博士論文

論文題目

Numerical Analysis for Interface and Nonlinear Boundary Value Problems for the Stokes Equations

（ストークス方程式に対するインターフェースおよび非線形境界値問題の数値解析）

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Contents

Chapter 1

Introduction

This thesis is comprised of this Introduction and the other three main chapters. In each chapter, we deal with some kind of boundary value problems for the Stokes equations and its numerical approximation by the finite element method. Well-posedness and error estimates are established in all cases. Theoretical convergence results are also verified by numerical experiments in every chapter. Chapter 2 is devoted to a nonlinear boundary value problem at artificial boundaries, which is proposed by Zhou and Saito [55] as a new mathematical model for the blood flow problem in arteries. Since this boundary condition is interpreted as an inequality, we introduce a *penalty* approximation for computation.

The latter two chapters are devoted to reformulations of an *interface* condition for the multiphase flow problems of viscous incompressible fluids. In Chapter 3, we study the *immersed boundary (IB) method* proposed by C. S. Peskin [46] in 1972, where the interface condition is interpreted as a singular outer force filed using the Dirac delta function. For computation, we introduce a regularized Dirac delta approximation and study the regularization and discretization errors separately. In Chapter 4, we study another reformulation method proposed by Fujita et. al [21] in 1995. This method is essentially equivalent to the IB method. Only the difference is an interpretation of the interface condition using a characteristic function instead of the Dirac delta, which makes analysis somewhat easier. Following the case of IB method, we study the regularization and discretization errors separately.

1.1 Artificial boundary condition setting

In numerical simulation of real-world flow problems, we often encounter some issues related to artificial boundary conditions. A typical and important example is the blood flow problem in the large arteries, where the blood is assumed to be a viscous incompressible fluid (see [20, 53]). The blood vessel is modeled as a branched pipe as illustrated, for example, in Fig. 1.1. In Fig. 1.1, the boundary S_2 represents a physical boundary as blood wall, while the boundaries S_1 and Γ are artificial boundaries standing for the *inflow* and *outflow* boundaries, respectively. In this domain (denoted by Ω), for $T > 0$, we consider the Navier–Stokes equations for velocity $v = (v_1, \ldots, v_d)$ and pressure q with the initial condition $v|_{t=0} = v_0$ and the boundary condition. We suppose that we are able to give a velocity profile $b = b(x, t)$ at the *inflow boundary* S_1, and

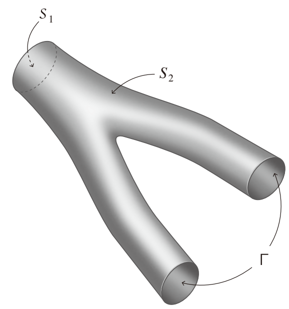

Figure 1.1: Example of domain (branched pipe)

that the flow is presumed to be a perfect non-slip flow on the wall S_2. That is, we consider

$$v_t + (v \cdot \nabla)v = \nabla \cdot \sigma(v, q) + f, \quad \nabla \cdot v = 0 \qquad \text{in } \Omega \times (0, T), \tag{1.1a}$$
$$v = b \qquad \text{on } S_1 \times (0, T), \tag{1.1b}$$
$$v = 0 \qquad \text{on } S_2 \times (0, T), \tag{1.1c}$$

where $\sigma(u, p) = (\sigma_{ij}(u, p))_{1\le i,j\le d} = -pI + 2\nu D(u)$ denotes the stress tensor, $D(u) = (D_{ij}(u))_{1\le i,j\le d} = \frac{1}{2}\left(\nabla u + \nabla u^{\mathrm{T}}\right)$ the deformation-rate tensor and I the identity matrix. Then, the blood flow simulation is highly dependent on the choice of artificial boundary conditions posed on the *outflow boundary* Γ.

In mathematics, one of the common boundary conditions is the Dirichlet boundary condition which poses a prescribed pressure and velocity profiles. However, the flow distribution and pressure field are unknown themselves and cannot be prescribed at the outflow boundary in many simulations. Another common boundary condition is a kind of the Neumann boundary condition named the free-traction condition

$$\tau(v, q) = 0 \quad \text{on} \quad \Gamma, \tag{1.2}$$

where $\tau(u, p) = \sigma(u, p)n$ denotes the traction vector on $\partial\Omega$ with n the outward normal vector to $\partial\Omega$. In fact, for easy implementation, the free-traction condition is often employed in many simulations (see [24, 27]). However, the free-traction condition (1.2) may cause a risk of failure of computation because of the lack of the *energy inequality*.

1.1.1 Unilateral boundary condition

Recently, Zhou and Saito [55] proposed a new outflow boundary condition as

$$v_n \ge 0, \quad \tau_n(v, q) \ge 0, \quad v_n\tau_n(v, q) = 0, \quad \tau_T(v) = 0 \quad \text{on } \Gamma. \tag{1.3}$$

for a generalization of the free-traction condition (1.2). Herein, $v_n = v \cdot n$, $\tau_n = \tau \cdot n$ are the normal components of v, τ and $\tau_T = \tau - \tau_n n$ the tangential one of τ. This is an analogy to Signorini's unilateral boundary condition in the theory of elasticity (see [32]). A benefit of using (1.3) is that the Navier–Stokes equations (1.1) admits energy inequality, which is connected with the stability of numerical schemes from the view-point of numerical computation. The purpose of Chapter 2 is to establish the well-posedness of a model Stokes problem using (1.3) and study its numerical calculations by the finite element method. For the numerical treatment of the inequality condition (1.3), we use its penalty approximation

$$\tau_n(v,q) = \frac{1}{\varepsilon}[v_n]_-, \quad \tau_T(v) = 0 \quad \text{on } \Gamma, \tag{1.4}$$

where $0 < \varepsilon \ll 1$ and

$$[s]_\pm = \max\{0, \pm s\}, \quad s = [s]_+ - [s]_- \quad (s \in \mathbb{R}). \tag{1.5}$$

Then, introducing a C^1 regularization ϕ_δ of $[\cdot]_-$, we can solve the Stokes/Navier–Stokes equations with (1.4) by using, for example, Newton's iteration.

[Main Result in Chapter 2] The well-posedness and error estimate of the unilateral problem for the Stokes equations

For theoretical treatment, we introduce a *reference flow* (g, π) such that

$$\nabla \cdot \sigma(g,\pi) = 0, \quad \nabla \cdot g = 0 \quad \text{in } \Omega, \tag{1.6a}$$
$$g = b \text{ on } S_1, \quad g = 0 \text{ on } S_2. \tag{1.6b}$$

Function g is nothing but a lifting function of inflow b. Using this, we will find (v, q) of the form

$$v = u + g, \qquad q = p + \pi.$$

Thus, we consider the Stokes equations with homogenius boundary condition on $S_1 \cup S_2$ and corresponding unilateral boundary condition on Γ

$$\begin{aligned}
&-\nu\Delta u + \nabla p = f, \quad \nabla \cdot u = 0 && \text{in } \Omega, && (1.7a)\\
&u = 0 && \text{on } S_1 \cup S_2, && (1.7b)\\
&u_n + g_n \geq 0, && \text{on } \Gamma, && (1.7c)\\
&\tau_n(u,p) + \alpha_n \geq 0 && \text{on } \Gamma, && (1.7d)\\
&(u_n + g_n)(\tau_n(u,p) + \alpha_n) = 0 && \text{on } \Gamma, && (1.7e)\\
&\tau_T(u) + \alpha_T = 0 && \text{on } \Gamma && (1.7f)
\end{aligned}$$

where f, g and $\alpha(\equiv 2\nu D(g)n)$ are prescribed functions. We also consider its penalty approximation

$$\begin{aligned}
&-\nu\Delta u + \nabla p = f, \quad \nabla \cdot u = 0 && \text{in } \Omega, && (1.8a)\\
&u = 0 && \text{on } S_1 \cup S_2, && (1.8b)\\
&\tau_n(u,p) + \alpha_n = \frac{1}{\varepsilon}\phi_\delta(u_n + g_n) && \text{on } \Gamma, && (1.8c)\\
&\tau_T(u) + \alpha_T = 0 && \text{on } \Gamma && (1.8d)
\end{aligned}$$

for penalty parameter ε and regularization parameter δ.

In Chapter 2, we show the well-posedness of (1.7) in terms of variational inequalities.

Theorem 1.1.1. *There exists a unique weak solution (u, p) to (1.7) and it holds that*

$$\|u\|_{H^1} + \|p\|_{L^2} \leq C_*, \tag{1.9}$$

where C_* denotes a positive constant depending only on Ω, $\|f\|_{L^2}$, $\|\alpha\|_{(H^{1/2}_{00}(\Gamma)^d)'}$ and $\|g\|_{H^1}$.

We also show the well-posedness of the P1b/P1 (MINI element) finite element approximation to (1.8) applying the theory of monotone operators.

Theorem 1.1.2. *Assume some nonrestrictive assumptions* (A1)–(A2) *are satisfied. Then, there exists a unique finite element approximated solution* (u_h, p_h) *of* (1.8)*, and we have*

$$\|u_h\|_{H^1} + \|p_h\|_{L^2} + \left\|\frac{1}{\varepsilon}\phi_\delta(u_{hn} + g_n)\right\|_{M_h'} \leq C_*; \tag{1.10a}$$

$$\left\|\frac{1}{\varepsilon}\phi_\delta(u_{hn} + g_n)\right\|_{H_{00}^{1/2}(\Gamma)^{n\prime}} \leq C_*\left(1 + \frac{h}{\varepsilon}\right); \tag{1.10b}$$

$$\frac{1}{\sqrt{\varepsilon}}\|[u_{hn} + g_n]_-\|_{L^2(\Gamma)^n} \leq C_*\left(1 + \frac{\delta}{\varepsilon}\right). \tag{1.10c}$$

where $M_h \subset H_{00}^{1/2}(\Gamma)^d$ *is a finite element approximated function space.*

As a main result in Chapter 2, we derive the following error estimate of optimal order $O(h)$.

Theorem 1.1.3. *Assume some nonrestrictive assumptions* (A1)–(A2) *are satisfied. Let* (u, p) *and* (u_h, p_h) *be above solutions, respectively, and suppose that* $(u, p) \in H^2(\Omega)^d \times H^1(\Omega)$ *and* $\tau_n(u, p) + \alpha_n \in H_{00}^{1/2}(\Gamma)^d$*. Moreover, assume that* h, ε, δ *are sufficiently small such that* $c_2\varepsilon \leq h \leq c_1\varepsilon$ *and* $\delta \leq c_3 h^{\frac{3}{2}}$ *with constants* $c_1 > c_2, c_3 > 0$*. Then, we have the optimal-order error estimate*

$$\|u - u_h\|_{H^1} + \|p - p_h\|_{L^2} \leq C_{**}h.$$

where C_{**} *denotes a positive constant depending only on* c_1*,* c_2*,* c_3*,* Ω*,* $|u|_{H^2}$*,* $|p|_{H^1}$*,* $\|\tau_n(u, p) + \alpha_n\|_{H_{00}^{1/2}(\Gamma)}$*,* $\|f\|_{L^2}$*,* $\|g\|_{H^1}$ *and* $\|\alpha\|_{(H_{00}^{1/2}(\Gamma)^d)'}$*.*

1.2 Interface problem for viscous incompressible fluids

For latter two chapters, we consider the Navier-Stokes equations with an interface condition

$$\frac{\partial u}{\partial t} + (u \cdot \nabla)u - \nu\Delta u + \frac{1}{\rho}\nabla\pi = h(x, t), \quad \nabla \cdot u = 0 \qquad \text{in } \Omega,\ t > 0, \tag{1.11a}$$

$$u = 0 \qquad \text{on } \partial\Omega,\ t > 0, \tag{1.11b}$$

$$[u] = 0, \quad [\tau] = g(x, t) \qquad \text{on } \Gamma,\ t > 0, \tag{1.11c}$$

$$u(x, 0) = u^{(0)}(x) \qquad \text{in } \Omega. \tag{1.11d}$$

for velocity $u(x, t)$ and pressure $\pi(x, t)$. Herein, Ω denotes a fixed bounded domain in $\mathbb{R}^d$ ($d = 2, 3$) with the boundary $\partial\Omega$, Γ is a surface/curve included in Ω which implies the interface of two-phase flow. See Fig. 1.2 as an example of Ω and Γ. The coefficients of kinetic viscosity ν and density ρ are assumed to be a constant for simplicity. The traction (or stress) vector is denoted by τ. Moreover, $[\cdot]$ stands for a jump across the interface Γ. We assume that $h(x, t), g(x, t)$ and $u^{(0)}(x)$ are given functions. There are number of literature devoted to numerical methods for these kinds of interface problems. For the finite element method, for example, you need to calculate the the (moving) boundary integral term $\int_\Gamma g(x, t)v(x)\, dx$ and the presence of it makes discretization somewhat technical. In order to avoid this difficulties, some kind of reformulation methods are proposed so far.

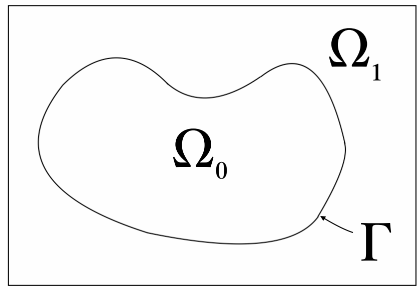

Figure 1.2: Example of Ω and Γ; $\overline{\Omega} = \overline{\Omega_0 \cup \Omega_1}$.

1.2.1 Immersed boundary method

The IB method is originally proposed by C. S. Peskin [46] for solving a class of fluid-structure interaction problems [44, 45]. In the IB method, the interface problem (1.11) is equivalently reformulated to classical partial differential equations as follows. Let $\Gamma(t)$ be parameterized as $\Gamma(t) = \{X(\theta, t) = (X_1(\theta, t), \dots, X_n(\theta, t)) \mid \theta \in \Theta\}$ for the Lagrangian coordinate $\theta \in \mathbb{R}^{d-1}$. Then, the interface condition (1.11c) is interpreted as an outer force filed f defined on Ω and putted in the Navier-Stokes equations such that

$$\frac{\partial u}{\partial t} + (u \cdot \nabla)u - \nu\Delta u + \frac{1}{\rho}\nabla\pi = h + f, \quad \nabla \cdot u = 0 \qquad \text{in } \Omega,\ t > 0, \tag{1.12a}$$

$$u = 0 \qquad \text{on } \partial\Omega,\ t > 0, \tag{1.12b}$$

$$u(x, 0) = u^{(0)}(x) \qquad \text{in } \Omega, \tag{1.12c}$$

$$f(x, t) = \int_\Theta F(\theta, t)\delta(x - X(\theta, t))\ d\theta. \tag{1.12d}$$

Herein, F denotes the force density distributed along $\Gamma(t)$, and δ is the Dirac delta function. Then, the IB formulation (1.12) coincides with (1.11) if $F(\theta, t) \equiv g(X(\theta), t)J_X(\theta, t)$ with the Jacobian J_X of X. For numerical computation, we solve (1.12) with the equation of the interface motion $\frac{\partial X}{\partial t} = u(X, t)$ after introducing regularization using a smooth Dirac delta approximation.

The main advantage of this method is that we can use fixed uniform meshes. In contrast to a huge number of applications, however, it seems that there are only a few results about theoretical convergence analysis. As previous works, convergences of the finite difference scheme for the steady Stokes/Poisson equations under periodic boundary condition are studied in [35, 42]. On the other hand, there is nothing for the finite element scheme. Moreover, convergence analysis in [35, 42] are based on the explicit formula of the Green function associated with the periodic boundary condition. Hence, it is difficult to apply those methods to more standard settings. In Chapter 3, in order to deal with the problem more generally, we take a different approach and study the convergence of the IB finite element method to the Dirichlet boundary value problem for the Stokes equations.

[Main Result in Chapter 3] Convergence of the IB finite-element method for the Stokes problem

For geometry setting of the interface, we assume the following:

- Γ is a C^1 boundary ($X(\theta)$ is a C^1 function);
- $\mathrm{dist}(\Gamma, \partial\Omega) > 0$;
- $J_X(\theta) \neq 0 \quad (\theta \in \Theta)$.

Then, we consider the IB formulation to the Stokes equations for the velocity u and pressure π,

$$-\nu\Delta u + \nabla\pi = f \quad \text{in } \Omega, \quad \nabla \cdot u = 0 \quad \text{in } \Omega,\ u = 0 \quad \text{on } \partial\Omega, \tag{1.13a}$$

$$f(x) = \int_\Theta F(\theta)\delta(x - X(\theta))\ d\theta \tag{1.13b}$$

and its regularized problem for u^ε and π^ε,

$$-\nu\Delta u^\varepsilon + \nabla\pi^\varepsilon = f^\varepsilon \quad \text{in } \Omega, \quad \nabla \cdot u^\varepsilon = 0 \quad \text{in } \Omega,\ u^\varepsilon = 0 \quad \text{on } \partial\Omega, \tag{1.14a}$$

$$f^\varepsilon(x) = \int_\Theta F(\theta)\delta^\varepsilon(x - X(\theta))\ d\theta \tag{1.14b}$$

where a Dirac delta approximation δ^ε satisfies that, for some $K > 0$,

$$\delta^\varepsilon(x) = \frac{1}{\varepsilon^n}\prod_{i=1}^n \phi\left(\frac{x_i}{\varepsilon}\right) \quad (x = (x_1, \ldots, x_n)); \tag{1.15a}$$

$$\phi \text{ is a continuous function in } \mathbb{R}, \quad \operatorname{supp}\phi \subset B(0, K\varepsilon), \quad \int_\mathbb{R} \phi(s)\ ds = 1. \tag{1.15b}$$

In Chapter 3, we derive the following error estimates for the IB finite element method of sub-optimal order in the $W^{1,q} \times L^q$ and L^r norms ($1 \leq q, r < \frac{n}{n-1}$).

Theorem 1.2.1. *Suppose that Ω is a convex polyhedral domain in $\mathbb{R}^n$ with $n = 2, 3$. Assume that $\{\mathcal{T}_h\}_h$ is a family of quasi-uniform triangulations. Let $F \in L^\infty(\Theta)$. Let (u, π) and $(u_h^\varepsilon, \pi_h^\varepsilon)$ be weak solutions of* (1.13) *and the P1b/P1 finite element approximation to* (1.14) *with* (1.15), *respectively. Further, let $\varepsilon = \gamma_1 h$ with a positive constant γ_1. Then, for any $0 < \alpha < 1$, there exists a positive constant C depending only on γ_1, n, α, Ω, K, $\|\phi\|_{L^\infty(\mathbb{R})}$, $\|J_X\|_{L^\infty(\Theta)}$,* meas($\Theta$), *and $\|F\|_{L^\infty(\Theta)}$ such that*

$$\|u - u_h^\varepsilon\|_{W^{1,q}} + \|\pi - \pi_h^\varepsilon\|_{L^q} \leq Ch^{1-\alpha} \quad \textit{with any } 1 \leq q \leq \frac{n}{n-\alpha} \tag{1.16}$$

and

$$\|u - u_h^\varepsilon\|_{L^r} \leq Ch^{1-\alpha} \quad \textit{with} \quad r = \frac{n}{n-\alpha-1}. \tag{1.17}$$

1.2.2 Reformulation using the characteristic function

Another reformulation method for (1.11) was proposed by H. Fujita et. al. [21] in 1995. Their reformulation reads

$$\frac{\partial u}{\partial t} + (u \cdot \nabla)u - \nu\Delta u + \frac{1}{\rho}\nabla\pi = h + \tilde{g}(\nabla\chi \cdot \tilde{n}), \quad \nabla \cdot u = 0 \qquad \text{in } \Omega,\ t > 0, \tag{1.18a}$$

$$u = 0 \qquad \text{on } \partial\Omega,\ t > 0, \tag{1.18b}$$

$$u(x, 0) = u^{(0)}(x) \qquad \text{in } \Omega. \tag{1.18c}$$

Herein, χ denotes the characteristic function of an internal area Ω_0 surrounded by $\Gamma(t)$ in Ω, and n is the unit normal vector on $\Gamma(t)$. Function $\tilde{g}$ and $\tilde{n}$ stand for smooth extensions into Ω of $g(x,t)$ and $n(x,t)$. The reformulation (1.18) is discretized by the finite element and finite difference methods using fixed uniform meshes as well as the IB reformulation. Actually, formulation (1.18) is essentially equivalent to the IB formulation (1.12), whereas reformulation (1.18) makes analysis somewhat simpler than the others.

The purpose of Chapter 4 is to study convergence of the finite element approximation to a model Stokes problem based on the reformulation (1.18). In [21], the derivation of reformulation and some numerical results are presented; no mathematical analysis including convergence are given. Since the derivative of characteristic function $\nabla\chi$ has singularities on Γ, regularization problem is required again. Following the case of the IB method, the regularization error and the discretization error are studied separately.

[Main Result in Chapter 4] Numerical analysis of a Stokes interface problem based on formulation using the characteristic function

We assume that Ω is a convex polyhedral domain in $\mathbb{R}^d$ with $d = 2, 3$, and Γ is a C^2 boundary. Then, we consider a model Stokes problem for the velocity u and pressure π,

$$-\nu\Delta u + \nabla\pi = \tilde{g}(\nabla\chi\cdot\tilde{n}) \quad \text{in } \Omega, \quad \nabla\cdot u = 0 \quad \text{in } \Omega,\ u = 0 \quad \text{on } \partial\Omega, \tag{1.19}$$

and its regularized problem for u^ε and π^ε,

$$-\nu\Delta u^\varepsilon + \nabla\pi^\varepsilon = \tilde{g}(\nabla\chi^\varepsilon\cdot\tilde{n}) \quad \text{in } \Omega, \quad \nabla\cdot u^\varepsilon = 0 \quad \text{in } \Omega,\ u^\varepsilon = 0 \quad \text{on } \partial\Omega, \tag{1.20}$$

where $g \in W^{2-\frac{1}{p},p}$ is a given function for some $p > d$, and χ^ε is defined by

$$\chi^\varepsilon(x) = \begin{cases} 1 & (x \in \Omega_0) \\ \max\{0, 1 - \frac{\mathrm{dist}(x,\Gamma)}{\varepsilon}\} & (x \notin \Omega_0). \end{cases} \tag{1.21}$$

In Chapter 4, we derive the following error estimates in the $H^1 \times L^2$ and L^2 norm.

Theorem 1.2.2. *Let* (u,π) *and* $(u_h^\varepsilon, \pi_h^\varepsilon)$ *be respectively the solution to* (1.19) *and* (1.20) *with* (1.21). *In particular, if* $\varepsilon = c_1 h$ *with a positive constant* c_1*, then we have*

$$\|u - u_h^\varepsilon\|_{H^1} + \|\pi - \pi_h^\varepsilon\|_{L^2} \le C\sqrt{h}. \tag{1.22}$$

Else if $\varepsilon = c_1 h^2$ *then we have*

$$\|u - u_h^\varepsilon\|_{L^2} \le Ch. \tag{1.23}$$

where C *denotes a positive constant depending only on* $\Omega, \Gamma, \|g\|_{W^{2-\frac{1}{p},p}}$ *and* c_1.

In this thesis, there are some situations when we want to use some different notations such as n or d for the number of dimension, p or π for the pressure solution, and so on, which are suitable for the arguments in each chapter. In order that we want readers to avoid to confuse the formulations, we explicitly write the definitions in each chapter. Although multiple same definitions may appear through the thesis, the arguments in each chapter become self contained. This helps readers understand the detailed content of each chapter separately.

Chapter 2

Unilateral problem for the Stokes equations: the well-posedness and finite element approximation

Abstract

We consider the stationary Stokes equations under a unilateral boundary condition of Signorini's type, which is one of artificial boundary conditions in flow problems. Well-posedness is discussed through its variational inequality formulation. We also consider the finite element approximation for a regularized penalty problem. The well-posedness, stability and error estimates of optimal order are established. The lack of a coupled Babuška and Brezzi's condition makes analysis difficult. We offer a new method of analysis. Particularly, our device to treat the pressure is novel and of some interest. Numerical examples are presented to validate our theoretical results.

2.1 Introduction

We suppose that Ω is a bounded domain in $\mathbb{R}^d$ with $d = 2, 3$ and that the boundary $\partial\Omega$ is comprises of three parts S_1, S_2 and Γ. Those S_1, S_2 and Γ are assumed to be smooth but the whole boundary $\partial\Omega$ is not necessarily smooth. One might imagine a branched pipe resembling that depicted in Fig. 2.1. The first purpose of this chapter is to study the well-posedness of the following unilateral boundary value problem for the Stokes equations

$$
\begin{aligned}
&-\nu\Delta u + \nabla p = f, \quad \nabla\cdot u = 0 && \text{in } \Omega, && (2.1a)\\
&u = 0 && \text{on } S_1 \cup S_2, && (2.1b)\\
&u_n + g_n \geq 0, && \text{on } \Gamma, && (2.1c)\\
&\tau_n(u,p) + \alpha_n \geq 0 && \text{on } \Gamma, && (2.1d)\\
&(u_n + g_n)(\tau_n(u,p) + \alpha_n) = 0 && \text{on } \Gamma, && (2.1e)\\
&\tau_T(u) + \alpha_T = 0 && \text{on } \Gamma && (2.1f)
\end{aligned}
$$

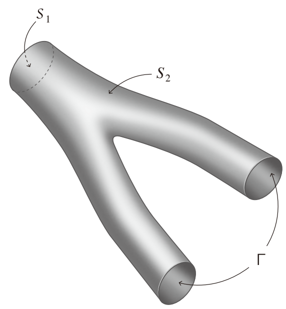

Figure 2.1: Example of Ω (branched pipe)

for velocity $u = (u_1, \ldots, u_d)$ and pressure p with density $\rho = 1$ and kinematic viscosity ν of the viscous incompressible fluid under consideration. Therein,

$$\tau(u, p) = \sigma(u, p)n$$

denotes the traction vector on $\partial\Omega$, where n is the outward normal vector to $\partial\Omega$, $\sigma(u, p) = (\sigma_{ij}(u, p))_{1\le i,j\le d} = -pI + 2\nu D(u)$ the stress tensor, $D(u) = (D_{ij}(u))_{1\le i,j\le d} = \frac{1}{2}\left(\nabla u + \nabla u^{\mathrm{T}}\right)$ the deformation-rate tensor and I the identity matrix. For a vector-valued function v on $\partial\Omega$, its normal and tangential components are denoted, respectively, as

$$v_n = v \cdot n, \quad v_T = v - v_n n.$$

Particularly, $\tau_n(u, p) = \tau(u, p) \cdot n$ and $\tau_T(u) = \tau(u, p) - \tau_n(u, p)n$ respectively denote normal and tangential traction vectors. Moreover, f, g and α are prescribed functions. We also consider the finite element approximation for a regularized penalty problem to (2.1) which is given as

$$-\nu\Delta u + \nabla p = f, \quad \nabla \cdot u = 0 \qquad \text{in } \Omega, \tag{2.2a}$$

$$u = 0 \qquad \text{on } S_1 \cup S_2, \tag{2.2b}$$

$$\tau_n(u, p) + \alpha_n = \frac{1}{\varepsilon}\phi_\delta(u_n + g_n) \qquad \text{on } \Gamma, \tag{2.2c}$$

$$\tau_T(u) + \alpha_T = 0 \qquad \text{on } \Gamma \tag{2.2d}$$

with $0 < \varepsilon \ll 1$ and $0 < \delta \ll 1$. Therein, $\phi_\delta(s)$ is a C^1 regularization of $[s]_- = \max\{0, -s\}$. We can take, for example,

$$\phi_\delta(s) = \begin{cases} 0 & (s \ge 0) \\ (\sqrt{s^2 + \delta^2} - \delta) & (s < 0). \end{cases} \tag{2.3}$$

First, we explain our motivation for studying (2.1) and (2.2). In numerical simulation of real-world flow problems, we often encounter some issues related to artificial boundary conditions. A typical and important example is the blood flow problem in the large arteries, where the blood is assumed to be a viscous incompressible fluid (see [20, 53]). The blood vessel is modeled as a branched pipe as illustrated, for example, in Fig. 2.1. Then, for $T > 0$, we consider the Navier–Stokes equations for velocity $v = (v_1, \ldots, v_d)$ and pressure q,

$$v_t + (v \cdot \nabla)v = \nabla \cdot \sigma(v, q) + f, \quad \nabla \cdot v = 0 \qquad \text{in } \Omega \times (0, T), \tag{2.4a}$$

$$v = b \qquad \text{on } S_1 \times (0, T), \tag{2.4b}$$

$$v = 0 \qquad \text{on } S_2 \times (0, T) \tag{2.4c}$$

with the initial condition $v|_{t=0} = v_0$. We are able to give a velocity profile $b = b(x,t)$ at the *inflow boundary* S_1. The flow is presumed to be a perfect non-slip flow on the wall S_2. Then, the blood flow simulation is highly dependent on the choice of artificial boundary conditions posed on the *outflow boundary* Γ.

