博士論文（要約）

論文題目　　Scaling limits in stochastic heat equation and stochastic chain model

（確率熱方程式および確率鎖模型に対するスケール極限）

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0 Introduction

The space-time scaling limit is a widely used method in mathematics and physics to study the macroscopic behavior of a microscopically described system. On the microscopic level, the system consists of a huge number of small objects whose time evolutions are govern by basic physical laws and certain interactions. With the spacial volume growing together with the time at a proper speed, such dynamics can appear macroscopically as real-wold processes.

Heuristically, consider a stochastic process $\{Y(t); t \in I\}$ evolving with discrete time $I = \mathbb{N}$ or continuous time $I = \{0\} \cup \mathbb{R}_+$. For each t, $Y(t)$ is a random field on some X, where X is usually taken as the lattice space, torus or Euclid space. To fix ideas, here we suppose that $X = \mathbb{R}^d$ and $I = \mathbb{N}$. Define a rescaled process $Y_\epsilon = \{Y_\epsilon(t); t \in [0,T]\}$ for each $\epsilon > 0$ as

$$Y_\epsilon(t) = Y_\epsilon(t,\cdot);\ Y_\epsilon(t,x) = Y([\epsilon^{-\alpha}t], \epsilon^{-1}x),\ \forall t \in [0,T],\ \forall x \in \mathbb{R}^d,$$

where $\alpha > 0$ is some fixed number. The terminology *space-time scaling limit* refers to the limit (in law) of Y_ϵ when ϵ goes to zero.

The study of scaling limit can go back to Donsker's invariance principle of simple random walks. Indeed, consider the simple random walk $S_n = \sum_{i=1}^n X_i$ on $\mathbb{Z}$, where X_i's are identical and independently distributed random variables such that $P(X_1 = 1) = P(X_1 = 0) = \frac{1}{2}$. For $n \in \mathbb{N}$ and $x \in \mathbb{R}$ define $Y(n,x) = \mathbf{1}_{\{S_n = x\}}$. Rescale Y with $\alpha = 2$ and we obtain that

$$Y_\epsilon(t,x) = \mathbf{1}_{\{S_t^\epsilon = x\}},\ \text{where } S_t^\epsilon = \epsilon S_{[\epsilon^{-2}t]},\ \forall \epsilon > 0.$$

Donsker's invariance principle yields that S^ϵ converges weakly to a one dimensional Brownian motion as $\epsilon \to 0$. More generally, one can derive, under adequate space-time scaling, particular nonlinear partial differential equation from a large scale interacting particle system by taking scaling limit of certain physical quantities, which is called hydrodynamics limit. On the other hand, when the system stays in its local stationary state, of course no evolution can be observed at the macroscopic level. In this case, the scaling limit of the vibration of the system around the equilibrium state appears as stochastic differential equations. This procedure is called equilibrium fluctuation. In this sense, the classical central limit theorem can be viewed as a special case of the equilibrium fluctuations.

In this thesis we consider two relatively independent models and study their scaling limit respectively. The first model is stochastic heat equations in random environment, and the second one is stochastic chains of anharmonic oscillators.

In the first part, we introduce a new model called stochastic partial differential equations in random environment. Consider a one dimensional stochastic heat equation on $[0,1]$ driven by space-time white noise and a random nonlinear term satisfying nice smooth properties. The homogenization of its mild solution is investigated under the diffusive time scaling. Under the condition that the distribution of the nonlinear term is stationary and ergodic with respect to a certain group of transformations, we extend the well-known central limit theorem for finite dimensional diffusions to this infinite dimensional setting, and prove a central limit theorem in L^1 sense with respect to the randomness of the environment. The limit distribution is a centered Gaussian law whose covariance operator is explicitly described through the resolvent equation. It concentrates only on the space of constant functions. Furthermore if the random environment degenerates to a deterministic, periodic and smooth nonlinear term, we also prove an invariance principle. This is based on the results in [23], [24] and [25].

This model attracts our interest because of several features. First, stochastic heat equation is a typical example of infinite dimensional diffusion. To view the environment from the moving particle, which is the basic approach to the homogenization of stochastic processes in random medias, has been proved powerful for random walks and finite dimensional diffusions in ergodic environments ([3], [10]), and our model can be viewed as a natural extension of the latter. To our knowledge, it is a new attempt of applying this approach to infinite dimensional nonlinear systems. On the other hand, the Laplacian in the stochastic heat equation makes the system not translation invariant, which becomes the main obstacle. To overcome this difficulty, a different strategy to define the transformation group on the path space of the nonlinear term is adopted. We modify the general theory on infinite dimensional diffusions to show the ergodicity and construct the generator of the moving environment, which is the hard part. From a more physical point of view, the model considered here can be used to describe the evolution of a flexible Brownian string in a random potential field ([5]), and the periodic case is closely related to the dynamical sine-Gordon equation ([7]).

