# 博士論文

論文題目　Topological invariants and localized wave functions for some topological phases

（ある種のトポロジカル相に対する位相不変量と局在化した波動関数の関係について）

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# Topological invariants and localized wave functions for some topological phases

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# 1 Introduction

In this thesis, we study some topological invariants which corresponds to wave functions localized in some sense from the point of view of $K$-theory and index theory. The contents of this paper is divided into two parts. In the first part (Sect. 3), we give a proof of the bulk-edge correspondence for type A and AII topological insulators based on the cobordism invariance of the index. In the second part (Sect .4), we consider periodic Hamiltonians on a three dimensional lattice which has two edges and the corner (the intersection of two edges), and showed that, if our Hamiltonian has a common spectral gap at two edges, then we can define a topological invariant for such systems. We also show that this topological invariant corresponds to wave functions localized near the corner. In this part, no additional symmetry is assumed.

## 1.1 Background

In condensed matter physics, a correspondence between two topological invariants is well-known. Such correspondence was obtained by the theoretical study of the quantum Hall effect. The quantum Hall effect was discovered by K. von Klitzing, G. Dorda and M. Pepper in 1980 [KDP80]. They observed a quantization of the Hall conductance of a two-dimensional electron gas in a strong magnetic field. Such quantization of the Hall conductance was explained by D. J. Thouless, M. Kohmoto, M. P. Nightingale and M. den Nijs from the topological point of view. They considered such effect on an infinite system without edge, and showed that the Hall conductance of the system coincides with the first Chern number (called the TKNN number) of the Bloch bundle on the Brillouin torus by using the Kubo formula [TKNdN82, Koh85]. B. I. Halperin considered such phenomena on a system with edge [Hal82]. Y. Hatsugai showed that the Hall conductance of a system with edge corresponds to a winding number (or a spectral flow) counted on a Riemann surface [Hat93a]. The equality between the first Chern number defined for such infinite system without edge (called the *bulk index*) and the winding number defined for such system with edge (called the *edge index*) was proved by Hatsugai [Hat93b]. This correspondence between two integer-valued topological invariants is called the *bulk-edge correspondence*.

In 2005, C. L. Kane and E. J. Mele proposed the quantum spin Hall effect [KM05a]. They defined a topological invariant (called the Kane-Mele index) which takes value in $\mathbb{Z}_2 = \mathbb{Z}/2\mathbb{Z}$, for two-dimensional systems with odd time-reversal symmetry [KM05b]. By using this invariant, they distinguished the quantum spin Hall state and the insulating state. On the quantum spin Hall state, there is a spin current carried on the edge. The quantum spin Hall effect was observed experimentally by König *et al.* [Kön07] based on the model proposed by B. A. Bernevig T. L. Hughes and S.-C. Zhang [BHZ06].

### 1.1.1 Hatsugai's proof

Hatsugai considered the quantum Hall effect on a square lattice in a uniform rational magnetic field, and showed the bulk-edge correspondence by using a Riemann surface [Hat93b].

### 1.1.2 Kellendonk–Richter–Schulz-Baldes' proof

By the pioneering work of J. Bellissard, a mathematical approach to the quantum Hall effect from the point of view of the noncommutative geometry was developed [Bel86, BvESB94]. J. Kellendonk, T. Richter and H. Schulz-Baldes gave a proof of the bulk-edge correspondence also for disordered systems [SBKR00, KRSB02]. They used $K$-theory for $C^*$-algebras, especially the boundary homomorphism of the six-term exact sequence associated to the Toeplitz extension,

$$0 \to K(l^2(\mathbb{Z})) \to \mathcal{T} \to C(\mathbb{T}) \to 0,$$

where $K(l^2(\mathbb{Z}))$ is the $C^*$-algebra of compact operators on $l^2(\mathbb{Z})$, $\mathcal{T}$ is the Toeplitz algebra and $C(\mathbb{T})$ is the $C^*$-algebra consists of continuous functions on the unit circle $\mathbb{T}$ in the complex plane with uniform norm. Since they used the pairing of $K$-theory and cyclic cohomology in the definition of bulk and edge invariants, this method cannot be applied to $\mathbb{Z}_2$-valued invariants as the quantum spin Hall effect. C. Bourne, A. L. Carey, Kellendonk and R. Rennie overcome such difficulty by using Kasparov's $KK$-theory [BCR15, BCR16, BKR16]. Y. Kubota studied topological phases and the bulk-edge correspondence from the point of view of the coarse geometry developed by J. Roe. He used the coarse Mayer-Vietoris exact sequence to prove the bulk-edge correspondence [Kub16a]. Their method is applied to any symmetry and to disordered systems.

### 1.1.3 Graf–Porta's proof

G. M. Graf and M. Porta proved the bulk-edge correspondence in [GP13] by using another vector bundle. They defined a vector bundle over some torus for a system without edge by using (formal) solutions of the Hamiltonian which decays as it goes to a specific direction. This vector bundle is closely related to the edge index. By using this bundle, they showed the bulk-edge correspondence for the quantum spin Hall system. They also gave a proof of the bulk-edge correspondence for the quantum Hall system. Note that such vector bundles consist of decaying solutions were also considered in [Tho83]. Since this vector bundle is defined without using the translation invariance in the direction across the edge, disordered systems can also be treated (the translation invariance along the edge is assumed). They called topological invariants for such disordered systems obtained by using such vector bundles as bulk indices in the sense that these invariants are defined by the bulk Hamiltonian and do not use the Dirichlet boundary condition. Graf–Porta showed that such topological invariant defined by decaying solutions equals to the ($\mathbb{Z}$ or $\mathbb{Z}_2$-valued) spectral flow. For periodic systems, they also showed that such topological invariant equals to the bulk index (the TKNN number or the Kane-Mele invariant) by some consideration on a Riemann surface. Graf–Porta's proof is based on functional analysis and basic homotopy theory. Graf and Porta gave in [GP13] another proof of the bulk-edge correspondence for quantum Hall systems based on Levinson's theorem in scattering theory.

### 1.1.4 Other works

There are many other works about the bulk-edge correspondence, especially form the point of view of $K$-theory and index theory. Avron–Seiler–Simon considered an index theorem for a pair of projections in this context [ASS94], and Elbau–Graf proved the bulk-edge correspondence for the quantum Hall system by generalizing their method [EG02]. Mathai–Thiang used T-duality to prove the bulk-edge correspondence. The case of systems with symmetry are also treated in their paper with K. C. Hannabuss [MT15, MT16a, MT16b, HMT16]. Kaufmann–Li–Wehefritz-Kaufmann studied the $\mathbb{Z}_2$-valued invariant from the point of view of $K$-theory and index theory [KLWK15a, KLWK15b]. In [KLWK15b], a proof of the bulk-edge correspondence for topological insulators is given by using the Baum–Connes isomorphism. In [ASBVB13] Avila–Schulz-Baldes–Villegas-Blas proved the bulk-edge correspondence for type AII topological insulators by using transfer matrix methods as in [Hat93b]. In [CMT16], Cornean–Monico–Teufel gave a constructive proof of the bulk-edge correspondence for type AII topological insulators . Note that $K$-theory is also used in the classification of topological phases [Kit09, FM13, Kel15, Thi16, Kub16b].

### 1.1.5 Quarter-plane Toeplitz operators

Apart from the study of topological phases, the analysis of Toeplitz operators was developed (see [Dou88, Dou98, BS06], for example). We here focus on the theory of quarter-plane Toeplitz operators. Such operators were first studied by R. G. Douglas and R. Howe, and many results were obtained [DH71, CDSS71, CDS72]. Among other things, E. Park showed that there is a short exact sequence for $C^*$-algebras,

$$0 \to \mathcal{K} \to \mathcal{T}^{\alpha,\beta} \to \mathcal{S}^{\alpha,\beta} \to 0. \tag{1}$$

where $\mathcal{T}^{\alpha,\beta}$ is the quarter-plane Toeplitz algebra (the meaning of each symbols are explained in Sect. 2.4). Park studied quarter-plane Toeplitz algebras by using $K$-theory for $C^*$-algebras [Par90, PS91].

## 1.2 Bulk-edge correspondence and the cobordism invariance of the index

In section 3, we study Graf–Porta's proof of the bulk-edge correspondence to use another vector bundle, from the point of view of $K$-theory and index theory. We give a proof of the bulk-edge correspondence for some two-dimensional type A and type AII topological insulators (Theorem 3.2.7 and Theorem 3.5.7) in the Altland-Zirnbauer classification [AZ97]. Our proof is based on the cobordism invariance of the index.

The proof goes as follows. We first construct two elements of some compactly supported $K$-groups, and see that their family indices are the $K$-class of the Bloch bundle (Lemma 3.4.2 and Lemma 3.5.11) and the $K$-class of the difference of the class of vector bundle consists of decaying solutions and that of a trivial bundle (Lemma 3.4.3 and Lemma 3.5.12), respectively. We then construct a cobordism between them. The cobordism is given by an element of a compactly supported $K$-group whose support is a Riemann surface considered in [GP13]. By using the cobordism invariance of the index (Lemma 3.4.4 and Lemma 3.5.13), we obtain a relation between the bulk index and the first Chern number of the vector bundle consists of decaying solutions (Proposition 3.4.1 and Proposition

3.5.10). We next show a relation between the latter invariant and the edge index by using the localization of the $K$-class and the excision property of the index (Proposition 3.4.5 and Proposition 3.5.14). This part is just a computation of some element of a $K$-group.

### 1.3 Topological invariants and corner states for Hamiltonians on a three dimensional lattice

In section 4, we consider a system with corner which appears as the intersection of two edges. We consider a periodic Hamiltonian on the three dimensional lattice $\mathbb{Z}\times\mathbb{Z}\times\mathbb{Z}$, which has a spectral gap not only on the bulk (which means that our Hamiltonian is gapped) but also on two edges (which means that restrictions of our Hamiltonian onto two semigroups of our lattice which correspond to two edges, are gapped) at the common Fermi level. For such systems, we define two topological invariants. One is defined for the bulk and edges (Definition 4.1.3). This definition says that there are topological phases for such "gapped" Hamiltonians. The other invariant is defined for the corner (Definition 4.1.5). This invariant is directly related to wave functions localized near the corner (Remark 4.2.2). Both invariants are defined as elements of some $K$-groups. Note that, in our settings, topological invariants considered in the case of the bulk-edge correspondence are zero since our edges are also gapped. In this sense, compared to the TKNN number and the Kane-Mele invariant, invariants considered in this part can be seen as secondary invariants. We next show a relation of these two invariants (Theorem 4.2.1), by using the six-term exact sequence associated to the quarter-plane Toeplitz extension (1) obtained by Park [Par90]. Although we mainly consider some three dimensional systems, our method can also be applied to some systems of other dimensions (Remark 4.2.5). Note that, except for the use of the sequence (1), our invariants are defined and the relation is proved as in the case of the bulk-edge correspondence [KRSB02].

### 1.4 Organization of this thesis

This thesis is organized as follows. In Sect. 2, some basic facts needed in this thesis is collected. In Sect. 3, a proof of the bulk-edge correspondence for type A and type AII topological insulators based on the cobordism invariance of the index is given. In Sect. 4, systems with corner are considered. A topological invariant for a "gapped" Hamiltonian is defined and a relation with localized wave functions near its corner is proved.


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## 2 Preliminaries

### 2.1 Topological $K$-theory

In this section, we collects some basic facts about topological $K$-theory (complex $K$-theory and $KSp$-theory) needed in this paper. For the later use, we give here an explicit construction of the isomorphism $\widetilde{KSp}^0(\mathbb{S}^{2,1}) \cong \mathbb{Z}_2$.

#### 2.1.1 Complex $K$-theory

We here corrects some basic facts about complex $K$-theory. Details can be found in [Hus94, Ati67, AS68a, Kar78, LM89, Fri00, Par08].

Let $X$ be a compact Hausdorff space. The isomorphism classes of finite rank complex vector bundles over $X$ makes an abelian monoid by taking direct sum as its binary operation. We define the $K$-group $K^0(X)$ for $X$ as the Grothendieck group of this abelian monoid. We denote $[E]$ for the class of a vector bundle $E$. $K^0(X)$ is a commutative ring with unit by taking tensor product as its multiplication. For a finite dimensional complex vector space $V$, we denote $\underline{V}$ for the product bundle $X \times V$ over $X$.[1] Let $x$ be a point of $X$. We define the reduced $K$-group for a based space $(X, x)$ by $\widetilde{K}^0(X) := \mathrm{Ker}(i^* \colon K^0(X) \to K^0(\{x\}))$, where $i \colon \{x\} \hookrightarrow X$ is a natural inclusion. Let $A$ be a closed subspace of $X$. We define the $K$-group for a pair $(X, A)$ by $K^0(X, A) := \widetilde{K}^0(X/A)$, where we take $A/A$ as a base point of $X/A$. For a locally compact Hausdorff space $Y$, we take its one-point compactification $Y^+$. Then $(Y^+, +)$ is a based compact Hausdorff space. We define $K^0_{\mathrm{cpt}}(Y) := \widetilde{K}^0(Y)$, which is called the compactly supported $K$-group for $Y$.

Let $n$ be a positive integer. We define $K^{-n}(X, A) := \widetilde{K}^0(\sum^n(X/A))$ where $\sum^n(X/A)$ is the $n$-fold reduced suspension. Let $\emptyset$ be the empty set, then we have $K^{-n}(X) = K^{-n}(X, \emptyset)$. We have the following long exact sequence.

$$\cdots \to K^{-1}(X, A) \to K^{-1}(X) \to K^{-1}(A) \xrightarrow{\partial} K^0(X, A) \to K^0(X) \to K^0(A).$$

There is an alternative description of $K$-groups. $K^0_{\mathrm{cpt}}(Y)$ can be defined as an equivalence classes of the isomorphism classes of triples $(E, F; f)$, where $E$ and $F$ are finite rank complex vector bundles over $Y$ and $f \colon E \to F$ is a bundle homomorphism invertible outside a compact set. Its equivalence relation is generated by stabilization and homotopy. We denote $[E, F; f]$ for its class.

Let $\mathbf{GL}(\infty, \mathbb{C})$ be the inductive limit of a sequence $\mathbf{GL}(1, \mathbb{C}) \to \mathbf{GL}(2, \mathbb{C}) \to \cdots$, where $\mathbf{GL}(n, \mathbb{C}) \to \mathbf{GL}(n+1, \mathbb{C})$ is given by $\mathbf{A} \mapsto \mathrm{diag}(\mathbf{A}, \mathbf{1})$. Then we have a natural isomorphism $[X, \mathbf{GL}(\infty, \mathbb{C})] \cong K^{-1}(X)$. We denote $[f]$ for the homotopy class of a continuous map $f \colon X \to \mathbf{GL}(\infty, \mathbb{C})$.

Let $H$ be a separable Hilbert space. We denote $\mathrm{Fred}(H)$ for the space of bounded linear Fredholm operators on $H$ with norm topology. Then there is a natural bijection $\mathrm{index} \colon [X, \mathrm{Fred}(H)] \to K^0(X)$ given by taking the family index, which was shown by Atiyah [Ati67] and K. Jänich [Jän65] independently. We refer the reader to [Ati67] for the construction of this map. Instead of explaining the construction, we note that, if a continuous map $f \colon X \to \mathrm{Fred}(H)$ consists of a family $f(x)$ of Fredholm operators whose

[1] Note that the notation $\underline{V}$ can mean product bundles over different base spaces. We use this notation for simplicity. Its base space will be clear from the context.

dimension of kernels are constant, then $\mathrm{index}(f) = [\mathrm{Ker}(f)] - [\mathrm{Coker}(f)]$, where $\mathrm{Ker}(f)$ and $\mathrm{Coker}(f)$ are vector bundles over $X$ whose fibers at a point $x$ in $X$ are $\mathrm{Ker}(f(x))$ and $\mathrm{Coker}(f(x))$ respectively.

Let $E$ be an even dimensional spin$^c$ vector bundle on $Y$. Then we have an isomorphism $K^0_{\mathrm{cpt}}(Y) \to K^0_{\mathrm{cpt}}(E)$, called the *Thom isomorphism*, given by taking a cup product with the Thom class associated to the spin$^c$ structure of $E$. If $E = Y \times \mathbb{C}$ and if we consider for $E$ a spin$^c$ structure naturally induced by the complex structure of $E$, the Thom isomorphism (or Bott isomorphism) $K^0_{\mathrm{cpt}}(Y) \to K^0_{\mathrm{cpt}}(Y \times \mathbb{C})$ is given by taking a cup product with the Bott class $[\underline{\mathbb{C}}, \underline{\mathbb{C}}; z] \in K^0_{\mathrm{cpt}}(\mathbb{C})$.

Let $X$ and $Y$ be manifolds possibly with boundary and let $f\colon X \to Y$ be a neat embedding.[2] Let $N$ be the normal bundle of this embedding. We assume that $N$ is even dimensional and equipped with a spin$^c$ structure. We take a tubular neighborhood of the embedded manifold, and identify it with $N$. Then we can define a *push-forward map* $f_!\colon K^0_{\mathrm{cpt}}(X) \to K^0_{\mathrm{cpt}}(Y)$ by the composition of the Thom isomorphism $K^0_{\mathrm{cpt}}(X) \to K^0_{\mathrm{cpt}}(N)$ and the map $K^0_{\mathrm{cpt}}(N) \to K^0_{\mathrm{cpt}}(Y)$ induced by the collapsing map $Y^+ \to Y^+/(Y^+ - N) \cong N^+$. The push-forward map $f_!$ is independent of the choice of a tubular neighborhood.

Let $X$ be an even-dimensional spin$^c$ manifold without boundary. We take an embedding $j$ of $X$ into an even dimensional Euclidean space $\mathbb{R}^{2n}$. We fix a spin$^c$ structure on $\mathbb{R}^{2n}$. Then the normal bundle $N$ of this embedding has a naturally induced spin$^c$ structure, and we have a push-forward map $j_!\colon K^0_{\mathrm{cpt}}(X) \to K^0_{\mathrm{cpt}}(\mathbb{R}^{2n})$. We take an orientation preserving linear isomorphism $\mathbb{R}^{2n} \cong \mathbb{C}^n$, and consider the inverse of Thom isomorphism $\beta^{-1}\colon K^0_{\mathrm{cpt}}(\mathbb{R}^{2n}) \cong K^0_{\mathrm{cpt}}(\mathbb{C}^n) \to K^0(\{\mathrm{pt}\}) \cong \mathbb{Z}$. We define a homomorphism $\mathrm{ind}_X\colon K^0(X) \to \mathbb{Z}$ by $[E] \mapsto \beta^{-1} \circ j_!([E] \otimes [L(X)])$, where $L(X)$ is the determinant line bundle associated to the spin$^c$ structure of $X$.[3]

Remark 2.1.1. The map $\mathrm{ind}_X$ coincides with the composite of the Thom isomorphism $K^0(X) \to K^0_{\mathrm{cpt}}(TX)$ associated to the spin$^c$ structure of $X$, and the *topological index* $K^0_{\mathrm{cpt}}(TX) \to \mathbb{Z}$ defined by Atiyah and I. M. Singer in [AS68a].[4]

### 2.1.2 $KSp$-theory

We here collects basic properties of $KSp$-theory which was introduced by J. D. Dupont [Dup69]. The definition of Real vector bundle is also included for the later use. Details can be found in [Ati66, Dup69, DNG14, DNG15].

Let $X$ be a topological space. A homeomorphism $\tau_X\colon X \to X$ of period 2 (i.e. $\tau_X^2 = \mathrm{id}_X$) is called an involution on $X$. We call the pair $(X, \tau_X)$ an *involutive space*. Let $X^\tau$ be the fixed point set of this involution.

Example 2.1.2. We denote by $\mathbb{R}^{p,q}$ the involutive space $\mathbb{R}^q \oplus i\mathbb{R}^p$ whose involution is given by id on the first $\mathbb{R}^q$ and $-\mathrm{id}$ on the second $\mathbb{R}^p$. We also denote by $\mathbb{D}^{p,q}$ the unit ball in $\mathbb{R}^{p,q}$ and $\mathbb{S}^{p,q}$ for the unit sphere in $\mathbb{R}^{p,q}$ with induced involutions. Note that the involutive space $\mathbb{S}^{p,q}$ is topologically $\mathbb{S}^{p+q-1}$.

[2] An embedding $f\colon X \hookrightarrow Y$ is said to be *neat* if $X \cap \partial Y = \partial X$ and the space $T_xX$ is not contained in $T_x(\partial Y)$ for any point $x \in \partial X$ (see [Hir94]).

[3] The determinant line bundle $L(X)$ associates to the principal spin$^c$ bundle of the spin$^c$ structure by the homomorphism $\mathbf{Spin^c}(2n) = (\mathbf{Spin}(2n) \times \mathbf{U}(1))/\{\pm 1\} \ni [\lambda, z] \mapsto z^2 \in \mathbf{U}(1)$.

[4] There exists different sign conventions in $K$-theory. If we use another conventions, the expression of $\mathrm{ind}_X$ can be different, for example. In this paper, we follow the one used in [AS68a].

Example 2.1.3. The complex projective space $\mathbb{C}\mathrm{P}^n = (\mathbb{C}^{n+1}\backslash\{0\})/\sim$ has an involution $\tau_{\mathbb{C}\mathrm{P}}$ induced by the complex conjugation on $\mathbb{C}^{n+1}$. The pair $(\mathbb{C}\mathrm{P}^n, \tau_{\mathbb{C}\mathrm{P}})$ is an involutive space.

A pair $(E, \Theta)$ is called a *Quaternionic* (resp. *Real*) *vector bundle* over the involutive space $(X, \tau_X)$, if $E$ is a complex vector bundle over the space $X$ and $\Theta\colon E \to E$ is a (topological) homeomorphism which satisfies, (i) the projection $E \to X$ is $\mathbb{Z}_2$-equivariant, (ii) $\Theta\colon E_x \to E_{\tau_X(x)}$ is anti-linear and (iii) $\Theta^2 = -\mathrm{id}_E$ (resp. $\Theta^2 = \mathrm{id}_E$). A homomorphism between Quaternionic (resp. Real) vector bundles $(E, \Theta_E)$ and $(F, \Theta_F)$ over $(X, \tau_X)$ is a homomorphism $\varphi\colon E \to F$ of complex vector bundles which satisfies $\varphi \circ \Theta_E = \Theta_F \circ \varphi$.

Example 2.1.4 (Hopf bundle). The Hopf bundle (or the tautological bundle) $H$ over $\mathbb{C}\mathrm{P}^n$ has an involution $\tau_H$ induced by the complex conjugation on $\mathbb{C}^{n+1}$. Then $(H, \tau_H)$ is a Real vector bundle over the involutive space $(\mathbb{C}\mathrm{P}^n, \tau_{\mathbb{C}\mathrm{P}})$.

Example 2.1.5 (Product bundle). Let $(X, \tau_X)$ be an involutive space. Let $\mathbb{H} = \mathbb{C} \oplus \mathbb{C}j$ be the quaternion. We consider $\mathbb{C}^{2m} = \mathbb{H}^m$. Then the projection onto the first component $\underline{\mathbb{C}^{2m}} = X \times \mathbb{C}^{2m} \to X$ is a complex vector bundle. Consider an anti-linear map $\Theta_0$ on the product space $X \times \mathbb{C}^{2m}$ given by $\Theta_0(x, v) = (\tau_X(x), jv)$. Then the pair $(\underline{\mathbb{C}^{2m}}, \Theta_0)$ is a Quaternionic vector bundle over $(X, \tau_X)$. A vector bundle which is isomorphic to some product bundle is called a trivial bundle.

