博士論文（要約）

論文題目　Fukaya categories, surface Lefschetz fibrations, and Koszul duality theory

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Abstract

This is a resume of the doctoral thesis of the author. In this paper, we summarise the results on the Fukaya-Seidel categories of PALFs and an application of Fukaya categories to the theory of the Koszul duality.

In Chapter 2, we study the Fukaya-Seidel categories of PALFs. We study examples and show that the derived Fukaya-Seidel categories have more information than the Milnor lattices of the PALFs.

In Chapter 3, we study an application of Fukaya categories to the theory of the Koszul duality. First, we define A_∞-Koszul duals for directed A_∞-categories in terms of twists in their A_∞-derived categories. Then, we compute a concrete formula of A_∞-Koszul duals for path algebras with directed A_n-type Gabriel quivers. The formula unveils all the ext groups of simple modules of the algebras and their higher composition structures.

1 Introduction

The results of this paper is based on the symplectic geometry, especially, the theory of the Fukaya categories. The Fukaya categories are A_∞-categories defined for symplectic manifolds [FOOO10$_1$], [FOOO10$_2$]. Those categories are studied in the context of homological mirror symmetry. The celebrated homological mirror symmetry conjecture was first proposed by Kontsevich [Ko94] and predicts the equivalence of two triangulated categories $D^\pi Fuk(M)$ and $D^b coh(X)$ for certain pairs of Calabi-Yau manifolds (M, X), called mirror pair. Here the former category $D^\pi Fuk(M)$ is the split closure of the derived Fukaya category of M as a symplectic manifold [FOOO10$_1$], [FOOO10$_2$] and the latter category $D^b coh(X)$ is the derived category of the category of coherent sheaves on X as a complex manifold. This conjecture is proved for several pairs of Calabi-Yau manifolds. See, for example, [PZ01], [Fu02], [Se15] and so on.

The concept of Fukaya-Seidel categories is defined for exact Lefschetz fibrations. Here, exact Lefschetz fibrations are, roughly speaking, Lefschetz fibrations with suitable exact symplectic structure [Se08]. The Fukaya-Seidel categories appear when we consider the case that X is a Fano manifold. In this case, the mirror partner of X is a Landau-Ginzburg model W [HV00]. Roughly speaking, the Landau-Ginzburg model is a holomorphic function W on a Kähler manifold, called potential function, with isolated singularities. The derived Fukaya-Seidel category $D\mathcal{F}(W)^{\rightarrow}$ is the triangulated category defined by using the data of the singularities, with the techniques of symplectic geometry [Se08], expected to be equivalent to $D^b coh(X)$.

As is common in mathematics, the theory of Fukaya categories is developed in the interaction with the other fields. The first one which is relevant to this paper is the 4-dimensional topology. The concept of PALFs is one of the most studied geometric structure in 4-dimensional topology. It is known that the Lefschetz fibrations are completely determined in terms of the monodromy operator on a regular fibre, so it is related to the study of mapping class groups of oriented surfaces. Hence, it has a combinatorial nature [Ka80]. If a given 4-manifold admits a structure of Lefschetz fibration, we can compute its homology groups, fundamental groups, and (some part of) the intersection forms by the data of monodromy. Moreover, if a 4-manifold X admits a structure of closed Lefschetz fibration, we can compute the signature of X [EN05].

There are two very fundamental results which relate the PALFs and symplectic geometry. The first result due to Donaldson shows that every symplectic 4-manifold admits a structure of Lefschetz fibration after sufficiently many times of blow-ups [Do99]. The second result due to Gompf [Go05] is that every positive Lefschetz fibration admits a symplectic structure. After those two papers, there are many studies involving techniques of both symplectic geometry and PALFs, see e.g. [DS03], [Au06], [AS08], [In15].

Along this context, the author proposes a new method to study the PALFs with a symplectic technique, the derived Fukaya-Seidel categories. The author proved that any PALF admits a structure of exact Lefschetz fibration. Thereafter, the author proved that the derived Fukaya-Seidel category of a PALF is independent of the choice of the exact symplectic structure attached to the PALF.