In the second part, we consider a chain of oscillators with interactions between the nearest neighbors. Microscopically the chain evolves with a system of Hamilton equations perturbed stochastically by continuous noises locally preserving the total length, momentum and energy. We investigate the fluctuation fields of the conserved profiles under the hyperbolic space-time scaling when the dynamics is in its equilibrium. We show in this part that the macroscopic dynamics of the conserved profiles evolves deterministically with a linear Euler system, whose initial condition is randomly distributed as a white noise. To prove the scaling limit, we need to assume a uniform spectral gap for the infinitesimal generator of the system. We also discuss such spectral gap estimate for both harmonic and anharmonic chains. This is based on the joint work [16] with Professor Stefano Olla.

The macroscopic behaviors of the conserved quantities in a microscopic Hamiltonian dynamics disturbed by noises is one of the central topics in statistical mechanics. For the harmonic case, superdiffusive scaling limits is observed when the noises conserve both energy and momentum ([8], [11]), otherwise the macroscopic energy profile would evolve diffusively ([2]). On the other hand, for anharmonic chains few rigorous mathematical results have been obtained until now, although many numerical experiments and heuristic calculations have been made and suggest similar phenomenon ([1]). One obstacle lies in the anharmonic models is the difficulty in presenting explicit computation for non quadratic potential functions and non Gaussian equilibrium measures. We refer to [15] for the diffusive behavior of the equilibrium fluctuation in an anharmonic stochastic chain with noises conserving only energy. Our aim is to promote their strategies to momentum and energy-conserving models.

1 Stochastic heat equation in random environment

Homogenization of finite dimensional diffusions in ergodic environments is a well-studied topic, including periodic and quasi-periodic environments as special cases. In the early works [12] and [20], central limit theorems are established for diffusions driven by random, self-adjoint operators of divergence type. Diffusions without drift is considered in [21]. In [17], an invariance principle in a quenched sense is obtained for diffusions in ergodic, almost surely C^2-smooth environment, through a study on the fundamental solutions corresponding to its generator.

A good review of finite dimensional results can be found in [10, §9]. The simplest example of their model can be described as follows. Suppose that on a probability space $(\Sigma, \mathscr{A}, \mathbb{Q})$ there is a group of stationary and ergodic transformation $\{\tau_c; c \in \mathbb{R}\}$, and for each fixed $\sigma \in \Sigma$, $X_t^\sigma \in \mathbb{R}$ is a diffusion process driven by the gradient-type operator

$$\mathcal{K}^\sigma f = \frac{1}{2} e^{V(\sigma,x)} \frac{d}{dx} \left(e^{-V(\sigma,x)} \frac{d}{dx} f \right), \ \forall f \in C_0^\infty(\mathbb{R}),$$

where $V(\sigma, x)$ is the potential function defined as $V(\sigma, x) = \mathbf{V}(\tau_x \sigma)$ for some good random variable $\mathbf{V}$ on Σ. Let σ be chosen randomly from Σ, and call $X_t = X_t^\cdot$ a diffusion in random environment. A central limit theorem for X_t is proved by the approach of environment process, namely, the Σ-valued stochastic process $\xi_t = \tau_{[X_t^\sigma]}\sigma$ which records the environment σ viewed from X_t^σ. The key fact is that $\{\xi_t, t \geq 0\}$ is a reversible Markov process possessing a stationary measure which is ergodic and absolutely continuous with respect to $\mathbb{Q}$. Notice that each sample path of $X_t^\cdot$ can be decomposed into

$$X_t = X_0 + \int_0^t \mathbf{V}(\xi_s) ds + B_t.$$

The general method can be applied to the additive functional $\int_0^t \mathbf{V}'(\xi_s) ds$. Heuristically speaking, denote by $\mathcal{K}$ the generator of ξ_t and consider the cell problem $-\mathcal{K}\mathbf{f} = \mathbf{V}$ associated with

$\mathcal{K}$. If it has a solution $\mathbf{f}$ which is square integrable with respect to the ergodic measure, with Itô's formula one can obtain that

$$X_t = X_0 + \mathbf{f}(X_0) - \mathbf{f}(X_t) + \int_0^t (\mathcal{D}\mathbf{f}(X_s) + 1) dB_s,$$

where $\mathcal{D}$ is the derivative operator generated by τ_c on Σ. Since the functional central limit theorem for martingales can be applied to the last term above and other terms all vanishes, we can prove the central limit theorem for X_t easily. If the cell problem does not have such a square integrable solution, a martingale approximation argument is firstly established in [**?**] for reversible process, and is extended later in [18], [19] and [22] to non-reversible process with a sector condition. We rely on these results in this part.