We now assume $X$ to be a compact Hausdorff space. The isomorphism classes of Quaternionic vector bundles[5] $\mathrm{Vect}_Q(X, \tau_X)$ over the involutive space $(X, \tau_X)$ makes an abelian monoid by taking direct sum as its binary operation. We define the $KSp$-group $KSp^0(X, \tau_X)$ for $(X, \tau_X)$ as the Grothendieck group of this abelian monoid. We denote by $[(E, \Theta)]$ the class of a Quaternionic vector bundle $(E, \Theta)$ over $(X, \tau_X)$ in $KSp^0(X, \tau_X)$. As in the case of $K$-groups, we can define the reduced $KSp$-group $\widetilde{KSp}^0(X, \tau_X)$ for an involutive space $((X, \tau_X), x)$ with a base point $x \in X^\tau$, the $KSp$-group $KSp^0((X, \tau_X), (A, \tau_X))$ for a pair of $(X, \tau_X)$ and its closed subspace $A$ with $\tau_X(A) = A$, and the compactly supported $KSp$-group $KSp^0_{\mathrm{cpt}}(Y, \tau_Y)$ for a locally compact Hausdorff involutive space $(Y, \tau_Y)$. Let $n$ be a positive integer. For such a pair $(X, \tau_X)$ and $A$, we define,

$$KSp^{-n}((X, \tau_X), A) := KSp^0((X, \tau_X) \times \mathbb{D}^{0,n}, (X, \tau_X) \times \mathbb{S}^{0,n} \cup (A, \tau_X) \times \mathbb{D}^{0,n}).$$

Then there is a long exact sequence associated to $((X, \tau_X), (A, \tau_X))$. By the *Bott periodicity theorem* [Ati66], we have the following natural isomorphism,

$$KSp^0_{\mathrm{cpt}}(Y, \tau_Y) \cong KSp^0_{\mathrm{cpt}}((Y, \tau_Y) \times \mathbb{R}^{1,1}).$$

There also is an alternative description of $KSp$-groups. A pair of Quaternionic vector bundles $(E, \Theta_E)$ and $(F, \Theta_F)$ over $(Y, \tau_Y)$ and a ($\mathbb{Z}_2$-equivariant) bundle homomorphism $f\colon (E, \Theta_E) \to (F, \Theta_F)$ which is invertible outside a compact set defines an element of the $KSp$-group $KSp^0_{\mathrm{cpt}}(Y, \tau_Y)$. We write its class by $[(E, \Theta_E), (F, \Theta_F); f]$.

Let $(H, \Theta)$ be a pair of a separable complex Hilbert space $H$ and an anti-linear map $\Theta\colon H \to H$ which satisfies $\Theta\Theta^* = \mathrm{id}$ and $\Theta^2 = -\mathrm{id}$. The map $\Theta$ induces an involution $\tau_\Theta$ on $B(H)$ given by $\tau_\Theta(G) = \Theta G \Theta^*$. $\tau_\Theta$ is closed on $\mathrm{Fred}(H)$ and the pair $(\mathrm{Fred}(H), \tau_\Theta)$ is an involutive space. Then we have a natural isomorphism index: $[(X, \tau_X), (\mathrm{Fred}(H), \tau_\Theta)]_{\mathbb{Z}_2} \to KSp^0(X, \tau_X)$ where $[-,-]_{\mathbb{Z}_2}$ is the homotopy classes of $\mathbb{Z}_2$-equivariant maps.

[5] We consider only such bundles of finite rank underlying complex vector bundles.

Let $\mathbb{C}^{2m} = \mathbb{H}^m$. The action of $j$ induces an involution $\tau_j$ on $\mathbf{GL}(2,\mathbb{C})$. By taking the inductive limit of a sequence $(\mathbf{GL}(2,\mathbb{C}),\tau_j) \to (\mathbf{GL}(4,\mathbb{C}),\tau_j) \to \cdots$, we have an involutive space $(\mathbf{GL}(\infty,\mathbb{C}),\tau_j)$. Then we have a natural isomorphism $[(X,\tau_j),(\mathbf{GL}(\infty,\mathbb{C}),\tau_j)]_{\mathbb{Z}_2} \cong KSp^{-1}(X,\tau_X)$.

Let $(X,\tau_X)$ and $(T,\tau_Y)$ be involutive manifolds possibly with boundary. We assume that its boundaries $\partial X$ and $\partial Y$ are closed under involutions $\tau_X$ and $\tau_Y$, respectively. We further assume that there is a $\mathbb{Z}_2$-equivariant neat embedding $f\colon (X,\tau_X) \to (T,\tau_Y)$ whose normal bundle $(N,\tau_N)$ is a trivial bundle of fiber $\mathbb{R}^{n,n}$. Then we can define a map from $(X,\tau_X)$ to $(T,\tau_Y)$ as in the case of complex $K$-theory. We identify the normal bundle $(N,\tau_N)$ with a ($\mathbb{Z}_2$-equivariant) tubular neighborhood of this embedding. We then define the map $f_!^{\mathrm{AII}}\colon (X,\tau_X) \to (T,\tau_Y)$ by the composition of the Bott periodicity isomorphism $KSp^0_{\mathrm{cpt}}(X,\tau_X) \to KSp^0_{\mathrm{cpt}}((X,\tau_X)\times\mathbb{R}^{n,n}) \cong KSp^0_{\mathrm{cpt}}(N,\tau_N)$ and the map $\mathrm{ext}\colon KSp^0_{\mathrm{cpt}}(N,\tau_N) \to KSp^0_{\mathrm{cpt}}(Y,\tau_Y)$ induced by the collapsing map $Y^+ \to Y^+/(Y^+ - N) \cong N^+$.

### 2.1.3 An explicit construction of the isomorphism $\widetilde{KSp}^0(\mathbb{S}^{2,1}) \cong \mathbb{Z}_2$.

We first note that the reduced $KSp$-group $\widetilde{KSp}(\mathbb{S}^{2,1})$ is calculated as follows,

$$\widetilde{KSp}(\mathbb{S}^{2,1}) \cong KSp(\mathbb{D}^{2,0},\mathbb{S}^{2,0}) \cong KSp_{\mathrm{cpt}}(\mathbb{R}^{2,0}) \cong \mathbb{Z}_2.$$

Thus there is the unique group isomorphism between $\widetilde{KSp}(\mathbb{S}^{2,1})$ and $\mathbb{Z}_2$. In this subsection, we give an explicit construction of this map and give an explicit expression of the non-zero element in $\widetilde{KSp}(\mathbb{S}^{2,1})$. Let $(E,\Theta)$ be a Quaternionic vector bundle over the involutive space $\mathbb{S}^{2,1}$. The space $\mathbb{S}^{2,1}$ has two fixed point sets of its involution for which we denote $0$ and $\infty$. Since $\mathbb{S}^{2,1}$ has fixed points, the complex rank of $E$ is even, for which we write $2n$. We consider a (complex) determinant $\det E$. Then $\Theta$ induces an anti-linear involution $\det\Theta$ on $\det E$ in a natural way, and the pair $(\det E, \det\Theta)$ is a Real vector bundle over $\mathbb{S}^{2,1}$. Let us divide $\mathbb{S}^{2,1} = \mathbb{D}^{2,0}_0 \cup_{\mathbb{S}^{2,0}} \mathbb{D}^{2,0}_\infty$ into an upper hemisphere $\mathbb{D}^{2,0}_0 \ni 0$ and a lower hemisphere $\mathbb{D}^{2,0}_\infty \ni \infty$. Since $\mathbb{D}^{2,0}$ is $\mathbb{Z}_2$-equivariant contractible, restrictions of $(E,\Theta)$ and $(\det E,\det\Theta)$ onto $\mathbb{D}^{2,0}_0$ and $\mathbb{D}^{2,0}_\infty$ are trivial (see [DNG14] [DNG15], for example). Since $Sp(n)$ is connected, the bundle $(E,\Theta)$ has trivializations on $0$ and $\infty$ which are unique up to homotopy, that is,

$$(E_0,\Theta) \cong \{0\}\times\mathbb{H}^n, \quad (E_\infty,\Theta) \cong \{\infty\}\times\mathbb{H}^n.$$

Above trivializations induce canonical trivializations on $(\det E_0,\det\Theta)$ and $(\det E_\infty,\det\Theta)$ which are unique up to homotopy. These trivializations extend to trivializations of $(\det E,\det\Theta)$ on $\mathbb{D}^{2,0}_0$ and $\mathbb{D}^{2,0}_\infty$, respectively. Such trivializations are unique up to homotopy. By using these trivializations, we obtain a clutching function $f\colon \mathbb{S}^{2,0} \to (\mathbf{U}(1),\tau)$ of the bundle $E$, where $\tau(z) = \bar{z}$. It is easily checked that $f$ is a $\mathbb{Z}_2$-equivariant continuous map. Thus we have a map,

$$\mathrm{Vect}_Q(\mathbb{S}^{2,1}) \to [\mathbb{S}^{2,0},(\mathbf{U}(1),\tau)]_{\mathbb{Z}_2}.$$

We give a group structure on $[\mathbb{S}^{2,0},(\mathbf{U}(1),\tau)]_{\mathbb{Z}_2}$ by using the multiplication on $\mathbf{U}(1)$. Since $\det(f\oplus g) = \det f\cdot\det g$, the above map is a monoid homomorphism.

**Lemma 2.1.6.** *For a $\mathbb{Z}_2$-equivariant map $f\colon \mathbb{S}^{2,0} \to (\mathbf{U}(1),\tau)$, there exists some point $t \in \mathbb{S}^{2,0}$ such that $f(t) \in \{\pm 1\}$.*

*Proof.* If not, the set $f(\mathbb{S}^{2,0})$ is contained in either the interior of the upper hemisphere of $\mathbf{U}(1)$ or that of the lower hemisphere. Since $f$ is $\mathbb{Z}_2$-equivariant, we have $f(-s) = \overline{f(s)}$ for $s \in \mathbb{S}^{2,0}$. These two conditions are inconsistent. $\square$

Let $f\colon \mathbb{S}^{2,0} \to (\mathbf{U}(1), \tau)$ be a $\mathbb{Z}_2$-equivariant map. By Lemma 2.1.6, there exists some $t \in \mathbb{S}^{2,0}$ such that $f(t) \in \{\pm 1\}$. We choose a path $l$ in $\mathbb{S}^{2,0}$ which connects $t$ and $-t$, and fix an orientation on $l$ and $\mathbf{U}(1)$. Since $f(-t) = \overline{f(t)} = f(t)$, the map $f|_l\colon l \to \mathbf{U}(1)$ defines a loop in $\mathbf{U}(1)$, and we consider the winding number for this loop. It is easily checked that the mod 2 of this winding number (for which we denote $w(f) \in \mathbb{Z}_2$) does not depend on the choice made. Thus we have a map,

$$w\colon [\mathbb{S}^{2,0}, (\mathbf{U}(1), \tau)]_{\mathbb{Z}_2} \to \mathbb{Z}_2.$$

It is easily checked that the map $w$ is a group homomorphism.

By the composition of the above maps, we obtain a monoid homomorphism $\mathrm{Vect}_Q(\mathbb{S}^{2,1}) \to \mathbb{Z}_2$. Thus by the universality of the Grothendieck construction, we obtain a group homomorphism, $KSp^0(\mathbb{S}^{2,1}) \to \mathbb{Z}_2$. It is clear by construction that this map maps classes of product bundles to zero. Thus the following group homomorphism on the reduced $KSp$-group is induced.

$$B\colon \widetilde{KSp}(\mathbb{S}^{2,1}) \to \mathbb{Z}_2.$$

Example 2.1.7. We give explicit examples of the element of $\widetilde{KSp}(\mathbb{S}^{2,1}) \cong KSp(\mathbb{D}^{2,0}, \mathbb{S}^{2,0})$ which maps to $1 \in \mathbb{Z}_2$. On $\mathbb{C}^2$, we consider a left $\mathbb{H}$-moudle structure given by,

$$I = \begin{pmatrix} i & 0 \\ 0 & i \end{pmatrix}, \quad J = \begin{pmatrix} 0 & -\tau \\ \tau & 0 \end{pmatrix},$$

where $\tau$ denotes for the complex conjugation on $\mathbb{C}$. Then the quaternionic structure $J$ on $\mathbb{C}^2$ defines an involution $\tau_J$ on $M(2, \mathbb{C})$ given by

$$\tau_J \begin{pmatrix} x & y \\ z & w \end{pmatrix} = J \begin{pmatrix} x & y \\ z & w \end{pmatrix} J^* = \begin{pmatrix} \bar{w} & -\bar{z} \\ -\bar{y} & \bar{x} \end{pmatrix}.$$

We now identify $[0, 2\pi]/\{0, 2\pi\}$ with $\mathbb{S}^{2,0}$ by $\theta \mapsto e^{i\theta}$. Then the involution on $\mathbb{S}^{2,0}$ is translated into the involution on $[0, 2\pi]/\{0, 2\pi\}$ given by $\theta \mapsto \theta \pm \pi$. Let $k$ be an integer. By using this expression, we define a map $g_k\colon [0, 2\pi]/\{0, 2\pi\} \to (M(2, \mathbb{C}), \tau_J)$ by,

$$g_k(\theta) = \begin{cases} \begin{pmatrix} e^{2ik\theta} & 0 \\ 0 & 1 \end{pmatrix} & \text{if} \quad 0 \leq \theta \leq \pi, \\ \begin{pmatrix} 1 & 0 \\ 0 & e^{-2ik\theta} \end{pmatrix} & \text{if} \quad \pi \leq \theta \leq 2\pi. \end{cases}$$

It is easily checked that the map $g_k$ is $\mathbb{Z}_2$-equivariant. Thus the triple $[(\underline{\mathbb{C}}^2, J), (\underline{\mathbb{C}}^2, J); g_k]$ defines an element in $KSp(\mathbb{D}^{2,0}, \mathbb{S}^{2,0})$. The determinant of $g_k$ is the following map.

$$\det g_k(\theta) = \begin{cases} e^{2ik\theta} & \text{if} \quad 0 \leq \theta \leq \pi, \\ e^{-2ik\theta} & \text{if} \quad \pi \leq \theta \leq 2\pi, \end{cases}$$

which is $\mathbb{Z}_2$-equivariant. It is easily checked that $w(\det g_k) = k \mod 2 \in \mathbb{Z}_2$.

By this example, the following holds.

**Proposition 2.1.8.** *The map $B\colon \widetilde{KSp}^0(\mathbb{S}^{2,1}) \to \mathbb{Z}_2$ is a group isomorphism.*

## 2.2 Spectral flow

The spectral flow was considered by Atiyah and G. Lusztig, and first appeared in Atiyah-Patodi-Singer's work [APS75, APS76]. In this subsection, we review one definition of the $\mathbb{Z}$-valued spectral flow by Phillips [Phi96]. We discuss some relationship between the spectral flow and the winding number [AS69, Phi96]. For the later use, we give a definition of the $\mathbb{Z}_2$-valued spectral flow following [Phi96].

### 2.2.1 $\mathbb{Z}$-valued spectral flow

Spectral flow is, roughly speaking, the net number of crossing points of eigenvalues of the family of self-adjoint Fredholm operators with zero.[6]

Definition 2.2.1 (Spectral Flow [Phi96]). Let $\mathrm{Fred}(H)^{\mathrm{s.a.}}$ be the space of self-adjoint Fredholm operators on a fixed separable complex Hilbert space with norm topology. Let $B\colon [-\pi,\pi] \to \mathrm{Fred}(H)^{\mathrm{s.a.}}$ be a continuous path. We choose a partition $-\pi = t_0 < t_1 < \cdots < t_n = \pi$ and positive numbers $c_1, c_2, \ldots, c_n$ so that for each $i = 1, 2, \ldots, n$, the function $t \mapsto \chi_{[-c_i,c_i]}(B_t)$ is continuous and finite rank on $[t_{i-1}, t_i]$, where $\chi_{[a,b]}$ denotes the characteristic function of $[a, b]$. We define the spectral flow of $B$ by

$$\mathrm{sf}(B) := \sum_{i=1}^{n}(\mathrm{rank}_{\mathbb{C}}(\chi_{[0,c_i]}(B_{t_i})) - \mathrm{rank}_{\mathbb{C}}(\chi_{[0,c_i]}(B_{t_{i-1}})) \in \mathbb{Z}.$$

Spectral flow is independent of the choice made and depends only on the homotopy class of the path $B$ leaving the endpoints fixed.

Let $\mathbb{R}_{>0}$ (resp. $\mathbb{R}_{<0}$) be the set of strictly positive (resp. negative) real numbers. For a separable Hilbert space $H$, we consider the space of bounded linear operators $B(H)$ on $H$ with norm topology. Let $\mathrm{Fred}(H)^{\mathrm{s.a.}}_*$ be the subspace of $B(H)$ consists of all self-adjoint Fredholm operators whose essential spectrum[7] does not contained neither $\mathbb{R}_{>0}$ nor $\mathbb{R}_{<0}$. We consider the following subspace of $\mathrm{Fred}(H)^{\mathrm{s.a.}}_*$,

$$\hat{F}^{\infty}_* := \{B \in \mathrm{Fred}(H)^{\mathrm{s.a.}}_* \mid ||B|| = 1,\ \mathrm{sp}(B) \text{ is finite and } \mathrm{ess\text{-}sp}(B) = \{\pm 1\}\}.$$

Let $i\colon \hat{F}^{\infty}_* \to \mathrm{Fred}(H)^{\mathrm{s.a.}}_*$ be the inclusion. Then $i$ is a homotopy equivalence. Let $\mathbf{U}(\infty)$ be the inductive limit of a sequence
$U(1) \to \mathbf{U}(2) \to \cdots$, where $\mathbf{U}(n)$ is the unitary group of degree $n$, and the map $\mathbf{U}(n) \to \mathbf{U}(n+1)$ is given by $\mathbf{A} \mapsto \mathrm{diag}(\mathbf{A}, \mathbf{1})$. We have a map $j\colon \hat{F}^{\infty}_* \to \mathbf{U}(\infty)$ given by $j(B) = \exp(i\pi(B+1))$. The map $j$ also is a homotopy equivalence.[8] Thus we have $K_1(C(\mathbb{T})) = [\mathbb{T}, \mathbf{U}(\infty)] \cong [\mathbb{T}, \hat{F}^{\infty}_*] \cong [\mathbb{T}, \mathrm{Fred}(H)^{\mathrm{s.a.}}_*]$. For a continuous loop in $\mathrm{Fred}(H)^{\mathrm{s.a.}}_*$, we can define its spectral flow, which is, roughly speaking, the net number of crossing points of eigenvalues with zero counted with multiplicity. Note that we have the following

[6] Counted positively for increasing eigenvalue crossings with respect to the parameter.

[7] In this paper, we denote the spectrum of an operator $T$ by $\mathrm{sp}(T)$, and the essential spectrum of $T$ by $\mathrm{ess\text{-}sp}(T)$. For a self-adjoint operator $T$, the essential spectrum of $T$ consists of accumulation points of $\mathrm{sp}(T)$ and isolated points of $\mathrm{sp}(T)$ with infinite multiplicity (see Proposition 2.2.2 of [HR00], for example).

[8] The subspace $\hat{F}^{\infty}_*$ and the commutativity of the diagram (2) was discussed explicitly in [Phi96], while the homotopy equivalence between $\hat{F}^{\infty}_*$ and $\mathbf{U}(\infty)$ was essentially proved in [AS69].

commutative diagram.

$$\begin{tikzcd}
{[\mathbb{T}, \mathrm{Fred}(H)^{\mathrm{s.a.}}_*]} \arrow[r, "\mathrm{sf}"] & \mathbb{Z} \\
{[\mathbb{T}, \hat{F}^\infty_*]} \arrow[u, "i_*"] \arrow[r, "j_*"] & {[\mathbb{T}, \mathbf{U}(\infty)],} \arrow[u]
\end{tikzcd} \tag{2}$$

where the map $\mathrm{sf}\colon [\mathbb{T}, \mathrm{Fred}(H)^{\mathrm{s.a.}}_*] \to \mathbb{Z}$ is given by the spectral flow, and the map $[\mathbb{T}, \mathbf{U}(\infty)] \to \mathbb{Z}$ is given by taking the winding number of the determinant. All arrows are group isomorphisms.

### 2.2.2 $\mathbb{Z}_2$-valued spectral flow

In this subsection, we give a definition of $\mathbb{Z}_2$-valued spectral flow. Note that De-Nittis–Schulz-Baldes considered in [DNSB15] such $\mathbb{Z}_2$-valued spectral flow for wider class of operators. They also treat the case with other symmetries. Since we later use a concrete definition of $\mathbb{Z}_2$-valued spectral flow as in Definition 2.2.4, we give here one definition following [Phi96]. The well-definedness and the homotopy invariance of $\mathbb{Z}_2$-valued spectral flow are proved in the same way as in [Phi96].

Let $(H, \Theta)$ be a pair of a separable complex Hilbert space $H$ and an anti-linear operator $\Theta\colon H \to H$ which satisfies $\Theta\Theta^* = \mathrm{id}$ and $\Theta^2 = -\mathrm{id}$. Note that the space of self-adjoint Fredholm operators $\mathrm{Fred}(H)^{\mathrm{s.a.}}$ is closed under the involution $\tau_\Theta$ given by $\tau_\Theta(G) = \Theta G \Theta^*$. The involutive space $(\mathrm{Fred}(H)^{\mathrm{s.a.}}, \tau_\Theta)$ is decomposed into following three components which are closed under the involution,

$$(\mathrm{Fred}(H)^{s.a.}, \tau_\Theta) = (\mathrm{Fred}(H)^{s.a.}_+, \tau_\Theta) \sqcup (\mathrm{Fred}(H)^{s.a.}_-, \tau_\Theta) \sqcup (\mathrm{Fred}(H)^{s.a.}_*, \tau_\Theta),$$

where,

(1) $(\mathrm{Fred}(H)^{s.a.}_+, \tau_\Theta) := \{f \in (\mathrm{Fred}(H)^{s.a.}, \tau_\Theta) \mid \sigma_{\mathrm{ess}}(f) \subset (0, \infty)\}$,

(2) $(\mathrm{Fred}(H)^{s.a.}_-, \tau_\Theta) := \{f \in (\mathrm{Fred}(H)^{s.a.}, \tau_\Theta) \mid \sigma_{\mathrm{ess}}(f) \subset (-\infty, 0)\}$,

(3) $(\mathrm{Fred}(H)^{s.a.}_*, \tau_\Theta) := \{f \in (\mathrm{Fred}(H)^{s.a.}, \tau_\Theta) \mid \sigma_{\mathrm{ess}}(f) \not\subseteq (0, \infty) \text{ nor } \sigma_{\mathrm{ess}}(f) \not\subseteq (-\infty, 0)\}$.

**Lemma 2.2.2.** $(\mathrm{Fred}(H)^{s.a.}_+, \tau_\Theta)$ *and* $(\mathrm{Fred}(H)^{s.a.}_-, \tau_\Theta)$ *are $\mathbb{Z}_2$-equivariant contractible.*

*Proof.* For $(T, t) \in (\mathrm{Fred}(H)^{s.a.}_+, \tau_\Theta) \times [0, 1]$, the element $(1 - t)T + t\,\mathrm{id}$ is contained in $(\mathrm{Fred}(H)^{s.a.}_+, \tau_\Theta)$. Thus $(\mathrm{Fred}(H)^{s.a.}_+, \tau_\Theta)$ is an $\mathbb{Z}_2$-equivariant contractible to id. Note that we consider on $[0, 1]$ the trivial involution. Similarly, $(\mathrm{Fred}(H)^{s.a.}_-, \tau_\Theta)$ is contractible to $-\mathrm{id}$ by the homotopy $(1 - t)T + t(-\mathrm{id})$. □

The following Lemma is proved in [Phi96] (Lemma, p462).

**Lemma 2.2.3.** *Given $B \in \mathrm{Fred}(H)^{\mathrm{s.a.}}$, there is a positive number $c$ and an open neighborhood $N$ of $B$ in $\mathrm{Fred}(H)^{\mathrm{s.a.}}$ so that $S \mapsto \chi_{[-c,c]}(S)$ is a continuous, finite-rank projection-valued function on $N$. Where $\chi_{[-c,c]}$ denotes the characteristic function of $[-c, c]$.*

We now consider an involution $\tau$ on the interval $[-1, 1]$ given by $\tau(t) = -t$. Then $([-1, 1], \tau)$ is an involutive space. We define a $\mathbb{Z}_2$-valued spectral flow $\mathrm{sf}_2(B)$ for equivariant continuous maps $B\colon ([-1, 1], \tau) \to (\mathrm{Fred}(H)^{s.a.}, \tau_\Theta)$.

