection of non-empty, non-dense, open intervals of S^1. For $I \in \mathcal{J}$, we denote as I' its causal complement *i.e.* the interior of $S^1 \setminus I$ (notice that $I' \in \mathcal{J}$). The **distance** between two intervals $I_1, I_2 \in \mathcal{J}$ is their angular distance, i.e., the infimum of a value $|theta|$ such that $e^{i\theta}I_1$ intersects I_2. It vanishes if the two intervals intersect, and is positive if their closures don't intersect. Also, for two intervals $I_1, I_2 \in \mathcal{J}$, we say that $I_1 \Subset I_2$ if the closure $\overline{I_1}$ is contained in I_2, that is, if I_1 and I_2' have a positive distance.

The **Möbius symmetry group**, denoted as Möb, acts geometrically on this spacetime. It is convenient to present it as two isomorphic projective matrix groups, one being $\mathrm{PSL}(2,\mathbb{R}) := \mathrm{SL}(2,\mathbb{R})/\{\pm 1\}$ acting on $\overline{\mathbb{R}}$, and the other being $\mathrm{PSU}(1,1) := \mathrm{SU}(1,1)/\{\pm 1\}$ acting on S^1. In the following, we indicate the mentioned actions.

$$\mathrm{PSL}(2,\mathbb{R}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix},\ \begin{matrix} a,b,c,d \in \mathbb{R}, \\ ad - bc = 1. \end{matrix} \right\}/\{\pm 1\}, \qquad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot x = \frac{ax+b}{cx+d} \quad (x \in \overline{\mathbb{R}}).$$

$$\mathrm{PSU}(1,1) = \left\{ \begin{pmatrix} \alpha & \beta \\ \bar\beta & \bar\alpha \end{pmatrix},\ \begin{matrix} \alpha,\beta \in \mathbb{C}, \\ |\alpha|^2 - |\beta|^2 = 1. \end{matrix} \right\}/\{\pm 1\}, \quad \begin{pmatrix} \alpha & \beta \\ \bar\beta & \bar\alpha \end{pmatrix} \cdot z = \frac{\alpha z+\beta}{\bar\beta z+\bar\alpha} \quad (z \in S^1).$$

An identification between the two representations can be acquired using the Cayley map and its inverse, resulting in one possible alternative as follows:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \text{with} \begin{cases} a = & \frac{\alpha+\bar\alpha-\beta-\bar\beta}{2} \\ b = & \frac{\alpha-\bar\alpha+\beta-\bar\beta}{2i} \\ c = & \frac{-\alpha+\bar\alpha+\beta-\bar\beta}{2i} \\ d = & \frac{\alpha+\bar\alpha+\beta+\bar\beta}{2} \end{cases} \leftrightarrow \begin{pmatrix} \alpha & \beta \\ \bar\beta & \bar\alpha \end{pmatrix} \text{with} \begin{cases} \alpha = & \frac{a+d+i(b-c)}{2} \\ \beta = & \frac{-a+d+i(b+c)}{2} \end{cases}$$

Next, we identify three important one parameter subgroups of Möb. Most importantly, the **rotations** are denoted by $\{\rho_\theta \in \text{Möb}\}_{\theta\in\mathbb{R}}$, and are such that $\rho_\theta \cdot z = e^{i\theta}z$ for $z \in S^1$. This is clearly periodic with period 2π. The **dilations** are denoted by $\{\delta_s \in \text{Möb}\}_{s\in\mathbb{R}}$, and are such that $\delta_s \cdot x = e^s x$ for $x \in \overline{\mathbb{R}}$. This s an important subgroup which leaves the positive semi-line in $\overline{\mathbb{R}}$ (respectively, the "positive semi-arc" $\{z \in S^1, \Im(z) > 0\}$). Finally, the **translations** are denoted by $\{\tau_a \in \text{Möb}\}_{a\in\mathbb{R}}$, and are such that $\tau_a \cdot x = x + a$ for $x \in \overline{\mathbb{R}}$. We give their matrix representatives as follows:

		$\mathrm{PSL}(2,\mathbb{R})$	$\mathrm{PSU}(1,1)$
rotations	ρ_θ	$\begin{pmatrix} \cos(\theta/2) & \sin(\theta/2) \\ -\sin(\theta/2) & \cos(\theta/2) \end{pmatrix}$	$\begin{pmatrix} e^{i\theta/2} & 0 \\ 0 & e^{-i\theta/2} \end{pmatrix}$
dilations	δ_s	$\begin{pmatrix} e^{s/2} & 0 \\ 0 & e^{-s/2} \end{pmatrix}$	$\begin{pmatrix} \cosh(s/2) & -\sinh(s/2) \\ -\sinh(s/2) & \cosh(s/2) \end{pmatrix}$
translations	τ_a	$\begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix}$	$\begin{pmatrix} 1+ia/2 & ia/2 \\ -ia/2 & 1-ia/2 \end{pmatrix}$

Structural remarks. The abovementioned three subgroups come into play on the Iwasawa decomposition KAN of Möb. It basically states that any element of Möb can be expressed as a product of elements of those subgroups, that is, if $g \in$ Möb then $g = kan$, where k, a, n are respectively a rotation, dilation, and a translation. Also with the aid of those subgroups, one can easily prove the following transitivity property of Möb: any three points (z_1, z_2, z_3) anticlockwise oriented in S^1 can be mapped by Möb to the triple $(1, i, -1)$. Indeed, the product of a rotation that sends z_3 to -1, followed by a translation to send the image of z_1 to 1 fixes the endpoints to the upper semicircle. Then, an appropriate dilation keeps those fixed, and can then appropriately allocate z_2 to the point i. We thus have that Möb acts transitively on $\mathcal{J}$.

Lie algebra. The Lie algebra of Möb is the Lie algebra $\mathfrak{sl}(2, \mathbb{R})$, consisting of traceless 2×2 real matrices. We consider the real generators R, D, and T of the one parameter subgroups of rotations, dilations and translations, respectively (*i.e.* $\rho_\theta = e^{\theta R}$, $\delta_s = e^{sD}$ and $\tau_a = e^{aT}$). Their matricial form is the following:

$$R = \frac{1}{2}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad D = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad T = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

$$[R, D] = R - T, \qquad [R, T] = D, \qquad [D, T] = T.$$

Positive energy representations. A strongly continuous projective unitary representation of Möb can be lifted to a "true" strongly continuous unitary representation of $\widetilde{\text{Möb}}$, which shall be one of the ingredients of a Möbius covariant net later. Consider one such strongly continuous unitary representation U of $\widetilde{\text{Möb}}$ on a Hilbert space $\mathcal{H}$.

Recall that it induces a Lie representation dU of the complexified Lie algebra $\mathfrak{sl}_{\mathbb{C}}(2, \mathbb{R})$, where for any $A \in \mathfrak{sl}(2, \mathbb{R})$ (an element of the *real* Lie algebra), the operator $-i\, dU(A)$ is a self adjoint operator (densely defined in $\mathcal{H}$), and generates the one-parameter subgroup $t \mapsto U(e^{tA})$. It is also customary to introduce the Gårding subspace $\mathcal{H}^\infty$ of vectors smooth for the action U. By the Dixmier-Malliavin theorem, all such vectors can be written as linear sums of mollified vectors of the form $\int_{\widetilde{\text{Möb}}} f(g)U(g)\Psi dg$, where f is a compactly supported smooth function, $\Psi \in \mathcal{H}$ and dg is the left invariant Haar measure. This subspace is then an invariant core for all lie algebra self-adjoint generators.

There are three operators which play an important role in representations, which are denoted L_0 and $L_\pm$. Although conventions for the definition may vary (but the commutation rules don't), we present them as follows:

$$L_0 := -i\, dU(R), \qquad \text{and} \qquad L_\pm := dU\big(\pm D + i(R - T)\big) = L_\mp^*.$$

From the adjointness property stated above, the linearity of dU implies that $dU(A + iB)^* = dU(-A + iB)$ on $\mathcal{H}^\infty$, for $A, B \in \mathfrak{sl}(2, \mathbb{R})$. This implies that $dU(L_\pm)^* = dU(L_\mp)$, a property which is also called the "unitarity" of the representation of $\{L_0, L_\pm\}$. It is also straightforward to check that the following commutation rules apply:

$$[L_0, L_\pm] = \mp L_\pm, \qquad [L_+, L_-] = 2L_0.$$

From those facts, we can draw a parallel with spectral rising/lowering operations, where L_0 is an energy observable, and L_- and L_+ are respectively the creation and annihilation operators. Indeed, we call the generator L_0 the **conformal Hamiltonian** and it will be used frequently. Since $e^{2\pi i L_0}$ is a multiple of the identity, L_0 has a discrete spectrum of eigenvalues: $\mathrm{sp}(L_0) \subset h + \mathbb{Z}$, for some $h \in \mathbb{R}$. Furthermore, for U to have fixed vectors (*e.g.* a vacuum state), then $h = 0 \in \mathrm{sp}(L_0)$. A physically natural assumption is to require that L_0 is a positive operator, and in this case, the representation is said to be a **positive energy representation**.

At first sight, the nomenclature of "positive energy" might seem mysterious, since the conformal Hamiltonian L_0 is the generator of the rotation subgroup, a group of transformations with not so much clear geometrical interpretation on the uncompactified light-ray $\mathbb{R}$. It is however well known that the positivity of L_0 is equivalent to the positivity of the self-adjoint generator of the translation subgroup τ. We summarize this fact in the following Proposition.

Proposition 2.10. Let U be a strongly continuous unitary representation of $\widetilde{\text{Möb}}$ in a Hilbert space $\mathcal{H}$. Then, the conformal Hamiltonian L_0 is positive iff the generator $P := -i\,dU(T)$ of the translations is also positive. Moreover, it suffices to show that any of the two operators is bounded bellow. Then, we recall, the representation is called a "positive energy representation".

Proof. Define the "π-rotated translation subgroup" $\tau_{\pi;\cdot}$ and its generator T_π as

$$\tau_{\pi;\cdot} : a \in \mathbb{R} \mapsto \tau_{\pi;a} := \rho_\pi \tau_a \rho_{-\pi} = \begin{pmatrix} 1 & 0 \\ a & 1 \end{pmatrix} \in PSL(2,\mathbb{R}), \quad T_\pi := \begin{pmatrix} 0 & 0 \\ -1 & 0 \end{pmatrix} \in \mathfrak{sl}(2,\mathbb{R}).$$

Accordingly, define the self-adjoint generators $P := -i\,dU(T)$ and $P_\pi = -i\,dU(T_\pi)$. Then,

$$P_\pi = U(\rho_\pi)PU(\rho_\pi)^*$$

and by the Lie algebra relations,

$$L_0 = (P + P_\pi)/2.$$

Assuming that P is positive, then so is P_π, and hence so is L_0.

For the converse implication, we first notice that the dilation group has the following geometrical action on P and P_π:

$$U(\delta_s)PU(\delta_s)^* = e^s P \quad \text{and} \quad U(\delta_s)P_\pi U(\delta_s)^* = e^{-s}P_\pi \qquad (s \in \mathbb{R}).$$

And hence $U(\delta_s)L_0U(\delta_s)^* = (e^sP + e^{-s}P_\pi)/2$. Therefore, for any vector $\Psi \in \mathcal{H}^\infty$, one has the inequality

$$\langle \Psi, P\Psi \rangle = \lim_{s\to+\infty} e^{-s}2\big\langle \Psi, U(\delta_s)L_0U(\delta_s)^*\Psi \big\rangle.$$

And hence, positivity of L_0 implies the positivity of the above for all vectors in the core $\mathcal{H}^\infty$, which means P is positive.

The equation above also shows that if L_0 is bounded below, then P is bounded below too. Now, $\mathrm{sp}(P) = \mathrm{sp}[U(\delta_s)PU(\delta_s)^*] = e^s\mathrm{sp}(P)$ for all $s \in \mathbb{R}$, *i.e.* the spectrum of P invariant for dilations, and therefore, P is only bounded below if it is positive, which is equivalent to the positivity of S as seen before. This concludes the proof. □

2.3 Möbius covariant local nets

Definition

As mentioned in the Introduction, the present work focuses on chiral nets, theories living in the one dimensional lightray and that composes 2 dimensional theories by tensor products. Following the philosophy of local quantum physics [Haa96], those should be described by local nets on the lightray $\mathbb{R}$, covariant for the Poincaré group. As seen before in Subsection 2.2, the Poincaré group acting on $\mathbb{R}$ is a "restriction" of the larger group Möb acting on $\overline{\mathbb{R}} \cong S^1$. In fact, a local net on $\mathbb{R}$ that is covariant for the Poincaré group is a restriction of a Möb covariant local net on S^1 (provided it satisfies the Reeh-Schlieder and the Bisognano-Wichmann property, see [GLW98, Theorem 1.4]). Thus, our main focus shall be on Möb *covariant local nets over* S^1.

In this subsection, we introduce the concept of a Möbius covariant net with its standard axioms, and later we discuss some additional properties. Although we shall only deal with general ideas, a model satisfying these assumptions, the $U(1)$-current model, will be presented in the subsection 2.6. We start with the main definition.

Definition 2.11. A **Möbius covariant local net** consists of a quadruple $(\mathfrak{A}, U, \Omega, \mathcal{H})$, where $\mathcal{H}$ is the Hilbert space of the theory, $\Omega \in \mathcal{H}$ is a unit vector corresponding to the vacuum state, U is a strongly continuous unitary representation of $\widetilde{\text{Möb}}$ in $\mathcal{H}$, and $\mathfrak{A}$ is a family of von Neumann algebras acting on $\mathcal{H}$ and indexed by elements of $\mathcal{J}$. Those are supposed to satisfy the following properties.

Assumption A **Isotony.** For any intervals $I, K \in \mathcal{J}$, if $I \subset K$, then $\mathfrak{A}(I) \subset \mathfrak{A}(K)$.

Assumption B **Locality.** For any intervals $I, K \in \mathcal{J}$, if $I \cap K = \emptyset$, then $[\mathfrak{A}(I), \mathfrak{A}(K)] = 0$.

Assumption C **Covariance.** For any interval $I \in \mathcal{J}$ and any symmetry $g \in \text{Möb}$, it holds that $\mathrm{Ad}_{U_g} \mathfrak{A}(I) = \mathfrak{A}(gI)$.

Assumption D **Positivity.** The generator of the rotation one-parameter group $U(\rho.)$ is denoted as L_0 and is positive.

Assumption E **Uniqueness of the vacuum.** Ω is the unique (up to phase) unit vector $\mathcal{H}$ which is invariant for the unitary representation U of Möb.

Assumption F **Cyclicity of the vacuum.** The vacuum vector Ω is cyclic for the algebra $\mathfrak{A}(S^1)$, where $\mathfrak{A}(S^1) = \vee_{I \in \mathcal{J}} \mathfrak{A}(I)$.

We briefly discuss the basic assumptions. Assumptions (A) to (C) are basic axioms that establish the locality the theory. In particular, (B) states that any two measurements done on causally disconnected regions do not interfere (their associated operators commute), and (C) implements the geometrical symmetry group into the picture. Assumption (D), as discussed in the subsection 2.2, is equivalent to the positivity of the self-adjoint generator $P = -i\, dU(T)$ of the translations, which itself is connected to the physical concept of positivity of energy. Assumption (E) introduces the concept of a vacuum state, unique up to phase, and assumption (F) just states that the Hilbert space is not unnecessarily too big.

Main properties

There are many interesting consequences that hold from those simple assumptions. In the following, we shall discuss some of them. It is worth mentioning that only the item 1 will be effectively used in this work, but nevertheless we state many other properties (used indirectly) for the sake of completeness. The complete proofs will, however, not be presented, and the reader will be pointed to references accordingly.

Property 1 - Discrete spectrum of L_0. It holds that $\mathrm{sp}(L_0) \subset \mathbb{N}$. This follows from the same argument from subsection 2.2, together with the fact that the vacuum vector Ω is associated to the lowest weight eigenvalue $l = 0$.

Property 2 - Positivity of P. The positivity of the self adjoint generator $P := -i\, dU(T)$ of the translations (*i.e.* $U(\tau_a) = e^{iaP}$ for $a \in \mathbb{R}$) is equivalent to the condition of positivity of L_0 (see Proposition 2.10).

Property 3 - Cores. The Gårding space $\mathcal{H}^\infty$ is generated by smooth local operators acting on Ω. Here, a smooth local operator is an element of $\mathfrak{A}(I)$ for some $I \in \mathcal{J}$ which is smooth for the rotation action $\theta \in \mathbb{R} \to \mathrm{Ad}_{U(\rho_\theta)} \in \mathrm{Aut}(\mathfrak{A}(S^1))$, with respect to the uniform norm topology. Moreover, all the positive integer powers of L_0 are positive operators with $\mathcal{H}^\infty$ as a core (see [Wei07, Theorem 2.1.3]).

Property 4 - Reeh-Schlieder property: *"The vacuum vector Ω is cyclic for any local algebra $\mathfrak{A}(I)$, for $I \in \mathcal{J}$."* This property expands the cyclicity property (Assumption F) from the total algebra to arbitrary local algebras, using the positivity property (assumption D). For Haag-Kastler nets, one also needs the assumption of weak additivity (cf. [Ara99, Theorem 4.14]), but for Möbius covariant nets, no additional assumptions are needed (for a proof, see [GF93, Corollary 2.8]). On the interpretation side, this property shows aspects of non-independence of the vacuum state, as any other vector state can be approximated by local operations on it. Another consequence is the impossibility of existence of an local number operator (as it would have the vacuum vector as an eigenvector), which brings "depth" into the discussion of local detectors.

Property 5 - Bisognano-Wichmann property. Let S^1_+ be the positive arc $S^1_+ = \{z \in S^1\,,\, \Im(z) > 0\}$. By the Reeh-Schlieder property, the vacuum sate Ω is cyclic and separating for $\mathfrak{A}(S^1_+)$, and hence the modular data $(J_{S^1_+}, \Delta_{S^1_+})$ can be defined. *The Bisognano-Wichmann property states that $\Delta^{it}_{S^1_+} = U(\delta_{(-2\pi t)})$ for $t \in \mathbb{R}$, and that $J_{S^1_+} \mathfrak{A}(I) J_{S^1_+} = \mathfrak{A}(\iota(I))$*

where $I \in \mathcal{J}$ and $\iota : z \in S^1 \mapsto \bar{z} \in S^1$ is the complex conjugation (reflection transformation). Putting in words, the modular data have a geometrical interpretation, where $\Delta^{it}_{S^1_+}$ acts as dilations by a factor of $2\pi t$, and $J_{S^1_+}$ acts as the reflection $\iota := z \in S^1 \mapsto \bar{z} \in S^1$ (equivalently, $x \in \overline{\mathbb{R}} \mapsto -x \in \overline{\mathbb{R}}$). The proof of this property relies on a Borchers's theorem [Bor92, Theorem II.9], which shall not be explored in this work. For a discussion on the subject, the reader is refereed to [GF93, Theorem 2.19]. Furthermore, by the Möbius covariance, this property generalizes to the modular data of any local algebra: Let $I \in \mathcal{J}$ be an interval, $\mathfrak{A}(I)$ its local algebra, and let $\phi \in$ Möb such that $\phi(I) = S^1_+$. Then, Δ^{it}_I corresponds to the dilations associated to I (that is, the one parameter group $t \mapsto \delta_{I,t} = \phi^{-1} \circ \delta_t \circ \phi$ of dilations preserving I), and J_I corresponds to the conjugation associated to I (that is, $\iota_I : \phi^{-1} \circ \iota \circ \phi$).

