博士論文

論文題目　Representation theory of Drinfeld doubles

（ドリンフェルトダブルの表現論）

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REPRESENTATION THEORY OF DRINFELD DOUBLES

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ABSTRACT. We determine a substantial part of the unitary representation theory of the Drinfeld double of a q-deformation of a compact Lie group in terms of the complexification of the compact Lie group. Using this, we show that the dual of every q-deformation of a higher rank compact Lie group has central property (T). We also determine the unitary dual of $SL_q(n, \mathbb{C})$.

1. INTRODUCTION

As has been observed by many authors (see, e.g., [14], [16]), the Drinfeld double of the q-deformation of a compact Lie group can be regarded as a quantization of the complexification of the original Lie group. Using this point of view, in this paper we study irreducible unitary representations of these Drinfeld doubles.

In [12], Joseph and Letzter defined a notion of quantum Harish-Chandra module, which also can be seen as a certain representation of a q-deformation of a complex semisimple Lie groups. In this paper, we compare these two notions and show that the quantum Harish-Chandra modules are nothing but the admissible representations of quantum doubles, which already has been implicitly prospected in [22]. Then we use deep analysis on quantum Harish-Chandra modules to compare the representation theory of the quantum doubles and that of the classical case. Our main theorem is as follows.

Main Theorem. *Let K be a connected simply connected compact Lie group and fix $0 < q < 1$. Consider the q-deformation K_q. Let $Q^\vee$ (resp. P) be the coroot lattice (resp. the weight lattice) and W the Weyl group.*

(1) The K-finite part of a unitary irreducible representation of the quantum double G_q of K_q is admissible.

(2) The irreducible admissible representations of G_q are parametrized by $(P \times X)/W$ where $X = \mathfrak{h}^*/2\pi i \log(q)^{-1} Q^\vee$ and W acts on $P \times X$ by the diagonal action.

(3) For $(\lambda, \nu) \in P \times \mathfrak{h}^*$ such that $\mathrm{Im}(\nu)$ is small enough, the corresponding irreducible admissible representation of G_q is unitary if and only if the corresponding irreducible representation of the complexification G of K is unitary.

This result allows us to:

- classify a substantial amount of unitary representations of such doubles in terms of those of complex semisimple Lie groups,
- prove central property (T) for the duals of general q-deformations of compact simple Lie groups with rank equal or larger than 2 and
- classify all unitary representations of the quantum doubles of $SU_q(n)$.

This work is motivated by the theory of operator algebras as follows. The study of central multipliers on compact quantum groups has been started by De Commer, Freslon and Yamashita [5]. It was already implicitly appeared in the study of approximation properties by Brannan [4] and Freslon [8]. In [5], it is also shown that the central multipliers are the same as the multipliers of quantum doubles, hence have a strong relationship with the unitary representation theory of quantum doubles, which was already studied by Pusz [19] and Voigt [21] in the case of $SU_q(2)$.

There is also a strong relationship with the theory of subfactors. Popa and Vaes [18] introduced a notion of multipliers for tensor categories, which appeared to be the same as the corresponding central multipliers in the quantum group case and also the multipliers for standard invariants in the case of subfactors in [17]. Neshveyev-Yamashita [15] and Ghosh-Jones [9] also introduced equivalent notions in different approaches. Together with central property (T) of quantum groups, this notion eventually lead us to the first example of non group-like subfactors with property (T) standard invariant.

This is an expository of [1] and [2] presented as a Ph.D. thesis of the author. Some results are overlapped with the independent work [22], but we included proofs.

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2. Preliminaries

2.1. **Quantized enveloping algebra.** Let K be a connected simply connected compact Lie group. Take its complexification $G = K_{\mathbb{C}}$ with its Iwasawa decomposition $G = KAN$. Take the Lie algebra $\mathfrak{g}$ of G and a Cartan algebra $\mathfrak{h}$. Let $(\cdot, \cdot)$ be the natural bilinear form on $\mathfrak{h}$, which is normalized as $(\alpha, \alpha) = 2$ for a short root α. Take the set of roots $\Delta \subset \mathfrak{h}^*$, the (co)root lattice Q ($Q^\vee$) and the (co)weight lattice P ($P^\vee$). For each $\alpha \in \Delta$, let $\alpha^\vee := 2\alpha/(\alpha, \alpha)$ be the coroot. Fix a set of simple roots $\Pi \subset \Delta$ and let Q_+, $Q^\vee_+$, P_+ and $P^\vee_+$ be the positive parts of corresponding lattices. Put $q_\alpha := q^{(\alpha,\alpha)/2}$,

$$n_q := \frac{q^n - q^{-n}}{q - q^{-1}},$$

$$n_q! := n_q(n-1)_q \dots 1_q,$$

$$\binom{n}{m}_q := \frac{n_q!}{m_q!(n-m)_q!}.$$

Definition 2.1. The *quantized enveloping algebra* $U_q(\mathfrak{g})$ is the Hopf $*$-algebra generated by $\{K_\lambda, E_\alpha, F_\alpha \mid \lambda \in P, \alpha \in \Pi\}$ with the relations

$$K_0 = 1, \quad K_\lambda K_\mu = K_{\lambda+\mu},$$
$$K_\lambda E_\alpha K_{-\lambda} = q^{(\alpha,\lambda)} E_\alpha, \quad K_\lambda F_\alpha K_{-\lambda} = q^{-(\alpha,\lambda)} F_\alpha,$$
$$[E_\alpha, F_\beta] = \delta_{\alpha,\beta} \frac{K_\alpha - K_{-\alpha}}{q_\alpha - q_\alpha^{-1}},$$
$$\sum_{r=0}^{1-(\alpha,\beta^\vee)} (-1)^r \binom{1-(\alpha,\beta^\vee)}{r}_{q_\beta} E_\beta^r E_\alpha E_\beta^{1-(\alpha,\beta^\vee)-r} = 0,$$
$$\sum_{r=0}^{1-(\alpha,\beta^\vee)} (-1)^r \binom{1-(\alpha,\beta^\vee)}{r}_{q_\beta} F_\beta^r F_\alpha F_\beta^{1-(\alpha,\beta^\vee)-r} = 0,$$
$$K_\lambda^* = K_\lambda, \quad E_\alpha^* = F_\alpha K_\alpha, \quad F_\alpha^* = K_{-\alpha} E_\alpha,$$
$$\widehat{\Delta}(K_\lambda) = K_\lambda \otimes K_\lambda, \quad \widehat{\varepsilon}(K_\lambda) = 1, \quad \widehat{S}(K_\lambda) = K_{-\lambda},$$
$$\widehat{\Delta}(E_\alpha) = E_\alpha \otimes 1 + K_\alpha \otimes E_\alpha, \quad \widehat{\varepsilon}(E_\alpha) = 0, \quad \widehat{S}(E_\alpha) = -K_{-\alpha} E_\alpha,$$
$$\widehat{\Delta}(F_\alpha) = F_\alpha \otimes K_{-\alpha} + 1 \otimes F_\alpha, \quad \widehat{\varepsilon}(F_\alpha) = 0, \quad \widehat{S}(F_\alpha) = -F_\alpha K_\alpha.$$

Let $U_q(\mathfrak{h})$ (resp. $U_q(\mathfrak{n}^+)$, $U_q(\mathfrak{n}^-)$) be the subalgebra generated by K_λ (resp. E_α, F_α).

For each $\lambda \in \mathfrak{h}^*$, let $M_q(\lambda)$ be the Verma module of highest weight λ and $V_q(\lambda)$ its unique irreducible quotient. If $\lambda \in P_+$, then $V_q(\lambda)$ is finite dimensional. We say that a $U_q(\mathfrak{g})$-module is of *type 1* if it decomposes into a direct sum of $V_q(\lambda)$'s for $\lambda \in P_+$. Notice that any subquotient of a module of type 1 is also of type 1.

Consider the adjoint action of $U_q(\mathfrak{g})$ on itself

$$\mathrm{ad}(x)(y) := x_{(1)} y \widehat{S}(x_{(2)}).$$

Here we used the sumless Sweedler notation:

$$\widehat{\Delta}(x) = x_{(1)} \otimes x_{(2)}.$$

We denote the type 1 part of $U_q(\mathfrak{g})$ with respect to the adjoint action. In [11, Theorem 7.1.6], it is shown that this is a left coideal algebra.

Since we need to deal with all (possibly) non-real weights in $\mathfrak{h}^*$, we use some terminologies, which are used only in this article.

Definition 2.2. We say that $\nu \in \mathfrak{h}^*$ is *dominant* (with respect to q) if $(\nu, \alpha^\vee) \notin \mathbb{Z}_{<0} + 2\pi i \log(q_\alpha)^{-1}\mathbb{Z}$.