Definition 2.2.4. ($\mathbb{Z}_2$-valued spectral flow) Let $B\colon ([-1,1],\tau) \to (\mathrm{Fred}(H)^{s.a.}, \tau_\Theta)$ be a $\mathbb{Z}_2$-equivariant continuous map. By Lemma 2.2.3, we choose a partition $0 = t_0 < t_1 < \cdots < t_n = 1$ of $[0,1]$ and positive numbers $c_1, c_2, \ldots, c_n$ so that for each $i = 1, 2, \ldots, n$, the map $t \mapsto \chi_{[-c_i, c_i]}(B_t)$ is continuous and finite rank on $[t_{i-1}, t_i]$. We then define the $\mathbb{Z}_2$-*valued spectral flow* $\mathrm{sf}_2(B)$ of $B$ as follows.

$$\mathrm{sf}_2(B) := \sum_{k=1}^{n} (\mathrm{rank}_{\mathbb{C}}(\chi_{[0,c_k]}(B_{t_k})) + \mathrm{rank}_{\mathbb{C}}(\chi_{[0,c_k]}(B_{t_{k-1}}))) \mod 2.$$

Note that $\mathrm{sf}_2(B)$ takes value in $\mathbb{Z}_2$.

In order to show the well-definedness and the homotopy invariance of $\mathrm{sf}_2$, we next consider the disjoint union of intervals $[-2,-1] \sqcup [1,2]$ with an involution $\tau$ given by $\tau(t) = -t$.

Definition 2.2.5. Let $B\colon ([-2,-1] \sqcup [1,2], \tau) \to (\mathrm{Fred}(H)^{s.a.}, \tau_\Theta)$ be a $\mathbb{Z}_2$-equivariant continuous map. By Lemma 2.2.3 we choose a partition $1 = t_0 < t_1 < \cdots < t_n = 2$ of $[1,2]$ and positive numbers $c_1, c_2, \ldots, c_n$ so that for each $i = 1, 2, \ldots, n$, the map $t \mapsto \chi_{[-c_i, c_i]}(B_t)$ is continuous and finite rank on $[t_{i-1}, t_i]$. We then define $\mathrm{sf}_2(B)$ of $B$ as follows.

$$\mathrm{sf}_2(B) := \sum_{k=1}^{n} (\mathrm{rank}_{\mathbb{C}}(\chi_{[0,c_k]}(B_{t_k})) + \mathrm{rank}_{\mathbb{C}}(\chi_{[0,c_k]}(B_{t_{k-1}}))) \mod 2.$$

The next Lemma follows easily.

**Lemma 2.2.6.** *Let $B\colon ([-2,-1] \sqcup [1,2], \tau) \to (\mathrm{Fred}(H)^{s.a.}, \tau_\Theta)$ be an equivariant continuous map. We have a $\mathbb{Z}$-valued spectral flow $\mathrm{sf}(B|_{[1,2]})$ for the restriction of $B$ onto $[0,1]$. Then we have*

$$\mathrm{sf}_2(B) = \mathrm{sf}(B|_{[1,2]}) \mod 2.$$

We now consider the concatenation of such $\mathbb{Z}_2$-equivariant maps. Let $B\colon ([-1,1],\tau) \to (\mathrm{Fred}(H)^{s.a.}, \tau_\Theta)$ and $C\colon ([-2,-1] \sqcup [1,2], \tau) \to (\mathrm{Fred}(H)^{s.a.}, \tau_\Theta)$ be $\mathbb{Z}_2$-equivariant continuous maps with $B_1 = C_1$. Then we define an equivariant continuous map $C \tilde{\circ} B\colon ([-2,2],\tau) \to (\mathrm{Fred}(H)^{s.a.}, \tau_\Theta)$ given by

$$(C \tilde{\circ} B)(t) = \begin{cases} B(t) & \text{if } -1 \leq t \leq 1 \\ C(t) & \text{if } -2 \leq t \leq -1 \text{ or } 1 \leq t \leq -2, \end{cases}$$

where an involution $\tau$ on $[-2,2]$ is defined by $\tau(t) = -t$. We then obtain the following formula.

**Lemma 2.2.7.** *Let $B\colon ([-1,1],\tau) \to (\mathrm{Fred}(H)^{s.a.}, \tau_\Theta)$ and $C\colon ([-2,-1] \sqcup [1,2], \tau) \to (\mathrm{Fred}(H)^{s.a.}, \tau_\Theta)$ be $\mathbb{Z}_2$-equivariant continuous maps with $B_1 = C_1$. Then we have,*

$$\mathrm{sf}_2(C \tilde{\circ} B) = \mathrm{sf}_2(B) + \mathrm{sf}_2(C).$$

We now prove the well-definedness (Proposition 2.2.8) and the homotopy invariance (Proposition 2.2.12) of the $\mathbb{Z}_2$-valued spectral flow.

**Proposition 2.2.8.** $\mathrm{sf}_2(B)$ *is well-defined, that is it depends only on the $\mathbb{Z}_2$-equivariant continuous map $B\colon ([-1,1],\tau) \to (\mathrm{Fred}(H)^{s.a.}, \tau_\Theta)$.*

*Proof.* In order to prove this proposition, it is enough to show that $\mathrm{sf}_2(B)$ does not change under the change of the choice of a partition $0 = t_0 < t_1 < \cdots < t_n = 1$ (Step 1), and that of positive numbers $c_1, c_2, \ldots, c_n$ (Step 2).

Step 1. We add a new point $t_*$ and consider the partition $0 = t_0 < t_1 < \cdots < t_j < t_* < t_{j+1} < t_n = 1$ and take $c_* = c_{j+1}$. Then $\mathrm{sf}_2(B)$ defined by this choice is,

$$\begin{aligned}\mathrm{sf}_2(B) = \ & \mathrm{rank}_{\mathbb{C}}(\chi_{[0,c_1]}(B_1)) + \mathrm{rank}_{\mathbb{C}}(\chi_{[0,c_1]}(B_0)) \\ & + \mathrm{rank}_{\mathbb{C}}(\chi_{[0,c_2]}(B_2)) + \mathrm{rank}_{\mathbb{C}}(\chi_{[0,c_2]}(B_1)) \\ & \quad \cdots \\ & + \mathrm{rank}_{\mathbb{C}}(\chi_{[0,c_j]}(B_j)) + \mathrm{rank}_{\mathbb{C}}(\chi_{[0,c_j]}(B_{j-1})) \\ & + \mathrm{rank}_{\mathbb{C}}(\chi_{[0,c_*]}(B_*)) + \mathrm{rank}_{\mathbb{C}}(\chi_{[0,c_*]}(B_j)) \\ & + \mathrm{rank}_{\mathbb{C}}(\chi_{[0,c_{j+1}]}(B_{j+1})) + \mathrm{rank}_{\mathbb{C}}(\chi_{[0,c_{j+1}]}(B_*)) \\ & \quad \cdots \\ & + \mathrm{rank}_{\mathbb{C}}(\chi_{[0,c_n]}(B_n)) + \mathrm{rank}_{\mathbb{C}}(\chi_{[0,c_n]}(B_{n-1})) \mod 2.\end{aligned}$$

Since $c_* = c_{j+1}$, we have

$$\mathrm{rank}_{\mathbb{C}}(\chi_{[0,c_*]}(B_*)) + \mathrm{rank}_{\mathbb{C}}(\chi_{[0,c_{j+1}]}(B_*)) = 0 \mod 2.$$

Thus $\mathrm{sf}_2(B)$ does not depend on the choice of a partition.

Step 2. Let us choose one subinterval $[t_{i-1}, t_i]$ in $[0, 1]$. We take $c'_i$ so that the map $t \mapsto \chi_{[-c'_i, c'_i]}(B_t)$ is continuous and finite rank on $[t_{i-1}, t_i]$. In what follows, we assume that $c'_i < c_i$ holds (the case of $c'_i > c_i$ is proved in the same way). Then, we have

$$\chi_{[0,c_i]}(B_t) - \chi_{[0,c'_i]}(B_t) = \chi_{(c'_i,c_i]}(B_t).$$

Since $\chi_{(c'_i,c_i]}(B_t)$ is continuous and finite rank on $[t_{i-1}, t_i]$, we have,

$$\mathrm{rank}_{\mathbb{C}}(\chi_{(c'_i,c_i]}(B_{t_{i-1}})) = \mathrm{rank}_{\mathbb{C}}(\chi_{(c'_i,c_i]}(B_{t_i})).$$

Thus we have,

$$\begin{aligned}\mathrm{rank}_{\mathbb{C}}(\chi_{[0,c_i]}(B_{t_i})) + \mathrm{rank}_{\mathbb{C}}(\chi_{[0,c_i]}(B_{t_{i-1}})) &= \mathrm{rank}_{\mathbb{C}}(\chi_{[0,c'_i]}(B_{t_i})) + \mathrm{rank}_{\mathbb{C}}(\chi_{(c'_i,c_i]}(B_{t_i})) \\ &\quad + \mathrm{rank}_{\mathbb{C}}(\chi_{[0,c'_i]}(B_{t_{i-1}})) + \mathrm{rank}_{\mathbb{C}}(\chi_{(c'_i,c_i]}(B_{t_{i-1}})) \\ &= \mathrm{rank}_{\mathbb{C}}(\chi_{[0,c'_i]}(B_{t_i}) + \mathrm{rank}_{\mathbb{C}}(\chi_{[0,c'_i]}(B_{t_{i-1}})) \mod 2\end{aligned}$$

Thus $\mathrm{sf}_2(B)$ does not depend on the choice of positive numbers $c_1, \ldots, c_n$. $\square$

We next show the homotopy invariance of the $\mathbb{Z}_2$-valued spectral flow (Proposition 2.2.12). For the proof we need some lemmas.

**Lemma 2.2.9.** *If the image of an equivariant map* $B\colon ([-1, 1], \tau) \to (\mathrm{Fred}(H)^{s.a.}, \tau_\Theta)$ *is contained in the fixed point set* $(\mathrm{Fred}(H)^{s,a})^{\tau_\Theta}$*, then* $\mathrm{sf}_2(B) = 0$.

*Proof.* In this case, $B_t$ satisfies $\Theta B_t \Theta^* = B_t$ for each $t \in [0, 1]$, so the image of $\chi_{[0,a]}(B_t)$ has a quaternionic vector space structure. Thus $\mathrm{rank}_{\mathbb{C}}(\chi_{[0,a]}(B_t))$ is even and $\mathrm{sf}_2(B) = 0$. $\square$

**Lemma 2.2.10.** *Let* $B\colon ([-2, -1] \sqcup [1, 2], \tau) \to (\mathrm{Fred}(H)^{\mathrm{s.a.}}, \tau_\Theta)$ *be an equivariant continuous map which satisfies* $B(\frac{3}{2} + t) = B(\frac{3}{2} - t)$ *for* $-\frac{1}{2} \le t \le \frac{1}{2}$*. Then* $\mathrm{sf}_2(B) = 0$.

*Proof.* In this case, we have $\mathrm{sf}(B|_{[1,2]}) = 0$. Thus by Lemma 2.2.6, we have,

$$\mathrm{sf}_2(B) = \mathrm{sf}(B|_{[1,2]}) \mod 2 \;=\; 0. \qquad \square$$

**Lemma 2.2.11.** *Let $N$ be a neighborhood of some element of* $\mathrm{Fred}(H)^{\mathrm{s.a.}}$ *considered in Lemma 2.2.3.*

*(1) Let $B$, $B' \colon ([-1,1], \tau) \to (\mathrm{Fred}(H)^{s.a.}, \tau_\Theta)$ be $\mathbb{Z}_2$-equivariant continuous maps which satisfies $B_0 = B'_0$, $B_1 = B'_1$ and that their images are contained in $N$. Then we have* $\mathrm{sf}_2(B) = \mathrm{sf}_2(B')$.

*(2) Let $B$, $B' \colon ([-2,-1] \sqcup [1,2], \tau) \to (\mathrm{Fred}(H)^{s.a.}, \tau_\Theta)$ be $\mathbb{Z}_2$-equivariant continuous maps which satisfies $B_1 = B'_1$, $B_2 = B'_2$ and that their images are contained in $N$. Then we have* $\mathrm{sf}_2(B) = \mathrm{sf}_2(B')$.

*Proof.* We consider the case of (1). Since images of $B$ and $B'$ are contained in such $N$, we can give a definition of both $\mathrm{sf}_2(B)$ and $\mathrm{sf}_2(B')$ by taking a partition $0 = t_0 < t_1 = 1$ and taking $c_1$ as in Lemma 2.2.3. Thus we have

$$\begin{aligned}
\mathrm{sf}_2(B) &= \mathrm{rank}_{\mathbb{C}}(\chi_{[0,c_1]})(B_1) + \mathrm{rank}_{\mathbb{C}}(\chi_{[0,c_1]}(B_0)) \mod 2 \\
&= \mathrm{rank}_{\mathbb{C}}(\chi_{[0,c_1]})(B'_1) + \mathrm{rank}_{\mathbb{C}}(\chi_{[0,c_1]}(B'_0)) \mod 2 \\
&= \mathrm{sf}_2(B').
\end{aligned}$$

The case of (2) is proved in a same way. $\square$

**Proposition 2.2.12.** *$\mathbb{Z}_2$-valued spectral flow is homotopy invariant, that is, if $B$ and $B'$ are $\mathbb{Z}_2$-equivariant continuous maps with $B_1 = B'_1$ which are homotopic via a $\mathbb{Z}_2$-equivariant homotopy leaving the endpoints fixed. Then* $\mathrm{sf}_2(B) = \mathrm{sf}_2(B')$.

*Proof.* Let $h \colon ([-1,1], \tau) \times ([0,1], \mathrm{id}) \to (\mathrm{Fred}(H)^{s.a.}, \tau_\Theta)$ be a $\mathbb{Z}_2$-equivariant continuous map with $h(t,0) = B_t$, $h(t,1) = B'_t$ and $h(1,s) = B_1 = B'_1$ for any $t \in [-1,1]$ and $s \in [0,1]$. We note that since these maps are $\mathbb{Z}_2$-equivariant, we have $h(-1,s) = B_{-1} = B'_{-1}$. The proof is divided into 4 steps.

Step 1. We first consider the grid in $[-1,1] \times [0,1]$ formed by cubes of sufficiently small side length such that the image of each cube by $h$ is contained in some neighborhood $N$ of Lemma 2.2.3. The existence of such side length is proved by first taking a finite cover of $[-1,1] \times [0,1]$ by such neighborhoods and next considering a Lebesgue number. We can take such grid such that it is symmetric about the involution $\tau \times \mathrm{id}$. We take such a grid (see Figure 1.). Then we have a partition $-1 = t_{-n} < \cdots < t_{-1} < 0 = t_0 < t_1 < \cdots < t_n = 1$ of $[-1,1]$ which satisfies $\tau(t_i) = t_{-i}$ for each $i = 0, \ldots n$ and a partition $0 = s_0 < s_1 < \cdots < s_l = 1$ of $[0,1]$ given by this grid. Each restriction of $h$ onto $[-1,1] \times \{s_i\}$ for $i = 0, \cdots, l$ gives an $\mathbb{Z}_2$-equivariant map, and each restriction of $h$ onto $[-1,1] \times [s_i, s_{i+1}]$ for $i = 0, \cdots, l-1$ (one of them corresponds to the shaded area of Figure 1) gives a $\mathbb{Z}_2$-equivariant homotopy between such maps. In order to prove this proposition, it is enough to show that $\mathrm{sf}_2([-1,1] \times \{s_i\}) = \mathrm{sf}_2([-1,1] \times \{s_{i+1}\})$ for each $i = 0, \cdots, l-1$. Thus, in what follows, we assume that the partition of $[0,1]$ is just $0 = s_0 < s_1 = 1$.

Step 2. We consider a $\mathbb{Z}_2$-equivariant map $C \colon ([-1,1], \tau) \to (\mathrm{Fred}(H)^{s.a.}, \tau_\Theta)$ given by

$$C_t = \begin{cases} h(0,t) & \text{if} \quad 0 \le t \le 1 \\ h(0,-t) & \text{if} \quad -1 \le t \le 0. \end{cases}$$

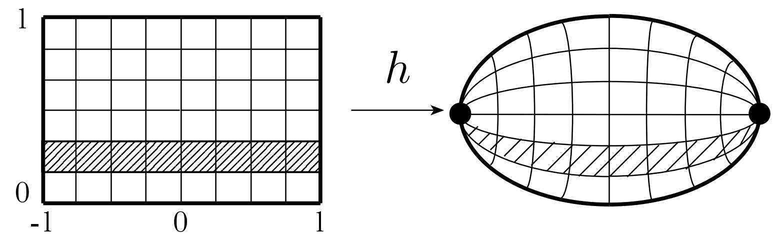


Figure 1:

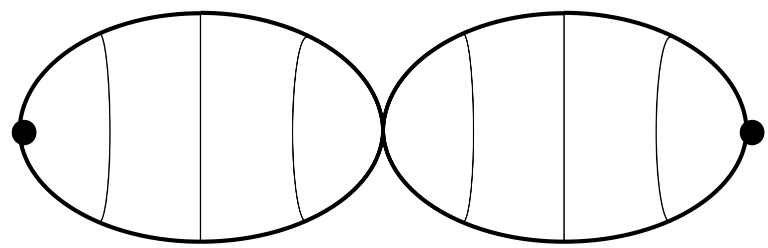

Figure 2:

The image of $C$ is contained in the fixed point set $(\mathrm{Fred}(H)^{s.a.})^{\tau_\Theta}$. By Lemma 2.2.9, we have $\mathrm{sf}_2(C) = 0$. By Lemma 2.2.7, we have $\mathrm{sf}_2(C\tilde{\circ}B') = \mathrm{sf}_2(B')$. Thus it is enough to show that $\mathrm{sf}_2(C\tilde{\circ}B') = \mathrm{sf}_2(B)$, and so we can assume further that our homotopy satisfies $h(0,s) = h(0,0)$ for any $s \in [0,1]$. In other words, we can "collapse" the line $h|_{\{0\}\times[0,1]}$ to a point as in Figure 2.

Step 3. Next we show that we can "collapse" each line $h_{\{t_i\}\times[0,1]}$ where $i = 1,\cdots,n$ to a point as in Figure 3. This is proved by almost the same argument as step 2, by using Lemma 2.2.10 instead of Lemma 2.2.9.

Step 4. By construction, the image of each disk in Figure 3 is contained in some neighborhood $N$ of Lemma 2.2.3. Thus by Lemma 2.2.11, we have $\mathrm{sf}_2(C\tilde{\circ}B') = \mathrm{sf}_2(B)$. $\square$

## 2.3 $K$-theory for $C^*$-algebras

In this subsection, we collects some basic facts from $K$-theory for $C^*$-algebras. We refer the reader to [Mur90, WO93, Bla98, HR00, RLL00] for the details.

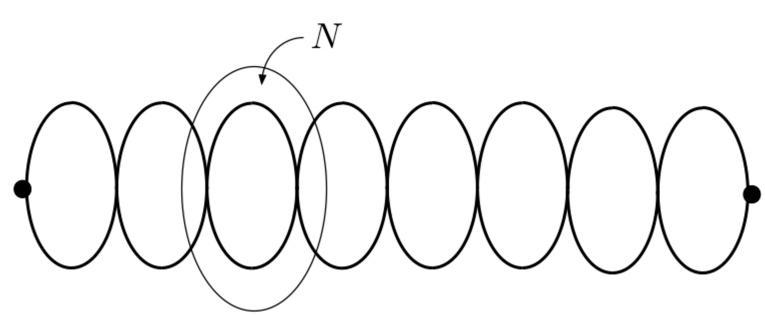


Figure 3:

Let $A$ be a unital $C^*$-algebra, that is a Banach $*$-algebra with multiplicative unit which satisfies $||a^*a|| = ||a||^2$ for any $a$ in $A$. An element $p$ in $A$ is called a projection if $p = p^* = p^2$, and an element $u$ in $A$ is said to be unitary if $u^*u = uu^* = 1$. For a positive integer $n$, let $M_n(A)$ be the matrix algebra of all $n \times n$ matrices with entries in $A$. As in the case of $M_n(\mathbb{C})$, the matrix algebra $M_n(A)$ has a natural $*$-algebra structure. We know that $M_n(A)$ has a (unique) norm making it a $C^*$-algebra. We denote the set of all projections in $M_n(A)$ by $P_n(A)$ and let $P_\infty(A) := \bigsqcup_{n=1}^\infty P_n(A)$. For $p \in P_n(A)$ and $q \in P_m(A)$, we denote $p \sim_0 q$ if and only if there exists some $v \in M_{m,n}(A)$ such that $p = v^*v$ and $q = vv^*$. The relation $\sim_0$ is an equivalence relation on $P_\infty(A)$. We define a binary operation $\oplus$ on $P_\infty(A)$ by $p \oplus q := \mathrm{diag}(p, q)$. Then $\oplus$ induces an addition $+$ on equivalence classes $D(A) := P_\infty(A)/\sim_0$, and $(D(A), +)$ is an abelian monoid. The $K_0$-group $K_0(A)$ for a unital $C^*$-algebra $A$ is defined to be the group completion (Grothendieck group) of the abelian monoid $(D(A), +)$. We denote by $[p]_0$ the class of $p \in P_\infty(A)$ in $K_0(A)$. For a non-unital $C^*$-algebra $I$, we define its $K_0$-group $K_0(I)$ to be the kernel of the map $K_0(\tilde{I}) \to K_0(\mathbb{C})$, where $\tilde{I} = I \oplus \mathbb{C}$ is the unitization of $I$, and the map is induced by the projection onto the second component. Let $\mathcal{U}(A)$ be the group of unitary elements in $A$, and let $\mathcal{U}_n(A) := \mathcal{U}(M_n(A))$. We consider the set $\mathcal{U}_\infty(A) := \bigsqcup_{n=1}^\infty \mathcal{U}_n(A)$. Let $\oplus$ be a binary operation on $\mathcal{U}_\infty(A)$ defined as above. For $u \in \mathcal{U}_n(A)$ and $v \in \mathcal{U}_m(A)$, we denote $u \sim_1 v$ if and only if there exists some $k \geq \max\{m, n\}$ such that $u \oplus 1_{k-n}$ and $v \oplus 1_{k-m}$ are homotopic in $\mathcal{U}_k(A)$. The relation $\sim_1$ is an equivalence relation on $\mathcal{U}_\infty(A)$, and $\oplus$ induces an addition $+$ on equivalence classes $\mathcal{U}_\infty(A)/\sim_1$. Then $K_1(A) := (\mathcal{U}_\infty(A)/\sim_1, +)$ is an abelian group. We denote by $[u]_1$ the class of $u \in \mathcal{U}_\infty(A)$ in $K_1(A)$. For a non-unital $C^*$-algebra $I$, its $K_1$-group is defined by using its unitization, that is, $K_1(I) := K_1(\tilde{I})$. Note that, by using the polar decomposition, an invertible element in $M_n(A)$ also defines an element in $K_1(A)$.

$*$-homomorphisms $\varphi, \psi \colon A \to B$ between $C^*$-algebras are said to be homotopic if there exists a path of $*$-homomorphisms $\varphi_t \colon A \to B$ for $t \in [0, 1]$ such that the map $[0, 1] \to B$ defined by $t \mapsto \varphi_t(a)$ is continuous for each $a \in A$, $\varphi_0 = \varphi$ and $\varphi_1 = \psi$. In this sense, $K_0$ and $K_1$ are the additive (covariant) homotopy functor from the category of $C^*$-algebras to the category of abelian groups. Let $H$ be a separable complex Hilbert space, and let $K(H)$ be the set of compact operators on $H$. For a $C^*$-algebra $A$, we have the following stability property, that is, $K_i(K(H) \otimes A) \cong K_i(A)$ where $i = 0, 1$ and $\otimes$ denotes for the $C^*$-algebraic tensor product.

