博士論文

論文題目
Semigroups generated by higher order elliptic operators and the Stokes operators in end point spaces
(端点型空間上の高階楕円型作用素やストークス作用素により生成される半群)

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Semigroups generated by higher order elliptic operators and the Stokes operators in end point spaces

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Contents

Chapter 1

Introduction

In this work, we would like to consider semigroups generated by higher order divergence type elliptic operators and the Stokes operators in end point spaces, for example, L^∞ spaces and BMO-type spaces. We consider a divergence type differential operator L of the form $-\sum_{|\alpha|,|\beta|\leq m}(-1)^m\partial^\beta a_{\alpha,\beta}\partial^\alpha$ with a positive integer m. We will establish L^∞ a priori estimates together with the existence and uniqueness of the resolvent equation

$$\begin{cases}(\lambda - L)u = f & \text{in } \Omega\\ u = \partial_\nu u = \cdots = \partial_\nu^{m-1}u = 0 & \text{on } \partial\Omega,\end{cases}$$

where u are unknown functions, f is a given external force term, and the boundary conditions are of Dirichlet type. We also consider the analyticity of the Stokes semigroup $S(t)$, the solution operator of the Stokes equations with the Dirichlet boundary condition

$$\begin{cases}v_t - \Delta v + \nabla q = 0 & \text{in} \quad \Omega\times(0,T),\\ \operatorname{div} v = 0 & \text{in} \quad \Omega\times(0,T),\\ v = 0 & \text{on} \quad \partial\Omega\times(0,T),\\ v|_{t=0} = v_0 & \text{on} \quad \Omega,\end{cases}$$

where v is an unknown vector field, q is an unknown scaler pressure, v_0 is a given initial data. The system of Stokes equations is the linearization of the Navier-Stokes equations which describes the flow of incompressible Newtonian fluids. There is a number of literature on these equations, and we will explain related literature in each chapter.

The main purpose of this work is to construct the foundations of the linear problem in order to analyse the partial differential equations which, for example, describe fluid mechanics, especially in end point spaces on various kinds of unbounded domains. The Masuda-Stewart's method established in [Mas272], and [Ste74],[Ste80] is a well-known method to show the analyticity of semigroups generated by higher order elliptic operators in L^∞ type spaces. In this method, we need $W^{2m,p}$ estimates to obtain the analyticity, so we need the assumption that the boundary has the uniformly $C^{2m-1,1}$ regularity. Therefore, we want to know whether we can relax the regularity assumptions on the boundary. For example, we would like to treat Δ^2 on a domain with the uniformly C^1 boundary. Regarding the Stokes semigroups, It is well known that $S(t)$ forms an analytic semigroup in $L^p_\sigma(\Omega)$ $(1 < p < \infty)$ for various kind of domains Ω including smoothly bounded domains [Gig81], [Sol77], where $L^p_\sigma(\Omega)$ is the L^p-closure of $C^\infty_{c,\sigma}(\Omega)$, the space of all solenoidal vector fields with compact support in Ω. In fact, the analyticity of $S(t)$ in $L^p_\sigma(\Omega)$ holds for any uniformly C^2-domain Ω provided that $L^p(\Omega)$ admits a topological direct sum decomposition, called the Helmholtz decomposition:

$$L^p(\Omega) = L^p_\sigma(\Omega) \oplus G^p(\Omega),\ G^p(\Omega) = \left\{\nabla q \in L^p(\Omega) \mid q \in L^p_{loc}(\Omega)\right\}.$$

This is recently proved in [GHHS12], where the maximal regularity in $L^p_\sigma(\Omega)$ is also established. The Helmholtz decomposition holds for any domain if $p = 2$ and for various kind of domains like bounded or exterior domains with smooth boundary for $1 < p < \infty$ [Gal94]. However, for any $p > 2$ there is an improper smooth sector-like domain such that the L^p-Helmholtz decomposition fails to satisfy [Bog86], [MB86]. Therefore, it is important to consider whether the Stokes semigroup forms an analytic semigroup in $L^p_\sigma(\Omega)$ for domains for which the L^p-Helmholtz decomposition may fail. This shows that the existence of the L^p-Helmholtz decomposition may not be necessary for the analyticity of $S(t)$ in $L^p_\sigma(\Omega)$.

Results in this thesis is summarised as follows,

1. Analyticity of the semigroups generated by higher order elliptic operators in L^∞ spaces on uniformly C^1 domains,

2. Analyticity of the Stokes semigroups in spaces of bounded functions on sector like domains,

3. Stokes resolvent estimates in spaces of bounded functions on admissible domains and its application to the case on cylindrical domains,

4. Analyticity of the Stokes semigroups in BMO-type spaces on admissible domains

5. Analyticity of the Stokes semigroups in L^p_σ spaces on sector-like domains,

6. Equivalence of BMO-type norms and its applications to the heat semigroups,

7. Large time behaviors of solutions of the Stokes and Navier-Stokes problems in BMO-type spaces,

8. Gaussian bounds for the semigroups generated by higher order elliptic operators and application to L^p maximal regularity.

We have mensioned first three results and 5th results. The 4th result is applied to get 5 by interpolation. The 6th result is established to clarify BMO spaces in a domain. The 7th result is an application of BMO theory to the Navier-tokes equations. The 8th result is for study of maximal regularity in L^p for higher order problem when the coefficients are relatively regular.

This work is organized as follows. In Chapter 2, after stating our main theorem, we give a proof of the a priori L^∞-resolvent estimate by a blow-up argument. We also establish a solvability result to prove the sectoriality of the higher order elliptic operator. The results of Chapter 2 has been published in [Suz16].

In Chapter 3, we consider the Stokes resolvent estimates in spaces of bounded functions on admissible domains in order to obtain the analyticity of the Stokes semigroups on sector like domains, and cylindrical domains. Although cylindrical domains are not strictly admissible, we can show that sector like domains, and cylindrical domains are admissible by considering the following Neumann problem of the pressure terms.

$$\Delta u = 0 \quad \text{in} \quad \Omega, \quad \frac{\partial u}{\partial n_\Omega} = 0 \quad \text{on} \quad \partial\Omega.$$

We consider this Neumann problem by a polar coordinate in the case of sector like domains and by dealing with decays at infinity of the space valuables in the case of cylindrical domains. It should be noted that this chapter consists of joint works with Professor Ken Abe, Professor Yoshikazu Giga, and Dr. Katharina Schade [AGSS15], [AGSS16].

In Chapter4, we establish the analyticity of the Stokes semigroups in BMO-type spaces on admissible domains. The gradient L^∞-BMO-type estimate is established in the paper [BG15], and we have established the analyticity of the Stokes semigroups in BMO-type spaces by a Poincaré type estimates from the results of [BG15]. We also establish John-Nirenberg type estimates near the boundary. This chapter consists of a joint work with Dr. Martin Bolkart and Professor Yoshikazu Giga [BGS15].

In Chapter5, we consider the analyticity of the Stokes semigroups in L^p_σ spaces on sector-like domains. We have established L^∞ analyticity of the Stokes semigroups in the joint work [AGSS15], but this result does not imply L^p analyticity of the Stokes semigroups because it would not be certain whether the interpolation space between L^2_σ and $C_{0,\sigma}$ is equal to L^p_σ. We deal with this difficulty by constructing a projection operator different from the Helmholtz projection and a Campanato-Stampaccia type interpolation. This chapter consists of a joint work with Dr. Martin Bolkart, Professor Yoshikazu Giga, Mr. Tatsuhiko Miura, and Professor Yohei Tsutsui [BGMST16].

In Chapter 6, we discuss the equivalence of BMO-type norms in domains and application to the heat semigroup. We define for $f \in L^1_{loc}(\Omega)$, $\nu \in (0,\infty]$ and $p \in [1,\infty)$ the seminorm

$$[f]_{b^\nu p} := \sup\left\{ \left(r^{-n} \int_{\Omega\cap B_r(x)} |f(y)|^p \, dy \right)^{1/p} : x \in \partial\Omega, 0 < r < \nu \right\},$$

and for $\mu \in (0, \infty]$,

$$[f]_{BMO^{\mu}p} := \sup \left\{ \left(\frac{1}{|B_r(x)|} \int_{B_r(x)} |f(y) - f_{B_r(x)}|^p \, dy \right)^{1/p} : B_r(x) \subset \Omega, r < \mu \right\},$$

where for any ball $B \subset \mathbb{R}^n$,

$$f_B := \frac{1}{|B|} \int_B f(y) dy.$$

Then our BMO space is defined by the norm

$$\|f\|_{BMO_b^{\mu,\nu}p} := [f]_{BMO^{\mu}p} + [f]_{b^{\nu}p}.$$

We mainly consider the equivalence with respect to μ, ν, and p, moreover we establised the equivalence to Miyachi's BMO space defined in [Miy90]. This chapter consists of a joint work with Dr. Martin Bolkart, Professor Yoshikazu Giga, and Professor Yohei Tsutsui [BGST16].

In Chapter 7, we consider the large time behaviors of solutions of the Stokes and Navier-Stokes problems in BMO-type spaces. In [Bol16], [BGST16], it is shown that the Stokes semigroup is bounded for large time in BMO-type spaces on the whole space $\mathbb{R}^n$, the half space $\mathbb{R}^n_+$, and bounded domains. However, there seems to be no result which establishes large time behaviors of a solution of nonlinear problems in BMO-type spaces. Our argument is a duality argument based on L^1-L^q estimates in [Mar11] and the BMO-L^q embeddings in [Bol16], and we apply the Kato's method to the Navier-Stokes problem. We remark that we can obtain L^q-BMO type estimates directly from L^q-L^∞ estimates in [Mar11], but the gradient L^q-BMO estimates obtained by this way seem not to be applied to the Navier-Stokes problem by Kato's method because we need shaper decay rates of the time valuable. Therefore, we will establish a shaper gradient L^q-BMO type estimate in an indirect way to avoid this difficulty. This chapter consists of a joint work with Dr. Martin Bolkart in preparation.

In Chapter 8, we established the Gaussian bounds for the semigroups generated by higher order elliptic operators and an application to L^p maximal regularity. It is claimed in [AQ00], [Dav97], [KW04] that semigroups generated by divergence type operators fails to have Gaussian bounds when p is larger than the Sobolev exponent. However, we would like to establish the Gaussian bounds for the semigroups generated by higher order divergence type operators for large p provided that the coefficients of the leading terms of the operator has C_{bu} regularity and the boundary of the domain is Lipschitz provided that the Lipschitz constant is small enough where $C_{bu}(\overline{\Omega})$ is the spaces of bounded uniformly continuous functions in $\overline{\Omega}$. Our argument is based on a Davies perturbation, operator theories, and R-boundedness. This chapter consists of a joint work with Professor Mattias Hieber, Mr. Tomoya Kenmochi, and Dr. Patrick Tolksdorf in preparation.

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Chapter 2

Higher order elliptic resolvent problems in L^∞ spaces on C^1 domains

2.1 Introduction

The goal of this chapter is to establish the analyticity of semigroups generated by divergence type elliptic operators of even order $2m$ with the Dirichlet condition in L^∞ spaces on a domain with the C^1 boundary. The analyticity results in L^∞ type spaces are often proved in a domain with the C^{2m} regularity, for example, as in [Lun95], [Mas72], [Mas272], [Ste74], [Ste80]. The point of this chapter is that we only assume that the domain has the C^1 boundary independent of the order of the operator. Our argument is based on a blow-up method. Instead of stating results for general operators we first discuss the bi-Laplace operator Δ^2 as the simplest example. Let Ω be a uniformly C^1 domain in $\mathbb{R}^n$. We consider the resolvent equation

$$\begin{cases} (\lambda + \Delta^2)u = f & \text{in } \Omega \\ u = \partial_\nu u = 0 & \text{on } \partial\Omega, \end{cases}$$

where ∂_ν is the normal derivative. We will state our main results regarding this operator. We define $N(u, \lambda)$ by

$$N(u, \lambda) = \sup_{x \in \Omega} (|\lambda||u(x)| + |\lambda|^{\frac{3}{4}} |\nabla u(x)|).$$

Theorem 2.1.1 (L^∞ a priori estimates for Δ^2 on uniformly C^1 domains)**.** *Let $\Omega \subset \mathbb{R}^n$ be a domain with uniformly C^1 boundary $\partial\Omega$, $p > n$. Then, for $0 < \varepsilon < \frac{\pi}{2}$ there exist $C > 0$, $M > 0$ such that*

$$N(u, \lambda) \leq C\|f\|_{L^\infty(\Omega)},$$

for all $\lambda \in \Sigma_{\pi-\varepsilon} \cap \{|z| \geq M\} = \{z \in \mathbb{C} : |\arg z| < \pi - \varepsilon\} \cap \{|z| \geq M\}$, $f \in L^\infty(\Omega)$, *and weak solutions* $u_\lambda \in W^{2,p}_{0,loc}(\Omega) \cap W^{1,\infty}(\Omega)$.

Let us illustrate our proof to establish the a priori estimate. Our method is a contradiction argument based on a blow-up method which was first introduced to analyze non-linear partial differential equations by E. De Giorgi. We have four steps to show the a priori estimate. The crucial steps are Step 3, the compactness of a blow-up sequence constructed by normalization and rescaling, and Step 4, the uniqueness of the blow-up limit. We shall briefly explain each step.

Outline of the proof of Theorem 2.1.1.

Step 1 (Normalization)

First of all, we argue by contradiction, then we can take a blow-up sequence of weak solutions such that there exist $\varepsilon > 0$ such that for all $k \in \mathbb{N}$ there exist $\lambda_k \in \Sigma_{\pi-\varepsilon} \cap \{|z| \geq k\}$, $f_k \in L^\infty(\Omega)$, $u_k \in W^{2,p}_{0,loc}(\Omega) \cap L^\infty(\Omega)$ $(p > n)$ which is a weak solution of the resolvent equation

$$\begin{cases} (\lambda_k + \Delta^2)u_k = f_k & \text{in } \Omega \\ u_k = \partial_\nu u_k = 0 & \text{on } \partial\Omega, \end{cases}$$

with L^∞ estimates $N(u_k, \lambda_k) > k\|f_k\|_\infty$. We set $v_k = |\lambda_k|u_k$ with $\lambda_k = |\lambda_k|e^{i\theta_k}$ and we normalize the resolvent equation as

$$\tilde{v}_k = \frac{v_k}{N(u_k, \lambda_k)}, \ \tilde{f}_k = \frac{f_k}{N(u_k, \lambda_k)}.$$

Then, we get the normalized resolvent equation in the weak sense

$$\begin{cases} (e^{i\theta_k} + \dfrac{\Delta^2}{|\lambda_k|})\tilde{v}_k = \tilde{f}_k & \text{in } \Omega \\ \tilde{v}_k = \partial_\nu \tilde{v}_k = 0 & \text{on } \partial\Omega, \end{cases}$$

with the estimates $\|\tilde{f}_k\|_\infty < \frac{1}{k}$, $|\lambda_k| \geq k$, $|\theta_k| < \pi - \varepsilon$, and $N(\frac{\tilde{v}_k}{|\lambda_k|}, \lambda_k) = 1$.

Step 2 (Rescaling)

We next take a sequence of points at which the solution $\tilde{v}_k$ takes a value close to its maximum. Note that $\tilde{v}_k$ is included in some Hölder space by the Sobolev embedding theorem. Thus, there exists $x_k \in \Omega$ such that

$$|\tilde{v}_k(x_k)| + |\lambda_k|^{-\frac{1}{4}}|\nabla\tilde{v}_k(x_k)| > \frac{1}{2}.$$

Set the rescaled functions as

$$\tilde{w}_k = \tilde{v}_k(x_k + \frac{x}{|\lambda_k|^{\frac{1}{4}}}), \ \tilde{g}_k = \tilde{f}_k(x_k + \frac{x}{|\lambda_k|^{\frac{1}{4}}}).$$

Then the rescaled domain Ω_k on which $\tilde{w}_k$ and $\tilde{g}_k$ are defined is represented as $|\lambda_k|^{\frac{1}{4}}(\Omega - x_k)$. By changing the variables, we can show that $\tilde{w}_k$ is a weak solution of the rescaled resolvent equation

$$\begin{cases} (e^{i\theta_k} + \Delta^2)\tilde{w}_k = \tilde{g}_k & \text{in } \Omega_k \\ \tilde{w}_k = \partial_\nu \tilde{w}_k = 0 & \text{on } \partial\Omega_k, \end{cases}$$

with $|\tilde{w}_k(0)| + |\nabla \tilde{w}_k(0)| > \frac{1}{2}$, $\sup_{x\in\Omega_k}(|\tilde{w}_k(x)| + |\nabla \tilde{w}_k(x)|) = 1$. Finally, we apply compactness and uniqueness to get a contradiction.

Step 3 (Compactness)

This step needs local L^p a priori estimates up to boundary for the problem. The actual proof is very involved. We use C^1 regularity to derive such estimates up to the boundary. In the compactness step we show equicontinuity of $\{\tilde{w}_k\}_{k\in\mathbb{N}}$ on some open neighborhood near the origin. Take a smooth cut off function $\rho \in C_0^\infty(B_2(0))$ and localize $\tilde{w}_k$ as $\tilde{w}_k^\rho = \rho\tilde{w}_k$. Then, $\tilde{w}_k^\rho$ is a weak solution of the localized resolvent equation

$$\begin{cases} (e^{i\theta_k} + \Delta^2)\tilde{w}_k^\rho = \tilde{g}_k\rho + (\text{lower order terms of } \tilde{w}_k) & \text{in } \Omega_k \cap B_2(0) \\ \tilde{w}_k^\rho = \partial_\nu \tilde{w}_k^\rho = 0 & \text{on } \partial\Omega_k \text{ and near } \partial B_2(0). \end{cases}$$

In order to apply $W^{2,p}$ estimates, we modify the lower order term by Leibniz' rule so that the lower order term is included in $W^{-2,p}(\Omega_k \cap B_2(0))$, and we have to mollify $\partial(\Omega_k \cap B_2(0))$ on some open neighborhood of $\partial(\Omega_k) \cap \partial(B_2(0))$ so that the boundary becomes uniformly C^1. By local $W^{2,p}$ estimates obtained by general results, we can show that $\{\tilde{w}_k^\rho\} \subset W_0^{2,p}(\Omega_k \cap B_2(0))$ is uniformly bounded. By the zero extension from $\Omega_k \cap B_2(0)$ to $B_2(0)$ and a compact embedding to some Hölder space, there exist a subsequence $\{\tilde{w}_{k_l}^\rho\}$ of $\{\tilde{w}_k^\rho\}$ and a limit w such that

$$\tilde{w}_{k_l}^\rho \to w \quad \text{in } C^1(B_2(0)) \ (l \to \infty).$$

Since $|\tilde{w}_{k_l}^\rho(0)| + |\nabla \tilde{w}_{k_l}^\rho(0)| > \frac{1}{2}$ and $\{\tilde{w}_{k_l}^\rho\}$ converges in $C^1(B_2(0))$, we can show that the limit function w of the blow-up sequence satisfies

$$|w(0)| + |\nabla w(0)| \geq \frac{1}{2}.$$

Step 4 (Uniqueness)

By similar arguments as in the compactness step, we can show that $\tilde{w}_{k_l}$ converges to w in $\Omega_\infty \cap Q$ $(l \to \infty)$ for each compact set Q, where Ω_∞ is determined by the way that the subsequence $\{x_{k_l}\}$ tends to $\partial\Omega$, i.e.,

$$\Omega_\infty = \begin{cases} \mathbb{R}^n & \text{if } \liminf\limits_{l\to\infty} |\lambda_{k_l}|^{\frac{1}{4}} d(x_{k_l}, \partial\Omega) = \infty \\ \mathbb{R}^n_+ & \text{if } d = \limsup\limits_{l\to\infty} |\lambda_{k_l}|^{\frac{1}{4}} d(x_{k_l}, \partial\Omega) < \infty. \end{cases}$$

Let l tend to ∞, then the resolvent equation of $\tilde{w}_{k_l}$ tends to the limit equation

$$\begin{cases} (e^{i\theta_\infty} + \varDelta^2)w = 0 & \text{in } \Omega_\infty \\ w = \partial_\nu w = 0 & \text{on } \partial\Omega_\infty, \end{cases}$$

where w satisfies this equation in the weak sense. Then, we obtain

$$\int_{\Omega_\infty} w(e^{i\theta_\infty} + \varDelta^2)\phi dx = 0 \text{ for all smooth test functions } \phi \in C_0^\infty(\Omega_\infty).$$

So, we consider the dual problem of this limit equation, and we can solve the dual problem by Fourier transform or partial Fourier transform. Then we can substitute $\psi \in C_0^\infty(\Omega_\infty)$ for $(e^{i\theta_\infty} - \varDelta^2)\phi$ in the limit equation, and we get

$$\int_{\Omega_\infty} w\psi dx = 0 \text{ for all test functions } \psi \in C_0^\infty(\Omega_\infty).$$

Therefore, we get the uniqueness result $w = 0$. This contradicts the result $|w(0)| + |\nabla w(0)| \geq \frac{1}{2}$ in the compactness step.

Although there are many articles stating resolvent estimates in L^∞ spaces as discussed below, [Suz16] seems to be the first paper to obtain a resolvent estimate by a blow-up argument. We note that a blow-up argument is used to get analyticity of the Stokes semigroup in L^∞ type space in K. Abe and Y. Giga [AG13] but for the semigroup itself not for the resolvent. We note that our method also applies to the L^∞-Stokes resolvent problem studied in K. Abe, Y. Giga and M. Hieber [AGH15] as outlined in K. Abe, Y. Giga, K. Schade and the author [AGSS16].

We would like to introduce literatures on the analyticity of semigroups generated by elliptic operators. In the case of L^p spaces, Y. Miyazaki [Miy06] established the analyticity of semigroups generated by higher order divergence type elliptic operators in L^p spaces on arbitrary C^1 domains. In the case of L^∞ spaces, there are some known works. In the work of K. Yosida [Yos66], analyticity for second order elliptic operators in $C_\infty(-\infty, \infty)$ space, the space of all continuous functions which vanish at infinity, is first established when the domain is the one-dimensional interval. K. Masuda [Mas272] considered the case when the operator is a higher order elliptic operator on uniformly C^{2m} domains with the Dirichlet condition, and H. B. Stewart [Ste74], [Ste80] extended this method. According to the book of E. Davies [Dav89], Gaussian estimates are valid for any domain when the operators are second order divergence type elliptic operators with L^∞ coefficients, M. Hieber [Hie96] established the analyticity in the case when the operator is an elliptic operator of order 2 and the domain is an arbitrary open set by means of Gaussian estimates. After these results, many researchers dealt with this problem in order to relax the continuity assumption of coefficients. For example, H. Heck, M. Hieber, and K. Stavrakidis [HHS10] show analyticity for higher order elliptic operators with VMO coefficients in $L^\infty(\mathbb{R}^n)$. There are more bibliographical remarks as seen in the book of A. Lunardi [Lun95]. As to other boundary value problems, K. Taira [Tai09] considers boundary value problems of second order elliptic operators with various boundary conditions. When we consider a higher order elliptic operator, Masuda-Stewart's method is a well-known method to show the analyticity of semigroups in L^∞ type spaces. In this method, we need $W^{2m,p}$ space estimates to obtain the analyticity, so we need the assumption that the boundary has the uniformly $C^{2m-1,1}$ regularity. Therefore, the analyticity results of semigroups is well-known when the boundary of domains is uniformly $C^{2m-1,1}$, and we want to know whether we can relax the regularity assumptions on the boundary. In our study, we apply $W^{m,p}$ estimates to get L^∞ estimates by a contradiction argument, and we relax the smoothness assumption on the boundary. For example, we can treat Δ^2 on a domain with the uniformly C^1 boundary.

This chapter is organized as follows. After stating our main theorem, we give a proof of the a priori L^∞-resolvent estimate by a blow-up argument in section 2. In section 3 we prove a solvability result to prove the sectoriality of the higher order elliptic operator.

2.2 Main results

We would like to state main results. Let $\Omega \subset \mathbb{R}^n$ be a domain, ν be the exterior unit normal vector of Ω. Let m be a positive integer, and $L = L_0 + L_1$ be a divergence type differential operator of order $2m$ with the leading term $L_0 = -\sum_{|\alpha|,|\beta|=m}(-1)^m\partial^\beta a_{\alpha,\beta}\partial^\alpha$ of L and the lower order term $L_1 = -\sum_{|\alpha|+|\beta|\le 2m-1}(-1)^{|\beta|}\partial^\beta a_{\alpha,\beta}\partial^\alpha$ of L. We mainly consider L^∞ a priori estimates together with the existence and uniqueness of the resolvent equation

$$\begin{cases}(\lambda - L)u = f & \text{in } \Omega\\ u = \partial_\nu u = \cdots = \partial_\nu^{m-1}u = 0 & \text{on } \partial\Omega,\end{cases} \tag{2.2.1}$$

where the boundary conditions are of Dirichlet type. We assume the following condition. Let the coefficients $a_{\alpha,\beta}$ of L be complex-valued. Let us denote the principal symbol of L by $b(x,\xi) = \sum_{|\alpha|,|\beta|=m} a_{\alpha,\beta}(x)\xi^{\alpha+\beta}$.

(E1) $a_{\alpha,\beta} \in \begin{cases} W^{1,\infty}(\overline{\Omega}) & \text{if } |\alpha| = m\\ L^\infty(\Omega) & \text{if } |\alpha| \le m-1,\end{cases}$

(E2) L is uniformly strongly elliptic, i.e., there exists $\delta_L > 0$ such that

$$\mathrm{Re}(b(x,\xi)) \ge \delta_L|\xi|^{2m} \text{ for } x \in \Omega,\ \xi \in \mathbb{R}^n.$$

(E3) L satisfies the root condition: for every pair of linearly independent real vectors $\xi,\ \eta$ the polynomial $b(x,\xi+\tau\eta)$ of the variable τ has equal number of roots with positive imaginary part and negative imaginary part.

The condition (E3) is also given in [Tan97, p.130]. As remarked in [Ste74], [Ste80], [Tan97], (E2) implies (E3) when $n > 2$ or the coefficients are real. Let us state resolvent estimates for L in L^∞ spaces. The definitions of uniformly C^1 domains, $W^{m,p}_{0,loc}(\Omega)$ and weak solutions will be given in section 2. We define a sectorial set to state L^∞ a priori estimates :

$$\Sigma_{\pi-\varepsilon} \cap \{|z| \geq M\} = \{\lambda \in \mathbb{C} : |\arg \lambda| < \pi - \varepsilon\} \cap \{|z| \geq M\},$$

$$\kappa_L = \sup_{x\in\Omega} \sup_{\xi\in\mathbb{R}^n,\ \xi\neq 0} |\arg b(x,\xi)|,$$

$$N(u,\lambda) = \sup_{x\in\Omega}\Big(\sum_{|\alpha|\leq m-1} |\lambda|^{1-\frac{|\alpha|}{2m}} |\partial^\alpha u(x)|\Big).$$

Theorem 2.2.1. *Let $\Omega \subset \mathbb{R}^n$ be a uniformly C^1 domain, $p > n$. Assume that L satisfies the conditions (E1), (E2), and (E3). Then, for $\kappa_L < \varepsilon < \frac{\pi}{2}$ there exist $C > 0$ (independent of λ, f, u_λ), and $M > 0$ such that*

$$N(u_\lambda, \lambda) \leq C\|f\|_{L^\infty(\Omega)}, \tag{2.2.2}$$

for $\lambda \in \Sigma_{\pi-\varepsilon} \cap \{|z| \geq M\}$, $f \in L^\infty(\Omega)$, and $u_\lambda \in W^{m,p}_{0,loc}(\Omega) \cap W^{m-1,\infty}(\Omega)$ which is a weak solution of the resolvent equation.

Remark 2.2.1. *There are two remarks about our main theorem.*

(1) If coefficients $a_{\alpha,\beta}$ of L are real-valued, then $\kappa_L = 0$.

(2) The smoothness of coefficients is not optimal in our results, although the smoothness of the boundary is weakened from C^{2m} to C^1. Analyticity results in L^∞ spaces on Lipschitz domains remain unknown.

We would like to state analyticity of semigroups generated by L. Let $D(L)$ denote the operator domain of L,

$$D(L) = \{u \in \cap_{p>\frac{n}{m}} W^{m,p}_{0,loc}(\Omega) :\ u,\ Lu \in L^\infty(\Omega)\}.$$

Theorem 2.2.2. *Let $\Omega \subset \mathbb{R}^n$ be a domain with uniformly C^1 boundary. Then, the operator L: $D(L) \to L^\infty(\Omega)$ generates an analytic semigroup in $L^\infty(\Omega)$.*

2.3 A priori estimates

In this section, we give a proof of Theorem 2.2.1. We begin with the definitions of a function space and weak solutions.

Definition 2.3.1. *We say that a function $u \in W^{m,p}_{loc}(\Omega)$ is in $W^{m,p}_{0,loc}(\Omega)$ if $u\eta \in W^{m,p}_0(\Omega)$ for all smooth functions $\eta \in C^\infty_0(\mathbb{R}^n)$.*

Definition 2.3.2. *Let $0 < \epsilon < \frac{\pi}{2}$, $M > 0$, $f \in L^\infty(\Omega)$, and $\lambda \in \Sigma_{\pi-\varepsilon} \cap \{|z| \geq M\}$. Then, we say that $u_\lambda \in W^{m,p}_{0,loc}(\Omega) \cap L^\infty(\Omega)$ is a weak solution of the resolvent equation (2.2.1) if u_λ satisfies*

$$\lambda(u_\lambda,\phi)_{L^2(\Omega)} + \sum_{|\alpha|,|\beta|\leq m} (a_{\alpha,\beta}\partial^\alpha u_\lambda, \partial^\beta\phi)_{L^2(\Omega)} = (f,\phi)_{L^2(\Omega)} \quad \text{for all } \phi \in C^\infty_0(\Omega).$$

Let us recall a uniformly C^1 domain and a Reifenberg flat domain for later arguments. The definition of a C^1 domain is found in Y. Miyazaki [Miy06], Chapter 6, and that of a Reifenberg flat domain is found in S-S. Byun and S. Ryu [BR11, Chapter 2].

Definition 2.3.3. *Let $\Omega \subset \mathbb{R}^n$ be a domain.*

(i) We say that Ω is a uniformly C^1 domain if there exist a family of open sets $\{U_s\}_{s\in\Gamma}$ with countable index set Γ, $N \in \mathbb{N}_{>0}$, $d > 0$, $M_\Omega > 0$, and a non-decreasing function $\omega_\Omega : [0,\infty) \to [0, 2M_\Omega]$ satisfying $\lim_{x\to 0}\omega_\Omega(x) = 0$ such that the following conditions hold:

(a) Any $N+1$ distinct sets of $\{U_s\}_{s\in\Gamma}$ have an empty intersection.

(b) For each $x \in \partial\Omega$ there exists $s \in \Gamma$ such that

$$B_d(x) = \{y \in \mathbb{R}^n : \ |x-y|_{\mathbb{R}^n} < d\} \subset U_s.$$

(c) For each $s \in \Gamma$ there exist a transformation $T_s : \mathbb{R}^n \to \mathbb{R}^n$ which is composed of a rotation and a translation of a coordinate system, and a uniform C^1 function $\phi_s : \mathbb{R}^{n-1} \to \mathbb{R}$ such that

$$|\partial_j\phi_s(x')| \le M_\Omega \ \ for \ x' \in \mathbb{R}^{n-1} \ and \ 1 \le j \le n,$$

$$|\partial_j\phi_s(x') - \partial_j\phi_s(y')| \le \omega_\Omega(|x'-y'|) \ for \ x', y' \in \mathbb{R}^{n-1} \ and \ 1 \le j \le n,$$

and $T_s(U_s\cap\Omega) = T_s(U_s)\cap\{x = (x', x_n) \in \mathbb{R}^n : \ x' \in \mathbb{R}^{n-1}, \ x_n > \phi_s(x')\}$.

(ii) We say that Ω is a (δ, R) Reifenberg flat domain if for all $x \in \partial\Omega$ and $r \in (0, R]$, there exists a coordinate system $\{z_1, \cdots, z_n\}$, which can depend on r and x so that $x = 0$ in this coordinate system and that

$$B_r(0) \cap \{z_n > \delta r\} \subset B_r(0) \cap \Omega \subset B_r(0) \cap \{z_n > -\delta r\}.$$

We need the following lemma in the uniqueness step. This lemma is obtained by a straightforward calculation.

Lemma 2.3.1. *Let Ω be a uniformly C^1 domain. Then, for each $0 < R < d$ Ω is a $(\omega_\Omega(R), R)$ Reifenberg flat domain.*

This implies that as R becomes smaller δ becomes smaller when the domain is a uniformly C^1 domain. In other words, the boundary $\partial\Omega$ becomes flat by an enlargement such as a rescaling.

We are now in position to prove a priori resolvent estimates (Theorem 2.2.1.)

Proof of Theorem 2.2.1. We argue by contradiction. Suppose that the L^∞ estimates were false. Then there would exist $\varepsilon > 0$ such that for $k \in \mathbb{N}$ there exist $\lambda_k \in \Sigma_{\pi-\varepsilon}\cap\{|z| \ge k\}$, $f_k \in L^\infty(\Omega)$, a weak solution $u_k \in W^{m,p}_{0,loc}(\Omega)\cap L^\infty(\Omega)$ of (2.2.1) with $N(u_k, \lambda_k) > k\|f_k\|_\infty$ where $N(u,\lambda) = \sup_{x\in\Omega}(\sum_{|\alpha|\le m-1} |\lambda|^{1-\frac{|\alpha|}{2m}}|\partial^\alpha u(x)|)$.

2.3.1 Normalizing and rescaling

We shall normalize and rescale the blow-up sequences $\{u_k\}$. We set $v_k = |\lambda_k|u_k$ with $\lambda_k = |\lambda_k|e^{i\theta_k}$ and normalize the resolvent equation by setting $\tilde{v}_k = \frac{v_k}{N(u_k,\lambda_k)}$ and $\tilde{f}_k = \frac{f_k}{N(u_k,\lambda_k)}$. Then, we get the following normalized resolvent equations in the weak sense

$$\begin{cases} (e^{i\theta_k} - \dfrac{L}{|\lambda_k|})\tilde{v}_k = \tilde{f}_k & \text{in } \Omega \\ \tilde{v}_k = \partial_\nu \tilde{v}_k = \cdots = \partial_\nu^{m-1}\tilde{v}_k = 0 & \text{on } \partial\Omega, \end{cases}$$

with the estimates $\|\tilde{f}_k\|_\infty < \frac{1}{k}$, $|\lambda_k| \geq k$, $|\arg\theta_k| \leq \pi - \varepsilon$, and $N(\frac{\tilde{v}_k}{|\lambda_k|}, \lambda_k) = 1$. We next take a sequence of points at which the solution $\tilde{v}_k$ takes a value close to its maximum. Since $N(\frac{\tilde{v}_k}{|\lambda_k|}, \lambda_k) = 1$, there exists $x_k \in \Omega$ such that

$$\sum_{|\alpha|\leq m-1} |\lambda|^{-\frac{|\alpha|}{2m}}|\partial^\alpha \tilde{v}_k(x_k)| > \frac{1}{2}.$$

Set the rescaled function as

$$\tilde{w}_k = \tilde{v}_k(x_k + \frac{x}{|\lambda_k|^{\frac{1}{2m}}}) \text{ and } \tilde{g}_k = \tilde{f}_k(x_k + \frac{x}{|\lambda_k|^{\frac{1}{2m}}}).$$

Then the rescaled domain Ω_k of $\tilde{w}_k$ and $\tilde{g}_k$ is represented as $|\lambda_k|^{\frac{1}{2m}}(\Omega - x_k)$. By changing the variables $x \in \Omega$ for $y = |\lambda_k|^{\frac{1}{2m}}(x - x_k)$, for all $\phi \in C_0^\infty(\Omega)$

$$\begin{aligned}
&\sum_{|\alpha|,|\beta|\leq m} (a_{\alpha,\beta}\partial^\alpha \tilde{v}_k, \partial^\beta\phi)_{L^2(\Omega)} \\
&= \sum_{|\alpha|,|\beta|\leq m} \int_\Omega a_{\alpha,\beta}(x)\partial^\alpha \tilde{v}_k(x)\partial^\beta \phi(x)dx \\
&= \sum_{|\alpha|,|\beta|\leq m} \int_{\Omega_k} a_{\alpha,\beta}(x_k + \frac{y}{|\lambda_k|^{\frac{1}{2m}}})(\partial^\alpha \tilde{v}_k)(x_k + \frac{y}{|\lambda_k|^{\frac{1}{2m}}})(\partial^\beta\phi)(x_k + \frac{y}{|\lambda_k|^{\frac{1}{2m}}})\frac{1}{|\lambda_k|^{\frac{n}{2m}}}dy \\
&= \sum_{|\alpha|,|\beta|\leq m} \int_{\Omega_k} b_{\alpha,\beta,k}(y)|\lambda_k|^{\frac{|\alpha|}{2m}}\partial^\alpha(\tilde{v}_k(x_k + \frac{y}{|\lambda_k|^{\frac{1}{2m}}}))|\lambda_k|^{\frac{|\beta|}{2m}}\partial^\beta(\phi(x_k + \frac{y}{|\lambda_k|^{\frac{1}{2m}}}))\frac{1}{|\lambda_k|^{\frac{n}{2m}}}dy \\
&= \sum_{|\alpha|,|\beta|\leq m} |\lambda_k|^{\frac{|\alpha|+|\beta|}{2m} - \frac{n}{2m}}(b_{\alpha,\beta,k}\partial^\alpha \tilde{w}_k, \partial^\beta\eta)_{L^2(\Omega_k)},
\end{aligned}$$

where $b_{\alpha,\beta,k}(y) = a_{\alpha,\beta}(x_k + \frac{y}{|\lambda_k|^{\frac{1}{2m}}})$, and $\eta(y) = \phi(x_k + \frac{y}{|\lambda_k|^{\frac{1}{2m}}}) \in C_0^\infty(\Omega_k)$. Similarly,

$$(\tilde{f}_k, \phi)_{L^2(\Omega)} = \frac{1}{|\lambda_k|^{\frac{n}{2m}}}(\tilde{g}_k, \eta)_{L^2(\Omega_k)}, \quad e^{i\theta_k}(\tilde{v}_k, \phi)_{L^2(\Omega)} = \frac{1}{|\lambda_k|^{\frac{n}{2m}}}e^{i\theta_k}(\tilde{w}_k, \eta)_{L^2(\Omega_k)}.$$

Thus, $\tilde{w}_k$ satisfies for all $\eta \in C_0^\infty(\Omega_k)$

$$e^{i\theta_k}(\tilde{w}_k, \eta) + \sum_{|\alpha|,|\beta|\leq m} |\lambda_k|^{\frac{|\alpha|+|\beta|}{2m}-1}(b_{\alpha,\beta,k}\partial^\alpha \tilde{w}_k, \partial^\beta\eta) = (\tilde{g}_k, \eta), \tag{2.3.1}$$

$$\|\tilde{g}_k\|_\infty < \frac{1}{k},\ \|\tilde{N}(\tilde{w}_k)\|_\infty = 1,\ \tilde{N}(\tilde{w}_k)(0) > \frac{1}{2}, \tag{2.3.2}$$

where $\tilde{N}(\tilde{w}_k)(x) = \sum_{|\alpha|\leq m-1} |\partial^\alpha \tilde{w}_k(x)|$.

2.3.2 Compactness

In the compactness step we show the equicontinuity of the rescaled blow-up sequence $\{\tilde{w}_k\}_{k\in\mathbb{N}}$ on some open neighborhood near the origin. We recall that $L = L_0 + L_1$ is a divergence type differential operator of order $2m$ with the leading term L_0 of L and the lower order term L_1 of L. We give assumptions in order to state $W^{m,p}$ estimates obtained by Y. Miyazaki [Miy06].
Let $b(x,\xi) = \sum_{|\alpha|,|\beta|=m} a_{\alpha,\beta}(x)\xi^{\alpha+\beta}$ denote the principal symbol of L, and let $C_{bu}(\overline{\Omega})$ be the spaces of bounded uniformly continuous functions in $\overline{\Omega}$. We recall Miyazaki's conditions:

(M1) $a_{\alpha,\beta} \in \begin{cases} C_{bu}(\overline{\Omega}) & \text{if } |\alpha| = |\beta| = m \\ L^\infty(\Omega) & \text{if } |\alpha| + |\beta| \leq 2m-1, \end{cases}$

(M2) L is uniformly strongly elliptic, i.e., there exists $\delta_L > 0$ such that

$$\mathrm{Re} b(x,\xi) \geq \delta_L |\xi|^{2m} \text{ for } x \in \Omega,\ \xi \in \mathbb{R}^n.$$

We remark that our conditions (E1), (E2), and (E3) are stronger than Miyazaki's condition (M1), (M2). When we cut off the solution $\tilde{w}_k$ by ρ and obtain the uniformly boundedness of $\tilde{w}_k^\rho$, we need the extra conditions (E1), (E2) to deal with lower order terms such as $(b_{\alpha,\beta,k}(\partial^\alpha \tilde{w}_k)(\partial^\sigma \rho), \partial^{\beta-\sigma}\phi)_{L^2(\Omega_k)}$ for $|\alpha| = m$, $0 < \sigma \leq \beta$, see the proof of Proposition 2.3.1. We need the condition $(E3)$ in the uniqueness step. A freezing method is sometimes helpful to deal with such terms, but this method may not work in our case since the domain Ω_k changes depending on k.

Theorem 2.3.1. (Y. Miyazaki [Miy06]) *Let $\Omega \subset \mathbb{R}^n$ be a uniformly C^1 domain, $p \in (1,\infty)$, and $\epsilon \in (\kappa_L, \frac{\pi}{2})$. Assume L satisfies (M1) and (M2). Then, there exist $M > 0$, $C > 0$ depending on C^1 regularity such that for all $\lambda \in \Sigma_{\pi-\epsilon} \cap \{|z| \geq M\}$ the resolvent $(\lambda - L)^{-1}$ exists with the estimate*

$$\|(\lambda - L)^{-1}\|_{L(W^{-i,p},W^{j,p})} \leq C|\lambda|^{-1+\frac{i+j}{2m}}.$$

Remark 2.3.1. *In the book of G. Simader [Sim72], $W^{m,p}$-estimates on bounded C^m domains are established. In the paper of Y. Miyazaki [Miy06], $W^{m,p}$-estimates on uniformly C^1 domains are established. In the paper of S-S. Byun and S. Ryu [BR11], estimates in the setting of Orlicz space are established. $W^{m,p}$ estimates are obtained as a corollary of the results of S-S. Byun and S. Ryu [BR11] together with uniqueness results. In the paper of V. Maz'ya, M. Mitrea and T. Shaposhnikova [MMS10], the cases of higher order elliptic systems with bounded coefficients in bounded Lipschitz domains are considered.*

In order to apply Theorem 2.3.1 to our case, we need to rescale the situation in Theorem 2.3.1. The rescaled estimate will be given in the proof of Proposition 2.3.1.

We would like to state the equicontinuity in the compactness step. We recall the setting in subsection 2.3.1. Assume that a rescaled uniformly C^1 domain Ω_k is of the form $\Omega_k = |\lambda_k|^{\frac{1}{2m}}(\Omega - x_k)$, where $x_k \in \Omega$ is a point at which each solution of the resolvent equation takes a value close to its maximum, and that the rescaled sequences $\{\lambda_k\}_{k\in\mathbb{N}} = \{|\lambda_k|e^{i\theta_k}\}_{k\in\mathbb{N}}$, $\{\tilde{w}_k\}_{k\in\mathbb{N}} \subset L^\infty(\Omega_k) \cap W^{m,p}_{0,loc}(\Omega_k)$, and $\{\tilde{g}_k\}_{k\in\mathbb{N}} \subset L^\infty(\Omega_k)$ satisfy (2.3.1), (2.3.2), and $\{|\lambda_k|\}_{k\in\mathbb{N}}$ tends to ∞ as k tends to ∞. We remark that the coefficients of L are also rescaled, and we denote the rescaled coefficients as $b_{\alpha,\beta,k}(y) = a_{\alpha,\beta}(x_k + \frac{y}{|\lambda_k|^{\frac{1}{2m}}})$. Then, the rescaled operator L_k is

$$L_k = \sum_{|\alpha|,|\beta|\le m} |\lambda_k|^{\frac{|\alpha|+|\beta|}{2m}-1}\partial^\beta b_{\alpha,\beta,k}\partial^\alpha.$$

Then, the equicontinuity is stated as follows.

Proposition 2.3.1. *Let $p > n$, $\{\lambda_k\}_{k\in\mathbb{N}}$, $\{\tilde{w}_k\}_{k\in\mathbb{N}}$, and $\{\tilde{g}_k\}_{k\in\mathbb{N}} \subset L^\infty(\Omega_k)$ satisfy (2.3.1), (2.3.2), and $\lim_{k\to\infty}|\lambda_k| = \infty$. Then, there exists a subsequence $\{\tilde{w}_{k_l}\}_{l\in\mathbb{N}}$ of $\{\tilde{w}_k\}_{k\in\mathbb{N}}$ which converges to some function $\tilde{w}$ uniformly on some open neighborhood of the origin. In particular, we obtain $\tilde{N}(w)(0) \ge \frac{1}{2}$.*

Proof. Take a cut off function $\rho \in C_0^\infty(\mathbb{R}^n)$ which satisfies

$$\rho = \begin{cases} 1 & \text{on } B_1(0) \\ 0 & \text{outside of } B_{\frac{3}{2}}(0), \end{cases} \qquad \|\rho\|_{m,\infty} \le K.$$

Localize $\tilde{w}_k$ by setting $\tilde{w}_k^\rho = \rho\tilde{w}_k$. By (2.3.1), $\tilde{w}_k^\rho$ is then a weak solution of the localized resolvent equation

$$\begin{cases} (e^{i\theta_k} - L_k)\tilde{w}_k^\rho = \tilde{g}_k\rho + I & \text{in } \Omega_k \cap B_2(0) \\ \tilde{w}_k^\rho = \partial_\nu\tilde{w}_k^\rho = \cdots = \partial_\nu^{m-1}\tilde{w}_k^\rho = 0 & \text{on } \partial\Omega_k \cup (\partial\overline{B_2(0)} \cap \Omega_k), \end{cases} \tag{2.3.3}$$

where I consists of lower order terms of $\tilde{w}_k$. Now, we calculate the term I more precisely in the weak sense. By Leibniz' rule

$$\begin{aligned} &\sum_{|\alpha|,|\beta|\le m} |\lambda_k|^{\frac{|\alpha|+|\beta|}{2m}-1}(b_{\alpha,\beta,k}\partial^\alpha\tilde{w}_k^\rho, \partial^\beta\phi)_{L^2(\Omega_k)} \\ &= \sum_{|\alpha|,|\beta|\le m}\sum_{\gamma<\alpha}\binom{\alpha}{\gamma}|\lambda_k|^{\frac{|\alpha|+|\beta|}{2m}-1}(b_{\alpha,\beta,k}\partial^\gamma\tilde{w}_k\partial^{\alpha-\gamma}\rho, \partial^\beta\phi)_{L^2(\Omega_k)} \\ &+ \sum_{|\alpha|,|\beta|\le m}|\lambda_k|^{\frac{|\alpha|+|\beta|}{2m}-1}(b_{\alpha,\beta,k}(\partial^\alpha\tilde{w}_k)\rho, \partial^\beta\phi)_{L^2(\Omega_k)}, \end{aligned}$$

where $\binom{\alpha}{\gamma} = \frac{\alpha!}{(\alpha-\gamma)!\gamma!}$.

Similarly, we get

$$\begin{aligned}
&\sum_{|\alpha|,|\beta|\le m} |\lambda_k|^{\frac{|\alpha|+|\beta|}{2m}-1}(b_{\alpha,\beta,k}\partial^\alpha \tilde{w}_k, \partial^\beta(\phi\rho))_{L^2(\Omega_k)} \\
&= \sum_{|\alpha|,|\beta|\le m}\sum_{0<\sigma\le\beta}\binom{\beta}{\sigma}|\lambda_k|^{\frac{|\alpha|+|\beta|}{2m}-1}(b_{\alpha,\beta,k}\partial^\alpha \tilde{w}_k, (\partial^\sigma\rho)(\partial^{\beta-\sigma}\phi))_{L^2(\Omega_k)} \\
&+ \sum_{|\alpha|,|\beta|\le m} |\lambda_k|^{\frac{|\alpha|+|\beta|}{2m}-1}(b_{\alpha,\beta,k}\partial^\alpha \tilde{w}_k, \rho\partial^\beta\phi)_{L^2(\Omega_k)}.
\end{aligned}$$

This implies by some calculations

$$\begin{aligned}
&e^{i\theta_k}(\tilde{w}_k^\rho, \phi) + \sum_{|\alpha|,|\beta|\le m} |\lambda_k|^{\frac{|\alpha|+|\beta|}{2m}-1}(b_{\alpha,\beta,k}\partial^\alpha \tilde{w}_k^\rho, \partial^\beta\phi)_{L^2(\Omega_k)} \\
&= (\tilde{g}_k\rho, \phi) + \sum_{|\alpha|,|\beta|\le m}\sum_{\gamma<\alpha}\binom{\alpha}{\gamma}|\lambda_k|^{\frac{|\alpha|+|\beta|}{2m}-1}(b_{\alpha,\beta,k}\partial^\gamma \tilde{w}_k\partial^{\alpha-\gamma}\rho, \partial^\beta\phi)_{L^2(\Omega_k)} \\
&- \sum_{|\alpha|,|\beta|\le m}\sum_{0<\sigma\le\beta}\binom{\beta}{\sigma}|\lambda_k|^{\frac{|\alpha|+|\beta|}{2m}-1}(b_{\alpha,\beta,k}\partial^\alpha \tilde{w}_k, (\partial^\sigma\rho)(\partial^{\beta-\sigma}\phi))_{L^2(\Omega_k)}. \\
&= (\tilde{g}_k\rho, \phi) + I_1 - I_2. \qquad (2.3.4)
\end{aligned}$$

So we obtain the term $I_1 - I_2$ in the weak sense. We can mollify $\partial(\Omega_k \cap B_2(0))$ on an open neighborhood of $\partial(\Omega_k) \cap \partial(B_2(0))$ so that the boundary becomes uniformly C^1. We claim the following estimate which is called the generalized Gårding's inequality in [Sim72, Chapter 6]: there exists $C'' > 0$ such that

$$\|\tilde{w}_k^\rho\|_{W^{m,p}(B_2(0)\cap\Omega_k)} \le C''(\sup_{\|\phi\|_{m,q}=1}|B[\tilde{w}_k^\rho, \phi]| + \|\tilde{w}_k^\rho\|_{L^p(B_2(0)\cap\Omega_k)}),$$

where $B[\cdot,\cdot]$ is the bilinear form $\sum_{|\alpha|,|\beta|\le m} |\lambda_k|^{\frac{|\alpha|+|\beta|}{2m}-1}(b_{\alpha,\beta,k}\partial^\alpha\cdot, \partial^\beta\cdot)_{L^2(B_2(0)\cap\Omega_k)}$ and q is the Hölder conjugate of p. We need this Gårding's inequality in order to obtain local $W^{m,p}$ bounds which imply the desired compactness. We would like to show this estimate by rescaling $W^{m,p}$ estimates in Theorem 2.3.1 and the L^p version of the Lax-Milgram type arguments based on the Hahn-Banach theorem. We apply the Hahn-Banach theorem to $\sum_{|\alpha|,|\beta|\le m} |\lambda_k|^{\frac{|\alpha|+|\beta|}{2m}-1}(b_{\alpha,\beta,k}\partial^\alpha \tilde{w}_k^\rho, \partial^\beta\phi)_{L^2(\Omega_k)}$, to conclude that there exists $h_k \in (W^{m,q}(B_2(0)\cap\Omega_k))^*$ such that

$$\|h_k\|_{(W^{m,q}(B_2(0)\cap\Omega_k))^*} = \sup_{\|\phi\|_{m,q}=1}|B[\tilde{w}_k^\rho, \phi]|,$$

$$h_k|_{W_0^{m,q}}(\phi) = \sum_{|\alpha|,|\beta|\le m} |\lambda_k|^{\frac{|\alpha|+|\beta|}{2m}-1}(b_{\alpha,\beta,k}\partial^\alpha \tilde{w}_k^\rho, \partial^\beta\phi)_{L^2(\Omega_k)}.$$

We thus obtain $h_k|_{W_0^{m,q}}(\phi) = -e^{i\theta_k}(\tilde{w}_k^\rho, \phi) + (\tilde{g}_k\rho, \phi) + I_1 - I_2$ and we can treat $\tilde{w}_k$ in I_1 and I_2 as a fixed external force function. We would like to obtain the rescaled version of Theorem 2.3.1, and apply rescaled estimates to our case. We come back to the resolvent equation

$$\begin{cases} (\lambda_k - L)u_k = f_k & \text{in } \Omega \\ u_k = \partial_\nu u_k = \cdots = \partial_\nu^{m-1} u_k = 0 & \text{on } \partial\Omega, \end{cases}$$

We normalize u_k by $N(u_k, \lambda_k)$, and $\tilde{u}_k$ denotes the normalization of u_k. We cut off $\tilde{u}_k$ by τ so that ρ is the rescaled cut off function of τ. Then we obtain

$$\begin{cases} (\lambda_k - L)\tilde{u}_k^\tau = \tilde{s}_k & \text{in } \Omega \\ \tilde{u}_k^\tau = \partial_\nu \tilde{u}_k^\tau = \cdots = \partial_\nu^{m-1} \tilde{u}_k^\tau = 0 & \text{on } \partial\Omega, \end{cases}$$

where $\tilde{s}_k$ is the sum of external force terms and lower order terms so that $h_k|_{W_0^{m,q}} + e^{i\theta_k}\tilde{w}_k^\rho$ is the rescaled functional of $\tilde{s}_k$. Then, $\tilde{s}_k$ can be calculated in the weak sense

$$\begin{aligned} < \tilde{s}_k, \phi > = (\tilde{f}_k \tau, \phi) &+ \sum_{|\alpha|,|\beta| \le m} \sum_{\gamma < \alpha} \binom{\alpha}{\gamma} (a_{\alpha,\beta} \partial^\gamma \tilde{u}_k \partial^{\alpha-\gamma}\tau, \partial^\beta \phi)_{L^2(\Omega)} \\ &- \sum_{|\alpha|,|\beta| \le m} \sum_{0 < \sigma \le \beta} \binom{\beta}{\sigma} (a_{\alpha,\beta} \partial^\alpha \tilde{u}_k, (\partial^\sigma \tau)(\partial^{\beta-\sigma}\phi))_{L^2(\Omega)}. \end{aligned}$$

Applying Theorem 2.3.1 yields $\|(\lambda_k - L_k)^{-1}\|_{L(W^{-i,p}, W^{j,p})} \le C|\lambda_k|^{-1+(i+j)/2m}$. Since $(\lambda_k - L)\tilde{u}_k^\tau = \tilde{s}_k$, we obtain the estimate

$$\begin{aligned} \|\nabla^j \tilde{u}_k^\tau\|_{L^p(\Omega)} \le & C|\lambda_k|^{-1+\frac{j}{2m}} (\|\tilde{f}_k \tau\|_{L^p}^p + \sum_{|\alpha|,|\beta| \le m} \sum_{\gamma < \alpha} |\lambda_k|^{\frac{|\beta|}{2m}} \binom{\alpha}{\gamma} \|a_{\alpha,\beta} \partial^\gamma \tilde{u}_k \partial^{\alpha-\gamma}\tau\|_{L^p(\Omega)}^p \\ &+ \sum_{|\alpha|,|\beta| \le m} \sum_{0 < \sigma \le \beta} |\lambda_k|^{\frac{|\beta-\sigma|}{2m}} \binom{\beta}{\sigma} \|a_{\alpha,\beta} \partial^\alpha \tilde{u}_k (\partial^\sigma \tau)\|_{L^p}^p)^{\frac{1}{p}}. \end{aligned}$$

We remark that the terms $|\lambda_k|^{|\beta|/2m}$ and $|\lambda_k|^{|\beta-\sigma|/2m}$ in the right hand side come from the term $|\lambda_k|^{\frac{i}{2m}}$ in the right hand side of the inequality in Theorem 2.3.1 where i is determined by the order of spaces $W^{-i,p}$ containing the component of external force functionals. By the rescaling as in subsection 2.3.1, we obtain

$$\|\nabla^j \tilde{u}_k^\tau\|_{L^p(\Omega)} = |\lambda_k|^{-1+(j-n)/2m} \|\nabla^j \tilde{w}_k^\rho\|_{L^p(\Omega_k)}.$$

We set a rescaled test function $\eta(y) = \phi(x_k + \frac{y}{|\lambda_k|^{1/2m}})$, then we obtain

$$
\begin{aligned}
< \tilde{s}_k, \phi > &= (\tilde{f}_k \tau, \phi) + \sum_{|\alpha|,|\beta| \le m} \sum_{\gamma < \alpha} \binom{\alpha}{\gamma} (a_{\alpha,\beta} \partial^\gamma \tilde{u}_k \partial^{\alpha-\gamma} \tau, \partial^\beta \phi)_{L^2(\Omega)} \\
&- \sum_{|\alpha|,|\beta| \le m} \sum_{0 < \sigma \le \beta} \binom{\beta}{\sigma} (a_{\alpha,\beta} \partial^\alpha \tilde{u}_k, (\partial^\sigma \tau)(\partial^{\beta-\sigma} \phi))_{L^2(\Omega)} \\
&= |\lambda_k|^{-\frac{n}{2m}} \{ (\tilde{g}_k \rho, \eta) + \sum_{|\alpha|,|\beta| \le m} \sum_{\gamma < \alpha} \binom{\alpha}{\gamma} |\lambda_k|^{\frac{|\alpha|+|\beta|}{2m} - 1} (b_{\alpha,\beta,k} \partial^\gamma \tilde{w}_k \partial^{\alpha-\gamma} \rho, \partial^\beta \eta)_{L^2(\Omega_k)} \\
&- \sum_{|\alpha|,|\beta| \le m} \sum_{0 < \sigma \le \beta} \binom{\beta}{\sigma} |\lambda_k|^{\frac{|\alpha|+|\beta|}{2m} - 1} (b_{\alpha,\beta,k} \partial^\alpha \tilde{w}_k, (\partial^\sigma \rho)(\partial^{\beta-\sigma} \eta))_{L^2(\Omega_k)} \}.
\end{aligned}
$$

By rescaling we obtain $\|\tilde{f}_k \tau\|_{L^p}^p \le \|\tilde{g}_k \rho\|_{L^p}^p$, and we also obtain

$$
\begin{aligned}
&\sum_{|\alpha|,|\beta| \le m} \sum_{\gamma < \alpha} |\lambda_k|^{\frac{|\beta|}{2m}} \binom{\alpha}{\gamma} \|a_{\alpha,\beta} \partial^\gamma \tilde{u}_k \partial^{\alpha-\gamma} \tau\|_{L^p(\Omega)}^p \\
&\le C \sum_{|\alpha|,|\beta| \le m} \sum_{\gamma < \alpha} |\lambda_k|^{p(\frac{|\alpha|+|\beta|}{2m} - 1 + \frac{|\beta|}{2m})} \sup_{\|\nabla^{|\beta|} \eta\|_q = |\lambda_k|^{-\frac{|\beta|}{2m}}} |(b_{\alpha,\beta,k} \partial^\gamma \tilde{w}_k \partial^{\alpha-\gamma} \rho, \partial^\beta \eta)_{L^2(\Omega_k)}|^p \\
&\le C \sum_{|\alpha|,|\beta| \le m} \sum_{\gamma < \alpha} |\lambda_k|^{p(\frac{|\alpha|+|\beta|}{2m} - 1)} \sup_{\|\nabla^{|\beta|} \eta\|_q = 1} |(b_{\alpha,\beta,k} \partial^\gamma \tilde{w}_k \partial^{\alpha-\gamma} \rho, \partial^\beta \eta)_{L^2(\Omega_k)}|^p \\
&\le C \sum_{|\alpha|,|\beta| \le m} \sum_{\gamma < \alpha} |\lambda_k|^{p(\frac{|\alpha|+|\beta|}{2m} - 1)} \|b_{\alpha,\beta,k} \partial^\gamma \tilde{w}_k \partial^{\alpha-\gamma} \rho\|_{L^p(\Omega_k)}^p.
\end{aligned}
$$

The last terms are also estimated similarly

$$
\begin{aligned}
&\sum_{|\alpha|,|\beta| \le m} \sum_{0 < \sigma \le \beta} |\lambda_k|^{\frac{|\beta-\sigma|}{2m}} \binom{\beta}{\sigma} \|a_{\alpha,\beta} \partial^\alpha \tilde{u}_k (\partial^\sigma \tau)\|_{L^p}^p \\
&\le C \sum_{|\alpha|,|\beta| \le m} \sum_{0 < \sigma \le \beta} |\lambda_k|^{p(\frac{|\alpha|+|\beta|}{2m} - 1 + \frac{|\beta|-|\sigma|}{2m})} \sup_{\|\nabla^{|\beta|-|\sigma|} \eta\|_q = |\lambda_k|^{-\frac{|\beta|-|\sigma|}{2m}}} |(b_{\alpha,\beta,k} \partial^\alpha \tilde{w}_k, (\partial^\sigma \rho)(\partial^{\beta-\sigma} \eta))_{L^2(\Omega_k)}|^p \\
&\le C \sum_{|\alpha|,|\beta| \le m} \sum_{0 < \sigma \le \beta} |\lambda_k|^{p(\frac{|\alpha|+|\beta|}{2m} - 1)} \sup_{\|\nabla^{|\beta|-|\sigma|} \eta\|_q = 1} |(b_{\alpha,\beta,k} \partial^\alpha \tilde{w}_k, (\partial^\sigma \rho)(\partial^{\beta-\sigma} \eta))_{L^2(\Omega_k)}|^p \\
&\le C \sum_{|\alpha|,|\beta| \le m} \sum_{0 < \sigma \le \beta} |\lambda_k|^{p(\frac{|\alpha|+|\beta|}{2m} - 1)} \|b_{\alpha,\beta,k} \partial^\alpha \tilde{w}_k (\partial^\sigma \rho)\|_{L^p(\Omega_k)}^p.
\end{aligned}
$$

Since the decomposition of $h_k|_{W_0^{m,q}} + e^{i\theta_k}(\tilde{w}_k^\rho, \cdot) \in W^{-m,p}(B_2(0) \cap \Omega_k)$ is unique by [AF03, Theorem 3.9] for $1 < p < \infty$, we obtain the following identity by

[AF03, (4), Theorem 3.9]

$$
\begin{aligned}
&\|h_k|_{W_0^{m,q}} + e^{i\theta_k}(\tilde{w}_k^\rho, \cdot)\|_{W^{-m,p}}^p \\
&= \|\tilde{g}_k\rho\|_{L^p}^p + \sum_{|\alpha|,|\beta|\le m} \sum_{\gamma<\alpha} \binom{\alpha}{\gamma} |\lambda_k|^{p(\frac{|\alpha|+|\beta|}{2m}-1)} \|b_{\alpha,\beta,k}\partial^\gamma \tilde{w}_k \partial^{\alpha-\gamma}\rho\|_{L^p}^p \\
&+ \sum_{|\alpha|,|\beta|\le m} \sum_{0<\sigma\le\beta} \binom{\beta}{\sigma} |\lambda_k|^{p(\frac{|\alpha|+|\beta|}{2m}-1)} \|b_{\alpha,\beta,k}\partial^\alpha \tilde{w}_k(\partial^\sigma\rho)\|_{L^p}^p.
\end{aligned}
$$

We remark that the right hand side of this equality can be compared with the right hand side of the previous inequality. Therefore, we conclude that

$$\|\tilde{w}_k^\rho\|_{W^{m,p}(B_2(0)\cap\Omega_k)} \le C'(\|h_k|_{W_0^{m,q}}\|_{W^{-m,p}(B_2(0)\cap\Omega_k)} + \|\tilde{w}_k^\rho\|_{L^p(B_2(0)\cap\Omega_k)}).$$

Since $\|h_k|_{W_0^{m,q}}\|_{W^{-m,p}(B_2(0)\cap\Omega_k)} \le \|h_k\|_{(W^{m,q}(B_2(0)\cap\Omega_k))^*} = \sup_{\|\phi\|_{m,q}=1} |B[\tilde{w}_k^\rho, \phi]|$, we finally obtain the desired generalized Gårding estimate

$$\|\tilde{w}_k^\rho\|_{W^{m,p}(B_2(0)\cap\Omega_k)} \le C(\sup_{\|\phi\|_{m,q}=1} |B[\tilde{w}_k^\rho, \phi]| + \|\tilde{w}_k^\rho\|_{L^p(B_2(0)\cap\Omega_k)}).$$

We return to the proof of Proposition 2.3.1. We have to deal with the first term $\sup_{\|\phi\|_{m,q}=1} |B[\tilde{w}_k^\rho, \phi]|$. By (2.3.4),

$$
\begin{aligned}
B[\tilde{w}_k^\rho, \phi] &= \sum_{|\alpha|,|\beta|\le m} |\lambda_k|^{\frac{|\alpha|+|\beta|}{2m}-1} (b_{\alpha,\beta,k}\partial^\alpha \tilde{w}_k^\rho, \partial^\beta, \phi)_{L^2(B_2(0)\cap\Omega_k)} \\
&= -e^{i\theta_k}(\tilde{w}_k^\rho, \phi) + (\tilde{g}_k\rho, \phi) + I_1 - I_2.
\end{aligned}
$$

Since $I_1 = \sum_{|\alpha|,|\beta|\le m} \sum_{\gamma<\alpha} \binom{\alpha}{\gamma} |\lambda_k|^{\frac{|\alpha|+|\beta|}{2m}-1} (b_{\alpha,\beta,k}\partial^\gamma \tilde{w}_k \partial^{\alpha-\gamma}\rho, \partial^\beta\phi)_{L^2}$, $\sup_{\|\phi\|_{m,q}=1} |I_1|$ is estimated as $\sup_{\|\phi\|_{m,q}=1} |I_1| \le C(m, n, \|b_{\alpha,\beta}\|_{L^\infty}, \rho)\|\tilde{w}_k\|_{W^{m-1,p}(B_2(0)\cap\Omega_k)}$.

For the estimate of $\sup_{\|\phi\|_{m,q}=1} |I_2|$, we need to modify I_2 appropriately. Since $b_{\alpha,\beta} \in W^{1,\infty}(B_2(0)\cap\Omega_k)$ and for $1 \le k_0 \le n$ so that $\alpha_{k_0} \neq 0$

$$(b_{\alpha,\beta,k}(\partial^{\alpha-e_{k_0}}\tilde{w}_k)(\partial^\sigma\rho), \partial^{\beta-\sigma+e_{k_0}}\phi)_{L^2} = -(\partial_{k_0}\{b_{\alpha,\beta,k}(\partial^{\alpha-e_{k_0}}\tilde{w}_k)(\partial^\sigma\rho)\}, \partial^{\beta-\sigma}\phi)_{L^2},$$

I_2 can be calculated as

$$
\begin{aligned}
I_2 &= \sum_{|\alpha|,|\beta|\le m} \sum_{0<\sigma\le\beta} \binom{\beta}{\sigma} |\lambda_k|^{\frac{|\alpha|+|\beta|}{2m}-1} (b_{\alpha,\beta,k}(\partial^\alpha \tilde{w}_k)(\partial^\sigma\rho), \partial^{\beta-\sigma}\phi)_{L^2(\Omega_k)} \\
&= \sum_{|\alpha|,|\beta|\le m} \sum_{0<\sigma\le\beta} \binom{\beta}{\sigma} |\lambda_k|^{\frac{|\alpha|+|\beta|}{2m}-1} \{-((\partial_{k_0}b_{\alpha,\beta,k})(\partial^{\alpha-e_{k_0}}\tilde{w}_k)(\partial^\sigma\rho), \partial^{\beta-\sigma}\phi)_{L^2} \\
&- (b_{\alpha,\beta,k}(\partial^{\alpha-e_{k_0}}\tilde{w}_k)(\partial^{\sigma+e_{k_0}}\rho), \partial^{\beta-\sigma}\phi)_{L^2} - (b_{\alpha,\beta,k}(\partial^{\alpha-e_{k_0}}\tilde{w}_k)(\partial^\sigma\rho), \partial^{\beta-\sigma+e_{k_0}}\phi)_{L^2}\}.
\end{aligned}
$$

Thus, we obtain the estimate of $\sup_{\|\phi\|_{m,q}=1} |I_2|$

$$\sup_{\|\phi\|_{m,q}=1} |I_2| \leq C(m, n, \|b_{\alpha,\beta}\|_{W^{1,\infty}}, \rho)\|\tilde{w}_k\|_{W^{m-1,p}(B_2(0)\cap\Omega_k)}.$$

Therefore, $\sup_{\|\phi\|_{m,q}=1} |B[\tilde{w}_k^\rho, \phi]|$ is estimated by

$$\sup_{\|\phi\|_{m,q}=1} |B[\tilde{w}_k^\rho, \phi]| \leq C(\|\tilde{w}_k^\rho\|_p + \|\tilde{g}_k\rho\|_p + \|\tilde{w}_k\|_{W^{m-1,p}(B_2(0)\cap\Omega_k)}).$$

As a consequence, we obtain by (2.3.2) and the Gårding type estimate

$$\begin{aligned}
\|\tilde{w}_k^\rho\|_{W^{m,p}(B_2(0)\cap\Omega_k)} &\leq C(\sup_{\|\phi\|_{m,q}=1} |B[\tilde{w}_k^\rho, \phi]| + \|\tilde{w}_k^\rho\|_{L^p(B_2(0)\cap\Omega_k)}) \\
&\leq C(\|\tilde{w}_k^\rho\|_{L^p(B_2(0)\cap\Omega_k)} + \|\tilde{g}_k\rho\|_{L^p(B_2(0)\cap\Omega_k)} + \|\tilde{w}_k\|_{W^{m-1,p}(B_2(0)\cap\Omega_k)}) \\
&\leq C|B_2(0)|^{\frac{1}{p}}(\|\tilde{g}_k\|_\infty + \|\tilde{N}(\tilde{w}_k)\|_\infty) \\
&\leq C|B_2(0)|^{\frac{1}{p}}(\frac{1}{k} + 1) < \infty.
\end{aligned}$$

Therefore, the sequence $\{\tilde{w}_k^\rho\} \subset W_0^{m,p}(B_2(0) \cap \Omega_k)$ is uniformly bounded. By the zero extension from $\Omega_k \cap B_2(0)$ to $B_2(0)$, we get the uniform boundedness of $\{\tilde{w}_k^\rho\} \subset W^{m,p}(B_2(0))$. By Rellich's compactness theorem and $p > n$, there exists a subsequence $\{\tilde{w}_{k_l}^\rho\}$ of $\{\tilde{w}_k^\rho\}$ and a limit function w such that

$$\tilde{w}_{k_l}^\rho \to w \quad \text{uniformly on } B_2(0) \ (l \to \infty).$$

Since $\tilde{N}(\tilde{w}_{k_l})(0) > \frac{1}{2}$, we get $\tilde{N}(w)(0) \geq \frac{1}{2}$. Thus we complete the compactness step. □

2.3.3 Uniqueness of the limit problem

In this section, we discuss the uniqueness step to get the uniqueness result $w = 0$ which contradicts the results of the compactness step. We need to consider the uniqueness problem of the limit equation. Set $d_k = d_{\Omega_k}(0, \partial\Omega_k) = |\lambda_k|^{\frac{1}{2m}} d_\Omega(x_k, \partial\Omega)$. As explained in the outline of the proof in the introduction, there are two cases we have to consider. Let w be the limit equation obtained in the compactness step.

Case (i) $\tilde{d} = \liminf_{k\to\infty} d_k = \liminf_{k\to\infty} |\lambda_k|^{\frac{1}{2m}} d(x_k, \partial\Omega) = \infty$.

