博士論文 (要約)

論文題目

On simultaneous approximation to a pair of powers of a real number by rational numbers

（実数の冪の対の有理数による同時近似について）

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On simultaneous approximation to a pair of powers of a real number by rational numbers (Web summary)

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Abstract

For integers $1 \le l < k$ with $k \ge 3$ and a real number ξ such that $1, \xi^l, \xi^k$ are linearly independent over $\mathbb{Q}$, we obtain two results on upper bound of the uniform exponent of simultaneous approximation to $(1, \xi^l, \xi^k)$ by rational numbers, which improve the results in our previous paper. Our first result treats the case of arbitrary k, and our second and the main result, which gives a better bound, treats the case $k = 5, 7, 9$.

This paper treats the problem of uniform simultaneous approximation by rational numbers to certain set of real numbers, which was initiated by Davenport-Schmidt [1] in the study of approximation to real numbers by algebraic integers of bounded degree. Fix a positive integer n and an element $\Xi = (\xi_0, ..., \xi_n) \in \mathbb{R}^{n+1}$ with $\xi_0 \neq 0$. We say that a real number λ is a uniform exponent of approximation to Ξ by rational numbers if there exists a constant $c > 0$ such that the system of inequalities

$$\max_{0 \le i \le n} (|x_i|) \le X, \quad \max_{0 \le i \le n} (|x_0 \xi_i - x_i \xi_0|) \le c X^{-\lambda}$$

admits a non-zero solution $(x_0, ..., x_n) \in \mathbb{Z}^{n+1}$ for each real number $X > 1$. Let $\lambda(\Xi)$ denote the supremum of such exponents λ. First we explain some known results. Suppose that we are given integers $1 \le l < k$ and $\Xi = (1, \xi^l, \xi^k) \in \mathbb{R}^3$ with $1, \xi^l, \xi^k$ linearly independent over $\mathbb{Q}$. Then Theorem 1a of [1] is described as follows:

Theorem 1 (Davenport-Schmidt 1969). *If $(l, k) = (1, 2)$, we have $\lambda(\Xi) \le (\sqrt{5} - 1)/2 \cong 0.618$.*

This bound of $\lambda(\Xi)$ is optimal in the case $k = 2$ in the sense that the supremum of $\lambda(\Xi)$'s for ξ's in the theorem is equal to $(\sqrt{5} - 1)/2 \cong 0.618$ (Roy [5]). Lozier-Roy [3] studied a similar problem in the case $(l, k) = (1, 3)$ and obtained the following result:

Theorem 2 (Lozier-Roy 2012). *We have $\lambda(\Xi) \le 2(9 + \sqrt{11})/35 \cong 0.7038$, when $(l, k) = (1, 3)$.*

The optimal bound in the case $(l, k) = (1, 3)$ is conjectured to be $(1 + 3\sqrt{5})/11 \cong 0.7007$ in [3], but it has not been proven yet. In the paper [4], we studied a similar problem for general (l, k) and proved the following result.

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Theorem 3. (1) $\lambda(\Xi) \leq \frac{\sqrt{(k-1)(k+3)}-(k-1)}{2}$.
(2) $\lambda(\Xi) \leq \frac{(k-1)(k+2)}{k^2+2k-1}$ *if* k *is odd.*

Note that the bound in (2) is better than that in (1) for odd k.

As before, Theorem 1.3 would not be still optimal. The main goal of the present paper is to improve the upper bound of Theorem 1.3 in some cases.

This paper gives us two kind of results, and our first result of this paper is as follows:

Theorem 4. *We have* $\lambda(\Xi) \leq (k^2-1)/(k^2+k-1)$, *when* $k \geq 3$.

This upper bound is stronger than Theorem 1.3 (1). But Theorem 1.3 (2) is stronger than this bound for $k \geq 3$ odd. The main purpose of this paper is to improve the estimate of Theorem 1.3 (2) for $k \in \{5, 7, 9\}$. To describe our main result, we define the constants μ_5, μ_7, μ_9 as follows:

- μ_5 is the largest root of $31t^3 + 120t^2 - 144t + 20 \in \mathbb{Q}[t]$.
- μ_7 is the largest root of $278t^3 + 171t^2 - 432t + 63 \in \mathbb{Q}[t]$.
- μ_9 is the largest root of $983t^3 + 8t^2 - 960t + 144 \in \mathbb{Q}[t]$.

Then the main result of this paper is as follows:

Theorem 5. (1) *We have* $\lambda(\Xi) \leq \mu_5 \cong 0.822586$, *when* $k = 5$.
(2) *We have* $\lambda(\Xi) \leq \mu_7 \cong 0.870696$, *when* $k = 7$.
(3) *We have* $\lambda(\Xi) \leq \mu_9 \cong 0.897852$, *when* $k = 9$.

Note that the bound in Theorem 5 is stronger than that in Theorem 3 for $k \in \{5, 7, 9\}$. Our proof of Theorem 5 is a generalization of that in Lozier-Roy [3] in some sense. Lozier-Roy proved Theorem 2 by studying asymptotic behavior and divisibility property of certain polynomials ($D^{(2)}, D^{(3)}, D^{(6)}$ in their notation). We develop a certain generalization of their technique and construct certain polynomial D with good asymptotic behavior and divisibility property which are much more complicated than theirs.