Let $A$ be a $C^*$-algebra. The suspension of $A$ is the $C^*$-algebra $SA = \{f \in C([0, 1], A) \mid f(0) = f(1) = 0\} \cong A \otimes C_0((0, 1))$, where $C([0, 1], A)$ is the $C^*$-algebra of continuous functions from $[0, 1]$ to $A$, and $C_0((0, 1))$ is the $C^*$-algebra of complex valued continuous functions which vanish at infinity. Then there is an isomorphism $\theta_A \colon K_1(A) \to K_0(SA)$. We also have a map $\beta_A \colon K_0(A) \to K_1(SA)$, called the Bott map. If $A$ is a unital $C^*$-algebra, $\beta_A$ is given by $\beta_A[p]_0 = [f_p]_1$ where $f_p(t) = \exp(2\pi i t p)$. By the Bott periodicity theorem, $\beta_A$ is an isomorphism. For a short exact sequence of $C^*$-algebras $0 \to I \to A \to B \to 0$, there associates the following six-term exact sequence.

$$\begin{tikzcd}
K_1(I) \arrow[r] & K_1(A) \arrow[r] & K_1(B) \arrow[d, "\delta_1"] \\
K_0(B) \arrow[u, "\delta_0"] & K_0(A) \arrow[l] & K_0(I). \arrow[l]
\end{tikzcd}$$

The map $\delta_0$ is called the exponential map, and $\delta_1$ is called the index map. If $A$ and $B$

are unital $C^*$-algebras, and $I$ is a closed ideal in $A$, then $\delta_0$ is expressed in the following way. For $p \in P_n(B)$, we can take its self-adjoint lift $\hat{p} \in M_n(A)$, and then we have $\delta_0[p]_0 = [\exp(-2\pi i\hat{p})]_1$. Note that the following diagram is commutative.

$$\begin{CD} K_1(B) @>{\theta_B}>> K_0(SB) \\ @V{\delta_1}VV @VV{\delta_0}V \\ K_0(I) @>{\beta_I}>> K_1(SI). \end{CD}$$

### 2.4 Quarter-plane Toeplitz $C^*$-algebras

In this subsection, we collects basic facts about quarter-plane Toeplitz $C^*$-algebras which are used later.

Let us denote the Hilbert space $l^2(\mathbb{Z}\times\mathbb{Z})$ by $\mathcal{H}$. For a pair of integers $(m,n)$, let $e_{m,n}$ be the element of $\mathcal{H}$ that is 1 at $(m,n)$ and 0 elsewhere. For such $(m,n)$, let $M_{m,n}\colon \mathcal{H}\to\mathcal{H}$ be the translation operator defined by $(M_{m,n}\varphi)(k,l) = \varphi(m+k, n+l)$. We choose real numbers $\alpha < \beta$, and define the following subspaces of $\mathcal{H}$ (see Figure 4.),

$$\mathcal{H}^\alpha := \text{closed span of } \{e_{m,n} \mid -\alpha m + n \geq 0\},$$

$$\mathcal{H}^\beta := \text{closed span of } \{e_{m,n} \mid -\beta m + n \leq 0\},\ \ \mathcal{H}^{\alpha,\beta} := \mathcal{H}^\alpha \cap \mathcal{H}^\beta.$$

We could take $\alpha = -\infty$ or $\beta = +\infty$, but not both. Let $P^\alpha$ and $P^\beta$ be orthogonal

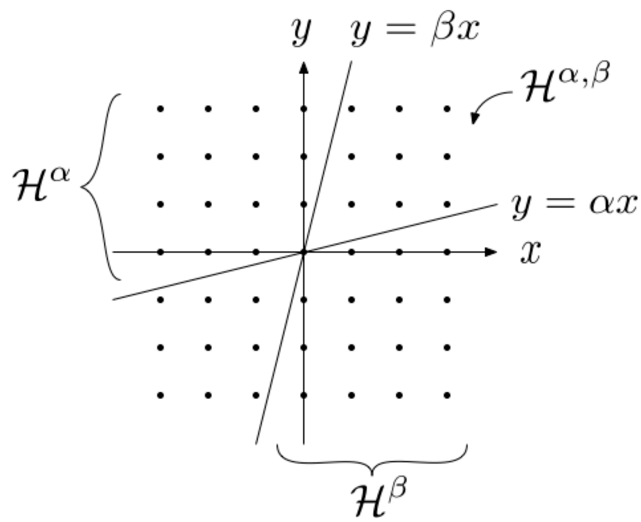


Figure 4:

projections of $\mathcal{H}$ onto $\mathcal{H}^\alpha$ and $\mathcal{H}^\beta$, respectively. Then $P^\alpha P^\beta$ be the orthogonal projection of $\mathcal{H}$ onto $\mathcal{H}^{\alpha,\beta}$. We define the *quarter-plane Toeplitz $C^*$-algebra* to be the $C^*$-algebra $\mathcal{T}^{\alpha,\beta}$ generated by $\{P^\alpha P^\beta M_{m,n} P^\alpha P^\beta \mid (m,n) \in \mathbb{Z}\times\mathbb{Z}\}$. We also define *half-plane Toeplitz $C^*$-algebras* $\mathcal{T}^\alpha$ and $\mathcal{T}^\beta$ to be $C^*$-algebras generated by $\{P^\alpha M_{m,n} P^\alpha \mid (m,n)\in\mathbb{Z}\times\mathbb{Z}\}$ and $\{P^\beta M_{m,n} P^\beta \mid (m,n)\in\mathbb{Z}\times\mathbb{Z}\}$, respectively. We have surjective $*$-homomorphisms $\gamma^\alpha\colon \mathcal{T}^{\alpha,\beta}\to\mathcal{T}^\alpha$ and $\gamma^\beta\colon \mathcal{T}^{\alpha,\beta}\to\mathcal{T}^\beta$, which map $P^\alpha P^\beta M_{m,n} P^\alpha P^\beta$ to $P^\alpha M_{m,n} P^\alpha$ and

$P^\beta M_{m,n} P^\beta$, respectively. We also have surjective $*$-homomorphisms $\sigma^\alpha\colon \mathcal{T}^\alpha \to C(\mathbb{T}\times\mathbb{T})$ and $\sigma^\beta\colon \mathcal{T}^\beta \to C(\mathbb{T}\times\mathbb{T})$ which map $P^\alpha M_{m,n} P^\alpha$ to $\chi_{m,n}$ and $P^\beta M_{m,n} P^\beta$ to $\chi_{m,n}$, respectively, where $\chi_{m,n}(z_1,z_2) = z_1^m z_2^n$. Well-definedness of $\gamma^\alpha$ and $\gamma^\beta$ is proved in [Par90], and that of $\sigma^\alpha$ and $\sigma^\beta$ is proved in [CD71]. Let $\rho^\alpha\colon \mathcal{T}^\alpha \to \mathcal{T}^{\alpha,\beta}$, $\rho^\beta\colon \mathcal{T}^\beta \to \mathcal{T}^{\alpha,\beta}$ and $\xi^{\alpha,\beta}\colon C(\mathbb{T}\times\mathbb{T}) \to \mathcal{T}^{\alpha,\beta}$ be bounded linear maps given by compression, that is, $\rho^\alpha(X) = P^\beta X P^\beta$, $\rho^\beta(Y) = P^\alpha Y P^\alpha$ and $\xi^{\alpha,\beta}(Z) = P^\alpha P^\beta Z P^\alpha P^\beta$, respectively. We define a $C^*$-algebra $\mathcal{S}^{\alpha,\beta}$ to be the pullback of $\mathcal{T}^\alpha$ and $\mathcal{T}^\beta$ along $C(\mathbb{T}\times\mathbb{T})$, that is,

$$\mathcal{S}^{\alpha,\beta} := \{(T^\alpha, T^\beta) \in \mathcal{T}^\alpha \oplus \mathcal{T}^\beta \mid \sigma^\alpha(T^\alpha) = \sigma^\beta(T^\beta)\}.$$

There is a surjective $*$-homomorphism $\gamma\colon \mathcal{T}^{\alpha,\beta} \to \mathcal{S}^{\alpha,\beta}$ given by $\gamma(T) = (\gamma^\alpha(T), \gamma^\beta(T))$. Let $\mathcal{K}$ be the $C^*$-algebra of compact operators on $\mathcal{H}^{\alpha,\beta}$.

**Theorem 2.4.1** (Park[Par90]). *There is the following short exact sequence for $C^*$-algebras,*

$$0 \to \mathcal{K} \to \mathcal{T}^{\alpha,\beta} \xrightarrow{\gamma} \mathcal{S}^{\alpha,\beta} \to 0,$$

*which has a linear splitting $\rho\colon \mathcal{S}^{\alpha,\beta} \to \mathcal{T}^{\alpha,\beta}$ given by $\rho(T^\alpha, T^\beta) = \rho^\alpha(T^\alpha) + \rho^\beta(T^\beta) - \xi^{\alpha,\beta}\sigma^\beta(T^\beta)$.*

The following theorem follows immediately by using Atkinson's theorem.[9]

**Theorem 2.4.2** (Douglas-Howe[DH71], Park[Par90]). *An operator $T$ in $\mathcal{T}^{\alpha,\beta}$ is a Fredholm operator if and only if $\gamma^\alpha(T)$ and $\gamma^\beta(T)$ are both invertible elements in $\mathcal{T}^\alpha$ and $\mathcal{T}^\beta$, respectively.*

By taking a tensor product of the above sequence and $C(\mathbb{T})$, we have the following short exact sequence,[10]

$$0 \to \mathcal{K}\otimes C(\mathbb{T}) \to \mathcal{T}^{\alpha,\beta}\otimes C(\mathbb{T}) \to \mathcal{S}^{\alpha,\beta}\otimes C(\mathbb{T}) \to 0.$$

Associated to this sequence, we have the following six-term exact sequence.

$$\begin{tikzcd}
K_1(\mathcal{K}\otimes C(\mathbb{T})) \arrow[r] & K_1(\mathcal{T}^{\alpha,\beta}\otimes C(\mathbb{T})) \arrow[r] & K_1(\mathcal{S}^{\alpha,\beta}\otimes C(\mathbb{T})) \arrow[d, "\delta_1"] \\
K_0(\mathcal{S}^{\alpha,\beta}\otimes C(\mathbb{T})) \arrow[u, "\delta_0"] & K_0(\mathcal{T}^{\alpha,\beta}\otimes C(\mathbb{T})) \arrow[l] & K_0(\mathcal{K}\otimes C(\mathbb{T})). \arrow[l]
\end{tikzcd}$$

Note that $K_1(\mathcal{K}\otimes C(\mathbb{T}))$ is isomorphic to $K_1(C(\mathbb{T}))$.

**Lemma 2.4.3.** *The map $\delta_0\colon K_0(\mathcal{S}^{\alpha,\beta}\otimes C(\mathbb{T})) \to K_1(C(\mathbb{T}))$ is surjective.*

*Proof.* Let us take a base point of $\mathbb{T}$, then we have isomorphisms[11] $K_0(\mathcal{S}^{\alpha,\beta}\otimes C(\mathbb{T})) \cong K_0(\mathcal{S}^{\alpha,\beta}) \oplus K_1(\mathcal{S}^{\alpha,\beta})$ and $K_1(\mathcal{K}\otimes C(\mathbb{T})) \cong K_1(\mathcal{K}) \oplus K_0(\mathcal{K}) \cong 0 \oplus \mathbb{Z}$. Consider the following

[9] Such a necessary and sufficient condition was first obtained by Douglas-Howe [DH71] for the special case $\mathcal{T}^{0,\infty}$ by expressing the algebra $\mathcal{T}^{0,\infty}$ as a tensor product of two Toeplitz algebras. Park obtained such conditions for the general $\mathcal{T}^{\alpha,\beta}$ in a different way [Par90].

[10] Since $C(\mathbb{T})$ is an abelian $C^*$-algebra, $C(\mathbb{T})$ is a nuclear $C^*$-algebra by Takesaki's theorem. Since $C(\mathbb{T})$ is nuclear, this sequence is exact (see [Mur90], for example).

[11] See Example 7.5.1 of [Mur90], for example.

commutative diagram,

$$\begin{tikzcd}
K_0(\mathcal{S}^{\alpha,\beta}\otimes C(\mathbb{T})) \arrow[r, "\cong"] \arrow[d, "\delta_0"] & K_0(\mathcal{S}^{\alpha,\beta})\oplus K_1(\mathcal{S}^{\alpha,\beta}) \arrow[d, "\delta_0\oplus\delta_1"] \\
K_1(\mathcal{K}\otimes C(\mathbb{T})) \arrow[r, "\cong"] & 0\oplus\mathbb{Z}.
\end{tikzcd}$$

The group $K_1(\mathcal{S}^{\alpha,\beta})$ is known to be isomorphic to $\mathbb{Z}$ [Par90], and the map $\delta_1\colon K_1(\mathcal{S}^{\alpha,\beta})\to K_0(\mathcal{K})$ is an isomorphism [Par90, Jia95]. Thus the left map is surjective. □

Remark 2.4.4. Note that the group $K_0(\mathcal{S}^{\alpha,\beta})$ is calculated as follows [Par90].

$$K_0(\mathcal{S}^{\alpha,\beta})\cong\begin{cases}\mathbb{Z} & \text{if } \alpha \text{ and } \beta \text{ both rational,}\\ \mathbb{Z}^2 & \text{if one of } \alpha \text{ and } \beta \text{ is rational and the other is irrational,}\\ \mathbb{Z}^3 & \text{if } \alpha \text{ and } \beta \text{ both irrational.}\end{cases}$$

In this sense, $K_0(\mathcal{S}^{\alpha,\beta})$ depends delicately on angles $\alpha$, $\beta$ of edges. By the proof of Lemma 2.4.3, this component maps to zero by $\delta_0$.

# 3 Bulk-edge correspondence and the cobordism invariance of the index

In this section, we give a $K$-theoretic proof of the bulk-edge correspondence based on the cobordism invariance of the index. We first consider type A topological insulators. The bulk-edge correspondence for type AII topological insulators is proved in a same way in Sect. 3.5.

## 3.1 Settings

Let $\mathbb{T}$ be the unit circle in the complex plane. We denote $t$ for an element of $\mathbb{T}$. We fix the counter-clockwise orientation on the circle $\mathbb{T}$. This oriented circle has a unique spin$^c$ structure up to isomorphism. Let $V$ be a finite dimensional Hermitian vector space. We denote by $N$ the complex dimension of $V$, and $||\cdot||_V$ for its norm. Let $l^2(\mathbb{Z};V)$ be the space of sequences $\varphi=\{\varphi_n\}_{n\in\mathbb{Z}}$ where each $\varphi_n$ is an element of $V$, and satisfies $\sum_{n\in\mathbb{Z}}||\varphi_n||_V^2<+\infty$. Let $S\colon l^2(\mathbb{Z};V)\to l^2(\mathbb{Z};V)$ be the shift operator defined by $(S\varphi)_n=\varphi_{n-1}$. Let $A_j\colon\mathbb{T}\to\mathrm{End}_{\mathbb{C}}(V)$ $(j\in\mathbb{Z})$ be continuous maps which satisfies $\sum_{j\in\mathbb{Z}}||A_j||_\infty<+\infty$, where $||A_j||_\infty=\sup_{t\in\mathbb{T}}||A_j(t)||_V$. We consider a tight-binding Hamiltonian on the lattice $\mathbb{Z}\times\mathbb{Z}$ with a periodic potential.

Definition 3.1.1 (Bulk Hamiltonian). For each $t$ in $\mathbb{T}$, we define a linear map,

$$H(t)\colon l^2(\mathbb{Z};V)\to l^2(\mathbb{Z};V),$$

by $(H(t)\varphi)_n=\sum_{j\in\mathbb{Z}}A_j(t)\varphi_{n-j}$. We call $H(t)$ a *bulk Hamiltonian*. $H(t)$ is a bounded linear operator and commutes with the shift operator $S$.

We assume that $H(t)$ is a self-adjoint operator for any $t$ in $\mathbb{T}$. Thus its spectrum $\mathrm{sp}(H(t))$ is contained in the real axis $\mathbb{R}$. We assume further that our bulk Hamiltonian has a spectral gap at the Fermi level $\mu$ for any $t$ in $\mathbb{T}$, i.e. $\mu$ is not contained in $\mathrm{sp}(H(t))$. We call this condition as a *spectral gap condition*. We take a positively oriented simple closed smooth loop $\gamma$ through $\mu$ in the complex plane, which satisfies that, for any $t$ in $\mathbb{T}$, the set $\mathrm{sp}(H(t)) \cap (-\infty, \mu)$ is contained inside the loop $\gamma$, and $\mathrm{sp}(H(t)) \cap (\mu, +\infty)$ is contained outside $\gamma$ (see Figure 5). This oriented loop has a unique spin$^c$ structure up to isomorphism. Let $\mathbb{S}^1_\eta$ be the unit circle in the complex plane, and $H(\eta, t) \in \mathrm{End}_{\mathbb{C}}(V)$ be a continuous family of Hermitian endomorphisms on $\mathbb{S}^1_\eta \times \mathbb{T}$ given by $H(\eta, t) = \sum_{j \in \mathbb{Z}} A_j(t)\eta^j$. By the Fourier transform $l^2(\mathbb{Z}; V) \cong L^2(\mathbb{S}^1_\eta; V)$, $H(t)$ is a multiplication operator $M_{H(\eta,t)}$ on $L^2(\mathbb{S}^1_\eta; V)$ generated by $H(\eta, t)$. Note that the shift operator $S$ corresponds to $M_\eta$ by the Fourier transform.[12] By our choice of the loop $l$, the following lemma holds.

**Lemma 3.1.2.** *For $(\eta, z, t) \in \mathbb{S}^1_\eta \times \gamma \times \mathbb{T}$, $H(\eta, t) - z$ is invertible.*

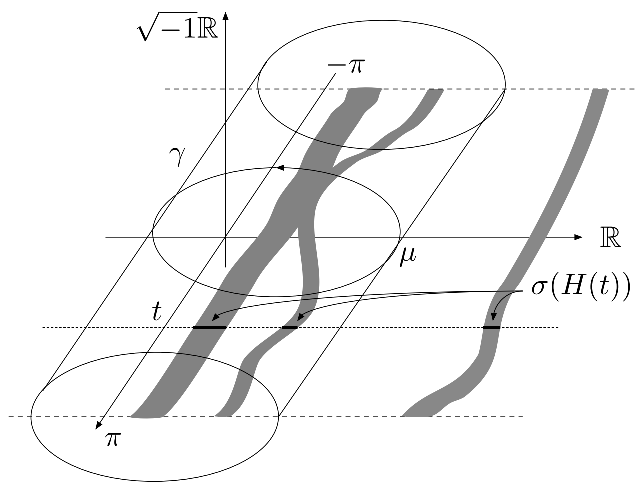


Figure 5: For an element $t$ in $\mathbb{T}$, bold black lines indicate the spectrum of the bulk Hamiltonian $\mathrm{sp}(H(t))$. The family of spectrums of bulk Hamiltonians makes a gray area. In order to make this picture simple, we here consider a family of spectrums parametrized by an interval $[-\pi, \pi]$ by considering the pullback by the map $[-\pi, \pi] \to [-\pi, \pi]/\{-\pi, \pi\} \cong \mathbb{T}$. Therefore dotted lines at $\pm\pi$ should be glued. The loop $\gamma$ (and the torus $\gamma \times \mathbb{T}$) is indicated as a (family of) circle(s)

Remark 3.1.3. As we will see later, taking this loop $\gamma$ is substantial for the definition of the bulk index and the edge index. In this sense, we do not need to choose this loop in order to formulate the bulk-edge correspondence for our system. We need to take this loop in order to define a vector bundle consists of decaying solutions.

[12]Note that, Graf and Porta used in [GP13], the operator $\tilde{S}$ defined by $(\tilde{S}\varphi)_n = \varphi_{n+1}$ instead of our shift operator $S$. Consequently, our choice of the orientation of $\mathbb{S}^1_\eta$ (in Sect. 3.2) is different from [GP13]. Although our choice of $S$ causes such confusing differences with [GP13], we decided to use this $S$ for the shift operator (and the orientation on $\mathbb{S}^1_\eta$), since this choice is consistent with the one used in the field of Toeplitz operators [BS06], and we use a family index of a family of Toeplitz operators in the proof of Lemma 3.4.3.

### 3.2 Bulk index, edge index and the bulk-edge correspondence

We define the bulk index and the edge index, and formulate our main theorem.

Definition 3.2.1 (Bloch bundle). By the spectral gap condition, Riesz projections give a continuous family of projections on $V$ parametrized by $\mathbb{S}^1_\eta \times \mathbb{T}$. Such projections are given by $\frac{1}{2\pi i}\int_\gamma (\lambda \cdot 1 - H(\eta, t))^{-1} d\lambda$. The images of this family makes a complex subvector bundle $E_{\mathrm{B}}$ of the product bundle $\underline{V} = (\mathbb{S}^1_\eta \times \mathbb{T}) \times V$. $E_{\mathrm{B}}$ is called the *Bloch bundle*.

The Bloch bundle defines an element $[E_{\mathrm{B}}]$ of a $K$-group $K^0(\mathbb{S}^1_\eta \times \mathbb{T})$. We fix a counterclockwise orientation on $\mathbb{S}^1_\eta$. This oriented circle has a unique spin$^c$ structure up to isomorphism. We consider the product spin$^c$ structure on $\mathbb{S}^1_\eta \times \mathbb{T}$. By using this spin$^c$ structure, we have the map $\mathrm{ind}_{\mathbb{S}^1_\eta \times \mathbb{T}} \colon K^0(\mathbb{S}^1_\eta \times \mathbb{T}) \to \mathbb{Z}$.

Definition 3.2.2 (Bulk Index). We define the *bulk index* of our system by

$$\mathcal{I}_{\mathrm{Bulk}} := -\,\mathrm{ind}_{\mathbb{S}^1_\eta \times \mathbb{T}}([E_{\mathrm{B}}]).$$

Remark 3.2.3. Note that the determinant line bundle $L$ associated to this spin$^c$ structure on $\mathbb{S}^1_\eta \times \mathbb{T}$ is trivial. By using the Atiyah-Singer index formula [AS68b], we have

$$\begin{aligned}
\mathrm{ind}_{\mathbb{S}^1_\eta \times \mathbb{T}}([E_{\mathrm{B}}]) &= \langle \mathrm{ch}(E_{\mathrm{B}}) e^{\frac{c_1(L)}{2}} \hat{A}(\mathbb{S}^1_\eta \times \mathbb{T}), [\mathbb{S}^1_\eta \times \mathbb{T}] \rangle \\
&= \langle c_1(E_{\mathrm{B}}), [\mathbb{S}^1_\eta \times \mathbb{T}] \rangle,
\end{aligned}$$

where $\mathrm{ch}(E_{\mathrm{B}})$ is the Chern character of $E_{\mathrm{B}}$, $\hat{A}(\mathbb{S}^1_\eta \times \mathbb{T})$ is the $\hat{A}$-genus of $\mathbb{S}^1_\eta \times \mathbb{T}$ and $[\mathbb{S}^1_\eta \times \mathbb{T}]$ is the fundamental class of $\mathbb{S}^1_\eta \times \mathbb{T}$. Thus our bulk index coincides with the minus of the first Chern number of the Bloch bundle. This minus sign is caused by our choice of the orientation of $\mathbb{S}^1_\eta$. Note that our bulk index equals to the one considered by Graf–Porta [GP13] and Hatsugai [Hat93b], and also the TKNN number [TKNdN82].

For each $k \in \mathbb{Z}$, let $\mathbb{Z}_{\geq k} := \{k, k+1, k+2, \ldots\}$ be the set of all integers greater than or equal to $k$. We regard $l^2(\mathbb{Z}_{\geq k}; V)$ as a closed subspace of $l^2(\mathbb{Z}; V)$ in a natural way. For each $k \in \mathbb{Z}$, we denote $P_{\geq k}$ for the orthogonal projection of $l^2(\mathbb{Z}; V)$ onto $l^2(\mathbb{Z}_{\geq k}; V)$.

Definition 3.2.4 (Edge Hamiltonian). For each $t$ in $\mathbb{T}$, we consider an operator $H^\#(t)$ given by the compression of $H(t)$ onto $l^2(\mathbb{Z}_{\geq 0}; V)$, that is,

$$H^\#(t) := P_{\geq 0} H(t) P_{\geq 0} \colon l^2(\mathbb{Z}_{\geq 0}; V) \to l^2(\mathbb{Z}_{\geq 0}; V).$$

We call $H^\#(t)$ an *edge Hamiltonian*.