An earlier paper by Zhou and Saito [55] presented discussion of the free-traction condition

$$\tau(v,q) = 0 \quad \text{on} \quad \Gamma, \tag{2.5}$$

which is one of the common outflow boundary conditions (see [24, 27]), and some nonlinear energy-preserving boundary conditions (see [4, 9, 10, 13, 14]) from the view-point of energy inequality. Moreover, we proposed a new outflow boundary condition as

$$v_n \geq 0, \quad \tau_n(v,q) \geq 0, \quad v_n\tau_n(v,q) = 0, \quad \tau_T(v) = 0 \quad \text{on } \Gamma. \tag{2.6}$$

This is an analogy to Signorini's condition in the theory of elasticity (see [32]). It is indeed a generalization of the free-traction condition (2.5), as

$$\text{if } v_n > 0 \text{ on } \omega \subset \Gamma, \text{ then } \tau_n(v,q) = 0 \text{ on } \omega;$$
$$\text{if } v_n = 0 \text{ on } \omega \subset \Gamma, \text{ then } \tau_n(v,q) \geq 0 \text{ on } \omega.$$

A benefit of using (2.6) is that (2.4) admits energy inequality, although it is not guaranteed under (2.5). To describe it more specifically, we take a *reference flow* (g,π), which is the solution of the Stokes system

$$\nabla\cdot\sigma(g,\pi) = 0, \quad \nabla\cdot g = 0 \quad \text{in } \Omega, \tag{2.7a}$$

$$g = b \text{ on } S_1, \quad g = 0 \text{ on } S_2, \quad g = -g_0(x)\int_{S_1} b\cdot n \, dS_1 \text{ on } \Gamma \tag{2.7b}$$

for all $t \in [0,T]$, where $g_0 = g_0(x) \in C_0^\infty(\Gamma)^d$ is a prescribed function satisfying

$$\int_\Gamma g_0\cdot n \, d\Gamma = 1, \quad g_0\cdot n \geq 0 \ \text{ on } \Gamma. \tag{2.8}$$

(Function g is nothing but a lifting function of b.) Using this, we will find (v,q) of the form

$$v = u + g, \qquad q = p + \pi.$$

Assuming that (2.4) admits a smooth solution $(v,q) = (u+g, p+\pi)$ in $0 \leq t \leq T$ and multiplying both sides of (2.4a) by u, by the integration by parts, we have

$$\frac{d}{dt}\|u\|^2_{L^2(\Omega)^d} + \underbrace{2\nu\int_\Omega D_{ij}(u)D_{ij}(u)\,dx + \frac{1}{2}\int_\Gamma v_n|u|^2\,d\Gamma - \int_\Gamma \tau(v,q)\cdot u\,d\Gamma}_{=I}$$
$$= \int_\Omega [f - g_t - (g\cdot\nabla)g]\cdot u\,dx - \int_\Omega (u\cdot\nabla)g\cdot u\,dx. \tag{2.9}$$

Using (2.6), we derive $I \geq 0$. Consequently,

$$\sup_{t\in[0,T]} \|u\|^2_{L^2(\Omega)^d} + 2\nu\int_0^T D_{ij}(u)D_{ij}(u) \leq C, \tag{2.10}$$

where C denotes a positive constant depending only on f, u_0, b and T (see [55, Theorem 4]). This inequality is useful. It plays a crucial role in the construction of a solution of the Navier–Stokes equations (see [55]). Moreover, it is connected with the stability of numerical schemes from the view-point of numerical computation. That is, it is preferred that energy inequality not be spoiled after discretizations (see [54]). With (2.5), we do not know whether $I \geq 0$ or not. Therefore, energy inequality cannot be derived even for the continuous case.

There are a lot of strategies to treat the inequality condition (2.6) in numerical calculations. For example, we can use its penalty approximation

$$\tau_n(v,q) = \frac{1}{\varepsilon}[v_n]_-, \quad \tau_T(v) = 0 \quad \text{on } \Gamma, \tag{2.11}$$

where $0 < \varepsilon \ll 1$ and

$$[s]_\pm = \max\{0, \pm s\}, \quad s = [s]_+ - [s]_- \quad (s \in \mathbb{R}). \tag{2.12}$$

We also obtain energy inequality with (2.11) for a sufficiently small ε (see [55, Theorem 5]). Moreover, after introducing a C^1 regularization ϕ_δ of $[\cdot]_-$, we can solve (2.4) with (2.11) by using, for example, Newton's iteration.

Our ultimate objective is to develop the theory for the initial-boundary value problems for the Navier–Stokes equations (2.4) with (2.6) or with (2.11) from the dual standpoints of analysis and numerical computations. As a primary step, we studied the well-posedness of these problems in Ladyzhenskaya's class in [55]. That is, we studied the unique existence of a solution of

$$\begin{aligned}
&u_t + ((u+g)\cdot\nabla)u + (u\cdot\nabla)g - \nabla\cdot\sigma(u,p) = F, \quad \nabla\cdot u = 0 && \text{in } \Omega,\\
&u = 0 && \text{on } S_1 \cup S_2,\\
&u_n + g_n \geq 0, \quad \tau_n(u,p) + \tau_n(g,\pi) \geq 0 && \text{on } \Gamma,\\
&(u_n + g_n)(\tau_n(u,p) + \tau_n(g,\pi)) = 0, \quad \tau_T(u) + \tau_T(g) \geq 0 && \text{on } \Gamma,
\end{aligned}$$

where $F = f - g_t - (g\cdot\nabla)g$.

For the analyses described herein, we devote our attention to the discretization of the space variable. Therefore, we study the finite element approximation using model Stokes problems. Consequently, we come to consider Problems (2.1) and (2.2).

As a matter of fact, (2.1) and (2.2) themselves are not new problems (see [5, 15, 18, 32] for example). In a classical monograph [32], Chapter 7 is devoted to similar problems, say Signorini's problem for incompressible materials. However, their problem includes the traction condition $\tau(u,p) = h$. More precisely, they assume that S_2 is divided into two parts S_{21}, S_{22} and consider

$$u = 0 \quad \text{on } S_{21}, \quad \tau(u,p) = h \quad \text{on } S_{22}$$

instead of (2.1b). Then, assuming

$$\overline{\Gamma} \cap \overline{(S_1 \cup S_{21})} = \emptyset, \tag{2.13}$$

we can prove that there exists a domain constant $C > 0$ satisfying

$$C\left[\|q\|_{L^2(\Omega)} + \|\tau\|_{H^{-\frac{1}{2}}(\Gamma)}\right] \leq \sup_{v\in H^1(\Omega)^d, v|_{S_1\cup S_{21}}=0} \frac{\int_\Omega q(\nabla\cdot v)\ dx + \int_\Gamma \tau v_n\ d\Gamma}{\|v\|_{H^1(\Omega)}} \tag{2.14}$$

for any $(q,\tau) \in L^2(\Omega) \times H^{-\frac{1}{2}}(\Gamma)$ (see [32, Theorem 7.2]). This inequality is usually designated as the coupled Babuška-Brezzi condition. The well-posedness and error estimates of the corresponding penalty problem (with no regularization) are direct consequences of this result from the general theory (see [6]). [15, 18] study similar unilateral problems and they also assume $S_{22} \neq \emptyset$. In contrast, we are interested in establishing a formulation without the traction boundary condition. Unfortunately, if $S_{22} = \emptyset$, then (2.14) is not available. It makes analysis somewhat difficult. Moreover, we do not prefer assuming (2.13). Consequently, we must develop a completely new method of analysis in this work. Particularly, we offer a new device to treat the pressure part. As a result, we succeed in deriving the optimal-order error esimate for the finite element approximation for the penalty problem using the MINI (P1b/P1) element (see Thereom 2.6.1). Our method is different from that of [18] in which the optimal-order error estimates are proved for linear and quadratic finite element approximations for Signorini's contact problem (with no penalization). We first derive non-optimal error estimates and then apply them to improve several estimates. Our error estimates are novel even in the literature of elasticity theory (see [15]).

Finite element approximation of another class of unilateral boundary value problems for the Stokes equations, say unilateral problems of *friction type*, are discussed, for example, in several reports of the literature [2, 30, 33, 34].

This chapter is composed of 7 sections. After introducing basic definitions and recalling some standard results in Section 2.2, we state two variational formulations, (PDE) and (VI), for (2.1) in Section 2.3. The equivalence of these formulations and well-posedness of (VI) are also established there (see Theorems 2.3.1 and 2.3.2). Section 2.4 is devoted to presentation of the finite element approximation for (2.2). We consider only the MINI (P1b/P1) element approximation. The well-posedness and error estimates are proved, respectively, in Sections 2.5 and 2.6 (see Theorems 2.5.1, 2.5.2 and 2.6.1). Finally, we confirm our results by numerical experiments in Section 2.7.

2.2 Preliminaries

Geometry We recall that $\Omega \subset \mathbb{R}^d$, $d = 2, 3$, is a bounded domain and the boundary $\partial\Omega$ comprised of three parts S_1, S_2 and Γ. We address the following two cases:

(G1) S_1, S_2 and Γ are smooth surfaces (curves) and Ω is a Lipschitz domain (see Fig. 2.1);

(G2) S_1, S_2 are polygon (line segment) and Ω is a polyhedral (polygonal) domain.

For the following, we assume that Ω is given as (G1) or (G2) unless otherwise stated explicitly. Moreover, we set

$$S = S_1 \cup S_2$$

and assume that the $d-1$ dimensional Lebesgue measure $|S|$ is positive.

Throughout this chapter, C denotes various positive constants depending on Ω.

Remark 2.2.1. *Although we mostly address the case presented in Fig. 2.1, our discussion is also valid for the case in which $\partial\Omega$ is smooth with $\overline{\Gamma} \cap \overline{S}_2 = \emptyset$, $\overline{S}_2 \cap \overline{S}_1 = \emptyset$, and $\overline{S}_1 \cap \overline{\Gamma} = \emptyset$.*

Function spaces and forms We use the standard Lebesgue and Sobolev spaces, for example, $L^2(\Omega)$, $H^1(\Omega)$, $L^2(\Gamma)$, $H^{\frac{1}{2}}(\Gamma)$. (We follow the notation of [38] as for function spaces and their norms.) The abbreviations

$$(v, w) = (v, w)_\Omega = (v, w)_{L^2(\Omega)}, \quad (v, w)_\Gamma = (v, w)_{0,\Gamma} = (v, w)_{L^2(\Gamma)},$$
$$\|v\| = \|v\|_\Omega = \|v\|_{0,\Omega} = \|v\|_{L^2(\Omega)}, \quad \|v\|_1 = \|v\|_{1,\Omega} = \|v\|_{H^1(\Omega)}, \quad \|v\|_\Gamma = \|v\|_{0,\Gamma} = \|v\|_{L^2(\Gamma)}$$

will be used. Moreover,

$$|v|_m = |v|_{m,\Omega} = |v|_{H^m(\Omega)}, \qquad |v|_{m,\Gamma} = |v|_{H^m(\Gamma)}$$

are the semi-norms of $H^m(\Omega)$, $H^m(\Gamma)$.

For a vector-valued function space, we use the same symbol to denote its norm as

$$\|v\| = \|v\|_{L^2(\Omega)^d} \quad (v \in L^2(\Omega)^d), \qquad \|v\|_1 = \|v\|_{H^1(\Omega)^d} \quad (v \in H^1(\Omega)^d).$$

The basic function spaces of our consideration are

$$V = \{v \in H^1(\Omega)^d \mid v = 0 \text{ on } S\} \quad \text{and} \quad Q = L^2(\Omega).$$

They are, respectively, Hilbert spaces equipped with the norms $\|v\|_1$ and $\|q\|$. We use closed subspaces of V,

$$V^\sigma = \{v \in V \mid \nabla \cdot v = 0 \text{ in } \Omega\}, \quad V_0 = H_0^1(\Omega)^d, \quad V_0^\sigma = \{v \in V_0 \mid \nabla \cdot v = 0 \text{ in } \Omega\},$$

and that of Q as

$$Q_0 = \left\{ q \in Q \mid \int_\Omega q \ dx = 0 \right\}.$$

Convex subsets

$$K = \{v \in V \mid v_n + g_n \geq 0 \text{ on } \Gamma\} \quad \text{and} \quad K^\sigma = \{v \in V^\sigma \mid v_n + g_n \geq 0 \text{ on } \Gamma\}$$

of V and V^σ, respectively, play important roles.

We recall the so-called *Lions–Magenes space* $H_{00}^{\frac{1}{2}}(\Gamma)$. It is defined as (see [38, §11.5, Ch. 1])

$$H_{00}^{\frac{1}{2}}(\Gamma) = \{\mu \in H^{\frac{1}{2}}(\Gamma) \mid \rho^{-1/2}\mu \in L^2(\Gamma)\}$$

which is a Hilbert space equipped with the norm

$$\|\mu\|_{H_{00}^{\frac{1}{2}}(\Gamma)} = \left(\|\mu\|^2_{H^{\frac{1}{2}}(\Gamma)} + \|\rho^{-1/2}\mu\|^2_\Gamma \right)^{\frac{1}{2}}.$$

Here, $\rho \in C^\infty(\overline{\Gamma})$ denotes any positive function satisfying $\rho|_{\partial\Gamma} = 0$, and for $x_0 \in \partial\Gamma$,

$$\lim_{x \to x_0} \frac{\rho(x)}{\text{dist}\ (x, \partial\Gamma)} = d' > 0$$

with some $d' > 0$. Moreover, we know (see [38, Theorem 11.7, Ch. 1])

$$H_{00}^{\frac{1}{2}}(\Gamma) = (H_0^1(\Gamma), L^2(\Gamma))_{\frac{1}{2},2} \quad \text{(algebraically and topologically)}, \tag{2.15}$$

where the right-hand side denotes the real interpolation space between $L^2(\Gamma)$ and $H_0^1(\Gamma)$ with the exponent $1/2$ and $p = 2$. Particularly $H_{00}^{\frac{1}{2}}(\Gamma)$ is *strictly* included in $H^{\frac{1}{2}}(\Gamma)$.

Below we set

$$M = H_{00}^{\frac{1}{2}}(\Gamma), \quad \|\mu\|_{\frac{1}{2},\Gamma} = \|\mu\|_{H_{00}^{\frac{1}{2}}(\Gamma)}$$

and

$$M_0 = \left\{ \mu \in M \mid \int_\Gamma \mu \ d\Gamma = 0 \right\}.$$

In general, X' denotes the topological dual space of a Banach space X. The norm of X' is defined as

$$\|\varphi\|_{X'} = \sup_{v\in X, v\neq 0} \frac{\langle \varphi, v\rangle_{X',X}}{\|v\|_X},$$

where $\langle\cdot,\cdot\rangle_{X',X}$ is the duality pairing between X' and X. For a closed subspace Y of X and $\varphi \in Y'$, we mean by $\|\varphi\|_{Y'}$

$$\|\varphi\|_{Y'} = \sup_{v\in Y,\ v\neq 0} \frac{\langle \varphi, v\rangle_{X',X}}{\|v\|_X}.$$

Set

$$\begin{aligned}
\langle\cdot,\cdot\rangle = \langle\cdot,\cdot\rangle_{V',V} &= \text{ the duality pairing between } V' \text{ and } V,\\
[\cdot,\cdot] = [\cdot,\cdot]_{M',M} &= \text{ the duality pairing between } M' \text{ and } M,\\
[[\cdot,\cdot]] = [[\cdot,\cdot]]_{(M^d)',M^d} &= \text{ the duality pairing between } (M^d)' \text{ and } M^d.
\end{aligned}$$

We use the following forms:

$$\begin{aligned}
a(u,v) &= 2\nu \int_\Omega D_{ij}(u) D_{ij}(v)\ dx && (u, v \in H^1(\Omega)^d);\\
b(p,u) &= -\int_\Omega p(\nabla\cdot u)\ dx && (p\in Q,\ u\in H^1(\Omega)^d).
\end{aligned}$$

Trace and lifting operators Let $\mathrm{Tr} = \mathrm{Tr}(\Omega,\Gamma)$ be a trace operator from $H^1(\Omega)$ into $H^{\frac{1}{2}}(\Gamma)$. The meaning of $\mathrm{Tr}(\Omega, S)$ is the same.

Lemma 2.2.1. *The trace operator $v \mapsto \mu = \mathrm{Tr}\, v$ is linear and continuous of $V \to M^d$. Conversely, there exists a linear and bounded operator $\mathcal{E}$ of $M^d \to V$, which is called a lifting operator, such that $\mathcal{E}\mu = \mu$ on Γ for all $\mu \in M^d$.*

This result follows directly from [25, Theorem 2.5] and [26, Theorem 1.5.2.3]. A partial result is also reported in [48, Theorems 1.1 and 5.1]. As a consequence of Lemma 2.2.1, we obtain a lifting operator $\mathcal{E}_n : M \to V$ such that

$$(\mathcal{E}_n\mu)_n = \mu, \quad (\mathcal{E}_n\mu)_T = 0 \ \text{ on } \Gamma, \qquad \|\mathcal{E}_n\mu\|_1 \leq C\|\mu\|_{\frac{1}{2},\Gamma}$$

for any $\mu \in M$.

Below, we will often write as $v|_\Gamma = \mathrm{Tr}\, v$ if there is no fear of confusion.

Remark 2.2.2. *In view of Lemma 2.2.1 and the standard trace/lifting theorem, the zero extension $\hat{\mu}$ of $\mu \in M^d$ into $\partial\Omega$;*

$$\hat{\mu} = \begin{cases} \mu & \text{on } \Gamma,\\ 0 & \text{on } \partial\Omega\backslash\Gamma \end{cases}$$

belongs to $H^{\frac{1}{2}}(\partial\Omega)^d$.

Remark 2.2.3. *Another definition of $H^{\frac{1}{2}}_{00}(\Gamma)$ is given by Baiocchi and Capelo [3, Page 379]. That is,*

$$H^{\frac{1}{2}}_{00}(\Gamma) = \{\mathrm{Tr}\, v \mid v \in H^1(\Omega),\ \ \mathrm{Tr}(\Omega, S)v = 0\}$$

which is a Hilbert space equipped with the norm

$$\|\mu\|_{H^{\frac{1}{2}}_{00}(\Gamma)} = \inf\{\|v\|_1 \mid v\in H^1(\Omega),\ \ \mathrm{Tr}(\Omega,S)v = 0,\ \ \mathrm{Tr}\, v = \mu\}.$$

Redefinition of traction vectors Next we propose the redefinition of $\tau(u,p)$. If a smooth vector field u and scalar field p satisfy the Stokes equation

$$-\nu\Delta u + \nabla p = f, \quad \nabla \cdot u = 0 \quad \text{in } \Omega$$

for a given $f \in L^2(\Omega)^d$, they satisfy

$$a(u,v) + b(p,v) + \int_\Gamma \tau(u,p) \cdot v = (f,v) \qquad (\forall v \in V) \tag{2.16}$$

and

$$a(u,v) + b(p,v) = (f,v) \qquad (\forall v \in V_0). \tag{2.17}$$

(In (2.16), $\tau(u,p)$ is understood as a usual function defined on Γ.) Based on those identities, for functions $(u,p) \in V^\sigma \times Q$ satisfying (2.17), we redefine the traction vector $\tau(u,p)$ as a functional over M^d by

$$[[\tau(u,p),\mu]] = a(u,w_\mu) + b(p,w_\mu) - (f,w_\mu) \qquad (\mu \in M^d), \tag{2.18}$$

where $w_\mu = \mathcal{E}\mu \in V$. Actually, the right-hand side of (2.18) is independent of the way of extension. Therefore, this definition is well-defined. Similarly, we redefine

$$[[\tau_T(u),\mu]] = a(u,w_\mu) + b(p,w_\mu) - (f,w_\mu) \qquad (\mu \in M^d \text{ with } \mu_n = 0;\ w_\mu = \mathcal{E}\mu) \tag{2.19}$$

and

$$[\tau_n(u,p),\mu] = a(u,w_\mu) + b(p,w_\mu) - (f,w_\mu) \qquad (\mu \in M;\ w_\mu = \mathcal{E}_n\mu). \tag{2.20}$$

Then, we deduce an expression

$$[[\tau(u,p),\mu]] = [\tau_n(u,p),\mu_n] + [[\tau_T(u),\mu_T]] \qquad (\mu \in M^d). \tag{2.21}$$

2.3 Variational formulations and well-posedness

From this point forward in our discussion, we always assume

$$f \in Q^d, \qquad b \in M^d, \qquad \beta \equiv -\int_S b \cdot n \, dS > 0.$$

We take $g \in H^1(\Omega)^d$ satisfying

$$\nabla \cdot g = 0 \text{ in } \Omega, \quad g|_S = b, \quad g|_C = 0, \quad g|_\Gamma = \beta g_0,$$

where g_0 is the function defined as (2.8). Then, we have

$$g_n \geq 0 \text{ on } \Gamma, \qquad g_n \in M, \qquad \alpha \equiv 2\nu D(g)n \in (M^d)'$$

Under those assumptions and redefinitions in the previous section, we interpret (2.1) precisely as follows.

(PDE) Find $(u,p) \in V \times Q$ such that

$$a(u,v) + b(p,v) = (f,v) \qquad (\forall v \in V_0), \tag{2.22a}$$

$$b(q,u) = 0 \qquad (\forall q \in Q), \tag{2.22b}$$

$$u_n + g_n \geq 0 \qquad \text{a.e. on } \Gamma, \tag{2.22c}$$

$$[\tau_n(u,p) + \alpha_n, \mu] \geq 0 \qquad (\forall \mu \in M,\ \mu \geq 0), \tag{2.22d}$$

$$[\tau_n(u,p) + \alpha_n, u_n + g_n] = 0 \tag{2.22e}$$

$$[[\tau_T(u) + \alpha_T, \mu]] = 0 \qquad (\forall \mu \in M^d,\ \mu_n = 0). \tag{2.22f}$$

If a solution (u,p) of (PDE) is sufficiently smooth, it solves (2.1) in the classical sense. Actually, (PDE) is equivalent to the following variational inequality.

(VI) Find $(u,p) \in K \times Q$ such that

$$a(u, v-u) + b(p, v-u) \geq (f, v-u) - [[\alpha, v-u]] \qquad (\forall v \in K), \tag{2.23a}$$
$$b(q,u) = 0 \qquad (\forall q \in Q). \tag{2.23b}$$

In this section, we prove the following two theorems.

Theorem 2.3.1. *Problems* (VI) *and* (PDE) *are equivalent.*

Theorem 2.3.2. *There exists a unique solution* $(u,p) \in K \times Q$ *of* (VI) *and it holds that*

$$\|u\|_1 + \|p\| \leq C_*, \tag{2.24}$$

where C_* *denotes a positive constant depending only on* Ω*,* $\|f\|$*,* $\|\alpha\|_{(M^d)'}$ *and* $\|g\|_1$*.*

Remark 2.3.1. *The boundary condition* (2.22f) *is nothing but one alternative. One can pose*

$$u_T + \alpha'_T = 0 \quad \text{a.e. on } \Gamma \tag{2.25}$$

with a prescribed α'_T *instead of* (2.22f)*. Actually, the discussion presented below remains true if we re-choose a suitable lifting function* g *and replace the original* V *with*

$$V = \{v \in H^1(\Omega)^d \mid v = 0 \text{ on } \Gamma,\ v_T = 0 \text{ on } \Gamma\}.$$

Proof of Theorem 2.3.1. (PDE)$\Rightarrow$(VI). Let $(u,p) \in V \times Q$ be a solution of (PDE). We verify (u,p) is a solution of (VI). First, we have $u \in K$ by (2.22c) and (2.22b). By using (2.18), (2.21), and (2.22c)–(2.22f), we have for any $v \in K$

$$\begin{aligned}
&a(u, v-u) + b(p, v-u) - (f, v-u) + [[\alpha, v-u]] \\
&= [[\tau(u,p), v-u]] + [[\alpha, v-u]] \\
&= [\tau_n(u,p), v_n - u_n] + \underbrace{[[\tau_T(u) + \alpha_T, v_T - u_T]]}_{=0} + [\alpha_n, v_n - u_n] \\
&= \underbrace{[\tau_n(u,p) + \alpha_n, v_n + g_n]}_{\geq 0} - \underbrace{[\tau_n(u,p) + \alpha_n, u_h + g_n]}_{=0} \geq 0.
\end{aligned}$$

(VI)$\Rightarrow$(PDE). Let $(u,p) \in K \times Q$ be a solution of (VI). We now verify (u,p) actually satisfies (PDE). First, (2.22b) and (2.22c) are obvious.

Let $v \in V_0$ be arbitrary. Substituting $v = u \pm v \in K$ into (2.23a), we have (2.22a).

We recall $\tau(u,p)$ is defined as (2.18). Consequently, (2.23a) implies

$$[[\tau(u,p), v-u]] \geq -[[\alpha, v-u]] \qquad (\forall v \in K).$$

Moreover, by (2.21)

$$[\tau_n(u,p) + \alpha_n, v_n - u_n] + [[\tau_T(u) + \alpha_T, v_T - u_T]] \geq 0 \qquad (\forall v \in K). \tag{2.26}$$

Let $\psi \in C_0^\infty(\Gamma)^d$ with $\psi_n = 0$ on Γ. Substituting $v = u \pm \mathcal{E}\psi \in K$ into (2.26), we have $[[\tau_T(u) + \alpha_T, \psi]] = 0$. By density, this implies (2.22f).

Next, let $\psi \in C_0^\infty(\Gamma)^d$ with $\psi_n \geq 0$ and $\psi_T = 0$ on Γ. Substituting $v = u + \mathcal{E}\psi \in K$ into (2.26), we have $[\tau_n(u,p) + \alpha_n, \psi_n] \geq 0$. By density, this implies (2.22d).

Combining (2.26) and (2.22f), we have

$$[\tau_n(u,p)+\alpha_n, v_n - u_n] \geq 0 \qquad (\forall v \in K).$$

At this stage, we introduce $w^* \in V$ satisfying

$$w^* = g \text{ on } \Gamma, \quad \|w^*\|_1 \leq C\|g\|_1. \tag{2.27}$$

Since $g\,|_\Gamma \in M^d$, such w^* truly exists in view of the trace theorem. However, it does not satisfy the divergence-free condition. Consequently, $w^* \notin V^\sigma$. We now have $-w_n^* + g_n = -g_n + g_n \geq 0$ and $2u_n + w_n^* + g_n = 2(u_n + g_n) \geq 0$. Therefore, we can choose as $v = -w^*$ and $v = 2u + w^*$ above and obtain (2.22e). □

Proof of Theorem 2.3.2. Since a is a coercive bilinear form in $V^\sigma \times V^\sigma$ by virtue of Korn's inequality (see [32, Lemma 6.2]), we can apply Stampacchia's theorem (see [12, Theorem 5.6]) to conclude that there exists a unique $u \in K^\sigma$ satisfying the following:

$$a(u, v-u) \geq (f, v-u) - [[\alpha, v-u]] \qquad (\forall v \in K^\sigma). \tag{2.28}$$

Taking $v = u \pm \varphi$ with $\varphi \in V_0^\sigma$ in (2.28), we deduce

$$a(u,\varphi) = (f,\varphi) \qquad (\forall \varphi \in V_0^\sigma). \tag{2.29}$$

Therefore, according to [23, Lemma I.2.1], there exists a unique $\hat{p} \in Q_0$ satisfying

$$-b(\hat{p}, v) = a(u,v) - (f,v) \qquad (\forall v \in V_0). \tag{2.30}$$

Now we set, for $k \in \mathbb{R}$,

$$p_k = \hat{p} + k \tag{2.31}$$

and verify, with an appropriate choice of k, that (u, p_k) is a solution of (VI). To this end, it suffices to check that (u, p_k) is a solution of (PDE).

We have by (2.18) and (2.28)

$$[\tau_n(u,p_k), v_n - u_n] + [[\tau_T(u), v_T - u_T]] \geq -[[\alpha, v-u]] \qquad (v \in K^\sigma). \tag{2.32}$$

Letting $\psi \in C_0^\infty(\Gamma)^d$ with $\psi_n = 0$ on Γ, then since $\int_\Gamma \psi_n \, d\Gamma = 0$, there is a function $w \in V$ satisfying $w = \psi$ on Γ, $\nabla \cdot w = 0$ in Ω and $\|w\|_1 \leq C\|\psi\|_{M^d}$. Substituting $v = u \pm w \in K^\sigma$ into (2.32), one obtains $[[\tau_T(u) + \alpha_T, \psi_T]] = 0$. By density, this implies (2.22f).

Therefore, it follows from (2.32) that

$$[\tau_n(u,p_k)+\alpha_n, v_n - u_n] \geq 0 \qquad (v \in K^\sigma). \tag{2.33}$$

We set

$$\gamma = \inf_{\mu \in Y} [\tau_n(u,\hat{p}) + \alpha_n, \mu],$$

where

$$Y = \left\{ \mu \in M \mid \mu \geq 0,\ \mu \not\equiv 0,\ \int_\Gamma \mu \, d\Gamma = 1 \right\}. \tag{2.34}$$

For any $\mu \in M$ with $\mu \geq 0$ and $\mu \not\equiv 0$, we have

$$[\tau_n(u,p_k)+\alpha_n, \mu] = [\tau_n(u,\hat{p}) + \alpha_n, \mu] - k\int_\Gamma \mu \geq \gamma \int_\Gamma \mu - k \int_\Gamma \mu$$

Therefore, we deduce (2.22d) if $k \le \gamma$.

For the time being, we admit that

$$\gamma = \frac{[\tau_n(u,\hat{p}) + \alpha_n, u_n + g_n]}{\beta}, \tag{2.35}$$

When $u_n + g_n = 0$ on Γ, we have $\gamma = 0$, but this is impossible because $u_n + g_n \ge 0$ and $\int_\Gamma g_n \, d\Gamma > 0$. Therefore, we have by (2.35)

$$[\tau_n(u,\hat{p}) + \alpha_n, u_n + g_n] = \gamma\beta = \gamma \int_\Gamma g_n \, d\Gamma = \gamma \int_\Gamma (u_n + g_n) \, d\Gamma.$$

Hence, taking

$$k = \gamma,$$

we obtain

$$[\tau_n(u,p_k) + \alpha_n, u_n + g_n] = [\tau_n(u,\hat{p}) + \alpha_n, u_n + g_n] - \gamma \int_\Gamma (u_n + g_n) = 0.$$

Therefore, we have verified (2.22e).

To show (2.35), we use $w^* \in V$ defined as (2.27) again. From (2.33) with $k = 0$,

$$[\tau_n(u,\hat{p}) + \alpha_n, v_n + w_n^*] \ge [\tau_n(u,\hat{p}) + \alpha_n, u_n + w_n^*] \qquad (v \in K^\sigma).$$

Since $w^* = g$ on Γ, this is expressed equivalently as

$$[\tau_n(u,\hat{p}) + \alpha_n, v_n + g_n] \ge [\tau_n(u,\hat{p}) + \alpha_n, u_n + g_n] \qquad (v \in K^\sigma).$$

Moreover, we obtain

$$\left[\tau_n(u,\hat{p}) + \alpha_n, \frac{v_n + g_n}{\beta}\right] \ge \left[\tau_n(u,\hat{p}) + \alpha_n, \frac{u_n + g_n}{\beta}\right] \qquad (v \in K^\sigma). \tag{2.36}$$

Now let $\mu \in Y$ be arbitrary and set $\tilde{\mu} = \beta\mu - g_n \in M$. Since $\int_\Gamma \tilde{\mu} \, d\Gamma = 0$, there exists $\tilde{v} \in V^\sigma$ such that $\tilde{v}_n = \tilde{\mu}$ on Γ according to Remark 2.2.2 and [23, Lemma I.2.2]. Then, the function $\tilde{v}$ satisfies that $\tilde{v}_n + g_n = \beta\mu \ge 0$ on Γ. Consequently, $\tilde{v} \in K^\sigma$. Therefore, we have by (2.36) that

$$\begin{aligned}
[\tau_n(u,\hat{p}) + \alpha_n, \mu] &= \left[\tau_n(u,\hat{p}) + \alpha_n, \frac{\tilde{\mu} + g_n}{\beta}\right] \\
&= \left[\tau_n(u,\hat{p}) + \alpha_n, \frac{\tilde{v}_n + g_n}{\beta}\right] \ge \frac{1}{\beta}\left[\tau_n(u,\hat{p}) + \alpha_n, u_n + g_n\right],
\end{aligned}$$

which implies (2.35).