Property 6 - Haag duality: *"For any interval $I \in \mathcal{J}$, the commutant of $\mathfrak{A}(I)$ is the local algebra associated to the causal complement of I, viz. $\mathfrak{A}(I)' = \mathfrak{A}(I')$."* Recall that the locality (Assumption B) only implies the inclusion $\mathfrak{A}(I') \subset \mathfrak{A}(I)'$. The reverse inclusion is a consequence of the geometrical action of modular conjugations on the Bisognano-Wichmann property.

Property 7 - Additivity: *"If $\{I_n \in \mathcal{J}\}_{n\in\mathbb{N}}$ is a covering for an interval $I = \cup_{n\in\mathbb{N}} I_n \in \mathcal{J}$, then $\vee_{n\in\mathbb{N}} \mathfrak{A}(I_n) = \mathfrak{A}(I)$."* This is a stronger version of the **inner continuity** property, where there is an additional requirement that the family I_n is increasing. Recall that isotony implies the inclusion $N := \vee_{n\in\mathbb{N}} \mathfrak{A}(I_n) \subset \mathfrak{A}(I)$. Using the Bisognano-Wichmann property, one can prove that N is invariant for the modular automorphism group of $(\mathfrak{A}(I), \Omega)$, and hence, by a theorem of Takesaki [Tak03, Vol. II, Chap. XI, Theorem 4.6], we have the equality of the sets. For a detailed proof, see [FJ96, page 545]. Lastly, this property is usually strengthened by an additional assumption called "strong additivity", which will be discussed in the following.

Property 8 - Factoriality. *"Local algebras are factors of type* III_1*."* For $\mathfrak{A}(S^1_+)$, the reason lies on the ergodicity of the modular automorphism group associated to the vacuum vector, see [Lon79, Theorem 3] [GF93, Lemma 2.9]. With the Möbius covariance, the same holds for other local algebras. Furthermore, if the "split property" holds (extra property 2.18(f), discussed in Definition 2.16), all local algebras are hyperfinite, and hence isomorphic to the unique hyperfinite type III_1 factor (see [GF93, Theorem 2.13]).

Property 9 - Irreducibility. *"The quasilocal algebra $\mathfrak{A}(S^1)$ is irreducible on the vacuum representation"*. In simpler terms, $\mathfrak{A}(S^1) = \mathcal{B}(\mathcal{H})$. This property is equivalent to factoriality, and also to the uniqueness of the vacuum [GL96, Proposition 1.2].

Other assumptions

As mentioned above, further additional assumptions can be added to the set of assumptions (A)-(F) of Definition 2.11. We shall mainly be interested in extra assumptions of Definition 2.18, but aside from those, we shall here briefly discuss other properties, the **strong additivity**, the **complete rationality**, and the **conformal covariance**.

Strong additivity, as the name implies, is a stronger version of the additivity property. For an interval $I \in \mathcal{J}$, one can remove a point inside it, and write the remaining set as a disjoint union of two intervals I_1 and I_2 (so one has $I, I_1, I_2 \in \mathcal{J}$, $z \in I$, and $I_1 \cup I_2 = I \setminus \{z\}$). A Möbius covariant net $(\mathfrak{A}, U, \Omega, \mathcal{H})$ is said to be strongly additive if $\mathfrak{A}(I) = \mathfrak{A}(I_1) \vee \mathfrak{A}(I_2)$. Although many important models satisfy this property, there are known counter examples, such as the derivatives of the $U(1)$-current (cf. [GLW98]).

Complete rationality is a condition which makes the category of superselection sectors "well behaved". A Möbius covariant net $(\mathfrak{A}, U, \Omega, \mathcal{H})$ is said to be completely rational if it is strongly additive, satisfies the split property, and has a finite μ-index, where the μ-index is the index of inclusion of subfactors generated by local algebras in a certain configuration. A net that satisfy this property has a finite number of irreducible representations (up to unitary equivalence), all of which with finite dimension. The precise notions involved in this property are out of the scope of this work, and we reefer the reader to [KLM01].

We make a final remark about *conformal nets*. Conformal QFT has a much larger symmetry group than standard "relativistic" QFT, and as for chiral components, this larger group is i the group $\mathrm{Diff}_+(S^1)$ of all order preserving diffeomorphisms of S^1. Thus, a Möbius covariant local net over S^1 is called $\mathrm{Diff}_+(S^1)$ *covariant local net*, or *local conformal net* if its unitary representation U of $\widetilde{\text{Möb}}$ extends to a projective unitary representation of $\mathrm{Diff}_+(S^1)$, such that the net still is covariant for U (see e.g. [Kaw15, Definition 3.1, item 4]). Whereas in the case of the symmetry group Möb any projective representation could be lifted to a true unitary representation of $\widetilde{\text{Möb}}$, the same if not valid for $\mathrm{Diff}_+(S^1)$. Nevertheless, in such representation, the generators can be fixed up to additive constants. The result is a (unitary, positive energy) representation of the Virasoro algebra, an infinite dimensional Lie algebra generated by a central element c and generators $\{L_n\}_{n\in\mathbb{Z}}$, subject to certain commutation relations. On conformal local nets, the central element c is associated to a scalar, which is denoted by the same notation c and refereed to as *central charge*. It is an invariant of the net, with admissible values ranging inside a discrete subset $(0,1)$, or on the continuum $[1,+\infty)$ (see [Kaw15, Section 3.8]). We mention that the $U(1)$ current model, introduced later in subsection 2.6, is in fact a conformal local net with central charge $c=1$, although this shall not be addressed in our discussion.

2.4 Nuclear maps

In order to discuss the conformal nuclearity condition, we shall first make a brief review of the notions of nuclear maps, p-nuclear maps, and their nuclearity indices, together with some of their basic properties. Those notions extend the Schatten class of L^p operators on a Hilbert space.[4] For this subsection, we shall take [BDL90a,BDL90b,FOP05] as reference. We also start with a more general setting, using capital letters $A, B, C, \dots$ for Banach spaces, and lowercase letters $a, b, c, \dots$ for operators. We later state as an example our case of interest, maps between a von Neumann algebra and a Hilbert space.

Our interest is only in the case $p \le 1$, where the discussion is much simpler than the general case. For a generalization to the case $1 \le p \le \infty$, see [FOP05].

Definition 2.12. Let A, B be Banach spaces and p a parameter with $0 < p \le 1$. A bounded linear operator $x : A \to B$ is said to be **p-nuclear** if there are families $\{\phi_k\}_{k\in\mathbb{N}} \subset A^*$ of linear functionals on X and $\{\xi_k\}_{k\in\mathbb{N}} \subset B$ of vectors in B such that the following decomposition holds:

$$x(\cdot) = \sum_{k\in\mathbb{N}} \phi_k(\cdot)\xi_k, \qquad \text{and} \qquad \sum_{k\in\mathbb{N}} \big(\|\phi_k\| \cdot \|\xi_k\|\big)^p \le +\infty.$$

Furthermore, any such decomposition is called a **p-nuclear decomposition**, and we define $\nu_p(x)$ as the **p-nuclearity index of** x given by

$$\nu_p(x) := \inf \sum_{k=0}^{\infty} (\|\phi_k\| \cdot \|\xi_k\|)^p,$$

with the infimum taken over all the p-nuclear decompositions as above.

Proposition 2.13. For operators in $\mathcal{B}(A, B)$, the following hold.

- Consider $0 < p \le 1$. Then, $\nu_p(\cdot)$ is p-homogeneous and subadditive, namely if x_1, x_2 are two operators and λ is a scalar, then

$$\nu_p(\lambda x_1) = |\lambda|^p \cdot \nu_p(x_1), \quad \text{and} \quad \nu_p(x_1 + x_2) \le \nu_p(x_1) + \nu_p(x_2).$$

- Consider $0 < p \le 1$. Then, $\nu_p(\cdot)^{(1/p)}$ is a quasi-norm, *viz.* all the axioms for norm holds except for the triangle inequality, which is replaced by

$$\sum_{k=1}^{N} \nu_p(x_k)^{(1/p)} \le \nu_p\Big(\sum_{k=1}^{N} x_k\Big)^{(1/p)} \le N^{\frac{1-p}{p}} \sum_{k=1}^{N} \nu_p(x_k)^{(1/p)}.$$

[4] To be precise, one should introduce the more technical concept of l^p operators. We shall not discuss those notions in our work, since we are only interested on the case $p \le 1$. See [FOP05] for a more detailed account.

- Consider $0 < p \leq q \leq 1$. Then, p-nuclearity implies q-nuclearity.

Example. Consider $0 < p \leq 1$. For $\mathcal{H}$ a Hilbert space, the operators t of the Schatten class $L^p(\mathcal{H})$ are p-nuclear with $\nu_p(t)^{(1/p)} \leq \|t\|_{L^p(\mathcal{H})}$.

All the Schatten classes $L^p(\mathcal{H})$ are ideals on $\mathcal{B}(\mathcal{H})$. The following proposition generalizes this assertion to p-nuclear maps.

Proposition 2.14. Consider $0 < p \leq 1$. For Banach spaces A_0, B_0, A, B, let $a \in \mathcal{B}(A_0, A), x \in \mathcal{B}(A, B), b \in \mathcal{B}(B, B_0)$ be bounded linear operators. If x is a p-nuclear map, then axb is also a p-nuclear map. Moreover, $\nu_p(axb) \leq \|a\|^p \cdot \nu_p(x) \cdot \|b\|^p$.

Example. Consider $0 < p \leq 1$. For $\mathcal{H}$ a Hilbert space, $\Omega \in \mathcal{H}$ a unit vector, $\mathfrak{M}$ a von Neumann algebra, and $t \in L^p(\mathcal{H})$ an L^p-operator, the map Θ defined by

$$\Theta : a \in \mathfrak{M} \mapsto t\, a\, \Omega \in \mathcal{H}$$

is a p-nuclear map with $\nu_p(\Theta)^{(1/p)} \leq \|t\|_{L^p(\mathcal{H})}$.

The later example is the prototype for maps appearing in the nuclearity conditions on the following subsection.

2.5 Nuclearity conditions and the split property

In this section, we discuss the conformal nuclearity condition and the split property, as well as other related conditions. We gather all the relevant notions in a list of extra assumptions for Möbius covariant local nets given in Definition 2.18. For our work, we shall mainly need the conformal nuclearity condition (Definition 2.15((2)) and extra assumption 2.18(d)), although we shall also need a stronger one, extra condition 2.18(b), on our later calculations of entropy in Section 3.

For introducing the nuclearity conditions, we first address its physical motivations while briefly digressing on the broader context of Haag-Kastler nets. Here, $\mathcal{O}$ are connected, bounded, open, globally hyperbolic spacetime regions (typically double cones), $\mathfrak{A}$ is a net of operator algebras over the Hilbert space of the theory $\mathcal{H}$, and H denotes generically the Hamiltonian. Our discussion starts with extra assumptions on Haag-Kastler nets to rule out certain unphysical models, *e.g.* models with an infinite number of particles on the same mass multiplet (in which the spin-statistics theorem does not hold). On that direction, one of the first results was the Haag and Swieca compactness criterion [HS65], a criterion for quantum field theories to describe particles. By analyzing the analogous of a phase space volume in QFT, they input restrictions to it for a theory to have the same "number of degrees of freedom" as a free theory, thus having a particle interpretation. More specifically, in $\mathcal{H}$, they define the subset $\mathcal{L}_{\mathcal{O}}$ of vectors localized in a certain region $\mathcal{O}$, and require $P_E\mathcal{L}_r$ to be compact (here, P_E is the projection on the subspace of energy $\leq E$). The Buchholz-Wichmann energy nuclearity condition [BW86] is a strengthened version of the above. They replace the sharp cutoff by a smooth damping, requiring nuclearity instead of compactness, *viz.* the set $e^{-\beta H}\mathcal{L}_{\mathcal{O}}$ is nuclear, with its nuclearity index satisfying $\nu(e^{-\beta H}\mathcal{L}_{\mathcal{O}}) \leq e^{-(\beta/\beta_0)^{-n}}$ for some parameters β_0 and n dependent on $\mathcal{O}$. The reasoning is the following. Assuming that boundary effects are negligible, considering $\mathcal{O}$ as a bounded region, there is a finite volume V which approximates the local theory. In this finite volume theory, call H_V the Hamiltonian, $\mathcal{H}_V$ the Hilbert space, and $\mathcal{L}_V$ the unit ball of $\mathcal{H}_V$. Then, $e^{-\beta H_V}$ is the density matrix of Gibbs states, a trace class operator, and therefore $e^{-\beta H_V}\mathcal{L}_V$ is a nuclear set in $\mathcal{H}_V$. We can thus expect $e^{-\beta H}\mathcal{L}_{\mathcal{O}}$ to be a nuclear set in the full theory also. Moreover, the trace of $e^{-\beta H_V}$ is the grand partition function (with $\mu = 0$), and there should be a pressure function $p_V(\beta)$ finite at the thermodynamical limit p as $V \to \infty$, where typically $p(\beta) = c\beta^{-n}$. Therefore $\nu(e^{-\beta H}\mathcal{L}_{\mathcal{O}}) = \exp(cV\beta^{-(n-1)})$. Later, the same condition was translated to an equivalent formulation with the nuclearity conditions for maps instead of for sets, which is now the standard formulation.

For chiral nets, we restate the nuclearity condition using L_0 instead of H, and adopt the nomenclature "conformal nuclearity condition", which was used in [BDL07]. Our precise definition follows.

Definition 2.15 (Conformal nuclearity conditions). Let $(\mathfrak{A}, U, \Omega, \mathcal{H})$ be a Möbius covariant local net. For $I \in \mathcal{J}$ and $\beta > 0$, define the damping map $\Theta_{I,\beta} : \mathfrak{A}_I \to \mathcal{H}$ by the following formula:

$$\Theta_{I,\beta} : a \in \mathfrak{A}(I) \mapsto e^{-\beta L_0} a\Omega \in \mathcal{H}. \tag{2}$$

With those maps, we define the conformal nuclearity conditions:

(1) Let $p \in (0, 1]$. The net satisfies the **conformal p-nuclearity condition** if the map $\Theta_{I,\beta}$ is p-nuclear for any $I \in \mathcal{J}$ and $\beta > 0$, and such that the inequality $\nu_p(\Theta_{I,\beta}) \leq \exp\big((c_{I,p}/\beta)^{n_{I,p}}\big)$ holds for positive constants $c_{I,p}$ and $n_{I,p}$ depending on I.

(2) The net satisfies the **conformal nuclearity condition** if the above holds for $p = 1$.

Remark. For $p \leq 1$, it is clear that conformal p-nuclearity implies conformal nuclearity. Our result regarding the UV-regularized entanglement entropy (Theorem 3.9) depends only on conformal nuclearity, although an intermediate result (Proposition 3.4) requires a condition stronger than conformal p-nuclearity (namely, extra assumption 2.18(b), explained later).

We now address one of the by-products of the conformal nuclearity condition, the *split property*. It is an algebraic property that relates to the independence of two "separated" local algebras, and implies the existence of intermediate type I factors, which shall be used later on our analysis. We first give a definition.

Definition 2.16 (Split property, or funnel property). [Kaw15, Definition 3.8] [GF93, Definition 2.11] Let $(\mathfrak{A}, U, \Omega, \mathcal{H})$ be a Möbius covariant local net. The net satisfies the **split property** if, for any any $I_1, I_2 \in \mathcal{J}$ such that $\overline{I_1} \cap \overline{I_2} = \emptyset$ (*i.e.* $I_1 \Subset I_2'$), the following equivalent properties hold

- the (algebraic) $*$-homomorphism $a \otimes b \in \mathfrak{A}(I_1) \otimes_{\text{alg}} \mathfrak{A}(I_2) \mapsto a \cdot b \in \mathfrak{A}(I_1) \vee \mathfrak{A}(I_2)$ extends to an $*$-isomorphism of von Neumann algebras $\mathfrak{A}(I_1) \bar{\otimes} \mathfrak{A}(I_2) \cong \mathfrak{A}(I_1) \vee \mathfrak{A}(I_2)$.
- the inclusion $\mathfrak{A}(I_1) \subset \mathfrak{A}(I_2)'$ is a standard split inclusion of von Neumann algebras (with respect to Ω), *i.e.* Ω is a cyclic vector for $\mathfrak{A}(I_1)$, $\mathfrak{A}(I_2)'$ and $\mathfrak{A}(I_1)' \cap \mathfrak{A}(I_2)'$, and there is a von Neumann algebra $\mathfrak{R}$ which is an intermediate factor type I, *viz.* $\mathfrak{A}(I_1) \subset \mathfrak{R} \subset \mathfrak{A}(I_2)'$.

Remark. The reason for the equivalence above is the following (cf. [DL83a, Lemma 2]). Consider two intervals $I_1, I_2 \in \mathcal{J}$ such that $I_1 \Subset I_2$, and define $I_3 := I_2'$. Recall that $\mathfrak{A}(I_3)' = \mathfrak{A}(I_2)$, by the Haag duality. The existence of a type I factor $\mathfrak{R}$ with $\mathfrak{A}(I_1) \subset \mathfrak{R} \subset \mathfrak{A}(I_3)$ implies that $x \otimes y \in \mathfrak{R} \otimes_{\text{alg}} \mathfrak{R}' \mapsto xy \in \mathfrak{R} \vee \mathfrak{R}' = \mathcal{B}(\mathcal{H})$ extends to an $*$-isomorphism $\mathfrak{R} \bar{\otimes} \mathfrak{R}' \cong \mathcal{B}(\mathcal{H})$, whose restriction to $\mathfrak{A}(I_1) \otimes_{\text{alg}} \mathfrak{A}(I_2)$ hence is a faithful $*$-homomorphism onto $\mathfrak{A}(I_1) \vee \mathfrak{A}(I_2)$. Conversely, an $*$-isomorphism $\phi : \mathfrak{A}(I_1) \vee \mathfrak{A}(I_2) \to \mathfrak{A}(I_1) \bar{\otimes} \mathfrak{A}(I_2)$ is spatial (this holds by [SZ79, Cor. 5.25], since both $\mathfrak{A}(I_1) \bar{\otimes} \mathfrak{A}(I_2)$ and $\mathfrak{A}(I_1) \vee \mathfrak{A}(I_2)$ have cyclic and separating vectors, respectively Ω and $\Omega \otimes \Omega$), and thus implemented by an unitary operator $u : \mathcal{H} \to \mathcal{H} \otimes \mathcal{H}$. The type I factor $\mathfrak{R} := u^*(\mathcal{B}(\mathcal{H}) \otimes 1)u$ satisfies $\mathfrak{A}(I_1) \subset \mathfrak{R} \subset \mathfrak{A}(I_2)'$.

Remark. As a consequence of the split property, together with additivity, it holds that *local algebras* $\mathfrak{A}(I)$ *are hyperfinite*, for $I \in \mathcal{J}$. Indeed, one can approximate any interval $I \in \mathcal{J}$ by an increasing sequence of intervals $\{I_n \in \mathcal{J} \,|\, I_n \subset I\}_{n \in \mathbb{N}}$ such that $I_n \Subset I_{n+1}$ for all $n \in \mathbb{N}$. The split condition implies the existence of a increasing family of type I factors $\{\mathfrak{R}_n\}_{n \in \mathbb{N}}$ with $\mathfrak{A}(I_n) \subset \mathfrak{R}_n \subset \mathfrak{A}(I_{n+1})$, and additivity implies that $\mathfrak{A}(I)$ is generated by $\vee_{n \in \mathbb{N}} \mathfrak{R}_n$, thus proving hyperfiniteness. Together with the factoriality condition, all local algebras are then isomorphic to the unique hyperfinite type III_1 factor [GF93, Theorem 2.13].