It is known that the concept of Fukaya-Seidel categories is a "categorification" of the Milnor lattices i.e. the K-groups of the derived Fukaya-Seidel categories coincide with the Milnor lattices. Thus, we naturally expect that the derived Fukaya-Seidel categories catch sensitive information that we cannot capture it by the Milnor lattices. In Section 2, we study examples showing that this is true (Theorem 2.2), i.e. the Fukaya-Seidel categories do have more information than the Milnor lattices. In this theorem, we distinguish three PALFs which share the same Milnor lattice by their Hochschild cohomology groups of Fukaya-Seidel categories. Hence, we have a new method to distinguish PALFs.

The second field which is relevant to this paper is the theory of the Koszul duality. Let us review the fundamental results about Koszul duality in [Lö86]. (The results presented here is a simplified version.) Let $A_0 = k$ be a field, A_1 be a finite dimensional vector space and I be a subspace of $A_1 \otimes A_1$. Define $A \coloneqq T(A_1)/I$ as the quotient algebra of the tensor algebra of A_1 over $A_0 = k$. Then, we have $E \coloneqq \mathrm{Ext}_A(k,k) \cong T(A_1^*)/I^\perp$, where $(-)^*$ is the linear dual over k and $I^\perp \subset A_1^* \otimes A_1^*$ is the annihilating submodule of $I \subset A_1 \otimes A_1$ (we use the natural isomorphism between $A_1^* \otimes A_1^*$ and $(A_1 \otimes A_1)^*$). Let us fix an isomorphism between A_1 and A_1^*. Then, I and $I^\perp$ are mutually complemental. Hence, we can say that the products and relations interchange between A and E. By the above computation, $\mathrm{Ext}_E(k,k)$ is naturally isomorphic to A. This is what we call Koszul duality and we can say that Koszul duality is a duality between products and relations represented by the Yoneda Ext algebra. Moreover, it is known that the certain derived categories of A and E are equivalent [BGS96].

Nowadays, many phenomena related to the Koszul duality are widely observed, for example, the Koszul duality for Koszul algebras [Pr70], [Lö86], [BGS96], its generalisation to augumented-A_∞ algebras [LPWZ04], a generalisation to Koszul operads [GK94], [Va07], [LV12], and its relation to mirror symmetry [AKO08]. The concept of Fukaya categories emerges in the context of Koszul duality in the paper of A. J. Blumberg, R. L. Cohen, and C. Teleman [BCT09] and the paper of T. Etgü and Y. Lekili [EL16]. These papers state that End A_∞-algebras of two certain objects in some Fukaya categories are Koszul dual to each other.

In this paper, we are interested in the case that there exist higher degree (homogenous) relation, i.e. for the algebra $A = T_{A_0}(A_1)/I$ with $I(\not\subset A_1^{\otimes 2}) \subset \bigoplus_{d\geq 2} A_1^{\otimes d}$. Here, $A_0 = k \oplus k \oplus \cdots \oplus k$, A_1 is a finitely generated A_0-module, and $\otimes$ is taken over A_0. In general, there is no easy description of E. Moreover, the ext algebra $\mathrm{Ext}_E(A_0, A_0)$ and A are no longer isomorphic. However, we can overcome this difficulty by referring the results in [LPWZ04]. They generalise the concept of Koszul dual to the augmented A_∞-algebras. After that, they prove that the twice dual is quasi-isomorphic to the original augmented A_∞-algebra and their derived categories are equivalent (under some finiteness condition). The above algebra A is an example of an augmented A_∞-algebra when $A_0 = k$, so we have its dual. But the description is too complicated and we can not interpret the Koszul dual as the duality between products and relations.

The results appear in section 3 gives a new expression of A_∞-Koszul duals of certain path algebras with relations (Theorem 3.5). We use the technique of the Fukaya categories and Dehn twists to compute A_∞-Koszul duals. Our expression of A_∞-Koszul duals does not contain anything new in the standpoint of the abstract theory of Koszul duality. However, we show that the technique of the Fukaya categories can be used for a concrete computation of an algebraic problem. Moreover, our description computed via the Fukaya categories provides a new way of understanding of Koszul duality as a duality between higher products and relations. Namely, we present an explicit description of A_∞-Koszul dual of certain class of path algebras with relations (Theorem 3.5) which enable us to understand the Koszul duality as a duality between higher products and relations. The notion of A_∞-Koszul dual is a natural generalisation. This is supported by the following two corollaries: an A_∞-Koszul dual C of $\mathcal{B}$ is naturally quasi-isomorphic to $\mathcal{A}$ (Corollary 3.3); $\mathcal{A}$ and its Koszul dual $\mathcal{B}$ are A_∞-derived equivalent, i.e. $Tw\mathcal{A} \cong Tw\mathcal{B}$ (hence, in particular, they are derived equivalent , i.e. $D\mathcal{A} \cong D\mathcal{B}$) (Corollary 3.2).