It remains to derive (2.24). First, from (2.30), we have

$$\|\hat{p}\| \le C \sup_{v \in V_0} \frac{|(f,v) - a(u,v)|}{\|v\|_1} \le C(\|f\| + \|u\|), \tag{2.37}$$

where we have used the standard infsup (Babuška and Brezzi's) condition (see [23, Corollary I.2.4])

$$\inf_{q \in Q_0} \sup_{v \in V_0} \frac{b(q,v)}{\|q\|\|v\|_1} \ge C.$$

Equation (2.22e), together with (2.20), implies

$$a(u, u+g) + b(p, u+g) - (f, u+g) + [\alpha_n, u_n + g_n] = 0.$$

Therefore, by virtue of Korn's inequality (see [32, Lemma 6.2]),

$$C\|u+g\|_1^2 \le C(\|f\| + \|\alpha\|_{(M^d)'} + \|g\|_1)\|u+g\|_1$$

and, consequently,

$$\|u+g\|_1 \le C_*, \qquad \|u\|_1 \le C_*. \tag{2.38}$$

Finally, because of the expression (2.35), we can estimate as

$$|\gamma| \le \frac{1}{\beta}\|\tau_n(u,\hat{p}) + \alpha_n\|_{M'}\|u_n + g_n\|_{\frac{1}{2},\Gamma} \le C(\|u\|_1 + \|\hat{p}\|)\|u+g\|_1. \tag{2.39}$$

Combining this with (2.37) and (2.38), we obtain (2.24). □

2.4 Finite element approximation

While there are a lot of strategies for solving the variational inequality problem (VI), we concentrate our attention to its penalty approximation.

As a regularization of $[s]_-(s \in \mathbb{R})$, we introduce a function $\phi_\delta : \mathbb{R} \to \mathbb{R}$ that satisfies

$$\phi_\delta \text{ is a non-increasing } C^1(\mathbb{R}) \text{ function;} \tag{2.40a}$$

$$|\phi_\delta(s) - [s]_-| \le C\delta \quad (s \in \mathbb{R}); \tag{2.40b}$$

$$\phi_\delta(s) = 0 \quad (s \ge 0), \qquad 0 \le \phi_\delta(s) \le -s \quad (s < 0); \tag{2.40c}$$

$$\left|\frac{d}{ds}\phi_\delta(s)\right| \le C \qquad (s \in \mathbb{R}), \tag{2.40d}$$

where $\delta \in (0,1]$ is a regularized parameter and C's are independent of δ. As described in the Introduction, we can take, for example, the function $\phi_\delta(s)$ defined as (2.3).

For penalty parameter $\varepsilon \in (0,1]$, we consider the following penalty problem,

(PE$_{\varepsilon,\delta}$) Find $(u,p) \in V \times Q$ such that

$$a(u,v) + b(p,v) - \frac{1}{\varepsilon}\int_\Gamma \phi_\delta(u_n + g_n)v_n \, d\Gamma = (f,v) - [[\alpha, v]] \qquad (\forall v \in V), \tag{2.41a}$$

$$b(q,u) = 0 \qquad (\forall q \in Q). \tag{2.41b}$$

This and the subsequent sections are devoted to the finite element approximation of (PE$_{\varepsilon,\delta}$). To avoid unimportant difficulties related to the "curved boundary", we consider only the case (G2). Consequently, the unit outer normal vector n to Γ is a constant vector over Γ.

We use the so-called MINI (P1b/P1) elements for discretization. Let $\{\mathcal{T}_h\}_h$ be a *regular* family of triangulations of Ω. As the granularity parameter, we used $h = \max\{h_T \mid T \in \mathcal{T}_h\}$, where h_T denotes the diameter of T. We introduce the following function spaces:

$$V_h = \{v_h \in C^0(\overline{\Omega}) \mid v_h = 0 \text{ on } S,\ v_h|_T \in [\mathcal{P}_1^{(d)} \oplus \mathrm{span}\{\varphi_T\}]^d \ (\forall T \in \mathcal{T}_h)\},$$
$$V_{0h} = V_h \cap H_0^1(\Omega)^d, \qquad V_h^\sigma = \{v_h \in V_h \mid b(q_h, v_h) = 0 \ (\forall q_h \in Q_h)\},$$
$$Q_h = \{q_h \in C^0(\overline{\Omega}) \mid q_h|_T \in \mathcal{P}_1^{(d)} \ (\forall T \in \mathcal{T}_h)\}, \quad Q_{0h} = Q_h \cap Q_0,$$
$$M_h = \{\mu_h = v_{hn}|_\Gamma \mid v_h \in V_h\}, \quad M_{0h} = \left\{\mu_h \in M_h \mid \int_\Gamma \mu_h \, d\Gamma = 0\right\}.$$

Therein, $\mathcal{P}_k^{(d)}$ denotes the set of all polynomials in $x_1, \ldots, x_d$ of degree $\le k$, and $\varphi_T = \prod_{i=1}^{d+1} \lambda_{T,i}$, with $\lambda_{T,1}, \ldots, \lambda_{T,d+1}$ the barycentric coordinates of T.

We denote by $\mathcal{S}_h$ the $d-1$ dimensional triangulation of Γ inherited from $\mathcal{T}_h$. We have

$$M_h = \{\mu_h \in C(\overline{\Gamma}) \mid \mu_h|_S \in \mathcal{P}_1^{(d-1)} \ (\forall S \in \mathcal{S}_h),\ \mu_h|_{\partial\Gamma} = 0\} \quad \text{(algebraically)}. \tag{2.42}$$

Moreover, we introduce a projection operator $\Lambda : Q \to Q_0$ by

$$\Lambda q = q - m(q) \quad \text{with} \quad m(q) = \frac{1}{|\Omega|}\int_\Omega q \, dx \qquad (q \in Q). \tag{2.43}$$

It seems readily apparent that $\|\Lambda q\| \le C\|q\|$ for $q \in Q$ and $\Lambda q_h \in Q_{0h}$ for $q_h \in Q_h$.

Then, the finite element approximation for $(\mathrm{PE}_{\varepsilon,\delta})$ reads as follows.

$(\mathbf{PE}_{\varepsilon,\delta,h})$ Find $(u_h, p_h) \in V_h \times Q_h$ such that

$$a(u_h, v_h) + b(p_h, v_h) - \frac{1}{\varepsilon}\int_\Gamma \phi_\delta(u_n + g_n) v_{hn} \, d\Gamma = (f, v_h) - [[\alpha, v_h]] \qquad (\forall v_h \in V_h), \tag{2.44a}$$

$$b(q_h, u_h) = 0 \qquad (\forall q_h \in Q_h). \tag{2.44b}$$

Before considering the well-posedness and error estimates, we recall here basic results on the finite element method.

Babuška-Brezzi condition As is well-known, the Babuška-Bezzi condition is well-known to hold true in $V_{0h} \times Q_{0h}$. There is a constant $\gamma' > 0$, which is independent of h, such that

$$\inf_{q_h \in Q_{0h}} \sup_{v_h \in V_{0h}} \frac{b(q_h, v_h)}{\|v_h\|_1 \|q_h\|} \ge \gamma'. \tag{2.45}$$

Discrete lifting operators and discrete traction vectors The following is a discrete analogue of Lemma 2.2.1.

Lemma 2.4.1 ([30, Lemma 2.1]). (i) *There is a continuous linear operator $\mathcal{E}_h$ from M_h^d to V_h such that $\mathcal{E}_h \mu_h = \mu_h$ on Γ and $\|\mathcal{E}_h \mu_h\|_1 \le C\|\mu_h\|_{\frac{1}{2},\Gamma}$ for any $\mu_h \in M_h^d$, where C is independent of h.*
(ii) *There is a continuous linear operator $\mathcal{E}_{nh}$ from M_h to V_h such that $(\mathcal{E}_{nh}\mu_h)_n = \mu_h$ and $(\mathcal{E}_{nh}\mu_h)_T = 0$ on Γ and $\|\mathcal{E}_{nh}\mu_h\|_1 \le C\|\mu_h\|_{\frac{1}{2},\Gamma}$ for any $\mu_h \in M_h$, where C is independent of h.*
(iii) *For $\mu_h \in M_{0h}$, the above $w_h = \mathcal{E}_{nh}\mu_h$ can be chosen in such way that $w_h \in V_h^\sigma$.*

As the continuous case, we define traction vectors $\tau(u_h, p_h) \in (M_h^d)'$, $\tau_T(u_h) \in (M_h^d)'$ and $\tau_n(u_h, p_h) \in M_h'$ for a solution $(u_h, p_h) \in V_h^\sigma \times Q_h$ of

$$a(u_h, v_h) + b(p_h, v_h) = (f, v_h) \qquad (v_h \in V_{0h}) \tag{2.46}$$

as follows:

$$[[\tau(u_h, p_h), \mu_h]] = a(u_h, w_h) + b(p_h, w_h) - (f, w_h) \quad (\mu_h \in M_h^d,\ w_h = \mathcal{E}_h \mu_h); \tag{2.47a}$$

$$[\tau_T(u_h), \mu_h] = a(u_h, w_h) + b(p_h, w_h) - (f, w_h) \quad (\mu_h \in M_h^d \text{ with } \mu_{hn} = 0,\ w_h = \mathcal{E}_h \mu_h); \tag{2.47b}$$

$$[\tau_n(u_h, p_h), \mu_h] = a(u_h, w_h) + b(p_h, w_h) - (f, w_h) \quad (\mu_h \in M_h,\ w_h = \mathcal{E}_{nh} \mu_h) \tag{2.47c}$$

These definitions are independent of the way of extensions. In fact, for any μ_h, let $w_h \in V_h$ and $\tilde{w}_h \in V_h$ be both extension of λ_h; $w_{hn} = \tilde{w}_{hn} = \lambda_h$ on Γ. Set $v_h = w_h - \tilde{w}_h$. Then, since $v_h \in V_{0h}$, we deduce, by (2.46),

$$\begin{aligned} a(u_h, w_h) + b(p_h, w_h) - (f, w_h) - [a(u_h, \tilde{w}_h) + b(p_h, \tilde{w}_h) - (f, \tilde{w}_h)] \\ = a(u_h, v_h) + b(p_h, v_h) - (f, v_h) = 0. \end{aligned}$$

Consequently, (2.47c) is well-defined.

2.5 Well-posedness of $(\mathrm{PE}_{\varepsilon,\delta,h})$

In this section, we establish the well-posedness of $(\mathrm{PE}_{\varepsilon,\delta,h})$. Thus, we shall prove the following two theorems. Recall that C_* denotes a positive constant depending only on Ω, $\|f\|$, $\|g\|_1$ and $\|\alpha\|_{(M^d)'}$.

Theorem 2.5.1. *There exists a unique solution $(u_h, p_h) \in V_h \times Q_h$ of $(\mathrm{PE}_{\varepsilon,\delta,h})$, and we have*

$$\|u_h\|_1 + \|\hat{p}_h\| + \left\| \frac{1}{\varepsilon}\phi_\delta(u_{hn} + g_n) + k_h \right\|_{M_h'} \le C_*, \tag{2.48}$$

where $\hat{p}_h = \Lambda p_h$ and $k_h = m(p_h)$.

Theorem 2.5.2. *Assume*

(A1) *the family $\{\mathcal{S}_h\}_h$ is quasi-uniform;*

(A2) *there exists $\Gamma_1 \subset \Gamma$ with $|\Gamma_1| > 0$ which is independent of h, ε, δ and Ω such that $u_{hn}+g_n > 0$ on Γ_1.*

Then, the solution $(u_h, p_h) \in V_h \times Q_h$ of $(\mathrm{PE}_{\varepsilon,\delta,h})$ admits the following estimates:

$$\|u_h\|_1 + \|p_h\| + \left\| \frac{1}{\varepsilon}\phi_\delta(u_{hn} + g_n) \right\|_{M_h'} \le C_*; \tag{2.49a}$$

$$\left\| \frac{1}{\varepsilon}\phi_\delta(u_{hn} + g_n) \right\|_{M'} \le C_* \left(1 + \frac{h}{\varepsilon}\right); \tag{2.49b}$$

$$\frac{1}{\sqrt{\varepsilon}}\|[u_{hn} + g_n]_-\|_\Gamma \le C_* \left(1 + \frac{\delta}{\varepsilon}\right). \tag{2.49c}$$

Remark 2.5.1. *Condition (A2) is not restrictive. It is natural to presume this condition if β is sufficiently large and h, ε, δ are suitably small.*

Remark 2.5.2. *If $\delta \le c_0\varepsilon$ with some $c_0 > 0$, we have, from (2.49c), $\|[u_{hn} + g_n]_-\|_\Gamma \to 0$ as $\varepsilon \to 0$.*

To prove Theorem 2.5.1, we apply the following fundamental result.

Lemma 2.5.1 ([36, Theorem 2.1]). *Let X be a separable reflexive Banach space and let $T : X \to X'$ be a (possibly nonlinear) operator satisfying the following conditions:*

1. *(boundness) There exist $C, C', m > 0$ s.t. $\|Tu\|_{X'} \le C\|u\|_X^m + C'$ for all $u \in X$;*
2. *(monotonicity) $\langle Tu - Tv, u - v\rangle_{X',X} \ge 0$ for all $u, v \in X$;*

3. *(hemicontinuity) For any $u, v, w \in X$, the function $\lambda \mapsto \langle T(u+\lambda v), w\rangle_{X',X}$ is continuous on $\mathbb{R}$;*

4. *(coerciveness) $\dfrac{\langle Tu, u\rangle_{X',X}}{\|u\|_X} \to \infty$ as $\|u\|_X \to \infty$.*

Then, for any $\varphi \in X'$, there exists $u \in X$ such that $Tu = \varphi$. Furthermore, if T is strictly monotone:

$$\langle Tu - Tv, u - v\rangle_{X',X} > 0 \qquad (\forall u, v \in X, u \neq v),$$

then the solution is unique.

We set $\rho_\delta : V \to V'$ by

$$\langle \rho_\delta(u), v\rangle = -\int_\Gamma \phi_\delta(u_n + g_n) v_n \, d\Gamma \qquad (v \in V).$$

Lemma 2.5.2. *ρ_δ is a bounded, monotone and hemicontinuous operator from V to V'.*

Proof. (boundness) By using (2.40c) and the trace theorem, we have

$$\langle \rho_\delta(u), v\rangle \leq \int_\Gamma |u_n + g_n| \cdot |v_n| \, d\Gamma \leq (\|u\|_1 + \|g_n\|_\Gamma)\|v\|_1$$

for $u, v \in V$. Hence,

$$\|\rho_\delta(u)\|_{V'} \leq \|u\|_1 + \|g_n\|_\Gamma \qquad (u \in V).$$

(monotonicity) Since $-\phi_\delta(s)$ is non-decreasing function, we have

$$\langle \rho_\delta(u) - \rho_\delta(v), u - v\rangle = -\int_\Gamma (\phi_\delta(u_n + g_n) - \phi_\delta(v_n + g_n))(u_n + g_n - (v_n + g_n)) \, d\Gamma \geq 0$$

for $u, v \in V$.
(hemicontinuity) Let $u, v, w \in V$. Then, a real-valued function

$$\langle \rho_\delta(u + \lambda v), w\rangle = -\int_\Gamma \phi_\delta(u_n + \lambda v_n) w_n \, d\Gamma$$

of $\lambda \in \mathbb{R}$ is a continuous function, since the function ϕ_δ is continuous. □

Proof of Theorem 2.5.1. It is divided into three steps.
Step 1. First, we prove that there exists a unique $u_h \in V_h^\sigma$ satisfying

$$a(u_h, v_h) + \frac{1}{\varepsilon}\langle \rho_\delta(u_h), v_h\rangle = (f, v_h) - [[\alpha, v_h]] \qquad (\forall v_h \in V_h^\sigma) \tag{2.50}$$

by using Lemma 2.5.1.

To do this, we introduce a nonlinear operator $A_\varepsilon : V_h \to V_h'$ by setting

$$\langle A_\varepsilon u_h, v_h\rangle = a(u_h, v_h) + \frac{1}{\varepsilon}\langle \rho_\delta(u_h), v_h\rangle \qquad (u_h, v_h \in V_h).$$

and verify the conditions of Lemma 2.5.1.
(boundness) For $u_h \in V_h^\sigma$, we have immediately

$$\|A_\varepsilon u_h\|_{(V_h^\sigma)'} \leq \left(\|a\| + \frac{1}{\varepsilon}\right)\|u_h\|_1 + \frac{1}{\varepsilon}\|g_n\|_\Gamma.$$

(strictly monotonicity) By virtue of Korn's inequality and monotonicity of ρ_δ,

$$\begin{aligned}\langle A_\varepsilon u_h - A_\varepsilon v_h, u_h - v_h\rangle &= a(u_h - v_h, u_h - v_h) + \frac{1}{\varepsilon}\langle \rho_\delta(u_h) - \rho_\delta(v_h), u_h - v_h\rangle \\ &\geq C\|u_h - v_h\|_1^2 > 0\end{aligned}$$

for $u_h, v_h \in V_h^\sigma$, $u_h \neq v_h$.
(hemicontinuity) Let $u_h, v_h, w_h \in V_h^\sigma$. Then, a real-valued function

$$\langle A_\varepsilon(u_h + \lambda v_h), w_h\rangle = a(u_h + \lambda v_h, w_h) + \frac{1}{\varepsilon}\langle \rho_\delta(u_h + \lambda v_h), w_h\rangle$$

of $\lambda \in \mathbb{R}$ is continuous, since $a(\cdot, w_h)$ is continuous and $\rho_\delta(\cdot)$ is hemicontinuous.
(coerciveness) For $u_h \in V_h^\sigma$, we have by (2.40c)

$$\begin{aligned}\langle \rho_\delta(u_h), u_h\rangle &= -\int_\Gamma \phi_\delta(u_{hn} + g_n)u_{hn}\ d\Gamma \\ &= -\int_\Gamma \phi_\delta(u_{hn} + g_n)\left([u_{hn} + g_n]_+ - [u_{hn} + g_n]_- - [g_n]_+ + [g_n]_-\right)\ d\Gamma \\ &\geq -\int_\Gamma \phi_\delta(u_n + g_n)[g_n]_-\ d\Gamma \\ &\geq -C\left(\|u_h\|_1 + \|g_n\|_\Gamma\right)\|g_n\|_\Gamma.\end{aligned}$$

This gives

$$\frac{\langle A_\varepsilon u_h, u_h\rangle}{\|u_h\|_1} = \frac{a(u_h, u_h)}{\|u_h\|_1} + \frac{1}{\varepsilon}\frac{\langle \rho_\delta(u_h), u_h\rangle}{\|u_h\|_1} \geq C\|u_h\|_1 - \frac{C}{\varepsilon}\frac{\left(\|u_h\|_1 + \|g_n\|_\Gamma\right)}{\|u_h\|_1}\|g_n\|_\Gamma,$$

and, hence,

$$\frac{\langle A_\varepsilon u_h, u_h\rangle}{\|u_h\|_1} \to \infty \quad \text{as} \quad \|u_h\|_1 \to \infty.$$

As a consequence, we can apply Lemma 2.5.1 to conclude that there exists a unique $u_h \in V_h^\sigma$ satisfying $A_\varepsilon u_h = F_h$, where $F_h \in (V_h^\sigma)'$ is defined as $\langle F, v_h\rangle = (f, v_h) - [[\alpha, v_h]]$ for $v_h \in V_h^\sigma$. Therefore, we have proved a unique existence of the solution $u_h \in V_h^\sigma$ of (2.50).

Step 2. We verify the unique existence of $p_h \in Q_h$ such that (u_h, p_h) is a solution of (PE$_{\varepsilon,\delta,h}$). In view of (2.45), there exists a unique $\hat{p}_h \in Q_{0h}$ satisfying

$$a(u_h, v_h) + b(\hat{p}_h, v_h) = (f, v_h) \qquad (v_h \in V_{0h}). \tag{2.51}$$

Now, we find a constant k_h such that (u_h, p_h) is a solution of (PE$_{\varepsilon,\delta,h}$), where $p_h = \hat{p}_h + k_h$. (In fact, with any $k_h \in \mathbb{R}$, (u_h, p_h) also solves (2.46).) To do this, we first rewrite (2.44a) as by using (2.47a)–(2.47c),

$$[\tau_n(u_h, p_h) - \varepsilon^{-1}\phi_\delta(u_{hn} + g_n) - \alpha_n, v_{hn}] + [[\tau_T(u_h) - \alpha_T, v_{hT}]] = 0 \quad (v_h \in V_h).$$

Consequently, in view of Lemma 2.4.1, it suffices to prove the following two equations:

$$[\tau_n(u_h, p_h) - \varepsilon^{-1}\phi_\delta(u_{hn} + g_n) - \alpha_n, \mu_h] = 0 \quad (\mu_h \in M_h); \tag{2.52a}$$

$$[[\tau_T(u_h) - \alpha_T, \mu_h]] = 0 \quad (\mu_h \in M_h^d \text{ with } \mu_{hn} = 0). \tag{2.52b}$$

It might be readily apparent that $v_h = \mathcal{E}_h\mu_h \in V_h$ belongs to V_h^σ for any $\mu_h \in M_h^d$ with $\mu_{hn} = 0$. Hence, (2.50) and (2.47b) immediately implies (2.52b).

On the other hand, combining (2.50) and (2.47c), we have

$$[\tau_n(u_h, p_h) + \alpha_n - \varepsilon^{-1}\phi_\delta(u_{hn} + g_n), \lambda_h] = 0 \qquad (\forall \lambda_h \in M_{0h}). \tag{2.53}$$

At this stage, let us take

$$\tilde{\mu}_h \in Y_h = \left\{ \mu_h \in M_h \mid \mu_h \geq 0,\ \mu_h \not\equiv 0,\ \int_\Gamma \mu_h \, d\Gamma = 1 \right\}.$$

Then, for any $\mu_h \in M_h$, the function $\mu_h - \kappa_h \tilde{\mu}_h$ belongs to M_{0h}, where $\kappa_h = \int_\Gamma \mu_h \, d\Gamma$.

Therefore, for any $\mu_h \in M_h$,

$$\begin{aligned}
[\tau_n(u_h, p_h) + \alpha_n - \varepsilon^{-1}\phi_\delta(u_{hn} + g_n), \mu_h] &= [\tau_n(u_h, p_h) + \alpha_n - \varepsilon^{-1}\phi_\delta(u_{hn} + g_n), \mu_h - \kappa_h \tilde{\mu}_h] \\
&\quad + [\tau_n(u_h, p_h) + \alpha_n - \varepsilon^{-1}\phi_\delta(u_{hn} + g_n), \kappa_h \tilde{\mu}_h] \\
&= \kappa_h [\tau_n(u_h, p_h) + \alpha_n - \varepsilon^{-1}\phi_\delta(u_{hn} + g_n), \tilde{\mu}_h]
\end{aligned} \tag{2.54}$$

Now, choosing

$$k_h = [\tau_n(u_h, \hat{p}_h) - \varepsilon^{-1}\phi_\delta(u_{hn} + g_n), \tilde{\mu}_h], \tag{2.55}$$

we have

$$\begin{aligned}
[\tau_n(u_h, p_h) + \alpha_n - \varepsilon^{-1}\phi_\delta(u_{hn} + g_n), \tilde{\mu}_h] &= [\tau_n(u_h, \hat{p}_h) + \alpha_n - \varepsilon^{-1}\phi_\delta(u_{hn} + g_n), \tilde{\mu}_h] - k_h \\
&= 0.
\end{aligned}$$

Hence, we get (2.52a) by (2.54).

It remains to be verified that (2.55) is independent of the choice of $\tilde{\mu}_h$. We let $\tilde{\mu}_h, \tilde{\mu}'_h \in Y_h$ with $\tilde{\mu}_h \neq \tilde{\mu}'_h$ and let the corresponding k_h be denoted by $\tilde{k}_h$, $\tilde{k}'_h$, respectively. Then, since $\lambda_h = \mu_h - \mu'_h$ satisfies $\int_\Gamma \lambda_h \, d\Gamma = 0$, we have by (2.53),

$$\tilde{k}_h - \tilde{k}'_h = [\tau_n(u_h, \hat{p}_h) + \alpha_n - \varepsilon^{-1}\phi_\delta(u_{hn} + g_n), \lambda_h] = 0,$$

which means that k_h is uniquely determined by (2.55).

Step 3. Finally, we derive the stability result (2.48). Substituting $v_h = u_h \in V_h^\sigma$ into (2.50), we obtain,

$$a(u_h, u_h) - \frac{1}{\varepsilon} \int_\Gamma \phi_\delta(u_{hn} + g_n) u_{hn} \, d\Gamma = (f, u_h) - [[\alpha, u_h]]. \tag{2.56}$$

Noting that, by (2.40c)

$$\begin{aligned}
-\frac{1}{\varepsilon} \int_\Gamma \phi_\delta(u_{hn} + g_n) u_{hn} \, d\Gamma &= -\frac{1}{\varepsilon} \int_\Gamma \phi_\delta(u_{hn} + g_n)(u_{hn} + g_n) \, d\Gamma + \frac{1}{\varepsilon} \int_\Gamma \phi_\delta(u_{hn} + g_n) g_n \, d\Gamma \\
&\geq \frac{1}{\varepsilon} \int_\Gamma \phi_\delta(u_{hn} + g_n)[u_{hn} + g_n]_- \, d\Gamma \geq 0,
\end{aligned} \tag{2.57}$$

we get

$$a(u_h, u_h) \leq (f, u_h) - [[\alpha, u_h]].$$

Hence, by virtue of Korn's inequality ([32, Lemma 6.2]),

$$\|u_h\|_1 \leq C(\|f\| + \|\alpha\|_{(M^d)'}). \tag{2.58}$$

Moreover, according to (2.45) and (2.51),

$$\|\hat{p}_h\| \leq \sup_{v_h \in V_{0h}} \frac{b(\hat{p}_h, v_h)}{\|v_h\|_1} = \sup_{v_h \in V_{0h}} \frac{(f, v_h) - a(u_h, v_h)}{\|v_h\|_1} \leq C(\|f\| + \|u_h\|_1). \tag{2.59}$$

Since (2.44a) is expressed as

$$\int_\Gamma (\varepsilon^{-1}\phi_\delta(u_{hn}+g_n)+k_h)\mu_h \, d\Gamma$$
$$= a(u_h,v_h)+b(\hat{p}_h,v_h)-(f,v_h)+[[\alpha,v_h]] \qquad (\forall \mu_h \in M_h,\ v_h=\mathcal{E}_{nh}\mu_h \in V_h),$$

we deduce

$$\left\| \frac{1}{\varepsilon}\phi_\delta(u_{hn}+g_n)+k_h \right\|_{M_h'} \le C(\|u_h\|_1+\|\hat{p}_h\|+\|f\|+\|\alpha\|_{(M^d)'}). \tag{2.60}$$

Summing up (2.58), (2.59) and (2.60), we obtain (2.48). □

We proceed to the proof of Theorem 2.5.2. We use the standard Lagrange interpolation operator $i_h : C(\overline{\Gamma}) \to M_h$ defined by

$$i_h\mu(P)=\mu(P) \quad \text{(every node } P \text{ of } \mathcal{S}_h) \tag{2.61}$$

and the L^2 projection operator $\pi_h : L^2(\Gamma) \to M_h$ defined by

$$\int_\Gamma (\pi_h\mu-\mu)\mu_h \, d\Gamma = 0 \qquad (\mu_h \in M_h). \tag{2.62}$$

The following results are well-known.

$$\mu \ge 0 \quad \Rightarrow \quad i_h\mu \ge 0, \tag{2.63a}$$
$$\|i_h\mu-\mu\|_\Gamma + h\|i_h\mu-\mu\|_{1,\Gamma} \le Ch^2|\mu|_{2,\Gamma} \qquad (\mu \in H^2(\Gamma)\cap H_0^1(\Gamma)), \tag{2.63b}$$
$$\|\pi_h\mu\|_\Gamma \le C\|\mu\|_\Gamma \qquad (\mu \in L^2(\Gamma)), \tag{2.63c}$$
$$\|\pi_h\mu\|_{1,\Gamma} \le C\|\mu\|_{1,\Gamma} \qquad (\mu \in H_0^1(\Gamma)), \tag{2.63d}$$
$$\|\pi_h\mu-\mu\|_\Gamma \le Ch\|\mu\|_{1,\Gamma} \qquad (\mu \in H_0^1(\Gamma)). \tag{2.63e}$$

In fact, (2.63a), (2.63b), (2.63c), (2.63e) are standard. On the other hand, (2.63d) holds true if $\{\mathcal{S}_h\}_h$ is quasi-uniform (see [8, 16, 29]).

Remark 2.5.3. *According to* (2.63b), $\|i_h\mu\|_{1,\Gamma}$ *is bounded by a positive constant depending only on* μ *if* $\mu \in C_0^\infty(\Gamma)$.

Lemma 2.5.3. $\|\pi_h\mu-\mu\|_{M'} \le Ch\|\mu\|_{\frac{1}{2},\Gamma}$ *for any* $\mu \in M$.

