博士論文

論文題目　A Categorical Approach for Freeness of Group Actions on C*-algebras

（C*-環への群作用の自由性に対する圏論的アプローチ）

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A Categorical Approach for Freeness of Group Actions on C*-algebras

C*-環への群作用の自由性に対する圏論的アプローチ

by

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Abstract

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We study freeness of group actions on C*-algebras from the viewpoint of homological algebra of triangulated category and relate it with dynamics.

This thesis consists of two parts. In Part I, we investigate the homological ideal $\mathfrak{J}_G^H$, the kernel of the restriction functors in compact Lie group equivariant Kasparov categories. Applying the relative homological algebra developed by Meyer and Nest, we relate the Atiyah-Segal completion theorem with the comparison of $\mathfrak{J}_G^H$ with the augmentation ideal of the representation ring. As an application, we refine a known result on permanence property of the Baum-Connes conjecture under extensions of groups.

In Part II, we study the continuous Rokhlin property of C*-dynamical systems, which is an analogue of freeness in dynamical system. We relate it with relative injectivity studied in Part I. In addition to that, we determine the KK-equivalence class of a G-C*-algebra with the continuous Rokhlin property and give a classification of continuour Rokhlin actions on Kirchberg algebras when the G is a compact Lie group with the Hodgkin condition.

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For the work in Part I, I am deeply grateful to Georges Skandalis and N. Christopher Phillips. Their initial suggestions on the Rokhlin property and the classifying space triggered this research. I also thank Kang Li for his helpful comments.

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Finally, I cordially thank my friend Yuki Arano, who is also the collaborator of this series of researches. Our research started when he brought to me a vague idea about the way to study the Rokhlin property of quantum group actions. His ideas, in particular the use of quantum group theory, saved dying proofs of some theorems.

Contents

Chapter 1

Introduction

Equivariant KK-theory is one of main subjects in the noncommutative topology (the area of mathematics which deals with topological properties of C*-algebras). In the study of the ordinal equivariant topology, freeness of group actions is one of the most fundamental properties. However, it is not trivial to give an appropriate definition of freeness for group actions on C*-algebras. This problem was first studied by Phillips by using equivariant K-theory for C*-algebras and summarized in the book [Phi87]. On the other hand, the Rokhlin property of finite group actions introduced by Izumi [Izu04a] is able to be regarded as a dynamical definition of freeness.

In this thesis, we give a new approach for freeness from the viewpoint of the homological algebra of triangulated categories and study the relation with the previous researches. A modified version of the Rokhlin property, the *continuous Rokhlin property*, is compatble with this viewpoint. It enables us to apply the homological algebra to classification theory.

Part I. A categorical perspective of freeness and the Atiyah–Segal completion theorem

The first part is based on the paper [AK15].

The main subject here is the homological ideal

$$\mathfrak{I}_G^H(A,B) := \mathrm{Ker}(\mathrm{Res}_G^H \colon \mathrm{KK}^G(A,B) \to \mathrm{KK}^H(A,B))$$

of the Kasparov category $\mathfrak{KK}^G$. Here, $\mathfrak{KK}^G$ is the category whose objects are separable G-C*-algebras, morphisms are equivariant KK-groups [Kas80b] and composition is given by the Kasparov product.

In [MN06], Meyer and Nest introduced a new approach to study the homological algebra of the Kasparov category. They observed that the Kasparov category has a canonical structure of the triangulated category. Moreover, they applied the Verdier localization for $\mathfrak{KK}^G$ in order to give a categorical formulation of the Baum–Connes assembly map. Actually, they prove that an analogue of the simplicial approximation in the Kasparov category is naturally isomorphic to the assembly map. Their argument is refined in [Mey08] in terms of relative homological algebra of the projective class developed by Christensen [Chr98]. Moreover, it is proved that the ABC spectral sequence (a generalization of Adams spectral sequence in relative homological algebra) for the functor $\mathrm{K}_*(G \ltimes \text{\textvisiblespace})$ and an object A converges to the domain of the assembly map.

These results are essentially based on the fact that the induction functor Ind_H^G is the left adjoint of the restriction functor Res_G^H when $H \leq G$ is an open subgroup. On the other hand, it is also known that when $H \leq G$ is a cocompact subgroup, Ind_H^G is the right adjoint of Res_G^H. This relation enables us to apply the homological algebra of the injective class for KK-theory. It should be noted that the category of separable G-C*-algebras is not closed under countable direct product although the fact that $\mathfrak{KK}^G$ have countable direct sums plays an essential role in [MN06, MN10, Mey08]. Therefore, we replace the category G-$\mathfrak{C}^*\mathfrak{sep}$ of separable G-C*-algebras with its (countable) pro-category. Actually, the category $\mathrm{Pro}_{\mathbb{Z}_{>0}} G$-$\mathfrak{C}^*\mathfrak{sep}$ is naturally equivalent to the category σG-$\mathfrak{C}^*\mathfrak{sep}$ of σ-G-C*-algebras, which is dealt with by Phillips in his study of the Atiyah–Segal completion theorem. Fortunately, KK-theory for (non-equivariant) σ-C*-algebras are investigated by Bonkat [Bon02]. We check that his definition is generalized for equivariant KK-theory and get the following theorem.

Theorem 3.16 and Theorem 4.4. *For a compact group G, the equivariant Kasparov category $\sigma\mathfrak{KK}^G$ of σ-G-C*-algebras has a structure of the triangulated* *category. Moreover, for a family $\mathcal{F}$ of G (Definition 4.1), the pair of thick subcategories $(\mathcal{FC}, \langle \mathcal{FI} \rangle^{\mathrm{loc}})$ is complementary. Here $\mathcal{FC}$ is the full subcategory of $\mathcal{F}$-contractible objects and $\mathcal{FI}$ is the class of $\mathcal{F}$-induced objects (for more details, see Definition 4.3).*

By investigating an explicit construction of the semi-orthogonal decomposition associated to the above homological algebra by using the phantom castle, we observe that the Milnor construction $\{E_{\mathcal{F},n}G\}$ of the universal $\mathcal{F}$-free proper G-space $E_{\mathcal{F}}G$ arises in a canonical way. For example, compornents $\tilde{A}_n$ of the phantom castle is the tensor product with $C(E_{\mathcal{F},n}G)$ and the right component $\tilde{A}$ of A with respect to the semi-orthogonal decomposition is nothing but $A \otimes C(E_{\mathcal{F}}G)$. For this reason, the subcategory $\langle \mathcal{FI} \rangle^{\mathrm{loc}}$ can be regarded as the category of C*-algebras with $\mathcal{F}$-free G-actions. In particular, we are interested in the case that $\mathcal{F} = \mathcal{T}$. Indeed, this characterization of freeness is compatible with the previous studty in [Phi87] (see Remark 4.8).

Moreover, this construction is parallel to the proof of a classical theorem in the ordinal equivariant topology, the *Atiyah–Segal completion theorem*. In the theory of equivariant cohomology, there is a canonical way to construct an equivariant general cohomology theory from a non-equivariant cohomology theory. Actually, the general cohomology group of the new space given by the Borel construction $X \times_G EG$ is regarded as the equivariant version of the given cohomology group of X. On the other hand, equivariant K-theory is defined in terms of equivariant vector bundles by Atiyah and Segal in [AS68, Seg68]. This group has a structure of modules over the representation ring $R(G)$ and hence is related to the representation theory of compact Lie groups. In 1969, Atiyah and Segal discovered a beautiful relation between them [AS69]. When the equivariant K-group $\mathrm{K}_G^*(X)$ of a compact G-space is finitely generated as an $R(G)$-module, then the completion of the equivariant K-group by the augmentation ideal is actually isomorphic to the (representable) K-group pf the Borel construction of X.

There are two generalization of this theorem. In [AHJM88], it is proved that the completion of $\mathrm{K}_G^*(X)$ by the family of ideals of the form I_G^H (where H is in a given family of subgroups of G) is isomorphic to the equivariant K-group $\mathrm{K}_G(X \times E_{\mathcal{F}}G)$, where $E_{\mathcal{F}}G$ is the universal $\mathcal{F}$-free G-space. In [Phi89a], the Atiyah–Segal completion theorem is generalized for K-theory of C*-algebras. In order to formulate the statement, he generalizes operator K-theory for σ-C*-algebras in

[Phi89b]. Actually, this contains the Atiyah–Segal completion theorem for twisted K-theory because the twisted equivariant K-group is isomorphic to the K-group of certain C*-algebra bundles with (twisted) group actions.

Here, we prove the following categorical counterpart of the Atiyah–Segal completion theorem.

Theorem 4.16. *Let G be a compact Lie group and let A, B be σ-C*-algebras such that $\mathrm{KK}^G_*(A,B)$ are finitely generated for $*=0,1$. Then the filtrations $(\mathfrak{J}_G^{\mathcal{F}})^*(A,B)$ and $(I_G^{\mathcal{F}})^*\mathrm{KK}^G(A,B)$ are equivalent.*

Applying it for the relative homological algebra of the injective class, we obtain the following generalization of the Atiyah–Segal completion theorem.

Theorem 4.22. *When $\mathrm{KK}^G_*(A,B)$ are finitely generated $R(G)$-modules for $*=0,1$, the following $R(G)$-modules are canonically isomorphic:*

$$\mathrm{KK}^G(A,B)^{\wedge}_{I_G^{\mathcal{F}}} \cong \mathrm{KK}^G(A,\tilde{B}) \cong \mathrm{RKK}^G(E_{\mathcal{F}}G;A,B) \cong \sigma\mathfrak{KK}^G/\mathcal{FC}(A,B).$$

We remark that the Atiyah–Segal completion theorem has been already generalized for equivariant KK-theory by Uuye [Uuy12]. He assumes that $\mathrm{KK}^H_*(A,B)$ are finitely generated for all subgroups H of G in order to regard the correspondence $X \mapsto \mathrm{KK}^G(A, B\otimes C(X))$ as an equivariant cohomology theory of finite type. In Theorem 4.22, we succeed to weaken this assumption.

It is remarkable that in some special cases we need not to assume that $\mathrm{KK}^G_*(A,B)$ are finitely generated.

Corollary 4.14. *Let $\mathcal{Z}$ be the family generated by all cyclic subgroups of G. Then, there is $n>0$ such that $(\mathfrak{J}_G^{\mathcal{Z}})^n = 0$.*

It immediately follows from Corollary 4.14 that if $\mathrm{Res}_G^H A$ is KK^H-contractible for any cyclic subgroup H of G, then A is KK^G-contractible. This is a variation of McClure's restriction map theorem [McC86] which is generalized by Uuye [Uuy12] for equivariant KK-theory. Since we improve the Atiyah–Segal completion theorem, the assumption in Theorem 0.1 of [Uuy12] is also weakened (Corollary 4.24).

We apply Corollary 4.14 for the study of the complementary pair $(\langle \mathcal{CI}\rangle_{\mathrm{loc}}, \mathcal{CC})$ of the Kasparov category $\sigma\mathfrak{KK}^G$ and the Baum–Connes conjecture (BCC). Oue main interest here is permanence property of the BCC under group extensions, which is studied by Chabert, Echterhoff and Oyono-Oyono in [OO01, CE01b, CE01a] with the use of the partial assembly map. Let $1 \to N \to G \xrightarrow{\pi} G/N \to 1$ be an extension of groups. It is proved in Corollary 3.4 of [CE01a] and Theorem 10.5 of [MN06] that if G/N and $\pi^{-1}(F)$ for any compact subgroup F of G/N satisfy the (resp. strong) BCC, then so does G. Here, the assumption that $\pi^{-1}(F)$ satisfy the BCC is related to the fact that the assmebly map is defined in terms of the complementary pair $(\langle \mathcal{CI} \rangle_{\mathrm{loc}}, \mathcal{CC})$ (this assumption is refined by Schick [Sch07] when G is discrete, H is cohomologically complete and has enough torsion-free amenable quotients by group-theoretic arguments). On the other hand, Corollary 4.14 implies that the subcategories $\mathcal{CC}$ and $\mathcal{CZC}$ coincide in $\sigma\mathfrak{KK}^G$. As a consequence we refine their results as following.

Theorem 4.29. *Let $1 \to N \to G \to G/N \to 1$ be an extension of second countable groups such that all compact subgroups of G/N are Lie groups and let A be a G-C*-algebra. Then the following holds:*

1. *If $\pi^{-1}(H)$ satisfies the (resp. strong) BCC for A for any compact cyclic subgroup H of G/N, then G satisfies the (resp. strong) BCC for A if and only if G/N satisfies the (resp. strong) BCC for $N \ltimes_r^{\mathrm{PR}} A$.*
2. *If $\pi^{-1}(H)$ and G/N have the γ-element for any compact cyclic subgroup H of G/N, then so does G. Moreover, in that case $\gamma_{\pi^{-1}(H)} = 1$ and $\gamma_{G/N} = 1$ if and only if $\gamma_G = 1$.*

Part II. Continuous Rokhlin Property for compact group actions

The second part is based on the paper [AK17].

After the initial work by Connes [Con75], the study of group actions on C*-algebras and von Neumann algebras, particularly their classification, is a fundamental subject in the theory of operator algebras, as well as the classification of operator algebras themselves. In this paper, we focus on actions of compact Lie groups on C*-algebras.

In the study of von Neumann algebras, the classification of compact group actions on factors are studied in Kawahigashi-Takesaki [KT92], Masuda-Tomatsu [MT10] and so on. In the context of C*-algebras, Izumi [Izu04a, Izu04b] introduced the Rokhlin property for finite group actions on C*-algebras, extracting an essential property which is used to classify group actions in the von Neumann algebra theory. Izumi classified finite group actions with the Rokhlin property, while also showing that many of fundamental C*-dynamical systems such as infinite tensor products of left regular actions of a finite group G on $\mathbb{M}_{|G|}$ and the quasi-free action of a finite group G on $\mathcal{O}_{|G|}$ by a regular representation have this property. After his work, the study of this kind of actions has been attracting attention among C*-algebraists [Phi11, OP12, HWZ15, HP15]. Among those, recently, Gardella [Gar14c] initiated the study of Rokhlin actions for general compact groups.

Roughly speaking, the Rokhlin property is an analogue in an approximate sense of freeness of product type (that is, G-actions on $G \times X$) in topological dynamics. For a G-C*-algebra A with the Rokhlin property, there is a fundamental technique that allows us to replace a projection or a unitary with a G-invariant one, which is called an *averaging technique* by Gardella [Gar14d]. More precisely, there is a sequence of completely positive maps $A \to A^\alpha$ which is approximately a $*$-homomorphism preserving A^α. This gives a strong restriction on the structure of C*-dynamical systems, its K-groups, crossed products and so on. For example, we can prove certain "approximate cohomology vanishing" type theorems, which play an essential role in the study of actions on von Neumann algebras.

Here, we study a variation of the Rokhlin property, the continuous Rokhlin property [Gar14e], replacing the parameter $\mathbb{N}$ of approximation with $\mathbb{R}_{\geq 0}$. Although the continuous Rokhlin property is strictly stronger than the Rokhlin property, many known examples of Rokhlin actions have the continuous Rokhlin property. Actually, a necessary and sufficient condition for a Rokhlin action on a unital UCT-Kirchberg algebra to have the continuous Rokhlin property is given in terms of equivariant KK-theory (Theorem 6.15).

In connection with the homological algebra discussed in Part I, it is remarkable that this replacement is compatible with Thomsen's picture of equivariant KK-theory [Tho99]. In fact, we immediately get the following theorem.

Theorem 5.9. *Let G be a second countable compact group and let A be a separable unital G-C*-algebra. If A has the continuous Rokhlin property, then A is*

$\mathfrak{I}_G$*-injective. Moreover, if* A *has continuous Rokhlin dimension with commuting towers at most* $d-1$*, then* A *is* $\mathfrak{I}_G^d$*-injective.*

This theorem is also available for the study of the Rokhlin property. Indeed, the Rokhlin dimension with commuting towers is finite if and only if so is the continuous Rokhlin dimension with commuting towers (Proposition 5.3).

Compatibility of the continuous Rokhlin property with equivariant KK-theory enables us to apply KK-theory for classification theory of C*-dynamical systems. Although KK-theory is a kind of (co)homology theory and hence does not distinguish two homotopic C*-algebras, it is also a powerful tool for classifications of C*-algebras up to isomorphism. For example, the Kirchberg-Phillips classification [Kir, Phi00] asserts that two unital Kirchberg algebras (separable, nuclear, simple and purely infinite C*-algebras) are isomorphic if and only if there is a KK-equivalence preserving the unit classes in the K_0-groups. It is essential in the proof of [Phi00] that the KK-group has a presentation as the set of homotopy classes of asymptotic morphisms. Actually, in [Izu04a] Izumi gives a necessary and sufficient condition for two actions with the Rokhlin property to be conjugate by using an intertwining argument. With the same argument, we prove that two unital Kirchberg G-algebras with the Rokhlin property are G-equivariantly isomorphic if and only if they are KK^G-equivalent (Proposition 6.12).

Now, let us focus on the case that G is a Lie group with the Hodgkin condition (connected and $\pi_1(G)$ is torsion-free) and A is a unital Kirchberg algebra. Then, the following theorem enables us to determine the structure of Kirchberg G-algebras with the continuous Rokhlin property.

Theorem 6.3. *Let* G *be a compact Lie group with the Hodgkin condition. Then, for any separable unital* G-C*-*algebra* (A,α) *with the continuous Rokhlin property, there is a* KK^G*-equivalence from* A *to* $A^\alpha \otimes C(G)$ *mapping* $[1_A] \in \mathrm{K}_0^G(A)$ *to* $[1_{A^\alpha \otimes C(G)}]$.

For the proof, we use the strong Baum–Connes isomorphism for an arbitrary coefficient of the dual quantum group $\hat{G}$ [MN07], which is rephrased in terms of the crossed product functor since $\hat{G}$ is torsion-free in the sense of [Mey08].

Consequently, together with a construction of the model action $\mathcal{O}(G)$ of a unital Kirchberg G-algebra with the continuous Rokhlin property which is KK^G-equivalent to $C(G)$ (Corollary 6.10), we get the following complete classification of unital Kirchberg G-algebras in terms of equivariant KK-theory.

Theorem 6.13. *Let G be a Hodgkin Lie group.*

- *A unital Kirchberg G-algebra (A, α) with the continuous Rokhlin property is G-equivariantly isomorphic to $A^\alpha \otimes \mathcal{O}(G)$.*
- *Two unital Kirchberg G-algebras (A, α) and (B, β) with the continuous Rokhlin property are isomorphic if and only if the fixed point algebras A^α and B^β are isomorphic. Moreover, if the underlying C^*-algerbas A and B are in the UCT class, then (A, α) and (B, β) are conjugate if and only if*

$$(\mathrm{K}_0(A^\alpha), [1_{A^\alpha}], \mathrm{K}_1(A^\alpha)) \cong (\mathrm{K}_0(B^\beta), [1_{B^\beta}], \mathrm{K}_1(B^\beta)).$$

- *A unital UCT-Kirchberg algebra (A, α) in the Cuntz standard form (i.e. $[1_A] = 0 \in K_0(A)$) admits a G-action with the continuous Rokhlin property if and only if there is a countable abelian group M such that $K_i(A)$ ($i = 0, 1$) are isomorphic to $M^{\oplus n}$, where $n = 2^{\mathrm{rank}\, G - 1}$. In this case, $M \cong \mathrm{K}_0(A^\alpha) \oplus \mathrm{K}_1(A^\alpha)$.*

It generalizes Theorem 5.3 of [Gar14b] (for $G = \mathbb{T}^1$) and Theorem 6.6 of [IM10] (for $G = \mathbb{T}^N$).

Part I

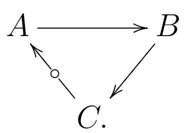
A categorical perspective of freeness and the Atiyah–Segal completion theorem

Chapter 2

Preliminaries in the relative homological algebra

In this chapter, we briefly summarize some terminologies and basic facts on the relative homological algebra of triangulated categories. The readers can find more details in [MN10] and [Mey08]. We modify a part of the theory in order to deal with the relative homological algebra of the injective class for countable families of homological ideals.

A triangulated category is an additive category together with the category automorphism Σ called the suspension and the class of triangles (a sequence $A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{h} \Sigma A$ such that $g \circ f = h \circ g = \Sigma f \circ h = 0$) which satisfies axioms [TR0]-[TR4] (see Chapter 1 of [Nee01]). We often write an exact triangle $A \to B \to C \to \Sigma A$ as



Here the symbol $A \mathrel{-\!\circ\!\!\to} B$ represents a morphism from A to ΣB.

Let $\mathfrak{T}$ be a triangulated category. An *ideal* $\mathfrak{J}$ of $\mathfrak{T}$ is a family of subgroups $\mathfrak{J}(A, B)$ of $\mathfrak{T}(A, B)$ such that $\mathfrak{T}(A, B) \circ \mathfrak{J}(B, C) \circ \mathfrak{T}(C, D) \subset \mathfrak{J}(A, D)$. A typical example is the kernel of an additive functor $F \colon \mathfrak{T} \to \mathfrak{A}$. We say that an ideal is a *homological ideal* if it is the kernel of a stable homological functor from $\mathfrak{T}$ to an abelian category $\mathfrak{A}$ with the suspension automorphism. Here a covariant functor F is homological if $F(A) \to F(B) \to F(C)$ is exact for any exact triangle $A \to B \to C \to \Sigma A$

and stable if $F \circ \Sigma = \Sigma \circ F$. Note that the kernel of an exact functor between triangulated categories is a homological ideal by Proposition 20 of [MN10].

For a homological ideal $\mathfrak{J}$ of $\mathfrak{T}$, an object A is $\mathfrak{J}$-*contractible* if id_A is in $\mathfrak{J}$ and is $\mathfrak{J}$-*injective* if $f^* \colon \mathfrak{T}(D, A) \to \mathfrak{T}(B, A)$ is zero for any $f \in \mathfrak{J}(B, D)$. The triangulated category $\mathfrak{T}$ *has enough* $\mathfrak{J}$-*injectives* if for any object $A \in \mathrm{Obj}\,\mathfrak{T}$ there is a $\mathfrak{J}$-injective object I and a $\mathfrak{J}$-monic morphism $A \to I$ (i.e. the morphism ι in the exact triangle $N \xrightarrow{\iota} A \to I \to \Sigma N$ is in $\mathfrak{J}$). Note that the morphism ι is $\mathfrak{J}$-*coversal*, that is, an arbitrary morphism $f \colon B \to A$ in $\mathfrak{J}$ factors through ι (see Lemma 3.5 of [Mey08]).

More generally, we consider the above homological algebra for a countable family $\mathfrak{J} = \{\mathfrak{J}_k\}_{k \in \mathbb{Z}_{>0}}$ of homological ideals of $\mathfrak{T}$. For example, we say an object A is $\mathfrak{J}$-contractible if A is $\mathfrak{J}_k$-contractible for any $k \in \mathbb{Z}_{>0}$.