The computation of the A_∞-Koszul dual takes place in the Fukaya categories of exact Riemann surfaces. In general, the Koszul dual can be computed by the operation in the derived category called the twist. Seidel proved in [Se08] that the twists are "quasi-isomorphic" to the Dehn twists in the Fukaya category. Thus, we first find a directed subcategory of the Fukaya category $Fuk(M)$ which is isomorphic to our directed A_∞-category $\mathcal{A} = \mathcal{A}(R)$. Then, we compute the Dehn twists of the objects which are lying in a directed subcategory of $Fuk(M)$ which is A_∞-derived equivalent to the category of modules of the algebra which is in our consideration. Finally, we investigate how the resulting curves intersect and encircle polygons to compute the morphism spaces and their higher compositions. After that, we find that there is a $(d+1)$-gon in M corresponding to a degree d relation, and the $(d+1)$-gon generates the d-th higher composition μ^d. This is our geometric explanation of the duality between higher products and relations. Some typical example is presented in Corollary 3.7.

Here, we fix some notations and assumptions we often use. In this paper, k is a fixed field; all categories are of over k; all graded vector spaces are assumed to have the property that their total dimensions are finite; modules are always right modules; manifolds are oriented; all the additional structures on manifolds are assumed to be compatible with their orientations; the support of any auto-diffeomorphism is away from the boundary of the manifold; the character $\mathcal{F}$ always stands for the Fukaya category $Fuk(M)$ of M where M is "the" exact symplectic manifold we consider in each paragraph; unless otherwise stated.

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2 Results on the Fukaya-Seidel categories of surface Lefschetz fibrations

The whole definitions and proofs can be found in [Su16].

In this subsection, we consider three Lefschetz fibrations π_1, π_2, and π_3 with regular fibre $\Sigma_{3,1}$. Those Lefschetz fibrations are defined by specifying their vanishing cycles $L_1, L_2,$ and L_3 as in Figure 1.

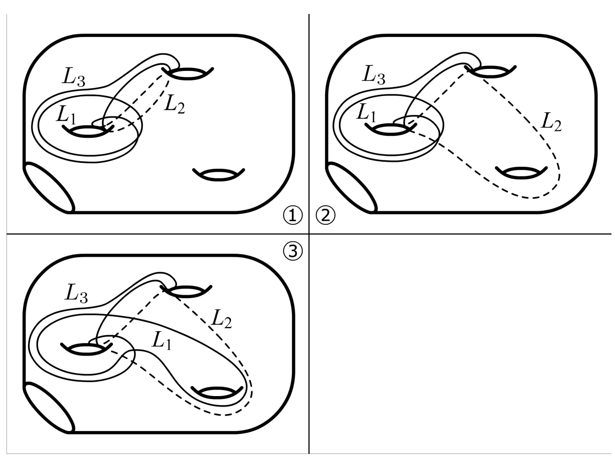

Figure 1: Vanishing cycles of π_i

These three Lefschetz fibrations have isomorphic Milnor lattices, namely $M_{\pi_i} \cong \mathbb{Z}^3$, $([L_j], [L_j]) = 0$, and $([L_j], [L_l]) = \pm 1$ for $j \neq l$ (the sign depends on the orientations of the vanishing cycles which we do not define them here).

Lemma 2.1 *The above three PALFs $\pi_i \colon E_i \to D$ satisfy the condition of vanishing of the two-fold first Chern class, i.e.* $2c_1(E_i) = 0$.

From the above lemma, we can define the Fukaya-Seidel category of π_i. These three PALFs can be distinguished by Hochschild cohomology groups as follows.