The intermediate factors of type I will be essential in our later definitions involving entropy. Since there are many choices of such factors (all of them defined via certain unitary operators), we shall frequently adopt the following notation:

Definition 2.17 (Intermediate pairs $(u, \mathfrak{R}_u)$). For $(\mathfrak{A}, U, \Omega, \mathcal{H})$ a Möbius covariant local net satisfying the split property, and $I_1, I_2 \in \mathcal{J}$ such that $I_1 \Subset I_2$, we use the symbol $(u, \mathfrak{R}_u)$ to denote an *intermediate pair*, *i.e.* a generic pair consisting of a unitary operator $u : \mathcal{H} \to \mathcal{H} \otimes \mathcal{H}$ implementing the $*$-isomorphism $\mathfrak{A}(I_1) \vee \mathfrak{A}(I_2) \cong \mathfrak{A}(I_1) \bar{\otimes} \mathfrak{A}(I_2)$, and an intermediate type I factor $\mathfrak{R}_u = u^*(\mathcal{B}(\mathcal{H}) \otimes 1)u$.

Having stated the nuclearity conditions and the split property, we now make a list of useful extra assumptions for Möbius covariant local nets, together with their implication chart.

Definition 2.18 (List of Extra Assumptions). For a Möbius covariant local net, we define the following additional properties:

Assumption 2.18(a) $\dim \ker(L_0 - N) = p(N)$, where p is the partition function [AS64, 24.2.1.I].

Assumption 2.18(b) $\dim \ker(L_0 - N) \leq C \exp(N^\kappa)$ for constants $\kappa \in (0, 1)$ and $C > 0$.

Assumption 2.18(c) Trace class condition: there are positive parameters a, b, c such that

$$\mathrm{Tr}(e^{-\beta L_0}) \leq a \exp\left(b\beta^{-c}\right) \qquad (\text{for } \beta > 0).$$

Assumption 2.18(d) Conformal p-nuclearity condition (Definition 2.15((1))).

Assumption 2.18(e) Conformal nuclearity condition (Definition 2.15((2))).

Assumption 2.18(f) Split property (Definition 2.16).

Proposition 2.19. The extra assumptions in the above Definition 2.18 satisfy (a) $\Rightarrow$ (b) $\Rightarrow$ (c) $\Rightarrow$ (d) $\Rightarrow$ (e) $\Rightarrow$ (f).

Proof. The implication (a) $\Rightarrow$ (b) follows from the asymptotic behavior of the partition function, given by $p(N) \sim \frac{1}{4\sqrt{3}N} e^{\pi\sqrt{2/3}\sqrt{N}}$ as $N \to \infty$ [AS64, 24.2.1.III]. We shall then work on the proof of the implication (b) $\Rightarrow$ (c). Consider the parameters $C > 0$ and $\kappa \in (0, 1)$ such that $\dim \ker(L_0 - N) \leq C \exp(N^\kappa)$. Then,

$$\mathrm{Tr}(e^{-\beta L_0}) = \sum_{N \geq 0} \dim \ker(L_0 - N) e^{-\beta N} \leq \sum_{N \geq 0} C e^{-\beta N + N^\kappa}.$$

Since $-\beta N$ eventually dominates N^κ, the trace is always finite. All that is left is to verify the dependence on β. Our strategy is to divide the sum in three parts, the first term, a finite sum, and a infinite sum with exponential decrease.

The exponent can be expressed as $-\beta N + N^\kappa = -\beta N^\kappa (N^{(1-\kappa)} - 1/\beta)$. Notice that there is a number A such that $N^{(1-\kappa)} - 1/\beta \geq A^{(1-\kappa)} - 1/\beta > 0$ whenever $N \geq A$. Indeed, one can take any A such that $A > \beta^{-1/(1-k)}$, but we shall fix this value later. Defining $B := A^{(1-\kappa)} - 1/\beta > 0$, one has that $-\beta N + N^\kappa \geq -B\beta N^\kappa$ for $N \geq A$. Hence, dividing the sum in $\{N = 0\}$, $\{0 < N \leq A\}$ and $\{N > A\}$, the first and the last sums can be bound using the following inequality:

$$1 + \sum_{N > A} e^{-\beta N + N^\kappa} = 1 + \sum_{N > A} e^{-\beta N^\kappa (N^{(1-\kappa)} - 1/\beta)} \leq 1 + \sum_{N > A} e^{-B\beta N^\kappa} \leq \sum_{N \in \mathbb{N}} e^{-B\beta N^\kappa}.$$

Now, to turn the last term above in a quantity independent of β, we pick A as following:

$$A := (2/\beta)^{1/(1-\kappa)}, \qquad B = 1/\beta.$$

Therefore, sum $\{N = 0\} \cup \{N > A\}$ is then bounded by a constant expressed in the following:

$$C\Big(1 + \sum_{N > A} e^{-\beta N + N^\kappa}\Big) \leq C \sum_{N \in \mathbb{N}} e^{N^\kappa}.$$

The remaining finite sum can be bounded by the number of terms times the supremum of the function. We first analyze the exponent $-\beta N + N^\kappa$. By elementary calculus, it takes its maximal value at $N_0 = (\kappa/\beta)^{1/(1-\kappa)}$, and hence one has the supremum bound

$$\sup_{N\geq 0} \left|e^{-\beta N+N^\kappa}\right| = \exp\left(\left((1-\kappa)\,\kappa^{-\frac{\kappa}{1-\kappa}}\right)\beta^{-\frac{\kappa}{1-\kappa}}\right).$$

Moreover, the number of terms is $\#\{0 < N \leq A\} = \lfloor A \rfloor \leq A = (2/\beta)^{1/(1-\kappa)}$. One has the following bound for the finite part of the sum:

$$\begin{aligned} C \sum_{0<N\leq A} e^{-\beta N+N^\kappa} \leq CA\|e^{-\beta N+N^\kappa}\|_\infty &\leq \frac{C2^{1/(1-\kappa)}}{\beta^{1/(1-\kappa)}} \exp\left(\left((1-\kappa)\,\kappa^{-\frac{\kappa}{1-\kappa}}\right)\beta^{-\frac{\kappa}{1-\kappa}}\right) \\ &= \frac{a_0}{\beta^{c_0}} \exp\left(b_1\beta^{-c_1}\right) \\ &\leq a_2 \exp\left(\beta^{-c_2}\right) \qquad (\beta > 0). \end{aligned}$$

where the constants a_0, c_0, b_1, c_1 are easily identifiable. To identify a_2, c_2, we notice that $\beta^{-c_0} \leq \exp(\beta^{-c_0})$, and put $c_2 = \max\{c_0, c_1\}$. Then, a_2 is such that $\beta^{-c_0} + b_1\beta^{-c_1} \leq \log(a) + \beta^{-c_2}$ holds for all $\beta > 0$ (*e.g.* $log(a_2) = 1 + b_1^{(c_2-c_1+1)/(c_2-c_1)}$).

Therefore, the trace satisfies the following inequality.

$$\mathrm{Tr}(e^{-\beta L_0}) \leq C \sum_{N\in\mathbb{N}} e^{N^\kappa} + a_2 \exp\left(\beta^{-c_2}\right) \leq a \exp\left(b\beta^{-c}\right) \qquad (\beta > 0),$$

where $a = C\sum_{N\in\mathbb{N}} e^{N^\kappa} + a_2$, $b = 1$, and $c = c_2$. This concludes the proof of (b) $\Rightarrow$ (c).

The implication (c) $\Rightarrow$ (d) follows since, for any $I \in \mathcal{J}$, $0 < p \leq 1$ and $\beta > 0$, the inequality $\nu_p(\Theta_{I,\beta}) \leq \mathrm{Tr}(e^{-p\beta L_0}) \leq a\exp\left((b/p^c)\,\beta^{-c}\right)$ holds. The implication (d) $\Rightarrow$ (e) is trivial. Finally, we refer the proof of (e) $\Rightarrow$ (f) to references, see [GF93, Lemma 2.12], which translates the arguments of [BDF87, Section 2] to the chiral setting. See also [BDL07, Corollary 6.4] for a different proof that holds also in a "distal" case, involving concepts of modular nuclearity and L^2-nuclearity. This concludes the proof. □

Remark. It is well known that the contextualization of split inclusions for local algebras of QFT was an idea introduced by Borchers. It was established for free field theories in [Buc74, Sum82, DL83a]. There are, however, models which do not obey the split property, such as those containing an infinite number of particles in some erratic manner (see [DL84, Section 10]). The energy nuclearity condition was introduced in [BW86], as a condition with physical motivations that would imply the split property. It was proved for free fields in [BW86, BJ87]. That it implies the split property was shown in [BW86] in a "distal" manner, which was subsequently improved in [BDF87]. Since then, nuclearity and split properties have been studied in a close manner. They have also been investigated in curved spacetimes [Ver93, DH06, Few15, LS16], and have been used in an abundant number of ways, for example, in analysis of thermodynamical properties [BJ89], as well as in construction of local current algebras and a quantum Noether's theorem [Dop82, DL83b, BDL86, DDFL87]. Connections between energy nuclearity condition and modular theory of von Neumann algebras have been studied in [BDL90a, BDL90b, BL04], where the concept of modular nuclearity was introduced. The finer property of p-nuclearity condition appeared in [BP90], regarding investigations of the phase space in AQFT. Later, in [FOP05], the condition was put into better formulation and meaningfully defined for $p > 1$. For chiral nets, the conformal nuclearity condition was studied in [BDL07], together with other notions of nuclearity. Also, in [MTW16] it was proved that the split property follows automatically if the chiral net is $\mathrm{Diff}_+(S^1)$-covariant.

2.6 The $U(1)$-current model

To finish this section, we introduce a commonly used model of a Möbius covariant net. In summary, we first show the construction the one-particle Hilbert space $\mathcal{H}_1$ with a local structure compatible with a positive energy irreducible unitary representation U_1 of $\widetilde{\text{Möb}}$, of lowest weight $l = 1$. Then, using the construction of second quantization, the local algebras are defined as the von Neumann algebras generated by Weyl unitaries localized in intervals of S^1. Finally, we will see that such model satisfies all the extra assumptions 2.18(a-f).

We start by considering the real vector space $C^\infty(S^1, \mathbb{R})$ of real valued smooth functions on S^1. An element of this space is said to be localized in $I \in \mathcal{J}$ if its support lies inside I. This gives a localization structure on $C^\infty(S^1, \mathbb{R})$, which is compatible with the (real bilinear) antisymmetric form σ defined by

$$\sigma(f, g) := \frac{1}{2\pi} \int_0^{2\pi} f(e^{i\theta}) \left.\frac{dg(e^{i\alpha})}{d\alpha}\right|_{\alpha=\theta} d\theta, \qquad \big(f, g \in C^\infty(S^1, \mathbb{R}) \big).$$

Notice that a generic element $f \in C^\infty(S^1, \mathbb{R})$ is described by its Fourier components, given by the formula $\hat{f}_k := \frac{1}{2\pi} \int_0^{2\pi} f(e^{i\theta}) e^{-ik\theta} d\theta$. Introducing complex structure J such that $\widehat{Jf}_k := -i \operatorname{sign}(k) \hat{f}_k$, the antisymmetric form induces a sesquilinear form $\langle \cdot, \cdot \rangle$ given by

$$\langle f, g \rangle := \sigma(f, Jg) + i\sigma(f, g) = \sum_{k=0}^{\infty} 2k\, \hat{f}_k\, \hat{g}_{-k} \qquad \big(f, g \in C^\infty(S^1, \mathbb{R}) \big).$$

Such structure is motivated by the fact that the imaginary part of the sesquilinear form is precisely the antisymmetric form, *viz.* $\Im\langle f, g \rangle = \sigma(f, g)$ for any $f, g \in C^\infty(S^1, \mathbb{R})$.

The complex structure together with the sesquilinear form thus induces a seminorm to which the kernel is composed of constant functions (since $\hat{f}_0$ needs not to vanish for $\|f\| = 0$ to hold). By taking quotient $C^\infty(S^1, \mathbb{R})/\mathbb{R}$, the sesquilinear form becomes an inner product and one may complete the space to a (complex) Hilbert space, which shall be denoted as $\mathcal{H}_1$. This is the **one particle Hilbert space** of the model.

Acting on such a space is the strongly continuous unitary representation U_1 of $\widetilde{\text{Möb}}$, which is induced by $U_1(\tilde{g})f := f \circ g^{-1}$ for $\tilde{g} \in \widetilde{\text{Möb}}$, $g = p(\tilde{g}) \in \text{Möb}$ and $f \in C^\infty(S^1, \mathbb{R})$. Unitarity follows from the following the reasoning. First, $U_1(\tilde{g})$ preserves the symplectic form σ, and hence is an isometry, thus extending naturally to an isometry on $\mathcal{H}_1$, in which it is also clearly a bijection. Furthermore, it can be proved that it commutes with the complex structure J, hence being $\mathbb{C}$-linear, and thus a unitary operator for all $\tilde{g} \in \widetilde{\text{Möb}}$. That it is a strongly continuous representation then follows from straightforward arguments. This allows for the definition of the operators L_0 and $L_\pm$ as in Subsection 2.2. The vector e_1 induced by $z \mapsto \Re(z)$ is an eigenvector of L_0 associated to the eigenvalue $l = 1$, and is annihilated by L_+. Furthermore, the eigenspace of $l = 1$ is spanned by it alone, and we thus have the **irreducibility of the representation with lowest weight** $l = 1$.

With the one particle space and symmetry group in hands, the full model is acquired via the very known method of **bosonic second quantization**. For a throughout exposition, see e.g. [BR97, Chapter 5]. Here, we only point out the main ideas, without any proofs. Set $\mathcal{H}$ as the symmetrized Fock space $\mathcal{F}_+(\mathcal{H}_1)$ based on the one particle Hilbert space $\mathcal{H}_1$. The unitary representation U_1 gives rise, through second quantization, to a strongly continuous unitary representation $U := \Gamma(U_1)$ acting on $\mathcal{H}$, to which the associated operators L_0 and $L_\pm$ extend the ones defined in $\mathcal{H}_1$. One can also define the annihilation, creation, and Segal field operators, denoted respectively by $a^-(f), a^+(f)$, and $J(f)$, where $f \in \mathcal{H}_1$. One can then finally construct the Weyl unitaries $W(f)$ and the local algebras $\mathfrak{A}(I) = \{W(f), \operatorname{supp}(f) \subset I\}''$. Together with the symmetry group and the vacuum vector, this satisfied the axioms for a Möbius covariant net (Definition 2.11).

With the basic structure of the model in hands, we now turn our attention to the extra assumptions of Definition 2.18.

Proposition 2.20. For the $U(1)$-current model, it holds that $\dim \operatorname{Ker}(L_0 - N) = p(N)$ is such that for all $N \in \mathbb{N}$, or in other words, the dimension of the N-eigenspace of L_0 is equal to the partition function of N. Hence, the model satisfies the extra assumption 2.18(a).

Proof. For positive integers $k > 0$, define the vectors $e_k \in \mathcal{H}_1$ by $e(z) := \Re(z^k) = \cos\big(k \cdot \arg(z)\big)$. Those, being linearly independent, form a basis for $\mathcal{H}_1$, and explicit calculations show that e_k are eigenvectors of L_0 with eigenvalues k.

In the full Hilbert space $\mathcal{H}$, the eigenspace $\ker(L_0 - N)$ is spanned by vectors of the form $S_+(e_{k_1} \otimes \ldots \otimes e_{k_j})$ with $\sum_{l=1}^{j} k_l = N$ (here, S_+ is the symmetrizing operator). Those are linearly independent for different choices of positive integers $\{k_l\}_{l=1}^{j}$ (provided $0 < j \leq n$ and $k_1 \leq k_2 \leq \ldots \leq k_j$, with $\sum_{l=1}^{j} k_l = N$). Thus, the dimension of the eigenspace is given by the number of different ways to arrange positive integers to sum up to N, which is given by the partition function $p(N)$. This concludes the proof. □

Since this is the most strict of the extra assumptions of Definition 2.18, the Proposition 2.19 implies the following.

Corollary 2.21. The $U(1)$-current model satisfies all the extra assumption 2.18(a - f), in particular, the conformal p-nuclearity condition and the split property.

Remark. Although not needed for our work, we now discuss some further properties of this model. The $U(1)$-current model, besides satisfying the required axioms of a Möbius covariant net (cf. Definition 2.11) and the extra assumptions 2.18(a-f), also satisfies strong additivity. It can be proven by approximating functions supported in an interval $I \in \mathcal{J}$ by a sum of functions supported in the subintervals of I. There is, however, an important property which it lacks, called complete rationality. This means that its μ-index infinite, which in turns implies that the model has an infinite number of superselection sectors. For a more detailed account on complete rationality, see [BMT88]. Also worth mentioning is the fact that the unitary representation of the Möbius group extends to a projective unitary representation of the group $\mathrm{Diff}_+(S^1)$, thus rendering it as a full fledged conformal net. For a further discussion, see *e.g.* [Wei07, Sec.2.4 §3]. Lastly, we mention that the construction of the $U(1)$-current model is a particular construction of loop group models, in the case where the underling group is, by no coincidence, the $U(1)$ group. For an exposition on loop group models, see [PS88].

3 Entropy cutoff for a chiral net

Throughout this section, let $(\mathfrak{A}, U, \Omega, \mathcal{H})$ be a Möbius covariant local net satisfying the extra assumption 2.18(f) (*i.e.* the split property, Definition 2.16). Let $\omega = \langle \Omega, \cdot\, \Omega \rangle$ be the vacuum state, and L_0 the conformal Hamiltonian. We also fix an interval $I \in \mathcal{J}$.

We shall try to define the entanglement entropy of the vacuum with respect to I, that is, the quantum entropy of ω as a state restricted to $\mathfrak{A}(I)$. Since $\mathfrak{A}(I)$ is a von Neumann algebra of type III, the von Neumann entropy is divergent (see Appendix A.1, Proposition A.5). We therefore present an alternative definition.

In Subsection 3.2, we present our definition for the entanglement entropy in three steps. (1.) Let $\delta > 0$ be a parameter, and $I_\delta = \cup_{-\delta<\theta<\delta}\, \rho_\theta(I)$ the "augmentation of I by δ". It then holds that $I \Subset I_\delta$, and the split property asserts that there are intermediate pairs $(u, \mathfrak{R}_u)$, where $\mathfrak{R}_u$ are factors of type I such that $\mathfrak{A}(I) \subset \mathfrak{R}_u \subset \mathfrak{A}(I_\delta)$. We define the entropy $H_{I,\delta}(\omega)$ with the aid of intermediate type I factors $\mathfrak{R}_u$ (note that, however, this will *not* be $S_{\mathfrak{R}_u}$ of Definition 2.7). (2.) Provided extra assumption 2.18(b) holds, our given definition $H_{I,\delta}(\omega)$ has an upper bound, which however diverges as δ approaches zero. We then regularize the states by a cutoff parameter E, and define the regularized entropy $H^E_{I,\delta}(\omega)$. (3.) Finally, we define the entanglement entropy $H_I(\omega)$ as the limit of the former as δ goes to zero. We also state our main result, Theorem 3.9, stating the finiteness of it, with an upper bound given in terms of the dimensions of eigenspaces of the conformal Hamiltonian. The proof of it is spread in the later subsections.