We say that $\nu \in \mathfrak{h}^*_\mathbb{R}$ is *small* if $(\nu, \alpha) < 1$ for any $\alpha \in \Delta$.

We say that $\nu \in \mathfrak{h}^*$ is *almost real* (with respect to q) if $\frac{\log(q)}{2\pi}\mathrm{Im}(\nu)$ is small.

For each $\lambda \in \mathfrak{h}^*_\mathbb{R}$, let $|\lambda|$ be the unique dominant element in the Weyl group orbit of λ.

Note that the set of small (resp. almost real) weights is open and invariant under the Weyl group action. For the later use, we state several lemmas. The following argument was suggested by Hironori Oya.

Lemma 2.3. *Let $\mu, \nu \in \mathfrak{h}^*_\mathbb{R}$ be small. Then $\mu - \nu \in Q^\vee$ implies $\mu = \nu$.*

Proof. Since $(\mu - \nu, \alpha) < 2$ for any $\alpha \in \Delta$, it suffices to show the following:
For any $x \in Q^\vee$, $(x, \alpha) < 2$ for all $\alpha \in \Delta$ implies $x = 0$.

To show this, after conjugating by the Weyl group action if necessary, we may assume $x \in Q^\vee_+$ and has a minimal height among $Wx \cap Q^\vee_+$. Then, since $x \in Q^\vee_+$, there exists $\alpha \in \Delta_+$ such that $(x, \alpha) > 0$. Since $(x, \alpha) < 2$, we get $(x, \alpha) = 1$. This asserts $s_\alpha(x) = x - \alpha^\vee$.

Now, since we have assumed that x has a minimal height, we get $x - \alpha^\vee \notin Q^\vee_+$, which means

$$x = \sum_{\beta \in \Pi, \beta \neq \alpha} n_\beta \beta^\vee.$$

In particular, $(x, \alpha) < 0$, which is a contradiction. □

All the extraordinariness of type A case in this paper comes from the following easy lemma.

Lemma 2.4. *Let $K = SU(n)$. Then for any $\nu \in \mathfrak{h}^*_\mathbb{R}$, there exists $\lambda \in P^\vee$ such that $\nu - \lambda$ is small.*

Proof. We identify $\mathfrak{h}^*_\mathbb{R} \simeq \mathbb{R}^n/\mathbb{R}(1, 1, \dots, 1)$ with the weight lattice $\mathbb{Z}^n/\mathbb{Z}(1, 1, \dots, 1)$. Write $\nu = (\nu_1, \nu_2, \dots, \nu_n) \in \mathfrak{h}^*_\mathbb{R}$. Let λ_i be the integer such that $0 \le \nu_i - \lambda_i < 1$. Then $\lambda = (\lambda_i)$ is the desired element in $P = P^\vee$. □

For the later use, we consider the subalgebra J of $U_q(\mathfrak{g})$ generated by $K_{2\lambda}, E_\alpha, F_\alpha K_\alpha$. Remark that J is the localization of the adjoint finite part $F(U_q(\mathfrak{g}))$ with respect to the Ore set $\{K_{-2\lambda} \mid \lambda \in P_+\}$. Consider the category $\mathcal{O}$ over J. This is essentially the same as the usual category $\mathcal{O}$ [11, 4.1.4], but the weight only makes sense as an element in $\mathfrak{h}^*/\pi i \log(q)^{-1} Q^\vee = \frac{1}{2}X$. For each $\Lambda \in \mathfrak{h}^*$, we put

$$\mathcal{O}^\Lambda := \{M \in \mathcal{O} \mid \mathrm{wt}(M) \subset \Lambda + P\}.$$

The following lemma may be well-known to experts, but we could not find any references.

Lemma 2.5. *For $0 < q < 1$ and any $\lambda \in \mathfrak{h}^*$ such that 2λ is almost real, we have* $\mathrm{ch}\, V_q(\lambda) = \mathrm{ch}\, V_1(\lambda)$.

Proof. Thanks to [7, Corollary 4.8], we have this equality for generic q.

In general, consider $A = \mathbb{Q}[q^{\pm 1}, q^{(\lambda, \alpha)} \mid \alpha \in Q]$, which is isomorphic to the Laurent polynomial algebra of several variables over $\mathbb{Q}$. Let us recall that we have an invariant form S_q called the Shapovalev form on $M_q(\lambda)$, which is defined over A and

$$V_q(\lambda) = M_q(\lambda)/\mathrm{Ann}(S_q).$$

Then, [11, Theorem 4.1.16] shows that the order of zeros of the determinant of $S_q^\mu = S_q|_{M_q(\lambda)_\mu}$ is constant along $[q, 1]$, hence we get

$$\dim V_q(\lambda)_\mu = \dim V_1(\lambda)_\mu.$$

□

For $\lambda \in P_+$, we regard $V_q(\lambda)$ as a family of representation of $U_q(\mathfrak{g})$ on a single vector space $V(\lambda)$ such that weight spaces are the same and each $U_{q^\alpha}(\mathfrak{sl}_{2,\alpha})$-isotypical components varies continuously with respect to q. (This is possible, for example, via the global base.)

2.2. **Quantum coordinate algebra.** Let A, B be Hopf algebras. A *skew pairing* between A and B is a map

$$A \times B \to \mathbb{C}$$

such that

$$\begin{aligned}(ab, c) &= (a \otimes b, \Delta_B(c)),\\ (a, cd) &= (\Delta_A(a), d \otimes c),\\ (1, c) &= \varepsilon_A(c),\\ (a, 1) &= \varepsilon_B(a),\end{aligned}$$

for $a, b \in A$, $c, d \in B$.

If A, B are Hopf $*$-algebras, we also assume

$$(a^*, b) = \overline{(a, S(b)^*)}.$$

For a pair of Hopf algebras with a skew pairing, one defines the following actions: for $a \in A$ and $b \in B$

$$\begin{aligned}a \triangleright b &:= (a, b_{(2)})b_{(1)}, & b \triangleleft a &:= (a, b_{(1)})b_{(2)},\\ b \triangleright a &:= (a_{(1)}, b)a_{(2)}, & a \triangleleft b &:= (a_{(2)}, b)a_{(1)}.\end{aligned}$$

Definition 2.6. The *quantum coordinate algebra* $\mathcal{O}(K_q) \subset U_q(\mathfrak{g})^*$ is the subspace of matrix coefficients of type 1 representations. Then $\mathcal{O}(K_q)$ carries a unique Hopf $*$-algebra structure which makes the pairing

$$U_q(\mathfrak{g}) \times \mathcal{O}(K_q) \to \mathbb{C}$$

skew.

Let $\mathcal{O}(T) := \mathcal{O}(K_q)/\operatorname{Ann}(U_q(\mathfrak{h}))$. Then $\mathcal{O}(T)$ can be identified with the algebra of regular functions on the maximal torus T of K. Denote the canonical surjection $\mathcal{O}(K_q) \to \mathcal{O}(T)$ by π_T.

Definition 2.7. Let $\mathcal{U}(K_q) := \prod_{\lambda \in P_+} \operatorname{End}(V_q(\lambda))$ be the full dual of $\mathcal{O}(K_q)$ and $c_c(\widehat{K}_q) := \bigoplus_{\lambda \in P_+} \operatorname{End}(V_q(\lambda)) \subset \mathcal{U}(K_q)$. Then one can embed $U_q(\mathfrak{g})$ into $\mathcal{U}(K_q)$ and $c_c(\widehat{K}_q)$ is an ideal of $\mathcal{U}(K_q)$.

One can easily show that there is a one-to-one correspondence among

(1) type 1 representations of $U_q(\mathfrak{g})$,
(2) nondegenerate representations of $c_c(\widehat{K}_q)$ and
(3) continuous representations of $\mathcal{U}(\widehat{K}_q)$.

Remark 2.8. For any $\nu \in \mathfrak{h}^*$, the symbol K_ν makes sense as an element in $\mathcal{U}(K_q)$ by the formula

$$K_\nu v = q^{(\nu, \mathrm{wt}(v))} v$$

for each weight vector $v \in V_q(\lambda)$. Then we again have

$$K_\nu K_\mu = K_{\nu+\mu}$$

for any $\nu, \mu \in \mathfrak{h}^*$. Moreover $K_{2\pi i \log(q)^{-1}\mu} = 1$ for any $\mu \in Q^\vee$ shows K_ν actually makes sense for any $\nu \in X := \mathfrak{h}^*/2\pi i \log(q)^{-1} Q^\vee$. Then elements in X are in a one-to-one correspondence with 1-dimensional representations on $\mathcal{O}(T)$. The character K_ν is a $*$-character if and only if $\nu \in i\mathfrak{h}^*_{\mathbb{R}}$. Notice that the Weyl group W acts on X in a natural way.