The aim of this part is to extend these strategies to an infinite dimensional, nonlinear system. We study the homogenization of a stochastic partial differential equation in random environment. The equation considered here is a stochastic heat equation on 1 dimensional unit interval $[0, 1]$, driven by a standard space-time white noise and having a random nonlinear term. Different from the finite dimensional model, the nonlinear term is supposed to be generated by a random field with stationary and ergodic law under only constant-shifts, that is, a group of transformations indexed by $\mathbb{R}$. We adopt this setting because that the Laplacian in the equation is preserved only by these transformations, which is necessary for obtaining the Markov property of the environment process.

The nonlinear term in the equation is supposed to be decomposed to a gradient-type part and a divergence-free part. It can be viewed as the evolution equation of a flexible Brownian string in some random potential field, see [5]. When the environment degenerates to a periodic nonlinear term, the model is closely related to the dynamical sine-Gordon equation (see, e.g., [7]), and in Theorem 1.3 we formulate an invariance principle for it, see also [23]. The divergence-free part can be preserves the equilibrium state and thus can be added to the model. For homogenization of finite dimensional diffusions in divergence-free random field, we refer to [14].

To state our model, we introduce some notations here. Throughout this part H stands for the Hilbert space $L^2[0, 1]$, with its inner product and norm denoted by $\langle \cdot, \cdot \rangle_H$ and $\| \cdot \|_H$, respectively. Let E be the Banach space $C[0, 1]$ equipped with the uniform topology. Denote by E_0 the subspace of E consisting of functions which vanishes at 0. Denote by μ_0 the Wiener measure on E_0 induced by a standard Brownian motion. Since the sample path of a Brownian motion is almost surely Hölder continuous with any order less than $\frac{1}{2}$, we fix some $\alpha \in (0, \frac{1}{2})$ and introduce E^α as the space consisting of all α-Hölder continuous functions defined on $[0, 1]$. With the natural embeddings, E, E^α and E_0 are treated as subspaces of H.

Now we state our model precisely. Suppose $(\Sigma, \mathscr{A}, \mathbb{Q})$ and $(\Omega, \mathscr{F}, \mathbb{P})$ to be two complete probability spaces, the latter of which is equipped with a filtration of sub σ-fields $\{\mathscr{F}_t \subseteq \mathscr{F}; t \geq 0\}$ satisfying the usual conditions formulated in [9, pp. 10, Definition 2.25]. Let $W(t, x)$ be a standard cylindrical Brownian motion defined on $(\Omega, \mathscr{F}, \mathbb{P})$ adapted to $\mathscr{F}_t$, and

$$\{(V(\sigma, u), B(\sigma, u)) \in \mathbb{R} \times H; (\sigma, u) \in \Sigma \times H\}$$

be an $\mathbb{R} \times H$-valued random field over H on $(\Sigma, \mathscr{A}, \mathbb{Q})$. Assume that V is Fréchet differentiable in u for almost all σ, and let U be the H-valued random field defined by $U = DV + B$. For fixed $\sigma \in \Sigma$, consider a 1-dimensional stochastic heat equation with homogeneous Neumann boundary conditions and initial condition $v \in E$, written as the following:

$$\begin{cases} \partial_t u(t, x) = \dfrac{1}{2} \partial_x^2 u(t, x) - U(\sigma, u(t)) + \dot{W}(t, x), & t > 0, x \in (0, 1); \\ \partial_x u(t, x)|_{x=0} = \partial_x u(t, x)|_{x=1} = 0, & t > 0; \\ u(0, x) = v(x), & x \in [0, 1]. \end{cases} \tag{1.1}$$

To make sure that for fixed σ, (1.1) has a strong solution in the space of continuous functions, assume that

(A1) $U(\sigma,\cdot)=DV(\sigma,\cdot)+B(\sigma,\cdot)$ is a bounded and Lipschitz continuous map from H to H for $\mathbb{Q}$-almost all $\sigma\in\Sigma$.

(A2) $\sup_{\Sigma\times H}\{|V|+\|DV\|_H+\|B\|_H\}<\infty$.