3.1 The energy function f

We reproduce [BDF87, Lemma 2.3], in which an auxiliary energy function f is constructed assuming only the positivity of the conformal Hamiltonian L_0 (replacing the usual Hamiltonian H used in the original paper). This function will be necessary for the calculations in Section 3.

We first gather some basic results pertaining functions of almost exponential decay in the following lemma.

Lemma 3.1. Let $0 < \beta < 1$ and $f : \mathbb{R} \to \mathbb{C}$ be a function such that

$$\sup_{x \in \mathbb{R}} \left| e^{|t|^\beta} f(t) \right| < +\infty.$$

Suppose $0 < \alpha < \beta$, $c > 0$, $\delta > 0$ and $p > 0$. Then, the following inequalities hold.

$$\begin{aligned}
\sup_{t\in\mathbb{R}} \left| e^{c|t|^\alpha} f(t) \right| &\leq \left[\sup_{t\in\mathbb{R}} \left| e^{|t|^\beta} f(t) \right| \right] \times \exp\left[\frac{c(\beta-\alpha)}{\beta} \left(\frac{c\alpha}{\beta} \right)^{\frac{\alpha}{\beta-\alpha}} \right], \\
\sup_{t\in\mathbb{R}} \left| e^{|t|^\alpha} f(\delta t) \right| &\leq \left[\sup_{y\in\mathbb{R}} \left| e^{(1/\delta)^\alpha |y|^\alpha} f(y) \right| \right], \\
\sup_{t\in\mathbb{R}} \left| e^{|t|^\alpha} f(t)^p \right| &\leq \left[\sup_{t\in\mathbb{R}} \left| e^{(1/p)|t|^\alpha} f(t) \right|^p \right], \\
\sup_{t\in\mathbb{R}} \left| e^{|t|^\alpha} \eta(|f(t)|) \right| &\leq \sup \left[\left\{ \left| e^{|t|^\alpha - 1} \right| \ , \ |f(t)| \leq 1 \right\} \cup \left\{ \left| e^{|t|^\alpha} f(t)^2 \right| \ , \ |f(t)| \geq 1 \right\} \right].
\end{aligned}$$

In particular, $t \in \mathbb{R} \mapsto e^{c|t|^\alpha} f(t) \in \mathbb{C}$ is integrable (and hence summable for $t \in \mathbb{N}$).

Proof. The inequalities follow from elementary calculus. We show the integrability of $t \mapsto e^{|t|^\alpha} f(t)$. One has that $\exp(c|t|^\alpha) f(t) = (e^{-|t|^\beta} e^{|t|^\alpha})(e^{t^\beta} f(t))$. The second factor in the RHS is bounded, and the first is integrable for the following reason. Fix a value $t_0 > c^{1/(\beta-\alpha)}$. On $\{|t| < |t_0|\}$, the function $(e^{|t|^{-\beta}} e^{|t|^\alpha})$ is bounded. For $|t| \geq |t_0|$, then $(e^{|t|^{-\beta}} e^{|t|^\alpha}) = \exp(-|t|^\alpha (t^{(\beta-\alpha)} - c)) \leq \exp(-|t|^\alpha (t_0^{(\beta-\alpha)} - c))$, which is integrable with respect to t. ☐

Example. Let $f : t \in \mathbb{R} \mapsto f(t) := \exp^{-b|t|^\beta} \in \mathbb{R}$. Then, for all $\alpha \in (0, \beta)$ and all $a > 0$, f satisfies the inequality

$$\sup_{t\in\mathbb{R}} \left| e^{a|t|^\alpha} f(t) \right| \leq +\infty.$$

The reason is the following. Set $t_0 = (a/b)^{1/(\beta-\alpha)}$. Then $a|t|^\alpha \leq b|t|^\beta$ when $|t| > t_0$ and $a|t|^\alpha < at_0^\alpha$ when $|t| < t_0$, and hence for every $t \in \mathbb{R}$ it holds that $e^{a|t|^\alpha} \leq ce^{b|t|^\beta}$, where $log(c) = at_0^\alpha$. Thus, $|e^{a|t|^\alpha} f(t)| \leq c$.

Lemma 3.2. (cf. [BDF87, Lemma 2.3]). Let $(\mathfrak{A}, U, \Omega, \mathcal{H})$ be a Möbius covariant local net. For a parameter $p \in (0, 1)$, there is an energy function $f : t \in \mathbb{R} \mapsto f(t) \in \mathbb{R}$ with the following properties:

1. The function f is such that $f(0) = 1/2$, and such that $\sup_{t\in\mathbb{R}} \left| f(t) \exp(|t|^p) \right| < +\infty$.
2. If a, b are local operators such that $[e^{-iL_0\theta}(a)e^{iL_0\theta}, b] = 0$ whenever $|\theta| < 1$, then the vacuum expectation value $\omega(ab)$ can be expressed as:
$$\langle \Omega, ab\, \Omega \rangle = \langle \Omega, \left(a\, f(L_0)\, b + b\, f(L_0)\, a \right) \Omega \rangle.$$
3. Let f_δ be the δ-scaled f, that is, $f_\delta(t) := f(\delta t)$. Then, for any pair of local operators a, b such that $[e^{-iL_0\theta}(a)e^{iL_0\theta}, b] = 0$ holds whenever $|\theta| < \delta$ (*i.e.* the distance of their respective local algebras is larger than δ), the vacuum expectation value of ab can be expressed as:
$$\langle \Omega, ab\, \Omega \rangle = \langle \Omega, \left(a\, f_\delta(L_0)\, b + b\, f_\delta(L_0)\, a \right) \Omega \rangle.$$

Proof. **(1,2) Existence of the function.** For the sake of readability, we shall reproduce the proof of Lemma 2.3 in [BDF87], with the conformal Hamiltonian L_0 taking place instead of the Hamiltonian H. Consider any two local operators a, b satisfying the commutation rule $[e^{iL_0 t} a e^{-iL_0 t}, b] = 0$ whenever $t \in (-1, 1)$. This implies that

$$\langle \Omega, a e^{i\theta L_0} b\Omega \rangle = \langle \Omega, b e^{-i\theta L_0} a\Omega \rangle, \quad (\theta \in (-1,1)). \tag{3}$$

By the positivity of L_0, the LHS extends to $\{\theta \in \mathbb{C}, \Im(\theta) > 0\}$, and likewise, the RHS extends to $\{\theta \in \mathbb{C}, \Im(\theta) < 0\}$. Therefore, there is a holomorphic function h defined on $\mathcal{P} = \mathbb{C} \setminus \big((-\infty, -1] \cup [1, +\infty)\big)$ such that on $(-1, 1)$ it approaches the function expressed in (3).

Next, fix a constant $\tau \in (0, 1)$, and consider the conformal map that takes the disc $\mathbf{D} = \{w \in \mathbb{C}, |w| < 1\}$ onto $\mathcal{P}_\tau := \mathbb{C} \setminus ((-\infty, -\tau] \cup [\tau, +\infty))$, given by $z_\tau(w) = 2\tau w/(w^2 + 1)$, As $\mathcal{P}_\tau \subset \mathcal{P}$ for $\tau \in (0, 1)$, the function $h_\tau(w) := h(z_\tau(w))$ is holomorphic on $\mathbf{D}$, and it is easy to see that h_τ is continuous and bounded (by $\|a\| \cdot \|b\|$) on $\overline{\mathbf{D}} \setminus \{\pm 1\}$. Therefore, by integrating $\frac{1}{w} h_\tau(w)$ on a circular path $w(s) = re^{is} \in \mathbf{D}$ with a fixed radius $r < 1$ and parameter $s \in (0, 2\pi)$, one can invoke Cauchy's residue theorem and take $r \nearrow 1$ to get the following equality for all $\tau \in (0, 1)$:

$$\langle \Omega, ab\Omega \rangle = \frac{1}{2\pi} \int_{t=0}^{\pi} \big\langle \Omega, (a e^{iL_0 \tau/\cos(t)} b + b e^{iL_0 \tau/\cos(t)} a)\Omega \big\rangle \, dt. \tag{4}$$

For $p \in (0, 1)$ given as in the statement, fix a value q such that $p < q < 1$. There exists a function g such that $\tilde{g}$ is smooth and supported inside $(0, 1)$ and such that g decays as $e^{-|t|^q}$ for large $|t|$ (see [Jaf67] and references therein for the existence of functions of almost exponential decay with compact Fourier transform, and [Joh15] for more concrete functions with weaker requirements needed here). One can then multiply the above equality by $\tilde{g}(\tau)$ and integrate it against $d\tau$ to then obtain

$$\langle \Omega, ab\Omega \rangle = \big\langle \omega, (a f(L_0) b + b f(L_0) a)\Omega \big\rangle, \tag{5}$$

where f is defined by $f(\tau) = (2\pi g(0))^{-1} \int_0^\pi g\big(\tau/\cos(t)\big) \, dt$. Putting $\tau = 0$ shows that $f(0) = 1/2$. Furthermore, f inherits the decay property from g.

Until now, f is a complex-valued function. Yet, it is immediate that the function $\bar{f}(t) = \overline{f(t)}$ has the same property:

$$\begin{aligned}
\big\langle \Omega, (a\bar{f}(L_0)b + b\bar{f}(L_0)a)\Omega \big\rangle &= \big\langle (b^* f(L_0) a^* + a^* f(L_0) b^*)\Omega, \Omega \big\rangle \\
&= \overline{\big\langle \Omega, (b^* f(L_0) a^* + a^* f(L_0) b^*)\Omega \big\rangle} \\
&= \overline{\big\langle \Omega, (a^* b^*)\Omega \big\rangle} \\
&= \big\langle \Omega, ab\Omega \big\rangle.
\end{aligned}$$

Therefore, the real part of f does the same job. In the following, we assume that f is real. This concludes the proof of items 1 and 2.

(3)Scaling. Now, considering the parameter δ, consider two local operators a, b satisfying the commutation rule $[e^{iL_0 t} a e^{-iL_0 t}, b] = 0$ whenever $t \in (-\delta, \delta)$. The previous discussion follows analogously, except that the equality given by equation (4) holds only for $\tau \in (-\delta, \delta)$. In following, using $\tilde{g}(\tau/\delta)$ instead of $\tilde{g}$ results in the equality

$$\langle \Omega, ab\Omega \rangle = \big\langle \omega, (a f_\delta(L_0) b + b f_\delta(L_0) a)\Omega \big\rangle, \tag{6}$$

where now $f_\delta(\tau) = f(\delta\tau)$, thus proving the item 3.

This concludes the proof of the lemma. □

3.2 Definitions

In this subsection, we go through three incremental steps into defining the entanglement entropy H_I^E with conformal energy cutoff at a value $E > 0$.

First, regarding the von Neumann entropy (as introduced in Subsection 2.1), we make an observation which will stand as motivation of our definition.

Remark. Given two normal states ϕ, ψ on $\mathcal{B}(\mathcal{H})$, we say $\phi \succeq \psi$ if there is a positive number $t > 0$ such that $t\phi \geq \psi$, and equivalently, if there is a positive number $\lambda \in (0,1]$ such that $\phi \geq \lambda\psi$ (here, $t = 1/\lambda$). The concavity of the von Neumann entropy asserts that $S_{\mathrm{vN}}(\phi) \geq \lambda S_{\mathrm{vN}}(\psi)$. We therefore have

$$S_{\mathrm{vN}}(\psi) = \inf_{\phi} \frac{1}{\lambda_\phi} S_{\mathrm{vN}}(\phi),$$

where the infimum considers all states ϕ to which there is a positive parameter $\lambda_\phi \in (0,1]$ such that $\phi \geq \lambda_\phi \psi$. Clearly, equality holds since $\psi \succeq \psi$ with $\lambda_\psi = 1$.

Turning back to the framework AQFT, we recall the split property (Definition 2.16) for the Möbius covariant local net $(\mathfrak{A}, U, \Omega, \mathcal{H})$. Consider an interval $I \in \mathcal{J}$ and a positive parameter $\delta > 0$. Let $I_\delta = \cup_{-\delta<\theta<\delta}\, \rho_\theta(I)$ be the the "augmentation of I by δ" (we suppose that δ is sufficiently small so that $I_\delta \in \mathcal{J}$, since otherwise I_δ might cover all S^1). It then holds that $I \Subset I_\delta$ as stated before (since $\rho_t(I) \subset I_2$ holds for all t such that $|t| < \delta$). By the split property, there are intermediate pairs $(u, \mathfrak{R}_u)$ as in Definition 2.17, *viz.* $u : \mathcal{H} \to \mathcal{H} \otimes \mathcal{H}$ is unitary and such that $u(xy)u^* = x \otimes y$ for any pair $(x, y) \in \mathfrak{A}(I) \times \mathfrak{A}(I_\delta)'$, and $\mathfrak{R}_u = u^*(\mathcal{B}(\mathcal{H}) \otimes 1)u)$ is an intermediate type I factor.

Also, for $x \in \mathcal{B}(\mathcal{H})$, one has that $x \otimes 1_{\mathcal{H}} \in \mathcal{B}(\mathcal{H}) \otimes 1_{\mathcal{H}}$, and hence $u^*(x \otimes 1_{\mathcal{H}})u \in \mathfrak{R}_u$. This gives an identification of operators in $\mathcal{H}$ to elements in $\mathfrak{R}_u$.

Now we address the entropy of a state ω with respect to I and I'_δ. Since the von Neumann entropy is well defined in the intermediate type I factors $\mathfrak{R}_u$ (denoted accordingly as $S_{\mathfrak{R}_u}$, see Definition 2.7), one can use the following definition of entropy of entanglement of a state ω between two algebras $\mathfrak{A}(I)$ and $\mathfrak{A}(I_\delta)'$:

Definition 3.3. Consider ψ a state on $\mathfrak{A}(S^1) = \mathcal{B}(\mathcal{H})$. For $I \in \mathcal{J}$ and $\delta > 0$ (such that $I_\delta \in \mathcal{J}$), we define the entropy $H_{I,\delta}$ as

$$H_{I,\delta}(\psi) := \inf_{(u,\mathfrak{R}_u)} \inf_{\phi} \frac{1}{\lambda_\phi} S_{\mathfrak{R}_u}(\phi),$$

Here, the first infimum takes into account all intermediate pairs $(u, \mathfrak{R}_u)$ as in Definition 2.17. The second infimum considers all normal states ϕ over $\mathfrak{A}(S^1) = \mathcal{B}(\mathcal{H})$ to which there is a positive number $\lambda_\phi \in (0,1]$ such that $\phi \geq \lambda_\phi \psi$ holds when restricted to $\mathfrak{A}(I) \vee \mathfrak{A}(I_\delta)'$.

Proposition 3.4. Suppose $(\mathfrak{A}, U, \Omega, \mathcal{H})$ is a Möbius covariant net satisfying the extra assumption 2.18(b), *i.e.* $\dim\ker(L_0 - N) \leq Ce^{N^\kappa}$ for some $C > 0$ and $\kappa \in (0,1)$. Then, entropy $H_{I,\delta}(\omega)$ is finite. In particular,

$$H_{I,\delta}(\omega) \leq S_\delta - \eta(C_\delta) < +\infty,$$

where C_δ and S_δ are given by

$$C_\delta = \sum_{N\geq 0} 2\dim\ker(L_0 - N)|f_\delta(N)|,$$

$$S_\delta = \sum_{N>0} 4\dim\ker(L_0 - N)\Big(\eta(|f_\delta(N)/2|)\Big),$$

with f the energy function as in Lemma 3.2, satisfying $\sup_{t\in\mathbb{R}} |e^{|t|^q} f(t)| < +\infty$ for some $q \in (\kappa, 1)$.

This will be proved by the end of Subsection 3.4.

We now consider a conformal energy cutoff parameter. This is done since the above quantity is expected to diverge when the spatial separation δ, taken as a variable parameter, approaches zero. We first define the regularization of states, and with those, we define the regularized entropy.

For any $E > 0$, let $P_E := \chi_{[0,E]}(L_0)$ denote the spectral projection of the conformal Hamiltonian L_0 with respect to the set $\{0, 1, 2, \dots, E\}$. The set $\{P_E\}_{E>0}$ is then an increasing family of projections (acting on $\mathcal{H}$) indexed by a parameter $E > 0$, such that P_E strongly converges to the identity as E goes to infinity.

Definition 3.5. Let ϕ be a normal positive functional on $\mathfrak{A}(S^1) = \mathcal{B}(\mathcal{H})$, and let $(u, \mathfrak{R}_u)$ be an intermediate pair as in Definition 2.17. For $E > 0$, the regularized functional $\phi^{E,u}$ is defined as

$$\phi^{E,u} := x \in \mathcal{B}(\mathcal{H}) \mapsto \phi\big((u^*(P_E \otimes 1)u)\, x\, (u^*(P_E \otimes 1)u)\big) \in \mathbb{C}.$$

For φ a normal positive functional on $\mathcal{B}(\mathcal{H} \otimes \mathcal{H})$ (*e.g.* for ϕ a normal state on $\mathcal{B}(\mathcal{H})$ as above, and $\varphi = \phi \circ \mathrm{Ad}_u$), the regularized functional φ^E (here independent of $(u, \mathfrak{R}_u)$) is defined as

$$\varphi^E := x \in \mathcal{B}(\mathcal{H} \otimes \mathcal{H}) \mapsto \varphi\big((P_E \otimes 1)\, x\, (P_E \otimes 1)\big) \in \mathbb{C}.$$

Remark. For a fixed normal state ϕ and fixed intermediate pair $(u, \mathfrak{R}_u)$, the regularized functionals $\phi^{E,u}$ are normal positive contractions, and $\phi^{E,u}/\|\phi^{E,u}\|$ are again normal states. As ϕ is normal, both $\phi^{E,u}$ and $\phi^{E,u}/\|\phi^{E,u}\|$ converge, as $E \to +\infty$, to the original state ϕ in the weak* topology. Indeed, the projections $u^*(P_E \otimes 1_{\mathcal{H}})u$ form an increasing family converging to the identity, and hence by normality, $\phi^{E,u}(x) \to \phi(x)$ for all positive operators $x \in \mathcal{B}(\mathcal{H})_+$, and hence for any operator $x \in \mathcal{B}(\mathcal{H})$. The same reasoning holds analogously for φ a normal state on $\mathcal{B}(\mathcal{H} \otimes \mathcal{H})$. In the case of $\varphi = \phi \circ \mathrm{Ad}_u$, the restriction of $\phi^{E,u}$ to $\mathfrak{R}_u$ "corresponds" to the restriction of φ^E to the first tensor component (denoted as $(\varphi^E)_1$), which in turns is equal to $\varphi_1(P_E \cdot P_E)$. The last converges to φ_1 in the weak* topology, as $E \to +\infty$.

Definition 3.6. Consider ψ a state on $\mathfrak{A}(S^1) = \mathcal{B}(\mathcal{H})$. For $I \in \mathcal{J}$, $\delta > 0$ (with $I_\delta \in \mathcal{J}$) and $E > 0$, the regularized entropy $H^E_{I,\delta}$ of ψ is defined by

$$H^E_{I,\delta}(\psi) := \inf_{(u,\mathfrak{R}_u)} \inf_{\phi} \frac{1}{\lambda_\phi} S_{\mathfrak{R}_u}\big(\phi^{E,u}/\|\phi^{E,u}\|\big).$$

Here, the first infimum takes into account all intermediate pairs $(u, \mathfrak{R}_u)$ as in Definition 2.17. The second infimum considers all normal states ϕ over $\mathcal{B}(\mathcal{H})$ to which there is a parameter $\lambda_\phi \in (0,1]$ such that $\phi^{E,u} \geq \lambda_\phi\, \psi^{E,u}$ holds when restricted to $\mathfrak{A}(I) \vee \mathfrak{A}(I_\delta)'$.