Proof. It follows from (2.63c) that $\|\pi_h\mu-\mu\|_\Gamma \le C\|\mu\|_\Gamma$. Combining this with (2.63e), (2.15) and applying the interpolation theorem (see [38, Theorem 5.1, Ch. 1]), we obtain

$$\|\pi_h\mu-\mu\|_\Gamma \le Ch^{\frac{1}{2}}\|\mu\|_{\frac{1}{2},\Gamma} \qquad (\mu \in M).$$

We can use this in the following way. That is, noting (2.62),

$$\|\pi_h\mu-\mu\|_{M'} = \sup_{\lambda\in M}\frac{[\pi_h\mu-\mu,\lambda]}{\|\lambda\|_{\frac{1}{2},\Gamma}} = \sup_{\lambda\in M}\frac{[\pi_h\mu-\mu,\pi_h\lambda-\lambda]}{\|\lambda\|_{\frac{1}{2},\Gamma}}$$
$$\le \sup_{\lambda\in M}\frac{\|\pi_h\mu-\mu\|_\Gamma\|\pi_h\lambda-\lambda\|_\Gamma}{\|\lambda\|_{\frac{1}{2},\Gamma}} \le Ch\|\mu\|_{\frac{1}{2},\Gamma}.$$

□

Lemma 2.5.4. $\|\phi_\delta(\mu)\|_{\frac{1}{2},\Gamma} \le C\|\mu\|_{\frac{1}{2},\Gamma}$ *for any* $\mu \in M$.

Proof. Using (2.40c) and (2.40d), we have $\|\phi_\delta(\mu)\|_\Gamma \le C\|\mu\|_\Gamma$ for $\mu \in Q$ and $\|\phi_\delta(\mu)\|_{1,\Gamma} \le C\|\mu\|_{1,\Gamma}$ for $\mu \in H_0^1(\Gamma)$. Hence, we can apply the (nonlinear) interpolation theorem (see [37, Theorem 3.1]) and (2.15) to get the desired result. □

Proof of Theorem 2.5.2. First, we derive an estimation for k_h. We take $\tilde{\mu} \in C_0^\infty(\Gamma)$ satisfying

$$\tilde{\mu} \ge 0,\ \tilde{\mu} \not\equiv 0 \text{ in } \Gamma, \quad \operatorname{supp} \tilde{\mu} \subset \Gamma_1.$$

Then, setting $\tilde{\mu}_h = i_h \mu \in M_h$, we have

$$\tilde{\mu}_h \ge 0,\ \tilde{\mu} \not\equiv 0 \text{ in } \Gamma, \quad \tilde{\mu}_h = 0 \text{ in } \Gamma\backslash\Gamma_1, \quad \|\tilde{\mu}_h\|_M \le C, \quad \left|\int_\Gamma \tilde{\mu}_h\ d\Gamma - \int_\Gamma \tilde{\mu}\ d\Gamma\right| \le Ch^2, \tag{2.64}$$

where those C's depend on μ.

Since (A2) gives

$$\phi_\delta(u_{hn} + g_n) = 0 \quad \text{ on } \Gamma_1,$$

we deduce from (2.44a) and (2.64)

$$\begin{aligned} k_h \int_\Gamma \tilde{\mu}_h\ d\Gamma &= a(u_h, \tilde{v}_h) + b(\hat{p}_h, \tilde{v}_h) - \frac{1}{\varepsilon}\int_\Gamma \phi_\delta(u_{hn} + g_n)\tilde{\mu}_h\ d\Gamma - (f, \tilde{v}_h) + [[\alpha, \tilde{v}_h]] \\ &= a(u_h, \tilde{v}_h) + b(\hat{p}_h, \tilde{v}_h) - (f, \tilde{v}_h) + [[\alpha, \tilde{v}_h]], \end{aligned} \tag{2.65}$$

where $\tilde{v}_h = \mathcal{E}_{nh}\tilde{\mu}_h \in V_h$.

This leads to

$$|k_h| \le C_*, \qquad \|p_h\| \le C_*. \tag{2.66}$$

Hence, we have proved (2.49a).

Using (2.62) and (2.44a), we can make the following calculation.

$$\begin{aligned} \int_\Gamma \varepsilon^{-1}\pi_h\phi_\delta(u_{hn} + g_n)\mu_h\ d\Gamma &= \int_\Gamma \varepsilon^{-1}\phi_\delta(u_{hn} + g_n)\mu_h\ d\Gamma \\ &= a(u_h, v_h) + b(p_h, v_h) - (f, v_h) + [[\alpha, v_h]] \quad (\forall \mu_h \in M_h,\ v_h = \mathcal{E}_{nh}\mu_h \in V_h). \end{aligned}$$

Therefore,

$$\|\varepsilon^{-1}\pi_h\phi_\delta(u_{hn} + g_n)\|_{M_h'} \le (\|u_h\|_1 + \|\hat{p}_h\| + \|f\| + \|\alpha\|_{(M^d)'} + \|g\|_1). \tag{2.67}$$

We write as

$$\begin{aligned} &\sup_{\mu\in M} \frac{[\varepsilon^{-1}\phi_\delta(u_{hn} + g_n), \mu]}{\|\mu\|_{\frac{1}{2},\Gamma}} \\ &\quad = \underbrace{\frac{1}{\varepsilon}\sup_{\mu\in M} \frac{[\phi_\delta(u_{hn} + g_n) - \pi_h\phi_\delta(u_{hn} + g_n), \mu]}{\|\mu\|_{\frac{1}{2},\Gamma}}}_{=I_1} + \underbrace{\sup_{\mu\in M} \frac{[\varepsilon^{-1}\pi_h\phi_\delta(u_{hn} + g_n), \mu]}{\|\mu\|_{\frac{1}{2},\Gamma}}}_{=I_2}. \end{aligned}$$

Using Lemmas 2.5.3 and 2.5.4,

$$\begin{aligned} \|\pi_h\phi_\delta(u_{hn} + g_n) - \phi_\delta(u_{hn} + g_n)\|_{M'} &\le Ch\|\phi_\delta(u_{hn} + g_n)\|_{\frac{1}{2},\Gamma} \\ &\le Ch\|u_{hn} + g_n\|_{\frac{1}{2},\Gamma} \\ &\le Ch(\|u_h\|_1 + \|g\|_1). \end{aligned}$$

Consequently,

$$|I_1| \le C\frac{h}{\varepsilon}(\|u_h\|_1 + \|g\|_1).$$

On the other hand, by virtue of (2.62), (2.63d) and (2.67), we have

$$\begin{aligned} I_2 &= \sup_{\mu\in M} \frac{[\varepsilon^{-1}\pi_h\phi_\delta(u_{hn}+g_n), \mu]}{\|\mu\|_{\frac{1}{2},\Gamma}} \le C \sup_{\mu\in M} \frac{[\varepsilon^{-1}\pi_h\phi_\delta(u_{hn}+g_n), \pi_h]}{\|\pi_h\mu\|_{\frac{1}{2},\Gamma}} \\ &\le C \sup_{\mu_h\in M_h} \frac{[\varepsilon^{-1}\pi_h\phi_\delta(u_{hn}+g_n), \mu_h]}{\|\mu_h\|_{\frac{1}{2},\Gamma}} \end{aligned}$$

Therefore, from (2.67),

$$|I_2| \le C(\|u_h\|_1 + \|p_h\| + \|f\| + \|\alpha\|_{(M^d)'} + \|g\|_1)$$

Summing up those estimates, we get (2.49b).

Finally, using (2.40b),

$$\begin{aligned} -\frac{1}{\varepsilon}\int_\Gamma \phi_\delta(u_{hn}+g_n)u_{hn}\, d\Gamma &\ge \frac{1}{\varepsilon}\int_\Gamma \phi_\delta(u_{hn}+g_n)[u_{hn}+g_n]_-\, d\Gamma \ge 0 \\ &\ge \frac{1}{\varepsilon}\int_\Gamma ([u_{hn}+g_n]_-^2 - C\delta[u_{hn}+g_n]_-)\, d\Gamma \ge 0. \end{aligned}$$

We apply this to (2.56) and obtain

$$\begin{aligned} \frac{1}{\varepsilon}\int_\Gamma [u_{hn}+g_n]_-^2\, d\Gamma &\le (f,u_h) - [[\alpha, u_h]] + C\frac{\delta}{\varepsilon}\int_\Gamma [u_{hn}+g_n]_-\, d\Gamma \\ &\le \|f\|\cdot\|u_h\|_1 + \|\alpha\|_{(M^d)'}\|u_h\|_1 + C\frac{\delta}{\varepsilon}(\|u_h\|_1 + \|g\|_1), \end{aligned}$$

which implies (2.49c). □

2.6 Error estimate

We are now ready to state the error estimates between (PDE) and ($\mathrm{PE}_{\varepsilon,\delta,h}$).

Theorem 2.6.1. *Assume that* (A1) *and* (A2) *are satisfied. Let* (u,p) *and* (u_h,p_h) *be solutions of* (PDE) *and* ($\mathrm{PE}_{\varepsilon,\delta,h}$)*, respectively, and suppose that* $(u,p)\in H^2(\Omega)^d \times H^1(\Omega)$ *and* $\tau_n(u,p)+\alpha_n \in M$*. Moreover, assume that* h,ε,δ *are sufficiently small and* $h \le c_1\varepsilon$ *with a constant* $c_1 > 0$*, then we have*

$$\|u-u_h\|_1 + \|p-p_h\| \le C_{**}\left(h + \varepsilon + \sqrt{\frac{\delta^2}{\varepsilon}}\right) \tag{2.68}$$

and

$$\begin{aligned} \left\|\tau_n(u,p) + \alpha_n - \frac{1}{\varepsilon}\phi_\delta(u_{hn}+g_n)\right\|_{M'} &+ \sqrt{\varepsilon}\left\|\tau_n(u,p) + \alpha_n - \frac{1}{\varepsilon}\phi_\delta(u_{hn}+g_n)\right\|_\Gamma \\ &\le C_{**}\left(h + \varepsilon + \sqrt{\frac{\delta^2}{\varepsilon}}\right), \end{aligned} \tag{2.69}$$

where C_{**} denotes a positive constant depending only on c_1, Ω, $|u|_2$, $|p|_1$, $\|\tau_n(u,p)+\alpha_n\|_M$, $\|f\|$, $\|g\|_1$ and $\|\alpha\|_{(M^d)'}$.

Particularly, if taking as $c_2\varepsilon \le h$ and $\delta \le c_3 h^{\frac{3}{2}}$ with constants $(c_1 >)c_2, c_3 > 0$, we have the optimal-order error estimate

$$\|u-u_h\|_1 + \|p-p_h\| \le C_{**}h.$$

We use the standard Lagrange interpolation operator $I_h : C(\Omega)^d \to V_h$ and the L^2 projection operator $\Pi_h : Q \to Q_h$. Actually, the following are well known.

$$\|v - I_h v\|_1 \le Ch|v|_2 \qquad (v \in [H^2(\Omega)\cap H^1_0(\Omega)]^d), \tag{2.70a}$$

$$\|q - \Pi_h q\| \le Ch|q|_1 \qquad (q \in H^1(\Omega)). \tag{2.70b}$$

Proof of Theorem 2.6.1. It is divided into three steps.
Step 1. Let us show that the following non-optimal error estimate holds true:

$$\|u-u_h\|_1 + \|\hat{p}-\hat{p}_h\| \le C_{**}\left(\sqrt{\varepsilon}+\sqrt{\delta}+\sqrt{h}\right), \tag{2.71}$$

where $\hat{p} = \Lambda p$, $\hat{p}_h = \Lambda p_h$.

We recall that (2.22) together with (2.20) give

$$a(u,v) + b(p,v) - [\tau_n(u,p), v_n] = (f,v) - [[\alpha_T, v_T]] \qquad (v \in V). \tag{2.72}$$

Hence, errors $u-u_h$ and $p-p_h$ satisfy

$$a(u-u_h,v_h) + b(p-p_h,v_h) - [\tau_n(u,p) + \alpha_n - \varepsilon^{-1}\phi_\delta(u_{hn}+g_n), v_{hn}] = 0 \quad (v_h \in V_h).$$

Setting $\hat{p} = \Lambda p$, $\hat{p}_h = \Lambda p_h$, $k = m(p)$ and $k_h = m(p_h)$, we can write as

$$\begin{aligned} a(u-u_h,v_h) = &\underbrace{-b(\hat{p}-\hat{p}_h,v_h)}_{=J_1(v_h)} \\ &\underbrace{+[\tau_n(u,p)+\alpha_n-\varepsilon^{-1}\phi_\delta(u_{hn}+g_n)+k-k_h, v_{hn}]}_{=J_2(v_h)} \qquad (v_h \in V_h). \end{aligned} \tag{2.73}$$

Particularly we have

$$b(\Lambda q_h - \hat{p}_h, v_h) = -a(u-u_h,v_h) - b(\hat{p}-\Lambda q_h, v_h) \qquad (v_h \in V_{0h},\ q_h \in Q_h).$$

and, by application of (2.45),

$$\begin{aligned} \|\Lambda q_h - \hat{p}_h\| &\le C \sup_{v_h\in V_{0h}} \frac{-a(u-u_h,v_h) - b(\hat{p}-\Lambda q_h,v_h)}{\|v_h\|_1} \\ &\le C(\|u-u_h\|_1 + \|\hat{p}-\Lambda q_h\|) \\ &\le C(\|u-u_h\|_1 + \|p-q_h\|) \qquad (q_h \in Q_h). \end{aligned} \tag{2.74}$$

At this stage, we set

$$v_h = I_h u - u_h \in V_h, \qquad q_h = \Pi_h p \in Q_h, \qquad \hat{q}_h = \Lambda q_h \in Q_{0h}.$$

Then,

$$\begin{aligned} \|\hat{p}-\hat{p}_h\| &\le \|\hat{p}-\hat{q}_h\| + \|\hat{q}_h - \hat{p}_h\| \\ &\le \|p - \Pi_h p\| + C(\|u-u_h\|_1 + \|p-\Pi_h p\|) \\ &\le C_{**}h + C\|u-u_h\|_1. \end{aligned} \tag{2.75}$$

Using (2.70), (2.74) and $\|\hat{p} - \hat{q}_h\| \leq C\|p - q_h\|$, we estimate as

$$\begin{aligned}
|J_1(I_h u - u_h)| &\leq |b(\hat{p} - \hat{p}_h, I_h u - u)| + |b(\hat{p} - \hat{q}_h, u - u_h)| + |b(\hat{q} - \hat{p}_h, u - u_h)| \\
&\leq \|b\| \cdot \|\hat{p} - \hat{p}_h\| \cdot \|I_h u - u\|_1 + \|b\| \cdot \|\hat{p} - \hat{q}_h\| \cdot \|u - u_h\|_1 + 0 \\
&\leq C(C_{**}h + \|u - u_h\|_1) \cdot h|u|_2 + Ch|p|_1\|u - u_h\|_1 \\
&\leq C_{**}h^2 + C_{**}h\|u - u_h\|_1 \\
&\leq C_{**}h^2 + C_{**}h\|I_h u - u_h\|_1. \qquad (2.76)
\end{aligned}$$

To perform an estimation for J_2, we divide it as

$$\begin{aligned}
J_2(I_h u - u_h) = &\underbrace{[\tau_n(u, p) + \alpha_n - \varepsilon^{-1}\phi_\delta(u_{hn} + g_n) + k - k_h, (I_h u)_n - u_n]}_{=J_{21}} \\
&+ \underbrace{[\tau_n(u, p) + \alpha_n - \varepsilon^{-1}\phi_\delta(u_{hn} + g_n) + k - k_h, u_n - u_{hn}]}_{=J_{22}}. \qquad (2.77)
\end{aligned}$$

According to stability results (2.48) and (2.49b), we deduce

$$\begin{aligned}
|J_{21}| &\leq (\|\tau_n(u, \hat{p})\|_{M'} + \|\varepsilon^{-1}\phi_\delta(u_{hn} + g_n)\|_{M'} + \|\alpha_n\|_{M'} + |k_h|)\|(I_h u)_n - u_n\|_{\frac{1}{2},\Gamma} \\
&\leq C_*\|I_h u - u\|_1 \leq C_{**}h.
\end{aligned}$$

Noting

$$\int_\Gamma (u_n - u_{hn})\ d\Gamma = \int_\Omega \nabla \cdot (u - u_h)\ dx = 0$$

and using (2.22c), (2.22d), (2.22e), (2.40b), (2.40c) and (2.49b), we can calculate as:

$$\begin{aligned}
J_{22} &= [\tau_n(u, p) + \alpha_n - \varepsilon^{-1}\phi_\delta(u_{hn} + g_n), u_n + g_n - (u_{hn} + g_n)] \\
&= -[\varepsilon^{-1}\phi_\delta(u_{hn} + g_n), u_n + g_n] - [\tau_n(u, p) + \alpha_n, u_{hn} + g_n] \\
&\qquad + [\varepsilon^{-1}\phi_\delta(u_{hn} + g_n), u_{hn} + g_n] \\
&= \underbrace{-[\varepsilon^{-1}\phi_\delta(u_{hn} + g_n), u_n + g_n]}_{\leq 0} \underbrace{-[\tau_n(u, p) + \alpha_n, [u_{hn} + g_n]_+]}_{\leq 0} \\
&\qquad + [\tau_n(u, p) + \alpha_n, [u_{hn} + g_n]_-] \underbrace{-[\varepsilon^{-1}\phi_\delta(u_{hn} + g_n), [u_{hn} + g_n]_-]}_{\leq 0} \\
&\leq [\tau_n(u, p) + \alpha_n, [u_{hn} + g_n]_- - \phi_\delta(u_{hn} + g_n)] + \varepsilon[\tau_n(u, p) + \alpha_n, \varepsilon^{-1}\phi_\delta(u_{hn} + g_n)] \\
&\leq \|\tau_n(u, p) + \alpha_n\|_\Gamma \|[u_{hn} + g_n]_- - \phi_\delta(u_{hn} + g_n)\|_\Gamma \\
&\qquad + \varepsilon\|\tau_n(u, p) + \alpha_n\|_M \|\varepsilon^{-1}\phi_\delta(u_{hn} + g_n)\|_{M'} \\
&\leq C_*\left[\delta + \varepsilon\left(1 + \frac{h}{\varepsilon}\right)\right] \leq C_*(\delta + \varepsilon + h). \qquad (2.78)
\end{aligned}$$

Summing up those estimates, we obtain

$$\begin{aligned}
C\|I_h u - u_h\|_1^2 &\leq a(I_h u - u_h, I_h u - u_h) \\
&= a(I_h u - u, I_h u - u_h) + a(u - u_h, I_h u - u_h) \\
&= a(I_h u - u, I_h u - u_h) + J_1(I_h u - u_h) + J_2(I_h u - u_h) \\
&\leq C_{**}h\|I_h u - u_h\|_1 + C_{**}h^2 + C_{**}h\|I_h u - u_h\|_1 + C_{**}h + C_*(\delta + \varepsilon + h). \qquad (2.79)
\end{aligned}$$

Therefore, we deduce

$$\|I_h u - u_h\|_1 \le C_{**}\left(\sqrt{h} + \sqrt{\varepsilon} + \sqrt{\delta}\right)$$

and

$$\|u - u_h\|_1 \le \|u - I_h u\|_1 + \|I_h u - u_h\|_1 \le C_{**}\left(\sqrt{h} + \sqrt{\varepsilon} + \sqrt{\delta}\right).$$

This, together with (2.75), implies (2.71).

Step 2. We derive an estimation for $|k_h - k|$ by using (2.71). For sufficiently small ε, h, δ with $h \le c_1\varepsilon$, according to (2.71) and (A2), there exists $\Gamma_0 \subset \Gamma$ with $|\Gamma_0| > 0$ such that $u_n + g_n > 0$ and $u_{hn} + g_n > 0$ on Γ_0. As in the proof of Theorem 2.5.2 (see (2.64)), we take $\tilde{\mu} \in C_0^\infty(\Gamma)$ satisfying $\tilde{\mu} \ge 0$, $\tilde{\mu} \not\equiv 0$ in Γ and $\operatorname{supp}\tilde{\mu} \subset \Gamma_0$. Then, setting $\tilde{\mu}_h = i_h\mu \in M_h$, we have

$$\tilde{\mu}_h \ge 0,\ \tilde{\mu}_h \not\equiv 0 \text{ in } \Gamma, \quad \tilde{\mu}_h = 0 \text{ in } \Gamma\backslash(\Gamma_0), \quad \|\tilde{\mu}_h\|_M \le C, \quad \left|\int_\Gamma \tilde{\mu}_h\, d\Gamma - \int_\Gamma \tilde{\mu}\, d\Gamma\right| \le Ch^2.$$

Since $u_n + g_n > 0$ on Γ_0, we have $\tau_n(u,p) + \alpha_n = 0$ on Γ_0 in view of (2.22e). Substituting $\tilde{v}_h = \mathcal{E}_{nh}\tilde{\mu}_h \in V_h \subset V$ into (2.72) and using (2.22d), we have

$$\begin{aligned} k\int_\Gamma \tilde{\mu}_h\, d\Gamma &= a(u,\tilde{v}_h) + b(\hat{p},\tilde{v}_h) - (f,\tilde{v}_h) + [[\alpha,\tilde{v}_h]] - \int_\Gamma (\tau_n(u,p) + \alpha_n)\tilde{\mu}_h\, d\Gamma \\ &= a(u,\tilde{v}_h) + b(\hat{p},\tilde{v}_h) - (f,\tilde{v}_h) + [[\alpha,\tilde{v}_h]]. \end{aligned}$$

This, together with (2.65) and (2.75), gives

$$\begin{aligned} |k_h - k| &\le |a(u_h - u,\tilde{v}_h)| + |b(\hat{p}_h - \hat{p},\tilde{v}_h)| \\ &\le C(\|u_h - u\|_1 + \|\hat{p}_h - \hat{p}\|) \le C_{**}(\|I_h u - u_h\|_1 + h). \end{aligned} \tag{2.80}$$

Step 3. We proceed to the proof of (2.68) and (2.69). We will prove

$$J_2(I_h u - u_h) + \frac{\varepsilon}{2}\|\lambda\|_\Gamma^2 \le C_{**}\left[h^2 + \frac{h^3}{\varepsilon} + \varepsilon h + \frac{\delta^2}{\varepsilon} + (\varepsilon + h)\|I_h u - u_h\|_1\right], \tag{2.81}$$

where

$$\lambda = \tau_n(u,p) + \alpha_n - \frac{1}{\varepsilon}\phi_\delta(u_{hn} + g_n).$$

Recall that $J_2(I_h u - u_h)$ is divided into $J_{21} + J_{22}$ as in (2.77). We have, by (2.80) and the standard trace theorem,

$$J_{21} \le (\|\lambda\|_{M'} + |k - k_h|)\|(I_h u)_n - u_n\|_{\frac{1}{2},\Gamma} \le C_{**}h(\|\lambda\|_{M'} + h + \|I_h u - u_h\|_1). \tag{2.82}$$

We derive an estimation for $\|\lambda\|_{M'}$. First,

$$\|\lambda\|_{M'} = \sup_{\mu\in M}\left(\frac{[\lambda,\mu - \mu_h]}{\|\mu\|_{\frac{1}{2},\Gamma}} + \frac{[\lambda,\mu_h]}{\|\mu\|_{\frac{1}{2},\Gamma}}\right) \le C\|\lambda\|_\Gamma \sup_{\mu\in M}\frac{\|\mu - \mu_h\|_\Gamma}{\|\mu\|_{\frac{1}{2},\Gamma}} + \sup_{\mu\in M}\frac{[\lambda,\mu_h]}{\|\mu\|_{\frac{1}{2},\Gamma}}, \tag{2.83}$$

where μ_h is an arbitrary element of M_h.

In order to set μ_h appropriately, we use Scott and Zhang's projection $\tilde{\Pi}_h : H^1(\Omega)^d \to Q_h^d$ ([51]). The projection $\tilde{\Pi}_h$ satisfies the same stability and interpolation error estimates as Π_h. For example, we have $\|\tilde{\Pi}_h v\| \le C\|v\|$, $\|\tilde{\Pi}_h v\|_1 \le C\|v\|_1$ and $\|v - \tilde{\Pi}_h v\| \le Ch\|v\|_1$ (see [11, §4.8]).

Particularly, we have $\|v - \tilde{\Pi}_h v\|_{H^{\frac{1}{2}}(\Omega)} \le Ch^{\frac{1}{2}}\|v\|_1$ by the interpolation. Furthermore, $\tilde{\Pi}_h$ preserves the boundary condition; $\tilde{\Pi}_h v \in V_h$ for $v \in V$. At this stage, for $\mu \in M$, we set

$$\mu_h = (w_h \cdot n)|_\Gamma, \qquad w_h = \tilde{\Pi}_h \mathcal{E}_n \mu.$$

Then, again by the standard trace theorem,

$$\|\mu - \mu_h\|_\Gamma \le C\|\mathcal{E}_n\mu - \tilde{\Pi}_h\mathcal{E}_n\mu\|_1 \le Ch^{\frac{1}{2}}\|\mathcal{E}_n\mu\|_1 \le Ch^{\frac{1}{2}}\|\mu\|_{\frac{1}{2},\Gamma}.$$

In view of (2.73),

$$[\lambda, \mu_h] = a(u - u_h, w_h) + b(\hat{p} - \hat{p}_h, w_h) - (k - k_h)\int_\Gamma \mu_h \, d\Gamma$$

and, by (2.75) and (2.80),

$$\begin{aligned}[\lambda, \mu_h] &\le C(\|u - u_h\|_1 + \|\hat{p} - \hat{p}_h\| + |k - k_h|)\|w_h\|_1 \\ &\le C_{**}(h + \|I_h u - u_h\|_1)\|\mu\|_{\frac{1}{2},\Gamma}.\end{aligned}$$

Putting those estimates together in (2.83), we obtain

$$\|\lambda\|_{M'} \le C_{**}(h^{\frac{1}{2}}\|\lambda\|_\Gamma + h + \|I_h u - u_h\|_1). \tag{2.84}$$

Substituting this into (2.82) and applying Schwarz's inequality, we deduce

$$\begin{aligned}J_{21} &\le C_{**}(h^2 + Ch^{\frac{3}{2}}\|\lambda\|_\Gamma + h\|I_h u - u_h\|_1) \\ &\le C_{**}\left(h^2 + \frac{h^3}{\varepsilon} + h\|I_h u - u_h\|_1\right) + \frac{\varepsilon}{6}\|\lambda\|_\Gamma^2.\end{aligned} \tag{2.85}$$

Next, we derive a sharp estimate for J_{22}. According to the third equality of (2.78), we have

$$J_{22} \le [\lambda, [u_{hn} + g_n]_-] = \underbrace{[\lambda, [u_{hn} + g_n]_- - \phi_\delta(u_{hn} + g_n)]}_{=J_{221}} - \varepsilon[\lambda, \lambda] + \underbrace{\varepsilon[\lambda, \tau_n(u, p) + \alpha_n]}_{=J_{222}}.$$

From (2.40b),

$$J_{221} \le \|\lambda\|_\Gamma\|[u_{hn} + g_n]_- - \phi_\delta(u_{hn} + g_n)\|_\Gamma \le C\delta\|\lambda\|_\Gamma \le C\frac{\delta^2}{\varepsilon} + \frac{\varepsilon}{6}\|\lambda\|_\Gamma^2.$$

Using (2.84), we get the estimation of J_{222} as follows:

$$\begin{aligned}J_{222} &\le \varepsilon\|\lambda\|_{M'}\|\tau_n(u, p) + \alpha_n\|_{\frac{1}{2},\Gamma} \\ &\le C_{**}\varepsilon(h^{\frac{1}{2}}\|\lambda\|_\Gamma + h + \|I_h u - u_h\|_1) \le C_{**}(\varepsilon h + \varepsilon\|I_h u - u_h\|_1) + \frac{\varepsilon}{6}\|\lambda\|_\Gamma^2.\end{aligned}$$

Summing up those estimates, we obtain

$$J_{22} \le C_{**}\left(\varepsilon h + \frac{\delta^2}{\varepsilon} + \varepsilon\|I_h u - u_h\|_1\right) - \frac{2\varepsilon}{3}\|\lambda\|_\Gamma^2.$$

Combing this with (2.85), we deduce (2.81).

Now, instead of (2.79), we have by (2.76) and (2.81)

$$\begin{aligned} C\|I_hu-u_h\|_1^2 &\le a(I_hu-u, I_hu-u_h)+J_1(I_hu-u_h)+J_2(I_hu-u_h) \\ &\le C_{**}(h+\varepsilon)\|I_hu-u_h\|_1 + C\left(h^2+\frac{h^3}{\varepsilon}+\varepsilon h+\frac{\delta^2}{\varepsilon}\right)-\frac{\varepsilon}{2}\|\lambda\|_\Gamma^2. \end{aligned}$$

This yields

$$\|I_hu-u_h\|_1 \le C_{**}\left(h+\sqrt{\frac{h^3}{\varepsilon}}+\sqrt{\varepsilon h}+\sqrt{\frac{\delta^2}{\varepsilon}}\right) \le C_{**}\left(h+\varepsilon+\sqrt{\frac{\delta^2}{\varepsilon}}\right),$$

since we chose as $h \le c_1\varepsilon$.

Finally, we obtain (2.68) and (2.69) by combining (2.75), (2.80), (2.81) and (2.84). This completes the proof of Theorem 2.6.1. □

Remark 2.6.1. *In addition to the basic assumption of Theorem 2.6.1, we suppose that*

$$u_{hn}+g_n > 0 \quad \text{on} \quad \Gamma. \tag{2.86}$$

Then, for sufficiently small h, ε, δ with $c_2\varepsilon \le h \le c_1\varepsilon$, we have

$$\|u-u_h\|_1 + \|p-p_h\| \le C_{**}h. \tag{2.87}$$

In particular, we do not need to choose as $\delta \le c_3h^{\frac{3}{2}}$. Inequality (2.87) *is derived by noting $J_{22}=0$ under* (2.86).