$H^\#(t)$ is the Toeplitz (or the discrete Wiener-Hopf) operator with continuous symbol $H(\eta, t)$. Since $H(\eta, t)$ is Hermitian, $\{H^\#(t) - \mu\}_{t \in \mathbb{T}}$ is a norm-continuous family of self-adjoint Fredholm operators on the Hilbert space $l^2(\mathbb{Z}_{\geq 0}; V)$.

Definition 3.2.5 (Edge Index). We define the *edge index* of our system as the minus of the spectral flow of the family $\{H^\#(t) - \mu\}_{t \in \mathbb{T}}$.

$$\mathcal{I}_{\mathrm{Edge}} := -\mathrm{sf}(\{H^\#(t) - \mu\}_{t \in \mathbb{T}}).$$

We denote by $\mathcal{L}(a, b)$ (resp. $\mathcal{L}^{\mathrm{o}}(a, b)$) the closed (resp. open) interval in $\mathbb{R}$ between $a$ and $b$. We now revisit the definition of the spectral flow explained in Sect. 2.2. In order to count $\mathrm{sf}(\{H^\#(t) - \mu\}_{t \in \mathbb{T}})$, we need to choose $t_i$ and $c_i$. These data is used to prove our main theorem, and we need to choose such data in the following specific way. We regard $\{H^{\#}(t)-\mu\}_{t\in\mathbb{T}}$ as a family parametrized by $[-\pi,\pi]$, and consider the following path in $\mathbb{C}\times[-\pi,\pi]$ (or a loop in $\mathbb{C}\times\mathbb{T}$),

$$l := \bigcup_{i=0}^{n-1} \mathcal{L}(\tilde{c}_i,\tilde{c}_{i+1})\times\{t_i\}\cup\{\tilde{c}_{i+1}\}\times[t_i,t_{i+1}]\bigcup\mathcal{L}(\tilde{c}_n,\tilde{c}_0)\times\{t_n\}$$

where $c_0 := 0$, and we denote $\tilde{c}_i := c_i+\mu$, for simplicity. The loop $l$ may have intersections with spectrums of edge Hamiltonians. If we choose $t_i$ and $c_i$ as in Sect. 2.2, crossing points appear only at intervals of the form $\mathcal{L}(\tilde{c}_i,\tilde{c}_{i+1})\times\{t_i\}$. We take $t_i$ and $c_i$ so that crossing points appear only at *open* intervals of the form $\mathcal{L}^{\mathrm{o}}(\tilde{c}_i,\tilde{c}_{i+1})\times\{t_i\}$ $(i=1,\dots,n-1)$ or at half-open intervals $[\tilde{c}_0,\tilde{c}_1)\times\{t_0\}$ and $[\tilde{c}_0,\tilde{c}_n)\times\{t_n\}$. We see that, when we consider $\mathrm{sf}(\{H^{\#}(t)-\mu\}_{t\in\mathbb{T}})$, we do not need to consider about crossing points at $\{\tilde{c}_0\}\times\{t_0\}$ and $\{\tilde{c}_0\}\times\{t_n\}$. This is because crossing points at these points, if exist, do not contribute to the spectral flow (actually, they cancel out, in our case). Thus, we assume that there are no crossing points at these points, for simplicity. Then, $\mathrm{sf}(\{H^{\#}(t)-\mu\}_{t\in\mathbb{T}})$ is the net number of crossing points $\{u_1,\dots,u_m\}$ counted with multiplicity, where crossing points in $\mathcal{L}(\tilde{c}_i,\tilde{c}_{i+1})\times\{t_i\}$ with $c_i>c_{i+1}$ are counted with positive sign, and that of $c_i<c_{i+1}$ are counted with negative sign. We fix an orientation on $l$ which is compatible with the natural one on the interval $\{\tilde{c}_{i+1}\}\times[t_i,t_{i+1}]$ (see Figure 6).

Remark 3.2.6. At each crossing point, there exist eigenvectors of our edge Hamiltonian whose eigenvalue is $\mu$. Such eigenvectors are wave functions localized near the edge (called *edge states*). In this sense, the edge index is defined by counting edge states.

The bulk-edge correspondence for two-dimensional type A topological insulators is the following:

**Theorem 3.2.7.** *The bulk index coincides with the edge index. That is,*

$$\mathcal{I}_{\mathrm{Bulk}}=\mathcal{I}_{\mathrm{Edge}}.$$

We give a proof of this theorem based on the cobordism invariance of the index.

## 3.3 A vector bundle consists of decaying solutions

In this section, we define a vector bundle over $\gamma\times\mathbb{T}$, which is a generalization of the one considered by Graf and Porta in [GP13].

**Lemma 3.3.1.** *There exists a positive integer $K$ such that for any integer $k\geq K$ and $(z,t)$ in $\gamma\times\mathbb{T}$, the following map is surjective*

$$P_{\geq k}(H(t)-z)P_{\geq 0}\colon l^2(\mathbb{Z}_{\geq 0};V)\to l^2(\mathbb{Z}_{\geq k};V).$$

*Proof.* For a non-negative integer $k$, we define a linear map $H_k(t)\colon l^2(\mathbb{Z};V)\to l^2(\mathbb{Z};V)$ by $(H_k(t)\varphi)_n=\sum_{j=-k}^{j=k}A_j(t)\varphi_{n-j}$. Let $s_k := H_k(\eta,t)-z$, then $H_k(t)-z=M_{s_k}$. We first show that there exists some integer $K'$ such that for any $k\geq K'$ and for any $(z,t)\in\gamma\times\mathbb{T}$, the operator $P_{\geq k}(H_k(t)-z)P_{\geq 0}\colon l^2(\mathbb{Z}_{\geq 0};V)\to l^2(\mathbb{Z}_{\geq k};V)$ is surjective.

Note that $||(H(\eta,t)-z)-(H_k(\eta,t)-z)||\leq\sum_{|j|>k}||A_j||_\infty\to 0$ as $k\to\infty$. Since $H(\eta,t)-z$ is invertible by Lemma 3.1.2, there exists a positive integer $K'$ such that for any $k \geq K'$ and any $(\eta, z, t) \in \mathbb{S}^1_\eta \times \gamma \times \mathbb{T}$, $H_k(\eta, t) - z$ is invertible. Note that $SP_{\geq k} = P_{\geq k+1}S$, and we have $P_{\geq 0} = (S^*)^k P_{\geq k} S^k$. Since $s_k \eta^{-k}$ does not contain the term of $\eta^m$ $(m > 0)$, $(s_k\eta^{-k})^{-1}$ does not contain the term of $\eta^m$ $(m < 0)$, and so $M_{(s_k\eta^{-k})^{-1}}$ maps $l^2(\mathbb{Z}_{\geq k}; V)$ to $l^2(\mathbb{Z}_{\geq k}; V)$. Thus for $\varphi \in l^2(\mathbb{Z}_{\geq k}; V)$,

$$\begin{aligned}
(P_{\geq k} M_{s_k} P_{\geq 0})(P_{\geq 0} M_{s_k^{-1}} P_{\geq k})(\varphi) &= P_{\geq k} M_{s_k} (S^*)^k P_{\geq k} S^k M_{s_k^{-1}}(\varphi) \\
&= P_{\geq k} M_{s_k} M_{\eta^{-k}} P_{\geq k} M_{\eta^k} M_{s_k^{-1}}(\varphi) \\
&= P_{\geq k} M_{s_k\eta^{-k}} P_{\geq k} M_{(s_k\eta^{-k})^{-1}}(\varphi) \\
&= P_{\geq k} M_{s_k\eta^{-k}} M_{(s_k\eta^{-k})^{-1}}(\varphi) \\
&= \varphi.
\end{aligned}$$

Thus $P_{\geq k} M_{s_k} P_{\geq 0}$ has a right inverse, and it is surjective.[13]

As is shown above, the composite of $P_{\geq 0} M_{s_k^{-1}} P_{\geq k}$ and $P_{\geq k} M_{s_k} P_{\geq 0}$ is the identity map. Thus $P_{\geq 0} M_{s_k^{-1}} P_{\geq k}$ is bounded below, and so has a closed range. We also have

$$\mathrm{Im}(P_{\geq 0} M_{s_k^{-1}} P_{\geq k}) \cap \mathrm{Ker}(P_{\geq k} M_{s_k} P_{\geq 0}) = \{0\},$$

and

$$\mathrm{Im}(P_{\geq 0} M_{s_k^{-1}} P_{\geq k}) \oplus \mathrm{Ker}(P_{\geq k} M_{s_k} P_{\geq 0}) = l^2(\mathbb{Z}_{\geq 0}; V).$$

Let $X_k := \mathrm{Im}(P_{\geq 0} M_{s_k^{-1}} P_{\geq k})$, and denote $P_{X_k}$ for the orthogonal projection onto $X_k$. Then by the open mapping theorem, $P_{\geq k} M_{s_k} P_{X_k} \colon X_k \to l^2(\mathbb{Z}_{\geq k}; V)$ is invertible. On the other hand, we have $P_{\geq 0} M_{s_k^{-1}} P_{\geq k} = P_{X_k} M_{s_k^{-1}} P_{\geq k}$, and

$$\mathbf{1} = (P_{\geq k} M_{s_k} P_{\geq 0})(P_{\geq 0} M_{s_k^{-1}} P_{\geq k}) = (P_{\geq k} M_{s_k} P_{X_k})(P_{X_k} M_{s_k^{-1}} P_{\geq k}).$$

Therefore, the inverse of $P_{\geq k} M_{s_k} P_{X_k}$ is $P_{X_k} M_{s_k^{-1}} P_{\geq k}$.

Since $s_k$ converges to $H(\eta, t) - z$ as $k \to \infty$ uniformly with respect to $(\eta, z, t)$ in $\mathbb{S}^1_\eta \times \gamma \times \mathbb{T}$, the operator $M_{s_k}$ converges to $H(t) - z$ uniformly, and so $M_{s_k^{-1}}$ converges to $(H(t) - z)^{-1}$ uniformly with respect to $(z, t)$. Thus there exist some $L > 0$ and an integer $K''$ such that for any $k \geq K''$ and $(z, t) \in \gamma \times \mathbb{T}$ we have $||M_{s_k^{-1}}|| < L$. For $k \geq K''$, we have $||(P_{\geq k} M_{s_k} P_{X_k})^{-1}||^{-1} > ||M_{s_k^{-1}}||^{-1} > L^{-1} > 0$. On the other hand, we have

$$||P_{\geq k}(H(t) - z)P_{X_k} - P_{\geq k} M_{s_k} P_{X_k}|| \leq ||H(t) - H_k(t)|| \leq \sum_{|j|>k} ||A_j||_\infty \to 0.$$

Thus there exists an positive integer $K$ such that for any $k \geq K$ and $(z, t)$ in $\gamma \times \mathbb{T}$, we have

$$||P_{\geq k}(H(t) - z)P_{X_k} - P_{\geq k} M_{s_k} P_{X_k}|| \leq ||(P_{\geq k} M_{s_k} P_{X_k})^{-1}||^{-1},$$

which means that $P_{\geq k}(H(t) - z)P_{X_k}$ also is invertible. Thus $P_{\geq k}(H(t) - z)P_{\geq 0}$ is surjective for such $k \geq K$ and $(z, t)$ in $\gamma \times \mathbb{T}$. □

We choose such an integer $k \geq K$, and denote

$$(E_{\mathrm{GP}})_{z,t} := \mathrm{Ker}(P_{\geq k}(H(t) - z)P_{\geq 0}).$$

[13] Such argument was used by Atiyah and Bott to compute the family index of a family of Toeplitz operators [Ati68, AB64b].

Remark 3.3.2. For $n \geq k$, we have

$$((P_{\geq k}(H(t)-z)P_{\geq 0})(\varphi))_n = ((H(t)-z)(\varphi))_n.$$

Thus $(E_{\mathrm{GP}})_{z,t}$ consists of sequences $\varphi = \{\varphi_n\}_{n\geq 0}$ in $l^2(\mathbb{Z}_{\geq 0}; V)$ which satisfies $(H(t)(\varphi))_n = z\varphi_n$ for $n \geq k$.

We define an operator $H^\flat(z,t)\colon l^2(\mathbb{Z}_{\geq 0}; V) \to l^2(\mathbb{Z}_{\geq 0}; V)$ to be the composite of $P_{\geq k}(H(t)-z)P_{\geq 0}$ and the inclusion $l^2(\mathbb{Z}_{\geq k}; V) \hookrightarrow l^2(\mathbb{Z}_{\geq 0}; V)$.

**Lemma 3.3.3.** *For any $(z,t)$ in $\gamma \times \mathbb{T}$, $H^\flat(z,t)$ is a Fredholm operator whose Fredholm index is zero. Moreover,* $\operatorname{Ker} H^\flat(z,t)$ *and $(E_{\mathrm{GP}})_{z,t}$ are the same, and* $\operatorname{Coker} H^\flat(z,t)$ *is naturally isomorphic to $V^{\oplus k}$. Thus the dimension of $(E_{\mathrm{GP}})_{z,t}$ is $kN$ which is constant with respect to the parameter.*

*Proof.* For $(z,t) \in \gamma \times \mathbb{T}$, $(H^{\#}(t) - z) - H^\flat(z,t)$ is a finite rank operator, and $H^{\#}(t) - z$ is a Fredholm operator. Thus, $H^\flat(z,t)$ also is a Fredholm operator, and their Fredholm indices are the same. Since $H^{\#}(t) - z$ is connected to a self-adjoint Fredholm operator $H^{\#}(t) - \mu$ by a continuous path of Fredholm operators, the homotopy invariance of the Fredholm index says that the Fredholm index of $H^{\#}(t) - z$ is zero. Thus the Fredholm index of $H^\flat(z,t)$ is zero, and so the dimensions of its kernel and cokernel are the same. On the other hand, $\operatorname{Ker} H^\flat(z,t)$ coincides with $(E_{\mathrm{GP}})_{z,t}$ by definition. By Lemma 3.3.1, $\operatorname{Coker} H^\flat(z,t)$ is naturally isomorphic to the closed subspace of $l^2(\mathbb{Z}_{\geq 0}; V)$ which consists of the sequences with $\varphi_n = 0$ for $n \geq k$, which is naturally isomorphic to $V^{\oplus k}$. $\square$

Definition 3.3.4. By Lemma 3.3.3, we have a vector bundle over $\gamma \times \mathbb{T}$ whose fiber at a point $(z,t)$ is given by $(E_{\mathrm{GP}})_{z,t}$. We denote this vector bundle by $E_{\mathrm{GP}}$. The rank of this vector bundle is $kN$.

Note that the family index of a family of Fredholm operators $\{H^\flat(z,t)\}_{\gamma\times\mathbb{T}}$ is an element of the $K$-group $K^0(\gamma \times \mathbb{T})$ which is given by $[E_{\mathrm{GP}}] - [\underline{V}^{\oplus k}]$. Thus $E_{\mathrm{GP}}$ appears as a family index. As in the definition of the bulk index, we consider the product spin$^c$ structure on $\gamma\times\mathbb{T}$. By using this spin$^c$ structure, we have the map $\mathrm{ind}_{\gamma\times\mathbb{T}}\colon K^0(\gamma\times\mathbb{T}) \to \mathbb{Z}$.

Definition 3.3.5 (Graf–Porta [GP13]). We define *Graf–Porta's index*[14] by

$$\mathcal{I}_{\mathrm{GP}} := \mathrm{ind}_{\gamma\times\mathbb{T}}([E_{\mathrm{GP}}] - [\underline{V}^{\oplus k}]).$$

Remark 3.3.6. The conclusion of Lemma 3.3.3 also holds for $(z,t) \in \mathbb{C} \times \mathbb{T}$ which satisfies $z \notin \mathrm{sp}(H(t))$. Thus we can define a vector bundle in the same way on the space $\{(z,t) \in \mathbb{C} \times \mathbb{T} \mid z \notin \mathrm{sp}(H(t))\}$. We denote this vector bundle as $\tilde{E}_{\mathrm{GP}}$. Then we have $E_{\mathrm{GP}} = \tilde{E}_{\mathrm{GP}}|_{\gamma\times\mathbb{T}}$. Such extension is used in Sect. 3.4.2.

Remark 3.3.7. The determinant line bundle associated to the spin$^c$ structure on $\gamma \times \mathbb{T}$ is trivial. Thus, as in Remark 3.2.3, we can show that $\mathrm{ind}_{\gamma\times\mathbb{T}}([\underline{V}^{\oplus k}]) = 0$, and $\mathcal{I}_{\mathrm{GP}}$ coincides with the first Chern number of the bundle $E_{\mathrm{GP}}$.

Remark 3.3.8. If, for some $k$, we have $A_j = 0$ for $j < -k$ and $k < j$, and if $A_{-k}$ and $A_k$ are invertible, then the difference equation $H(t)\varphi = z\varphi$ is solved by an initial condition. In this case, each fiber $(E_{\mathrm{GP}})_{z,t}$ is identified with the space of formal solutions of the equation $H(t)\psi = z\psi$ which decays as $n \to +\infty$. This is Graf and Porta's original idea to define such a vector bundle [GP13].

[14] For a periodic system, the Bloch bundle is defined and its first Chern number also is defined (we call this invariant the bulk index). In order to distinguish this invariant and the invariant defined in this way, we call this invariant *Graf–Porta's index* in this paper.

## 3.4 Proof of the bulk-edge correspondence

In this section, we prove Theorem 3.2.7 by first showing that the bulk index coincides with the minus of the Graf–Porta's index (Sect. 3.4.1) and next showing that the minus of the Graf–Porta's index coincides with the edge index (Sect. 3.4.2).

### 3.4.1 Bulk index and Graf–Porta's index

We show the bulk index coincides with Graf–Porta's index. The key ingredient is the cobordism invariance of the index.

**Proposition 3.4.1.** $\mathcal{I}_{\mathrm{Bulk}} = -\mathcal{I}_{\mathrm{GP}}$.

Let $\mathbb{D}^2_\eta$ be the closed unit disk in the complex plane whose boundary is $\mathbb{S}^1_\eta$, and let $\mathbb{D}^2_z$ be the closed domain of the complex plane surrounded by $\gamma$. We fix, on $\mathbb{D}^2_\eta$ and $\mathbb{D}^2_z$, spin$^c$ structures naturally induced by the spin$^c$ structure of the complex structure of the complex plane. Let $X := (\mathbb{D}^2_\eta \times \mathbb{D}^2_z) \setminus (\mathbb{S}^1_\eta \times \gamma) \times \mathbb{T}$ and $Y := \partial X = \mathbb{S}^1_\eta \times (\mathbb{D}^2_z \setminus \gamma) \times \mathbb{T} \sqcup (\mathbb{D}^2_\eta \setminus \mathbb{S}^1_\eta) \times \gamma \times \mathbb{T}$. We consider the product spin$^c$ structure on $X$ and the boundary spin$^c$ structure on $Y$. We take an extension $f = f(\eta, z, t)\colon X \to \mathrm{End}_{\mathbb{C}}(V)$ of $H(\eta, t) - z$, which is defined on $\mathbb{S}^1_\eta \times \gamma \times \mathbb{T} = \partial Y$. The map $f$ defines an endomorphism of $\underline{V} = X \times V$ for which we also denote $f$.

We now consider restrictions of the endomorphism $f$ of $\underline{V}$ onto $\mathbb{S}^1_\eta \times (\mathbb{D}^2_z \setminus \gamma) \times \mathbb{T}$ and $(\mathbb{D}^2_\eta \setminus \mathbb{S}^1_\eta) \times \gamma \times \mathbb{T}$. By Lemma 3.1.2, we define elements of $K$-groups,

$$\alpha_z := [\underline{V}, \underline{V}; f|_{\mathbb{S}^1_\eta \times (\mathbb{D}^2_z \setminus \gamma) \times \mathbb{T}}] \in K^0_{\mathrm{cpt}}(\mathbb{S}^1_\eta \times (\mathbb{D}^2_z \setminus \gamma) \times \mathbb{T}),$$

and

$$\alpha_\eta := [\underline{V}, \underline{V}; f|_{(\mathbb{D}^2_\eta \setminus \mathbb{S}^1_\eta) \times \gamma \times \mathbb{T})}] \in K^0_{\mathrm{cpt}}((\mathbb{D}^2_\eta \setminus \mathbb{S}^1_\eta) \times \gamma \times \mathbb{T}).$$

We first see that each element maps, through Thom isomorphisms and push-forward maps, to the bulk index (Lemma 3.4.2) and Graf–Porta's index (Lemma 3.4.3). We then prove Proposition 3.4.1 by using the cobordism invariance of the index (Lemma 3.4.4).

We denote by $F$ the composite of the inverse of the Thom isomorphism $\beta_z^{-1}\colon K^0_{\mathrm{cpt}}(\mathbb{S}^1_\eta \times (\mathbb{D}^2_z \setminus \gamma) \times \mathbb{T}) \to K^0(\mathbb{S}^1_\eta \times \mathbb{T})$ and $\mathrm{ind}_{\mathbb{S}^1_\eta \times \mathbb{T}}\colon K^0(\mathbb{S}^1_\eta \times \mathbb{T}) \to \mathbb{Z}$. The relation between $\alpha_z$ and $\mathcal{I}_{\mathrm{Bulk}}$ is stated as follows. This lemma follows easily from Atiyah–Bott's elementary proof of the Bott periodicity theorem [AB64b].

**Lemma 3.4.2.** $\beta_z^{-1}(\alpha_z) = [E_{\mathrm{B}}]$. *Thus we have* $F(\alpha_z) = \mathcal{I}_{\mathrm{Bulk}}$.

We denote by $F'$ the composite of the inverse of the Thom isomorphsim $\beta_\eta^{-1}\colon K^0_{\mathrm{cpt}}((\mathbb{D}^2_\eta \setminus \mathbb{S}^1_\eta) \times \gamma \times \mathbb{T}) \to K^0(\gamma \times \mathbb{T})$ and $\mathrm{ind}_{\gamma \times \mathbb{T}}\colon K^0(\gamma \times \mathbb{T}) \to \mathbb{Z}$. The relation between $\alpha_\eta$ and $\mathcal{I}_{\mathrm{GP}}$ is stated as follows.

**Lemma 3.4.3.** $\beta_\eta^{-1}(\alpha_\eta) = -[E_{\mathrm{GP}}] + [\underline{V}^{\oplus k}]$. *Thus we have* $F'(\alpha_\eta) = -\mathcal{I}_{\mathrm{GP}}$.

*Proof.* We have the following commutative diagram (see Sect. 7 of [Ati68]),

$$\begin{tikzcd}
K^{-1}(\mathbb{S}^1_\eta \times \gamma \times \mathbb{T}) \arrow[r, "\partial"] \arrow[d, "\mathcal{T}"'] & K^0(\mathbb{D}^2_\eta \times \gamma \times \mathbb{T}, \mathbb{S}^1_\eta \times \gamma \times \mathbb{T}) \arrow[d, "-\beta_\eta^{-1}"] \\
{[\gamma \times \mathbb{T}, \mathrm{Fred}(l^2(\mathbb{Z}_{\geq 0}))]} \arrow[r, "\mathrm{index}"] & K^0(\gamma \times \mathbb{T})
\end{tikzcd}$$

where $\partial$ is the boundary map of the long exact sequence for the pair $(\mathbb{S}^1_\eta\times\mathbb{D}^2_z\times\mathbb{T}, \mathbb{S}^1_\eta\times\gamma\times\mathbb{T})$, and a map $\mathcal{T}$ from $K^{-1}(\mathbb{S}^1_\eta\times\gamma\times\mathbb{T})$ to $[\gamma\times\mathbb{T}, \mathrm{Fred}(l^2(\mathbb{Z}_{\geq 0}))]$ is given by taking a family of Toeplitz operators. Note that $-\beta_\eta$ is given by taking a cup product with the element $[\underline{\mathbb{C}}, \underline{\mathbb{C}}; \bar{z}] \in K^0_{\mathrm{cpt}}(\mathbb{D}^2_\eta \setminus \mathbb{S}^1_\eta)$. By Lemma 3.1.2, we have an element $[H(\eta,t)-z] \in K^{-1}(\mathbb{S}^1_\eta \times \gamma\times\mathbb{T})$. This element maps to $\alpha_\eta \in K^0_{\mathrm{cpt}}((\mathbb{D}^2_\eta\setminus\mathbb{S}^1_\eta)\times\gamma\times\mathbb{T})$ by the boundary map $\partial$. On the other hand, we have $\mathcal{T}([H(\eta,t)-z]) = \{H^{\#}(t)-z\}_{\gamma\times\mathbb{T}}$. Now since $H^\flat(z,t)$ is a finite rank perturbation of $H^{\#}(t)-z$, we have $\{H^{\#}(t)-z\}_{\gamma\times\mathbb{T}} = \{H^\flat(z,t)\}_{\gamma\times\mathbb{T}}$ as an element of the set $[\gamma\times\mathbb{T}, \mathrm{Fred}(l^2(\mathbb{Z}_{\geq 0}))$. Thus, by Lemma 3.3.3, we have $\mathrm{index}(\{H^\flat(z,t)\}_{\gamma\times\mathbb{T}}) = [E_{\mathrm{GP}}] - [\underline{V}^{\oplus k}]$. By the commutativity of the above diagram, we have $-\beta_\eta^{-1}(\alpha_\eta) = [E_{\mathrm{GP}}] - [\underline{V}^{\oplus k}]$. □

*Proof of Proposition 3.4.1.* By Lemma 3.1.2, the endomorphism $f$ defines the following element $\alpha$ of the compactly supported $K$-group,

$$\alpha := [\underline{V}, \underline{V}; f(\eta,z,t)] \in K^0_{\mathrm{cpt}}(X).$$

The element $\alpha$ gives a cobordism between $\alpha_z$ and $\alpha_\eta$, that is, let $i\colon Y \hookrightarrow X$ be the inclusion, then we have $i^*(\alpha) = (\alpha_z, \alpha_\eta)$. By Lemma 3.4.2 and Lemma 3.4.3, we have $(F\oplus F')(\alpha_z,\alpha_\eta) = -\mathcal{I}_{\mathrm{Bulk}} - \mathcal{I}_{\mathrm{GP}}$. Thus the following lemma, which states the cobordism invariance of the index, is suffice to prove Proposition 3.4.1.