Theorem 2.2 *The above three Lefschetz fibrations can be distinguished in terms of Fukaya-Seidel categories. Namely, their Hochschild cohomology groups are as follows:*

$$\begin{aligned} &HH^0(\mathcal{F}(\pi_1)^{\to}) \cong k, && HH^1(\mathcal{F}(\pi_1)^{\to}) \cong k, \\ &HH^0(\mathcal{F}(\pi_2)^{\to}) \cong k, && HH^1(\mathcal{F}(\pi_2)^{\to}) \cong 0, \\ &HH^0(\mathcal{F}(\pi_3)^{\to}) \cong k^2, && HH^1(\mathcal{F}(\pi_3)^{\to}) \cong k. \end{aligned}$$

The Milnor lattices of the above three Lefschetz fibrations all agree, so this is an example that the Fukaya-Seidel categories do have more information than the Milnor lattices. But in fact, the total space of π_1, π_2, π_3 are not homeomorphic each other, so there leave a lot to be desired. So, there emerges a natural question:

Problem 2.3 *Is there two PALFs π_1, π_2 with vanishing of the two-fold first Chern class such that the total spaces are homeomorphic (or diffeomorphic), the Milnor lattices are isomorphic, but their category of twisted complexes of Fukaya-Seidel categories are not equivalent?*

The category of twisted complexes has more information than the derived category of a given A_∞-category. Namely, Kajiura [Kaj13] proved that there are two A_∞-categories C_0 and C_1 in such that $DC_0 \cong DC_1$ but $TwC_0 \not\cong TwC_1$. So there emerges another question:

Problem 2.4 *Is there a geometric example of this? This asks that whether there exist two Lefschetz fibrations π_1, π_2 such that $D\mathcal{F}(\pi_1)^{\to} \cong D\mathcal{F}(\pi_2)^{\to}$ but $Tw\mathcal{F}(\pi_1)^{\to} \not\cong Tw\mathcal{F}(\pi_2)^{\to}$.*

3 Results on the Koszul duality theory

The whole definitions and proofs can be found in [Su17]. This paper is under review of the Journal of Symplectic Geometry.

The first theorem gives us a method to compute A_∞-Koszul duals.

Theorem 3.1 *Let $\mathcal{A}$ be a directed A_∞-category with the object set $Ob(\mathcal{A}) = \{X_0 < X_1 < \cdots < X_n\}$. Suppose that there exist an exact Riemann surface M and a collection of Lagrangian branes $\boldsymbol{L}^\# = (L_0^\#, L_1^\#, \ldots, L_n^\#)$ such that $\mathcal{A}$ and $\mathcal{F}^\rightarrow(\boldsymbol{L}^\#)$ are quasi-isomorphic. Then, $\mathcal{F}^\rightarrow(\boldsymbol{S}^\#)$ is an A_∞-Koszul dual of $\mathcal{A}$, where $\boldsymbol{S}^\# = (S_n^\#, S_{n-1}^\#, \ldots, S_0^\#)$ is a collection of objects defined by the iteration of Dehn twists $S_j^\# := \tau_{L_0^\#}\tau_{L_1^\#}\cdots\tau_{L_{j-1}^\#}L_j^\#$.*

The proof heavily use Theorem 17.16 in [Se08] which states that some equivalence between algebraic twists and Dehn twists in the Fukaya categories.

Thanks to the above theorem, we can compute an A_∞-Koszul dual via the Fukaya categories and Dehn twists. Moreover, we obtain the following corollaries.

Corollary 3.2 *Let $\mathcal{A}$ and $\mathcal{F}^\rightarrow(\boldsymbol{S}^\#)$ be as in Theorem 3.1. Then, there exists a quasi-isomorphism between $Tw\mathcal{A}$ and $Tw(\mathcal{F}^\rightarrow(\boldsymbol{S}^\#))$, hence there exists a equivalence of derived categories between $D\mathcal{A}$ and $D(\mathcal{F}^\rightarrow(\boldsymbol{S}^\#))$ as triangulated categories.*

Corollary 3.3 *Let $\mathcal{A}$ and $\mathcal{F}^\rightarrow(\boldsymbol{S}^\#)$ be as in Theorem 3.1. Then, $\mathcal{A}$ is an A_∞-Koszul dual of $\mathcal{F}^\rightarrow(\boldsymbol{S}^\#)$.*