2.7 Numerical examples

In this section, we present some results of numerical experiments to confirm our theoretical results. We prefer the original setting (2.4) with (2.6), (2.11) to (2.1) and (2.2). Therefore, we consider a model Stokes problem with a nonlinear Robin condition as

$$-\nu\Delta v+\nabla q = f, \quad \nabla\cdot v = 0 \qquad \text{in } \Omega, \tag{2.88a}$$

$$v = b \qquad \text{on } S_1, \tag{2.88b}$$

$$v = 0 \qquad \text{on } S_2, \tag{2.88c}$$

$$\tau_n(v,q) = \frac{1}{\varepsilon}\phi_\delta(v_n), \quad v_T = 0 \qquad \text{on } \Gamma, \tag{2.88d}$$

where ϕ_δ represents the regularized function defined as (2.3).

Remark 2.7.1. *As described in Introduction, we are interested in computing v and q in* (2.4). *The unknown functions u and p in* (2.1) *and* (2.2) *are introduced as "perturbations" of those target variables. They clarify analysis. Moreover, the reference flow (g,π) plays an important role in theoretical considerations, although it is not readily apparent that it is always available in actual computations.*

Remark 2.7.2. *In* (2.88), *we take $v_T = 0$ instead of $\tau_T(u)=0$ as a boundary condition for the tangential component of v on Γ. See Remark 2.3.1.*

The finite element approximation for (2.88) reads as follows.

$(\mathbf{PE}'_{\varepsilon,\delta,h})$ Find $(v_h, q_h) \in W_h \times Q_h$ such that $v_h = i_h b$ on S_1 and

$$\begin{aligned} &a(v_h, w_h) + b(q_h, w_h) - \frac{1}{\varepsilon}\int_\Gamma \phi_\delta(v_n) w_{hn}\, d\Gamma = (f, w_h) && (\forall w_h \in V_h),\\ &b(r_h, v_h) = 0 && (\forall r_h \in Q_h), \end{aligned}$$

where

$$W_h = \{v_h \in C^0(\overline{\Omega}) \mid v_h = 0 \text{ on } S_2,\ v_{hT} = 0 \text{ on } \Gamma,\ v_h|_T \in \left[\mathcal{P}_1^{(d)} \oplus \operatorname{span}\{\varphi_T\}\right]^d\ (\forall T \in \mathcal{T}_h)\}.$$

First, we deal with a simple example, setting $\Omega = \{(x,y) \mid 0 \le x \le L, -R \le y \le R\}$, $S_1 = \{0\} \times [-R, R]$, and $\Gamma = \{L\} \times [-R, R]$, we impose

$$b(x,y) = (C_0(R^2 - y^2), 0), \quad f \equiv 0 \tag{2.89}$$

with $C_0 > 0$. Then, (2.88) has the exact solution, which is given explicitly as

$$v(x,y) = \left(C_0(R^2 - y^2), 0\right), \qquad q(x,y) = 2\nu C_0 L\left(1 - \frac{x}{L}\right). \tag{2.90}$$

This is the well-known Poiseuille flow.

Details of our computation are the following. Set $L = 15$, $R = 5$, $\nu = 1/50$, and $C_0 = 5/(\nu L)$. For the triangulation of Ω, we use a uniform mesh composed of $12N^2$ congruent right-angle triangles; The rectangle is divided into $3N \times 2N$ squares. Then, each small square is decomposed into two equal triangles by a diagonal. Consequently, $h = \sqrt{2}/N$. Since we have employed the C^1 regularization ϕ_δ, Newton's method is available for computing the nonlinear equation $(\mathrm{PE}'_{\varepsilon,\delta,h})$. Penalty parameters are chosen as $\varepsilon = \delta = h/20$. Hence, it is ensured by Theorem 2.6.1 and Remark 2.6.1 that

$$\|v - v_h\|_1 + \|q - q_h\| \le Ch. \tag{2.91}$$

To verify this point, we set

$$E_h^{(1)} = \|v - v_h\|, \quad E_h^{(2)} = \|v - v_h\|_1, \quad E_h^{(3)} = \|q - q_h\|,$$

and observe that

$$\rho_h^{(i)} = \frac{\log E_{h'}^{(i)} - \log E_h^{(i)}}{\log h' - \log h} \qquad (i = 1, 2, 3)$$

with $h' \approx 2h$.

The result is reported in Tab. 2.1 and support our theoretical result (2.91). We were unable to derive the L^2 error for v_h. From Tab. 2.1, we observe that second-order convergence actually occurs.

Next, we consider a two-dimensional branched pipe as portrayed in Fig. 2.2. Since this Ω is not a polygon, we approximate it by a polygon Ω_h with vertices located on $\partial\Omega$. On S_1, we impose a parabolic inflow similarly to (2.89). Fig. 2.3 shows the state of a numerical flow velocity v_h.

As before, we observe $\rho_h^{(1)}$, $\rho_h^{(2)}$ and $\rho_h^{(3)}$. Since, in this case, we are unable to obtain the (explicit) exact solution, we use numerical solutions with extra fine mesh. Tab. 2.2 presents the results. We observe that convergence rates of the H^1 error for v_h and the L^2 error for q_h are close to unity even in the curved domain. Moreover, that of the L^2 error for v_h is close to 2.

h	$E_h^{(1)}$	$\rho_h^{(1)}$	$E_h^{(2)}$	$\rho_h^{(2)}$	$E_h^{(3)}$	$\rho_h^{(3)}$
1.0743	13.9	–	$1.20 \cdot 10^2$	–	$2.07 \cdot 10^{-1}$	–
0.5371	3.47	2.001	$5.96 \cdot 10^1$	1.010	$6.57 \cdot 10^{-2}$	1.656
0.2685	0.87	2.000	$2.97 \cdot 10^1$	1.003	$2.18 \cdot 10^{-2}$	1.594
0.1342	0.21	2.000	$1.48 \cdot 10^1$	1.001	$7.42 \cdot 10^{-3}$	1.553
0.0665	0.052	2.000	7.17	1.000	$2.56 \cdot 10^{-3}$	1.527

Table 2.1: Numerical convergence rates of $(\mathrm{PE}'_{\varepsilon,\delta,h})$ for (2.90).

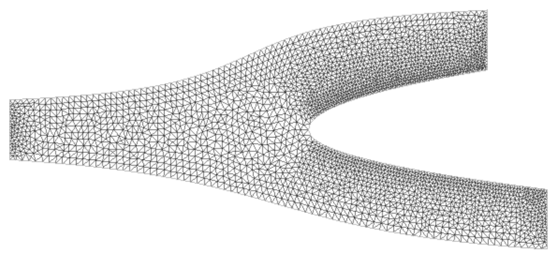

Figure 2.2: A branched pipe and an example of triangulation.

Finally, we examine the convergence in terms of the penalty parameter ε. We now consider a non-trivial external force f and the resulting flow velocity is showed in 2.4.

Letting $\delta = \varepsilon^{\frac{3}{2}}$ (see Theorem 2.6.1), we set

$$E_\varepsilon^{(1)} = \|v - v_h\|, \quad E_\varepsilon^{(2)} = \|v - v_h\|_1, \quad E_\varepsilon^{(3)} = \|q - q_h\|,$$

and observe

$$\rho_\varepsilon^{(i)} = \frac{\log E_{\varepsilon'}^{(i)} - \log E_\varepsilon^{(i)}}{\log \varepsilon' - \log \varepsilon} \qquad (i = 1, 2, 3),$$

where $\varepsilon' \approx 2\varepsilon$ and (v, q) is the numerical solution with extra small ε.

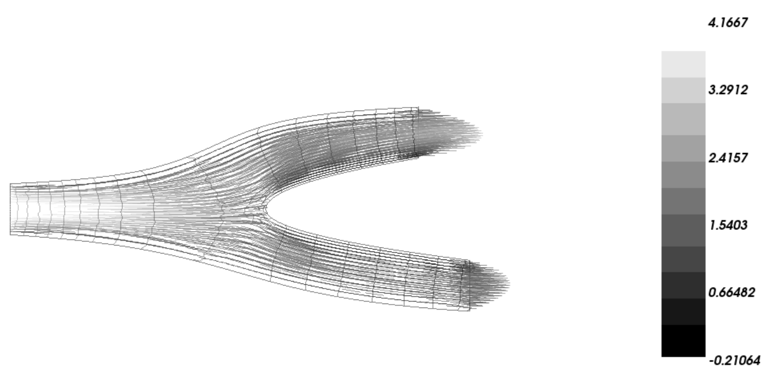

Figure 2.3: Velocity and pressure field in branched pipe.

h	$E_h^{(1)}$	$\rho_h^{(1)}$	$E_h^{(2)}$	$\rho_h^{(2)}$	$E_h^{(3)}$	$\rho_h^{(3)}$
0.69279	$2.497 \cdot 10^{-1}$	—	5.941	—	$1.786 \cdot 10^{-1}$	—
0.33353	$7.767 \cdot 10^{-2}$	1.552	3.359	0.780	$5.909 \cdot 10^{-2}$	1.513
0.17571	$2.044 \cdot 10^{-2}$	2.083	1.768	1.001	$3.069 \cdot 10^{-2}$	1.022

Table 2.2: Numerical convergence rates of $(\mathrm{PE}'_{\varepsilon,\delta,h})$ for branched pipe.

Results are reported in Tab. 2.3 and 2.4, respectively, for $h = 0.6928$ and $h = 0.1757$. We observe from these tables that the first order convergence with respect to ε actually occurs.

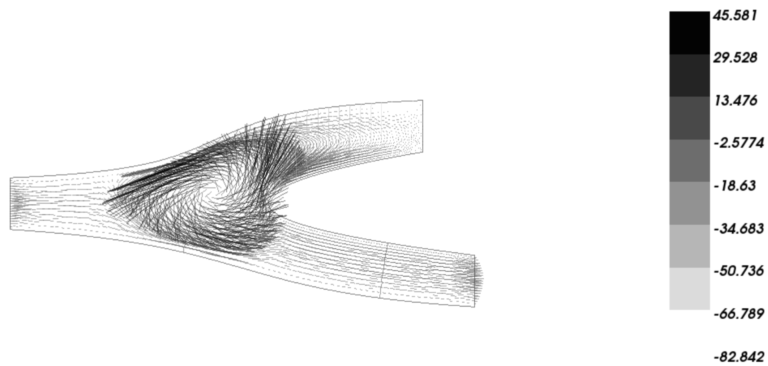

Figure 2.4: Velocity and pressure fields.

ε	$E_\varepsilon^{(1)}$	$\rho_\varepsilon^{(1)}$	$E_\varepsilon^{(2)}$	$\rho_\varepsilon^{(2)}$	$E_\varepsilon^{(3)}$	$\rho_\varepsilon^{(3)}$
0.034639	7.315	—	22.400	—	9.359	—
0.017319	3.649	1.003	11.177	1.002	4.668	1.003
0.008659	1.749	1.060	5.358	1.060	2.238	1.060

Table 2.3: Numerical convergence rate of $(\mathrm{PE}'_{\varepsilon,\delta,h})$ for $h = 0.6928$.

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ε	$E_\varepsilon^{(1)}$	$\rho_\varepsilon^{(1)}$	$E_\varepsilon^{(2)}$	$\rho_\varepsilon^{(2)}$	$E_\varepsilon^{(3)}$	$\rho_\varepsilon^{(3)}$
0.0087854	2.088	—	6.421	—	2.469	—
0.0043927	1.024	1.028	3.149	1.027	1.210	1.028
0.0021964	0.486	1.073	1.497	1.072	0.575	1.073

Table 2.4: Numerical convergence rate of $(\mathrm{PE}'_{\varepsilon,\delta,h})$ for $h = 0.1757$.

Chapter 3

Convergence of the immersed-boundary finite-element method for the Stokes problem

Abstract

Convergence results for the immersed boundary method applied to a model Stokes problem with the homogeneous Dirichlet boundary condition are presented. As a discretization method, we deal with the finite element method. First, the immersed force field is approximated using a regularized delta function and its error in the $W^{-1,p}$ norm is examined for $1 \leq p < n/(n-1)$, n being the space dimension. Then, we consider the immersed boundary discretization of the Stokes problem and study the regularization and discretization errors separately. Consequently, error estimate of order $h^{1-\alpha}$ in the $W^{1,1} \times L^1$ norm for the velocity and pressure is derived, where α is an arbitrarily small positive number. Error estimate of order $h^{1-\alpha}$ in the L^r norm for the velocity is also derived with $r = n/(n-1-\alpha)$. The validity of those theoretical results are confirmed by numerical examples.

3.1 Introduction

The immersed boundary (IB) method is a powerful method for solving a class of fluid-structure interaction problems originally proposed by Peskin [44, 45] to simulate the blood flow through artificial heart valves. For later developments, see [46]. The IB method is also successfully applied to multi-phase flow problems, elliptic interface problems, and so on.

In contrast to a huge number of applications, it seems that there are only a few results about theoretical convergence analysis. The pioneering work was done by Y. Mori in 2008 (see [42]). He studied a model (stationary) Stokes problem for the velocity u and pressure q in an n dimensional torus $U = [\mathbb{R}/(2\pi\mathbb{Z})]^n \subset \mathbb{R}^n$,

$$-\Delta u + \nabla q = f - g \text{ in } U, \quad \nabla \cdot u = 0 \text{ in } U, \tag{3.0}$$

with

$$f(x) = \int_{\Xi} F(\theta)\delta(x - X(\theta))\ d\theta, \quad g = \frac{1}{(2\pi)^n}\int_{\Theta} F(\theta)\ d\theta.$$

Herein, the *immersed boundary* $\Gamma \subset U$, which is assumed to be a hypersurface of $\mathbb{R}^n$, is parameterizaed as

$$\Gamma = \{X(\theta) = (X_1(\theta), \ldots, X_n(\theta)) \mid \theta \in \Theta\},$$

where Θ denotes a subset of $\mathbb{R}^{n-1}$; see Figure 3.1. The function $F = F(\theta)$ denotes the force distributed along Γ and $\delta = \delta(x)$ the (scalar-valued) Dirac delta function. (In [42], the case $n = 2$ was explicitly mentioned.) Introducing the regularized delta function $\delta^h \approx \delta$ with a parameter $h > 0$, he considered the regularized Stokes problem

$$-\Delta\tilde{u} + \nabla\tilde{q} = \int_{\Xi} F(\theta)\delta^h(x - X(\theta))\ d\theta - g \text{ in } U, \quad \nabla \cdot \tilde{u} = 0 \text{ in } U.$$

The regularized problem was discretized by the finite difference method using a uniform Eulerian grid with grid size h. Then, he succeeded in deriving the maximum norm error estimate for the velocity of the form

$$\|u - \tilde{u}_h\|_{L^\infty(U)} \le C(h + h^\alpha)|\log h| \quad (\alpha > 0 \text{ suitable constant})$$

under regularity assumptions on Γ and F together with structural assumptions on δ^h. Herein, $\tilde{u}_h$ denotes the finite difference solution. After that, the method and results were extended to several directions (see [39, 40]). For example, several L^p-error estimates, $1 \le p \le \infty$, were obtained in [40]. A typical result is given as

$$\|u - \tilde{u}_h\|_{L^p(U)} + h\|q - \tilde{q}_h\|_{L^p(U)} \le Ch^2|\log h|^\eta \quad (\eta > 0 \text{ suitable constant}).$$

Similar results for the Poisson interface problem was presented in [35]. On the other hand, we observe from numerical experiments that the IB method has a first order accuracy for the velocity in the L^∞ norm. Therefore, those estimates are only sub-optimal and the proof of optimal-order error estimate is still open at present. Moreover, the explicit formula of the Green function associated with (3.0) was used to derive error estimates in [39, 40, 42]. Hence, it is difficult to apply those methods to more standard settings, for example, to the Dirichlet boundary value problem.

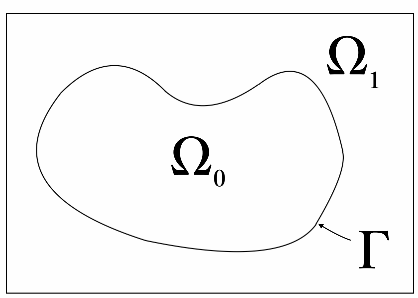

Figure 3.1: $\overline{\Omega} = \overline{\Omega_0 \cup \Omega_1}$, Ω_0, Ω_1 and Γ.

In this work, we take a different approach. We consider the Dirichlet boundary value problem for the Stokes equations (3.5) below and study the *regularization error and discretized error* *separately* in Sections 3.2 and 3.3. To this end, we first give interpretations of the immersed outer force f above as an $\mathbb{R}^n$-Lebesgue measure and as a functional over $W_0^{1,p}(\Omega)^n$; see Propositions 3.2.1 and 3.2.2. (The meaning of mathematical symbols will be mentioned in Paragraph 3.2.1.) Then, we introduce a regularized delta function δ^ε with a parameter $\varepsilon > 0$ and examine the error between f and its regularization

$$f^\varepsilon(x) = \int_\Xi F(\theta)\delta^\varepsilon(x - X(\theta))\ d\theta$$

in the $W^{-1,p}(\Omega)^n$ norm for $1 \le p < \frac{n}{n-1}$; see Proposition 3.2.3. Estimate for the regularization error (see Proposition 3.2.4) is a direct consequence of Proposition 3.2.3 and the stability result of [41] (or (A1_p) below). After introducing structural assumptions on δ^ε,

$$\delta^\varepsilon(x) = \frac{1}{\varepsilon^n}\prod_{i=1}^n \phi\left(\frac{x_i}{\varepsilon}\right),$$

that is essentially the same as that of [39, 40, 42], we show that the $W^{1,p} \times L^p$ error estimate for the velocity and pressure is of order $\varepsilon^{1-n+\frac{n}{p}}$ if $1 \le p < \frac{n}{n-1}$; see Proposition 3.2.5.

Then, we proceed to the study of discretization in Section 3.3 . We are concerned with the finite element method rather than the finite difference method. This enable us to apply several sharp $W^{1,p} \times L^p$ stability and error estimates due to [22] (or (A2_p) below). Finally, we obtain several (still sub-optimal but nearly-optimal) error estimates in several norms; see Theorem 3.3.1 which is the main result of this chapter. The effect of numerical integration for computing f^ε is discussed in Section 3.4. Actually, a simple numerical integration formula does not spoil the accuracy of the IB method (see Proposition 3.4.1 and Theorem 3.4.1). The validity of those theoretical results are confirmed by numerical examples in Section 3.5.

We only assume that ϕ is a continuous function in $\mathbb{R}$ with compact support and with the unit mean value (see (3.12)). On the other hand, several conditions on moment and smoothing orders of ϕ were assumed in [39, 40, 42]; we are able to remove those restrictions.

It should be kept in mind that our aim is to reveal the accuracy of the regularization and discretization procedures and is not to propose a new computational method; see also Remark 3.2.2. We consider the finite element method only as a model discretization method.

3.2 Immersed boundary formulation

3.2.1 Geometry and notation

Suppose that Ω is a polyhedral domain in $\mathbb{R}^n$, $n = 2, 3$, with the boundary $\partial\Omega$. The domain Ω is divided into two disjoint components Ω_0 and Ω_1 by a simple closed curve ($n = 2$) or surface ($n = 3$) which is designated by Γ. The curve (surface) Γ is called the *immersed boundary* and is supposed to be parametrized as $\Gamma = \{X(\theta) = (X_1(\theta), \ldots, X_n(\theta)) \mid \theta \in \Theta\}$ where Θ is a bounded subset of $\mathbb{R}^{n-1}$ for the Lagrangian coordinate. See Fig. 3.1 for example. We set

$$J_X(\theta) = \begin{cases} \sqrt{\left|\frac{\partial X_1}{\partial\theta}\right|^2 + \left|\frac{\partial X_2}{\partial\theta}\right|^2} & \text{if } n = 2, \\ \sqrt{\left|\frac{\partial(X_2,X_3)}{\partial(\theta_1,\theta_2)}\right|^2 + \left|\frac{\partial(X_3,X_1)}{\partial(\theta_1,\theta_2)}\right|^2 + \left|\frac{\partial(X_1,X_2)}{\partial(\theta_1,\theta_2)}\right|^2} & \text{if } n = 3. \end{cases}$$

Throughout this chapter, we assume the following:

- Γ is a C^1 boundary ($X(\theta)$ is a C^1 function);

- $\mathrm{dist}(\Gamma, \partial\Omega) > 0$;
- $J_X(\theta) \neq 0 \quad (\theta \in \Theta)$.

We collect here the notation used in this chapter. We follow the notation of [1] for function spaces and their norms. For a function space X, the space X^n stands for a product space $X \times \cdots \times X$. For abbreviations, we write as, for example,

$$\|u\|_{W^{1,p}} = \|u\|_{W^{1,p}(\Omega)^n}, \qquad \|\pi\|_{L^p} = \|\pi\|_{L^p(\Omega)}.$$

We set $W_0^{1,p}(\Omega) = \{v \in W^{1,p}(\Omega) \mid v|_{\partial\Omega} = 0\}$ and $W^{-1,p}(\Omega)$ the topological dual of $W_0^{1,p}(\Omega)$. The dual product between $W^{-1,p}(\Omega)^n$ and $W_0^{1,p}(\Omega)^n$ is denoted by $\langle \cdot, \cdot \rangle_{W^{-1,p}, W_0^{1,p}}$. We let $L_0^p(\Omega) = \{q \in L^p(\Omega) \mid \int_\Omega q \, dx = 0\}$. Set $B(a,r) = \{x \in \mathbb{R}^n \mid |x-a| < r\}$ for $a \in \mathbb{R}^n$ and $r > 0$.

For $1 \leq p \leq \infty$, let p' be the conjugate exponent of p; $1 \leq p' \leq \infty$ and $\frac{1}{p} + \frac{1}{p'} = 1$.

For vectors $a = (a_1, \ldots, a_n),\ b = (b_1, \ldots, b_n) \in \mathbb{R}^n$, let us denote by $a \cdot b = a_1 b_1 + \cdots + a_n b_n$ the scalar product.

3.2.2 Immersed boundary force

We set (formally at this stage) the immersed boundary force field $f : \Omega \to \mathbb{R}^n$ as

$$f = \int_\Theta F(\theta) \delta_{X(\theta)} \, d\theta \tag{3.1}$$

for $F \in L^1(\Theta)^n$. Hereinafter, we set $\delta_a(x) = \delta(x-a)$ for $a \in \mathbb{R}^n$. We have (still formally)

$$\int_\Omega f(x) \cdot \varphi(x) \, dx = \int_\Theta F(\theta) \cdot \varphi(X(\theta)) \, d\theta \qquad (\varphi \in C_0^\infty(\Omega)^n). \tag{3.2}$$

We state two interpretations of (3.2).

Proposition 3.2.1. *Let $F \in L^1(\Theta)^n$. Then, f defined as* (3.1) *is a finitely signed measure on Ω, with which the integration is defined for any (vector-valued) measurable function φ on Ω. In particular, if $F \in L^p(\Theta)^n$ for $1 \leq p \leq \infty$, the integrant is given by*

$$\langle f, \varphi \rangle = \int_\Omega \varphi \, df = \int_\Theta F(\theta) \cdot \varphi(X(\theta)) \, d\theta$$

for any $\varphi \in W^{1,p'}(\Omega)^n$. Moreover, f is a singular measure against the Lebesgue measure on Ω and, consequently, $f \notin L^1(\Omega)^n$.

Proof. We identify $\delta_a(x) = \delta(x-a)$ with the Dirac measure concentrated at $a \in \mathbb{R}^n$. Then, for any measurable set $B \subset \mathbb{R}^n$ and $\theta \in \Theta$, we have

$$\delta_{X(\theta)}(B) = 1_{X^{-1}(B)}(\theta) = \begin{cases} 1 & (X(\theta) \in B) \\ 0 & (X(\theta) \notin B), \end{cases}$$

where $1_{X^{-1}(B)}$ denotes the indicator function of $X^{-1}(B)$ on Θ. By virtue of Lebesgue's dominated convergence theorem, we derive for any disjoint measurable sets $\{B_n\}_n$

$$
\begin{aligned}
f\left(\bigcup_{n=1}^{\infty} B_n\right) &= \int_\Theta F(\theta)\sum_{n=1}^{\infty}\delta_{X(\theta)}(B_n)\ d\theta = \int_\Theta F(\theta)\sum_{n=1}^{\infty} 1_{X^{-1}(B_n)}(\theta)\ d\theta \\
&= \sum_{n=1}^{\infty}\int_\Theta F(\theta)1_{X^{-1}(B_n)}(\theta)\ d\theta = \sum_{n=1}^{\infty}\int_\Theta F(\theta)\delta_{X(\theta)}(B_n)\ d\theta \\
&= \sum_{n=1}^{\infty} f(B_n).
\end{aligned}
$$

Herein, note that $F(\theta)\sum_{n=1}^{N} 1_{X^{-1}(B_n)}(\theta)$ is integrable for any $N \in \mathbb{N}$ since $F \in L^1(\Theta)^n$ and B_n is disjoint. It follows $f(\emptyset) = 0$ from $\delta_a(\emptyset) = 0$ for all $a \in \mathbb{R}^n$. Thus, f is a finitely signed measure on Ω so that the integral $\int_\Omega \varphi\ df$ is well-defined for all measurable function φ. According to an integral with the Dirac measure, we have

$$
\int_\Omega \varphi\ df = \int_\Theta F(\theta)\cdot\varphi(X(\theta))\ d\theta,
$$

where the right hand side is meaningful for $F \in L^p(\Theta)^n$ and $\varphi \in W^{1,p'}(\Omega)^n$. Although the $\mathbb{R}^n$-Lebesgue measure $m(\Gamma)$ of Γ vanishes (note that Γ is "very thin"), we have $f(\Gamma) \neq 0$. Hence, f is singular against m. Finally, the fact $f \notin L^1(\Omega)$ follows from the Lebesgue decomposition theorem. $\square$

Although f does not belong to any $L^p(\Omega)$ spaces as is mentioned in Proposition 3.2.1, it is well-defined as a functional on $W^{1,p}(\Omega)^n$.

Proposition 3.2.2. *Let $1 \le p < \infty$ and $F \in L^p(\Theta)^n$. Then, the functional*

$$
\langle f, \varphi\rangle = \int_\Theta F(\theta)\cdot\varphi(X(\theta))\ d\theta \quad (\varphi \in C_0^\infty(\Omega)^n)
$$

is extended by continuity to a bounded linear functional on $W_0^{1,p}(\Omega)^n$, which will be denoted by $\langle f, \cdot\rangle_{W^{-1,p},W_0^{1,p}}$ below. That is, we have $f \in W^{-1,p}(\Omega)^n$.

Proof. Let $\varphi \in C_0^\infty(\Omega)^n$. Since

$$
\int_\Gamma |\ \varphi|_\Gamma\ |^{p'}\ d\Gamma = \int_\Theta |\ \varphi(X(\theta))\ |^{p'}\ |J_X(\theta)|\ d\theta,
$$

we have by the trace theorem

$$
\begin{aligned}
\langle f, \varphi\rangle &\le \|F\|_{L^p(\Theta)}\left(\int_\Theta |\ \varphi(X(\theta))\ |^{p'}\ d\theta\right)^{\frac{1}{p'}} \\
&\le \|F\|_{L^p(\Theta)}\|J_X\|_{L^\infty(\Theta)}^{-\frac{1}{p'}}\|\varphi\|_{L^{p'}(\Gamma)} \le C\|F\|_{L^p(\Theta)}\|\varphi\|_{W^{1,p'}(\Omega)}.
\end{aligned}
$$

$\square$

Let $\varepsilon > 0$ be a regularized parameter. Take a continuous function $\delta^\varepsilon = \delta^\varepsilon(x)$ satisfying

$$
\operatorname{supp}\delta^\varepsilon \subset B(0, K\varepsilon) \tag{3.3}
$$

with $K > 0$.

Setting $\delta_a^\varepsilon(x) = \delta^\varepsilon(x-a)$ for $a \in \mathbb{R}^n$, we introduce the *regularized immersed force field* as

$$f^\varepsilon = \int_\Theta F(\theta)\delta^\varepsilon_{X(\theta)}\ d\theta. \tag{3.4}$$

Since $\delta_a^\varepsilon \in L^\infty(\Omega)$, we have $f^\varepsilon \in L^\infty(\Omega)$ for $F \in L^1(\Theta)$. The following result plays the most crucial role in this study.

Proposition 3.2.3. *Suppose that we are given a continuous function δ^ε satisfying* (3.3). *Then, for $1 \le p < \frac{n}{n-1}$ and $F \in L^p(\Theta)$, we have*

$$\|f - f^\varepsilon\|_{W^{-1,p}} \le C_0 \|F\|_{L^p(\Theta)} \left[\left|1 - \int_{\mathbb{R}^n} \delta^\varepsilon(y)\ dy\right| + \|\rho\delta^\varepsilon\|_{L^p(\mathbb{R}^n)}\right],$$

where $\rho(x) = x$ and C_0 denotes a positive constant depending only on n, p and $\|J_X\|_{L^\infty(\Theta)}$.