Lemma 2.3.2. *Let $p > n$. The limit w is a weak solution of the limit resolvent equation*

$$(e^{i\theta_\infty} - \tilde{L}_0)w = 0 \quad in\ \mathbb{R}^n, \tag{2.3.5}$$

where $\tilde{a}_{\alpha,\beta}$ are the limit constants of coefficients $a_{\alpha,\beta}$, w is the limit function, and the limit operator $\tilde{L}_0$ is defined in the weak form

$$-(-1)^m \sum_{|\alpha|=|\beta|=m} \tilde{a}_{\alpha,\beta}(w, \partial^{\alpha+\beta}\eta)_{L^2(\mathbb{R}^n)} \text{ for all test functions } \eta \in C_0^\infty(\mathbb{R}^n).$$

Proof. First of all, we show Ω_k tends to $\mathbb{R}^n$. Since $\tilde{d} = \infty$, for $r > 0$ there exists $k_0 \in \mathbb{N}$ such that $B_r(0) \subset \Omega_k$ for all $k \geq k_0$. We fix a smooth test function $\eta \in C_0^\infty(\mathbb{R}^n)$, then we substitute a larger bounded sub domain Ω_η with C^1 boundary for supp η because supp η is compact. Since Ω_η is bounded, there exists $k_\eta \in \mathbb{N}$ such that $\Omega_\eta \subset \Omega_k$ for $k \geq k_\eta$. We next consider the convergence of the most crucial rescaled term $|\lambda_k|^{\frac{|\alpha|+|\beta|}{2m}-1}(b_{\alpha,\beta,k}\partial^\alpha \tilde{w}_k, \partial^\beta \eta)_{L^2(\Omega_k)}$. Then, for $k \geq k_\eta$

$$(b_{\alpha,\beta,k}\partial^\alpha \tilde{w}_k, \partial^\beta \eta)_{L^2(\Omega_k)} = (b_{\alpha,\beta,k}\partial^\alpha \tilde{w}_k, \partial^\beta \eta)_{L^2(\Omega_\eta)}.$$

Let us consider the convergence of the terms

$$|\lambda_k|^{\frac{|\alpha|+|\beta|}{2m}-1} b_{\alpha,\beta,k}(x) = |\lambda_k|^{\frac{|\alpha|+|\beta|}{2m}-1} a_{\alpha,\beta}(x_k + \frac{x}{|\lambda_k|^{\frac{1}{2m}}}) \text{ on } \Omega_\eta.$$

For $|\alpha| + |\beta| \leq 2m - 1$,

$$|\lambda_k|^{\frac{|\alpha|+|\beta|}{2m}-1} \|b_{\alpha,\beta,k}\|_\infty = |\lambda_k|^{\frac{|\alpha|+|\beta|}{2m}-1} \|a_{\alpha,\beta}\|_\infty \to 0 \ (k \to \infty).$$

For $|\alpha| = |\beta| = m$, since $a_{\alpha,\beta}$ are uniformly continuous, for $\epsilon > 0$ there exists $\delta > 0$ such that if $x \in \Omega_\eta$ and $|x_k + \frac{x}{|\lambda_k|^{\frac{1}{2m}}} - x_k| = |\frac{x}{|\lambda_k|^{\frac{1}{2m}}}| < \delta$ then $|b_{\alpha,\beta,k}(x) - b_{\alpha,\beta,k}(0)| < \epsilon$. Since $|\lambda_k| \to \infty$, there exists k_0 such that $|\frac{x}{|\lambda_k|^{\frac{1}{2m}}}| < \delta$ for $k \geq k_0$. Since $|a_{\alpha,\beta}(x_k)| \leq \|a_{\alpha,\beta}\|_\infty$, there exist a constant $\tilde{a}_{\alpha,\beta}$ and a subsequence $\{b_{\alpha,\beta,k_l}\}$ such that $b_{\alpha,\beta,k_l}(0) = a_{\alpha,\beta}(x_{k_l}) \to \tilde{a}_{\alpha,\beta}$ $(l \to \infty)$. Then, for $k_0 \leq k_l$

$$\begin{aligned}|b_{\alpha,\beta,k_l}(x) - \tilde{a}_{\alpha,\beta}| &\leq |b_{\alpha,\beta,k_l}(x) - b_{\alpha,\beta,k_l}(0)| + |b_{\alpha,\beta,k_l}(0) - \tilde{a}_{\alpha,\beta}| \\ &\leq \epsilon + |b_{\alpha,\beta,k_l}(0) - \tilde{a}_{\alpha,\beta}| \to \epsilon \ (l \to \infty).\end{aligned}$$

Since ϵ and x are arbitrary, we get the pointwise convergence

$$b_{\alpha,\beta,k_l}(x) \to \tilde{a}_{\alpha,\beta} \ (l \to \infty).$$

Furthermore, $|b_{\alpha,\beta,k}\partial^\beta \eta|$ is estimated by the L^q integrable function $\|a_{\alpha,\beta}\|_\infty |\partial^\beta \eta|$. Therefore, the L^q version of the Lebesgue's dominated convergence theorem implies

$$\|b_{\alpha,\beta,k_l}\partial^\beta \eta - \tilde{a}_{\alpha,\beta}\partial^\beta \eta\|_{L^q(\Omega_\eta)} \to 0.$$

The argument of Proposition 2.3.1 implies $\|\tilde{w}_k\|_{W^{m,p}(Q)} \leq C$ for all $k \geq k_\eta$ and all compact sets $Q \subset \mathbb{R}^n$ where $C < \infty$ is depend on Q but independent of k. By this uniformly boundedness and $p > n$, we obtain the locally uniform convergence of $\tilde{w}_{k_l}$ in C^{m-1} spaces, i.e., $\tilde{w}_{k_l} \to w$ in $C^{m-1}(Q)$ for all compact set $Q \subset \mathbb{R}^n$. Since Ω_η is compact, we obtain for $k_l \geq k_\eta$

$$\begin{aligned}&|(b_{\alpha,\beta,k_l}\partial^\alpha \tilde{w}_{k_l}, \partial^\beta \eta)_{L^2(\Omega_\eta)} - (\tilde{a}_{\alpha,\beta}\partial^\alpha w, \partial^\beta \eta)_{L^2(\Omega_\eta)}| \\ &\leq |((b_{\alpha,\beta,k} - \tilde{a}_{\alpha,\beta})\partial^\alpha \tilde{w}_{k_l}, \partial^\beta \eta)_{L^2(\Omega_\eta)}| + |(\tilde{a}_{\alpha,\beta}(\partial^\alpha \tilde{w}_{k_l} - \partial^\alpha w), \partial^\beta \eta)_{L^2(\Omega_\eta)}| \\ &\leq \|\tilde{w}_{k_l}\|_{W^{m,p}(\Omega_\eta)} \|b_{\alpha,\beta,k_l}\partial^\beta \eta - \tilde{a}_{\alpha,\beta}\partial^\beta \eta\|_{L^q(\Omega_\eta)} + |(\tilde{a}_{\alpha,\beta}(\partial^{\alpha-e_i}\tilde{w}_{k_l} - \partial^{\alpha-e_i}w, \partial^{\beta+e_i}\eta)_{L^2(\Omega_\eta)}| \\ &\leq C\|b_{\alpha,\beta,k_l}\partial^\beta \eta - \tilde{a}_{\alpha,\beta}\partial^\beta \eta\|_{L^q(\Omega_\eta)} + C'\|\tilde{w}_{k_l} - w\|_{C^{m-1}(\Omega_\eta)} \to 0.\end{aligned}$$

The other terms can be treated similarly, and as l tends to ∞ the rescaled equation

$$e^{i\theta_k}(\tilde{w}_k,\eta)_{L^2(\Omega_k)}+\sum_{|\alpha|,|\beta|\le m}|\lambda_k|^{\frac{|\alpha|+|\beta|}{2m}-1}(b_{\alpha,\beta,k}\partial^\alpha\tilde{w}_k,\partial^\beta\eta)_{L^2(\Omega_k)}=(\tilde{g}_k,\eta)_{L^2(\Omega_k)}$$

tends to

$$e^{i\theta_\infty}(w,\eta)_{L^2(\mathbb{R}^n)}+\sum_{|\alpha|=|\beta|=m}\tilde{a}_{\alpha,\beta}(\partial^\alpha w,\partial^\beta\eta)_{L^2(\mathbb{R}^n)}=0,$$

where $|\theta_\infty|\le\pi-\epsilon$. Therefore, w is a weak solution of the limit resolvent equation (2.3.5). $\square$

Let $\tilde{a}_{\alpha,\beta}\in\mathbb{C}$, and we consider the limit resolvent equation of the following uniformly elliptic operator with constant coefficients:

$$\tilde{L}_0=\sum_{|\alpha|,|\beta|=m}(-1)^{m+1}\tilde{a}_{\alpha,\beta}\partial^{\alpha+\beta}.$$

We can show that $\tilde{L}_0$ satisfies the ellipticity condition (E2) and the root condition (E3) by a continuity argument because L_{k_l} satisfies (E2), (E3) and $b_{\alpha,\beta,k_l}(x)$ converges to $\tilde{a}_{\alpha,\beta}$ pointwisely. Let $S(\mathbb{R}^n)$ be the space of all rapidly decreasing functions in the sense of L. Schwartz.

Lemma 2.3.3 (Dual problem when Ω_∞ is the whole space $\mathbb{R}^n$)**.** *Let $|\theta_\infty|\le\pi-\epsilon$ with $\epsilon\in(\kappa_L,\pi/2)$. Let ψ be in $C_0^\infty(\mathbb{R}^n)$. Then there exists a solution ϕ in $S(\mathbb{R}^n)$ such that*

$$(e^{i\theta_\infty}-\tilde{L}_0)\phi=\psi\quad\text{ in }\mathbb{R}^n.$$

Proof. Let $\hat{\phi}$ denote the Fourier transform of ϕ. By the Fourier transform we get

$$(e^{i\theta_\infty}-(-1)^{m+1}\sum_{|\alpha|,|\beta|=m}\tilde{a}_{\alpha,\beta}(i\xi)^{\alpha+\beta})\hat{\phi}=\hat{\psi}\quad\text{ in }\mathbb{R}^n.$$

Since $|\theta_\infty|\le\pi-\epsilon$ with $\epsilon\in(\kappa_L,\pi/2)$, if $|\sum_{|\alpha|,|\beta|=m}\tilde{a}_{\alpha,\beta}\xi^{\alpha+\beta}|=1$ then

$$\begin{aligned}&\mathrm{Re}(e^{i\theta_\infty}+\sum_{|\alpha|,|\beta|=m}\tilde{a}_{\alpha,\beta}\xi^{\alpha+\beta})\\&=\cos\theta_\infty+\cos(\arg\sum_{|\alpha|,|\beta|=m}\tilde{a}_{\alpha,\beta}\xi^{\alpha+\beta})\\&>-\cos\kappa_L+\cos(\sup_{x\in\Omega}\sup_{\xi\in\mathbb{R}^n,\ \xi\ne0}\arg\sum_{|\alpha|,|\beta|=m}a_{\alpha,\beta}(x)\xi^{\alpha+\beta})=0.\end{aligned}$$

If $|\sum_{|\alpha|,|\beta|=m} \tilde{a}_{\alpha,\beta}\xi^{\alpha+\beta}| \neq 1$, then we also get $\sum_{|\alpha|,|\beta|=m} \tilde{a}_{\alpha,\beta}\xi^{\alpha+\beta} \neq -e^{i\theta_\infty}$. This implies for all ξ. Thus, we get

$$\hat{\phi} = \frac{1}{e^{i\theta_\infty} + \sum_{|\alpha|,|\beta|=m} \tilde{a}_{\alpha,\beta}\xi^{\alpha+\beta}} \hat{\psi} \in S(\mathbb{R}^n).$$

Since $\hat{\phi} \in S(\mathbb{R}^n)$, we get $\phi \in S(\mathbb{R}^n)$. □

We are able to complete the proof in the case (i) by the following uniqueness lemma.

Lemma 2.3.4. *Let $|\theta_\infty| \leq \pi - \epsilon$, $w \in W^{m,p}_{0,loc}(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$ be a weak solution of the limit equation*

$$(e^{i\theta_\infty} - \tilde{L}_0)w = 0 \quad \text{ in } \mathbb{R}^n,$$

that is, for all $\phi \in C_0^\infty(\mathbb{R}^n)$ w satisfies

$$e^{i\theta_\infty}(w,\phi)_{L^2(\mathbb{R}^n)} + \sum_{|\alpha|,|\beta|=m} \tilde{a}_{\alpha,\beta}(\partial^\alpha w, \partial^\beta \phi)_{L^2(\mathbb{R}^n)} = 0.$$

Then, $w = 0$.

Proof. Integrating by parts, we obtain

$$\int_{\mathbb{R}^n} w(e^{i\theta_\infty} - \tilde{L}_0)\phi dx = 0 \text{ for all smooth test functions } \phi \in C_0^\infty(\mathbb{R}^n).$$

Since $C_0^\infty(\mathbb{R}^n)$ is dense in $S(\mathbb{R}^n)$, we can take ϕ in $S(\mathbb{R}^n)$ as a test function. So, we consider the dual problem of the limit equation. For all smooth ψ in $C_0^\infty(\mathbb{R}^n)$ we want to find a solution ϕ in $S(\mathbb{R}^n)$ such that

$$(e^{i\theta_\infty} - \tilde{L}_0)\phi = \psi \quad \text{ in } \mathbb{R}^n.$$

By Lemma 2.3.3, for all test function ψ in $C_0^\infty(\mathbb{R}^n)$ we can choose ϕ in $S(\mathbb{R}^n)$ such that

$$(e^{i\theta_\infty} - \tilde{L}_0)\phi = \psi \quad \text{ in } \mathbb{R}^n.$$

and substitute ψ for $(e^{i\theta_\infty} - \tilde{L}_0)\phi$ in the limit equation, then we get

$$\int_{\mathbb{R}^n} w\psi dx = 0 \text{ for all test functions } \psi \in C_0^\infty(\mathbb{R}^n).$$

By the fundamental lemma of calculus of variation, we get the uniqueness result $w = 0$. □

Case (ii) $\tilde{d} = \liminf_{k\to\infty} d_k = \liminf_{k\to\infty} |\lambda_k|^{\frac{1}{2m}} d(x_k, \partial\Omega) < \infty$.

In this case, we consider the case that Ω_k tends to $\mathbb{R}^n_+$.

Lemma 2.3.5. *The limit w is a weak solution of the limit resolvent equation*

$$\begin{cases}(e^{i\theta_\infty}-\tilde{L_0})w=0 & \text{in } \mathbb{R}^n_+\\ w=\partial_\nu w=\cdots=\partial_\nu^{m-1}w=0 & \text{on } \partial\mathbb{R}^n_+.\end{cases}$$

Proof. By $\tilde{d}=\liminf_{k\to\infty} d_k=\liminf_{k\to\infty}|\lambda_k|^{\frac{1}{2m}}d(x_k,\partial\Omega)<\infty$, there exists a subsequence $\{d_{k_l}\}$ of $\{d_k\}$ such that $\lim_{l\to\infty} d_{k_l}=\tilde{d}$. Since rotations and translations preserve the ellipticity and Ω is uniformly C^1, without loss of generality, we may assume that the perpendicular from x_{k_l} to $\partial\Omega$ coincides with the x_n-axis, i.e., we may assume Ω_{k_l} tends to $\mathbb{R}^n_{\tilde{d}}$ as $l\to\infty$ where $\mathbb{R}^n_{\tilde{d}}$ is a half space $\mathbb{R}^n_{\tilde{d}}=\{x=(x_1,\cdots,x_n)\in\mathbb{R}^n : x_n>-\tilde{d}\}$. By parallel translations, we can treat $\mathbb{R}^n_{\tilde{d}}$ as $\mathbb{R}^n_+$, and $\tilde{N}(w)(0,\cdots,0,\tilde{d})\geq\frac{1}{2}$ in this situation. Let y_l be the intersection point of the perpendicular from x_{k_l} and $\partial\Omega$. For each smooth test function $\eta\in C_0^\infty(\mathbb{R}^n_+)$, $d(\text{supp}\eta,\partial\mathbb{R}^n_+)=d_\eta>0$. Since y_l tends to $0\in\overline{\mathbb{R}^n_+}$, we can take large $R_\eta>0$ so that for large l $B_{R_\eta}(y_l)\supset$supp η. By Lemma 2.3.1, Ω is a $(\omega_\Omega(R_\eta),\, R_\eta)$ Reifenberg flat domain. Since $\Omega_l=|\lambda_l|^{\frac{1}{2m}}(x_l-\Omega)$, Ω_l is a $(\omega_\Omega(\frac{R_\eta}{|\lambda_l|^{\frac{1}{2m}}}),\, R_\eta)$ Reifenberg flat domain, by the Reifenberg flat condition,

$$B_{R_\eta}(y_l)\cap\Omega_k\supset B_{R_\eta}(y_l)\cap\{x_n>\omega_\Omega(\frac{R_\eta}{|\lambda_l|^{\frac{1}{2m}}})R_\eta\}$$

Since $\omega_\Omega(|h|)\to 0$ as $|h|\to 0$, we can take large l_0 so that for $l\geq l_0$

$$\begin{aligned}B_{R_\eta}(y_l)\cap\Omega_k &\supset B_{R_\eta}(y_l)\cap\{x_n>d_\eta\}\\ &\supset B_{R_\eta}(y_l)\cap\text{supp }\eta.\end{aligned}$$

As a consequence, for $l_0\leq l$ supp η is included in Ω_l. So, we can apply a similar argument as in case (i), w is a weak solution of the limit resolvent equation

$$\begin{cases}(e^{i\theta_\infty}-\tilde{L_0})w=0 & \text{in } \mathbb{R}^n_+\\ w=\partial_\nu w=\cdots=\partial_\nu^{m-1}w=0 & \text{on } \partial\mathbb{R}^n_+,\end{cases}$$

that is, for all $\phi\in C_0^\infty(\mathbb{R}^n_+)$ w satisfies

$$e^{i\theta_\infty}(w,\phi)_{L^2(\mathbb{R}^n_+)}+\sum_{|\alpha|,|\beta|=m}\tilde{a}_{\alpha,\beta}(\partial^\alpha w,\partial^\beta\phi)_{L^2(\mathbb{R}^n_+)}=0.$$

□

We can solve this dual problem by the general solvability results for example found in the book of H. Tanabe [Tan97], but, in our case, we can solve our dual problem explicitly.

Lemma 2.3.6 (Dual problem when Ω_∞ is the half space $\mathbb{R}^n_+$)**.** *Let ψ be in $C_0^\infty(\mathbb{R}^n_+)$. Then there exists a solution ϕ in $S(\mathbb{R}^n_+)$ such that*

$$\begin{cases}(e^{i\theta_\infty}-\tilde{L_0})\phi=\psi & \text{in } \mathbb{R}^n_+\\ \phi=\partial_\nu\phi=\cdots=\partial_\nu^{m-1}\phi=0 & \text{on } \partial\mathbb{R}^n_+.\end{cases}$$

Proof. By the partial Fourier transform with respect to the variable $x' = (x_1, \cdots, x_{n-1})$ in $\mathbb{R}^{n-1}$, we get the ordinary differential equation of $\hat{\phi}$ for $\alpha = (\alpha', \alpha_n)$, $\beta = (\beta', \beta_n)$,

$$\begin{cases} (e^{i\theta_\infty} - (-1)^{m+1} \sum_{|\alpha|,|\beta|=m} \tilde{a}_{\alpha,\beta}(i\xi)^{\alpha'+\beta'} \partial^{\alpha_n+\beta_n})\hat{\phi} = \hat{\psi} & \text{in } \mathbb{R}^n_+ \\ \hat{\phi} = \partial_\nu \hat{\phi} = \cdots = \partial_\nu^{m-1}\hat{\phi} = 0 & \text{on } \mathbb{R}^{n-1} \times \{0\}. \end{cases}$$

First of all, let ψ_0 be the zero extension of ψ from $\mathbb{R}^n_+$ to $\mathbb{R}^n$, and consider the whole space case. This case is already solved in Lemma 2.3.3 and we get a solution

$$\phi_0 = F^{-1}\left(\frac{1}{e^{i\theta_\infty} + \sum_{|\alpha|,|\beta|=m} \tilde{a}_{\alpha,\beta}\xi^{\alpha+\beta}}\right) * \psi_0 \in S(\mathbb{R}^n).$$

Let $h(x') = \phi_0(x', 0) \in S(\mathbb{R}^{n-1})$, and define $\eta = \phi - \phi_0$. We next consider the following boundary value problem:

$$\begin{cases} (e^{i\theta_\infty} - (-1)^{m+1} \sum_{|\alpha|,|\beta|=m} \tilde{a}_{\alpha,\beta}(i\xi)^{\alpha'+\beta'} \partial^{\alpha_n+\beta_n})\hat{\eta} = 0 & \text{in } \mathbb{R}^n_+ \\ \partial_\nu^k \hat{\eta}(x', 0) = -\partial_\nu^k \hat{h}(x') & \text{on } \partial\mathbb{R}^n_+, k = 0, 1, \cdots, m-1. \end{cases}$$

We want to determine the characteristic roots of this ODE. The characteristic equation is

$$e^{i\theta_\infty} - (-1)^{m+1} \sum_{|\alpha|,|\beta|=m} \tilde{a}_{\alpha,\beta}(i\xi)^{(\alpha_1+\beta_1,\cdots,\alpha_{n-1}+\beta_{n-1},0)} t^{\alpha_n+\beta_n} = 0.$$

Since L_0 is strongly uniformly elliptic,

$$e^{i\theta_\infty} - (-1)^{m+1} \sum_{|\alpha|,|\beta|=m} \tilde{a}_{\alpha,\beta}(i\xi)^{(\alpha_1+\beta_1,\cdots,\alpha_{n-1}+\beta_{n-1},0)} (is)^{\alpha_n+\beta_n} \neq 0 \text{ for } s \in \mathbb{R},$$

$t = is$ are not characteristic roots for $s \in \mathbb{R}$. By the root condition of $e^{i\theta_\infty} - \tilde{L_0}$, we get the characteristic roots with nonzero real part

$$t_j = -p_j + iq_j,\ p_{j+m} + iq_{j+m} \text{ for } 1 \le j \le m, \text{ and } p_j > 0,\ q_j \in \mathbb{R}.$$

So, we take $t_j = -p_j + iq_j$ for $1 \le j \le m$ and let $z_j \ge 1$ be the multiplicity of t_j for $1 \le j \le N(m)$, then $\sum_{j=1}^{N(m)} z_j = m$.

In this notation, $\hat{\eta}(\xi', x_n) = \sum_{j=1}^{N(m)} \sum_{k=1}^{z_j} c_{k,j}(\xi')(x_n)^{k-1} e^{t_j(\xi')x_n}$ is a general solution which belongs to $S(\mathbb{R}_+)$ with respect to x_n. If $z_j = 1$ for all j, then $N = m$ and by the boundary condition we get

$$\hat{\eta}(\xi', 0) = \sum_{j=1}^{m} c_j(\xi') = -\hat{h}(\xi') \in S(\mathbb{R}^{n-1}).$$

$$(\partial_\nu^l \hat{\eta})(\xi', 0) = \sum_{j=1}^{m} c_j t_j(\xi')^l = -(\partial_\nu^l \hat{h})(\xi') \in S(\mathbb{R}^{n-1}) \text{ for all } l \leq m-1.$$

$$AC = \begin{pmatrix} 1 & \cdots & 1 \\ t_1 & \cdots & t_m \\ \vdots & \vdots & \vdots \\ t_1^m & \cdots & t_m^m \end{pmatrix} \begin{pmatrix} c_1 \\ \vdots \\ \vdots \\ c_m \end{pmatrix} = \begin{pmatrix} -\hat{h}(\xi') \\ \vdots \\ \vdots \\ -(\partial_\nu^l \hat{h})(\xi') \end{pmatrix}.$$

Since A is a Vandermonde matrix and $t_i \neq t_j$, A^{-1} exists and

$$\begin{pmatrix} c_1 \\ \vdots \\ c_m \end{pmatrix} = -A^{-1} \begin{pmatrix} \hat{h}(\xi') \\ \vdots \\ (\partial_\nu^l \hat{h})(\xi') \end{pmatrix}.$$

Thus, we get the explicit form of c_j and

$$\phi = \phi_0 + \sum_{j=1}^{m} F_{\xi'}^{-1}(c_j(\xi') e^{t_j(\xi') x_n}).$$

Since we obtain a pointwise decay of kernels at large $|x|$ by the argument of [Tan97, Lemma 4.5], and since $\hat{h} \in S(\mathbb{R}_+^n)$ is smooth and rapidly decreasing, we can show that the term $F_{\xi'}^{-1}(c_j(\xi') e^{t_j(\xi') x_n})$ is in $S(\mathbb{R}_+^{n-1})$ by similar arguments seen in [Tan97, Theorem 4.4, Lemma 4.5], and we get the desired solution. If $z_j > 1$ for some j, although the matrix used for calculations of $c_{k,j}$ is different from A, we can argue similarly as in the case when $z_j = 1$ for all j, see [Tan97, Chapter 4]. $\square$

Lemma 2.3.7. *Let $|\theta_\infty| \leq \pi - \epsilon$, $w \in W_{0,loc}^{m,p}(\mathbb{R}_+^n) \cap L^\infty(\mathbb{R}_+^n)$ be a weak solution of the resolvent limit equation*

$$\begin{cases} (e^{i\theta_\infty} - \tilde{L}_0) w = 0 & \text{in } \mathbb{R}_+^n \\ w = \partial_\nu w = \cdots = \partial_\nu^{m-1} w = 0 & \text{on } \partial\mathbb{R}_+^n, \end{cases}$$

that is, for all $\phi \in C_0^\infty(\mathbb{R}_+^n)$ $\tilde{w}$ satisfies

$$e^{i\theta_\infty}(\tilde{w}, \phi)_{L^2(\mathbb{R}_+^n)} + \sum_{|\alpha|,|\beta|=m} \tilde{a}_{\alpha,\beta}(\partial^\alpha \tilde{w}, \partial^\beta \phi)_{L^2(\mathbb{R}_+^n)} = 0.$$

Then, $w = 0$.

Proof. By a similar argument as in the proof of Lemma 2.3.4, we can prove this lemma by solvability of the dual problem. $\square$

As a consequence, we obtain the uniqueness of limit equation $w = 0$ which contradicts $|\tilde{N}(w)(0, \cdots, 0, \tilde{d})| \geq \frac{1}{2}$ in the compactness step. This contradiction implies a priori resolvent estimates. The proof of Theorem 2.2.1 is now complete. $\square$

2.4 Construction of weak solutions

In this section, we prove the sectoriality of L, i.e., we obtain uniqueness and existence of the resolvent equation by a priori estimates. Our construction of weak solutions is based on an approximation method given in the book of A. Lunardi [Lun95]. There, strong solutions of resolvent equations are constructed when the operator is a non divergence type operator. We recall the definition of a sectorial operator.

Definition 2.4.1. *Let X be a complex Banach space with norm $\|\cdot\|$ and $L : D(L) \subset X \to X$ be a linear operator, with not necessarily dense domain. Then, we say L is sectorial if there exist constants $\omega \in \mathbb{R}, 0 \leq \epsilon \leq \frac{\pi}{2}$, $C > 0$ such that*

(i) The resolvent set $\rho(L)$ of L contains

$$S_{\pi-\epsilon,\omega} = \{\lambda \in \mathbb{C} : \ \lambda \neq \omega, \ |\arg(\lambda - \omega)| < \pi - \epsilon\},$$

(ii)

$$\|(\lambda - L)^{-1}\|_{L(X)} \leq \frac{C}{|\lambda - \omega|} \ for \ all \ \lambda \in S_{\pi-\epsilon,\omega}.$$

Theorem 2.4.1. *Let $\Omega \subset \mathbb{R}^n$ be a uniformly C^1 domain, $p > n$. Assume that L satisfies the conditions (E1), (E2), and (E3). Then, for $\kappa_L < \varepsilon < \frac{\pi}{2}$ there exists $M > 0$ such that for $\lambda \in \Sigma_{\pi-\varepsilon} \cap \{|z| \geq M\}$, $f \in L^\infty(\Omega)$ there exists a unique solution $u_\lambda \in W^{m,p}_{0,loc}(\Omega) \cap W^{m-1,\infty}(\Omega)$ of the resolvent equation (1).*

Proof. First of all, we use an approximation method to get a weak solution of the resolvent equation. For $k \in \mathbb{N}$, let $\phi_k \in C_0^\infty(\mathbb{R}^n)$ be a cut off function such that

$$0 \leq \phi_k \leq 1, \ \phi_k = \begin{cases} 1 & \text{in } B_k(0) \\ 0 & \text{outside of } B_{2k}(0). \end{cases}$$

For arbitrary $f \in L^\infty(\Omega)$, we define $f_k = \phi_k f \in L^p(\Omega)$ $(1 \leq p \leq \infty)$. Since $W^{1,\infty}(\overline{\Omega}) \subset C^{0,\mu}(\overline{\Omega}) \cap L^\infty(\overline{\Omega}) \subset C_{bu}(\overline{\Omega})$, (E1), (E2) imply (M1), (M2). Thus we can apply Theorem 2.3.1, then we get a weak solution $u_k \in W_0^{m,p}(\Omega)$ of

$$\begin{cases} (\lambda - L)u_k = f_k & \text{in } \Omega \\ u_k = \partial_\nu u_k = \cdots = \partial_\nu^{m-1} u_k = 0 & \text{on } \partial\Omega, \end{cases}$$

with the estimate $\|u_k\|_{W^{m,p}(\Omega)} \leq C_p \|f_k\|_{L^p(\Omega)}$. Since $W_0^{m,p} \hookrightarrow W^{m-1,\infty}$ by Sobolev embedding type theorems, u_k is in $W^{m,p}_{0,loc}(\Omega) \cap W^{m-1,\infty}(\Omega)$. So, we can apply Theorem 2.2.1 to

$$\begin{cases} (\lambda - L)u_k = f_k & \text{in } \Omega \\ u_k = \partial_\nu u_k = \cdots = \partial_\nu^{m-1} u_k = 0 & \text{on } \partial\Omega. \end{cases}$$

We obtain

$$N(u_k,\lambda) \le C\|f_k\|_{L^\infty} \le C\|f\|_{L^\infty}.$$

Therefore, $\{u_k\} \subset W^{m,p}_{0,loc}(\Omega) \cap W^{m-1,\infty}(\Omega)$ is uniformly bounded. Furthermore, we also can show that $\{u_k\} \subset W^{m,p}(M)$ is uniformly bounded for each compact set $M \subset \overline{\Omega}$ by almost the same way as in the compactness step. So, there exist a subsequence $\{u_{k_j}\}$ in $W^{m-1,\infty}(\Omega) \cap C^{m-1,-\frac{n}{p}+1}(\Omega)$ and a limit function u such that

$$u_{k_j} \to u \in W^{m-1,\infty}(M) \cap C^{m-1,-\frac{n}{p}+1}(M) \text{ uniformly on each compact set } M$$

with the estimate $N(u,\lambda) \le C\|f\|_{L^\infty}$. We are going to show that u is in $W^{m,p}_{0,loc}(\Omega)$ and is a weak solution of

$$\begin{cases} (\lambda - L)u = f & \text{in } \Omega \\ u = \partial_\nu u = \cdots = \partial_\nu^{m-1} u = 0 & \text{on } \partial\Omega. \end{cases}$$

We fix any closed ball $B_R(0)$. We can apply the similar argument seen in Proposition 2.3.1, then $\{u_{k_j}\} \subset W^{m,p}(B_R(0) \cap \Omega)$ is uniformly bounded. This implies $u \in W^{m,p}(B_R(0) \cap \Omega)$. Since R is arbitrarily large, u is in $W^{m,p}_{loc}(\Omega)$. Furthermore, for each smooth function $\phi \in C_0^\infty(\mathbb{R}^n)$ $\phi u_k \in W^{m,p}_0(\Omega)$. Since ϕ has compact support, ϕu_k converges to ϕu in $W^{m,p}_0(\Omega)$. So $u \in W^{m,p}_{0,loc}(\Omega)$. Take large $j, l \in \mathbb{N}$, then we get

$$\begin{cases} (\lambda - L)(u_{k_j} - u_{k_l}) = 0 & \text{in } B_R(0) \cap \Omega \\ u_{k_j} - u_{k_l} = \partial_\nu (u_{k_j} - u_{k_l}) = \cdots = \partial_\nu^{m-1}(u_{k_j} - u_{k_l}) = 0 & \text{on } \partial(B_R(0) \cap \Omega). \end{cases}$$

By cut off method and Theorem 2.3.1 we get the local $W^{m,p}$ estimate,

$$\|u_{k_j} - u_{k_l}\|_{W^{m,p}(B_{\frac{R}{2}}(0)\cap\Omega)} \le C(\lambda)\|u_{k_j} - u_{k_l}\|_{m-1,\infty} \to 0 \quad (j,l \to \infty).$$

Thus, we get

$$u_{k_j} \to u \quad \text{in } W^{m,p}(B_{\frac{R}{2}}(0) \cap \Omega).$$

For each fixed test function $\eta \in C_0^\infty(\Omega)$ take large R so that $B_{\frac{R}{2}}(0) \supset \operatorname{supp} \eta$

$$\begin{aligned} &|(a_{\alpha,\beta}\partial^\alpha u_{k_l}, \partial^\beta \eta)_{L^2(\Omega)} - (a_{\alpha,\beta}\partial^\alpha u, \partial^\beta \eta)_{L^2(\Omega)}| \\ &\le |(a_{\alpha,\beta}(\partial^\alpha u_{k_l} - \partial^\alpha u), \partial^\beta \eta)_{L^2(\operatorname{supp}\eta)}| \\ &\le C\|u_{k_l} - u\|_{W^{m,p}(\operatorname{supp}\eta)} \to 0. \end{aligned}$$

This implies the convergence of the main term, and other terms can be dealt with similarly. So, as j tends to ∞, the resolvent equation tends to the equation $(\lambda - L)u = f$ in Ω in weak sense. Thus, we get a weak solution of the resolvent equation. We get the uniqueness by linearity of the resolvent equation and L^∞ a priori estimates. $\square$

Theorem 2.2.1 and Theorem 2.4.1 imply that L is sectorial. Let us define e^{tL} as follows;

$$\begin{cases} e^{0L}x = x & \text{for all } x \in L^\infty(\Omega) \\ e^{tL}x = \dfrac{1}{2\pi i}\displaystyle\int_\gamma e^{t\lambda}(\lambda - L)^{-1}x d\lambda & \text{for all } t > 0 \text{ and } x \in L^\infty(\Omega), \end{cases}$$

where $r > M > 0$, $\theta < \epsilon$, and γ is the curve

$$(\partial\Sigma_{\pi-\theta} \cap \{|\lambda| \geq r\}) \cup \{\lambda \in \mathbb{C} : |\arg\lambda| \geq \pi - \theta, |\lambda| = r\},$$

oriented counter clockwise. Since L is sectorial, e^{tL} forms an analytic semigroup in $L^\infty(\Omega)$ by applying abstract semigroup theories. So the proof of Theorem 2.2.2 is complete.

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Chapter 3

On the Stokes resolvent estimate in spaces of bounded functions

3.1 Introduction and main results

It is important to consider the Navier-Stokes equations in various types of domains since there is a huge variety of domains that a fluid might occupy. The analysis of the Stokes equations, a linearized version of the Navier-Stokes equations is fundamental. Especially, analyticity of the Stokes semigroup $S(t)$, the solution operator of the Stokes equation

$$\begin{cases} v_t - \Delta v + \nabla q = 0 & \text{in} \quad \Omega \times (0,T), \\ \operatorname{div} v = 0 & \text{in} \quad \Omega \times (0,T), \\ v = 0 & \text{on} \quad \partial\Omega \times (0,T), \\ v|_{t=0} = v_0 & \text{on} \quad \Omega, \end{cases}$$

i.e., $S(t)v_0 = v(\cdot,t)$, is important to measure a regularizing effect of the Stokes flow, where Ω is a uniformly C^2 domain in $\mathbf{R}^n$ $(n \geq 2)$. Here we impose the Dirichlet boundary condition to fix the idea.

It is well-known that $S(t)$ forms an analytic semigroup in $L^p_\sigma(\Omega)$ $(1 < p < \infty)$ for various kind of domains Ω including smoothly bounded domains [Gig81], [Sol77], where $L^p_\sigma = L^p_\sigma(\Omega)$ is the L^p-closure of $C^\infty_{c,\sigma}(\Omega)$, the space of all solenoidal vector fields with compact support in Ω. By now analyticity results are known for various type of unbounded domain not necessarily an exterior domain. For example, L^p-analyticity is proved for a layer domain [AS03], an aperture domain [FS96]. Moreover, these results are extended for the case of variable viscosity coefficients [AT09], [Abel10]. In fact, the analyticity of $S(t)$ in $L^p_\sigma(\Omega)$ holds for any uniformly C^2 domain provided that $L^p(\Omega)$ admits a topological direct sum decomposition, called the Helmholtz decomposition

[GHHS12]. It is also known that $S(t)$ forms analytic semigroup for any uniformly C^2 domain if one considers $\tilde{L}^p$ spaces, i.e., $\tilde{L}^p_\sigma = L^p_\sigma \cap L^2_\sigma$ (for $p > 2$) and $\tilde{L}^p_\sigma = L^p_\sigma + L^2_\sigma$ ($1 < p \leq 2$) as developed by Farwig, Kozono and Sohr [FKS05], [FKS07], [FKS09].

The Helmholtz decomposition holds for any domain if $p = 2$ and for various kind of domains like bounded or exterior domains with smooth boundary for $1 < p < \infty$ [Gal94]. However, for any $p > 2$ there is an improper smooth sector-like domain such that the L^p-Helmholtz decomposition fails to satisfy [Bog86], [MB86].

Furthermore,, if one considers the case $p = \infty$, the results are still limited since the Helmholtz projection is not bounded. If $\Omega = \mathbf{R}^n_+$, it is proved in [DHP01], [Sol03] that $S(t)$ forms an analytic semigroup in $C_{0,\sigma} = C_{0,\sigma}(\Omega)$, the L^∞-closure of $C^\infty_{c,\sigma}(\Omega)$ based on an explicit representation formula. For a general domain it is proved [AG13] that the semigroup is analytic in $C_{0,\sigma}$ provided that the domain is "admissible" in the sense of [AG13]. Since it turns out that a bounded domain [AG13] and an exterior domain [AG14] are admissible (even strictly admissible), we conclude that $S(t)$ forms an analytic semigroup in such a domain; for improvement of these results see [AGH15] where only C^2 regularity is used.

Recently, it turns out that $S(t)$ does not form an analytic semigroup on $C_{0,\sigma}(\Omega)$ in a layer domain $\{(x', x_n) \in \mathbf{R}^n \mid a < x_n < b\}$, $a, b \in \mathbf{R}$, $a < b$ provided that $n \geq 3$ as proved by von Below [Bel14]; in the case $n = 2$ he proved the analyticity of $S(t)$. His result in particular implies that a layer domain ($n \geq 3$) is not admissible. Since a layer domain allows an L^r-Helmholtz decomposition [Miy94], the Helmholtz decomposition does not imply admissibility. On the other hand, there is a planar non Helmholtz domain which is admissible so admissibility and Helmholtz domain is a different notion.

The goal of this chapter is to prove that the Stokes semigroup forms an analytic semigroup in $L^p_\sigma(\Omega)$ for a C^3 sector-like domain for which the L^p-Helmholtz decomposition may fail. We also would like to show that a cylindrical domain including a two-dimensional layer domain is admissible in the sense of [AG13]. By a cylindrical domain we mean there are finitely many outlets which are a half part of infinite cylinder. An infinite cylinder $\mathbf{R} \times D$ with a bounded domain $D \subset \mathbf{R}^{n-1}$ is a typical example of a cylindrical domain. For this purpose, we consider the Neumann problem of the Laplace equation of the form

$$\begin{cases} \Delta u = 0 & \text{in} \quad \Omega, \\ \frac{\partial u}{\partial n_\Omega} = \operatorname{div}_{\partial\Omega} g & \text{on} \quad \partial\Omega, \end{cases} \tag{3.1.1}$$

where n_Ω is the exterior unit normal of $\partial\Omega$ and $\operatorname{div}_{\partial\Omega}$ is the surface divergence [Gig06]. Admissibility (see Section 3.2.3) easily follows from the next weighted L^∞-estimate for the gradient of a solution of (3.1.1).

Theorem 3.1.1. *Let Ω be a C^2 sector-like domain or a C^2 cylindrical domain in $\mathbf{R}^n$ $(n \geq 2)$. Then there exists a constant C such that*

$$\|d_\Omega \nabla u\|_{L^\infty(\Omega)} \leq C\|g\|_{L^\infty(\partial\Omega)} \tag{3.1.2}$$

holds for all weak solution u of (3.1.1) with $\nabla u \in L^2(\Omega) \cap L^r(\Omega)$ for some $r \geq n$, where $g \in C(\partial\Omega) \cap L^\infty(\partial\Omega)$ with $\mathrm{div}_{\partial\Omega}\, g \in C(\partial\Omega)$ satisfies $g \cdot n_\Omega = 0$ on $\partial\Omega$. Here $d_\Omega(x)$ is the distance from $x \in \Omega$ to the boundary $\partial\Omega$.

We shall prove this result in Section 3.4 for sector-like domains, Theorem 3.2.2 for an infinite layer and in Theorem 3.3.2 for a general cylindrical domain.

This type of estimate is first proved in [AG13] for a C^3-bounded domain and a half space. For a C^3 exterior domain, a similar estimate is proved in [AG14]. In both cases we need not assume that $\nabla u \in L^2(\Omega) \cap L^r(\Omega)$. We only need to assume that u is a (very) weak solution. Moreover, we need not assume the regularity of g. If (3.1.2) holds for all very weak solutions u having finite left-hand side of (3.1.2), we say that Ω is *strictly admissible* [AG14]. Note that an infinite cylinder is not strictly admissible because a linear function $u(x_1, x') = x_1$ solves (3.1.1) with $g = 0$. The estimate (3.1.2) is independently proved by Kenig, Lin and Shen [KLS13] for a bounded $C^{1,\gamma}$ domain for their study on homogenization of the Neumann problem.

Let us explain our idea of the proof of Theorem 3.1.1 when Ω is an infinite layer. We derive an estimate for $d_\Omega \nabla u$ in a domain $\Omega(S) = (-S, S) \times D$ uniformly in $S \geq 1$ of the form

$$d_{\Omega(S)}(x)\,|\nabla u(x)| \leq C \left(\|g\|_{L^\infty((-S,S)\times\partial D)} + \sum_{x_1 = \pm S} \left\| \frac{\partial u}{\partial x_1} \right\|_{L^{n-1}(D)} (x_1) \right), \tag{3.1.3}$$

where $\|f\|^p_{L^{n-1}(D)}(x_1) = \int_D |f(x_1, x')|^p \, dx'$ for $f = f(x_1, x')$. We shall establish the estimate (3.1.3) by a contradiction argument and derive a contradiction with uniqueness result of the Neumann problem under no flux condition. As we proved in Theorem 3.2.1, a solution u of (3.1.1) with $g = 0$ satisfying $d_\Omega \nabla u \in L^\infty(\Omega)$ must be constant provided that there is no flux $\int_{x_1 = R} \frac{\partial u}{\partial x_1} dx' = 0$. The no flux condition is essential since otherwise $u = x_1$ is a nontrivial solution which breaks the uniqueness. We shall prove such uniqueness essentially by strong maximum principles. Theorem 3.1.1 formally follows from (3.1.3) by sending $S \to \infty$ since $\nabla u \in L^2(\Omega)$. For rigorous argument, see the proof of Theorem 3.2.2.

Once admissibility has been established, we have analyticity of the semigroup $S(t)$ by applying the main result of [AG13] if Ω is C^3. If one applies the resolvent estimate [AGH15], we only need C^2 regularity. Although it is in fact possible to extend the arguments outlined in [AGH15] to admissible domains, we will establish the resolvent estimate for general admissible domains without appealing Masuda-Stewart arguments that were used in [AGH15]. We apply a blow-up argument to derive necessary resolvent estimates. We consider the resolvent equation

$$\begin{cases} (\lambda - \Delta)v + \nabla p &= f \quad \text{in} \quad \Omega, \\ \mathrm{div}\, v &= 0 \quad \text{in} \quad \Omega, \\ v &= 0 \quad \text{on} \quad \partial\Omega, \end{cases} \tag{3.1.4}$$

with $\|d_\Omega \nabla p\|_\infty < \infty$, where $\lambda \in \Sigma_{\pi-\varepsilon}$ for a fixed $\varepsilon > 0$. Here $\Sigma_\varphi = \{\lambda \in \mathbf{C} \mid |\arg \lambda| < \varphi\}$. We set

$$N(v,\lambda) = |\lambda|\,|v(x)| + |\lambda|^{1/2}\,|\nabla v(x)|\,. \tag{3.1.5}$$

Theorem 3.1.2. *Let Ω be an admissible, uniformly C^2 domain in $\mathbf{R}^n$. For $\varepsilon \in (0,\pi/2)$ there exists a constant C and M (independent of f and λ) such that*

$$\|N(v,\lambda)\|_{L^\infty(\Omega)} \leq C\|f\|_{L^\infty(\Omega)}$$

for all $f \in C^\infty_{c,\sigma}(\Omega)$ and $\tilde{L}^r$-solution v ($r > n$) of (3.1.4) with $\lambda \in \Sigma_{\pi-\varepsilon}$ and $|\lambda| \geq M$.

As we mentioned before, we appeal to a blow-up argument to prove Theorem 3.1.2 which was developed by the last author [Suz16] for higher order elliptic problems under C^1-regularity of Ω. Since we have to control the pressure from vorticity, the present method seems to need C^2-regularity of Ω. The key equation to control the pressure from (3.1.4) is

$$\Delta p = 0 \quad \text{in} \quad \Omega, \quad \frac{\partial p}{\partial n_\Omega} = -\operatorname{div}_{\partial\Omega}(\omega \times n_\Omega) \quad \text{on} \quad \partial\Omega,$$

where $\omega = \operatorname{curl} v$ is the vorticity. Here we assume $n = 3$ for simplicity. The harmonicity of p is clear by taking the divergence of (3.1.4) while the boundary condition is obtained by taking inner product of (3.1.4) with n_Ω and use $n_\Omega \cdot \Delta v = -\operatorname{div}_{\partial\Omega}(\omega \times n)$ when $\operatorname{div} v = 0$. This relation may be well known but it is often implicit e.g. [JL04, (1.3b)]. This pressure is sometimes called the Stokes pressure [LLP07] because it reflects the effect of viscosity.

Theorem 3.1.2 is considered as an extension of the main resolvent estimate [AGH15, Theorem 1.1] where the strict admissibility of Ω is assumed. As in [AGH15] one is able to assert that the Stokes operator A defined as in [AGH15] generates a C_0-analytic semigroup $S(t)$ on $C_{0,\sigma}(\Omega)$ of angle $\pi/2$. We shall state this result which follows from Theorem 3.1.2 as in [AGH15].

Corollary 3.1.1. *Let Ω be an admissible, uniformly C^2 domain in $\mathbf{R}^n$. Then the Stokes semigroup $S(t)$ forms an analytic semigroup on $C_{0,\sigma}(\Omega)$ of angle $\pi/2$. In particular, for a C^2 cylindrical domain this analyticity holds.*

The analyticity in $C_{0,\sigma}(\Omega)$ is one of key tools to estimate the lifespan from below of a solution to the Navier-Stokes equations starting from L^∞ type data as shown in Abe [A15] for bounded or exterior domains.

Note that the extension to L^∞_σ space is nontrivial in our setting. It is not clear whether or not the approximate limit is uniquely defined since its pressure is not well-controlled which is different from the case of a strictly admissible domain.

This type of result as well as [AG13], [AG14], [AGH15] is concerned with regularizing effect locally-in-time. The boundedness of the Stokes semigroup $S(t)$ for $t > 0$ is a different question. For a bounded domain, the semigroup is even exponentially decaying by the Poincaré estimate [AG13]. For exterior domains the global boundedness of $S(t)$ was proved in [Mar14], which was extended to a global time derivative estimate in [HM16]. Moreover, both results are even further extended to global boundedness in sectors of angle less than $\pi/2$ in [BH15]. For the case of the half space, these results are already established in [Sol03] and [DHP01].

This chapter is organized as follows. In Section 2 we establish (3.1.3) and prove Theorem 3.1.1 for an infinite cylinder to clarify the idea. In Section 3 we extend this idea to a cylindrical domain. In Section 4 we establish the admissibility of a sector-like domain. In Section 5 we prove Theorem 3.1.2 by a blow-up argument as well as Corollary 3.1.1.

Our method is a contradiction argument based on blow-up. There are four major steps to show our a priori estimate. Let us explain rough ideas of our methods.

The first step of our method is normalizing the Stokes resolvent equations. Assume that the L^∞ resolvent apriori estimate does not holds. Then, for each $C = M = k \in \mathbb{N}$, there are sequences $\{u_k\}_k,\ \{p_k\}_k,\ \{\lambda_k\}_k,\ \{f_k\}_k$ which satisfy (1) and $N(u_k, \lambda_k) > k\|f_k\|_{L^\infty(\Omega)}$. We set $v_k = |\lambda_k| u_k$ in order to deal with λ_k in (1), and we normalize sequences by $N(\frac{v_k}{|\lambda_k|}, \lambda_k)$. Then we obtain normalized Stokes resolvent equations

$$\begin{cases} (e^{i\theta_k} - \frac{\Delta}{|\lambda_k|})\tilde{v_k} + \nabla \tilde{p_k} = \tilde{f_k} & \text{in } \Omega \\ \mathrm{div}\tilde{v_k} = 0 & \text{in } \Omega \\ \tilde{v_k} = 0 & \text{on } \partial\Omega \end{cases}$$

where $\{\tilde{v_k}\}_k,\ \{\tilde{p_k}\}_k,\ \{\tilde{f_k}\}_k$ are normalized sequences with $\frac{1}{k} > \|\tilde{f_k}\|_\infty$, $|\lambda_k| \geq k$, $|\theta_k| \leq \pi - \varepsilon$, and $N(\frac{\tilde{v_k}}{|\lambda_k|}, \lambda_k) = 1$.

The second step is rescaling the resolvent equations. Since $N(\frac{\tilde{v_k}}{|\lambda_k|}, \lambda_k) = 1$, there is a sequence points in Ω at which $|\tilde{v_k}| + |\lambda_k|^{-\frac{1}{2}}|\nabla\tilde{v_k}|$ takes a value close to 1. We rescale each normalized function at x_k. Then we get rescaled Stokes resolvent equations

$$\begin{cases} (e^{i\theta_k} - \Delta)\tilde{w_k} + \nabla \tilde{q_k} = \tilde{g_k} & \text{in } \Omega_k \\ \mathrm{div}\tilde{w_k} = 0 & \text{in } \Omega_k \\ \tilde{w_k} = 0 & \text{on } \partial\Omega_k \end{cases}$$

where $\{\tilde{w_k}\}_k,\ \{\tilde{q_k}\}_k,\ \{\tilde{g_k}\}_k$ are rescaled sequences with $\Omega_k = |\lambda_k|^{\frac{1}{2}}(\Omega - x_k)$, $\frac{1}{k} > \|\tilde{g_k}\|_\infty$, and $\sup_{x\in\Omega_k}(|\tilde{w_k}(x)| + |\nabla\tilde{w_k}(x)|) = 1$, $|\tilde{w_k}(0)| + |\nabla\tilde{w_k}(0)| > \frac{1}{2}$.

The third step is compactness of the rescaled sequences considered. In this step, we show the equicontinuity of sequence $\{\tilde{w_k}\}_{k\in\mathbb{N}}$ on some open neighborhood near the origin. We consider localized resolvent equation to get local $W^{2,r}$ estimates up to the boundary. By this estimates, the local $W^{2,r}$ norms of the rescaled strong solutions are estimated by $\|\tilde{g_k}\|_\infty < \frac{1}{k}$, and $\sup_{x\in\Omega_k}(|\tilde{w_k}(x)| + |\nabla\tilde{w_k}(x)|) = 1$, and some pressure terms. If $\tilde{d} = \liminf_{k\to\infty} d_k = \liminf_{k\to\infty}|\lambda_k|^{\frac{1}{2}} d(x_k, \partial\Omega) = \infty$, this pressure terms can be controlled easily. However, if $\tilde{d} < \infty$, we need the Neumann admissible condition and a Poincaré type inequality to control the pressure terms. By the local $W^{2,r}$ estimates, we get uniform control of the rescaled strong solutions. Therefore, there exists a subsequence $\{\tilde{w_{k_l}}\}_{l\in\mathbb{N}}$ of $\{\tilde{w_k}\}_{k\in\mathbb{N}}$ which converges to some function $\tilde{w}$ uniformly on some open neighborhood near the origin. In Particular,

$$|\tilde{w}(0)| + |\nabla \tilde{w}(0)| \geq \frac{1}{2}. \tag{3.1.6}$$

The final step is uniqueness of the limit resolvent equation considered. For each fixed $\eta \in C_0^\infty(\mathbb{R}^n)$ we need the convergence of

$$e^{i\theta_k}(\tilde{w_k}, \eta)_{L^2(\Omega_k)} + (\tilde{w_k}, \Delta\eta)_{L^2(\Omega_k)} + (\nabla \tilde{q_k}, \eta)_{L^2(\Omega_k)} = (\tilde{g_k}, \eta)_{L^2(\Omega_k)}.$$

We have to consider two cases as in the compactness step. If $\tilde{d} = \infty$, the rescaled domain Ω_k tends to the whole space $\mathbb{R}^n$. Fortunately, the sequence of $\{\nabla \tilde{q_k}\}_k$ tends to 0 by the Neumann admissible condition in this case. So, the limit equation is

$$e^{i\theta_\infty}(\tilde{w}, \eta)_{L^2(\mathbb{R}^n)} - (\tilde{w}, \Delta\eta)_{L^2(\mathbb{R}^n)} = 0,$$

where $|\theta_\infty| \leq \pi - \epsilon$. Considering the dual problem of this limit equation, we get the uniqueness result

$$w = 0,$$

If $\tilde{d} < \infty$, by appropriate rotations and translations, the rescaled domain Ω_k tends to the half space $\mathbb{R}^n_+$. Unfortunately, the situation is complicated in this case. The limit equation is the Stokes resolvent equation in weak sense on half space $\mathbb{R}^n_+$. Analyzing the dual Stokes resolvent problem by the Neumann admissible condition, we get the uniqueness result

$$w = 0,$$

which contradicts (3.1.6).

We would like to state the analyticity of the Stokes semigroup on a L^∞ type space. We begin with the definition of a cylindrical domain.

Definition 3.1.1. *A domain $\Omega \subset \mathbb{R}^n$ is a cylindrical domain if there is an $m \in \mathbb{N}$ such that $\Omega = \bigcup_{i=0}^m \Omega^i$ where Ω^0 is a bounded domain and Ω^i are disjoint semi-infinite cylinders.*

Our method to show the analyticity of the Stokes semigroups on L^∞ type spaces is divided into two parts, (i) the admissibility of a cylindrical domain for weighted gradient pressure estimates, (ii) Stokes resolvent estimates. When we consider Stokes resolvent estimates in L^∞ type spaces, we need a weighted gradient pressure estimate to avoid Poiseuille type flows. So, the admissibility of the domain is a crucial condition. Let P be the $\tilde{L}^p$ Helmholtz projection in [FKS05], [FKS07], [FKS09], $Q = I - P$, $d_\Omega(x) = \inf_{y\in\partial\Omega}|x-y|$.

Definition 3.1.2. *A domain $\Omega \subset \mathbb{R}^n$ is admissible if there exists $r \geq n$ and $C > 0$ such that*

$$\|d_\Omega Q[\nabla \cdot f](x)\|_\infty \leq C\|f\|_{L^\infty(\partial\Omega)}$$

holds for matrix-valued functions $f = (f_{ij}) \in C^1(\Omega)$ satisfying $\nabla \cdot f \in (L^r \cap L^2)(\Omega)$ with $trf = 0$ and $\partial_l f_{ij} = \partial_j f_{il}$ for $1 \leq i, j, l \leq n$.

$\nabla u = Q[\nabla \cdot f]$ satisfies the Neumann problem,

$$\Delta u = 0 \text{ in } \Omega,\ \frac{\partial u}{\partial n_\Omega} = \mathrm{div}_{\partial\Omega} g = \mathrm{div}_{\partial\Omega}(f - f^T) \cdot n \text{ on } \partial\Omega.$$

So, weighted estimates for Neumann problem imply the admissibility. Let Ω be a infinite cylinder $\mathbb{R} \times D \subset \mathbb{R}^n$ with a bounded C^2 domain $D \subset \mathbb{R}^{n-1}$ for the sake of simplicity, $(\partial\Omega)(S) = (-S, S) \times \partial D$ for $S \geq 0$, $\Omega(S) = (-S, S) \times D$.

Theorem 3.1.3 (Weighted estimate for Neumann problem)**.** *Let $h = div_{\partial\Omega} g \in C(\partial\Omega)$ be a boundary data with $g \cdot n_\Omega = 0$ on $\partial\Omega$, $g \in (C \cap L^\infty)(\partial\Omega)$. Then,*

$$\|d_\Omega \nabla u\|_\infty \leq C\|g\|_{L^\infty(\partial\Omega)}$$

holds for all $\dot{H}^1$ weak solutions $u \in L^1_{loc}(\Omega)$ with $\nabla u \in \tilde{L}^r(\Omega)$.

Weighted estimate for Neumann problem is obtained by a local weighted pointwise estimate of Neumann problem with L^2 decay of ∇u. We obtain a local weighted pointwise estimate by a contradiction argument with uniqueness theorem of Neumann problem.

Theorem 3.1.4 (Uniqueness of Neumann problem under no flux condition)**.** *Let u be a solution with zero boundary data, $\|d_\Omega \nabla u\|_\infty < \infty$, and $\int_D \frac{\partial u}{\partial x_1}(s_1, x')dx' = 0$ for some $s_1 \geq 0$. Then $u \equiv c$.*

Theorem 3.1.5 (Local weighted pointwise estimate of Neumann problem)**.** *There exists a constant $C > 0$ such that*

$$d_\Omega(x)|\nabla u(x)| \leq C(\|g\|_{L^\infty((\partial\Omega)(S))} + \Sigma_{x_1=\pm S}\|\frac{\partial u}{\partial x_1}\|_{L^{n-1}(D)}(x_1) \text{ for } x \in \Omega(S),\ S \geq 1,$$

holds for all $\dot{H}^1$ weak solutions u with $\nabla u \in L^2(\Omega(S))$, and a boundary data $h = \mathrm{div}_{\partial\Omega} g \in C(\partial\Omega(S))$, $h = \pm\frac{\partial u}{\partial x_1}$ at $x_1 = \pm S$, where $g \in C(\partial\Omega(S))$ satisfies $g \cdot n_\Omega = 0$ provided that $\|d_\Omega \nabla u\|_{L^\infty(\Omega(S))} < \infty$ and that $\frac{\partial u}{\partial x_1} \in L^{n-1}(D)$.

This admissibility implies a-priori L^∞ Stokes resolvent estimates. Let us consider the Stokes resolvent equation with Dirichlet condition under the weighted gradient pressure estimate.

$$\begin{cases} (\lambda - \Delta)u_\lambda + \nabla q_\lambda = f,\ \mathrm{div} u_\lambda = 0 & \text{in } \Omega \\ u_\lambda = 0 & \text{on } \partial\Omega. \end{cases}$$

$$N(u_\lambda,\lambda) = \||\lambda||u_\lambda| + |\lambda|^{\frac{1}{2}}|\nabla u_\lambda|\|_\infty,\ \Sigma_{\pi-\varepsilon} = \{|\arg\lambda| < \pi-\varepsilon\},$$

$L^\infty_d(\Omega) = \{f \in L^1_{loc}(\Omega); \|f\|_{L^\infty_d} < \infty\}$, where $d_\Omega(x) = \inf_{y\in\partial\Omega}|x-y|$ and $\|f\|_{L^\infty_d} = \sup_{x\in\Omega} d_\Omega(x)|f(x)|$. In the next section we explain outlines of our proof of a-priori L^∞ Stokes resolvent estimates.

Theorem 3.1.6 (Stokes resolvent estimate)**.** *Let $\Omega \subset \mathbb{R}^n$ be a uniformly C^2 admissible domain, $r > n$. Then, for $0 < \varepsilon < \frac{\pi}{2}$ there exist C, $M > 0$ such that for $f \in C^\infty_{c,\sigma}(\Omega)$, $\lambda \in \Sigma_{\pi-\epsilon}$ $|\lambda| \geq M$, $\tilde{L}^r$ solution u_λ the following estimate holds*

$$N(u_\lambda,\lambda) \leq C\|f\|_{L^\infty(\Omega)}.$$

Let Ω be a domain in $\mathbb{R}^n$. In this section, we consider the a priori L^∞ estimate for $\tilde{L}^r$ strong solutions of the resolvent Stokes equation with zero Dirichlet boundary condition in the sense of traces.

$$\begin{cases} (\lambda - \mathit{\Delta})u_\lambda + \nabla p_\lambda = f & \text{in } \Omega \\ \mathrm{div} u_\lambda = 0 & \text{in } \Omega \\ u_\lambda = 0 & \text{on } \partial\Omega. \end{cases} \tag{3.1.7}$$

Let us state the a priori L^∞ resolvent estimates. We define a norm $N(u_\lambda,\lambda)$, a sectorial region $\Sigma_{\pi-\varepsilon}$, and a function space $L^\infty_d(\Omega)$ by

$$N(u_\lambda,\lambda) = \sup_{x\in\Omega}(|\lambda||u_\lambda(x)| + |\lambda|^{\frac{1}{2}}|\nabla u_\lambda(x)|),$$

$$\Sigma_{\pi-\varepsilon} = \{\lambda \in \mathbb{C} : |\arg\lambda| < \pi-\varepsilon\},$$

$$L^\infty_d(\Omega) = \{f \in L^1_{loc}(\Omega); \|f\|_{L^\infty_d} < \infty\},$$

where $d_\Omega(x) = \inf_{y\in\partial\Omega}|x-y|$ and $\|f\|_{L^\infty_d} = \sup_{x\in\Omega} d_\Omega(x)|f(x)|$.

Theorem 3.1.7 (L^∞ a priori estimates)**.** *Let $\Omega \subset \mathbb{R}^n$ be a Neumann admissible domain with uniform C^2 boundary, and let $r > n$, $f \in \tilde{L}^r(\Omega) \cap L^\infty(\Omega)$. Then, for $0 < \varepsilon < \pi$ there exist $C > 0$, $M > 0$ such that*

$$N(u_\lambda,\lambda) \leq C\|f\|_{L^\infty(\Omega)},$$

for $\lambda \in \Sigma_{\pi-\varepsilon} \cap \{|z| \geq M\}$, and $(u_\lambda, \nabla p_\lambda) \in (\tilde{W}^{2,p}_\sigma \cap W^{1,\infty})(\Omega) \times (\tilde{L}^r \cap L^\infty_d)(\Omega)$ which is a strong solutions of the Stokes resolvent equation and satisfies $N(u_\lambda,\lambda) < \infty$ and $\|\nabla p_\lambda\|_{L^\infty_d(\Omega)} < \infty$.

3.2 Infinite cylinders

3.2.1 Uniqueness under no flux condition

We begin with a uniqueness result for the Neumann problem

$$-\Delta u = 0 \quad \text{in} \quad \Omega, \quad \frac{\partial u}{\partial n_\Omega} = 0 \quad \text{on} \quad \partial\Omega \tag{3.2.1}$$

in a C^2 infinite cylinder Ω in $\mathbf{R}^n$ which means in this section that $\Omega := \mathbf{R} \times D$ with a C^2 bounded domain D in $\mathbf{R}^{n-1}$ ($n \geq 2$).

Lemma 3.2.1. *Let $u \in C^2(\Omega) \cap C^1(\overline{\Omega})$ be a solution of* (3.2.1) *in a C^2 infinite cylinder Ω. Assume that u is bounded in Ω, i.e., $u \in L^\infty(\Omega)$. Then u is a constant function.*

Proof. We first observe that a flux

$$F(s) := \int_D \frac{\partial u}{\partial x_1}(s, x')\, dx' \tag{3.2.2}$$

is independent of $s \in \mathbf{R}$ since $0 = \int_{s_1}^{s_2} \int_D \Delta u \, dx = F(s_2) - F(s_1)$ for a solution u of (3.2.1). If we define

$$E(s) := \int_D u(s, x')\, dx', \tag{3.2.3}$$

this implies $dE(s)/ds$ is a constant function in s. Since u is assumed to be bounded, $E(s)$ must be a constant function, i.e., $E(s) \equiv c$. We may assume that $E(s) = 0$ for all $s \in \mathbf{R}$ by subtracting $c/|D|$, where $|D|$ is the Lebesgue measure of D.

We shall prove that $u \equiv 0$ by the strong maximum principle [PW67, Section 3]. Assume that $u \not\equiv 0$. Then we may assume that $\sup_\Omega u > 0$ by considering $-u$ if necessary. This supremum is NOT attained in $\overline{\Omega}$. Indeed, if it were attained in the interior, then the strong maximum principle would imply that $u \equiv \sup u > 0$ which contradicts $E(s) \equiv 0$ by (3.2.3). If the maximum were taken on the boundary, again we obtain $u \equiv \sup u$ since otherwise the Hopf (boundary) lemma implies $\partial u/\partial n_\Omega > 0$ on $\partial\Omega$. This again contradicts $E(s) \equiv 0$.

Since the supremum $\sup u$ is not attained in $\overline{\Omega}$, we may assume that there is a sequence $x_m = (s_m, x'_m)$ such that $u(x_m) \to \sup u$ and $|s_m| \to \infty$ as $m \to \infty$. We may assume (by taking a subsequence) that $s_m \to \infty$ since the case $s_m \to -\infty$ can be treated similarly. Since $\overline{D}$ is compact, we may assume that $x'_m \to x^*$ for some $x^* \in \overline{D}$ by taking a subsequence if necessary. We shift u by defining

$$u_m(x) := u(x_1 + s_m, x') \quad \text{for} \quad x = (x_1, x'). \tag{3.2.4}$$

Since u is bounded and satisfies (3.2.1), we observe that

$$\sup_{s \in \mathbf{R}} \left\| u_m : W^{2,q}((s, s+1) \times D) \right\| \le C_q \tag{3.2.5}$$

with $C_q > 0$ independent of m by elliptic regularity; see e.g. Theorem 3.5.2. By the Sobolev embedding and Rellich's compactness there is a subsequence of $\{u_m\}$ still denoted by $\{u_m\}$ such that u_m in (3.2.4) converges to some function v locally uniformly with its first derivatives so the boundary condition of (3.2.1) is inherited. Since v is weakly harmonic in Ω, Weyl's lemma implies that v is smooth and harmonic in Ω. Thus v is a solution of (3.2.1) with $v \in C^2(\Omega) \cap C^1(\overline{\Omega})$. Moreover, $E(s) \equiv 0$ implies

$$\int_D v(s, x')\, dx' = 0 \tag{3.2.6}$$

for all $s \in \mathbf{R}$. By definition of x_m we observe that v takes its maximum $\sup u$ at $(0, x^*)$. As before the Hopf lemma implies that $v \equiv \sup u$ is a positive constant which contradicts (3.2.6). We thus conclude $u \equiv 0$. ☐

We shall prove a uniqueness result without assuming the boundedness but assuming $\|d_\Omega \nabla u\|_\infty := \|d_\Omega \nabla u\|_{L^\infty(\Omega)} < \infty$. Of course, $u(x) = x_1$ is a solution of (3.2.1) satisfying $\|d_\Omega \nabla u\|_\infty < \infty$ so to exclude such a solution we need some extra condition. For example, if we assume the no flux condition $F(s) = 0$ for F defined by (3.2.2), we are able to exclude such a solution. We prepare a lemma which asserts that no flux condition with $\|d_\Omega \nabla u\|_\infty < \infty$ implies boundedness of u.

Lemma 3.2.2. *Let Ω be a C^2 infinite cylinder in $\mathbf{R}^n$. For $S \in \mathbf{R}$ let $u \in C^2(\Omega_{>S}) \cap C^1(\overline{\Omega}_{>S})$ satisfy*

$$-\Delta u = 0 \quad in \quad \Omega_{>S}, \qquad \frac{\partial u}{\partial n_\Omega} = 0 \quad on \quad (\partial\Omega)_{>S}, \tag{3.2.7}$$

where $U_{>S} = U \cap \{x_1 > S\}$ for a set $U \subset \mathbf{R}^n$. Assume that $d_\Omega \nabla u$ is bounded in $\Omega_{>S}$ and $F(s_1) = 0$ for some $s_1 > S$. Then u is bounded in $\Omega_{>S+\delta}$ for any $\delta > 0$.

Proof. As in the proof of Lemma 3.2.1 we observe that $F(s)$ is independent of s. Since $F(s_1) = 0$, $F(s) \equiv 0$ for $s > s_1$. Since $d_\Omega \nabla u$ is bounded and $F(s) = 0$ for all $s > s_1$, the integration of ∇u of one of x' variable implies

$$\sup_{s>S} \|u : L^q((s, s+1) \times D)\| < \infty$$

for any $q > 1$ (cf. [AG13], [AGH15, (2.1)]). In a similar way to deriving (3.2.5), since u solves (3.2.7) by elliptic regularity (Appendix A), this implies

$$\sup_{s>S} \left\| u : W^{2,q}((s+\delta, s+1-\delta) \times D) \right\| < \infty$$

for any $\delta \in (0, 1/2)$. By the Sobolev inequality (for $q > n/2$) this implies that u is bounded in $(S+\delta, \infty) \times D$. ☐

Theorem 3.2.1. *Let $u \in C^2(\Omega) \cap C^1(\overline{\Omega})$ satisfy (3.2.1) in a C^2 infinite cylinder Ω. Assume that $F(s_1) = 0$ for some $s_1 \in \mathbf{R}$. If $\|d_\Omega \nabla u\|_\infty < \infty$, then u is a constant function.*

Proof. By Lemma 3.2.2 we observe that u is bounded. (The boundedness of u in $\{x_1 < 0\}$ follows from Lemma 3.2.2 by reflection with respect to $x_1 = 0$.) Thus Theorem 3.2.1 follows from Lemma 3.2.1 ☐

Corollary 3.2.1. *Let $u \in C^2(\Omega_{>0}) \cap C^1(\overline{\Omega}_{>0})$ satisfy (3.2.1) in $\Omega_{>0}$ when Ω is a C^2 infinite cylinder. Assume that $\partial u/\partial x_1 = 0$ at $x_1 = 0$ for $x' \in D$. If $\|d_\Omega \nabla u\|_\infty < \infty$, then u is a constant function.*

Proof. Since $\partial u/\partial x_1 = 0$ at $x_1 = 0$, we extend u for $x_1 < 0$ as an even function, i.e., $u(x_1, x') = u(-x_1, x')$. Then this extended function fulfills all assumptions of Theorem 3.2.1 since $F(0) = 0$. We apply Theorem 3.2.1 to conclude that u is a constant function. ☐

3.2.2 Weighted estimate for the Neumann problem

Our goal in this subsection is to establish a weighted L^∞ estimate of the form $\|d_\Omega \nabla u\|_\infty \leq C\|g\|_\infty$ for a weak solution u of

$$\Delta u = 0 \quad \text{in} \quad \Omega, \quad \frac{\partial u}{\partial n_\Omega} = \operatorname{div}_{\partial\Omega} \, g \quad \text{on} \quad \partial\Omega, \tag{3.2.8}$$

with $\nabla u \in L^2(\Omega)$ when g is tangential, i.e., $g \cdot n_\Omega = 0$.