Let us define the following central projections on $\mathcal{O}(K_q)$

$$p^\lambda := 1_\lambda \in \operatorname{End}(V_q(\lambda)) \subset c_c(K_q)$$

and let $\varphi := p^0$ be the Haar state. For a type 1 $U_q(\mathfrak{g})$-module V, the element p^λ is nothing but the projection onto V^λ.

We have

$$\varphi\omega = \omega\varphi = \omega(1)\varphi$$

for any $\omega \in \mathcal{U}(\widehat{K}_q)$.

(The universal C*-completion of) $\mathcal{O}(K_q)$ is a compact quantum group in the sense of [23]. In our notation, the modular automorphism of $\mathcal{O}(K_q)$ is given by

$$\sigma_t(x) = K_{-2it\rho} \triangleright x \triangleleft K_{-2it\rho},$$

where ρ is the half sum of positive roots: $\rho = \dfrac{1}{2}\sum_{\alpha\in\Delta_+}\alpha$. We also have

$$S^2(x) = K_{-2\rho} \triangleright x \triangleleft K_{2\rho}.$$

In particular,

$$\sigma_i(x) = K_{2\rho} \triangleright x \triangleleft K_{2\rho} = S^2(K_{4\rho} \triangleright x)$$

and hence

$$\varphi(yx) = \varphi(S^2(K_{4\rho} \triangleright x)y),$$

which can be rewritten as

$$x \triangleright \varphi = \varphi \triangleleft (S^2(K_{4\rho} \triangleright x)).$$

3. Drinfeld doubles

Definition 3.1. For Hopf algebras A and B with a skew pairing, the *Drinfeld double* $A \bowtie B$ is the algebra generated by A and B with the commutation relation

$$ab = (a_{(1)} \triangleright b \triangleleft S(a_{(3)}))a_{(2)}$$

for $a \in A$ and $b \in B$. As a vector space, the multiplication map gives an isomorphism $A \otimes B \to A \bowtie B$.

If both A and B are Hopf $*$-algebras, $A \bowtie B$ is again a Hopf $*$-algebra.

Remark 3.2. It is not necessary for B to be a "genuine" Hopf algebra to define the Drinfeld double $A \bowtie B$ as an algebra, as long as the bimodule action of A on B makes sense. For example, one can define $D_c := c_c(\widehat{K}_q) \bowtie \mathcal{O}(K_q)$ and $\tilde{D} := \mathcal{U}(K_q) \bowtie \mathcal{O}(K_q)$ in the same manner.

Let $D = \mathcal{O}(\widehat{G}_q) := U_q(\mathfrak{g}) \bowtie \mathcal{O}(K_q)$. We are interested in the representation theory of D. We start with describing the algebra structure of D.

Let $U_q(\mathfrak{b}^+)$ (resp. $U_q(\mathfrak{b}^-)$) be the subalgebra of $U_q(\mathfrak{g})$ generated by K_λ and E_α (resp. K_λ and F_α). Take a universal R-matrix

$$R := q^{\sum_{\alpha,\beta\in\Pi}(B^{-1})_{\alpha,\beta}H_\alpha\otimes H_\beta} \prod_{\alpha\in\Delta_+} \exp_{q_\alpha}((1-q_\alpha^{-2})F_\alpha \otimes E_\alpha),$$

where B is the matrix $((\alpha^\vee,\beta^\vee))_{\alpha,\beta}$, H_α is the self-adjoint element which satisfies $q_\alpha^{H_\alpha}=K_\alpha$, E_α, F_α are the PBW basis corresponding to α and

$$\exp_q(x) := \sum_{n=0}^{\infty} q^{n(n+1)/2}\frac{x^n}{n_q!}.$$

Although R is an element in $\mathcal{U}(K_q\times K_q)$, the elements $(x\otimes \mathrm{id})(R)$ and $(\mathrm{id}\otimes x)(R^{-1})$ make sense as elements in $U_q(\mathfrak{g})$ for every $x\in\mathcal{O}(K_q)$.

Define Hopf algebras

$$\mathcal{O}(B_q^{\pm}) := \mathcal{O}(K_q)/\mathrm{Ann}(U_q(\mathfrak{b}^{\mp})).$$

Then the maps

$$l_+ : \mathcal{O}(B_q^+) \to U_q(\mathfrak{b}^+) : x\mapsto (x\otimes\mathrm{id})(R),$$
$$l_- : \mathcal{O}(B_q^-) \to U_q(\mathfrak{b}^-) : x\mapsto (\mathrm{id}\otimes x)(R^{-1})$$

are isomorphisms of Hopf algebras. The map

$$I(x) := l_-(x_{(1)})\widehat{S}^{-1}(l_+(x_{(2)})) = (\mathrm{id}\otimes x)(R_{12}^{-1}R_{21}^{-1})$$

is a $U_q(\mathfrak{g})$-module isomorphism from $\mathcal{O}(K_q)$ onto $F(U_q(\mathfrak{g}))$ [3, Theorem 3]. We put

$$\Psi(x) = (l_-\otimes l_+)\Delta(x) = (\mathrm{id}\otimes x\otimes\mathrm{id})(R_{23}R_{12}^{-1}).$$

The following result is first observed by Krähmer [14].

Theorem 3.3. *The map*

$$\widehat{\Delta}\times\Psi : U_q(\mathfrak{g})\bowtie\mathcal{O}(K_q)\to U_q(\mathfrak{g})\otimes U_q(\mathfrak{g})$$

is an injective algebra homomorphism.

We further describe the image of this map. Put

$$D' := (F(U_q(\mathfrak{g}))\otimes\widehat{S}(F(U_q(\mathfrak{g}))))\widehat{\Delta}(U_q(\mathfrak{g})).$$

Proposition 3.4. *We have*

$$(\widehat{\Delta}\times\Psi)(D) = D'.$$

Proof. First, notice that

$$D' = (F(U_q(\mathfrak{g}))\otimes 1)\widehat{\Delta}(U_q(\mathfrak{g}))$$

since $F(U_q(\mathfrak{g}))$ is a left coideal. We also remark that

$$\beta : U_q(\mathfrak{g})\otimes U_q(\mathfrak{g})\to U_q(\mathfrak{g})\otimes U_q(\mathfrak{g}) : x\otimes y\mapsto (x\otimes 1)\widehat{\Delta}(y)$$

is an isomorphism of vector spaces with inverse map

$$\beta^{-1}(x\otimes y) = x\widehat{S}(y_{(1)})\otimes y_{(2)}.$$

Combining these, it suffices to show that

$$\beta^{-1}(\widehat{\Delta}\times\Psi)(D) = F(U_q(\mathfrak{g}))\otimes U_q(\mathfrak{g}).$$

Now using

$$m(\mathrm{id}\otimes\widehat{S})\Psi(x) = I(x),$$

we get

$$\begin{aligned}\beta^{-1}(\widehat{\Delta}\times\Psi)(ax) &= \beta^{-1}(l_+(a_{(1)})\otimes l_-(a_{(2)}))(1\otimes x)\\ &= l_+(a_{(1)})\widehat{S}(l_-(a_{(2)}))\otimes l_-(a_{(3)})x\\ &= I(a_{(1)})\otimes l_-(a_{(2)})x.\end{aligned}$$

for $a\in\mathcal{O}(K_q)$ and $x\in U_q(\mathfrak{g})$. Hence we get the conclusion. □

4. Admissible representations

Definition 4.1. A *unitary representation* of G_q is a nondegenerate $*$-representation of D_c on a Hilbert space. The *K-finite* part of a unitary representation $\pi: D_c\to B(H)$ is the representation of D_c (or D) restricted to $\bigoplus_{\lambda\in P_+}\pi(p^\lambda)H$.

An *admissible representation* of G_q is a nondegenerate representation of D_c such that the multiplicity of each irreducible representation of $c_c(\widehat{K}_q)$ is finite. This is the same as a representation of D which is of type 1 as a representation of $U_q(\mathfrak{g})$ and whose multiplicity of each irreducible representation of $U_q(\mathfrak{g})$ is finite. The multiplicity as a $U_q(\mathfrak{g})$-representation is called the *K-type multiplicity.*

We start this section with showing the K-finite part of an irreducible unitary representation is admissible.

The following lemma has already appeared in the proof of [13, Theorem 8.1].