3.1 Combinatorial setup

In this subsection, we prepare notations to describe A_∞-Koszul duals of path algebras with relations whose Gabriel quiver is the directed A_n-quiver $\overrightarrow{\Delta}_n$,

$$\underset{0}{\bullet} \xrightarrow{\alpha_1} \underset{1}{\bullet} \xrightarrow{\alpha_2} \underset{2}{\bullet} \xrightarrow{\alpha_3} \underset{\cdots}{\cdots} \xrightarrow{\alpha_n} \underset{n}{\bullet}.$$

Let R be a path algebra with relations $R = k\overrightarrow{\Delta}/(\rho_1, \rho_2, \ldots, \rho_m)$, where each relation $\rho_j = \alpha_{t_j}\alpha_{t_j-1}\cdots\alpha_{s_j+1}$ is a path from s_j to t_j for $s_j, t_j \in [0,n]_\mathbb{Z} := \{0, 1, \ldots, n\}$. We call s_j and t_j a source point and a target point of ρ_j respectively. We assume that the length of any relation is greater or equal to two, i.e. $t_j - s_j \geq 2$. We call ρ_j a relation corresponds to $[s_j, t_j]_\mathbb{Z}$. We will sometimes identify relations and such subsets of $[0,n]_\mathbb{Z}$. Now, we can assume that $[s_i, t_i]_\mathbb{Z} \not\subset [s_j, t_j]_\mathbb{Z}$ for $i \neq j$ and $s_1 < s_2 < \cdots < s_m$ (hence $t_1 < t_2 < \cdots < t_m$), so we assume them. We write $S = \{s_1, s_2, \ldots s_m\}, T = \{t_1, t_2, \ldots, t_m\}$ and write $R = R_{S,T}$ to emphasise S and T. We write the associated directed A_∞-category as $\mathcal{A}_{S,G}$. From now, we fix n, S, and T.

We define key items to describe an A_∞-Koszul dual of $\mathcal{A}_{S,T}$. First, we define a map $d\colon [0,n]_\mathbb{Z} \to [0,n]_\mathbb{Z} \sqcup \{-\infty\}$ by $d(p) = \max\{s_j \,|\, t_j \leq p\} = \max\{s \,|\, s < p, \hom_{\mathcal{A}_{S,T}}(s,p) = 0\}$. This is the nearest point s smaller than p such that $\hom_{\mathcal{A}_{S,T}}(s,p)$ vanishes. We define a finite decreasing sequence $\{a_i^{(p)}\}_{0\leq i\leq l_p}$ as follows. First, we set $a_0^{(0)} = 0$ and $l_0 = 0$. For $p \geq 1$, we define $a_0^{(p)} = p, a_1^{(p)} = p-1$. Suppose we have defined $a_q^{(p)}$ for $q < i$. If $d(a_{i-2}^{(p)}) \neq d(a_{i-1}^{(p)})$, then we set $a_i^{(p)} := d(a_{i-2}^{(p)})$. If $d(a_{i-2}^{(p)}) = d(a_{i-1}^{(p)})$, then we set $l_p = i-1$ and finish the definition. Then, we have the following basic property by the elementary proof.

Lemma 3.4 *The sequences $\{a_i^{(p)}\}_{0\leq i\leq l_p}$ are strictly decreasing and non-negative, i.e. $0 \leq a_i^{(p)} < a_{i-1}^{(p)}$.*

3.2 A_∞-Koszul duals of path algebras

For n, S, T, we define a new directed A_∞-category $\mathcal{B}_{S,T}$ as follows. Define $Ob(\mathcal{B}) = \{B(n) < B(n-1) < \cdots < B(0)\}$, $\hom^i_{\mathcal{B}_{S,T}}(B(p), B(a_i^{(p)})) = k \cdot \eta_p^i$ (where η_p^i is a formal symbol), and other hom's are zero. Let us write $\eta_p^l = \tilde{\eta}_{p,a_l^{(p)}}$. Then, we have that $\tilde{\eta}_{p,q} \in \hom_{\mathcal{B}_{S,T}}(B(p), B(q))$. Finally, we define μ's as follows:

$$\mu^d(\tilde{\eta}_{j_{d-1},j_d}, \tilde{\eta}_{j_{d-2},j_{d-1}}, \cdots, \tilde{\eta}_{j_0,j_1}) = \begin{cases} (-1)^{\left(\left|\tilde{\eta}_{j_{d-1},j_d}\right|+1\right)\left|\tilde{\eta}_{j_0,j_d}\right|}\tilde{\eta}_{j_0,j_d} & \text{(if it can be non-zero)} \\ 0 & \text{(otherwise).} \end{cases}$$