For this purpose we recall a Neumann problem associated to the Laplace equation. For a bounded Lipschitz domain U in $\mathbf{R}^n$ there exists a weak solution $v \in H^1(U)$ (unique up to an additive constant) of

$$\Delta v = 0 \quad \text{in} \quad U, \quad \frac{\partial v}{\partial n_U} = h \quad \text{on} \quad \partial U, \tag{3.2.9}$$

for any $h \in H^{-1/2}(\partial U)$ provided that $\int_{\partial U} h \, d\mathcal{H}^{n-1} = 0$, where H^s denotes the L^2-Sobolev space of order s. Such a solution is called an $\dot{H}^1$ weak solution v of (3.2.9), which means that $v \in L^1_{loc}(\Omega)$ with $\nabla v \in L^2(\Omega)$ fulfills

$$\int_U \nabla v \cdot \nabla \psi \, dx = \int_{\partial U} h \, \psi \, d\mathcal{H}^{n-1}$$

for all $\psi \in H^1(U)$. This definition applies to the case when U is unbounded by replacing H^1 by the homogeneous Sobolev space $\dot{H}^1$. The existence of a unique $\dot{H}^1$ weak solution for a bounded Lipschitz domain is guaranteed by the Lax-Milgram theorem [Eva10] or the Riesz representation theorem for the Hilbert space

$$\mathring{H}^1(U) = \left\{ u \in H^1(U) \,\middle|\, \int_U u \, dx = 0 \right\}.$$

See e.g. [BF13, Theorem Ⅲ, 4.3]. For an unbounded domain the solution may not exist.

In our limiting procedure we have to handle these u such that ∇u may not be integrable near $\partial\Omega$. For this purpose it is convenient to recall the notion of a very weak solution of (3.2.9) for $U = \Omega(S) = (-S, S) \times D$ when $h = h_0$ in $\{\pm S\} \times D$ with $h_0 \in L^2(\{\pm S\} \times D)$ and $h = \operatorname{div}_{\partial\Omega} \; g \in C\left(\overline{\partial\Omega(S)}\right)$ on $(\partial\Omega)(S) = (-S, S) \times \partial D$ with $g = (g_1, g') \in C\left(\overline{(\partial\Omega)(S)}\right)$, where g is tangential. We say that $u \in L^1(\Omega(S))$ is a *very weak* solution of (3.2.9) in $\Omega(S)$ with such data h if

$$\begin{aligned}\int_{\Omega(S)} u \, \Delta\varphi \, dx &= \int_{(\partial\Omega)(S)} \nabla_{\partial\Omega}\varphi \cdot g \, d\mathcal{H}^{n-1} \\ &\quad - \sum_{x_1 = \pm S} \int_D (\varphi h_0)(x_1, x') dx' - \sum_{x_1 = \pm S} (\operatorname{sgn} x_1) \int_{\partial D} (g_1 \varphi)(x_1, x') d\mathcal{H}^{n-2}_{x'}\end{aligned}$$

for all $\varphi \in C^2\left(\overline{\Omega(S)}\right)$ with $\partial\varphi/\partial n_{\Omega(S)} = 0$ on $\partial(\Omega(S))$, where $\nabla_{\partial\Omega}$ denotes the tangential gradient [Gig06]. Since g is tangential, $\nabla_{\partial\Omega}$ can be replaced by ∇. Such a notion of a very weak solution for (3.2.8) is introduced by [AG13], [AG14]. These types of notions of a very weak solution are elaborated by [MR15] to handle the case when the Neumann data equals the Dirac measure which corresponds to the Green function of the Neumann problem.

Lemma 3.2.3. *Let Ω be a C^2 infinite cylinder in R^n $(n \geq 2)$. Let u be an $\dot{H}^1$ weak solution u with $\nabla u \in L^2(\Omega(S))$ of* (3.2.9) *in $\Omega(S)$ with the boundary data*

$$\begin{aligned} h &= \mathrm{div}_{\partial\Omega} g \in C\left(\overline{\partial\Omega(S)}\right) && \text{on} \quad (\partial\Omega)(S) \\ h &= \pm\partial u/\partial x_1 && \text{on} \quad \{x_1 = \pm S\}, \end{aligned}$$

where $g \in C\left(\overline{(\partial\Omega)(S)}\right)$ satisfies $g \cdot n_\Omega = 0$ on $\partial\Omega(S)$. Then there exists a constant (depending only on D) such that

$$d_{\Omega(S)}(x)\,|\nabla u(x)| \leq C\left(\|g\|_{L^\infty(\partial\Omega(S))} + \sum_{x_1=\pm S}\left\|\frac{\partial u}{\partial x_1}\right\|_{L^{n-1}(D)}(x_1)\right) \tag{3.2.10}$$

for all $x \in \Omega(S)$ provided that $\|d_\Omega \nabla u\|_{L^\infty(\Omega(S))}$ is finite and $\partial u/\partial x_1(\pm S, \cdot) \in L^{n-1}(D)$. The constant C is independent of translation and dilation of D.

Proof. We first prove that u is a very weak solution of (3.2.9) with the same data. Since $\partial\varphi/\partial n_{\Omega(S)} = 0$, integration by parts yields

$$\int_{\Omega(S)} u\,\Delta\varphi\,dx = -\int_{\Omega(S)} \nabla u \cdot \nabla\varphi\,dx.$$

Since u is an $\dot{H}^1$ weak solution of (3.2.9), we see that

$$\int_{\Omega(S)} \nabla u \cdot \nabla\varphi\,dx = \int_{|x_1|=S} h_0\varphi\,dx' + \int_{(\partial\Omega)(S)} (\mathrm{div}_{\partial\Omega}\ g)\varphi\,d\mathcal{H}^{n-1}.$$

Since g is tangential, integration by parts yields

$$\begin{aligned} \int_{(\partial\Omega)(S)} (\mathrm{div}_{\partial\Omega}\ g)\varphi\,d\mathcal{H}^{n-1} &= -\int_{(\partial\Omega)(S)} \nabla_{\partial\Omega}\varphi \cdot g\,d\mathcal{H}^{n-1} \\ &\quad + \sum_{x_1=\pm S} (\mathrm{sgn}\,x_1)\int_{\partial D} (g^1\varphi)(x_1, x')d\mathcal{H}^{n-2}_{x'}. \end{aligned}$$

Combining these three identities, we conclude that u is a very weak solution of (3.2.9).

As in [AG13], [AG14] we argue by contradiction. There exists a sequence $\{u_m, g_m, S_m\}_{m=1}^\infty$ such that

$$\begin{aligned} 1 &= \|d_{\Omega_m}\nabla u_m\|_{L^\infty(\Omega_m)} \\ &> m\left(\|g_m\|_{L^\infty(\partial\Omega_m)} + \sum_{x_1=\pm S}\|h_{0m}\|_{L^{n-1}(D)}(x_1)\right) \end{aligned} \tag{3.2.11}$$

with $\Omega_m = \Omega(S_m)$ such that $u_m \in L^1_{loc}(\overline{\Omega}_m)$ is a weak solution of (3.2.9) with $\Omega = \Omega_m$. Here $g_m \in L^\infty(\partial\Omega_m)$ is assumed to satisfy $g_m \cdot n_\Omega = 0$ on $\partial\Omega_m$ and $h_{0m} = \pm\partial u_m/\partial x_1$ on $\{x_1 = \pm S_m\}$. We take $x_m \in \overline{\Omega}_m$ such that

$$|d_{\Omega_m}(x_m)\nabla u_m(x_m)| > 1/2. \tag{3.2.12}$$

We may assume that $u_m(x_m) = 0$ by adding a constant.

There are two cases depending on the behavior of $\{x_m\}_{m=1}^\infty$.

Case 1 There exists a subsequence still denoted by $\{x_m\}$ which converges to $\hat{x} \in \overline{\Omega}_\infty$ as $m \to \infty$.

Case 2 The sequence $\{x_m\}$ tends to infinity, i.e., $|x_m| \to \infty$.

We discuss Case 1 which is divided into two cases, (a) $\hat{x} \in \Omega_\infty$ and (b) $\hat{x} \in \partial\Omega_\infty$. We first discuss case (a). We may assume that $\lim_{m\to\infty} S_m = S \in [1, \infty]$ by taking a subsequence. The estimate $\|d_{\Omega_m}\nabla u_m\|_{L^\infty(\Omega_m)} \leq 1$ guarantees that $\{u_m\}$ is bounded in $L^q_{loc}(\overline{\Omega}_\infty)$ for any $q > 1$ by (3.2.1). Indeed, since $u_m(x_m) = 0$ and $x_m \to \hat{x} \in \overline{\Omega}_\infty$, integrating $|\nabla u_m(x)| \leq 1/d_{\Omega_m}(x)$ from x_m yields a bound in $L^q_{loc}(\overline{\Omega})$ (cf. [AG13], [AGH15, (2.1)]). In fact, by [AGH15, Proposition 2.1] we are able to prove that $\|u_m\|_{L^q(\Omega(K))} \leq C_K$ for any $K (< S)$ since $u_m(x_m) = 0$. We shall discuss case (a) since $\{u_m\}$ is bounded in $L^q_{loc}(\overline{\Omega}_\infty)$. By a diagonal argument we see that $\{u_m\}$ converges to some $u \in L^q_{loc}(\overline{\Omega}_\infty)$ weakly for some $q > 1$ by taking a subsequence. It is easy to see that u is a very weak solution of (3.2.9) in $\Omega_\infty = (-S, S) \times D$ with $g = 0$ and $h_0 = 0$ since $\|g_m\|_\infty \to 0$ and $\|h_{0m}\|_{L^1} \to 0$ by (3.2.11). (One should be a little bit careful since φ cannot be taken uniformly with respect to Ω_m when S_m depends on m since we request $\partial\varphi/\partial n_{\Omega_m} = 0$. However, it is easy to construct a sequence $\{\varphi_m\}$ converging to φ uniformly up to second derivatives so one concludes that u is a very weak solution.)

Since $\{u_m\}$ is bounded in $L^q_{loc}(\overline{\Omega}_\infty)$ and each u_m is harmonic, the Cauchy estimates for harmonic functions [Eva10, 2.2.c] (quantitative version of Weyl's lemma) implies that all derivatives are locally bounded in Ω_∞. Thus the convergence $u_m \to u$ is locally uniform with its derivatives in Ω_∞ so that $u(\hat{x}) = 0$. This in particular implies $u \in C^\infty(\Omega_\infty)$. We have to prove that the limit u is C^1 up to the boundary. Since u is a very weak solution of (3.2.9) in Ω_∞ with zero boundary Neumann data, the elliptic regularity (Theorem 3.5.2) implies that $u \in C^1(\overline{\Omega}_\infty)$. Note that there is a corner point of Ω_∞ if S is finite. In this case we can interpret this point as a regular point by a reflection argument since the Neumann data at $x_1 = \pm S$ is zero.

If S is finite, the uniqueness of the homogeneous Neumann problem is still valid by a reflection argument since the problem is reduced to whole Ω with x_1-periodicity. Since $u(\hat{x}) = 0$, the uniqueness implies $u \equiv 0$. However, by (3.2.12) we have $|d_\Omega(\hat{x})\nabla u(\hat{x})| \geq 1/2$ which yields a contradiction.

If $S = \infty$ so that $\Omega_\infty = \mathbf{R} \times D$, we have to check the flux condition. We take a cut-off function $\chi \in C^2([0, \infty))$ such that $\chi(s) = 0$ on $[0, 1/2]$ and $\chi(s) = 1$ on

$[1,\infty)$ such that $0 \le \chi \le 1$ and $\chi' \ge 0$. We set $\chi_k(x_1) = \chi(kx_1)$ $(k = 1, 2, \dots)$ and take it as a test function in the definition of a weak solution to get

$$\int_{\Omega_{>0}} u_m \Delta\chi_k dx = \int_{\partial\Omega_{>0}} \nabla_{\partial\Omega}\chi_k \cdot g_m d\mathcal{H}^{n-1} - \int_{|x_1|=S_m} h_{0m}\ \chi_k\ dx' \\ - \int_{\partial D} g^1_m(S_m, x') d\mathcal{H}^{n-1}_{x'} + \int_{\partial D} g^1_m(-S_m, x') d\mathcal{H}^{n-1}_{x'}.$$

Since $\|g_m\|_\infty \to 0$ and $\|h_{0m}\|_{L^{n-1}} \to 0$ as $m \to \infty$, the right-hand side tends to zero. We thus obtain $\int_{\Omega_{>0}} u\Delta\chi_k dx = 0$. Integrating by parts yields

$$\int_0^{1/k} \left(\int_D \frac{\partial u}{\partial x_1}(x_1, x') dx' \right) \frac{\partial \chi_k}{\partial x_1} dx_1 = 0.$$

Letting $k \to \infty$ yields $F(0) = \int_D \frac{\partial u}{\partial x_1}(0, x') dx' = 0$. We are now able to apply Theorem 3.2.1 to get $u \equiv 0$ which is a contradiction.

The case (b) can be treated as in [AG13] by rescaling u_m as $v_m(x) = u_m(x_m + d_m x)$ with $d_m = d_{\Omega_m}(x_m)$. Thanks to the L^{n-1} norm of h in (3.2.10) the estimate (3.2.11) is preserved for v_m. Since v_m is bounded in $L^q_{loc}(\overline{\Omega'}_m)$ with $\Omega'_m = (\Omega_m - x_m)/d_m$, there is a weak limit v in $L^q_{loc}(\overline{\Omega})$ (for some $q > 1$) by taking a subsequence if necessary. We have to prove that v is a very weak solution with homogeneous data in a half space or a quadrant type space $\mathbf{R}^n_{+<S}$. We give a proof in the case of half space. In other words, we have to prove $\int_{\mathbf{R}^n_+} v\Delta\varphi\ dx = 0$ for all $\varphi \in C^2_c(\overline{\mathbf{R}^n_+})$ satisfying $\frac{\partial\varphi}{\partial x_n} = 0$ on the boundary in the case that the limit domain is a half space. In [AG13, Proof of Theorem 2.5, Case 2] we use C^3-regularity of Ω to construct a sequence of test functions $\{\varphi_m\}$ (approximating $\varphi \in C^2(\overline{\mathbf{R}^n_+})$ with $\partial\varphi/\partial x_n = 0$) satisfying $\partial\varphi_m/\partial n_{\Omega_m} = 0$ on $\partial\Omega_m$, where Ω_m is the rescaled domain. The reason is that we appealed to the normal coordinate in [AG13, p. 12]. Here we use a different construction of $\{\varphi_m\}$ in Lemma 3.5.11, which requires only C^2-regularity of Ω. In fact, by rotation and translation we may assume that the rescaled domain Ω_m converges to a half space $\mathbf{R}^n_{+,-1}$ of the form $\mathbf{R}^n_{+,-1} = \{(x', x_n) \in \mathbf{R}^n \mid x_n > -1\}$. Assume that $\varphi \in C^2_c(\overline{\mathbf{R}^n_{+,-1}})$ with $\operatorname{spt}\varphi \subset B_R(0)$ and $\partial\varphi/\partial x_n = 0$ on $x_n = -1$. Applying Lemma 3.5.11 yields a sequence of functions $\varphi_m \in C^2_c(\overline{\Omega_m})$ such that

$$\frac{\partial\varphi_m}{\partial n_{\Omega_m}} = 0 \quad \text{on} \quad \partial\Omega_m, \quad \operatorname{spt}\varphi_m \subset B_{4R/3}(0)$$

and that φ_m converges to φ uniformly in $\Omega_m \cap \mathbf{R}^n_{+,-1}$ up to second derivatives. The desired condition $\int_{\mathbf{R}^n_{+,-1}} v\Delta\varphi\ dx = 0$ follows from the fact that v_m is a very weak solution of (3.2.8) in Ω_m with corresponding data g_m converging to zero. We apply the uniqueness result in a half space [AG13, Lemma 2.9] to get a contradiction. If S is finite, then there might be a chance that the rescaled limit space is not a half space but a quadrant type space like $\mathbf{R}^n_{+<S}$. In this case we extend a solution by an even reflexion outside $x_1 = S$ and the reduce the problem into the half space.

We next study Case 2, i.e., $|x_m| \to \infty$. If one writes $x_m = (s_m, x'_m)$, we may assume that $s_m \to \infty$ and $x'_m \to x^* \in \overline{D}$ since the case $s_m \to -\infty$ can be treated similarly. We shift u_m as in (3.2.4), i.e., $w_m(x) := u_m(x_1 + s_m, x')$. If $d_m = d_{\Omega_m}(x_m) \to 0$ by taking a subsequence, we rescale w_m to consider $v_m(x) := w_m(d_m x_1, d_m x' + x'_m)$. We are able to reduce this case to Case 1 (b). It remains to discuss the case that $\inf d_m > 0$. We may assume that $\lim_{m\to\infty}(S_m - s_m) = S_* \in [0, \infty]$ exists by taking a subsequence. As in Case 1 (a), by (3.2.10) we observe that there is a subsequence still denoted by $\{w_m\}$ converging to a weak solution w of (3.2.8) in $\Omega_\infty = (-\infty, S_*) \times D$ with $g = 0$. Moreover, the convergence is locally uniform with its derivatives in Ω_∞ so that $w(0, x^*) = 0$ since $\{w_m\}$ is a bounded sequence in $L^q_{loc}(\overline{\Omega})$ and w_m is harmonic. As before, the elliptic regularity implies $w \in C^2(\Omega_\infty) \cap C^1(\overline{\Omega}_\infty)$. In the case $S_* = \infty$ since the flux condition $F(0) = 0$ is fulfilled as in the proof of Case 1 (a), we apply Theorem 3.2.1 to get $w \equiv 0$. This would contradict $|d_\Omega((0, x^*))\nabla w(0, x^*)| \geq 1/2$. The case $S_* < \infty$ is easier and get a contradiction. We thus proved (3.2.10). ☐

Remark 3.2.1. *Let $u \in L^1_{loc}(\Omega(S))$ be a very weak solution of* (3.2.9) *in* $\Omega(S)$ *with $g \in C\left(\overline{(\partial\Omega)(S)}\right)$ satisfying $g \cdot n_\Omega = 0$ with $\operatorname{div}_{\partial\Omega}\ g \in C\left(\overline{(\partial\Omega)(S)}\right)$ and $h_0 = 0$ on $\{x_1 = \pm S\}$. Then one can extend this solution u to $\tilde{u}$ by a reflection argument so that the extended $\tilde{u}$ is periodic in x_1 direction with period $4S$ and $\tilde{u}$ is a solution of* (3.2.8) *in Ω with periodic data $\tilde{g}$. Indeed, we extend u evenly in x_1 with respect to $x_1 = S$ so that the extended function still denoted u is a function on $\Omega(-S, 3S)$. We then extended it periodically with respect to x_1 to get the desired $\tilde{u}$. Since the Neumann data at $x_1 = \pm S$ is zero, $\tilde{u}$ is harmonic in Ω. Moreover, $\tilde{u}$ solves* (3.2.8) *in Ω with the extended data $\tilde{g}$. One is always able to reduce the problem to the whole domain Ω having a periodicity in x_1.*

Theorem 3.2.2. *Let Ω be a C^2 infinite cylinder in $\mathbf{R}^n$ $(n \geq 2)$. Then there exists a constant C (depending only on D and n) such that*

$$\|d_\Omega \nabla u\|_{L^\infty(\Omega)} \leq C\|g\|_{L^\infty(\partial\Omega)} \tag{3.2.13}$$

holds for all $\dot{H}^1$ weak solution u with $\nabla u \in L^2(\Omega) \cap L^r(\Omega)$ (with some $r \geq n$) of (3.2.9) *in Ω with $h = \operatorname{div}_{\partial\Omega}\ g \in C(\partial\Omega)$, where $g \in C(\partial\Omega) \cap L^\infty(\partial\Omega)$ satisfying $g \cdot n_\Omega = 0$ on $\partial\Omega$.*

Proof. We take $S > 1$. Since $\nabla u \in L^r(\Omega)$ with some $r \geq n$, as in [AG13] a mean value theorem implies that $\sup_{x\in\Omega(S)} d_\Omega(x)\,|\nabla u(x)| < \infty$.

Furthermore, we take S such that $\partial u/\partial x_1$ at $x_1 = \pm S$ is in $L^2(D)$. Since u is an $\dot{H}^1$ weak solution of (3.2.9) in $\Omega(S)$ with $h = \operatorname{div}_{\partial\Omega}\ g$ in $\partial\Omega$ and $h = \pm\partial u/\partial x_1$ at $x_1 = \pm S$ and since $\|d_\Omega \nabla u\|_{L^\infty(\Omega(S))}$ is finite, we are able to apply Lemma 3.2.3 to conclude that

$$\left\|d_{\Omega(S)}\nabla u\right\|_{L^\infty(\Omega(S))} \leq C_D\left(\|g\|_{L^\infty(\partial\Omega)(S)} + \sum_{x_1=\pm S}\left\|\frac{\partial u}{\partial x_1}\right\|_{L^{n-1}(D)}(x_1)\right)$$

for a.e. $S>0$. This implies

$$\left\|d_{\Omega(S)}\nabla u\right\|^n_{L^\infty(\Omega(S))} \le C_n C_D^n \left(\|g\|^n_{L^\infty(\partial\Omega(S))} + \sum_{x_1=\pm S} \left\|\frac{\partial u}{\partial x_1}\right\|^n_{L^{n-1}(D)}(x_1) \right)$$

for a.e. $S>0$ with some C_n depending only on n. By the Hölder inequality, we observe that

$$\left(\int_{|x_1|=S} \left|\frac{\partial u}{\partial x_1}\right|^{n-1} dx'\right)^{n/(n-1)} \le \left(\int_{|x_1|=S} \left|\frac{\partial u}{\partial x_1}\right|^{n} dx'\right) \mathcal{H}^{n-1}(D)^{1/(n-1)}.$$

This implies

$$\begin{aligned}&\left\|d_{\Omega(S)}\nabla u\right\|^n_{L^\infty(\Omega(S))}\\ &\qquad \le C_n C_D^n \left(\|g\|^n_{L^\infty(\partial\Omega)} + \sum_{x_1=\pm S} \left\|\frac{\partial u}{\partial x_1}\right\|^n_{L^{n}(D)}(x_1)\mathcal{H}^{n-1}(D)^{1/(n-1)} \right).\end{aligned}$$

Integrating over (S_0, S_0+1) with respect to S and noting that $\left\|d_{\Omega(S)}\nabla u\right\|^n_{L^\infty(\Omega(S))}$ is nondecreasing in S we obtain

$$\left\|d_{\Omega(S_1)}\nabla u\right\|^n_{L^\infty(\Omega(S_1))} \le C_n C_D^n \|g\|^n_{L^\infty(\partial\Omega)} + C' \int_{\Omega(S_0+1)\setminus\Omega(S_0)} \left|\frac{\partial u}{\partial x_1}\right|^n dx$$

with $C' = C_n C_D^n \mathcal{H}^{n-1}(D)^{1/(n-1)}$ for all $S_0>0$ and $S_1 \in (0, S_0]$. Since $\nabla u \in L^n(\Omega)$ sending S_0 to the infinity implies that

$$\left\|d_{\Omega(S_1)}\nabla u\right\|^n_{L^\infty(\Omega(S_1))} \le C_n C_D^n \|g\|^n_{L^\infty(\partial\Omega)}.$$

Sending S_1 to the infinity yields (3.2.13). □

Remark 3.2.2. *(i) The assumption* $\mathrm{div}_{\partial\Omega}\ g \in C^0(\partial\Omega)$ *is actually unnecessary if one observe that* $\mathrm{div}_{\partial\Omega}\ g \in H^{-1/2}((\partial\Omega)(2S))$ *for any* $S>1$ *which is enough to conclude that* u *is a very weak solution of* (3.2.9) *in* $\Omega(S)$ *with such a data.*

(ii) In the proof of Theorem 3.2.2 we actually use a weaker assumption for u*. We just invoke that* u *is an* $\dot{H}^1$ *weak solution of* (3.2.9) *in* $\Omega(S)$ *with* $h = \mathrm{div}_{\partial\Omega}\, g$ *on* $\partial\Omega$ *and* $h = \pm\partial u/\partial x_1$ *at* $x_1 = \pm S$ *for all* S *instead of assuming that* u *is an* $\dot{H}^1$ *weak solution of* (3.2.9) *in whole* Ω *with* $h = \mathrm{div}_{\partial\Omega}\, g$ *on* Ω*. This weaker assumption requiring only that* u *is a solution in* $\Omega(S)$ *with suitable boundary condition for all* S *is not enough to derive* (3.2.13) *if* Ω *is an aperture domain with* $n \ge 3$ *since there is a nontrivial* u *satisfying* $\nabla u \in L^2(\Omega) \cap L^r(\Omega)$ *(with some* $r \ge n$*) which is an* $\dot{H}^1$ *weak solution of* (3.2.9) *in* $\Omega(S)$ *with* $g=0$ *for all* S *as shown in [FS96]. In fact, this* u *is not an* $\dot{H}^1$ *solution in* Ω *with* $g=0$ *since*

such u must satisfy $\nabla u \in L^2_\sigma(\Omega)$ yielding $\nabla u \equiv 0$, by the L^2-Helmholtz decomposition.

In two-dimensional sector like domains one is able to prove that $\int_{\partial B_r \cap \Omega} \partial u/\partial r \, d\mathcal{H}^1 \to 0$ as $r \to \infty$ by taking a subsequence if $\nabla u \in L^2(\Omega)$ (Remark 3.4.2 (ii)), while in an aperture domain with $n \geq 3$ the corresponding flux may not decay under $\nabla u \in L^2(\Omega)$ (actually under $\nabla u \in L^p(\Omega)$ with $p > n/(n-1)$). In the two-dimensional setting it is likely that Theorem 3.2.2 extends to an aperture domain while for the higher dimensional aperture domain the present proof at least does not work although there might be a chance the statement still holds.

(iii) *(Strict admissibility) We have proved that a C^3 bounded domain, a C^3 exterior domain are strictly admissible in [AG14]. The C^3 regularity was invoked to construct sequence $\{\varphi_m\}$ converging to φ in the rescaling procedure (Case 1 (b) of the proof of Lemma 3.2.3) as well as the uniqueness of very weak solutions of* (3.2.8) *in a given domain. This last uniqueness is easy to generalize to C^2 domain in the argument of the present paper. In fact, we get the assertion of Lemma 3.2.1 for C^2 bounded domain and exterior domain for u with $\|d\nabla u\|_\infty < \infty$. The approximation of φ by $\{\varphi_m\}$ requires only C^2 regularity as in Case 1 (b) of the proof of Lemma 3.2.3. Thus we conclude that C^2 bounded domain and C^2 exterior domain are strictly admissible. In [Abe13] it is proved that a C^3 perturbed half space is also strictly admissible. This also requires only C^2 regularity by the same reason.*

3.2.3 Admissibility

Theorem 3.2.2 is enough to guarantee that Ω is admissible in the sense of [AG13]. We say that Ω is *admissible* if there exists $r \geq n$ and a constant C such that

$$\|d_\Omega \mathbf{Q}[\nabla \cdot f](x)\|_\infty \leq C\|f\|_{L^\infty(\partial\Omega)} \tag{3.2.14}$$

holds for all matrix-valued functions $f = (f_{ij})_{1\leq i,\, j\leq n} \in C^1(\overline{\Omega})$ satisfying $\nabla \cdot f = \left(\sum_{j=0}^n \partial_j f_{ij}\right) \in (L^r \cap L^2)(\Omega)$ with $\operatorname{tr} f = 0$ and $\partial_\ell f_{ij} = \partial_j f_{i\ell}$ for $1 \leq i, j, \ell \leq n$ where $I - Q$ is the L^2-Helmholtz orthogonal projection from $L^2(\Omega)$ to $L^2_\sigma(\Omega)$.

Theorem 3.2.3. *A C^2 infinite cylinder is admissible.*

Proof. Since $\nabla w = \mathbf{Q}[\nabla \cdot f]$ solves the Neumann problem

$$-\Delta w = 0 \quad \text{in} \quad \Omega, \quad \frac{\partial w}{\partial n_\Omega} = \operatorname{div}_{\partial\Omega} W$$

with $W = (f - f^T) \cdot n$ in $\dot{H}^1$ sense with $W \in C^1(\partial\Omega)$ we are able to apply Theorem 3.2.2 to deduce the estimate (3.2.14). □

Remark 3.2.3. *(i) To deduce analyticity of the Stokes semigroup in $C_{0,\sigma}$ it suffices to restrict the class of f as $f_{ij} = \partial_j v^i$ with* $\operatorname{div} v = 0$ *and $f \in (L^r \cap L^2)(\Omega)$.*

(ii) By Remark 3.2.2 we can prove (3.2.14) *for $f \in C(\overline{\Omega})$ which may not be C^1. For all domains which we have considered so far, the restriction $f \in C^1(\Omega)$ actually can be weakened for $f \in C(\Omega)$. This replacement enables us to conclude the analyticity of the Stokes semigroup in $C_{0,\sigma}$ even if Ω is C^2 under this stronger admissibility. For various domains we so far proved strict admissibility in the sense of [AG14] which then in turn yields admissibility of [AG13] with the replacement of $f \in C^1(\overline{\Omega})$ by $f \in C(\overline{\Omega})$ which is stronger than admissibility.*

3.3 Cylindrical Domains

We say a domain $\Omega \subset \mathbf{R}^n$ is a C^2 *cylindrical domain with several exits to infinity* or, for short, C^2 *cylindrical domain*, if it has a C^2-boundary and there is an $m \in \mathbf{N}$ such that $\Omega = \bigcup_{i=0}^m \Omega^i$, where Ω^0 is a bounded domain and Ω^i, $i = 1, \dots, m$ are disjoint semi-infinite cylinders, that is, up to rotation and translation,

$$\Omega^i = \{x^i = (x_1^i, \dots, x_n^i) \in \mathbf{R}^n : x_1^i > 0, [x^i]' = (x_2^i, \dots, x_n^i) \in D^i\}, \tag{3.3.1}$$

where $D^i \subseteq \mathbf{R}^{n-1}$, $i = 1, \dots, m$ are bounded domains of class C^2 and $\Omega^i \cap \Omega^j = \emptyset$ for $i \neq j$, $i, j \leq m$. We may assume that the lateral boundary $\partial D^i \times (0, \infty)$ is also a part of the boundary $\partial\Omega$.

Our goal in this section is again to show that cylindrical domains are admissible domains. Before we come to that, we first state some properties of this class of domains.

Remark 3.3.1. *1. For each C^2 cylindrical domain $\Omega \subseteq \mathbf{R}^n$, there is an $R_\Omega > 0$ and an $x_\Omega \in \Omega$ such that*

$$\Omega \setminus B_{R_\Omega}(x_\Omega) = \bigcup_{i=1}^m \Omega^i \setminus B_{R_\Omega}(x_\Omega), \tag{3.3.2}$$

where Ω^i for $i = 1, \dots, m$ are semi-infinite cylinders given in (3.3.1)*. Here $B_R(x)$ denotes the closed ball of radius R centered at x. Without loss of generality, we may assume $x_\Omega = 0$ by translation.*

2. The usual Sobolev embedding theorems hold, since Ω has minimally smooth boundary, and hence extension theorems for Sobolev spaces hold for Ω, see [Ama95, Ch. 5, Thm. 2.4.5] and [Tan97, Thm. 3.2.1].

3. For $1 < q < \infty$, Poincaré's inequality holds, namely there is a $C > 0$ such for all $u \in W^1_{q,0}(\Omega)$ it holds

$$\|u\|_{L_q(\Omega)} \leq C \|\nabla u\|_{L_q(\Omega)}. \tag{3.3.3}$$

This can be shown by using a suitable decomposition of unity and applying Poincaré's inequality in each cylinder and in the bounded part.

3.3.1 Uniqueness under no flux condition

Lemma 3.3.1. *Let $u \in C^2(\Omega) \cap C^1(\overline{\Omega})$ be a bounded solution of* (3.2.1) *in a C^2 cylindrical domain $\Omega \subseteq \mathbf{R}^n$. Then u is a constant function.*

Proof. The proof is similar to that of Lemma 3.2.1. For each semi-infinite cylinder Ω^i we define the flux by

$$F_i(s) := \int_{D^i} \frac{\partial u}{\partial x_1}(s, x')\, dx', \quad s > 0$$

where we write $x_1 = x_1^i$ and $x' = [x^i]'$ to increase readability. We also define

$$E_i(s) = \int_{D^i} u(s, x')\, dx'.$$

As in Lemma 3.2.1 $F_i(s)$ is independent of $s > 0$. Since u is bounded, this implies that $E_i(s)$ must be a constant c_i. We may assume that one of c_i, say $c_1 = 0$ by subtracting c_1/D from u.

Our goal is again to prove $u \equiv 0$ in Ω by the strong maximum principle [PW67, Section 3]. Assume that $u \not\equiv 0$. Then we may assume $\sup u > 0$ by considering $-u$ if necessary.

We first show again that $\sup u$ is not attained in $\overline{\Omega}$. If it is attained in Ω, then u must be a constant $\sup u$. However, this must be zero since $c_1 = 0$. Also, u cannot attain its supremum on $\partial\Omega$ since, by the Hopf (boundary) lemma, this would imply $\partial u / \partial n_\Omega > 0$.

Therefore, we may assume that there is a sequence $\{x_k\}$ with $x_k = (s_k, x'_k)$ for $k \in \mathbf{N}$ such that $u(x_k) \to \sup u$ and $|s_k| \to \infty$ for $k \to \infty$. By the pigeonhole principle there is a subsequence of $\{x_k\}$ also denoted by $\{x_k\}$ such that $\{x_k\} \subset \Omega^i$ for some $i = 1, \dots, m$. The assertion then follows by the proof of Lemma 3.2.1. □

Lemma 3.3.2. *Let Ω be a C^2 cylindrical domain in $\mathbf{R}^n$. For $S \geq 0$, let $u \in C^2(\Omega_{>S}) \cap C^1(\overline{\Omega}_{>S})$ satisfy*

$$-\Delta u = 0 \quad \text{in} \quad \Omega_{>S}, \quad \frac{\partial u}{\partial n_\Omega} = 0 \quad \text{on} \quad (\partial\Omega)_{>S}, \tag{3.3.4}$$

where $\Omega_{>S} := \bigcup_{i=1}^m \Omega^i \cap \{x_1^i > S\}$ and $\partial\Omega_{>S} := \bigcup_{i=1}^m \partial\Omega^i \cap \{x_1^i > S\}$. Assume that S is taken so that $\Omega_{>S}$ consists of mutually disjoint i semi (infinite) cylinder. Assume that $d_\Omega \nabla u$ is bounded in $\Omega_{>S}$ and $F_i(s_1) = 0$ for some $s_1 > S$. Then u is bounded in $\Omega_{>S+\delta}$ for any $\delta > 0$.

This is a trivial extension of Lemma 3.2.2.

Theorem 3.3.1. *Let $u \in C^2(\Omega) \cap C^1(\overline{\Omega})$ satisfy (3.2.1) in a C^2 cylindrical domain Ω in $\mathbf{R}^n$. Assume that $F_i(s_1) = 0$ for some $s_1 > 0$ and all $i = 1, \dots, m$. If $\|d_\Omega \nabla u\|_\infty < \infty$, then u is a constant function.*

Proof. By Lemma 3.3.2 we observe that u is bounded in $\Omega_{>s_1}$. In $\Omega(s_1) := \Omega \setminus \Omega_{>s_1}$, it is clear that by the regularity assumption on $u \in C^1(\overline{\Omega(s_1)})$ and hence it attains its maximum there. Therefore u is bounded in Ω and Lemma 3.3.2 with Elliptic Regularity in Appendix A yields boundedness in $W^2_q(\Omega(s_1))$. Thus Theorem 3.3.1 follows from Lemma 3.3.1. $\square$

Cylindrical domains have one disadvantage compared to infinite cylinders: They are not mirror-reflexive at $\{x_1 = S\}$ for any $S \in \mathbf{R}$. To overcome this disadvantage, we need to also give a cut-off uniqueness result.

Corollary 3.3.1. *Let $\Omega \subseteq \mathbf{R}^n$ $(n \geq 2)$ be a C^2 cylindrical domain. Let $S > 0$ be given. Let $u \in C^2(\Omega(S)) \cap C^1(\overline{\Omega(S)})$ be a classical solution of*

$$-\Delta u = 0 \quad in \quad \Omega(S), \qquad \frac{\partial u}{\partial n_{\Omega(S)}} = 0 \quad on \quad \partial(\Omega(S)), \tag{3.3.5}$$

in $\Omega(S)$, where $\Omega(S) := \Omega_0 \cup \bigcup_{i=1}^m \Omega^i \cap \{x_1^i \leq S\}$ for $S \geq 0$. Then u is a constant function.

Proof. Since $\Omega(S)$ is bounded, $u \in C^1(\overline{\Omega(S)})$ attains its supremum, $\sup u$, on $\overline{\Omega(S)}$. Let $x \in \overline{\Omega(S)}$ be such that $u(x) = \sup u$. If $x \in \Omega(S)$ then the strong maximum principle, implies the assertion. If $x \in \partial\Omega \cap \partial\Omega(S)$, which means the C^2-part of the boundary, Hopf's Lemma contradicts the Neumann boundary condition. It remains to check the case $x \in \partial\Omega(S) \setminus \partial\Omega$. This implies there is an $i \in \{1, \dots, m\}$ such that $x \in \{x_1 = S\} \times \partial D^i$. We may evenly reflect once at $\{x_1 = S\} \times \partial D^i$ to obtain an extension $\tilde{u}$ of u which is harmonic in $\Omega(2S)$ and fulfills zero Neumann boundary conditions at $(\partial\Omega)(2S)$ and conclude again with Hopf's Lemma, that $x \notin \partial(\Omega(S)) \setminus \partial\Omega$. $\square$

3.3.2 Weighted estimate for the Neumann problem and Admissibility

For cylindrical domains we would now like to prove the equivalent version of Theorem 3.2.2 in order to show that this class of domains is again an admissible. We continue to use the notation for $S \geq 0$,

$$\Omega_{>S} = \bigcup_{i=1}^m \Omega^i \cap \{x_1^i > S\}, \quad \Omega(S) = \Omega \setminus \Omega_{>S} = \Omega_0 \cup \bigcup_{i=1}^m \left(\Omega^i \cap \{x_1^i \leq S\}\right),$$

$$(\partial\Omega)(S) = \partial\Omega \setminus \partial\Omega_{>S} = \partial\Omega \cap \left(\partial\Omega_0 \cup \bigcup_{i=1}^m \left(\partial\Omega^i \cap \{x_1^i \leq S\}\right)\right).$$

Our goal in this subsection is to establish a weighted L^∞ estimate of the form $\|d_\Omega \nabla u\|_\infty \leq C\|g\|_\infty$ for a weak solution u of (3.2.8) where $\Omega \subset \mathbf{R}^n$ $(n \geq 2)$ is a cylindrical domain. For this class of domains, we also consider an $\dot{H}^1$ weak solutions of (3.2.9) bounded Lipschitz sub domains $U \subset \Omega$.

Also this time, in our limiting procedure we have to handle u such that ∇u may not be integrable near $\partial\Omega$. For this purpose it is convenient to recall notion of a very weak solution of (3.2.9) for $U = \Omega(S)$ when $h = h_0^i$ in $\{S\} \times D^i$ with $h_0^i \in L^2\left(\{\pm S\} \times D^i\right)$ for $i = 1, \dots, m$ and $h = \operatorname{div}_{\partial\Omega}\ g \in C\left(\overline{\partial\Omega(S)}\right)$ on $(\partial\Omega)(S)$ with $g = (g_1, g') \in C\left(\overline{(\partial\Omega)(S)}\right)$, where g is tangential. We say that $u \in L^1\left(\Omega(S)\right)$ is a *very weak* solution of (3.2.9) in $\Omega(S))$ with such data h if

$$\int_{\Omega(S)} u\ \Delta\varphi\ dx = \int_{(\partial\Omega)(S)} \nabla_{\partial\Omega}\varphi \cdot g\ d\mathcal{H}^{n-1} - \sum_{i=1}^{m} \int_D^i (\varphi h_0^i)(S, x')dx' - \sum_{i=1}^{m} \int_{\partial D^i} (g_1\varphi)(S, x')d\mathcal{H}_{x'}^{n-2}$$

for all $\varphi \in C^2\left(\overline{\Omega(S)}\right)$ with $\partial\varphi/\partial n_{\Omega(S)} = 0$ on $\partial\left(\Omega(S)\right)$. Since g is tangential, $\nabla_{\partial\Omega}$ can be replaced by ∇.

Lemma 3.3.3. *Let $\Omega \subset \mathbf{R}^n$ $(n \geq 2)$ be a C^2 cylindrical domain. Then there exists a constant $C = C(D^i, n) > 0$ such that for all $x \in \Omega_{\leq S}$*

$$d_{\Omega(S)}(x)\,|\nabla u(x)| \leq C\left(\|g\|_{L^\infty\left((\partial\Omega)_{\leq S}\right)} + \sum_{i=1}^{m} \left\|\frac{\partial u}{\partial x_1}\right\|_{L^{n-1}(D^i)}(S)\right) \tag{3.3.6}$$

holds for almost all $S \geq 1$ and all $\dot{H}^1$ weak solutions u of (3.2.9) *with boundary data $h \in H^{-1/2}(\partial(\Omega(S)))$, with*

$$h = \begin{cases} \operatorname{div}_{\partial\Omega}\ g & \text{in } (\partial\Omega)(S) \\ h_0^i := \partial u/\partial x_1 & \text{on } \{S\} \times D^i, (i \in \{1, \dots, m\}) \end{cases} \tag{3.3.7}$$

where $g \in C\left(\overline{(\partial\Omega(S)}\right)$ satisfies $g \cdot n_\Omega = 0$ on $(\partial\Omega)(S)$ and $\operatorname{div}_{\partial\Omega}\ g \in C(\overline{\partial\Omega(S)})$, provided that $\nabla u \in L^2\left(\Omega(S)\right)$ in $\Omega(S)$ and $\|d_\Omega \nabla u\|_{L^\infty(\Omega(S))} < \infty$ and that $\partial u/\partial x_1(S, \cdot) \in L^{n-1}(D^i)$.

By a similar argumentation as in the proof of Lemma 3.2.3, u is also a very weak solution of (3.2.9).

Proof. We argue again by contradiction. Assume there is a sequence $\{u_k, g_k, S_k\}_{k=1}^\infty$ such that $1 = \|d_\Omega \nabla u_k\|_{L^\infty(\Omega_k)} > k\|g_k\|_{L^\infty(\partial\Omega_k)}$, $\Omega_k := \Omega(S_k)$. Let $\{x_k\}_{k=1}^\infty$ be a sequence such that $x_k \in \overline{\Omega_k}$, $|d_\Omega \nabla u_k(x_k)| > 1/2$ and again, without loss of generality $u_k(x_k) = 0$, $\lim_{k\to\infty} S_k = S_\infty \in [1, \infty]$. We begin to treat Case 1: $x_k \to \hat{x} \in \overline{\Omega}_\infty$, more specifically Case 1 (a) $\hat{x} \in \Omega_\infty$.

Similarly to Lemma 3.2.3, we obtain a weakly convergent subsequence converging to some u of (3.2.8) in $\Omega_\infty = \Omega(S_\infty)$ with $g = 0$ and $h_0 = 0$ as before.

The convergence is again locally uniform with its derivatives in Ω: Again since $u_k(x_k) = 0$ and $x_m \to \hat{x} \in \Omega$ integrating over $|\nabla u_m(x)| \leq 1/d_\Omega(x)$ from x_k yields a bound in $L^q_{loc}(\Omega_k)$. By a diagonal argument, we see that $\{u_k\}$ converges to some $u \in L^q_{loc}(\overline{\Omega})$. This u is a very weak solution of (3.2.9) in $\Omega_\infty = \Omega(S_\infty)$ with zero Neumann data. Let $\phi \in C^2(\overline{\Omega_\infty})$ with $\partial\phi/\partial n_\Omega = 0$ on $\partial\Omega_\infty$. Let $(\chi_i)_{0\leq i\leq m_\Omega} \in C^\infty(\overline{\Omega})$ be a partition of unity in Ω, such that $\sum_{i=0}^m \chi_i = 1$ in $\overline{\Omega}$ with $0 \leq \chi^i \leq 1$, spt $\chi_i \subseteq \Omega^i, i \in \{1,\ldots,m\}$ and spt $\chi_0 \subseteq \Omega(1)$. We write $\phi = \sum_{i=0}^m \chi_i\phi =: \sum_{i=0}^{m_\Omega} \phi^i$. For $i > 0$, it follows as in the infinite cylinder case that $\int_{\Omega_\infty} u_\infty \Delta\phi^i dx = 0$. For $i = 0$ it holds

$$\int_{\Omega_\infty} u_\infty \Delta\phi^0 dx = \int_{\Omega(1)} (u_\infty - u_m)\Delta\phi^0 dx + \int_{\Omega(1)} u_m \Delta\phi^0 dx.$$

For $m \to \infty$, the first summand tends to zero, since $u_m \to u_\infty$ weakly and the second summand tends to zero, since $[h_0]_m$ and g_m tend to zero for $m \to \infty$.

By the Cauchy estimates for harmonic functions, [Eva10, 2.2.c], $u \in C^\infty(\Omega_\infty) \cap C^1(\overline{\Omega_\infty})$ and $u(\hat{x}) = 0$ and $|d_{\Omega_\infty}\nabla u(\hat{x})| > 1/2$.

For finite S_∞, Corollary 3.3.1 yields a contradiction to $|d_\Omega \nabla u(\hat{x})| > 1/2$. For $S_\infty = \infty$; checking the flux condition can be done analogously to Lemma 3.2.3 by replacing D by D^i to obtain $F(0) = 0$, then Theorem 3.3.1 yields the contradiction.

In the case (b), we can use the same argumentation of Lemma 3.2.3. If $\hat{x} \in \partial\Omega_\infty$, note that by the argumentation given in the aforementioned proof, C^2-regularity of the boundary is enough to deduce convergence to the half space or quadrant type space.

Case 2 can be dealt with analogously to the one in the proof of Lemma 3.2.3, since there is an $1 \leq i \leq m$ such that $x_k \in \Omega^i$ for sufficiently large k. $\square$

Theorem 3.3.2. *Let Ω be a C^2 cylindrical domain in $\mathbf{R}^n$ $(n \geq 2)$. Then there exists a constant C (depending only on D^i, Ω_0 and n) such that*

$$\|d_\Omega \nabla u\|_{L^\infty(\Omega)} \leq C\|g\|_{L^\infty(\partial\Omega)} \tag{3.3.8}$$

holds for all $\dot{H}^1$ weak solution $u \in L^1_{loc}(\overline{\Omega})$ of (3.2.8) in Ω with $\nabla u \in L^2(\Omega) \cap L^r(\Omega)$ (with some $r \geq n$) and $h = \operatorname{div}_{\partial\Omega} g \in C(\partial\Omega)$, where $g \in C(\partial\Omega) \cap L^\infty(\partial\Omega)$ satisfying $g \cdot n_\Omega = 0$ on $\partial\Omega$.

Proof. The proof is again analogous to the one given for Theorem 3.2.2. Instead of Lemma 3.2.3 we apply its cylindrical counterpart, Lemma 3.3.3 in order to estimate (3.3.8). $\square$

Subsequently, we obtain analogously to Theorem 3.2.2 the following result.

Theorem 3.3.3. *A C^2 cylindrical domain Ω in $\mathbf{R}^n$ $(n \geq 2)$ is admissible.*

Proof. The proof is analogous to the infinite cylinder case by applying Theorem 3.3.2 instead of Theorem 3.2.2. $\square$

3.4 Sector-like domains

Let S_θ denote $S_\theta = \{x = (x_1, x_2) \mid |\arg x| < \theta/2\}$, which is a sector in the plane $\mathbf{R}^2$ with opening angle $0 < \theta < 2\pi$. We say that a planar domain Ω is a sector-like domain with opening angle θ if $\Omega \backslash D_R = S_\theta \backslash D_R$ for some $R > 0$ (up to rotation and translation), where D_R is an open disk of radius R centered at the origin. According to [MB86, Example 2, Fig. 5], the L^p-Helmholtz decomposition fails for a sector-like domain when $p > q'_\theta$ or $p < q_\theta$ with $q_\theta = 2/(1+\pi/\theta)$ and $1/q_\theta + 1/q'_\theta = 1$ even if it is smooth. (For $p \in (q_\theta, q'_\theta)$ the L^p-Helmholtz decomposition holds [MB86].) Note that if the opening angle is larger than π, there always exists $p > 2$ such that the L^p-Helmholtz decomposition fails.

We will show that the existence of the L^p-Helmholtz decomposition may not be necessary for the analyticity of $S(t)$ in $L^\infty_\sigma(\Omega)$ type spaces.

Theorem 3.4.1. *Let Ω be a C^2 sector-like domain. Then $S(t)$ is a C_0-analytic semigroup in $C_{0,\sigma}(\Omega)$, the L^∞-closure of $C^\infty_{c,\sigma}(\Omega)$. (Moreover, $t\|\nabla^2 S(t)v_0\|_\infty / \|v_0\|_\infty$ is bounded in $(0,T)$ where $\|\cdot\|_\infty$ denotes the supremum norm in Ω.)*

Theorem 3.4.2. *Let Ω be a C^2 sector-like domain. Then Ω is admissible in the sense of [AG13, Definition 2.3].*

3.4.1 Uniqueness for the Neumann problem

We consider the uniqueness of the homogeneous Neumann problem

$$\Delta u = 0 \quad \text{in} \quad \Omega, \qquad \frac{\partial u}{\partial n_\Omega} = 0 \quad \text{on} \quad \partial\Omega, \tag{3.4.1}$$

where n_Ω is the unit exterior normal vector field of $\partial\Omega$.

Lemma 3.4.1. *Let Ω be a C^2 sector-like domain. Let $u \in C^2(\Omega) \cap C^1(\overline{\Omega})$ be a solution of* (3.4.1) *satisfying*

$$\|d_\Omega \nabla u\|_\infty < \infty \tag{3.4.2}$$

where $d_\Omega(x) = \inf_{y\in\partial\Omega} |x-y|$. Then u is a constant function.

Lemma 3.4.2. *Let $\Omega = S_\theta$. Let $u \in C^2(\Omega) \cap C^1\left(\overline{\Omega}\backslash \{0\}\right)$ be a solution of* (3.4.1) *(except $x = 0$) satisfying* (3.4.2)*. Assume that for some $R > 0$*

$$F(R) := \int_{\Gamma_R \cap \Omega} \frac{\partial u}{\partial r} d\mathcal{H}^1 = 0 \tag{3.4.3}$$

where $\Gamma_R = \partial D_R$ and $\partial/\partial r$ is the radial derivative. Then u is a constant function.

Remark 3.4.1. *The no flux condition* (3.4.3) *is necessary in Lemma 3.4.2. In fact $u = \log|x|$ solves* (3.4.1) *with* (3.4.2) *since $d_\Omega(x) = |x| \sin\left(\min\left((\theta/2 - \varphi), \pi/2\right)\right)$ for $\varphi = \arg x > 0$.*

A key step for the proof of both Lemmas is to show boundedness of a solution.

Lemma 3.4.3 (Boundedness)**.** *For $R > 0$ let $u \in C^2(S_\theta \backslash D_R) \cap C^1\left(\overline{S}_\theta \backslash D_R\right)$ satisfy*

$$\Delta u = 0 \quad in \quad S_\theta \backslash D_R, \quad \frac{\partial u}{\partial n_\Omega} = 0 \quad on \quad (\partial S_\theta) \backslash D_R. \tag{3.4.4}$$

Assume that $\|d_{S_\theta} \nabla u\|_\infty < \infty$ and $F(R_1) = 0$ for some $R_1 > R$. Then u is bounded in $S_\theta \backslash D_{R+\delta}$ for any $\delta > 0$.

Proof of Lemma 3.4.3. We may assume $R = 1$ by dilation. We use polar coordinates $x_1 = e^s \cos\varphi$, $x_2 = e^s \sin\varphi$ so that $S_\theta \backslash D_R$ is transformed to a region $\{(s, \varphi) \mid s \geq 0, |\varphi| < \theta\}$. The transformed dependent variable is denoted by U, i.e. $U(s, \varphi) = u(x_1, x_2)$. Then U solves

$$\Delta U = 0 \text{ in } \mathbf{R}_+ \times (-\theta, \theta), \ \frac{\partial U}{\partial \varphi} = 0 \text{ on } \mathbf{R}_+ \times \{\pm\theta\} = \Gamma \ \ (\mathbf{R}_+ = (0, \infty)) \tag{3.4.5}$$

and satisfies

$$|\nabla U| \leq C'/d(\varphi), \quad d(\varphi) = \operatorname{dist}((s, \varphi), \Gamma) \tag{3.4.6}$$

since $d_{S_\theta} \nabla u$ is bounded. Since (3.4.4) holds, integration by parts shows that the flux $F(R^*)$ is independent of R^*. Thus $F(R^*) = 0$ holds for all $R^* > 1$, which yields $F(e^s) = dE(s)/ds = 0$ for all $s > 0$ with $E(s) = \int_{-\theta}^{\theta} U(s, \varphi) d\varphi$ since $r\partial/\partial r = \partial/\partial s$. Thus $E(s)$ is a constant c independent of $s > 0$. We may assume $c = 0$ by subtracting c from U. Then (3.4.6) implies by the Poincaré type inequality [Eva10] that

$$\sup_{S_0 > 0} \|U : L^q\left((S_0, S_0 + 1) \times (-\theta, \theta)\right)\| < \infty$$

for any $q > 1$ (cf. [AGH15]). By a standard elliptic regularity theory this implies that U is bounded in $(\delta, \infty) \times (-\theta, \theta)$; see e.g. appendix of our companion paper [AGSS15]. $\square$

Proof of Lemma 3.4.2. As in the proof of Lemma 3.4.3 we use the polar coordinates. We observe that U satisfies (3.4.5) in $\mathbf{R} \times (-\theta, \theta)$. By Lemma 3.4.3 we observe that U is bounded for $s > 1$. A similar argument implies that U is also bounded for $s < -1$. Moreover, we may assume $E(s) = 0$ for all $s > 0$.

We shall prove that $U \equiv 0$ by the strong maximum principle [PW67, Section 3]. Assume that $U \not\equiv 0$. Then we may assume that $\sup U > 0$ by considering $-U$ if necessary. This supremum is not attained in $\mathbf{R} \times [-\theta, \theta]$. Indeed, if it is attained in the interior, then the strong maximum principle implies that $U \equiv \sup U > 0$ which contradicts the property $E(s) = 0$. If the maximum is taken on the boundary, $U \equiv \sup U$ since otherwise the Hopf lemma implies $\partial U/\partial\varphi > 0$ at that point which contradicts the Neumann condition. This again contradicts the property $E(s) = 0$.

We may assume that there is a sequence $z_m = (s_m, \varphi_m)$ such that $U(z_m) \to \sup U$ and $|s_m| \to \infty$ as $m \to \infty$. We may assume that $s_m \to \infty$ since the case $s_m \to -\infty$ can be treated similarly. We may assume that $\varphi_m \to \varphi_*$ for some $\varphi_* \in [-\theta, \theta]$ by taking a subsequence. We shift U and define $U_m(z) := U(s + s_m, \varphi)$ for $z = (s, \varphi)$ and observe that $\{U_m\}$ is a bounded sequence of solutions of (3.4.5) in $\mathbf{R} \times [-\theta, \theta]$. By elliptic regularity up to boundary for harmonic functions with homogeneous Neumann data U_m converges to some solution V of (3.4.5) in $\mathbf{R} \times [-\theta, \theta]$ with its first derivatives locally uniformly in $\mathbf{R} \times [-\theta, \theta]$. Then V satisfies $\int_{-\theta}^{\theta} V(s, \varphi) d\varphi = 0$ for all $s > 0$ and $V(0, \varphi_*) = \max V = \sup U > 0$. As before the strong maximum principle and the Hopf lemma implies that $V \equiv \sup U > 0$ which contradicts $\int_{-\theta}^{\theta} V(s, \varphi) d\varphi = 0$. We thus conclude that $U \equiv 0$. □

Proof of Lemma 3.4.1. By the assumption $\Omega \backslash D_{R_0} = S_\theta \backslash D_{R_0}$ for some $R_0 > 0$. By (3.4.1) we observe that $F(R) = 0$ for all $R > R_0$. By Lemma 3.4.3 we see that u is bounded in $\overline{\Omega}$. As in the proof of Lemma 3.4.3 we use the polar coordinates. We observe that U satisfies (3.4.5) in $(\log R_0, \infty) \times (-\theta, \theta)$. By the no flux condition $F(R) = 0$ we may assume that $E(s) = 0$ for $s \in (\log R_0, \infty)$ by adding a constant.

We shall prove that $u \equiv 0$ by the strong maximum principle. Assume that $u \not\equiv 0$. We may assume that $\sup u > 0$. If $\sup u$ is achieved in $\overline{\Omega}$, the strong maximum principle and the Hopf lemma implies $u \equiv \sup u$ as in the proof of Lemma 3.4.2. This contradicts $E(s) = 0$. If $\sup u$ is not achieved in $\overline{\Omega}$, then there is a sequence $z_m = (s_m, \varphi_m)$ such that $U(z_m) \to \sup U$ and $s_m \to \infty$, $\varphi_m \to \varphi_*$ for some $\varphi_* \in [-\theta, \theta]$. We define U_m as in the proof of Lemma 3.4.2 to observe that U_m solves (3.4.5) in $(\log R_0 - s_m, \infty) \times (-\theta, \theta)$. By elliptic regularity U_m converges to some solution V of (3.4.5) in $\mathbf{R} \times (-\theta, \theta)$ with its first derivatives locally uniformly in $\mathbf{R} \times [-\theta, \theta]$. This V fulfills the same property of V in the proof of Lemma 3.4.2 which contradicts $\int_{-\theta}^{\theta} V d\varphi = 0$. We thus conclude that $u \equiv 0$. □

3.4.2 Weighted L^∞ estimates for the Neumann problem

The idea to prove that an infinite cylinder is admissible by establishing Theorem 3.2.2 applies to a sector-like domain in $\mathbf{R}^2$ as discussed in [AGSS15]. We say that Ω is a sector-like domain if there is a closed ball B_R of radius R centered at the origin such that

$$\Omega \backslash B_R = S_\theta \backslash B_R$$

where $S_\theta = \left\{ x = (x_1, x_2) \in \mathbf{R}^2 \;\middle|\; |\arg x| < \theta/2 \right\}$. The point is that by a conformal transformation $x_1 = e^s \cos\varphi$, $x_2 = e^s \sin\varphi$ (often called the Kondrachev transformation) S_θ becomes an infinite cylinder. As proved in [AGSS15], we are able to prove that a C^2 sector-like domain is admissible.

It turns out that the proof of [AGSS15, Theorem 3.2] should be modified with slight modification of the statement. We shall give its rigorous statement.

Theorem 3.4.3. *Let Ω be a C^2 sector-like domain in $\mathbf{R}^2$. Then there exists a constant C such that the estimate* (3.2.13) *holds for all $\dot{H}^1$ weak solution*

u with $\nabla u \in L^2(\Omega)$ of (3.2.9) *in Ω with $h = \mathrm{div}_{\partial\Omega}\ g \in C(\partial\Omega)$, where $g \in C(\partial\Omega) \cap L^\infty(\partial\Omega)$ satisfies $g \cdot n_\Omega = 0$ on $\partial\Omega$.*

This is enough to prove the main theorem [AGSS15, Theorem 1.3] claiming that Ω is admissible.

To show Theorem 3.4.3 the lemma [AGSS15, Lemma 3.1] is too weak. It should be stated as Lemma 3.2.3. To state its explicit form let us fix the notations for a sector-like domain Ω with an opening angle θ. We may assume that $\Omega \backslash B_R = S_\theta \backslash B_R$ with some $R \in (0,1)$ by dilation, rotation and translation. We set $\Omega_R = \mathrm{int}\, B_{2R} \cap \Omega$. Let us state the statement which is stronger than the lemma [AGSS15, Lemma 3.1] and enough to prove Theorem 3.4.3. This lemma is considered as a variant of Lemma 3.2.3.

Lemma 3.4.4. *Let Ω be a C^2 sector-like domain in $\mathbf{R}^2$. Then there exists a constant C (depending only on Ω) such that*

$$\min\left(d_\Omega(x), \log(2R/|x|)\right)|\nabla u(x)| \leq C\|g\|_{L^\infty(\partial\Omega\cap B_{2R})} + \int_{\partial\Omega_R\cap\Omega}\left|\frac{\partial u}{\partial r}\right| d\mathcal{H}^1 \quad \text{for all} \quad x \in \Omega_R \tag{3.4.7}$$

holds for all $R \geq 1$ and all $\dot{H}^1$ weak solution u with $\nabla u \in L^2(\Omega_R)$ of (3.2.9) *in Ω_R with $h = \mathrm{div}_{\partial\Omega}\, g \in C(\partial\Omega)$ in $\partial\Omega \cap B_{2R}$, $h = \partial u/\partial r$ on $\partial\Omega_R \cap \Omega$, where $g \in C\left(\overline{\partial\Omega \cap B_{2R}}\right)$ satisfies $g \cdot n_\Omega = 0$ on $\partial\Omega \cap \Omega_R$ provided that $\|d\nabla u\|_{L^\infty(\Omega_R)}$ is finite and that $\partial u/\partial r|_{r=R} \in L^1(\partial\Omega_R \cap \Omega)$.*

Remark 3.4.2. *(i) The proof is parallel to that of Lemma 3.2.3 with modification needed in the proof of [AGSS15, Lemma 3.1], where $\partial u/\partial r = 0$ at $|x| = 2R$ is assumed. (In this case, $\min\left(d_\Omega(x), \log(2R/|x|)\right)$ can be replaced to $d_\Omega(x)$.) The quantity $\log(2R/|x|)$ is the distance from the boundary $S = \log 2R$ to a point $S_1 = \log|x|$ if we introduce the coordinates (s, φ) defined by $x_1 = e^s\cos\varphi$, $x_2 = e^s \sin\varphi$.*

(ii) If $\nabla u \in L^2(\Omega)$, then (3.4.7) *yields* (3.2.9) *as in the proof of Theorem 3.2.2. In fact, by the transformation $x_1 = e^s\cos\varphi$, $x_2 = e^s\sin\varphi$ we observe that*

$$\int_{\Omega\cap\{|x|>2R\}} |\nabla u|^2 dx_1 dx_2 = \int_S^\infty \int_I |\nabla_{s,\varphi}\overline{u}|^2 ds d\varphi, \quad I = (-\theta/2, \theta/2)$$

where $S = \log 2R$ and $\overline{u}(s,\varphi) = u(x_1, x_2)$. Thus if $\nabla u \in L^2(\Omega)$, then

$$\int_{\Omega\cap\{|x|>2R\}} |\nabla u|^2 dx_1 dx_2 \to 0$$

as $R \to \infty$. By this observation (3.2.9) *follows from* (3.4.7) *by taking square and integrating from $2R$ to $2R+1$.*

3.5 Stokes resolvent estimate

Let Ω be a uniformly C^2 domain in $\mathbf{R}^n$. In this section we establish an a priori L^∞ estimate for solutions of the resolvent Stokes equations under a zero Dirichlet condition (3.1.4) by a blow-up argument. The version we establish here is stronger than Theorem 3.1.2. We set

$$L^\infty_d(\Omega) = \left\{ f \in L^1_{loc}(\Omega) \;\middle|\; \|f\|_{L^\infty_d(\Omega)} := \operatorname{ess.sup}_{x\in\Omega} |d_\Omega(x) f(x)| < \infty \right\}.$$

We also use a standard notation $W^{m,r}$ for describing L^r-Sobolev space of order m and $W^{m,r}_{loc}$ for its localized version.

Theorem 3.5.1 (L^∞ a priori estimate)**.** *Let Ω be a uniformly C^2 domain in $\mathbf{R}^n$. Let $c_* > 0$. For $\varepsilon \in (0, \pi/2)$ there exist positive constants C and M depending only on Ω, c_*, ε such that*

$$\|N(v,\lambda)\|_\infty \le C\|f\|_\infty \tag{3.5.1}$$

with N as defined in (3.1.5) *for all $(v,p) \in \left(W^{1,\infty}(\Omega) \cap W^{2,r}_{loc}(\Omega)\right) \times L^\infty_d(\Omega)$ (with some $r > 1$) solving (3.1.4) with $\lambda \in \Sigma_{\pi-\varepsilon} \cap \{|\lambda| \ge M\}$ and $f \in L^\infty_\sigma(\Omega)$ provided that*

$$\|\nabla p\|_{L^\infty_d(\Omega)} \le c_* \|\nabla v\|_\infty. \tag{3.5.2}$$

Since admissibility of Ω implies (3.5.2), Theorem 3.5.1 yields Theorem 3.1.2. To prove (3.5.1) we argue by contradiction and apply a blow up argument as in [AG13], [AG14] but not for evolution equation but the resolvent equations (3.1.4) as in [Suz16] where elliptic problems are discussed by a blow-up argument. We construct a blow up sequence and prove strong convergence to a nontrivial solution with homogeneous problem which contradicts the uniqueness.

3.5.1 Construction of a blow up sequence

We argue by contradiction to prove (3.5.1). Suppose that (3.5.1) were not valid. Then there are $\varepsilon > 0$ and $\lambda_k \in \Sigma_{\pi-\varepsilon}$, $|\lambda_k| \ge k$, $f_k \in L^\infty_\sigma(\Omega)$, $(v_k, \nabla p_k)$ which solve (3.1.4) with

$$k\,\|f_k\|_{L^\infty(\Omega)} < \|N(v_k,\lambda_k)\|_\infty < \infty, \quad \|\nabla p_k\|_{L^\infty_d(\Omega)} \le c_*\,\|\nabla v_k\|_\infty\,.$$

We normalize (v_k, q_k) as

$$u_k = |\lambda_k| v_k / \|N(v_k,\lambda_k)\|_\infty\,, \quad q_k = p_k / \|N(v_k,\lambda_k)\|_\infty \quad \text{and} \quad \lambda_k = |\lambda_k| e^{i\theta_k}$$

and observe that (u_k, q_k) solves

$$\begin{aligned} \left(e^{i\theta_k} - \frac{\Delta}{|\lambda_k|}\right) u_k + \nabla q_k &= \tilde{f}_k \quad \text{in} \quad \Omega \\ \operatorname{div} u_k &= 0 \quad \text{in} \quad \Omega \\ u_k &= 0 \quad \text{on} \quad \partial\Omega \end{aligned}$$

where $\tilde{f}_k = f_k / \|N(v_k, \lambda_k)\|_\infty$. By definition we notice that

$$\left\| N\left(\frac{u_k}{|\lambda_k|}, \lambda_k\right)\right\|_\infty = 1, \quad \left\|\tilde{f}_k\right\|_\infty < 1/k, \quad |\lambda_k| \geq k, \quad |\theta_k| \leq \pi - \varepsilon$$

We next rescale (u_k, q_k) around the point x_k where $N(u_k/|\lambda_k|, \lambda_k)(x_k)$ is close to 1. We take a sequence $\{x_k\} \subset \Omega$ such that

$$|u_k(x_k)| + |\lambda_k|^{-1/2} |\nabla u_k(x_k)| > 1/2.$$

We rescale (u_k, q_k) and $\tilde{f}_k$ as

$$w_k(x) = u_k\left(x_k + \frac{x}{|\lambda_k|^{1/2}}\right) \qquad \varpi_k = |\lambda_k|^{1/2} q_k \left(x_k + \frac{x}{|\lambda_k|^{1/2}}\right)$$

$$g_k = \tilde{f}_k \left(x_k + \frac{x}{|\lambda_k|^{1/2}}\right).$$

Then we observe that rescaled (w_k, ϖ_k) solves

$$\begin{cases} \left(e^{i\theta_k} - \Delta\right) w_k + \nabla \varpi_k = g_k & \text{in} \quad \Omega_k \\ \operatorname{div} w_k = 0 & \text{in} \quad \Omega_k \\ w_k = 0 & \text{on} \quad \partial\Omega_k. \end{cases} \tag{3.5.3}$$

One can translate the estimates for (u_k, q_k) by (w_k, ϖ_k).

$$\sup_{x \in \Omega_k} |w_k(x)| + |\nabla w_k(x)| = 1, \quad |w_k(0)| + |\nabla w_k(0)| > 1/2, \quad \|g_k\|_\infty \leq 1/k, \tag{3.5.4}$$

where $\Omega_k = |\lambda_k|^{1/2}(\Omega - x_k)$. The pressure estimates becomes

$$\|d_{\Omega_k} \nabla \varpi_k\| \leq c_* \|\nabla w_k\|_\infty. \tag{3.5.5}$$

3.5.2 Convergence

We shall divide the situation into two cases depending on whether or not $\hat{d} = \limsup_{k\to\infty} d_k$ with $d_k = d(0, \partial\Omega_k) = |\lambda_k|^{1/2} d(x_k, \partial\Omega)$ is infinite.

Lemma 3.5.1 (Case 1, $\hat{d} = \infty$)**.** *Let $\{w_k\}$ be the blow up sequence. Then w_k converges to some $w \in W^{1,\infty}(\mathbf{R}^n)$ locally uniformly as $k \to \infty$ with $\|\nabla w_k\|_\infty \leq 1$ by taking a subsequence. Moreover, w solves a resolvent Laplace equation in the sense that there exists θ_m with $|\theta_\infty| \leq \pi - \varepsilon$ such that w satisfies*

$$\int_{\mathbf{R}^n} \nabla w \cdot \nabla \varphi \, dx + \int_{\mathbf{R}^n} e^{i\theta_\infty} w \varphi \, dx = 0 \tag{3.5.6}$$

for all $\varphi \in C_c^\infty(\mathbf{R}^n)$.

Proof. We may assume that $d_k \to \infty$ and $\theta_k \to \theta_\infty$ by taking a subsequence if necessary. Since Ω_k converges to $\mathbf{R}^n$, for $\varphi \in C_c^\infty(\mathbf{R}^n)$ we may assume that φ vanishes in neighborhood of $\partial\Omega_k$ if k is sufficiently large. For such k since (w_k, ϖ_k) solves (3.5.3), we see that

$$\int_{\mathbf{R}^n} \nabla w_k \cdot \nabla\varphi \, dx + \int_{\mathbf{R}^n} e^{i\theta_k} w_k \varphi \, dx + \int_{\mathbf{R}^n} \nabla\pi_k \cdot \varphi \, dx = \int_{\mathbf{R}^n} g_k \cdot \varphi \, dx. \quad (3.5.7)$$

By (3.5.4) we note that g_k converges to zero uniformly. Moreover, since $\|\nabla w_k\|_\infty \leq 1$ by (3.5.4), the estimate (3.5.5) implies that $\nabla\varpi_k$ tends to zero locally uniformly in $\mathbf{R}^n$. This implies that

$$\int_{\mathbf{R}^n} \nabla w_k \cdot \nabla\varphi \, dx + \int_{\mathbf{R}^n} e^{i\theta_k} w_k \cdot \varphi \, dx \to 0 \quad \text{as} \quad k \to \infty. \quad (3.5.8)$$

Since $\{w_k\}$ is bounded in $W^{1,\infty}(\Omega_k)$ by (3.5.4), there is a limit w such that $\nabla w_k \to \nabla w$ $*$ weakly in L^∞ and $w_k \to w$ locally uniformly by taking a subsequence; the latter convergence is by Ascoli-Arzelà theorem. The convergence (3.5.8) now yields (3.5.6). $\square$

The case $\hat{d} < \infty$ (Case 2) is more involved. The rescaled domain Ω_k converges to a half space of the form (up to rotation)

$$\mathbf{R}^n_{+,-c} = \{(x', x_n) \in \mathbf{R}^n \mid x_n > -c\}$$

with some $c > 0$. Indeed, we take a nearest boundary point $y_k \in \partial\Omega_k$ from 0. This y_k converges to some $\hat{y}$ and $y_k/|y_k|$ converges to some unit vector e by taking a subsequence. We rotate the coordinate so that $e = (0, \dots, -1)$ and that $\hat{y}_n = -c$ with $c \geq 0$. By this choice of coordinates one can prove that $\partial\Omega_k$ converges to $\partial\mathbf{R}^n_{+,-c}$ in the sense of C^2 graphs in a big cube.