Lemma 4.2. *Let A be a $*$-algebra and $N\in\mathbb{Z}_{\geq 0}$. Suppose A is a subalgebra (with the $*$-structure ignored) of $\prod_{i\in I}\mathrm{End}(V_i)$, where $(V_i)_{i\in I}$ is a family of vector spaces with dimensions at most N. Then the dimension of any irreducible $*$-representation of A is at most N.*

Theorem 4.3. *Let π be an irreducible $*$-representation of D_c on a Hilbert space H. Then the multiplicity of $V_q(\lambda)$ in $\pi|_{c_c(\widehat{K}_q)}$ is at most $\dim V_q(\lambda)$. In particular, the K-finite part is an irreducible admissible D-module.*

Proof. For each $\mu=(\mu_1,\mu_2)\in P_+\times P_+$, one can define a finite dimensional representation π^μ of D by

$$\pi^\mu=(\pi^{\mu_1}\otimes\pi^{\mu_2})(\widehat{\Delta}\times\Psi).$$

Then since $\widehat{\Delta}\times\Psi$ is injective, we get an embedding

$$\bigoplus_{\mu\in P_+\times P_+}\pi^\mu : D\hookrightarrow \prod_{\mu\in P_+\times P_+}\mathrm{End}(V_q(\mu_1)\otimes V_q(\mu_2)).$$

Fix $\lambda\in P_+$. By cutting the embedding above by p^λ, we get an embedding

$$p^\lambda D_c p^\lambda\hookrightarrow\prod_{\mu\in P_+\times P_+}\mathrm{End}(\pi^\mu(p^\lambda)(V_q(\mu_1)\otimes V_q(\mu_2))).$$

Let $v_{w_0\mu_2}$ be the lowest weight vector in $V_q(\mu_2)$. Since $v_{\mu_1}\otimes v_{w_0\mu_2}$ is cyclic for the diagonal action of $U_q(\mathfrak{g})$ on $V_q(\mu_1)\otimes V_q(\mu_2)$, the map

$$\mathrm{Hom}_{U_q(\mathfrak{g})}(V_q(\mu_1)\otimes V_q(\mu_2),V_q(\lambda))\to V_q(\lambda): f\mapsto f(v_{\mu_1}\otimes v_{w_0\mu_2})$$

is injective. Hence we get

$$[V_q(\mu_1) \otimes V_q(\mu_2) : V_q(\lambda)] \leq \dim V_q(\lambda).$$

Therefore $\dim \pi^{\mu}(p^{\lambda})(V_q(\mu_1) \otimes V_q(\mu_2)) \leq (\dim V_q(\lambda))^2$. Now we can apply Lemma 4.2 to get the desired conclusion. □

Let $\mathrm{Adm}(G_q)$ be the category of admissible representations. By Proposition 3.4, we see that the quantum Harish-Chandra module by Joseph and Letzter [12] is nothing but the admissible representation of D. Namely for an admissible representation of D',

$$xv := (x \otimes 1)v, vy := (1 \otimes \widehat{S}(y))v, \mathrm{ad}(a)v := \widehat{\Delta}(a)v$$

is a quantum Harish-Chandra module. Hence we can apply the categorical equivalence between the quantum Harish-Chandra modules and the category $\mathcal{O}$.

Let κ be the involutive antiautomorphism on $U_q(\mathfrak{g})$ defined by

$$\kappa(E_\alpha) = K_{-\alpha}F_\alpha, \kappa(F_\alpha) = E_\alpha K_\alpha, \kappa(K_\lambda) = K_\lambda.$$

For $\Lambda \in \mathfrak{h}^*$ and $V \in \mathcal{O}^\Lambda$, we define a $U_q(\mathfrak{g}) \otimes U_q(\mathfrak{g})$-module structure on $(M_q(\Lambda) \otimes V)^*$ by

$$(v, xf) := ((\kappa \otimes \widehat{S}^{-1})(x)v, f)$$

for $x \in U_q(\mathfrak{g}) \otimes U_q(\mathfrak{g})$, $v \in M_q(\Lambda) \otimes V$ and $f \in (M_q(\Lambda) \otimes V)^*$. Let $\Psi_\Lambda(V)$ be the finite part with respect to the action of $\Delta(U_q(\mathfrak{g}))$. Then $\Psi_\Lambda(V)$ is a D'-module in a natural fashion. Via the isomorphism $(\widehat{\Delta} \times \Psi)^{-1}$, we regard $\Psi_\Lambda(V)$ as a D-module.

Notice that the center of D is isomorphic to $Z(U_q(\mathfrak{g})) \otimes Z(U_q(\mathfrak{g}))$ via $\widehat{\Delta} \times \Psi$. Via the Harish-Chandra isomorphism $\psi : Z(U_q(\mathfrak{g})) \to \mathrm{span}\{K_{2\lambda} \mid \lambda \in P\}^W$, we know that 1-dimensional representations of the center are parametrized by $Y \times Y$, where Y is quotient of $\frac{1}{2}X$ by the dot-action of Weyl group

$$w.\Lambda = w(\Lambda + \rho) - \rho$$

as follows:

For $\Lambda \in \mathfrak{h}^*$, define the linear functional χ_Λ on $U_q(\mathfrak{g})$ by

$$\chi_\Lambda(aK_\lambda b) = \widehat{\varepsilon}(a)q^{(\lambda,\Lambda)}\widehat{\varepsilon}(b)$$

for $a \in U_q(\mathfrak{n}^-)$, $b \in U_q(\mathfrak{n}^+)$ and $\lambda \in P$. For $\Lambda, \Lambda' \in \mathfrak{h}^*$,

$$Z(D) \to \mathbb{C} \colon x \mapsto \chi_{(\Lambda,\Lambda')}(x) = (\chi_\Lambda \otimes \chi'_\Lambda)((\widehat{\Delta} \otimes \Psi)(x))$$

is a 1-dimensional representation, which depends only on the equivalence class in $(\Lambda, \Lambda') \in Y \times Y$.

This gives a decomposition

$$\mathrm{Adm}(G_q) = \bigoplus_{(\Lambda,\Lambda') \in Y \times Y} \mathrm{Adm}(G_q)_{(\Lambda,\Lambda')},$$

where $\mathrm{Adm}(G_q)_{(\Lambda,\Lambda')}$ is the subcategory of admissible modules such that the center acts as a scalar $\chi_{(\Lambda,\Lambda')}$. For $V \in \mathrm{Adm}(G_q)_{(\Lambda,\Lambda')}$, we say that V has the *central character* (Λ, Λ') (resp. *left central character* Λ, *right central character* Λ'). Let $\mathrm{Adm}(G_q)_\Lambda$ be the subcategory of $\mathrm{Adm}(G_q)$ with the left central character Λ. The following theorem is a direct translation of [11, Section 8.4] in our setting.

Theorem 4.4. *For every dominant weight $\Lambda \in \mathfrak{h}^*$, we have the following.*

(i) We have a contravariant exact functor

$$\Psi_\Lambda : \mathcal{O}^\Lambda \to \mathrm{Adm}(G_q)_\Lambda : V \mapsto F(M_q(\Lambda) \otimes V)^*.$$

(ii) There exists a contravariant functor $T_\Lambda : \mathrm{Adm}(G_q)_\Lambda \to \mathcal{O}^\Lambda$ such that $\Psi_\Lambda \circ T_\Lambda = \mathrm{id}$ and

$$\mathrm{Hom}(V, T_\Lambda(X)) \simeq \mathrm{Hom}(X, \Psi_\Lambda(V)).$$

(iii) If Λ is regular, the functor Ψ_Λ is a categorical equivalence.

(iv) For every irreducible $V \in \mathcal{O}^\Lambda$, the admissible module $\Psi_\Lambda(V)$ is either 0 or irreducible. Moreover this exhausts all irreducibles.

5. Parabolic inductions

Fix a subset $\Sigma \subset \Pi$ and let $(\mathfrak{h}^\Sigma)^*$ be the linear span of Σ. Then Σ can be regarded as the set of simple roots of a Lie subalgebra $\mathfrak{g}^\Sigma \subset \mathfrak{g}$. Take a short root α in Σ and set $q^\Sigma := q^{(\alpha,\alpha)/2}$. Let $U_q(\mathfrak{l}^\Sigma)$ be the subalgebra of $U_q(\mathfrak{g})$ generated by $E_\alpha, F_\alpha, K_\lambda$'s where $\alpha \in \Sigma$ and $\lambda \in P$. Then we have a quotient map $\pi^\Sigma : \mathcal{O}(K_q) \to \mathcal{O}(L_q^\Sigma) = \mathcal{O}(K_q)/\mathrm{Ann}(U_q(\mathfrak{l}^\Sigma))$. Let ρ^Σ be the half-sum of positive roots in Δ^Σ and $\rho^{\perp\Sigma} = \rho - \rho^\Sigma$.