Proof. Let $\varphi \in C_0^\infty(\Omega)^n$ and express it as

$$\varphi(x) = \varphi(X(\theta)) + (x - X(\theta)) \cdot \int_0^1 \nabla\varphi(t(x - X(\theta)) + X(\theta))\ dt \quad (x \in \mathbb{R}^n).$$

Then, applying Fubini's lemma, we have

$$\begin{aligned}\langle f - f^\varepsilon, \varphi\rangle = &\underbrace{\int_\Theta F(\theta)\varphi(X(\theta))\left(1 - \int_\Omega \delta^\varepsilon_{X(\theta)}(x)\ dx\right)\ d\theta}_{=I_1} \\ &\underbrace{- \int_0^1 \int_\Theta F(\theta) \int_\Omega \delta^\varepsilon_{X(\theta)}(x)(x - X(\theta)) \cdot \nabla\varphi(t(x - X(\theta)) + X(\theta))\ dx d\theta dt}_{=I_2}.\end{aligned}$$

For a sufficiently small ε, we have $B(X(\theta), K\varepsilon) \subset \Omega$ and

$$\int_\Omega \delta^\varepsilon_{X(\theta)}\ dx = \int_{B(X(\theta),K\varepsilon)} \delta^\varepsilon_{X(\theta)}(x)\ dx = \int_{B(0,K\varepsilon)} \delta^\varepsilon(y)\ dy = \int_{\mathbb{R}^n} \delta^\varepsilon(y)\ dy.$$

Hence,

$$\begin{aligned}|I_1| &\le \left|1 - \int_{\mathbb{R}^n} \delta^\varepsilon(y)\ dy\right| \int_\Theta |F(\theta)| \cdot |\varphi(X(\theta))|\ d\theta \\ &\le \|F\|_{L^p(\Theta)} \left(\int_\Theta |\varphi(X(\theta))|^{p'}\ d\theta\right)^{\frac{1}{p'}} \left|1 - \int_{\mathbb{R}^n} \delta^\varepsilon(y)\ dy\right| \\ &\le \frac{\|F\|_{L^p(\Theta)}}{\|J_X\|^{1/p'}_{L^\infty(\Theta)}} \|\varphi\|_{L^{p'}(\Gamma)} \left|1 - \int_{\mathbb{R}^n} \delta^\varepsilon(y)\ dy\right| \\ &\le C\|F\|_{L^p(\Theta)} \left|1 - \int_{\mathbb{R}^n} \delta^\varepsilon(y)\ dy\right| \|\varphi\|_{W^{1,p'}(\Omega)}.\end{aligned}$$

By virtue of Hölder's inequality, we have

$$
\begin{aligned}
|I_2| \leq & \int_0^1 \int_\Theta |F(\theta)| \cdot \|(x - X(\theta))\delta^\varepsilon_{X(\theta)}\|_{L^p(\Omega)} \cdot \\
& \cdot \left[\int_\Omega |\nabla \varphi(t(x - X(\theta)) + X(\theta))|^{p'} \, dx \right]^{\frac{1}{p'}} \, d\theta dt \\
\leq & \|\rho\delta^\varepsilon\|_{L^p(\mathbb{R}^n)} \int_0^1 \int_\Theta |F(\theta)| \left[\int_{\mathbb{R}^n} |\nabla \tilde{\varphi}(t(x - X(\theta)) + X(\theta))|^{p'} \, dx \right]^{\frac{1}{p'}} \, d\theta \\
\leq & \|\rho\delta^\varepsilon\|_{L^p(\mathbb{R}^n)} \int_0^1 \int_\Theta |F(\theta)| \, d\theta \left[\frac{1}{t^n} \int_{\mathbb{R}^n} |\nabla \tilde{\varphi}(z)|^{p'} \, dz \right]^{\frac{1}{p'}} \\
\leq & \|\rho\delta^\varepsilon\|_{L^p(\mathbb{R}^n)} \|F\|_{L^1(\Theta)} \left(\int_0^1 t^{-\frac{n}{p'}} \, dt \right) \|\tilde{\varphi}\|_{W^{1,p'}(\mathbb{R}^n)} \\
\leq & \frac{p'}{p' - n} \|F\|_{L^1(\Theta)} \|\rho\delta^\varepsilon\|_{L^p(\mathbb{R}^n)} \|\varphi\|_{W^{1,p'}},
\end{aligned}
$$

where $\tilde{\varphi}$ denotes the zero extension of φ into $\mathbb{R}^n$ and $z = t(x - X(\theta)) + X(\theta)$. (Note that $n < p' \leq \infty$ by $1 \leq p < \frac{n}{n-1}$.) □

Remark 3.2.1. *We take $\varphi_0 \in C_0^\infty(\Omega)$ satisfying*

$$\varphi_0(x) = 1 \text{ if } x \in \Gamma(\varepsilon) = \{x \in \Omega \mid dist(x, \Gamma) < \varepsilon\} \cup \Omega_0.$$

Then, I_2 in the proof above vanishes and

$$\|f - f^\varepsilon\|_{W^{-1,p}} \geq \frac{\langle f - f^\varepsilon, \varphi_0 \rangle}{\|\varphi_0\|_{W^{1,p'}}} = \frac{1}{\|\varphi_0\|_{W^{1,p'}}} \int_\Theta F(\theta) \, d\theta \left[1 - \int_{\mathbb{R}^n} \delta^\varepsilon(x) \, dx \right].$$

Hence,

$$\int_{\mathbb{R}^n} \delta^\varepsilon(x) \, dx \to 1 \quad (\varepsilon \to 0)$$

is a necessary condition for $\|f - f^\varepsilon\|_{W^{-1,p}} \to 0$ to hold.

3.2.3 Target and regularized problems

We proceed to the formulation of the immersed boundary method. Using f and f^ε defined by (3.1) and (3.4), we consider, respectively, the immersed boundary formulation to the Stokes equations for the velocity u and pressure π,

$$-\nu \Delta u + \nabla \pi = f \text{ in } \Omega, \quad \nabla \cdot u = 0 \text{ in } \Omega, \quad u = 0 \text{ on } \partial\Omega \tag{3.5}$$

and its regularized problem for u^ε and π^ε,

$$-\nu \Delta u^\varepsilon + \nabla \pi^\varepsilon = f^\varepsilon \text{ in } \Omega, \quad \nabla \cdot u^\varepsilon = 0 \text{ in } \Omega, \quad u^\varepsilon = 0 \text{ on } \partial\Omega. \tag{3.6}$$

By a *weak solution* $(u, \pi) \in W_0^{1,p}(\Omega)^n \times L_0^p(\Omega)$ of (3.5) for example, we mean a solution of the following variational equations: Find $(u, \pi) \in W_0^{1,p}(\Omega)^n \times L_0^p(\Omega)$ such that

$$a(u, v) + b(\pi, v) = \langle f, v \rangle_{W^{-1,p}, W_0^{1,p}} \qquad (\forall v \in W_0^{1,p'}(\Omega)^n), \tag{3.7a}$$

$$b(q, u) = 0 \qquad (\forall q \in L_0^{p'}(\Omega)), \tag{3.7b}$$

where

$$a(u,v) = \frac{\nu}{2}\int_\Omega \left(\frac{\partial u_j}{\partial x_i} + \frac{\partial u_i}{\partial x_j}\right)\left(\frac{\partial v_j}{\partial x_i} + \frac{\partial v_i}{\partial x_j}\right)\, dx, \tag{3.8a}$$

$$b(\pi,u) = -\int_\Omega \pi(\nabla\cdot u)\, dx. \tag{3.8b}$$

Remark 3.2.2. *Problem* (3.7) *can be directly discretized by the finite element method with no regularization of f. Such methods were studied in [7, 52] for nonstationary Navier-Stokes equations. However, our aim here is to reveal the accuracy of the regularization and discretization procedures as is mentioned in Introduction.*

Remark 3.2.3. *The bilinear form a defined by* (3.8a) *is based on the deformation-rate tensor* $[(1/2)(u_{i,j}+u_{j,i})]_{1\le i,j\le n}$. *Another definition*

$$a(u,v) = \nu\int_\Omega \frac{\partial u_j}{\partial x_i}\frac{\partial v_j}{\partial x_i}\, dx$$

is also available. However, with (3.8a), *our problem is (essentially) equivalent to a two-phase Stokes problem considered in [52] for example.*

We make the following assumption for $1\le p<\infty$:

(A1$_p$) For a given $g\in W^{-1,p}(\Omega)^n$, there exists a unique weak solution $(w,r)\in W_0^{1,p}(\Omega)\times L_0^p(\Omega)$ of the Stokes problem,

$$-\nu\Delta w + \nabla r = g \text{ in } \Omega,\quad \nabla\cdot w = 0 \text{ in } \Omega,\quad w=0 \text{ on } \partial\Omega \tag{3.9}$$

satisfying

$$\|w\|_{W^{1,p}} + \|r\|_{L^p} \le C_1\|g\|_{W^{-1,p}}. \tag{3.10}$$

Moreover, if $g\in W^{-1,2}(\Omega)^n\cap L^p(\Omega)$, we have $(w,r)\in W^{2,p}(\Omega)\times W^{1,p}(\Omega)$ and

$$\|w\|_{W^{2,p}} + \|r\|_{W^{1,p}} \le C_2\|g\|_{L^p}. \tag{3.11}$$

Herein, C_1 and C_2 denote positive constants depending only on p and Ω.

Remark 3.2.4. *If Ω is a convex Lipschitz domain and $1<p\le 2$, (A1$_p$) is satisfied in view of [41, Example 5.5] and Lemma 3.2.2. However, we directly assume (A1$_p$) instead of the shape condition on Ω. Below, p will be restricted as $p<n/(n-1)$.*

The following result is a direct consequence of Lemma 3.2.3 and (A1$_p$).

Proposition 3.2.4. *Let $1\le p<\frac{n}{n-1}$ and suppose that* (A1$_p$) *is satisfied. Let $F\in L^p(\Theta)$. Let (u,π) and $(u^\varepsilon,\pi^\varepsilon)$ be the weak solutions of* (3.5) *and* (3.6), *respectively. Then, we have*

$$\|u-u^\varepsilon\|_{W^{1,p}} + \|\pi-\pi^\varepsilon\|_{L^p} \le C_0C_1\|F\|_{L^p(\Theta)}\left[\left|1-\int_{\mathbb{R}^n}\delta^\varepsilon(y)\, dy\right| + \|\rho\delta^\varepsilon\|_{L^p(\mathbb{R}^n)}\right].$$

The most familiar choice of δ^ε is given by a product of one variable functions:

$$\delta^\varepsilon(x) = \frac{1}{\varepsilon^n}\prod_{i=1}^n \phi\left(\frac{x_i}{\varepsilon}\right) \quad (x=(x_1,\ldots,x_n)); \tag{3.12a}$$

$$\phi \text{ is continuous in } \mathbb{R},\quad \operatorname{supp}\phi\subset B(0,K\varepsilon),\quad \int_{\mathbb{R}}\phi(s)\, ds = 1 \tag{3.12b}$$

with $K > 0$. In (3.12a), the function $(1/\varepsilon)\phi(x_i/\varepsilon)$ is an approximation of the one-dimensional Dirac delta. Then, we can calculate as:

$$\int_{\mathbb{R}^n} \delta^\varepsilon(y)\ dy = 1; \tag{3.13a}$$

$$\int_{\mathbb{R}^n} |y|^p |\delta^\varepsilon(y)|^p\ dy \le C_3 \varepsilon^{p-pn+n}; \tag{3.13b}$$

$$\int_{\mathbb{R}^n} |\delta^\varepsilon(y)|^p\ dy \le C_3' \varepsilon^{-pn+n}, \tag{3.13c}$$

where C_3 and C_3' denote positive constants depending only on p, n, K and $\|\phi\|_{L^\infty(\mathbb{R})}$. For example, if $n = 3$,

$$\begin{aligned}
&\int_{\mathbb{R}^n} |y|^p |\delta^\varepsilon(y)|^p\ dy \\
&\quad \le \varepsilon^{p-pn+n} \int_0^{\sqrt{n}K} \int_0^{2\pi} \int_0^{\pi} s^{p+n-1} |\phi(s\cos\varphi\sin\theta)|^p \cdot \\
&\qquad\qquad \cdot |\phi(s\sin\varphi\sin\theta)|^p |\phi(s\cos\theta)|^p \sin\theta\ ds d\varphi d\theta \\
&\quad \le \frac{4\pi}{p+n} (\sqrt{3}K)^{p+n} \|\phi\|_{L^\infty(\mathbb{R})}^{3p} \varepsilon^{p-pn+n}.
\end{aligned}$$

Similarly, we can take $C_3 = \frac{2\pi}{p+2}(\sqrt{2}K)^{p+2}\|\phi\|_{L^\infty(\mathbb{R})}^{2p}$ if $n = 2$.

Therefore, our error estimate for the regularized problem is given as follows.

Proposition 3.2.5. *Let $1 \le p < \frac{n}{n-1}$ and suppose that* (A1$_p$) *is satisfied. Let $F \in L^p(\Theta)$. Let (u, p) and $(u^\varepsilon, p^\varepsilon)$, respectively, be the weak solutions of* (3.5) *and* (3.6) *with* (3.12)*. Then, we have*

$$\|u - u^\varepsilon\|_{W^{1,p}} + \|p - p^\varepsilon\|_{L^p} \le C\|F\|_{L^p(\Theta)} \varepsilon^{1-n+\frac{n}{p}}, \tag{3.14}$$

where C denotes a positive constant depending only on n, p, $\|J_X\|_{L^\infty(\Theta)}$, K, $\|\phi\|_{L^\infty(\mathbb{R})}$ and Ω.

Remark 3.2.5. *Proposition 3.2.5 remains valid for a bounded Lipschitz domain Ω.*

3.3 Discretization by finite element method

This section is devoted to a study of the finite element approximation applied to (3.6). We introduce a family of regular triangulations $\{\mathscr{T}_h\}_h$ of Ω (see [11, (4.4.16)]). Hereinafter, we set $h = \max\{h_T \mid T \in \mathscr{T}_h\}$, where h_T denotes the diameter of T. For any $T \in \mathscr{T}_h$, let $\mathscr{P}_1(T)$ be the set of all polynomials defined on T of degree ≤ 1, and let $\mathscr{B}(T) = [\mathscr{P}_1(T) \oplus \mathrm{span}\{\lambda_1\lambda_2\cdots\lambda_{n+1}\}]^n$, where λ_i are the barycentric coordinates of T. Below, we consider the P1-b/P1 element (MINI element) approximation. That is, set

$$\begin{aligned}
V_h &= \{v_h \in C(\overline{\Omega})^n \cap W_0^{1,2}(\Omega)^n \mid v_h|_T \in \mathscr{B}(T)\ (\forall T \in \mathscr{T}_h)\}, \\
Q_h &= \{q_h \in C(\overline{\Omega}) \cap L_0^2(\Omega) \mid q_h|_T \in \mathscr{P}_1(T)\ (\forall T \in \mathscr{T}_h)\}.
\end{aligned}$$

It is well-known that (see [23, Lemma II.4.1]) a pair of V_h and Q_h satisfies the uniform Babuška–Brezzi (inf–sup) condition

$$\sup_{v_h \in V_h} \frac{b(v_h, q_h)}{\|v_h\|_{W^{1,2}}} \ge \beta\|q_h\|_{L^2} \qquad (q_h \in Q_h),$$

where $\beta > 0$ is independent of h.

Remark 3.3.1. *We deal with the P1-b/P1 element only for the sake of simple presentation. An arbitrary pair of conforming finite element spaces $V_h \subset W_0^{1,2}(\Omega)^n$ and $Q_h \subset L_0^2(\Omega)$ satisfying the uniform Babuška–Brezzi condition is available.*

We state the finite element approximation to (3.6): Find $(u_h^\varepsilon, \pi_h^\varepsilon) \in V_h \times Q_h$ such that

$$a(u_h^\varepsilon, v_h) + b(\pi_h^\varepsilon, v_h) = (f^\varepsilon, v_h)_{L^2} \qquad (\forall v_h \in V_h), \tag{3.15a}$$

$$b(q_h, u_h^\varepsilon) = 0 \qquad (\forall q_h \in Q_h). \tag{3.15b}$$

The finite element approximation $(w_h, r_h) \in V_h \times Q_h$ of (3.9) is defined similarly.

We make the following assumption:

(A2$_p$) For a given $g \in L^p(\Omega)^n$, the finite element approximation $(w_h, r_h) \in V_h \times Q_h$ of (3.9) admits

$$\|w - w_h\|_{W^{1,p}} + \|r - r_h\|_{L^p} \le C_4 \inf_{(v_h, q_h) \in V_h \times Q_h} \left(\|w - v_h\|_{W^{1,p}} + \|r - q_h\|_{L^p} \right),$$

where C_4 denotes a positive constant depending only on p and Ω, and $(w, r) \in W_0^{1,p}(\Omega)^n \times L_0^p(\Omega)$ the weak solution of (3.9).

Remark 3.3.2. *If Ω is a convex polyhedral domain in $\mathbb{R}^n$ with $n = 2, 3$ and $\{\mathcal{T}_h\}_h$ is quasi-uniform (see [11, (4.4.15)]), then (A2$_p$) is actually satisfied for $1 < p \le \infty$; see Corollaries 4, 5, 6 and Remark 4 of a sophisticated paper [22]. However, we directly assume (A2$_p$) instead of the shape condition on Ω as before.*

Applying the standard interpolation/projection error estimates, we obtain the following.

Proposition 3.3.1. *Let $1 \le p < \infty$ and suppose that* (A1$_p$) *and* (A2$_p$) *are satisfied. Let $(u^\varepsilon, \pi^\varepsilon)$ and $(u_h^\varepsilon, \pi_h^\varepsilon)$ be solutions of* (3.6) *and* (3.15), *respectively. Then, we have*

$$\|u^\varepsilon - u_h^\varepsilon\|_{W^{1,p}} + \|\pi^\varepsilon - \pi_h^\varepsilon\|_{L^p} \le Ch\|f^\varepsilon\|_{L^p}, \tag{3.16}$$

where C denotes a positive constant depending only on p and Ω.

Putting together those results, we deduce the following error estimate.

Proposition 3.3.2. *Let $1 \le p < \frac{n}{n-1}$ and suppose that* (A1$_p$) *and* (A2$_p$) *are satisfied. Assume $F \in L^{p'}(\Theta)$. Let (u, π) and $(u_h^\varepsilon, \pi_h^\varepsilon)$ be solutions of* (3.5) *and* (3.15) *with* (3.12), *respectively. Then, we have*

$$\|u - u_h^\varepsilon\|_{W^{1,p}} + \|\pi - \pi_h^\varepsilon\|_{L^p} \le C\varepsilon^{-n+\frac{n}{p}}(\varepsilon + h), \tag{3.17}$$

where C denotes a positive constant depending only on n, p, $\|J_X\|_{L^\infty(\Theta)}$, $\mathrm{meas}(\Theta)$, K, $\|\phi\|_{L^\infty(\mathbb{R})}$, $\|F\|_{L^p(\Theta)}$, $\|F\|_{L^{p'}(\Theta)}$ and Ω.

Proof. Since f^ε is defined in terms of δ^ε given by (3.12), we have by (3.13c)

$$\begin{aligned} \|f^\varepsilon\|_{L^p} &\le \mathrm{meas}(\Theta)^{1/p}\|F\|_{L^{p'}(\Theta)}\|\delta^\varepsilon\|_{L^p(\mathbb{R}^n)} \\ &\le C\,\mathrm{meas}(\Theta)^{1/p}\|F\|_{L^{p'}(\Theta)}\varepsilon^{-n+\frac{n}{p}}, \end{aligned}$$

where $C > 0$ is a constant depending only on p, n, K, and $\|\phi\|_{L^\infty(\mathbb{R})}$. Hence, in view of Lemmas 3.2.5 and 3.3.1,

$$\begin{aligned} \|u - u_h^\varepsilon\|_{W^{1,p}} + \|\pi - \pi_h^\varepsilon\|_{L^p} &\le \|u - u^\varepsilon\|_{W^{1,p}} + \|\pi - \pi^\varepsilon\|_{L^p} \\ &\quad + \|u^\varepsilon - u_h^\varepsilon\|_{W^{1,p}} + \|\pi^\varepsilon - \pi_h^\varepsilon\|_{L^p} \\ &\le C\varepsilon^{1-n+\frac{n}{p}} + Ch \cdot C\varepsilon^{-n+\frac{n}{p}}. \end{aligned}$$

□

We usually take as $\varepsilon = h$ in the immersed boundary method. Therefore, applying Proposition 3.3.2 with $p = 1$, we obtain the optimal order error estimate

$$\|u - u_h^\varepsilon\|_{W^{1,1}} + \|\pi - \pi_h^\varepsilon\|_{L^1} \le Ch. \tag{3.18}$$

It should be kept in mind that this estimate is available only if (A1$_p$) and (A2$_p$) are true. However, the case $p = 1$ is excluded both in [41] and [22] (see Remarks 3.2.4 and 3.3.2). In conclusion, we offer the following theorem as the final error estimate in this chapter.

Theorem 3.3.1. *Suppose that Ω is a convex polyhedral domain in $\mathbb{R}^n$ with $n = 2, 3$. Assume that $\{\mathcal{T}_h\}_h$ is a family of quasi-uniform triangulations. Let $F \in L^\infty(\Theta)$. Let (u, π) and $(u_h^\varepsilon, \pi_h^\varepsilon)$ be solutions of* (3.5) *and* (3.15) *with* (3.12)*, respectively. Further, let $\varepsilon = \gamma_1 h$ with a positive constant γ_1. Then, for any $0 < \alpha < 1$, there exists a positive constant C depending only on γ_1, n, α, Ω, K, $\|\phi\|_{L^\infty(\mathbb{R})}$, $\|J_X\|_{L^\infty(\Theta)}$, $\mathrm{meas}(\Theta)$, and $\|F\|_{L^\infty(\Theta)}$ such that*

$$\|u - u_h^\varepsilon\|_{W^{1,q}} + \|\pi - \pi_h^\varepsilon\|_{L^q} \le Ch^{1-\alpha} \quad \text{with any } 1 \le q \le \frac{n}{n-\alpha} \tag{3.19}$$

and

$$\|u - u_h^\varepsilon\|_{L^r} \le Ch^{1-\alpha} \quad \text{with} \quad r = \frac{n}{n-\alpha-1}. \tag{3.20}$$

Proof. As was pointed out in Remarks 3.2.4 and 3.3.2, (A1$_p$) and (A2$_p$) are true for a convex polyhedral domain. Setting $\alpha = n(1-1/p)$ in (3.17) and applying an obvious inequality $\|\psi\|_{L^q} \le C\|\psi\|_{L^p}$ for $1 \le q \le n/(n-\alpha)$, we deduce (3.19). Inequality (3.20) is a consequence of (3.17) and the Sobolev embedding theorem (see [1, Theorem 4.12, Part I, Case C]). $\square$

Remark 3.3.3. *The exponent r in* (3.20) *is included in $2 < r < \infty$ if $n = 2$ and in $3/2 < r < 3$ if $n = 3$.*

3.4 Numerical integration

In this section, we study the error caused by numerical integrations for computing f^ε. As will be stated below, a simple numerical integration formula does not spoil the accuracy of the immersed boundary method described in Theorem 3.3.1.

First, we deal with the case $n = 2$. Suppose that we are given a continuous function $F(\theta)$ in $\Theta = (c_1, d_1)$ with $c_1 < d_1$. Let us introduce a partition $c_1 = \theta_0 < \theta_1 < \cdots < \theta_M = d_1$. Moreover, letting $\theta_{-\frac{1}{2}} = \theta_0$, $\theta_{i-\frac{1}{2}} = (\theta_i + \theta_{i-1})/2$ for $1 \le i \le M$, and $\theta_{M+\frac{1}{2}} = \theta_M$, we set $\zeta_i = \theta_{i+\frac{1}{2}} - \theta_{i-\frac{1}{2}}$ for $0 \le i \le M$. Further, set $\zeta = \max_{0 \le i \le M} \zeta_i$. Then, we employ the midpoint rule to compute f^ε, that is,

$$f^{\varepsilon,\zeta}(x) = \sum_{i=0}^{M} F(\theta_i)\delta^\varepsilon_{X(\theta_i)}(x)\zeta_i = \sum_{i=0}^{M} F(\theta_i)\delta^\varepsilon(x - X(\theta_i))\zeta_i. \tag{3.21}$$

It is useful to express $f^{\varepsilon,\zeta}$ as

$$f^{\varepsilon,\zeta}(x) = \int_\Theta \hat{F}^\zeta(\theta)\delta^\varepsilon(x - \hat{X}^\zeta(\theta))\, d\theta, \tag{3.22}$$

where $\hat{F}^\zeta(\theta)$ and $\hat{X}^\zeta(\theta) = (\hat{X}_1^\zeta(\theta), \hat{X}_2^\zeta(\theta))$ are piecewise constant functions such that

$$\hat{F}^\zeta(\theta) = F(\theta_i), \quad \hat{X}^\zeta(\theta) = X(\theta_i) \qquad (\theta_{i-\frac{1}{2}} < \theta \le \theta_{i+\frac{1}{2}},\ 0 \le i \le M).$$

From the standard theory, we know

$$\|F - \hat{F}^\zeta\|_{L^\infty(\Theta)} \le C\zeta |F|_{W^{1,\infty}(\Theta)}, \tag{3.23a}$$

$$\|X - \hat{X}^\zeta\|_{L^\infty(\Theta)^n} \le C\zeta |X|_{W^{1,\infty}(\Theta)^n}, \tag{3.23b}$$

where $|F|_{W^{1,\infty}(\Theta)}$ denotes the seminorm in $W^{1,\infty}(\Theta)$ for example.

For the case of $n = 3$, $f^{\varepsilon,\zeta}$ is defined similary. We introduce a suitable partition of $\Theta = (c_1, d_1) \times (c_2, d_2)$ with $c_l < d_l$, $l = 1, 2$, and the size parameter $\zeta > 0$. Let $\hat{F}^\zeta(\theta)$ and $\hat{X}^\zeta(\theta)$ be piecewise constant interpolations of $F(\theta)$ and $X(\theta)$, respectively. Then, f^ε is approximated by $f^{\varepsilon,\zeta}$ defined as (3.22). For the partition of Θ, we only assume so that (3.23) hold true.

Remark 3.4.1. *Let $n = 2$. If $F(\theta)$ is a periodic function, $F(c_1) = F(d_1)$, and the partition is uniform, $f^{\varepsilon,\zeta}$ is coincide with the trapezoidal rule for $F(\theta)\delta^\varepsilon(x - X(\theta))$. However, we here do not explicitly assume the periodicity for $F(\theta)$.*

Proposition 3.4.1. *Let δ^ε be a continuous function satisfying* (3.3). *Suppose that $F \in C^1(\Theta)$. Then, for $1 \le p < \frac{n}{n-1}$, we have*

$$\|f - f^{\varepsilon,\zeta}\|_{W^{-1,p}} \le C \left[\left| 1 - \int_{\mathbb{R}^n} \delta^\varepsilon(y)\ dy \right| + \|\rho\delta^\varepsilon\|_{L^p(\mathbb{R}^n)} + \zeta + \zeta^{1-n+\frac{n}{p}} \right],$$

where $\rho(x) = x$ and C denotes a positive constant depending only on n, p, $\|J_X\|_{L^\infty(\Theta)}$, $|F|_{W^{1,\infty}(\Theta)}$ and $|X|_{W^{1,\infty}(\Theta)^n}$.

Proof. It is a just modification of the proof of Proposition 3.2.3. Let $\varphi \in C_0^\infty(\Omega)^n$ and express it as

$$\varphi(x) = \varphi(\hat{X}^\zeta(\theta)) + (x - \hat{X}^\zeta(\theta)) \cdot \int_0^1 \nabla\varphi(t(x - \hat{X}^\zeta(\theta)) + \hat{X}^\zeta(\theta))\ dt$$

for $x \in \mathbb{R}^n$. Using this, we have

$$\begin{aligned}
&\langle f - f^{\varepsilon,\zeta}, \varphi \rangle_{W^{-1,p}, W_0^{1,p}} \\
&\quad = \underbrace{\int_\Theta F(\theta)\varphi(X(\theta))\ d\theta - \int_\Theta F(\theta)\varphi(\hat{X}^\zeta(\theta)) \left(\int_\Omega \delta^\varepsilon(x - \hat{X}^\zeta(\theta))\ dx \right) d\theta}_{=I_1} \\
&\quad \underbrace{- \int_0^1 \int_\Theta \hat{F}^\zeta(\theta) \int_\Omega \delta^\varepsilon_{\hat{X}^\zeta(\theta)}(x)(x - \hat{X}^\zeta(\theta)) \cdot \nabla\varphi(t(x - \hat{X}^\zeta(\theta)) + \hat{X}^\zeta(\theta))\ dxd\theta dt}_{=I_2}.
\end{aligned}$$

To estimate $|I_1|$, we further divide it as follows:

$$\begin{aligned}
I_1 = &\underbrace{\int_\Theta [F(\theta) - \hat{F}^\zeta(\theta)]\varphi(X(\theta))\ d\theta}_{=I_{11}} + \underbrace{\int_\Theta \hat{F}^\zeta(\theta)[\varphi(X(\theta)) - \varphi(\hat{X}^\zeta(\theta))]\ d\theta}_{=I_{12}} \\
&+ \underbrace{\int_\Theta \hat{F}^\zeta(\theta)\varphi(\hat{X}^\zeta(\theta)) \left(1 - \int_\Omega \delta^\varepsilon(x - \hat{X}^\zeta(\theta))\ dx \right) d\theta}_{=I_{13}}.
\end{aligned}$$

As in the proof of Proposition 3.2.3, we derive

$$|I_{13}| \le C\|\hat{F}^\zeta\|_{L^p(\Theta)} \left| 1 - \int_{\mathbb{R}^n} \delta^\varepsilon(y)\ dy \right| \|\varphi\|_{W^{1,p'}(\Omega)}.$$

By (3.23), we have

$$|I_{11}| \le C\|F - \hat{F}^\zeta\|_{L^p(\Theta)}\|\varphi\|_{L^{p'}(\Theta)}$$
$$\le C\zeta|F|_{W^{1,\infty}(\Theta)}\|\varphi\|_{W^{1,p'}(\Omega)}.$$

We apply Morrey's inequality to obtain

$$|I_{12}| \le C\int_\Theta |\hat{F}^\zeta(\theta)| \cdot |X(\theta) - \hat{X}^\zeta(\theta)|^{1-n/p'} \cdot \|\varphi\|_{W^{1,p'}}\ d\theta$$
$$\le C\zeta^{1-n/p'}\|\hat{F}^\zeta\|_{L^1(\Theta)}|X|_{W^{1,\infty}(\Theta)^n}\|\varphi\|_{W^{1,p'}}.$$

Estimation for $|I_2|$ is done in exactly the way as the proof of Proposition 3.2.3, that is, we deduce

$$|I_2| \le C\|\hat{F}^\zeta\|_{L^1(\Theta)}\|\rho\delta^\varepsilon\|_{L^p(\mathbb{R}^n)}\|\varphi\|_{W^{1,p'}}.$$

Finally, noting $\|\hat{F}^\zeta\|_{L^1(\Theta)} \le C\|F\|_{L^\infty(\Theta)}$, we get the disired inequality. □

Applying Proposition 3.4.1 instead of Proposition 3.2.3, we obtain the following result.