Proposition 3.7. For a Möbius covariant net with the split property (extra assumption 2.18(f)), the entropy $H^E_{I,\delta}(\omega)$ is finite, and independent of δ. In particular,

$$H^E_{I,\delta}(\omega) \leq S_E - \eta(C_E) < +\infty,$$

where C_E and S_E are given by

$$C_E = 2 \sup_{t \geq 0}\{\, |f(t)|\, \} \sum_{N=0}^{E} \dim\ker(L_0 - N),$$

$$S_E = 4 \sup_{t \geq 0}\left\{ \Big|\eta\big(|f(t)|\big)\Big| \right\} \sum_{N=1}^{E} \dim\ker(L_0 - N),$$

with f an energy function as in Lemma 3.2.

This will be proved in Subsection 3.5

Finally, we consider the regularized entanglement entropy as a limit of the above when taking δ as a parameter approaching zero (and hence I_δ approaching I).

Definition 3.8. Consider ψ a state on $\mathfrak{A}(S^1) = \mathcal{B}(\mathcal{H})$. For $I \in \mathcal{J}$ and a cutoff parameter $E > 0$, the regularized entanglement entropy H^E_I of ψ with respect to the interval I and with cutoff E is defined as:

$$H^E_I(\psi) := \lim_{\delta \searrow 0} H^E_{I,\delta}(\psi)$$

where $H^E_{I,\delta}$ is as in Definition 3.6.

Theorem 3.9. For a Möbius covariant net with the split property (extra assumption 2.18(f)), the entropy $H_I^E(\omega)$ is finite. In particular,

$$H_I^E(\omega) \leq S_E - \eta(C_E) < +\infty,$$

where C_E and S_E are given by

$$C_E = 2 \sup_{t \geq 0} \{ \, |f(t)| \, \} \sum_{N=0}^{E} \dim \ker(L_0 - N),$$

$$S_E = 4 \sup_{t \geq 0} \left\{ \left| \eta\big(|f(t)|\big) \right| \right\} \sum_{N=1}^{E} \dim \ker(L_0 - N),$$

with f an energy function as in Lemma 3.2.

The theorem is actually a mere corollary of Proposition 3.7, but is indeed the main result of this work.

In the following, we shall utilize an auxiliary functional $\theta_{\delta,+}$ and prove the finiteness of the regularized entanglement entropy defined above.

3.3 Calculations

On the upcoming, we shall work incrementally throughout the proofs of Propositions 3.4, 3.7 and Theorem 3.9.

The definitions 3.3, 3.6 and 3.8 all start with a fixed interval $I \in \mathcal{J}$, and for a given parameter $\delta > 0$ (sufficiently small), an augmented interval $I_\delta = \cup_{-\delta<\theta<\delta}\, \rho_\theta(I)$. Also, they all rely on intermediate pairs $(u, \mathfrak{R}_u)$ and the von Neumann entropy of states ϕ restricted to $\mathfrak{R}_u$. The calculations are, however, done with aid of the unitary u, considering $\phi \circ \mathrm{Ad}_{u^*}$ as a state on $\mathcal{H} \otimes \mathcal{H}$ and restricting it to the first tensor component. We state this fact in the following lemma.

Lemma 3.10. Consider ψ a normal state in $\mathcal{B}(\mathcal{H})$. For $\delta > 0$ fixed, let $(u, \mathfrak{R}_u)$ be an intermediate pair as in Definition 2.17. Let φ be a normal positive functional on $\mathcal{B}(\mathcal{H} \otimes \mathcal{H})$ such that $\varphi \circ \mathrm{Ad}_{u^*} \geq \psi$ on $\mathfrak{A}(I) \vee \mathfrak{A}(I_\delta)'$. Then,

$$\|\varphi\| S_1\big(\varphi/\|\varphi\|\big) \geq H_{I,\delta}(\psi) \qquad \text{and} \qquad \|\varphi^E\| S_1\big(\varphi^E/\|\varphi^E\|\big) \geq H_{I,\delta}^E(\psi),$$

where E is any cutoff value, and S_1 is the von Neumann entropy of a state in $\mathcal{B}(\mathcal{H}) \otimes \mathcal{B}(\mathcal{H})$ restricted to the first tensor component.

Proof. Call $\tilde{\varphi} = \varphi \circ \mathrm{Ad}_{u^*}$ and $\varphi_1 = \varphi(\,\cdot \otimes 1)$. Then, with $\phi = \tilde{\varphi}/\|\tilde{\varphi}\| \geq \lambda_\phi \psi$, with $\lambda_\phi = 1/\|\tilde{\varphi}\| = 1/\|\varphi\|$. From definitions 3.3 and 3.6, we have $\|\varphi\| S_{\mathfrak{R}_u}(\tilde{\varphi}/\|\tilde{\varphi}\|) \geq H_{I,\delta}(\psi)$, and likewise for the regularized entropies. It suffices to show that $S_{\mathfrak{R}_u}(\tilde{\varphi}) = S_1(\varphi) = S_{\mathrm{vN}}(\varphi_1)$, and by Definition 2.7, it suffices to show that there is an $*$-isomorphism $f : \mathfrak{R}_u \to \mathcal{B}(\mathcal{H})$ such that $\tilde{\varphi} = \varphi_1 \circ f$. If f is defined by $f^{-1} : x \in \mathcal{B}(\mathcal{H}) \mapsto u^*(x \otimes 1)u \in \mathfrak{R}_u$, then it satisfies the requirement. This proves the lemma. □

With the above lemma stated, we now expose our strategy for our proofs.

Suppose the net $(\mathfrak{A}, U, \Omega, \mathcal{H})$ satisfies extra assumption 2.18(a) with $\dim \ker(L_0 - N) \leq Ce^{N^\kappa}$ for some $C > 0$ and $\kappa \in (0,1)$. Fix $\tilde{\kappa} \in (\kappa, 1)$.

The split property guarantees the existence of a unitary operator $u_\delta : \mathcal{H} \to \mathcal{H} \otimes \mathcal{H}$ intertwining $\mathfrak{A}(I) \vee \mathfrak{A}(I_\delta)'$ and $\mathfrak{A}(I) \bar{\otimes} \mathfrak{A}(I_\delta)'$. Also, there is an energy function f (and scaled functions f_δ) such as in Lemma 3.2, with decay property $\sup_{t\in\mathbb{R}} |e^{|t|^{\tilde{\kappa}}} f(t)| < +\infty$. Recall that the following holds:

$$\omega(ab) = \omega(\, a f_\delta(L_0) b + b f_\delta(L_0) a \,) \qquad (\; a \in \mathfrak{A}(I), b \in \mathfrak{A}(I_\delta)' \;)$$

Define θ_δ as the self-adjoint linear functional on $\mathcal{B}(\mathcal{H}\otimes\mathcal{H})$ given by the following formula:

$$\theta_\delta(a\otimes b) := \omega(af_\delta(L_0)b + bf_\delta(L_0)a) \tag{7}$$

The vacuum state ω and the functional $\theta_\delta \circ \mathrm{Ad}_{u_\delta^*}$ coincide in $\mathfrak{A}(I)\vee\otimes\mathfrak{A}(I_\delta)'$, so one might be tempted to invoke Lemma 3.10 and state that $H_{I,\delta}(\omega) \leq S_1(\theta_\delta)$. However, one should notice that θ_δ is only positive when restricted to the above-mentioned algebra. Nevertheless, one can decompose it as $\theta_\delta = \theta_{\delta,+} - \theta_{\delta,-}$, where $\theta_{\delta,\pm}$ are positive functionals on $\mathcal{B}(\mathcal{H}\otimes\mathcal{H})$ (this will *not* be the Jordan decomposition, the detailed construction of $\theta_{\delta,\pm}$ will be explained on the following section, in particular on equation (12)). Hence, restricted to $\mathfrak{A}(I)\otimes\mathfrak{A}(I_\delta)'$, one has $\theta_{\delta,+} = \theta_\delta + \theta_{\delta,-} = \omega\circ\mathrm{Ad}_{u_\delta^*} + \theta_{\delta,-}$, and therefore, after normalizing it to $\hat{\theta}_{\delta,\pm} = \theta_{\delta,\pm}/\|\theta_{\delta,\pm}\|$, one has that $\hat{\theta}_{\delta,+}\circ\mathrm{Ad}_{u_\delta} \geq (1/\|\theta_{\delta,+}\|)\omega$. By Lemma 3.10, one has the inequality

$$H_{I,\delta}(\omega) \leq \|\theta_{\delta,+}\| S_{\mathfrak{R}_u}(\hat{\theta}_{\delta,+}\circ Ad_{u^*}) = \|\theta_{\delta,+}\| S_1(\hat{\theta}_{\delta,+}). \tag{8}$$

It suffices then to identify the positive functional $\theta_{\delta,+}$ and find and an upper bound for the entropy $S_1(\hat{\theta}_{\delta,+})$. This will be addressed on the following.

3.4 The auxiliary functional $\theta_{\delta,+}$ and its entropy $S_1(\theta_{\delta,+})$

We first further analyze the properties of θ_δ to appropriately define a decomposition $\theta_\delta = \theta_{\delta,+} - \theta_{\delta,-}$ The conformal Hamiltonian L_0 has discrete eigenvalues $N\in\mathbb{N}$ with eigenspaces $\mathcal{H}_N = \ker(L_0 - N)$ of finite dimension $\dim(\mathcal{H}_N) = P(N)$. Let $\{\Phi_n\}_n$ be a normalized basis of eigenvectors with eigenvalues $l_n\in\mathbb{N}$. Then, following the definition of θ_δ as in Equation (7), one has

$$\theta_\delta(x\otimes y) = \sum_{n\leq 0} f_\delta(l_n)\Big(\langle\Omega, x\Phi_n\rangle\langle\Phi_n, y\Omega\rangle + \langle\Omega, y\Phi_n\rangle\langle\Phi_n, x\Omega\rangle\Big). \tag{9}$$

We proceed by decomposing the terms $\langle\Omega, x\Phi_n\rangle\langle\Phi_n, y\Omega\rangle + \langle\Omega, y\Phi_n\rangle\langle\Phi_n, x\Omega\rangle$ as a linear combination of positive terms. First, we notice that the n-sum has a special value at $n = 0$, which account for the state $\omega\otimes\omega$, with multiplicity one since $f_\delta(0) = 1/2$. We therefore focus on the terms corresponding to $n > 0$.

We introduce $\phi_{l,n}$ as pure states on $\mathcal{H}$, defined as the following for $l\in\mathbb{Z}$ and $n > 0$:

$$\phi_{l,n} := \Big\langle \frac{(\Omega + i^l\Phi_n)}{\|\Omega + i^l\Phi_n\|},\ \cdot\ \frac{(\Omega + i^l\Phi_n)}{\|\Omega + i^l\Phi_n\|}\Big\rangle = \frac{1}{2}\big\langle(\Omega + i^l\Phi_n),\ \cdot\ (\Omega + i^l\Phi_n)\big\rangle. \tag{10}$$

where the second equality follows since Ω and Φ_n are orthogonal to each other.

Standard algebraic manipulations show that the following polarizations hold for $n > 0$:

$$\langle\Omega, x\Phi_n\rangle = \sum_{l=0}^{3}\frac{i^{-l}}{2}\phi_{l,n}(x) \qquad\text{and}\qquad \langle\Phi_n, y\Omega\rangle = \sum_{k=0}^{3}\frac{i^m}{2}\phi_{k,n}(y)$$

For $n > 0$, the terms $\langle\Omega, x\Phi_n\rangle\langle\Phi_n, y\Omega\rangle$ and $\langle\Omega, y\Phi_n\rangle\langle\Phi_n, x\Omega\rangle$ appearing in θ_δ can then be written as a linear sum of positive functionals as follows:

$$\langle\Omega, x\Phi_n\rangle\langle\Phi_n, y\Omega\rangle = \frac{1}{4}\sum_{l=0}^{3}\phi_{l,n}(x)\cdot\Big(\phi_{l,n}(y) - \phi_{l+2,n}(y) + i\phi_{l+1,n}(y) - i\phi_{l-1,n}(y)\Big)$$
$$\langle\Omega, y\Phi_n\rangle\langle\Phi_n, x\Omega\rangle = \frac{1}{4}\sum_{l=0}^{3}\phi_{l,n}(x)\cdot\Big(\phi_{l,n}(y) - \phi_{l+2,n}(y) - i\phi_{l+1,n}(y) + i\phi_{l-1,n}(y)\Big)$$

And hence:

$$\langle\Omega, x\Phi_n\rangle\langle\Phi_n, y\Omega\rangle + \langle\Omega, y\Phi_n\rangle\langle\Phi_n, x\Omega\rangle = \frac{1}{2}\sum_{l=0}^{3}\phi_{l,n}(x)\cdot\Big(\phi_{l,n}(y) - \phi_{l+2,n}(y)\Big)$$

In Equation (9), the above terms show up in θ_δ multiplied by $f_\delta(l_n)$. Aside from the value $l_n = 0$, for which we know $f_\delta(0) = 1/2$, each $f_\delta(l_n)$ might be positive or negative (as we noted, we can and do take a real f). We then just need to be cautious about the sign of $f_\delta(l_n)$. Thus, we define:

$$a_\delta(k) := \begin{cases} 1 & \text{if } f_\delta(k) > 0 \\ 0 & \text{otherwise} \end{cases} \qquad b_\delta(k) := \begin{cases} 1 & \text{if } f_\delta(k) < 0 \\ 0 & \text{otherwise} \end{cases} \tag{11}$$

Then, for each n at most one of the indices $a_\delta(l_n)$ and $b_\delta(l_n)$ is 1, and it holds that $f_\delta(k) = (a_\delta(k) - b_\delta(k)) \cdot \big|f_\delta(k)\big|$. Summing all terms $f_\delta(l_n)\Big(\langle\Omega, x\Phi_n\rangle\langle\Phi_n, y\Omega\rangle + \langle\Omega, y\Phi_n\rangle\langle\Phi_n, x\Omega\rangle\Big)$, we obtain

$$\begin{aligned}\theta_\delta = {} & \underbrace{\omega\otimes\omega + \sum_{n>0}\sum_{l=0}^{3}\frac{|f_\delta(l_n)|}{2}\,\phi_{l,n}\otimes\Big(a_\delta(l_n)\phi_{l,n} + b_\delta(l_n)\phi_{l+2,n}\Big)}_{=:\,\theta_{\delta,+}} \\ & - \underbrace{\sum_{n>0}\sum_{l=0}^{3}\frac{|f_\delta(l_n)|}{2}\,\phi_{l,n}\otimes\Big(a_\delta(l_n)\phi_{l+2,n} + b_\delta(l_n)\phi_{l,n}\Big)}_{=:\,\theta_{\delta,-}}\end{aligned}$$

Hence we get the desired decomposition $\theta_\delta = \theta_{\delta,+} - \theta_{\delta,-}$ with $\theta_{\delta,\pm}$ defined as

$$\begin{aligned}\theta_{\delta,+} &:= \omega\otimes\omega + \sum_{n>0}\sum_{l=0}^{3}\frac{|f_\delta(l_n)|}{2}\,\phi_{l,n}\otimes\Big(a_\delta(l_n)\phi_{l,n} + b_\delta(l_n)\phi_{l+2,n}\Big). \\ \theta_{\delta,-} &:= \sum_{n>0}\sum_{l=0}^{3}\frac{|f_\delta(l_n)|}{2}\,\phi_{l,n}\otimes\Big(a_\delta(l_n)\phi_{l+2,n} + b_\delta(l_n)\phi_{l,n}\Big).\end{aligned} \tag{12}$$

With the definition of equation (12) in hands, we now focus on estimating the entropy $S_1(\theta_{\delta,+}/\|\theta_{\delta,+}\|)$.

Define τ_δ as the positive functional $\theta_{\delta,+}$ restricted to the first tensor component. It is then expressed as follows:

$$\tau_\delta(x) := \theta_{\delta,+}(x\otimes 1) = \omega + \sum_{n>0}\sum_{l=0}^{3}\frac{|f_\delta(l_n)|}{2}\,\phi_{l,n}(x) \qquad (\,x\in\mathcal{B}(\mathcal{H})\,). \tag{13}$$

This decomposition of τ_δ into pure states $\phi_{k,n}$ is indeed convergent in norm, because we assume that the net satisfies the condition 2.18(b), that is, $\dim(\mathcal{H}_N)$ grows bounded by an almost exponential function $C\exp(N^\kappa)$ for some $C > 0$ and $\kappa \in (0, 1)$.

The above "positive decomposition in pure states" enables us to invoke Corollary 2.6. We first notice that since the operator $\theta_{\delta,+}$ is not normalized, so is τ_δ not normalized. One can calculate its norm $C_\delta := \|\tau_\delta\| = \|\theta_{\delta,+}\| = \theta_{\delta,+}(1\otimes 1)$ in terms of f_δ and $\dim(\mathcal{H}_N)$. Likewise, it will be useful to give an upper bound S_δ for the "non-normalized entropy $S_{\mathrm{vN}}(\tau)$" in terms of the same quantities. We have:

$$C_\delta = \sum_{N\geq 0} 2\dim(\mathcal{H}_N)\,|f_\delta(N)|, \tag{14}$$

$$S_\delta = \sum_{N>0} 4\dim(\mathcal{H}_N)\,\eta\Big(\,|f_\delta(N)/2|\,\Big). \tag{15}$$

In the above, every factor $|f_\delta(N)|/2$ appears $4\times\dim(\mathcal{H}_N)$ times, where the factor 4 is because of the sum in l. The sum on S_δ on Equation (15) has no term $N = 0$, for it can be dropped because it corresponds to ω with multiplicity one.

Hence, we have the upper bound for the normalized entropy $S(\tau_\delta/\|\tau_\delta\|)$, and therefore for $H_{I,\delta}(\omega)$ as follows:

$$H_{I,\delta}(\omega) \le \|\theta_{\delta,+}\| S_1\left(\frac{\theta_{\delta,+}}{\|\theta_{\delta,+}\|}\right) = \|\tau_\delta\| S_{\mathrm{vN}}\left(\frac{\tau_\delta}{\|\tau_\delta\|}\right) \le S_\delta - \eta(C_\delta).$$

We make a remark on the convergence of above formulae determining C_δ and S_δ. By the hypotheses of Proposition 3.4, we assume that the net satisfies the extra assumption 2.18(b), that is, $\dim(\mathcal{H}_N)$ grows bounded by an almost exponential function $C\exp(N^\kappa)$ for some $C > 0$ and $\kappa \in (0,1)$. Recall that we chose f with the decay property $\sup_{t\in\mathbb{R}} |e^{|t|^{\tilde{\kappa}}} f(t)| < +\infty$, where $0 < \kappa < \tilde{\kappa} < 1$. Both $|f|$ and $\eta \circ |f|$ then work as damping factors, resulting in summable expressions (see Lemma 3.1). This entails the finiteness of $S_1(\theta_{\delta,+}/\|\theta_{\delta,+}\|)$, thus proving Proposition 3.4.

Notice that the used arguments were for a given fixed distance δ between the intervals I and I'_δ. As δ approaches zero, the damping factor $f_\delta(N)$ approaches a constant, and hence the values of c_E and $S_1(\theta_{\delta,+})$ diverge. For controlling such quantity as δ goes to zero, one then needs to implement a cutoff, which will be introduced in the following.

3.5 Implementing the cutoff

We now implement an cutoff based on the conformal Hamiltonian. Let the "conformal energy cutoff" be denoted by E, a positive integer. As before, let $P_E := \chi_{[0,E]}(L_0)$ be the spectral projection of L_0 with respect to the set $\{0, 1, 2, \dots, E\}$.