Lemma 5.1. *Set $B_\Sigma := U_q(\mathfrak{l}^\Sigma) \bowtie \mathcal{O}(K_q) \subset D$. Take $\nu \in \mathfrak{h}^*$ such that $\nu \perp \alpha$ for any $\alpha \in \Sigma$. Then for each admissible $D^\Sigma := U_q(\mathfrak{l}^\Sigma) \bowtie \mathcal{O}(L_q^\Sigma)$-module V, one can define a B_Σ-module structure on V by*

- *For $x \in \mathcal{O}(K_q)$,*
$$xv := \pi^\Sigma(x \triangleleft K_\nu)v,$$
- *For $a \in U_q(\mathfrak{l}^\Sigma)$,*
$$av := av.$$

Proof. We check each commutation relations. It is easy to show the above formula gives a $U_q(\mathfrak{l}^\Sigma)$-module structure and an $\mathcal{O}(K_q)$-module structure. Therefore, we only need to examine the commutation relation for $x \in \mathcal{O}(K_q)$ and $a \in U_q(\mathfrak{l}^\Sigma)$.

For $a \in U_q(\mathfrak{l}^\Sigma)$, notice that a commutes with K_ν. Hence

$$xav = \pi^\Sigma(x \triangleleft K_\nu)av = a_{(2)}\pi^\Sigma(\widehat{S}^{-1}(a_{(1)}) \triangleright x \triangleleft K_\nu a_{(3)})v = a_{(2)}(\widehat{S}^{-1}(a_{(1)}) \triangleright x \triangleleft a_{(3)})v.$$

□

We denote the B_Σ-module given in the lemma above by $V_{(\nu)}$.

For an admissible D^Σ-module V, define a D-module $\mathrm{Ind}_\Sigma^\Pi V$ by

$$\mathrm{Ind}_\Sigma^\Pi V := D_c \otimes_{B_\Sigma} V_{(-2\rho^{\perp\Sigma})}.$$

Recall that $X = \mathfrak{h}^*/2\pi i \log(q)^{-1}Q^\vee$, which is canonically identified with the space of 1-dimensional representations of $\mathcal{O}(K_q)$. In the case of $\Sigma = \emptyset$, since $D^\emptyset = U_q(\mathfrak{h}) \otimes \mathcal{O}(T)$, the $D^\emptyset$-module structures on $\mathbb{C}$ are parametrized by the element $(\lambda, \nu) \in P \times X$, which we denote by $\mathbb{C}_{(\lambda,\nu)}$ Put

$$L_q(\lambda, \nu) := \mathrm{Ind}_\emptyset^\Pi \mathbb{C}_{(\lambda,\nu)}.$$

Let Λ be the map

$$c_c(\widehat{K}_q) \to L_q(\lambda, \nu) : \omega \mapsto \omega \otimes 1.$$

Then Λ gives a $U_q(\mathfrak{g})$-module isomorphism

$$L_\lambda := \{\omega \in c_c(\widehat{K}_q) \mid \omega K_\mu = q^{(\lambda,\mu)} K_\mu \text{ for any } \lambda \in P\} \to L_q(\lambda, \nu).$$

In particular,

- The module $L_q(\lambda, \nu)$ only depends on λ as a $U_q(\mathfrak{g})$-module .
- The K-type $|\lambda|$ is minimal among all K-types, that is, all other K-type μ satisfies $\mu \in |\lambda| + Q_+$.
- We have $[L_q(\lambda, \nu) : V_q(\mu)] = \dim V_q(\mu)_\lambda$.

Since the multiplicity of the minimal K-type is 1, there exists a unique irreducible subquotient of $L_q(\lambda, \nu)$ whose minimal K-type is $|\lambda|$, which we denote by $V_q(\lambda, \nu)$.

Similarly, one can define a D^Σ-module

$$L_q^\Sigma(\lambda, \nu) := D_c^\Sigma \otimes_{B^\Sigma} \mathbb{C}_{(\lambda, \nu - 2\rho^\Sigma)},$$

where $B^\Sigma = U_q(\mathfrak{h}) \bowtie \mathcal{O}(L_q^\Sigma) \subset D^\Sigma$.

Lemma 5.2. *We have an isomorphism*

$$L_q(\lambda, \nu) \simeq \mathrm{Ind}_\Sigma^\Pi(L_q^\Sigma(\lambda, \nu)).$$

Proof. By definition, we have

$$\mathrm{Ind}_\Sigma^\Pi(L_q^\Sigma(\lambda, \nu)) = D_c \otimes_{B_\Sigma} D_c^\Sigma \otimes_{B^\Sigma} \mathbb{C}_{(\lambda, \nu - 2\rho)}.$$

We claim

$$D_c \otimes_{B_\Sigma} D_c^\Sigma \otimes_{B^\Sigma} \mathbb{C}_{(\lambda, \nu-2\rho)} \to D_c \otimes_B \mathbb{C}_{(\lambda,\nu-2\rho)} : \omega \otimes \mu \otimes 1 \mapsto \omega\mu \otimes 1$$

is an isomorphism.

To construct the inverse, for each $\omega \in c_c(\widehat{K}_q)$, we can find an idempotent $\mu \in c_c(\widehat{L}_q^\Sigma)$ such that $\omega\mu = \omega$. Now one can define a map

$$D_c \otimes_B \mathbb{C}_{(\lambda,\nu-2\rho)} \to D_c \otimes_{B_\Sigma} D_c^\Sigma \otimes_{B^\Sigma} \mathbb{C}_{(\lambda,\nu-2\rho)} : \omega \otimes 1 \mapsto \omega \otimes \mu \otimes 1.$$

Here we notice it does not depend on the choice of μ. In fact, for $\mu_1, \mu_2 \in c_c(\widehat{L}_q^\Sigma)$ with $\omega\mu_i = \omega$, one can find an idempotent μ_0 such that $\mu_i\mu_0 = \mu_i$ for $i = 1, 2$. Then

$$\omega \otimes \mu_1 = \omega \otimes \mu_1\mu_0 = \omega\mu_1 \otimes \mu_0 = \omega \otimes \mu_0 = \omega \otimes \mu_2.$$

Therefore this map is well-defined.

These maps are inverses to each other. □

Next, we construct an invariant sesquilinear pairing on principal series modules. Define a functional $\widehat{\varphi} \in c_c(\widehat{K}_q)$ by

$$\widehat{\varphi}(x) = \sum_{\lambda \in P_+} \mathrm{Tr}_\lambda(K_{2\rho}) \mathrm{Tr}_\lambda(K_{-2\rho} x),$$

where Tr_λ is the non-normalized trace on $V_q(\lambda)$. In [20], it is shown that this is the left invariant weight on $c_c(\widehat{K}_q)$, that is, a positive functional on $c_c(\widehat{K}_q)$ such that

$$\widehat{\varphi}(x \triangleleft a) = \widehat{\varphi}(x)\varepsilon(a).$$

We also have

$$\widehat{\varphi}(a \triangleright x) = \widehat{\varphi}(x) K_{4\rho}(a).$$

Therefore we get

$$\widehat{\varphi}((a \triangleright x)y) = \widehat{\varphi}(a_{(2)} \triangleright (x(S^{-1}(a_{(1)}) \triangleright y))) = K_{4\rho}(a_{(2)})\widehat{\varphi}(x(S^{-1}(a_{(1)}) \triangleright y)),$$
$$\widehat{\varphi}((x \triangleleft a)y) = \widehat{\varphi}(x(y \triangleleft S(a_{(1)})) \triangleleft a_{(2)}) = \widehat{\varphi}(x(y \triangleleft S(a))).$$

Now, $\widehat{\varphi}$ defines a natural inner product on L_λ by

$$(x, y)_q^0 := \widehat{\varphi}(y^* x).$$

This inner product satisfies

$$\begin{aligned}(x, a^* \triangleright y)_q^0 &= \widehat{\varphi}((a^* \triangleright y)^* x) = \widehat{\varphi}((S(a) \triangleright y^*)x)\\ &= K_{-4\rho}(a_{(1)})\widehat{\varphi}(y^*(a_{(2)} \triangleright x)) = K_{-4\rho}(a_{(1)})(a_{(2)} \triangleright x, y)_q^0,\end{aligned}$$

$$(x, y \triangleleft a^*)_q^0 = \widehat{\varphi}((y \triangleleft a^*)^* x) = \widehat{\varphi}((y^* \triangleleft S(a))x) = \widehat{\varphi}(y^*(x \triangleleft S^2(a)) = (x \triangleleft S^2(a), y)_q^0.$$

We regard this inner product as a sesquilinear pairing

$$L_q(\lambda, \nu) \times L_q(\lambda, -\overline{\nu}) \to \mathbb{C}.$$

Proposition 5.3. *This sesquilinear pairing is invariant, i.e.*

$$(ax, y)_q^0 = (x, a^* y)_q^0$$

for any $a \in D$, $x \in L_q(\lambda, \nu)$ and $y \in L_q(\lambda, -\overline{\nu})$.