Theorem 3.4.1. *Let* (u, π) *and* $(u_h^\varepsilon, \pi_h^\varepsilon)$ *be solutions of* (3.5) *and* (3.15) *with* (3.12), *respectively, where* f^ε *is replaced by* $f^{\varepsilon,\zeta}$ *defined as* (3.22). *In addition to assumptions of Theorem 3.3.1, we assume that* $F \in C^1(\Theta)$. *Further, let* $\zeta = \gamma_2 h$ *with a positive constant* γ_2. *Then, error estimates* (3.19) *and* (3.20) *remain true.*

3.5 Numerical experiments

Throughout this section, we let $\Omega = (-1, 1)^2 \subset \mathbb{R}^2$ and $\Gamma = B(0, 1/2)$. We consider the stationary Stokes problem

$$-\nu\Delta u + \nabla\pi = f + g \text{ in } \Omega, \quad \nabla \cdot u = 0 \text{ in } \Omega, \quad u = 0 \text{ on } \partial\Omega.$$

Herein, we have added an extra outer force $g = (1, 0)$ in order to illustrate a pressure jump across Γ. We also set $\Theta = [0, 2\pi]$ and

$$X(\theta) = \frac{1}{2}(\cos(\theta), \sin(\theta)).$$

In accordance to the simplest elasticity modeling (see [7, 46]), we take $F(\theta) = \kappa\partial^2 X/\partial\theta^2$ with κ is a suitable positive constant. Specifically, taking $\kappa = 2$, we set

$$F(\theta) = -(\cos(\theta), \sin(\theta)).$$

We deal with the following problem: Find $(u_h^\varepsilon, p_h^\varepsilon) \in V_h \times Q_h$ such that

$$\begin{aligned} &a(u_h^\varepsilon, v_h) + b(p_h^\varepsilon, v_h) = (f^{\varepsilon,\zeta} + g, v_h)_{L^2} && (\forall v_h \in V_h), \\ &b(q_h, u_h^\varepsilon) = 0 && (\forall q \in Q_h). \end{aligned}$$

Herein, $f^{\varepsilon,\zeta}$ is defined as (3.22) and $\zeta = 2\pi/M$, $\theta_i = i\zeta$, $0 \le i \le M$. As a choice of δ^ε, we examine the following discrete delta function

$$\phi(s) = \begin{cases} \frac{1}{2}(1 + \cos(\pi s)) & (|s| \le 1) \\ 0 & (\text{otherwise}). \end{cases}$$

For discretization, we take $\mathscr{T}_h$ as a uniform mesh composed of $2N^2$ congruent right-angle triangles; each side of Ω is divided into N intervals of the same length. Then each small square is decomposed into two equal triangles by a diagonal. Each parameters are set as follows:

$$h = \frac{\sqrt{2}}{N}, \quad \varepsilon = h, \quad M = N, \quad \text{and} \quad \zeta = \frac{2\pi}{M} = \sqrt{2}\pi\varepsilon.$$

To confirm convergence results described in Theorems 3.3.1 and 3.4.1, we compute the following quantities:

$$E_h^{r(0)} = \|\tilde{u} - u_h\|_{L^r}, \quad E_h^{r(1)} = \|\tilde{u} - u_h\|_{W^{1,r}}, \quad \text{and} \quad E_h^{r(3)} = \|\tilde{\pi} - \pi_h\|_{L^r},$$

where $(\tilde{u}, \tilde{\pi}) \in V_{h'} \times Q_{h'}$ denotes the numerical solution using a finer triangulation $\mathcal{T}_{h'}$. Moreover, we compute

$$\rho_h^{r(i)} = \frac{\log E_{2h}^{r(i)} - \log E_h^{r(i)}}{\log 2h - \log h} \quad (i = 1, 2, 3).$$

The result is reported in Table 3.1–3.3. We observe from Table 3.1 that convergence rates of the $W^{1,1}$-error for velocity is first-order while that of the L^1-error for pressure is larger than 1. It is also observed that as p becomes larger, each convergence rate becomes worse. Nevertheless, the rate of the L^2-error for velocity is still larger than 1; see Table 3.3. All of those numerical results support our theoretical results. From those numerical observations, we infer that the following optimal-order error estimate,

$$\|u^\varepsilon - u_h^\varepsilon\|_{W^{1,r}} + h\|u^\varepsilon - u_h^\varepsilon\|_{L^r} \leq Ch^{1-\alpha}$$

actually holds true. However, we postpone the proof of this conjecture for future study.

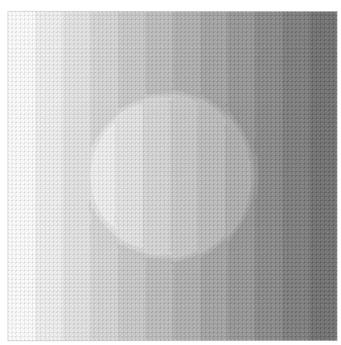

Figure 3.2: Profile of pressure π_h^ε for $N = 80$. A jump of pressure is observed across Γ so that π_h^ε becomes a discontinuous function.

h	$E_h^{1(1)}$	$\rho_h^{1(1)}$	$E_h^{1(2)}$	$\rho_h^{1(2)}$	$E_h^{1(3)}$	$\rho_h^{1(3)}$
0.2828	0.000436582	—	0.0104959	—	0.0508396	—
0.1414	0.00010817	2.0129	0.00525413	0.9983	0.0195045	1.382
0.0707	2.61239e-05	2.0498	0.00262956	0.9986	0.00892026	1.128
0.0353	7.315e-06	1.8364	0.0012387	1.086	0.00269059	1.729

Table 3.1: Convergence rates in $W^{1,r} \times L^r$ with $r = 1$.

h	$E_h^{1.5(1)}$	$\rho_h^{1.5(1)}$	$E_h^{1.5(2)}$	$\rho_h^{1.5(2)}$	$E_h^{1.5(3)}$	$\rho_h^{1.5(3)}$
0.2828	5.92276e-06	—	0.000648725	—	0.00932687	—
0.1414	9.07843e-07	1.803	0.000298948	0.745	0.00291411	1.118
0.0707	1.51842e-07	1.719	0.000150538	0.659	0.00113506	0.906
0.0353	2.37657e-08	1.783	7.14263e-05	0.717	0.000222016	1.569

Table 3.2: Convergence rates in $W^{1,r} \times L^r$ with $r = 3/2$.

h	$E_h^{2(1)}$	$\rho_h^{2(1)}$	$E_h^{2(2)}$	$\rho_h^{2(2)}$	$E_h^{2(3)}$	$\rho_h^{2(3)}$
0.2828	8.89239e-08	—	4.5878e-05	—	0.00196196	—
0.1414	8.78242e-09	1.669	1.98618e-05	0.60390	0.000535873	0.9361
0.0707	1.03983e-09	1.539	1.02922e-05	0.47422	0.000176548	0.8009
0.0353	1.05306e-10	1.651	5.33301e-06	0.47426	2.25745e-05	1.4836

Table 3.3: Convergence rates in $W^{1,r} \times L^r$ with $r = 2$.

Chapter 4

Numerical analysis of a Stokes interface problem based on formulation using the characteristic function

Abstract

Numerical analysis of a model Stokes interface problem with the homogeneous Dirichlet boundary condition is considered. The interface condition is interpreted as an additional singular force field to the Stokes equations using characteristic function. As a discretization method, the finite element method is applied after introducing regularization of the singular source term. Consequently, the error is divided into the regularization and discretization parts which will be studied separately. As results, error estimates of order $h^{\frac{1}{2}}$ in $H^1 \times L^2$ norm for the velocity and pressure, and of order h in L^2 norm for the velocity are derived. Those theoretical results are also verified by numerical examples.

4.1 Introduction

In study of multi-phase flow problems of viscous incompressible fluids, we often encounter the Navier-Stokes equations with an interface condition

$$\frac{\partial u}{\partial t} + (u \cdot \nabla)u - \nu \Delta u + \frac{1}{\rho}\nabla p = h(x,t), \quad \nabla \cdot u = 0 \qquad \text{in } \Omega,\ t > 0, \tag{4.1a}$$

$$u = 0 \qquad \text{on } \partial\Omega,\ t > 0, \tag{4.1b}$$

$$[u] = 0, \quad [\tau] = g(x,t) \qquad \text{on } \Gamma,\ t > 0, \tag{4.1c}$$

$$u(x,0) = u^{(0)}(x) \qquad \text{in } \Omega. \tag{4.1d}$$

for velocity $u(x,t)$ and pressure $p(x,t)$. Herein, Ω denotes a fixed bounded domain in $\mathbb{R}^d$ ($d = 2,3$) with the boundary $\partial\Omega$, Γ is a surface/curve included in Ω which implies the interface.

The coefficient of kinetic viscosity ν is assumed to be a piecewise constant function ($\nu = \nu_1$ inside Γ and $\nu = \nu_2$ outside Γ for example). The traction (or stress) vector is denoted by τ; see Section 4.2 for the precise definition. Moreover, $[\cdot]$ stands for a jump across the interface Γ. We assume that $h(x,t), g(x,t)$ and $u^{(0)}(x)$ are given functions. There are number of literature devoted to numerical methods for these kinds of interface problems (see [19,28,50]) for example. In particular, the variational formulation of (4.1) is directly discretized by the finite element method (see [7] and [52]). However, the variational formulation could not be applied to the finite difference and finite volume methods because of the presence of the boundary integral term $\int_\Gamma g(x,t)v(x)\ dx$. Even if we use the finite element method, the approximation of the boundary integral term is quite technical. In order to avoid those difficulties, the *immersed boundary (IB) method* is frequently applied in many applications. It is the method proposed by C. S. Peskin [46] originally for solving a class of fluid-structure interaction problems [44,45]. In the IB method, the interface problem (4.1) is equivalently reformulated to classical partial differential equations as follows. Let $\Gamma(t)$ be parameterized as $\Gamma(t) = \{X(\theta,t) = (X_1(\theta,t),\dots,X_n(\theta,t)) \mid \theta \in \Theta\}$ for the Lagrangian coordinate $\theta \in \mathbb{R}^{d-1}$. Then, the interface condition (4.1c) is interpreted as an outer force filed f defined on Ω and putted in the Navier-Stokes equations such that

$$\frac{\partial u}{\partial t} + (u\cdot\nabla)u - \nu\Delta u + \nabla p = h + f, \quad \nabla\cdot u = 0 \qquad \text{in } \Omega,\ t>0, \tag{4.2a}$$

$$u = 0 \qquad \text{on } \partial\Omega,\ t>0, \tag{4.2b}$$

$$u(x,0) = u^{(0)}(x) \qquad \text{in } \Omega, \tag{4.2c}$$

$$f(x,t) = \int_\Theta F(\theta,t)\delta(x - X(\theta,t))\ d\theta. \tag{4.2d}$$

Herein, F denotes the force density distributed along $\Gamma(t)$, and δ is the Dirac delta function. For computation, we solve (4.2) with the equation of the interface motion $\frac{\partial X}{\partial t} = u(X,t)$. The main advantage of this method is that we can use fixed uniform meshes. Consequently, the equation is discretized not only by the finite element method but also the finite difference method. Moreover, $f(x,t)$ is replaced by a regularized outer force $f^\varepsilon(x,t)$ which is defined using a smooth delta approximation. Then, the value of $f^\varepsilon(x,t)$ is calculated by simple quadrature formulas. Actually, the IB method is recognized to be one of most powerful methods for the interface problems and it is widely applied at present. However, it seems that there are only a few results about theoretical convergence analysis in contrast to a huge number of applications. In a previous paper, Saito and Sugitani [49], we have studied the convergence of the IB method for a model stationary Stokes problem. That is, the immersed force field is approximated using a regularized delta function and its error in the $W^{-1,p}$ norm is examined for $1 \le p \le d/(d-1)$. Then, we consider the immersed boundary discretization of the Stokes problem and study the regularization and discretization errors separately. Consequently, error estimate of order $h^{1-\alpha}$ in the $W^{1,1}\times L^1$ norm for the velocity and pressure is derived, where α is an arbitrarily small positive number. Error estimate of order $h^{1-\alpha}$ in the L^r norm for the velocity is also derived with $r = d/(d-1-\alpha)$. However, optimal order and L^2 error estimates are still open at present.

At this stage, it is worth recalling that a simpler reformulation method for (4.1) was proposed by H. Fujita et. al. [21] in 1995. Their reformulation reads

$$\frac{\partial u}{\partial t} + (u\cdot\nabla)u - \nu\Delta u + \nabla p = h + \tilde{g}(\nabla\chi\cdot\tilde{n}), \quad \nabla\cdot u = 0 \qquad \text{in } \Omega,\ t>0, \tag{4.3a}$$

$$u = 0 \qquad \text{on } \partial\Omega,\ t>0, \tag{4.3b}$$

$$u(x,0) = u^{(0)}(x) \qquad \text{in } \Omega. \tag{4.3c}$$

Herein, χ denotes the characteristic function of an internal area surrounded by $\Gamma(t)$ in Ω, and n is the unit normal vector on $\Gamma(t)$. Function $\tilde{g}$ and $\tilde{n}$ stand for smooth extensions into Ω of $g(x,t)$

and $n(x, t)$. The reformulation (4.3) is discretized by the finite element and finite difference methods using fixed uniform meshes as well as the IB reformulation. Actually, formulation (4.3) is essentially equivalent to the IB formulation (4.2), whereas (4.3) seems to be easier to deal with both mathematically and practically since there are no Lagrangian coordinate and no need to generate lots of meshes along $\Gamma(t)$ for each time step like moving mesh. In [21], the derivation of reformulation and some numerical results are presented; no mathematical analysis including convergence are given.

The purpose of this chapter is to study convergence of reformulation using the characteristic function. To this end, following [49], we consider a model interface problem for the stationary Stokes equations (Stokes interface problem) and study the regularization and discretization errors separately. We state our model problem in the classical form and its *weak formulation* in Section 4.2. Then, since the derivative of characteristic function $\nabla\chi$ has singularities on Γ, regularization is required. We sate our regularization procedure and examine its error in Section 4.3. As a matter of fact, the $H^1 \times L^2$ error estimate between regularized and original problems is estimated of order $\varepsilon^{\frac{1}{2}}$ (see Proposition 4.3.2). Section 4.4 is devoted to the finite element approximation by MINI (P1b/P1) finite elements. Theorem 4.4.1, the main result of this chapter, offers the error estimates for discretization parameter $h > 0$. That is, the $H^1 \times L^2$ error for velocity and pressure converges at order $h^{\frac{1}{2}}$, while the L^2 error for velocity has a first order convergence. Finally, we confirm our results by numerical experiments in Section 4.5. We verify that desired convergence rates are obtained with uniform mesh.

4.2 Stokes interface problem

4.2.1 Geometry and Notation

Let Ω be a polyhedral domain in $\mathbb{R}^d (d = 2, 3)$ with the boundary $\partial\Omega$. We suppose that Ω is divided into two disjoint subdomains Ω_0 and Ω_1 by a simple Lipschitz curve ($d = 2$) or surface ($d = 3$) denoted by Γ. We assume that the interface Γ is closed ($\partial\Omega \cap \bar{\Gamma} = \emptyset$), or goes across Ω ($\partial\Omega \cap \bar{\Gamma} \neq \emptyset$). For example, see Fig. 4.1. In both cases, the boundaries $\partial\Omega_i (i = 0, 1)$ are Lipschitz boundaries. As for function spaces and their norms, we follow the notation of

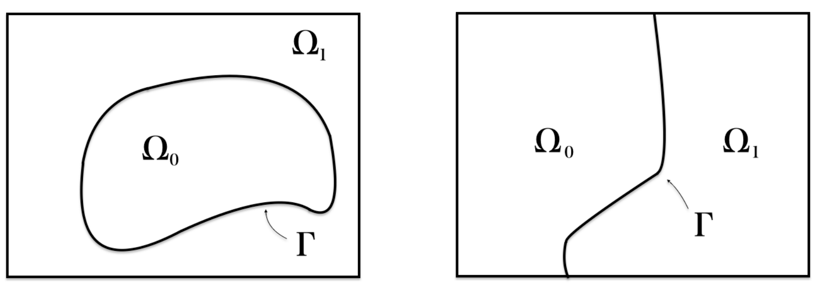

Figure 4.1: Example of Ω as $\partial\Omega \cap \bar{\Gamma} = \emptyset$ (left) and $\partial\Omega \cap \bar{\Gamma} \neq \emptyset$ (right).

[1]. The standard Lebesgue and Sobolev spaces such as $L^2(\Omega), H^1(\Omega), W^{1,\infty}(\Omega)$, $L^2(\Gamma)$, and $W^{2-\frac{1}{p},p}(\Gamma)$ with some $p > d$ will be used. We set $H_0^1 = \{v \in H^1(\Omega) \mid v = 0 \text{ on } \partial\Omega\}$ and $L_0^2(\Omega) = \{q \in L^2(\Omega) \mid \int_\Omega q \, dx = 0\}$. For a function space X, the space X^d denotes a product

space $X \times \cdots \times X$. The norms are denoted by

$$\|u\|_{H^1} = \|u\|_{H^1(\Omega)^d}, \quad \|p\|_{L^2} = \|p\|_{L^2(\Omega)} \tag{4.4}$$

for abbreviations. H^{-1} stands for the dual space of $H_0^1(\Omega)$ and the dual product between $H^{-1}(\Omega)^d$ and $H_0^1(\Omega)^d$ is written as $\langle\cdot,\cdot\rangle = \langle\cdot,\cdot\rangle_{H^{-1},H_0^1}$. Also, the inner product of $L^2(\Omega)^d$ is denoted by $(\cdot,\cdot)_{L^2}$.

4.2.2 Model Stokes interface problem and equivalent formulations

We consider the following model Stokes interface problem

$$
\begin{aligned}
& -\nu\Delta u_i + \nabla p_i = 0, \quad \nabla\cdot u_i = 0 && \text{in } \Omega_i, && \text{(4.5a)}\\
& u_i = 0 && \text{on } \partial\Omega_i \setminus \Gamma \quad (i=0,1), && \text{(4.5b)}\\
& u_0 = u_1, \quad \tau_0 + \tau_1 = g && \text{on } \Gamma, && \text{(4.5c)}
\end{aligned}
$$

for velocity u_i and pressure p_i with density $\rho = 1$ and kinetic viscosity $\nu > 0$, respectively, in $\Omega_i(t)$ $(i = 0,1)$. Herein, τ_i denotes the traction vector defined by

$$\tau_i = \sigma(u_i, p_i)n_i \tag{4.6}$$

where $\sigma(u,p) = (\sigma_{jk}(u,p))_{1\le j,k\le n} = -pI + \nu(\nabla u + \nabla u^{\mathbf{T}})$ is called the stress tensor, I the identity matrix, and n_i the outward normal vector to $\partial\Omega_i$. Moreover, g is a prescribed function standing for a jump of tractions across Γ. We assume $g \in L^2(\Gamma)^d$ for the time being.

To deal with the problem precisely, we introduce a *weak solution*. By a *weak solution* to (4.5), we mean a solution of the following variational equations are: Find $(u,p) \in H_0^1(\Omega)^d \times L_0^2(\Omega)$ such that

$$
\begin{aligned}
& a(u,v) + b(p,v) = \int_\Gamma g\cdot v \; d\Gamma && (\forall v \in H_0^1(\Omega)^d), && \text{(4.7a)}\\
& b(q,u) = 0 && (\forall q \in L_0^2(\Omega)), && \text{(4.7b)}
\end{aligned}
$$

where

$$
\begin{aligned}
a(u,v) &= \frac{\nu}{2}\int_\Omega \left(\frac{\partial u_j}{\partial x_i} + \frac{\partial u_i}{\partial x_j}\right)\left(\frac{\partial v_j}{\partial x_i} + \frac{\partial v_i}{\partial x_j}\right)\, dx, && \text{(4.8a)}\\
b(p,u) &= -\int_\Omega p\,(\nabla\cdot u)\; dx. && \text{(4.8b)}
\end{aligned}
$$

Indeed, if there exists a smooth solution (u_i, p_i) to (4.5), then (u_i,p_i) satisfies (4.7) as

$$u = \begin{cases} u_0 & \text{in } \Omega_0 \\ u_1 & \text{in } \Omega_1 \end{cases} \quad \text{and} \quad p = \begin{cases} p_0 & \text{in } \Omega_0 \\ p_1 & \text{in } \Omega_1. \end{cases}. \tag{4.9}$$

Proof. It is obvious that (u,p) defined by (4.9) belongs to $H_0^1(\Omega)^d \times L^2(\Omega)$ since u is continuous on Γ and vanishes on $\partial\Omega$. Further, since $\nabla\cdot u_i = 0$ in Ω_i, u satisfies (4.7b). In order to derive (4.7a), multiply (4.5a) by $v \in C_0^\infty(\Omega)^d$ and integrate over Ω_i, respectively. Then we have

$$-\nu\int_{\Omega_i} \Delta u_i \cdot v|_{\Omega_i}\; dx + \int_{\Omega_i} \nabla p_i \cdot v|_{\Omega_i}\; dx = 0 \quad \forall v \in C_0^\infty(\Omega)^d,\ (i=0,1). \tag{4.10}$$

Using density result $C_0^\infty(\Omega)^d \subset H_0^1(\Omega)^d$, Green's formula, and summing up both equations, we obtain

$$a(u,v) + b(p,v) = \int_\Gamma (\sigma(u_0,p_0)n_0 + \sigma(u_1,p_1)n_1) \cdot v \; d\Gamma, \; \forall v \in H_0^1(\Omega)^d. \tag{4.11}$$

Because of the jump condition (4.5c), the right hand side equals to $\int_\Gamma g \cdot v \; d\Gamma$. This discussion remains true if p_i is replaced by $p_i + c$ with some $c \in \mathbb{R}$. Finally, we can choose $c \in \mathbb{R}$ such that $\int_\Omega (p+c) \; dx = 0$. $\square$

Since $v \mapsto \int_\Omega g \cdot v \; d\Gamma$ by $g \in L^2(\Gamma)^d$ is a bounded linear functional on $H_0^1(\Omega)^d$, the well-posedness of (4.7) is proved by the standard theory. Actually, we recall the following result.

Lemma 4.2.1 (cf. [31] and [17])**.** *Let Ω be a connected, bounded, convex polyhedral domain of $\mathbb{R}^d$, and let h be in $H^{-1}(\Omega)^d$. Then, there exists a unique weak solution $(w,r) \in H_0^1(\Omega)^d \times L_0^2(\Omega)$ of the Stokes problem*

$$-\nu \Delta w + \nabla r = h \text{ in } \Omega, \quad \nabla \cdot w = 0 \text{ in } \Omega, \quad w = 0 \text{ on } \partial\Omega \tag{4.12}$$

satisfying

$$\|w\|_{H^1} + \|r\|_{L^2} \leq C_1 \|h\|_{H^{-1}}. \tag{4.13}$$

Moreover, if $h \in L^2(\Omega)^d$, we have $(w,r) \in H^2(\Omega)^d \times H^1(\Omega)$ and

$$\|w\|_{H^2} + \|r\|_{H^1} \leq C_2 \|h\|_{L^2}. \tag{4.14}$$

Herein, C_1 and C_2 denote positive constants depending only on Ω.

Now we proceed to derive an equivalent formulation to (4.7). To this end, we assume

$$\Gamma \text{ is of class } C^2, \tag{4.15}$$

$$g \in W^{2-\frac{1}{p},p}(\Gamma)^d \text{ with some } p > d. \tag{4.16}$$

According to [21, (1.17)], we have

$$\int_\Gamma g \cdot \varphi \; d\Gamma = \langle \tilde{g}(\nabla\chi \cdot \tilde{n}), \varphi \rangle \quad \forall \varphi \in C_0^\infty(\Omega)^d \tag{4.17}$$

where χ is the characteristic function of Ω_0 in Ω;

$$\chi(x) = \begin{cases} 1 & x \in \Omega_0, \\ 0 & x \notin \Omega_0. \end{cases} \tag{4.18}$$

Moreover, $\tilde{n}$ is a C^1 extension of n_1 into Ω and $\tilde{g}$ is the extension of g given by the following lemma. For the reader's convenience, we recall the proof of (4.17) in Appendix.

Lemma 4.2.2. *Suppose that* (4.15) *and* (4.16) *are satisfied. Then, there exists $\tilde{g} \in W^{2,p}(\Omega)^d \cap W^{1,\infty}(\Omega)^d$ such that*

$$\|\tilde{g}\|_{W^{1,\infty}(\Omega)^d} \leq C_0 \|g\|_{W^{2-\frac{1}{p},p}(\Gamma)^d} \tag{4.19}$$

where C_0 denotes a positive constant depending only on Γ and Ω.

The proof is a consequence of the lifting-extension theorem (see [43, Theorem 2-5.8. and Theorem 2-3.9.]), and Sobolev embedding theorem (cf. [1, Theorem 4.12]).

At this stage, we set

$$f = \tilde{g}(\nabla\chi \cdot \tilde{n}). \tag{4.20}$$

Then, we still have $f \in H^{-1}(\Omega)^d$ and state an equivalent formulation to (4.7): Find $(u, p) \in H_0^1(\Omega)^d \times L_0^2(\Omega)$ such that

$$a(u, v) + b(p, v) = \langle f, v\rangle_{H^{-1},H_0^1} \qquad (\forall v \in H_0^1(\Omega)^d), \tag{4.21a}$$

$$b(q, u) = 0 \qquad (\forall q \in L_0^2(\Omega)). \tag{4.21b}$$

Finally, writing down the strong form of (4.21), we obtain the Stokes equations with singular source term defined by (4.20):

$$-\nu\Delta u + \nabla p = f \text{ in } \Omega,\ \nabla \cdot u = 0 \text{ in } \Omega,\ u = 0 \text{ on } \partial\Omega. \tag{4.22}$$

which is equivalent to our problem (4.5) in the distribution sense.

Remark 4.2.1. *Problem* (4.7) *can be directly discretized by the finite element method using the boundary integral on* Γ. *Such methods were studied in [7], [52] for nonstationary Navier-Stokes equations. However, in order to avoid moving mesh problem, we study the formulation* (4.21) *and apply the finite element method to it using uniform mesh.*

4.3 Regularization to the distribution form with characteristic function

As explained in Introduction, for computation using uniform mesh, we introduce a regularized force field $f^\varepsilon \in L^2(\Omega)^d$ as

$$f^\varepsilon = \tilde{g}\left(\nabla\chi^\varepsilon \cdot \tilde{n}\right). \tag{4.23}$$

Herein, $\varepsilon > 0$ is a regularization parameter and χ^ε is an appropriate approximation to the characteristic function χ. We assume that χ^ε is a Lipschitz function. For f^ε given as (4.23), let us consider

$$-\nu\Delta u^\varepsilon + \nabla p^\varepsilon = f^\varepsilon \text{ in } \Omega,\ \nabla \cdot u^\varepsilon = 0 \text{ in } \Omega,\ u^\varepsilon = 0 \text{ on } \partial\Omega. \tag{4.24}$$

Since $f^\varepsilon \in L^2$, there exists a unique weak solution $(u^\varepsilon, p^\varepsilon) \in H_0^1(\Omega)^d \times L_0^2(\Omega)$ for all $\varepsilon > 0$. Then, the error of regularization is estimated using Lemma 4.2.1.

Proposition 4.3.1. *Let* (u, p) *and* $(u^\varepsilon, p^\varepsilon)$ *be the weak solution of* (4.22) *and* (4.24), *respectively. Then we have*

$$\|u - u^\varepsilon\|_{H^1} + \|p - p^\varepsilon\|_{L^2} \leq C^*\|\chi - \chi^\varepsilon\|_{L^2} \tag{4.25}$$

where $C^* > 0$ *is a positive constant depending only on* Ω, Γ, *and* $\|g\|_{W^{2-\frac{1}{p},p}}$.

Proof. By virtue of Lemma 4.2.1, we have

$$\|u - u^\varepsilon\|_{H^1} + \|p - p^\varepsilon\|_{L^2} \leq C_1\|f - f^\varepsilon\|_{H^{-1}}$$

It remains to bound $\|f - f^\varepsilon\|_{H^{-1}}$ by $\|\chi - \chi^\varepsilon\|_{L^2}$. Indeed, we obtain for all $v \in H_0^1(\Omega)^d$

$$\begin{aligned}
\langle f - f^\varepsilon, v\rangle_{H^{-1},H_0^1} &= \langle \tilde{g}\nabla(\chi - \chi^\varepsilon) \cdot \tilde{n}, v\rangle_{H^{-1},H_0^1} \\
&= -(\chi - \chi^\varepsilon, \nabla \cdot (\tilde{n}(\tilde{g} \cdot v)))_{L^2} \\
&\leq \|\chi - \chi^\varepsilon\|_{L^2}\|\tilde{n}(\tilde{g} \cdot v))\|_{H^1} \\
&\leq \|\tilde{n}\|_{W^{1,\infty}}\|\tilde{g}\|_{W^{1,\infty}}\|\chi - \chi^\varepsilon\|_{L^2}\|v\|_{H^1}.
\end{aligned}$$

Hence, desired result holds as $C^* = C_0C_1\|\tilde{n}\|_{W^{1,\infty}}\|g\|_{W^{2-\frac{1}{p},p}}$. $\square$

4.3.1 Construction of χ^ε

Now we choose χ^ε, for example, as a polyline for outward direction to Γ

$$\chi^\varepsilon(x) = \begin{cases} 1 & (x \in \Omega_0) \\ \max\{0, 1 - \frac{\mathrm{dist}(x,\Gamma)}{\varepsilon}\} & (x \notin \Omega_0). \end{cases} \tag{4.26}$$

Then, χ^ε is a Lipschitz function on Ω, i.e. $\chi \in W^{1,\infty}(\Omega)$ and satisfies that

$$\|\chi - \chi^\varepsilon\|_{L^2(\Omega)} \leq C_3\sqrt{\varepsilon} \tag{4.27}$$

where C_3 is a positive constant depending only on Γ. To verify this, we set $\Gamma^\varepsilon = \{x \in \Omega_1 \mid \mathrm{dist}(x,\Gamma) \leq \varepsilon\}$ and calculate as (noting that $\chi - \chi^\varepsilon$ equals to χ^ε in Γ^ε and vanishes outside)

$$\begin{aligned} \|\chi - \chi^\varepsilon\|_{L^2(\Omega)} &= \|\chi^\varepsilon\|_{L^2(\Gamma^\varepsilon)} \\ &\leq \underbrace{\|\chi^\varepsilon\|_{L^\infty}}_{\leq 1} \mathrm{meas}(\Gamma^\varepsilon)^{\frac{1}{2}} \leq C_3\sqrt{\varepsilon}. \end{aligned}$$

Therefore, we obtain the regularization error estimate as follows.