Recall the functionals $\theta_{\delta,+}$ and τ_δ defined by equations (12) and (13), respectively. The functional $\theta^E_{\delta,+} = \theta_{\delta,+}\big((P_E \otimes 1) \cdot (P_E \otimes 1)\big)$, regularized as in Definition 3.5, is a normal positive functional which converges $\theta_{\delta,+}$ in the weak* topology, as $E \to \infty$. Also, its restriction to the first tensor component is just $\tau_\delta\big(P_E \cdot P_E\big)$, which we denote by $\tau_{\delta,E}$. Whereas $\theta_{\delta,+}$ and τ_δ are only guaranteed to be well defined if the net satisfies condition 2.18(b), the regularized functionals $\theta^E_{\delta,+}$ and $\tau_{\delta,E}$ are well defined even if the net only satisfied the split property. Furthermore, $\theta^E_{\delta,+}$ is stable with respect to regularization, *i.e.* $(\theta^E_{\delta,+})^E = \theta^E_{\delta,+}$. And since $\theta^E_{\delta,+} \circ \mathrm{Ad}_u \ge \omega^E$, by the Lemma 3.10, it follows that $\|\tau_{\delta,E}\| S_{\mathrm{vN}}(\tau_{\delta,E}/\tau_{\delta,E})$ is an upper bound for the regularized entropy given by Definition 3.6. By these reasons, from hereon we drop the requirement of condition 2.18(b), and only require the net to satisfy the split property.

We now focus on estimating $S_{\mathrm{vN}}(\tau_{\delta,E}/\|\tau_{\delta,E}\|)$. From $\tau_\delta = \theta_{+,1}(\cdot \otimes 1)$ as expressed in equation (13) and considering that all Φ_n are eigenvectors of L_0 with eigenvalue l_n, the only non-vanishing terms of $\tau_{\delta,E}$ are those corresponding to $\phi_{l,n}$ such that $l_n \le E$. One then has, after the cutoff

$$\tau_{\delta,E}(x) = \omega + \sum_{n>0}^{l_n \le E} \sum_{l=0}^{3} \frac{|f_\delta(l_n)|}{2}\, \phi_{l,n}(x) \qquad (\, x \in \mathcal{B}(\mathcal{H})\,).$$

The formula above allows us to use Corollary 2.6, and therefore, the entropy $S_{\mathrm{vN}}(\tau^E/\|\tau^E\|)$ of the normalized state can be estimated by the following:

$$S_{\mathrm{vN}}\Big(\frac{\tau_{\delta,E}}{\|\tau_{\delta,E}\|}\Big) \le \log(c_{\delta,E}) + \frac{1}{c_{\delta,E}} S_{\delta,E},$$

where $c_{\delta,E}$ and $S_{\delta,E}$ are respectively the norm and the "non normalized entropy" defined by

$$c_{\delta,E} := \|\tau_{\delta,E}\| = \sum_{N=0}^{E} 2\dim(\mathcal{H}_N)\,|f_\delta(N)|$$

$$S_{\delta,E} := \sum_{N=1}^{E} 4\dim(\mathcal{H}_N)\,\eta\Big(|f_\delta(N)|/2\Big)$$

As currently presented, the upper bound for $S_{\mathrm{vN}}(\tau_{\delta,E}/\|\tau_{\delta,E}\|)$ still depends on δ. However, $c_{\delta,E}$ and $S_{\delta,E}$ can be respectively bounded by constants C_E and S_E that are independent of

δ, given by the following:

$$C_E := 2\,\|f\|_\infty \sum_{N=0}^{E} \dim(\mathcal{H}_N), \tag{16}$$

$$S_E := 4\,\|\eta\circ f\|_\infty \sum_{N=1}^{E} \dim(\mathcal{H}_N). \tag{17}$$

Therefore, using $1 < c_{\delta,E} < C_E$, one has $\log c_{\delta,E} \le \log C_E$, and also $S_{\delta,E} \le S_E$. Joining those on the bound of $S_{\mathrm{vN}}(\tau_{\delta,E}/\|\tau_{\delta,E}\|)$, one finally has:

$$S_{\mathrm{vN}}\Big(\frac{\tau_{\delta,E}}{\|\tau_{\delta,E}\|}\Big) \le \log C_E + \frac{S_E}{c_{\delta,E}} \le +\infty.$$

Hence, by Lemma 3.10, the entropy $H_{I,\delta}(\omega)$ is bounded by

$$H^E_{I,\delta}(\omega) \le S_E - \eta(C_E).$$

The upper bound $S_E - \eta(C_E)$ above is finite and independent from δ, thus proving Proposition 3.7. Taking δ vanishing as in Definition 3.8, the same result can be interpreted as an upper bound for the entanglement entropy (with cutoff E) of ω between $\mathfrak{A}(I)$ and its commutant. We finally achieve our desired estimate:

$$H^E_I(\omega) \le S_E - \eta(C_E). \tag{18}$$

This proves Theorem 3.9. We have thus established the finiteness of the regularized entanglement entropy for a Möbius covariant local net satisfying the split property, that is, a relativistic chiral component of a quantum field in the algebraic setting.

Remark. In addition to Theorem 3.9, suppose that the net satisfies the extra assumption 2.18(a), *viz.* $\dim(\mathcal{H}_N) = p(N)$. Asymptotically as $N \to +\infty$, the partition function behaves like $\sim \frac{1}{4N\sqrt{3}} e^{\pi\sqrt{2/3}\,\sqrt{N}}$ [AS64, 24.2.1.III]. Then, the $\sum_{N=0}^{E}\dim(\mathcal{H}_N)$ is asymptotically bounded by $(12)^{-1/2}\mathrm{Ei}(\pi\sqrt{2/3}\,\sqrt{E})$, where Ei is the exponential integral function [AS64, 5.1.2]. Therefore:

$$H^E_I(\omega) \lesssim -\eta\Big((1/\sqrt{3})\|f\|_\infty \mathrm{Ei}(\pi\sqrt{2/3}\,\sqrt{E})\Big) + (2/\sqrt{3})\|\eta\circ f\|_\infty \mathrm{Ei}(\pi\sqrt{2/3}\,\sqrt{E}).$$

A much more palatable estimate can be done considering extra assumption 2.18(b), by considering $\dim(\mathcal{H}_N) \le Ce^{N^\kappa} \le Ce^N$, since $\kappa < 1$. Then, $\sum_{N=0}^{N}\dim(\mathcal{H}_N) \sim e^E$ which in turns implies that $H^E_I(\omega) \sim Ee^E$.

We finish this section with a recapitulation of the proofs of Proposition 3.4, Proposition 3.7, and Theorem 3.9.

Proof (Summary of proofs). Let $(u_\delta, \mathfrak{R}_{u_\delta})$ be an intermediate pair of the split inclusion $\mathfrak{A}(I) \subset \mathfrak{A}(I_\delta)'$. Let f (and scaled function f_δ) be the energy function such as in Lemma 3.2 and as we remarked, we may assume f is real. Define θ_δ as the self-adjoint linear functional on $\mathcal{B}(\mathcal{H}\otimes\mathcal{H})$ by Equation (7), *viz.* $\theta_\delta(x\otimes y) := \omega(xf_\delta(L_0)y + yf_\delta(L_0)x)$.

Define $\phi_{l,n}$ as in Equation (10), a_δ and b_δ as in Equation (11), and $\theta_\pm$ as in Equations (12). Then $\theta_{\delta,\pm}$ are positive functionals on $\mathcal{H}\otimes\mathcal{H}$ and $\theta_\delta = \theta_{\delta,+} - \theta_{\delta,-}$. Hence $\theta_{\delta,+}$ is a normal positive functional on $\mathcal{B}(\mathcal{H}\otimes\mathcal{H})$ such that $\theta_{\delta,+}\circ \mathrm{Ad}_{u_\delta} \ge \omega$ on $\mathfrak{A}(I)\vee\mathfrak{A}(I_\delta)'$. Also, let τ_δ be the restriction of $\theta_{\delta,+}$ to the first tensor component. Then, by Lemma 3.10, $H_{I,\delta}(\omega) \le \|\tau_\delta\| S_{\mathrm{vN}}(\tau_\delta/\|\tau_\delta\|) \le S_\delta - \eta(C_\delta)$, where C_δ and S_δ are defined by Equations (14) and (15) respectively. Provided the net satisfies extra assumption 2.18(b), the latter is finite. This proves Proposition 3.4.

For a given cutoff parameter $E > 0$, define $\tau_{\delta,E} := \tau_\delta \circ \mathrm{Ad}_{P_E}$, where $P_E = \chi_{[0,E]}(L_0)$. Still by Lemma 3.10, $H^E_{I,\delta}(\omega) \le \|\tau_{\delta,E}\| S_{\mathrm{vN}}(\tau_{\delta,E}/\|\tau_{\delta,E}\|)$, and the latter is bounded by $S_E -$

$\eta(C_E)$, where C_E and S_E are defined by Equations (16) and (17), respectively. This proves Proposition 3.7, and taking the limit $\delta \to 0$ proves Theorem 3.9.

Summing up,

$$\begin{cases} C_\delta = \sum_{N\geq 0} 2\dim(\mathcal{H}_N)|f_\delta(N)|, \\ S_\delta = \sum_{N>0} 4\dim(\mathcal{H}_N) \times \eta\Big(|f_\delta(N)/2|\Big), \\ H_{I,\delta}(\omega) \leq S_\delta - \eta(C_\delta), \end{cases} \quad \begin{cases} C_E := 2\|f\|_\infty \sum_{N=0}^{E} \dim(\mathcal{H}_N), \\ S_E := 4\|\eta\circ|f|\|_\infty \sum_{N=1}^{E} \dim(\mathcal{H}_N), \\ H_I^E(\omega) \leq S_E - \eta(C_E) < +\infty. \end{cases}$$

This concludes the summary. □

4 Conclusions and final remarks

In the present work, we focused our attention on chiral components of two dimensional theories, namely the Möbius covariant local nets. Provided the split property holds, we have given a sensible definition for regularized local entropic quantities restricted to an interval $I \in \mathcal{J}$. Considering the vacuum state, we also provided an upper bound with a conformal energy cutoff E.

We recapitulate our definitions and comment on them a little further. Taking an interval I and a small separation parameter δ, we consider all intermediate pairs $(u, \mathfrak{R}_u)$ between $\mathfrak{A}(I)$ and $\mathfrak{A}(I_\delta)$, and all states ϕ that majorize ω when restricted to $\mathfrak{A}(I) \vee \mathfrak{A}(I'_\delta)$, that is, everywhere besides a vicinity of the boundary of the intervals I and I'_δ. The quantity $H_{I,\delta}$ regularized by δ (Definition 3.3) considers the infimum of all von Neumann entropies $S_{\mathfrak{R}_u}$ of the states ϕ. With a cutoff E, our quantity $H^E_{I,\delta}$ (Definition 3.6) considers the infimum of all von Neumann entropies of the states ϕ restricted to $\mathfrak{R}_u$ but "adjoined" by the projection $u^*(P_E \otimes 1)u$. Lastly, the quantity H^E_I with cutoff E is obtained by the limit $\delta \searrow 0$ (Definition 3.8). As to why we consider the infima, as δ approaches zero, the continuity of the net implies that the local algebras of I and I_δ are very close together, and hence the quantities calculated here should reflect the properties of entanglement between $\mathfrak{A}(I)$ and its commutant $\mathfrak{A}(I')$, and so the choice of the intermediate pairs $(u, \mathfrak{R}_u)$ should not influence the calculations very much. The infimum on the majorizing states, however, excludes those with too much aberrant behavior near the boundary, and is precisely what we want. Furthermore, on the lattice case, the algebras are of type I, and on the limit of $\delta \to 0$, all the involved algebras coincide, *viz.* $\mathfrak{A}(I) = \mathfrak{R}_u = \mathfrak{A}(I_\delta)$. It is clear then that our definitions recover the original ones on those cases. The only admittedly "weak" part of the reasoning is the dependence of the implementation of the cutoff with respect to the intermediate pair $(u, \mathfrak{R}_u)$. But as our reasoning goes, since $u^*(P_E \otimes 1)u$ is close to 1 (in the operator strong topology, as E gets large) and effectively regularizes the entropy, it is expected to filter out the high energy fluctuations.

The proof of finiteness was done by the construction of a positive functional θ_+ on $\mathcal{B}(\mathcal{H}) \otimes \mathcal{B}(\mathcal{H})$, with a known bound for its entropy, and with the property that $\theta_+ \circ \mathrm{Ad}_u \geq \omega$ on $\mathfrak{A}(I) \vee \mathfrak{A}(I'_\delta)$ for any intermediate pair $(u, \mathfrak{R}_u)$. Such functional was crafted using an energy function f as in Lemma 3.2, which is independent on the particular structure of the local net. We also remark that the majorizing property is valid for any intermediate pair dismisses the infimum criteria utilized in the definition (it could as well be taken as a supremum).

Hence, we have a definition and an upper bound. It is not to say, though, that our results have no drawbacks. Let us dwell on those now. Mainly, we could say that the framework of only chiral nets is quite restrictive, and even there, our methods did not allow us to provide any lower bounds, and the upper bounds are very rough when comparing to results in obtained in lattices. Considering the $U(1)$-current model, our estimates are of the order of Ee^E, which has a much worse divergence than the Lagrangian estimates $(c/3)\log(l/a)$ of Holzhey, Larsen and Wilczek [HLW94, CC04], where l is the length of the interval (in the real line picture) and $1/a$ corresponds to the energy cutoff E, as a is the lattice spacing. Not only that, our results do not even display a dependence on the interval length l. The technical reason for such aspects of our result is that our estimates depend only on an orthonormal basis of eigenvectors of the conformal Hamiltonian, a "very global" operator. Here, sharper estimates ought to take into consideration the characteristics of each local algebra $\mathfrak{A}(I)$, to bring a bound dependent in l. Regarding the central charge, the behavior of $\dim L_0 - n$ is related to it if one assumes modularity, hence the central charge c appears in a natural way, in accordance with the physics literature.

We remind ourselves, though, that we were not expecting to reproduce Holzhey's formula. For one, we are not dealing with conformal nets (so there is no need for a central charge to appear), and our interval I is in the ligthray, whereas theirs is a spatial interval $I \in \{(t = 0, x) \in \mathbb{R}^2\}$ of length l. Thus, further sharpening the estimates (or even the definitions) and analyzing it against the lattice methods is a valid research area to be explored. Yet, also in another operator-algebraic work on entanglement entropy [HS17], the $\log(l/a)$-dependence cannot be obtained. Actually, in general, there are several possible definitions of entanglement entropy which coincide with each other when the state is pure (see e.g. [VP98, Theorem 3]). It is unclear to which definition the lattice approach corresponds. This suggests that the expression of the entanglement entropy in the physics literature is specific to the lattice regularization, and it is difficult to reproduce it from the approach in the continuum.

Another interesting question would be on how to adapt the methods provided here to theories in higher dimensions. What we instantly loose, on the general setting, is the conformal Hamiltonian L_0 and its very convenient spectral properties. In contrast to the conformal Hamiltonian L_0, the usual Hamiltonian H does not have a discrete spectrum, yet energy nuclearity conditions and some ideas from the present paper might help defining an appropriate cutoff. As the energy nuclearity index contains naturally a dependence on the size of the region, a successfull approach should lead to an estimate of entropy depending also the region. Another further possible approach would be as in [Nar94], where a quantity that should correspond to our $H_{\mathcal{O},\delta}(\omega)$ is defined, and expected to be finite (provided some conditions hold). In fact, at finite separation δ, this might even be preferable to our Definition 3.3, since it does not have to consider all pairs $(u, \mathfrak{R}_u)$ of the split property (it does rely, though, on a stronger nuclearity condition). The problem in this setting, however, is how to include a true "energy cutoff" that tames the divergence in δ. We reproduce the result in Appendix A, and discuss the possibilities of implementations of an energy regularization.

In closing, we would like to point that, although the framework used is restricted to nets living in a one-dimensional spacetime, our method was able to point out a way to regularize both the UV divergence and the "non type I"-ness of local algebras. Though sharper estimations are needed, the methods described here may help to elucidate how cutoff prescriptions might be implemented in the calculation of entanglement entropy for more general Haag-Kastler nets.

Appendix A Narnhofer's result

In this appendix, we shall reproduce Narnhofer's result [Nar94] on the local entropy of the vacuum state for Haag-Kastler nets satisfying a certain nuclearity condition. First, in Subsection A.1, we review generalized notions of entropy. Then, in Subsection A.2, we utilize those notions to define the local entropy.

A.1 One-subalgebra entropy

The von Neumann entropy as seen in Subsection 2.1 requires a trace and density matrices, which are not necessarily available for normal states in von Neumann algebras. In this subsection, we present the generalized von Neumann entropy [NT85] in terms of relative entropy [Ara76, Ara77]. This turns out to be divergent for algebras of type III_1 (Proposition A.5), so we also introduce the one-subalgebra entropy [CNT87] as an alternative regularized entropy.

The relative entropy for normal states of von Neumann algebras was introduced in [Ara76, Ara77] using the relative modular operator. We recall, when Ω a cyclic and separating vector for the von Neumann algebra $\mathfrak{M}$, that the relative modular operator $\Delta_{\Phi,\Omega}$ comes from the polar decomposition of the $S_{\Phi,\Omega}$ operator, given by the closure of $x\Omega \in \mathfrak{M}\Omega \mapsto x^*\Psi \in \mathcal{H}$. For the general case, see *e.g.* [Ara77, §2].

Definition A.1 (Relative entropy). Let $\mathfrak{M}$ be a von Neumann algebra with standard form $(\mathfrak{M}, \mathcal{H}, J, \mathcal{P})$, and ϕ, ψ be two normal positive functionals on it, with vector representatives $\Phi, \Psi \in \mathcal{P}$, respectively. Then, the relative entropy is defined as

$$S_{\mathfrak{M}}(\phi\,;\,\psi) = \begin{cases} \left\langle \Phi, (-\log \Delta_{\Psi,\Phi})\,\Phi \right\rangle & \text{if } \Phi \in \overline{\mathfrak{M}'\Psi} \\ +\infty & \text{otherwise} \end{cases}$$

Remark. The definition above uses the convention of [OP04], which is opposite to [Ara76, Ara77, CNT87] (in the sense that Araki's $S(\psi|\phi)$ corresponds to $S(\phi, \psi)$ in our definition).

Example. For the algebra $\mathfrak{M} = \mathcal{B}(\mathcal{H})$ and normal states ϕ, ψ with corresponding density matrices ρ_ϕ, ρ_ψ, the relative entropy agrees with the form

$$S_{\mathfrak{M}}(\phi\,;\,\psi) = \phi(\log \rho_\phi - \log \rho_\psi).$$

Moreover, when $\psi = \sum_n \lambda_n \psi_n$ is a convex decomposition of ψ into an orthogonal family of pure states ψ_n, then $S_{\mathfrak{M}}(\psi_n\,;\,\psi) = -\log \lambda_n$.

The following properties regarding relative entropy will be of use. We shall not present the proofs, and refer the reader to [OP04, Chapter 5].