Proof. Trivial for $a \in U_q(\mathfrak{g})$.

The assertion follows for $a \in \mathcal{O}(K_q)$ also by a calculation:

$$\begin{aligned}(x, a^* y)_q^0 &= \overline{K_{\nu-2\rho}(a^*_{(2)})}(x, a^*_{(3)} \triangleright y \triangleleft S(a^*_{(1)})))_q^0\\ &= K_{-\overline{\nu}+2\rho}(a_{(2)})K_{-4\rho}(a_{(3)})(a_{(4)} \triangleright x \triangleleft S(a_{(1)}), y)_q^0\\ &= K_{-\overline{\nu}-2\rho}(a_{(2)})(a_{(3)} \triangleright x \triangleleft S(a_{(1)}), y)_q^0\\ &= (ax, y)_q^0.\end{aligned}$$

□

We also need to consider the case $q = 1$. For $q = 1$, let $L_1(\lambda, \nu)$ be the Harish-Chandra module of the (nonunitary) principal series of G, that is, the K-finite part of

$$\{f \in C^\infty(G) \mid f(gtan) = t^\lambda a^{\frac{1}{2}\nu - \rho} f(g)\}$$

for $t \in T$, $a \in A$ and $n \in N$. Again $L_1(\lambda, \nu) \simeq L_\lambda$ as K-module. The invariant inner product is given by

$$(f, g)_1^0 := \int_K f(k)\overline{g(k)}dk.$$

Notice that by the Schur orthogonality theorem, the inner products $(\cdot, \cdot)_q^0$ forms a continuous family for $0 < q \leq 1$ on L_λ.

From now on, let $0 < q \leq 1$. We describe the irreducible subquotient using this sesquilinear form. Let $L_q^0(\lambda, \nu)$ be the submodule of $L_q(\lambda, \nu)$ generated by its minimal K-type.

Lemma 5.4. *We have*

$$V_q(\lambda, \nu) = L_q^0(\lambda, \nu) / \operatorname{Ann} L_q^0(\lambda, -\overline{\nu}).$$

Proof. Set $L_q^{00}(\lambda,\nu) := L_q^0(\lambda,\nu)/\operatorname{Ann} L_q^0(\lambda,-\overline{\nu})$. Then this is an admissible D-module with a D-invariant nondegenerate pairing

$$L_q^{00}(\lambda,\nu) \times L_q^{00}(\lambda,-\overline{\nu}) \to \mathbb{C}.$$

Notice that their minimal K-types are cyclic both in $L_q^{00}(\lambda,\nu)$ and $L_q^{00}(\lambda,-\overline{\nu})$.

Take a D-submodule K of $L_q^{00}(\lambda,\nu)$. If $L_q^{00}(\lambda,\nu)^{|\lambda|} \subset K$, then $K = L_q^{00}(\lambda,\nu)$ since the minimal K-type is cyclic. If $L_q^{00}(\lambda,\nu)^{|\lambda|} \not\subset K$, since the pairing is $U_q(\mathfrak{g})$-invariant, $L_q^{00}(\lambda,\nu)^{|\lambda|} \subset \operatorname{Ann} K$. Hence $\operatorname{Ann} K = L_q^{00}(\lambda,-\overline{\nu})$. Since the pairing is nondegenerate, $K = 0$. Therefore $L_q^{00}(\lambda,\nu)$ is irreducible. ☐

As a variation of sesquilinear pairing above, we also get the following, which has been shown in [22].

Lemma 5.5. *For $(\lambda,\nu) \in P \times \mathfrak{h}^*$ and $\Lambda,\Lambda' \in \mathfrak{h}^*$ such that $(\lambda,\nu) = (\Lambda - \Lambda', -\Lambda - \Lambda' - 2\rho)$, we have an isomorphism*

$$\Psi_\Lambda(M_q(\Lambda')) \simeq L_q(\lambda,\nu).$$

Proof. For $q = 1$, see [6, Lemma II.3.5].

Let $0 < q < 1$. Consider the Verma module $M_q(\Lambda) \otimes M_q(\Lambda')$ of $U_q(\mathfrak{g}) \otimes U_q(\mathfrak{g})$. Notice that the map

$$U_q(\mathfrak{g}) \to M_q(\Lambda) \otimes M_q(\Lambda') \colon x \mapsto (\kappa \otimes \widehat{S}^{-1})\widehat{\Delta}(x)(v_\Lambda \otimes v_{\Lambda'})$$

is surjective and its kernel is the right ideal generated by $\{K_\lambda - q^{(\lambda,\Lambda-\Lambda')} \mid \lambda \in P\}$. Hence the bilinear pairing

$$M_q(\Lambda) \otimes M_q(\Lambda') \times L_q(\lambda,\nu) \to \mathbb{C} \colon ((\kappa \otimes \widehat{S}^{-1})\widehat{\Delta}(x)(v_\Lambda \otimes v_{\Lambda'}), \Lambda(y)) = \widehat{\varphi}(yx).$$

is well-defined and nondegenerate.

We claim that the bilinear pairing satisfies

$$(v, xw) = ((\kappa \otimes \widehat{S}^{-1})(\widehat{\Delta} \times \Psi)(x)v, w)$$

for any $x \in D$, $v \in M_q(\Lambda) \otimes M_q(\Lambda')$ and $w \in L_q(\lambda,\nu)$. For $x \in U_q(\mathfrak{g})$, this follows from the definition. For $a \in \mathcal{O}(K_q)$, notice that

$$(\kappa \otimes \widehat{S}^{-1})\Psi(a) \in U_q(\mathfrak{b}^+) \otimes U_q(\mathfrak{b}^+).$$

Therefore, by definition of Ψ, we get

$$(\kappa \otimes \widehat{S}^{-1})\Psi(a)(v_\Lambda \otimes v_{\Lambda'}) = K_{-\Lambda-\Lambda'}(a).$$

Using this, we compute

$$\begin{aligned}
&((\kappa \otimes \widehat{S}^{-1})\Psi(a)(\kappa \otimes \widehat{S}^{-1})\widehat{\Delta}(x)(v_\Lambda \otimes v_{\Lambda'}), \Lambda(y)) \\
&= ((\kappa \otimes \widehat{S}^{-1})(\widehat{\Delta} \times \Psi)(xa)(v_\Lambda \otimes v_{\Lambda'}), \Lambda(y)) \\
&= ((\kappa \otimes \widehat{S}^{-1})(\widehat{\Delta} \times \Psi)(a_{(2)}(S(a_{(1)}) \triangleright x \triangleleft a_{(3)}))(v_\Lambda \otimes v_{\Lambda'}), \Lambda(y)) \\
&= K_{-\Lambda-\Lambda'}(a_{(2)})\widehat{\varphi}((S(a_{(1)}) \triangleright x \triangleleft a_{(3)})y) \\
&= K_{-\Lambda-\Lambda'-4\rho}(a_{(2)})\widehat{\varphi}(x(a_{(1)} \triangleright y \triangleleft S(a_{(3)}))) \\
&= ((\kappa \otimes \widehat{S}^{-1})(\widehat{\Delta} \times \Psi)(x)(v_\Lambda \otimes v_{\Lambda'}), a\Lambda(y)),
\end{aligned}$$

hence we have proven the claim.

Now, by definition of $\Psi_\Lambda(M_q(\Lambda'))$ and the fact that $L_q(\lambda,\nu)$ is admissible, we get an injective D-module map

$$L_q(\lambda,\nu)\to\Psi_\Lambda(M_q(\Lambda')).$$

By [11, Proposition 3.8.7], we get the comparison of the K-type multiplicities:

$$[\Psi_\Lambda(M_q(\Lambda')):V_q(\mu)]=V_q(\mu)_\lambda=[L_q(\lambda,\nu):V_q(\mu)]$$

for any $\mu\in P_+$. Hence this is an isomorphism. □

Proposition 5.6. *We have the following.*

- *For $0<q\le 1$ and each $w\in W$, there exists a meromorphic family of intertwining operators*

  $$T_q^w: L_q(\lambda,\nu)\to L_q(w\lambda,w\nu).$$

  Moreover T_q^w is continuous with respect to q as long as ν is almost real and $-\frac{1}{2}(\lambda-\nu)-\rho$ is dominant.
- *For a simple reflection $w=s_\alpha$, the operator $T_q^\alpha:=T_q^{s_\alpha}$ is given by the following.*

  For each $v\in L_q(\lambda,\nu)$ such that $v\in V(\lambda)\otimes V(s)^*_m$ as an $\mathfrak{sl}_{2,\alpha}$-module,

  $$T_q^\alpha v=\prod_{k=|m|+1}^{s}\frac{(k-z)_{q_\alpha}}{(k+z)_{q_\alpha}}v,$$

  where $z=\frac{1}{2}(\nu,\alpha^\vee)$, $m=\frac{1}{2}(\lambda,\alpha^\vee)$.