Definition 2.1. A *filtration* associated to $\mathfrak{J}$ is a filtration of the morphism sets of $\mathfrak{T}$ coming from the composition of ideals $\{\mathfrak{J}_{i_1} \circ \mathfrak{J}_{i_2} \circ \cdots \circ \mathfrak{J}_{i_r}\}_{r \in \mathbb{Z}_{>0}}$ where $\{i_1, i_2, \dots\}$ is a sequence of positive integers such that each $k \in \mathbb{Z}_{>0}$ arises infinitely many times.

Note that two filtrations associated to $\mathfrak{J}$ are equivalent (here, we say that two filtrations A_* and A'_* of an abelian group A are equivalent if for any $n \in \mathbb{Z}_{>0}$ there is $m \in \mathbb{Z}_{>0}$ such that $A_m \subset A'_n$ and $A'_m \subset A_n$). For simplicity of notation, we use the letter $\mathfrak{J}^r$ for the r-th component of a (fixed) filtration associated to $\mathfrak{J}$ unless otherwise noted.

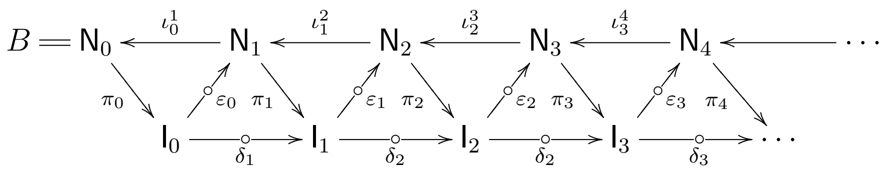
The relative homological algebra is related to complementary pairs (or semi-orthogonal decompositions) of the triangulated categories. For a thick triangulated subcategory $\mathfrak{C}$ of $\mathfrak{T}$ (Definition 1.5.1 and Definition 2.1.6 of [Nee01]), there is a natural way to obtain a new triangulated category $\mathfrak{T}/\mathfrak{C}$ called the Verdier localization (see Section 2.1 of [Nee01]). A pair $(\mathfrak{N}, \mathfrak{I})$ is a *complementary pair* if $\mathfrak{T}(N, I) = 0$ for any $N \in \mathrm{Obj}\,\mathfrak{N}$, $I \in \mathrm{Obj}\,\mathfrak{I}$ and for any $A \in \mathrm{Obj}\,\mathfrak{T}$ there is an exact triangle $\mathsf{N}_A \to A \to \mathsf{I}_A \to \Sigma \mathsf{N}_A$ such that $\mathsf{N}_A \in \mathrm{Obj}\,\mathfrak{N}$ and $\mathsf{I}_A \in \mathrm{Obj}\,\mathfrak{I}$. Actually, such an exact triangle is unique up to isomorphism for each A and there are functors $N \colon \mathfrak{T} \to \mathfrak{N}$ and $I \colon \mathfrak{T} \to \mathfrak{I}$ that maps A to N_A and I_A respectively. We say that N (resp. I) the *left* (resp. *right*) *approximation functor* with respect to the complementary pair $(\mathfrak{N}, \mathfrak{I})$. These functors induces the category equivalence $\mathsf{I} \colon \mathfrak{T}/\mathfrak{N} \to \mathfrak{I}$ and $\mathsf{N} \colon \mathfrak{T}/\mathfrak{I} \to \mathfrak{N}$.

Moreover we assume that a triangulated category $\mathfrak{T}$ admits countable direct sums and direct products. A thick triangulated subcategory of $\mathfrak{T}$ is *colocalizing* (resp. *localizing*) if it is closed under countable direct products (resp. direct sums). For a class $\mathcal{C}$ of objects in $\mathfrak{T}$, let $\langle \mathcal{C} \rangle^{\mathrm{loc}}$ (resp. $\langle \mathcal{C} \rangle_{\mathrm{loc}}$) denote the smallest colocalizing (resp. localizing) thick triangulated subcategory which includes all objects in $\mathcal{C}$. We say that an ideal $\mathfrak{J}$ is *compatible with countable direct products* if the canonical isomorphism $\mathfrak{T}(A, \prod B_n) \cong \prod \mathfrak{T}(A, B_n)$ restricts to $\mathfrak{J}(A, \prod B_n) \cong \prod \mathfrak{J}(A, B_n)$.

We write $\mathfrak{N}_{\mathfrak{J}}$ for the thick subcategory of objects which is $\mathfrak{J}_k$-contractible for any k. If each $\mathfrak{J}_k$ is compatible with countable direct products, $\mathfrak{N}_{\mathfrak{J}}$ is colocalizing. We write $\mathfrak{I}_{\mathfrak{J}}$ for the class of $\mathfrak{J}_k$-injective objects for some k.

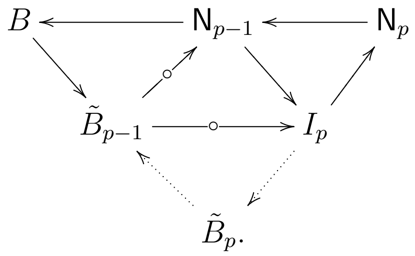
Theorem 2.2 (Theorem 3.21 of [Mey08])**.** *Let $\mathfrak{T}$ be a triangulated category with countable direct product and let $\mathfrak{J} = \{\mathfrak{J}_i\}$ be a family of homological ideals with enough $\mathfrak{J}_i$-injective objects which are compatible with countable direct products. Then, the pair $(\mathfrak{N}_{\mathfrak{J}}, \langle \mathfrak{I}_{\mathfrak{J}} \rangle^{\mathrm{loc}})$ is complementary.*

We review the explicit construction of the left and right approximation in Theorem 3.21 of [Mey08]. We start with the following diagram called the *phantom tower* for B:

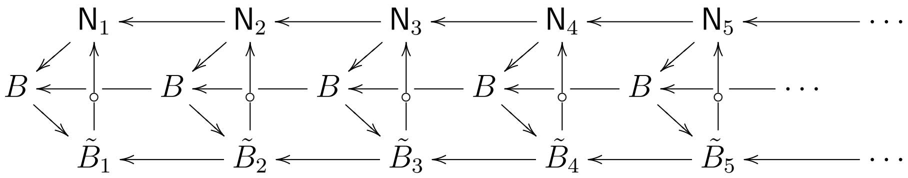


where ι_k^{k+1} are in $\mathfrak{J}_{i_k}$ and I_k are $\mathfrak{J}_{i_k}$-injective (here $\{i_k\}_{k \in \mathbb{Z}_{>0}}$ is the same as in Definition 2.1). There exists such a diagram for any B since $\mathfrak{T}$ has enough $\mathfrak{J}$-injectives. We write ι_k^l for the composition $\iota_{l-1}^l \circ \iota_{l-2}^{l-1} \circ \cdots \circ \iota_k^{k+1}$. Since each ι_k^{k+1} is $\mathfrak{J}_{i_k}$-coversal, we obtain $\mathfrak{J}^p(A, B) = \mathrm{Im}(\iota_0^p)_*$ for any A.

Next we extend this diagram to the *phantom castle*. Due to the axiom [TR1], there is a (unique) object $\tilde{B}_p$ in $\mathfrak{T}$ and an exact triangle $\mathsf{N}_p \to B \to \tilde{B}_p \to \Sigma \mathsf{N}_p$ for each p. By the axiom [TR4], we can complete the following diagram by dotted morphisms



and hence $\tilde{B}_p$ is $\mathfrak{J}^p$-injective. Moreover, we obtain a projective system



of exact triangles. Now we take the homotopy projective limit $\mathsf{I}_B := \text{ho-}\varprojlim_p \tilde{B}_p$ (we also use the symbol $\tilde{B}$ for this object) and $\mathsf{N}_B := \text{ho-}\varprojlim \mathsf{N}_p$. Here the homotopy projective limit of a projective system (B_p, φ_p^{p+1}) is the third part of the exact triangle

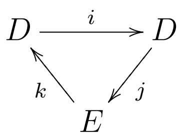
$$\Sigma^{-1} \prod B_p \to \text{ho-}\varprojlim B_p \to \prod B_p \xrightarrow{\text{id}-S} \prod B_p$$

where $S := \prod \varphi_m^{m+1}$. Then, the axiom [TR4] implies that the homotopy projective limit $\mathsf{N}_B \to B \to \mathsf{I}_B \to \Sigma \mathsf{N}_B$ of the projective system of exact triangles is also exact. In fact, it can be checked that I_B is in $\langle \mathfrak{I}_\mathfrak{J} \rangle^{\text{loc}}$ and N_B is in $\mathfrak{N}_\mathfrak{J}$ and hence N_B and I_B gives the left and right approximation of B.

At the end of this section, we review the ABC spectral sequence, introduced in [Mey08] and named after Adams, Brinkmann and Christensen. Let B be an object in $\mathfrak{T}$, let $\mathfrak{J}$ be a countable family of homological ideals with a fixed filtration and let $F\colon \mathfrak{T} \to \mathfrak{Ab}$ be a homological functor. Set

$$\begin{cases} D = \bigoplus D^{p,q}, & D^{p,q} := F_{p+q+1}(\mathsf{N}_{p+1}), \\ E = \bigoplus E^{p,q}, & E^{p,q} := F_{p+q+1}(\mathsf{I}_{p+1}), \end{cases} \quad \begin{cases} i_{p,q} := (\iota_{p+1}^{p+2})^* & : D^{p,q} \to D^{p+1,q-1}, \\ j_{p,q} := (\varepsilon_{p+1})^* & : D^{p,q} \to E^{p-1,q+1}, \\ k_{p,q} := (\pi_p)^* & : E^{p,q} \to D^{p,q+1}, \end{cases}$$

where $\mathsf{N}_p = A$ and $\mathsf{I}_p = 0$ for $p < 0$. Then the diagram



forms an exact couple. We call the associated spectral sequence is the *ABC spectral sequence* for B and F.

Proposition 2.3 (Proposition 4.3 of [Mey08])**.** *Let B be an object in $\mathfrak{T}$ and let F be a homological functor. Set $D_r^{pq} = D_r^{pq}(B) := i^{r-1}(D^{p-r+1,p+r-1})$ and $E_r^{pq} = E_r^{pq}(B) := k^{-1}(D_r^{p,q})/j(\operatorname{Ker} i^r)$. Then the following hold:*

1. $$D_r^{p-1,q} = \begin{cases} \mathfrak{J}^{r-1}F_{p+q+1}(\mathsf{N}_p) & \text{if } p \geq 0, \\ \mathfrak{J}^{p+r-1}F_{p+q+1}(B) & \text{if } -r+1 \leq p \leq 0, \\ F_{p+q+1}(B) & \text{if } p \leq -r+1, \end{cases}$$

   where $\mathfrak{J}^p F(B)$ denotes the subgroup $\{f_*\xi \mid \xi \in F(A), f \in \mathfrak{J}^p(A,B)\}$ of $F(B)$.

2. *The E_2-page E_2^{pq} is isomorphic to the right derived functor $\mathbb{R}^p F^q(B) := H_p(F_q(\mathsf{I}_*), (\delta_i)_*)$.*

3. *There is an exact sequence*

   $$0 \to \frac{\mathfrak{J}^p F_{p+q+1}(B)}{\mathfrak{J}^{p+1}F_{p+q+1}(B)} \to E_\infty^{p,q} \to \mathsf{Bad}^{p+1,p+q+1} \xrightarrow{i} \mathsf{Bad}^{p,p+q+1}$$

   where $\mathsf{Bad}^{p,q}(B) = \mathsf{Bad}^{p,q} := \mathfrak{J}^\infty F_q(\mathsf{N}_p)$.

Lemma 2.4. *Assume that $i\colon \mathsf{Bad}^{p+1,p+q+1}(B) \to \mathsf{Bad}^{p,p+q+1}(B)$ is injective. Then, the ABC spectral sequence E^r_{pq} converges to $F(B)$ with the filtration $\mathfrak{J}^*F(B)$. Moreover, $\alpha_*\colon F(B) \to F(\tilde{B})$ induces an isomorphism of graded quotients with respect to the filtration $\mathfrak{J}^*F$.*

Proof. The convergence of the ABC spectral sequence follows from Proposition 2.3 (3). By (the dual of) Proposition 3.27 of [Mey08], we have the morphism between exact couples and hence the commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \frac{\mathfrak{J}^p F_{p+q+1}(B)}{\mathfrak{J}^{p+1}F_{p+q+1}(B)} & \longrightarrow & E_\infty^{pq}(B) & \xrightarrow{\chi} & \mathsf{Bad}^{p+q+1,p}(B) & \xrightarrow{i} & \mathsf{Bad}^{p,q}(B) \\ & & \downarrow{\scriptstyle \alpha_*} & & \downarrow{\scriptstyle \alpha_*} & & \downarrow{\scriptstyle \alpha_*} & & \downarrow{\scriptstyle \alpha_*} \\ 0 & \longrightarrow & \frac{\mathfrak{J}^p F_{p+q+1}(\tilde{B})}{\mathfrak{J}^{p+1}F_{p+q+1}(\tilde{B})} & \longrightarrow & E_\infty^{pq}(\tilde{B}) & \xrightarrow{\tilde{\chi}} & \mathsf{Bad}^{p+q+1,p}(\tilde{B}) & \xrightarrow{i} & \mathsf{Bad}^{p,q}(\tilde{B}). \end{array}$$

Now, by Proposition 2.3 (2), the map $\alpha_*\colon E_2^{pq}(B) \to E_2^{pq}(\tilde{B})$ is an isomorphism and hence so is $\alpha_*\colon E_\infty^{pq}(B) \to E_\infty^{pq}(\tilde{B})$. Therefore, injectivity of $i\colon \mathsf{Bad}^{p+1,p+q+1}(B) \to \mathsf{Bad}^{p,p+q+1}(B)$ implies $\chi = 0$. Consequently we get $\tilde{\chi} = 0$, which gives the conclusion. $\square$

Chapter 3

Equivariant KK-theory for σ-C*-algebras

In this chapter, we summarize basic properties of equivariant KK-theory for σ-C^*-algebras for the convenience of readers. Most of them are obvious generalizations of equivariant KK-theory for C*-algebras (a basic reference is [Bla98]) and non-equivariant KK-theory for σ-C*-algebras by Bonkat [Bon02]. Throughout this section we assume that G is a second countable locally compact topological group.

3.1 Generalized operator algebras and Hilbert C*-modules

Topological properties of inverse limits of C*-algebras was studied by Phillips in [Phi88a, Phi88b, Phi89a, Phi89b]. He introduced the notion of representable K-theory for σ-C*-algebras in order to formulate the Atiyah-Segal completion theorem for C*-algebras.

Definition 3.1. A *pro-G-C*-algebra* is a complete locally convex $*$-algebra with continuous G-action whose topology is determined by its G-invariant continuous C*-seminorms. A pro-G-C*-algebra is a *σ-G-C*-algebra* if its topology is generated by countably many G-invariant C*-seminorms.

In other words, a pro-G-C*-algebra is a projective limit of G-C*-algebras. Actually, a pro-G-C*-algebra A is isomorphic to $\varprojlim_{p \in \mathcal{S}(A)} A_p$, where $\mathcal{S}(A)$ is the net of G-invariant continuous seminorms and

$$A_p := A/\{x \in A \mid p(x^*x) = 0\}$$

is the completion of A by the seminorm $p \in \mathcal{S}(A)$. A pro-G-C*-algebra is *separable* if A_p are separable for any $p \in \mathcal{S}(A)$. If A is a separable σ-G-C*-algebra, then it is separable as a topological space. Basic operations (full and reduced tensor products, free products and crossed products) are also well-defined for pro-C*-algebras. When G is compact, any σ-C*-algebras with continuous G-action are actually σ-G-C*-algebras.

We write $\sigma G\text{-}\mathfrak{C}^*\mathfrak{sep}$ for the category of separable σ-G-C*-algebras and equivariant $*$-homomorphisms. Then we have the category equivalence

$$\varprojlim \colon \mathrm{Pro}_{\mathbb{Z}_{>0}} G\text{-}\mathfrak{C}^*\mathfrak{sep} \to \sigma G\text{-}\mathfrak{C}^*\mathfrak{sep}$$

where $\mathrm{Pro}_{\mathbb{Z}_{>0}} G\text{-}\mathfrak{C}^*\mathfrak{sep}$ is the category of surjective projective systems of separable G-C*-algebras indexed by $\mathbb{Z}_{>0}$ with the morphism set $\mathrm{Hom}(\{A_n\}, \{B_m\}) := \varprojlim_n \varinjlim_m \mathrm{Hom}(A_n, B_m)$. Actually, a $*$-homomorphism $\varphi \colon A \to B$ induces a morphism between projective systems since each composition $A \xrightarrow{\varphi} B \to B_p$ factors through some A_q.

Next we introduce the notion of Hilbert module over pro-C*-algebras.

Definition 3.2. A *G-equivariant pre-Hilbert B-module* is a locally convex B-module together with the B-valued inner product $\langle \cdot, \cdot \rangle : E \times E \to B$ and the continuous G-action with

$$\begin{gathered} \langle e_1, e_2 b \rangle = \langle e_1, e_2 \rangle b, \langle e_1, e_2 \rangle^* = \langle e_2, e_1 \rangle, \\ g(\langle e_1, e_2 \rangle) = \langle g(e_1), g(e_2) \rangle, g(eb) = g(e)g(b), \end{gathered}$$

and the topology of E is induced by seminorms $p_E(e) := p(\langle e, e \rangle)^{1/2}$ for $p \in \mathcal{S}(B)$. A G-equivariant pre-Hilbert B-module is a *G-equivariant Hilbert B-module* if it is complete with respect to these seminorms.

Basic operations (direct sums, interior and exterior tensor products and crossed products) are also well-defined (see Section 1 of [Sch94]).

As a locally convex space, E is isomorphic to the projective limit $\varprojlim_{p\in\mathcal{S}(B)} E_p$ where $E_p := E/\{e \in E \mid p(\langle e, e\rangle) = 0\}$. A G-equivariant Hilbert B-module E is *countably generated* if E_p is countably generated for any $p \in \mathcal{S}(B)$.

Let $\mathbb{L}(E)$ and $\mathbb{K}(E)$ be the algebra of adjointable bounded and compact operators on E respectively. They are actually pro-G-C*-algebras since we have isomorphisms

$$\mathbb{L}(E) \cong \varprojlim_{p\in\mathcal{S}(B)} \mathbb{L}(E_p),\ \mathbb{K}(E) \cong \varprojlim_{p\in\mathcal{S}(B)} \mathbb{K}(E_p).$$

In particular, $\mathbb{L}(E)$ and $\mathbb{K}(E)$ are σ-G-C*-algebra if so is B. Note that $\mathbb{L}(E)$ is not separable and the canonical G-action on $\mathbb{L}(E)$ is not continuous in norm topology.

Kasparov's stabilization theorem is originally introduced in [Kas80a] and generalized by Mingo-Phillips [MP84] and Meyer [Mey00] for equivariant cases. Bonkat [Bon02] also gives a generalization for σ-C^*-algebras. Let $\mathcal{H}$ be a separable infinite dimensional Hilbert space and we write $\mathcal{H}_B$, $\mathcal{H}_{G,B}$ and $\mathbb{K}_G$ for $\mathcal{H} \otimes B$, $\mathcal{H} \otimes L^2(G) \otimes B$ and $\mathbb{K}(L^2G \otimes \mathcal{H})$ respectively.

Theorem 3.3. *Let B be a σ-unital σ-G-C*-algebra and let E be a countably generated G-equivariant Hilbert B-module together with an essential G-equivariant $*$-homomorphism $\varphi\colon \mathbb{K}_G \to \mathbb{L}(E)$. Then there is an isomorphism*

$$E \oplus \mathcal{H}_{G,B} \cong \mathcal{H}_{G,B}$$

as G-equivariant Hilbert B-modules.

Proof. In non-equivariant case, the proof is given in Section 1.3 of [Bon02]. In fact, we have a sequence $\{e^i\}$ in E such that $\sup_n \|e^i_n\| \leq 1$ and $\{\pi(e^i)\}$ generates E_p for any $p \in \mathcal{S}(B)$ since the projection $(E_p)_1 \to (E_q)_1$ between unit balls is surjective for any $p \geq q$. Now we obtain the desired unitary U as the unitary factor in the polar decomposition of the compact operator

$$T\colon \mathcal{H}_B \to E \oplus \mathcal{H}_B;\ T(\xi^i) = 2^{-i}e^i \oplus 4^{-i}\xi^i$$

where $\{\xi^i\}$ is a basis of $\mathcal{H}_B$. Actually the range of $|T|$ is dense because $T^*T = \mathrm{diag}(4^{-2}, 4^{-4}, \dots) + (2^{-i-j}\langle e_i, e_j\rangle)_{ij}$ is strictly positive.

In equivariant case, we identify E with $L^2(G, \mathcal{H}) \otimes (L^2(G, \mathcal{H})^* \otimes_{\mathbb{K}_G} E)$ and set $E_0 := \mathcal{H} \otimes_{\mathbb{C}} (L^2(G, \mathcal{H})^* \otimes_{\mathbb{K}_G} E)$. Let U be the (possibly non-equivariant) unitary from $\mathcal{H}_B$ to $E_0 \oplus \mathcal{H}_B$ as above. Then we obtain

$$\tilde{U}(g) := g(U) \colon C_c(G, \mathcal{H}_B) \to C_c(G, E_0 \oplus \mathcal{H}_B)$$

which extends to a G-equivariant unitary $\tilde{U} \colon \mathcal{H}_{G,B} \cong L^2(G, \mathcal{H}_B) \to L^2(G, E_0 \oplus \mathcal{H}_B) \cong E \oplus \mathcal{H}_{G,B}$. More detail is found in Section 3 of [Mey00]. □

A pro-C*-algebra is *σ-unital* if there is a strictly positive element $h \in A$. Here, we say that an element $h \in A$ is strictly positive if $\overline{hA} = \overline{Ah} = A$. A pro-C*-algebra A is σ-unital if and only if it has a countable approximate unit. A separable σ-C*-algebra is σ-unital and moreover has a countable increasing approximate unit (Lemma 5 of [Hen89]).

Lemma 3.4. *Let B be a σ-C*-algebra with G-action, $A \subset B$ a σ-G-C*-algebra, Y a σ-compact locally compact space, $\varphi \colon Y \to B$ a function such that $y \mapsto [\varphi(y), a]$ are continuous functions which take values in A. Then there is a countable approximate unit $\{u_i\}$ for A that is quasi-central for $\varphi(Y)$ and quasi-invariant, that is, the sequences $[u_i, \varphi(y)]$ $(y \in Y)$ and $g(u_i) - u_i$ converge to zero.*

Proof. Let $\{p_n\}_{n \in \mathbb{Z}_{>0}}$ be an increasing sequence of invariant C*-seminorms on B generating the topology of B and let $\{v_m\}$ be a countable increasing approximate unit for A and $h := \sum 2^{-k} v_k$. By induction, we can choose an increasing sequence $\{u_n\}$ given by convex combinations of v_i's such that

1. $p_n(u_n h - h) \leq 1/n$,
2. $p_n([u_n, \varphi(y)]) \leq 1/n$ for any $y \in \overline{Y_n}$,
3. $p_n(g(u_n) - u_n) \leq 1/n$ for any $g \in \overline{X_n}$.