Lemma 3.5.2 (Case 2, $\hat{d} < \infty$). *Assume that Ω_k converges to $\mathbf{R}^n_{+,-c}$. Let $\{(w_k, \varpi_k)\}$ be the blow up sequence. Then w_k converges to some $w \in W^{1,\infty}(\mathbf{R}^n_{+,-c})$ locally uniformly as $k \to \infty$ with $\|\nabla w_k\|_\infty \leq 1$ and $\nabla\varpi_k$ converges to some $\nabla\varpi \in L^\infty_d(\mathbf{R}^n_{+,-c})$ $*$ weakly in $L^\infty(\mathbf{R}^n_{+,b})$ (with $\varpi \in L^1_{loc}(\mathbf{R}^n_{+,-c})$) for any $b > -c$ by taking a subsequence. Moreover, (w, ϖ) solves the resolvent Stokes equation in $\mathbf{R}^n_{+,-c}$ in the sense that there exists θ_∞ with $|\theta_\infty| \leq \pi - \varepsilon$ such that (w, ϖ) satisfies*

$$\int_{\mathbf{R}^n_{+,-c}} \nabla w \cdot \nabla\varphi \, dx + \int_{\mathbf{R}^n_{+,-c}} e^{i\theta_\infty} w \cdot \varphi \, dx + \int_{\mathbf{R}^n_{+,-c}} \nabla\varpi \cdot \varphi \, dx = 0 \quad (3.5.9)$$

for all $\varphi \in C_c^\infty(\mathbf{R}^n_+)$. Furthermore, w satisfies $\operatorname{div} w = 0$ *in $\mathbf{R}^n_{+,-c}$ as well as $w = 0$ on $\partial\mathbf{R}^n_{+,-c}$.*

Proof. The proof parallels that of Lemma 3.5.1. We may assume that $d_k \to \hat{d}$, $\theta_k \to \theta_\infty$. Moreover, for $\varphi \in C_c^\infty(\mathbf{R}^n_{+,-c})$ we may assume that φ vanishes in a neighborhood of $\partial\Omega_k$. Since (3.5.5) (with (3.5.4)) now only yields a bound on $\|\nabla\varpi_k\|_{L^\infty(\operatorname{spt}\varpi)}$ uniformly for large k, the last term of the left-hand side of (3.5.7) does not vanish. Letting $k \to \infty$ in (3.5.7) yields (3.5.9) for a limit of (w_k, ϖ_k). The remaining assertion is easy to verify. $\square$

3.5.3 Uniform convergence near the origin

We shall prove that w_k converges to w uniformly together with its first derivative near the origin. For this purpose we recall $W^{2,r}$ estimates for the generalized Stokes system. Here L^r_{av} is the space of all L^r-function in Ω whose average over Ω equals zero.

Proposition 3.5.1 ($W^{2,r}$ estimates)**.** *Let Ω be a C^2 bounded domain in $\mathbf{R}^n$ and let $r \in (1,\infty)$. Then there exists a constant $c > 0$ depending only on r, n and the C^2-regularity of $\partial\Omega$ such that*

$$\|u\|_{W^{2,r}(\Omega)} + \|p\|_{W^{2,r}(\Omega)} \leq C\left(\|f\|_{L^r(\Omega)} + \|g\|_{W^{1,r}(\Omega)}\right)$$

for all solutions $(u,p) \in \left(W^{2,r}(\Omega) \cap W^{1,r}_0(\Omega)\right) \times \left(W^{1,r}(\Omega) \cap L^r_{\mathrm{av}}(\Omega)\right)$ of the Stokes system

$$-\Delta u + \nabla p = f, \quad \operatorname{div} u = g \quad \textit{in} \quad \Omega$$

with $u = 0$ on $\partial\Omega$.

This type of L^r estimates is by now very popular [BF13, Section b.3], [Gal11, Section 4.6]. It dates back to R. Farwig and H. Sohr [FS94], where the existence of a solution is also established.

Lemma 3.5.3. *Let $\{w_k\}$ and its limit w be as in Lemma 3.5.1 or Lemma 3.5.2. Then ∇w_k converges to ∇w locally uniformly near the origin as $k \to \infty$. In particular, $|w(0)| + |\nabla w(0)| \geq 1/2$.*

Proof. Let $\zeta \in C^\infty_c(\mathbf{R}^n)$ be a cut-off function of the form such that $\zeta(x) = 1$ on $B_1(0)$ and $\zeta(x) = 0$ outside $B_2(0)$ where $B_r(x)$ denotes the closed ball of radius r centered at x. We localize w_k by defining $W_k = \zeta w_k$. Then W_k solves

$$\begin{cases} -\Delta W_k + \nabla \Pi_{k,0} = \zeta g_k - e^{i\theta_k} W_k + E & \text{in} \quad \Omega_k \cap B_2(0) \\ \operatorname{div} W_k = w_k \cdot \nabla\zeta & \text{in} \quad \Omega_k \cap B_2(0) \\ W_k = 0 & \text{on} \quad (\partial\Omega_k \cap B_2(0)) \cup (\partial B_2(0) \cap \Omega_k), \end{cases}$$

where $\varpi_{k,0} = \varpi_k - \fint_{\Omega_k \cap B_2(0)} \varpi_k dx$, $\Pi_{k,0} = \zeta\varpi_{k,0}$ and E is the lower order terms of w_k and $\varpi_{k,0}$. It can be calculated as

$$E = -2\operatorname{div}(w\nabla\zeta) + w_k\Delta\zeta + \varpi_{k,0}\nabla\zeta$$

since $\Delta W_k = (\Delta w)\zeta + 2\operatorname{div}(w_k\zeta) - w_k\Delta\zeta$ and $\nabla(\varpi_{0,k}\zeta) = (\nabla\pi_{0,k})\zeta + \pi_{0,k}\nabla\zeta$ by Leibniz rule.

We mollify $\partial(\Omega_k \cap B_k(0))$ in an open neighborhood of $\partial(\Omega_k) \cap \partial(B_k(0))$ so that the boundary $\partial\Omega'_k$ of Ω'_k is C^2 and its C^2-regularity is uniform in k. Moreover, we may take Ω'_k so that

$$\Omega_k \cap B_2(0) \subset \Omega'_k \subset \Omega_k \cap B_3(0).$$

We now apply the $W^{2,r}$ estimate (Proposition 3.5.1) to get

$$\begin{aligned}
&\|W_k\|_{W^{2,r}(\Omega'_k)} + \|\nabla\Pi_k\|_{L^r(\Omega'_k)} \\
&\qquad \le C\left(\|-e^{i\theta_k}W_k + \zeta g_k + E\|_{L^r(\Omega'_k)} + \|w_k\nabla\zeta\|_{W^{1,r}(\Omega'_k)}\right) \\
&\qquad\qquad \le C\,|B_2(0)|^{\frac{1}{r}}\left(\|g_k\|_\infty + \|w_k\|_{W^{1,\infty}(\Omega_k)}\right) + C\|\varpi_{k,0}\|_{L^r(\Omega'_k)}.
\end{aligned}$$

Here we modify the definition of $\varpi_{k,0}$ by $\varpi_k - \fint_{\Omega'_k}\varpi_k dx$. By (3.5.4) we observe that

$$\|g_k\|_\infty + \|w_k\|_{W^{1,\infty}} \le 1 + 1/k \le 2.$$

By the Poincaré type inequality [AGH15, (2.1)] we have

$$\|\varpi_{k,0}\|_{L^r(\Omega'_k)} \le C\|\nabla\varpi_k\|_{L^\infty_d(\Omega'_k)}.$$

This together with (3.5.5) and bound on $\|\nabla W_k\|_\infty$ yields $\|\varpi_{k,0}\|_{L^r(\Omega'_k)} \le M$ with M independent of k. We thus observe that $\|W_k\|_{W^{2,r}(\Omega'_k)} + \|\nabla\Pi_k\|_{L^r(\Omega'_k)}$ is bounded in k. We take $r > n$ to get uniform bound for $\|w_k\|_{C^{1+\mu}(\Omega_k\cap B_1(0))}$ with $\mu = 1 - n/r$ from the above $W^{2,r}$-bound (Proposition 3.5.1) by Morrey's inequality [Eva10, 5.6.2]. By Ascoli-Arzelà theorem we conclude that $w_k \to w$ uniformly (with its first derivative) in $B_1(0)\cap\mathbf{R}^n_{+,-c}$ (Case 2) or in $B_2(0)$ (Case 1) since w_k itself converges to w locally uniformly.

Since $|w_k(0)|+|\nabla w_k(0)| \ge 1/2$ by (3.5.4), this implies that $|w(0)|+|\nabla w(0)| \ge 1/2$. □

3.5.4 Uniqueness of the limit problem

In this subsection we give a uniqueness result for the resolvent Laplace equation in $\mathbf{R}^n$ and the resolvent Stokes equations in $\mathbf{R}^n_+$ so that the limit w in Lemma 3.5.1 and Lemma 3.5.2 is identically zero.

Lemma 3.5.4 (Uniqueness in $\mathbf{R}^n$). *For $\mu \in \mathbf{C}\backslash(-\infty,0]$ assume that $w \in L^\infty(\mathbf{R}^n)$ satisfies*

$$\int_{\mathbf{R}^n} w(\mu-\Delta)\eta\,dx = 0 \tag{3.5.10}$$

for all $\eta \in C^\infty_c(\mathbf{R}^n)$. Then $w = 0$.

Lemma 3.5.5 (Uniqueness in $\mathbf{R}^n_+$). *For $\mu \in \mathbf{C}\backslash(-\infty,0]$ assume that $w \in W^{1,\infty}(\mathbf{R}^n_+)$ satisfies*

$$\int_{\mathbf{R}^n_+} w(\mu-\Delta)\eta\,dx + \int_{\mathbf{R}^n_+} \nabla q\cdot\eta\,dx = 0$$

with some $q \in L^1_{loc}(\overline{\mathbf{R}^n_+})$ such that $\nabla q \in L^\infty_d(\mathbf{R}^n_+)$ for all $\eta \in C^\infty_c(\mathbf{R}^n_+)$. If $w = 0$ on $\partial\mathbf{R}^n_+$, then $w = 0$, $\nabla q = 0$ in $\mathbf{R}^n_+$.

Proof of Theorem 3.5.1 admitting Lemma 3.5.4 and 3.5.5. Suppose that (3.5.1) were false. Then we have w as a blow up limit of a sequence of solutions as in Lemma 3.5.1 and 3.5.2. By Lemma 3.5.3 we know that $|w(0)| + |\nabla w(0)| \geq 1/4$. However, since w solves the resolvent Laplace equation (3.5.6) in $\mathbf{R}^n$ or the resolvent Stokes equation (3.5.9), we are able to apply the uniqueness results (Lemma 3.5.4 for the case $\hat{d} = \infty$ and Lemma 3.5.5 for the case $\hat{d} < \infty$) to get $w = 0$ which is a contradiction. The proof is now complete. □

The rest of this subsection is devoted to the proof of Lemma 3.5.4 and 3.5.5.

Proof of Lemma 3.5.4. Since $C_c^\infty(\mathbf{R}^n)$ is dense in $S(\mathbf{R}^n)$, the space of rapidly decaying functions in the sense of Schwartz, one may assume $\eta \in S(\mathbf{R}^n)$ in (3.5.10).

For a given $\psi \in C_c^\infty(\mathbf{R}^n)$ there is a solution $\eta \in S(\mathbf{R}^n)$ of $(\mu - \Delta)\eta = \psi$. This is easy to prove since the Fourier transform $\hat{\eta}$ of η is given by $\hat{\eta}(\zeta) = \left(\mu + |\zeta|^2\right)^{-1} \hat{\psi}$ and the Fourier transform is bijective on $S(\mathbf{R}^n)$. We thus observe that

$$\int_{\mathbf{R}^n} w \, \psi \, dx = 0$$

for all $\psi \in C_c^\infty(\mathbf{R}^n)$. By a fundamental theorem of calculus of variation this implies that $w = 0$ a.e. □

Lemma 3.5.5 can be proved as in [Sol03], where the uniqueness of the non-stationary problem has been proved. Note that Lemma 3.5.5 can be proved directly by a duality argument by the following existence result which can be proved by the Laplace transform of L^1 theory of the Stokes flow in a half space [GMS99].

Lemma 3.5.6 (Existence in L^1)**.** *Let $f \in C_{c,\sigma}^\infty(\mathbf{R}_+^n)$ be a vector field of the form $f(x) = \sum_{j=1}^{n-1} \partial_j \psi_j(x)$ for $\psi = (\psi_j) \in C_{c,\sigma}^\infty(\mathbf{R}_+^n)$. For $\mu \in \mathbf{C} \backslash (-\infty, 0]$ there exists $\eta \in W^{2,1}(\mathbf{R}_+^n)$, $\pi \in L_{loc}^1(\mathbf{R}_+^n)$ with $\nabla \pi \in L^1(\mathbf{R}_+^n)$ satisfying*

$$(\mu - \Delta)\eta + \nabla \pi = f, \quad \operatorname{div} \eta = 0 \quad \textit{in} \quad \mathbf{R}_+^n$$

with $\eta(x', 0) = 0$. Moreover, there exists a constant $C = C(\mu)$ such that

$$\|\eta\|_{W^{2,1}} \leq C \|\psi\|_{W^{2,1}}.$$

3.5.5 Elliptic local regularity

We shall prove a local regularity up to the boundary for an L^r (very) weak solution of the Neumann problem. Such regularity results are more or less known but it is not easy to find exact reference to our setting so we give a proof for the reader's convenience and completeness.

Theorem 3.5.2. *Let Ω be a domain in $\mathbf{R}^n$ $(n \geq 2)$. Let x_0 be a point on $\partial\Omega$. Then there exists a constant C depending only on $n, r \in (1,\infty)$, $R > 0$ and C^2-regularity of $\partial\Omega$ in $B_{2R}(x_0)$ such that*

$$\|\nabla^2 u\|_{L^r(\Omega_R)} \leq C\|u\|_{L^r(\Omega_{2R})}$$

for all $u \in L^r(\Omega_{2R})$ which satisfies $\Delta u = 0$ in Ω_{2R} and $\partial u/\partial n_\Omega = 0$ on $\operatorname{int} B_{2R}(x_0) \cap \partial\Omega$ in a (very) weak sense provided that $\partial\Omega$ is C^2 in $B_{2R}(x_0)$. Here $\Omega_R = \operatorname{int} B_R(x_0) \cap \Omega$.

To prove this statement we need to recall unique solvability of $W^{2,q}$ solution and uniqueness of L^r (very) weak solution.

Lemma 3.5.7. *Let Ω be a C^1 bounded domain in $\mathbf{R}^n$ and $q \in (1,\infty)$.*

(i) *For a given $f \in W^{-1,q}(\Omega)$ satisfying $\int_\Omega f\ dx = 0$ there exists a unique solution $u \in W^{1,q}(\Omega)$ with $\int_\Omega u\ dx = 0$ (a standard weak solution based on bilinear form) of*

$$-\Delta u = f \quad in \quad \Omega \quad and \quad \frac{\partial u}{\partial n_\Omega} = 0 \quad on \quad \partial\Omega.$$

Moreover, there exists a constant C depending only on q, n and C^1-regularity of $\partial\Omega$ such that

$$\|\nabla u\|_{L^q(\Omega)} \leq C\|f\|_{W^{-1,q}(\Omega)}.$$

(ii) *Assume that Ω is C^2. If f is in $L^q(\Omega)$, then $u \in W^{2,q}(\Omega)$. Moreover,*

$$\|\nabla^2 u\|_{L^q(\Omega)} \leq C\|f\|_{L^q(\Omega)}.$$

The first statement under C^1-regularity assumption is taken from Simader and Sohr [SS92]. Both statements with C^2-regularity is rather standard and classical; see e.g. [LM68, Teor. 4.1].

Lemma 3.5.8 (Uniqueness of L^r-solution)**.** *Let Ω be a C^2 bounded domain in $\mathbf{R}^n$. For $r \in (1,\infty)$ let $u \in L^r(\Omega)$ be a (very) weak solution to the homogeneous Neumann problem, i.e., $\int_\Omega v\Delta u dx = 0$ for all $v \in C^2(\overline{\Omega})$ such that $\partial v/\partial n_\Omega = 0$. Then u must be a constant.*

Proof of Lemma 3.5.8. This can be proved by a duality argument. We may assume $r < n/(n-1)$ since Ω is bounded. For a given $f \in C_c^\infty(\Omega)$ with $\int_\Omega f\ dx = 0$, there is a unique $W^{2,r'}$ solution v for $-\Delta v = f$ in Ω and $\partial v/\partial n_\Omega = 0$ on $\partial\Omega$ with $\int_\Omega v\ dx = 0$ by Lemma 3.5.7 (ii) where r' is the conjugate exponent of r, i.e., $1/r + 1/r' = 1$. By definition of a very weak solution we see that

$$\int_\Omega u\Delta v\ dx = 0.$$

In the definition we need $v \in C^2(\overline{\Omega})$ but it can be replaced by $v \in C^1(\overline{\Omega}) \cap W^{2,r'}(\Omega)$ by an approximation. Indeed, for $v \in C^1(\overline{\Omega}) \cap W^{2,r'}(\Omega)$ with $\partial v/\partial n_\Omega =$

0 on $\partial\Omega$ and $\int_\Omega v\,dx = 0$, we approximate the boundary value $v \in W^{2-1/r',r'}(\partial\Omega)$ by $w_m \in C^2(\partial\Omega)$ such that $\|v - w_m\|_{W^{2-1/r',r'}} \to 0$. Since the boundary is C^2, there is a bounded linear extension operator E from $W^{2-1/r',r'}(\partial\Omega) \times W^{1-1/r',r'}(\partial\Omega)$ to $W^{2,r'}(\Omega)$ such that $E(w,g)|_{\partial\Omega} = w$, $(\partial/\partial n_\Omega)\ E(w,g)|_{\partial\Omega} = g$ and moreover, $E(w,g) \in C^2(\overline{\Omega})$ if $w \in C^2(\partial\Omega)$, $g \in C^1(\partial\Omega)$; see [AF03]. (Such an extension operator is for example found in Lemma 3.5.9 and Lemma 3.5.10. We set $v_m = E(w_m, 0) - \fint E(w_m,0)dx$ and observe that $\|v_m - v\|_{W^{2,r'}} \to 0$ and that $v_m \in C^2(\overline{\Omega})$ with $\int_\Omega v_m dx = 0$. The desired identity $\int_\Omega u\Delta v\,dx = 0$ follows from $\int_\Omega u\Delta v_m dx = 0$ since $v_m \to v$ in $W^{2,r'}(\Omega)$. We thus conclude that $\int_\Omega uf\,dx = 0$ for all $f \in C_c^\infty(\Omega)$ with the average zero condition. This implies that u is a constant. $\square$

Proof of Theorem 3.5.2. We use a cut-off function φ satisfying the homogeneous Neumann condition constructed in Lemma 3.5.10. Since Ω_{2R} is not C^2, we consider a C^2 bounded domain which is slightly larger domain $\tilde{\Omega}_{2R}$. We consider $v = u\varphi$ and observe that v is an L^r very weak solution of

$$-\Delta v = f \quad \text{in} \quad \tilde{\Omega}_{2R} \quad \text{and} \quad \partial v/\partial n_\Omega = 0 \quad \text{on} \quad \partial\tilde{\Omega}_{2R}$$

with $f = -2\,\mathrm{div}(u\nabla\varphi) - u\Delta\varphi \in W^{-1,r}(\tilde{\Omega}_{2R})$. By the existence (Lemma 3.5.7 (i)) and the uniqueness (Lemma 3.5.8) there exists a constant C' depending on φ such that

$$\|\nabla u\|_{L^r(\tilde{\Omega}_{2R})} \le C'\|u\|_{L^r(\tilde{\Omega}_{2R})}.$$

In particular, $v \in W^{1,r}(\tilde{\Omega}_{2R})$. We may take the cut-off function so that $\varphi \equiv 1$ on $\Omega_{4R/3}$, then we observe that $u \in W^{1,r}(\Omega_{4R/3})$.

We repeat this argument in $\tilde{\Omega}_{4R/3}$ with a cut-off function such that $\varphi \equiv 1$ on $\tilde{\Omega}_R$ with $\varphi \in C_c^2(B_{4R/3})$ satisfying $\partial\varphi/\partial n_\Omega = 0$ on $\partial\Omega$ (Lemma 3.5.10). We now apply Lemma 3.5.7 (ii) to get the desired $W^{2,r}$-estimate in Ω_R. Such an argument is often called a bootstrap argument. $\square$

3.5.6 Construction of cut-off functions

We shall construct cut-off functions which keep the Neumann boundary condition. We begin with an extension lemma. This is by now a standard way to extend which is for example found in [BF13, 2.4].

Lemma 3.5.9. *Let $k \ge 1$ be a natural number and R, $R' > 0$. Let $f \in C_c^k(\mathbf{R}^{n-1})$, $g \in C_c^{k-1}(\mathbf{R}^{n-1})$ satisfy $\mathrm{spt}\, f$, $\mathrm{spt}\, g \subset \mathrm{int}\, B_{2R}^{n-1}$ and $g = 0$ on $B_{4R/3}^{n-1}$. Then there exists $\psi \in C_c^k(\mathbf{R}^n)$ such that*

$$\psi(x',0) = f(x'), \quad \frac{\partial\psi}{\partial x_n}(x',0) = g(x'), \quad x' = (x_1,\ldots,x_{n-1}) \in \mathbf{R}^{n-1}$$

and $\mathrm{spt}\,\psi \subset \mathrm{int}\, B_{2R}^{n-1} \times (-2R', 2R')$ and $\psi(x',x_n) = f(x')$ on $B_R^{n-1} \times [-R', R']$. If $0 \le f \le 1$, then ψ can be taken such that $0 \le \psi \le 1$.

Proof. Let ρ_ε be a symmetric Friedrichs' mollifier in $\mathbf{R}^{n-1}$. In other words, we take a non-increasing function $\eta \in C_c^\infty[0,\infty)$ such that $\eta = 1$ on $[0,1/2]$ and $\eta = 0$ on $[2,\infty)$ and $0 \leq \eta \leq 1$ and that $\int_0^2 \eta(s)ds = 1$ and define $\rho_\varepsilon(x') = \eta\left(|x'|/\varepsilon\right)\varepsilon^{1-n}$ for $\varepsilon > 0$. Let a and b be small positive parameters to be determined later. We set

$$\psi(x', x_n) := f(x')\eta(x_n/R') + \left(\rho_{a|x_n|} * g\right)(x')\eta(x_n/b)x_n, \ x_n \neq 0$$

and $\psi(x', 0) = f(x')$, when $*$ denotes the convolution in $\mathbf{R}^{n-1}$. The parameter $a > 0$ is taken small so that $\rho_{a|x_n|} * g = 0$ on $B_R^{n-1} \times [-R', R']$. The parameter $b > 0$ is taken small so that $0 \leq f \leq 1$ implies $0 \leq \psi \leq 1$. Since ψ is C^k outside $x_n = 0$, it suffices to prove that ψ is C^k at $x_n = 0$ and satisfies $\partial\psi/\partial x_n = g$ at $x_n = 0$.

To show C^1 property at $x_n = 0$ at $\partial\psi/\partial x_n = g$ it suffices to prove that

$$\lim_{x_n \to 0} |x_n| \left\| \partial_{x_n}\left(\rho_{a|x_n|} * g\right)\right\|_\infty = 0, \tag{3.5.11}$$

$$\lim_{x_n \to 0} |x_n| \left\| \partial_{x'}\left(\rho_{a|x_n|} * g\right)\right\|_\infty = 0, \tag{3.5.12}$$

where $\|\cdot\|_\infty$ denotes the L^∞ norm in $\mathbf{R}^{n-1}$. We first prove (3.5.11) when $g \in C_c^1(\mathbf{R}^{n-1})$. Since $\rho_s(x') = s^{1-n}\eta\left(|x'|/s\right)$, we observe that $\partial_s\rho_s = -s^{-1}\operatorname{div}'(x'\rho_s)$, where div' is the divergence in x' variable. This implies

$$|x_n|\partial_{x_n}(\rho_{a|x_n|} * g) = -\operatorname{div}'(x'\rho_s) * g = -x'\rho_s * \nabla' g, \ s = a|x_n|$$

where ∇' denotes the gradient in x' variable. Since $\|x'\rho_s\|_{L^1(\mathbf{R}^{n-1})} \leq C|x_n|$ with some C independent of x_n, we observe that

$$\left\| |x_n|\partial_{x_n}(\rho_{a|x_n|} * g)\right\|_\infty \leq C|x_n|\|\nabla' g\|_\infty \to 0 \quad \text{as} \quad x_n \to 0$$

so (3.5.11) is proved for $g \in C_c^1(\mathbf{R}^{n-1})$.

For general $g \in C_c(\mathbf{R}^{n-1})$, we approximate g by $g_m \in C_c^1(\mathbf{R}^{n-1})$ so that $\|g - g_m\|_\infty \to 0$ as $m \to 0$. Since $\left\| |x_n|\,\partial_{x_n}\left(\rho_{a|x_n|} * g\right)\right\|_\infty \leq C\|g\|_\infty$, with C independent of g and $x_n \neq 0$ and $\left\| |x_n|\,\partial_{x_n}\left(\rho_{a|x_n|} * g_m\right)\right\|_\infty \to 0$ as $x_n \to 0$ for $g_m \in C_c^1(\mathbf{R}^{n-1})$, we conclude that

$$\begin{aligned}\varlimsup_{x_n \to 0} \left\| |x_n|\,\partial_{x_n}\left(\rho_{a|x_n|} * g\right)\right\|_\infty &\leq \varlimsup_{x_n \to 0} \left\| |x_n|\,\partial_{x_n}\left(\rho_{a|x_n|} * g_m\right)\right\|_\infty \\ &+ \varlimsup_{x_n \to 0} \left\| |x_n|\,\partial_{x_n}\left(\rho_{a|x_n|}(g - g_m)\right)\right\|_\infty \leq 0 + C\left\|g - g_m\right\|_\infty.\end{aligned}$$

This yields (3.5.11) by sending $m \to \infty$. The proof for (3.5.12) is similar. A similar argument yields higher regularity if $k \geq 2$; we omit the detail. □

We now construct a cut-off function near the boundary such that the normal derivative equals zero. We are tempting to use the normal coordinates but to get a C^2 cut-off function we need C^3 regularity of the boundary. We won't use the normal coordinates here. For a given $h \in C^k\left(B_{2R}^{n-1}\right)$ with $h(0) = 0$ let

$$\Omega_h = \left\{(y', y_n) \mid y_n > h(y'), \ y' \in B_{2R}^{n-1}\right\}. \tag{3.5.13}$$

Lemma 3.5.10. *Let $k \geq 1$ be a natural number and $R > 0$. Let $\Omega = \Omega_h$ be as in (3.5.13). Then there exists $\varphi \in C_c^k\left(\operatorname{int} B_{2R} \cap \overline{\Omega}\right)$ such that $0 \leq \varphi \leq 1$ in $B_{2R} \cap \Omega$ and $\partial\varphi/\partial n_\Omega = 0$ on $\partial\Omega \cap B_{2R}$ with $\varphi = 1$ on $B_R \cap \Omega$, where $B_R = B_R^n$.*

Proof. We flatten the boundary by

$$x_n = y_n - h(y'), \quad x' = y'. \tag{3.5.14}$$

This transformation $x = T(y)$ is C^k. We write φ with new independent variables x and still denoted by φ. The condition $\partial\varphi/\partial n_\Omega = 0$ is transformed into

$$\frac{\partial\varphi}{\partial x_n} - \frac{\nabla' h}{\sqrt{1 + |\nabla' h|^2}} \cdot \nabla'\varphi = 0 \quad \text{at} \quad x_n = 0. \tag{3.5.15}$$

Let f be $C_c^k\left(\mathbf{R}^{n-1}\right)$ such that $f = 1$ on $T\left(B_{4R/3} \cap \Omega\right)$ and $f = 0$ on $T\left(B_{2R} \cap \Omega\right)$. We set

$$g = \frac{\nabla' h}{\sqrt{1 + |\nabla' h|^2}} \cdot \nabla' f \in C_c^{k-1}\left(\mathbf{R}^{n-1}\right).$$

Applying C^k-extension lemma (Lemma 3.5.9) yields desired φ by choosing R and R' appropriately. Note that the numbers R, $4R/3$, $2R$ do not have a particular meaning. We may take R, $C_1 R$, $C_2 R$ for $1 < C_1 < C_2$ in Lemma 3.5.9 to apply to construct φ so that φ satisfies $\varphi = 1$ in $B_R \cap \Omega$; we shall cut-off outside B_{2R} if necessary to fulfill $\operatorname{spt}\varphi \subset \operatorname{int} B_{2R} \cap \Omega$. ☐

We conclude this section to give a way to approximate test functions φ in $\mathbf{R}_+^n$ satisfying $\partial\varphi/\partial x_n = 0$ on the boundary by a similar test function in a domain approximating $\mathbf{R}_+^n$.

Lemma 3.5.11. *Assume the same hypothesis of Lemma 3.5.10. Let $\varphi \in C_c^k(\overline{\mathbf{R}_+^n})$ satisfy $\partial\varphi/\partial x_n = 0$ at $x_n = 0$ and $\operatorname{spt}\varphi \subset B_R$. Then there exists $\varphi_h \in C_c^k(\overline{\Omega}_h)$ which fulfills $\partial\varphi_h/\partial n_{\Omega_h} = 0$ on $\partial\Omega_h$ with $\operatorname{spt}\varphi_h \subset B_{4R/3}$ such that*

$$\|\varphi_h - \varphi\|_{L^\infty(\Omega_{h+})} \leq C_1 \|\nabla h\|_\infty$$
$$\|\nabla\varphi_h - \nabla\varphi\|_{L^\infty(\Omega_{h+})} \leq C_1 \|\nabla h\|_\infty$$

with a constant C_1 depending only on n and a bound for $\|\nabla h\|_\infty$. The mapping $\varphi \mapsto \varphi_h$ can be taken a linear operator. If $k \geq 2$, then

$$\|\nabla^2\varphi_h - \nabla^2\varphi\|_{L^\infty(\Omega_{h+})} \leq C_2\left(\|\nabla h\|_\infty + \|\nabla^2 h\|_\infty\right)$$

with C_2 depending only on a bound for $\|\nabla h\|_\infty$ and $\|\nabla^2 h\|_\infty$. Here $\Omega_{h+} = \Omega_h \cap \mathbf{R}_+^n$.

Proof. We again use the transformation T given in (3.5.14). Since T is a C^k-transformation, it suffices to construct $\Phi_h \in C_c^k(\overline{\mathbf{R}_+^n})$ satisfying (3.5.15) with $\varphi = \Phi_h$ and $\operatorname{spt}\Phi_h \subset B_{R_0}$ with R_0 slightly bigger than R so that $T^{-1}(B_{R_0}) \subset B_{4R/3}$ as well as following estimates:

$$\|\Phi_h - \varphi\|_{L^\infty(\mathbf{R}_+^n)} \leq C_1 \|\nabla h\|_\infty \tag{3.5.16}$$

$$\|\nabla\Phi_h - \nabla\varphi\|_{L^\infty(\mathbf{R}^n_+)} \leq C_1\|\nabla h\|_\infty \tag{3.5.17}$$

$$\|\nabla^2\Phi_h - \nabla^2\varphi\|_{L^\infty(\mathbf{R}^n_+)} \leq C_2\left(\|\nabla h\|_\infty + \|\nabla^2 h\|_\infty\right) \quad \text{if} \quad k \geq 2. \tag{3.5.18}$$

For this purpose we construct

$$\Phi_h(x', x_n) = \varphi(x', x_n) + \left(\rho_{a|x_n|} * g\right)\eta(x_n/b)x_n$$

with

$$g = \frac{\nabla h'}{\sqrt{1 + |\nabla' h|^2}} \cdot \nabla'\varphi \in C^{k-1}(\mathbf{R}^{n-1}),$$

where $\rho,\ \eta$ is taken as in the proof of Lemma 3.5.9. The positive parameter a and b are taken small so that $\operatorname{spt}\rho_{a|x_m|} * g \subset B_{R_0}$. This Φ_h which linearly depends on φ fulfills all desired properties as shown below.

Since $\partial\varphi/\partial x_n = 0$ and $\Phi_h = \varphi$ on the boundary $\{x_n = 0\}$, Φ_h fulfills

$$\frac{\partial\Phi_h}{\partial x_n} = g = \frac{\nabla h'}{\sqrt{1 + |\nabla' h|^2}} \cdot \nabla'\Phi_h \quad \text{at} \quad x_n = 0,$$

which is nothing but (3.5.15). By the Hausdorff-Young inequality one observes that the operator defined by $U(x_n)g = \rho_{a|x_n|} * g = g * \rho_{a|x_n|}$ fullfills

$$\|U(x_n)g\|_\infty \leq C\|g\|_\infty$$

$$\||x_n|\partial_{x_n}U(x_n)g\|_\infty \leq C\|g\|_\infty, \quad \||x_n|\nabla' U_{x_n}(x_n)g\|_\infty \leq C\|g\|_\infty \tag{3.5.19}$$

with C independent of $x_n \in \mathbf{R},\ x_n \neq 0$. This implies (3.5.16) and (3.5.17) since $\Phi_h - \varphi = (U(x_n)g)\,\eta(x_n/b)x_n$. Similarly, the estimate (3.5.18) follows from above estimate as well as

$$\||x_n|\nabla\nabla' U(x_n)g\|_\infty \leq C\|\nabla' g\|, \quad \left\||x_n|\partial^2_{x_n}U(x_n)g\right\|_\infty \leq C\|\nabla' g\|_\infty.$$

The first one follows from (3.5.19) while the second one follows from the observation that

$$\left(U(x_n)g\right)(x') = a^{1-n}\int_{\mathbf{R}^{n-1}} \eta\left(\frac{z'}{a}\right) g\left(x' - |x_n|z'\right) dz'.$$

As in the proof of (3.5.11) the function $U(x_n)g \in C^k_c(\overline{\mathbf{R}^n_+})$ which implies that $\Phi_h \in C^k_c(\overline{\mathbf{R}^n_+})$. □

3.5.7 Existence in L^1

In order to consider the existence problem in the uniqueness step, we introduce the solution formula in non-stationary case obtained by Ukai.

We begin with the statement of the uniqueness theorem.

Lemma 3.5.12. *Let $|\theta_\infty| \le \pi - \epsilon$, $w \in \tilde{W}^{2,r}_{loc} \cap W^{1,\infty}(\mathbb{R}^n_+)$, $q \in \tilde{W}^{1,r}_{loc} \cap L^\infty(\mathbb{R}^n_+)$, and $\|\nabla q\|_{L^\infty_d(\mathbb{R}^n_+)} < +\infty$. If w, q satisfy for $\eta \in C^\infty_0(\mathbb{R}^n_+)$*

$$\int_{\mathbb{R}^n_+} w(e^{i\theta_\infty} - \Delta)\eta + (\nabla q)\eta dx = 0,$$

then $w = 0, \nabla q = 0$.

We need a solution formula of the following dual problem in $\mathbb{R}^n_+$ in order to obtain the uniqueness result. Dual problems imply the uniqueness step in half space by the argument in [AG13]. Before stating our solution formula, we state the dual problem and prepare some notations for our solution formula.

Lemma 3.5.13 (Dual problem when Ω_∞ is the half space $\mathbb{R}^n_+$)**.** *Let $|\theta_\infty| \le \pi - \epsilon$, $f \in C^\infty_{c,\sigma}(\mathbb{R}^n_+)$ be a vector field of the form $f(x) = \sum_{j=1}^{n-1} \partial_j \psi_j(x)$ for $\psi_j \in C^\infty_{c,\sigma}(\mathbb{R}^n_+)$. Then, there exist $\eta \in W^{2,1}(\mathbb{R}^n_+)$, $\pi \in L^1_{loc}(\mathbb{R}^n_+)$ with $\nabla\pi \in L^1(\mathbb{R}^n_+)$ such that*

$$\begin{cases} (e^{i\theta_\infty} - \Delta)\eta + \nabla\pi = f & \text{in } \mathbb{R}^n_+ \\ \mathrm{div}\eta = 0 & \text{in } \mathbb{R}^n_+ \\ \eta(x', 0) = 0 & \text{on } \partial\mathbb{R}^n_+, \end{cases} \tag{3.5.20}$$

with the estimates

$$\|\eta\|_{W^{2,1}} \le C\|\Psi\|_{W^{2,1}}. \tag{3.5.21}$$

Let us introduce the solution formula in non-stationary case obtained in [Uka87]. Although we can consider this resolvent problem directly, it is very convenient for us to consider non-stationary case since non-stationary solutions can be written explicitly. We want to obtain resolvent estimates by the Laplace transform of non-stationary solutions. Let us define The Riesz operators R_j, S_j, and the operator $\Lambda = \Delta^{\frac{1}{2}}_{tan}$ by

$$F(R_j f)(\xi) = \frac{i\xi_j}{|\xi|} Ff(\xi),\ F(S_j f)(\xi) = \frac{i\xi_j}{|\xi'|} Ff(\xi),\ F(\Lambda f)(\xi) = |\xi'| Ff(\xi).$$

We set $R_{tan} = (R_1, \cdots, R_{n-1})$, $S_{tan} = (S_1, \cdots, S_{n-1})$, and define the operator U by

$$Uf = rR_{tan} \cdot S_{tan}(R_{tan} \cdot S_{tan} + R_n)E,$$

where $r = r_{\mathbb{R}^n \to \mathbb{R}^n_+}$ is the restriction operator from $\mathbb{R}^n$ to $\mathbb{R}^n_+$ and $E = E_{\mathbb{R}^n_+ \to \mathbb{R}^n}$ is the zero extension from $\mathbb{R}^n_+$ to $\mathbb{R}^n$. Let G_t be the Gauss kernel, then we can define the solution operators $E(t)$, $E_N(t)$ to the heat equation in $\mathbb{R}^n_+$ with Dirichlet (resp. Neumann) data by

$$E(t)f(x) = \int_{\mathbb{R}^n_+} \{G_t(x-y) - G_t(x'-y', x_n+y_n)\} f(y)dy,$$
$$E_N(t)f(x) = \int_{\mathbb{R}^n_+} \{G_t(x-y) + G_t(x'-y', x_n+y_n)\} f(y)dy.$$

And we define the operators V_1 and V_2 by

$$V_1 f = -S_{tan} \cdot f_{tan} + f^n,\ V_2 f = f_{tan} + S_{tan} f^n.$$

We recall the solution formula of non-stationary Stokes problem in $\mathbb{R}^n_+$ obtained in [Uka87].

Theorem 3.5.3 (The solution formula to non-stationary Stokes problem in $\mathbb{R}^n_+$)**.** *Let u be a vector field*

$$\begin{aligned} u_{tan} = u' &= E(t)V_2 u_0 - SUE(t)V_1 u_0, \\ u_n &= UE(t)V_1 u_0. \end{aligned} \tag{3.5.22}$$

Then, u satisfies the non-stationary Stokes problem in $\mathbb{R}^n_+$.

We also recall gradient L^1 estimates for the solution of non-stationary Stokes problem in $\mathbb{R}^n_+$ obtained in [GMS99].

Theorem 3.5.4 (Gradient L^1 estimates for non-stationary Stokes problem in $\mathbb{R}^n_+$)**.** *Let u be the solution of the form (3.5.7) for the Stokes equation in $\mathbb{R}^n_+$ under the zero Dirichlet boundary condition with a initial data $u_0 \in L^1(\mathbb{R}^n_+)$. Then there exists a constant C independent of u_0 such that*

$$\|\nabla u(t)\|_{L^1} \leq C t^{-\frac{1}{2}} \|u_0\|_{L^1}. \tag{3.5.23}$$

This estimate is obtained by a explicit calculation of the solution formula. Gradient L^1 estimates for the heat equation in $\mathbb{R}^n_+$ is the key in the explicit calculation.

Let us return to the resolvent problem. Let us define the Laplace transform with respact to the time valuable. Let u be a solution of non-stationary stokes problem in $\mathbb{R}^n_+$:

$$\begin{cases} \partial_t u - \Delta u + \nabla \Pi = g,\ \mathrm{div}\eta = 0 & \text{in } \mathbb{R}^n_+ \\ u(t,x) = 0 & \text{on } (0,\infty) \times \partial\mathbb{R}^n_+, \\ u(0,x) = f & \text{on } \mathbb{R}^n_+. \end{cases} \tag{3.5.24}$$

If $\mathrm{Re}\lambda > 0$, the Laplace transform L can be defined as a usual way,

$$Lg(\lambda) = \int_0^\infty g(t) e^{-\lambda t} dt.$$

But, in more general case, we have to change the integral route in order to obtain the convergence of the integral. For each $0 < \gamma < \frac{\pi}{2} - \epsilon$, $\|u(t)\|_{L^2(\mathbb{R}^n_+)}$ is uniformly bounded on $\{t\ ; |\arg t| \leq \gamma\}$ and strongly continuous at $t = 0$. This implies the convergence in L^2 sense of

$$Lu(\lambda) = \int_\Gamma e^{-\lambda t} u(t) dt,$$

where Γ is the half line $\{re^{\mp i\gamma};\ 0 < r < \infty\}$ when $\pm\mathrm{Im}(\lambda) > -\mathrm{Re}(\lambda)\cot\gamma$. Then we obtain a solution formula of the Stokes resolvent problem in $\mathbb{R}^n_+$ by the Laplace transform, and obtain gradient estimates in L^1 space.

Theorem 3.5.5 (Solution formula to Stokes resolvent equation in $\mathbb{R}^n_+$). *Let $(u,\ \pi)$ be a solution of (3.5.24). Then the Laplace transform ($\eta = Lu,\ \pi = L\Pi$) of $(u,\ \Pi)$ satisfy*

$$\begin{cases} (\lambda - \mathit{\Delta})\eta + \nabla\pi = f + Lg & \text{in } \mathbb{R}^n_+ \\ \mathrm{div}\eta = 0 & \text{in } \mathbb{R}^n_+ \\ \eta(x', 0) = 0 & \text{on } \partial\mathbb{R}^n_+. \end{cases} \tag{3.5.25}$$

Furthermore, we obtain a solution formula for Stokes resolvent equation and resolvent L^1 gradient estimates,

$$\begin{aligned} \eta_{tan} &= F(\lambda)V_2 f - SUF(\lambda)V_1 f, \\ \eta_n &= UF(\lambda)V_1, \end{aligned} \tag{3.5.26}$$

where $F(\lambda) = (\lambda - \mathit{\Delta}_D)^{-1}$ be a solution operator of the resolvent Laplace equation in $\mathbb{R}^n_+$ with zero Dirichlet condition,

$$\|\nabla\eta\|_{L^1} \leq C|\lambda|^{-\frac{1}{2}}\|f\|_{L^1}. \tag{3.5.27}$$

Proof. Let u be a solution of (3.5.24). Then we apply the Laplace transform to (3.5.24). Since $L(\partial_t u) = \lambda Lu - u(0) = \lambda Lu - f$, $\eta = Lu$ then satisfy (3.5.25). Furthermore, we consider L^1 gradient resolvent estimates. In order to obtain this L^1 estimates, we need to apply Theorem 3.5.4 when t is included in Γ, i.e., we need to show

$$\|\nabla u(t)\|_{L^1} \leq C|t|^{-\frac{1}{2}}\|u_0\|_{L^1} \text{ for } t \in \Gamma. \tag{3.5.28}$$

We remark that we only have to check gradient $\mathcal{H}^1(\mathbb{R}^n_+)$-$L^1(\mathbb{R}^n_+)$ estimates for the heat equation in $\mathbb{R}^n_+$ for $t \in \Gamma$ to obtain (3.5.28). Let us recall the Hardy spaces.

$$\mathcal{H}^1(\mathbb{R}^n) = \{f \in L^1(\mathbb{R}^n) : f^*(x) = \sup_{s>0}|G_s * f(x)| \in L^1(\mathbb{R}^n)\},\ \|f\|_{\mathcal{H}^1} = \|f^*\|_1,$$

$$\|f\|_{\mathcal{H}^1(\mathbb{R}^n_+)} = \inf\{\|\tilde{f}\|_{\mathcal{H}^1(\mathbb{R}^n)} : \tilde{f} \in \mathcal{H}^1(\mathbb{R}^n)\ \tilde{f}|_{\mathbb{R}^n_+} = f\}.$$

Lemma 3.5.14.

$$
\begin{aligned}
&\|\nabla E(t)a\|_{\mathcal{H}^1(\mathbb{R}^n_+)} \le C|t|^{-\frac{1}{2}}\|a\|_{L^1} \text{ for } t \in \Gamma\\
&\|\nabla\nabla_{tan}(\varDelta_{tan}^{\frac{1}{2}})^{-1}E(t)a\|_{\mathcal{H}^1(\mathbb{R}^n_+)} \le C|t|^{-\frac{1}{2}}\|a\|_{L^1} \text{ for } t \in \Gamma\\
&\|\varDelta_{tan}^{\frac{1}{2}}eE(t)a\|_{\mathcal{H}^1(\mathbb{R}^n_+)} \le C|t|^{-\frac{1}{2}}\|a\|_{L^1} \text{ for } t \in \Gamma\\
&\|\nabla\nabla_{tan}(\varDelta_{tan}^{\frac{1}{2}})^{-1}E_N(t)a\|_{\mathcal{H}^1(\mathbb{R}^n_+)} \le C|t|^{-\frac{1}{2}}\|a\|_{L^1} \text{ for } t \in \Gamma
\end{aligned}
\tag{3.5.29}
$$

where $a \in L^1(\mathbb{R}^n_+)$.

Let us explain the outline of our method to these estimates. This gradient $\mathcal{H}^1(\mathbb{R}^n_+)$-$L^1(\mathbb{R}^n_+)$ estimates are obtained by the next lemma seen in [GMS99].

Lemma 3.5.15. *Let K be an integral operator of the form*

$$Kf(x) = \int_{\mathbb{R}^n} k(x,y)f(y)dy \text{ for } x \in \mathbb{R}^n.$$

If the kernel $k(x,y)$ satisfies $\sup_{y\in\mathbb{R}^n}\|k(\cdot,y)\|_{\mathcal{H}^1} = k_0 < \infty$, then $K : L^1 \to \mathcal{H}^1$ is bounded with the estimate

$$\|Kf\|_{\mathcal{H}^1} \le k_0\|f\|_1.$$

To apply this lemma, We need the pointwise Gaussian kernel estimates. For the sake of simplicity, we only state the Gaussian kernel estimate for the second inequality in Lemma 3.5.14.

Lemma 3.5.16 (Estimates of the Gaussian kernel on $\mathbb{R}^n_+$)**.** *For $t \in \Gamma$, $0 \le l \le n$, $m \ge 0$, $j,k \le n-1$*

$$|\partial_j\partial_k\Lambda^{-1}G_t(x)| \le C|t|^{(l+m-n-1)/2}|x'|^{-l}|x_n|^{-m}.$$

Gaussian kernel estimates are obtained by a direct calculation. Since $\Lambda^{-1} = (-\varDelta')^{-\frac{1}{2}} = (-\sum_{k=1}^{n-1}\partial_k^2)^{-\frac{1}{2}}$, the integral kernel of Λ^{-1} is $c_n|x'|^{-n+2}$ for $n \ge 3$.

$$\Lambda^{-1}G_t(x) = c_n\int_{\mathbb{R}^{n-1}} |x'-y'|^{-n+2}G_t(y',x_n)dy'.$$

We set $x = t^{\frac{1}{2}}z$, then by changing the variables

$$\partial_j\partial_k\Lambda^{-1}G_t(x) = c_nt^{-\frac{3}{2}}|t|^{\frac{-n+2}{2}}\partial_{z_j}\partial_{z_k}(\Lambda^{-1}G_1)(z).$$

$\partial_{z_j}\partial_{z_k}(\Lambda^{-1}G_1)(z)$ is estimated in the proof of [GMS99, Lemma 2.3], by dividing the kernel by a cut off function.

$$|\partial_{z_j}\partial_{z_k}(\Lambda^{-1}G_1)(z)| \le C_{l,m}|z'|^{-l}|z_n|^{-m}$$

This implies Gaussian kernel estimates. Thus, we can apply Lemma 3.5.15 to obtain Lemma 3.5.14. (We remark that the argument seen in the proof of

[GMS99, Lemma 2.4] can also be applicable to estimates (3.5.29) since $t \in \Gamma$ can be written of the simple form $t = re^{\mp i\gamma}$ for a fixed γ.) Lemma 3.5.14 implies (4.9) since $f \in \mathcal{H}^1$ if and only if $R_j f \in L^1$ for all $1 \le j \le n$ and $\|f\|_{L^1(\mathbb{R}^n_+)} \le C\|f\|_{\mathcal{H}^1(\mathbb{R}^n_+)}$. By (4.9), we conclude that

$$\begin{aligned}\|\nabla\eta\|_{L^1} &= \int_\Gamma \|\nabla u(t)\|_{L^1} e^{-|\lambda t|}dt \\ &\le C\int_0^\infty r^{-\frac{1}{2}}e^{-|\lambda| r}dr\|f\|_{L^1} = C'|\lambda|^{-\frac{1}{2}}\|f\|_{L^1}.\end{aligned}$$

□

What is left to prove the dual problem in $\mathbb{R}^n_+$ is to show $\eta \in W^{2,1}(\mathbb{R}^n_+)$ by $L^1 - W^{-1,1}$type estimates.

Proof of Lemma 3.5.13(the dual problem). Let η be a solution of (3.5.20). Let us define μ be a solution constructed by Theorem 3.5.5 of the following resolvent equation

$$\begin{cases}(e^{i\theta_\infty} - \Delta)\mu + \nabla\sigma = \Psi & \text{in } \mathbb{R}^n_+\\ \mathrm{div}\mu = 0 & \text{in } \mathbb{R}^n_+\\ \mu(x',0) = 0 & \text{on } \partial\mathbb{R}^n_+,\end{cases} \tag{3.5.30}$$

Since $f = \nabla'\Psi$ and ∇' commute with the solution operators, we obtain $\eta = \nabla'\mu$. This implies by Theorem 3.5.5,

$$\|\eta\|_{L^1} \le C\|\nabla\mu\|_{L^1} \le C\|\Psi\|_{L^1}. \tag{3.5.31}$$

We also apply Theorem 3.5.5 to η, then we obtain the L^1 gradient estimate of η. Furthermore, by applying the same argument to η, we can obtain L^1 estimates of $\nabla\nabla'\eta$ since ∇' and solution operators are commutative with each other. So we only need to estimate the term $\partial_n^2\eta$ in order to obtain the estimates of the second order derivatives. We consider the non-stationary solution u when the time variable $t \in \Gamma$ since solutions $\eta = Lu$ of Stokes resolvent equation in $\mathbb{R}^n_+$ is hard to be written explicitly. Desired resolvent estimates are obtained by the estimates of non-stationary problem and the Laplace transform L. $\partial_n^2 u$ can be calculated by [GMS99, Lemma 1.2] and Fourier transforms. Let $\bar{u}$ satisfy $r\bar{u} = u$.

Lemma 3.5.17. *([GMS99, Lemma 1.2].)*

$$\begin{aligned}\partial_n\bar{u}_n = -R_n\{&R'\cdot\Lambda eE(t)g' - R_n\nabla'\cdot eE(t)g' \\ &+ R'\cdot\nabla' eE(t)g_n + R_n\Lambda eE(t)g_n\},\end{aligned}$$

$$\begin{aligned}\partial_n\bar{u}' = &\partial_n E(t)g' + \partial_n\nabla'\Lambda^{-1}E(t)g_n + R_nR'\nabla'\cdot eE(t)g' \\ &- R_n^2\nabla'\nabla'\Lambda^{-1}\cdot eE(t)g' - R_nR'\Lambda eE(t)g_n + R_n^2\nabla' eE(t)g_n.\end{aligned}$$

Lemma 3.5.18.

$$
\begin{aligned}
\partial_n^2 \bar{u}_n = -R_n\{&R' \cdot \Lambda e E_N(t)\partial_n g' - R_n \nabla' \cdot e E_N(t)\partial_n g' \\
&+ R' \cdot \nabla' e E_N(t)\partial_n g_n + R_n \Lambda e E_N(t)\partial_n g_n\},
\end{aligned}
$$

$$
\begin{aligned}
\partial_n^2 \bar{u}' = {}&\partial_n E_N(t)\partial_n g' + \nabla'(\nabla'\Lambda^{-1})E(t)\partial_n^2\phi_n + R_n R' \nabla' \cdot e E_N(t)\partial_n g' \\
&- R_n^2 \nabla'\nabla'\Lambda^{-1} \cdot e E_N(t)\partial_n g' - R_n R' \Lambda e E_N(t)\partial_n g_n + R_n^2 \nabla' e E_N(t)\partial_n g_n.
\end{aligned}
$$

Since the representations of $\partial_n \bar{u}_n, \partial_n \bar{u}'$ are already obtained by Lemma 3.5.17, we need only to apply ∂_n to $\partial_n \bar{u}_n, \partial_n \bar{u}'$ in Lemma 3.5.17. ∂_n commutes with differential operators $R_j, R', \Lambda, \nabla'$ since each symbol of the Fourier transform commutes with each other. By the Leibniz's rule for differentiation under the integral sign and the integration by parts formula, $\partial_n E(t)g = E_N(t)\partial_n g$ since g has compact support. Thus, we obtain the representation of terms with the exception of the second term $\partial_n^2(\nabla'\Lambda^{-1}E(t)g_n)$ of $\partial_n^2 \bar{u}'$. So, we shall calculate $\partial_n^2(\nabla'\Lambda^{-1}E(t)g_n)$.

Since ∂_n commutes with differential operators and $\partial_n E(t)g = E_N(t)\partial_n g$,

$$
\partial_n^2(\nabla'\Lambda^{-1}E(t)g_n) = \nabla'\Lambda^{-1}\partial_n E_N(t)\partial_n g_n.
$$

By the Leibniz ' s rule for differentiation under the integral sign

$$
\begin{aligned}
\partial_n E_N(t)\partial_n g_n &= \int_{\mathbb{R}^n_+} \{\partial_{x_n} G_t(x-y) + \partial_{x_n} G_t(x'-y', x_n+y_n)\}\partial_n g_n(y)dy \\
&= \int_{\mathbb{R}^n_+} \{-\frac{x_n - y_n}{2t} G_t(x-y) - \frac{x_n + y_n}{2t} G_t(x'-y', x_n+y_n)\}\partial_n g_n(y)dy
\end{aligned}
$$

Since g has compact support, integration by parts for the variable y_n implies

$$
\partial_n E_N(t)\partial_n g_n = -\int_{\mathbb{R}^n_+} \{-G_t(x-y) + G_t(x'-y', x_n+y_n)\}\partial_n^2 g_n(y)dy = E(t)\partial_n^2 g_n.
$$

Thus, we obtain

$$
\partial_n^2(\nabla'\Lambda^{-1}E(t)g_n) = \nabla'\Lambda^{-1}E(t)\partial_n^2 g_n = \nabla'\Lambda^{-1}E(t)\partial_n^2\nabla'\phi_n = \nabla'(\nabla'\Lambda^{-1})E(t)\partial_n^2\phi_n.
$$

So, we obtain Lemma 3.5.18, and then we obtain the L^1 estimate of $\partial_n^2\eta$ by Lemma 3.5.14 and the Laplace transform L. Therefore, we finish estimating $\|\eta\|_{W^{2,1}} \leq C(\lambda)\|\Psi\|_{W^{2,1}}$. □

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Chapter 4

Analyticity of the Stokes semigroups in BMO-type spaces

4.1 Introduction

4.1.1 Main results

We will investigate the homogeneous Stokes equations

$$\begin{array}{rcll} u_t - \Delta u + \nabla \pi & = & 0 & \text{in } \Omega \times (0,T) \\ \operatorname{div} u & = & 0 & \text{in } \Omega \times (0,T) \\ u & = & 0 & \text{on } \partial\Omega \times (0,T) \\ u(0) & = & u_0 & \end{array} \tag{4.1.1}$$

in a uniformly C^3-domain $\Omega \subset \mathbb{R}^n$ $(n \geq 2)$. The L^p-theory for $1 < p < \infty$ of the Stokes equations is quite well understood if the Helmholtz projection in L^p exists. For this let $L^p_\sigma(\Omega)$ be the closure of $C^\infty_{c,\sigma}(\Omega)$, the space of smooth solenoidal vector fields with compact support, in $L^p(\Omega)$. The Helmholtz projection is then the projection operator from $L^p(\Omega)$ into $L^p_\sigma(\Omega)$ derived from the Helmholtz decomposition. In [Gig81] the second author proved that the Stokes operator generates an analytic semigroup in $L^p_\sigma(\Omega)$ if Ω is a bounded domain. The same result was proved in [GHHS12] for general domains under the assumption that the Helmholtz decomposition of $L^p_\sigma(\Omega)$ exists. For domains not admitting the L^p-Helmholtz decomposition this result is still unknown.

In [AG13] and [AG14] K. Abe and the second author proved similar analyticity results in solenoidal subspaces of $L^\infty(\Omega)$ for a certain class of domains called admissible. Similar analyticity results in L^∞ by resolvent estimates were obtained in [AGH15].

In this work we want to generalize these analyticity results to a subspace of BMO. In order to do so we introduce a norm measuring the mean oscillation of the function inside the domain and the mean value of the function near the boundary. We define this BMO-type norm in the following way. Let for $f \in L^1_{loc}(\Omega)$ and $B \subset \Omega$ the mean value f_B be defined as

$$f_B := \frac{1}{|B|} \int_B f(y)\, dy.$$

For the parameter $\mu \in (0, \infty]$ we define the BMO-seminorm

$$[f]_{BMO^\mu(\Omega)} := \sup_{B_r(x) \subset \Omega, r < \mu} \frac{1}{|B_r(x)|} \int_{B_r(x)} |f(y) - f_{B_r(x)}|\, dy.$$

We will usually omit Ω in the notation of the seminorm if no confusion may arise. The space $BMO^\mu(\Omega)$ is then defined as

$$BMO^\mu(\Omega) := \{f \in L^1_{loc}(\Omega) : [f]_{BMO^\mu} < \infty\}.$$

We define for $\nu \in (0, \infty]$ the seminorm

$$[f]_{b^\nu} := \sup\{r^{-n} \int_{B_r(x_0) \cap \Omega} |f(y)|\, dy : x_0 \in \partial\Omega, 0 < r < \nu\}.$$

Then

$$\|f\|_{BMO_b^{\mu,\nu}} := [f]_{BMO^\mu} + [f]_{b^\nu}$$

will be called the BMO-type norm. The space $BMO_b^{\mu,\nu}(\Omega)$ is then defined as the space of all functions $f \in L^1_{loc}(\Omega)$ satisfying $\|f\|_{BMO_b^{\mu,\nu}} < \infty$. Let $VMO_{b,0}^{\mu,\nu}(\Omega)$ be the closure of $C_c^\infty(\Omega)$ and $VMO_{b,0,\sigma}^{\mu,\nu}(\Omega)$ the closure of $C_{c,\sigma}^\infty(\Omega)$ with respect to the norm $\|\cdot\|_{BMO_b^{\mu,\nu}}$. Furthermore, let $C_{0,\sigma}(\Omega)$ be the closure of $C_{c,\sigma}^\infty(\Omega)$ with respect to the L^∞-norm. It is obvious that $C_{0,\sigma}(\Omega) \hookrightarrow VMO_{b,0,\sigma}^{\mu,\nu}(\Omega)$.

Further we define for $p \in (1, \infty)$

$$[f]_{BMO^{\mu}p} := \sup_{B_r(x) \subset \Omega, r < \mu} (\frac{1}{|B_r(x)|} \int_{B_r(x)} |f(y) - f_{B_r(x)}|^p\, dy)^{1/p},$$

$$[f]_{b^\nu p} := \sup_{x_0 \in \partial\Omega, 0 < r < \nu} (r^{-n} \int_{B_r(x_0) \cap \Omega} |f(y)|^p\, dy)^{1/p},$$

$$\|f\|_{BMO_b^{\mu,\nu}p} := [f]_{BMO^\mu p} + [f]_{b^\nu p}.$$

Note that by the John-Nirenberg inequality the seminorm $[f]_{BMO^\mu p}$ is equivalent to $[f]_{BMO^\mu}$ provided that $p \in (1, \infty)$ and $\mu \in (0, \infty]$.

In [FKS05], [FKS07] it was proved that for the space $\tilde{L}^r := L^2 \cap L^r$ if $r \geq 2$, $\tilde{L}^r := L^2 + L^r$ otherwise, there is a bounded Helmholtz projection P_r from $\tilde{L}^r(\Omega)$ to $\tilde{L}^r_\sigma(\Omega)$ in uniformly C^2-domains. Furthermore, it was proved that the Stokes operator generates an analytic semigroup in $\tilde{L}^r_\sigma(\Omega)$. Here $\tilde{L}^r_\sigma(\Omega)$ is the closure of $C_{c,\sigma}^\infty(\Omega)$ in the $\tilde{L}^r$-norm. The Sobolev space $\tilde{W}_0^{1,r}$ is defined as the closure of $C_c^\infty(\Omega)$ with respect to the norm $\|\cdot\|_{\tilde{W}^{1,r}} := \|\cdot\|_{\tilde{L}^r} + \|\nabla\cdot\|_{\tilde{L}^r}$. In [FKS05], [FKS09] it was proved that for every $u_0 \in \tilde{L}^r(\Omega)$ there is a unique solution $u(t) \in \tilde{W}_0^{1,r}(\Omega) \cap \tilde{L}_\sigma^r(\Omega)$ with $\nabla^2 u(t), u_t(t), \nabla\pi(t) \in \tilde{L}^r(\Omega)$. We call such a solution $\tilde{L}^r$-solution.

We are now ready to define the notion of an admissible domain in the sense of [AG13]. Let Ω be a uniformly C^2-domain. The domain Ω is then called admissible if there are $r > n$ and a constant $C > 0$ such that for all matrix-valued functions $f \in C^1(\overline{\Omega})$ with $\text{div} f \in \tilde{L}^r(\Omega)$, $\text{tr} f = 0$ and $\partial_l f_{ij} = \partial_j f_{il}$ $(1 \leq i,j,l \leq n)$

$$\sup_{x\in\Omega} dist(x,\partial\Omega)|(I - P_r)(\nabla f)(x)| \leq C\|f\|_{L^\infty(\partial\Omega)}$$

holds. Examples of admissible domains are bounded domains, the half space ([AG13]) and exterior domains ([AG14]). A layer domain of dimension $n \geq 3$ is an example of a domain that is not admissible ([Bel14]) but has a Helmholtz decomposition in L^p ([Miy94]). Furthermore, there are also examples of admissible domains that do not have a Helmholtz decomposition in L^p as constructed in [AGSS15].

Our main result then states that in an admissible domain the Stokes operator generates an analytic semigroup in $VMO_{b,0,\sigma}^{\mu,\nu}(\Omega)$ for suitable choices of μ and ν. The constant $C_{n,L}$ denotes here a constant depending on the regularity of the domain which will be defined in section 4.2.3.

Theorem 4.1.1. *Let $\Omega \subset \mathbb{R}^n$ be an admissible, uniformly C^3-domain. Let $0 < \nu \leq R^*$ and $\mu \in (R^*, \infty]$. Then the Stokes operator generates a C_0-analytic semigroup in $VMO_{b,0,\sigma}^{\mu,\nu}(\Omega)$.*

The main idea of the proof is deriving estimates for

$$\int_{B_r(x)} (u(y,t) - u_{B_r(x)}(t))^2\, dy \quad \text{and} \quad \int_{B_r(x_0)\cap\Omega} u(y,t)^2\, dy$$

for $B_r(x) \subset \Omega$ and $x_0 \in \partial\Omega$. This can be done by using the fundamental theorem of calculus $u(t) = \int_0^t u_s(s)\, ds - u_0$, the equality $u_t = \Delta u - \nabla\pi$ and integration by parts such that we only need to estimate π and the gradient of u. Via an estimate on harmonic functions the pressure in this calculation is also controlled by the gradient of u. By the estimate

$$\sup_{0<t<T_0} (t^{\frac{1}{2}} \|\nabla u(t)\|_\infty) \leq C\|u_0\|_{BMO_b^{\mu,\nu}}$$

of Theorem 4.1.2 we then obtain for $t < T_0$ the inequality

$$\|u(t)\|_{BMO_b^{\mu,2\nu}2} \leq C\|u_0\|_{BMO_b^{\mu,\nu}}.$$

Finally, we will need equivalence results between different BMO_b-norms to compare these two norms and get the boundedness in $VMO_{b,0,\sigma}^{\mu,\nu}(\Omega)$. Together with the time derivative estimate of Theorem 4.1.2 this yields the analyticity of the Stokes operator in $VMO^{\mu,\nu}_{b,0,\sigma}(\Omega)$.

This chapter is organized as follows. In section 4.2.1 we will prove estimates that will be needed to get control of the pressure terms that will appear in our calculations. In section 4.2.2 we will prove that we can estimate the BMO-type norm of the solution by another BMO-type norm of the initial data and that the solution is in $VMO^{\mu,\nu}_{b,0,\sigma}(\Omega)$. In section 4.2.3 we will prove the required equivalence results of different BMO-type norms. In section 4.2.4 we will consider the Stokes semigroup in the half-space and prove the global boundedness of the semigroup and its derivatives.

4.1.2 Literature reviews of gradient L^∞-BMO resolvent estimates

In [BG15], it is proved that for the Stokes equations the L^∞-norm of the derivatives of the solution can be estimated by the BMO_b-norm of the initial data as in the following theorem.

Theorem 4.1.2. *Let $n \geq 2$, $r > n$ and*

$$\tilde{N}(u,\pi)(x,t) := t^{1/2}|\nabla u(x,t)| + t|\nabla^2 u(x,t)| + t|u_t(x,t)| + t|\nabla \pi(x,t)|.$$

Let Ω be an admissible, uniformly C^3-domain in $\mathbb{R}^n$, $\mu, \nu \in (0,\infty]$. Then there exist a solution operator S to (4.1.1) and constants $C, T_0 > 0$ depending only on μ, ν, n and Ω such that

$$\sup_{0<t<T_0} \|\tilde{N}(u,\pi)(\cdot,t)\|_\infty \leq C\|u_0\|_{BMO_b^{\mu,\nu}}$$

holds for every $\tilde{L}^r$-solution $(u,\nabla\pi)$ with $u_0 \in C^\infty_{c,\sigma}(\Omega)$. By density the estimate holds also for each $u_0 \in VMO^{\mu,\nu}_{b,0,\sigma}(\Omega)$ with $S(t)u_0 = u$ and a suitable choice of π. The solution operator S is taken so that it agrees with the L^2-Stokes semigroup on $C^\infty_{c,\sigma}(\Omega)$.

The estimate $t\|u_t(t)\|_{BMO_b^{\mu,\nu}} \leq C\|u_0\|_{BMO_b^{\mu,\nu}}$ for $t < T_0$ which is a consequence of the theorem is the estimate needed for proving the analyticity of a semigroup. Nevertheless, we have the required estimate but this is not enough to conclude that the Stokes operator actually generates a semigroup on $VMO^{\mu,\nu}_{b,0,\sigma}(\Omega)$ since the theorem does not give us sufficient control about the solution u itself. It is the aim of this chapter to close this gap and to show that the Stokes semigroup is analytic in $VMO^{\mu,\nu}_{b,0,\sigma}(\Omega)$.

For this we will need to assume some regularity at the boundary and will make use of the following property.

Lemma 4.1.1. *Let Ω be a uniformly C^2-domain. Then there exists a constant R such that for all $x \in \Omega$ with $dist(x,\partial\Omega) < R$ there is a unique projection to a boundary point $x_c \in \partial\Omega$ such that the line between x_c and x is normal to $\partial\Omega$ in x_c.*

Proof. For a proof see [GT77, appendix] and [KP02, §4.4]. □

We define then for a uniformly C^2-domain the number $R^* > 0$ to be the supremum of all R satisfying the above for Ω and its complement. This R^* is often called the reach of $\partial\Omega$ ([KP02]).

4.2 BMO estimates

4.2.1 Boundary estimate for the pressure

In this section we will prove estimates for harmonic functions in order to estimate the pressure terms in Section 4.2.2 in a suitable way.

Theorem 4.2.1. *Let Ω be a bounded C^2-domain and consider the equation*

$$\begin{array}{rcll} \Delta\pi & = & 0 & \text{in} \quad \Omega \\ \frac{\partial\pi}{\partial\mathbf{n}} & = & \operatorname{div}_{\partial\Omega} g & \text{on} \quad \partial\Omega \\ \int_\Omega \pi\, dx & = & 0. & \end{array}$$

Then there is a constant $C > 0$ depending only on C^2-regularity of Ω and the second eigenvalue of the Neumann Laplacian in Ω such that

$$\|\pi\|_{L^2(\partial\Omega)} \le C\|g\|_{L^2(\partial\Omega)} \tag{4.2.1}$$

holds for all $g \in L^2(\partial\Omega)$ with $g \cdot \mathbf{n} = 0$ on $\partial\Omega$. The constant C is additionally invariant under scaling transformations of the domain Ω.

We shall prove this theorem in several steps. For Lipschitz domains we consider the Sobolev space on the boundary $\partial\Omega$. Let $H^1(\partial\Omega)$ denote the space of all $f \in L^2(\partial\Omega)$ whose weak tangential derivative $\nabla_{\partial\Omega} f$ is also in $L^2(\partial\Omega)$. We equip this space with an inner product in the same way as in the definition of $H^1(\Omega)$. The space $H^s(\partial\Omega)$ $(0 \le s \le 1)$ is given as the complex interpolation space $\left[L^2(\partial\Omega), H^1(\partial\Omega)\right]_s$ based on fractional powers of the self-adjoint operator associated with the inner product of H^1 ([LM68]). It is well-known that the trace space $H^{1/2}(\partial\Omega)$ of $H^1(\Omega)$ agrees with this characterization of the interpolation ([LM68]). Let $H^{-s}(\partial\Omega)$ be the the dual space of $H^s(\partial\Omega)$.

Lemma 4.2.1. *Let $\Omega \subset \mathbb{R}^n$ be a bounded Lipschitz domain. Then*

$$\|\nabla_{\partial\Omega} f\|_{H^{-1}(\partial\Omega)} \le \|f\|_{L^2(\partial\Omega)} \tag{4.2.2}$$

for all $f \in L^2(\partial\Omega)$, where $\nabla_{\partial\Omega}$ denotes the weak tangential gradient.

Proof. This can be seen immediately from the definition of $H^{-1}(\partial\Omega)$. □

Lemma 4.2.2. *Let $\Omega \subset \mathbb{R}^n$ be a bounded Lipschitz domain. Then*

$$\|\nabla_{\partial\Omega} f\|_{H^{-s}(\partial\Omega)} \le \|f\|_{H^{1-s}(\partial\Omega)} \tag{4.2.3}$$

for all $f \in H^{1-s}(\partial\Omega)$ and $s \in [0,1]$. In particular,

$$\|\nabla_{\partial\Omega} f\|_{H^{-1/2}(\partial\Omega)} \leq \|f\|_{H^{1/2}(\partial\Omega)} \tag{4.2.4}$$

for all $f \in H^{1/2}(\partial\Omega)$.

Proof. We interpolate (4.2.2) with

$$\|\nabla_{\partial\Omega} f\|_{L^2(\partial\Omega)} \leq \|f\|_{H^1(\partial\Omega)}$$

to get (4.2.3) by complex interpolation theory ([LM68]). Note that $H^{-s}(\partial\Omega) = \left[L^2(\partial\Omega), H^{-1}(\partial\Omega)\right]_s$. □

We next recall the solvability of the Neumann problem

$$\begin{array}{rcll} \Delta u & = & 0 & \text{in } \ \Omega \\ \frac{\partial u}{\partial \mathbf{n}} & = & h & \text{on } \ \partial\Omega \end{array} \tag{4.2.5}$$

under the compatibility condition $\int_{\partial\Omega} h \, d\mathcal{H}^{n-1} = 0$. The Lax-Milgram theorem or even the Riesz representation theorem for a Hilbert space guarantees the existence of a solution $u \in H^1(\Omega)$ for $h \in H^{-1/2}(\partial\Omega)$. If h is regular, say $h \in H^{1/2}(\partial\Omega)$ and if $\partial\Omega$ is C^2, then u is H^2. This is also standard. We just summarize these results which are for example found in [BF13, Theorem III.4.3] including the case when the Laplace equation (4.2.5) is replaced by the Poisson equation $\Delta u = f$.