Here, "it can be non-zero" means that $\hom_{\mathcal{B}_{S,T}}(B(j_0), B(j_d))$ is non-zero and the relevant morphisms satisfy the degree condition $\left|\tilde{\eta}_{j_0,j_d}\right| = \left|\tilde{\eta}_{j_0,j_1}\right| + \left|\tilde{\eta}_{j_1,j_2}\right| + \cdots + \left|\tilde{\eta}_{j_{d-1},j_d}\right| + (2-d)$, where $|x|$ stands for the degree of x. Then, this defines a directed A_∞-category.

Now, the following theorem is the main theorem of this chapter:

Theorem 3.5 *$\mathcal{B}_{S,T}$ is an A_∞-Koszul dual of $\mathcal{A}_{S,T}$.*

Corollary 3.6 *An A_∞-algebra $B_{S,T} := \bigoplus_{i,j} \hom_{\mathcal{B}_{S,T}}(B(i), B(j))$ is quasi-isomorphic to $(R_{S,T})^!_{dg}$.*

The outline is as follows: first, we find an exact Riemann surface M and a collection of Lagrangian branes $\boldsymbol{L}^\#$ such that $\mathcal{A}_{S,T}$ and $\mathcal{F}^{\to}(\boldsymbol{L}^\#)$ are quasi-isomorphic; next, we compute the Dehn twists and obtain an A_∞-Koszul dual as $\mathcal{F}^{\to}(\boldsymbol{S}^\#)$.

Now, we see the structure of our A_∞-Koszul dual $\mathcal{B}_{S,T}$ with some concrete examples. First, we see the case when $R = R_{S,T}$ is a quadratic algebra, i.e. all the relations are of the form $[i, i+2]_\mathbb{Z}$. By easy calculation, we can show that $\hom_\mathcal{B}(B(p), B(q))$ for $p \geq q$ is non-zero only when $\{q, q+1, \cdots, p-2\} \subset S$. Moreover, when this is the case, the degree of the non-zero morphism is $p - q$. By the condition of degree, we can show that $\mu^d = 0$ except for $d = 2$. Finally, we can conclude that $\mathcal{B}$ is isomorphic to $\mathcal{A}((R_{S^c,T^c})^{op})$, where $S^c := \{0, 1, \ldots, n-2\} \setminus S$ and $T^c := \{2, 3, \ldots, n\} \setminus T$.

For example, if $\alpha_{j+1} \in \hom_\mathcal{A}(j, j+1)$ and $\alpha_{j+2} \in \hom_\mathcal{A}(j+1, j+2)$ satisfy that $\mu^2(\alpha_{j+2}, \alpha_{j+1}) = \mu^2_\mathcal{A}(\alpha_{j+2}, \alpha_{j+1}) = \alpha_{j+2}\alpha_{j+1} \neq 0$, then we have $\mu^2(\tilde{\eta}_{j+1,j}, \tilde{\eta}_{j+2,j+1}) = \mu^2_\mathcal{B}(\tilde{\eta}_{j+1,j}, \tilde{\eta}_{j+2,j+1}) = 0$. Conversely, if $\alpha_{j+1} \in \hom_\mathcal{A}(j, j+1)$ and $\alpha_{j+2} \in \hom_\mathcal{A}(j+1, j+2)$ satisfy that $\mu^2(\alpha_{j+2}, \alpha_{j+1}) = \mu^2_\mathcal{A}(\alpha_{j+2}, \alpha_{j+1}) = 0$, then we have $\mu^2(\tilde{\eta}_{j+1,j}, \tilde{\eta}_{j+2,j+1}) = \mu^2_\mathcal{B}(\tilde{\eta}_{j+1,j}, \tilde{\eta}_{j+2,j+1}) = \tilde{\eta}_{j+2,j} \neq 0$. Thus, we can observe that the products and relations of $\mathcal{A}$ and $\mathcal{B}$ are "reversed" as we have already seen.