To explain the proof in more detail, we fix some notations. Let $k \in \{5, 7, 9\}$. For $\mathbf{x} := (x_0, x_1, x_2) \in \mathbb{R}^3$, we put

$$\|\mathbf{x}\| := \max(|x_0|, |x_1|, |x_2|), \quad L(\mathbf{x}) := \max(|x_1 - x_0\xi^l|, |x_2 - x_0\xi^k|).$$

We fix $\lambda, c > 0$ with $\lambda > \mu_k$, assume for any real number $X > 1$ that the inequalities

$$\|\mathbf{x}\| \leq X \quad \text{and} \quad L(\mathbf{x}) \leq cX^{-\lambda}$$

have a non-zero solution $\mathbf{x}$ in $\mathbb{Z}^3$ and deduce a contradiction.

It is known that there exists a sequence of nice solutions $\{\mathbf{x}_i\}_i$ in $\mathbb{Z}^3$, which is called a sequence of minimal points. We define the set I by

$$I := \{i \mid i \geq 2, \ \mathbf{x}_{i-1}, \mathbf{x}_i, \mathbf{x}_{i+1} \text{ are linearly independent over } \mathbb{Q}\},$$

which is known to be an infinite set, and for each $i \in I$, let $n(i)$ be the next element of i in I. Also, we define the integers p_i and q_i $(i \in I)$ by $\mathbf{x}_{n(i)} = p_i\mathbf{x}_i + q_i\mathbf{x}_{i+1}$.

For triples of indeterminates $\mathbf{x} := (x_0, x_1, x_2)$ and $\mathbf{x}^{(a)} := (x_0^{(a)}, x_1^{(a)}, x_2^{(a)})$ $(1 \leq a \leq k)$, we define $\varphi(\mathbf{x}) := x_0^{k-l} x_2^l - x_1^k$ and define $\Phi(\mathbf{x}^{(1)}, ..., \mathbf{x}^{(k)})$ to be the unique symmetric multilinear form with the property $\Phi(\mathbf{x}, ..., \mathbf{x}) = \binom{k}{l}\varphi(\mathbf{x})$. For each $a \in \{0, 1, ..., k\}$, we define

$$\Phi_a(\mathbf{x}, \mathbf{y}) := \Phi(\mathbf{x}, ..., \mathbf{x}, \underbrace{\mathbf{y}, ..., \mathbf{y}}_{a}).$$

Also, let $G \in \mathbb{Z}[\mathbf{x}, \mathbf{y}]$ be the polynomial constructed in [4, Section 4] as a $\mathbb{Z}$-linear combination of the polynomials $\Phi_a(\mathbf{x}, \mathbf{y})\Phi_{k-1-a}(\mathbf{x}, \mathbf{y})$ $(0 \leq a \leq (k-1)/2)$.

Now we explain the proof of Theorem 5. We put $Q(\mathbf{x}, \mathbf{y}) := G(\mathbf{x}, \mathbf{y})\Phi_1(\mathbf{x}, \mathbf{y})\Phi_0(\mathbf{x}, \mathbf{y})^{k-1}\Phi_k(\mathbf{x}, \mathbf{y})^{k-2}$ and define the polynomial $D(\mathbf{x}, \mathbf{y}) \in \mathbb{Z}[\mathbf{x}, \mathbf{y}]$ of the form

$$D = aG^k + b\Phi_0^{k+1}\Phi_k^{k-1} + cQ + \sum_{\nu=1}^{m} d_\nu F_\nu \Phi_0^{k+1+\beta_\nu-\alpha_\nu}\Phi_k^{k-1-\beta_\nu}$$

for some F_ν's in $\mathbb{Z}[\mathbf{x}, \mathbf{y}]$ with $G|F_\nu$ and some $a, b, c, d_1, ..., d_m \in \mathbb{Z}_{\neq 0}, \alpha_\nu, \beta_\nu \in \mathbb{N}$.

For $f \in \mathbb{Z}[\mathbf{x}, \mathbf{y}]$, put $f(i) := f(\mathbf{x}_i, \mathbf{x}_{n(i)})$. We choose F_ν's suitably as $\mathbb{Z}$-linear combinations of certain products of polynomials of the form $\Phi_a(\mathbf{x}, \mathbf{y})$ so that $F_\nu(i)$'s have nice estimate from above as $i \in I$ goes to infinity, by the aid of computer. Then we prove that, if $D(i) = 0$ for infinitely many $i \in I$, the main term of $D(i)$ is $cQ(i)$. On the other hand, we can prove that $Q(i) \neq 0$ for any sufficiently large $i \in I$. So we conclude that $D(i) \neq 0$ for any sufficiently large $i \in I$.

Moreover, we choose the coefficients $a, b, c, d_1, ..., d_m$ suitably so that $D(i)$ is divisible by q_i^k, again by the aid of the computer. Combining it with the nonvanishing $D(i) \neq 0$, we conclude that $|q_i|^k \ll |D(i)|$ as $i \in I$ goes to infinity. On the other hand, we prove that the main term of $D(i)$ is $aG(i)^k + cQ(i)$ and that both $G(i)^k, Q(i)$ are $o(|q_i|^k)$ as $i \in I$ goes to infinity. Thus we deduce the contradiction and the proof of Theorem 5 is finished.

It would be an interesting future problem to find more conceptual way to construct the polynomial D which works for general k.

References

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