Each induction step is the same as in Section 1.4 of Kasparov [Kas88]. □

Theorem 3.5. *Let J be a σ-G-C*-algebra, A_1 and A_2 σ-unital closed subalgebras of $M(J)$ where G acts continuously on A_1, Δ a separable subset of $M(J)$ such that $[\Delta, A_1] \subset A_1$ and $\varphi \colon G \to M(J)$ a function such that $\sup_{g \in G, p \in \mathcal{S}(M(J))} p(\varphi(g))$ is bounded. Moreover we assume that $A_1 \cdot A_2$, $A_1 \cdot \varphi(G)$, and $\varphi(G) \cdot A_1$ are in J and $g \mapsto \varphi(g)a$ are continuous functions on G for any $a \in A_1 + J$. Then, there are G-continuous even positive elements $M_1, M_2 \in M(J)$ such that*

- $M_1 + M_2 = 1$,
- $M_i a_i$, $[M_i, d]$, $M_2\varphi(g)$, $\varphi(g)M_2$, $g(M_i) - M_i$ *are in* J *for any* $a_i \in A_i$, $d \in \Delta$, $g \in G$,
- $g \mapsto M_2\varphi(g)$ *and* $g \mapsto \varphi(g)M_2$ *are continuous.*

Proof. The proof is given by the combination of arguments in p.151 of [Kas88] and in Theorem 10 of [Hen89]. Actually, by Lemma 3.4 we get an approximate unit $\{u_n\}$ for A_1 and $\{v_n\}$ for J such that

1. $p_n(u_n h_1 - h_1) \leq 2^{-n}$,
2. $p_n([u_n, y]) \leq 2^{-n}$ for any $y \in Y$,
3. $p_n(g(u_n) - u_n) \leq 2^{-n}$ for any $g \in X_n$,
4. $p_n(v_n w - w) \leq 2^{-2n}$ for any $w \in W_n$,
5. $p_n([v_n, z])$ is small enough to $p_n([b_n, z]) \leq 2^{-n}$ for any $z \in \{h_1, h_2\} \cup Y \cup \varphi(\overline{X}_n)$,
6. $p_n(g(b_n) - b_n) \leq 2^{-n}$ for any $g \in \overline{X}_n$,

where h_1, h_2, k are strictly positive element in A_1, A_2 and J respectively such that $p_n(h_1), p_n(h_2), p_n(k) \leq 1$ for any n, $Y \subset \Delta$ is a compact subset whose linear span is dense in Δ, X_n is a increasing sequence of relatively compact open subsets of G whose union is dense in G, $W_n := \{k, u_n h_2, u_{n+1} h_2\} \cup u_n\varphi(\overline{X}_n) \cup u_{n+1}\varphi(\overline{X}_{n+1}) \cup \varphi(\overline{X}_n)u_n \cup \varphi(\overline{X}_{n+1})u_{n+1}$ and $b_n := (v_n - v_{n-1})^{1/2}$. Now, it can be checked that the finite sum $\sum b_n u_n b_n$ converges in the strict topology to the desired element $M_2 \in M(J)$. □

3.2 Equivariant KK-groups

A generalization of KK-theory for pro-C*-algebras was first defined by Weidner [Wei89] and was generalized for equivariant case by Schochet [Sch94]. Here the notion of coherent A-B bimodule is introduced in order to avoid Kasparov's technical theorem for pro-C*-algebras. On the other hand, Bonkat [Bon02] introduced a new definition of KK-theory for σ-C*-algebras applying the technical theorem 3.5 for σ-C*-algebras. In this paper we adopt the latter definition.

Definition 3.6. Let A and B be σ-unital $\mathbb{Z}/2$-graded σ-G-C*-algebras. A G-equivariant Kasparov A-B bimodule is a triplet (E, φ, F) where

- E is a $\mathbb{Z}/2$-graded countably generated G-equivariant Hilbert B-module,
- $\varphi \colon A \to \mathbb{L}(E)$ is a graded G-equivariant $*$-homomorphism,
- $F \in \mathbb{L}(E)$ is an odd self-adjoint operator such that $[F, \varphi(A)]$, $\varphi(A)(F^2 - 1)$, $\varphi(A)(g(F) - F)$ are in $\mathbb{K}(E)$ and $\varphi(a)F, F\varphi(a)$ are G-continuous.

Two G-equivariant Kasparov A-B bimodules (E_1, φ_1, F_1) and (E_2, φ_2, F_2) are *unitarily equivalent* if there is a unitary $u \in \mathbb{L}(E_1, E_2)$ such that $u\varphi_1 u^* = \varphi_2$ and $uF_1u^* = F_2$. Two G-equivariant Kasparov A-B bimodules (E_1, φ_1, F_1) and (E_2, φ_2, F_2) are *homotopic* if there is a Kasparov G-equivariant A-IB bimodule (E, φ, F) such that $(\mathrm{ev}_i)_*(E, \varphi, F)$ are unitarily equivalent to (E_i, φ_i, F_i) for $i = 0, 1$.

Definition 3.7. Let A and B be σ-unital $\mathbb{Z}/2$-graded σ-G-C*-algebras. The KK-group $\mathrm{KK}^G(A, B)$ is the set of homotopy equivalence classes of G-equivariant Kasparov A-B bimodules.

It immediately follows from the definition that $\mathrm{KK}^G(\mathbb{C}, A)$ is canonically isomorphic to the representable equivariant K-group $\mathcal{R}K_0^G(A)$ introduced in [Phi89b].

Definition 3.8. Let (E_1, φ_1, F_1) be a G-equivariant Kasparov A-B bimodule and (E_2, φ_2, F_2) a G-equivariant Kasparov B-C bimodule. A Kasparov product of (E_1, φ_1, F_1) and (E_2, φ_2, F_2) is a G-equivariant Kasparov A-C bimodule $(E_1 \otimes_B E_2, \varphi, F)$ that satisfies the following.

1. The operator $F \in \mathbb{L}(E_1 \otimes_B E_2)$ is an F_2-*connection*. That is, $T_x \circ F_2 - (-1)^{\deg x \cdot \deg F_2} F \circ T_x$ and $F_2 \circ T_x^* - (-1)^{\deg x \cdot \deg F_2} T_x^* \circ F$ are compact for any $x \in E_1$.
2. $\varphi(a)[F_1 \otimes 1, F]\varphi(a)^* \geq 0 \mod \mathbb{K}(E)$.

Theorem 3.9. *Let A, B, C and D be σ-unital σ-G-C*-algebras. Moreover we assume that A is separable. The Kasparov product gives a well-defined group homomorphism*

$$\mathrm{KK}^G(A, B) \otimes \mathrm{KK}^G(B, C) \to \mathrm{KK}^G(A, C)$$

which is associative, that is, $(x \otimes_B y) \otimes_C z = x \otimes_B (y \otimes_C z)$ for any $x \in \mathrm{KK}^G(A, B)$, $y \in \mathrm{KK}^G(B, C)$ and $z \in \mathrm{KK}^G(C, D)$ when B is also separable.

Proof. What we have to show is existence, uniqueness up to homotopy, well-definedness of maps between KK-groups and associativity of the Kasparov product. All of them are proved in the same way as in Theorem 12 and Theorem 21 of [Ska84] or Theorem 2.11 and Theorem 2.14 of [Kas88]. Note that we can apply the Kasparov technical theorem 3.5 since we may assume that $\sup_{p\in\mathcal{S}(\mathbb{L}(E))} p(F) \leq 1$ by a functional calculus and a separable σ-C*-algebra is separable as a topological algebra (see also Section 18.3 - 18.6 of [Bla98]). □

Moreover, we obtain the Puppe exact sequence (as Theorem 19.4.3 of [Bla98]) for a $*$-homomorphism between σ-C*-algebras and the six term exact sequences (Theorem 19.5.7 of [Bla98]) for a semisplit exact sequence of σ-C*-algebras by the same proofs.

Next we deal with the Cuntz picture [Cun83] (see also [Mey00]) of KK-theory for σ-G-C*-algebras.

Definition 3.10 (Definition 2.2 of [Cun83])**.** We say that $(\varphi_0, \varphi_1) \colon A \rightrightarrows D \rhd J \to B$ is an equivariant *prequasihomomorphism* from A to B if D is a σ-unital σ-C*-algebra with G-action, φ_0 and φ_1 are equivariant $*$-homomorphisms from A to D such that $\varphi_0(a) - \varphi_1(a)$ are in a separable G-invariant ideal J of D such that the restriction of the G-action on J is continuous, and $J \to B$ is an equivariant $*$-homomorphism. Moreover we say that (φ_0, φ_1) is *quasihomomorphism* if D is generated by $\varphi_0(A)$ and $\varphi_1(A)$, J is generated by $\{\varphi_0(a) - \varphi_1(a) \mid a \in A\}$ and $J \to B$ is injective.

The idea given in [Cun87] is also generalized for σ-G-C*-algebras.

Definition 3.11. Let A and B be σ-G-C*-algebras. The full free product $A * B$ is the σ-G-C*-algebra given by the completion of the algebraic free product $A *_{\mathrm{alg}} B$ by seminorms

$$p_{\pi_A, \pi_B}(a_1 b_1 \dots a_n b_n) = \|\pi_A(a_1)\pi_B(b_1) \dots \pi_A(a_n)\pi_B(b_n)\|$$

where π_A and π_B are $*$-representations of A and B on the same Hilbert space. In other words, when $A = \varprojlim A_n$ and $B = \varprojlim B_m$, the free product $A * B$ is the projective limit $\varprojlim (A_n * B_m)$.

By definition, any $*$-homomorphisms $\varphi_A\colon A \to D$ and $\varphi_B\colon B \to D$ are uniquely extended to $\varphi_A * \varphi_B\colon A * B \to D$. We denote by QA the free product $A * A$ and by qA the kernel of the $*$-homomorphism $\mathrm{id}_A * \mathrm{id}_A\colon QA \to A$.

Since we have the stabilization theorem 3.3 and the technical theorem 3.5 for σ-G-C*-algebras, the following properties of quasihomomorphisms and KK-theory is proved in the same way. We only enumerate their statements and references for the proofs. Here we write q_sA for the G-C*-algebra $q(A \otimes \mathbb{K}_G)$.

- The set of homotopy classes of G-equivariant quasihomomorphisms from $A \otimes \mathbb{K}_G$ to $B \otimes \mathbb{K}_G$ is isomorphic to $\mathrm{KK}^G(A, B)$ (Section 5 of [Cun83]).
- The functor
$$\mathrm{KK}^G\colon G\text{-}\mathfrak{C}^*\mathfrak{sep} \times G\text{-}\mathfrak{C}^*\mathfrak{sep} \to R(G)\text{-}\mathbf{Mod}$$
is stable and split exact in both variables (Proposition 2.1 of [Cun87]).
- For any σ-G-C*-algebras A and B, $A * B$ and $A \oplus B$ are KK^G-equivalent (proof of Proposition 3.1 of [Cun87]).
- The element $\pi_A := [\pi_0]$ in $\mathrm{KK}^G(qA, A)$ where $\pi_0 := (\mathrm{id}_A * 0)|_{qA}\colon qA \to A$ is the KK^G-equivalence (Proposition 3.1 of [Cun87]).
- There is a one-to-one correspondence between G-equivariant quasihomomorphisms from $A \otimes \mathbb{K}_G$ to $B \otimes \mathbb{K}_G$ and G-equivariant $*$-homomorphisms from q_sA to $B \otimes \mathbb{K}_G$ (Theorem 5.5 of [Mey00]).
- There is a canonical isomorphism $\mathrm{KK}^G(A, B) \cong [q_sA, B \otimes \mathbb{K}_G]^G$ (the stabilization theorem 3.3 and Proposition 1.1 of [Cun87]).
- The correspondence
$$[q_sA \otimes \mathbb{K}_G, q_sB \otimes \mathbb{K}_G]^G \to \mathrm{KK}^G(A, B),\ \varphi \mapsto \pi_B \circ \varphi \circ (\pi_A)^{-1}$$
induces a natural isomorphism (Theorem 6.5 of [Mey00]).

For a projective system $\{A_n, \pi_n\}$ of σ-C*-algebras, the homotopy projective limit $\text{ho-}\varprojlim A_n$ is actually isomorphic to the mapping telescope
$$\mathrm{Tel}\, A_n := \{f \in \prod C([0,1], A_n) \mid f_n(1) = \pi_n(f(0))\}.$$

The following theorem follows from the fact that the functor $\mathrm{KK}^G(A, \_)$ and $\mathrm{KK}^G(\_, B)$ is compatible with direct products when B is a G-C*-algebra.

Theorem 3.12. *The following holds:*

1. *Let* $\{A_n\}_{n\in\mathbb{Z}_{>0}}$ *be an inductive system of* σ-G-C*-*algebras and* $A := \text{ho-}\varinjlim A_n$. *For a* σ-G-C*-*algebra* B, *there is an exact sequence*

$$0 \to \varprojlim{}^1 \mathrm{KK}^G_{*+1}(A_n, B) \to \mathrm{KK}^G(A, B) \to \mathrm{KK}^G_*(A_n, B) \to 0.$$

2. *Let* $\{B_n\}_{n\in\mathbb{Z}_{>0}}$ *be a projective system of* σ-G-C*-*algebras and* $B := \text{ho-}\varprojlim B_n$. *For a* σ-G-C*-*algebra* B, *there is an exact sequence*

$$0 \to \varprojlim{}^1 \mathrm{KK}^G_{*+1}(A, B_n) \to \mathrm{KK}^G(A, B) \to \varprojlim \mathrm{KK}^G_*(A, B_n) \to 0$$

3. *Let* $\{A_n\}_{n\in\mathbb{Z}_{>0}}$ *be a projective system of* σ-G-C*-*algebras and* $A := \text{ho-}\varprojlim A_n$. *For a* G-C*-*algebra* B, *there is an isomorphism*

$$\mathrm{KK}^G(A, B) \cong \varinjlim \mathrm{KK}^G(A_n, B).$$

Corollary 3.13. *Let* $A = \text{ho-}\varprojlim A_n$ *and* $B = \text{ho-}\varprojlim B_m$ *be homotopy projective limits of* C*-*algebras. There is an exact sequence*

$$0 \to \varprojlim_m{}^1 \varinjlim_n \mathrm{KK}^G_{*+1}(A_n, B_m) \to \mathrm{KK}^G_*(A, B) \to \varprojlim_m \varinjlim_n \mathrm{KK}^G(A_n, B_m) \to 0.$$

In particular, if two homotopy projective limits $A = \text{ho-}\varprojlim A_n$ and $B = \text{ho-}\varprojlim B_m$ of G-C*-algebras are KK^G-equivalent, then we get a pro-isomorphism of projecctive systems $\{A_n\}_n \to \{B_m\}_m$ in $\mathfrak{KK}^G$.

3.3 The Kasparov category

Definition 3.14. We write $\sigma\mathfrak{KK}^G$ for the Kasparov category of σ-G-C*-algebras i.e. the additive category whose objects are separable σ-G-C*-algebras, morphisms from A to B are $\mathrm{KK}^G(A, B)$ and composition is given by the Kasparov product.

Note that the inclusion $G\text{-}\mathfrak{C}^*\mathfrak{sep} \subset \sigma G\text{-}\mathfrak{C}^*\mathfrak{sep}$ induces a full embedding $\mathfrak{KK}^G$ in $\sigma\mathfrak{KK}^G$. Additional structures of $\mathfrak{KK}^G$ such as tensor products, crossed products and countable direct sums are extended on $\sigma\mathfrak{KK}^G$. Moreover the category $\mathfrak{KK}^G$ has countably infinite direct products.

Theorem 3.15 (Theorem 2.2 of [Tho98], Satz 3.5.10 of [Bon02])**.** *The category $\sigma\mathfrak{KK}^G$ is an additive category that has the following universal property: there is the canonical functor $\mathrm{KK}^G\colon \sigma\mathfrak{C}^*\mathfrak{sep} \to \sigma\mathfrak{KK}^G$ such that for any* C*-*stable, split-exact, and homotopy invariant functor $F\colon \sigma\mathfrak{C}^*\mathfrak{sep} \to \mathfrak{A}$ there is a unique functor $\tilde{F}$ such that the diagram*

$$\begin{array}{ccc} \sigma\mathfrak{C}^*\mathfrak{sep}^G & \longrightarrow & \mathfrak{A} \\ \downarrow & \nearrow & \\ \sigma\mathfrak{KK}^G & & \end{array}$$

commutes.

This follows from the Cuntz picture introduced in the previous subsection.

A structure of the triangulated category on $\mathfrak{KK}^G$ is introduced in [MN06]. Let S be the suspension functor $SA := C_0(\mathbb{R}) \otimes A$ of C*-algebras. Roughly speaking, the inverse $\Sigma := S^{-1}$ and the mapping cone exact sequence

$$\Sigma B \to \mathrm{cone}(f) \to A \xrightarrow{f} B$$

determines a triangulated category structure of $\mathfrak{KK}^G$. More precisely we need to replace the category $\mathfrak{KK}^G$ with another one that is equivalent to $\mathfrak{KK}^G$, whose objects are pair (A, n) where A is a separable σ-G-C*-algebra and $n \in \mathbb{Z}$, morphisms from (A, n) to (B, m) are $\mathrm{KK}_{n-m}(A, B)$ and composition is given by the Kasparov product. In this category the functor $\Sigma\colon (A, n) \mapsto (A, n+1)$ is an category isomorphism (not only an equivalence) and $S \circ \Sigma = \Sigma \circ S$ are natural equivalent with the identity functor. A triangle $\Sigma(B, m) \to (C, l) \to (A, n) \to (B, m)$ is exact if there is a $*$-homomorphism from A' to B' and the isomorphism α, β and γ such that the diagram

$$\begin{array}{ccccccc} \Sigma B & \longrightarrow & C & \longrightarrow & A & \longrightarrow & B \\ \cong\downarrow \Sigma\beta & & \cong\downarrow\gamma & & \cong\downarrow\alpha & & \cong\downarrow\beta \\ \Sigma B' & \longrightarrow & \mathrm{cone}(f) & \longrightarrow & A' & \xrightarrow{f} & B' \end{array}$$

commutes. For simplicity of notation we use the same letter $\mathfrak{KK}^G$ for this category.

Theorem 3.16. *The category $\sigma\mathfrak{KK}^G$, with the suspension Σ and exact triangles as above, is a triangulated category.*

We omit the proof. Actually, the same proof as for $\mathfrak{KK}^G$ given in Appendix 1 of [MN06] works since we have the Cuntz picture of equivariant KK-theory introduced in the previous subsection.

Chapter 4

Equivariant topology in KK-theory

In this section we apply the relative homological algebra of the injective class introduced in Section 2 for equivariant KK-theory and relate it with the Atiyah-Segal completion theorem. We deal with the Kasparov category $\sigma\mathfrak{KK}^G$ of σ-G-C*-algebras, which is closed under countably infinite direct products. The definition and the basic properties of equivariant KK-theory for σ-G-C*-algebras are summarized in Appendix 3. In most part of this section we assume that G is a compact Lie group. We need not to assume that G is either connected or simply connected.

4.1 Semi-orthogonal decomposition in equivariant KK-theory

For a subgroup $H \leq G$, consider the homological ideal $\mathfrak{I}_G^H := \operatorname{Ker} \operatorname{Res}_G^H$ of $\sigma\mathfrak{KK}^G$. There are only countably many homological ideals of the form $\mathfrak{I}_G^H$ since $\mathfrak{I}_G^{H_1} = \mathfrak{I}_G^{H_2}$ when H_1 and H_2 are conjugate and the set of conjugacy classes of subgroups of a compact Lie group G is countable (Corollary 1.7.27 of [Pal60]),

Definition 4.1. Let $\mathcal{F}$ be a *family*, that is, a set of closed subgroups of a compact group G that is closed under subconjugacy. We write $\mathfrak{I}_G^{\mathcal{F}}$ for the countable family of homological ideals $\{\mathfrak{I}_G^H \mid H \in \mathcal{F}\}$.

In particular, we say that the family $\mathcal{T}$ consisting of the trivial subgroup $\{e\}$ is the trivial family.

Let us recall that the induction functor $\operatorname{Ind}_H^G \colon \sigma H\text{-}\mathfrak{C}^*\mathfrak{sep} \to \sigma G\text{-}\mathfrak{C}^*\mathfrak{sep}$ is given by

$$\operatorname{Ind}_H^G A := \{f \in C(G, A) \mid \alpha_h(f(g \cdot h)) = f(g)\}$$

with the left regular G-action $\lambda_g(f)(g') = f(g^{-1}g')$ when H is a cocompact subgroup of G. By the universal property of the Kasparov category (Theorem 3.15), it induces the functor between Kasparov categories. An important property of this functor is the following Frobenius reciprocity.

Proposition 4.2 (Section 3.2 of [MN06])**.** *Let G be a locally compact group and $H \leq G$ be a cocompact subgroup. Then the induction functor Ind_H^G is the right adjoint of the restriction functor Res_G^H. That is, for any σ-G-C^*-algebra A and σ-H-C^*-algebra B we have*

$$\mathrm{KK}^G(A, \operatorname{Ind}_H^G B) \cong \mathrm{KK}^H(\operatorname{Res}_G^H A, B).$$

Proof. The equivariant KK-cycles induced from the $*$-homomorphisms

$$\varepsilon_A \colon \operatorname{Res}_G^H \operatorname{Ind}_H^G A \cong C(G, A)^H \to A, \;\; f \mapsto f(e),$$
$$\eta_B \colon B \to \operatorname{Ind}_H^G \operatorname{Res}_G^H B \cong C(G/H) \otimes B; \;\; a \mapsto a \otimes 1_{G/H},$$

form a counit and a unit of an adjunction between Ind_H^G and Res_G^H. Actually it directly follows from the definition that the compositions

$$\operatorname{Res}_G^H A \xrightarrow{\operatorname{Res}_G^H \eta_A} \operatorname{Res}_G^H \operatorname{Ind}_H^G \operatorname{Res}_G^H A \xrightarrow{\varepsilon_{\operatorname{Res}_G^H A}} \operatorname{Res}_G^H A$$
$$\operatorname{Ind}_H^G B \xrightarrow{\eta_{\operatorname{Ind}_H^G B}} \operatorname{Ind}_H^G \operatorname{Res}_G^H \operatorname{Ind}_H^G B \xrightarrow{\operatorname{Ind}_H^G \varepsilon_B} \operatorname{Ind}_H^G B$$

are identities in $\sigma\mathfrak{KK}^G$. □

Definition 4.3. Let G be a compact group and let $\mathcal{F}$ be a family of G.

1. A separable σ-G-C^*-algebra A is *$\mathcal{F}$-induced* if A is isomorphic to the inductions $\operatorname{Ind}_H^G A_0$ where A_0 is a separable σ-H-C^*-algebra and $H \in \mathcal{F}$. We write $\mathcal{FI}$ for the class of $\mathcal{F}$-induced objects.

2. A separable σ-G-C^*-algebra A is *$\mathcal{F}$-contractible* if $\operatorname{Res}_G^H A$ is KK^H-contractible for any $H \in \mathcal{F}$. We write $\mathcal{FC}$ for the class of $\mathcal{F}$-contractible objects.