Lemma 4.2.3. *Let Ω be a bounded Lipschitz domain in $\mathbb{R}^n$. For a given $h \in H^{-1/2}(\partial\Omega)$ with $\int_{\partial\Omega} h \, d\mathcal{H}^{n-1} = 0$, there is a unique weak solution $u \in H^1(\Omega)$ of* (4.2.5) *satisfying $\int_\Omega u \, dx = 0$. This linear operator $h \mapsto u$ fulfills the estimate*

$$\|u\|_{H^1(\Omega)} \leq C\|h\|_{H^{-1/2}(\partial\Omega)} \tag{4.2.6}$$

with C depending only on Ω through its Lipschitz regularity of Ω as well as the second eigenvalue of the Laplacian with Neumann boundary conditions.

Moreover, if Ω is C^2 and $h \in H^{1/2}(\partial\Omega)$, then $u \in H^2(\Omega)$. The linear operator $h \mapsto u$ fulfills the estimate

$$\|u\|_{H^2(\Omega)} \leq C\|h\|_{H^{1/2}(\partial\Omega)}. \tag{4.2.7}$$

Here, the constant C depends in addition on C^2-regularity of Ω.

The dependence of C with respect to the second eigenvalue of the Laplacian with Neumann boundary condition appears when one uses the Poincaré type inequality to control the L^2-norm of u by the L^2-norm of ∇u. If the boundary regularity is fixed, then the constant decreases as the second eigenvalue increases.

The estimate (4.2.6) together with the well-known trace theorem [BF13, Theorem III, 2.2] and (4.2.4) yield key estimates for the boundary value of the solution π in Theorem 4.2.1.

Lemma 4.2.4. *Let Ω be a bounded Lipschitz domain in $\mathbb{R}^n$. Let $g \in H^{1/2}(\partial\Omega)$ satisfy $g \cdot n = 0$ on $\partial\Omega$ and let $\pi \in H^1(\Omega)$ with $\int_\Omega \pi\, dx = 0$ be the unique solution of* (4.2.5) *with $h = \mathrm{div}_{\partial\Omega} g$. Then*

$$\|\gamma\pi\|_{H^{1/2}(\partial\Omega)} \leq C\|g\|_{H^{1/2}(\partial\Omega)} \tag{4.2.8}$$

with C depending only on the Lipschitz regularity of Ω and the second eigenvalue of the Laplacian with Neumann boundary condition, where γ denotes the trace on $\partial\Omega$.

Proof. We first notice that $\int_{\partial\Omega} h\, d\mathcal{H}^{n-1} = 0$ because g is tangential. By the inequality (4.2.4) we observe that $\mathrm{div} g \in H^{-1/2}(\partial\Omega)$, which guarantees the existence of an H^1-solution π (Lemma 4.2.3). We now observe by the trace theorem, (4.2.6) and (4.2.4) that

$$\begin{aligned}
\|\gamma\pi\|_{H^{1/2}(\partial\Omega)} &\leq C_1\|\pi\|_{H^1(\Omega)} \\
&\leq C_2\|\mathrm{div}_{\partial\Omega} g\|_{H^{-1/2}(\partial\Omega)} \\
&\leq C_3\|g\|_{H^{1/2}(\partial\Omega)}
\end{aligned}$$

which yields (4.2.8) where C_j denotes a constant depending only on Ω. Here we only used Lipschitz regularity of the boundary. □

We finally apply a duality argument.

Lemma 4.2.5. *Assume that Ω is a bounded C^2-domain in $\mathbb{R}^n$. Let g and π be as in Lemma 4.2.4. Then*

$$\|\gamma\pi\|_{H^{-1/2}(\partial\Omega)} \leq C\|g\|_{H^{-1/2}(\partial\Omega)} \tag{4.2.9}$$

with C depending only on C^2-regularity of Ω as well as the second eigenvalue of the Laplacian with Neumann boundary condition in Ω.

Proof. Let u_h be the H^2-solution (satisfying $\int_\Omega u_h\, dx = 0$) of (4.2.5) with $h \in H^{1/2}(\partial\Omega)$ satisfying $\int_{\partial\Omega} h\, d\mathcal{H}^{n-1} = 0$. By the Green formula we have

$$\int_{\partial\Omega} (\gamma\pi) h\, d\mathcal{H}^{n-1} - \int_{\partial\Omega} \frac{\partial\pi}{\partial n} u_h\, d\mathcal{H}^{n-1} = \int_\Omega (\pi\Delta u_h - u_h\Delta\pi)\, dx = 0,$$

where γu_h is denoted simply by u_h. Thus

$$\int_{\partial\Omega} (\gamma\pi) h\, d\mathcal{H}^{n-1} = \int_{\partial\Omega} (\mathrm{div}_{\partial\Omega} g) u_h\, d\mathcal{H}^{n-1} = -\int_{\partial\Omega} g \cdot \nabla_{\partial\Omega} u_h\, d\mathcal{H}^{n-1}.$$

This representation yields

$$\left| \int_{\partial\Omega} (\gamma\pi) h\, d\mathcal{H}^{n-1} \right| \leq \|g\|_{H^{-1/2}(\partial\Omega)} \|\gamma\nabla_{\partial\Omega} u_h\|_{H^{1/2}(\partial\Omega)}.$$

By the trace theorem we get

$$\left| \int_{\partial\Omega} (\gamma\pi) h\, d\mathcal{H}^{n-1} \right| \leq \|g\|_{H^{-1/2}(\partial\Omega)} \|h\|_{H^{1/2}(\partial\Omega)}$$

which yields (4.2.9). □

Proof of Theorem 2.1. Since the estimate (4.2.9) guarantees that $g \mapsto \gamma\pi$ is extendable from tangential $H^{-1/2}(\partial\Omega)$ to $H^{-1/2}(\partial\Omega)$, interpolating (4.2.8) with (4.2.9) yields (4.2.1), where we suppress the trace symbol γ. Here we invoke the property that $\left[H^{1/2}(\partial\Omega), H^{-1/2}(\partial\Omega)\right]_{1/2} = L^2(\partial\Omega)$ ([LM68]). The scaling invariance follows directly from scaling Ω, π and g. □

Remark 4.2.1. *In the application of Theorem 4.2.1 in Section 4.2.2 we will need to consider this estimate for domains of the form $\Omega \cap B_r(x_0)$ ($x_0 \in \partial\Omega, r < 2c_0R^*$) with smoothed corners and for balls. For the balls we can invoke that the constant C is scaling invariant and thus depends only on dimension. For the other domains we will need some uniform control of the constant in this estimate. We can achieve this through control on the C^2-regularity of the domains and an estimate from below on the second eigenvalue λ_2 of the Neumann Laplacian. For convex domains $\tilde{\Omega}$ such an estimate is explicitly known by [PW60], where $\lambda_2 \geq \frac{\pi^2}{diam(\tilde{\Omega})^2}$ was shown. But not all of the domains we consider are convex. For these remaining domains one can obtain an estimate from below directly from [Che90], where this was proved for manifolds satisfying an "interior rolling ε-ball condition" which is satisfied for all C^2-domains. Another possibility is to use the result of [Arr95] where it was proved that small perturbations of the domain result in small changes of the eigenvalues. Thus, if we choose $0 < c_0 < \frac{1}{2}$ sufficiently small we get from the estimate $\lambda_2 \geq \frac{\pi^2}{r^2}$ for the upper half of a ball with radius r the estimate $\lambda_2 \geq \frac{\pi^2}{2r^2}$ for all $B_r(x_0) \cap \Omega$ with $x_0 \in \partial\Omega$, $r < 2c_0R^*$ and smoothed corners. From this we can conclude that the constant C of* (4.2.1) *is bounded from above in all applications of Theorem 4.2.1 if we choose $\nu < c_0R^*$ and have control on the C^2-regularity.*

4.2.2 Boundedness in BMO-type spaces

In this section we will prove that the solution operator maps $VMO^{\mu,\nu}_{b,0,\sigma}(\Omega)$ to $VMO^{\mu,\nu}_{b,0,\sigma}(\Omega)$ under suitable choices of μ and ν and finally conclude the analyticity of the Stokes semigroup in these BMO-type spaces. We will distinguish between small and large balls and use the derivative estimate of Theorem 4.1.2 in order to prove this boundedness. It will be easier to do most of the calculations with the BMO_b2-norms since in this case we do not have to take care of the absolute value in the definition and it enables us to integrate by parts in a way that fits to our needs.

Since we will also need some control over the mean values we will start with an estimate on mean values of the solution.

Lemma 4.2.6. *Let $\mu, \nu \in (0, \infty]$ and Ω an admissible uniformly C^3-domain. Then there are constants $C, T_0 > 0$ which are independent of r, u_0 and t such that*

$$\left|u_{B_r(x)}(t) - u_{0B_r(x)}\right| \leq C\frac{t^{1/2}}{r}\|u_0\|_{BMO_b^{\mu,\nu}}$$

holds for all solutions $u := S(t)u_0$ of (4.1.1) *with $u_0 \in VMO^{\mu,\nu}_{b,0,\sigma}(\Omega)$, $t \in (0,T_0)$ and $B_r(x) \subset \Omega$.*

Proof. By the fundamental theorem of calculus, equation $(4.1.1)_1$ and integration by parts we get

$$\begin{aligned}
&\frac{1}{|B_r(x)|}\int_{B_r(x)}(u(y,t)-u_0(y))\,dy\\
=&\frac{1}{|B_r(x)|}\int_{B_r(x)}\int_0^t\frac{\partial u}{\partial s}(y,s)\,ds\,dy\\
=&\frac{1}{|B_r(x)|}\int_0^t\int_{B_r(x)}(\Delta u(y,s)-\nabla\pi(y,s))\,dy\,ds\\
=&\frac{1}{|B_r(x)|}\int_0^t\int_{\partial B_r(x)}(\frac{\partial u}{\partial \mathbf{n}}(y,s)-\pi(y,s)\mathbf{n})\,d\mathcal{H}^{n-1}(y)\,ds.
\end{aligned}$$

Then we can estimate this in the following way by using the Hölder inequality $\|f\|_{L^1(\partial B_r)} \le \|1\|_{L^2(\partial B_r)}\|f\|_{L^2(\partial B_r)}$, where $\|1\|_{L^2(\partial B_r)} = Cr^{\frac{n-1}{2}}$

$$\begin{aligned}
&\left|\frac{1}{|B_r(x)|}\int_{B_r(x)}(u(y,t)-u_0(y))\,dy\right|\\
\le&\frac{\omega_n^{-1}}{r}\int_0^t\frac{1}{r^{n-1}}\int_{\partial B_r(x)}|\frac{\partial u}{\partial \mathbf{n}}(y,s)|+|\pi(y,s)\mathbf{n}|\,ds\,d\mathcal{H}^{n-1}(y)\\
\le&\frac{\omega_n^{-1}}{r}\int_0^t\frac{1}{r^{n-1}}(\int_{\partial B_r(x)}\|\nabla u(s)\|_{L^\infty(\Omega)}\,d\mathcal{H}^{n-1}(y)+\|\pi(s)\|_{L^1(\partial B_r(x))})\,ds\\
\le&\frac{C}{r}\int_0^t(\|\nabla u(s)\|_{L^\infty(\Omega)}+r^{-\frac{n-1}{2}}\|\pi(s)\|_{L^2(\partial B_r(x))})\,ds.
\end{aligned}$$

Here, we used that $\nabla u(t)$ is in $W^{1,\infty}(\Omega)$ by Theorem 4.1.2 and thus $\nabla u \in C(\overline{\Omega})$ such that we can estimate $\|\frac{\partial u}{\partial \mathbf{n}}\|_{L^\infty(\partial\Omega)}$ by $\|\nabla u\|_{L^\infty(\Omega)}$.

We get then by Theorem 4.1.2, Theorem 4.2.1 with choosing π such that $\int_{B_r(x)}\pi = 0$, $(4.1.1)_1$ and the Hölder inequality $\|f\|_{L^2(\partial B_r)} \le \|1\|_{L^2(\partial B_r)}\|f\|_{L^\infty(\partial B_r)}$

$$\begin{aligned}
&\left|\frac{1}{|B_r(x)|}\int_{B_r(x)}(u(y,t)-u_0(y))\,dy\right|\\
\le&\frac{C}{r}\int_0^t(\|\nabla u(s)\|_\infty+r^{-\frac{n-1}{2}}\|curl u(s)\times\mathbf{n}\|_{L^2(\partial B_r(x))})\,ds\\
\le&\frac{C}{r}\int_0^t\|\nabla u(s)\|_\infty\,ds\\
\le&\frac{C}{r}\int_0^t s^{-\frac{1}{2}}\|u_0\|_{BMO^{\mu,\nu}_b}\,ds\\
\le&C\frac{t^{1/2}}{r}\|u_0\|_{BMO^{\mu,\nu}_b}.
\end{aligned}$$

□

In the next theorem we obtain bounds for the mean oscillation of the solution in large balls.

Theorem 4.2.2. *Let $\Omega \subset \mathbb{R}^n$ be an admissible, uniformly C^3-domain, $\mu, \nu \in (0, \infty]$. Then there are constants $C, T_0 > 0$ depending only on Ω, n, μ and ν such that for all $0 < r < \mu$ and $x \in \Omega$ with $B_r(x) \subset \Omega$, $t \in (0, T_0)$ and all $u_0 \in VMO^{\mu,\nu}_{b,0,\sigma}(\Omega)$ with $u(t) = S(t)u_0$*

$$\frac{1}{|B_r(x)|}\int_{B_r(x)} |u(y,t) - u_{B_r(x)}(t)|^2 \, dy \le C(1 + \frac{t}{r^2})\|u_0\|^2_{BMO^{\mu,\nu}_b}.$$

Proof. By the fundamental theorem of calculus, $(4.1.1)_1$ and integration by parts we get

$$\begin{aligned}
&\int_{B_r} |u(y,t) - u_{B_r}(t)|^2 \, dy \\
=& \int_{B_r} (u(y,t) - u_{B_r}(t))(\int_0^t \frac{\partial (u - u_{B_r})}{\partial s}(y,s)\, ds - (u_0(y) - u_{0B_r}))\, dy \\
\le& \left| \int_{B_r} (u(y,t) - u_{B_r}(t)) \int_0^t (\Delta u(y,s) - \nabla \pi(y,s))\, ds\, dy \right| \\
&+ \left| \int_{B_r} (u(y,t) - u_{B_r}(t)) \int_0^t \frac{\partial u_{B_r}}{\partial s}(s)\, ds\, dy \right| \\
&+ \|u(y,t) - u_{B_r}(t)\|_{L^2(B_r)} \|u_0 - u_{0B_r}\|_{L^2(B_r)} \\
\le& \left| \int_0^t \int_{B_r} \nabla u(y,t) \nabla u(y,s)\, dy\, ds \right| + \left| \int_0^t \int_{\partial B_r} (u(y,t) - u_{B_r}(t)) \frac{\partial u}{\partial \mathbf{n}}(s)\, d\mathcal{H}^{n-1}(y)\, ds \right| \\
&+ \left| \int_0^t \int_{\partial B_r} (u(y,t) - u_{B_r}(t)) \pi(y,s) \mathbf{n})\, d\mathcal{H}^{n-1}(y)\, ds \right| \\
&+ \left| \int_{B_r} (u(y,t) - u_{B_r}(t))\, dy (u_{B_r}(t) - u_{0B_r}) \right| \\
&+ \frac{1}{2} \int_{B_r} |u(y,t) - u_{B_r}(t)|^2\, dy + \frac{1}{2} \int_{B_r} |u_0(y,t) - u_{0B_r}|^2\, dy,
\end{aligned}$$

where we can take the second to last term to the left hand side and the last term can be estimated by $r^n [u_0]^2_{BMO^{\mu}2}$ which by the John-Nirenberg inequality is equivalent to $r^n [u_0]^2_{BMO^{\mu}}$. We will use the derivative estimate of Theorem 4.1.2 for estimating the other parts. The first term can be estimated by

$$\begin{aligned}
\left| \int_0^t \int_{B_r} \nabla u(y,t) \nabla u(y,s)\, dy\, ds \right| &\le \int_{B_r} \|\nabla u(t)\|_\infty \int_0^t \|\nabla u(s)\|_\infty\, ds\, dy \\
&\le C r^n t^{-1/2} \|u_0\|_{BMO^{\mu,\nu}_b} \int_0^t s^{-1/2} \|u_0\|_{BMO^{\mu,\nu}_b}\, ds \\
&\le C r^n \|u_0\|^2_{BMO^{\mu,\nu}_b}.
\end{aligned}$$

For the second summand we get

$$
\begin{aligned}
&\left|\int_0^t \int_{\partial B_r} (u(y,t) - u_{B_r}(t)) \frac{\partial u}{\partial \mathbf{n}}(s)\, d\mathcal{H}^{n-1}(y)\, ds\right| \\
\leq &\int_{\partial B_r} |u(y,t) - u_{B_r}(t)| \int_0^t \|\nabla u(s)\|_\infty \, ds\, d\mathcal{H}^{n-1}(y) \\
\leq &C \int_{\partial B_r} |u(y,t) - u_{B_r}(t)|\, d\mathcal{H}^{n-1}(y) \int_0^t s^{-1/2} \|u_0\|_{BMO_b^{\mu,\nu}}\, ds \\
\leq &C t^{1/2} \|u_0\|_{BMO_b^{\mu,\nu}} \int_{\partial B_r} |u(y,t) - u_{B_r}(t)|\, d\mathcal{H}^{n-1}(y).
\end{aligned}
$$

In order to estimate the third term we estimate the pressure part by using Theorem 4.2.1, $(4.1.1)_1$ and Hölder's inequality.

$$
\begin{aligned}
&\left|\int_0^t \int_{\partial B_r} (u(y,t) - u_{B_r}(t)) \pi(y,s)\mathbf{n})\, d\mathcal{H}^{n-1}(y)\, ds\right| \\
\leq &\|u(t) - u_{B_r}(t)\|_{L^2(\partial B_r)} \int_0^t \|\pi(s)\|_{L^2(\partial B_r)}\, ds \\
\leq &C r^{(n-1)/2} \|u(t) - u_{B_r}(t)\|_{L^\infty(\partial B_r)} \int_0^t r^{(n-1)/2} \|curl u(s) \times \mathbf{n}\|_{L^\infty(\partial B_r)}\, ds \\
\leq &C r^n \|\nabla u(t)\|_\infty \int_0^t \|\nabla u(s)\|_\infty\, ds \\
\leq &C r^n \|u_0\|^2_{BMO_b^{\mu,\nu}},
\end{aligned}
$$

where we used Poincaré's inequality with constant Cr in B_r in the second to last line. For the fourth term we use Lemma 4.2.6

$$
\begin{aligned}
&\left|\int_{B_r} (u(y,t) - u_{B_r}(t))\, dy (u_{B_r}(t) - u_{0B_r})\right| \\
\leq &C\Big(\int_{B_r} |u(y,t) - u_{B_r}(t)|^2\, dy\Big)^{1/2} r^{n/2} \frac{t^{1/2}}{r} \|u_0\|_{BMO_b^{\mu,\nu}} \\
\leq &\varepsilon \int_{B_r} |u(y,t) - u_{B_r}(t)|^2\, dy + C_\varepsilon \frac{t}{r^2} r^n \|u_0\|^2_{BMO_b^{\mu,\nu}}.
\end{aligned}
$$

Thus we have the estimate

$$
\begin{aligned}
\int_{B_r} |u(y,t) - u_{B_r}(t)|^2\, dy \leq &C_\varepsilon r^n \Big(1 + \frac{t}{r^2}\Big) \|u_0\|^2_{BMO_b^{\mu,\nu}} + \varepsilon \int_{B_r} |u(y,t) - u_{B_r}(t)|^2\, dy \\
&+ C t^{1/2} \|u_0\|_{BMO_b^{\mu,\nu}} \int_{\partial B_r} |u(y,t) - u_{B_r}(t)|\, d\mathcal{H}^{n-1}(y).
\end{aligned}
$$

After taking the term containing ε to the left hand side it is left to estimate $\int_{\partial B_r} |u(y,t) - u_{B_r}(t)|\, d\mathcal{H}^{n-1}(y)$. By the trace theorem and Poincaré's inequality we obtain

$$\int_{\partial B_r} |u(y,t) - u_{B_r}(t)| \, d\mathcal{H}^{n-1}(y) \leq C_r(\int_{B_r} |\nabla u(y,t)|^2 + |u(y,t) - u_{B_r}(t)|^2 \, dy)^{1/2}$$
$$\leq C_r(\int_{B_r} |\nabla u(y,t)|^2 \, dy)^{1/2}.$$

We see by a scaling argument that $C_r = Cr^{n/2}$. Then

$$\begin{aligned} Ct^{1/2} \int_{\partial B_r} |u(y,t) - u_{B_r}(t)| \, d\mathcal{H}^{n-1}(y) &\leq Ct^{1/2} r^{n/2} \|\nabla u(t)\|_{L^2(B_r)} \\ &\leq Ct^{1/2} r^n \|\nabla u(t)\|_\infty \\ &\leq Cr^n \|u_0\|_{BMO_b^{\mu,\nu}} \end{aligned}$$

such that we finally obtain

$$\int_{B_r(x)} |u(y,t) - u_{B_r(x)}(t)|^2 \, dy \leq Cr^n(1 + \frac{t}{r^2}) \|u_0\|^2_{BMO_b^{\mu,\nu}}.$$

□

For boundedness we will need similar estimates for small r. These estimates can be proved in a much simpler way by using Poincaré's inequality.

Lemma 4.2.7. *Let $\Omega \subset \mathbb{R}^n$ be an admissible, uniformly C^3-domain, $\mu, \nu \in (0, \infty]$. There are constants $C, T_0 > 0$ depending only on Ω, n, μ and ν such that for all $r > 0$ and $x \in \Omega$ with $B_r(x) \subset \Omega$, $t \in (0, T_0)$ and all $u_0 \in VMO^{\mu,\nu}_{b,0,\sigma}(\Omega)$ with $u(t) = S(t)u_0$*

$$\frac{1}{|B_r(x)|} \int_{B_r(x)} |u(y,t) - u_{B_r(x)}(t)|^2 \, dy \leq C\frac{r^2}{t} \|u_0\|^2_{BMO_b^{\mu,\nu}}.$$

Proof. By Poincaré's inequality in B_r with constant Cr and Theorem 4.1.2 we can estimate

$$\begin{aligned} \int_{B_r(x)} |u(y,t) - u_{B_r(x)}(t)|^2 \, dy &\leq \int_{B_r(x)} \|u(t) - u_{B_r(x)}(t)\|^2_\infty \, dy \\ &\leq C \int_{B_r(x)} r^2 \|\nabla u(t)\|^2_\infty \, dy \\ &\leq Cr^n \frac{r^2}{t} \|u_0\|^2_{BMO_b^{\mu,\nu}}. \end{aligned}$$

□

We can now estimate the BMO-part of the BMO_b-norm in a suitable way. In a similar way we will get estimates for the boundary part of the norm. Since $B_r(x_0) \cap \Omega$ for $x_0 \in \partial\Omega$ is not a C^2-domain which we will need for the estimate of the pressure, we need to change the parameter ν in a certain way.

Theorem 4.2.3. *Let $\Omega \subset \mathbb{R}^n$ be an admissible, uniformly C^3-domain, $\mu \in (0,\infty]$, $0 < \nu \leq c_0 R^*$, where c_0 is the constant of Remark 4.2.1. There are constants $C, T_0 > 0$ depending only on Ω, n, μ and ν such that for all $x_0 \in \partial\Omega$, $r < \nu$, $t \in (0, T_0)$ and all $u_0 \in VMO^{\mu,\nu}_{b,0,\sigma}(\Omega)$ with $u(t) = S(t)u_0$*

$$\frac{1}{r^n}\int_{B_r(x_0)\cap\Omega} |u(y,t)|^2\,dy \leq C(\|u_0\|^2_{BMO^{\mu,\nu}_b} + [u_0]^2_{b^{2\nu}2}).$$

Proof. Let $B_r(x_0)\cap\Omega \subset \tilde{B} \subset B_{2r}(x_0)\cap\Omega$ be a domain with C^2-regularity, where the C^2-regularity of $\tilde{B}$ depends only on ν and the C^3-regularity of Ω. Again by the fundamental theorem of calculus and integration by parts we obtain

$$\begin{aligned}
\int_{\tilde{B}} |u(y,t)|^2\,dy &= \int_{\tilde{B}} (u(y,t)\int_0^t \frac{\partial u}{\partial s}(y,s)\,ds)\,dy - \int_{\tilde{B}} u(y,t)u_0(y)\,dy \\
&\leq \left|\int_{\tilde{B}} u(y,t)\int_0^t (\Delta u(y,s) - \nabla\pi(y,s))\,ds\,dy\right| + \|u(t)\|_{L^2(\tilde{B})}\|u_0\|_{L^2(\tilde{B})} \\
&\leq \left|\int_{\tilde{B}} u(y,t)\int_0^t (\Delta u(y,s) - \nabla\pi(y,s))\,ds\,dy\right| + \frac{1}{2}\int_{\tilde{B}} |u(y,t)|^2\,dy \\
&\quad + \frac{1}{2}\int_{B_{2r}(x_0)\cap\Omega} |u_0|^2\,dy,
\end{aligned}$$

where we take the second summand to the left hand side. The last summand can be estimated by $r^n[u_0]^2_{b^{2\nu}2}$. For the first summand we obtain by using the estimate $\|u\|_{L^\infty(B_{2r}(x_0)\cap\Omega)} \leq Cr\|\nabla u\|_\infty$, which follows from the homogeneous boundary condition, estimating the part with pressure π in the same way as in Theorem 4.2.2 and integrating by parts

$$\begin{aligned}
&\frac{C}{r^n}\left|\int_{\tilde{B}} u(y,t)\int_0^t (\Delta u(y,s) - \nabla\pi(y,s))\,ds\,dy\right| \\
\leq&\frac{C}{r^n}\left(\left|\int_{\tilde{B}} \nabla u(y,t)\int_0^t \nabla u(y,s)\,ds\,dy\right| + \left|\int_{\partial\tilde{B}} u(y,t)\int_0^t \frac{\partial u}{\partial\mathbf{n}}(y,s)\,ds\,d\mathcal{H}^{n-1}(y)\right|\right. \\
&\left.+ \left|\int_{\partial\tilde{B}} u(y,t)\int_0^t \pi(y,s)\mathbf{n}\,ds\,d\mathcal{H}^{n-1}(y)\right|\right) \\
\leq&\frac{C}{r^n}\left(\int_{\tilde{B}} \|\nabla u(t)\|_\infty \int_0^t \|\nabla u(s)\|_\infty\,ds\,dy\right. \\
&+ \int_{\partial\tilde{B}} r\|\nabla u(t)\|_\infty \int_0^t \|\nabla u(s)\|_\infty\,ds\,d\mathcal{H}^{n-1}(y) \\
&\left.+ \int_{\partial\tilde{B}} |u(y,t)|\int_0^t |\pi(y,s)\mathbf{n}|\,ds\,d\mathcal{H}^{n-1}(y)\right) \\
\leq&C\|u_0\|^2_{BMO^{\mu,\nu}_b} + \frac{C}{r^n}\|u(t)\|_{L^2(\partial\tilde{B})}\int_0^t \|\pi(s)\|_{L^2(\partial\tilde{B})}\,ds \\
\leq&C\|u_0\|^2_{BMO^{\mu,\nu}_b}.
\end{aligned}$$

Finally we obtain

$$\int_{\tilde{B}} |u(y,t)|^2 \, dy \leq C\|u_0\|^2_{BMO_b^{\mu,\nu}} + C[u_0]^2_{b^{2\nu}2}.$$

□

Let $C_{n,L}$ be a constant depending on Ω which will be defined in section 4.2.3. Roughly speaking, $C_{n,L}$ measures the degree of shrinkage of transforms from neighborhoods near the boundary to $\mathbb{R}^n_+$.

Theorem 4.2.4. *Let $\Omega \subset \mathbb{R}^n$ be an admissible, uniformly C^3-domain, $0 < \nu \leq R^*$, $\mu \in (R^*, \infty]$. Then there are constants $C, T_0 > 0$ depending only on Ω, n, μ and ν such that for all $t \in (0, T_0)$ and all $u_0 \in VMO^{\mu,\nu}_{b,0,\sigma}(\Omega)$ with $u(t) = S(t)u_0$*

$$\|u(t)\|_{BMO_b^{\mu,\nu}} \leq C\|u_0\|_{BMO_b^{\mu,\nu}}$$

holds.

Proof. Since the norms with different such ν are equivalent by Theorem 4.2.7 that will be proved later, we can assume that $\nu < \min\{R^*/(4C_{n,L}), c_0R^*\}$. By Theorem 4.2.2, Lemma 4.2.7 and Theorem 4.2.3 we obtain for some T_0 and C depending only on Ω, n, μ, ν

$$\|u(t)\|_{BMO_b^{\mu,\nu}2} \leq C\|u_0\|_{BMO_b^{\mu,\nu}} + C[u_0]_{b^{2\nu}2} \quad (t \in (0, T_0)). \tag{4.2.10}$$

We will now use two different equivalence results on the BMO_b-norms. At first note that it is immediate from the definition and Hölder's inequality that

$$\|u(t)\|_{BMO_b^{\mu,\nu}} \leq C\|u(t)\|_{BMO_b^{\mu,\nu}2}.$$

Since $2\nu < R^*/(2C_{n,L})$ we can use the equivalence between $\|\cdot\|_{BMO_b^{\mu,2\nu}2}$ and $\|\cdot\|_{BMO_b^{\mu,2\nu}}$ that will be proved in the next section to estimate $C[u_0]_{b^{2\nu}2}$ such that we get

$$\|u(t)\|_{BMO_b^{\mu,\nu}} \leq C\|u_0\|_{BMO_b^{\mu,\nu}} + C\|u_0\|_{BMO_b^{\mu,2\nu}} \quad (t \in (0, T_0)).$$

Now we will use the equivalence between $\|\cdot\|_{BMO_b^{\mu,\nu}}$ and $\|\cdot\|_{BMO_b^{\mu,2\nu}}$ (Theorem 4.2.7) which yields

$$\|u(t)\|_{BMO_b^{\mu,\nu}} \leq C\|u_0\|_{BMO_b^{\mu,\nu}} \quad (t \in (0, T_0)).$$

□

Now we have all estimates that are necessary to obtain a semigroup. However, as in the L^∞-case $C^\infty_{c,\sigma}(\Omega)$ is not dense in the largest solenoidal subspace of $BMO_b^{\mu,\nu}(\Omega)$. Thus, in order to get a semigroup on $VMO^{\mu,\nu}_{b,0,\sigma}(\Omega)$ we have to ensure that the solutions $u(t) \in VMO^{\mu,\nu}_{b,0,\sigma}(\Omega)$ for $u_0 \in VMO^{\mu,\nu}_{b,0,\sigma}(\Omega)$. This will be done in the appendix.

We are now able to show our main result, the analyticity of the $VMO^{\mu,\nu}_{b,0,\sigma}(\Omega)$-Stokes semigroup.

Proof of Theorem 4.1.1. By Theorem 4.1.2 and the embedding $L^\infty(\Omega) \hookrightarrow BMO_b^{\mu,\nu}(\Omega)$ we know that the solution operator $S(t)$ satisfies the estimate

$$\|\frac{\mathrm{d}}{\mathrm{d}t}S(t)u_0\|_{BMO_b^{\mu,\nu}} \leq \frac{C}{t}\|u_0\|_{BMO_b^{\mu,\nu}} \quad (t \in (0,T_0)).$$

Furthermore, we know by the previous theorem that

$$\|S(t)u_0\|_{BMO_b^{\mu,\nu}} \leq C_0\|u_0\|_{BMO_b^{\mu,\nu}} \quad (t \in (0,T_0)). \tag{4.2.11}$$

By the appendix we obtain that $S(t)u_0 \in VMO_{b,0,\sigma}^{\mu,\nu}(\Omega)$ for every $u_0 \in VMO_{b,0,\sigma}^{\mu,\nu}(\Omega)$ and $t \in (0,T_0)$. From this we can conclude that $S(t)u_0 \in VMO_{b,0,\sigma}^{\mu,\nu}(\Omega)$ for every $t > 0$. This together with the above estimates yields that S is an analytic semigroup. It is left to show that S is a C_0-semigroup. It was proved in Proposition 5.3 of [AG13] that for all $u_0 \in C_{c,\sigma}^\infty(\Omega)$

$$\lim_{t\to 0}\|S(t)u_0 - u_0\|_\infty = 0.$$

If we now take $u_0 \in VMO_{b,0,\sigma}^{\mu,\nu}(\Omega)$, then there exists by definition of $VMO_{b,0,\sigma}^{\mu,\nu}(\Omega)$ a sequence $u_0^m \in C_{c,\sigma}^\infty(\Omega)$ such that u_0^m converges to u_0 with respect to the $BMO_b^{\mu,\nu}$-norm. Then we have by (4.2.11) for $t < T_0$

$$\begin{aligned}
&\|S(t)u_0 - u_0\|_{BMO_b^{\mu,\nu}} \\
\leq&\|S(t)(u_0 - u_0^m)\|_{BMO_b^{\mu,\nu}} + \|S(t)u_0^m - u_0^m\|_{BMO_b^{\mu,\nu}} + \|u_0^m - u_0\|_{BMO_b^{\mu,\nu}} \\
\leq&(C_0+1)\|u_0^m - u_0\|_{BMO_b^{\mu,\nu}} + (2+\omega_n)\|S(t)u_0^m - u_0^m\|_\infty.
\end{aligned}$$

For given $\varepsilon > 0$ we choose then $m \in \mathbb{N}$ such that $\|u_0^m - u_0\|_{BMO_b^{\mu,\nu}} < \frac{\varepsilon}{2(C_0+1)}$ and then $t_0 > 0$ sufficiently small such that $\|S(t)u_0^m - u_0^m\|_\infty < \frac{\varepsilon}{2(2+\omega_n)}$ for all $0 < t < t_0$. Then

$$\|S(t)u_0 - u_0\|_{BMO_b^{\mu,\nu}} < \varepsilon$$

for $t < t_0$ which proves that S is a C_0-semigroup.

□

4.2.3 Remark on equivalences of BMO_b-norms

In this section we will prove the equivalence results for different BMO_b-norms that were used in the proof of Theorem 4.2.4.

For these equivalence results we will need a fundamental theorem on BMO-functions that states that the L^1-norm of a function in a large area can be controlled by the L^1-norm of the function in a small area and the BMO-seminorm of f.

Theorem 4.2.5. *Let $\mu \in (0,\infty]$ and $\Omega \subset \mathbb{R}^n$ be a domain. Then for all $f \in BMO^\mu(\Omega)$, $a > 1$, $r > 0$, $x_1, x_2 \in \Omega$ with $B_r(x_1) \subset B_{ar}(x_2) \subset \Omega$ and $ar \leq \mu$ holds the inequality*

$$\|f\|_{L^1(B_{ar}(x_2))} \leq |B_{ar}(x_2)|(1+a^n)[f]_{BMO^\mu(\Omega)} + a^n\|f\|_{L^1(B_r(x_1))}. \tag{4.2.12}$$

Proof. Let $B_1 := B_r(x_1)$, $B_2 := B_{ar}(x_2)$ and $\tilde f := f - f_{B_1}$. By $\int_{B_1} \tilde f - \tilde f_{B_2}\,dy = -|B_1|\tilde f_{B_2}$ we obtain

$$|B_1||\tilde f_{B_2}| \leq \int_{B_1} |\tilde f - \tilde f_{B_2}|\,dy$$

and thus

$$\begin{aligned}|B_2|[\tilde f]_{BMO^\mu} &\geq \int_{B_2} |\tilde f - \tilde f_{B_2}|\,dy\\ &\geq \int_{B_1} |\tilde f - \tilde f_{B_2}|\,dy\\ &\geq |B_1||\tilde f_{B_2}|.\end{aligned}$$

From this we can estimate the mean value of $\tilde f$ in B_2 by

$$|\tilde f_{B_2}| \leq a^n[\tilde f]_{BMO^\mu}.$$

Then we can estimate the L^1-norm of f by using estimates on the mean values together with the L^1-norm of f on a small ball.

$$\begin{aligned}\|f\|_{L^1(B_2)} &\leq \|f - f_{B_1}\|_{L^1(B_2)} + |B_2||f_{B_1}|\\ &= \|\tilde f\|_{L^1(B_2)} + |B_2||f_{B_1}|\\ &\leq \|\tilde f - \tilde f_{B_2}\|_{L^1(B_2)} + |B_2||\tilde f_{B_2}| + \frac{|B_2|}{|B_1|}\|f\|_{L^1(B_1)}\\ &\leq |B_2|[\tilde f]_{BMO^\mu} + |B_2|a^n[\tilde f]_{BMO^\mu} + a^n\|f\|_{L^1(B_1)}\\ &= |B_2|(1+a^n)[f]_{BMO^\mu} + a^n\|f\|_{L^1(B_r(x_1))}.\end{aligned}$$

□

Since we consider BMO-functions on domains it will be useful to extend those functions to the more classical BMO-functions on $\mathbb{R}^n$. P. W. Jones proved in [Jon80] the exact condition when this is possible. This condition is in particular satisfied if the domain is a bounded Lipschitz domain.

Theorem 4.2.6. *Let $\Omega \subset \mathbb{R}^n$ be a bounded Lipschitz domain. Then there is a constant C depending only on Lipschitz regularity of $\partial\Omega$ such that for each $f \in BMO^\infty(\Omega)$ there is an extension $\bar f \in BMO^\infty(\mathbb{R}^n)$ such that*

$$[\bar f]_{BMO^\infty(\mathbb{R}^n)} \leq C[f]_{BMO^\infty(\Omega)}.$$

Theorem 4.2.7. *Let $\Omega \subset \mathbb{R}^n$ be a uniformly C^2-domain, $\nu_1 < \nu_2 \leq R^*$ and $\mu \in [\nu_2, \infty]$. The norm $\|\cdot\|_{BMO_b^{\mu,\nu_1}}$ is then equivalent to $\|\cdot\|_{BMO_b^{\mu,\nu_2}}$.*

Proof. It follows immediately from the definition that for $\nu_1 < \nu_2$, $\|f\|_{BMO_b^{\mu,\nu_1}} \leq \|f\|_{BMO_b^{\mu,\nu_2}}$. Thus it is left to show that

$$\frac{1}{r^n}\int_{\Omega\cap B_r(x_0)} |f(y)|\,dy \leq C\|f\|_{BMO_b^{\mu,\nu_1}}$$

with a constant $C > 0$ independent of $x_0 \in \partial\Omega$ and $\nu_1 \leq r < \nu_2$. Since $\nu_1 \leq r < R^*$, every $B_{\frac{\nu_1}{2}}(x_0) \cap \Omega \subset B_r(x_0) \cap \Omega$ contains a closed ball B_0 of radius $\nu_1/4$ and the Lipschitz regularity of $\Omega \cap B_r(x_0)$ is uniform. Thus by Theorem 4.2.6 there is a uniform constant $C > 0$ such that for all $x_0 \in \partial\Omega$ and all $\nu_1 \leq r < \nu_2$ there is an extension of $f|_{\Omega \cap B_r(x_0)}$ to $\bar{f} \in BMO^\infty(\mathbb{R}^n)$ with

$$[\bar{f}]_{BMO^\infty(\mathbb{R}^n)} \leq C[f]_{BMO^\infty(\Omega \cap B_r(x_0))} \leq C[f]_{BMO^\mu(\Omega)}.$$

Since $\int_{B_0} |f(y)|\,dy \leq \nu_1^n [f]_{b^{\nu_1}}$ we obtain by Theorem 4.2.5 for $\nu_1 \leq r < \nu_2$ that

$$\begin{aligned}
\frac{1}{r^n} \int_{\Omega \cap B_r(x_0)} |f(y)|\,dy &\leq \frac{1}{r^n} \int_{B_r(x_0)} |\bar{f}(y)|\,dy \\
&\leq \omega_n (1 + (\frac{4\nu_2}{\nu_1})^n)[\bar{f}]_{BMO^\infty(\mathbb{R}^n)} + \frac{(\frac{4r}{\nu_1})^n}{r^n} \|f\|_{L^1(B_0)} \\
&\leq C[f]_{BMO^\mu(\Omega)} + C[f]_{b^{\nu_1}(\Omega)}
\end{aligned}$$

with a constant independent of r and x_0. □

We now want to prove the equivalence between $BMO_b^{\mu,\nu}p$ and $BMO_b^{\mu,\nu}$. Our proof is divided into two parts. One concerning Hölder type estimates and one concerning reverse Hölder type estimates which will be the crucial part.

Lemma 4.2.8 (Hölder type estimates)**.** *Let $\Omega \subset \mathbb{R}^n$ be a domain, $p \in [1, \infty)$, $\mu, \nu \in (0, \infty]$ and $f \in BMO_b^{\mu,\nu}p(\Omega)$. Then f satisfies the following estimate*

$$\|f\|_{BMO_b^{\mu,\nu}} \leq C\|f\|_{BMO_b^{\mu,\nu}p}$$

for some constant $C = C(n, p) > 0$.

Proof. This Lemma is easily obtained by the use of Hölder's inequality. □

For the reverse Hölder type inequality we need the John-Nirenberg inequality.

Theorem 4.2.8 (John-Nirenberg inequality)**.** *Let $\Omega \subset \mathbb{R}^n$ be a domain, $p \in [1, \infty)$, $f \in BMO^\mu(\Omega)$. Then, there exists $C = C(n, p) > 0$ such that*

$$[f]_{BMO^\mu p} \leq C[f]_{BMO^\mu}.$$

Proof. This inequality is rather different from the original John-Nirenberg inequality ([JN61]), but it can be obtained from this inequality. □

Let $\Omega \subset \mathbb{R}^n$ be a uniformly C^2-domain with Lipschitz constant L and let $x_0 \in \partial\Omega$. We define $\Phi_{x_0} : \Omega \cap \bar{B}_{R^*}(x_0) \to \mathbb{R}^n_+$ by $\Phi_{x_0}(x) = (x', x_n - \phi_{x_0}(x'))$ where ϕ_{x_0} is a Lipschitz function with Lipschitz constant L which is a local coordinate of $\partial\Omega$ at x_0. Let $d(A)$ denote the diameter of A. Then we define the degree of shrinkage of Ω (denoted by $C_{n,L}$) by

$$\sup\{\frac{d(\Phi_{x_0}(B_r(x) \cap \Omega))}{d(B_r(x) \cap \Omega)}, \frac{d(\Phi_{x_0}^{-1}(B_r(x) \cap \Omega))}{d(B_r(x) \cap \Omega)} : x \in \Omega, B_r(x) \subset B_{R^*}(x_0), x_0 \in \partial\Omega\}.$$

We remark that this degree depends only on n and L because Ω is uniformly Lipschitz. Now we want to state the reverse Hölder type estimates up to the boundary.

Lemma 4.2.9 (Reverse Hölder type estimates up to the boundary)**.** *Let $\Omega \subset \mathbb{R}^n$ be a uniformly C^2-domain with Lipschitz constant L. Let $C_{n,L}$ denote the degree of shrinkage of Ω. Let $\nu \in (0, R^*/(2C_{n,L}^2)]$, $\mu \in [R^*, \infty]$, $p \in [1, \infty)$, $f \in BMO_b^{\mu,\nu}(\Omega)$. Then there exists a constant $C = C(n, p, \Omega, \nu) > 0$ such that*

$$[f]_{b^\nu p} \leq C\|f\|_{BMO_b^{\mu,\nu}(\Omega)}.$$

Proof. Let $x_0 \in \partial\Omega$ and $r < \nu$ be given. We will then write Φ for Φ_{x_0}. Then, by changing variables

$$\begin{aligned}
&(r^{-n} \int_{\Omega \cap B_r(x_0)} |f(y)|^p \, dy)^{1/p} \\
=&(r^{-n} \int_{\Phi(\Omega \cap B_r(x_0))} |(f \circ \Phi^{-1})(z)|^p |J_{\Phi^{-1}}| \, dz)^{1/p} \\
\leq&(1 + L)(\frac{|\Phi(B_r)|}{r^n})^{\frac{1}{p}} (|\Phi(B_r)|^{-1} \int_{\Phi(\Omega \cap B_r(x_0))} |(f \circ \Phi^{-1})(z)|^p \, dz)^{1/p} \\
\leq&(1 + L)(\omega_n C_{n,L})^{\frac{1}{p}} (|\Phi(B_r)|^{-1} \int_{\Phi(\Omega \cap B_r(x_0))} |(f \circ \Phi^{-1})(z)|^p \, dz)^{1/p},
\end{aligned}$$

where $J_{\Phi^{-1}}$ denotes the Jacobian of Φ^{-1}. Let $E_{\mathbb{R}^n_+}$ be the x_n-odd extension from $\mathbb{R}^n_+$ to $\mathbb{R}^n$. We define the function g by $g = E_{\mathbb{R}^n_+}(f \circ \Phi^{-1})$ and set

$$Q_R = \Phi(\Omega \cap B_R(x_0)) \cup (-\Phi(\Omega \cap B_R(x_0))) \text{ for } R = r, R^*.$$

Then, $\int_{Q_R} g dx = 0$ for $R = r, R^*$. We want to apply Theorem 4.2.8, so we check that g satisfies the assumption of Theorem 4.2.8, i.e., $g \in BMO^{C_{n,L}\nu}(Q_{R^*})$. For this we will show that

$$[g]_{BMO^{C_{n,L}\nu}(Q_{R^*})} \leq C\|f\|_{BMO_b^{\mu,\nu}(\Omega)}. \tag{4.2.13}$$

Take $B_s(x) \subset Q_{R^*}$ with $s < C_{n,L}\nu < \mu/C_{n,L}$. There are two cases we have to consider.

1. $B_s(x) \cap \partial\mathbb{R}^n_+ = \emptyset$
2. $B_s(x) \cap \partial\mathbb{R}^n_+ \neq \emptyset$

In the case (1), we may assume $B_s(x) \subset \mathbb{R}^n_+$. We remark that $g = f \circ \Phi^{-1}$ in this case. We will show

$$\frac{1}{|B_s(x)|} \int_{B_s(x)} |g(y) - g_{B_s(x)}| \, dy \leq C\|f\|_{BMO_b^{\mu,\nu}(\Omega)}.$$

Take arbitrary $c \in \mathbb{R}$. Then, by changing variables

$$\begin{aligned}
&\int_{B_s(x)} |g(z) - g_{B_s(x)}|\, dz \\
\leq &\int_{B_s(x)} |f \circ \Phi^{-1}(z) - c|\, dz + |B_s(x)||c - (f \circ \Phi^{-1})_{B_s(x)}| \\
\leq &2 \int_{B_s(x)} |f \circ \Phi^{-1}(z) - c|\, dz \\
= &2 \int_{\Phi^{-1}(B_s(x))} |f(y) - c||J_\Phi|\, dy \\
\leq &2(1+L) \int_{\Phi^{-1}(B_s(x))} |f(y) - c|\, dy.
\end{aligned}$$

Let $d > 0$ be the distance from $\Phi^{-1}(B_s(x))$ to the boundary of $B_{R^*} \cap \Omega$. If the diameter of $\Phi^{-1}(B_s(x))$ is smaller than d, we can take the smallest ball $B_{s'}(z')$ with $s' < d < R^*$ and $z' \in \Omega$ so that $\Phi^{-1}(B_s(x)) \subset B_{s'}(z') \subset B_{R^*}(x_0) \cap \Omega$. Then $s' \leq C_{n,L} s < \mu$ and we obtain

$$\int_{\Phi^{-1}(B_s(x))} |f(y) - c|\, dy \leq \int_{B_{s'}(z')} |f(y) - c|\, dy.$$

Since c is arbitrary, this implies

$$\frac{1}{|B_s(x)|} \int_{B_s(x)} |g(y) - g_{B_s(x)}|\, dy \leq C[f]_{BMO^\mu} < +\infty.$$

If the diameter of $\Phi^{-1}(B_s(x))$ is bigger than d, then we take a perpendicular from $\Phi^{-1}(x)$ to $\partial\Omega$, and let x' denote a point at which the perpendicular intersects with $\partial\Omega$. Take the smallest ball $B_{s'}(x') \subset B_{R^*}(x_0)$ which contains $\Phi^{-1}(B_s(x))$. Then,

$$\frac{1}{|B_s(x)|} \int_{B_s(x)} |g(y) - g_{B_s(x)}|\, dy \leq C \frac{s'^n}{|B_s|} \frac{1}{s'^n} \int_{B_{s'}(x') \cap \Omega} |f(y) - c|\, dy.$$

By taking $c = 0$ in the integral,

$$\begin{aligned}
&\frac{1}{|B_s(x)|} \int_{B_s(x)} |g(y) - g_{B_s(x)}|\, dy \\
\leq &C \frac{s'^n}{|B_s|} \frac{1}{s'^n} \int_{B_{s'}(x') \cap \Omega} |f(y)|\, dy \leq C_{n,L,d} \frac{s'^n}{|B_s|} [f]_{b^{R^*}}.
\end{aligned}$$

We remark that $[f]_{b^{R^*}}$ is estimated by $C\|f\|_{BMO_b^{\mu,\nu}(\Omega)}$ because $f \in BMO_b^{\mu,\nu}(\Omega)$ and $BMO_b^{\mu,\nu}(\Omega)$ is equivalent to $BMO_b^{\mu,R^*}(\Omega)$ by Theorem 4.2.7. We also remark that $\frac{s'^n}{|B_s|}$ is finite because $d(\Phi^{-1}(B_s(x))) \leq C_{n,L} s$. In the case (2), $B_s(x)$ can be decomposed up to a null set as

$$B_s(x) = (B_s(x) \cap \mathbb{R}^n_+) \cup (B_s(x) \cap (-\mathbb{R}^n_+)) = B^1 \cup B^2.$$

Then, $\int_{B_s(x)} |g(z)-g_{B_s(x)}|\,dz \le 2\int_{B^1}|g(z)|\,dz+2\int_{B^2}|g(z)|\,dz$. Since the second term can be estimated in the same way as the first term, we only need to estimate the first term. By change of variables,

$$\int_{B^1}|g(z)|\,dz=\int_{\Phi^{-1}(B^1)}|f(z)||J_\Phi|\,dz\le(1+L)\int_{\Phi^{-1}(B^1)}|f(z)|\,dz.$$

Let us take a perpendicular from $\Phi^{-1}(x)$ to $\partial\Omega$, and let x' denote the point at which the perpendicular intersects with $\partial\Omega$. Take the smallest ball $B_{s'}(x')\subset B_{R^*}(x_0)$ which contains $\Phi^{-1}(B^1)$. Then,

$$\begin{aligned}&\int_{\Phi^{-1}(B^1)}|f(z)|\,dz\\ \le& Cs'^n\frac{1}{s'^n}\int_{B_{s'}(x')}|f(z)|\,dz\\ \le& Cs'^n[f]_{b^{R^*}}<+\infty.\end{aligned}$$

We have thus proved (4.2.13).

As a consequence, we can apply Theorem 4.2.8 to g and get for the largest ball $B_{\tilde r}(\tilde x)$ satisfying $B_{\tilde r}(\tilde x)\subset Q_r$ and the smallest ball $B_{r'}(x')$ satisfying $Q_r\subset B_{r'}(x')$

$$\begin{aligned}&(r^{-n}\int_{\Omega\cap B_r(x_0)}|f(y)|^p\,dy)^{1/p}\\ \le& C(|\Phi(B_r)|^{-1}\int_{\Phi(\Omega\cap B_r(x_0))}|(f\circ\Phi^{-1})(z)|^p\,dz)^{1/p}\\ =& C(2|Q_r|^{-1}\frac{1}{2}\int_{Q_r}|g(z)-g_{Q_r}|^p\,dz)^{1/p}\\ \le&(C|B_{\tilde r}(\tilde x)|^{-1}\int_{Q_r}|g(z)-g_{B_{r'}(x')}|^p\,dz)^{1/p}\\ \le&(C|B_{r'}(x')|^{-1}\int_{B_{r'}(x')}|g(z)-g_{B_{r'}(x')}|^p\,dz)^{1/p}\\ \le& C[g]_{BMO^{C_{n,L}\nu}(Q_{R^*})}\end{aligned}$$

Here, we used $r\le C_{n,L}\tilde r$ and $r'\le C_{n,L}r\le C_{n,L}\nu$. By (4.2.13) we obtain as a consequence for arbitrarily given $x_0\in\partial\Omega$ and $r<\nu$,

$$(r^{-n}\int_{\Omega\cap B_r(x_0)}|f(y)|^p\,dy)^{1/p}\le C\|f\|_{BMO_b^{\mu,\nu}(\Omega)}.$$

Therefore, we obtain the reverse Hölder type estimates up to the boundary. □

Theorem 4.2.9. *Let $\Omega\subset\mathbb{R}^n$ be a uniformly C^2-domain with Lipschitz constant L. Let $C_{n,L}$ denote the degree of shrinkage of Ω. Let $\nu\in(0,R^*/(2C_{n,L}^2)]$, $\mu\in[R^*,\infty]$, $p\in[1,\infty)$, $f\in BMO_b^{\mu,\nu}(\Omega)$. Then, $\|\cdot\|_{BMO_b^{\mu,\nu}p}$ is equivalent to $\|\cdot\|_{BMO_b^{\mu,\nu}}$.*

Proof. Lemma 4.2.8 and Theorem 4.2.9 imply the equivalence. □

4.2.4 Bounded Analyticity in the Half-Space

In this section we will prove that the Stokes semigroup is a bounded analytic semigroup in a solenoidal subspace of $BMO_b^{\infty,\infty}(\mathbb{R}^n_+)$. Furthermore, we will obtain global derivative estimates of the solution.

Theorem 4.2.10. *Let $\Omega = \mathbb{R}^n_+$ be the half-space. Then there is a constant C which only depends on the dimension n such that for all $u_0 \in VMO_{b,0,\sigma}^{\infty,\infty}(\mathbb{R}^n_+)$*

$$\sup_{t>0} \|u(t)\|_{BMO_b^{\infty,\infty}} \leq C\|u_0\|_{BMO_b^{\infty,\infty}}, \tag{4.2.14}$$

$$\sup_{t>0} t^{1/2}\|\nabla u(t)\|_\infty \leq C\|u_0\|_{BMO_b^{\infty,\infty}}, \tag{4.2.15}$$

$$\sup_{t>0} t\|\nabla^2 u(t)\|_\infty \leq C\|u_0\|_{BMO_b^{\infty,\infty}}, \tag{4.2.16}$$

$$\sup_{t>0} t\|u_t(t)\|_\infty \leq C\|u_0\|_{BMO_b^{\infty,\infty}}, \tag{4.2.17}$$

$$\sup_{t>0} t\|\nabla \pi(t)\|_\infty \leq C\|u_0\|_{BMO_b^{\infty,\infty}}, \tag{4.2.18}$$

where $(u, \nabla\pi)$ is the solution of the Stokes equations with $S(t)u_0 = u(t)$. In particular, S is a bounded analytic semigroup on $VMO_{b,0,\sigma}^{\infty,\infty}(\mathbb{R}^n_+)$.

Proof. We will use that the spaces $BMO_b^{\infty,\infty}(\mathbb{R}^n_+)$ and $L^\infty(\mathbb{R}^n_+)$ are scaling-invariant. By Theorem 4.1.2 and Theorem 4.2.4 we obtain the existence of some $T_0 > 0$ such that for all $u_0 \in VMO_{b,0,\sigma}^{\infty,\infty}(\mathbb{R}^n_+)$ the estimate

$$\sup_{0<t<T_0} (\|u(t)\|_{BMO_b^{\infty,\infty}} + \|\tilde{N}(u,\pi)(\cdot,t)\|_\infty) \leq C_{T_0}\|u_0\|_{BMO_b^{\infty,\infty}}$$

holds. By taking $u_0^\lambda(x) := u_0(\lambda x)$ as initial data for $\lambda > 0$ we obtain the same estimate for $u^\lambda(x,t) = u(\lambda x, \lambda^2 t)$ and $\pi^\lambda = \lambda\pi(\lambda x, \lambda^2 t)$ with the right hand side $C_{T_0}\|u_0^\lambda\|_{BMO_b^{\infty,\infty}}$ which is equal to $C_{T_0}\|u_0\|_{BMO_b^{\infty,\infty}}$. By the scaling-invariance of the spaces we can conclude from the estimate for (u^λ, π^λ) that

$$\sup_{0<t<\lambda^2 T_0} (\|u(t)\|_{BMO_b^{\infty,\infty}} + \|\tilde{N}(u,\pi)(\cdot,t)\|_\infty) \leq C_{T_0}\|u_0\|_{BMO_b^{\infty,\infty}}$$

with C_{T_0} independent of $\lambda > 0$. Since λ was arbitrary we can replace $\sup_{0<t<\lambda^2 T_0}$ by $\sup_{t>0}$ in the above inequality and get the desired estimates. The bounded analyticity follows then from the time derivative estimate. $\square$

4.2.5 Appendix

Our goal in this section is to prove a density result. Let $\tilde{A}_r$ be the Stokes operator in the space $\tilde{L}^r_\sigma$ which is constructed in [FKS05], [FKS07].

Theorem 4.2.11. *Let Ω be a uniformly C^2-domain in $\mathbb{R}^n$ ($n \geq 2$). For $f \in D(\tilde{A}_{r_0})$, $r_0 > 2$, there exists a sequence $\{f_m\} \subset C_{c,\sigma}^\infty(\Omega)$ such that $\|f - f_m\|_{\tilde{W}^{1,r}(\Omega)} \to 0$ as $m \to \infty$ for all $r \in [2, r_0)$.*

This density result yields the following property for the Stokes semigroup $S(t)$. Let $\tilde{W}^{1,r}_{\sigma,0}(\Omega)$ denote the $\tilde{W}^{1,r}$-closure of $C^\infty_{c,\sigma}(\Omega)$.

Corollary 4.2.1. *Let Ω be a uniformly C^2-domain and $u_0 \in C^\infty_{c,\sigma}(\Omega)$. Then $S(t)u_0 \in \tilde{W}^{1,r}_{\sigma,0}(\Omega)$ for all $r \geq 2$ and $t > 0$. In particular, $S(t)u_0 \in C_{0,\sigma}(\Omega) \subset VMO^{\mu,\nu}_{b,0,\sigma}(\Omega)$ with $\mu, \nu \in (0, \infty]$.*

This follows from Theorem 4.2.11. Indeed, since S is an analytic semigroup in $\tilde{L}^r_\sigma(\Omega)$ we observe that $S(t)u_0 \in D(\tilde{A}_r)$ for $t > 0$ and $u_0 \in \tilde{L}^r_\sigma(\Omega)$. If $u_0 \in C^\infty_{c,\sigma}(\Omega)$ so that $u_0 \in \tilde{L}^{r_0}_\sigma(\Omega)$ for any $r_0 \geq 2$, then we get $S(t)u_0 \in D(\tilde{A}_{r_0})$. Thus, applying Theorem 4.2.11 implies that $S(t)u_0 \in \tilde{W}^{1,r}_{\sigma,0}(\Omega)$ for any $r \geq 2$. The remaining assertion follows from the Sobolev embedding for $r > n$ and $L^\infty(\Omega) \hookrightarrow BMO^{\mu,\nu}_b(\Omega)$.

The rest of this section is devoted to the proof of Theorem 4.2.11. For this purpose we need an approximation of the domain Ω.

For a uniformly C^2-domain Ω of type (α, β, K) in the sense of [FKS05] one can easily construct a sequence of uniformly C^2-domains Ω_m of type (α, β, K) such that $\Omega_m \subset \Omega$, $dist(\Omega_m, \partial\Omega) \geq \frac{1}{m}$ and

$$\Omega \subset \{x \in \mathbb{R}^n : dist(x, \Omega_m) \leq \frac{2}{m}\}$$

for $m \in \mathbb{N}$.

Lemma 4.2.10. *For $f \in D(\tilde{A}_{r_0})$ with $r_0 > 2$ and $r \in [2, r_0)$ there exists a sequence $\{f_m\} \subset \tilde{W}^{1,r}_0(\Omega_m) \cap \tilde{L}^r_\sigma(\Omega_m)$ such that $\|f_m - f\|_{\tilde{W}^{1,r}(\Omega)} \to 0$ as $m \to \infty$. Here we interpret f_m as a function defined on Ω by extending via $f_m = 0$ in $\Omega \setminus \Omega_m$.*

Proof. Let $\tilde{A}_{r_0,m}$ be the Stokes operator in $\tilde{L}^{r_0}_\sigma(\Omega_m)$. By the construction of the operator there exists λ_0 such that if $\lambda \geq \lambda_0$, then $\lambda + \tilde{A}_{r_0,m}$ is invertible in $\tilde{L}^{r_0}_\sigma(\Omega_m)$, where λ_0 is independent of Ω_m since this property only depends on (α, β, K). We fix λ_0. For $f \in D(\tilde{A}_{r_0})$ we define $g \in \tilde{L}^{r_0}_\sigma(\Omega)$ by

$$g = (\lambda_0 + \tilde{A}_{r_0})f.$$

We approximate g by $g_m \in C^\infty_{c,\sigma}(\Omega)$ such that $\|g - g_m\|_{\tilde{L}^r(\Omega)} \to 0$ as $m \to \infty$. We may assume that supp$g_m \subset \Omega_m$ by taking a subsequence. We set

$$f_m = (\lambda_0 + \tilde{A}_{r_0,m})^{-1}(g_m|_{\Omega_m}).$$

Since $f_m \in D(\tilde{A}_{r_0,m})$, it is clear that $f_m \in \tilde{W}^{1,r}_0(\Omega_m) \cap \tilde{L}^r_\sigma(\Omega_m)$ for all $r \in [2, r_0]$. We extend f_m by 0 and obtain a sequence of functions f_m defined on Ω. By the a priori estimate of [FKS05], [FKS07] we see that

$$\|f_m\|_{\tilde{W}^{2,r}(\Omega_m)} \leq C\|g_m\|_{\tilde{L}^r(\Omega_m)} \quad (r \in [2, r_0))$$

with C depending only on (α,β,K). It is not difficult to show that $f_m \to f$ in the sense of distributions in Ω. Since $\|g_m\|_{\tilde{L}^r(\Omega)}$ is bounded by a constant multiple of $\|g\|_{\tilde{L}^r(\Omega)}$, this implies that $\|f_m\|_{\tilde{W}^{1,r_0}(\Omega)}$ is bounded. By

$$\|\nabla f - \nabla f_m\|_{L^r(\Omega)} \le \|\nabla f - \nabla f_m\|^{\theta}_{L^2(\Omega)} \|\nabla f - \nabla f_m\|^{1-\theta}_{L^{r_0}(\Omega)}$$

with $\frac{1}{r} = \frac{\theta}{r_0} + \frac{1-\theta}{2}$ and the same estimate for $f - f_m$ it suffices to prove that $f_m \to f$ strongly in $H^1(\Omega)$. We consider $H^1(\Omega)$ equipped with the scalar product $(f,g) = \int_\Omega (\lambda_0 + A_{r_0}) f \cdot g$ which is equivalent to the standard scalar product in $H^1(\Omega)$.

Since we already know that $f_m \to f$ in the sense of distributions and since $\|f_m\|_{H^1(\Omega)}$ is bounded, we can conclude that $f_m \to f$ weakly in $H^1(\Omega)$. To obtain strong convergence it remains to prove that $\|f_m\|_{H^1} \to \|f\|_{H^1}$. For this purpose we observe that

$$\|f_m\|^2_{H^1(\Omega)} = \int_{\Omega_m} (\lambda_0 + \tilde{A}_{r_0,m}) f_m \cdot f_m \, dx = \int_{\Omega_m} g_m \cdot f_m \, dx.$$

Since $f_m \to f$ weakly in $L^2(\Omega)$ and $g_m \to g$ strongly in L^2 we conclude that

$$\|f_m\|^2_{H^1} \to \int_\Omega f \cdot g \, dx \quad (m \to \infty).$$

The limit equals to

$$\|f\|^2_{H^1} = \int_\Omega (\lambda_0 + A_{r_0}) f \cdot f \, dx.$$

Thus $f_m \to f$ in H^1. The proof is now complete. □

Lemma 4.2.11. *Let $\Omega \subset \mathbb{R}^n$ be a domain and $1 < r < \infty$. Let $f \in \tilde{W}^{1,r}_0(\Omega) \cap \tilde{L}^r_\sigma(\Omega)$ with $c_0 := dist(\mathrm{supp} f, \partial\Omega) > 0$. Then there exists a sequence $f_m \in C^\infty_{c,\sigma}(\Omega)$ such that $\|f_m - f\|_{\tilde{W}^{1,r}} \to 0$ as $m \to \infty$.*

Proof. Let $\varepsilon > 0$ and take some $\delta < \min\{\varepsilon, c_0/2\}$. Let Ω' be defined by

$$\Omega' = \{x \in \Omega : dist(x, \partial\Omega) > c_0/2\}.$$

Since f is regarded as an element of $\tilde{L}^r_\sigma(\Omega')$, there exists a sequence $f_k \in C^\infty_{c,\sigma}(\Omega')$ such that $f_k \to f$ in $\tilde{L}^r_\sigma(\Omega')$. Let ϱ_δ be the standard mollifier whose support is contained in a ball of radius δ centered at zero. We define $f_\delta = f * \varrho_\delta$. We construct a sequence $f_{k,\delta} \in C^\infty_{c,\sigma}(\Omega)$ by $f_{k,\delta} = f_k * \varrho_\delta$ such that $f_{k,\delta}$ converges to f_δ in $\tilde{W}^{1,r}(\Omega)$. Note that the support of $f_{k,\delta}$ is contained in Ω by the choice of Ω' and ϱ. We observe that

$$\begin{aligned} \|f - f_{k,\delta}\|_{\tilde{W}^{1,r}} &\le \|f - f_\delta\|_{\tilde{W}^{1,r}} + \|f_\delta - f_{k,\delta}\|_{\tilde{W}^{1,r}} \\ &\le \|f - f_\delta\|_{\tilde{W}^{1,r}} + C_\delta \|f - f_k\|_{\tilde{L}^r}. \end{aligned}$$

For $\varepsilon > 0$ we take δ sufficiently small such that $\|f - f_\delta\|_{\tilde{W}^{1,r}} \le \varepsilon/2$ and then choose k_0 large enough to obtain for all $k \ge k_0$ that $C_\delta \|f - f_k\|_{\tilde{L}^r} \le \varepsilon/2$. □

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Chapter 5

On analyticity of the L^p-Stokes semigroup for some non-Helmholtz domains

5.1 Introduction

In this chapter, as a continuation of [AGSS15], [AGSS16] and [BG16], we study the Stokes semigroup, i.e., the solution operator $S(t) : v_0 \mapsto v(\cdot, t)$ of the initial-boundary problem for the Stokes system

$$v_t - \Delta v + \nabla q = 0, \quad \operatorname{div} v = 0 \quad \text{in} \quad \Omega \times (0, \infty)$$

with the zero boundary condition

$$v = 0 \quad \text{on} \quad \partial\Omega \times (0, \infty)$$

and the initial condition $v|_{t=0} = v_0$, where Ω is a domain in $\mathbf{R}^n$ with $n \geq 2$. It is by now well-known that $S(t)$ forms a C_0-analytic semigroup in L^p_σ ($1 < p < \infty$) for various domains like smooth bounded domains ([Gig81], [Sol77]). Here $L^p_\sigma = L^p_\sigma(\Omega)$ denotes the L^p-closure of $C^\infty_{c,\sigma}(\Omega)$, the space of all solenoidal vector fields with compact support in Ω. More recently, it has been proved in [GHHS12] that $S(t)$ always forms a C_0-analytic semigroup in $L^p_\sigma(\Omega)$ for any uniformly C^2-domain Ω provided that $L^p(\Omega)$ admits a topological direct sum decomposition called the Helmholtz decomposition of the form

$$L^p(\Omega) = L^p_\sigma(\Omega) \oplus G^p(\Omega)$$

where $G^p(\Omega) = \left\{ \nabla q \in L^p(\Omega) \mid q \in L^1_{loc}(\Omega) \right\}$. In [GHHS12] the L^q maximal regularity in time with values in $L^p_\sigma(\Omega)$ was also established.

The Helmholtz decomposition holds for any domain if $p = 2$. The L^p-Helmholtz decomposition holds for various domains like bounded or exterior domains with smooth boundary for $1 < p < \infty$ ([Gal11]). However, it is also known ([Bog86], [MB86]) that there is an improper smooth sector-like planar domain such that the L^p-Helmholtz decomposition fails to hold. Let us state one of the results in [MB86] more precisely. Let $C(\vartheta)$ denote the cone of the form

$$C(\vartheta) = \{x = (x', x_n) \in \mathbf{R}^n \mid -x_n \geq |x| \cos(\vartheta/2)\},$$

where $\vartheta \in (0, 2\pi)$ is the opening angle. When $n = 2$, we simply say that $C(\vartheta)$ is a sector. We say that a planar domain Ω is a *sector-like domain* with opening angle ϑ if $\Omega \backslash B_R(0) = C(\vartheta) \backslash B_R(0)$ for some $R > 0$ (up to rotation and translation), where $B_R(0)$ is a closed disk of radius R centered at the origin.

It is known that the L^p-Helmholtz decomposition fails for a sector-like domain Ω when $p > q'_\vartheta$ or $p < q_\vartheta$ with $q_\vartheta = 2/(1 + \pi/\vartheta)$, $1/q_\vartheta + 1/q'_\vartheta = 1$ even if the boundary $\partial\Omega$ is smooth [MB86, Example 2, Fig. 5], while for $p \in (q_\vartheta, q'_\vartheta)$ the L^p-Helmholtz decomposition holds. This means that if the opening angle ϑ is larger than π, there always exists $p > 2$ such that the L^p-Helmholtz decomposition fails.

It has been a longstanding open question whether or not the existence of the L^p-Helmholtz decomposition is necessary for L^p analyticity of $S(t)$. In this chapter, we give a negative answer for this question by proving that there is a domain Ω for which $S(t)$ is analytic in L^p_σ while the L^p-Helmholtz decomposition fails. This is a subtle problem since the existence of the L^p-Helmholtz projection is known to be necessary for L^p solvability of the resolvent equation ([Shi13]). However, in this statement the external force term is allowed to be in the more general space L^p instead of L^p_σ. Our problem is different from that in [Shi13].

We say that Ω has a C^k graph boundary if Ω is of the form

$$\Omega = \{(x', x_n) \in \mathbf{R}^n \mid x_n > h(x')\}$$

(up to translation and rotation) with some real-valued C^k function h with variable $x' \in \mathbf{R}^{n-1}$.

Theorem 5.1.1. *Let Ω be a sector-like domain in $\mathbf{R}^2$ having a C^3 graph boundary. Then $S(t)$ forms a C_0-analytic semigroup in $L^p_\sigma(\Omega)$ for all $p \in [2, \infty)$.*

Here is our strategy to prove Theorem 5.1.1. It is by now well-known that $S(t)$ forms an analytic semigroup in $\tilde{L}^p_\sigma$, i.e., $\tilde{L}^p_\sigma = L^p_\sigma \cap L^2_\sigma$ $(p \geq 2)$, $\tilde{L}^p_\sigma = L^p_\sigma + L^2_\sigma$ $(1 < p < 2)$ ([FKS05], [FKS07], [FKS09]). Thus $S(t)v_0$ is well-defined for $v_0 \in C^\infty_{c,\sigma}(\Omega)$. To show Theorem 5.1.1, a key step is to prove the two estimates

$$\|S(t)v_0\|_p \leq c\|v_0\|_p \tag{5.1.1}$$

$$t\left\|\frac{\mathrm{d}}{\mathrm{d}t}S(t)v_0\right\|_p \leq c\|v_0\|_p \tag{5.1.2}$$

for all $v_0 \in C^\infty_{c,\sigma}(\Omega)$, $t \in (0,1)$, where $\|v_0\|_p$ denotes the L^p-norm of v_0. The constant c should be taken independent of t and v_0. We shall establish (5.1.1) and (5.1.2) by interpolation since both estimates are known for $p = 2$.