Proof. First, observe that thanks to Theorem 4.4, for generic (Λ,Λ'), the module $\Psi_\Lambda(M_q(\Lambda'))\simeq L_q(\lambda,\nu)$ is the unique module such that the central character is (Λ,Λ') and the minimal K-type is $|\Lambda-\Lambda'|$. Hence

$$L_q(\lambda,\nu)\simeq\Psi_\Lambda(M_q(\Lambda'))\simeq\Psi_{w.\Lambda}(M_q(w.\Lambda'))\simeq L_q(w\lambda,w\nu).$$

Hence there exists an intertwining operator for generic (λ,ν).

For the computation, the rank 1 case and $0<q<1$. Let $\mathfrak{g}=\mathfrak{sl}_2$. We identify $\mathfrak{h}=\mathbb{C}$ with $\Pi=\{1\}$, so that $Q=\mathbb{Z}$ and $P=\frac{1}{2}\mathbb{Z}$. For generic $(\lambda,\nu)\in P\times\mathfrak{h}^*$, take the intertwining operator

$$T: L_q(\lambda,\nu)\to L_q(-\lambda,-\nu).$$

Since

$$[L_q(\lambda,\nu):V_q(\mu)]=V_q(\mu)_\lambda=\begin{cases}1 & \text{for } \lambda-\mu\in\mathbb{Z}\\ 0 & \text{otherwise}\end{cases},$$

the operator T is a scalar on each K-type. Let $T_\mu=T|_{L_q(\lambda,\nu)^\mu}$. We may assume $T_{|\lambda|}=1$.

We fix generators of $\mathcal{O}(SU_q(2))$ as follows. Fix an orthonormal basis $(\xi_{\pm1/2})$ of $V_q(1/2)$. Define $a,b,c,d\in\mathcal{O}(SU_q(2))$ by

$$(a,x):=(\pi^{1/2}(x)\xi_{1/2},\xi_{1/2}),\qquad (b,x):=q(\pi^{1/2}(x)\xi_{1/2},\xi_{-1/2}),$$
$$(c,x):=q^{-1}(\pi^{1/2}(x)\xi_{-1/2},\xi_{1/2}),\qquad (d,x):=(\pi^{1/2}(x)\xi_{-1/2},\xi_{-1/2}).$$

Then a,b,c,d generate $\mathcal{O}(SU_q(2))$ with defining relations

$$ab=qba, ac=qca, bc=cb, ad-qbc=da-q^{-1}bc=1,$$

$\Delta(a) = a\otimes a+b\otimes c, \Delta(b) = a\otimes b+b\otimes d, \Delta(c) = c\otimes a+d\otimes c, \Delta(d) = d\otimes d+c\otimes b.$

For any $r, l \in \mathbb{Z}_{\geq 0}$, the Clebsch-Gordan rule asserts that the element $c^r a^l$ is in $\bigoplus_{0\leq k\leq \frac{r+l}{2}} \mathcal{O}(SU_q(2))^k$, hence $\varphi \triangleleft (c^r a^l) \in \bigoplus_{0\leq k\leq \frac{r+l}{2}} L^k_{\frac{l-r}{2}}$. Moreover since $\varphi\triangleleft(c^r a^l) \in L_{\frac{l-r}{2}}$ is of weight $\frac{r+l}{2}$, the element $\varphi\triangleleft(c^r a^l)$ is in $(L_{\frac{l-r}{2}})^{\frac{r+l}{2}}$. Since φ is faithful on $\mathcal{O}(SU_q(2))$, the element $\varphi\triangleleft(c^r a^l)$ is nonzero. Moreover we have

$$\begin{aligned} c\Lambda(\varphi \triangleleft (c^r a^l)) &= q^{-1+\nu}\Lambda(a \triangleright \varphi \triangleleft (c^r a^l S(c))) + q^{1-\nu}\Lambda(c \triangleright \varphi \triangleleft (c^r a^l S(d))) \\ &= -q^{\nu}\Lambda(a \triangleright \varphi \triangleleft (c^r a^l c)) + q^{1-\nu}\Lambda(c \triangleright \varphi \triangleleft (c^r a^{l+1})) \\ &= (-q^{r+l+2+\nu} + q^{-\nu})\Lambda(\varphi \triangleleft (c^{r+1} a^{l+1})). \end{aligned}$$

Hence for $\mu \geq \lambda \geq 0$, putting $l = \lambda + \mu$ and $r = \mu - \lambda$,

$$\begin{aligned} &T_{\mu+1}(-q^{2\mu+2+\nu} + q^{-\nu})\Lambda(\varphi \triangleleft (c^{\mu-\lambda+1} a^{\lambda+\mu+1})) \\ &= Tc\Lambda(\varphi \triangleleft (c^r a^l)) \\ &= cT\Lambda(\varphi \triangleleft (c^r a^l)) \\ &= T_\mu(-q^{2\mu+2-\nu} + q^{\nu})\Lambda(\varphi \triangleleft (c^r a^r)). \end{aligned}$$

Therefore

$$\frac{T_{\mu+1}}{T_\mu} = \frac{q^{2\mu+2-\nu} - q^{\nu}}{q^{2\mu+2+\nu} - q^{-\nu}} = \frac{(\mu+1-\nu)_q}{(\mu+1+\nu)_q}.$$

Iterating use of this formula shows the desired formula for generic (λ, ν). Since the representations $L_q(\lambda, \nu)$ vary continuously with respect to ν, the operator defined by the same formula intertwines the representations, as long as the denominator is nonzero. In particular, this forms a meromorphic family.

In general, the above calculation shows that the operator T defined as above an intertwining operator

$$L_q^{\Pi}(\lambda, \nu) \to L_q^{\Pi}(s_\alpha\lambda, s_\alpha\nu)$$

for $\Pi = \{\alpha\}$. We may induce the intertwining operator using Lemma 5.2 to get the desired formula.

Also this formula tends to the one in the classical case ([6, Proposition III.3.7])

$$T_1^\alpha v = \prod_{k=|m|+1}^{s} \frac{k-z}{k+z} v,$$

so that we may form T_q^w as a continuous family with respect to q, as long as the denominator is nonzero (in particular, if ν is almost real and $-\frac{1}{2}(\lambda-\nu)-\rho$ is dominant). □

Lemma 5.7. *For a dominant weight $\Lambda \in \mathfrak{h}^*$, $\alpha \in \Delta_+$ and $\Lambda' \in \mathfrak{h}^*$ such that $(\Lambda' + \rho, \alpha^\vee) \geq 0$,*

$$\|\Lambda - \Lambda'\| \leq \|\Lambda - s_\alpha.\Lambda'\|,$$

where $\|\lambda\|$ is the square root of (λ, λ). Moreover the equality holds if and only if s_α stabilizes either Λ or Λ'.

Proof. Immediate from $\|\Lambda - s_\alpha.\Lambda'\|^2 - \|\Lambda-\Lambda'\|^2 = 4(\Lambda'+\rho, \alpha^\vee)(\Lambda+\rho, \alpha)$. □

Proposition 5.8. *We have the following.*

1. *The module $\Psi_\Lambda(V_q(\Lambda'))$ is isomorphic to $V_q(\lambda,\nu)$ if and only if there is no $\alpha\in\Delta_+$ such that*
$$(\Lambda+\rho,\alpha^\vee)=0,(\Lambda'+\rho,\alpha^\vee)\in\mathbb{Z}_{\geq 0}.$$
Otherwise $\Psi_\Lambda(V_q(\Lambda'))=0$.
2. *The set of modules $\{V_q(\lambda,\nu)\mid\lambda\in P,\nu\in X\}$ exhausts all irreducibles.*
3. *We have $V_q(\lambda,\nu)\simeq V_q(\lambda',\nu')$ if and only if there exists $w\in W$ such that $(\lambda,\nu)=(w\lambda',w\nu')$.*
4. *If $\frac{1}{2}(\lambda-\nu)-\rho$ is dominant, then $L_q(\lambda,\nu)$ contains $V_q(\lambda,\nu)$ as a submodule.*
5. *If $-\frac{1}{2}(\lambda-\nu)-\rho$ is dominant, then $V_q(\lambda,\nu)$ is a quotient of $L_q(\lambda,\nu)$.*

Proof. For (1), assume there is no $\alpha\in\Delta_+$ as above. Take a resolution of $V_q(\Lambda')$ by Verma modules. Then the Verma modules appearing in the resolution are of the form $M_q(w.\Lambda')$ such that
$$w=s_{\alpha_1}s_{\alpha_2}\dots s_{\alpha_k},$$
$$\Lambda'\geq s_{\alpha_k}.\Lambda'\geq s_{\alpha_{k-1}}s_{\alpha_k}.\Lambda'\geq\dots\geq s_{\alpha_1}s_{\alpha_2}\dots s_{\alpha_k}.\Lambda'=w.\Lambda'$$
for $\alpha_i\in\Delta_+$.