In particular, when $\mathcal{F} = \mathcal{T}$ we say that A is trivially induced and trivially contractible respectively.

Theorem 4.4. *Let G be a compact group and let $\mathcal{F}$ be a family G. The pair $(\mathcal{FC}, \langle \mathcal{FI} \rangle^{\mathrm{loc}})$ is complementary in $\sigma\mathfrak{KK}^G$.*

Proof. This is proved in the same way as Proposition 3.21 of [Mey08].

Note that $\mathcal{FC} = \mathfrak{N}_{\mathfrak{I}_G^{\mathcal{F}}}$ and $\mathcal{FI} \subset \mathfrak{I}_{\mathfrak{I}_G^{\mathcal{F}}}$. By Theorem 2.2, it suffices to show that $\sigma\mathfrak{KK}^G$ has enough $\mathfrak{I}_G^{\mathcal{F}}$-injectives and all $\mathfrak{I}_G^{\mathcal{F}}$-injective objects are in $\langle \mathcal{FI} \rangle^{\mathrm{loc}}$. The first assertion follows from the existence of the right adjoint functor of Res_G^H. Actually, for any $H \in \mathcal{F}$, the morphism $A \to I_1 := \mathrm{Ind}_H^G \mathrm{Res}_G^H A$ is $\mathfrak{I}_G^H$-monic and I_1 is $\mathfrak{I}_G^H$-injective. Moreover, the morphism A is a direct summand of I_1 when A is $\mathfrak{I}_G^H$-injective. This implies the second assertion. □

In particular, applying Theorem 4.4 for the case of $\mathcal{F} = \mathcal{T}$, we immediately get the following simple but non-trivial application.

Corollary 4.5. *Let A be a separable σ-C^*-algebra and let $\{\alpha_t\}_{t\in[0,1]}$ be a homotopy of G-actions on A. We write A_t for the σ-G-C^*-algebra (A, α_t). Then, A_0 and A_1 are equivalent in $\sigma\mathfrak{KK}^G/\mathcal{TC}$. In particular, if A_0 and A_1 are in $\langle \mathcal{TI} \rangle^{\mathrm{loc}}$, then they are KK^G-equivalent.*

Proof. Consider the σ-G-C^*-algebra $\tilde{A} := (A \otimes C[0,1], \tilde{\alpha})$ where $\tilde{\alpha}(a)(t) = \alpha_t(a(t))$. Since the evaluation maps $\mathrm{ev}_t \colon \tilde{A} \to A_t$ are non-equivariantly homotopy equivalent, they induce equivalences in $\sigma\mathfrak{KK}^G/\mathcal{TC}$. Consequently, $\mathrm{ev}_1 \circ (\mathrm{ev}_0)^{-1} \colon A_0 \to A_1$ is an equivalence in $\sigma\mathfrak{KK}^G/\mathcal{TC}$. The second assertion is obvious. □

4.2 The Borel construction

Next we study a canonical model of phantom towers and phantom castles. Actually, we observe that the cellular approximation tower obtained in the proof of Theorem 4.4 is nothing but the Milnor construction of the universal $\mathcal{F}$-free (i.e. every stabilizer subgroups are in $\mathcal{F}$) proper (in the sense of [Pal61]) G-space (see [Lüc05]). Hereafter, for a compact G-space X, we write $\mathcal{C}_X$ for the mapping cone

$$\{f \in C_0([0,\infty), C(X)) \mid f(0) = \mathbb{C} \cdot 1_X\}$$

of the $*$-homomorphism $\mathbb{C} \to C(X)$ induced from the collapsing map $X \to \mathrm{pt}$.

Definition 4.6. Let $\{H_p\}_{p \in \mathbb{Z}_{>0}}$ be a countable family of subgroups in $\mathcal{F}$ such that any $L \in \mathcal{F}$ are contained infinitely many H_p's. We call the phantom tower and the phantom castle determined inductively by

$$\mathsf{I}_p(B) := \mathrm{Ind}_{H_p}^G \mathrm{Res}_G^{H_p} \mathsf{N}_p(B) \cong \mathsf{N}_p(B) \otimes C(G/H_p)$$

is the *Milnor phantom tower* and the *Milnor phantom castle* (associated to $\{H_p\}$) respectively.

By definition, I_k and N_k in the Milnor phantom tower are explicitly of the form

$$\begin{aligned} \mathsf{N}_k &\cong A \otimes \mathcal{C}_{G/H_1} \otimes \cdots \otimes \mathcal{C}_{G/H_k} \\ \mathsf{I}_k &\cong A \otimes \mathcal{C}_{G/H_1} \otimes \cdots \otimes \mathcal{C}_{G/H_{k-1}} \otimes C(G/H_k) \end{aligned}$$

and ι_k^{k+1} is induced from the restriction (evaluation) $*$-homomorphism $\mathrm{ev}_0 \colon \mathcal{C}_{G/H_k} \to \mathbb{C}$ given by $f \mapsto f(0)$.

For G-spaces $X_1, \ldots, X_n$, the join $*_{k=1}^n X_k$ is defined to be the quotient of $\Delta^n \times (\prod X_k)$, where

$$\Delta^n := \{(t_1, \ldots, t_n) \in [0,1]^n \mid \sum t_i = 1\},$$

with the relation

$$(t_1, \ldots, t_n, x_1, \ldots, x_n) \sim (t_1, \ldots, t_n, y_1, \ldots, y_n) \text{ if } x_k = y_k \text{ for any } k \text{ such that } t_k \neq 0.$$

It is equipped with the G-action induced from the diagonal action on $\Delta^n \times \prod X_k$ (where G acts on Δ^n trivially). For $n \in \mathbb{Z}_{>0}$, $E_{\mathcal{F},n}G$ denotes the n-th step of the Milnor construction [?] $*_{k=1}^n G/H_k$.

Lemma 4.7. *The n-th step of the cellular approximation $\tilde{\mathbb{C}}_n$ of $\mathbb{C}$ is isomorphic to $C(E_{\mathcal{F},n}G)$.*

Proof. Let f be a function on the Gelfand-Naimark dual of $\mathcal{C}_{G/H_1} \otimes \cdots \otimes \mathcal{C}_{G/H_n}$ given by

$$f(((x_1, t_1), \ldots, (x_n, t_n))) = t_1 + \cdots + t_n.$$

Obviously, $f^{-1}((0,\infty))$ is G-homeomorphic to $(0,\infty) \times (* \, G/H_k)$ and $f^{-1}(0) = *$. Consequently, $\mathcal{C}_{G/H_1} \otimes \cdots \otimes \mathcal{C}_{G/H_n}$ is G-equivariantly isomorphic to the mapping cone $\mathcal{C}_{* \, G/H_k}$. □

Consequently, it can be seen that the right approximation $\tilde{A}$ is nothing but the tensor product $A \otimes C(E_{\mathcal{F}}G)$, where $E_{\mathcal{F}}G$ is the universal $\mathcal{F}$-free G-space. In other words, a G-C*-algebra A is in the full subcategory $\langle \mathcal{FI} \rangle^{\mathrm{loc}}$ if and only if it is KK^G-equivalent to $A \otimes C(E_{\mathcal{F}}G)$. It is analogous to the fact in the equivariant topology that a G-space X is $\mathcal{F}$-free if and only if it is G-equivariantly homotopy equivalent to $X \times E_{\mathcal{F}}G$.

Remark 4.8. In particular, when $\mathcal{F} = \mathcal{T}$, the subcategory $\langle \mathcal{TI} \rangle^{\mathrm{loc}}$ gives a characterization of freeness of group actions on C*-algebras. In fact, it is compatible with the previous approach for the definition of freeness of G-actions by Phillips [Phi87]. In Section 4.4 of [Phi87] (see in particular Proposition 4.4.4), a G-C*-algebra A is said to have discrete KK-theory if $I_G^n \mathrm{KK}^G(A, A) = 0$ for some $n > 0$ and be KK-free if all G-invariant ideals $I \lhd A$ has discrete KK-theory. We remark that a G-C*-algebra A is $\mathfrak{J}_G^k$-injective for some $k > 0$, then it has discrete KK-theory, which is obvious from the definition. Moreover, if G is a subgroup of $\mathbb{T}^1$ or $\mathrm{KK}_*^G(A, A)$ $(* = 0, 1)$ are finitely generated as $R(G)$-modules, then the converse follows from Theorem 4.12 or Theorem 4.16 in the next section.

More generally, let X be a $\mathcal{F}$-free finite G-CW-complex containing a point x whose stabilizer subgroup is H. By Proposition 2.2 of [Mey08], there is $n > 0$ such that $C(X)$ is $(\mathfrak{J}_G^{\mathcal{F}})^n$-injective. Moreover, the morphism $\mathrm{ev}_0 \colon \mathcal{C}_X \to \mathbb{C}$ is in $\mathfrak{J}_G^H$ since the path of H-equivariant $*$-homomorphisms $\mathrm{ev}_{(t,x)} \colon \mathcal{C}_X \to \mathbb{C}$ connects ev_0 and zero. Let $\{X_i\}$ be a family of $\mathcal{F}$-free compact G-CW-complexes such that for any $H \in \mathcal{F}$ there are infinitely many X_i's such that $X_i^H \neq \emptyset$. Then, in the same way as Theorem 2.2, the exact triangle

$$SC(\mathop{\ast}_{i=1}^{\infty} X_i) \to \bigotimes_{i=1}^{\infty} \mathcal{C}_{X_i} \to \mathbb{C} \to C(\mathop{\ast}_{i=1}^{\infty} X_i)$$

gives the approximations of $\mathbb{C}$ with respect to the complementary pair $(\mathcal{FC}, \langle \mathcal{FI} \rangle^{\mathrm{loc}})$.

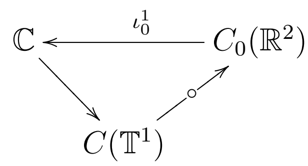
4.3 Comparison of ideals and the Atiyah–Segal completion theorem

Now we compare the filtration $(\mathfrak{J}_G^{\mathcal{F}})^*(A,B)$ with another one;

$$(I_G^{\mathcal{F}})^n \mathrm{KK}^G(A,B) := \{\sum_i \gamma_i^1 \cdots \gamma_i^n \xi_i \mid \gamma_i^k \in I_G^{H_k}, \xi_i \in \mathrm{KK}^G(A,B), \}$$

where I_G^H are the augmentation ideals $\operatorname{Ker}\operatorname{Res}_G^H$ of $R(G)$ and $\{H_i\}$ is the same as Definition 4.6. Obviously its equivalence class is independent of the choice of such $\{H_i\}$. We also remark that $(I_G^{\mathcal{F}})^n \mathrm{KK}^G(A,B) \subset (\mathfrak{J}_G^{\mathcal{F}})^n(A,B)$.

Example 4.9. We consider the case that $G = \mathbb{T}^1$ and $\mathcal{F} = \mathcal{T}$. The first triangle in the Milnor phantom tower is



where $\mathbb{T}^1 = U(1)$ acts on $\mathbb{R}^2 = \mathbb{C}$ canonically. By the Bott periodicity, $\mathrm{KK}^G(\mathsf{N}_1, \mathbb{C})$ is freely generated by the Bott generator $\beta \in \mathrm{KK}^G(\mathsf{N}_1,\mathbb{C})$ and $\mathfrak{J}_G(\mathsf{N}_1,\mathbb{C}) = I_G \cdot \beta$. Consequently, ι_0^1 is in $I_G\mathrm{KK}^G(A,B)$. More explicitly, $\iota_0^1 = \lambda \cdot \beta$ where $\lambda := [\Lambda^0\mathbb{C}] - [\Lambda^1\mathbb{C}]$. Since ι_0^1 is $\mathfrak{J}_G$-coversal, $\mathfrak{J}_G(A,B) = I_G\mathrm{KK}^G(A,B)$ holds for any A and B.

Example 4.10. Let G be a compact connected Lie group such that $\pi_1(G)$ has no 2-torsion element and let T be a maximal torus of G. In this case the following lemma shows that $\iota_0^1 = 0$ and hence $\mathfrak{J}_G^T = I_G^T \mathrm{KK}^G = 0$.

Lemma 4.11. *The morphism $\pi_0 \in \mathrm{KK}^G(\mathbb{C}, C(G/T))$ in the Milnor phantom tower has a left inverse.*

Proof. Let us fix a choice of positive roots $P \subset \Delta$ and $\rho := \sum_{\alpha\in P} \alpha/2$. By the assumption about 2-torsion elements of $\pi_1(G)$, the weight $i\rho$ is analytically integral, that is, $\langle X, i\rho\rangle \in 2\pi\mathbb{Z}$ for any $X \in i\mathfrak{t}$ such that $e^X = 1 \in T$ (in terms of bundles, this means that the flag manifold G/T has a homogeneous *Spin*^{c}-structure).

Let $\lambda \in i\mathfrak{t}^*$ be an analytically integral weight such that $\lambda + \rho$ is regular i.e. $\langle \alpha, \lambda + \rho\rangle \neq 0$ for all $\alpha \in P$. The Borel-Weil-Bott theorem (see for example

Theorem 8.7 of [BGV04]) says that the equivariant index of the twisted Dirac operator D_λ on G/T twisted by λ is the highest weight module $[V_\lambda] \in R(G)$. In particular when $\lambda = 0$, the index of the (untwisted) Dirac operator is $1 \in R(G)$. Therefore, the corresponding K-homology cycle $[D] \in \mathrm{KK}^G(C(G/T), \mathbb{C})$ satisfies

$$[\pi_0] \otimes_{C(G/T)} [D] = [\mathrm{Ind}\, D] = 1 \in R(G) \cong \mathrm{KK}^G(\mathbb{C}, \mathbb{C})$$

since π_0 is induced from the $*$-homomorphism mapping $1 \in \mathbb{C}$ to the identity element in $C(G/T)$. □

For a group homomorphism $L \to G$ and a family $\mathcal{F}$ of G, define the pull-back to be

$$\varphi^*\mathcal{F} := \{\varphi^{-1}(H) \mid H \in \mathcal{F}\}.$$

Then, the functor $\varphi^* \colon \sigma\mathfrak{KK}^G \to \sigma\mathfrak{KK}^L$ maps $\langle \mathcal{FI} \rangle$ and $\mathcal{FC}$ to $\langle \varphi^*\mathcal{FI} \rangle$ and $\varphi^*\mathcal{FC}$ respectively.

Theorem 4.12. *Let $H \leq G$ be compact connected Lie groups without any 2-torsion in their fundamental groups and $\mathrm{rank}\, G - \mathrm{rank}\, H \leq 1$. For a group homomorphism $\varphi \colon L \to G$, let $\mathcal{F} := \varphi^*\mathcal{F}_H$. Then, for any $r \in \mathbb{Z}_{>0}$ there is $k \in \mathbb{Z}_{>0}$ such that $(\mathfrak{J}_L^{\mathcal{F}})^k(A, B) \subset (I_L^{\mathcal{F}})^r \mathrm{KK}^L(A, B)$ for any $A, B \in \sigma\mathfrak{C}^*\mathfrak{sep}^L$.*

Proof. It suffices to find a compact $\mathcal{F}$-free proper L-space X such that the exact triangle

$$\mathcal{C}_X \xrightarrow{\iota} \mathbb{C} \xrightarrow{\pi} C(X) \to \Sigma\mathcal{C}_X$$

in $\sigma\mathfrak{KK}^L$ satisfies $\iota \in (I_L^{\mathcal{F}})^r \mathrm{KK}^L(\mathcal{C}_X, \mathbb{C})$ because

$$\mathrm{Im}(\iota \otimes \mathrm{id}_B)_* = \mathrm{Ker}(\pi \otimes \mathrm{id}_B)_* \supset (\mathfrak{J}_G^{\mathcal{F}})^k(A, B),$$

for any $A, B \in \sigma\mathfrak{C}^*\mathfrak{sep}^L$ and $k \in \mathbb{Z}_{>0}$ such that $X \subset E_{\mathcal{F},k}L$. Since $\varphi^* I_G^H \subset I_L^M$ for any $M \in \mathcal{F}$, we can reduce the problem for the case that $\varphi = \mathrm{id}$.

When $\mathrm{rank}\, G = \mathrm{rank}\, H$, it immediately follows from Example 4.10 (note that in this case $(\mathfrak{J}_G^{\mathcal{F}})^k = 0$ for some $k > 0$). To see the case that $\mathrm{rank}\, G - \mathrm{rank}\, H = 1$, choose an inclusion of maximal tori $T_H \subset T_G$. Consider the exact triangle

$$SC(T_G/T_H) \to \mathcal{C}_{T_G/T_H} \to \mathbb{C} \to C(T_G/T_H).$$

By Example 4.9, $\mathrm{Res}_G^{T_G} \iota_0^1$ is in $I_{T_G}^{T_H}\mathrm{KK}^{T_G}(\mathsf{N}_1, \mathbb{C})$. Since $(I_{T_G}^{T_H})^n \subset I_G^{T_H} R(T_G)$ for sufficiently large $n > 0$ (Lemma 3.4 of [AHJM88]), for any $l > 0$ there is $k > 0$ such that

$$\iota_0^k = \iota_0^1 \otimes \cdots \otimes \iota_0^1 \in (I_G^H)^l \mathrm{KK}^{T_G}(\mathsf{N}_k, \mathbb{C})$$

(note that $I_G^{T_H} = I_G^H$). Moreover, ι_0^k is actually in $(I_G^H)^l\mathrm{KK}^G(\mathsf{N}_k, \mathbb{C})$ since $\mathrm{KK}^G(\mathsf{N}_k, \mathbb{C})$ is a direct summand of $\mathrm{KK}^{T_G}(\mathsf{N}_k, \mathbb{C})$ by Example 4.10. Now $E_{\mathcal{F},k}G$ is the desired X (recall that $C(E_{\mathcal{F},k}G) \cong \tilde{\mathbb{C}}_k$). □

As a corollary, we obtain a generalization of Corollary 1.3 of [AHJM88]. For a family $\mathcal{F}$ of G, we write $\mathcal{F}_{\mathrm{cyc}}$ for the family generated by (topologically) cyclic subgroups in $\mathcal{F}$. In particular, let $\mathcal{Z}$ denote the family generated by all cyclic subgroups. Here, we say that $T \leq G$ is a cyclic subgroup of G if there is an element $g \in T$ such that $\overline{\{g^n\}} = T$. Note that T is cyclic if and only if $T \cong \mathbb{T}^m \times \mathbb{Z}/l\mathbb{Z}$.

Lemma 4.13. *Let $\mathcal{F} \subset \mathcal{F}'$ be families of G. If for any $H \in \mathcal{F}'$ there is $k \in \mathbb{Z}_{>0}$ such that $(\mathfrak{J}_H^{\mathcal{F}|_H})^k = 0$, then two filtrations $\mathfrak{J}_G^{\mathcal{F}}$ and $\mathfrak{J}_G^{\mathcal{F}'}$ are equivalent uniformly, that is, for any $k > 0$ there is $n > 0$ (independent of A and B) such that $(\mathfrak{J}_G^{\mathcal{F}})^n(A, B) \subset (\mathfrak{J}_G^{\mathcal{F}'})^k(A, B)$ for any $A, B \in \sigma\mathfrak{C}^*\mathfrak{sep}^G$.*

Proof. Pick $H_1, \dots, H_k \in \mathcal{F}'$. By assumption, we can choose $L_{i,1}, \dots, L_{i,j_i}$ ($i = 1, \dots, k$) such that $\mathfrak{J}_{H_i}^{L_{i,1}} \circ \cdots \circ \mathfrak{J}_{H_i}^{L_{i,j_i}} = 0$. Then, by definition

$$(\mathfrak{J}_G^{L_{1,1}} \circ \cdots \circ \mathfrak{J}_G^{L_{1,j_1}}) \circ \cdots \circ (\mathfrak{J}_G^{L_{i,1}} \circ \cdots \circ \mathfrak{J}_G^{L_{k,j_k}}) \subset \mathfrak{J}_G^{H_1} \circ \cdots \circ \mathfrak{J}_G^{H_k},$$

which is the conclusion. □

Corollary 4.14. *For a compact Lie group G, the following hold:*

1. *There is $n > 0$ such that $(\mathfrak{J}_G^{\mathcal{Z}})^n = 0$. In particular, the subcategory $\mathcal{Z}\mathcal{C}$ is zero in $\sigma\mathfrak{KK}^G$.*
2. *For any family $\mathcal{F}$ of G, the filtrations $(\mathfrak{J}_G^{\mathcal{F}})^*$ and $(\mathfrak{J}_G^{\mathcal{F}_{\mathrm{cyc}}})^*$ are equivalent. Moreover, $\mathcal{F}\mathcal{C} = \mathcal{F}_{\mathrm{cyc}}\mathcal{C}$ in $\sigma\mathfrak{KK}^G$.*

Note that the second assertion means that for any $n > 0$ we obtain $k > 0$ (which does not depend on A and B) such that $(\mathfrak{J}_G^{\mathcal{F}})^k(A, B) \subset (\mathfrak{J}_G^{\mathcal{F}_{\mathrm{cyc}}})^n(A, B)$.

Proof. First, we prove when G is abelian by induction with respect to the order of G/G^0. When G/G^0 is cyclic, then the assertion holds because G is also cyclic. Now we assume that G/G^0 is not cyclic (and hence any element in G/G^0 is contained in a proper subgroup). Let $\mathcal{P}$ be the family of G generated by pull-backs of proper subgroups of G/G^0. By the induction hypothesis and Lemma 4.13, it suffices to show that there is a large $n > 0$ such that $(\mathfrak{I}_G^{\mathcal{P}})^n = 0$. Because G is covered by finitely many subgroups in $\mathcal{P}$, we obtain a large $m > 0$ such that $(I_G^{\mathcal{P}})^m = 0$. Since G/G_0 is a direct product of finite cyclic groups, there is a nontrivial group homomorphism $f\colon G/G_0 \to \mathbb{T}^1$. Applying Theorem 4.12 for compositions of the quotient $G \to G/G^0$ and f, we get $n > 0$ such that $(\mathfrak{I}_G^{\mathcal{P}})^n \subset (I_G^{\mathcal{P}})^m \mathrm{KK}^G = 0$.

Fot general G, let $\pi\colon G \to U(n)$ be a faithful representation of G. As is pointed out in the proof of Theorem 4.12 for $T_{U(n)} \leq U(n)$ and π (in this case $\mathcal{F}$ is equal to the family of all abelian subgroups $\mathcal{AB}$ of G), Example 4.10 implies that there is $k \in \mathbb{Z}_{>0}$ such that $(\mathfrak{I}_G^{\mathcal{AB}})^k = 0$. Now, we get the conclusion by Lemma 4.13 for $\mathcal{Z} \subset \mathcal{AB}$.