**Lemma 3.4.4.** $(F\oplus F')(\alpha_z,\alpha_\eta) = 0$.

*Proof.* By the Whitney embedding theorem, there exists a neat embedding $E\colon X \to \mathbb{R}^M \times [0,+\infty)$ for sufficiently large even integer $M$, such that the boundary $Y$ of $X$ maps to $\mathbb{R}^M\times\{0\}$.[15] We fix a spin$^c$ structure on $\mathbb{R}^M\times[0,+\infty)$. Then a normal bundle of this embedding has a naturally induced spin$^c$ structure. Let $i'\colon \mathbb{R}^M \hookrightarrow \mathbb{R}^M\times[0,+\infty)$ be an inclusion given by $i'(x) = (x,0)$. We have the following commutative diagram.

$$\begin{tikzcd}
K^0_{\mathrm{cpt}}(X) \arrow[r, "i^*"] \arrow[d, "E_!"] & K^0_{\mathrm{cpt}}(Y) \arrow[r, "F\oplus F'"] \arrow[d, "(E|Y)_!"'] & \mathbb{Z} \\
K^0_{\mathrm{cpt}}(\mathbb{R}^M\times[0,+\infty)) \arrow[r, "i'^*"] & K^0_{\mathrm{cpt}}(\mathbb{R}^M) \arrow[ur, "\beta^{-1}"'] &
\end{tikzcd}$$

where $\beta$ is the Thom isomorphism. We have $i^*(\alpha) = (\alpha_z,\alpha_\eta)$ and $K^0_{\mathrm{cpt}}(\mathbb{R}^M\times[0,+\infty)) \cong \widetilde{K}^0(\mathbb{B}^{M+1}) = \{0\}$, where $\mathbb{B}^{M+1}$ is the $(M+1)$-dimensional closed ball which is contractible. By the commutativity of this diagram, we have $(F\oplus F')(\alpha_z,\alpha_\eta) = 0$. □

### 3.4.2 Graf–Porta's index and the edge index

We next show Graf–Porta's index coincides with the edge index. In order to prove this, we use a localization argument for a $K$-class. We localize the support of a $K$-class near the crossing points of the spectrums of edge Hamiltonians and the loop $l$ which we take in Sect. 3.2. We then use the excision property of the index.

[15] For the proof of the Whitney embedding theorem for an embedding of a non-compact smooth manifold with boundary into a half-space, see [Ada93, Hir94], for example. It is also easy to construct such embedding explicitly for our space $X$.

**Proposition 3.4.5.** $-\mathcal{I}_{\mathrm{GP}} = \mathcal{I}_{\mathrm{Edge}}$.

We identify $V^{\oplus k}$ with the image of the projection $1 - P_{\geq k}$, that is, the closed subspace of $l^2(\mathbb{Z}_{\geq 0}; V)$ consists of sequences $\varphi = \{\varphi_n\}$ which satisfies $\varphi_n = 0$ for $n \geq k$. For $(z,t)$ in $\gamma \times \mathbb{T}$, we consider the following map,

$$g_{z,t} := (H^{\#}(t) - z)|_{(E_{\mathrm{GP}})_{z,t}} \colon (E_{\mathrm{GP}})_{z,t} \to V^{\oplus k}.$$

By the definition of $(E_{\mathrm{GP}})_{z,t}$, the space $(H^{\#}(t) - z)((E_{\mathrm{GP}})_{z,t})$ is contained in $V^{\oplus k}$. Therefore we have a bundle homomorphism $g \colon \tilde{E}_{\mathrm{GP}}|_{\tilde{\mathbb{T}}} \to \underline{V}^{\oplus k}$.

**Lemma 3.4.6.** *For* $(z,t) \in \gamma \times \mathbb{T}$*, we have,* $\mathrm{Ker}\, g_{z,t} = \mathrm{Ker}(H^{\#}(t) - z)$*.*

*Proof.* $\varphi$ is an element of $\mathrm{Ker}\, g_{z,t}$ if and only if $\varphi$ is an element of $l^2(\mathbb{Z}_{\geq 0}; V)$ which satisfies $(H(t)\varphi)_n = z\varphi_n$ for $n \geq k$ (see Remark 3.3.2), and $(H(t)\varphi)_n = z\varphi_n$ for $0 \leq n \leq k-1$ (since $\varphi$ is an element of $\mathrm{Ker}\, g_{z,t}$). This is equivalent to saying that $\varphi$ is an element of $\mathrm{Ker}(H^{\#}(t) - z)$. □

Remark 3.4.7. If there is no edge states, then $g$ is a bundle isomorphism, and we have a trivialization of the bundle $E_{\mathrm{GP}}$.

We now deform the torus $\gamma \times \mathbb{T}$, so that, instead of the loop $\{\mu\} \times \mathbb{T}$, the deformed torus intersects $\mathbb{R} \times \mathbb{T} \subset \mathbb{C} \times \mathbb{T}$ on the loop $l$ which we take in Sect. 3.2, and that, at each crossing point, a neighborhood of the point in the deformed torus is contained in $\mathbb{C} \times \{t\}$ for some $t$ in $\mathbb{T}$ (see Figure 6). We denote this deformed torus by $\tilde{\mathbb{T}}$. Since $l$ is contained in the resolvent set of bulk Hamiltonians, such $\tilde{\mathbb{T}}$ is contained in the set $\{(z,t) \in \mathbb{C} \times \mathbb{T} \mid z \notin \mathrm{sp}(H(t))\}$. Thus we can consider the restriction of $\tilde{E}_{\mathrm{GP}}$ onto $\tilde{\mathbb{T}}$ (see Remark 3.3.6). $\tilde{\mathbb{T}}$ is deformed continuously to $\gamma \times \mathbb{T}$ (see Figure 6, and consider the *"crushing the ledge" map*). By using this deformation, $\tilde{\mathbb{T}}$ has a spin$^c$ structure naturally induced by that of $\gamma \times \mathbb{T}$. By using this spin$^c$ structure, we have the map $\mathrm{ind}_{\tilde{\mathbb{T}}} \colon K^0(\tilde{\mathbb{T}}) \to \mathbb{Z}$.

**Lemma 3.4.8.** $\mathcal{I}_{\mathrm{GP}} = \mathrm{ind}_{\tilde{\mathbb{T}}}([\tilde{E}_{\mathrm{GP}}|_{\tilde{\mathbb{T}}}, \underline{V}^{\oplus k}; g])$*.*

*Proof.* By the naturality of the index, the integer $\mathrm{ind}_{\gamma \times \mathbb{T}}([E_{\mathrm{GP}}] - [\underline{V}^{\oplus k}])$ coincides with $\mathrm{ind}_{\tilde{\mathbb{T}}}([\tilde{E}_{\mathrm{GP}}|_{\tilde{\mathbb{T}}}] - [\underline{V}^{\oplus k}])$. Thus, we have

$$\mathcal{I}_{\mathrm{GP}} = \mathrm{ind}_{\tilde{\mathbb{T}}}([\tilde{E}_{\mathrm{GP}}|_{\tilde{\mathbb{T}}}] - [\underline{V}^{\oplus k}]) = \mathrm{ind}_{\tilde{\mathbb{T}}}([\tilde{E}_{\mathrm{GP}}|_{\tilde{\mathbb{T}}}, \underline{V}^{\oplus k}; 0]) = \mathrm{ind}_{\tilde{\mathbb{T}}}([\tilde{E}_{\mathrm{GP}}|_{\tilde{\mathbb{T}}}, \underline{V}^{\oplus k}; g]). \quad \square$$

By Lemma 3.4.6, the support of the bundle homomorphism $g \colon \tilde{E}_{\mathrm{GP}}|_{\tilde{\mathbb{T}}} \to \underline{V}^{\oplus k}$ is contained in the spectrum of edge Hamiltonians, and coincides with the set of crossing points $\{u_1, \ldots, u_m\}$. For each crossing point $u_a$, we take an open disk neighborhood $U_a$ of $u_a$ in $\tilde{\mathbb{T}}$ which is also contained in $\mathbb{C} \times \{t\}$ for some $t$ in $\mathbb{T}$ (remember our choice of $t_i$ and $c_i$ in Sect. 3.2). We take $U_a$ small enough so that $U_a$ does not intersect one another. Then we have following elements of compactly supported $K$-groups,

$$[\tilde{E}_{\mathrm{GP}}|_{U_a}, \underline{V}^{\oplus k}; g] \in K^0_{\mathrm{cpt}}(U_a), \;\; a = 1, \ldots, m.$$

For each $a$, we consider a spin$^c$ structure on $U_a$ which is the restriction of the spin$^c$ structure of $\tilde{\mathbb{T}}$. By using this spin$^c$ structure, we have a homomorphism $\mathrm{ind}_{U_a, \tilde{\mathbb{T}}} \colon K^0_{\mathrm{cpt}}(U_a) \to \mathbb{Z}$.

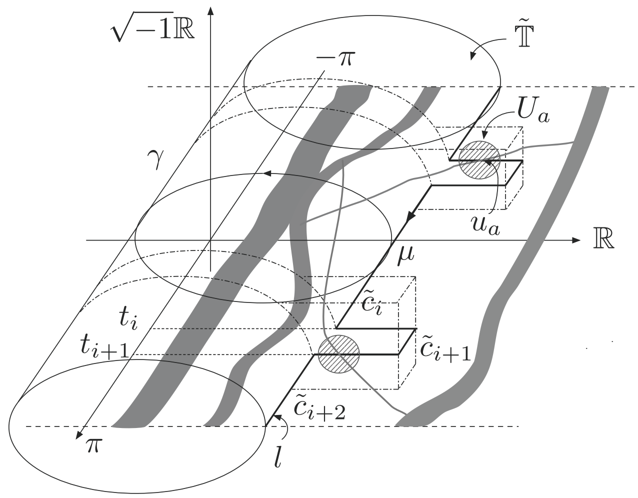


Figure 6: Eigenvalues which cross the Fermi level $\mu$, and the deformed torus $\tilde{\mathbb{T}}$

We later consider another spin$^c$ structure on $U_a$, so we write subscript $(U_a, \tilde{\mathbb{T}})$ in order to indicate which spin$^c$ structure is used. By the excision property of the index, we have,

$$\mathrm{ind}_{\tilde{\mathbb{T}}}([\tilde{E}_{\mathrm{GP}}|_{\tilde{\mathbb{T}}}, \underline{V}^{\oplus k}; g]) = \sum_{a=1}^{m} \mathrm{ind}_{U_a, \tilde{\mathbb{T}}}([\tilde{E}_{\mathrm{GP}}|_{U_a}, \underline{V}^{\oplus k}; g]). \tag{3}$$

In order to prove Proposition 3.4.5, it is enough to show the following Lemma.

**Lemma 3.4.9.** *For each crossing point $u_a$, the following holds.*

*(I) If $u_a$ is contained in the interval $\mathcal{L}(\tilde{c}_i, \tilde{c}_{i+1}) \times \{t_i\}$ where $\tilde{c}_i > \tilde{c}_{i+1}$ and if its multiplicity is $r_a$, then $\mathrm{ind}_{U_a,\tilde{\mathbb{T}}}([\tilde{E}_{\mathrm{GP}}|_{U_a}, \underline{V}^{\oplus k}; g]) = +r_a$.*

*(II) If $u_a$ is contained in the interval $\mathcal{L}(\tilde{c}_i, \tilde{c}_{i+1}) \times \{t_i\}$ where $\tilde{c}_i < \tilde{c}_{i+1}$ and if its multiplicity is $r_a$, then $\mathrm{ind}_{U_a,\tilde{\mathbb{T}}}([\tilde{E}_{\mathrm{GP}}|_{U_a}, \underline{V}^{\oplus k}; g]) = -r_a$.*

In order to prove Lemma 3.4.9, we need some Lemmas. We now consider on $U_a$ a complex structure induced by the inclusion $U_a \subset \mathbb{C} \times \{t\} \cong \mathbb{C}$.

**Lemma 3.4.10.** *$\tilde{E}_{\mathrm{GP}}|_{U_a} \to U_a$ is a holomorphic vector bundle.*

*Proof.* Let us consider the following operator,

$$D(z,t) := \begin{pmatrix} 0 & H^\flat(z,t)^* \\ H^\flat(z,t) & 0 \end{pmatrix}.$$

$D(z,t)$ is a bounded linear self-adjoint operator on $l^2(\mathbb{Z}_{\geq 0}; V) \oplus l^2(\mathbb{Z}_{\geq 0}; V)$. We have $\mathrm{Ker}\, D(z,t) = \mathrm{Ker}\, H^\flat(z,t) \oplus \mathrm{Ker}\, H^\flat(z,t)^* \cong (E_{\mathrm{GP}})_{z,t} \oplus V^{\oplus k}$. By Lemma 3.3.3, the rank of $\mathrm{Ker}\, D(z,t)$ is constant on $U_a$. By considering the spectral decomposition of the self-adjoint operator $D(z,t)$, it is easy to see that we can take a positively oriented smooth simple closed curve $C$ in $\mathbb{C}$, which does not intersects with $\mathrm{sp}(D(z,t))$ and contains just the zero eigenvalue inside $C$ for any $(z,t) \in U_a$. Then Riesz projections give a holomorphic family of projections $p(z,t) := \frac{1}{2\pi i}\int_C (\lambda 1 - D(z,t))^{-1} d\lambda$ parametrized by $U_a$. The images of this family is $\tilde{E}_{\mathrm{GP}}|_{U_a} \oplus \underline{V}^{\oplus k}$, and is a holomorphic vector bundle on $U_a$. Therefore, $\tilde{E}_{\mathrm{GP}}|_{U_a}$ is a holomorphic vector bundle. □

**Lemma 3.4.11.** *Let $z$ be the complex coordinate of $U_a$, where $U_a$ is considered as a subspace of $\mathbb{C}$. Then $[\tilde{E}_{\mathrm{GP}}|_{U_a}, \underline{V}^{\oplus k}; g]$, the element of the $K$-group $K^0_{\mathrm{cpt}}(U_a)$, is expressed as, $[\tilde{E}_{\mathrm{GP}}|_{U_a}, \underline{V}^{\oplus k}; g] = [\underline{\mathbb{C}}, \underline{\mathbb{C}}; z^{r_a}]$.*

*Proof.* Since $\tilde{E}_{\mathrm{GP}}|_{U_a}$ is a holomorphic vector bundle, $g \colon \tilde{E}_{\mathrm{GP}}|_{U_a} \to \underline{V}^{\oplus k}$ is a holomorphic bundle map. Thus $\det g \colon \det E_{\mathrm{GP}}|_{U_a} \to \underline{\mathbb{C}}$ is also holomorphic. We take a holomorphic trivialization of $\det \tilde{E}_{\mathrm{GP}}|_{U_a}$, then $\det g$ is a holomorphic map on $U_a$ which is zero at $u_a$ whose order is $r_a$. By considering the Taylor series, $\det g$ is expressed as $z^{r_a}h(z)$ where $h(z)$ is a nowhere vanishing function on $U_a$. Thus we have,

$$[\tilde{E}_{\mathrm{GP}}|_{U_a}, \underline{V}^{\oplus k}; g] = [\det \tilde{E}_{\mathrm{GP}}|_{U_a}, \underline{\mathbb{C}}; \det g] = [\underline{\mathbb{C}}, \underline{\mathbb{C}}; z^{r_a}h(z)] = [\underline{\mathbb{C}}, \underline{\mathbb{C}}; z^{r_a}]. \qquad \square$$

We remind the reader that $U_a$ is considered as a subspace in two different ways. One is as a subspace of $\mathbb{C} \cong \mathbb{C} \times \{t\}$, and the other is as a subspace of $\tilde{\mathbb{T}}$. Thus we have two induced spin$^c$ structures (and orientations) on $U_a$, which can be different.

*Proof of Lemma 3.4.9.* In the case where the assumption of (I) holds, these two orientations are the same, and in the case (II), they are the opposite. Thus, by Lemma 3.4.11, we have

$$\mathrm{ind}_{U_a,\tilde{\mathbb{T}}}([\tilde{E}_{\mathrm{GP}}|_{U_a}, \underline{V}^{\oplus k}; g]) = \begin{cases} +r_a, & \text{if (I) holds.} \\ -r_a, & \text{if (II) holds.} \end{cases}$$

*Proof of Proposition 3.4.5.* By Lemma 3.4.9 and the definition of the spectral flow given at Sect. 2.2, we have $\sum_{i=1}^m \mathrm{ind}_{U_a,\tilde{\mathbb{T}}}([\tilde{E}_{\mathrm{GP}}|_{U_a}, \underline{V}^{\oplus k}; g]) = \mathrm{sf}(H^{\#}(t) - \mu)$. Thus by Lemma 3.4.8 and the equation (3), we have

$$\begin{aligned} \mathcal{I}_{\mathrm{GP}} = \mathrm{ind}_{\tilde{\mathbb{T}}}([\tilde{E}_{\mathrm{GP}}|_{\tilde{\mathbb{T}}}, \underline{V}^{\oplus k}; g]) &= \sum_{a=1}^m \mathrm{ind}_{U_a,\tilde{\mathbb{T}}}([\tilde{E}_{\mathrm{GP}}|_{U_a}, \underline{V}^{\oplus k}; g]) \\ &= \mathrm{sf}(H^{\#}(t) - \mu) = -\mathcal{I}_{\mathrm{Edge}}. \end{aligned}$$

This completes the proof of Theorem 3.2.7.

Remark 3.4.12. Although our cobordism argument follows that of Graf–Porta's, but there is a little difference in the following sense. Graf–Porta extended the Bloch bundle to some neighborhood of $\mathbb{S}^1_\eta \times \mathbb{T}$ and relate wave functions which make this bundle and $E_{\mathrm{GP}}$ through the eigenfunction of $H(\eta,t) - z$. Relating these functions is not so easy in general, and so Graf–Porta assumed some non-degeneracy of the band in their proof (especially Lemma 5.8 of [GP13]). In our proof, we treat wave functions rather indirectly. Since the operation of taking the Fermi projection and the family index are understood $K$-theoretically (they are used in the proof of the Bott periodicity [AB64b, Ati68]), we defined elements $\alpha_z$ and $\alpha_\eta$ of $K$-groups and showed that they maps to $[E_{\mathrm{B}}]$ (Lemma 3.4.2) and $-[E_{\mathrm{GP}}] + [\underline{V}^{\oplus k}]$ (Lemma 3.4.3), respectively. Then we construct the element $\alpha$ by using $H(\eta,t) - z$ itself and gives a cobordism between $\alpha_z$ and $\alpha_\eta$. In this way, we avoid to treat wave functions directly, and so we do not need to assume some non-degenracy of the band.

### 3.5 Type AII : 2d topological insulator

In this subsection, we give a proof of the bulk-edge correspondence for the two-dimensional quantum spin Hall system in a same line as above.

#### 3.5.1 Settings, bulk index, edge index and the bulk-edge correspondence

On the unit circle $\mathbb{T}$, we consider an involution $\tau_{\mathbb{T}}\colon \mathbb{T} \to \mathbb{T}$ induced by the complex conjugation on $\mathbb{C}$. $(\mathbb{T}, \tau_{\mathbb{T}})$ is an involutive space. Let $V$ be a finite dimensional Hermitian vector space. We denote the complex dimension of $V$ by $N$. We assume that we have an anti-linear map $\Theta\colon V \to V$ which satisfies $\Theta\Theta^* = 1$ and $\Theta^2 = -1$.[16] In this case, $N$ is even. $\Theta$ induces an anti-linear map on $l^2(\mathbb{Z}; V)$ by $\{\varphi_n\}_{n\in\mathbb{Z}} \mapsto \{\Theta\varphi_n\}_{n\in\mathbb{Z}}$. We use the same symbol $\Theta$ for this map. Note that $\Theta$ commutes with $P_{\geq k}$ and acts on subspaces $l^2(\mathbb{Z}_{\geq k}; V)$ for $k \in \mathbb{Z}$. We define an involution $\tau_\Theta$ on $\mathrm{End}_{\mathbb{C}}(V)$ by $G \mapsto \Theta G \Theta^*$. The pair $(\mathrm{End}_{\mathbb{C}}(V), \tau_\Theta)$ is an involutive space. We also write $\tau_\Theta$ for the involution on $B(l^2(\mathbb{Z}; V))$ induced by $\Theta$ in the same way.

We consider a bulk Hamiltonian of Definition 3.1.1 (self-adjointness is assumed). We assume further that the map $H(t)\colon (\mathbb{T}, \tau_{\mathbb{T}}) \to (B(l^2(\mathbb{Z}; V)), \tau_\Theta)$ is $\mathbb{Z}_2$-equivariant, that is, $H(\tau_{\mathbb{T}}(t)) = \Theta H(t) \Theta^*$. In other words, our bulk Hamiltonian satisfies the *(odd) time-reversal symmetry*. In this case, the spectrum of our Hamiltonian has the following symmetry.

**Lemma 3.5.1.** *For $t \in \mathbb{T}$, we have $\mathrm{sp}(H(t)) = \mathrm{sp}(H(\tau_{\mathbb{T}}(t)))$.*

We assume the spectral gap condition and take a real number $\mu$ as in Sect. 3.1. We also take a simple closed loop $\gamma$ in the complex plane as in Sect. 3.1. We here take $\gamma$ to be reflection symmetric, $\gamma = \bar{\gamma}$. Let $\tau_\gamma$ be the involution on $\gamma$ induced by the complex conjugation.

By the Fourier transform, we obtain a continuous family of automorphisms $\{H(\eta, t) - z\}_{\mathbb{S}^{1,1}_\eta \times \gamma \times \mathbb{T}}$ which satisfies the following property.

**Lemma 3.5.2.** *The map $\mathbb{S}^{1,1}_\eta \times (\gamma, \tau_\gamma) \times (\mathbb{T}, \tau_{\mathbb{T}}) \to (\mathrm{End}_{\mathbb{C}}(V), \tau_\Theta)$ given by $(\eta, z, t) \mapsto H(\eta, t) - z$ is a $\mathbb{Z}_2$-equivariant map.*

We consider the Bloch bundle $E_{\mathrm{B}}$. By Lemma 3.5.2, $\Theta$ acts on $E_{\mathrm{B}}$ and we have the Quaternionic vector bundle $(E_{\mathrm{B}}, \Theta)$ over $\mathbb{S}^{1,1}_\eta \times (\mathbb{T}, \tau_{\mathbb{T}})$. The Bloch bundle defines an element $[(E_{\mathrm{B}}, \Theta)]$ of the $KSp$-group $KSp^0(\mathbb{S}^{1,1}_\eta \times (\mathbb{T}, \tau_{\mathbb{T}}))$.