We are tempted to interpolate the L^∞ type result obtained in [AGSS15] with the L^2-result. In fact, in [AGSS15] the estimates (5.1.1) and (5.1.2) with $p = \infty$ are established for all $v_0 \in C_{0,\sigma}(\Omega)$, the L^∞-closure of $C^\infty_{c,\sigma}(\Omega)$ for a C^2 sector-like domain Ω in $\mathbf{R}^2$. However, it is not clear that the complex interpolation space $\left[L^2_\sigma, C_{0,\sigma}\right]_\rho$ agrees with L^p_σ with $2/p = 1 - \rho$ although it is well-known as the Riesz-Thorin theorem that $\left[L^2, L^\infty\right]_\rho = L^p$. To interpolate, we would need a projection to solenoidal spaces which is almost impossible since such a projection involves the singular integral operator which is not bounded in L^∞.

To circumvent this difficulty, we consider the Stokes semigroup $S(t)$ in BMO-type spaces as studied in [BG16], [BGS16], [BGST16]. For $p \in [1, \infty)$, $\mu \in (0, \infty]$ we define the BMO seminorm

$$\left[f : BMO^\mu_p(\Omega)\right] := \sup \left\{ \left((\frac{1}{|B_r(x)|} \int_{B_r(x)} \left|f(y) - f_{B_r(x)}\right|^p \mathrm{d}y \right)^{1/p} \;\middle|\; B_r(x) \subset \Omega,\ r < \mu \right\},$$

where $f_B = (\frac{1}{|B|} \int_B f \mathrm{d}x$, the average of f over B and $B_r(x)$ denotes the closed ball of radius r centered at x. It is well-known that one gets an equivalent seminorm when the ball B_r is replaced by a cube. We also need to control the boundary behavior. For $\nu \in (0, \infty]$ we define

$$\left[f : b^\nu_p(\Omega)\right] := \sup \left\{ \left(\frac{1}{r^n} \int_{B_r(x_0) \cap \Omega} |f(y)|^p \mathrm{d}y \right)^{1/p} \;\middle|\; x_0 \in \partial\Omega,\ r > 0,\ B_r(x_0) \subset U_\nu(\partial\Omega) \right\},$$

where $U_\nu(E)$ is a ν-open neighborhood of E, i.e.,

$$U_\nu(E) = \{x \in \mathbf{R}^n \mid \operatorname{dist}(x, E) < \nu\}.$$

We shall often assume that $\nu < R^*$, where R^* is the reach from the boundary. The BMO norm we use is

$$\left\| f : BMO^{\mu,\nu}_{b,p}(\Omega) \right\| = \left[f : BMO^\mu_p(\Omega)\right] + \left[f : b^\nu_p(\Omega)\right].$$

If $p = 1$, we often drop p. The BMO space we consider is

$$BMO^{\mu,\nu}_{b,p}(\Omega) = \left\{ f \in L^1_{loc}(\Omega) \;\middle|\; \left\| f : BMO^{\mu,\nu}_{b,p}(\Omega) \right\| < \infty \right\}.$$

This space is independent of p for sufficiently small ν, i.e., $\nu < R^*$ ([BGS16], [BGST16]) and $BMO^{\infty,\infty}_b$ agrees with Miyachi's BMO space ([Miy90]) for various domains including a half-space and bounded C^2 domains ([BGST16]). Although the $BMO^{\infty,\nu}_b(\Omega)$ norm is equivalent to the $BMO^{\infty,\infty}_b(\Omega)$ norm when Ω is bounded, there are many unbounded domains for which the $BMO^{\infty,\nu}_b(\Omega)$ norm is actually weaker than the $BMO^{\infty,\infty}_b(\Omega)$ norm when ν is finite. We define the solenoidal space $VMO^{\mu,\nu}_{b,0,\sigma}$ as the $BMO^{\mu,\nu}_b$-closure of $C^\infty_{c,\sigma}(\Omega)$. In [BG16], [BGS16] among other results the analyticity of $S(t)$ in $VMO^{\infty,\nu}_{b,0,\sigma}$ has been established for a uniformly C^3 domain which is admissible in the sense of [AG13] provided that ν is sufficiently small.

Theorem 5.1.2 ([BG16], [BGS16])**.** *Let Ω be an admissible uniformly C^3 domain in $\mathbf{R}^n$. Then $S(t)$ forms a C_0-analytic semigroup in $VMO^{\mu,\nu}_{b,0,\sigma}$ for any $\mu \in (0,\infty]$ and $\nu \in (0,\nu_0)$ with some ν_0 depending only on μ and regularity of $\partial\Omega$.*

Moreover, we obtain not only estimates of the form (5.1.1) and (5.1.2), where we replace L^p by L^∞ or $BMO^{\infty,\nu}_b$, but even an estimate stronger than (5.1.2) with $p=\infty$, i.e.,

$$t\left\|\frac{\mathrm{d}S(t)}{\mathrm{d}t}v_0\right\|_\infty \leq c\left\|v_0 : BMO^{\mu,\nu}_b(\Omega)\right\|, \quad \mu,\nu\in(0,\infty] \tag{5.1.3}$$

which shows a regularizing effect.

It has been proved in [AGSS15] that a C^2 sector-like domain in $\mathbf{R}^2$ is admissible and thus Theorem 5.1.2 applies to the setting of Theorem 5.1.1. Note that a C^2 sector-like domain in $\mathbf{R}^2$ is expected to be not strictly admissible in the sense of [AG14]. In fact, a bounded domain ([AG13]), a half-space ([AG13]), an exterior domain ([AG14], [AGH15]) and a bent half-space ([Abe14]) are strictly admissible if the boundary is uniformly C^3. On the other hand, an infinite cylinder is admissible but not strictly admissible ([AGSS16]) and a layer domain with $n \geq 3$ is not admissible ([Bel14], [BB]).

5.2 Main results

In order to get the L^p estimates we need an interpolation result. Let $C_c(\Omega)$ denote the space of all continuous functions with compact support in Ω.

Theorem 5.2.1. *Let Ω be a Lipschitz half-space in $\mathbf{R}^n$, i.e., a domain having Lipschitz graph boundary. Let T be a linear operator from $C_c(\Omega)$ to $L^2(\Omega)$. Assume that there is a constant c such that*

$$\|Tu\|_2 \leq c\|u\|_2$$

$$[Tu : BMO^\infty(\Omega)] \leq c\|u\|_\infty$$

for $u \in C_c(\Omega)$. Then for $p \in (2,\infty)$, $\|Tu\|_p \leq c_*\|u\|_p$ for $u \in C_c(\Omega)$ with c_* depending only on c, h and p.

There are a couple of such interpolation results between BMO and L^2, which go back to Campanato and Stampacchia; in [Giu03, Theorem 2.14] the interpolation between L^p and BMO is discussed when Ω is a cube. However, in these results the original inequalities are assumed to hold for $L^2(\Omega)\cap BMO(\Omega)$ and not for $C_c(\Omega)$. Thus ours are not included in the literature. In [DY05]

Duong and Yan showed a similar result with $BMO_A(\mathcal{X})$, where A is some operator. They worked on metric measure spaces of homogeneous type $(\mathcal{X}, d, \mu)$. In particular, in the case $\mathcal{X} = \Omega, d(x,y) = |x-y|$ and $\mu(E) = |E|$, we can see that $BMO_A(\Omega) \subset BMO^\infty(\Omega)$.

Unfortunately, Theorem 5.1.2 and Theorem 5.2.1 are not enough to derive (5.1.1) and (5.1.2) by interpolation. Similarly to the L^∞ case we do not know whether or not the complex interpolation space $\left[L^2_\sigma, VMO^{\infty,\nu}_{b,0,\sigma}\right]_\rho$ with $2/p = 1-\rho$ agrees with L^p_σ, although we know that $\left[L^2, BMO\right]_\rho = L^p$ for $\Omega = \mathbf{R}^n$ as discussed in [JJ82].

To circumvent this difficulty, we construct the following projection operator.

Theorem 5.2.2. *Let Ω be a Lipschitz half-space in $\mathbf{R}^n$. Assume that $\nu \in (0,\infty]$. There is a linear operator $\mathbf{Q}$ from $C_c(\Omega)$ to $VMO^{\infty,\nu}_{b,0,\sigma}(\Omega) \cap L^2_\sigma(\Omega)$ such that*

$$\|\mathbf{Q}u\|_2 \le c\|u\|_2$$
$$\|\mathbf{Q}u : BMO^{\infty,\nu}_b(\Omega)\| \le c\|u\|_\infty$$

for all $u \in C_c(\Omega)$. Moreover, $\mathbf{Q}u = u$ for $u \in C_c(\Omega) \cap L^2_\sigma(\Omega)$.

Since there may be no L^p-Helmholtz decomposition, our $\mathbf{Q}$ should be different from the Helmholtz projection. We shall construct such an operator $\mathbf{Q}$ using the solution operator of the equation $\operatorname{div} u = f$ given by Solonnikov [Sol83]. Although deriving the L^2 estimate is easy, to derive the BMO estimate is more involved since we have to estimate the b^ν type seminorm.

To derive (5.1.1), we actually interpolate

$$\|S(t)\mathbf{Q}u\|_2 \le c\|u\|_2$$

and

$$\|S(t)\mathbf{Q}u : BMO^{\infty,\nu}_b\| \le c\|u\|_\infty$$

for $u \in C_c(\Omega)$. Similarly, we derive (5.1.2) by interpolating the estimate for $t\frac{\mathrm{d}S}{\mathrm{d}t}\mathbf{Q}$.

This chapter is organized as follows. In Section 5.3, we establish an interpolation inequality of Campanato-Stampacchia type. In Section 5.4, we construct the projection operator $\mathbf{Q}$. In Section 5.5, we give a complete proof of Theorem 5.1.1.

5.3 L^2-BMO interpolation on a Lipschitz half-space

In this section, we give a proof of Theorem 5.2.1 for a Lipschitz half-space, i.e.,

$$\Omega := \{(x', x_n) \in \mathbf{R}^n | x_n > h(x')\}$$

with a Lipschitz function h on $\mathbf{R}^{n-1}$.

By Q we mean a closed cube with sides parallel to the coordinate axes. Let $\ell(Q)$ be the side length of Q, and for $\tau > 0$, τQ a cube with the same center as Q and side length $\tau\ell(Q)$.

5.3.1 Reduction to the half-space and extension

Here, we prepare lemmas that are basic estimates for the proof. Since h is Lipschitz continuous, $F(x) := (x', x_n - h(x'))$ is a bi-Lipschitz map from Ω to $\mathbf{R}^n_+$. For a function u defined on $\mathbf{R}^n_+$ the pull-back function $F^*(u)$ of u on Ω is defined by $u \circ F$. We start with estimates for $(F^{-1})^*$ which is the pull-back function $(F^{-1})^*(v)$ of v on $\mathbf{R}^n_+$ defined by $v \circ F^{-1}$.

Lemma 5.3.1. *Let Ω be a Lipschitz half-space.*

(i):

$$\left[(F^{-1})^* v : BMO^\infty(\mathbf{R}^n_+)\right] \leq c\,[v : BMO^\infty(\Omega)]\,.$$

(ii):

$$\left\|(F^{-1})^* v\right\|_{L^2(\mathbf{R}^n_+)} \leq c\|v\|_{L^2(\Omega)}.$$

Here, c is a constant depending only on the Lipschitz bound of h and n.

Proof. (i): Because $\mathbf{R}^n_+$ is an open subset of $\mathbf{R}^n$, we know that for any $\tau > 2$,

$$\left[(F^{-1})^* v : BMO^\infty(\mathbf{R}^n_+)\right] \leq c_\tau \sup_{\tau Q \subset \mathbf{R}^n_+} \inf_{d \in \mathbf{R}} \int_Q \left|(F^{-1})^* v - d\right| \mathrm{d}y,$$

where the supremum is taken over cubes Q, for which τQ is contained in $\mathbf{R}^n_+$, see [Sim72]. Since F is a bi-Lipschitz map, it holds

$$c_1 \operatorname{dist}(y, \partial\mathbf{R}^n_+) \leq \operatorname{dist}(F^{-1}(y), \partial\Omega) \leq c_2 \operatorname{dist}(y, \partial\mathbf{R}^n_+)$$

with some constants $c_1, c_2 > 0$ for all $y \in \mathbf{R}^n_+$. Since $(\tau-1)\ell(Q)/2 \leq \operatorname{dist}(Q, \partial\mathbf{R}^n_+)$ for such cubes Q, we have the lower bound

$$c\tau\ell(Q) \leq \operatorname{dist}(F^{-1}(Q), \partial\Omega)$$

with some $c > 0$, which depends on n and h. Therefore, taking large τ, we can find cubes $\{R_k\}_{k=1}^{c_*} \subset \Omega$, which have no intersection of interiors, so that $\cup_{k=1}^{c_*} R_k$ is connected and

$$\begin{cases} \circ\ \ell(R_k) = \ell(Q), \\ \circ\ F^{-1}(Q) \subset \cup_{k=1}^{c_*} R_k, \text{ where } c_* \in \mathbf{N} \text{ depends only on } h, \text{ and} \\ \circ \text{ if } R_j \cap R_k \neq \emptyset, \text{ the smallest cube } R_{j,k} \text{ including } R_j \text{ and } R_k \text{ is in } \Omega. \end{cases}$$

From these, one obtains that for cubes Q with $\tau Q \subset \mathbf{R}^n_+$,

$$\inf_{d \in \mathbf{R}} \frac{1}{|Q|} \int_Q \left|(F^{-1})^* v - d\right| \mathrm{d}y \leq c \sum_{k=1}^{c_*} \frac{1}{|R_k|} \int_{R_k} |v - v_{R_1}| \mathrm{d}y.$$

It is enough to show that

$$\frac{1}{|R_k|} \int_{R_k} |v - v_{R_j}| \mathrm{d}y \leq c[v : BMO^\infty(\Omega)] \tag{5.3.1}$$

for the case $R_j \cap R_k \neq \emptyset$. To do this, we follow the argument of [Jon80, Lemma 2.2 and 2.3]. Let $\tilde{R}_k$ and $\tilde{R}_j$ be subcubes of R_k and R_j respectively so that $\ell(\tilde{R}_k) = \ell(R_k)/2$, $\ell(\tilde{R}_j) = \ell(R_j)/2$ and they touch each other. Moreover, denote by $\tilde{R}_{j,k}$ a cube satisfying $\ell(\tilde{R}_{j,k}) = \ell(\tilde{R}_j) + \ell(\tilde{R}_k)$ and $\tilde{R}_j \cup \tilde{R}_k \subset \tilde{R}_{j,k} \subset R_{j,k}$. Hence, we have

$$\begin{aligned}
\frac{1}{|R_k|}\int_{R_k} |v - v_{R_j}|\mathrm{d}y &\leq \frac{1}{|R_k|}\int_{R_k} |v - v_{R_k}|\mathrm{d}y + |v_{R_k} - v_{R_j}| \\
&\leq c[v : BMO^\infty(\Omega)] + c|v_{\tilde{R}_j} - v_{\tilde{R}_k}| \\
&\leq c[v : BMO^\infty(\Omega)] + c\frac{1}{|\tilde{R}_{j,k}|}\int_{\tilde{R}_{j,k}} |v - v_{\tilde{R}_{j,k}}|\mathrm{d}y \\
&\leq c[v : BMO^\infty(\Omega)].
\end{aligned}$$

(ii): This is verified as follows

$$\|(F^{-1})^* v\|^2_{L^2(\mathbf{R}^n_+)} = \int_\Omega |v|^2 J_F \mathrm{d}x \leq c\int_\Omega |v|^2 \mathrm{d}x,$$

where J_F is the modulus of the Jacobian of F which is bounded, because h is Lipschitz continuous. □

Next, we consider the even extension of functions on the half-space. For a function f on $\mathbf{R}^n_+$, we extend f outside $\mathbf{R}^n_+$ by

$$E[f](x', -x_n) := f(x', x_n) \text{ for } x_n > 0.$$

From an elementary geometrical observation, we can see that the extension operator E is a BMO-extension operator for $\mathbf{R}^n_+$.

Lemma 5.3.2.

$$[E[f] : BMO^\infty(\mathbf{R}^n)] \leq c\left[f : BMO^\infty(\mathbf{R}^n_+)\right].$$

Proof. It is sufficient to consider cubes $Q \subset \mathbf{R}^n$ with $Q \cap \mathbf{R}^n_+ \neq \emptyset$ and $Q \cap \mathbf{R}^n_- \neq \emptyset$. For such Q, let Q' be a cube so that its center lies on $\partial\mathbf{R}^n_+$, $\ell(Q') = 2\ell(Q)$ and $Q \subset Q'$. Further, let Q^* be the smallest cube in $\mathbf{R}^n_+$ containing the upper half of Q'. With these notations, the desired inequality is proved from

$$\inf_{d\in\mathbf{R}} \frac{1}{|Q|}\int_Q |E[f] - d|\,\mathrm{d}y \leq c \inf_{d\in\mathbf{R}} \frac{1}{|Q^*|}\int_{Q^*} |f - d|\mathrm{d}y.$$

□

5.3.2 Sharp maximal operator

For the proof of Theorem 1.3, we make use of the sharp maximal operator $M^\sharp$ due to Fefferman and Stein ([FS72]). We define for $x \in \mathbf{R}^n$ and $f \in L^1_{loc}(\mathbf{R}^n)$ the function $M^\sharp f$ by

$$M^\sharp f(x) := \sup_{Q \ni x} \frac{1}{|Q|} \int_Q |f(y) - f_Q| \mathrm{d}y.$$

It is immediate from the definition that $[f : BMO^\infty(\mathbf{R}^n)] = \|M^\sharp f\|_{L^\infty(\mathbf{R}^n)}$. It is well-known that if $f \in L^{p_0}(\mathbf{R}^n)$ for some $p_0 \in (1, \infty)$, then for $p \in [p_0, \infty)$

$$\|f\|_{L^p(\mathbf{R}^n)} \leq c\|M^\sharp f\|_{L^p(\mathbf{R}^n)}, \tag{5.3.2}$$

which is applied below. (Both sides of (5.3.2) may be infinite.) This follows from $\|f\|_{L^p(\mathbf{R}^n)} \leq \|Mf\|_{L^p(\mathbf{R}^n)}$ and $\|Mf\|_{L^p(\mathbf{R}^n)} \leq c\|M^\sharp f\|_{L^p(\mathbf{R}^n)}$, where M is the Hardy-Littlewood maximal operator [FS72].

5.3.3 Marcinkiewicz interpolation

Here, we give a variant of the Marcinkiewicz interpolation theorem.

Proposition 5.3.1. *Let D be an open subset of $\mathbf{R}^n$ and S a sublinear operator from $C_c(D)$ to $L^2(\mathbf{R}^n)$. If*

$$\|S[f]\|_{L^2(\mathbf{R}^n)} \leq c\|f\|_{L^2(D)}$$

$$\|S[f]\|_{L^\infty(\mathbf{R}^n)} \leq c\|f\|_{L^\infty(D)}$$

for $f \in C_c(D)$, then $\|S[f]\|_{L^p(\mathbf{R}^n)} \leq c_p\|f\|_{L^p(D)}$ for $f \in C_c(D)$ with c_p depending only on c and $p \in (2, \infty)$.

Proof. For $\lambda > 0$ and $\alpha > 0$, we decompose f into two parts; $f = f_2 + f_\infty$ where

$$f_2(x) = \begin{cases} 0 \quad \text{if} \quad |f(x)| \leq \alpha\lambda \\ f(x) - \alpha\lambda \mathrm{sign}(f(x)) \quad \text{if} \quad |f(x)| > \alpha\lambda, \end{cases}$$

where sign $\xi = \xi/|\xi|$ for $\xi \neq 0$ and sign $\xi = 0$ for $\xi = 0$. Observe that $f_2,\ f_\infty \in BC(D)$, and then $f_2,\ f_\infty \in C_c(D)$. Therefore, the two inequalities of our assumption hold for f_2 and f_∞, respectively. We set $\alpha = \left(2\|S\|_{L^\infty(D)\to L^\infty(\mathbf{R}^n)}\right)^{-1}$

and observe that $|\{x \in \mathbf{R}^n \mid |S[f_\infty](x)| > \lambda/2\}| = 0$. We now conclude that

$$\begin{aligned}
\int_{\mathbf{R}^n} |S[f]|^p \,\mathrm{d}x &\le p \int_0^\infty \lambda^{p-1} \left|\{x \in \mathbf{R}^n \mid |S[f](x)| > \lambda\}\right| \mathrm{d}\lambda \\
&\le p \int_0^\infty \lambda^{p-1} \left|\{x \in \mathbf{R}^n \mid |S[f_2](x)| > \lambda/2\}\right| \mathrm{d}\lambda \\
&\le p \int_0^\infty \lambda^{p-1} \left(\frac{2}{\lambda} \|S\|_{L^2(D)\to L^2(\mathbf{R}^n)} \|f_2\|_{L^2(D)}\right)^2 \mathrm{d}\lambda \\
&\le c \int_0^\infty \lambda^{p-3} \int_{\{|f|>\alpha\lambda\}} |f(x)|^2 \,\mathrm{d}x \,\mathrm{d}\lambda \\
&= 2c \int_0^\infty \lambda^{p-3} \left(\int_{\alpha\lambda}^\infty t \left|\{x \in \mathbf{R}^n \mid |f(x)| > t\}\right| \mathrm{d}t\right) \mathrm{d}\lambda \\
&= 2c \int_0^\infty t \left|\{x \in \mathbf{R}^n \mid |f(x)| > t\}\right| \left(\int_0^{t/\alpha} \lambda^{p-3} \mathrm{d}\lambda\right) \mathrm{d}t \\
&\le c \|f\|_{L^p(D)}^p.
\end{aligned}$$

□

5.3.4 Proof of Theorem 5.2.1

For simplicity, we write $g := Tf$. By changing variables, one obtains

$$\int_\Omega |g|^p \mathrm{d}x \le c \int_{\mathbf{R}^n_+} |(F^{-1})^* g|^p \mathrm{d}y \le c \int_{\mathbf{R}^n} |E[(F^{-1})^* g]|^p \mathrm{d}y \le c \int_{\mathbf{R}^n} |\Phi[f]|^p \mathrm{d}y,$$

where $\Phi[f] := M^\sharp \left(E[(F^{-1})^* g]\right)$. Here, because $E[(F^{-1})^* g] \in L^2(\mathbf{R}^n)$, we have applied (5.3.2) in the third inequality. With the help of Proposition 5.3.1, it is enough to see $L^2(\Omega)$-$L^2(\mathbf{R}^n)$ and $L^\infty(\Omega)$-$L^\infty(\mathbf{R}^n)$ estimates for Φ. The former estimate can be seen by L^2-boundedness of Hardy-Littlewood maximal operator and (ii) of Lemma 5.3.1. The latter one follows from (i) of Lemma 5.3.1 and Lemma 5.3.2. Then the proof of Theorem 5.2.1 is completed.

5.4 Non-Helmholtz projection

Our goal in this section is to prove Theorem 5.2.2.

5.4.1 A solution operator to the divergence problem

As in Section 5.3, let $\Omega = \{(x', x_n) \in \mathbf{R}^n \mid x' \in \mathbf{R}^{n-1}, x_n > h(x')\}$ be a Lipschitz half-space in $\mathbf{R}^n$ with a Lipschitz continuous function h on $\mathbf{R}^{n-1}$. Then, there is a closed cone of the form

$$C_1 = \{x = (x', x_n) \in \mathbf{R}^n \mid x' \in \mathbf{R}^{n-1}, -x_n \ge |x| \cos(2\theta)\}$$

with an angle $\theta \in (0, \pi/4)$ (depending on the Lipschitz constant of h) such that

$$x + C_1 = \{y \in \mathbf{R}^n \mid y - x \in C_1\} \subset \Omega^c \, (:= \mathbf{R}^n \setminus \Omega) \quad \text{for all} \quad x \in \Omega^c.$$

In the notation of the introduction $C_1 = C(4\theta)$ so that the opening angle equals 4θ. With this angle we define a closed cone $C_0 = C(2\theta)$, i.e.,

$$C_0 = \{x = (x', x_n) \in \mathbf{R}^n \mid x' \in \mathbf{R}^{n-1}, \, -x_n \geq |x| \cos\theta\}.$$

The closed cone C_0 also satisfies

$$x + C_0 \subset \Omega^c \quad \text{for all} \quad x \in \Omega^c. \tag{5.4.1}$$

Let $L \in C_c^\infty(\mathbf{R}^n)$ be a function such that

$$\operatorname{supp} L \subset (B_2(0) \setminus B_{1/2}(0)) \cap (-C_0), \qquad \int_{S^{n-1}} L(\sigma) \, \mathrm{d}\mathcal{H}^{n-1}(\sigma) = 1. \tag{5.4.2}$$

Here $-C_0 = \{-y \mid y \in C_0\}$ and S^{n-1} is the unit sphere in $\mathbf{R}^n$. Then we define a vector field $K = (K_1, \dots, K_n)$ as

$$K(x) := \frac{x}{|x|^n} L\left(\frac{x}{|x|}\right), \quad x \in \mathbf{R}^n \setminus \{0\}. \tag{5.4.3}$$

In Section 5.4, we always assume that Ω is a Lipschitz half-space in $\mathbf{R}^n$ unless otherwise indicated.

Definition 5.4.1. *For a scalar function $f \in C_c^\infty(\Omega)$, we define a vector field $u = Sf$ as*

$$u(x) = Sf(x) := (K * \bar{f})(x) = \int_{\mathbf{R}^n} K(x-y)\bar{f}(y) \, \mathrm{d}y, \quad x \in \mathbf{R}^n.$$

Here $\bar{f}$ denotes the zero extension of f to $\mathbf{R}^n$ given by

$$\bar{f}(x) := \begin{cases} f(x), & x \in \Omega, \\ 0, & x \in \Omega^c. \end{cases}$$

This operator was introduced by Solonnikov [Sol83]. For a fixed $x \in \mathbf{R}^n$, since

$$\frac{x-y}{|x-y|} \in \operatorname{supp} L|_{S^{n-1}} \subset S^{n-1} \cap (-C_0)$$

implies $y \in x + C_0$, we can write

$$u(x) = \int_{x + C_0} K(x-y)\bar{f}(y) \, \mathrm{d}y.$$

This formula and the property (5.4.1) of Ω imply that $u(x) = 0$ for all $x \in \Omega^c$. In particular, u vanishes on $\partial\Omega$. However, the support of u may become unbounded although f is compactly supported in Ω.

By the change of variables $x - y = r\sigma$ with $r > 0$ and $\sigma \in S^{n-1}$ we have

$$u(x) = \int_0^\infty \int_{S^{n-1}} \sigma L(\sigma) \bar{f}(x - r\sigma) \mathrm{d}\mathcal{H}^{n-1}(\sigma)\, \mathrm{d}r.$$

Hence if $f \in C_c^\infty(\Omega)$ is supported in $B_R(0)$ and $x \in B_a(0)$ $(R, a > 0)$, then

$$u(x) = \int_0^{R+a} \int_{S^{n-1}} \sigma L(\sigma) \bar{f}(x - r\sigma) \mathrm{d}\mathcal{H}^{n-1}(\sigma)\, \mathrm{d}r,$$

which implies that $u = Sf$ is smooth in Ω. Moreover, $u = Sf$ vanishes near $\partial\Omega$ and thus it is smooth in the whole space $\mathbf{R}^n$, since f is compactly supported in Ω.

Lemma 5.4.1. *Let $p \in (1, \infty)$. There exists a constant $c > 0$ such that*

$$\|\nabla u\|_{L^p(\Omega)} \leq c \|f\|_{L^p(\Omega)}$$

for all $f \in C_c^\infty(\Omega)$ and $u = Sf$.

Proof. Let u_i be the i-th component of u:

$$u_i(x) = (K_i * \bar{f})(x) = \int_{\mathbf{R}^n} K_i(z) \bar{f}(x - z)\, \mathrm{d}z.$$

Differentiating both sides with respect to the j-th variable, we have

$$\partial_j u_i(x) = \int_{\mathbf{R}^n} K_i(z)(\partial_j \bar{f})(x - z)\, \mathrm{d}z = \lim_{\varepsilon \to 0} \int_{\mathbf{R}^n \setminus B_\varepsilon(0)} K_i(z)(\partial_j \bar{f})(x - z)\, \mathrm{d}z$$

and, by changing variables $y = x - z$ and integrating by parts,

$$\begin{aligned}
&\partial_j u_i(x) \\
&= \lim_{\varepsilon \to 0} \left(\int_{\partial B_\varepsilon(x)} K_i(x - y) \frac{x_j - y_j}{|x - y|} \bar{f}(y)\, \mathrm{d}\mathcal{H}^{n-1}(y) + \int_{\mathbf{R}^n \setminus B_\varepsilon(x)} (\partial_j K_i)(x - y) \bar{f}(y)\, \mathrm{d}y \right).
\end{aligned}$$

On the one hand, we change variables $x - y = \varepsilon\sigma$ with $\sigma \in S^{n-1}$ to get

$$\begin{aligned}
&\lim_{\varepsilon \to 0} \int_{|x-y|=\varepsilon} K_i(x - y) \frac{x_j - y_j}{|x - y|} \bar{f}(y)\, \mathrm{d}\mathcal{H}^{n-1}(y) \\
&= \lim_{\varepsilon \to 0} \int_{|x-y|=\varepsilon} \frac{x_i - y_i}{|x - y|} \frac{x_j - y_j}{|x - y|} L\left(\frac{x - y}{|x - y|} \right) \bar{f}(y) \frac{1}{|x - y|^{n-1}}\, \mathrm{d}\mathcal{H}^{n-1}(y) \\
&= \lim_{\varepsilon \to 0} \int_{S^{n-1}} \sigma_i \sigma_j L(\sigma) \bar{f}(x - \varepsilon\sigma)\, \mathrm{d}\mathcal{H}^{n-1}(\sigma) \\
&= \bar{f}(x) \int_{S^{n-1}} \sigma_i \sigma_j L(\sigma)\, \mathrm{d}\mathcal{H}^{n-1}(\sigma),
\end{aligned}$$

where the last equality follows from the fact that L is integrable on S^{n-1} and $\bar{f}$ is continuous at x. On the other hand, we differentiate K_i to obtain

$$\begin{aligned} K_{ij}(z) &:= \partial_j K_i(z) = \frac{k_{ij}(z/|z|)}{|z|^n}, \\ k_{ij}(z) &:= (\delta_{ij} - n z_i z_j) L(z) + z_i (\partial_j L)(z) - z_i z_j \sum_{\ell=1}^{n} z_\ell (\partial_\ell L)(z) \end{aligned} \tag{5.4.4}$$

for $z \in \mathbf{R}^n \setminus \{0\}$. Then K_{ij} is homogeneous of degree $-n$ and there is a constant $c > 0$ such that

$$|K_{ij}(z)| \leq \frac{c}{|z|^n} \quad \text{for all} \quad z \in \mathbf{R}^n \setminus \{0\}$$

by the smoothness of L on S^{n-1}. Moreover, for every R_1 and R_2 with $0 < R_1 < R_2$,

$$\begin{aligned} &\int_{R_1<|z|<R_2} K_{ij}(z)\,\mathrm{d}z = \int_{R_1<|z|<R_2} \partial_j K_i(z)\,\mathrm{d}z \\ &= \int_{|z|=R_2} K_i(z) \frac{z_j}{|z|}\, d\mathcal{H}^{n-1}(z) - \int_{|z|=R_1} K_i(z) \frac{z_j}{|z|}\,\mathrm{d}\mathcal{H}^{n-1}(z) \\ &= \int_{|z|=R_2} \frac{z_i}{|z|} \frac{z_j}{|z|} L\left(\frac{z}{|z|}\right) \frac{1}{|z|^{n-1}}\,\mathrm{d}\mathcal{H}^{n-1}(z) - \int_{|z|=R_1} \frac{z_i}{|z|} \frac{z_j}{|z|} L\left(\frac{z}{|z|}\right) \frac{1}{|z|^{n-1}}\,\mathrm{d}\mathcal{H}^{n-1}(z) \\ &= \int_{S^{n-1}} \sigma_i \sigma_j L(\sigma)\,\mathrm{d}\mathcal{H}^{n-1}(\sigma) - \int_{S^{n-1}} \sigma_i \sigma_j L(\sigma)\,\mathrm{d}\mathcal{H}^{n-1}(\sigma) = 0. \end{aligned}$$

In the fourth equality we changed variables $z = R_2\sigma$ and $z = R_1\sigma$ with $\sigma \in S^{n-1}$, respectively. This equality is equivalent to

$$\int_{S^{n-1}} k_{ij}(\sigma)\,\mathrm{d}\mathcal{H}^{n-1}(\sigma) = 0. \tag{5.4.5}$$

Thus we can apply the Calderón-Zygmund theory (see eg. [Gra14, Theorem 5.2.7 and Theorem 5.2.10]) of singular integral operators to the kernel K_{ij} and obtain the formula

$$\partial_j u_i(x) = \bar{f}(x) \int_{S^{n-1}} \sigma_i \sigma_j L(\sigma)\,\mathrm{d}\mathcal{H}^{n-1}(\sigma) + \int_{\mathbf{R}^n} K_{ij}(x-y) \bar{f}(y)\,\mathrm{d}y, \tag{5.4.6}$$

where the second integral is considered in the sense of the Cauchy principal value.

Finally, the inequality

$$\left| \bar{f}(x) \int_{S^{n-1}} \sigma_i \sigma_j L(\sigma)\,\mathrm{d}\mathcal{H}^{n-1}(\sigma) \right| \leq |\bar{f}(x)| \int_{S^{n-1}} L(\sigma)\,\mathrm{d}\mathcal{H}^{n-1}(\sigma) = |\bar{f}(x)|$$

and the Calderón-Zygmund theory imply that

$$\|\partial_j u_i\|_{L^p(\Omega)} \leq c \|\bar{f}\|_{L^p(\mathbf{R}^n)} = c \|f\|_{L^p(\Omega)}$$

with a positive constant c independent of f. Hence the lemma follows. $\square$

Lemma 5.4.2. *For every $f \in C_c^\infty(\Omega)$ the vector field $u = Sf$ satisfies*

$$\operatorname{div} u = f \quad \text{in} \quad \Omega, \quad u = 0 \quad \text{on} \quad \partial\Omega.$$

Proof. We have already observed that u vanishes on the boundary. Let us compute $\operatorname{div} u = \sum_{i=1}^n \partial_i u_i$ in Ω. By the formula (5.4.6) in the proof of Lemma 5.4.1,

$$\operatorname{div} u(x) = \bar{f}(x) \int_{S^{n-1}} \sum_{i=1}^n \sigma_i^2 L(\sigma)\, \mathrm{d}\mathcal{H}^{n-1}(\sigma) + \int_{\mathbf{R}^n} \sum_{i=1}^n K_{ii}(x-y)\bar{f}(y)\, \mathrm{d}y.$$

In this formula, we have

$$\int_{S^{n-1}} \sum_{i=1}^n \sigma_i^2 L(\sigma)\, \mathrm{d}\mathcal{H}^{n-1}(\sigma) = \int_{S^{n-1}} L(\sigma)\, \mathrm{d}\mathcal{H}^{n-1}(\sigma) = 1$$

by (5.4.2) and, for all $z \in \mathbf{R}^n \setminus \{0\}$,

$$\begin{aligned}\sum_{i=1}^n K_{ii}(z) &= \frac{1}{|z|^n} L\left(\frac{z}{|z|}\right) \sum_{i=1}^n \left(1 - n\frac{z_i^2}{|z|^2}\right) \\ &\quad + \frac{1}{|z|^n} \sum_{i=1}^n \frac{z_i}{|z|} (\partial_i L)\left(\frac{z}{|z|}\right) - \sum_{i=1}^n \frac{z_i^2}{|z|^{n+2}} \sum_{k=1}^n \frac{z_k}{|z|} (\partial_k L)\left(\frac{z}{|z|}\right) = 0.\end{aligned}$$

Hence $\operatorname{div} u(x) = \bar{f}(x) = f(x)$ for all $x \in \Omega$. □

Lemma 5.4.2 means that the operator S is a solution operator to the divergence problem with Dirichlet boundary condition. Note that S is not a unique solution operator because a solution to the divergence problem is not unique.

Next we define a linear operator that plays a main role in this section.

Definition 5.4.2. *For a vector field $u \in C_c^\infty(\Omega)$, we define a vector field Tu as*

$$Tu(x) := \int_{\mathbf{R}^n} K(x-y)\overline{\operatorname{div} u}(y)\, \mathrm{d}y, \quad x \in \mathbf{R}^n.$$

Here K is given by (5.4.3) *and $\overline{\operatorname{div} u}$ denotes the zero extension of* $\operatorname{div} u$ *to* $\mathbf{R}^n$.

The above definition means that T is given by $T = S \circ \operatorname{div}$. Since the divergence of a vector field $u \in C_c^\infty(\Omega)$ is in $C_c^\infty(\Omega)$, Tu is smooth in the whole space $\mathbf{R}^n$ and vanishes outside of Ω, as discussed right after Definition 5.4.1. Also, by Lemma 5.4.2 we have

$$\operatorname{div} Tu = \operatorname{div} u \quad \text{in} \quad \Omega, \quad Tu = 0 \quad \text{on} \quad \partial\Omega.$$

Clearly $Tu = 0$ in $\mathbf{R}^n$ for $u \in C_{c,\sigma}^\infty(\Omega)$. Note that, as in the case of the operator S, the support of Tu may be unbounded.

Theorem 5.4.1. *Let $p \in (1, \infty)$. There exists a constant $c > 0$ such that*

$$\|Tu\|_{L^p(\Omega)} \leq c\|u\|_{L^p(\Omega)}$$

for all $u \in C_c^\infty(\Omega)$.

Proof. Let us compute the i-th component $(Tu)_i$ of Tu with $i = 1, \ldots, n$ for a compactly supported vector field u in Ω. As in the proof of Lemma 5.4.1, we integrate by parts to get

$$\begin{aligned}(Tu)_i(x) &= \lim_{\varepsilon\to 0} \int_{\partial B_\varepsilon(x)} K_i(x-y)\frac{x-y}{|x-y|} \cdot \bar{u}(y)\,\mathrm{d}\mathcal{H}^{n-1}(y) \\ &\quad + \lim_{\varepsilon\to 0} \int_{\mathbf{R}^n\setminus B_\varepsilon(x)} (\nabla K_i)(x-y) \cdot \bar{u}(y)\,\mathrm{d}y \\ &= \int_{S^{n-1}} \sigma_i L(\sigma)\{\sigma \cdot \bar{u}(x)\}\,\mathrm{d}\mathcal{H}^{n-1}(\sigma) + \int_{\mathbf{R}^n} (\nabla K_i)(x-y) \cdot \bar{u}(y)\,\mathrm{d}y,\end{aligned}$$

or equivalently,

$$(Tu)_i(x) = \sum_{j=1}^n \{a_{ij}\bar{u}_j(x) + S_{ij}\bar{u}_j(x)\}, \quad x \in \mathbf{R}^n. \tag{5.4.7}$$

Here u_j is the j-th component of u and

$$a_{ij} = \int_{S^{n-1}} \sigma_i\sigma_j L(\sigma)\,\mathrm{d}\mathcal{H}^{n-1}(\sigma), \quad S_{ij}\bar{u}_j(x) = \int_{\mathbf{R}^n} K_{ij}(x-y)\bar{u}_j(y)\,\mathrm{d}y,$$

where $K_{ij} = \partial_j K_i$ is given by (5.4.4). Since a_{ij} is a constant satisfying

$$|a_{ij}| \leq \int_{S^{n-1}} L(\sigma)\,\mathrm{d}\mathcal{H}^{n-1}(\sigma) = 1 \tag{5.4.8}$$

and $S_{ij}\bar{u} = K_{ij} * \bar{u}$ is a singular integral (see the proof of Lemma 5.4.1), the Calderón-Zygmund theory yields the boundedness of the operator T on $L^p(\Omega)$. □

By Theorem 5.4.1, the operator T extends uniquely to a bounded linear operator on $L^p(\Omega)$ with each $p \in (1, \infty)$, which we again refer to as T.

Our next goal is to estimate the $BMO_b^{\infty,\nu}(\Omega)$-norm of Tu for $u \in C_c^\infty(\Omega)$ and $\nu \in (0, \infty]$. To this end, we estimate each term of the right-hand side in (5.4.7) for $u = (u_1, \ldots, u_n) \in C_c^\infty(\Omega)$. By (5.4.8) we have

$$[a_{ij}\bar{u}_j : BMO^\infty(\Omega)] \leq [u_j : BMO^\infty(\Omega)], \quad [a_{ij}\bar{u}_j : b^\nu(\Omega)] \leq [u_j : b^\nu(\Omega)]$$

and thus

$$\|a_{ij}\bar{u}_j : BMO_b^{\infty,\nu}(\Omega)\| \leq \|u_j : BMO_b^{\infty,\nu}(\Omega)\|.$$

Moreover, since

$$[u_j : BMO^\infty(\Omega)] \le 2\|u_j\|_{L^\infty(\Omega)}, \quad [u_j : b^\nu(\Omega)] \le \omega_n \|u_j\|_{L^\infty(\Omega)},$$

where $\omega_n = 2\pi^{n/2}/n\Gamma(n/2)$ is the volume of the unit ball $B_1(0)$ in $\mathbf{R}^n$ with the Gamma function $\Gamma(z) := \int_0^\infty x^{z-1}e^{-x}\,\mathrm{d}x$, we have

$$\|a_{ij}\bar{u}_j : BMO_b^{\infty,\nu}(\Omega)\| \le (2+\omega_n)\|u_j\|_{L^\infty(\Omega)}. \tag{5.4.9}$$

Let us estimate $S_{ij}\bar{u}_j = K_{ij} * \bar{u}_j$, $i, j = 1, \dots, n$ in $BMO_b^{\infty,\nu}(\Omega)$. Recall that the integral kernel K_{ij} is of the form

$$K_{ij}(x) = \frac{k_{ij}(x/|x|)}{|x|^n}, \quad x \in \mathbf{R}^n \setminus \{0\},$$

where $k_{ij} \in C_c^\infty(\mathbf{R}^n)$ is given by (5.4.4) and satisfies

$$\operatorname{supp} k_{ij} \subset (B_2(0) \setminus B_{1/2}(0)) \cap (-C_0), \quad \int_{S^{n-1}} k_{ij}(\sigma)\,\mathrm{d}\mathcal{H}^{n-1} = 0,$$

see (5.4.2) and (5.4.5). We first estimate the BMO^∞-seminorm of $S_{ij}\bar{u}_j$.

Lemma 5.4.3. *Let K be a function defined on $\mathbf{R}^n \setminus \{0\}$ such that*

$$|K(x-y) - K(x)| \le A|y|^\delta |x|^{-n-\delta} \quad \textit{whenever} \quad |x| \ge 2|y| > 0 \tag{5.4.10}$$

for some $A, \delta > 0$. Suppose that a convolution operator S with K is bounded on $L^2(\mathbf{R}^n)$ with an operator norm B. Then, there exists a dimensional constant c_n such that

$$[Sf : BMO^\infty(\mathbf{R}^n)] \le c_n(A+B)\|f\|_{L^\infty(\mathbf{R}^n)}$$

for all $f \in L^2(\mathbf{R}^n) \cap L^\infty(\mathbf{R}^n)$.

Proof. See [Gra214, Theorem 3.4.9 and Corollary 3.4.10]. □

Lemma 5.4.4. *There exists a constant $c > 0$ such that*

$$[S_{ij}\bar{u}_j : BMO^\infty(\Omega)] \le c\|u_j\|_{L^\infty(\Omega)} \tag{5.4.11}$$

for all $u = (u_1, \dots, u_n) \in C_c^\infty(\Omega)$ and $i, j = 1, \dots, n$.

Proof. We shall apply Lemma 5.4.3 to $S = S_{ij}$. For this purpose it is sufficient to show that the function $K = K_{ij}$ satisfies (5.4.10), since we already know that the convolution operator S_{ij} is bounded on $L^2(\mathbf{R}^n)$, see the proof of Lemma 5.4.1. To this end, we differentiate K_{ij} to get

$$\nabla K_{ij}(x) = -\frac{nk_{ij}(x/|x|)}{|x|^{n+1}}\frac{x}{|x|} + \frac{1}{|x|^{n+1}}\left(I_n - \frac{1}{|x|^2}x \otimes x\right)\nabla k_{ij}\left(\frac{x}{|x|}\right)$$

for $x \in \mathbf{R}^n \setminus \{0\}$, where I_n is the identity matrix of size n and $x \otimes x := (x_i x_j)_{i,j}$ is the tensor product of x. Since k_{ij} is smooth on S^{n-1}, we have

$$|\nabla K_{ij}(x)| \le \frac{c}{|x|^{n+1}}, \quad x \in \mathbf{R}^n \setminus \{0\}.$$

Hence, for all $x, y \in \mathbf{R}^n \setminus \{0\}$ with $|x| \ge 2|y| > 0$,

$$\begin{aligned} |K(x-y) - K(x)| &= \left| \int_0^1 \frac{d}{dt}(K(x-ty))\,\mathrm{d}t \right| = \left| \int_0^1 (-y) \cdot \nabla K(x-ty)\,\mathrm{d}t \right| \\ &\le |y| \int_0^1 \frac{c}{|x-ty|^{n+1}}\,\mathrm{d}t \le |y| \int_0^1 \frac{c}{(|x|-|y|)^{n+1}}\,\mathrm{d}t \\ &\le \frac{c|y|}{(|x|-|x|/2)^{n+1}} = \frac{2^{n+1}c|y|}{|x|^{n+1}}. \end{aligned}$$

Thus K_{ij} satisfies (5.4.10) with $\delta = 1$ and we can apply Lemma 5.4.3 to obtain

$$[S_{ij}\bar{u}_j : BMO^\infty(\mathbf{R}^n)] \le c\|\bar{u}_j\|_{L^\infty(\mathbf{R}^n)} = c\|u_j\|_{L^\infty(\Omega)} \tag{5.4.12}$$

with some constant $c > 0$.

By the definition of the BMO^∞-seminorm, we have

$$[S_{ij}\bar{u}_j : BMO^\infty(\Omega)] \le [S_{ij}\bar{u}_j : BMO^\infty(\mathbf{R}^n)]\,.$$

Hence the inequality (5.4.11) follows from (5.4.12). □

Next, let us estimate the b^ν-part of $S_{ij}\bar{u}_j$. Recall the two closed cones

$$C_j = \{x = (x', x_n) \in \mathbf{R}^n \mid x' \in \mathbf{R}^{n-1},\ -x_n \ge |x|\cos(2^j\theta)\}, \quad j = 0, 1$$

with $\theta \in (0, \pi/4)$. For $r > 0$ and $x_0 \in \mathbf{R}^n$, we define

$$A_r(x_0) := \bigcup_{x \in B_r(x_0) \cap (x_0 + C_1)^c} (x + C_0) \cap (x_0 + C_1)^c \subset \mathbf{R}^n. \tag{5.4.13}$$

Here $x_0 + C_1 = \{y \in \mathbf{R}^n \mid y - x_0 \in C_1\}$ and $x + C_0$ is defined similarly.

Lemma 5.4.5. *For all $r > 0$ and $x_0 \in \mathbf{R}^n$ we have $A_r(x_0) \subset B_{r/\sin\theta}(x_0)$.*

Proof. By translation, we may assume that $x_0 = 0$. Let $a := (0, \dots, 0, r/\sin\theta) \in \mathbf{R}^n$. Suppose that

(i) $B_r(0) \subset a + C_0$,

(ii) $x + C_0 \subset a + C_0$ for all $x \in a + C_0$,

(iii) $(a + C_0) \cap C_1^c \subset B_{r/\sin\theta}(0)$.

Then, the statements (i) and (ii) imply

$$A_r(0) = \bigcup_{x \in B_r(0) \cap C_1^c} (x + C_0) \cap C_1^c \subset (a + C_0) \cap C_1^c.$$

Hence the statement (iii) yields $A_r(0) \subset B_{r/\sin\theta}(0)$. Now let us prove the statements (i)-(iii). Note that, since $\theta \in (0, \pi/4)$, the cones C_0 and C_1 are represented as

$$C_j = \{x = (x', x_n) \in \mathbf{R}^n \mid x' \in \mathbf{R}^{n-1},\ x_n \le 0,\ |x'| \le (-x_n)\tan(2^j\theta)\}, \quad j = 0, 1.$$

(i) Let $x = (x', x_n) \in B_r(0)$. Then, $x - a = (x', x_n - r/\sin\theta)$ satisfies

$$(x-a)_n = x_n - \frac{r}{\sin\theta} \le r - \frac{r}{\sin\theta} < 0$$

and

$$\Big(\frac{r}{\sin\theta} - x_n\Big)^2 \tan^2\theta - |x'|^2 \ge \frac{(r - x_n\sin\theta)^2}{\cos^2\theta} - (r^2 - x_n^2) = \frac{(r\sin\theta - x_n)^2}{\cos^2\theta} \ge 0,$$

or equivalently,

$$|x'| \le \Big(\frac{r}{\sin\theta} - x_n\Big)\tan\theta = -(x-a)_n\tan\theta.$$

Hence $x - a \in C_0$, that is, $x \in a + C_0$ and the statement (i) holds.

(ii) Let $x \in a + C_0$. If $y \in x + C_0$, then $(y-a)_n = (y-x)_n + (x-a)_n \le 0$ and

$$|y'| \le |x'| + |y' - x'| \le -(x-a)_n\tan\theta - (y-x)_n\tan\theta = -(y-a)_n\tan\theta,$$

which means that $y \in a + C_0$. Hence the statement (ii) holds.

(iii) Let $x \in (a + C_0) \cap C_1^c$. Then we have

$$(x-a)_n = x_n - r/\sin\theta \le 0, \quad |x'| \le \Big(\frac{r}{\sin\theta} - x_n\Big)\tan\theta. \tag{5.4.14}$$

Hence

$$|x|^2 \le \Big(\frac{r}{\sin\theta} - x_n\Big)^2\tan^2\theta + x_n^2 =: f(x_n).$$

To estimate the right-hand side in the above inequality for $x \in (a + C_0) \cap C_1^c$, we derive the range of x_n for $x \in (a + C_0) \cap C_1^c$. If $x_n \ge 0$, then $x \in (a + C_0) \cap C_1^c$ holds if and only if the condition (5.4.14) is satisfied. Thus x_n must satisfy

$$0 \le x_n \le \frac{r}{\sin\theta}.$$

On the other hand, if $x_n < 0$, then $x \in (a + C_0) \cap C_1^c$ holds if and only if

$$(-x_n)\tan(2\theta) < |x'| \le \Big(\frac{r}{\sin\theta} - x_n\Big)\tan\theta.$$

In particular, if $x \in (a + C_0) \cap C_1^c$ and $x_n < 0$, then x_n must satisfy

$$(-x_n)\tan(2\theta) < \Big(\frac{r}{\sin\theta} - x_n\Big)\tan\theta,$$

which yields the inequality

$$-\frac{r}{\cos\theta} < (\tan(2\theta) - \tan\theta)\, x_n.$$

Since

$$\begin{aligned}\tan(2\theta) - \tan\theta &= \tan(2\theta) - \frac{1}{2}\tan(2\theta)(1 - \tan^2\theta)\\ &= \frac{1}{2}\tan(2\theta)(1+\tan^2\theta) = \frac{\tan(2\theta)}{2\cos^2\theta} > 0 \quad \Big(0 < \theta < \frac{\pi}{4}\Big),\end{aligned}$$

the above inequality is equivalent to

$$-\frac{2r\cos\theta}{\tan(2\theta)} < x_n (< 0).$$

In summary, the range of x_n for $x \in (a + C_0) \cap C_1^c$ is

$$\alpha := -\frac{2r\cos\theta}{\tan(2\theta)} < x_n \le \frac{r}{\sin\theta} =: \beta$$

and thus we obtain

$$|x|^2 \le f(x_n) \le \sup_{s\in(\alpha,\beta]} f(s) = \max\{f(\alpha), f(\beta)\},$$

where the last equality follows from the fact that $f(x_n)$ is a concave parabola. On the one hand, we have $f(\beta) = \beta^2 = r^2/\sin^2\theta$. On the other hand, since

$$\alpha = -\frac{2r\cos\theta\cos(2\theta)}{\sin(2\theta)} = -\frac{r\cos(2\theta)}{\sin\theta} = \frac{r(1-2\cos^2\theta)}{\sin\theta},$$

we have

$$\begin{aligned}f(\alpha) &= \Big(\frac{r}{\sin\theta} - \frac{r(1-2\cos^2\theta)}{\sin\theta}\Big)^2\tan^2\theta + \frac{r^2\cos^2(2\theta)}{\sin^2\theta}\\ &= \frac{r^2}{\sin^2\theta}\{4\tan^2\theta\cos^4\theta + \cos^2(2\theta)\} = \frac{r^2}{\sin^2\theta}.\end{aligned}$$

Hence $|x|^2 \le r^2/\sin^2\theta$ and thus $x \in B_{r/\sin\theta}(0)$ for every $x \in (a + C_0) \cap C_1^c$. Therefore, the statement (iii) holds and the lemma follows. □

Now we can estimate the b^ν-part of $S_{ij}\bar{u}_j$.

Lemma 5.4.6. *Let $\nu \in (0, \infty]$. There exists a constant $c > 0$ such that*

$$[S_{ij}\bar{u}_j : b^\nu(\Omega)] \le \frac{c}{\sin^{n/2}\theta}\|u_j\|_{L^\infty(\Omega)} \tag{5.4.15}$$

for all $u = (u_1, \ldots, u_n) \in C_c^\infty(\Omega)$ and $i, j = 1, \ldots, n$.

Proof. First we note that for all $f \in L^1_{loc}(\Omega)$ the inequality

$$[f : b^\nu(\Omega)] \le \omega_n^{1/2}\,[f : b_2^\nu(\Omega)]$$

holds by Hölder's inequality. Hence, to prove (5.4.15), it is sufficient to show the inequality

$$[S_{ij}\bar{u}_j : b_2^\nu(\Omega)] \le \frac{c}{\sin^{n/2}\theta}\left[u_j : b_2^{\nu/\sin\theta}(\Omega)\right] \le \frac{c\omega_n^{1/2}}{\sin^{n/2}\theta}\|u_j\|_{L^\infty}. \qquad (5.4.16)$$

The second inequality of (5.4.16) follows from the definition of $[\,\cdot : b_2^{\nu/\sin\theta}(\Omega)]$. Let us show the first inequality. The singular integral $S_{ij}\bar{u}_j$ is of the form

$$S_{ij}\bar{u}_j(x) = (K_{ij} * \bar{u}_j)(x) = \int_{\mathbf{R}^n} K_{ij}(x-y)\bar{u}_j(y)\,\mathrm{d}y, \quad x \in \mathbf{R}^n.$$

Since $\operatorname{supp} K_{ij} \subset -C_0$ (see (5.4.2) and (5.4.4)) and $\operatorname{supp} u \subset \Omega$, we can write

$$S_{ij}\bar{u}_j(x) = \int_{(x+C_0)\cap\Omega} K_{ij}(x-y)\bar{u}_j(y)\,\mathrm{d}y, \quad x \in \mathbf{R}^n.$$

Hence, if we set

$$W_r(x_0) := \bigcup_{x \in B_r(x_0)\cap\Omega} (x + C_0)\cap\Omega$$

for each $x_0 \in \partial\Omega$ and $r > 0$ with $B_r(x_0) \subset U_\nu(\partial\Omega)$, then we have

$$S_{ij}\bar{u}_j(x) = \int_{(x+C_0)\cap\Omega} K_{ij}(x-y)(\bar{u}_j|_{W_r(x_0)})(y)\,dy = [K_{ij} * (\bar{u}_j|_{W_r(x_0)})](x)$$

for all $x \in B_r(x_0)\cap\Omega$, where

$$(\bar{u}_j|_{W_r(x_0)})(x) := \begin{cases} \bar{u}_j(x), & x \in W_r(x_0), \\ 0, & x \notin W_r(x_0). \end{cases}$$

Since K_{ij} is a singular kernel (see the proof of Lemma 5.4.1), the Calderón-Zygmund theory implies that

$$\begin{aligned}\int_{B_r(x_0)\cap\Omega} |S_{ij}\bar{u}_j(x)|^2\,\mathrm{d}x &= \int_{B_r(x_0)\cap\Omega} |[K_{ij} * (\bar{u}_j|_{W_r(x_0)})](x)|^2\,\mathrm{d}x \\ &\le c\int_{\mathbf{R}^n} |(\bar{u}_j|_{W_r(x_0)})(x)|^2\,\mathrm{d}x = c\int_{W_r(x_0)} |\bar{u}_j(x)|^2\,\mathrm{d}x\end{aligned}$$

with some constant $c > 0$. Now we recall the property of the infinite cone C_1:

$$x + C_1 \subset \Omega^c \Leftrightarrow \Omega \subset (x + C_1)^c \quad \text{for all} \quad x \in \Omega^c.$$

By this property we have

$$W_r(x_0) \subset \bigcup_{x \in B_r(x_0) \cap (x_0 + C_1)^c} (x + C_0) \cap ((x_0 + C_1)^c \cap \Omega) = A_r(x_0) \cap \Omega,$$

where $A_r(x_0)$ is given by (5.4.13), and thus Lemma 5.4.5 yields

$$W_r(x_0) \subset A_r(x_0) \cap \Omega \subset B_{r/\sin\theta}(x_0) \cap \Omega.$$

Hence we have

$$\begin{aligned}
&\frac{1}{r^n} \int_{B_r(x_0)\cap\Omega} |S_{ij}\bar{u}_j(x)|^2 \, \mathrm{d}x \leq \frac{c}{r^n} \int_{W_r(x_0)} |\bar{u}_j(x)|^2 \, \mathrm{d}x \\
&\leq \frac{c}{r^n} \int_{B_{r/\sin\theta}(x_0)\cap\Omega} |\bar{u}_j(x)|^2 \, \mathrm{d}x = \frac{c}{\sin^n \theta} \left(\frac{\sin\theta}{r} \right)^n \int_{B_{r/\sin\theta}(x_0)\cap\Omega} |u_j(x)|^2 \, \mathrm{d}x \\
&\leq \frac{c}{\sin^n \theta} \left[u_j : b_2^{\nu/\sin\theta}(\Omega) \right]^2
\end{aligned}$$

for every $x_0 \in \partial\Omega$ and $r > 0$ with $B_r(x_0) \subset U_\nu(\partial\Omega)$, which yields

$$[S_{ij}\bar{u}_j : b_2^\nu(\Omega)]^2 \leq \frac{c}{\sin^n \theta} \left[u_j : b_2^{\nu/\sin\theta}(\Omega) \right]^2.$$

The proof is complete. □

Now we obtain an estimate for the $BMO_b^{\infty,\nu}(\Omega)$-norm of Tu.

Theorem 5.4.2. *Let $\nu \in (0, \infty]$. There exists a constant $c > 0$ such that*

$$\|Tu : BMO_b^{\infty,\nu}(\Omega)\| \leq c\|u\|_{L^\infty(\Omega)}$$

for all $u \in C_c^\infty(\Omega)$.

Proof. Since the i-th component of Tu, $i = 1, \ldots, n$, is of the form (5.4.7), we have by (5.4.9), (5.4.11) and (5.4.15) that

$$\begin{aligned}
&\|Tu : BMO_b^{\infty,\nu}(\Omega)\| \\
&\leq c \sum_{i,j=1}^{n} (\|a_{ij}\bar{u}_j : BMO_b^{\infty,\nu}(\Omega)\| + [S_{ij}\bar{u}_j : BMO^\infty(\Omega)] + [S_{ij}\bar{u}_j : b^\nu(\Omega)]) \\
&\leq c \sum_{j=1}^{n} \|u_j\|_{L^\infty(\Omega)} \leq c\|u\|_{L^\infty(\Omega)}
\end{aligned}$$

with a positive constant c. □

5.4.2 Non-Helmholtz projection

As in the previous subsection, let Ω denote a Lipschitz half-space in $\mathbf{R}^n$.

Definition 5.4.3. *For a vector field $u \in C_c^\infty(\Omega)$, we define a vector field $Q'u$ on $\mathbf{R}^n$ as $Q'u := u - Tu$. Here the operator T is given in Definition 5.4.2.*

For a vector field $u \in C_c^\infty(\Omega)$, the vector field Tu is smooth in $\mathbf{R}^n$ and

$$\operatorname{div} Tu = \operatorname{div} u \quad \text{in} \quad \Omega, \quad Tu = 0 \quad \text{on} \quad \partial\Omega.$$

Moreover, $Tu = 0$ in Ω for all $u \in C_{c,\sigma}^\infty(\Omega)$, see the argument after Definition 5.4.2. Thus $Q'u = u - Tu$ is also smooth in $\mathbf{R}^n$ and

$$\operatorname{div} Q'u = 0 \quad \text{in} \quad \Omega, \quad Q'u = 0 \quad \text{on} \quad \partial\Omega \tag{5.4.17}$$

for all $u \in C_c^\infty(\Omega)$, and $Q'u = u$ in Ω for all $u \in C_{c,\sigma}^\infty(\Omega)$. Note that Q' is not a projection from $C_c^\infty(\Omega)$ onto $C_{c,\sigma}^\infty(\Omega)$, since the support of Tu may be unbounded and thus $Q'u$ is in general not in $C_{c,\sigma}^\infty(\Omega)$. However, Q' maps $C_c^\infty(\Omega)$ into $L_\sigma^p(\Omega)$.

Lemma 5.4.7. *For all $u \in C_c^\infty(\Omega)$ and $p \in (1, \infty)$, we have $Q'u \in L_\sigma^p(\Omega)$.*

We shall first prove an auxiliary proposition for the above lemma. For $p \in (1, \infty)$, let $G^p(\Omega) = \{\nabla q \in L^p(\Omega) \mid q \in L_{loc}^1(\Omega)\}$.

Proposition 5.4.1. *Let $p \in (1, \infty)$. For every $\nabla q \in G^p(\Omega)$, there exists a sequence $\{q_k\}_{k=1}^\infty$ of functions in $C_c^\infty(\mathbf{R}^n)$ such that*

$$\lim_{k\to\infty} \|\nabla q - \nabla q_k\|_{L^p(\Omega)} = 0. \tag{5.4.18}$$

Proof. Since the restriction of $C_c^\infty(\mathbf{R}^n)$ on Ω is dense in $W^{1,p}(\Omega)$, it is sufficient to show that for every $\nabla q \in G^p(\Omega)$ there is a sequence $\{q_k\}_{k=1}^\infty$ of functions in $W^{1,p}(\Omega)$ such that (5.4.18) holds. Let us prove this claim.

(i) First we assume that the claim is valid for the half-space $\mathbf{R}_+^n$ and show the claim for general Lipschitz half-spaces $\Omega = \{(x', x_n) \in \mathbf{R}^n \mid x_n > h(x')\}$. As in Section 5.3, let $F(x) := (x', x_n - h(x'))$ be a bi-Lipschitz map from Ω to $\mathbf{R}_+^n$. Let $\nabla q \in G^p(\Omega)$ and $\widetilde{q} := q \circ F^{-1}$, where $F^{-1}(y) := (y', y_n + h(y'))$ is the inverse mapping of F. Then, since $\nabla\widetilde{q}(y) = \nabla F^{-1}(y)\nabla q(F^{-1}(y))$ for $y \in \mathbf{R}_+^n$ and each component of ∇F^{-1} is bounded (because h is Lipschitz continuous), we have $\nabla\widetilde{q} \in G^p(\mathbf{R}_+^n)$. Hence, by our assumption that the claim is valid for $\mathbf{R}_+^n$, there is a sequence $\{\widetilde{q}_k\}_{k=1}^\infty$ of functions in $W^{1,p}(\mathbf{R}_+^n)$ such that $\lim_{k\to\infty} \|\nabla\widetilde{q} - \nabla\widetilde{q}_k\|_{L^p(\mathbf{R}_+^n)} = 0$.

Let $q_k := \widetilde{q}_k \circ F$ for each $k \in \mathbf{N}$. Then, since

$$\nabla q(x) = \nabla F(x)\nabla\widetilde{q}(F(x)), \quad \nabla q_k(x) = \nabla F(x)\nabla\widetilde{q}_k(F(x)), \quad x \in \Omega$$

and each component of ∇F is bounded, we have $q_k \in W^{1,p}(\Omega)$ and

$$\|\nabla q - \nabla q_k\|_{L^p(\Omega)} \le c\|\nabla\widetilde{q} - \nabla\widetilde{q}_k\|_{L^p(\mathbf{R}_+^n)} \to 0$$

as $k \to \infty$. Thus the claim is valid for general Lipschitz half-spaces Ω.

(ii) Now we prove the claim for $\Omega = \mathbf{R}^n_+$. We follow the idea of the proof of the claim in the case $\Omega = \mathbf{R}^n$, see [Sol01, Lemma 2.5.4]. Let $\varphi \in C_c^\infty(\mathbf{R}^n)$ be a function such that

$$0 \le \varphi \le 1 \quad \text{in } \mathbf{R}^n, \quad \varphi = 1 \quad \text{in } B_1(0), \quad \varphi = 0 \quad \text{in } \mathbf{R}^n \setminus B_2(0)$$

and $\varphi_k(x) := \varphi(k^{-1}x)$ for $k \in \mathbf{N}$ and $x \in \mathbf{R}^n$. Then, $\lim_{k\to\infty} \varphi_k(x) = 1$ for all $x \in \mathbf{R}^n$ and $\operatorname{supp} \varphi_k \subset B_{2k}(0)$, $\operatorname{supp} \nabla\varphi_k \subset B_{2k}(0) \setminus B_k(0)$ for $k \in \mathbf{N}$.

Let $\nabla q \in G^p(\mathbf{R}^n_+)$. Then $q \in W^{1,p}_{loc}(\overline{\mathbf{R}^n_+})$, that is, $q \in W^{1,p}(U)$ for every bounded subset U of $\mathbf{R}^n_+$; see the proof of [Nec67, Theorem 7.6 in Chapter 2]. Hence by setting $\Omega_k := \mathbf{R}^n_+ \cap (\operatorname{int} B_{2k}(0) \setminus B_k(0))$ for $k \in \mathbf{N}$, we have $q \in W^{1,p}(\Omega_k)$ and thus there is a constant a_k such that $\int_{\Omega_k} (q - a_k)\,\mathrm{d}x = 0$ for each $k \in \mathbf{N}$. From this equality and the change of variables $x = ky$ for $x \in \Omega_k$ and $y \in \Omega_1$ we have

$$\int_{\Omega_1} (q(ky) - a_k)\,\mathrm{d}y = k^{-n} \int_{\Omega_k} (q(x) - a_k)\,\mathrm{d}x = 0.$$

Hence we can apply Poincaré's inequality to $q(ky) - a_k$ on Ω_1 and get

$$\left(\int_{\Omega_1} |q(ky) - a_k|^p\,\mathrm{d}y \right)^{1/p} \le c \left(\int_{\Omega_1} |\nabla(q(ky))|^p\,\mathrm{d}y \right)^{1/p}$$

with a constant $c > 0$ independent of k. In this inequality, we observe that

$$\int_{\Omega_1} |q(ky) - a_k|^p\,\mathrm{d}y = k^{-n} \int_{\Omega_k} |q(x) - a_k|^p\,\mathrm{d}x,$$

$$\int_{\Omega_1} |\nabla(q(ky))|^p\,\mathrm{d}y = k^p \int_{\Omega_1} |(\nabla q)(ky)|^p\,\mathrm{d}y = k^{p-n} \int_{\Omega_k} |\nabla q(x)|^p\,\mathrm{d}x$$

by the change of variables $x = ky$ and thus

$$\|q - a_k\|_{L^p(\Omega_k)} \le ck\|\nabla q\|_{L^p(\Omega_k)}, \quad k \in \mathbf{N}. \tag{5.4.19}$$

For each $k \in \mathbf{N}$, let $q_k := \varphi_k(q - a_k)$ on $\mathbf{R}^n_+$. Then since $\operatorname{supp} q_k \subset \mathbf{R}^n_+ \cap B_{2k}(0)$ holds by the relation $\operatorname{supp} \varphi_k \subset B_{2k}(0)$, it follows that $q_k \in W^{1,p}(\mathbf{R}^n_+)$ and

$$\|\nabla q - \nabla q_k\|_{L^p(\mathbf{R}^n_+)} \le \|\nabla q - \varphi_k \nabla q\|_{L^p(\mathbf{R}^n_+)} + \|(\nabla\varphi_k)(q - a_k)\|_{L^p(\mathbf{R}^n_+)}. \tag{5.4.20}$$

Since $0 \le \varphi_k(x) \le 1$ and $\lim_{k\to\infty} \varphi_k(x) = 1$ for all $x \in \mathbf{R}^n_+$ and $\nabla q \in L^p(\mathbf{R}^n_+)$, the dominated convergence theorem yields

$$\lim_{k\to\infty} \|\nabla q - \varphi_k \nabla q\|_{L^p(\mathbf{R}^n_+)} = 0. \tag{5.4.21}$$

On the other hand, since $\nabla\varphi_k = k^{-1}(\nabla\varphi)_k$ and $\operatorname{supp} \nabla\varphi_k|_{\mathbf{R}^n_+} \subset \overline{\Omega_k}$ for each $k \in \mathbf{N}$, it follows from (5.4.19) and the dominated convergence theorem that

$$\|(\nabla\varphi_k)(q - a_k)\|_{L^p(\mathbf{R}^n_+)} \le ck^{-1}\|q - a_k\|_{L^p(\Omega_k)} \le c\|\nabla q\|_{L^p(\Omega_k)} \to 0 \tag{5.4.22}$$

as $k \to \infty$. Applying (5.4.21) and (5.4.22) to (5.4.20) we have

$$\lim_{k\to\infty} \|\nabla q - \nabla q_k\|_{L^p(\mathbf{R}^n_+)} = 0,$$

where $q_k \in W^{1,p}(\mathbf{R}^n_+)$ for all $k \in \mathbf{N}$. Hence the claim is valid when $\Omega = \mathbf{R}^n_+$ and the proposition follows. $\square$

Proof of Lemma 5.4.7. Let $u \in C_c^\infty(\Omega)$ and $p \in (1,\infty)$. Then, since $Tu \in L^p(\Omega)$ by Theorem 5.4.1, we have $Q'u = u - Tu \in L^p(\Omega)$. To show $Q'u \in L^p_\sigma(\Omega)$, we employ a characterization of elements of $L^p_\sigma(\Omega)$ ([Gal11, Lemma III.2.1]): a vector field $v \in L^p(\Omega)$ is in $L^p_\sigma(\Omega)$ if and only if

$$\int_\Omega v \cdot \nabla q \,\mathrm{d}x = 0 \quad \text{for all} \quad \nabla q \in G^{p'}(\Omega) \ \left(p' := \frac{p}{p-1}\right).$$

Let ∇q be any element of $G^{p'}(\Omega)$. From Proposition 5.4.1, there is a sequence $\{q_k\}_{k=1}^\infty$ of functions in $C_c^\infty(\mathbf{R}^n)$ such that the equality (5.4.18) with p replaced by p' holds. Since $Q'u$ is well-defined and smooth in $\mathbf{R}^n$ for $u \in C_c^\infty(\Omega)$ and $q_k \in C_c^\infty(\mathbf{R}^n)$, integration by parts yields

$$\int_\Omega Q'u \cdot \nabla q_k \,\mathrm{d}x = -\int_\Omega q_k \operatorname{div} Q'u \,\mathrm{d}x + \int_{\partial\Omega} q_k\, Q'u \cdot \nu \,\mathrm{d}\mathcal{H}^{n-1}$$

for all $k \in \mathbf{N}$, where ν denotes the unit outer normal vector field of $\partial\Omega$. We apply (5.4.17) to the right-hand side of this equality to get $\int_\Omega Q'u \cdot \nabla q_k \, dx = 0$ for all $k \in \mathbf{N}$. Since $Q'u \in L^p(\Omega)$ and (5.4.18) with p replaced by p' holds, the above equality implies that

$$\int_\Omega Q'u \cdot \nabla q \,\mathrm{d}x = \lim_{k\to\infty} \int_\Omega Q'u \cdot \nabla q_k \,\mathrm{d}x = 0.$$

Hence by the characterization of elements of $L^p_\sigma(\Omega)$ we conclude that $Q'u \in L^p_\sigma(\Omega)$ for all $u \in C_c^\infty(\Omega)$. The proof is complete. $\square$

Remark 5.4.1.