Now, the assertion (2) immediately follows from (1) and Lemma 4.13. □

Remark 4.15. Unfortunately, in contrast to Theorem 4.12, $\iota_0^k \in I_G^{\mathcal{F}} \mathrm{KK}^G(\mathsf{N}_k, \mathbb{C})$ does not hold for general compact Lie groups and families. For example, consider the case that $G = \mathbb{T}^2$ and $\mathcal{F} = \mathcal{T}$. Computing the six-term exact sequence of the equivariant K-homology groups associated to the exact triangle

$$SC(S^{2n-1} \times S^{2n-1}) \to \mathcal{C}_{S^{2n-1}\times S^{2n-1}} \to \mathbb{C} \to C(S^{2n-1} \times S^{2n-1}),$$

we obtain $\mathrm{KK}^G(\mathcal{C}_{S^{2n-1}\times S^{2n-1}}, \mathbb{C}) \cong R(G)\cdot\iota_0^k$ (note that $\mathrm{KK}_1^G(C(S^{2n-1}\times S^{2n-1}), \mathbb{C}) \cong \mathrm{K}_1(\mathbb{C}P^n \times \mathbb{C}P^n) = 0$ by Poincaré duality). By Theorem 3.12 (3), we obtain $\mathrm{KK}^G(\mathsf{N}_{\mathbb{C}}, \mathbb{C}) \cong R(G) \cdot \iota_0^\infty$ and hence ι_0^∞ is not in $I_G \mathrm{KK}^G(\mathsf{N}_{\mathbb{C}}, \mathbb{C})$.

Instead of Theorem 4.12, the following theorem holds for general compact Lie groups and families.

Theorem 4.16. *Let G be a compact Lie group and let A, B be σ-C^*-algebras such that $\mathrm{KK}_*^G(A, B)$ is finitely generated for $* = 0, 1$. Then the filtrations $(\mathfrak{I}_G^{\mathcal{F}})^*(A, B)$ and $(I_G^{\mathcal{F}})^* \mathrm{KK}^G(A, B)$ are equivalent.*

Note that this is a direct consequence of Lemma 4.7 and Corollary 2.5 of [Uuy12] when $\mathrm{KK}_*^H(A, B)$ are finitely generated for any $H \leq G$ and $* = 0, 1$.

In order to show Theorem 4.16, we prepare some lemmas.

Lemma 4.17. *Let G be a compact Lie group, let X be a compact G-space and let A, B be σ-$G \ltimes X$-C*-algebras. We assume that $\mathrm{KK}_*^{G \ltimes X}(A, B)$ are finitely generated for $* = 0, 1$. Then, the following holds:*

1. *Assume that G satisfies Hodgkin condition and let T be a maximal torus of G. Then $\mathrm{KK}_*^{T \ltimes X}(A, B)$ are finitely generated for $* = 0, 1$.*
2. *When $G = \mathbb{T}^n$, $\mathrm{KK}_*^{H \ltimes X}(A, B)$ are finitely generated for any $H \leq \mathbb{T}^n$.*
3. *For any cyclic subgroup H of G, there is a G-space Y such that $C(Y)$ is $(\mathfrak{J}_G^H)^k$-injective for some $k > 0$ and $\mathrm{KK}_*^{G \ltimes X}(A, B \otimes C(Y))$ are finitely generated for $* = 0, 1$.*

Proof. First, (1) follows from the fact that $C(G/T)$ is KK^G-equivalent to $\mathbb{C}^{|W_G|}$ (which is essentially proved in p.31 of [RS86]). To see (2), first we consider the case that $\mathbb{T}^n/H$ is isomorphic to $\mathbb{T}$. Then, the assertion follows from the six-term exact sequence of the functor $\mathrm{KK}^{\mathbb{T}^n \ltimes X}(A, B \otimes \text{␣})$ associated to the exact triangle

$$SC(\mathbb{T}^1) \to C_0(\mathbb{R}^2) \to \mathbb{C} \to C(\mathbb{T}^1).$$

In general $\mathbb{T}^n/H$ is isomorphic to $\mathbb{T}^m$. By iterating this argument m times, we immediately obtain the conclusion.

Finally we show (3). Since the space of conjugacy classes of G is homeomorphic to the quotient of a finite copies of the maximal torus T of G^0 by a finite group, there is a finite family of class functions separating conjugacy classes of G. A moment of thought will give you a finite faithful family of representations $\{\pi_i \colon G \to U(n_i)\}$ such that $\{\chi(\pi_i)\}$ separates the conjugacy classes of G. Then, two elements g_1 , g_2 in G are conjugate in G if and only if so are in $U := \prod U(n_i)$ (here G is regarded as a subgroup of U by $\prod \pi_i$). Set $\mathcal{F} := \{L \leq G \cap gHg^{-1} \mid g \in U\}$. Then G acts on U/H $\mathcal{F}$-freely and every subgroup in $\mathcal{F}_{\mathrm{cyc}}$ is contained in a conjugate of H. By Corollary 4.14 (2), $C(U/H)$ is $(\mathfrak{J}_G^H)^k$-injective for some $k > 0$. Moreover, $\mathrm{KK}_*^G(A, B \otimes C(U/H))$ are finitely generated $R(G)$-modules. To see this, choose a maximal torus T of U containing H. Then U/H is a principal T/H-bundle over U/T and we can apply the same argument as (2). □

Lemma 4.18. *Let X be a compact G-space and let $X_1, \ldots, X_n$ be closed G-subsets of X such that $X_1 \cup \cdots \cup X_n = X$. Then, in the category $\sigma\mathfrak{KK}^{G \ltimes X}$, the filtration* *associated to the family of ideals* $\mathfrak{J}_{X_1,\dots,X_n} := \{\operatorname{Ker} \operatorname{Res}^{G \ltimes X_i}_{G \ltimes X}\}$ *is trivial (i.e. there is* $k > 0$ *such that* $(\mathfrak{J}_{X_1,\dots,X_n})^k = 0$*).*

Proof. It suffices to show the following: Let X be a compact G-space and X_1, X_2 be a closed G-subspaces such that $X = X_1 \cup X_2$. For separable σ-$G \ltimes X$-C*-algebras A, B, D and $\xi_1 \in \mathrm{KK}^{G \ltimes X}(A, B)$, $\xi_2 \in \mathrm{KK}^{G \ltimes X}(B, D)$ such that $\operatorname{Res}^{G \ltimes X_1}_{G \ltimes X} \xi_1 = 0$ and $\operatorname{Res}^{G \ltimes X_2}_{G \ltimes X} \xi_2 = 0$ holds, we have $\xi_2 \circ \xi_1 = 0$.

To see this, we use the Cuntz picture. Let $\mathbb{K}_G := \mathbb{K}(L^2(G)^\infty)$ and let $q_{s,X}A$ be the kernel of the canonical $*$-homomorphism

$$((A \otimes \mathbb{K}_G) *_X (A \otimes \mathbb{K}_G)) \otimes \mathbb{K}_G \to (A \otimes \mathbb{K}_G) \otimes \mathbb{K}_G$$

for a $G \ltimes X$-C*-algebra A. Then, $\mathrm{KK}^{G \ltimes X}(A, B)$ is isomorphic to the set of homotopy classes of $G \ltimes X$-equivariant $*$-homomorphisms from $q_{s,X}A$ to $q_{s,X}B$ and the Kasparov product is given by the composition.

Let X' be the G-space $X_1 \times \{0\} \cup (X_1 \cap X_2) \times [0,1] \cup X_2 \times \{1\} \subset X \times [0,1]$ and let $p \colon X' \to X$ be the projection. Note that p is a homotopy equivalence. Let $\varphi_1 \colon q_{s,X}A \to q_{s,X}B$ be a $G \ltimes X$-equivariant $*$-homomorphism such that $[\varphi_1] = \xi_1$. By using a homotopy trivializing $\varphi_1|_{X_1}$, we obtain a $G \ltimes X'$-equivariant $*$-homomorphism $\varphi_1' \colon q_{s,X'}p^*A \to q_{s,X'}p^*B$ such that $[\varphi_1'] = \xi_1$ under the isomorphism $\mathrm{KK}^{G \ltimes X}(A, B) \cong \mathrm{KK}^{G \ltimes X'}(p^*A, p^*B)$ and $\varphi_1' = 0$ on $X' \cap X \times [0, 1/2]$. Similarly, we get $\varphi_2' \colon p^*q_sB \to p^*q_sD$ such that $[\varphi_2'] = \xi_2$ and $\varphi_2' = 0$ on $X' \cap X \times [1/2, 1]$. Then, $\xi_2 \circ \xi_1 = [\varphi_2' \circ \varphi_1'] = 0$. □

Proof of Theorem 4.16. By Corollary 4.14, it suffices to show the theorem for $\mathcal{F}_{\mathrm{cyc}}$. Hence we may assume that $\mathcal{F} = \mathcal{F}_{\mathrm{cyc}}$ without loss of generality. When $G = \mathbb{T}^n$, the conclusion follows from Lemma 4.17 (2) and Corollary 2.5 of [Uuy12].

For general G, let U be the Lie group as in the proof of Lemma 4.17 (3) and let T be a maximal torus of U. Consider the inclusion

$$\begin{aligned} \mathrm{KK}^G(A, B) &\cong \mathrm{KK}^{U \ltimes U/G}(\operatorname{Ind}_G^U A, \operatorname{Ind}_G^U B) \\ &\subset KK^{T \ltimes U/G}(\operatorname{Ind}_G^U A, \operatorname{Ind}_G^U B). \end{aligned}$$

Set $\tilde{\mathcal{F}}$ and $\mathcal{F}'$ the family of G and T respectively given by

$$\tilde{\mathcal{F}} := \{L \leq G \cap gHg^{-1} \mid H \in \mathcal{F}, g \in U\},$$
$$\mathcal{F}' := \{L \leq T \cap gHg^{-1} \mid H \in \mathcal{F}, g \in U\}.$$

Note that Corollary 4.14 implies that the filtration $(\mathfrak{J}_G^{\tilde{\mathcal{F}}})^*$ is equivalent to $(\mathfrak{J}_G^{\mathcal{F}})^*$ since $\mathcal{F}_{\mathrm{cyc}} = \tilde{\mathcal{F}}_{\mathrm{cyc}}$.

Consider the family of homological ideals

$$\mathfrak{J}_{T \ltimes U/G}^{\mathcal{F}'} := \{\operatorname{Ker} \operatorname{Res}_{T \ltimes U/G}^{H \ltimes U/G} \mid H \in \mathcal{F}'\}.$$

We claim that the restriction of the filtration $(\mathfrak{J}_{T \ltimes U/G}^{\mathcal{F}'})^*(\operatorname{Ind}_G^U A, \operatorname{Ind}_G^U B)$ on $\mathrm{KK}^G(A, B)$ is equivalent to $(\mathfrak{J}_G^{\mathcal{F}})^*(A, B)$.

Pick $L \in \mathcal{F}'$. The slice theorem (Theorem 2.4 of [Zun06]) implies that there is a family of closed L-subspaces $X_1, \ldots, X_n$ of U/G and $x_i \in X_i$ such that $\bigcup X_i = U/G$ and the inclusions $Lx_i \to X_i$ are L-equivariant homotopy equivalences. Now we have canonical isomorphisms

$$\begin{aligned}\mathrm{KK}^{L \ltimes X_i}(\operatorname{Ind}_G^U A|_{X_i}, \operatorname{Ind}_G^U B|_{X_i}) \xrightarrow{\operatorname{Res}_{X_i}^{Lx_i}} & \mathrm{KK}^{L \ltimes Lx_i}(\operatorname{Ind}_G^U A|_{Lx_i}, \operatorname{Ind}_G^U B|_{Lx_i}) \\ \to & \mathrm{KK}^{gLg^{-1} \cap G}(A, B)\end{aligned}$$

such that $\operatorname{Res}_G^{gLg^{-1} \cap G} = \operatorname{Res}_{U \ltimes U/G}^{L \ltimes X_i}$ under these identifications (here $g \in U$ such that $gL = x_i \in U/L$). Now, we have $gLg^{-1} \cap G \in \tilde{\mathcal{F}}$. Therefore, by Lemma 4.18, we obtain $(\mathfrak{J}_G^{\tilde{\mathcal{F}}})^k \subset \mathfrak{J}_{T \ltimes U/G}^{\mathcal{F}'}$ for some $k > 0$. Conversely since $\mathcal{F} = \mathcal{F}_{\mathrm{cyc}}$, for any $L \in \tilde{\mathcal{F}}$, we can take $g \in U$ such that $gLg^{-1} \in \mathcal{F}'$. Hence $\mathrm{KK}^G(A, B) \cap \mathfrak{J}_{T \rtimes U/G}^{\mathcal{F}'}(A, B) \subset \mathfrak{J}_G^{\tilde{\mathcal{F}}}(A, B)$.

Similarly, the filtration $(I_G^{\mathcal{F}})^* \mathrm{KK}^G(A, B)$ is equivalent to the restriction of

$$(I_T^{\mathcal{F}'})^* \mathrm{KK}^{T \ltimes U/G}(\operatorname{Ind}_G^U A, \operatorname{Ind}_G^U B).$$

Actually, by Lemma 3.4 of [AHJM88], the $I_G^{\mathcal{F}}$-adic and $I_U^{\mathcal{F}''}$-adic topologies on $\mathrm{KK}^G(A, B)$ (here $\mathcal{F}''$ is the smallest family of U containing $\mathcal{F}'$) coincide and so do the $I_U^{\mathcal{F}'}$-adic and $I_T^{\mathcal{F}'}$-adic topologies on $\mathrm{KK}^{T \ltimes U/G}(\operatorname{Ind}_G^U A, \operatorname{Ind}_G^U B)$.

Finally, the assertion is reduced to the case of $G = \mathbb{T}^n$. □

Theorem 4.16 can be regarded as a categorical counterpart of the Atiyah-Segal completion theorem. Since Theorem 4.16 holds without assuming that $\mathrm{KK}^H_*(A,B)$ are finitely generated for every $H \le G$, we also obtain a refinement of the Atiyah-Segal theorem (Corollary 2.5 of [Uuy12]).

Lemma 4.19. *Let A, B be separable σ-G-C^*-algebras such that $\mathrm{KK}^G_*(A,B)$ are finitely generated for $* = 0, 1$. Then there is a pro-isomorphism*

$$\{\mathrm{KK}^G(A,B)/(\mathfrak{J}^{\mathcal{F}}_G)^p(A,B)\}_{p\in\mathbb{Z}_{>0}} \to \{\mathrm{KK}^G(A,\tilde{B}_p)\}_{p\in\mathbb{Z}_{>0}}.$$

Proof. By Lemma 4.17 (3), there are compact G-spaces $\{X_k\}_{k\in\mathbb{Z}_{>0}}$ such that $\mathrm{KK}^G_*(A, B\otimes C(X_k))$ are finitely generated for $* = 0, 1$, each $C(X_i)$ is $(\mathfrak{J}^{\mathcal{F}}_G)^r$-injective for some $r > 0$ and for any $H \in \mathcal{F}$ there are infinitely many X_k's such that $X_k^H \neq \emptyset$. Set

$$\mathsf{N}'_p := B \otimes \bigotimes_{i=1}^{p} \mathcal{C}_{X_i},\ \mathsf{I}'_p := \mathsf{N}'_{p-1} \otimes C(X_p),\ \tilde{B}'_p := B \otimes C(\mathop{\ast}_{i=1}^{p} X_i)$$

and $\mathsf{N}'_B :=$ ho-$\varprojlim N'_p$, $\tilde{B}' :=$ ho-$\varprojlim \tilde{B}'_p$. By the same argument as Theorem 2.2, we obtain that

$$S\tilde{B} \to \mathsf{N}_B \to B \to \tilde{B}$$

is the approximation of B with respect to $(\mathcal{FC}, \langle\mathcal{FI}\rangle^{\mathrm{loc}})$. Moreover, by the six-term exact sequence, we obtain that $\mathrm{KK}^G_*(A,\tilde{B}'_p)$ are finitely generated $R(G)$-modules.

Consider the long exact sequence of projective systems

$$\{\mathrm{KK}^G_*(A, S\tilde{B}'_p)\}_p \xrightarrow{\partial_p} \{\mathrm{KK}^G_*(A,\mathsf{N}'_p)\}_p \xrightarrow{(\iota^p_0)_*} \{\mathrm{KK}^G_*(A,B)\} \xrightarrow{(\alpha^p_0)_*} \{\mathrm{KK}^G_*(A,\tilde{B}'_p)\}_p.$$

Then, $\{\mathrm{Im}(\iota^p_0)_*\}_p = \{\mathrm{Ker}(\alpha^p_0)_*\}_p$ is pro-isomorphic to $(\mathfrak{J}^{\mathcal{F}}_G)^*(A,B)$. Actually, for any $p > 0$ there is $r > 0$ such that $(\mathfrak{J}^{\mathcal{F}}_G)^r(A,B) \subset \mathrm{Ker}(\alpha^p_0)_* = \mathrm{Im}(\iota^p_0)_* \subset (\mathfrak{J}^{\mathcal{F}}_G)^p(A,B)$ since $\tilde{B}'_p$ is $(\mathfrak{J}^{\mathcal{F}}_G)^r$-injective for some $r > 0$.

Therefore, it suffices to show that the boundary map $\{\partial_p\}$ is pro-zero. Apply Theorem 4.16 and the Artin-Rees lemma for finitely generated $R(G)$-modules $M := \mathrm{KK}^G(A,\mathsf{N}'_p)$ and $L := \mathrm{Im}\,\partial_p$. Since $\tilde{B}'_p$ is $(\mathfrak{J}^{\mathcal{F}}_G)^r$-injective for some $r > 0$, there is $k > 0$ and $l > 0$ such that

$$\mathrm{Im}(\iota^{p+l}_p)_* \cap L = (\mathfrak{J}^{\mathcal{F}}_G)^l(A,\mathsf{N}'_p) \cap L \subset (I^{\mathcal{F}}_G)^k M \cap L \subset (I^{\mathcal{F}}_G)^r L = 0.$$

Consequently, for any $p > 0$ there is $l > 0$ such that $\operatorname{Im} \iota_p^{p+l} \circ \partial_{p+l} = 0$. □

Remark 4.20. It is also essential for Lemma 4.19 to assume that $\mathrm{KK}_*^G(A,B)$ are finitely generated. Actually, by Theorem 4.12, the pro-isomorphism in Lemma 4.19 implies the completion theorem when $G = \mathbb{T}^1$ and $\mathcal{F} = \mathcal{T}$. On the other hand, since the completion functor is not exact in general, there is a σ-C*-algebra A such that the completion theorem fails for $\mathrm{K}_*^G(A)$. For example, let A be the mapping cone of

$$\bigoplus_{n=0}^{\infty} \lambda^n \colon \bigoplus^{\infty} \mathbb{C} \to \bigoplus^{\infty} \mathbb{C}.$$

Then, the completion functor for the exact sequence $0 \to R(G)^\infty \to R(G)^\infty \to K_0^G(A) \to 0$ is not exact in the middle (cf. Example 8 of [Sta15, Chapter 86]).

Lemma 4.21. *Let A, B be separable σ-G-C*-algebras such that $\mathrm{KK}_*^G(A,B)$ are finitely generated for $* = 0,1$. Then, the ABC spectral sequence for $\mathrm{KK}^G(A, \text{␣})$ and B converges toward $\mathrm{KK}^G(A,B)$ with the filtration $(\mathfrak{J}_G^{\mathcal{F}})^*(A,B)$.*

Proof. According to Lemma 2.4, it suffices to show that $i \colon \mathsf{Bad}^{p+1,p+q+1} \to \mathsf{Bad}^{p,p+q+1}$ is injective. As is proved in Lemma 4.19, the boundary map ∂_p is pro-zero homomorphism and hence the projective system $\{\operatorname{Ker} \iota_0^p\} = \{\operatorname{Im} \partial_p\}$ is pro-zero. Therefore, for any $p > 0$ there is a large $q > 0$ such that

$$\operatorname{Ker} \iota_0^1 \cap (\mathfrak{J}_G^{\mathcal{F}})^\infty(A, \mathsf{N}_p) \subset \operatorname{Ker} \iota_0^p \cap (\mathfrak{J}_G^{\mathcal{F}})^q(A, \mathsf{N}_p) = \operatorname{Ker} \iota_0^p \cap \operatorname{Im} \iota_p^{p+q} = 0.$$

□

Theorem 4.22. *Let A and B be separable σ-G-C*-algebras such that $\mathrm{KK}_*^G(A,B)$ are finitely generated $R(G)$-modules ($* = 0,1$). Then, the morphisms*

- $\mathrm{KK}^G(A,B) \to \mathrm{KK}^G(A,\tilde{B})$,
- $\mathrm{KK}^G(A,B) \to \mathrm{RKK}^G(E_{\mathcal{F}}G; A, B)$,
- $\mathrm{KK}^G(A,B) \to \sigma\mathfrak{KK}^G/\mathcal{FC}(A,B)$

induce the isomorphism of graded quotients with respect to the filtration $(\mathfrak{J}_G^{\mathcal{F}})^*(A,B)$. In particular, we obtain isomorphisms

$$\mathrm{KK}^G(A,B)^\wedge_{I_G^{\mathcal{F}}} \cong \mathrm{KK}^G(A,\tilde{B}) \cong \mathrm{RKK}^G(E_{\mathcal{F}}G; A, B) \cong \sigma\mathfrak{KK}^G/\mathcal{FC}(A,B).$$

Proof. This is a direct consequence of Lemma 4.19 and Lemma 4.21. Note that Lemma 4.19 implies that the projective system $\{\mathrm{KK}^G(A, \tilde{B}_p)\}$ satisfies the Mittag-Leffler condition and hence the $\varprojlim^1$-term vanishes. □

Corollary 4.23. *Let A be a separable σ-C*-algebra and let β_t be a homotopy of continuous actions of a compact Lie group G on a σ-C*-algebra B. We write B_t for σ-G-C*-algebras (B, β_t). If $\mathrm{KK}^G_*(A, B_0)$ and $\mathrm{KK}^G_*(A, B_1)$ are finitely generated for $* = 0, 1$, there is an isomorphism*

$$\mathrm{KK}^G(A, B_0)^{\wedge}_{I_G^{\mathcal{T}}} \to \mathrm{KK}^G(A, B_1)^{\wedge}_{I_G^{\mathcal{T}}}.$$

We also weaken the assumption of Theorem 0.1 of Uuye [Uuy12], a generalization of McClure's restriction map theorem (Theorem A and Corollary C of [McC86]) for KK-theory.

Corollary 4.24. *Let G be a compact Lie group and let A and B separable G-C*-algebras. We assume that $\mathrm{KK}^G_*(A, B)$ are finitely generated for $* = 0, 1$. Then the following hold:*

1. *If $\mathrm{KK}^H(A, B) = 0$ holds for any finite cyclic subgroup H of G, then $\mathrm{KK}^G(A, B) = 0$.*
2. *If $\xi \in \mathrm{KK}^G(A, B)$ satisfies $\mathrm{Res}^H_G \xi = 0$ for any elementary finite subgroup H of G, then $\xi = 0$.*

Proof. It is proved in Theorem 0.1 of [Uuy12] under a stronger assumption that $\mathrm{KK}^H(A, B)$ are finitely generated $R(G)$-modules for any closed subgroup $H \leq G$. Applying Theorem 4.22, the same proof shows the conclusion. □

4.4 The Baum-Connes conjecture for group extensions

In this section we apply Corollary 4.14 for the study of the complementary pair $(\langle \mathcal{CI} \rangle_{\mathrm{loc}}, \mathcal{CC})$ of the Kasparov category $\sigma\mathfrak{KK}^G$ when G is a Lie group. As a consequence, we refine the theory of Chabert, Echterhoff and Oyono-Oyono [OO01, CE01b, CE01a] on permanence property of the Baum-Connes conjecture under extensions of groups.