We next construct a map from this $KSp$-group to $\mathbb{Z}_2$ as follows. We consider the $\mathbb{Z}_2$-equivariant embedding $m\colon \mathbb{S}^{1,1}_\eta \times (\mathbb{T}, \tau_{\mathbb{T}}) \hookrightarrow \mathbb{R}^{1,1} \times \mathbb{R}^{1,1} \times \mathbb{R}^{2,0}$ defined by $(\eta, t) \mapsto (\eta, t, 0)$. The normal bundle of this embedding is a trivial bundle whose fiber is $\mathbb{R}^{2,2}$. Thus we have $m_!^{\mathrm{AII}}\colon KSp^0(\mathbb{S}^{1,1}_\eta \times (\mathbb{T}, \tau_{\mathbb{T}})) \to KSp^0_{\mathrm{cpt}}(\mathbb{R}^{1,1} \times \mathbb{R}^{1,1} \times \mathbb{R}^{2,0})$. Since $KSp^0_{\mathrm{cpt}}(\mathbb{R}^{1,1} \times \mathbb{R}^{1,1} \times \mathbb{R}^{2,0}) \cong KSp^{-6}(\mathrm{pt}) \cong \mathbb{Z}_2$, we have the following map.

$$\mathrm{ind}^{\mathrm{AII}}_{\mathbb{S}^1_\eta \times \mathbb{T}}\colon KSp^0(\mathbb{S}^{1,1}_\eta \times (\mathbb{T}, \tau_{\mathbb{T}})) \to \mathbb{Z}_2.$$

This map $\mathrm{ind}^{\mathrm{AII}}_{\mathbb{S}^1_\eta \times \mathbb{T}}$ is defined in the same line as the topological index map associated to the spin$^c$ structure of the torus in $K$-theory. An easy calculation shows that the $KSp$-group

[16] Note that $\Theta^*$ is the adjoint of the anti-linear map $\Theta$ which satisfies $\langle \Theta x, y\rangle_V = \overline{\langle x, \Theta^* y\rangle_V}$ for any $x, y \in V$.

$KSp^{0,0}(\mathbb{S}^{1,1}_\eta \times (\mathbb{T}, \tau_\mathbb{T}))$ is isomorphic to $\mathbb{Z} \oplus \mathbb{Z}_2$, where the $\mathbb{Z}$ summand corresponds to the class of product bundles. We next show that $\mathrm{ind}^{\mathrm{AII}}_{\mathbb{S}^1_\eta \times \mathbb{T}}$ maps the class of product bundles to 0.

**Lemma 3.5.3.** *Let $[(\underline{\mathbb{C}^l}, \Theta_0)] \in KSp^{0,0}(\mathbb{S}^{1,1}_\eta \times (\mathbb{T}, \tau_\mathbb{T}))$ be the class of a product bundle. Then* $\mathrm{ext} \circ \beta([(\underline{\mathbb{C}^l}, \Theta_0)]) = 0$. *Thus* $\mathrm{ind}^{\mathrm{AII}}_{\mathbb{S}^1_\eta \times \mathbb{T}}([(\underline{\mathbb{C}^l}, \Theta_0)]) = 0$.

*Proof.* Following section 4 of [Ati66], we first give an explicit description of the map $\beta$. Let $\mathrm{Cliff}(\mathbb{R}^{2,2})$ be the Clifford algebra over $\mathbb{R}$ generated by $e_1$, $e_2$, $e_3$, $e_4$ with relations $e_i^2 = -1$ and anti-commutes each other, and an involution $\tau_{Cl} \colon \mathrm{Cliff}(\mathbb{R}^{2,2}) \to \mathrm{Cliff}(\mathbb{R}^{2,2})$ given by $e_i \mapsto -e_i$ for $i = 1, 2$ and $e_j \mapsto e_j$ for $j = 3, 4$. As an $\mathbb{R}$-algebra $\mathrm{Cliff}(\mathbb{R}^{2,2})$ is isomorphic to $\mathbb{H}(2)$. Let $\Delta = \mathbb{C}^4$. We give an complex $\mathrm{Cliff}(\mathbb{R}^{2,2})$-module structure on $\Delta$ as follows.

$$e_1 = \begin{pmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & -i & 0 \\ 0 & -i & 0 & 0 \\ -i & 0 & 0 & 0 \end{pmatrix}, \; e_2 = \begin{pmatrix} 0 & 0 & -i & 0 \\ 0 & 0 & 0 & i \\ -i & 0 & 0 & 0 \\ 0 & i & 0 & 0 \end{pmatrix},$$

$$e_3 = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}, \; e_4 = \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}.$$

Then we have $\epsilon := e_1 e_2 e_3 e_4 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$. The element $\epsilon$ defines a $\mathbb{Z}_2$-grading on $\Delta$, for which we denote $\Delta = \Delta^{\mathrm{even}} \oplus \Delta^{\mathrm{odd}}$. Let $\tau_\Delta \colon \Delta \to \Delta$ be an anti-linear involution given by the complex conjugation $(r_1 + is, r_2 + it, z_3, z_4) \mapsto (\overline{r_1 + is}, \overline{r_2 + it}, \bar{z_3}, \bar{z_4})$. We have $\tau_\Delta(s \cdot m) = \tau_{Cl}(s)\tau_\Delta(m)$ for $s \in \mathrm{Cliff}(\mathbb{R}^{2,2})$ and $m \in \Delta$. Thus $(\Delta, \tau_\Delta)$ is an irreducible Real $\mathbb{Z}_2$-graded $\mathrm{Cliff}(\mathbb{R}^{2,2})$-module in the sense of Atiyah [Ati68]. By [Ati68], the map $\beta$ can be expressed by taking an exterior tensor product with the Bott element. More explicitly,

$$\beta([(\underline{\mathbb{C}^l}, \Theta_0)]) = [(\underline{\mathbb{C}^l}, \Theta_0)] \cdot [(\underline{\Delta}^{\mathrm{even}}, \underline{\Delta}^{\mathrm{odd}}; \sigma)]$$

where $\sigma$ is the Clifford multiplication.

We next consider the map $\mathrm{ext} \colon KSp^0_{\mathrm{cpt}}(\mathbb{S}^{1,1} \times \mathbb{T} \times \mathbb{R}^{2,2}) \to KSp^0_{\mathrm{cpt}}(\mathbb{R}^{1,1} \times \mathbb{R}^{1,1} \times \mathbb{R}^{2,0})$. We now identify $\mathbb{R}^{1,1}$ with $\mathbb{C}$ and by use the polar coordinate. We write $(r_1 e^{i\theta_1}, r_2 e^{i\theta_2}, (s, t))$ for an element of $\mathbb{R}^{1,1} \times \mathbb{R}^{1,1} \times \mathbb{R}^{2,0}$. We here used the polar coordinate for two $\mathbb{R}^{1,1}$. We define $\tilde{\sigma} \colon \mathbb{R}^{1,1} \times \mathbb{R}^{1,1} \times \mathbb{R}^{2,0} \to \mathrm{Hom}(\Delta^{\mathrm{even}}, \Delta^{\mathrm{odd}})$ by

$$\begin{aligned} \tilde{\sigma}(r_1 e^{i\theta_1}, r_2 e^{i\theta_2}, (s, t)) &= s e_1 + t e_2 + (r_1 - 1) e_3 + (r_2 - 1) e_4 \\ &= \begin{pmatrix} r_2 - it - 1 & -r_1 - is + 1 \\ r_1 - is - 1 & r_2 + it - 1 \end{pmatrix}. \end{aligned}$$

By using this $\tilde{\sigma}$, we have

$$\mathrm{ext} \circ \beta([(\underline{\mathbb{C}^l}, \Theta_0)]) = [(\underline{\mathbb{C}^l}, \Theta_0)] \cdot [(\underline{\Delta}^{\mathrm{even}}, \underline{\Delta}^{\mathrm{odd}}; \tilde{\sigma})].$$

Thus, it is enough to show that $\tilde{\sigma}$ is $\mathbb{Z}_2$-equivariant homotopic to a bundle isomorphism between $\underline{\Delta}^{\mathrm{even}} = \mathbb{R}^{1,1} \times \mathbb{R}^{1,1} \times \mathbb{R}^{2,0} \times \Delta^{\mathrm{even}}$ and $\underline{\Delta}^{\mathrm{odd}} = \mathbb{R}^{1,1} \times \mathbb{R}^{1,1} \times \mathbb{R}^{2,0} \times \Delta^{\mathrm{odd}}$ in the space of compactly supported homomorphisms.

Let $c \in \mathbb{R}$ be a parameter, and consider a $\mathbb{Z}_2$-equivariant bundle homomorphism $\tilde{\sigma} + cI$. Then we have

$$\det(\tilde{\sigma} + cI)((r_1 e^{i\theta_1}, r_2 e^{i\theta_2}, (s,t))) = |r_1 + is - 1|^2 + |r_2 + it - 1|^2 + 2cr_1 - 2c + c^2.$$

If we take sufficiently large $c$, then $\tilde{\sigma} + cI$ is a bundle isomorphism, and $\tilde{\sigma}$ is $\mathbb{Z}_2$-equivariant homotopic to such a bundle isomorphism. □

We now define the $\mathbb{Z}_2$-valued topological invariant for our bulk system.

Definition 3.5.4 (Bulk Index). We define the *bulk index* of our system by,

$$\mathcal{I}^{\mathrm{AII}}_{\mathrm{Bulk}} := \mathrm{ind}^{\mathrm{AII}}_{\mathbb{S}^1_\eta \times \mathbb{T}}([(E_{\mathrm{B}}, \Theta)]) \in \mathbb{Z}_2.$$

Remark 3.5.5. This invariant is called the *Kane-Mele index* or the *FKMM-invariant* [FKMM00, KM05b, DNG15].

We next consider the edge Hamiltonian $H^{\#}(t)$. Since $\Theta$ commutes with $P_{\geq k}$, we have a continuous family of self-adjoint Fredholm operators $\{H^{\#}(t) - \mu\}_{t \in \mathbb{T}}$ which gives a $\mathbb{Z}_2$-equivariant map $(\mathbb{T}, \tau_{\mathbb{T}}) \to (\mathrm{Fred}(l^2(\mathbb{Z}_{\geq 0}; V)^{s.a.}, \tau_\Theta)$. The spectrum of our edge Hamiltonian also has the symmetry with respect to the involution $\tau_{\mathbb{T}}$ (see also Figure 7).

Definition 3.5.6 (Edge Index). We define the *edge index* of our system by

$$\mathcal{I}^{\mathrm{AII}}_{\mathrm{Edge}} := \mathrm{sf}_2(\{H^{\#}(t) - \mu\}_{t \in \mathbb{T}}) \in \mathbb{Z}_2.$$

We now revisit the definition of the $\mathbb{Z}_2$-valued spectral flow (Definition 2.2.4). In order to define $\mathrm{sf}_2(\{H^{\#}(t) - \mu\}_{t \in \mathbb{T}})$, we choose $t_i$ and $c_i$. We now consider a path $l \in \mathbb{C} \times [0, \pi]$ defined as in the Sect. 3.2. Then $\tilde{l} = l \cup (\tau \times \tau_{\mathbb{T}})(l)$ gives a loop in $\mathbb{C} \times \mathbb{T}$. By using this loop $\tilde{l}$, we consider a deformed torus $\tilde{\mathbb{T}}$ as in Sect. 3.4.2 in such a way that $\tilde{\mathbb{T}}$ is closed under the involution. $\mathrm{sf}_2(\{H^{\#}(t) - \mu\}_{t \in \mathbb{T}})$ is defined by counting the number of crossing points $\{u_1, \cdots, u_m\}$ of the spectrum of the edge Hamiltonian and the path $l$ with multiplicity, and taking mod 2. If we consider the path $(\tau \times \tau)(l)$, there also are crossing points $\{\hat{u}_1, \cdots, \hat{u}_m\}$ in such a symmetric way with respect to the involution (see Figure 7). Thus we can say that the $\mathbb{Z}_2$-valued spectral flow $\mathrm{sf}_2(\{H^{\#}(t) - \mu\}_{t \in \mathbb{T}})$ is defined by counting the *pair* of crossing points $(u_a, \hat{u}_a)$ with multiplicity and taking mod 2. We take this point of view in Proposition 3.5.14.

The following is the main theorem of this subsection, the bulk-edge correspondence for two-dimensional type AII topological insulators.

**Theorem 3.5.7** (Bulk-edge correspondence for quantum spin Hall systems)**.** *The bulk index coincides with the edge index. That is,*

$$\mathcal{I}^{\mathrm{AII}}_{\mathrm{Bulk}} = \mathcal{I}^{\mathrm{AII}}_{\mathrm{Edge}}.$$

In order to prove this theorem, we consider the decaying solutions as Graf–Porta [GP13].

### 3.5.2 Proof of the bulk-edge correspondence

As in the case of type A systems, we consider decaying solutions in order to prove Theorem 3.5.7. By Lemma 3.3.1 and Lemma 3.3.3, we can define $E_{\mathrm{GP}}$ as the kernels of a family of Fredholm operators. Since our Hamiltonian satisfies the time-reversal symmetry and the operator $\Theta$ acts on $l^2(\mathbb{Z}; V)$ point-wisely, the following Lemma follows.

**Lemma 3.5.8.** $\Theta P_{\geq k}(H(t) - z)P_{\geq 0}\Theta^* = P_{\geq k}(H(t) - z)P_{\geq 0}$.

By the above lemma, $\Theta$ acts on the bundle $E_{\mathrm{GP}}$, and the pair $(E_{\mathrm{GP}}, \Theta)$ is a Quaternionic vector bundle over the involutive space $(\gamma, \tau_\gamma) \times (\mathbb{T}, \tau_{\mathbb{T}})$. Note that the cokernels $\underline{V}^{\oplus k}$ also has a Quaternionic vector bundle structure over $(\gamma, \tau_\gamma) \times (\mathbb{T}, \tau_{\mathbb{T}})$ in a natural way. Thus $(E_{\mathrm{GP}}, \Theta)$ defines an element of $KSp^0((\gamma, \tau_\gamma) \times (\mathbb{T}, \tau_{\mathbb{T}}))$. We now construct a map from $KSp^0((\gamma, \tau_\gamma) \times (\mathbb{T}, \tau_{\mathbb{T}}))$ to $\mathbb{Z}_2$ as follows. We consider an $\mathbb{Z}_2$-equivariant embedding $m' \colon (\gamma, \tau_\gamma) \times (\mathbb{T}, \tau_{\mathbb{T}}) \hookrightarrow \mathbb{R}^{1,1} \times \mathbb{R}^{1,1} \times \mathbb{R}^{2,0}$ defined by $(z, t) \mapsto (z, t, 0)$. The normal bundle of this embedding is a trivial bundle whose fiber is $\mathbb{R}^{2,2}$. Thus we have, ${m'}_!^{\mathrm{AII}} \colon KSp^0((\gamma, \tau_\gamma) \times (\mathbb{T}, \tau_{\mathbb{T}})) \to KSp^0_{\mathrm{cpt}}((\gamma, \tau_\gamma) \times (\mathbb{T}, \tau_{\mathbb{T}}) \times \mathbb{R}^{2,2})$. Since $KSp^0_{\mathrm{cpt}}(\mathbb{R}^{1,1} \times \mathbb{R}^{1,1} \times \mathbb{R}^{2,0}) \cong \mathbb{Z}_2$, we have the map,

$$\mathrm{ind}^{\mathrm{AII}}_{\gamma \times \mathbb{T}} \colon KSp^0((\gamma, \tau_\gamma) \times (\mathbb{T}, \tau_{\mathbb{T}})) \to \mathbb{Z}_2,$$

defined by the composition of these maps.

Definition 3.5.9. We define the Graf-Porta's index of our system by,

$$\mathcal{I}^{\mathrm{AII}}_{\mathrm{GP}} := \mathrm{ind}^{\mathrm{AII}}_{\gamma \times \mathbb{T}}([(E_{\mathrm{GP}}, \Theta)]) \in \mathbb{Z}_2.$$

As in the type A case, the proof of Theorem 3.5.7 is divided into two parts. We first show that the bulk index coincides with Graf–Porta's index by using the cobordism invariance of the index (Proposition 3.5.10). We next show that Graf–Porta's index equals to the edge index by the localization of the $KSp$-class (Proposition 3.5.14).

**Proposition 3.5.10.** $\mathcal{I}^{\mathrm{AII}}_{\mathrm{Bulk}} = \mathcal{I}^{\mathrm{AII}}_{\mathrm{GP}}$.

We consider the space $X$ and $Y$ considered in the case of type A systems with involutions. Let $(X, \tau_X) := (\mathbb{D}^{1,1}_\eta \times \mathbb{D}^{1,1}_z) \setminus (\mathbb{S}^{1,1}_\eta \times (\gamma, \tau_\gamma)) \times (\mathbb{T}, \tau_{\mathbb{T}})$ and $(Y, \tau_Y) := \mathbb{S}^{1,1}_\eta \times (\mathbb{D}^{1,1}_z \setminus (\gamma, \tau_\gamma)) \times (\mathbb{T}, \tau_{\mathbb{T}}) \sqcup (\mathbb{D}^{1,1}_\eta \setminus \mathbb{S}^{1,1}_\eta) \times (\gamma, \tau_\gamma) \times (\mathbb{T}, \tau_{\mathbb{T}})$. Take an extension of $H(\eta, t) - z$ as in the case of type A, and we obtain an elements,

$$\alpha^{\mathrm{AII}}_z \in KSp^0_{\mathrm{cpt}}(\mathbb{S}^{1,1}_\eta \times (\mathbb{D}^{1,1}_z \setminus (\gamma, \tau_\gamma)) \times (\mathbb{T}, \tau_{\mathbb{T}})).$$

$$\alpha^{\mathrm{AII}}_\eta \in KSp^0_{\mathrm{cpt}}((\mathbb{D}^{1,1}_\eta \setminus \mathbb{S}^{1,1}_\eta) \times (\gamma, \tau_\gamma) \times (\mathbb{T}, \tau_{\mathbb{T}})).$$

Let $F_{\mathrm{AII}}$ be the composition of the inverse of the Bott periodicity isomorphism.

$$(\beta^{\mathrm{AII}}_z)^{-1} \colon KSp^0_{\mathrm{cpt}}(\mathbb{S}^{1,1}_\eta \times (\mathbb{D}^{1,1}_z \setminus (\gamma, \tau_\gamma)) \times (\mathbb{T}, \tau_{\mathbb{T}})) \to KSp^0(\mathbb{S}^{1,1}_\eta \times (\mathbb{T}, \tau_{\mathbb{T}})),$$

and $\mathrm{ind}^{\mathrm{AII}}_{\mathbb{S}^1_\eta \times \mathbb{T}} \colon KSp^0(\mathbb{S}^{1,1}_\eta \times (\mathbb{T}, \tau_{\mathbb{T}})) \to \mathbb{Z}_2$. The relation between $\alpha^{\mathrm{AII}}_z$ and $\mathcal{I}^{\mathrm{AII}}_{\mathrm{Bulk}}$ is stated as follows, which follows easily from Atiyah's proof of the Bott periodicity theorem for $KR$-theory [Ati66].

**Lemma 3.5.11.** $(\beta^{\mathrm{AII}}_z)^{-1}(\alpha^{\mathrm{AII}}_z) = [(E_{\mathrm{B}}, \Theta)]$. *Thus we have* $F_{\mathrm{AII}}(\alpha^{\mathrm{AII}}_z) = \mathcal{I}^{\mathrm{AII}}_{\mathrm{Bulk}}$.

Let $F'_{\mathrm{AII}}$ be the composition of the inverse of the Bott periodicity isomorphism,

$$(\beta_\eta^{\mathrm{AII}})^{-1}\colon KSp^0_{\mathrm{cpt}}((\mathbb{D}_\eta^{1,1}\setminus\mathbb{S}_\eta^{1,1})\times(\gamma,\tau_\gamma)\times(\mathbb{T},\tau_{\mathbb{T}}))\to KSp^0((\gamma,\tau_\gamma)\times(\mathbb{T},\tau_{\mathbb{T}})),$$

and $\mathrm{ind}^{\mathrm{AII}}_{\gamma\times\mathbb{T}}\colon KSp^0((\gamma,\tau_\gamma)\times(\mathbb{T},\tau_{\mathbb{T}}))\to\mathbb{Z}_2$. The relation between $\alpha_\eta^{\mathrm{AII}}$ and $\mathcal{I}^{\mathrm{AII}}_{\mathrm{GP}}$ is stated as follows.

**Lemma 3.5.12.** $(\beta_\eta^{\mathrm{AII}})^{-1}(\alpha_\eta)=[(E_{\mathrm{GP}},\Theta)]+[(\underline{V}^{\oplus k},\Theta)]$. *Thus we have* $F'_{\mathrm{AII}}(\alpha_\eta^{\mathrm{AII}})=\mathcal{I}^{\mathrm{AII}}_{\mathrm{GP}}$.

*Proof.* As in Lemma 3.4.3, the proof is given by considering the following commutative diagram [Ati66, Ati68].

$$\begin{tikzcd}
KSp^{-1}(\mathbb{S}_\eta^{1,1}\times(\gamma,\tau_\gamma)\times(\mathbb{T},\tau_{\mathbb{T}})) \arrow[r, "\partial"] \arrow[d, "\mathcal{T}"'] & KSp^0((\mathbb{D}_\eta^{1,1},\mathbb{S}_\eta^{1,1})\times(\gamma,\tau_\gamma)\times(\mathbb{T},\tau_{\mathbb{T}})) \arrow[d, "(\beta_\eta^{\mathrm{AII}})^{-1}"] \\
{[(\gamma,\tau_\gamma)\times(\mathbb{T},\tau_{\mathbb{T}}),(\mathrm{Fred}(l^2(\mathbb{Z}_{\geq 0};\mathbb{C}^2),\tau_\Theta)]} \arrow[r, "\mathrm{index}"] & KSp^0((\gamma,\tau_\gamma)\times(\mathbb{T},\tau_{\mathbb{T}})).
\end{tikzcd}$$

□

By these considerations, Proposition 3.5.10 follows by the following Lemma.