Now, by assumption, the iterated use of Lemma 5.7 shows that
$$\|\Lambda-w.\Lambda'\|>\|\Lambda-\Lambda'\|.$$
We apply Ψ_Λ to this resolution to get a resolution of $\Psi_\Lambda(V(\Lambda'))$ by principal series representations. Then, the above estimate shows that the minimal K-type of principal series representations appearing in the resolution is always strictly larger than $|\lambda|$. Hence the minimal K-type of $\Psi_\Lambda(V_q(\Lambda'))$ is $|\lambda|$. In particular, it is nonzero and hence irreducible, by (iv) of Theorem 4.4.

Conversely, assume that there exists such α. Then $V_q(\Lambda')$ is a quotient of the cokernel of $M_q(s_\alpha.\Lambda')\to M_q(\Lambda')$, which is injective. Applying Ψ_Λ, we get that $\Psi_\Lambda(V_q(\Lambda'))$ is a submodule of the kernel of the surjective map $L_q(\lambda,\nu)\to L_q(s_\alpha\lambda,s_\alpha\nu)\simeq L_q(\lambda,s_\alpha\nu)$. Now, the comparison of the K-type multiplicity gives that this map has to be injective.

(2) follows from (iv) of Theorem 4.4 and (1).

For (3), we use the functor T_Λ in (ii) of Theorem 4.4. We only need to show that $H:=\Psi_\Lambda(V_q(\Lambda'))\simeq\Psi_\Lambda(V_q(\Lambda''))$ implies $\Lambda=\Lambda''$. For this, assume $\Lambda\neq\Lambda'$ and put $M:=T_\Lambda(H)$. Then the adjoint property gives injections $V_q(\Lambda)\to M$ and $V_q(\Lambda')\to M$. This gives rise to a map $V_q(\Lambda)\oplus V_q(\Lambda')\to M$ which is injective since $V_q(\Lambda)$ and $V_q(\Lambda')$ are distinct irreducibles. Applying Ψ_Λ, this gives a surjection $H\to H\oplus H$, which is a contradiction.

(5) is a direct consequence of (1). (4) follows from (5) and Lemma 5.4. □

Corollary 5.9. *There exists an invariant sesquilinear form on $V_q(\lambda,\nu)$ if and only if there exists $w\in W$ such that $w\lambda=\lambda$ and $w\nu=-\overline{\nu}$ modulo $2\pi i\log(q)^{-1}Q^\vee$. Moreover if it exists an invariant sesquilinear form, it is unique and hermitian up to scalar multiple.*

In particular, if ν is almost real, $w\nu=-\overline{\nu}$ in $\mathfrak{h}^*$.

Proof. Notice that for any admissible D-modules V, W, there exists a one-to-one correspondence between invariant sesquilinear pairings and D-module homomorphisms $V \to W\check{}$, where $W\check{}$ is the K-finite part of the D-module of antilinear functionals on W. Now, Lemma 5.4 and Proposition 5.8.(3) shows $V_q(\lambda, -\overline{\nu})\check{} \simeq V_q(\lambda, \nu)$, Together with Schur's lemma, this shows the first statement and the unicity of invariant sesqulinear forms. Moreover the unicity shows the invariant sesquilinear form is hermitian up to scalar multiple.

The second part is a consequence of Lemma 2.3. □

Thanks to the corollary above, our task boils down to determine whether this unique invariant hermitian form is positive definite or not for each irreducible admissible D-module.

Corollary 5.10. *If $2\Lambda'$ is almost real, the representation $\Psi_\Lambda(V_q(\Lambda'))$ has the same K-type multiplicity as $\Psi_\Lambda(V_1(\Lambda'))$. In particular, $V_q(\lambda, \nu)$ has the same K-type multiplicity with $V_1(\lambda, \nu)$ if ν is almost real.*

Proof. Thanks to Lemma 2.5, the character of $V_q(\Lambda')$ is the same as $V_1(\Lambda')$. Hence we get the same factor in the resolution by the Verma modules.

Now applying Ψ_Λ to the resolution, we get the desired conclusion since the K-type multiplicities of $L_q(\lambda, \mu)$ are the same as in the classical case. □

The following lemma is a well-known result in linear algebra, so we omit the proof.

Lemma 5.11. *Let $(\cdot, \cdot)_q$ be a continuous path of hermitian forms on a finite dimensional vector space V for an interval $q \in [q_0, 1]$. Assume that the dimensions of annihilators are constant. Then $(\cdot, \cdot)_q$ is positive definite for all q if and only if it is for some q.*

Theorem 5.12. *For almost real ν, the representation $V_q(\lambda, \nu)$ is unitary if and only if $V_1(\lambda, \nu)$ is.*

Proof. We take $-1/2(\lambda - \nu) - \rho$ to be dominant, so that $V_q(\lambda, \nu)$ is a quotient of $L_q(\lambda, \nu) = L_\lambda$. Pick $w \in W$ such that $w\nu = -\overline{\nu}$. Then the invariant sesquilinear form $(\cdot, \cdot)_q : L_\lambda \times L_\lambda \to \mathbb{C}$ is given by

$$(x, y)_q = (x, T_q^w y)_q^0,$$

where T_q^w is the intertwining operator $L_q(\lambda, \nu) \to L_q(w\lambda, w\nu) = L_q(\lambda, -\overline{\nu})$. Hence it varies continuously with respect to q.

Since $L_q(\lambda, \nu) = L_q^0(\lambda, \nu)$, the image of the intertwining operator T_q^w is $L_q^0(w\lambda, w\nu)$. Thanks to Lemma 5.4, the annihilator of this sesquilinear form is the kernel of $L_q(\lambda, \nu) \to V_q(\lambda, \nu)$. In particular, this is the unique invariant hermitian form in Corollary 5.9 composited with the natural quotient map. Moreover, Corollary 5.10 shows that the dimensions of the annihilators restricted to each K-type are constant. Thus, we can apply Lemma 5.11 to each K-type to get the desired conclusion. □

For arbitrary algebra A, the set of all irreducible modules has a natural topology as follows:

A net of irreducible modules (V_i) converges to an irreducible representation V if for any $v \in V$ and $f \in V^*$, there exists $v_i \in V_i$ and $f_i \in V_i^*$ such that

$$(xv, f) = \lim_i (xv_i, f_i)$$

for any $x \in A$.

Notice that the Fell topology on the irreducible unitary representations of G_q is nothing but the restriction of topology on $\mathrm{Adm}(G_q)$ defined as above.

Proposition 5.13. *Let (λ_i, ν_i) be a net in $(P \times X)/W$ and $(\lambda, \nu) \in (P \times X)/W$. Take the corresponding parameter (Λ_i, Λ_i') and (Λ, Λ') such that Λ is dominant. Then the net of modules $V_q(\lambda_i, \nu_i)$ converges to $V_q(\lambda, \nu)$ if and only if*

$$\Lambda_i \to \Lambda, V_q(\Lambda_i') \to V_q(\Lambda) \text{ as } i \to \infty.$$

Proof. The assertion follows from the construction of Ψ_Λ and T_Λ in Theorem 4.4. □

In particular, the modules $V_q(\lambda_i, \nu_i)$ converges to the trivial representation if and only if $(\lambda_i, \nu_i) \to (0, 2\rho)$ in the usual topology on $(P \times X)/W$.

Corollary 5.14. *Let K be a connected simply connected compact simple Lie group whose rank is at least 2 and fix $0 < q < 1$. Then the discrete quantum group $\widehat{K_q}$ has central property* (T) *in the sense of* [1]*. Equivalently, the tensor category* $\mathrm{Rep}(K_q)$ *has property* (T) *in the sense of* [18].

Proof. Since the set of almost real weights is open, we get the conclusion. □

We conclude this section with the case of $K = SU(n)$. Let us remark that for $\chi \in 2\pi i \log(q)^{-1} P^\vee$, the module $V_q(\lambda, \nu - \chi)$ is unitary if and only if $V_q(\lambda, \nu)$ is. The following corollary is an immediate consequence of Lemma 2.4 and Theorem 5.12.

Corollary 5.15. *Let $K = SU(n)$. For $(\lambda, \nu) \in P \times \mathfrak{h}^*$, the representation $V_q(\lambda, \nu)$ is unitary if and only if there exists $\chi \in 2\pi i \log(q)^{-1} P^\vee$ such that $V_1(\lambda, \nu - \chi)$ is unitary.*

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