1. *Let $p \in (1,\infty)$. By Theorem 5.4.1 and Lemma 5.4.7, we have $Q'u \in L^p_\sigma(\Omega)$ and $\|Q'u\|_{L^p(\Omega)} \le c\|u\|_{L^p(\Omega)}$ for all $u \in C_c^\infty(\Omega)$. Moreover, $Q'u = u$ holds for all $u \in C_{c,\sigma}^\infty(\Omega)$. Hence, by the density argument, Q' extends uniquely to a bounded linear operator on $L^p(\Omega)$ that is a projection onto $L^p_\sigma(\Omega)$.*

2. *The projection onto $L^p_\sigma(\Omega)$ given as above is NOT the Helmholtz projection. Indeed, if it were the Helmholtz projection, then for each $u \in C_c^\infty(\Omega)$ there would exist $\pi \in L^1_{loc}(\Omega)$ such that $(I - Q')u = \nabla\pi$ holds. Since $(I - Q')u = Tu = K * \operatorname{div} u$ for $u \in C_c^\infty(\Omega)$, the existence of such π would imply that $\partial_j(K_i * \operatorname{div} u) = \partial_i(K_j * \operatorname{div} u)$ for all $i, j = 1, \dots, n$. For each $f \in C_c^\infty(\Omega)$ with $\int_\Omega f \mathrm{d}x = 0$ there is $u \in C_c^\infty(\Omega)$ satisfying $f = \operatorname{div} u$. This is possible since we are able to apply Bogovskiĭ's lemma to a bounded*

Lipschitz domain $D \subset \Omega$ containing the support of f (see [Gal11, Theorem III.3.3]). Thus the above equality would imply that $\partial_j K_i = \partial_i K_j + c$ with some constant c for all $i, j = 1, \dots, n$ as a distribution. This contradicts the fact that $\partial_j K_i \neq \partial_i K_j + c$ for $i \neq j$ as observed in (5.4.4).

3. *It is possible to prove the characterization*

$$L^p_\sigma(\Omega) = \{u \in L^p(\Omega) \mid \operatorname{div} u = 0 \ \textit{in}\ \Omega,\ u \cdot \nu = 0 \ \textit{on}\ \partial\Omega\}$$

if we use Proposition 5.4.1 and an integration by parts formula. This characterization is well-known for bounded ([FM77]), exterior and other domains (see [Gal11, Section III.2]). However, for a Lipschitz half-space, it is less popular. A proof can be found in [MM15, Lemma 2.1].

The linear operator Q' also maps $C_c^\infty(\Omega)$ into $VMO^{\infty,\nu}_{b,0,\sigma}(\Omega)$.

Lemma 5.4.8. *For all $u \in C_c^\infty(\Omega)$ and $\nu \in (0,\infty]$, we have $Q'u \in VMO^{\infty,\nu}_{b,0,\sigma}(\Omega)$.*

We shall prove two auxiliary propositions for the above lemma. For $p \in (1,\infty)$, let $W^{1,p}_{0,\sigma}(\Omega)$ be the $W^{1,p}$-closure of $C^\infty_{c,\sigma}(\Omega)$.

Proposition 5.4.2. *For all $p \in (1,\infty)$ we have $L^p_\sigma(\Omega) \cap W^{1,p}_0(\Omega) \subset W^{1,p}_{0,\sigma}(\Omega)$. Thus $L^p_\sigma(\Omega) \cap W^{1,p}_0(\Omega) = W^{1,p}_{0,\sigma}(\Omega)$.*

Proof. Let $\rho \in C_c^\infty(\mathbf{R}^n)$ be a function such that

$$0 \le \rho \le 1 \quad \text{in } \mathbf{R}^n, \quad \operatorname{supp} \rho \subset B_1(0), \quad \int_{B_1(0)} \rho \,\mathrm{d}x = 1$$

and $\rho_\delta(x) := \delta^{-n}\rho(\delta^{-1}x)$ for $\delta > 0$, $x \in \mathbf{R}^n$. Let $u \in L^p_\sigma(\Omega) \cap W^{1,p}_0(\Omega)$. Then there is a sequence $\{u_k\}_{k=1}^\infty$ of functions in $C^\infty_{c,\sigma}(\Omega)$ such that $\lim_{k\to\infty} \|u - u_k\|_{L^p(\Omega)} = 0$. For $a > 0$, we define a vector field u^a on Ω as

$$u^a(x) := \begin{cases} u(x', x_n - a), & x_n > h(x') + a, \\ 0, & h(x') < x_n \le h(x') + a \end{cases}$$

and $u^a_k = (u_k)^a$ similarly. Then it is clear that $u^a \in W^{1,p}_0(\Omega)$ and $u^a_k \in C^\infty_{c,\sigma}(\Omega)$ for all $a > 0$. Moreover, we have

$$\|u^a - u^a_k\|_{L^p(\Omega)} = \|u - u_k\|_{L^p(\Omega)} \quad \text{for all} \quad a > 0, \quad \lim_{a\to 0} \|u - u^a\|_{W^{1,p}(\Omega)} = 0.$$

By the second equality and the fact that $W^{1,p}_{0,\sigma}(\Omega)$ is closed in $W^{1,p}(\Omega)$, it is sufficient for showing $u \in W^{1,p}_{0,\sigma}(\Omega)$ to prove $u^a \in W^{1,p}_{0,\sigma}(\Omega)$ for all $a > 0$.

For each $a > 0$, there is a constant $d = d(a) > 0$ such that $\operatorname{dist}(\operatorname{supp} u^a_k, \partial\Omega) \ge d$ for all $k \in \mathbf{N}$. Then, for a given $\varepsilon > 0$, we can take $\delta \in (0, d/2)$ so small that

$$\|u^a - u^a * \rho_\delta\|_{W^{1,p}(\Omega)} < \frac{\varepsilon}{2},$$

since $u^a \in W_0^{1,p}(\Omega)$. Also, since $\nabla\rho_\delta = \delta^{-1}(\nabla\rho)_\delta$, we have

$$\begin{aligned}
&\|u^a * \rho_\delta - u_k^a * \rho_\delta\|_{W^{1,p}(\Omega)} \\
&\leq c(\|u^a * \rho_\delta - u_k^a * \rho_\delta\|_{L^p(\Omega)} + \|u^a * \nabla\rho_\delta - u_k^a * \nabla\rho_\delta\|_{L^p(\Omega)}) \\
&= c(\|(u^a - u_k^a) * \rho_\delta\|_{L^p(\Omega)} + \delta^{-1}\|(u^a - u_k^a) * (\nabla\rho)_\delta\|_{L^p(\Omega)}) \\
&\leq c(1+\delta^{-1})\|u^a - u_k^a\|_{L^p(\Omega)} = c(1+\delta^{-1})\|u - u_k\|_{L^p(\Omega)}
\end{aligned}$$

with a constant $c > 0$ independent of ε and δ. Hence by taking $k \in \mathbf{N}$ so large that

$$\|u - u_k\|_{L^p(\Omega)} < \frac{\varepsilon}{2c(1+\delta^{-1})},$$

we have $\|u^a * \rho_\delta - u_k^a * \rho_\delta\|_{W^{1,p}(\Omega)} < \varepsilon/2$ and thus

$$\|u^a - u_k^a * \rho_\delta\|_{W^{1,p}(\Omega)} \leq \|u^a - u^a * \rho_\delta\|_{W^{1,p}(\Omega)} + \|u^a * \rho_\delta - u_k^a * \rho_\delta\|_{W^{1,p}(\Omega)} < \varepsilon.$$

On the other hand, since $\operatorname{dist}(\operatorname{supp} u_k^a, \partial\Omega) > d$ and $\delta \in (0, d/2)$, the function $u_k^a * \rho_\delta$ is smooth and compactly supported in Ω. Moreover, we have

$$\operatorname{div}(u_k^a * \rho_\delta) = (\operatorname{div} u_k^a) * \rho_\delta = 0 \quad \text{in} \quad \Omega.$$

Thus $u_k^a * \rho_\delta \in C_{c,\sigma}^\infty(\Omega)$ and u^a is approximated by elements of $C_{c,\sigma}^\infty(\Omega)$ in $W^{1,p}(\Omega)$, which means that $u^a \in W_{0,\sigma}^{1,p}(\Omega)$. Hence $u \in W_{0,\sigma}^{1,p}(\Omega)$ and the proof is now complete. □

Proposition 5.4.3. *Let $\nu \in (0, \infty]$. If $p > n$, then $W_{0,\sigma}^{1,p}(\Omega) \subset VMO_{b,0,\sigma}^{\infty,\nu}(\Omega)$.*

Proof. Let $u \in W_{0,\sigma}^{1,p}(\Omega)$ and $u_k \in C_{c,\sigma}^\infty(\Omega)$ such that $\lim_{k\to\infty} \|u - u_k\|_{W^{1,p}(\Omega)} = 0$. Since $p > n$ and $u, u_k \in W_0^{1,p}(\Omega)$, Morrey's inequality (see e.g. [AF03, Theorem 4.12]) implies

$$\|u - u_k\|_{L^\infty(\Omega)} \leq c\|u - u_k\|_{W^{1,p}(\Omega)}$$

with a positive constant c independent of u and u_k. Thus we have

$$\|u - u_k : BMO_b^{\infty,\nu}(\Omega)\| \leq (2 + \omega_n)\|u - u_k\|_{L^\infty(\Omega)} \leq c\|u - u_k\|_{W^{1,p}(\Omega)} \to 0$$

as $k \to \infty$. Hence $u \in VMO_{b,0,\sigma}^{\infty,\nu}(\Omega)$ and the proof is now complete. □

Proof of Lemma 5.4.8. Since $u \in C_c^\infty(\Omega)$ and thus $\partial_i u \in C_c^\infty(\Omega)$ for all $i = 1, \ldots, n$, it follows from Lemma 5.4.7 that $Q'u \in L_\sigma^r(\Omega)$ and $\partial_i Q'u = Q'(\partial_i u) \in L^r(\Omega)$ for all $r \in (1, \infty)$ and $i = 1, \ldots, n$. From this fact and the equality (5.4.17), we have $Q'u \in L_\sigma^r(\Omega) \cap W_0^{1,r}(\Omega)$ for all $r \in (1, \infty)$. Hence, by taking $r > n$, we can apply Proposition 5.4.2 and Proposition 5.4.3 to obtain $Q'u \in VMO_{b,0,\sigma}^{\infty,\nu}(\Omega)$. □

Remark 5.4.2. *Let $\nu \in (0,\infty]$. Theorem 5.4.2 and Lemma 5.4.8 imply that $Q'u \in VMO^{\infty,\nu}_{b,0,\sigma}(\Omega)$ and*

$$\|Q'u : BMO^{\infty,\nu}_b(\Omega)\| \leq c\|u\|_{L^\infty(\Omega)}$$

for all $u \in C_c^\infty(\Omega)$. Also, we have $Q'u = u$ for all $u \in C^\infty_{c,\sigma}(\Omega)$. Hence Q' extends uniquely to a bounded linear operator (again referred to as Q') from $C_0(\Omega)$, which is the L^∞-closure of $C_c^\infty(\Omega)$, into $VMO^{\infty,\nu}_{b,0,\sigma}(\Omega)$ that satisfies $Q'u = u$ for all $u \in C_{0,\sigma}(\Omega)$.

Now let us extend Q' to a linear operator that gives the projection mentioned in Theorem 5.2.2. For $p \in (1,\infty)$, we define a Banach space X_p and its norm as

$$X_p := L^p(\Omega) \cap C_0(\Omega), \quad \|u\|_{X_p} := \max\{\|u\|_{L^p(\Omega)}, \|u\|_{L^\infty(\Omega)}\}.$$

Note that the Banach space $C_0(\Omega)$ consists of all continuous functions f on Ω such that the set $\{x \in \Omega \mid |f(x)| \geq \varepsilon\}$ is compact in Ω for every $\varepsilon > 0$ (see e.g. [Rud87, Theorem 3.17]).

Lemma 5.4.9. *For each $p \in (1,\infty)$, the linear subspace $C_c^\infty(\Omega)$ is dense in X_p.*

Proof. The proof is more or less standard (see e.g. [Jos05, Corollary 19.24]). We give it for completeness. Let $u \in X_p$ and $\Omega_k := \{x \in \Omega \mid |x| \leq k, \operatorname{dist}(x,\partial\Omega) \geq 1/k\}$ for $k \in \mathbf{N}$. For any given $\varepsilon > 0$, the set $\{x \in \Omega \mid |u(x)| \geq \varepsilon/2\}$ is compact in Ω since $u \in C_0(\Omega)$. Moreover, since $u \in L^p(\Omega)$, we can take $k \in \mathbf{N}$ so large that

$$\|u\|_{L^p(\Omega\setminus\Omega_k)} < \frac{\varepsilon}{2}, \quad \|u\|_{L^\infty(\Omega\setminus\Omega_k)} < \frac{\varepsilon}{2}. \tag{5.4.23}$$

Let $\varphi \in C_c^\infty(\Omega)$ be a continuous cut-off function such that

$$0 \leq \varphi \leq 1 \quad \text{in } \Omega, \quad \varphi = 1 \quad \text{in } \Omega_k, \quad \varphi = 0 \quad \text{in } \Omega \setminus \Omega_{2k}.$$

Since $u - \varphi u = 0$ in Ω_k and $|u - \varphi u| \leq |u|$ in $\Omega \setminus \Omega_k$, it follows from (5.4.23) that

$$\|u - \varphi u\|_{L^p(\Omega)} \leq \|u\|_{L^p(\Omega\setminus\Omega_k)} < \frac{\varepsilon}{2}, \quad \|u - \varphi u\|_{L^\infty(\Omega)} \leq \|u\|_{L^\infty(\Omega\setminus\Omega_k)} < \frac{\varepsilon}{2}. \tag{5.4.24}$$

Let ρ_δ be a mollifier as in the beginning of the proof of Proposition 5.4.2. Since

$$\varphi u \in L^p(\Omega), \quad \operatorname{dist}(\operatorname{supp}(\varphi u), \partial\Omega) \geq \frac{1}{2k},$$

we can take $\delta \in (0, 1/4k)$ so small that

$$u_\delta := \rho_\delta * (\varphi u) \in C_c^\infty(\Omega), \quad \|\varphi u - u_\delta\|_{L^p(\Omega)} < \frac{\varepsilon}{2}. \tag{5.4.25}$$

On the other hand, since φu is uniformly continuous on Ω_{4k}, we can again choose $\delta \in (0, 1/4k)$ so small that $\|\varphi u - u_\delta\|_{L^\infty(\Omega_{4k})} < \varepsilon/2$. Moreover, since

$\operatorname{supp}(\varphi u) \subset \Omega_{2k}$ and $\delta \in (0, 1/4k)$, we have $\varphi u = u_\delta = 0$ outside of Ω_{4k} and thus

$$\|\varphi u - u_\delta\|_{L^\infty(\Omega)} = \|\varphi u - u_\delta\|_{L^\infty(\Omega_{4k})} < \frac{\varepsilon}{2}. \tag{5.4.26}$$

Combining (5.4.24), (5.4.25) and (5.4.26), we obtain $u_\delta \in C_c^\infty(\Omega)$ and

$$\|u - u_\delta\|_{X_p} = \max\{\|u - u_\delta\|_{L^p(\Omega)}, \|u - u_\delta\|_{L^\infty(\Omega)}\} < \varepsilon.$$

Hence the lemma follows. □

Let $Y_p := L^p_\sigma(\Omega) \cap VMO^{\infty,\nu}_{b,0,\sigma}(\Omega)$ for $p \in (1, \infty)$, $\nu \in (0, \infty]$. Since $L^p_\sigma(\Omega)$ and $VMO^{\infty,\nu}_{b,0,\sigma}(\Omega)$ are closed in $L^p(\Omega)$ and $BMO^{\infty,\nu}_b(\Omega)$, respectively, Y_p becomes a Banach space under the norm

$$\|v\|_{Y_p} := \max\{\|v\|_{L^p(\Omega)}, \|v : BMO^{\infty,\nu}_b(\Omega)\|\}.$$

Theorem 5.4.3. *Let $p \in (1, \infty)$ and $\nu \in (0, \infty]$. The linear operator Q' given in Definition 5.4.3 extends uniquely to a bounded linear operator Q_p from X_p into Y_p. Moreover, there exists a constant $c > 0$ such that*

$$\|Q_p u\|_{L^p(\Omega)} \le c\|u\|_{L^p(\Omega)}, \quad \|Q_p u : BMO^{\infty,\nu}_b(\Omega)\| \le c\|u\|_{L^\infty(\Omega)} \tag{5.4.27}$$

for all $u \in X_p$ and $Q_p u = u$ holds for all u in the X_p-closure of $C^\infty_{c,\sigma}(\Omega)$.

Proof. Let $u \in C_c^\infty(\Omega)$. Then we have $Q'u \in Y_p$ by Lemma 5.4.7 and Lemma 5.4.8. Moreover, by Theorem 5.4.1 and Theorem 5.4.2, there is a constant $c > 0$ independent of u such that

$$\|Q'u\|_{L^p(\Omega)} \le c\|u\|_{L^p(\Omega)}, \quad \|Q'u : BMO^{\infty,\nu}_b(\Omega)\| \le c\|u\|_{L^\infty(\Omega)}. \tag{5.4.28}$$

Hence we have $Q'u \in Y_p$ and $\|Q'u\|_{Y_p} \le c\|u\|_{X_p}$ for all $u \in C_c^\infty(\Omega)$. Since $C_c^\infty(\Omega)$ is dense in X_p by Lemma 5.4.9, the operator Q' extends uniquely to a bounded linear operator Q_p from X_p into Y_p. Also, it follows from (5.4.28) that the inequality (5.4.27) holds for all $u \in X_p$. Since $Q'u = u$ holds for all $u \in C^\infty_{c,\sigma}(\Omega)$ as observed after Definition 5.4.3, by a density argument we have $Q_p u = u$ for all u in the X_p-closure of $C^\infty_{c,\sigma}(\Omega)$. The proof is complete. □

Finally, Theorem 5.2.2 follows from Theorem 5.4.3 with $p = 2$, that is, the linear operator $\mathbf{Q}$ in Theorem 5.2.2 is given by $\mathbf{Q} = Q_2$.

Remark 5.4.3. *The statements of Proposition 5.4.3 and Lemma 5.4.9 are still valid for an arbitrary domain Ω.*

5.5 Analyticity in L^p

In this section we shall give a complete proof of Theorem 5.1.1.

Proof of Theorem 1.1. Let $S(t)$ be the Stokes semigroup in $\tilde{L}^p_\sigma$ constructed by [FKS05], [FKS09]. To show that $S(t)$ forms an analytic semigroup in L^p_σ $(2 \leq p < \infty)$ it suffices to prove that there exists a constant c such that

$$\|S(t)v_0\|_p \leq c\|v_0\|_p \tag{5.5.1}$$

$$\left\|t\frac{\mathrm{d}}{\mathrm{d}t}S(t)v_0\right\|_p \leq c\|v_0\|_p \tag{5.5.2}$$

for all $v_0 \in C^\infty_{c,\sigma}(\Omega)$ and $t \in (0,1)$. Let $\mathbf{Q}$ be the operator in Theorem 5.2.2. Since $\mathbf{Q}$ is bounded in L^2 and maps L^2 to L^2_σ and $S(t)$ fulfills (5.5.1) and (5.5.2) for $p = 2$, we have

$$\|S(t)\mathbf{Q}u\|_2 \leq c\|u\|_2 \tag{5.5.3}$$

$$\left\|t\frac{\mathrm{d}}{\mathrm{d}t}S(t)\mathbf{Q}u\right\|_2 \leq c\|u\|_2 \tag{5.5.4}$$

for all $u \in C_c(\Omega)$ and $t \in (0,1)$. Since Ω is admissible as proved in [AGSS15], $S(t)$ forms an analytic semigroup in $VMO^{\infty,\nu}_{b,0,\sigma}$ by Theorem 5.1.2. We conclude that

$$\|S(t)\mathbf{Q}u : BMO^{\infty,\nu}_b(\Omega)\| \leq c\|u\|_\infty \tag{5.5.5}$$

$$\left\|t\frac{\mathrm{d}}{\mathrm{d}t}S(t)\mathbf{Q}u : BMO^{\infty,\nu}_b(\Omega)\right\| \leq c\|u\|_\infty \tag{5.5.6}$$

for all $u \in C_c(\Omega)$ and $t \in (0,1)$ since $\mathbf{Q}$ fulfills

$$\|\mathbf{Q}u : BMO^{\infty,\nu}_b(\Omega)\| \leq c\|u\|_\infty,\ \mathbf{Q}u \in VMO^{\infty,\nu}_{b,0,\sigma}$$

for all $u \in C_c(\Omega)$ by Theorem 5.2.2. (Note that we have a stronger statement than (5.5.6) by replacing the BMO_b type norm by the L^∞ norm since we have the regularizing estimate (5.1.3).) We apply an interpolation result (Theorem 5.2.1) to (5.5.3) and (5.5.5) and to (5.5.4) and (5.5.6) to get, respectively

$$\|S(t)\mathbf{Q}u\|_p \leq c\|u\|_p \tag{5.5.7}$$

$$\left\|t\frac{\mathrm{d}}{\mathrm{d}t}S(t)\mathbf{Q}u\right\|_p \leq c\|u\|_p \tag{5.5.8}$$

for all $u \in C_c(\Omega)$ and for all $t \in (0,1)$. Since $\mathbf{Q}u = u$ for $u \in C^\infty_{c,\sigma}(\Omega)$ this yields (5.5.1) and (5.5.2).

It remains to prove that $S(t)$ is a C_0-semigroup in L^p_σ. Since $C^\infty_{c,\sigma}(\Omega)$ is dense in L^p_σ, for $v_0 \in L^p_\sigma$ there is $v_{0m} \in C^\infty_{c,\sigma}$ such that $\|v_0 - v_{0m}\|_p \to 0$ as $m \to \infty$. By (5.5.1) we observe that

$$\begin{aligned}\|S(t)v_0 - v_0\|_p \leq& \|S(t)(v_0 - v_{0m})\|_p + \|S(t)v_{0m} - v_{0m}\|_p + \|v_{0m} - v_0\|_p \\ \leq& c\|v_0 - v_{0m}\|_p + \|S(t)v_{0m} - v_{0m}\|_p.\end{aligned}$$

Sending $t \downarrow 0$, we get

$$\overline{\lim_{t\downarrow 0}} \|S(t)v_0 - v_0\|_p \leq c\|v - v_{0m}\|_p,$$

since $S(t)v_{0m} \to v_{0m}$ in $\tilde{L}^p_\sigma$ as $t \downarrow 0$ by [FKS05], [FKS09]. Sending $m \to \infty$, we conclude that $S(t)v_0 \to v_0$ in L^p_σ as $t \downarrow 0$. $\square$

Remark 5.5.1. *In a similar way as we derived* (5.5.5) *and* (5.5.6) *we are able to derive from the L^∞-BMO estimates in [BG16] that*

$$t \left\|\nabla^2 S(t)\mathbf{Q}u : BMO_b^{\infty,\nu}(\Omega)\right\| \leq c\|u\|_\infty$$

$$t^{1/2} \left\|\nabla S(t)\mathbf{Q}u : BMO_b^{\infty,\nu}(\Omega)\right\| \leq c\|u\|_\infty$$

for all $u \in C_c(\Omega)$ and $t \in (0,1)$.

Note that L^2 results

$$t \left\|\nabla^2 S(t)\mathbf{Q}u\right\|_2 \leq c\|u\|_2$$

$$t^{1/2} \left\|\nabla S(t)\mathbf{Q}u\right\|_2 \leq c\|u\|_2$$

easily follow from the analyticity of $S(t)$ in L^2_σ and L^2-boundedness of $\mathbf{Q}$ if one observes that $\|\nabla u\|_2^2 = (Au, u)_{L^2}$ and

$$\|\nabla^2 u\|_2 \leq c\left(\|Au\|_2 + \|\nabla u\|_2 + \|u\|_2\right)$$

(see e.g. [Sol01, Chapter III, Theorem 2.1.1 (d)]), where A is the Stokes operator in L^2_σ.

Interpolating the L^2 results and the above L^∞-BMO results, we are able to prove that there is $c_p > 0$ satisfying

$$t \left\|\nabla^2 S(t)v_0\right\|_p \leq c_p\|v_0\|_p$$

$$t^{1/2} \left\|\nabla S(t)v_0\right\|_p \leq c_p\|v_0\|_p$$

for all $v_0 \in L^p_\sigma(\Omega)$ and $t \in (0,1)$ with $p \in (2,\infty)$.

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Chapter 6

Equivalence of BMO-type norms with applications to the heat semigroup

6.1 Introduction

In this article, we discuss equivalence norms of BMO type spaces in domains. Because we have to consider the behavior of functions near the boundary, our BMO norms consist of the interior and boundary parts. The reason we are interested in such problems is to prove analyticity of the heat and Stokes semigroups in domains.

$BMO(\mathbb{R}^n)$ space has previously been introduced by the seminal paper of John and Nirenberg [JN61]. Fefferman [Fef71] showed that $BMO(\mathbb{R}^n)$ is the dual of the Hardy space $H^1(\mathbb{R}^n)$ and a decomposition of functions in $BMO(\mathbb{R}^n)$ in terms of Riesz transforms. A constructive proof of the last result was given by Uchiyama [Uch82]. The theory of $BMO(\mathbb{R}^n)$ was developed in the remarkable paper of Fefferman and Stein [FS72]. BMO spaces play important roles in harmonic analysis and PDEs, as a substitute of L^∞. Several operators in them are not bounded on L^∞, but from L^∞ to BMO. Moreover, the real and complex interpolation theories can work with BMO space. For example, L^p coincides with interpolation spaces with BMO space, [Han77], [JJ82].

We already know ways to characterize functions in $BMO(\mathbb{R}^n)$. For instance, Carleson measures ([Car66], [FS72], [Ste93]), A_p weights ([GCRF85]) and Littlewood-Paley decomposition ([Tri83]). $BMO(\mathbb{R}^n)$ appeares in several problems in harmonic analysis; paraproduct [Bon81], commutator of singular integrals [CRW76], $T(1)$ theorem [DJ84], and in PDEs, especially in fluid dynamics; well-posedness for incompressible Navier-Stokes equations on the whole space [KT01] and a blow up criterion for the same equation [KT00].

If one considers BMO space in a domain Ω, the situation is less clear compared with the case of the whole space $\mathbb{R}^n$. To discuss possible definitions of BMO in a domains, we will define various type of BMO type (semi)norms. Some times, we have to be careful about the behavior near the boundary $\partial\Omega$. For the purpose we define for $f \in L^1_{loc}(\Omega)$, $\nu \in (0,\infty]$ and $p \in [1,\infty)$ the seminorm

$$[f]_{b^\nu p} := \sup \left\{ \left(r^{-n} \int_{\Omega \cap B_r(x)} |f(y)|^p \, dy \right)^{1/p} : x \in \partial\Omega, 0 < r < \nu \right\},$$

and for $\mu \in (0,\infty]$,

$$[f]_{BMO^\mu p} := \sup \left\{ \left(\frac{1}{|B_r(x)|} \int_{B_r(x)} |f(y) - f_{B_r(x)}|^p \, dy \right)^{1/p} : B_r(x) \subset \Omega, r < \mu \right\},$$

where for any ball $B \subset \mathbb{R}^n$,

$$f_B := \frac{1}{|B|} \int_B f(y) dy.$$

Then our BMO space is defined by the norm

$$\|f\|_{BMO_b^{\mu,\nu} p} := [f]_{BMO^\mu p} + [f]_{b^\nu p}.$$

If one replaces balls by cubes in the above definition one gets an equivalent seminorm. For a proof of this fact for general domains we refer to [Sta89]. We then let $BMO_b^{\mu,\nu}(\Omega)$ be the space of all functions $f \in L^1_{loc}(\Omega)$ satisfying $\|f\|_{BMO_b^{\mu,\nu}} < \infty$. Furthermore the space $VMO_{b,0}^{\mu,\nu}(\Omega)$ is defined as the closure of $C_c^\infty(\Omega)$ in $BMO_b^{\mu,\nu}(\Omega)$ and the solenoidal space $VMO_{b,0,\sigma}^{\mu,\nu}(\Omega)$ is defined as the closure of $C_{c,\sigma}^\infty(\Omega)$ in $BMO_b^{\mu,\nu}(\Omega)$. Similarly, $C_0(\Omega)$ and $C_{0,\sigma}(\Omega)$ are defined as the $L^\infty(\Omega)$-closure of $C_c^\infty(\Omega)$ and $C_{c,\sigma}^\infty(\Omega)$, respectively.

There exists a similar definition of the BMO_b-norm that was used by A. Miyachi in [Miy90]. We generalize his norm to $p \in [1,\infty)$ by

$$[f]_{BMO^M p} := \sup \left\{ \left(\frac{1}{|B_r(x)|} \int_{B_r(x)} |f - f_{B_r(x)}|^p \, dy \right)^{\frac{1}{p}} : B_{2r}(x) \subset \Omega \right\}$$

$$[f]_{b^M p} := \sup \left\{ \left(\frac{1}{|B_r(x)|} \int_{B_r(x)} |f| \, dy \right)^{\frac{1}{p}} : B_{2r}(x) \subset \Omega \text{ and } B_{5r}(x) \cap \Omega^c \neq \emptyset \right\}$$

and

$$\|f\|_{BMO_b^M p} := [f]_{BMO^M p} + [f]_{b^M p}.$$

For the case $p = 1$, we omit p in the definitions above.

Main results in this section consist of three type equivalences for $BMO_b^{\mu,\nu}p$ norms:

[I] equivalence for any $\mu,\ \nu \in (0,\infty]$ (Theorems 6.3.4, 6.3.5, 6.3.6, 6.3.7)

[II] equivalence for the power $p \in [1,\infty)$ (Theorems 6.3.12, 6.3.13)

[III] equivalence to $BMO_b^M p$ (Theorems 6.3.8, 6.3.9)

The main ingredients of the proofs of [I] and [III] are Jones' extension theorem (Theorem 6.2.1) and L^1 growth estimate for BMO functions (Theorem 6.3.2). The proof of [II] makes use of $L^1 - BMO$ interpolation in $\mathbb{R}^n$ (Lemma 6.3.3) and careful use of Jones' construction for his extension operator.

As it is mentioned above, some of our results make use of extension arguments. Although for any domains the extension of L^∞ functions by 0 does not cause problems, it is an interesting problem for BMO functions on domains. Jones [Jon80] gave a sufficient condition on domains for the existence of a bounded extension operator. Since his operator is needed in our aims, we recall its construction in the next section. But for some domains, the zero extension of BMO functions is useful, see Lemma 6.3.2. One can see that layer domains do not fulfill the Jones condition and have no extension operator, see Remark 6.2.1.

As an application we shall prove that the heat semigroup is bounded and analytic in BMO type spaces in $\mathbb{R}^n$ and $\mathbb{R}^n_+$ with the Dirichlet boundary condition:

$$\begin{cases} u_t - \Delta u = 0 & \text{in } \Omega \times (0,T) \\ u = 0 & \text{in } \partial\Omega \times (0,T) \\ u(0) = u_0. \end{cases}$$

We obtain the BMO^∞ estimate for the heat semigroup in $\mathbb{R}^n$ by a duality argument (Theorem 6.4.1), since $BMO^\infty(\mathbb{R}^n)^* = \mathcal{H}^1(\mathbb{R}^n)$. We also obtain the $BMO_b^{\infty,\infty}$ estimate in $\mathbb{R}^n_+$ by an odd extension argument (Theorem 6.4.2). Moreover, we shall prove that the heat semigroup is analytic in $BMO_b^{\mu,\nu}$ in a bounded C^3-domain for ν sufficiently small (Theorem 6.4.4). The $BMO_b^{\mu,\nu}$ estimate is derived from a gradient L^∞-BMO type estimate (Theorem 6.4.3), and the equivalence of BMO type spaces. We will obtain the gradient L^∞-BMO type estimate by a contradiction argument based on a blow up method without explicit calculations of the heat semigroup. Before we apply the equivalence theorems, we need to compare the gradient L^∞ norm of the solution with some BMO type norms of the solution (Lemma 6.4.3). Lemma 6.4.3 contains four steps. The first step, Lemma 6.4.3 (1), is a pointwise mean value estimate of the solution with respect to the time valuables. Lemma 6.4.3 (1) is obtained by the fundamental theorem of calculus and the gradient L^∞-BMO type estimate. The second step, Lemma 6.4.3 (2), is a L^2 mean oscillation estimate similar to the Poincaré's inequality. Lemma 6.4.3 (2) is obtained from combinations of calculations by Lemma 6.4.3 (1), the gradient estimates, the John-Nirenberg inequality, and the Poincaré's inequality. The third step, Lemma 6.4.3 (3), is a direct consequence of the Poincaré's inequality. The final step, Lemma 6.4.3 (4), is the L^2 mean estimate up to the boundary. The calculations of (1), (2), (3) up to the boundary imply Lemma 6.4.3 (4). By Lemma 6.4.3, we can conclude that the $BMO_b^{\mu,\nu}2$ norm of the solution is bounded by the $BMO_b^{\mu,\nu}$ norm of the initial data and L^2 mean of the initial data up to the boundary. By the equivalence theorem we can deal with L^2 terms, and we can get Theorem 6.4.4.

Let us review literature concerning BMO type estimates of the heat equation. A. Carpio [Car96] and the second author, S. Matsui, Y. Shimizu [GMS99] established $\mathcal{H}^1$-L^1 estimates which by duality imply gradient L^∞-BMO estimates:

$$t^{\frac{1}{2}}\|\nabla G_t * u_0\|_{L^\infty(\mathbb{R}^n)} \leq C[u_0]_{BMO(\mathbb{R}^n)},$$

where G_t denotes the Gaussian kernel and $*$ the convolution. We remark that $\|G_t * u_0\|_{L^\infty(\mathbb{R}^n)}$ is not bounded by $[u_0]_{BMO(\mathbb{R}^n)}$ which can be seen by taking u_0 constant. Moreover, this gradient L^∞-BMO estimates cannot be generalized to the case when a domain has nonempty boundary under the Dirichlet condition since u may not be spatially constant even if u_0 is a constant. In [KOT03] and also in [Lemma 14.4.1, Oga13], $BMO(\mathbb{R}^n)$ estimates and gradient L^∞-BMO estimates for $e^{t\Delta}u_0$, $\nabla e^{t\Delta}u_0$, $\nabla^2 e^{t\Delta}u_0$ were established:

$$[\nabla^k e^{t\Delta}u_0]_{BMO(\mathbb{R}^n)} \leq Ct^{-\frac{k}{2}}[u_0]_{BMO(\mathbb{R}^n)} \text{ for } k = 0, 1, 2,$$

$$\|\nabla^k e^{t\Delta}u_0\|_{L^\infty(\mathbb{R}^n)} \leq Ct^{-\frac{k}{2}}[u_0]_{BMO(\mathbb{R}^n)} \text{ for } k = 1, 2.$$

$BMO(\mathbb{R}^n)$ estimates are obtained by $\mathcal{H}^p$-$\mathcal{H}^q$ estimates, see [HO10], [Miy96], [Miy98], and a duality argument. Gradient L^∞-BMO estimates are also obtained by a duality argument.

6.2 Jones' extension theorem

We will need to consider certain classes of domains in order to compare different BMO-type norms or to prove embeddings from BMO-type spaces to L^p. For the existence of an extension operator on $BMO^\infty(\Omega)$ we will need the notion of a uniform domain. In some cases we will also need C^2-boundary to get control over the ratio $|B_r(x_0)|/|B_r(x_0) \cap \Omega|$ for small r and $x_0 \in \partial\Omega$. Both properties are crucial in several proofs.

Lemma 6.2.1. *Let Ω be a uniformly C^2-domain. Then there exists a constant R depending only on C^2-regularity of Ω such that there is a projection $P_{\partial\Omega} : \{x \in \mathbb{R}^n : dist(x, \partial\Omega) < R\} \to \partial\Omega$ to the boundary with $P_{\partial\Omega}x - x = dist(x, \partial\Omega)\mathbf{n}$, where $\mathbf{n}$ is the exterior normal of $\partial\Omega$ in $P_{\partial\Omega}x$ if $x \in \Omega$ and the interior normal of $\partial\Omega$ in $P_{\partial\Omega}x$ if $x \notin \Omega$.*

Proof. For a proof see [GT77, appendix] and [KP02, §4.4]. □

We define then for a C^2-domain the reach of Ω denoted by $R^* > 0$ to be the supremum of all R as in the above Lemma. The reach of Ω then depends only on C^2-regularity of Ω.

For several equivalence proofs we will need an extension theorem for BMO-functions on domains that is due to P. W. Jones ([Jon80]). Since the construction of this extension will be important for our needs, we will give a sketch of this construction. In order to do so we need to define the dyadic Whitney decomposition of a set A.

For a set $A \subset \mathbb{R}^n$ let $\mathcal{A} = \{Q_j\}_{j\in\mathbb{N}}$ be a set of dyadic closed cubes with side length $\ell(Q_j)$ contained in A such that

1. $A = \cup_j Q_j$
2. $\mathring{Q}_j \cap \mathring{Q}_k = \emptyset$ if $j \neq k$
3. $1 \leq \frac{d(Q_j, \mathbb{R}^n \setminus A)}{\ell(Q_j)} \leq 4\sqrt{n}$ $(j \in \mathbb{N})$
4. $\frac{1}{4} \leq \frac{\ell(Q_j)}{\ell(Q_j)} \leq 4$ if $Q_j \cap Q_k \neq \emptyset$.

Then $\mathcal{A}$ will be called a dyadic Whitney decomposition of A. For the existence of the Whitney decomposition for open setes we refer to Chapter VI, Theorem 1 in [Ste70].

We define two different distance functions on the Whitney decomposition. For $Q_j, Q_k \in \mathcal{A}$ we call $Q_j = Q(0) \to Q(1) \to Q(2) \ldots \to Q(m) = Q_k$ a Whitney chain of length m connecting Q_j and Q_k if $Q(l) \in \mathcal{A}$ and $Q(l) \cap Q(l+1) \neq \emptyset$ for each $0 \leq l \leq m-1$. The distance function $d_1(Q_j, Q_k)$ will then be defined as the length of the shortest Whitney chain connecting Q_j and Q_k.

For $Q_j, Q_k \in \mathcal{A}$ we define the second distance function as

$$d_2(Q_j, Q_k) := \log\left|\frac{\ell(Q_j)}{\ell(Q_k)}\right| + \log\left|\frac{d(Q_j, Q_k)}{\ell(Q_j) + \ell(Q_k)} + 2\right|,$$

where d denotes the Euclidean distance between the cubes. Note that d_1 and d_2 are scaling invariant.

A domain $A \subset \mathbb{R}^n$ will be called a uniform domain if there is some $K > 0$ such that

$$d_1(Q_j, Q_k) \leq K d_2(Q_j, Q_k) \tag{6.2.1}$$

for all $Q_j, Q_k \in \mathcal{A}$ and some dyadic Whitney decomposition $\mathcal{A}$. The name uniform is due to the following equivalent definition of this class of domains ([GO79]). A domain Ω is uniform if there exist constants $a, b > 0$ such that for all $x, y \in \Omega$ there is a rectifiable curve $\gamma \subset \Omega$ of length $s(\gamma) \leq a|x-y|$ with $\min\{s(\gamma(x,z)), s(\gamma(y,z))\} \leq b\ dist(z, \partial\Omega)$, where $\gamma(x,z)$ denotes the part of γ between x and z. Bounded Lipschitz domains are examples of uniform domains.

We are now able to formulate the extension theorem for BMO-functions.

Theorem 6.2.1. *Let $A \subset \mathbb{R}^n$ be a uniform domain. Then there is a constant $C(K)$ such that for each $f \in BMO^\infty(A)$ there is an extension $\bar{f} \in BMO^\infty(\mathbb{R}^n)$ such that*

$$[\bar{f}]_{BMO^\infty(\mathbb{R}^n)} \leq C(K)[f]_{BMO^\infty(A)}. \tag{6.2.2}$$

In particular, the theorem holds for bounded Lipschitz domains with a constant only depending on the Lipschitz regularity of A.

Proof. The theorem is due to [Jon80]. □

We will repeat the explicit construction of $\bar{f}$. Let A^c be the complement of A and $\mathcal{A}'$ be the Whitney decomposition of its interior. Choose for every $Q'_j \in \mathcal{A}'$ a corresponding $Q_j \in \mathcal{A}$ in the following way. If there are cubes $Q_j \in \mathcal{A}$ which satisfy $\ell(Q_j) \geq \ell(Q'_j)$ then choose the nearest cube $Q_j \in \mathcal{A}$ satisfying $\ell(Q_j) \geq \ell(Q'_j)$. For all other cubes choose some largest cube $Q_0 \in \mathcal{A}$ and let Q_0 be the cube corresponding to all $Q'_j \in \mathcal{A}'$ for which there are no cubes in $Q_j \in \mathcal{A}$ satisfying $\ell(Q_j) \geq \ell(Q'_j)$. The second case appears for example if A is a bounded domain. Then $\bar{f}$ is defined as

$$\bar{f}(x) := \begin{cases} f(x) & : x \in A \\ f_{Q_j} & : x \in Q'_j \end{cases} ,$$

where $Q_j \in A$ is the cube corresponding to Q'_j. Since by Corollary 2.9 of [Jon80] $|\partial\Omega| = 0$ for uniform domains, we can ignore the boundary of Ω in the construction.

Furthermore, we will need the following Lemma, compare [Jon80], Lemma 2.10.

Lemma 6.2.2. *Let $A \subset \mathbb{R}^n$ be a uniform domain, $\mathcal{A}$ and $\mathcal{A}'$ the Whitney decomposition of A and A^c respectively and let $Q'_j \in \mathcal{A}'$. If there exists a cube $Q \in \mathcal{A}$ with $\ell(Q) \geq \ell(Q'_j)$, then*

$$d(Q_j, Q'_j) \leq 65K^2 \ell(Q'_j) \leq 65K^2 \ell(Q_j) \tag{6.2.3}$$

with K the number obtained in condition (6.2.2) and Q_j the cube corresponding to Q'_j.

Remark 6.2.1. *Domains of the form $\mathbb{R}^k \times \Omega$ with $1 \leq k \leq n-1$ and $\Omega \subset \mathbb{R}^{n-k}$ bounded are examples of domains which are not uniform. We will show that for such domains Jones' extension theorem does not hold. Let $f(x) = x_1$, then for every cube Q in Ω*

$$\begin{aligned} \frac{1}{|Q|} \int_Q |f - f_Q| \, dy &= \frac{\ell(Q)^{n-1}}{|Q|} \int_{-\ell(Q)/2}^{\ell(Q)/2} |x_1| \, dx_1 \\ &= \frac{1}{4} \ell(Q). \end{aligned}$$

Thus $f \in BMO^\infty(\Omega)$ because the cubes in Ω have sidelength of at most $diam(\Omega)$ since Ω is up to translation contained in $\mathbb{R}^k \times (0, diam(\Omega))^{n-k}$. This function cannot be extended to a function $\bar{f} \in BMO^\infty(\mathbb{R}^n)$ since otherwise $BMO^\infty(\mathbb{R}^n)$ would contain functions of linear growth.

6.3 Embeddings and equivalences of BMO-type norms

Theorem 6.3.1. *Let $\Omega \subset \mathbb{R}^n$ be a domain and $\mu, \nu \in (0, \infty]$. Then the embeddings*

$$L^\infty(\Omega) \hookrightarrow BMO_b^{\mu,\nu}(\Omega), \tag{6.3.1}$$

$$C_0(\Omega) \hookrightarrow VMO_{b,0}^{\mu,\nu}(\Omega), \tag{6.3.2}$$

$$C_{0,\sigma}(\Omega) \hookrightarrow VMO_{b,0,\sigma}^{\mu,\nu}(\Omega) \tag{6.3.3}$$

hold with a constant only depending on n, i.e. independent of Ω, μ and ν.

Proof. It follows immediately from the definition of the norm that $\|f\|_{BMO_b^{\mu,\nu}} \leq (2 + \omega_n)\|f\|_\infty$, where $\omega_n = |B_1(0)|$ is the measure of the unit ball in $\mathbb{R}^n$. □

Remark 6.3.1. *It follows immediately from the definition that for $0 < \mu_1 \leq \mu_2 \leq \infty$ and $0 < \nu_1 \leq \nu_2 \leq \infty$ the estimates*

$$[f]_{BMO^{\mu_1}} \leq [f]_{BMO^{\mu_2}}, \quad [f]_{b^{\nu_1}} \leq [f]_{b^{\nu_2}}$$

and the embedding

$$BMO_b^{\mu_1,\nu_1}(\Omega) \hookrightarrow BMO_b^{\mu_2,\nu_2}(\Omega)$$

hold.

Theorem 6.3.2. *Let $\mu \in (0, \infty]$ and $\Omega \subset \mathbb{R}^n$ be a domain. Then for all $f \in BMO^\mu(\Omega)$, $a > 1$, $r > 0$, $x_1, x_2 \in \Omega$ with $B_r(x_1) \subset B_{ar}(x_2) \subset \Omega$ and $ar < \mu$ holds the inequality*

$$\|f\|_{L^1(B_{ar}(x_2))} \leq |B_{ar}(x_2)|(1 + a^n)[f]_{BMO^\mu(\Omega)} + a^n\|f\|_{L^1(B_r(x_1))}. \tag{6.3.4}$$

The same statement holds for cubes in Ω of side length r and ar, respectively.

Proof. Let $\tilde{f} := f - f_{B_r(x_1)}$. By $\int_{B_r(x_1)}(\tilde{f} - \tilde{f}_{B_{ar}(x_2)})\,dy = -|B_r(x_1)|\tilde{f}_{B_{ar}(x_2)}$ we obtain

$$|B_r(x_1)||\tilde{f}_{B_{ar}(x_2)}| \leq \int_{B_r(x_1)} |\tilde{f} - \tilde{f}_{B_{ar}(x_2)}|\,dy$$

and thus

$$\begin{aligned} |B_{ar}(x_2)|[\tilde{f}]^\mu_{BMO} &\geq \int_{B_{ar}(x_2)} |\tilde{f} - \tilde{f}_{B_{ar}(x_2)}|\,dy \\ &\geq |B_r(x_1)||\tilde{f}_{B_{ar}(x_2)}| \end{aligned}$$

which can be rewritten as

$$|\tilde{f}_{B_{ar}(x_2)}| \leq a^n[\tilde{f}]_{BMO^\mu}. \tag{6.3.5}$$

Then we are able to estimate

$$\begin{aligned}
&\|f\|_{L^1(B_{ar}(x_2))}\\
\leq&\|f - f_{B_r(x_1)}\|_{L^1(B_{ar}(x_2))} + |B_{ar}(x_2)||f_{B_r(x_1)}|\\
=&\|\tilde{f}\|_{L^1(B_{ar}(x_2))} + |B_{ar}(x_2)||f_{B_r(x_1)}|\\
\leq&\|\tilde{f} - \tilde{f}_{B_{ar}(x_2)}\|_{L^1(B_{ar}(x_2))} + |B_{ar}(x_2)||\tilde{f}_{B_{ar}(x_2)}| + \frac{|B_{ar}(x_2)|}{|B_r(x_1)|}\|f\|_{L^1(B_r(x_1))}\\
\leq&|B_{ar}(x_2)|[\tilde{f}]_{BMO^\mu} + |B_{ar}(x_2)|a^n[\tilde{f}]_{BMO^\mu} + a^n\|f\|_{L^1(B_r(x_1))}.\\
=&|B_{ar}(x_2)|(1 + a^n)[f]_{BMO^\mu} + a^n\|f\|_{L^1(B_r(x_1))}.
\end{aligned}$$

□

Theorem 6.3.3. *Let $\Omega \subset \mathbb{R}^n$ be an arbitrary domain. Let $0 < \mu_1 < \mu_2 < \infty$. Then the seminorms $[\cdot]_{BMO^{\mu_1}}$ and $[\cdot]_{BMO^{\mu_2}}$ are equivalent.*

Proof. We prove this theorem by using cubes instead of balls. Let $Q_r(x)$ be a cube of sidelength $r < \mu_1$ centered at x. We will prove that the BMO-seminorm in $Q_{2r}(x)$ is controlled by the BMO^{μ_1}-seminorm and a constant only depending on the dimension n provided that $Q_{2r}(x) \subset \Omega$. By iteration and Remark 6.3.1 we then get the stated result. Divide $Q_{2r}(x)$ into 2^n cubes Q_i of sidelength r with disjoint interior such that each cube has one corner in x.

Assume without loss of generality that $f_{Q_r(x)} = 0$. Then

$$\|f\|_{L^1(Q_r(x))} \leq [f]_{BMO^{\mu_1}}|Q_r(x)|.$$

By using Theorem 6.3.2

$$\begin{aligned}
&\frac{1}{|Q_{2r}(x)|}\int_{Q_{2r}(x)}|f - f_{Q_{2r}(x)}|\,dy\\
\leq&\frac{2}{|Q_{2r}(x)|}\|f\|_{L^1(Q_{2r}(x))}\\
\leq&\frac{2}{|Q_{2r}(x)|}\sum_{i=1}^{2^n}\|f\|_{L^1(Q_i)}\\
\leq&\frac{2}{|Q_{2r}(x)|}\sum_{i=1}^{2^n}(|Q_i|(1+2^n)[f]_{BMO^{\mu_1}} + 2^n\|f\|_{L^1(Q_i\cap Q_r(x))})\\
\leq&\frac{2}{|Q_{2r}(x)|}((1+2^n)[f]_{BMO^{\mu_1}}|Q_{2r}(x)| + 2^{n+1}|Q_r(x)|[f]_{BMO^{\mu_1}})\\
\leq&2(1 + 2\cdot 2^n)[f]_{BMO^{\mu_1}}
\end{aligned}$$

and thus

$$[f]_{BMO^{2\mu_1}} \leq 2(1 + 2^{n+1})[f]_{BMO^{\mu_1}}.$$

□

Lemma 6.3.1. *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain, $\mu, \nu \in (1, \infty]$. Then there exists a constant only depending on n, μ, ν and Ω such that for all $f \in BMO_b^{\mu,\nu}(\Omega)$*

$$\|f\|_{L^1(\Omega)} \leq c\|f\|_{BMO_b^{\mu,\nu}(\Omega)}.$$

Proof. Let $(B_i)_{i\in I}$ be a cover of $\overline{\Omega}$ consisting of balls $B_r(x) \subset \Omega$ with $r < \mu$ and balls $B_r(x)$ with $x \in \partial\Omega$ and $r < \nu$. Then there is a finite subcover of $\overline{\Omega}$ of balls $(B_i)_{1\leq i\leq N}$. This subcover contains at least one ball centered at some point on the boundary. Since there are only finitely many balls in the subcover the number

$$r_0 := \min_{B_i\cap B_j\cap\Omega\neq\emptyset} \ \sup_{B_r(x)\subset B_i\cap B_j\cap\Omega} r$$

exists and is positive. For the balls centered at the boundary we can estimate $\|f\|_{L^1(B_i\cap\Omega)} \leq |B_i|[f]_{b^\nu}$ and for all neighbouring balls B_j that are contained in Ω there is a ball $B_{r_0}^{i,j} \subset B_i \cap B_j$ of radius r_0 with $\|f\|_{L^1(B_{r_0}^{i,j})} \leq |B_i|[f]_{b^\nu}$. By Theorem 6.3.2 we obtain then for the neighbouring balls the estimate

$$\|f\|_{L^1(B_j)} \leq |B_j|(1 + (\frac{\mu}{r_0})^n)[f]_{BMO^\mu} + (\frac{\mu}{r_0})^n|B_i|[f]_{b^\nu} \leq c\|f\|_{BMO_b^{\mu,\nu}}$$

and can continue this strategy until we estimated $\|f\|_{L^1(B_j)}$ on all balls $B_j \subset \Omega$. Thus

$$\|f\|_{L^1(\Omega)} \leq c\sum_{i=1}^N \|f\|_{L^1(B_i)} \leq c\|f\|_{BMO_b^{\mu,\nu}}$$

with some constant c depending only on n and the subcover $(B_i)_{1\leq i\leq N}$, i.e. depending only on n, μ, ν and Ω. □

Theorem 6.3.4. *Let Ω be a bounded domain and $\mu_1, \mu_2, \nu_1, \nu_2 \in (0, \infty]$. Then the norms $\|\cdot\|_{BMO_b^{\mu_1,\nu_1}}$ and $\|\cdot\|_{BMO_b^{\mu_2,\nu_2}}$ are equivalent.*

Proof. Assume that $\nu_1 < \nu_2$. By the boundedness of Ω we have that $[f]_{BMO^\infty}$ is equal to $[f]_{BMO^{diam(\Omega)}}$ such that we can assume that μ_1 and μ_2 are finite. By Theorem 6.3.3 we obtain the equivalence of $[f]_{BMO^{\mu_1}}$ and $[f]_{BMO^{\mu_2}}$. For $\nu_1 \leq r < \nu_2$ and $x_0 \in \partial\Omega$ we obtain by the inequality $\|f\|_{L^1(\Omega)} \leq c\|f\|_{BMO_b^{\mu_1,\nu_1}}$ of Lemma 6.3.1 the estimate

$$\begin{aligned}\frac{1}{r^n}\int_{B_r(x_0)\cap\Omega} |f(y)|\, dy &\leq \nu_1^{-n}\|f\|_{L^1(\Omega)} \\ &\leq c\|f\|_{BMO_b^{\mu_1,\nu_1}}\end{aligned}$$

which completes the proof. □

Example 6.3.1. *For the unbounded domain $\mathbb{R}_+ = (0, \infty)$ we will give some examples that the BMO_b-norms may differ for different values of μ or ν. For domains which contain arbitrarily large balls similar examples give that the spaces $BMO_b^{\mu,\nu}(\Omega), BMO_b^{\infty,\nu}(\Omega)$ and $BMO_b^{\infty,\infty}(\Omega)$ are different because they allow different kinds of growth at infinity.*

- *Let* $f_1(x) = x$. *Then* $[f_1]_{b^\nu} = \frac{1}{2}\nu$ *and* $[f_1]_{BMO^\mu} = \frac{\mu}{4}$. *This gives us that* $f_1 \in BMO_b^{\mu,\nu}(\mathbb{R}_+)$ *for* $\mu, \nu < \infty$ *but* $f_1 \notin BMO_b^{\mu,\nu}(\mathbb{R}_+)$ *if* $\mu = \infty$ *or* $\nu = \infty$.

- *Let* $f_2(x) = \log(x+1)$. *It is well known that* $[f_2]_{BMO^\infty} < \infty$, *thus* $[f_2]_{BMO^\mu} < \infty$ *for all* $\mu \in (0,\infty]$. *Furthermore,* $[f_2]_{b^\nu} = \frac{1}{\nu}\int_1^{\nu+1} \log(x)\,dx = \log(\nu+1) + \frac{\log(\nu+1)}{\nu} - 1$. *Thus* $f_2 \in BMO^{\infty,\nu}(\mathbb{R}_+)$ *for* $\nu \in (0,\infty)$ *but* $f_2 \notin BMO^{\infty,\infty}(\mathbb{R}_+)$.

- *Let*

$$f_3(x) := \begin{cases} x - 2^n & : x \in [2^n, 2^n + \frac{1}{4}2^{\frac{n}{2}})\ (n \in \mathbb{N}_0) \\ 2^n + \frac{1}{2}2^{\frac{n}{2}} - x & : x \in [2^n + \frac{1}{4}2^{\frac{n}{2}}, 2^n + \frac{1}{2}2^{\frac{n}{2}})\ (n \in \mathbb{N}_0) \\ 0 & : \textit{otherwise.} \end{cases}$$

  Then $[f_3]_{BMO^\mu(\mathbb{R}_+)} \le \mu$ *and* $[f_3]_{b^\infty(\mathbb{R}_+)} \le \sup_{n\in\mathbb{N}_0} \frac{1}{2^n}\int_0^{2^{n+1}} f_3(y)\,dy \le \frac{1}{8}$ *follow from a direct calculation. Thus* $f_3 \in BMO_b^{\mu,\infty}(\mathbb{R}_+)$ *for* $\mu < \infty$ *but* $f_3 \notin BMO_b^{\infty,\infty}(\mathbb{R}_+)$ *which can be seen by calculating the bounded mean oscillation in every interval* $(2^n, 2^n + \frac{1}{2}2^{\frac{n}{2}})$.

Theorem 6.3.5. *Let* $\Omega \subset \mathbb{R}^n$ *be a uniform domain with uniformly* C^2*-boundary and let* $0 < \nu_1 < \nu_2 < \infty$. *Then the norms* $\|\cdot\|_{BMO_b^{\infty,\nu_1}}$ *and* $\|\cdot\|_{BMO_b^{\infty,\nu_2}}$ *are equivalent.*

Proof. We extend f by Theorem 6.2.1 to $\bar{f} \in BMO^\infty(\mathbb{R}^n)$. For $\nu_0 := \min\{\frac{\nu_1}{8}, \frac{R^*}{8}\}$ and $\tilde{x}$ such that $B_{\nu_0}(\tilde{x}) \subset B_{\frac{r}{2}}(x_0) \cap \Omega$ for $\nu_1 \le r < \nu_2$ and $x_0 \in \partial\Omega$ we obtain by Theorem 6.3.2

$$\begin{aligned} &\sup_{\nu_1 \le r < \nu_2, x_0 \in \partial\Omega} \frac{1}{r^n}\int_{B_r(x_0)\cap\Omega} |f|\,dy \\ \le &\sup_{\nu_1 \le r < \nu_2, x_0 \in \partial\Omega} \frac{1}{r^n}\int_{B_r(x_0)} |\bar{f}|\,dy \\ \le &\sup_{\nu_1 \le r < \nu_2, x_0 \in \partial\Omega} \frac{c}{r^n}|B_r(x_0)|[\bar{f}]_{BMO^\infty(\mathbb{R}^n)} + \frac{c}{r^n}\int_{B_{\nu_0}(\tilde{x})} |f|\,dy \\ \le &c[f]_{BMO^\infty(\Omega)} + \frac{c}{\nu_1^n}\int_{B_{\frac{\nu_1}{2}}(\tilde{x})} |f|\,dy \\ \le &c\|f\|_{BMO^{\infty,\nu_1}} \end{aligned}$$

and for $r < \nu_1$ follows the estimate directly from the definition. □

Theorem 6.3.6. *Let* Ω *be a uniformly* C^2*-domain,* $\mu \in (0,\infty]$, $\nu_1, \nu_2 \in (0, R^*]$ *and* $\nu_1 < \nu_2 < \infty$. *Then* $\|\cdot\|_{BMO_b^{\mu,\nu_1}}$ *and* $\|\cdot\|_{BMO_b^{\mu,\nu_2}}$ *are equivalent.*

Proof. By Theorem 6.3.3 and Remark 6.3.1 we can assume without loss of generality that $2\nu_1 = \nu_2 < \mu$. Each $B_r(x) \cap \Omega$ with $x_0 \in \partial\Omega$ and $\nu_1 \le r < \nu_2$ is a Lipschitz-domain with uniform Lipschitz-regularity, where $[f]_{BMO^\mu(B_r(x)\cap\Omega)}$

equals $[f]_{BMO^\infty(B_r(x)\cap\Omega)}$. Furthermore every $B_r(x)\cap\Omega$ contains a ball $B_{\nu_1/4}(x_1)$ such that there is $x_0 \in \partial\Omega$ and $r_0 < \nu_1$ with $B_{\nu_1/4}(x_1) \subset B_{r_0}(x_0)$. By Theorem 6.2.1 we obtain for every $\Omega \cap B_r(x)$ an extension $\bar{f}$ of f such that

$$\begin{aligned}\|f\|_{L^1(B_r(x)\cap\Omega)} &\leq \|\bar{f}\|_{L^1(B_r(x))} \\ &\leq c|B_r(x)|(1+8^n)[f]_{BMO^\mu} + 8^n\|f\|_{L^1(B_{\nu_1/4}(x_1))}\end{aligned}$$

with a uniform constant c since we have control on the Lipschitz regularity of $B_r(x_0) \cap \Omega$. Thus

$$[f]_{b^{\nu_2}} \leq c\|f\|_{BMO_b^{\mu,\nu_1}}.$$

□

Theorem 6.3.7. *Let $\Omega := G \times \mathbb{R}^{n-k}$, where $G \subset \mathbb{R}^k$ is a bounded Lipschitz domain and $1 \leq k \leq n-1$. Let $\mu_1, \mu_2, \nu_1, \nu_2 \in (0,\infty]$. Then $\|\cdot\|_{BMO_b^{\mu_1,\nu_1}}$ and $\|\cdot\|_{BMO_b^{\mu_2,\nu_2}}$ are equivalent.*

Proof. Let $\delta := diam(G)$. The seminorms $[\cdot]_{BMO^{\mu_1}}$ and $[\cdot]_{BMO^{\mu_2}}$ are equivalent by $[\cdot]_{BMO^\delta} = [\cdot]_{BMO^\infty}$ for $\mu \geq \delta$ and Theorem 6.3.3. We can assume without loss of generality that $\nu_1 < \nu_2$. Let $\{\Omega_i\}_{i\in\mathbb{Z}^{n-k}}$ be the collection of domains

$$G \times (i_{k+1}\delta, (i_{k+1}+1)\delta) \times \cdots \times (i_n\delta, (i_n+1)\delta)$$

with $i \in \mathbb{Z}^{n-k}$ such that Ω is the interior of the closure of the disjoint union of all Ω_i. Each Ω_i is then just the translation of the bounded Lipschitz domain Ω_0. Since $\partial\Omega_i \cap \partial\Omega \neq \emptyset$ for every $i \in \mathbb{Z}^{n-k}$ we obtain by a similar argumentation as in the proof of Theorem 6.3.1 that there is a constant C depending on ν_1, μ_1 and n and the shape of Ω_0 but independent of i such that

$$\|f\|_{L^1(\Omega_i)} \leq C\|f\|_{BMO_b^{\mu_1,\nu_1}(\Omega)}.$$

The number of Ω_i for which $\Omega_i \cap (B_r(x) \cap \Omega) \neq \emptyset$ is at most $(\frac{2r+2\delta}{\delta})^{n-k}$ such that we can estimate for $\nu_1 \leq r < \nu_2$ (where $\nu_2 = \infty$ is allowed)

$$\begin{aligned}\frac{1}{r^n}\int_{B_r(x)\cap\Omega} |f(y)|\,dy &\leq \frac{1}{r^n}\sum_{\Omega_i\cap(B_r(x)\cap\Omega)\neq\emptyset} \|f\|_{L^1(\Omega_i)} \\ &\leq \frac{(2r+2\delta)^{n-k}}{\delta^{n-k}r^n}C\|f\|_{BMO_b^{\mu_1,\nu_1}} \\ &\leq C(\mu_1,\nu_1,n,\delta)\|f\|_{BMO_b^{\mu_1,\nu_1}}\end{aligned}$$

and thus

$$\|f\|_{BMO_b^{\mu_2,\nu_2}} \leq C(\mu_1,\mu_2,\nu_1,n,\delta)\|f\|_{BMO_b^{\mu_1,\nu_1}}$$

which was left to prove. □

We have shown that Jones' extension theorem does not hold for layer domains and other domains of the form $G \times \mathbb{R}^{n-k}$, where G is bounded. Nevertheless, by the introduction of the BMO_b-norms which do not allow the linear growth of f as in Remark 6.2.1 we can construct a simple extension operator for BMO_b-functions.

Lemma 6.3.2. *Let $\Omega := G \times \mathbb{R}^{n-k}$ with $G \subset \mathbb{R}^k$ a bounded Lipschitz domain and $\mu, \nu \in (0, \infty]$. Then there is a constant C depending only on n, Ω, μ, ν such that for each $f \in BMO_b^{\mu,\nu}(\Omega)$ the extension by 0 which we will denote by $\bar{f} \in BMO^\infty(\mathbb{R}^n)$ satisfies*

$$[\bar{f}]_{BMO^\infty(\mathbb{R}^n)} \leq C \|f\|_{BMO_b^{\mu,\nu}(\Omega)}.$$

Proof. By Theorem 6.3.7 we can assume that $\mu = \nu = \infty$. It is immediate by construction that if $B \subset \Omega$, then $\frac{1}{|B|} \int_B |\bar{f}(y) - \bar{f}_B| \, dy \leq [f]_{BMO^\infty(\Omega)}$ and that for $B \subset \Omega^c$, $\frac{1}{|B|} \int_B |\bar{f}(y) - \bar{f}_B| \, dy = 0$. Thus it is only left to estimate the mean oscillation in balls which have nonempty intersection with the boundary. For each $B_r(x)$ which satisfies $B_r(x) \cap \partial\Omega \neq \emptyset$ we take $x_0 \in B_r(x) \cap \partial\Omega$, then $B_r(x) \subset B_{2r}(x_0)$ and we have

$$\frac{1}{|B_r(x)|} \int_{B_r(x)} |\bar{f}(y) - \bar{f}_{B_r(x)}| \, dy \leq \frac{2}{|B_r(x)|} \int_{B_{2r}(x_0)} |\bar{f}(y)| \, dy \leq \frac{2^{n+1}}{\omega_n} [f]_{b^\infty}.$$

□

Theorem 6.3.8. *Miyachi's definition of the BMO_b-norm $\|\cdot\|_{BMO_b^M}$ is equivalent to $\|\cdot\|_{BMO_b^{\mu,\nu}}$ for $\mu, \nu \in (0, \infty]$ if Ω is a bounded Lipschitz domain.*

Proof. The seminorms $[f]_{BMO^M}$ and $[f]_{BMO^\infty}$ are equivalent by [Sta89, Corollary 2.26].

For $x \in \Omega$ and $r > 0$ with $B_{2r}(x) \subset \Omega$ and $B_{5r}(x) \cap \Omega^c \neq \emptyset$ let $x_0 \in \partial\Omega \cap B_{5r}(x)$. Then $B_r(x) \subset B_{6r}(x_0) \cap \Omega$ and

$$\begin{aligned} \frac{1}{|B_r(x)|} \int_{B_r(x)} |f| \, dy &\leq \frac{1}{|B_r(x)|} \int_{B_{6r}(x_0) \cap \Omega} |f| \, dy \\ &\leq 6^n \omega_n^{-1} [f]_{b^\infty}. \end{aligned}$$

We have now proved that

$$\|f\|_{BMO_b^M} \leq C \|f\|_{BMO_b^{\infty,\infty}} \leq C \|f\|_{BMO_b^{\mu,\nu}},$$

where the last inequality follows from Theorem 6.3.4.

It is now just left to estimate $[\cdot]_{b^\nu}$ by Miyachi's norm. First note that $f \in BMO_b^{\mu,\nu}(\Omega) = BMO_b^{\infty,\infty}(\Omega)$ can be extended by 0 to $\bar{f} \in BMO^\infty(\mathbb{R}^n)$ with $[\bar{f}]_{BMO^\infty(\mathbb{R}^n)} \leq C \|f\|_{BMO_b^{\mu,\nu}}$ by the same argument as in the proof of Lemma 6.3.2. Since Ω is a bounded Lipschitz domain there exists a finite cone K of height h and angle θ with vertex 0 such that for every $x_0 \in \partial\Omega$ there exists a rotation R_{x_0} such that the cone $x_0 + K_{x_0} := x_0 + R_{x_0} K$ is contained in Ω. By

Theorem 6.3.4 we can assume that $\nu < h$. Then there is a constant $0 < c_\theta < 1$ such that for all $x_0 \in \partial\Omega$ and $0 < r < \nu$ there is a ball of radius $c_\theta r$ with center $x \in x_0 + K_{x_0}$ such that $B_{2c_\theta r}(x) \subset B_r(x_0) \cap (x_0 + K_{x_0}) \subset B_r(x_0) \cap \Omega$. We choose then a possibly larger ball $B_{r_M}(x)$ with radius $r_M \geq c_\theta r$ such that $B_{c_\theta r}(x) \subset B_{r_M}(x)$, $B_{2r_M}(x) \subset \Omega$ and $B_{5r_M}(x) \cap \Omega^c \neq \emptyset$. Then by Theorem 6.3.2

$$\begin{aligned}\frac{1}{r^n}\int_{B_r(x_0)\cap\Omega} |f|\,dy &\leq \frac{1}{r^n}\|\bar{f}\|_{L^1(B_r(x_0))}\\ &\leq \frac{1}{r^n}(|B_r(x)|(1+\frac{1}{c_\theta^n})[\bar{f}]_{BMO^\infty(B_r(x))} + \frac{1}{c_\theta^n}\|f\|_{L^1(B_{r_M}(x))})\\ &\leq C([f]_{BMO^\infty(\Omega)} + [f]_{b^M})\\ &\leq C\|f\|_{BMO_b^M}.\end{aligned}$$

□

Remark 6.3.2. *If we consider general domains, f_2 of Example 6.3.1 illustrates that in general $BMO_b^{\mu,\nu}$ may only correspond to the Miyachi norm if $\mu = \nu = \infty$ or if $BMO_b^{\mu,\nu}$ and $BMO_b^{\infty,\infty}$ are equivalent. It is easy to see that $f_2 \notin BMO_b^M(\mathbb{R}_+)$.*

Theorem 6.3.9. *Let $\Omega \subset \mathbb{R}^n$ be a Lipschitz half-space, i.e. a domain lying above the graph of some Lipschitz function. Then $\|\cdot\|_{BMO_b^M}$ and $\|\cdot\|_{BMO_b^{\infty,\infty}}$ are equivalent.*

Proof. By the same arguments as in the proof of Theorem 6.3.8 one obtains that the BMO-seminorms are equivalent and that $[f]_{b^M} \leq C[f]_{b^\infty}$. It is now left to prove that for all $f \in BMO_b^{\infty,\infty}(\Omega)$ the estimate $[f]_{b^\infty} \leq C\|f\|_{BMO_b^M}$ holds. This is done similarly to the argument of Theorem 6.3.8. At first we see that we can extend $f \in BMO_b^{\infty,\infty}$ to a BMO^∞-function $\bar{f}$ defined on $\mathbb{R}^n$. Since Ω is a Lipschitz half-space, there exists an infinite cone K of angle θ such that for all $x_0 \in \partial\Omega$ the relation $x_0 + K \subset \Omega$ holds. Then there exists a constant c_θ such that for all $x_0 \in \partial\Omega$ and $r > 0$ there exists a ball $B_{r_M}(x)$ such that $B_{2r_M}(x) \subset \Omega$, $B_{5r_m}(x) \cap \Omega^c \neq \emptyset$ and $B_{c_\theta r}(x) \subset B_{r_M}(x) \cap (B_r(x_0) \cap \Omega)$ with $B_{2c_\theta r}(x) \subset B_r(x_0) \cap (x_0 + K)$. Thus by Theorem 6.3.2

$$\begin{aligned}\frac{1}{r^n}\int_{B_r(x_0)\cap\Omega} |f|\,dy &\leq \frac{1}{r^n}(|B_r(x)|(1+\frac{1}{c_\theta^n})[\bar{f}]_{BMO^\infty(B_r(x))} + \frac{1}{c_\theta^n}\|f\|_{L^1(B_{r_M}(x))})\\ &\leq C([f]_{BMO^\infty(\Omega)} + [f]_{b^M}).\end{aligned}$$

□

Remark 6.3.3. *The equivalence proofs of Theorem 6.3.9 and Theorem 6.3.8 can be extended to a large class of other domains by using similar ideas. The equivalence of BMO_b^M and $BMO_b^{\infty,\infty}$ for example also holds in exterior Lipschitz domains and domains of the form $G \times \mathbb{R}^{n-k}$, where $G \subset \mathbb{R}^k$ is a bounded C^2-domain, where the higher boundary regularity is needed since there is no extension operator from $BMO^\infty(\Omega)$ to $BMO^\infty(\mathbb{R}^n)$ (cf. Remark 6.2.1) such that we need to consider extension operators on subsets of Ω.*

We now want to prove an interpolation result that shows that if a function is in BMO and L^1 it is also in L^p for a large class of domains and that we can estimate it in a certain way. We will start with the result in $\mathbb{R}^n$.