For $\mathbb{Q}$-almost every σ (1.1) has a unique solution $u^{\sigma,v}(t,x)$ which is continuous in t and α-Hölder continuous in x (see [5, 6]) for any $\alpha<\frac{1}{2}$. Hence $\{u^{\sigma,v}(t);t\geq 0\}$ forms a continuous Markov process taking values in E^α. To continue, consider the path space

$$C^{1,0}(H;\mathbb{R}\times H)\triangleq\left\{\phi=(v,b):H\to\mathbb{R}\times H\;\middle|\;v\in C^1(H;\mathbb{R}),b\in C(H;H)\right\}.$$

Due to the regularity of $u^{\sigma,v}(t,\cdot)$, the distribution of $u^{\sigma,v}(t)$ depends only on the law of the sub-field $\{\tilde{V},\tilde{B}\}\triangleq\{V(u),B(u)\}_{u\in E^\alpha}$. Hence let

$$\Sigma^0_{path}\triangleq\left\{\tilde{\phi}:E_\alpha\to\mathbb{R}\times H\;\middle|\;\exists\phi\in C^{1,0}(H;\mathbb{R}\times H),\ \text{s.t. }\tilde{\phi}=\phi|_{E^\alpha}\right\}.$$

Denote by Σ_{path} the completion of Σ^0_{path} under the Fréchet metric

$$d_\Sigma(\sigma_1,\sigma_2)=\sum_{k=1}^{\infty}\frac{1}{2^k}\cdot\frac{d_{\Sigma,k}(\sigma_1,\sigma_2)}{1+d_{\Sigma,k}(\sigma_1,\sigma_2)},\tag{1.2}$$

where for $\sigma_1=(\tilde{v}_1,\tilde{b}_1)$ and $\sigma_2=(\tilde{v}_2,\tilde{b}_2)\in\Sigma$,

$$d_{\Sigma,k}(\sigma_1,\sigma_2)=\sup_{u\in E^\alpha_k}\left\{|\tilde{v}_1(u)-\tilde{v}_2(u)|+\|D\tilde{v}_1(u)-D\tilde{v}_2(u)\|_H+\|\tilde{b}_1(u)-\tilde{b}_2(u)\|_H\right\},$$

$$E^\alpha_k=\left\{u\in E^\alpha,|u(x)|\leq k,\frac{|u(x)-u(y)|}{|x-y|^\alpha}\leq k,\forall x,y\in[0,1]\right\}.$$

(Σ_{path},d_Σ) is a Polish metric space. Equip it with the Borel σ-field and adopt it as the path space of $\{\tilde{V},\tilde{B}\}$. For $c\in\mathbb{R}$, let τ_c be the transformation on Σ_{path} defined by

$$\tau_c\circ\tilde{\phi}=\tilde{\phi}(\cdot+c\mathbf{1}),\ \forall\tilde{\phi}\in\Sigma_{path},\tag{1.3}$$

where $\mathbf{1}$ stands for the constant function $\mathbf{1}(x)\equiv 1$ on $[0,1]$. Let $P_{\tilde{V},\tilde{B}}$ be the distribution of $\{\tilde{V},\tilde{B}\}$ on Σ_{path} and assume further that

(A3) $P_{\tilde{V},\tilde{B}}\circ(\tau_c)^{-1}=P_{\tilde{V},\tilde{B}}$, $\forall c\in\mathbb{R}$ and $P_{\tilde{V},\tilde{B}}$ is ergodic, i.e., if a measurable set $A\subseteq\Sigma_{path}$ satisfies that $P_{\tilde{V},\tilde{B}}(A\Delta\tau_c[A])=0$, $\forall c\in\mathbb{R}$, then $P_{\tilde{V},\tilde{B}}(A)=1$ or 0. Moreover,

$$P_{\tilde{V},\tilde{B}}(\tau_c\tilde{\phi}=\tilde{\phi},\ \forall c\in\mathbb{R})<1.\tag{1.4}$$

(A4) $\exists$ a measurable set Σ_1 such that $\tau_c[\Sigma_1]=\Sigma_1$, $\mathbb{Q}(\Sigma_1)=1$ and for all $\sigma\in\Sigma_1$,

$$\int_{E_0}e^{-2V(\sigma,u)}\langle Df(u),B(\sigma,u)\rangle_H\mu_0(du)=0\tag{1.5}$$

holds for all f on H such that $f(u)=f^\dagger(\langle u,\varphi_1\rangle_H,\ldots,\langle u,\varphi_M\rangle_H)$ for some $M\geq 1$, $f^\dagger\in C^1_b(\mathbb{R}^M)$ and $\varphi_1,\ldots,\varphi_M\in H$.