Let G be a second countable locally compact group such that any compact subgroup of G is a Lie group. We bear the case that G is a real Lie group in mind. We write $\mathcal{C}$ and $\mathcal{CZ}$ for the family of compact and compact cyclic subgroups of G respectively.

Corollary 4.25. *We have $\mathcal{CC} = \mathcal{CZC}$ and $\langle \mathcal{CI} \rangle_{\mathrm{loc}} = \langle \mathcal{CZI} \rangle_{\mathrm{loc}}$.*

Proof. Since $\mathcal{CZ} \subset \mathcal{C}$, we have $\mathcal{CZI} \subset \mathcal{CI}$ and $\mathcal{CC} \subset \mathcal{CZC}$. Hence it suffices to show $\mathcal{CC} = \mathcal{CZC}$, which immediately follows from Corollary 4.14 (2). □

Corollary 4.26 (cf. Theorem 1.1 of [MM04])**.** *The canonical map $f \colon E_{\mathcal{CZ}}G \to E_{\mathcal{C}}G$ induces the KK^G-equivalence $f^* \colon C(E_{\mathcal{CZ}}G) \to C(E_{\mathcal{C}}G)$.*

Note that the topological K-homology group $\mathrm{K}_*^{\mathrm{top}}(G; A)$ is isomorphic to the KK-group $\mathrm{KK}^G(C(E_{\mathcal{C}}G), A)$ of σ-C*-algebras for any G-C*-algebra A.

Proof. Since f is a T-equivariant homotopy equivalence between $E_{\mathcal{C}}G$ and $E_{\mathcal{CZ}}G$ for any $T \in \mathcal{CZ}$, f^* is an equivalence in $\sigma\mathfrak{KK}^G/\mathcal{CZC}$. The conclusion follows from Corollary 4.25 because $C(E_{\mathcal{CZ}}G)$ and $C(E_{\mathcal{C}}G)$ are in $\langle \mathcal{CI} \rangle_{\mathrm{loc}} = \langle \mathcal{CZI} \rangle_{\mathrm{loc}}$. □

Next we review the Baum-Connes conjecture for extensions of groups. Let $1 \to N \to G \to G/N \to 1$ be an extension of second countable locally compact groups. We assume that any compact subgroup of G/N is a Lie group. As in Subsection 5.2 of [EM07], we say that a subgroup H of G is *N-compact* if $\pi(H)$ is compact in G/N. We write $\mathcal{C}_N$ for the family of N-compact subgroups of G. Then, we have the complementary pair $(\langle \mathcal{C}_N \mathcal{I} \rangle_{\mathrm{loc}}, \mathcal{C}_N \mathcal{C})$. It is checked as following. First, in the same way as Lemma 3.3 of [MN06], for a large compact subgroup H of G/N we have

$$\mathrm{KK}^G(\mathrm{Ind}_{\tilde{H}}^G A, B) \cong \mathrm{KK}^{\tilde{H}}(\mathrm{Res}_{\tilde{U}_H}^{\tilde{H}} \mathrm{Ind}_{\tilde{H}}^{\tilde{U}_H} A, \mathrm{Res}_G^{\tilde{H}} B)$$

where $\tilde{H} := \pi^{-1}(H)$ for any $H \leq G/N$ and U_H is as Section 3 of [MN06]. Hence $\mathrm{KK}^G(Q, M) = 0$ for any $Q \in \mathcal{C}_N \mathcal{I}$ and $M \in \mathcal{C}_N \mathcal{C}$. Let $S\mathsf{M} \to \mathsf{Q} \to \mathbb{C} \to \mathsf{M}$ be the approximation exact triangle of $\mathbb{C}$ in $\sigma\mathfrak{KK}^{G/N}$ with respect to $(\langle \mathcal{CI} \rangle_{\mathrm{loc}}, \mathcal{CC})$. Since the functor $\pi^* \colon \sigma\mathfrak{KK}^{G/N} \to \sigma\mathfrak{KK}^G$ maps $\mathcal{CI}$ to $\mathcal{C}_N\mathcal{I}$ and $\mathcal{CC}$ to $\mathcal{C}_N\mathcal{C}$ respectively, $S\pi^*\mathsf{M} \to \pi^*\mathsf{Q} \to \mathbb{C} \to \pi^*\mathsf{M}$ gives the approximation of $\mathbb{C}$ in $\sigma\mathfrak{KK}^G$ with respect to $(\langle \mathcal{C}_N\mathcal{I}\rangle_{\mathrm{loc}}, \mathcal{C}_N\mathcal{C})$. Hereafter, for simplicity of notations we omit π^* for σ-(G/N)-C*-algebras which are regarded as σ-G-C*-algebras.

Since $\mathcal{CI} \subset \mathcal{C}_N\mathcal{I}$ and $\mathcal{C}_N\mathcal{C} \subset \mathcal{CC}$, we obtain the diagram of semi-orthogonal decompositions

$$\begin{array}{ccccc} \langle \mathcal{CI}\rangle_{\mathrm{loc}} & = & \langle \mathcal{CI}\rangle_{\mathrm{loc}} & \to & 0 \\ \downarrow & & \downarrow & & \downarrow \\ \langle \mathcal{C}_N\mathcal{I}\rangle_{\mathrm{loc}} & \to & \mathfrak{KK}^G & \to & \mathcal{C}_N\mathcal{C} \\ \downarrow & & \downarrow & & \| \\ \langle \mathcal{C}_N\mathcal{I}\rangle_{\mathrm{loc}} \cap \mathcal{CC} & \to & \mathcal{CC} & \to & \mathcal{C}_N\mathcal{C}, \end{array} \qquad \begin{array}{ccccc} \mathsf{P} & = & \mathsf{P} & \to & 0 \\ \downarrow{\scriptstyle \mathsf{D}_G^{G/N}} & & \downarrow{\scriptstyle \mathsf{D}_G} & & \downarrow \\ \mathsf{Q} & \xrightarrow{\mathsf{D}_{G/N}} & \mathbb{C} & \to & \mathsf{M} \\ \downarrow & & \downarrow & & \| \\ \mathsf{Q}\otimes\mathsf{N} & \to & \mathsf{N} & \to & \mathsf{M}. \end{array} \tag{4.27}$$

For a σ-G-C*-algebra A, the (full or reduced) crossed product $N \ltimes A$ is a twisted σ-G/N-C*-algebra (Definition 2.1 of [PR89]). By the Packer-Raeburn stabilization trick (Theorem 1 of [Ech94]), it is Morita equivalent to the untwisted G/N-C*-algebra

$$N \ltimes^{\mathrm{PR}} A := C_0(G/N, N \ltimes A) \rtimes_{\tilde{\alpha},\tilde{\tau}} (G/N)$$

where $\tilde{\alpha}$ and $\tilde{\tau}$ are induced from the canonical G-action on $C_0(G/N, N \ltimes A)$. The *Packer-Raeburn crossed product* $N \ltimes^{\mathrm{PR}} \llcorner\!\lrcorner$ is a functor from G-$\mathfrak{C}^*\mathfrak{sep}$ to G/N-$\mathfrak{C}^*\mathfrak{sep}$, which induces the *partial descent functor* (Section 4 of [CE01b])

$$j_G^{G/N} : \sigma\mathfrak{KK}^G \to \sigma\mathfrak{KK}^{G/N}$$

by universality of $\sigma\mathfrak{KK}^G$ (Theorem 3.15).

Lemma 4.28. *The functor $j_G^{G/N}$ maps $\langle \mathcal{C}_N\mathcal{I}\rangle_{\mathrm{loc}}$ to $\langle \mathcal{CI}\rangle_{\mathrm{loc}}$ and $\mathcal{C}_N\mathcal{C}$ to $\mathcal{CC}$.*

Proof. Let H be a N-compact subgroup of G and let A be a σ-H-C*-algebra. Then, $N \ltimes^{\mathrm{PR}} \mathrm{Ind}_H^G A$ admits a canonical σ-$G/N \ltimes ((G/N \times H\backslash G)/G)$-C*-algebra structure. Since the G/N-action on $(G/N \times H\backslash G)/G$ is proper, $N \ltimes^{\mathrm{PR}} \mathrm{Ind}_H^G A$ is in $\langle \mathcal{CI}\rangle_{\mathrm{loc}}$. Consequently we obtain $j_G^Q(\langle \mathcal{C}_N\mathcal{I}\rangle_{\mathrm{loc}}) \subset \langle \mathcal{CI}\rangle_{\mathrm{loc}}$.

Let A be a $\mathcal{C}_N$-contractible σ-C*-algebra. Then, for any compact subgroup H of G/N, $\mathrm{Res}_{G/N}^H(N \ltimes^{\mathrm{PR}} A) = N \ltimes \mathrm{Res}_G^{\pi^{-1}(K)} A$ is KK^H-contractible. Hence we obtain $j_G^{G/N}(\mathcal{C}_N\mathcal{C}) \subset \mathcal{CC}$. $\square$

Consider the partial assembly map

$$\mu_{G,A}^{G/N}\colon \mathrm{K}_*^{\mathrm{top}}(G;A) \to \mathrm{K}_*^{\mathrm{top}}(G/N; N \ltimes A)$$

constructed in Definition 5.14 of [CE01a]. Then, in the same way as Theorem 5.2 of [MN04], we have the commutative diagram

$$\begin{array}{ccccc}
\mathrm{K}_*^{\mathrm{top}}(G;\mathsf{P}\otimes A) & \xrightarrow{\cong} & \mathrm{K}_*^{\mathrm{top}}(G;\mathsf{Q}\otimes A) & \xrightarrow{\cong} & \mathrm{K}_*^{\mathrm{top}}(G;A) \\
\downarrow{\cong} & & \downarrow & & \downarrow{\mu_{G,A}^{G/N}} \\
\mathrm{K}_*^{\mathrm{top}}(G/N; N \ltimes^{\mathrm{PR}} (\mathsf{P}\otimes A)) & \longrightarrow & \mathrm{K}_*^{\mathrm{top}}(Q; N \ltimes^{\mathrm{PR}} (\mathsf{Q}\otimes A)) & \xrightarrow{\cong} & \mathrm{K}_*^{\mathrm{top}}(G/N; N \ltimes^{\mathrm{PR}} A) \\
\downarrow{\cong} & & \downarrow{\cong} & & \downarrow{\mu_{G/N,N\ltimes^{\mathrm{PR}}A}} \\
\mathrm{K}_*(G \ltimes (\mathsf{P}\otimes A)) & \xrightarrow{j_G(\mathsf{D}_G^{G/N})} & \mathrm{K}_*(G \ltimes (\mathsf{Q}\otimes A)) & \xrightarrow{j_G(\mathsf{D}_{G/N})} & \mathrm{K}_*(G \ltimes A)
\end{array}$$

and hence the composition of partial assembly maps

$$\mu_{G,A} = \mu_{G/N,N\ltimes^{\mathrm{PR}}A} \circ \mu_{G,A}^{G/N}\colon \mathrm{K}_*^{\mathrm{top}}(G;A) \to \mathrm{K}_*^{\mathrm{top}}(G/N; N \ltimes^{\mathrm{PR}} A) \to \mathrm{K}_*(G \ltimes A)$$

is isomorphic to the canonical map $\mathrm{K}_*(G \ltimes (\mathsf{P}\otimes A)) \to \mathrm{K}_*(G \ltimes (\mathsf{Q}\otimes A)) \to \mathrm{K}_*(G \ltimes A)$. In other words, the partial assembly map $\mu_{G,A}^{G/N}$ is isomorphic to the assembly map $\mu_{G,\mathsf{Q}\otimes A}$ for $\mathsf{Q}\otimes A$.

We say that a separable σ-G-C*-algebra A satisfies the (resp. strong) Baum-Connes conjecture (BCC) if $j_G(\mathsf{D}_G)$ induces the isomorphism of K-groups (resp. the KK-equivalence).

Theorem 4.29. *Let $1 \to N \to G \to G/N \to 1$ be an extension of second countable groups such that all compact subgroups of G/N are Lie groups and let A be a separable σ-G-C*-algebra. Then the following holds.*

1. *If $\pi^{-1}(H)$ satisfies the (resp. strong) BCC for A for any $H \in \mathcal{CZ}$, then G satisfies the (resp. strong) BCC for A if and only if G/N satisfies the (resp. strong) BCC for $N \ltimes_r^{\mathrm{PR}} A$.*
2. *If $\pi^{-1}(H)$ for any $H \in \mathcal{CZ}$ and G/N have the γ-element, then so does G. Moreover, in that case $\gamma_{\pi^{-1}(H)} = 1$ for any $H \in \mathcal{CZ}$ and $\gamma_{G/N} = 1$ if and only if $\gamma_G = 1$.*

Proof. To see (1), it suffices to show that G satisfies the (resp. strong) BCC for $\mathsf{Q}\otimes A$. Consider the full subcategory $\mathfrak{N}$ of $\sigma\mathfrak{KK}^G$ consisting of objects D such that G satisfies the (resp./ strong) BCC for $D \otimes A$. Set $\mathcal{CZI}_1$ be the family of all G-C*-algebras of the form $C_0((G/N)/H)$ for $H \in \mathcal{CZ}$. By assumption, $\mathfrak{N}$ contains $\pi^*\mathcal{CZI}_1$. Since $\mathfrak{N}$ is localizing and colocalizing, $\mathfrak{N}$ contains $\pi^* \langle \mathcal{CZI}_1 \rangle_{\mathrm{loc}}^{\mathrm{loc}}$, which is equal to $\pi^* \langle \mathcal{CI}_1 \rangle_{\mathrm{loc}}^{\mathrm{loc}}$ because $C_0(G/N)/H)$ are KK^G-equivalent to $C_0((G/N)/H) \otimes C(E_{\mathcal{CZ}} H) \in \pi^* \langle \mathcal{CZI}_1 \rangle^{\mathrm{loc}}$. By Proposition 9.2 of [MN06], we obtain $\mathsf{Q} \in \mathfrak{N}$.

The assertion (2) is proved in the same way as Theorem 33 of [EM07]. Actually, since we may assume without loss of generality that G/N is totally disconnected by Corollary 34 of [EM07], the homomorphism

$$\mathsf{D}_G^* \colon \mathrm{KK}^G(A, \mathsf{P}) \to \mathrm{KK}^G(\mathsf{P} \otimes A, \mathsf{P})$$

is an isomorphism if $A \in \pi^* \langle \mathcal{CZI} \rangle_{\mathrm{loc}}$ and in particular when $A = \mathsf{Q}$ (note that any compact subgroup is contained in an open compact subgroup which is also a Lie group by assumption). Consequently we obtain a left inverse $\eta_G^{G/N} \colon \mathsf{Q} \to \mathsf{P}$ of $\mathsf{D}_G^{G/N}$. Now, the composition $\eta_G^{G/N} \circ \pi^* \eta_{G/N} \colon \mathbb{C} \to \mathsf{P}$ is a dual Dirac morphism of G. Of course $\eta_G \circ \mathsf{D}_G = \mathrm{id}_{\mathbb{C}}$ if $\eta_G^{G/N} \circ \mathsf{D}_G^{G/N} = \mathrm{id}_{\mathsf{Q}}$ and $\eta_{G/N} \circ \mathsf{D}_{G/N} = \mathrm{id}_{\mathbb{C}}$. $\square$

Part II

Continuous Rokhlin property for compact group actions

Chapter 5

Continuous Rokhlin property

5.1 Definition and Examples

Let G be a second countable compact group and let (A, α) be a unital G-C*-algebra. Set

$$\mathfrak{T}_\alpha A := \{a = (a_t) \in C_b([0, \infty), A) : g \mapsto \alpha_g(a) \text{ is norm continuous}\},$$
$$\mathfrak{A}_\alpha A := \mathfrak{T}_\alpha A / C_0([0, \infty), A), \ \mathfrak{C}_\alpha A := \mathfrak{A}_\alpha A \cap A'.$$

They are equipped with the canonical G-C*-algebra structure. We say the (non-separable) C*-algebra $\mathfrak{C}_\alpha A$ is the *central path algebra* of (A, α).

Definition 5.1. Let G be a compact group, let X be a compact G-space and let (A, α) be a unital G-C*-algebra.

1. We say that (A, α) has the *continuous Rokhlin property* if there is a G-equivariant unital $*$-homomorphism $\varphi\colon C(G) \to \mathfrak{C}_\alpha A$.

2. We say that (A, α) has the *continuous X-Rokhlin property* if there is a G-equivariant unital $*$-homomorphism $\varphi\colon C(X) \to \mathfrak{C}_\alpha A$.

3. We say that (A, α) has *continuous Rokhlin dimension with commuting towers* at most d, and write as $\dim_{\mathrm{cRok}}^c(A, \alpha) \leq d$, if there are G-equivariant completely positive contractive order zero maps

$$\varphi^{(0)}, \dots, \varphi^{(d)}\colon C(G) \to \mathfrak{C}_\alpha A$$

with commuting ranges such that $\varphi^{(0)}(1) + \cdots + \varphi^{(d)}(1) = 1$.

These are variations of the Rokhlin property (Definition 3.1 of [Izu04a]) and the Rokhlin dimension (Definition 2.3 of [HWZ15]), which are defined by using $\mathbb{Z}_{>0}$ instead of $[0, \infty)$.

Remark 5.2. Here we claim that a unital G-C*-algebra (A, α) has continuous Rokhlin dimension with commuting towers at most $d-1$ if and only if it has the continuous E_dG-Rokhlin property (cf. Lemma 1.7 of [HP15] and Lemma 4.4 of [Gar14e]). To see this, recall that a completely positive contractive map of order zero from A to B is of the form $\varphi(ta)$, where $t \in C_0(0,1]$ is the identity function and φ is a $*$-homomorphism from the cone $\mathcal{C}A := C_0(0,1] \otimes A$ to B (Corollary 4.1 of [WZ09]). We remark that $\varphi(ta)$ is G-equivariant if and only if so is φ. Hence we get a unital G-equivariant $*$-homomorphism

$$\mathcal{C}C(G)^+ \otimes \cdots \otimes \mathcal{C}C(G)^+ \to \mathfrak{C}_\alpha A.$$

On the other hand, by the same argument as the proof of Lemma 4.7, we get an isomorphism

$$\mathcal{C}C(X * Y)^+ \cong \mathcal{C}(X)^+ \otimes \mathcal{C}C(Y)^+$$

for any G-spaces X and Y. In particular, we get $\mathcal{C}C(G)^+ \otimes \dots \mathcal{C}C(G)^* \cong \mathcal{C}C(E_dG)^+$. Now, the relation $\varphi^{(1)}(1) + \cdots + \varphi^{(n)}(1) = 1$ implies that the $*$-homomorphism $\mathcal{C}C(E_dG)^+ \to \mathfrak{C}_\alpha A$ factors through the restriction $\mathcal{C}C(E_dG)^+ \to C(E_dG)$.

Conversely, if (A, α) has the X-Rokhlin property for some compact free G-space X which is proper in the sense of Palais [Pal61] (in other words $X \to X/G$ is a principal G-bundle), then A has finite continuous Rokhlin dimension since there is a continuous G-map $X \to E_dG$ for some d by universality of EG and compactness of X.

Proposition 5.3. *For a unital G-C*-algebra (A, α), we have*

$$\dim_{\mathrm{Rok}}^c(A, \alpha) \le \dim_{\mathrm{cRok}}^c(A, \alpha) \le 2 \dim_{\mathrm{Rok}}^c(A, \alpha) + 1.$$

In particular, we have $\dim_{\mathrm{cRok}}^c(A, \alpha) \le 1$ if (A, α) has the Rokhlin property.

Proof. We write $F_\alpha(A) := \ell_\alpha^\infty(\mathbb{N}, A)/c_0(\mathbb{N}, A) \cap A'$ as in [Gar14c]. Let $\varphi^{(i)} : C(G) \to F_\alpha(A)$ $(i = 0, \dots, d)$ be completely positive contractive maps of order zero such that $\varphi^{(0)}(1) + \cdots + \varphi^{(d)}(1) = 1$ and choose completely positive contractive lifts $(\varphi_n^{(i)})_{n \in \mathbb{N}}$ by applying the Choi-Effros lifting theorem [CE76]. Let F_n be an increasing sequence of finite subsets of $C(G)$ such that $\overline{\bigcup F_n} = C(G)$ and choose φ_n inductively such that $\|[\varphi_n^{(i)}(f), \varphi_m^{(i)}(f)]\| \leq 2^{-n-m}$ for any $m \geq n$, $f \in F_n$ and $i = 0, \ldots, d$.

Then, we obtain $(2d+2)$ G-equivariant completely positive contractive maps of order zero from $C(G)$ to $\mathfrak{C}_\alpha A$ of order zero given by

$$\tilde{\varphi}^{(i.j)}_{2n+j+t}(f) = (1-|t|)\varphi_n^{(i)}(f) \quad \text{for } t \in [-1,1]$$

for $i = 0, \ldots, d$ and $j = 0, 1$. By definition, the images in $\mathfrak{C}_\alpha A$ commute and $\sum \tilde{\varphi}_t^{(i,j)}(1) = 1$. □

Remark 5.4. The continuous Rokhlin property is actually strictly stronger than the Rokhlin property. See Subsection 6.4 for more details.

Example 5.5. The following examples are pointed out to us by Eusebio Gardella. Let G be a finite group. The UHF algebra $\mathbb{M}_{|G|^\infty}$ with the G-action $\alpha_g := \bigotimes^\infty \mathrm{Ad}\,\lambda_g$ has the continuous Rokhlin property. Actually, we obtain an asymptotically central path of mutually orthogonal projections $\{p_t^g\}_{g \in G}$ satisfying $\alpha_h(p_t^g) = p_t^{hg}$ given by

$$p_t^g := 1 \otimes \cdots \otimes 1 \otimes u_{t-n}(p_g \otimes 1)u_{t-n}^* \otimes 1 \otimes \cdots$$

for $t \in [n, n+1]$. Here $p_g \in \mathbb{M}_{|G|} \cong \mathbb{B}(\ell^2 G)$ is the projection onto $\mathbb{C}\delta_g$ for $g \in G$ and u_t is a homotopy of G-invariant unitaries in $\mathbb{M}_{|G|}^{\otimes 2}$ such that $u_0 = 1$ and u_1 is the flip on $\mathbb{C}^{|G|} \otimes \mathbb{C}^{|G|}$.

Since $\mathcal{O}_n \otimes \mathbb{M}_{n^\infty} \cong \mathcal{O}_n$, we obtain an example of a continuous Rokhlin action on the Cuntz algebras $\mathcal{O}_{|G|}$. On the other hand, since any automorphism on $\mathcal{O}_{|G|}$ is approximately inner (Theorem 3.6 of [Rør93]), Theorem 3.5 of [Izu04a] implies that every Rokhlin action on $\mathcal{O}_{|G|}$ is conjugate to the above action and hence has the continuous Rokhlin property. In particular, the quasi-free action with respect to the left regular representation has the continuous Rokhlin property (Proposition 5.6 of [Izu04a]). Similarly, by the above example and Kirchberg's absorption theorem (Theorem 3.2 of [KP00]), the unique Rokhlin action of G on $\mathcal{O}_2$ (Theorem 4.2 of [Izu04a]) has the continuous Rokhlin property.