Next, we see the case that $n = 3$ and we have only one relation corresponding to $[0, 3]_\mathbb{Z}$. The algebra $R = R_{S,T}$ is no longer quadratic. For this algebra, the duality emerges as the following form. In this case, the formula defines $\mathcal{B}$ as follows: hom spaces are all zero but $\hom^0_\mathcal{B}(B(j), B(j)) = k \cdot \tilde{\eta}_{j,j}$, $\hom^1_\mathcal{B}(B(j+1), B(j)) = k \cdot \tilde{\eta}_{j+1,j}$, $\hom^2_\mathcal{B}(B(3), B(0)) = \tilde{\eta}_{3,0}$; μ's are all zero but $\mu^3(\tilde{\eta}_{1,0}, \tilde{\eta}_{2,1}, \tilde{\eta}_{3,2}) = \tilde{\eta}_{3,0}$. This is nothing but the duality between product and relation. This phenomenon cannot be captured in the dg settings because the dg-structure lacks the structure of higher composition maps.

Let us see the general cases of $R_{S,T}$. We can show $l_p \geq 2 \Leftrightarrow p \in T$ and when this is the case, there exists a relation corresponding to $[a_2^{(p)}, p]_\mathbb{Z}$. We can see that the relation corresponds to $[s_j, t_j]_{\mathbb{Z}}$ in $R_{S,T}$ emerges in the structure of $\mathcal{B}$ as the degree two morphism $\tilde{\eta}_{t_j,s_j}$ with nontrivial higher composition $\mu^{t_j-s_j}$.

These are the typical examples:

Corollary 3.7 *Define $S_{n,k}$ and $T_{n,k}$ for $n > k$ by $S_{n,k} = \{0, 1, \cdots, n-k\}$ and $\{k, k+1, \cdots, n\}$. Let us write $A_{n,k} \coloneqq A_{S_{n,k},T_{n,k}}$, $\mathcal{A}_{n,k} \coloneqq \mathcal{A}_{S_{n,k},T_{n,k}}$, and $\mathcal{B}_{n,k} \coloneqq \mathcal{B}_{S_{n,k},T_{n,k}}$. Then, we have the following:*

1. *For $\mathcal{B}_{n,k}$, we have*

$$\hom^d_{\mathcal{B}_{n,k}}(B(p), B(q)) = \begin{cases} k \cdot \tilde{\eta}_{p,q} & (d \geq 0 \text{ is even and } p-q = kl \text{ for } d = 2l) \\ k \cdot \tilde{\eta}_{p,q} & (d \geq 0 \text{ is odd and } p-q = kl+1 \text{ for } d = 2l+1) \\ 0 & (\text{otherwise}) \end{cases}$$

and $\mu^k(\eta^1_p, \eta^1_{p-1}, \ldots, \eta^1_{p-k+1}) = \eta^2_p \colon S^{\#}_p \to S^{\#}_{p-k}[2]$. (There are many other collections of morphisms with non-vanishing higher compositions, but we omit to write.)

2. *Especially, for $n = k$, our category $\mathcal{B}_n \coloneqq \mathcal{B}_{n,n}$ is described as follows: $Ob(\mathcal{B}_n) = \{B(n) > B(n-1) > \cdots > B(0)\}$,*

$$\hom^d_{\mathcal{B}_n}(B(p), B(q)) = \begin{cases} k \cdot \eta^0_p & (d = 0, p = q) \\ k \cdot \eta^1_p & (d = 1, p-q = 1) \\ k \cdot \eta^2_n & (d = 2, p = n, q = 0) \\ 0 & (\text{otherwise}), \end{cases}$$

and μ's are all zero but μ^2 with identity morphisms and $\mu^n(\eta^1_n, \eta^1_{n-1}, \ldots, \eta^1_1) = \eta^2_n \colon S^{\#}_n \to S^{\#}_0[2]$.

It is remarkable that the whole information of relations of $R_{S,T}$ can be recovered (by hand) by the morphisms of $\mathcal{B}$ with degree less than or equal to two and relevant higher compositions. Thus, there emerges a natural question.

Problem 3.8 *Find the properties of directed A_∞-categories that determine $\mathcal{B}$ from its objects, morphisms with degree less than or equals to 2, and μ's between such a morphisms.*

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