(A4) is equivalent to say that $\delta(e^{-2V}B)=0$ holds for all $\sigma\in\Sigma_1$, where δ is the divergence operator adjoint to the Malliavin derivative (see, e.g., [13, pp. 35, Definition 1.3.1]). Our main result, a central limit theorem for $u^v(t)$, is stated as follows.

Theorem 1.1. *Under **(A1)** to **(A4)**, $u(t)/\sqrt{t}$ satisfies the central limit theorem in L^1 sense with respect to the environment and the limit distribution concentrates on the space of constant functions, i.e. for any bounded continuous function f on E,*

$$\lim_{t\to\infty}E_{\mathbb{Q}}\left|E_{\mathbb{P}}\left[f\left(\frac{u(t)}{\sqrt{t}}\right)\right]-\int_{\mathbb{R}}f(y\mathbf{1})\Phi_a(y)dy\right|=0,\tag{1.6}$$

where a is a constant, $y\mathbf{1}$ is the function on $[0,1]$ taking constant value $y\in\mathbb{R}$, and Φ_a is the probability density function of a 1-d centered Gaussian law with variance a^2. Furthermore there exists some strictly positive constant C depending only on V such that $C\leq a^2\leq 1$.

The following example shows that periodic non linear term is included in our model as a special case.

Example 1.2. Take $\Sigma = [0,1]$, $\mathscr{A} = \mathcal{B}(\Sigma)$ and $\mathbb{Q}$ to be the Lebesgue measure. Suppose V to be a measurable function on $[0,1] \times \mathbb{R}$ such that

$$V(x,\cdot) \in C^1(\mathbb{R}),\ V(x,y) = V(x,y+1),\ \forall x \in [0,1],\ y \in \mathbb{R}.$$

Define the random field (V,B) for all $\sigma \in \Sigma$ and $u \in E$ as

$$V(\sigma,u) = \int_0^1 V(x,u(x)+\sigma)dx$$

and $B(\sigma,u) \equiv 0$. Assume that both V and $\frac{d}{dy}V$ are uniformly bounded, then **(A1)** to **(A4)** are fulfilled. This gives us the periodic model.

For the deterministic periodic environment defined in Example 1.2, with a little more effort one is able to derive a limit theorem stronger than (1.6). Let $u^{\sigma,v}(t)$ be the solution to (1.1) with V and B defined in Example 1.2, then $u^v(t,x) \triangleq u^{\sigma,v-\sigma}(t,x) + \sigma$ solves the following stochastic heat equation with a periodic, gradient-type nonlinear term:

$$\begin{cases} \partial_t u(t,x) = \frac{1}{2}\partial_x^2 u(t,x) - \frac{d}{dy}V(x,u(t,x)) + \dot{W}(t,x), & t>0, x\in(0,1), \\ \partial_x u(t,0) = \partial_x u(t,1) = 0, & t>0, \\ u(0,x) = v(x), & x\in[0,1]. \end{cases} \tag{1.7}$$

For any $\epsilon > 0$, consider the E-valued process

$$\left\{u^{(\epsilon)}(t) = \epsilon u(\epsilon^{-2}t); t\in[0,T]\right\}. \tag{1.8}$$

In view of Theorem 1.1, the law of $u^{(\epsilon)}(t)$ converges weakly to a normal distribution as $\epsilon \to 0$ for any fixed $t \in [0,T]$. By showing the tightness, one can prove further an invariance principle, stating that $u^{(\epsilon)}(\cdot)$ as a process, converges weakly to a Brownian motion.

Theorem 1.3. *Suppose that the initial distribution of $u(t)$ is absolutely continuous with respect to μ, which is the infinite measure on E such that $u(0)$ subjects to the Lebesgue measure on $\mathbb{R}$ and $u(\cdot) - u(0)$ subjects to μ_0. Then $\{u^{(\epsilon)}(t), t\in[0,T]\}$ converges weakly to a Gaussian process $\{\sigma\mathbf{1}\cdot B_t, t\in[0,T]\}$ as $\epsilon\to 0$, where B_t is a 1-dimensional Brownian motion on $[0,T]$ and the coefficient σ is the same as in Theorem 1.1.*

2 Chain of oscillators with conservative noise

The macroscopic scaling limit in Hamiltonian systems defined on lattice spaces is an important topic in the study of stochastic mechanics. One simple and useful example is the 1 dimensional chain of coupled oscillators. It is clear that in Hamiltonian system with nonlinear interactions, the local ergodicity given by certain stochastic perturbation is crucial for rigorous mathematical approach to macroscopic dynamics. In particular, the ballistic, superdiffusive or diffusive behaviors in 1 dimensional chain of oscillators with different kinds of stochastic perturbations are considered in many articles. Here to present a mathematical discussion, we first introduce some notations.