Example 5.6. Fix $\theta, \omega \in \mathbb{R} \setminus \mathbb{Q}$ such that $\theta - \omega \in \mathbb{R} \setminus \mathbb{Q}$. Let us consider the $\mathbb{Z}$-actions α_θ and α_ω on $C(\mathbb{T})$ induced by the rotation by $e^{2\pi i \theta}$ and $e^{2\pi i \omega}$ respectively.

Let

$$B := \mathbb{Z}_{\alpha_\omega} \ltimes C(\mathbb{T}),\ A := \mathbb{Z}_{\alpha_\theta} \ltimes B = \mathbb{Z}_{\alpha_\theta} \ltimes (\mathbb{Z}_{\alpha_\omega} \ltimes C(\mathbb{T}))$$

be noncommutative tori (here we use the same letter α_θ for the automorphism $\mathbb{Z} \ltimes \alpha_\theta$ on B). Then, the automorphism α_θ on B is approximately representable. Actually, by choosing a subsequence $\{n_k\}$ of $\mathbb{Z}_{>0}$ such that $n_k\omega \to \theta$, we get a sequence of unitaries (u^{n_k}) in B approximating α_θ where u denotes the canonical unitary implementing α_ω in the crossed product B. Hence the dual action $\hat{\alpha}_\theta$ of $\mathbb{T}$ on A has the Rokhlin property (Proposition 3.6 of [Gar14a]).

It will be seen in Remark 6.6 and Example 6.14 that $(A, \hat{\alpha})$ does not have the contionuous Rokhlin property although $(A \otimes \mathcal{O}_\infty, \hat{\alpha} \otimes \mathrm{id}_\infty)$ has the contionuous Rokhlin property.

5.2 Averaging technique via equivariant KK-theory

A fundamental technique for C*-dynamical systems with the (continuous) Rokhlin property is an averaging process.

Let G be a second countable compact group, let X be a compact G-space. Let A be a separable unital G-C*-algebra with the continuous X-Rokhlin property. Take a completely positive contractive lift of φ using the Choi-Effros lifting theorem, which is a path of completely positive contractive maps $\varphi_t \colon C(X) \to A$.

Let us choose an increasing sequence F_n of finite subsets of $A \otimes C(X) \cong C(X, A)$ such that $\overline{\bigcup F_n} = A \otimes C(X)$ and a sequence of open coverings $\{U_{n,i}\}_{i \in I_n}$ such that

$$\|a(x) - a(y)\| < 2^{-n} \text{ for all } a \in F_n \text{ and } x, y \in U_{n,i}.$$

Let us choose points $x_{n,i} \in U_{x,i}$ and partition of unities $(\{f_{n,i}\}_{i \in I_n})_n$ associated to $\{U_{n,i}\}_{i \in I_n}$. Then, for any $a \in F_n$, we have

$$\left\| a(x) - \sum\nolimits_{i \in I_n} a(x_{n,i}) f_{n,i}(x) \right\| < \sum\nolimits_{i \in I_n} f_{n,i}(x) \, \|a(x) - a(x_{n,i})\| < 2^{-n}$$

uniformly on $x \in X$.

Now, we construct a path of completely positive maps $\psi_t \colon A \otimes C(X) \to A$ parametrized by $t \in [0, \infty)$ as

$$\begin{aligned}\psi_t(a) :=&(t-n) \sum_{i \in I_n} \varphi_{\chi(t)}(f_{n,i})^{1/2} a(x_{n,i}) \varphi_{\chi(t)}(f_{n,i})^{1/2} \\ &+ (n+1-t) \sum_{i \in I_{n+1}} \varphi_{\chi(t)}(f_{n+1,i})^{1/2} a(x_{n+1,i}) \varphi_{\chi(t)}(f_{n+1,i})^{1/2}\end{aligned} \tag{5.7}$$

for $t \in [n, n+1]$, where $\chi \colon \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is a homeomorphism such that

$$\| [\varphi_t(f_n), a(x_{n,i})] \| < 2^{-n} |I_n|^{-1} \text{ for any } t \geq \chi(n) \text{ and } a \in F_n.$$

Since $\pi(\psi_t(a))$, where $\pi \colon \mathfrak{T}_\alpha A \to \mathfrak{A}_\alpha A$ is the quotient map, coincides with $(\mathrm{id}_A \otimes \varphi_\chi)(a)$ in $\mathfrak{A}_\alpha A$ for any $a \in A \otimes C(X)$, this ψ determines a G-equivariant completely positive asymptotic morphism in the sense of [Tho99].

This averaging map is compatible with the picture of KK-theory given in [HLT99, Tho99] using completely positive asymptotic morphisms. Actually, we have the isomorphism

$$\mathrm{KK}^G(A, B) \cong [\![SA \otimes \mathbb{K}_G, SB \otimes \mathbb{K}_G]\!]^G_{\mathrm{cp}}, \tag{5.8}$$

where $[\![A, B]\!]^G_{\mathrm{cp}}$ is the set of homotopy classes of completely positive G-equivariant asymptotic morphisms from A to B. Moreover the Kasparov product is given by the composition of asymptotic morphisms. Hence ψ gives an element $\mathrm{KK}^G(A \otimes C(X), A)$. Let $\iota_X \colon \mathbb{C} \to C(X)$ denote the inclusion (we simply write it as ι when $X = G$) and $\iota_{X,A} := \mathrm{id}_A \otimes \iota_X$. Then, the KK^G-cycle represented by ψ is a left inverse of $[\iota_{X,A}]$.

As an immediate consequence of the above construction, Lemma 4.7 and Remark 5.2, we get the following theorem.

Theorem 5.9. *Let G be a second countable compact group and let A be a separable unital G-C^*-algebra. If A has the continuous Rokhlin property, then A is $\mathfrak{I}_G$-injective. Moreover, if A has continuous Rokhlin dimension with commuting towers at most $d-1$, then A is $\mathfrak{I}_G^d$-injective.*

Corollary 5.10. *Let A and B be separable unital G-C^*-algebras with finite Rokhlin dimension with commuting towers. For $\phi \in \mathrm{KK}^G(A, B)$, ϕ is a KK^G-equivalence if and only if $\mathrm{Res}_G\, \phi$ is a KK-equivalence. Moreover, if A and B are in the UCT* *class, ϕ is a* KK^G*-equivalence if and only if* $(\mathrm{Res}_G\,\phi)_*\colon \mathrm{K}_*(A) \to \mathrm{K}_*(B)$ *is an isomorphism.*

Proof. It follows from Theorem 5.9, Proposition 5.3, Theorem 4.4 and the universal coefficient theorem (Proposition 7.3 of [RS87]). □

Lemma 5.11. *Let (A, α) be a separable unital G-*C^**-algebra with the Rokhlin property and let $p \in C^*_\lambda(G) = G \ltimes \mathbb{C} \subset G \ltimes A$ be the projection corresponding to the trivial representation. Then, p is a full projection in $G \ltimes A$ such that the map*

$$j\colon A^\alpha \to p(G \ltimes A)p,\ j(a) := pa = pap$$

is an isomorphism. In particular, $G \ltimes A$ is Morita equivalent to A^α and j induces the KK*-equivalence.*

Proof. It is proved in [Ros79] that j is isomorphic for arbitrary action α. We remark that p corresponds to the constant function $1 \in L^1(G, A, \alpha) \subset G \ltimes A$. According to Proposition 2.6 of [Gar14c], α is saturated in the sense of Definition 7.1.4 of [Phi87]. Now, fullness of p is shown in the proof of Proposition 7.1.8 of [Phi87]. □

Corollary 5.12. *Let A be a* C^**-algebra and let $\{\alpha_t\}_{t\in[0,1]}$ be a homotopy of G-actions on A such that α_0 and α_1 have finite Rokhlin dimension with commuting towers. Then (A, α_0) and (A, α_1) are* KK^G*-equivalent. Moreover, if α_0 and α_1 have the Rokhlin property, the fixed point subalgebras A^{α_0} and A^{α_1} are* KK*-equivalent.*

Proof. It follows from Theorem 5.9, Proposition 5.3, Proposition 5.11 and Corollary 4.5. □

Chapter 6

Classification results

In this chapter, we focus on G-C*-algebras with the continuous Rokhlin property and present the results on the classification. Our main result is a complete classification of unital Kirchberg G-algebras (i.e. unital Kirchberg algebras with G-actions) with the continuous Rokhlin property up to G-equivariant isomorphism in terms of (equivariant) KK-theory.

6.1 Equivariant KK classification

In this section, we focus on the case that G is a compact Lie group with the Hodgkin condition, that is, G is connected and $\pi_1(G)$ is torsion-free. An important feature of Hodgkin Lie groups is that the dual quantum group $\hat{G}$ is torsion-free in the sense of Section 7.2 of [Mey08]. Together with the strong Baum-Connes conjecture for $\hat{G}$ (proved in Corollary 3.4 of [MN07] and rephrased in terms of crossed products in Section 7.2 of [Mey08]), we obtain the following.

Proposition 6.1. *Let G be a Hodgkin Lie group. A KK^G-morphism $\xi \in \mathrm{KK}^G(A, B)$ is a KK^G-equivalence if and only if $G \ltimes \xi \in KK(G \ltimes A, G \ltimes B)$ is a* KK*-equivalence.*

Remark 6.2. A typical use of Proposition 6.1 is the KK-equivalence of noncommutative tori. More strongly, here we show that $A_t := \mathbb{Z}_{\alpha_t} \ltimes C(\mathbb{T})$ (where α_t is as in Example 5.6) with the dual $\mathbb{T}$-action are $\mathrm{KK}^{\mathbb{T}}$-equivalent.

Let $\tilde{\alpha}$ be the automorphism on $C([0,1], C(\mathbb{T}))$ given by $\tilde{\alpha}(x)(t) := \alpha_t(x(t))$ for $x \in C([0,1], C(\mathbb{T}))$ and $t \in [0,1]$. Set $\tilde{A} := \mathbb{Z}_{\tilde{\alpha}} \ltimes (C([0,1], C(\mathbb{T})))$. Then, the equivariant $*$-homomorphisms $\mathrm{ev}_t \colon \tilde{A} \to A_t$ induce $\mathrm{KK}^{\mathbb{T}}$-equivalences since $\mathbb{T} \ltimes (\mathrm{ev}_t)$ are homotopy equivalences by the Takesaki-Takai duality. We remark that the $\mathrm{KK}^{\mathbb{T}}$-equivalence constructed here preserves the unit classes in the K_0-groups.

Theorem 6.3. *Let G be a compact Lie group with the Hodgkin condition. Then, for any separable unital G-C^*-algebra (A, α) with the continuous Rokhlin property, there is a KK^G-equivalence from A to $A^\alpha \otimes C(G)$ mapping $[1_A] \in \mathrm{K}_0^G(A)$ to $[1_{A^\alpha \otimes C(G)}]$.*

Proof. Let ψ be as in (5.7). By Proposition 6.1, it suffices to show that $G \ltimes \xi$ is a KK-equivalence for

$$\xi := [\psi|_{A^\alpha \otimes C(G)}] \in \mathrm{KK}^G(A^\alpha \otimes C(G), A).$$

Let $i_1 \colon A^\alpha \otimes \mathbb{C} \to A^\alpha \otimes C(G)$ and $i_2 \colon A^\alpha \to A$ be canonical inclusions. Since $\psi \circ i_1$ is asymptotically equivalent to i_2, we obtain

$$(G \ltimes \xi) \circ (G \ltimes [i_1]) = G \ltimes [i_2] \in \mathrm{KK}(C^*_\lambda(G) \otimes A^\alpha, G \ltimes A). \tag{6.4}$$

Let p be the projection in $C^*_\lambda(G)$ corresponding to the trivial representation and let $j \colon A^\alpha \to C^*_\lambda(G) \otimes A^\alpha$ be the inclusion given by $j(a) := p \otimes a$. Then, by Lemma 5.11, both $(G \ltimes i_1) \circ j$ and $(G \ltimes i_2) \circ j$ induce KK-equivalences. Now we obtain the conclusion by composing $[j]$ with the left and right hand sides of (6.4) from the right. □

Remark 6.5. In fact, KK^G-equivalence of A with $B \otimes C(G)$ for some B holds under weaker assumptions for the action. As in [Yam11], we can apply the Baum-Connes conjecture for duals of Hodgkin Lie groups to the path of cocycle actions using the duality [VV03] and we obtain that any G-C^*-algebra is KK^G-equivalent to $B \otimes C(G)$ for some B if its dual action is asymptotically inner.

Remark 6.6. It is remarkable that the isomorphism

$$\xi_* \colon \mathrm{K}_0(A^\alpha \otimes C(G)) \to \mathrm{K}_0(A)$$

is actually an ordered map. It is because an asymptotic morphism maps projection to projection. This fact gives a stronger obstruction for a G-C^*-algebra to have the continuous Rokhlin property.

For example, let (A, α) be as in 5.6. Let τ be the unique trace on B whose image is $\mathbb{Z} + \omega\mathbb{Z}$. If A has the continuous Rokhlin property, the composition

$$\mathrm{K}_0(A) \xrightarrow{\xi_*} \mathrm{K}_0(B \otimes C(\mathbb{T})) \xrightarrow{\mathrm{ev}_*} \mathrm{K}_0(B) \xrightarrow{\tau_*} \mathbb{Z} + \omega\mathbb{Z} \subset \mathbb{R}$$

is an ordered map, where $\mathrm{ev}\colon B \otimes C(\mathbb{T}) \to B$ is the evaluation at a point in $\mathbb{T}$. It contradicts with the fact that there is a unique ordered map from $(\mathrm{K}_0(A), \mathrm{K}_0(A)_+)$ to $(\mathbb{R}, \mathbb{R}_{\geq 0})$ and its image is $\mathbb{Z} + \theta\mathbb{Z} + \omega\mathbb{Z}$.

The following corollary is an analogue of Theorem 5.5 of [Gar14a] for $\mathbb{T}$-C*-dynamical systems.

Corollary 6.7. *Let G be a Hodgkin Lie group and let (A, α) be a separable unital G-C*-algebra with the continuous Rokhlin property. Set $n := 2^{\mathrm{rank}\, G - 1}$, where $\mathrm{rank}\, G$ is the dimension of the maximal torus of G. Then, there is a countable abelian group M such that $\mathrm{K}_i(A)$ is isomorphic to M^n for $i = 0, 1$. Moreover, in this case $M \cong \mathrm{K}_0(A^\alpha) \oplus \mathrm{K}_1(A^\alpha)$.*

Proof. It follows from Theorem 6.3. Note that $C(G)$ is KK-equivalent to $\mathbb{C}^k \oplus C_0(\mathbb{R})^l$ for some $k, l \in \mathbb{Z}_{>0}$ by the universal coefficient theorem [RS87] because $\mathrm{K}^*(G)$ is torsion-free (Theorem A (i) of [Hod67]). By taking tensor product with $\mathbb{Q}$ and using the Chern character isomorphism, we conclude that $k = l = n$ (Hopf's theorem, see for example Theorem 1.34 of [FOT08]). □

Corollary 6.8. *Let G be a Hodgkin Lie group. Two separable unital G-C*-algebras with the continuous Rokhlin property are KK^G-equivalent if and only if their fixed point algebras are KK-equivalent. In particular, when these C*-algebras are in the UCT-class, then they are KK^G-equivalent if and only if the K_*-groups of fixed point algebras are isomorphic.*

Proof. It follows from Theorem 6.3 and Proposition 5.11. Note that for a separable unital G-C*-algebra (A, α) with the continuous Rokhlin property, A is in the UCT class if and only if so is A^α because $C(G)$ is KK-equivalent to $\mathbb{C}^n \oplus C_0(\mathbb{R})^n$. □

6.2 Model actions

Theorem 6.9. *Let X be a G-space. Assume that for any $x, y \in X$ there is a G-equivariant continuous map $F\colon X \to X$ with $F(x) = y$. Then, there is a Kirchberg*

G-algebra $\mathcal{O}(X)$ with the X-Rokhlin property which is KK^G*-equivalent to $C(X)$. Moreover, if such F can be taken to be homotopic to the identity, $\mathcal{O}(X)$ has the continuous Rokhlin property.*

Proof. The construction is in the same way as Lemma 5.2 of [Izu04b].

Let us fix a dense subset $\{x_1, x_2, \dots\}$ of X and a sequence of continuous maps $\{F_n \colon X \to X\}_{n\in\mathbb{N}}$ such that $\{F_n(x_m) \mid n \in \mathbb{N}\}$ is dense in X. For each $n > 0$, choose a family of mutually orthogonal projections $\{p_{n,i}\}_{i=0}^n$ in $\mathcal{O}_\infty$ with $\sum_{i=0}^n p_{n,i} = 1$ such that

$$[p_{n,0}] = [1], [p_{n,i}] = 0 \in K_0(\mathcal{O}_\infty).$$

Let us consider the following G-equivariant $*$-homomorphism

$$\iota_n \colon C(X, \mathcal{O}_\infty^{\otimes n}) \to C(X, \mathcal{O}_\infty^{\otimes n}) \otimes \mathcal{O}_\infty \cong C(X, \mathcal{O}_\infty^{\otimes(n+1)}),$$
$$\varphi_n(f) := f \otimes p_{n,0} + \sum_{i=1}^n F_i^*(f) \otimes p_{n,i}$$

and set $\mathcal{O}(X) := \varinjlim(C(X, \mathcal{O}_\infty^{\otimes n}), \iota_n)$. Obviously the compositions

$$\varphi_n \colon C(X) \hookrightarrow C(X, \mathcal{O}_\infty^{\otimes n}) \hookrightarrow \mathcal{O}(X)$$

form a X-Rokhlin map. Therefore, a non-equivariant KK-equivalence $\varphi_1 \colon C(X) \to \mathcal{O}(X)$ is actually an equivariant KK-equivalence by Corollary 5.10.

To see that $\mathcal{O}(X)$ is simple and purely infinite, fix a nonzero element $f \in C(X, \mathcal{O}_\infty^{\otimes n})$. Then, we can choose an open subset $U \subset X$ and $h, k \in C(X, \mathcal{O}_\infty^{\otimes n})$ such that $hxk|_U = 1$. By assumption, there is a large m such that $\bigcup_{i=1}^m F_i^{-1}(U) = X$. By choosing a partition of unity $\{\phi_i\}_{i=1}^m$ of X with respect to this covering and a family of isometries $v_1, \dots, v_m \in \mathcal{O}_\infty^{\otimes(m-n)}$ with

$$v_i v_i^* \leq p_{n+1,0} \otimes \dots \otimes p_{m-1,0} \otimes p_{m,i},$$

we get

$$(\sum \phi_i^{1/2} F_i^* h \otimes v_i^*)(\varphi_{m-1} \dots \circ \varphi_n(f))(\sum \phi_i^{1/2} F_i^* k \otimes v_i) = 1 \in C(X, \mathcal{O}_\infty^{\otimes m}).$$

Assume that there are pathes $F_{n,t}\colon X \to X$ of continuous maps with $F_{n,0} = F_n$ and $F_{n,1} = \mathrm{id}$. Then, for $t \in [n, n+1]$ we get a path of $*$-homomorphisms

$$\varphi_t\colon C(X) \to C(X, \mathcal{O}_\infty^{\otimes(n+1)}) \hookrightarrow \mathcal{O}(X),$$
$$\varphi_t(f) := f \otimes p_{n,0} + \sum_{i=1}^{n} F^*_{i,t-n}(f) \otimes p_{n,i}$$

connecting φ_n and φ_{n+1}. By construction $(\varphi_t)_{t\in[0,\infty)}$ forms a continuous X-Rokhlin map. □

Corollary 6.10. *Let G be a compact second countable group.*

1. *For any $n \geq 1$, $\dim^c_{\mathrm{cRok}}(\mathcal{O}(E_{n+1}G)) = n$.*
2. *If G is path-connected, $\mathcal{O}(G)$ has the continuous Rokhlin property.*

Proof. First we show (1). By Theorem 5.9 and Theorem 6.9, it suffices to show that for any $x, y \in E_nG$ there is a G-equivariant continuous map $F\colon E_nG \to E_nG$ with $F(x) = y$ which is homotopic to the identity.

Fix $x, y \in E_nG$. We may assume that $s_1 \geq (n+1)^{-1}$ when we write as $x = [s_1, \dots, s_n, h_1, \dots, h_n]$. Choose a homotopy $\gamma\colon [0,1] \to E_nG$ with

$$\gamma(0) = [1, 0, \dots, 0, e, \dots, e],\ \ \gamma(1) = h_1^{-1} \cdot y.$$

(Note that E_nG is connected.) Then, the path of continuous maps

$$F_s([t_1, \dots, t_n, g_1, \dots, g_n]) = \begin{cases} [\frac{t_1 - s + 1}{s}, \frac{t_2}{s}, \dots, \frac{t_n}{s}, g_1, \dots, g_n] & 0 \leq t_1 \leq s \\ g_1 \cdot \gamma((t_1 - s)/(1 - (n+1)^{-1})) & s \leq t_1 \leq 1 \end{cases}$$

is well-defined. Since it satisfies $F_{1-(n+1)^{-1}}(x) = y$ and $F_1 = \mathrm{id}$, we get the conclusion.

For (2), the multiplications of $g \in G$ from the right are desired G-equivariant continuous maps. Since there is a path connecting g with e, they are homotopic to the identity. □

In Section 4.2 of [AK17], another construction of $\mathcal{O}(G)$ for path-connected compact groups is given by using the crossed product by the dual discrete quantum group $\hat{G}$.

We also remark that $\mathcal{O}(G)$ has the continuous Rokhlin property. It is a consequence of Theorem 6.15.

6.3 Classification of Kirchberg G-algebras

An important feature of Kirchberg algebras is the Kirchberg-Phillips classification: two unital Kirchberg algebras are isomorphic if and only if there is a KK-equivalence between them preserving unit classes (Theorem 4.2.4 of [Phi00]). It is generalized to the equivariant setting when the actions have the Rokhlin property.

Lemma 6.11. *Let G be a compact group and let (A,α) and (B,β) be unital separable G-C*-algebras. Assume β has the Rokhlin property and there exists a $*$-homomorphism $\varphi\colon A\to B$ such that there exists a sequence of unitaries $(u_n)_{n\in\mathbb{N}}$ in $B\otimes C(G)$ such that*

$$(\mathrm{Ad}(u_n(g))\circ\varphi)(x)\to(\beta_g\circ\varphi\circ\alpha_g^{-1})(x)\ \text{as}\ n\to\infty.$$

Then, there exists a G-equivariant $*$-homomorphism $\psi\colon A\to B$ which is approximately unitarily equivalent to φ. Moreover, if φ is an isomorphism and α also has the Rokhlin property, then ψ can be taken to be an isomorphism.