Proposition A.2. Consider a von Neumann algebra $\mathfrak{M}$, and normal positive linear functionals ϕ, ψ on $\mathfrak{M}$. The relative entropy satisfies the following properties:

1. Scaling property. For $\alpha, \beta > 0$, one has

$$S_{\mathfrak{M}}(\alpha\phi\,;\,\beta\psi) = \alpha S_{\mathfrak{M}}(\phi\,;\,\psi) - \alpha\|\phi\|(\log \beta - \log \alpha).$$

2. Positivity. If ϕ, ψ are states, then $S_{\mathfrak{M}}(\phi, \psi) \geq 0$, and the equality holds iff $\phi = \psi$.
3. Weak* lower semicontinuity of $(\phi, \psi) \in \mathfrak{M}_*^+ \times \mathfrak{M}_*^+ \mapsto S_{\mathfrak{M}}(\phi, \psi) \in \mathbb{R}_{\geq 0} \cup \{+\infty\}$.
4. Joint convexity. For any two weak* converging sums of positive functionals, one has $S_{\mathfrak{M}}(\sum_k \phi_k\,;\,\sum_k \psi_k) \leq \sum_k S_{\mathfrak{M}}(\phi_k\,;\,\psi_k)$.
5. Monotone properties:
   (a) Superadditivity in first argument: $S_{\mathfrak{M}}(\sum_k \phi_k\,;\,\psi) \leq \sum_k S_{\mathfrak{M}}(\phi_k\,;\,\psi)$.
   (b) Decrease in second argument: If $\psi_1 \leq \psi_2$ then $S_{\mathfrak{M}}(\phi, \psi_1) \geq S_{\mathfrak{M}}(\phi, \psi_2)$.
6. If $\mathfrak{N} \subset \mathfrak{M}$ is a inclusion of von Neumann algebras with $E : \mathfrak{M} \to \mathfrak{N}$ a conditional expectation, then, for ϕ, ψ normal positive functionals over $\mathfrak{M}$, it holds that

$$S_{\mathfrak{N}}(\phi, \psi) \leq S_{\mathfrak{N}}(\phi, \psi) + S_{\mathfrak{M}}(\phi, \phi \circ E) = S_{\mathfrak{M}}(\phi, \psi).$$

The concept of relative entropy allows one to introduce a quantity that recovers the von Neumann entropy (Definition 2.4) for general von Neumann algebras [NT85]:

Definition A.3 (Entropy). Let $\mathfrak{M}$ be a von Neumann algebra, and ϕ a normal state on it. The entropy $S_{\mathfrak{M}}(\phi)$ is then defined as

$$S_{\mathfrak{M}}(\phi) := \sup_{\phi = \sum_k \lambda_k \phi_k} \lambda_k\, S_{\mathfrak{M}}(\phi_k\,;\,\phi),$$

where the supremum is taken over all convex decompositions $\phi = \sum_k \lambda_k \phi_k$ in the algebra $\mathfrak{M}$.

Proposition A.4. The entropy of Definition A.1 satisfies the following properties:

1. $S_{\mathfrak{M}}(\phi) \geq 0$ and vanishes iff ϕ is a pure state.
2. Invariance under automorphisms.

3. Weak* lower semicontinuity of $\phi \in \mathfrak{S}(\mathfrak{M}) \mapsto S_{\mathfrak{M}}(\phi) \in \mathbb{R} \cup \{+\infty\}$.
4. Concavity. If $\phi = \sum_n \lambda_n \phi_n$ is a convex decomposition of the normal state ϕ into normal states ϕ_n, then
$$\sum_n \lambda_n S_{\mathfrak{M}}(\phi_n) \leq S_{\mathfrak{M}}(\phi) \leq \sum_n \lambda_n S_{\mathfrak{M}}(\phi_n) + \sum_n \eta(\lambda_n).$$
5. If there is $E : \mathfrak{M} \to \mathfrak{N}$ a conditional expectation preserving a state ϕ, then $S_{\mathfrak{M}}(\phi) \geq S_{\mathfrak{N}}(\phi)$.

The proofs hold most directly from Proposition A.2 (although concavity requires more work), for details and further discussions, see [OP04, Chapter 6].

Next, we present a known fact that the entropy of a faithful normal state on a von Neumann algebra of type III_1 (*e.g.* the vacuum state on a local algebra) is divergent. This is one of the key motivations for the need of a regularized definition of entropy. We also remark that this is but a case of a general result that concerns all von Neumann algebras of type II and III, see [OP04, Lemma 6.9].

Proposition A.5. If $\mathfrak{M}$ is a von Neumann algebra of type III_1, then the entropy of Definition A.1 is such that $S_{\mathrm{vN}}(\phi) = +\infty$ for any faithful normal state ϕ on $\mathfrak{M}$.

Proof. Recall that for algebras of type III_1, any two faithful normal states are approximately unitarily equivalent (homogeneity of the state space, see [CS78]). From ϕ, it then suffices to create a certain faithful normal state ψ with arbitrarily large entropy, and then approximate the entropy of ϕ.

For any positive integer $n > 0$, there is a family of mutually orthogonal projections $\{e_k\}_{k=1}^n \subset \mathfrak{M}$ such that $\sum_{k=1}^n e_k = 1$ and $\phi(e_k) = 1/n$ for all $k \in \{1, 2, \ldots n\}$. One can than define the faithful normal state ψ on $\mathfrak{M}$ by
$$\psi(x) := \sum_{k=1}^{n} \phi(e_k \, x \, e_k) \qquad (x \in \mathfrak{M}).$$

Now we can consider the algebra $\mathfrak{N}$ generated by the projections $\{e_k\}_{k=1}^n$. It holds that $\mathfrak{N}$ is an abelian subalgebra of the centralizer subalgebra $\mathfrak{M}_\psi$ given by
$$\begin{aligned} \mathfrak{M}_\psi &:= \{ x \in \mathfrak{M} \mid \sigma_t^{\mathfrak{M},\psi}(x) = x \;\; (\forall t \in \mathbb{R}) \} \\ &= \{ x \in \mathfrak{M} \mid \psi(xy) = \psi(yx) \;\; (\forall y \in \mathfrak{M}) \}, \end{aligned}$$

where the equality holds by the Pedersen-Takesaki theorem [BR97, Proposition 5.3.28]. Then by Takesaki's theorem [Tak03, Theorem IX.4.2], since $\mathfrak{N}$ is invariant by the modular automorphism group $\sigma^{\mathfrak{M},\psi}$, there is a conditional expectation $E : \mathfrak{M} \to \mathfrak{N}$ preserving ψ. Hence, by Proposition A.4, $S_{\mathfrak{M}}(\psi) \geq S_{\mathfrak{A}}(\psi) = \log n$.

Finally, by the homogeneity of the state space, there are unitary operators $u_m \in \mathfrak{M}$ such that $\|\psi - \phi(u_m \,\cdot\, u_m^*)\| \to 0$. By the lower semicontinuity and unitary invariance of the entropy, we have
$$S_{\mathfrak{M}}(\phi) = \liminf_{n \to +\infty} S_{\mathfrak{M}}\big(\, \phi(u_m \,\cdot\, u_m^*) \,\big) \geq S_{\mathfrak{M}}(\psi) \geq \log n.$$

Since this holds for all $n > 0$, we have $S_{\mathfrak{M}}(\phi) = +\infty$. This concludes the proof. ∎

Alternative regularized definitions are available, and in the following, we describe one connected to Narnhofer's work.

Definition A.6 (One-subalgebra entropy). Let $\mathfrak{M}$ be a von Neumann algebra, and ϕ a normal state on it. For $\mathfrak{N}$ a von Neumann subalgebra of $\mathfrak{M}$, the *one-subalgebra entropy* $H_\phi(\mathfrak{N})$ of ϕ with respect to $\mathfrak{N}$ is defined as
$$H_\phi(\mathfrak{N}) := \sup_{\phi = \sum_k \lambda_k \phi_k} \lambda_k \, S_{\mathfrak{N}}(\phi_k \,;\, \phi),$$

where the supremum is taken over all convex decompositions of the state on the larger algebra $\mathfrak{M}$, whereas the relative entropy is taken with respect to $\mathfrak{N}$.

Remark (i). The one-subalgebra entropy is a special case of the Connes-Narnhofer-Thirring entropy $H_\phi(\gamma_1, \gamma_2, \ldots, \gamma_n)$ with $n = 1$ and γ_1 set as the inclusion map $\mathfrak{N} \hookrightarrow \mathfrak{M}$ [CNT87, Remark III.5.3]. However, the one-subalgebra entropy lifts the requirement of $\mathfrak{N}$ being finite dimensional.

Remark (ii). The one-subalgebra entropy recovers the generalized von Neumann entropy of Definition A.3 when $\mathfrak{N} = \mathfrak{M}$. In general, one has that $S_{\mathfrak{N}\subset\mathfrak{M}}(\phi) \leq S_{\mathfrak{N}}(\phi|_{\mathfrak{N}})$, since the collection of convex decompositions of ϕ on the larger algebra $\mathfrak{M}$ is a subset of the collection of convex decompositions on the smaller subalgebra $\mathfrak{N}$. Equality also holds when there is a conditional expectation $E : \mathfrak{M} \to \mathfrak{N}$ preserving the state ϕ.

Proposition A.7 (Concavity). [CNT87, Proposition III.6.e] Let ϕ be a normal state on a von Neumann algebra $\mathfrak{M}$ and $\phi = \sum_k \lambda_k \phi_k$ be a finite convex decomposition of it. Then, for a *finite dimensional* subalgebra $\mathfrak{N} \subset \mathfrak{M}$, the one-subalgebra entropy satisfies the following concavity property:

$$\sum_k \lambda_k H_{\phi_k}(\mathfrak{N}) \leq H_\phi(\mathfrak{N}) \leq \sum_k \lambda_k H_{\phi_k}(\mathfrak{N}) + \sum_k \eta(\lambda_k).$$

Remark. It is not clear to the authors if the concavity property holds when the subalgebra $\mathfrak{N}$ is infinite dimensional. This is needed, however, for the result concerning the local entropy of the vacuum.

A.2 Narnhofer entropy

Here we re-derive Narnhofer's local entropy of the vacuum [Nar94]. In this section, let $(\mathfrak{A}, U, \Omega, \mathcal{H})$ be a Haag-Kastler net [Haa96] on a $1 + d$-dimensional Minkowski spacetime. For every vector $x \in \mathbb{R}^{1+d}$, the transformation τ_x denotes the translation by x. The Hamiltonian H is the positive operator that generates the one-parameter unitary group of time translations $t \in \mathbb{R} \mapsto U(\tau_{(t,\mathbf{0})})$.

Notation. We shall denote (open) double cones in the spacetime by $\mathcal{O} \subset \mathbb{R}^{1+d}$. For $\mathcal{O}$ a double cone, $\mathcal{O}'$ denotes its causal complement (notice, however, that $\mathcal{O}'$ is *not* a double cone). For $\mathcal{O}$ a double cone and $\delta > 0$ a positive parameter, the δ-enlarged double cone $\mathcal{O}_\delta$ is defined by

$$\mathcal{O}_\delta := \bigcup_{|x|<\delta} \tau_x(\mathcal{O}),$$

where $|x| = \sqrt{\sum_{\mu=0}^d (x^\mu)^2}$ is the "Euclidean norm" of the translation vector $x \in \mathbb{R}^{1+d}$.

Before enunciating Narnhofer's estimate, we first review the concept of *energy p-nuclearity condition* [BW86, BDL90b] for some $p \in (0, 1)$. For any double cone $\mathcal{O}$ in the spacetime and for any positive value of $\beta > 0$, the map $\Theta_{\mathcal{O},\beta}$ is defined by

$$\Theta_{\mathcal{O},\beta} : a \in \mathfrak{A}(\mathcal{O}) \mapsto e^{-\beta H} a\Omega, \quad (a \in \mathfrak{A}(\mathcal{O})).$$

Such map is supposed to be p-nuclear. Also, for a fixed region $\mathcal{O}$, there should be a positive constant β_0 and a positive integer n such that the p-nuclearity index of $\Theta_{\mathcal{O},\beta}$ has the following β-dependency:

$$\nu_p(\Theta_\beta) \leq e^{(\beta_o/\beta)^n}.$$

From the positivity of the Hamiltonian H, there exists an energy function f as in Lemma 3.2 (where H substitutes L_0). With $a \in \mathfrak{A}(\mathcal{O})$ and $b \in \mathfrak{A}(\mathcal{O}_\delta)'$, one has

$$\omega(ab) = \langle f_\delta(H) a\Omega, b\Omega \rangle + \langle b^*\Omega, f_\delta(H) a\Omega \rangle.$$

With the function f, we define the maps $\Upsilon_{\mathcal{O},\delta}$ with a nuclearity property as follows.

Lemma A.8. [Nar94, Lemma 4.5], [BDF87, Lemma 2.2] Let $(\mathfrak{A}, U, \Omega, \mathcal{H})$ be a Haag-Kastler net, and suppose it satisfies the energy p-nuclearity condition for some $p \in (0,1)$. Fix a double cone $\mathcal{O}$ and positive number $\delta > 0$. Fix $q = n/(n+1)$ (where β_0, n was such that $\nu_p(\Theta_{\mathcal{O},\beta}) \leq \exp[(\beta/\beta_0)^n]$). Let f be an energy function as in Lemma 3.2 with decay property $\sup_{t\in\mathbb{R}} |e^{|t|^q} f(t)| < +\infty$. Define the map $\Upsilon_{\mathcal{O},\delta}$ by

$$\Upsilon_{\mathcal{O},\delta} : a \in \mathfrak{A}(\mathcal{O}) \mapsto f_\delta(H)a\Omega \in \mathcal{H}.$$

Then $\Upsilon_{\mathcal{O},\delta}$ is p-nuclear. Moreover, there is a positive constant $c > 0$ such that $\nu_p(\Upsilon_{\mathcal{O},\delta}) \leq \frac{c}{\delta}$ for $\delta > 0$.

Proof. The operator $\Upsilon_{I,\delta}$ can be expressed as:

$$f_\delta(H)a\Omega = f_\delta(H)\Big(\underbrace{\sum_{m=0}^{\infty} e^{+C(\delta m)^\kappa} E_{\delta,m}}_{v_\delta}\Big)\Big(\underbrace{\sum_{j=1}^{\infty} e^{-C(\delta j)^\kappa} \overbrace{E_{\delta,j}e^{\beta_j H}e^{-\beta_j H}a\Omega}^{\Xi_{\delta,j}(a)}}_{\Xi_\delta(a)}\Big)$$

We take a moment to digress on the meaning of all the terms introduced above. For any positive integer m, we set $E_{\delta,m}$ as the spectral projection of the Hamiltonian H on the space of $(\delta(m-1), \delta m)$ (that is, $E_{\delta,m} = \chi_{\delta(m-1,m)}(H)$). Given the truncated operator $\Xi_{\delta,j}(a) = E_{\delta,j}a\Omega$, we can squeeze in the exponential factors $e^{\beta H}e^{-\beta H}$ which leads us to an bounded operator times the p-nuclear $\Theta_{\mathcal{O},\beta}$ operator. Hence, $\Xi_{\delta,j}$ will be p-nuclear. We then introduce an exponential damping factor, which will allow us to sum all $\Xi_{\delta,j}$'s in the nuclear operator Ξ_δ (the parameters C and κ will be calculated afterwards). And to counterbalance it, there appears an exponential factor, which we called v_δ. This is an unbounded operator, but because of the decay property of f_δ, the operator $f_\delta(H)v_\delta$ shall be bounded. Hence, Θ_{f_δ} shall be a bounded operator times a p-nuclear one, and therefore, p-nuclear itself. We shall now work on the details.

First, we shall estimate the p-nuclearity indices of each $\Xi_{\delta,j}$. We first notice that $\Xi_{\delta,j} = E_{\delta,j}e^{+\beta H}\Theta_{\mathcal{O},\beta}$ holds for all $\beta > 0$. By the energy p-nuclearity condition, there are constants β_0 and n such that $\nu_p(\Theta_{\mathcal{O},\beta}) \leq \exp(\beta_0/\beta)^n$. Hence, for any j one has:

$$\nu_p(\Xi_{\delta,j}) \leq \inf_{\beta>o} \|E_{\delta,j}e^{\beta H}\|^p \cdot \nu_p(\Theta_{\mathcal{O},\beta}) \leq \inf_{\beta>0} e^{\beta\delta pj+(\beta_0/\beta)^n} \leq e^{2(\beta_0\delta pj)^{n/n+1}}.$$

With the above bound on each nuclearity index of $\Xi_{\delta,j}$, we can choose the parameters C and κ such that Ξ_δ also has a finite nuclearity index. Set $\kappa = n/(n+1)$. For a small $\epsilon > 0$, set $C := \frac{2p^\kappa+\epsilon}{p}\beta_0^\kappa$ as the exponential damping factor. Then, Ξ_δ is nuclear, since

$$\nu_p(\Xi_\delta) \leq \sum_{j=1}^{\infty} \left|e^{-C(\delta j)^\kappa}\right|^p \nu_p(\Xi_{\delta,j}) \leq \sum_{j=1}^{\infty} e^{-\epsilon(\beta_0\delta j)^\kappa} \leq \frac{1}{\delta}\int_{-1}^{\infty} e^{-\epsilon(\beta_0 j)^\kappa}dj.$$

Lastly, $f_\delta(H)v_\delta$ is bounded, with norm independent of δ. Hence, the following holds:

$$\nu_p(\Upsilon_{\mathcal{O},\delta}) \leq \frac{1}{\delta}\text{const}.$$

This concludes the proof of the lemma. □

Definition A.9 (Narnhofer's local entropy). For $\mathcal{O}$ a double cone and $\delta > 0$, Narnhofer's localized entropy for a normal state ψ is defined by the formula:

$$S_{\mathcal{O},\delta}(\psi) := \sup_{\psi=\lambda_k\psi_k} \sum_i \lambda_k S_{\mathfrak{A}(\mathcal{O})}(\psi_k\,;\,\psi),$$

where the supremum is taken over the all convex decompositions on the larger algebra $\mathfrak{A}(\mathcal{O}_\delta)$.

Remark. Narnhofer's entropy is the one-subalgebra entropy of Definition A.6 for the state $\psi|_{\mathfrak{A}(\mathcal{O}_\delta)}$ with respect to the inclusion $\mathfrak{A}(\mathcal{O}) \subset \mathfrak{A}(\mathcal{O}_\delta)$.

Theorem A.10. Let $(\mathfrak{A}, U, \Omega, \mathcal{H})$ be a Haag-Kastler net satisfying the energy p-nuclearity condition for some $p \in (0,1)$. Also suppose that the *concavity property* (Proposition A.7) holds for $S_{\mathcal{O},\delta}$.

Let $\mathcal{O}$ be a double cone and $\delta > 0$. Let f and $\Upsilon_{\mathcal{O},\delta}$ be as in Lemma A.8. Then, for the vacuum state $\omega = \langle \Omega, \cdot\, \Omega\rangle$, Narnhofer's entropy is bounded as follows:

$$S_{\mathcal{O},\delta}(\omega) \leq \frac{16}{e(1-p)}\, \nu_p(\Upsilon_{\mathcal{O},\delta}).$$

Remark. To the knowledge of the authors, it is not clear if the concavity property (Proposition A.7) holds as mentioned in the hypothesis of Theorem A.10, since the subalgebra $\mathfrak{A}(\mathcal{O})$ is not finite dimensional. We thus add those as extra requirements.

Our proof is divided in two parts. First, in Lemma A.11 we use the f function to construct a majorizing positive functional τ, and then in Lemma A.12 we use it to estimate the entropy of ω.