Consider an infinite particle system, where each particle is numbered with $x \in \mathbb{Z}$ and is described by its momentum $p_x \in \mathbb{R}$ and position $q_z \in \mathbb{R}$. The mass of each particle is set to be 1. The particle x and its nearest neighbor $x+1$ is connected by an spring with potential $V(q_{x+1} - q_x)$. To avoid technical difficulties, we assume that V is smooth, non-negative, strictly convex and $\int_{\mathbb{R}} \exp(-V(r)+\lambda r)dr < \infty$ holds for all $\lambda \in \mathbb{R}$. When V is quadratic, the model is called a harmonic chain, otherwise it is called anharmonic.

Define the inter-particle distances $\{r_x = q_x - q_{x-1}; x \in \mathbb{Z}\}$ and consider a deterministic Hamiltonian $\mathscr{H}$ formally defined as the total energy of the system

$$\mathscr{H} = \sum_{x\in\mathbb{Z}} e_x, \text{ where } e_x = \frac{p_x^2}{2} + V(r_x), \ \forall x \in \mathbb{Z}. \tag{2.1}$$

The macroscopic dynamics of $\{(p_x(t), r_x(t))\}_{x\in\mathbb{Z}}$ is given by the Hamiltonian system

$$\begin{cases} p_x'(t) = V'(r_{x+1}(t)) - V'(r_x(t)), & x \in \mathbb{Z}; \\ r_x'(t) = p_x(x) - p_{x-1}(t), & x \in \mathbb{Z}. \end{cases} \tag{2.2}$$

The dynamics (2.2) has three (locally) conserved physical quantities: the total length $\sum r_x$, the total momentum $\sum p_x$ and the total energy $\mathscr{H} = \sum e_x$. Both mathematicians and physicians are interested in the macroscopic behavior of the conserved profile in (2.2). Generally speaking, for chains with nonlinear interactions, if only energy is preserved by the noise, the equilibrium fluctuation evolves diffusively ([4], [15]). To see other macroscopic scaling, we need to adopt momentum-conservative noise ([8]).

In this part we deal with the energy and momentum-conservative noise for harmonic or anharmonic chains. We adopt continuous type noises involving both the momentum $\{p_x\}$ and the positions $\{r_x\}$, since it gives us better mixing property. It is easy to see that the way to define such a conservative noise between two oscillators is unique, see (2.3). With these noises we can define a stochastic dynamics satisfying all the three conservation laws of (2.2). In this sense, our model is nearer to the Hamiltonian system than the models in [4] and [15].

To state the model, we introduce some basic notations. Let $C_0^\infty(\mathbb{R})$ and $S(\mathbb{R})$ be the class of compactly supported smooth functions and Schwartz functions on $\mathbb{R}$, the latter of which is equipped with the Fréchet metric. Denote by $S'(\mathbb{R})$ the dual space of $S(\mathbb{R})$. Given any metric space M, let $C([0,T]; M)$ and $C([0,\infty); M)$ denote the the collections of all continuous trajectories on M with finite and infinite time interval respectively.

Now we describe our model precisely. Consider a stochastic particle system consisting of infinite many particles in 1-dimensional space. Its configuration space is denoted by $\Omega = (\mathbb{R}^2)^{\mathbb{Z}}$. For a specific configuration $\omega = \{(p_x, r_x)\}_{x\in\mathbb{Z}}$, p_x stands for the velocity of the particle numbered with x, and r_x stands for the distance between the particles x and $x-1$. Consider the Markov process in the configuration space, formally given by the infinitesimal generator

$$\mathcal{L}^\gamma = \mathcal{A} + \gamma\mathcal{S}.$$

Here $\mathcal{A}$ is the Liouville generator

$$\mathcal{A} = \sum_{x\in\mathbb{Z}} (p_x - p_{x-1})\partial_{r_x} + (V'(r_{x+1}) - V'(r_x))\partial_{p_x},$$

$\gamma > 0$ is the strength of noise and $\mathcal{S} = \frac{1}{2}\sum_{x\in\mathbb{Z}} \mathcal{X}_{x,x+1}^2$ where

$$\mathcal{X}_{x,y} = (p_y - p_x)(\partial_{r_y} - \partial_{r_x}) - (V'(r_y) - V'(r_x))(\partial_{p_y} - \partial_{p_x}), \tag{2.3}$$

with some potential function V. Throughout this part we assume that

(A1) V is a smooth, non-negative function on $\mathbb{R}$ and $V(0) = 0$.