Proof. Put $\theta\colon A\to B\otimes C(G)$ as $\theta(x)(g):=(\beta_g\circ\varphi\circ\alpha_g^{-1})(x)$ for $g\in G$ and $x\in A$ by identifying $B\otimes C(G)$ with the space of continuous B-valued functions on G. Then θ is a G-equivariant $*$-homomorphism. Fix a G-invariant compact set $F\subset A$ and $\varepsilon>0$. By assumption, we may take a unitary $u\in B\otimes C(G)$ such that

$$\|u(\varphi(x)\otimes 1)u^*-\theta(x)\|<\varepsilon$$

for $x\in F$.

Let $\varphi\colon C(G)\to F_\alpha(B)$ be a Rokhlin map. Let us construct a completely positive lift of $\varphi\otimes\mathrm{id}_B\colon C(G)\otimes B\to\ell^\infty_\beta(\mathbb{N},B)/c_0(\mathbb{N},B)$ in the same way as ψ_t in (5.7) (see also the proof of Theorem 2.11 in [Gar14c]). Then we get a completely positive map $\chi\colon B\otimes C(G)\to B$ such that

- $\|\chi(u)^*\chi(u)-1\|<\varepsilon$, $\|\chi(u)\chi(u)^*-1\|<\varepsilon$,
- $\|\chi(u)\chi(\varphi(x)\otimes 1)\chi(u)^*-\chi\circ\theta(x)\|<\varepsilon$ for $x\in F$,

- $\|\chi(\varphi(x) \otimes 1) - \varphi(x)\| < \varepsilon$ for $x \in F$,
- $\|\beta_g(\chi(x)) - \chi((\beta_g \otimes \lambda_g)(x))\| < \varepsilon$ for $g \in G$ and $x \in F \cup \{u\}$.

The last condition can be replaced with a stronger one:

- χ is G-equivariant,

by replacing χ with

$$x \mapsto \int_G \beta_{g^{-1}}(\chi((\beta_g \otimes \lambda_g)(x)))dg.$$

Take the unitary $v := \chi(u)|\chi(u)|^{-1}$. Then we have $\|\chi(u) - v\| < \varepsilon/2$ and hence

$$\|v\varphi(x)v^* - \chi \circ \theta(x)\| < \frac{3}{2}\varepsilon.$$

Since $\chi \circ \theta$ is G-equivariant, we get

$$\|v\varphi(\alpha_g(x))v^* - \beta_g(v\varphi(x)v^*)\| < 3\varepsilon.$$

Moreover since

$$\|\theta(x) - \varphi(x) \otimes 1\| = \sup_{g \in G} \|(\beta_g \circ \varphi \circ \alpha_g^{-1})(x) - x\|,$$

we get

$$\|(\chi \circ \theta)(x) - \varphi(x)\| < \sup_{g \in G} \|(\beta_g \circ \varphi \circ \alpha_g^{-1})(x) - x\| + \varepsilon.$$

Therefore, the intertwining argument in Theorem 3.5 of [Izu04a] and Lemma 5.1 of [Izu04b] works for this situation and we obtain the conclusion. □

Proposition 6.12. *Let G be a second countable compact group and let (A, α) and (B, β) be unital Kirchberg G-algebras with the Rokhlin property. If there is a* KK^G*-equivalence from A to B mapping $[1_A]$ to $[1_B]$, then they are conjugate.*

Proof. First, we observe that a $*$-homomorphism $\varphi\colon A \to B$ satisfies $[\varphi \circ \alpha] = [\beta \circ \varphi] \in \mathrm{KK}(A, B \otimes C(G))$ if $[\varphi]$ is in the image of the functor Res_G. To see this, take a G-equivariant quasihomomorphism $[\varphi_0', \varphi_1']\colon A \to B \otimes \mathbb{K}_G$ such that $\mathrm{Res}_G[\psi_0, \psi_1] = [\varphi]$. Then, we get

$$[\varphi] \circ [\alpha] = [\varphi_0', \varphi_1'] \circ [\alpha] = [\varphi_0' \circ \alpha, \varphi_1' \circ \alpha] = [\beta \circ \varphi_0', \beta \circ \varphi_1'] = [\beta] \circ [\varphi].$$

Now, take a KK^G-equivalence $\xi \in \mathrm{KK}^G(A, B)$ preserving unit classes and a $*$-isomorphism $\varphi \colon A \to B$ such that $[\varphi] = \mathrm{Res}_G \xi$. Then, $\varphi \circ \alpha$ and $\beta \circ \varphi$ are equivalent in $\mathrm{KK}(A, B \otimes C(G))$ by the above argument, and hence asymptotically unitarily equivalent thanks to Theorem 4.1.1 of [Phi00]. Now we get the conclusion by Lemma 6.11. □

Theorem 6.13. *Let G be a Hodgkin Lie group.*

- *A unital Kirchberg G-algebra (A, α) with the continuous Rokhlin property is G-equivariantly isomorphic to $A^\alpha \otimes \mathcal{O}(G)$.*
- *Two unital Kirchberg G-algebras (A, α) and (B, β) with the continuous Rokhlin property are isomorphic if and only if the fixed point algebras A^α and B^β are isomorphic. Moreover, if the underlying C^*-algerbas A and B are in the UCT class, then (A, α) and (B, β) are conjugate if and only if*

$$(\mathrm{K}_0(A^\alpha), [1_{A^\alpha}], \mathrm{K}_1(A^\alpha)) \cong (\mathrm{K}_0(B^\beta), [1_{B^\beta}], \mathrm{K}_1(B^\beta)).$$

- *A unital UCT-Kirchberg algebra (A, α) in the Cuntz standard form (i.e. $[1_A] = 0 \in K_0(A)$) admits a G-action with the continuous Rokhlin property if and only if there is a countable abelian group M such that $K_i(A)$ ($i = 0, 1$) are isomorphic to $M^{\oplus n}$, where $n = 2^{\mathrm{rank}\, G - 1}$. In this case, $M \cong \mathrm{K}_0(A^\alpha) \oplus \mathrm{K}_1(A^\alpha)$.*

Proof. By Theorem 6.3, for every G-C^*-algebra (A, α) with the continuous Rokhlin property, there is a KK^G-equivalence from $A^\alpha \otimes \mathcal{O}(G)$ to A preserving unit elements. Moreover, as is shown in Corollary 6.10 (2), both $(A^\alpha \otimes \mathcal{O}(G), \mathrm{id}_{A^\alpha} \otimes \gamma)$ and (A, α) have the continuous Rokhlin property. Since the fixed point algebra A^α of a (continuous) Rokhlin action on Kirchberg algebras is again a Kirchberg algebra (Corollary 3.20 of [Gar14c]), A and $A^\alpha \otimes \mathcal{O}(G)$ are G-equivariantly isomorphic by Proposition 6.12.

The second assertion follows from Theorem 4.2.4 of [Phi00], Corollary 6.8 and Proposition 6.12. The third assertion follows from the KK-equivalence $C(G) \sim \mathbb{C}^n \oplus C_0(\mathbb{R})^n$ as in the proof of Corollary 6.7. □

Example 6.14. Let $(A, \hat{\alpha})$ be as in Example 5.6 and let us consider the tensor product $(A \otimes \mathcal{O}_\infty, \hat{\alpha} \otimes \mathrm{id}_{\mathcal{O}_\infty})$. It is a unital Kirchberg algebra since A is simple (this is because A is the crossed product of $C(\mathbb{T})$ by $\mathbb{Z}^2$ by a free minimal action [AS94]).

It is shown in Remark 6.2 that there is a $\mathrm{KK}^{\mathbb{T}}$-equivalence between A and $B \otimes C(\mathbb{T})$ (on which $\mathbb{T}$ acts by $\mathrm{id}_B \otimes \lambda$ where λ is the regular action) preserving the unit classes in the K_0-groups. Therefore, we can apply Proposition 6.12 to see that $A \otimes \mathcal{O}_\infty$ is $\mathbb{T}$-equivariantly isomorphic to $\mathcal{O}(\mathbb{T}) \otimes D$, where D is the UCT-Kirchberg algebra with $\mathrm{K}_0(D) \cong \mathbb{Z}^2 \cong \mathrm{K}_1(D)$. In particular, $A \otimes \mathcal{O}_\infty$ has the continuous Rokhlin property.

6.4 Rokhlin property vs. continuous Rokhlin property

We conclude the article by comparing the Rokhlin and continuous Rokhlin properties.

Theorem 6.15. *Let G be a second countable compact group and let A be a unital UCT-Kirchberg G-algebra with the Rokhlin property. Then, A has the continuous Rokhlin property if (and only if) it is $\mathfrak{J}_G$-injective.*

For the proof, it suffices to prove that there exists a G-equivariant asymptotic morphism $\psi_t \colon A \otimes C(G) \to A$ such that $\psi_t \circ \iota_A$ is asymptotically equal to id_A (recall that $\iota \colon \mathbb{C} \to C(G)$ is the canonical inclusion and $\iota_A := \iota \otimes \mathrm{id}_A$), so that $f \mapsto \psi_t(1 \otimes f)$ gives the desired continuous Rokhlin approximation. Now, by assumption of $\mathfrak{J}_G$-injectivity, there exists a left inverse $\xi \in KK^G(A \otimes C(G), A)$ of ι_A in the level of KK-theory. In order to construct such ψ_t starting from this ξ, first we prepare some lemmas.

Lemma 6.16. *For any second countable compact group G, there is a unital UCT-Kirchberg G-algebra D with a unital G-equivariant $*$-homomorphism $j \colon C(G) \to D$ inducing a KK^G-equivalence.*

Proof. Let us define D to be the Cuntz–Pimsner construction [Pim97, Kum04] for the Hilbert $C(G)$-bimodule $\mathcal{H}_G \otimes C(G)$, where $\mathcal{H}_G$ is the Hilbert space $L^2(G)^\infty$ together with the natural $C(G)$-action from the left. Now, it is shown in Proposition 2.1 of [Kum04] that D is a unital Kirchberg algebra. Moreover, the canonical inclusion $j\colon C(G) \to D$ induces a KK^G-equivalence by Theorem 4.4 of [Pim97], whose proof also works for the equivariant setting. □

We write ι' for the canonical inclusion $\mathbb{C} \to D$ (in other words, $\iota' := j \circ \iota$) and $\iota'_A := \iota' \otimes \mathrm{id}_A$.

Lemma 6.17. *Let A be a $\mathfrak{J}_G$-injective unital UCT-Kirchberg G-algebra with the Rokhlin property and let $j\colon C(G) \to D$ be as in Lemma 6.16. Then, there is a G-equivariant $*$-homomorphism $\theta\colon A \otimes D \to A$ such that $[\theta] \in \mathrm{KK}^G(A \otimes D, A)$ is the left inverse of $[\iota'_A]$.*

Proof. Set $\xi' := \xi \circ (\mathrm{id}_A \otimes [j]^{-1})$, which is a left inverse of $[\iota'_A]$. Thanks to Theorem 4.1.1 of [Phi00], we can take a unital $*$-homomorphism $\phi\colon A \otimes D \to A$ with $[\phi] = \mathrm{Res}_G\, \xi$ (note that $\xi_*[1_{A\otimes D}] = \xi_* i_*[1_A] = [1_A]$). Let β denote the G-action on $A \otimes D$. Then, by the same argument as the proof of Lemma 6.12, we get $[\phi \circ \beta] = [\alpha \circ \phi] \in \mathrm{KK}(A \otimes D, A \otimes C(G))$. Applying Lemma 6.11, we get a G-equivariant unital $*$-homomorphism $\phi'\colon A \otimes D \to A$ approximately unitarily equivalent to ϕ.

Since $[\phi' \circ \iota'_A] \in \mathrm{KK}^G(A, A)$ induces the identity on $\mathrm{K}_*(A)$, it is a KK^G-equivalence by Corollary 5.10. Now, we observe that $[\phi' \circ \iota'_A]^{-1} \otimes \mathrm{id}_D$ is represented by a G-equivariant $*$-homomorphism. To see this, recall that there is an isomorphism

$$\mathrm{Ind}^G\colon \mathrm{KK}(A \otimes D, A) \cong \mathrm{KK}^G(A \otimes D, A \otimes C(G))$$

(see for example Proposition 3.2 of [AK15]), which is compatible with the isomorphism between the spaces of $*$-homomorphisms

$$\mathrm{Ind}^G\colon \mathrm{Hom}(A \otimes D, A) \cong \mathrm{Hom}^G(A \otimes D, A \otimes C(G)),$$

defined by $((\mathrm{Ind}^G(\varphi))(x))(g) := \alpha_g(\varphi(x))$ for any $x \in A \otimes D$ and $g \in G$. Now, let $\sigma\colon A \otimes D \to A$ be a unital $*$-homomorphism representing

$$((\mathrm{id}_A \otimes j)_* \circ \mathrm{Ind}^G)^{-1}([\phi' \circ \iota'_A]^{-1} \otimes \mathrm{id}_D) \in \mathrm{KK}(A \otimes D, A).$$

Then, the $*$-homomorphism $\tilde{\sigma} := (\mathrm{id}_A \otimes j)_* \circ \mathrm{Ind}^G(\sigma)$ satisfies $[\tilde{\sigma}] = [\phi' \circ \iota'_A]^{-1} \otimes \mathrm{id}_D$.

Now, $\theta := \phi' \circ \tilde{\sigma}$ is the desired $*$-homomorphism because

$$[\tilde{\sigma}] \circ [\iota'_A] = ([\phi' \circ \iota'_A]^{-1} \otimes \mathrm{id}_D) \circ (\mathrm{id}_A \otimes [\iota']) = [\iota'_A] \circ [\phi' \circ \iota'_A]^{-1}.$$

□

Proof of Theorem 6.15. Let θ be the G-equivariant $*$-homomorphism constructed in Lemma 6.17. Thanks to Theorem 4.1.1 of [Phi00], there is a path of unitaries $(u_t) \in A$ such that

$$u_t \theta(a \otimes 1) u_t^* \to a$$

for any $a \in A$.

Choose an increasing sequence $\{F_n\}$ of self-adjoint compact G-invariant subset of A which satisfies

- $A = \overline{\bigcup_n F_n}$,
- $A \otimes D = \overline{\bigcup_n \theta^{-1}(F_n)}$ and
- $\{u_s : s \leq n-1\} \subset F_n$.

By an inductive reparametrization, we may assume for any $n \in \mathbb{N}$,

$$\|u_t \theta(a \otimes 1) u_t^* - a\| < 2^{-n} \|a\|$$

for any $t \geq n$ and $a \in F_n$. Note that we also get

(i) $\|\alpha(u_t)(\theta(a \otimes 1) \otimes 1)\alpha(u_t)^* - a \otimes 1\|_{A \otimes C(G)} < 2^{-n} \|a\|$ for $a \in F_n$,

(ii) $\|\operatorname{Ad}((u_{n+1} \otimes 1)\alpha(u_{n+1})^*)\alpha(u_n) - \alpha(u_n)\| < 2^{-n}$,

since

$$\begin{aligned}
&\|\alpha(u_t)(\theta(a \otimes 1) \otimes 1)\alpha(u_t)^* - a \otimes 1\|_{A \otimes C(G)} \\
=& \sup_{g \in G} \|\alpha_g(u_t) \theta(a \otimes 1) \alpha_g(u_t)^* - a\| \\
=& \sup_{g \in G} \|u_t \theta(\alpha_{g^{-1}}(a) \otimes 1) u_t^* - \alpha_{g^{-1}}(a)\| \leq 2^{-n} \|a\|,
\end{aligned}$$

and

$$\| \mathrm{Ad}(u_n^*)(a) - \theta(a \otimes 1)\| \leq \| \mathrm{Ad}(u_n^*)(a - u_n\theta(a \otimes 1)u_n^*)\| \leq 2^{-n}\|a\|$$

for any $a \in F_n$ (now the second inequality follows from them).

Thanks to the inequalities (i) and (ii), again by an induction, in the same way as the proof of Lemma 6.11 we may take G-equivariant averaging maps $\chi_k \colon A \otimes C(G) \to A$ such that

1. $\|\chi_n(\alpha(u_t))\theta(a \otimes 1)\chi_n(\alpha(u_t))^* - a\| < 2^{-n}\|a\|$,
2. $\| \mathrm{Ad}(u_{n+1}\chi_n(\alpha(u_{n+1}))^*)(\chi_n(\alpha(u_n))) - \chi_n(\alpha(u_n))\| < 2^{-n}$,
3. $\| \mathrm{Ad}(\chi_n(\alpha(u_n))\chi_{n-1}(\alpha(u_n))^*)(\chi_{n-1}(\alpha(u_{n-1}))) - \chi_{n-1}(\alpha(u_{n-1}))\| < 2^{-n+1}$,
4. $\|\chi_n(\alpha(u_t))^*\chi_n(\alpha(u_t)) - 1\|, \|\chi_n(\alpha(u_t))^*\chi_n(\alpha(u_t)) - 1\| < 2^{-n}$.

for $n \leq t \leq n+1$ and $a \in F_n$. (Note that for (3), we use (2) in the previous step. Actually we do not need (2) for the later argument, but we put this to get (3) in the induction.) Then due to the condition (4), for $n \leq t \leq n+1$,

$$v_{n,t} := \chi_n(\alpha(u_t))|\chi_n(\alpha(u_t))|^{-1}$$

are G-invariant unitaries such that

$$\|v_{n,t} - \chi_n(\alpha(u_t))\| < 2^{-n-1}.$$

Hence we rewrite (1) and (3) as

(1') $\|v_{n,t}\theta(a \otimes 1)v_{n,t}^* - a\| < 2^{-n+1}\|a\|$,

(3') $\| \mathrm{Ad}(v_{n+1,n+1}v_{n,n+1}^*)(v_{n,n}) - v_{n,n}\| < 2^{-n+2}$,

for any $n \leq t \leq n+1$, $a \in F_n$. Note that we can check in the same way as (ii) that (1') implies $\| \mathrm{Ad}(v_{n,t}v_{n+1,t}^*)(a) - a\| < 2^{-n+2}\,\|a\|$ for any $a \in F_n$.

Now, for $n \in \mathbb{Z}_{>0}$ and $t \in [0, 1]$, set

$$w_t^{(n)} := v_{n+1,n+1}v_{n,n+1}^*v_{n,t-n}$$

(in particular, $w_0^{(n)} = v_{n,n}$ and $w_{2n} = v_{n+1,n+1}v_{n,n+1}^* v_{n,t-n}$). We construct a desired path ψ_t to be

$$\psi_{2n+t}(x) := \begin{cases} t\,\mathrm{Ad}(w_0^{(n)})\theta(x) + (1-t)\,\mathrm{Ad}(w_1^{(n-1)})\theta(x) & \text{for } t \in [0,1], \\ \mathrm{Ad}(w_t^{(n)})\theta(x) & \text{for } t \in [1,2]. \end{cases}$$

Indeed, from (1') and (3'), ψ_t is a G-equivariant asymptotic morphism since

$$\begin{aligned} &\| \mathrm{Ad}((v_{n+1,n+1}v_{n,n+1}^*)v_{n,n}) \circ \theta(x) - \mathrm{Ad}(v_{n,n}) \circ \theta(x)\| \\ <&\| \mathrm{Ad}(v_{n,n}(v_{n+1,n+1}v_{n,n+1}^*)) \circ \theta(x) - \mathrm{Ad}(v_{n,n}) \circ \theta(x)\| + 2^{-n+3} < 2^{-n+4} \end{aligned}$$

for any $x \in \theta^{-1}(F_n)$. Moreover again from (1') and (3'), $\|\psi_t(a \otimes 1) - a\| \to 0$ as $t \to \infty$, as desired. □

We remark that Theorem 6.15 holds for general second countable compact groups, in particular for finite groups. As a concluding remark, we discuss on the connection of our results with Izumi's classification of finite group actions on unital UCT-Kirchberg algebras with the Rokhlin property [Izu04b]. For any finite group G, a G-module (i.e., a $\mathbb{Z}[G]$-module) is called *relatively projective* if it is a direct summand of the module of the form $M \otimes_{\mathbb{Z}} \mathbb{Z}[G]$ for some countable abelian group M. The class of G-modules which is isomorphic to an inductive limit of relatively projective modules is characterized by its cohomology groups and called *CCT G-modules* (Theorem 3.15 of [Izu04b]). A complete classification of unital UCT-Kirchberg G-algebras with the Rokhlin property is given in Corollary 5.4 of [Izu04b] by their K_*-groups as CCT G-modules and $[1_A] \in \mathrm{K}_0(A)^G$.

In fact, it is an immediate consequence of Theorem 5.9 that the K_*-group of G-C*-algebras with the continuous Rokhlin property is relatively projective. The class of relatively projective modules is strictly smaller than the class of CCT modules although they coincide under some reasonable assumptions. Actually, Katsura [Kat07] shows that every CCT module is given by the third term of a pure exact sequence whose first and second terms are relatively projective (in other words, its "relatively projective dimension" is at most 1, cf. Proposition 5.3). Hence we obtain a unital UCT-Kirchberg G-algebra with the Rokhlin property which does not have the continuous Rokhlin property. Moreover, relative projectivity is also a sufficient condition for the continuous Rokhlin property.

Lemma 6.18. *Let G be a finite group and let A be a unital UCT-Kirchberg G-algebra such that $\mathrm{K}_*(A)$ is a relatively projective G-module. Then, A is $\mathfrak{J}_G$-injective.*

Proof. Let $\mathcal{O}(G)$ denote the model action in Theorem 6.9, which is KK^G-equivalent to $C(G)$ by the inclusion. Choose an isomorphism $M_* \oplus M'_* \cong N_* \otimes_{\mathbb{Z}} \mathbb{Z}[G]$ where N_* are abelian groups. Let A, A' be the unital UCT-Kirchberg G-algebras in the Cuntz standard form with the Rokhlin property corresponding to M_*, M'_* respectively and let B be the unital UCT-Kirchberg algebra in the Cuntz standard form corresponding to N_*. By Lemma 5.1 of [Izu04b], we obtain G-equivariant $*$-homomorphisms $\varphi\colon A \to B \otimes \mathcal{O}(G)$ and $\varphi'\colon A' \to B \otimes \mathcal{O}(G)$ such that $[\varphi] \oplus [\varphi'] \in \mathrm{KK}^G(A \oplus A', B \otimes \mathcal{O}(G))$ induces an isomorphism of K_*-groups and hence a KK^G-equivalence by Corollary 5.10. Since $B \otimes \mathcal{O}(G)$ is $\mathfrak{J}_G$-injective, so are the direct summands A and A'. □

Corollary 6.19. *Let G be a finite group. Under the one-to-one correspondence given in Corollary 5.4 of* [Izu04b]*, any triplet (M_0, x, M_1) such that M_0 and M_1 are relatively projective corresponds to a unital UCT-Kirchberg G-algebra with the continuous Rokhlin property.*

Proof. This follows from Lemma 6.18 and Theorem 6.15. □

Corollary 6.20. *Let G be a finite group and let A be a unital UCT-Kirchberg G-algebra with the Rokhlin property. If both $\mathrm{K}_0(A)$ and $\mathrm{K}_1(A)$ are either finitely generated groups or bounded p-groups, then A has the continuous Rokhlin property.*

Proof. This follows from the above corollary and Lemma 3.12 and Lemma 3.13 of [Izu04b]. □

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