**Lemma 3.5.13.** $(F_{\mathrm{AII}}\oplus F'_{\mathrm{AII}})(\alpha_z^{\mathrm{AII}},\alpha_\eta^{\mathrm{AII}})=0$

*Proof.* We first construct an explicit $\mathbb{Z}_2$-equivariant embedding $E^{\mathrm{AII}}\colon X\to\mathbb{R}^{6,4}\times([0,+\infty),\mathrm{id})$ which maps the boundary to the boundary and whose normal bundle is a trivial bundle of fiber $\mathbb{R}^{3,3}$. Let $\theta\colon(\mathbb{D}_\eta^{1,1}\times\mathbb{D}_z^{1,1})\setminus(\mathbb{S}_\eta^{1,1}\times(\gamma,\tau_\gamma))\to\mathbb{R}^{0,1}$ be a function given by $\theta(\eta,z)=\arg((1-|\eta|)+\sqrt{-1}(1-|z|))$, where $\arg(x+\sqrt{-1}y)$ is the argument of $x+\sqrt{-1}y$. We here consider $\mathbb{D}_\eta^{1,1}$ and $\mathbb{D}_z^{1,1}$ as a unit disc in the complex plane and $|\cdot|$ is the absolute value in the complex plane. By using this, we define a map $f\colon(\mathbb{D}_\eta^{1,1}\times\mathbb{D}_z^{1,1})\setminus(\mathbb{S}_\eta^{1,1}\times(\gamma,\tau_\gamma))\to\mathbb{R}^{0,1}\times([0,+\infty),\mathrm{id})$ by $f(\eta,z)=(\cos(2\theta(\eta,z)),\sin(2\theta(\eta,z)))$, where $\mathbb{R}^{0,1}\times([0,+\infty),\mathrm{id})$ is identified with the upper half-plane in $\mathbb{R}^{0,2}$. $\theta$ and $f$ are $\mathbb{Z}_2$-equivariant maps. We consider the map,

$$\begin{aligned}
e\colon(X,\tau_X)&=(\mathbb{D}_\eta^{1,1}\times\mathbb{D}_z^{1,1})\setminus(\mathbb{S}_\eta^{1,1}\times(\gamma,\tau_\gamma))\times(\mathbb{T},\tau_{\mathbb{T}})\\
&\longrightarrow\mathbb{R}^{1,1}\times\mathbb{R}^{1,1}\times(\mathbb{R}^{0,1}\times([0,+\infty),\mathrm{id}))\times\mathbb{R}^{1,1}=\mathbb{R}^{3,4}\times([0,+\infty),\mathrm{id}),
\end{aligned}$$

given by $e(\eta,z,t)=(\eta,z,f(\eta,z),t)$. The map $e$ is a $\mathbb{Z}_2$-equivariant neat embedding which maps the boundary to the boundary whose normal bundle is a trivial bundle of fiber $\mathbb{R}^{0,3}$. By using this, we define an embedding $E^{\mathrm{AII}}\colon(X,\tau_X)\to(\mathbb{R}^{3,4}\times([0,+\infty),\mathrm{id}))\times\mathbb{R}^{3,0}$ by $E^{\mathrm{AII}}(\eta,z,t)=(e(\eta,z,t),0)$. This map $E^{\mathrm{AII}}$ has the desired properties. Let $i'\colon\mathbb{R}^{6,4}\hookrightarrow\mathbb{R}^{6,4}\times([0,+\infty),\mathrm{id})$ be an inclusion given by $i'(x)=(x,0)$. We have the following commutative diagram.

$$\begin{tikzcd}
KSp^0_{\mathrm{cpt}}(X,\tau_X) \arrow[r, "i^*"] \arrow[d, "(E^{\mathrm{AII}})_!"] & KSp^0_{\mathrm{cpt}}(Y,\tau_Y) \arrow[r, "F_{\mathrm{AII}}\oplus F'_{\mathrm{AII}}"] \arrow[d, "(E^{\mathrm{AII}}|Y)_!"'] & \mathbb{Z}_2 \\
KSp^0_{\mathrm{cpt}}(\mathbb{R}^{6,4}\times([0,+\infty),\mathrm{id})) \arrow[r, "i'^*"] & KSp^0_{\mathrm{cpt}}(\mathbb{R}^{6,4}) \arrow[ur, "(\beta^{\mathrm{AII}})^{-1}"'] &
\end{tikzcd}$$

We have $i^*(\alpha^{\mathrm{AII}})=(\alpha_z^{\mathrm{AII}},\alpha_\eta^{\mathrm{AII}})$ and $KSp^0_{\mathrm{cpt}}(\mathbb{R}^{6,4}\times([0,+\infty),\mathrm{id}))=0$ since the one point compactification of $\mathbb{R}^{6,4}\times([0,+\infty),\mathrm{id})$ is a 11 dimensional closed ball with some involution, which is $\mathbb{Z}_2$-equivariant contractible. By the commutativity of this diagram, we have $(F_{\mathrm{AII}}\oplus F'_{\mathrm{AII}})(\alpha_z^{\mathrm{AII}},\alpha_\eta^{\mathrm{AII}})=0$. □

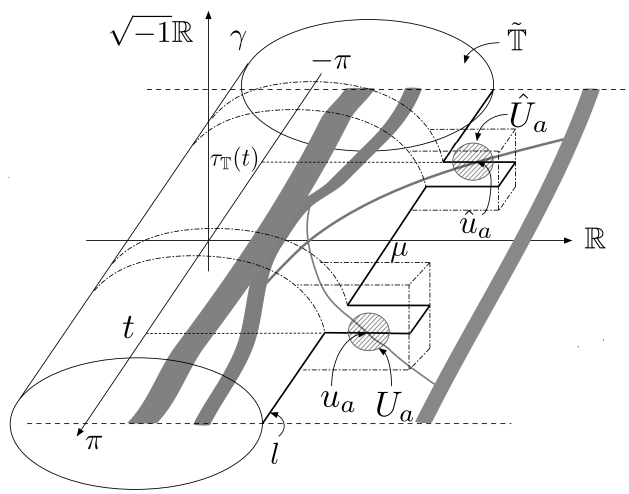


Figure 7: Under the time-reversal symmetry, the spectrum of edge Hamiltonians (gray area and gray line) are symmetric with respect to the involution $\tau_{\mathbb{T}}$ on $\mathbb{T}$. The loop $l$ and the deformed torus $\tilde{\mathbb{T}}$ are taken in such a symmetric way. Thus crossing points appear as a pair $(u_a, \hat{u}_a)$. We take neighborhoods $U_a$ and $\hat{U}_a$ in $\tilde{\mathbb{T}}$ of these points in such a symmetric way.

**Proposition 3.5.14.** $\mathcal{I}_{\mathrm{GP}}^{\mathrm{AII}} = \mathcal{I}_{\mathrm{Edge}}^{\mathrm{AII}}$.

This Proposition follows by the localization of the $KSp$-class by using the Dirichlet boundary condition as in the case of type A (Proposition 3.4.5). We take a deformed torus $\tilde{\mathbb{T}}$ such that $\tilde{\mathbb{T}}$ is closed under the involution of $(\mathbb{C}, \tau) \times \mathbb{R}^{1,1}$. We write $\tau_{\tilde{\mathbb{T}}}$ for the involution on $\tilde{\mathbb{T}}$ induced by this involution. Then we define a map $\mathrm{ind}_{\tilde{\mathbb{T}}}^{\mathrm{AII}} \colon KSp^0(\tilde{\mathbb{T}}, \tau_{\tilde{\mathbb{T}}}) \to \mathbb{Z}_2$ in the same way. The bundle homomorphism $g \colon (\tilde{E}_{\mathrm{GP}}|_{\tilde{\mathbb{T}}}, \Theta) \to (\underline{V}^{\oplus k}, \Theta)$ is a $\mathbb{Z}_2$-equivariant homomorphism. The following statement holds which corresponds to Lemma 3.4.8.

**Lemma 3.5.15.** $\mathcal{I}_{\mathrm{GP}}^{\mathrm{AII}} = \mathrm{ind}_{\tilde{\mathbb{T}}}^{\mathrm{AII}}([(\tilde{E}_{\mathrm{GP}}|_{\tilde{\mathbb{T}}}, \Theta), (\underline{V}^{\oplus k}, \Theta); g])$.

The support of $g$ corresponds to crossing points. By using excision property, we obtain the following equation which corresponds to the equation (3).

$$\mathrm{ind}_{\tilde{\mathbb{T}}}^{\mathrm{AII}}([(\tilde{E}_{\mathrm{GP}}|_{\tilde{\mathbb{T}}}, \Theta), (\underline{V}^{\oplus k}, \Theta); g]) = \sum_{a=1}^{m} \mathrm{ind}_{U_a \cup \hat{U}_a}^{\mathrm{AII}}([(\tilde{E}_{\mathrm{GP}}|_{U_a \cup \hat{U}_a}, \Theta), (\underline{V}^{\oplus k}, \Theta); g]).$$

Now the problem is how to count the contribution of each crossing point to $\mathcal{I}_{\mathrm{GP}}^{\mathrm{AII}}$ by using $KSp$-theory. In the type AII case, we have the following Lemma which corresponds to Lemma 3.4.9. Note that since the spectrum of edge Hamiltonians (and our choice of loop $l$ and deformed torus $\tilde{\mathbb{T}}$) are symmetric with respect to the involution on $\mathbb{T}$, each crossing point appears as a pair.

**Lemma 3.5.16.** *For each pair of crossing points $u_a$ and $\hat{u}_a$, the following holds. If the multiplicity of the crossing point $u_a$ is $r_a$, then*

$$\mathrm{ind}_{U_a \cup \hat{U}_a}^{\mathrm{AII}}([(\tilde{E}_{\mathrm{GP}}|_{U_a \cup \hat{U}_a}, \Theta), (\underline{V}^{\oplus k}, \Theta); g]) = r_a \mod 2.$$

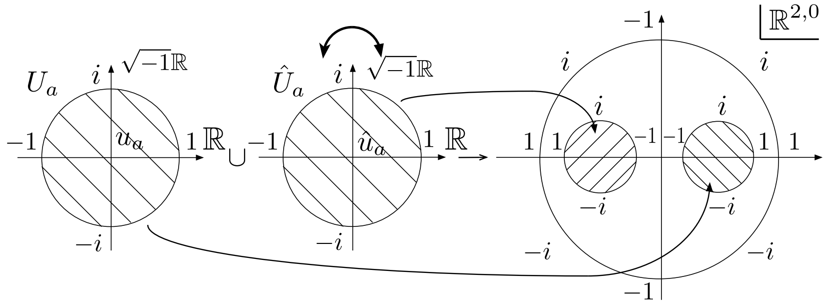


Figure 8: The embedding $U_a \cup \hat{U}_a \to \mathbb{R}^{2,0}$ is indicated at this figure. An example of the value of the determinant of the clutching function and that of the one on $\mathbb{S}^{2,0}$ is indicated by the numbers $1, i, -1, -i$. We here consider the case $r_a = 1$.

*Proof.* At each pair of crossing points $u_a$ and $\hat{u}_a$, We take neighborhoods $U_a$ and $\hat{U}_a$ in $\tilde{\mathbb{T}}$ of these points such that they maps to each other by the involution on the deformed torus $\tilde{\mathbb{T}}$. We then embeds the involutive space $U_a \cup \hat{U}_a$ into $\mathbb{D}^{2,0}$ in a $\mathbb{Z}_2$-equivariant way. More explicitly, we take two small balls in the unit ball in $\mathbb{D}^{2,0}$ in such a way that they maps to each other by the involution on $\mathbb{R}^{2,0}$. We then consider such an embedding as in Figure 8. We embeds $U_a$ into the right ball in $\mathbb{D}^{2,0}$ and $\hat{U}_a$ by first flipping $\hat{U}_a$ around the imaginary axis and embedding into the left ball in $\mathbb{D}^{2,0}$. By using this embedding, we obtain an homomorphism, $KSp^0(U_a \cup \hat{U}_a, \tau_{\tilde{\mathbb{T}}}) \to KSp^0(\mathbb{D}^{2,0}, \mathbb{S}^{2,0})$. The (complex) determinant of the clutching function of a representative of $\mathrm{ext}([(\tilde{E}_{\mathrm{GP}}|_{U_a \cup \hat{U}_a}, \Theta), (\underline{V}^{\oplus k}, \Theta); g])$ is $\mathbb{Z}_2$-equivariant homotopic to $g_{r_a}$ considered in Example 2.1.7. Thus by using the isomorphism $KSp^0(\mathbb{D}^{2,0}, \mathbb{S}^{2,0}) \cong \mathbb{Z}_2$ of Proposition 2.1.8, we have,

$$\mathrm{ind}^{\mathrm{AII}}_{U_a \cup \hat{U}_a}([(\tilde{E}_{\mathrm{GP}}|_{U_a \cup \hat{U}_a}, \Theta), (\underline{V}^{\oplus k}, \Theta); g]) = w(g_{r_a}) = r_a \mod 2. \qquad \square$$

This completes the proof of Theorem 3.5.7.

# 4 Topological invariants and corner states for Hamiltonians on a three dimensional lattice

## 4.1 Bulk-edge invariant and corner invariant

In this subsection, we consider some “gapped” Hamiltonians, and define two topological invariants for them. A correspondence between these two is proved.

Let $V$ be a finite dimensional Hermitian vector space and denote $N$ for the complex dimension of $V$. Let us denote the Hilbert space $l^2(\mathbb{Z} \times \mathbb{Z}; V) = \mathcal{H} \otimes V$ by $\mathcal{H}_V$, and let $\mathcal{H}^\alpha_V := \mathcal{H}^\alpha \otimes V$, $\mathcal{H}^\beta_V := \mathcal{H}^\beta \otimes V$, $\mathcal{H}^{\alpha,\beta}_V := \mathcal{H}^{\alpha,\beta} \otimes V$, $P^\alpha_V := P^\alpha \otimes 1$ and $P^\beta_V := P^\beta \otimes 1$. We consider a continuous family of bounded linear operators,

$$H \colon \mathbb{T} \to B(L^2(\mathbb{T} \times \mathbb{T}; V)),$$

where, for each $t$ in $\mathbb{T}$, the operator $H(t)$ is a self-adjoint multiplication operator generated by a continuous map.[17] We call $H(t)$ *bulk Hamiltonians*. By using the Fourier transform

[17]Let $f \colon \mathbb{T} \times \mathbb{T} \to \mathrm{End}(V)$ be a continuous map. Then the operator on $L^2(\mathbb{T} \times \mathbb{T}; V)$ defined by $g \mapsto fg$ is called the multiplication operator generated by $f$.

$L^2(\mathbb{T}\times\mathbb{T};V)\cong\mathcal{H}_V$, we have a continuous family of self-adjoint bounded linear operators on $\mathcal{H}_V$. We denote by $H(t)\colon \mathcal{H}_V\to\mathcal{H}_V$ this family of operators, by using the same symbol.

Example 4.1.1. We assume that we are given, for each $(p,q,r)\in\mathbb{Z}\times\mathbb{Z}\times\mathbb{Z}$, an endomorphism $A_{p,q,r}$ on $V$, which satisfies $\sum_{(p,q,r)\in\mathbb{Z}\times\mathbb{Z}\times\mathbb{Z}}||A_{p,q,r}||_{\mathrm{op}}<+\infty$. We consider a linear operator, $H_1\colon l^2(\mathbb{Z}\times\mathbb{Z}\times\mathbb{Z};V)\to l^2(\mathbb{Z}\times\mathbb{Z}\times\mathbb{Z};V)$ defined by,

$$(H_1\varphi)_{(k,l,m)}=\sum_{(p,q,r)\in\mathbb{Z}\times\mathbb{Z}\times\mathbb{Z}}A_{p,q,r}\varphi_{k-p,l-q,m-r}.$$

We assume that $H_1$ is a self-adjoint operator. Then its partial Fourier transform gives an example of our family.

We consider half-plane Toeplitz operators, that is,

$$H^\alpha(t):=P_V^\alpha H(t)P_V^\alpha\colon \mathcal{H}_V^\alpha\to\mathcal{H}_V^\alpha,\ \ H^\beta(t):=P_V^\beta H(t)P_V^\beta\colon \mathcal{H}_V^\beta\to\mathcal{H}_V^\beta.$$

They are bounded self-adjoint operators. We call $H^\alpha(t)$ and $H^\beta(t)$ *edge Hamiltonians.* We take an orthonormal frame of $V$. Then, since $H^\alpha(t)$ and $H^\beta(t)$ are compression of the same operator $H(t)$, the pair $(H^\alpha(t),H^\beta(t))$ defines a self-adjoint element of the $C^*$-algebra $M_N(\mathcal{S}^{\alpha,\beta}\otimes C(\mathbb{T}))$. We now take some real number $\mu$ which satisfies that $\mu$ does not contained neither $\mathrm{sp}(H^\alpha(t))$ nor $\mathrm{sp}(H^\beta(t))$ for any $t$ in $\mathbb{T}$. Then the element $(H^\alpha(t)-\mu,H^\beta(t)-\mu)$ is invertible in the $C^*$-algebra $M_N(\mathcal{S}^{\alpha,\beta}\otimes C(\mathbb{T}))$.

Remark 4.1.2. Such $\mu$ does exists. Actually we can take $\mu$ sufficiently large or small. However, if we choose such $\mu$, our topological invariants is zero (see also Remark 4.2.4). Non-trivial invariants appear if operators $H^\alpha(t)$ and $H^\beta(t)$ for $t$ in $\mathbb{T}$ have a common spectral gap at the Fermi level $\mu$. Note that, in this case, our Hamiltonian $H(t)$ also has a spectral gap at $\mu$ since $\mathrm{sp}(H(t))$ is contained in $\mathrm{sp}(H^\alpha(t),H^\beta(t))$.

By Remark 4.1.2, we further assume that the spectrum of $(H^\alpha(t)-\mu,H^\beta(t)-\mu)$ does not contained neither $\mathbb{R}_{>0}$ nor $\mathbb{R}_{<0}$. We refer to our assumption about the choice of $\mu$ as the *spectral gap condition.*

We next define some topological invariants for our operators. We consider the following subspaces of the complex plane.

$$\Pi^+:=\{z\in\mathbb{C}\mid \mathrm{Re}(z)>0\},\quad \Pi^-:=\{z\in\mathbb{C}\mid \mathrm{Re}(z)<0\},\quad \Pi:=\Pi^+\sqcup\Pi^-.$$

Let $h\colon\Pi\to\mathbb{C}$ be a continuous function which takes value at 0 on $\Pi^+$ and 1 on $\Pi^-$.

Definition 4.1.3. By using the continuous functional calculous, we have a projection $p:=h(H^\alpha(t)-\mu,H^\beta(t)-\mu)$ in $M_N(\mathcal{S}^{\alpha,\beta}\otimes C(\mathbb{T}))$. We denote by $\mathcal{I}_{\mathrm{Bulk-Edge}}$ the element $[p]_0$ in the $K$-group $K_0(\mathcal{S}^{\alpha,\beta}\otimes C(\mathbb{T}))$, and call the *bulk-edge invariant.*[18]

Remark 4.1.4. Note that, in order to define the bulk-edge invariant, we use just the information of "bulk and edge", and do not use the information of "corner". This is a justification of the name of the "bulk-edge invariant".

We next define the corner invariant for our system. We consider following quarter-plane Toeplitz operators,

$$H^{\alpha,\beta}(t):=P_V^\alpha P_V^\beta H(t)P_V^\alpha P_V^\beta\colon \mathcal{H}_V^{\alpha,\beta}\to\mathcal{H}_V^{\alpha,\beta}.$$

We call $H^{\alpha,\beta}(t)$ *corner Hamiltonians.*

[18]In order to define the element $[p]_0$ of the $K$-group $K_0(\mathcal{S}^{\alpha,\beta}\otimes C(\mathbb{T}))$, we took the orthonormal frame of $V$. However, this element does not depend on the choice.

Definition 4.1.5. By Theorem 2.4.2, we have a continuous family $\{H^{\alpha,\beta}(t) - \mu\}_{t\in\mathbb{T}}$ of bounded self-adjoint Fredholm operators. By our spectral gap condition, this family defines the element $\mathcal{I}_{\mathrm{Corner}}$ of the $K$-group $K_1(C(\mathbb{T})) \cong [\mathbb{T}, \mathrm{Fred}(H)_*^{\mathrm{s.a.}}]$. We call this element $\mathcal{I}_{\mathrm{Corner}}$ the *corner invariant*.

## 4.2 Correspondence

The following is the main theorem of this part.

**Theorem 4.2.1.** *The map $\delta_0\colon K_0(\mathcal{S}^{\alpha,\beta}\otimes C(\mathbb{T})) \to K_1(C(\mathbb{T}))$ maps the bulk-edge invariant to the corner invariant, that is,*

$$\delta_0(\mathcal{I}_{\mathrm{Bulk-Edge}}) = \mathcal{I}_{\mathrm{Corner}}.$$

*Proof.* Let $q := (1-h)(H^\alpha(t) - \mu, H^\beta(t) - \mu)$. Since $\delta_0([p]_0 + [q]_0) = \delta_0[1_N]_0 = 0$, we have $\delta_0(\mathcal{I}_{\mathrm{Bulk-Edge}}) = -\delta_0[q]_0 = [\exp(2\pi i\hat{q})]_1$, where $\hat{q} := \rho(q)$ is a self-adjoint lift of $q$. By our spectral gap condition and Theorem 2.4.1, we have ess-sp$(\hat{q}(t)) = \mathrm{sp}(q(t)) = \{0, 1\}$. By considering a spectral deformation which collapses eigenvalues in some small neighborhoods of 0 and 1 to points 0 and 1, respectively, we can deform $\hat{q}(t)$ into an element $\tilde{q}(t)$ of $\hat{F}_*^\infty$. Thus we obtain $[\exp(2\pi i\hat{q})]_1 = [\exp(2\pi i\tilde{q})]_1$. Let us consider the isomorphisms $K_1(C(\mathbb{T})) = [\mathbb{T}, \mathbf{U}(\infty)] \cong [\mathbb{T}, \hat{F}_*^\infty] \cong [\mathbb{T}, \mathrm{Fred}(H)_*^{\mathrm{s.a.}}]$. Then we have the following correspondence of elements.

$$\delta_0(\mathcal{I}_{\mathrm{Bulk-Edge}}) = [\exp(2\pi i\tilde{q})]_1 = [2\tilde{q} - 1] = [2\tilde{q} - 1].$$

Since two loops $\{2\tilde{q}(t) - 1\}_{t\in\mathbb{T}}$ and $\{H^{\alpha,\beta}(t) - \mu\}_{t\in\mathbb{T}}$ are homotopic in $\mathrm{Fred}(H)_*^{\mathrm{s.a.}}$, we have $[2\tilde{q} - 1] = [\{H^{\alpha,\beta}(t) - \mu\}_{t\in\mathbb{T}}] = \mathcal{I}_{\mathrm{Corner}}$. □

Remark 4.2.2. If we consider the diagram (2), we see that our invariants maps to the spectral flow of $\{H^{\alpha,\beta}(t) - \mu\}_{t\in\mathbb{T}}$ by the map $\mathrm{sf}\colon K_1(C(\mathbb{T})) \to \mathbb{Z}$. The spectral flow of $\{H^{\alpha,\beta}(t) - \mu\}_{t\in\mathbb{T}}$ is defined by counting wave functions localized near the corner. Thus, if the corner invariant is non-trivial, there exists topologically protected *corner states*. Note that the bulk-edge invariant cannot change unless the spectral gap of edges closes. By Theorem 4.2.1, this stability also holds for corner invariants.

Remark 4.2.3. Since the map $\delta_0\colon K_0(\mathcal{S}^{\alpha,\beta} \otimes C(\mathbb{T})) \to K_1(C(\mathbb{T}))$ is not an isomorphism (Remark 2.4.4), bulk-edge invariants may have more information than corner invariants.

Remark 4.2.4. If $\mu$ satisfies that $(H^\alpha(t) - \mu, H^\beta(t) - \mu)$ is contained in $\mathbb{R}_{>0}$ or $\mathbb{R}_{<0}$, then we can still define the bulk-edge invariant in the same way. The $\mathbb{Z}$-valued spectral flow of the family $\{H^{\alpha,\beta}(t) - \mu\}_{t\in\mathbb{T}}$ can also be defined.[19] In this case, it is easy to see that $\delta_0(\mathcal{I}_{\mathrm{Bulk-Edge}}) = 0$ (see the proof of Theorem 4.2.1) and $\mathrm{sf}(\{H^{\alpha,\beta}(t) - \mu\}_{t\in\mathbb{T}}) = 0$. Thus we still have some kind of "bulk-edge and corner" correspondence for such $\mu$.

Remark 4.2.5. Since $\mathbb{T}$ is just a parameter space in our formulation, we can generalize the parameter space $\mathbb{T}$ to other spaces. Let $X$ be a compact Hausdorff space, and consider a continuous family of bounded self-adjoint multiplication operators generated by continuous functions $H\colon X \to B(L^2(\mathbb{T}\times\mathbb{T}; V))$. Then we have edge Hamiltonians and

[19] In this case, the family $\{H^{\alpha,\beta}(t) - \mu\}_{t\in\mathbb{T}}$ is not contained in $\mathrm{Fred}(H)_*^{\mathrm{s.a.}}$, and does not define an element of the $K$-group $K_1(C(\mathbb{T}))$.

the corner Hamiltonian. If we assume our spectral gap condition, we can define the bulk-edge invariant and the corner invariant in the same way as elements of $K_0(\mathcal{S}^{\alpha,\beta} \otimes C(X))$ and $K_1(C(X))$, respectively. As in Theorem 4.2.1, it is easily checked that the bulk-edge invariant maps to the corner invariant by the exponential map $\delta_0$. In this case we lack the understanding of the corner invariant as the spectral flow, but we still have a relation with our invariants and corner states. It is easily checked that if the corner invariant is non-trivial, then there are topologically protected corner states. In particular, if we take $X = \mathbb{T} \times \mathbb{T}$, then this argument gives a "bulk-edge and corner" correspondence for such four dimensional systems.

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