(A2) $\delta_- \triangleq \inf_{r\in\mathbb{R}} V''(r) > 0$ and $\delta_+ \triangleq \sup_{r\in\mathbb{R}} V''(r) < \infty$.

An easy observation from (A1) and (A2) is that $\delta_- r^2 < 2V(r) < \delta_+ r^2$ for all $r \in \mathbb{R}$.

Define the energy associated to the particle x to be

$$e_x = \frac{p_x^2}{2} + V(r_x), \ \forall x \in \mathbb{Z}.$$

Formally, the dynamics generated by $\mathcal{L}^\gamma$ conserves three physical quantities: the total momentum $\sum p_x$, total length $\sum r_x$ and total energy $\sum e_x$. For a given parameter $c = (\beta, \mu, \lambda) \in \mathbb{R}_+ \times \mathbb{R}^2$, define the grand canonical Gibbs measure π^c on the configuration space Ω as

$$\pi^c(d\vec{p}d\vec{r}) = \prod_{x\in\mathbb{Z}} \left[\frac{\exp(-\mu^2(2\beta)^{-1})}{\sqrt{2\pi\beta^{-1}} Z_{\beta,\lambda}} e^{-\beta e_x + \mu p_x + \lambda r_x} dp_x dr_x \right],$$

where the normalization constant $Z_{\beta,\lambda}$ is

$$Z_{\beta,\lambda} = \int_{\mathbb{R}} e^{-\beta V(r) + \lambda r} dr < \infty.$$

Denote by E^c the expectation corresponding to π^c. For a fixed $c = (\beta, \mu, \lambda)$, let $(\mathcal{H}, \langle \cdot, \cdot \rangle)$ be the Hilbert space $L^2(\Omega, \pi^c)$, and denote by $\| \cdot \|$ the corresponding norm. The measure π^c is invariant for the dynamics generated by $\mathcal{L}^\gamma$ for all $c \in \mathbb{R}_+ \times \mathbb{R}^2$. It is easy to observe that

$$\langle -\mathcal{L}^\gamma f, f \rangle = \gamma \langle -\mathcal{S} f, f \rangle = \frac{\gamma}{2} \sum_{x\in\mathbb{Z}} E^c \left[(\mathcal{X}_{x,x+1} f)^2 \right], \tag{2.4}$$

which is the Dirichlet form associated with the generator $\mathcal{L}^\gamma$.

Denote by $\{\omega(t) = (\vec{p}(t), \vec{r}(t))\}_{[0,\infty)}$ the Markov process generated by $\mathcal{L}^\gamma$. The existence of the infinite dynamics can be proved for a wide class of initial conditions, in particular for a set of configurations that has measure one for any Gibbs measure π^c defined above. The probability measure on the path space $C([0,\infty); \Omega)$ induced by $\omega(\cdot)$ with the initial condition $\omega(0)$ subjecting to π^c is denoted by $\mathbb{P}_{\pi^c}$. Let $\mathbb{E}_{\pi^c}$ be the expectations corresponding to $\mathbb{P}_{\pi^c}$.

To introduce the condition of spectral gap, we also consider the system of finite chain. For $N \geq 1$, let $\Omega_N = (\mathbb{R}^2)^N$, and π_N^c be the probability measure on Ω_N which coincides with the marginal distribution of π^c on $\{(p_x, r_x); 1 \leq x \leq N\}$. Denote the expectation with respect to π_N^c by E_N^c. For a deterministic vector $w = (p, r, e) \in \mathbb{R}^2 \times \mathbb{R}_+$, we define

$$\Sigma_N^w = \left\{ (p_x, r_x)_{1\leq x\leq N} \in \Omega_N \;\middle|\; \frac{1}{N} \sum_{x=1}^N (p_x, r_x, e_x) = w \right\}. \tag{2.5}$$

Due to the conditions on V, Σ_N^w is not empty if and only if $e \geq p^2/2 + V(r)$. Denote by π_N^w the uniform measure on Σ_N^w, and let E_N^w be the expectation with respect to π_N^w. The model is said to posses a uniform spectral gap if the following condition holds.

(SP) For each $N \geq 3$, there is a constant C_N independent of w such that

$$E_N^w \left[(f - E_N^w[f])^2 \right] \leq C_N \sum_{x=1}^{N-1} E_N^w \left[(\mathcal{X}_{x,x+1} f)^2 \right] \tag{2.6}$$

holds for every bounded and smooth function f on Ω_N.