Proposition 4.3.2. *Let (u,p) and $(u^\varepsilon, p^\varepsilon)$ be respectively the weak solution of* (4.22) *and* (4.24) *with* (4.26)*. Then we have*

$$\|u - u^\varepsilon\|_{H^1} + \|p - p^\varepsilon\|_{L^2} \leq C\sqrt{\varepsilon}. \tag{4.28}$$

Remark 4.3.1. *Any other choices of χ^ε, such as C^1 approximation, are of course possible for implementation. For our error estimates, however, it is enough to suppose $\chi^\varepsilon \in W^{1,\infty}$. The order of error is independent of χ^ε.*

4.4 Discretization by finite element method

This section is dedicated to a study of the finite element approximation to (4.24). Let $\{\mathcal{T}_h\}_h$ be a family of *regular* triangulations of Ω, i.e., there exists $\kappa > 0$ satisfying $h_T \leq \kappa\rho_T$ for all $T \in \mathcal{T}_h$. Herein, h_T denotes the diameter of T, ρ_T the diameter of the inscribed ball of T, and $h = \max\{h_K \mid K \in \mathcal{T}_h\}$.

We employ the P1-b/P1 (MINI) element for discretization as setting

$$\begin{aligned} V_h &= \{v_h \in C(\overline{\Omega})^d \cap H_0^1(\Omega)^d \mid v_h|_T \in [\mathcal{P}_1(T) \oplus \mathcal{B}(T)]^d \ (\forall T \in \mathcal{T}_h)\}, \\ Q_h &= \{q_h \in C(\overline{\Omega}) \cap L_0^2(\Omega) \mid q_h|_T \in \mathcal{P}_1(T) \ (\forall T \in \mathcal{T}_h)\}. \end{aligned}$$

Therein, $\mathcal{P}_k(T)$ is the set of all polynomials defined on $T \in \mathcal{T}_h$ of degree $\leq k$, and $\mathcal{B}(T) = \mathrm{span}\{\lambda_1\lambda_2\cdots\lambda_{d+1}\}$ is so-called *bubble* function with λ_i the barycentric coordinates of T. It is well-known that a pair of V_h and Q_h satisfies the uniform Babuška–Brezzi (inf–sup) condition

$$\sup_{v_h \in V_h} \frac{b(v_h, q_h)}{\|v_h\|_{H^1}} \geq \kappa_2\|q_h\|_{L^2} \qquad (q_h \in Q_h),$$

where $\kappa_2 > 0$ is independent of h.

Remark 4.4.1. *We deal with the P1-b/P1 element only for the sake of simple presentation. An arbitrary pair of conforming finite element spaces $V_h \subset H^1(\Omega)^d$ and $Q_h \subset L_0^2(\Omega)$ satisfying the uniform Babuška–Brezzi condition is available.*

The finite element approximation to (4.24) is given as follows: Find $(u_h^\varepsilon, p_h^\varepsilon) \in V_h \times Q_h$ such that

$$
\begin{aligned}
& a(u_h^\varepsilon, v_h) + b(p_h^\varepsilon, v) = (f^\varepsilon, v_h)_{L^2} && (\forall v_h \in V_h), && \text{(4.29a)}\\
& b(q_h, u_h^\varepsilon) = 0 && (\forall q_h \in Q_h). && \text{(4.29b)}
\end{aligned}
$$

The well-posedness of (4.29) is a standard result, for example, refer to [47, Theorem 15.3.].

4.4.1 Error estimate

We are now ready to state the error estimates. First, discretization error is bounded by the following.

Proposition 4.4.1. *Let $(u^\varepsilon, p^\varepsilon)$ and $(u_h^\varepsilon, p_h^\varepsilon)$ be the solution to* (4.24) *and* (4.29)*, respectively. Then, we have*

$$
\begin{aligned}
\|u^\varepsilon - u_h^\varepsilon\|_{H^1} + \|p^\varepsilon - p_h^\varepsilon\|_{L^2} &\leq C^{**} h \|\chi^\varepsilon\|_{H^1}, && \text{(4.30a)}\\
\|u^\varepsilon - u_h^\varepsilon\|_{L^2} &\leq C^{**} h^2 \|\chi^\varepsilon\|_{H^1}. && \text{(4.30b)}
\end{aligned}
$$

where C^{**} denotes a positive constant depending only on $\Omega, \Gamma, \|g\|_{W^{2-\frac{1}{p},p}}$. Moreover, if χ^ε is given by (4.26)*, then we derive*

$$
\begin{aligned}
\|u^\varepsilon - u_h^\varepsilon\|_{H^1} + \|p^\varepsilon - p_h^\varepsilon\|_{L^2} &\leq C^{**} \frac{h}{\sqrt{\varepsilon}}, && \text{(4.31a)}\\
\|u^\varepsilon - u_h^\varepsilon\|_{L^2} &\leq C^{**} \frac{h^2}{\sqrt{\varepsilon}}. && \text{(4.31b)}
\end{aligned}
$$

Proof. It is well known that the finite element approximation makes the optimal approximation. That is,

$$
\|u^\varepsilon - u_h^\varepsilon\|_{H^1} + \|p^\varepsilon - p_h^\varepsilon\|_{L^2} \leq C_4 \inf_{(v_h, q_h) \in V_h \times Q_h} \left(\|u^\varepsilon - v_h\|_{H^1} + \|p^\varepsilon - q_h\|_{L^2} \right),
$$

where $C_4 > 0$ depends only on Ω. Refer to [47, Theorem 15.3], for example. Applying the standard interpolation error estimates and stability result (4.14), we obtain that

$$
\|u^\varepsilon - u_h^\varepsilon\|_{H^1} + \|p^\varepsilon - p_h^\varepsilon\|_{L^2} \leq C_5 h \|f^\varepsilon\|_{L^2}.
$$

Furthermore, by virtue of the duality technique in [23, Theorem 1.9. §1. Chapter II], we have

$$
\|u^\varepsilon - u_h^\varepsilon\|_{L^2} \leq C_5 h^2 \|f^\varepsilon\|_{L^2}.
$$

Therein, C_5 depends only on Ω. Then, estimates (4.30) is consequently obtained as $C^{**} = C_0 C_5 \|\tilde{g}\|_{W^{2-\frac{1}{p},p}} \|\tilde{n}\|_{L^\infty}$ since $\|f^\varepsilon\|_{L^2} = \|\tilde{g}(\nabla\chi^\varepsilon \cdot \tilde{n})\|_{L^2} \leq \|\tilde{g}\|_{L^\infty} \|\tilde{n}\|_{L^\infty} \|\nabla\chi^\varepsilon\|_{L^2}$. When χ^ε is given by (4.26), we continue to calculate as

$$
\begin{aligned}
\|\nabla\chi^\varepsilon\|_{L^2(\Omega)} &= \left\| \nabla \frac{\mathrm{dist}(x, \Gamma)}{\varepsilon} \right\|_{L^2(\Gamma^\varepsilon)} \\
&\leq \frac{1}{\varepsilon} \mathrm{meas}(\Gamma^\varepsilon)^{\frac{1}{2}} \|\mathrm{dist}(x, \Gamma)\|_{W^{1,\infty}} = \frac{1}{\sqrt{\varepsilon}} \|\mathrm{dist}(x, \Gamma)\|_{W^{1,\infty}}.
\end{aligned}
$$

□

At this stage, we apply Propositions 4.3.2 and 4.4.1 to deduce the total error estimate which is the main theorem in this chapter.

Theorem 4.4.1. *Let (u,p) and $(u_h^\varepsilon, p_h^\varepsilon)$ be respectively the solution to* (4.22) *and* (4.29) *with* (4.26)*. Then we have*

$$\|u - u_h^\varepsilon\|_{H^1} + \|p - p_h^\varepsilon\|_{L^2} \le C\left(\sqrt{\varepsilon} + \frac{h}{\sqrt{\varepsilon}}\right), \tag{4.32a}$$

$$\|u - u_h^\varepsilon\|_{L^2} \le C\left(\sqrt{\varepsilon} + \frac{h^2}{\sqrt{\varepsilon}}\right). \tag{4.32b}$$

where C denotes a positive constant depending only on $\Omega, \Gamma, \|g\|_{W^{2-\frac{1}{p},p}}$. In particular, if $\varepsilon = c_1 h$ with a positive constant c_1, then

$$\|u - u_h^\varepsilon\|_{H^1} + \|p - p_h^\varepsilon\|_{L^2} \le C\sqrt{h}. \tag{4.33}$$

Else if $\varepsilon = c_1 h^2$ then

$$\|u - u_h^\varepsilon\|_{L^2} \le Ch. \tag{4.34}$$

where C denotes a positive constant depending only on $\Omega, \Gamma, \|g\|_{W^{2-\frac{1}{p},p}}$ and c_1.

Remark 4.4.2. *As will be observed in next section, we infer that*

$$\|u - u_h^\varepsilon\|_{L^2} \le Ch$$

actually holds true both as $\varepsilon = h$ and $\varepsilon = h^2$. In order to prove this, there are some difficulties in constructing appropriate duality problem to the regularized equations (4.24)*.*

4.5 Numerical examples

In this section, we show some results of numerical experiments to verify our theoretical results. We consider the following Stokes interface problem

$$-\nu\Delta u_i + \nabla p_i = \iota, \quad \nabla\cdot u_i = 0 \qquad \text{in } \Omega_i \quad (i = 0,1), \tag{4.35a}$$

$$u_i = 0 \qquad \text{on } \partial\Omega_i \setminus \Gamma, \tag{4.35b}$$

$$u_0 = u_1, \quad \tau_0 + \tau_1 = g \qquad \text{on } \Gamma, \tag{4.35c}$$

for $\nu > 0$. Here, ι is a an extra outer force field added in order to illustrate a pressure jump across Γ. We want to obtain the solution of (4.35) numerically. To do this, for f^ε given as (4.23), we consider the stationary Stokes problem

$$-\nu\Delta u^\varepsilon + \nabla p^\varepsilon = f^\varepsilon + \iota \text{ in } \Omega,\ \nabla\cdot u^\varepsilon = 0 \text{ in } \Omega,\ u^\varepsilon = 0 \text{ on } \partial\Omega, \tag{4.36}$$

and its regularized problem

$$-\nu\Delta u^\varepsilon + \nabla p^\varepsilon = f^\varepsilon + \iota \text{ in } \Omega,\ \nabla\cdot u^\varepsilon = 0 \text{ in } \Omega,\ u^\varepsilon = 0 \text{ on } \partial\Omega. \tag{4.37}$$

Then, we solve the following finite element approximation

$$a(u_h^\varepsilon, v_h) + b(p_h^\varepsilon, v_h) = (f^\varepsilon + \iota, v_h)_{L^2} \qquad (\forall v_h \in V_h), \tag{4.38a}$$

$$b(q_h, u_h^\varepsilon) = 0 \qquad (\forall q_h \in Q_h). \tag{4.38b}$$

First example corresponds to the case $\partial\Omega \cap \overline{\Gamma} = \emptyset$. Setting as $\Omega = (-1,1)^2 \subset \mathbb{R}^2$ and $\Gamma = \{(x,y) \in \Omega \mid r = (x^2+y^2)^{\frac{1}{2}} = 1/2\}$, we impose $\nu = 1$ and

$$g = n = -\left(\frac{x}{r}, \frac{y}{r}\right) \text{ and } \iota = (1,0).$$

Then, $g, n \in C^\infty(\Gamma)^2$ have canonical extension to $C^\infty(\Omega\backslash\{0\})^2$. Indeed, we have $\tilde{g}, \tilde{n} \in W^{1,\infty}(\Omega)^2$. We simply use χ^ε as described in (4.25). For discretization, we employ the uniform triangulation over Ω, which is divided into N^2 isosceles right triangles with $h = \sqrt{2}/N$. Hence, it is ensured by Theorem 4.4.1 that

- if $\varepsilon = h$ then $\|u^\varepsilon - u_h^\varepsilon\|_{H^1} + \|p^\varepsilon - p_h^\varepsilon\|_{L^2} \leq C\sqrt{h}$,
- if $\varepsilon = h^2$ then $\|u - u_h^\varepsilon\|_{L^2} \leq Ch$.

To verify this point, we compute the following quantities:

$$\begin{aligned} E_h^{(1)} &= \|\hat{u} - u_h^{\varepsilon_1}\|_{L^2}, \quad E_h^{(2)} = \|\hat{u} - u_h^{\varepsilon_1}\|_{H^1}, \quad E_h^{(3)} = \|\hat{p} - p_h^{\varepsilon_1}\|_{L^2}, \\ G_h^{(1)} &= \|\hat{u} - u_h^{\varepsilon_2}\|_{L^2}, \quad G_h^{(2)} = \|\hat{u} - u_h^{\varepsilon_2}\|_{H^1}, \quad G_h^{(3)} = \|\hat{p} - p_h^{\varepsilon_2}\|_{L^2}. \end{aligned}$$

Herein, ε_i is chosen as $\varepsilon_i = h^i$ $(i = 1,2)$ and $(\hat{u}, \hat{p}) \in V_{\hat{h}} \times Q_{\hat{h}}$ denotes the numerical solution using a finer triangulation $\mathcal{T}_{\hat{h}}$. Then, we observe

$$\rho_h^{(i)} = \frac{\log E_{2h}^{(i)} - \log E_h^{(i)}}{\log 2h - \log h}, \quad \text{and} \quad \mu_h^{(i)} = \frac{\log G_{2h}^{(i)} - \log G_h^{(i)}}{\log 2h - \log h} \quad (i = 1,2,3).$$

The results are reported in Table 4.1-4.2. It is showed in Table 4.1 that $H^1 \times L^2$ error of (u,p) is of order more than 1/2, and in Table 4.2 that L^2 error of u is of order more than 1. These numerical results support our theoretical results. Note that every order of error with $\varepsilon = h$ is higher than with $\varepsilon = h^2$.

h	$E_h^{(1)}$	$\rho_h^{(1)}$	$E_h^{(2)}$	$\rho_h^{(2)}$	$E_h^{(3)}$	$\rho_h^{(3)}$
0.1414	1.422e-03	—	0.0316202	—	0.246641	—
0.0707	5.056e-04	1.492	0.0174619	0.856	0.170653	0.531
0.0353	1.462e-04	1.790	0.0092711	0.913	0.093534	0.867
0.0176	3.601e-05	2.021	0.0051765	0.840	0.022952	2.026

Table 4.1: Convergence rates of example 1 as $\varepsilon = h$.

h	$G_h^{(1)}$	$\mu_h^{(1)}$	$G_h^{(2)}$	$\mu_h^{(2)}$	$G_h^{(3)}$	$\mu_h^{(3)}$
0.1414	1.509e-03	—	3.771e-02	—	1.508e-01	—
0.0707	6.081e-04	1.312	2.524e-02	0.579	8.616e-02	0.807
0.0353	2.261e-04	1.427	1.843e-02	0.453	5.234e-02	0.718
0.0176	9.610e-05	1.234	1.230e-02	0.583	2.432e-02	1.105

Table 4.2: Convergence rates of example 1 as $\varepsilon = h^2$.

Second example corresponds to the case $\partial\Omega \cap \overline{\Gamma} \neq \emptyset$. We set $\Omega = (-1,1) \times (0,1) \subset \mathbb{R}^2$ and Γ equal to y axis. In this simple case, we have a exact solution

$$u = (0,0) \text{ and } p = \begin{cases} y - 1 & (x > 0) \\ y & (x \leq 0) \end{cases} \quad \text{in } \Omega$$

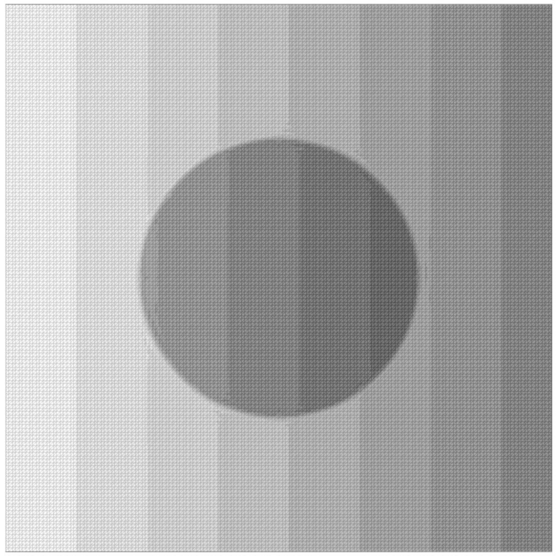

Figure 4.2: Magnitude of pressure $p_h^{\varepsilon_1}$ of example 1 for N = 160. The pressure solution becomes discontinuous function across interface Γ.

to (4.35) for $\iota = (0,1)$ and $g = n = (-1,0)$. Therefore, the numerical error is measured precisely. We compute same quantities in example 1 and reports in Table 4.3-4.4. In this case, we observe there are almost no differences between $\varepsilon = h$ and $\varepsilon = h^2$. $H^1 \times L^2$ error of (u,p) is of order 1/2, and L^2 error of u is of order 1.5, which suggests that the optimal-order error estimate

$$\|u - u_h^\varepsilon\|_{H^1} + \|u - u_h^\varepsilon\|_{L^2} \le C\sqrt{h}.$$

holds true when $\varepsilon = c_1 h^\alpha$ with $\alpha \ge 1$. The proof of this conjecture remains for future study.

h	$E_h^{(1)}$	$\rho_h^{(1)}$	$E_h^{(2)}$	$\rho_h^{(2)}$	$E_h^{(3)}$	$\rho_h^{(3)}$
0.0707	2.94849e-05	—	2.824e-03	—	4.853e-02	—
0.0353	1.07469e-05	1.456	2.028e-03	0.477	3.341e-02	0.538
0.0176	3.85533e-06	1.478	1.445e-03	0.489	2.329e-02	0.520
0.0088	1.37281e-06	1.489	1.025e-03	0.494	1.635e-02	0.510

Table 4.3: Convergence rates of example 2 as $\varepsilon = h$.

h	$G_h^{(1)}$	$\mu_h^{(1)}$	$G_h^{(2)}$	$\mu_h^{(2)}$	$G_h^{(3)}$	$\mu_h^{(3)}$
0.0707	1.250e-04	—	1.198e-02	—	5.491e-02	—
0.0353	4.559e-05	1.456	8.606e-03	0.477	3.816e-02	0.525
0.0176	1.635e-05	1.478	6.131e-03	0.489	2.674e-02	0.512
0.0088	5.824e-06	1.489	4.351e-03	0.494	1.882e-02	0.506

Table 4.4: Convergence rates of example 2 as $\varepsilon = h^2$.

Acknowledgement

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Appendix A. Proof of (4.17)

In this appendix, we prove the equation

$$\int_\Gamma g \cdot \varphi \, d\Gamma = \langle \tilde{g}(\nabla\chi \cdot \tilde{n}), \varphi \rangle \quad \forall \varphi \in C_0^\infty(\Omega)^d. \tag{4.17}$$

Proof. First, we note that the assumptions (4.15) and (4.16) imply $\tilde{g} \in W^{1,\infty}(\Omega)^d$ and $\tilde{n} \in H^1(\Omega)^d$. Then, we have $\tilde{g}(\nabla\chi \cdot \tilde{n}) \in H^{-1}(\Omega)^d$ by the representation

$$\tilde{g}(\nabla\chi \cdot \tilde{n}) = \sum_{i=0}^{d} \frac{\partial}{\partial x_i}(\tilde{g}\chi\tilde{n}_i) - \sum_{i=0}^{d} \chi \frac{\partial}{\partial x_i}(\tilde{g}\tilde{n}_i).$$

Furthermore, for all $\varphi \in C_0^\infty(\Omega)^d$, the function $(\tilde{g} \cdot \varphi)\tilde{n}$ belongs to $H_0^1(\Omega)^d$. Thus, we have

$$\begin{aligned} \langle \tilde{g}(\nabla\chi \cdot \tilde{n}), \varphi \rangle &= \langle \nabla\chi, (\tilde{g} \cdot \varphi)\tilde{n} \rangle = -\langle \chi, \operatorname{div}((\tilde{g} \cdot \varphi)\tilde{n}) \rangle \\ &= -\int_{\Omega_0} \operatorname{div}((\tilde{g} \cdot \varphi)\tilde{n}) \, dx = -\int_\Gamma (g \cdot \varphi) n_1 \cdot (-n_1) \, d\Gamma. \end{aligned}$$

□

Bibliography

[1] R. A. Adams and J. J. F. Fournier. *Sobolev spaces*, volume 140 of *Pure and Applied Mathematics (Amsterdam)*. Elsevier/Academic Press, Amsterdam, second edition, 2003.

[2] M. Ayadi, L. Baffico, M. K. Gdoura, and T. Sassi. Error estimates for stokes problem with Tresca friction conditions. *ESAIM Math. Model. Numer. Anal.*, 48:1413–1429, 2014.

[3] C. Baiocchi and A. Capelo. *Variational and Quasivariational Inequalities. Applications to Free Boundary Problems.* Wiley, 1984.

[4] Y. Bazilevs, J. R. Gohean, T. J. R. Hughes, R. D. Moser, and Y. Zhang. Patient-specific isogeometric fluid-structure interaction analysis of thoracic aortic blood flow due to implantation of the Jarvik 2000 left ventricular assist device. *Comput. Methods Appl. Mech. Engrg.*, 198:3534–355, 2009.

[5] F. Ben Belgacem, Y. Renard, and L. Slimane. A mixed formulation for the signorini problem in nearly incompressible elasticity. *Appl. Numer. Math.*, 54:1–22, 2005.

[6] D. Boffi, F. Brezzi, and M. Fortin. *Mixed Finite Element Methods and Applications.* Springer, 2013.

[7] D. Boffi, L. Gastaldi, and L. Heltai. Numerical stability of the finite element immersed boundary method. *Mathematical Models and Methods in Applied Sciences*, 17(10):1479–1505, 2007.

[8] M. Boman. Estimates for the L_2-projection onto continuous finite element spaces in a weighted L_p-norm. *BIT Numer. Math.*, 46:249–260, 2006.

[9] F. Boyer and P. Fabrie. Outflow boundary conditions for the incompressible non-homogeneous Navier-Stokes equations. *Discrete Contin. Dyn. Syst. Ser. B*, 7:219–250, 2007.

[10] F. Boyer and P. Fabrie. *Mathematical Tools for the Study of the Incompressible Navier-Stokes Equations and Related Models.* Springer, 2013.

[11] S. C. Brenner and L. R. Scott. *The Mathematical Theory of Finite Element Methods.* Springer, 2008.

[12] H. Brezis. *Functional Analysis, Sobolev Spaces and Partial Differential Equations.* Springer, 2011.

[13] C. H. Bruneau and P. Fabrie. Effective downstream boundary conditions for incompressible Navier-Stokes equations. *Internat. J. Numer. Methods Fluids*, 19:693–705, 1994.

[14] C. H. Bruneau and P. Fabrie. New efficient boundary conditions for incompressible Navier-Stokes equations: a well-posedness result. *RAIRO Modél. Math. Anal. Numér.*, 30:815–840, 1996.

[15] F. Chouly and P. Hild. On convergence of the penalty method for unilateral contact problems. *Appl. Numer. Math.*, 65:27–40, 2013.

[16] M. Crouzeix and V. Thomée. The stability in L_p and W_p^1 of the L_2-projection onto finite element function spaces. *Math. Comp.*, 48:521–532, 1987.

[17] M. Dauge. Stationary stokes and navier-stokes systems on two-or three-dimensional domains with corners. part i. linearized equations. *SIAM Journal on Mathematical Analysis*, 20(1):74–97, 1989.

[18] G. Drouet and P. Hild. Optimal convergence for discrete variational inequalities modelling signorini contact in 2d and 3d without additional assumptions on the unknown contact set. *SIAM J. Numer. Anal.*, 53:27–40, 2015.

[19] J. M. Floryan and H. Rasmussen. Numerical methods for viscous flows with moving boundaries. *Applied Mechanics Reviews*, 42(12):323–341, 1989.

[20] L. Formaggia, A. Quarteroni, and A. Veneziani. *Cardiovascular Mathematics.* Springer, 2009.

[21] H. Fujita, H. Kawahara, and H. Kawarada. Distribution theoretic approach to fictitious domain method for Neumann problems. *East-West J. Numer. Math.*, 3(2):111–126, 1995.

[22] V. Girault, R. H. Nochetto, and L. R. Scott. Max-norm estimates for Stokes and Navier-Stokes approximations in convex polyhedra. *Numer. Math.*, 131(4):771–822, 2015.

[23] V. Girault and P. A. Raviart. *Finite Element Methods for Navier-Stokes Equations.* Springer, 1986.

[24] P. M. Gresho and R. L. Sani. *Incompressible Flow and the Finite Element Method, Volume 2, Isothermal Laminar Flow.* Wiley, 2000.

[25] P. Grisvard. Behavior of the solutions of an elliptic boundary value problem in a polygonal or polyhedral domain. *Numerical Solution of P.D.E's III, Proc. Third Sympos. (SYNSPADE)*, pages 207–274, 1976.

[26] P. Grisvard. *Elliptic Problems in Nonsmooth Domains.* Pitman, 1985.

[27] J. G. Heywood, R. Rannacher, and S. Turek. Artificial boundaries and flux and pressure conditions for the incompressible Navier-Stokes equations. *Internat. J. Numer. Methods Fluids*, 22:325–352, 1996.

[28] T. Y. Hou. Numerical solutions to free boundary problems. *Acta numerica*, 4:335–415, 1995.

[29] J. Douglas Jr., T. Dupont, and L. Wahlbin. The stability in L^q of the L^2-projection into finite element function spaces. *Numer. Math.*, 23:193–197, 1975.

[30] T. Kashiwabara. Finite element method for Stokes equations under leak boundary condition of friction. *SIAM J. Numer. Anal.*, 51:2448–2469, 2013.

[31] R. B. Kellogg and J. E. Osborn. A regularity result for the stokes problem in a convex polygon. *Journal of Functional Analysis*, 21(4):397–431, 1976.

[32] N. Kikuchi and J. T. Oden. *Contact Problems in Elasticity.* SIAM, 1988.

[33] Y. Li and R. An. Semi-discrete stabilized finite element methods for Navier-Stokes equations with nonlinear slip boundary conditions based on regularization procedure. *Numer. Math.*, 117:1–36, 2011.

[34] Y. Li and R. An. Penalty finite element method for Navier-Stokes equations with nonlinear slip boundary conditions. *Internat. J. Numer. Methods Fluids*, 69:550–576, 2012.

[35] Z. Li. On convergence of the immersed boundary method for elliptic interface problems. *Math. Comp.*, 84(293):1169–1188, 2015.

[36] J. L. Lions. *Quelques Méthodes de Résolution des Problèmes aux Limites Non Linéaires.* Dunod, 1969.

[37] J. L. Lions. Some remarks on variational inequalities. *Proc. Internat. Conf. on Functional Analysis and Related Topics*, pages 269–282, 1969.

[38] J. L. Lions and E. Magenes. *Non-homogeneous Boundary Value Problems and Applications, I.* Springer, 1972.

[39] Y. Liu and Y. Mori. Properties of discrete delta functions and local convergence of the immersed boundary method. *SIAM J. Numer. Anal.*, 50(6):2986–3015, 2012.

[40] Y. Liu and Y. Mori. L^p convergence of the immersed boundary method for stationary Stokes problems. *SIAM J. Numer. Anal.*, 52(1):496–514, 2014.

[41] V. Maz'ya and J. Rossmann. L_p estimates of solutions to mixed boundary value problems for the Stokes system in polyhedral domains. *Math. Nachr.*, 280(7):751–793, 2007.

[42] Y. Mori. Convergence proof of the velocity field for a Stokes flow immersed boundary method. *Comm. Pure Appl. Math.*, 61(9):1213–1263, 2008.

[43] J. Necas. *Direct methods in the theory of elliptic equations.* Springer Science & Business Media, 2011.

[44] C. S. Peskin. *Flow Patterns Around Heart Valves: A Digital Computer Method for Solving the Equations of Motion.* PhD thesis, Albert Einstein College of Medicine of Yeshiva University, 7 1972. order number: 7230378.

[45] C. S. Peskin. Numerical analysis of blood flow in the heart. *J. Computational Phys.*, 25(3):220–252, 1977.

[46] C. S. Peskin. The immersed boundary method. *Acta Numer.*, 11:479–517, 2002.

[47] A. Quarteroni. *Numerical models for differential problems*, volume 2. Springer Science & Business Media, 2010.

[48] N. Saito and H. Fujita. Remarks on traces of H^1-functions defined in a domain with corners. *J. Math. Sci. Univ. Tokyo*, 7:325–345, 2000.

[49] N. Saito and Y. Sugitani. Convergence of the immersed-boundary finite-element method for the stokes problem. *arXiv preprint arXiv:1611.07172*, 2016.

[50] R. Scardovelli and S. Zaleski. Direct numerical simulation of free-surface and interfacial flow. *Annual review of fluid mechanics*, 31(1):567–603, 1999.

[51] L. R. Scott and S. Zhang. Finite element interpolation of non-smooth functions satisfying boundary conditions. *Math. Comp.*, 54:483–493, 1990.

[52] M. Tabata. Finite element schemes based on energy-stable approximation for two-fluid flow problems with surface tension. *Hokkaido Mathematical Journal*, 36(4):875–890, 2007.

[53] C. A. Taylor, T. J. Hughes, and C. K. Zarins. Finite element modeling of blood flow in arteries. *Comput. Methods Appl. Mech. Engrg.*, 158:155–196, 1998.

[54] R. Temam. *Navier-Stokes Equations: Theory and Numerical Analysis.* AMS, AMS Chelsea Publishing, 2001.

[55] G. Zhou and N. Saito. The Navier–Stokes equations under a unilateral boundary condition of signorini's type. *Journal of Mathematical Fluid Mechanics*, pages 1–30, 2015.