Lemma A.11 (Splitting state). Consider $\mathcal{O}$ a double cone, $\delta > 0$, f and $\Upsilon_{\mathcal{O},\delta}$ as in Theorem A.10. Fix a value $\epsilon > 0$. For the vacuum state ω, denote by $\omega_\otimes$ the state on $\mathfrak{A}(\mathcal{O})\bar{\otimes}\mathfrak{A}(\mathcal{O}_\delta)$ defined by

$$\omega_\otimes(a \otimes b) = \omega(ab) \qquad (a \in \mathfrak{A}(\mathcal{O}), b \in \mathfrak{A}(\mathcal{O}_\delta)').$$

Then, there are families of normal positive functionals $\{\phi_k^\pm \in \mathfrak{A}(\mathcal{O})^*\}_{k\in\mathbf{Z}}$ and $\{\psi_k^\pm \in \mathfrak{A}(\mathcal{O}_\delta)^*\}_{k\in\mathbf{Z}}$ such that τ defined by

$$\tau := \sum_k \phi_k^+ \otimes \psi_k^+ = \omega_\otimes + \sum_k \phi_k^- \otimes \psi_k^-$$

is a normal positive functional on $\mathfrak{A}(\mathcal{O})\bar{\otimes}\mathfrak{A}(\mathcal{O}_\delta)$. Moreover,

$$\sum_k \left(\|\phi_k^\pm\|\,\|\psi_k^\pm\|\right)^q \leq 8(\nu_q(\Upsilon_{\mathcal{O},\delta}) + \epsilon), \quad (q \in \{p, 1\}).$$

Proof. Recall that the p-nuclearity of $\Upsilon_{\mathcal{O},\delta}$ implies its 1-nuclearity (Proposition 2.13). By the nuclearity of $\Upsilon_{\mathcal{O},\delta}$, for such fixed value of $\epsilon > 0$, there are normal functionals $\{\chi_m \in \mathfrak{A}(\mathcal{O})^*\}_{m\in\mathbf{Z}}$ and vectors $\{\Phi_m \in \mathcal{H}\}_{m\in\mathbf{Z}}$ such that

$$\Upsilon_{\mathcal{O},\delta}(a) = \sum_m \chi_m(a)\, \Phi_m, \qquad (a \in \mathfrak{A}(\mathcal{O}))$$
$$\sum_k \left(\|\chi_m\|\,\|\Phi_m\|\right)^q \leq \nu_q(\Upsilon_{\mathcal{O},\delta}) + \epsilon, \qquad (q \in \{p, 1\}).$$

For $a \in \mathfrak{A}(\mathcal{O})$ and $b \in \mathfrak{A}(\mathcal{O}_\delta)'$, one has:

$$\begin{aligned}
\omega(ab) &= \langle \Omega, a f_\delta(H) b\Omega\rangle + \langle \Omega, b f_\delta(H) a\Omega\rangle \\
&= \langle f_\delta(H) a^*\Omega, b\Omega\rangle + \langle \Omega, b f_\delta(H) a\Omega\rangle \\
&= \sum_m \underbrace{\overline{\chi_m(a^*)}}_{\phi_{(2m)}(a)} \underbrace{\langle \Phi_k, b\Omega\rangle}_{\psi_{(2m)}(b)} + \sum_m \underbrace{\chi_m(a)}_{\phi_{(2m+1)}(a)} \underbrace{\langle \Omega, b\Phi_k\rangle}_{\psi_{(2m+1)}(b)} = \sum_k \phi_k(a)\psi_k(b).
\end{aligned}$$

$$\text{with } \sum_k \left(\|\phi_k\|\,\|\psi_k\|\right)^q \leq 2(\nu_p(\Upsilon_{\mathcal{O},\delta}) + \epsilon), \quad (q \in \{p, 1\}).$$

We will now address the (lack of) positivity of the functionals ϕ_k. Bear in mind that the same argumentation holds for the functionals ψ_k. Recall that each ϕ_k a normal linear operator, but not necessarily positive. Each of them, though, splits to a linear combination of four positive functionals, say $\phi_k = \sum_{\alpha=0}^3 (i)^\alpha \phi_k^\alpha$. One has also, that for every index

α, the inequality $\|\phi_k^\alpha\| \leq \|\phi_k\|$ holds. Adopting the same notation for ψ_k, one has $\psi_k = \sum_{\alpha=0}^3 (i)^\alpha \psi_k^\alpha$. This gives the $\phi_k \otimes \psi_k = \sum_{\alpha,\beta=0}^3 (i)^{\alpha+\beta} \phi_k^\alpha \otimes \psi_k^\beta = \sum_{\gamma=0}^3 (i)^\gamma \sum_{\alpha=0}^3 \phi_k^\alpha \otimes \psi_k^{\gamma-\alpha}$. With that in mind, the splitting of the vacuum state ω in $\mathfrak{A}(\mathcal{O}) \otimes \mathfrak{A}(\mathcal{O}_\delta)'$ becomes:

$$\omega_\otimes = \sum_k \phi_k \otimes \psi_k = \sum_{\gamma=0}^3 (i)^\gamma \sum_k \sum_{\alpha=0}^3 \phi_k^\alpha \otimes \psi_k^{\gamma-\alpha} = \sum_k \sum_{\alpha=0}^3 \phi_k^\alpha \otimes \psi_k^{4-\alpha} - \sum_k \sum_{\alpha=0}^3 \phi_k^\alpha \otimes \psi_k^{2-\alpha},$$

$$\text{with } \sum_k \sum_{\alpha=0}^3 \left(\|\phi_k^\alpha\| \, \|\psi_k^{\gamma-\alpha}\|\right)^q \leq 8(\nu_p(\Upsilon_{\mathcal{O},\delta}) + \epsilon), \quad (q \in \{p, 1\},\ \gamma \in \{0,1,2,3\}).$$

In the last equality, we have ignored the imaginary terms corresponding to $\gamma = 1, 3$. This is because ω is self-adjoint (positive, indeed), and the sum of the imaginary terms is then zero. We are then left with the positive ($\gamma = 0$) and negative ($\gamma = 2$) parts.

The indices might then be rearranged as

$$\phi_{4k+\alpha}^+ = \phi_k^\alpha, \quad \psi_{4k+\alpha}^+ = \psi_k^{4-\alpha}, \quad \phi_{4k+\alpha}^- = \phi_k^\alpha, \quad \psi_{4k+\alpha}^+ = \psi_k^{2-\alpha},$$

and we get the following:

$$\omega_\otimes = \sum_k \phi_k^+ \otimes \psi_k^+ - \sum_k \phi_k^- \otimes \psi_k^-,$$
$$\text{with } \sum_k \left(\|\phi_k^\pm\| \, \|\psi_k^\pm\|\right)^q \leq 8(\nu_p(\Upsilon_{\mathcal{O},\delta}) + \epsilon), \quad (q \in \{p, 1\}).$$

We now can define τ to have the following equality of positive functionals:

$$\tau := \omega_\otimes + \sum_k \phi_k^- \otimes \psi_k^- = \sum_k \phi_k^+ \otimes \psi_k^+$$
$$\text{with } \sum_k \left(\|\phi_k^\pm\| \, \|\psi_k^\pm\|\right)^q \leq 8(\nu_p(\Upsilon_{\mathcal{O},\delta}) + \epsilon), \quad (q \in \{p, 1\}).$$

This concludes the proof of the lemma. □

Lemma A.12 (Upper bound for entropy). Consider $\mathcal{O}$ a double cone, $\delta > 0$, f and $\Upsilon_{\mathcal{O},\delta}$ as in Theorem A.10. Then, it holds that

$$S_{\mathcal{O},\delta}(\omega) \leq \frac{16}{e(1-p)} \nu_p(\Upsilon_{\mathcal{O},\delta}).$$

Proof. Choose any $p \in (0, 1)$ and fix a value $\epsilon > 0$. By the previous lemma, we have families of positive functionals $\{\phi_k^\pm \in \mathfrak{A}(\mathcal{O})^*\}_{k\in\mathbf{Z}}$ and $\{\psi_k^\pm \in \mathfrak{A}(\mathcal{O}_\delta)'^*\}_{k\in\mathbf{Z}}$, and the positive functional τ defined by

$$\tau := \sum_k \phi_k^+ \otimes \psi_k^+ = \omega_\otimes + \sum_k \phi_k^- \otimes \psi_k^-.$$

Moreover, τ is a positive functional on $\mathfrak{A}(\mathcal{O}) \bar{\otimes} \mathfrak{A}(\mathcal{O}_\delta)'$, and

$$\sum_k \left(\|\phi_k^\pm\| \, \|\psi_k^\pm\|\right)^q \leq 8(\nu_p(\Upsilon_{\mathcal{O},\delta}) + \epsilon), \quad (q \in \{p, 1\}).$$

The norm of τ is given by:

$$\|\tau\| = \sum_k \left(\|\phi_k^+\| \, \|\psi_k^+\|\right) \leq 8(\nu_1(\Upsilon_{\mathcal{O},\delta}) + \epsilon).$$

Furthermore, since $\tau = \omega + (\sum_k \phi_k^- \otimes \psi_k^-)$ is a sum of positive operators, it holds that $1 \leq \|\tau\|$. It is simple to see that, since $0 \leq \|\tau\| - 1 = \sum_k \|\phi_k^- \otimes \psi_k^-\| \leq 8(\nu_1(\Upsilon_{\mathcal{O},\delta}) + \epsilon)$.

We can normalize τ to a state $\bar{\tau} = \tau/\|\tau\|$, which gives us the following convex decomposition in states:

$$\bar{\tau} = \frac{1}{\|\tau\|}\,\omega + \frac{\|\tau\|-1}{\|\tau\|}\,\frac{\sum_k \phi_k^- \otimes \psi_k^-}{\|\tau\|-1}.$$

Supposing the concavity of $S_{\mathcal{O},\delta}$ holds, one has

$$S_{\mathcal{O},\delta}(\bar{\tau}) \geq \frac{1}{\|\tau\|} H_{\mathcal{O},\delta}(\omega) + \frac{\|\tau\|-1}{\|\tau\|} S_{\mathcal{O},\delta}\Big(\frac{\sum_k \phi_k^- \otimes \psi_k^-}{\|\tau\|-1}\Big).$$

With the above expressions in hand, we reproduce Narnhofer's calculations (Theorem 4.6 of [Nar94]) to determine an upper bound for the entropy $H_{\mathcal{O},\delta}(\bar{\tau})$.

Since we shall need convex decompositions of the state τ, it will be useful to again change some notations. We rescale ϕ_k^+ and ψ_k^+ to ϕ_k and ψ_k (without superscripts) as follows:

$$\phi_k = \frac{1}{\|\phi_k^+\|}\phi_k^+, \qquad \psi_k = \frac{\|\phi_k^+\|}{\|\tau\|}\psi_k^+, \qquad (k \in \mathbb{N})$$

so that

$$\bar{\tau} = \sum_k \phi_k \otimes \psi_k, \qquad \|\phi_k\| = 1\ (\forall k \in \mathbb{N}), \qquad \sum_k \|\psi_k\|^q \leq \frac{8}{\|\tau\|}(\nu_q(\Upsilon_{\mathcal{O},\delta}) + \epsilon)\ (\forall q \in \{p, 1\}).$$

Now, to every convex decomposition of $\bar{\tau}$ in $\sum_l \lambda_l \tau_l$ in the algebra $\mathfrak{A}(\mathcal{O}_\delta)$, there is a family $\{b_l\}_{l\in\mathbb{N}}$ which is a "decomposition of the identity in its commutant" (*i.e.* every operator b_l is a positive operator in $\mathfrak{A}(\mathcal{O}_\delta)'$ and $\sum_{l=0}^{\infty} b_l = 1$), such that each component is given by $\lambda_l \tau_l = \bar{\tau}(\,\cdot\, b_l)$ (see *e.g.* [BR87, Theorem 2.3.19]). Furthermore, substituting $\bar{\tau} = \sum_k \phi_k \otimes \psi_k$, we have the following possible decompositions in the smaller algebra $\mathfrak{A}(\mathcal{O})$:

$$\bar{\tau}\big|_{\mathfrak{A}(\mathcal{O})} = \sum_l \lambda_l\, \tau_l\big|_{\mathfrak{A}(\mathcal{O})} = \sum_{k,l} \psi_k(b_l)\phi_k = \sum_k \|\psi_k\|\phi_k$$

We have to analyze $\sum_l \lambda_l S_{\mathfrak{A}(\mathcal{O})}(\tau_l\,;\,\bar{\tau})$. By the scaling of relative entropy, this sum can be represented as

$$\sum_l \lambda_l S_{\mathfrak{A}(\mathcal{O})}(\tau_l\,;\,\bar{\tau}) = H\big(\{\lambda_l\}_l\big) + \sum_l S_{\mathfrak{A}(\mathcal{O})}(\lambda_l\tau_l\,;\,\bar{\tau}),$$

where $H(\{\lambda_l\}) = \sum_l \eta(\lambda_l)$ is the entropy for the abelian distribution "$1 = \sum_l \lambda_l$". Now, on the algebra $\mathfrak{A}(\mathcal{O})$ we can substitute $\bar{\tau} = \sum_k \|\psi_k\|\phi_k$ and $\tau_l = \sum_k \psi_k(b_l)\phi_k$, and use the joint concavity of the relative entropy to obtain

$$S_{\mathfrak{A}(O)}(\lambda_l\,\tau_l\,;\,\bar{\tau}) \leq \sum_k S_{\mathfrak{A}(O)}\big(\psi_k(b_l)\phi_k\,;\,\|\psi_k\|\phi_k\big) = \sum_k \psi_k(b_l)\log\frac{\psi_k(b_l)}{\|\psi_k\|}$$

Substituting this in the previous equation, we have

$$\sum_l \lambda_l S_{\mathfrak{A}(\mathcal{O})}(\tau_l\,;\,\bar{\tau}) \leq H\big(\{\lambda_l\}_l\big) + H\big(\{\|\psi_k\|\}_k\big) - H\big(\{(\psi_k(b_l)\}_{k,l}\big) \leq 2H\big(\{\|\psi_k\|\}_k\big)$$

where in the last passage, we used the triangular inequality $H_L - H_K \leq H_{KL}$ (Proposition 2.9) with $H_L = H\big(\{\lambda_l\}_l\big)$, $H_K = H\big(\{\|\psi_k\|\}_k\big)$, and $H_{KL} = H\big(\{\psi_k(b_l)\}_{k,l}\big)$.

With the above fixed value of $p \in (0,1)$, by Proposition 2.3 one has $\eta(\phi_k) \leq c_p\|\phi_k\|^p$, where $c_p = 1/[(1-p)e]$. Hence, the RHS above is bounded by $2c_p\sum_k \|\psi_k\|^p \leq 16c_p(\nu_p(\Upsilon_{\mathcal{O},\delta}) + \epsilon)/\|\tau\|$. Since this holds for all convex decompositions, we have

$$S_{\mathcal{O},\delta}(\bar{\tau}) \leq \frac{16c_p(\nu_p(\Upsilon_{\mathcal{O},\delta}) + \epsilon)}{\|\tau\|}.$$

Supposing the concavity of the entropy holds, we then have:

$$S_{\mathcal{O},\delta}(\omega) \leq \|\tau\|\, S_{\mathcal{O},\delta}(\bar{\tau}) \leq 16c_p(\nu_p(\Upsilon_{\mathcal{O},\delta}) + \epsilon).$$

The former holds for all $\epsilon > 0$. Hence, one finally has

$$S_{\mathcal{O},\delta}(\omega) \leq \frac{16}{e(1-p)}\,\nu_p(\Upsilon_{\mathcal{O},\delta}).$$

This proves the lemma, and also Theorem A.10. $\square$

Recapitulating, Narnhofer's entropy $S_{\mathcal{O},\delta}(\omega)$ can be considered a δ-regularized entanglement entropy of the vacuum state ω restricted to $\mathfrak{A}(\mathcal{O})$, or as the entanglement entropy of $\mathfrak{A}(\mathcal{O})$ with respect to the algebra $\mathfrak{A}(\mathcal{O}_\delta)'$ ("slightly smaller" than $\mathfrak{A}(\mathcal{O})'$). For finite $\delta > 0$, upper bounds are given by Theorem A.10, and lower bounds can be computed by taking specific convex decompositions. As δ approaches zero, the entropy approaches the generalized von Neumann entropy in Definition A.3, which by Proposition A.5, diverges as $1/\delta$. The geometric properties of the entropy are encoded in the nuclearity indices $\nu_p(\Upsilon_{\mathcal{O},\delta})$.

One would like to regularize the entropy $S_{\mathcal{O},\delta}$ in terms of an "energy cutoff" such that the limit $\delta \to 0$ is finite. For a method that can use an adaptation of the proof by Narnhofer, the difficulties are twofold: first, there needs to be a meaningful regularization prescription for all normal states (not only the vacuum), and second, concavity should still hold. Unfortunately, no such result is available, to the knowledge of the authors. We present some possible formulations. In the following, let H be the Hamiltonian and, for a parameter $E > 0$, let $P_E = \chi_{[0,E]}(H)$ be the spectral projection of H with respect to the interval $[0, E]$.

First, one can try to use an intermediate pair $(u, \mathfrak{R}_u)$ given by the split property (which follows from the energy nuclearity condition) and "migrate" the projection P_E to $\mathfrak{R}_u$ (similar to our Definition 3.6). One could then try to estimate the entropy of a regularized state $\omega_\otimes^E = \langle (P_E \otimes 1)u\Omega, \cdot\, (P_E \otimes 1)u\Omega\rangle$. The problem is that, since P_E is not an operator localized in $\mathfrak{A}(\mathcal{O})$, the auxiliary state $\bar{\tau}$ can no longer be used. It was crucial that the (tensor) vacuum state $\omega_\otimes = \langle u\Omega, \cdot\, u\Omega\rangle$ could be split as $\omega_\otimes(a \otimes b) = \sum_k \psi_k^+(a)\phi_k^+(b) - \sum_k \psi_k^-(a)\phi_k^-(b)$, and this holds true only for localized tensor elements $a \otimes b \in \mathfrak{A}(\mathcal{O}) \otimes \mathfrak{A}(\mathcal{O}_\delta)'$, but not necessarily for $P_E a P_E \otimes b \in \mathcal{B}(\mathcal{H}) \otimes \mathfrak{A}(\mathcal{O}_\delta)'$. Thus, one needs new techniques for efficiently splitting $\omega_\otimes^E$.

A second, more physically intuitive notion would be to cutoff the excitations of the convex decompositions considered in Definition A.9. Every convex decomposition $\omega = \sum_k \lambda_k \omega_k$ in $\mathfrak{A}(\mathcal{O}_\delta)$ is associated to a decomposition of unity $\{b_k \in \mathfrak{A}(\mathcal{O}_\delta)'_+\}$ with $\sum_k b_k = 1$, such that, in $\mathfrak{A}(\mathcal{O}_\delta)$, $\lambda_k\omega_k = \omega(\,\cdot\, b_k) = \langle \sqrt{b_k}\Omega, \cdot \sqrt{b_k}\Omega\rangle$. One can interpret each $\sqrt{b_k}$ as an "excitation generated outside $\mathcal{O}$", which would be effectively "cutoff" by considering instead the positive functional $\lambda_k\omega_k^E = \langle P_E\sqrt{b_k}\Omega, \cdot\, P_E\sqrt{b_k}\Omega\rangle$. A reasonable cutoff entropy would then be the supremum of $\sum_k \lambda_k S_{\mathfrak{A}(\mathcal{O})}\big(\omega_l^E/\|\omega_l^E\|\,;\,\omega\big)$, that is, Definition A.9 with regularized "substates" instead. However, Narnhofer's method involving the splitting of the state is not applicable, and thus one needs new ways to find an upper bound.

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