To define the equilibrium fluctuation field of the conserved quantities, let $w_x = w_x(\omega) = (p_x, r_x, e_x)$ be a random vector on Ω. For each $N \geq 1$ and test function $G \in S(\mathbb{R})$, define a random vector $Y_N(t, G) \in \mathbb{R}^3$ by

$$Y_N(t, G) = \frac{1}{\sqrt{N}} \sum_{x\in\mathbb{Z}} G\left(\frac{x}{N}\right) (w_x(\omega(Nt)) - E^c[w_0]). \tag{2.7}$$

Observe that when $\omega(\cdot)$ is in its stationary state $\mathbb{P}_{\pi^c}$,

$$Y_N(t, G) \Rightarrow \mathcal{N}\left(0, \chi_c \int_{\mathbb{R}} G^2(y) dy\right), \ \forall G \in S(\mathbb{R}), \ \forall t \geq 0,$$

where the limit is a 3 dimensional centered Gaussian law, and χ_c is the covariance of w_0:

$$\chi_c \triangleq E^c\left[(w_0 - E^c[w_0]) \cdot (w_0 - E^c[w_0])'\right]. \tag{2.8}$$

To investigate the macroscopic dynamics of the equilibrium fluctuation defined in (2.7), we introduce the *thermodynamic entropy* of the system as follows. Rewrite the marginal density of π^c as $\exp\{-\beta e_x + \mu p_x + \lambda r_x - g(\beta, \mu, \lambda)\}$, where

$$g(\beta, \mu, \lambda) = \frac{\mu^2}{2\beta} + \frac{1}{2}\ln\left(\frac{2\pi}{\beta}\right) + \ln Z_{\beta,\lambda}.$$

Given $(p, r, e) \in \mathbb{R}^2 \times \mathbb{R}_+$, the thermodynamic entropy $S(p, r, e)$ is defined as

$$S(p, r, e) = \sup_{c=(\beta,\mu,\lambda)} \left\{-\beta e + \mu p + \lambda r - g(\beta, \mu, \lambda)\right\}, \tag{2.9}$$

where the superior is taken for all $c \in \mathbb{R}_+ \times \mathbb{R}^2$. It is not hard to observe from (A1) and (A2) that $S(p, r, e) < \infty$ and the superior can be reached at some unique $c = (\beta, \mu, \lambda)$. Define

$$A(p, r, e) = \left(\frac{\lambda(p, r, e)}{\beta(p, r, e)}, p, \frac{p\lambda(p, r, e)}{\beta(p, r, e)}\right).$$

Now we define $\{w(t, \cdot) = (p(t, \cdot), r(t, \cdot), e(t, \cdot)); t \in [0, T]\}$ to be the stochastic process taking values in $(S'(\mathbb{R}))^3$ with the initial distribution

$$w(0, y) = \chi_c^{1/2} \dot{B}_y, \ \forall\, y \in \mathbb{R}, \tag{2.10}$$

where $\{\dot{B}_y\}$ stands for the three dimensional standard white noise on $\mathbb{R}$, and satisfying the deterministic evolution equation written as

$$\partial_t w(t, y) = D\partial_y w(t, y), \tag{2.11}$$

where $D = \mathcal{D}A(\bar{w})$ is the Jacobin matrix of $A = A(p, r, e)$ at $\bar{w} = E^c[w_0]$.

Denote by $\mathbb{Q}_N$ the probability distribution induced by $Y_N(t)$ in (2.7) on $C([0, T]; (S'(\mathbb{R}))^3)$ when $\omega(t)$ starts from one of its equilibrium measure π^c. Denote by $\mathbb{Q}$ the probability measure determined by the process $w(t, \cdot)$ defined through (2.10) and (2.11) on the same space. Now we are prepared to state our main result, concerning the hyperbolic scaling limit of the equilibrium fluctuation field of the conserved quantities of $\omega(t)$.

Theorem 2.1. *Suppose that (A1), (A2) and (SP) hold. Then $\mathbb{Q}_N$ converges weakly to $\mathbb{Q}$.*

Concerning the condition (SP), we have the following result.

Theorem 2.2. *Suppose that (A1) and (A2) hold. There exists $\epsilon > 0$ such that if V satisfies that $\delta_+/\delta_- \in [1, 1 + \epsilon)$, then the uniform spectral gap estimate (SP) holds with $C_N = CN^2$, where C is some constant depending only on V.*

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