Lemma 6.3.3. *Let $f \in BMO^\infty(\mathbb{R}^n) \cap L^1(\mathbb{R}^n)$ and $1 < p < \infty$. Then $f \in L^p(\mathbb{R}^n)$ and the estimate*

$$\|f\|_{L^p(\mathbb{R}^n)} \leq Cp\|f\|_{L^1(\mathbb{R}^n)}^{\frac{1}{p}}[f]_{BMO^\infty(\mathbb{R}^n)}^{1-\frac{1}{p}}$$

holds, where the constant $C > 0$ only depends on the dimension n.

Proof. Compare e.g. [Han77] and [KW08]. □

We will later use this Lemma together with Jones' extension theorem for BMO-functions.

Lemma 6.3.4. *Let $A \subset \mathbb{R}^n$ be a bounded uniform domain, $f \in L^1(A) \cap BMO^\infty(A)$ and $1 < p < \infty$. Then $\bar{f} - f_{Q_0} \in L^p(\mathbb{R}^n)$ and*

$$\|\bar{f} - f_{Q_0}\|_{L^p(\mathbb{R}^n)} \leq C(\|\bar{f}\|_{L^1(B)} + |B||f_{Q_0}|)^{\frac{1}{p}}[f]_{BMO^\infty(A)}^{1-\frac{1}{p}}$$

for any point $x \in \overline{A}$, where $B = B_{(4\sqrt{n}+2)diam(A)}(x)$. If A is a bounded Lipschitz-domain the constant depends only on n, p and the Lipschitz-regularity of A.

Proof. We can use Theorem 6.2.1 to get $\bar{f} \in BMO^\infty(\mathbb{R}^n)$ with $[\bar{f}]_{BMO^\infty(\mathbb{R}^n)} \leq C[f]_{BMO^\infty(A)}$. Furthermore adding constants will not change the BMO-seminorm. By condition (3) on the Whitney decomposition we can see that the ball B contains all cubes in $\mathcal{A}'$ for which there exists a larger cube in $\mathcal{A}$. Thus if $y \notin B$ every cube containing y corresponds to Q_0. From this we can see that $\bar{f}$ is on B^C constantly equal to f_{Q_0}. The function $\bar{f} - f_{Q_0}$ has then compact support and is locally integrable, thus in $\bar{f} - f_{Q_0} \in L^1(\mathbb{R}^n)$. Lemma 6.3.3 then yields

$$\begin{aligned}
\|\bar{f} - f_{Q_0}\|_{L^p(\mathbb{R}^n)} &\leq C\|\bar{f} - f_{Q_0}\|_{L^1(\mathbb{R}^n)}^{\frac{1}{p}}[\bar{f} - f_{Q_0}]_{BMO^\infty(\mathbb{R}^n)}^{1-\frac{1}{p}} \\
&\leq C\|\bar{f} - f_{Q_0}\|_{L^1(B)}^{\frac{1}{p}}[\bar{f}]_{BMO^\infty(\mathbb{R}^n)}^{1-\frac{1}{p}} \\
&\leq C(\|\bar{f}\|_{L^1(B)} + |B||f_{Q_0}|)^{\frac{1}{p}}[f]_{BMO^\infty(A)}^{1-\frac{1}{p}}.
\end{aligned}$$

□

Theorem 6.3.10. *Let A be a bounded uniform domain, $f \in L^1(A) \cap BMO^\infty(A)$ and $1 < p < \infty$. Then $f \in L^p(A)$ with*

$$\|f\|_{L^p(A)} \leq C((1 + \frac{diam(A)^n}{|Q_0|})\|f\|_{L^1(A)})^{\frac{1}{p}}[f]_{BMO^\infty(A)}^{1-\frac{1}{p}} + \frac{|A|^{\frac{1}{p}}}{|Q_0|}\|f\|_{L^1(A)}.$$

If A is a bounded Lipschitz-domain the constant depends only on n, p and the Lipschitz-regularity of A.

Proof. Note that by Lemma 6.2.2 all $Q'_j \in \mathcal{A}'$ that correspond to $Q_j \neq Q_0$ are contained in a cube of side length $(130K^2+2)\ell(Q_j)$ with the same center as Q_j. Thus for $B := B_{(4\sqrt{n}+2)diam(A)}(x)$ we have

$$\|\bar{f}\|_{L^1(B)} \leq ((130K^2+2)^n + 1)\|f\|_{L^1(A)}$$

because there are at most $(130K^2+2)^n$ cubes outside of Q_j, in which f may be defined as f_{Q_j}. By the previous Lemma we get

$$\begin{aligned}
&\|f\|_{L^p(A)} \\
\leq&\|\bar{f} - f_{Q_0}\|_{L^p(\mathbb{R}^n)} + |A|^{\frac{1}{p}}|f_{Q_0}| \\
\leq&C(\|\bar{f}\|_{L^1(B)} + |B||f_{Q_0}|)^{\frac{1}{p}}[f]^{1-\frac{1}{p}}_{BMO^\infty(A)} + |A|^{\frac{1}{p}}|f_{Q_0}| \\
\leq&C(((130K^2+2)^n+1)\|f\|_{L^1(A)} + \frac{|B|}{|Q_0|}\|f\|_{L^1(A)})^{\frac{1}{p}}[f]^{1-\frac{1}{p}}_{BMO^\infty(A)} + \frac{|A|^{\frac{1}{p}}}{|Q_0|}\|f\|_{L^1(A)} \\
\leq&C(K,n,p)((1+\frac{diam(A)^n}{|Q_0|})\|f\|_{L^1(A)})^{\frac{1}{p}}[f]^{1-\frac{1}{p}}_{BMO^\infty(A)} + \frac{|A|^{\frac{1}{p}}}{|Q_0|}\|f\|_{L^1(A)}.
\end{aligned}$$

□

Theorem 6.3.11. *Let $\Omega \subset \mathbb{R}^n$ be a bounded uniform domain. Let $\mu, \nu \in (0,\infty]$ and $p \in [1,\infty)$. Then the embeddings*

$$BMO^{\mu,\nu}_b(\Omega) \hookrightarrow L^p(\Omega), \tag{6.3.6}$$

$$VMO^{\mu,\nu}_{b,0,\sigma}(\Omega) \hookrightarrow L^p_\sigma(\Omega) \tag{6.3.7}$$

hold.

Proof. From Lemma 6.3.1 we see that $\|f\|_{L^1(\Omega)} \leq C\|f\|_{BMO^{\mu,\nu}_b}$ and by definition $[f]_{BMO^\mu(\Omega)} \leq \|f\|_{BMO^{\mu,\nu}_b}$. By the equivalence result for different finite μ of Theorem 6.3.3 we get that we can replace $[f]_{BMO^\mu(\Omega)}$ by $[f]_{BMO^{diam(\Omega)}(\Omega)} = [f]_{BMO^\infty(\Omega)}$. Then we can use Theorem 6.3.10 in order to get

$$\begin{aligned}
\|f\|_p &\leq C(\|f\|^{\frac{1}{p}}_{L^1(\Omega)}[f]^{1-\frac{1}{p}}_{BMO^\infty} + \|f\|_{L^1(\Omega)}) \\
&\leq C\|f\|_{BMO^{\mu,\nu}_b}
\end{aligned}$$

□

Finally we will give an equivalence result of $BMO^{\mu,\nu}_b p$ for different p. Our proof here will be based on Jones' extension theorem for BMO-functions. Another proof for the same fact can be found in [BGS15].

Theorem 6.3.12. *Let $\mu \in (0,\infty]$ and $\Omega \subset \mathbb{R}^n$ be a uniformly C^2-domain. Let $\nu \in (0,R^*)$ and $p \in (1,\infty)$. If $\mu < \infty$, then we assume additionally $\nu < \infty$. Then $\|\cdot\|_{BMO^{\mu,\nu}_b p}$ and $\|\cdot\|_{BMO^{\mu,\nu}_b}$ are equivalent.*

Proof. The seminorms $[\cdot]_{BMO^{\mu}p}$ and $[\cdot]_{BMO^{\mu}}$ are equivalent by the John-Nirenberg inequality ([JN61]) and Hölder's inequality. By Theorem 6.3.3 we can furthermore assume that $\mu > \nu$ if ν is finite. By Hölder's inequality

$$\begin{aligned}[f]_{b^{\nu}} &= \sup_{x\in\partial\Omega, r<\nu} \frac{1}{r^n}\int_{B_r(x)\cap\Omega}|f|\,dy\\ &\leq \sup_{x\in\partial\Omega, r<\nu} \omega_n^{\frac{1}{p'}}\Big(\frac{1}{|B_r(x)|}\int_{B_r(x)\cap\Omega}|f|^p\,dy\Big)^{1/p}\\ &= \omega_n^{\frac{1}{p'}}[f]_{b^{\nu}p}.\end{aligned}$$

Thus it is left to show that there is a constant C such that for all $x\in\partial\Omega, r<\nu$ and $f\in BMO_b^{\mu,\nu}(\Omega)$ the estimate

$$\frac{1}{r^n}\int_{B_r(x)\cap\Omega}|f|^p \leq C\|f\|^p_{BMO_b^{\mu,\nu}}$$

holds. By the assumption $r<\nu<R^*$ we see that all domains $B_r(x_0)\cap\Omega$ with $r<\nu$ and $x_0\in\partial\Omega$ are Lipschitz-domains, where we can estimate the Lipschitz-regularity uniformly in r and x_0. Since we assumed $\mu>\nu$ the seminorms $[\cdot]_{BMO^{\mu}(B_r(x_0)\cap\Omega)}$ and $[\cdot]_{BMO^{\infty}(B_r(x_0)\cap\Omega)}$ coincide. Then for every $f\in BMO_b^{\mu,\nu}(\Omega)$ the restriction $f|_{B_r(x_0)\cap\Omega}$ is in $BMO^{\infty}(B_r(x_0)\cap\Omega))\cap L^1(B_r(x_0)\cap\Omega))$ such that we can apply Theorem 6.3.10 to obtain

$$\begin{aligned}&\|f\|_{L^p(B_r(x_0)\cap\Omega)}\\ \leq& C(((1+\frac{r^n}{|Q_0|})\|f\|_{L^1(B_r(x_0)\cap\Omega))})^{\frac{1}{p}}[f]^{1-\frac{1}{p}}_{BMO^{\infty}(B_r(x_0)\cap\Omega)}+\frac{r^{\frac{n}{p}}}{|Q_0|}\|f\|_{L^1(B_r(x_0)\cap\Omega)}).\end{aligned}$$

By the assumption on the Whitney decomposition and $r<R^*$ we obtain that Q_0 is at least of sidelength $\frac{r}{16\sqrt{n}}$ and thus we can rewrite the above inequality by

$$\begin{aligned}&\|f\|_{L^p(B_r(x_0)\cap\Omega))}\\ \leq& C\|f\|^{\frac{1}{p}}_{L^1(B_r(x_0)\cap\Omega)}[f]^{1-\frac{1}{p}}_{BMO^{\infty}(B_r(x_0)\cap\Omega)}+Cr^{n(\frac{1}{p}-1)}\|f\|_{L^1(B_r(x_0)\cap\Omega)}\\ \leq& Cr^{\frac{n}{p}}[f]^{\frac{1}{p}}_{b^{\nu}}[f]^{1-\frac{1}{p}}_{BMO^{\mu}(\Omega)}+Cr^{\frac{n}{p}}[f]_{b^{\nu}}\end{aligned}$$

from which we can conclude that

$$\begin{aligned}[f]_{b^{\nu}p} &= \sup_{x_0\in\partial\Omega, r<\nu} r^{-\frac{n}{p}}\|f\|_{L^p(B_r(x_0)\cap\Omega)}\\ &\leq C\|f\|_{BMO_b^{\mu,\nu}}.\end{aligned}$$

□

Remark 6.3.4. *The function f_3 of Example 6.3.1 shows that it is in fact necessary to exclude the case $\mu<\infty$ and $\nu=\infty$ in the case of the half-space since $[f_3]_{b^{\infty}p(\mathbb{R}_+)}=\infty$ for $p\in(1,\infty)$.*

Theorem 6.3.13. *Let $\Omega \subset \mathbb{R}^n$ be an arbitrary domain and let $p \in (1,\infty)$. Then the norms $\|\cdot\|_{BMO_b^M}$ and $\|\cdot\|_{BMO_b^M p}$ are equivalent.*

Proof. The proof of this Theorem uses the same ideas as the proof of Theorem 6.3.12. By the John-Nirenberg inequality $[\cdot]_{BMO^M}$ and $[\cdot]_{BMO^M p}$ are equivalent and it follows from Hölder's inequality that $[f]_{b^M} \le C[f]_{b^M p}$. We have now a look at all balls $B := B_r(x)$ such that $B_{2r}(x) \subset \Omega$ and $B_{5r}(x) \cap \Omega^c \neq \emptyset$. Since the constant of Theorem 6.3.10 and the ratio $\frac{|B|}{|Q_0|}$ are scaling invariant and we are only considering balls here we have

$$\begin{aligned}\frac{1}{|B|^{\frac{1}{p}}}\|f\|_{L^p(B)} &\le C((1+\frac{r^n}{|Q_0|})\frac{1}{|B|}\|f\|_{L^1(B)})^{\frac{1}{p}}[f]^{1-\frac{1}{p}}_{BMO^\infty(B)} + \frac{1}{|Q_0|}\|f\|_{L^1(B)}\\ &\le C[f]^{\frac{1}{p}}_{b^M}[f]^{1-\frac{1}{p}}_{BMO^M} + C[f]_{b^M}.\end{aligned}$$

Thus we have proved that $[f]_{b^M p} \le C\|f\|_{BMO_b^M}$. □

6.4 The heat semigroup in BMO-type spaces

In this section we will prove several properties of the heat semigroup with respect to the considered BMO_b spaces, i.e., we consider the equation

$$\begin{cases} u_t - \Delta u = 0 & \text{in } \Omega \times (0,T)\\ u = 0 & \text{in } \partial\Omega \times (0,T)\\ u(0) = u_0. \end{cases} \tag{6.4.1}$$

We will start with the case $\Omega = \mathbb{R}^n$ and $T = \infty$.

Theorem 6.4.1. *Let $u_0 \in BMO^\infty(\mathbb{R}^n)$. Then there is a solution u to* (6.4.1) *which satisfies the estimate*

$$\sup_{t>0}([u(t)]_{BMO^\infty} + t^{\frac{1}{2}}\|\nabla u(t)\|_\infty + t\|\nabla^2 u(t)\|_\infty + t\|u_t(t)\|_\infty) \le C[u_0]_{BMO^\infty} \tag{6.4.2}$$

with a constant C just depending on n.

Proof. We will derive the estimate $\sup_{t>0}[u(t)]_{BMO^\infty} \le C[u_0]_{BMO^\infty}$ by duality. Let $\varphi \in \mathcal{H}^1(\mathbb{R}^n)$, then since $(\mathcal{H}^1)^* = BMO^\infty$, where $\mathcal{H}^1(\mathbb{R}^n)$ is the Hardy space which is defined as

$$\{f \in L^1(\mathbb{R}^n) : \|f\|_{\mathcal{H}^1} := \|\sup_{s>0}|G_s * f|\|_1 < \infty\}.$$

we have

$$\begin{aligned}|(u(t),\varphi)| = \left|\int_{\mathbb{R}^n}(G_t * u_0)\varphi\,dy\right| &= \left|\int_{\mathbb{R}^n} u_0(G_t*\varphi)\,dy\right|\\ &\le [u_0]_{BMO^\infty}\|G_t * \varphi\|_{\mathcal{H}^1}\\ &= [u_0]_{BMO^\infty}\sup_{s>0}\|G_s * G_t * \varphi\|_{L^1}\\ &\le [u_0]_{BMO^\infty}\|\varphi\|_{\mathcal{H}^1}.\end{aligned}$$

The inequality can also be derived from the estimate

$$[f * g]_{BMO^\infty} \leq \|g\|_1 [f]_{BMO^\infty} \quad (f \in BMO^\infty(\mathbb{R}^n), g \in L^1(\mathbb{R}^n)),$$

which was proved in [Gul02] (equation (41)) by a similar duality argument.

The derivative estimates are also proved via a duality argument. The gradient estimate $\|\nabla u(t)\|_\infty \leq t^{1/2}[u_0]_{BMO^\infty}$ has already been proved in the appendix of [BG15]. We will here just prove the estimate for the second derivative $\nabla^2 u$, which is done by using the same ideas as the proof in the appendix of [BG15]. The estimate for the first derivative can be proved in a similar way. The time derivative estimate follows then from the estimate on the second derivative by $u_t = \Delta u$. As a first step we prove the estimate

$$t\|\nabla^2 G_t * u_0\|_{\mathcal{H}^1} \leq C\|u_0\|_1 \tag{6.4.3}$$

for all $u_0 \in L^1(\mathbb{R}^n)$ and with $t = 1$. By the definition of the $\mathcal{H}^1$-norm

$$\begin{aligned}
\|\partial_i\partial_j G_1 * u_0\|_{\mathcal{H}^1} &= \|\sup_{s>0} |G_s * \partial_i\partial_j G_1 * u_0|\|_1 \\
&\leq \|\sup_{s>0}(|\partial_i\partial_j G_{s+1}| * |u_0|)\|_1 \\
&\leq \|(\sup_{s>0} |\partial_i\partial_j G_{s+1}|) * |u_0|\|_1 \\
&\leq \|\sup_{s>0} |\partial_i\partial_j G_{s+1}|\|_1 \|u_0\|_1.
\end{aligned}$$

Since $\partial_i\partial_j G_t = -\delta_{ij}\frac{G_t}{2t} + x_i x_j \frac{G_t}{4t^2}$, we obtain for $\varrho = \frac{|x|^2}{4t}$ the estimate

$$|\partial_i\partial_j G_t(x)| \leq \frac{2\delta_{ij}}{\pi^{\frac{n}{2}}}\frac{1}{|x|^{n+2}}\mathrm{e}^{-\varrho}\varrho^{\frac{n}{2}+1} + \frac{4}{\pi^{\frac{n}{2}}}\frac{|x_i||x_j|}{|x|^{n+4}}\mathrm{e}^{-\varrho}\varrho^{\frac{n}{2}+2}$$

from which we can conclude that

$$|\partial_i\partial_j G_t(x)| \leq \frac{C_0}{|x|^{n+2}}.$$

Furthermore, for $t \geq 1$ we can estimate $|\partial_i\partial_j G_t(x)| \leq 2(4\pi)^{-\frac{n}{2}}$ such that we have

$$|\partial_i\partial_j G_{s+1}(x)| \leq \min\{2(4\pi)^{-\frac{n}{2}}, \frac{C_0}{|x|^{n+2}}\} =: a(x) \text{ for } s > 0, x \in \mathbb{R}^n.$$

Since $a \in L^1(\mathbb{R}^n)$, we get with $C_* = \int_{\mathbb{R}^n} a(x)\,dx$ the estimate

$$\|\partial_i\partial_j G_1 * u_0\|_{\mathcal{H}^1} \leq C_*\|u_0\|_1,$$

which is (6.4.3) for $t = 1$. In order to generalize this to arbitrary time $t > 0$ we rescale u by the scaling transformation $u_\lambda(x) = \lambda^n u(\lambda x)$ for $\lambda > 0$. The norms in $L^1(\mathbb{R}^n)$ and $\mathcal{H}^1(\mathbb{R}^n)$ are invariant under this transformation and thus we get from the equality $(\partial_i \partial_j G_1) * (u_0)_\lambda = \lambda^2((\partial_i \partial_j G_{\lambda^2}) * u_0)_\lambda$ and the estimate for $t = 1$ that

$$\lambda^2 \|(\partial_i \partial_j G_{\lambda^2} * u_0)\|_{\mathcal{H}^1} \leq C_* \|u_0\|_1.$$

We obtain now (6.4.3) for $t > 0$ by taking $\lambda = t^{\frac{1}{2}}$. Then by duality

$$\|\partial_i \partial_j G_t * u_0\|_\infty \leq C t^{-1} [u_0]_{BMO^\infty}$$

for all $t > 0$. □

Similar estimates can be obtained for the half-space via an odd extension and reduction to the case $\Omega = \mathbb{R}^n$. We will first formulate the extension argument.

Lemma 6.4.1. *Let $\mu > 0$ and $\nu \geq 2\mu$. Then there exists a dimensional constant C such that for all $f \in BMO_b^{\mu,\nu}(\mathbb{R}^n_+)$ the odd extension $\bar{f} \in BMO^\mu(\mathbb{R}^n)$ satisfies*

$$[\bar{f}]_{BMO^\mu(\mathbb{R}^n)} \leq C \|f\|_{BMO_b^{\mu,\nu}(\mathbb{R}^n_+)},$$

where the odd extension $\bar{f}$ is defined by $\bar{f}(x) = f(x)$ if $x_n > 0$, $\bar{f}(x) = -f(x_1, \ldots, x_{n-1}, -x_n)$ if $x_n < 0$ and $\bar{f}(x) = 0$ if $x_n = 0$.

Proof. Let $x \in \mathbb{R}^n$ and $r < \mu$. We distinguish between two cases. If $B_r(x) \subset \mathbb{R}^n_+$ or $B_r(x) \subset (\mathbb{R}^n_+)^c$, then $\frac{1}{|B_r(x)|} \int_{B_r(x)} |\bar{f}(y) - \bar{f}_{B_r(x)}|\, dy \leq [f]_{BMO^\mu(\mathbb{R}^n_+)}$. If $B_r(x) \cap \partial\mathbb{R}^n_+ \neq \emptyset$, then there is $\tilde{x} \in B_r(x)$ with $\tilde{x}_n = 0$. Since $\tilde{x} \in B_r(x)$ the relation $B_r(x) \subset B_{2r}(\tilde{x})$ holds and thus by $2r < \nu$

$$\begin{aligned}\frac{1}{|B_r(x)|} \int_{B_r(x)} |\bar{f}(y) - \bar{f}_{B_r(x)}|\, dy &\leq \frac{2}{\omega_n r^n} \int_{B_{2r}(\tilde{x})} |\bar{f}(y)|\, dy \\ &\leq \frac{2^{n+1}}{\omega_n} [f]_b^\nu.\end{aligned}$$

□

The Lemma holds in particular for the odd extension from $BMO_b^{\infty,\infty}(\mathbb{R}^n_+)$ to $BMO^\infty(\mathbb{R}^n)$. For the case $\mu = \infty$, $\nu < \infty$ this extension does not hold. The function $f(x) := \log|x + (0, \ldots, 0, 1)|$ is in $BMO_b^{\infty,\nu}(\mathbb{R}^n_+)$ for finite ν but the odd extension is not in $BMO^\infty(\mathbb{R}^n)$ since for x with $x_n = 0$

$$\frac{1}{|B_r(x)|} \int_{B_r(x)} |\bar{f}(y) - \bar{f}_{B_r(x)}|\, dy = \frac{1}{|B_r(x)|} \int_{B_r(x)} |\bar{f}(y)|\, dy \to \infty \quad (r \to \infty).$$

Theorem 6.4.2. *Let $u_0 \in BMO_b^{\infty,\infty}(\mathbb{R}^n_+)$. Then there is a solution u to (6.4.1) which satisfies the estimate*

$$\sup_{t>0}([u(t)]_{BMO_b^{\infty,\infty}} + t^{1/2}\|\nabla u(t)\|_\infty + t\|\nabla^2 u(t)\|_\infty + t\|u_t(t)\|_\infty) \leq C[u_0]_{BMO_b^{\infty,\infty}} \tag{6.4.4}$$

with a constant C just depending on n.

Proof. By Lemma 6.4.1 we can extend u_0 to $\bar{u}_0 \in BMO^\infty(\mathbb{R}^n)$, which is a function that is odd with respect to the last component. We can now use Theorem 6.4.1 to get a solution $\bar{u}$ to (6.4.1) with $\Omega = \mathbb{R}^n$ and initial data $\bar{u}_0$. The solution $\bar{u}$ then is also an odd function in the last component and satisfies the estimate

$$\begin{aligned}
&\sup_{t>0}([\bar{u}(t)]_{BMO^\infty(\mathbb{R}^n)} + t^{\frac{1}{2}}\|\nabla\bar{u}(t)\|_{L^\infty(\mathbb{R}^n)} + t\|\nabla^2\bar{u}(t)\|_{L^\infty(\mathbb{R}^n)} + t\|\bar{u}_t(t)\|_{L^\infty(\mathbb{R}^n)})\\
\le& C[\bar{u}_0]_{BMO^\infty(\mathbb{R}^n)}\\
\le& C\|u_0\|_{BMO_b^{\infty,\infty}(\mathbb{R}^n_+)}.
\end{aligned}$$

Then $u(t) := \bar{u}(t)|_{\mathbb{R}^n_+}$ satisfies (6.4.1) with initial data u_0. The boundary condition is satisfied because $\bar{u}$ is an odd function in the last component. It is immediate from the definition that $[u(t)]_{BMO^\infty(\mathbb{R}^n_+)} \le [\bar{u}(t)]_{BMO^\infty(\mathbb{R}^n)}$. Furthermore, we obtain for $r > 0$ and $x_0 \in \partial\mathbb{R}^n_+$

$$\begin{aligned}
\frac{1}{r^n}\int_{B_r(x_0)\cap\mathbb{R}^n_+} |u(y,t)|\,dy &= \frac{1}{2r^n}\int_{B_r(x_0)} |\bar{u}(y,t)|\,dy\\
&= \frac{1}{2r^n}\int_{B_r(x_0)} |\bar{u}(y,t) - \bar{u}_{B_r(x)}(t)|\,dy\\
&\le \frac{\omega_n}{2}[\bar{u}(t)]_{BMO^\infty(\mathbb{R}^n)}
\end{aligned}$$

and thus $[u(t)]_{b^\infty(\mathbb{R}^n_+)} \le C\|u_0\|_{BMO_b^{\infty,\infty}(\mathbb{R}^n_+)}$. □

If the underlying geometry of the domain is more complicated or one of the parameters is finite, we need a different method to prove similar estimates.

Lemma 6.4.2. *Let $n < p < \infty$. If $u_0 \in C_c^\infty(\Omega)$ and Ω is a uniformly C^3-domain, then there is a solution u of* (6.4.1) *with $u(t) \in C^2(\overline{\Omega}) \cap W_0^{1,p}(\Omega)$ for all $t > 0$ satisfying*

$$\sup_{0<t<T_0}(\|u(t)\|_\infty + t^{\frac{1}{2}}\|\nabla u(t)\|_\infty + t\|\nabla^2 u(t)\|_\infty + t\|u_t(t)\|_\infty) < \infty \tag{6.4.5}$$

for every $0 < T_0 < \infty$.

Proof. By Theorem 3.1.2 in [Lun95] there exists an analytic semigroup H in $L^p(\Omega)$ to (6.4.1). We define $u(t) := H(t)u_0$. We argue in a similar way as in the proof of Proposition 5.2 in [AG13], where a similar property was shown for the Stokes semigroup. By the semigroup properties we obtain an estimate

$$\sup_{0<t<T_0}(\|u(t)\|_{D(\Delta_p)} + t\|u_t(t)\|_{D(\Delta_p)}) \le C_{T_0}\|u_0\|_{D(\Delta_p)},$$

where $\|f\|_{D(\Delta_p)} = \|f\|_p + \|\Delta f\|_p$. This norm is equivalent to $\|f\|_{W^{2,p}}$ and thus we have by $u_0 \in C_c^\infty(\Omega) \subset D(\Delta_p)$

$$\sup_{0<t<T_0}(\|u(t)\|_{1,p} + t^{\frac{1}{2}}\|\nabla u(t)\|_{1,p} + t\|u_t(t)\|_{1,p}) < \infty.$$

For estimating $\|\nabla^2 u\|_{W^{1,p}}$ we note that u solves the equation $\Delta u = u_t$ in Ω with $u = 0$ on $\partial\Omega$. Since Ω is a C^3-domain we obtain by higher regularity theory for elliptic systems as in Theorem 8.13 of [GT77] for $t \leq T_0$

$$\begin{aligned}\|u(t)\|_{3,p} &\leq C(\|u_t(t)\|_{1,p} + \|u(t)\|_p) \\ &\leq \frac{1}{t} C_{T_0} \|u_0\|_{D(\Delta_p)}.\end{aligned}$$

In summary we have that

$$\begin{aligned}&\sup_{0<t<T_0} (\|u(t)\|_\infty + t^{\frac{1}{2}} \|\nabla u(t)\|_\infty + t\|\nabla^2 u(t)\|_\infty + t\|u_t(t)\|_\infty) \\ \leq &\sup_{0<t<T_0} (\|u(t)\|_{1,p} + t^{\frac{1}{2}} \|\nabla u(t)\|_{1,p} + t\|\nabla^2 u(t)\|_{1,p} + t\|u_t(t)\|_{1,p}) \\ <&\infty,\end{aligned}$$

$u(t) \in C^2(\overline{\Omega})$ by the Sobolev embedding theorem and $u(t) \in W_0^{1,p}(\Omega)$ by the boundary conditions on u. $\square$

Theorem 6.4.3. *Let Ω be a domain with uniformly C^3-boundary. Let $\mu, \nu \in (0, \infty]$. Then there exist constants $C > 0$ and $T_0 > 0$ such that for all $u_0 \in VMO_{b,0}^{\mu,\nu}(\Omega)$ there is a solution u to* (6.4.1) *satisfying*

$$\sup_{0<t<T_0} (t^{\frac{1}{2}} \|\nabla u(t)\|_\infty + t\|\nabla^2 u(t)\|_\infty + t\|u_t(t)\|_\infty) \leq C\|u_0\|_{BMO_b^{\mu,\nu}}. \tag{6.4.6}$$

Proof. The proof is similar to the proof of the same estimate for the Stokes equations (cf. [AG13], [BG15]). By Lemma 6.4.2 there are solutions satisfying (6.4.5) for every $u_0 \in C_c^\infty(\Omega)$. Let

$$N(u)(x,t) := t^{\frac{1}{2}} |\nabla u(x,t)| + t|\nabla^2 u(x,t)| + t|u_t(x,t)|.$$

We assume for these solutions that the estimate does not hold. Then there is a sequence of solutions u^m to initial data $u_0^m \in C_c^\infty(\Omega)$ and a sequence $t_m \to 0$ such that

$$\|N(u^m)(\cdot, t_m)\|_\infty > m\|u_0^m\|_{BMO_b^{\mu,\nu}}.$$

We normalize u^m by $\tilde{u}^m := u^m / (\sup_{0<t<t_m} \|N(u^m)(\cdot,t)\|_\infty)$ and thus obtain

$$\sup_{0<t<t_m} \|N(\tilde{u}^m)(\cdot,t)\|_\infty = 1 \tag{6.4.7}$$

and

$$\|\tilde{u}_0^m\|_{BMO_b^{\mu,\nu}} < 1/m. \tag{6.4.8}$$

Thus there exist $x_m \in \Omega$ and $\tau_m < t_m$ such that $N(\tilde{u})(x_m, \tau_m) \geq 1/2$. Then we rescale the solution with respect to (x_m, τ_m) by

$$v^m(x,t) := \tilde{u}(\tau_m^{1/2} x + x_m, \tau_m t) \tag{6.4.9}$$

and obtain by (6.4.7)

$$\sup_{0<t<1} \|N(v^m)(\cdot,t)\|_\infty = 1 \tag{6.4.10}$$

and

$$\|N(v^m)(0,1)\|_\infty \geq 1/2. \tag{6.4.11}$$

Furthermore, by (6.4.8)

$$\|v_0^m\|_{BMO_b^{\mu_m,\nu_m}} \to 0 \quad (m \to \infty), \tag{6.4.12}$$

where $\mu_m = \tau_m^{-1/2}\mu$, $\nu_m = \tau_m^{-1/2}\nu$ and

$$\Omega_m := \{x \in \mathbb{R}^n : x = (y - x_m)/\tau_m^{1/2}, y \in \Omega\}.$$

Then v^m solves the heat equation (6.4.1) in the rescaled domain Ω_m.

Now let $c_m := dist(x_m, \partial\Omega)/(\tau_m^{1/2}) = dist(0, \partial\Omega_m)$. We distinguish between the two cases $\limsup_{m\to\infty} c_m = \infty$ and $\limsup_{m\to\infty} c_m < \infty$. If $\limsup_{m\to\infty} c_m = \infty$ we can take a subsequence such that $\lim_{m\to\infty} c_m = \infty$. Then Ω_m expands to $\mathbb{R}^n$. Thus we obtain for every function $\varphi \in C_c^\infty(\mathbb{R}^n \times [0,1))$

$$\int_0^1 \int_{\mathbb{R}^n} v^m(x,t)(\Delta\varphi(x,t) + \varphi_t(x,t))\,dx\,dt = -\int_{\mathbb{R}^n} v_0^m(x)\varphi(x,0)\,dx$$

and the same equality for the partial derivatives

$$\int_0^1 \int_{\mathbb{R}^n} \partial_i v^m(x,t)(\Delta\varphi(x,t) + \varphi_t(x,t))\,dx\,dt = \int_{\mathbb{R}^n} v_0^m(x)\partial_i\varphi(x,0)\,dx, \tag{6.4.13}$$

where the right hand side converges to zero by (6.4.12) and $\int_{\mathbb{R}^n} \partial_i\varphi(x,0)\,dx = 0$. By local Hölder estimates (cf. [LSU68, Chapter IV, Theorem 10.1]) we obtain that v_m satisfies not only (6.4.10) but also Hölder estimates in the second derivative and time derivative. Therefore, we can obtain a subsequence again denoted by v^m such that $\nabla v^m, \nabla^2 v^m, v_t^m$ converge locally uniformly to some $g, \nabla g, h$. In the limit the equation (6.4.13) becomes

$$\int_0^1 \int_{\mathbb{R}^n} g(\Delta\varphi(x,t) + \varphi_t(x,t))\,dx\,dt = 0$$

with $t^{1/2}\|g\|_\infty \leq c$ by (6.4.10). By the uniqueness result of Chung on the heat equation (cf. [Chu99, Theorem 3.1 and Theorem 3.2]) we get that $g = 0$. Then $\nabla g = 0$ as well. By $\lim_{m\to\infty} \nabla^2 v^m = \nabla g = 0$ and $v_t^m = \Delta v^m$ we see that h needs to vanish, too. We now have proved that $N(v^m)$ converges locally uniformly to 0 which is a contradiction to (6.4.11).

Now we have to consider the case $\limsup_{m\to\infty} c_m < \infty$. Then there is a subsequence satisfying $\lim_{m\to\infty} c_m = c_0 \in [0,\infty)$. Then Ω_m expands to a half-space $\mathbb{R}^n_{+,-c_0} := \{x \in \mathbb{R}^n : x_n > -c_0\}$ (cf. [AG13] and [BG15]). Again, by local Hölder estimates we obtain that v^m satisfies Hölder estimates in the second derivative and time derivative together with (6.4.10). Furthermore, by the boundary condition and (6.4.10) we can see that v^m is locally bounded and we thus get that $v^m, \nabla v^m, \nabla^2 v^m, v_t^m$ converge locally uniformly to some $v, \nabla v, \nabla^2 v, v_t$. The limit v then satisfies for all $\varphi \in C_c^\infty(\mathbb{R}^n_{+,-c_0} \times [0,1))$ the equation

$$\int_0^1 \int_{\mathbb{R}^n_{+,-c_0}} v(x,t)(\Delta\varphi(x,t) + \varphi_t(x,t))\, dx\, dt = -\lim_{m\to\infty} \int_{\mathbb{R}^n_{+,-c_0}} v_0^m(x,t)\varphi(x,0)\, dx,$$

where the right hand side is equal to 0 by (6.4.12). Thus v satisfies the homogeneous heat equation (6.4.1) in $\mathbb{R}^n_{+,-c_0}$. By (6.4.10) and the boundary condition we know that v is bounded by $Ct^{1/2}(x_n + c_0)$. If we take the odd extension $\bar{v}$ of v to $\mathbb{R}^n$, the extension still satisfies the heat equation with initial data $\bar{v}_0 = 0$ and the estimate $\bar{v}(x,t) \le Ct^{1/2}(|x_n| + c_0)$. By the uniqueness result of Chung (cf. [Chu99, Theorem 3.1 and Theorem 3.2]) we obtain that $v = 0$. Thus v and its derivatives converge locally uniformly to 0 which is again a contradiction to (6.4.10).

We have now proved that the statement holds for all $u_0 \in C_c^\infty(\Omega)$. By density we can extend the estimate to $VMO_{b,0}^{\mu,\nu}$ □

We will now present the key steps for proving the boundedness of $\|u(t)\|_{BMO_b^{\mu,\nu}}$.

Lemma 6.4.3. *Let Ω be a domain with uniformly C^3-boundary. Let $\mu,\nu \in (0,\infty]$. Then there exist constants $C > 0$ and $T_0 > 0$ such that for all $u_0 \in VMO_{b,0}^{\mu,\nu}(\Omega)$ there is a solution u to (6.4.1) such that*

1. *For all $x \in \Omega$, $r > 0$ with $B_r(x) \subset \Omega$ and $t \in (0,T_0)$*

$$|u_{B_r(x)}(t) - u_{0B_r(x)}|\, dy \le C\frac{t^{\frac{1}{2}}}{r}\|u_0\|_{BMO_b^{\mu,\nu}}.$$

2. *For all $x \in \Omega$, $0 < r < \mu$ with $B_r(x) \subset \Omega$ and $t \in (0,T_0)$*

$$\frac{1}{|B_r(x)|}\int_{B_r(x)} |u(y,t) - u_{B_r(x)}|^2\, dy \le C(1 + \frac{t}{r^2})\|u_0\|^2_{BMO_b^{\mu,\nu}}.$$

3. *For all $x \in \Omega$, $0 < r < \mu$ with $B_r(x) \subset \Omega$ and $t \in (0,T_0)$*

$$\frac{1}{|B_r(x)|}\int_{B_r(x)} |u(y,t) - u_{B_r(x)}|^2\, dy \le C\frac{r^2}{t}\|u_0\|^2_{BMO_b^{\mu,\nu}}.$$

4. *If $\nu \le R^*$, then for all $x_0 \in \partial\Omega$, $0 < r < \nu$ and $t \in (0,T_0)$*

$$\frac{1}{r^n}\int_{B_r(x_0)\cap\Omega} |u(y,t)|^2\, dy \le C(\|u_0\|^2_{BMO_b^{\mu,\nu}} + [u_0]_{b^\nu 2}).$$

Proof. We will only give the key steps of the proof since the statements mainly follow from Theorem 6.4.3 and standard calculations. In Section 3 of [BGS15] this argument has been carried out in detail for the Stokes equations and by ignoring the pressure term there one gets the result for the heat equation.

For proving (1) we use the equality $\int_0^t u_s(s)\,ds - u_0 = u(t)$, $(6.4.1)_1$, integration by parts and the estimate of Theorem 6.4.3 on ∇u.

For proving (2) we again use the equality $\int_0^t u_s(s)\,ds - u_0 = u(t)$, $(6.4.1)_1$, integration by parts and the estimate of Theorem 6.4.3 on ∇u and combine it with the estimate of (1). The statement (3) follows directly from Poincaré's inequality.

In order to prove (4) we use again the equality $\int_0^t u_s(s)\,ds - u_0 = u(t)$, $(6.4.1)_1$, integration by parts and the estimate of Theorem 6.4.3 on ∇u. Compared to Theorem 3.4 in [BGS15], where the estimate was proved for the Stokes equations and the smallness assumption on ν was also necessary for obtaining control on the constants that appear in estimating the pressure term, the assumption here is only necessary for ensuring that integration by parts is possible. Thus ν can be taken larger if for all $B_r(x_0) \cap \Omega$ with $x_0 \in \partial\Omega$ and $r < \nu$ integration by parts is possible.

□

By the equivalence between $BMO_b^{\mu,\nu}$ and $BMO_b^{\mu,\nu}2$ of Theorem 6.3.12 the following theorem follows.

Theorem 6.4.4. *Let Ω be a domain with uniformly C^3-boundary. Let $\mu \in (0,\infty]$, $\nu \in (0, R^*]$. Let ν be finite if μ is finite. Then there exist constants $C > 0$ and $T_0 > 0$ such that for all $u_0 \in VMO_{b,0}^{\mu,\nu}(\Omega)$ there is a solution u to (6.4.1) satisfying*

$$\sup_{0<t<T_0} \left(\|u(t)\|_{BMO_b^{\mu,\nu}} + t^{\frac{1}{2}}\|\nabla u(t)\|_\infty + t\|\nabla^2 u(t)\|_\infty + t\|u_t(t)\|_\infty\right) \le C\|u_0\|_{BMO_b^{\mu,\nu}}.$$

By Lemma 6.4.2 we can see that $u(t) \in W_0^{1,p}(\Omega) \subset VMO_{b,0}^{\mu,\nu}(\Omega)$ with $p > n$ such that we can choose an arbitrary $T_0 \in (0,\infty)$ by iteration and get the same estimate with a different constant C_{T_0}. In particular, the Laplacian generates an analytic semigroup in $VMO_{b,0}^{\mu,\nu}(\Omega)$.

Remark 6.4.1. *If one replaces $VMO_{b,0}^{\mu,\nu}(\Omega)$ by $VMO_{b,0}^{M}(\Omega)$ in the above Theorem the statement still holds.*

6.5 Applications to the Stokes semigroup

In this section we will give some applications of the results for the Stokes semigroup which is the solution operator $S : u_0 \to S(t)u_0 = u(t)$ of the equation

$$\begin{cases} u_t - \Delta u + \nabla\pi = 0 & \text{in} \quad \Omega \times (0,T) \\ \mathrm{div} u = 0 & \text{in} \quad \Omega \times (0,T) \\ u = 0 & \text{on} \quad \partial\Omega \times (0,T) \\ u(0) = u_0 \end{cases}. \tag{6.5.1}$$

It was proved in [BGS15] that in bounded domains the Stokes semigroup is analytic in $VMO^{\mu,\nu}_{b,0,\sigma}(\Omega)$ for sufficiently small ν. By the equivalence result of Theorem 6.3.4 we can extend this result to general $\mu, \nu \in (0,\infty]$. We can further prove by the embedding theorem that the semigroup has the following property.

Theorem 6.5.1. *Let Ω be a bounded C^3-domain $\mu, \nu \in (0,\infty]$. Let S be the Stokes semigroup on $VMO^{\mu,\nu}_{b,0,\sigma}(\Omega)$. Then S is a bounded semigroup.*

Proof. For every $T_0 \in (0,\infty)$ there is some constant C_{T_0} such that

$$\sup_{0<t<T_0} \|u(t)\|_{BMO_b^{\mu,\nu}} \leq C\|u_0\|_{BMO_b^{\mu,\nu}}$$

by [BGS15, Theorem 3.5] and the equivalence result of Theorem 6.3.4. Thus we can now assume $t \geq 1$. Let $p > n$. By the embedding of Theorem 6.3.11 we obtain that $u_0 \in VMO^{\mu,\nu}_{b,0,\sigma}(\Omega) \subset L^p_\sigma(\Omega)$. Let $u(t) := S(t)u_0$ be the solution of the homogeneous Stokes equations with initial data u_0. By L^p-theory (see e.g. [Gig81]) we obtain for $t \geq 1$

$$\begin{aligned} \|u(t)\|_{1,p} &\leq \|u(t)\|_p + t^{1/2}\|\nabla u(t)\|_p \\ &\leq C\|u_0\|_p \\ &\leq C\|u_0\|_{BMO_b^{\mu,\nu}}. \end{aligned}$$

By the embedding $W^{1,p}(\Omega) \hookrightarrow L^\infty(\Omega) \hookrightarrow BMO_b^{\mu,\nu}(\Omega)$ we can conclude

$$\|u(t)\|_{BMO_b^{\mu,\nu}} \leq C\|u_0\|_{BMO_b^{\mu,\nu}}.$$

□

We can further extend the main result of [BGS15] to the analyticity of the Stokes semigroup S in $VMO^M_{b,0,\sigma}$ in a suitable class of domains.

Theorem 6.5.2. *Let Ω be a uniformly C^3-domain that is admissible in the sense of [AG13]. Then the solution operator S to* (6.5.1) *is an analytic semigroup in $VMO^M_{b,0,\sigma}(\Omega)$.*

Proof. The proof can be copied almost verbatim from [BGS15]. We will just give a short sketch on the main ideas and main differences. The gradient and time derivative estimate with respect to the BMO^M_b-norm have been proved in Remark 6.4 of [BG15]. For the estimate $\|u(t)\|_{BMO_b^M} \leq C\|u_0\|_{BMO_b^M}$ for $t < T_0$ we use similar ideas to those of Section 6.4 and [BGS15], i.e., integrating by parts, fundamental theorem of calculus, using the gradient estimate and applying an equivalence result as well as an estimate for controlling the pressure term. The equivalence result we need to apply in this case is the statement of Theorem 6.3.13. The pressure estimate needs then only to be considered in balls since all sub domains of Ω appearing in the definition of the BMO_b^M-norm are balls. The constant in this estimate (cf. Theorem 2.1 in [BGS15]) is scaling invariant such that we have suitable control about the pressure term in every ball. □

Remark 6.5.1. 1. *This Theorem avoids the previously necessary assumption to consider only small domains in the boundary seminorm, which in [BGS15] was ensured by taking ν small.*

2. *By Theorem 6.3.9 one obtains that if Ω is a Lipschitz half-space that is admissible and uniformly C^3, then S is an analytic semigroup in $VMO^{\infty,\infty}_{b,0,\sigma}(\Omega)$. Except for the case of the half-space this result was not included in the main result of [BGS15] since in all other cases ν needed to be finite.*

3. *The analyticity in $VMO^{\infty,\infty}_{b,0,\sigma}(\Omega)$ for a sector-like domain lying above a C^3-graph boundary as considered in [BGMST16] provides another approach to the proof of the L^p-analyticity for $p \in (2,\infty)$ in this domain. This was done there by interpolating $L^2 - L^2$ estimates with $L^\infty - BMO^{\infty,\nu}_b$ estimates for $S(t)Qu_0$ and $\frac{\mathrm{d}}{\mathrm{d}t}S(t)Qu_0$. Here, ν could be chosen arbitrarily, $u_0 \in C_c(\Omega)$ and Q is a projection operator from $L^2 \cap L^\infty$ onto $L^2_\sigma \cap VMO^{\infty,\nu}_{b,0,\sigma}$. For the proof in [BGMST16] it was assumed that ν is sufficiently small. By using the analyticity result of Theorem 6.5.2 one can now also assume $\nu = \infty$ in the proof. Note that for some of these domains for sufficiently large p the L^p-Helmholtz decomposition fails to hold.*

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Chapter 7

On large time behaviors of solutions of the Stokes and Navier-Stokes problems in BMO-type spaces

7.1 Introduction

The main purpose of this chapter is to study the behavior for large time of a solution of the Stokes and Navier-Stokes problem in BMO-type spaces. We also consider local energy decay estimates of the Stokes semigroups for large time in BMO-type spaces on some unbounded domains. Let Ω be a domain in $\mathbb{R}^n$. We first consider the initial value problem of the Stokes equation with the Dirichlet boundary conditions.

$$\begin{cases} \partial_t v - \Delta v + \nabla \pi = 0 & \text{in } \Omega \times (0, \infty) \\ \mathrm{div} v = 0 & \text{in } \Omega \times (0, \infty) \\ v = 0 & \text{on } \partial\Omega \times (0, \infty) \\ v(0) = v_0 & \text{in } \Omega, \end{cases}$$

where v_0 is a known initial data, v is an unknown vector field, π is an unknown scalar pressure. As an application, we also consider the initial value problem of the Navier-Stokes equation with the Dirichlet boundary condition:

$$\begin{cases} \partial_t V - \Delta V + (V \cdot \nabla) V + \nabla \Pi = 0 & \text{in } \Omega \times (0, \infty) \\ \mathrm{div} V = 0 & \text{in } \Omega \times (0, \infty) \\ V = 0 & \text{on } \partial\Omega \times (0, \infty) \\ V(0) = V_0 & \text{in } \Omega, \end{cases}$$

where V_0 is a known initial data, V is an unknown vector field, Π is an unknown scalar pressure, ∂_t is the time derivative, and $(V \cdot \nabla)V$ is the nonlinear term whose the ith component is $\displaystyle\sum_{k=1}^{n} V^k \frac{\partial V_i}{\partial x_k}$. The large time behavior of a solution of Navier-Stokes problem are often established in various kinds of spaces on some domains. Energy decay estimates of the Stokes semigroups are also established in order to obtain the large time behavior of a solution of Navier-Stokes problem. We would like to introduce literatures on L^p-L^q estimates.

In the case of the whole space $\mathbb{R}^n$, the Young's inequality implies L^p-L^q estimates for $1 < p \le q \le \infty (p \ne \infty)$. In the case of the half space $\mathbb{R}^n_+$, there are some known works, for example, as seen in [BM88], [Uka87]. In the work of [AS03], L^p-L^q estimates are established when the domain is a layer domains for $1 < p \le q < \infty$. In [Gig85], L^p-L^q estimates are established by applying the Stokes operator of fractional orders in the case when the domain is a bounded domain for $1 < p \le q < \infty$. This result is extended by [Mar11] to the case $p = 1,\ q = \infty$. There is a number of literature in which L^p-L^q estimates are considered for $1 \le p \le q \le \infty$ on exterior domains, for example, as seen in [BV93], [Che93], [DS99], [DS299], [DKS98], [Iwa89], [Mar11], [MS97], [Shi99]. In the case of the bent half space, L^p-L^q estimates are established in [Kub07] and [KS05]. In the case of an aperture domain, there are also some known works, for example, as seen in[Abe02], [His04].

In [Bol16], [BGST16], it is shown that the Stokes semigroup is bounded for large time in BMO-type spaces on various kinds of domains, such as, the whole space $\mathbb{R}^n$, the half space $\mathbb{R}^n_+$, and bounded domains.

However, there seems to be no result which establishes large time behaviors of a solution of nonlinear problems in BMO-type spaces.

This is because the nonlinear term $(V \cdot \nabla)V$ on BMO-type spaces is hard to deal with, i.e., it is not certain whether inequalities like $\|v^2\|_{BMO} \le \|v\|^2_{BMO}$ are valid. Our argument is based on the Kato's method. we first would like to discuss the outline of our argument.

There are literatures concerning the Navier-Stokes problem in L^r type spaces, for example, [FK62], [FK64], [GM85], [Kat84], [Miy81]. When the domain is bounded, these known results imply the existence and uniqueness of the Navier-Stokes problem with VMO initial data by the embedding theorem [Bol16, Theorem 5.44]. In this chapter, we will give an alternative proof of large time behaviors of a solution of nonlinear problems in BMO-type spaces.

Let Ω be a domain in $\mathbb{R}^n$, $n \ge 3$. We would like to state our main theorem. In Section 7.2, we will describe the statements more precisely.

Remark 7.1.1. *We will state L^q-BMO type estimates based on a duality argument with L^1-L^q estimates. We will establish a shaper gradient L^q-BMO type estimates which enable us to consider the Navier-Stokes problem by a successive approximation. We should like to remark that we can obtain L^q-BMO type estimates directly from L^q-L^∞ estimates in [Mar11], but the gradient L^q-BMO estimates obtained by this way seem not to be applied to the Navier-Stokes problem because we need shaper decay rates of the time valuable t. Therefore, we will* *establish gradient L^q-BMO type estimates in an indirect way in order to avoid this difficulty.*

Theorem 7.1.1. *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain, $n \geq 3$, with the C^m boundary $\partial\Omega$ for $2m > n$.*

(i) Let $\mu, \nu \in (0, \infty]$, $q \in (n/(n-1), \infty)$, $v_0 \in VMO^{\mu,\nu}_{b,0,\sigma}(\Omega)$ be a initial data, $S(t)$ be the solution operator of the Stokes problem. Then there exists a constant $C_{n,\Omega,q} > 0$ such that

$$\|S(t)v_0\|_{BMO_b^{\mu,\nu}} \leq Ct^{-\frac{n}{2q}}\|v_0\|_{L^q},$$
$$\|\nabla S(t)v_0\|_{BMO_b^{\mu,\nu}} \leq Ct^{-\frac{n}{2q}-\frac{1}{2}}\|v_0\|_{L^q}.$$

(ii) For all $\epsilon \in (0, 1)$, $\mu, \nu \in (0, \infty]$, $U, \nabla W \in VMO^{\mu,\nu}_{b,0,\sigma}$, there exists a constant $C > 0$ depending on $\epsilon, n, R_0^\epsilon$ such that

$$\|S(t)P(U \cdot \nabla W)\|_{BMO_b^{\mu,\nu}} \leq Ct^{-\frac{\epsilon}{2}}\|U\|_{BMO_b^{\mu,\nu}}\|\nabla W\|_{BMO_b^{\mu,\nu}}.$$

(iii) For all $\delta \in (0, 1/2)$ there exists a constant $\gamma = \gamma(\Omega, \delta, n) > 0$ such that if the initial data $V_0 \in VMO^{\mu,\nu}_{b,0,\sigma}(\Omega)$ satisfies $\|V_0\|_{BMO_b^{\mu,\nu}(\Omega)} \leq \gamma$, the Navier-Stokes problem admits a unique strong solution $V(t)$ on $(0, \infty)$. Moreover, for large $t > 0$

$$\|V(t)\|_{BMO_b^{\mu,\nu}} = o(t^{-\frac{\delta}{2}}), \|\nabla V(t)\|_{BMO_b^{\mu,\nu}} = o(t^{-\frac{\delta+1}{2}}).$$

Remark 7.1.2. *In Theorem 7.1.1 (i), we establish a shaper gradient L^q-BMO type estimates especially when t is larger than 1. We remark that we can obtain a gradient L^q-BMO type estimate directly from gradient L^q-L^∞ estimates in [Mar11] for $t \leq 1$ and for $t > 1$ with the decay rate $-\frac{n}{2q}$, but we need to check the decay rate of the time valuable t carefully when we apply L^q-BMO type estimates to the Navier-Stokes problem by a successive approximation.*

Let us illustrate our proof to establish Theorem 7.1.1. We have four steps to prove Theorem 7.1.1. The first step is L^q-$\mathcal{H}^1$ estimates of the Stokes semigroups, and the second step is a duality argument in order to obtain Theorem 7.1.1 (i). The third step is the application of Theorem 7.1.1 (i) and Hölder type inequalities to Theorem 7.1.1 (ii). The final step is based on the Kato's method to finish our proof of Theorem 7.1.1. We shall briefly explain each step.

Step 1 (L^q-$\mathcal{H}^1$ estimates of the Stokes semigroups)

We consider the initial value problem of the Stokes equation with the Dirichlet boundary conditions.

$$\begin{cases} \partial_t v - \Delta v + \nabla\pi = 0 & \text{in } \Omega \times (0, \infty) \\ \mathrm{div} v = 0 & \text{in } \Omega \times (0, \infty) \\ v = 0 & \text{on } \partial\Omega \times (0, \infty) \\ v(0) = v_0 & \text{in } \Omega, \end{cases}$$

Let $S(t)(v_0) = v$ be the solution operator of the Stokes problem. Step1 is mainly due to the following L^1-L^q estimates for $q \in (1, n]$.

$$\|S(t)v_0\|_{L^q} \leq Ct^{-\frac{n}{2}(1-\frac{1}{q})}\|v_0\|_{L^1}, \tag{7.1.1}$$

$$\|\nabla S(t)v_0\|_{L^q} \leq Ct^{-\frac{n}{2}(1-\frac{1}{q})-\frac{1}{2}}\|v_0\|_{L^1}. \tag{7.1.2}$$

L^p-L^q estimates of the Stokes semigroups are established on various kinds of domains.

In order to obtain BMO-type estimates by a duality argument, we need estimates in Hardy spaces on domains. Hardy spaces on domains are introduced in [Miy90]. We denote this Hardy spaces by $\mathcal{H}^1_M(\Omega)$. We will recall the definition of $\mathcal{H}^1_M(\Omega)$ in Subsection 7.3.1. As in the case of $\mathcal{H}^1(\mathbb{R}^n)$, we can get the following inequality

$$\|f\|_{L^1(\Omega)} \leq \|f\|_{\mathcal{H}^1_M(\Omega)}. \tag{7.1.3}$$

By (7.1.1), (7.1.3), we conclude that

$$\|S(t)v_0\|_{L^q(\Omega)} \leq Ct^{-\frac{n}{2}(1-\frac{1}{q})}\|v_0\|_{\mathcal{H}^1_M(\Omega)}. \tag{7.1.4}$$

Step 2 (Duality arguments)

In this step, we obtain BMO-type estimates from Hardy type estimates in Step1 by a duality argument. For our duality arguments, we also recall the Stokes resolvent estimates which are obtained together with L^p-L^q estimates. Let u_λ be a solution of the Stokes resolvent equation with the Dirichlet boundary condition

$$\begin{cases} (\lambda - \Delta)u_\lambda + \nabla p_\lambda = f & \text{in } \Omega \\ \mathrm{div} u_\lambda = 0 & \text{in } \Omega \\ u_\lambda = 0 & \text{on } \partial\Omega. \end{cases}$$

where f is a given external force term and p_λ is an unknown pressure term. Then u_λ satisfies the Stokes resolvent estimate for $\mathrm{Re}\lambda \geq 0$ ($\lambda \neq 0$)

$$|\lambda|^{-\frac{n}{2}(1-\frac{1}{q})+1}\|u_\lambda\|_{L^q} \leq C\|f\|_{\mathcal{H}^1_M}. \tag{7.1.5}$$

The resolvent estimate (7.1.5) is obtained from (7.1.4) by arguments with the Laplace transform, for example, as in [His04, Lemma 3.3].

Since $BMO^{\mu,\nu}_b(\Omega) = BMO^M_b(\Omega) = (\mathcal{H}^1_M(\Omega))^*$ by the equivalence of BMO spaces in [Bol16], [BGST16], and the duality theorem in [Miy90], we obtain the following equality

$$\|u_\lambda\|_{BMO^{\mu,\nu}_b} = \sup\{|(u_\lambda, g)| : \ g \in \mathcal{H}^1_M(\Omega), \ \|g\|_{\mathcal{H}^1_M(\Omega)} \leq 1\}. \tag{7.1.6}$$

Let ψ_λ be a solution of the Stokes problem with the external force g,

$$\begin{cases} (\lambda - \Delta)\psi_\lambda + \nabla q_\lambda = g & \text{in } \Omega \\ \mathrm{div} \psi_\lambda = 0 & \text{in } \Omega \\ \psi_\lambda = 0 & \text{on } \partial\Omega. \end{cases}$$

Then we can calculate as

$$\begin{aligned}(u_\lambda, g) &= (u_\lambda, (\lambda - \Delta)\psi_\lambda + \nabla q_\lambda)\\ &= ((\lambda - \Delta)u_\lambda, \psi_\lambda) - (\mathrm{div} u_\lambda, q_\lambda)\\ &= ((\lambda - \Delta)u_\lambda, \psi_\lambda)\\ &= ((\lambda - \Delta)u_\lambda, \psi_\lambda) - (p_\lambda, \mathrm{div}\psi_\lambda)\\ &= ((\lambda - \Delta)u_\lambda + \nabla p_\lambda, \psi_\lambda)\\ &= (f, \psi_\lambda).\end{aligned}$$

Therefore, we obtain the following estimate by [Bol16, Theorem 5.44], and (7.1.5)

$$\begin{aligned}|(u_\lambda, g)| &\le \|f\|_{L^{q'}(\Omega)}\|\psi_\lambda\|_{L^q(\Omega)}\\ &\le C_{n,p,diam(\Omega)}\|f\|_{BMO_b^{\mu,\nu}(\Omega)}\|\psi_\lambda\|_{L^q(\Omega)}\\ &\le C_{n,p,q,diam(\Omega)}|\lambda|^{\frac{n}{2}(1-\frac{1}{q})-1}\|f\|_{BMO_b^{\mu,\nu}(\Omega)}\|g\|_{\mathcal{H}^1_M(\Omega)}\end{aligned}$$

where q' is the Hölder conjugate of q. By (7.1.6), this implies that

$$\|u_\lambda\|_{BMO_b^{\mu,\nu}} \le C_{n,q,diam(\Omega)}|\lambda|^{\frac{n}{2q'}-1}\|f\|_{BMO_b^{\mu,\nu}(\Omega)}. \tag{7.1.7}$$

We also need decay estimates of $u_{s+h} - u_s$ to finish our duality argument by the inverse Laplace transform formula as in [His04, Lemma 5.2]. This decay estimates of $u_{s+h} - u_s$ in BMO is obtained from a duality argument and the following deformations of equality

$$\begin{aligned}(u_{s+h} - u_s, g) &= (u_{s+h}, g) - (u_s, g)\\ &= (u_{s+h}, (s + h - \Delta)\psi_{s+h} + \nabla q_{s+h}) - (u_s, (s - \Delta)\psi_s + \nabla q_s)\\ &= (f, \psi_{s+h}) - (f, \psi_s) = (f, \psi_{s+h} - \psi_s).\end{aligned}$$

These resolvent estimates imply for $\epsilon = n/q'$

$$\|S(t)v_0\|_{BMO_b^{\mu,\nu}} \le C_{n,\epsilon,diam(\Omega)}t^{-\frac{\epsilon}{2}}\|v_0\|_{BMO_b^{\mu,\nu}(\Omega)}. \tag{7.1.8}$$

We can also obtain the following estimates by similar arguments

$$\begin{aligned}&\|S(t)v_0\|_{BMO_b^{\mu,\nu}} \le Ct^{-\frac{n}{2q'}}\|v_0\|_{L^{q'}}\\ &\|\nabla S(t)v_0\|_{BMO_b^{\mu,\nu}} \le Ct^{-\frac{n}{2q'}-\frac{1}{2}}\|v_0\|_{L^{q'}}\end{aligned} \tag{7.1.9}$$

In particular, we obtain a shaper gradient L^q-BMO type estimates for large t, and $q \in (n/(n-1), \infty)$.

Step 3 (Application to estimates of nonlinear terms)

In this section, we consider the estimate of the nonlinear term of the integral equation

$$V(t) = S(t)V_0 - \int_0^t T(t-s)P(V \cdot \nabla V)(s)ds. \tag{7.1.10}$$

By (7.1.9) and L^q boundedness of the Helmholtz projection on bounded domains, and [Bol16, Theorem 5.44], for $q \in (n/(n-1), \infty)$

$$\begin{aligned}\|S(t)P(U\cdot\nabla W)\|_{BMO_b^{\mu,\nu}} &\le C_{n,q,diam(\Omega)} t^{-\frac{nq}{2}} \|P(U\cdot\nabla W)\|_{L^q}\\ &\le C_{n,q,diam(\Omega)} t^{-\frac{n}{2q}} \|U\cdot\nabla W)\|_{L^q}\\ &\le C_{n,p,diam(\Omega)} t^{-\frac{n}{2q}} \|U\|_{L^{2q}} \|\nabla W\|_{L^{2q}}\\ &\le Ct^{-\frac{\epsilon}{2}} \|U\|_{BMO_b^{\mu,\nu}} \|\nabla W\|_{BMO_b^{\mu,\nu}}.\end{aligned}$$

As a consequence, we conclude that

$$\begin{aligned}&\|S(t)P(U\cdot\nabla W)\|_{BMO_b^{\mu,\nu}} \le Ct^{-\frac{\epsilon}{2}} \|U\|_{BMO_b^{\mu,\nu}} \|\nabla W\|_{BMO_b^{\mu,\nu}},\\ &\|\nabla S(t)P(U\cdot\nabla W)\|_{BMO_b^{\mu,\nu}} \le Ct^{-\frac{\epsilon+1}{2}} \|U\|_{BMO_b^{\mu,\nu}} \|\nabla W\|_{BMO_b^{\mu,\nu}},\end{aligned} \tag{7.1.11}$$

for $\mu, \nu \in (0, \infty]$, $\epsilon \in (0, 1)$.

Step 4 (Kato's arguments)

In this step, we would like to start from the existence and uniqueness of a solution of the integral equation (7.1.10). Subsequently, we establish that This solution is a solution of the Navier-Stokes problem. Step 4 is based on Kato's arguments, for example, as in [Kat84] in which the Navier-Stokes problem on L^p spaces are considered. We shall explain the outline of this argument. We construct a solution of (7.1.10) by a successive approximation. Let $X_0 = S(t)V_0$, and let us consider $X_{m+1} = X_0 + GX_m$ where $GU(t) = -\int_0^t T(t-s)P(U\cdot\nabla U)(s)ds$. By (7.1.11),

$$\begin{aligned}&\|GU(t)\|_{BMO_b^{\mu,\nu}} \le C\int_0^t (t-s)^{-\frac{\epsilon}{2}} \|U\|_{BMO_b^{\mu,\nu}} \|\nabla U\|_{BMO_b^{\mu,\nu}} ds,\\ &\|\nabla GU(t)\|_{BMO_b^{\mu,\nu}} \le C\int_0^t (t-s)^{-\frac{\epsilon+1}{2}} \|U\|_{BMO_b^{\mu,\nu}} \|\nabla U\|_{BMO_b^{\mu,\nu}} ds\end{aligned} \tag{7.1.12}$$

We argue by induction. We take $K_0 = K_0' = c\|V_0\|_{BMO_b^{\mu,\nu}}$, and we fix an arbitrary $\delta \in (0, \frac{1}{2})$. We assume that there exist small constant K_m, K_m'

$$\|X_m\|_{BMO_b^{\mu,\nu}} \le ct^{-\frac{\delta}{2}} K_m,\ \|\nabla X_m\|_{BMO_b^{\mu,\nu}} \le ct^{-\frac{\delta+1}{2}} K_m'. \tag{7.1.13}$$

The continuity of $t^{\frac{\delta}{2}} X_0$ at $t = 0$, and the decay estimates of X_0 are corollaries of the boundedness of the Stokes semigroups on $BMO_{b,\sigma}^{\mu,\nu}(\Omega)$ in [Bol16, Theorem 6.50]. By (7.1.8), we can also obtain the decay estimates for large time of X_0. By assumption and the estimates (7.1.12), we obtain the estimate of the nonlinear terms

$$\begin{aligned}&\|GX_m\|_{BMO_b^{\mu,\nu}} \le C\int_0^t (t-s)^{-\frac{\epsilon}{2}} s^{-\frac{2\delta+1}{2}} K_m K_m' ds,\\ &\|\nabla GX_m\|_{BMO_b^{\mu,\nu}} \le C\int_0^t (t-s)^{-\frac{\epsilon+1}{2}} s^{-\frac{2\delta+1}{2}} K_m K_m' ds\end{aligned} \tag{7.1.14}$$

The two integrals $\int_0^t (t-s)^{-\frac{\epsilon}{2}} s^{-\frac{2\delta+1}{2}} ds = t^{-\frac{\epsilon+2\delta-1}{2}} \int_0^1 (1-z)^{-\frac{\epsilon}{2}} z^{-\frac{2\delta+1}{2}} dz$ and $\int_0^t (t-s)^{-\frac{\epsilon+1}{2}} s^{-\frac{2\delta+1}{2}} ds$ converge for $\epsilon \in (0,1),\ \delta \in (0, \frac{1}{2})$. Since $\epsilon \in (0,1)$ is arbitrary, we choose $\epsilon = \delta$ so that we can compare the local decays with respect to t, i.e., $t^{-\frac{\epsilon}{2}} + t^{-\frac{\epsilon+2\delta-1}{2}} \leq 2t^{-\frac{\delta}{2}}$ for $0 < t \leq 1$. For the case of $1 \leq t$, we choose $\epsilon = 1-\delta$ so that $t^{-\frac{\epsilon}{2}} + t^{-\frac{\epsilon+2\delta-1}{2}} \leq 2t^{-\frac{\delta}{2}}$ for $1 \leq t$. As a consequence, we obtain the decay of $t^{\frac{\delta}{2}} X_{m+1}$ and $t^{\frac{\delta+1}{2}} X'_{m+1}$

$$
\begin{aligned}
K_{m+1} &\leq K_0 + \|GX_m\|_{BMO_b^{\mu,\nu}} \\
&\leq C_{n,\delta,diam(\Omega)} t^{-\frac{\delta}{2}} \|V_0\|_{BMO_b^{\mu,\nu}(\Omega)} + C_{n,\delta,diam(\Omega)} t^{-\frac{\epsilon+2\delta-1}{2}} \\
&\leq C_{n,\delta,diam(\Omega)} t^{-\frac{\delta}{2}} (K_0 + K_m K'_m), \\
K'_{m+1} &\leq K'_0 + \|\nabla GX_m\|_{BMO_b^{\mu,\nu}} \leq C_{n,\delta,diam(\Omega)} t^{-\frac{\delta+1}{2}} (K_0 + K_m K'_m). \quad (7.1.15)
\end{aligned}
$$

These imply the continuity of $t^{\frac{\delta}{2}} X_{m+1}$ and $t^{\frac{\delta+1}{2}} X'_{m+1}$ at $t = 0$. As was shown in [Kat84], We can take small $\gamma > 0$ depending on $n,\ \delta,\ diam(\Omega)$ so that if $\|V_0\|_{BMO_{b,\sigma}^{\mu,\nu}} \leq \gamma$ then $K_m,\ K'_m$ are uniformly bounded by a fixed constant K.

Therefore, we conclude that $\{X_m\},\ \{\nabla X_m\}$ are uniformly bounded in $BMO_b^{\mu,\nu}$.

$$
\|X_m\|_{BMO_b^{\mu,\nu}} \leq ct^{-\frac{\delta}{2}} K,\ \|\nabla X_m\|_{BMO_b^{\mu,\nu}} \leq ct^{-\frac{\delta+1}{2}} K'. \quad (7.1.16)
$$

By carrying on with an argument based on [Kat84], we obtain a unique global solution $V(t)$ of the integral equation $V(t) = S(t)V_0 - \int_0^t T(t-s)P(V\cdot\nabla V)(s)ds$. By a Hölder estimate, it turns out that $V(t)$ is a strong solution of the Navier-Stokes problem.

7.2 Main theorems

We would like to state main theorem of this chapter.

Let us recall the definition of BMO-type spaces. For $f \in L^1_{loc}(\Omega)$, $\mu, \nu \in (0,\infty]$, $p \in [1,\infty)$, we define the seminorms $[f]_{BMO^\mu}$ and $[f]_{b^\nu}$ by

$$
\begin{aligned}
[f]_{BMO^\mu p} &= \sup\{(\frac{1}{|B_r(x)|} \int_{B_r(x)} |f(y) - f_{B_r(x)}|^p dy)^{\frac{1}{p}} :\ B_r(x) \subset \Omega,\ 0 < r < \mu\}, \\
[f]_{b^\nu p} &= \sup\{(r^{-n} \int_{\Omega\cap B_r(x)} |f(y)|^p dy)^{\frac{1}{p}} :\ x \in \partial\Omega,\ 0 < r < \nu\},
\end{aligned}
$$

where $B_r(x)$ denotes the closed ball of radius $r > 0$ centered at $x \in \Omega$ and $f_{B_r(x)} = 1/|B_r(x)| \int_{B_r(x)} f(y)dy$. Let $\|f\|_{BMO_b^{\mu,\nu}} = [f]_{BMO^\mu p} + [f]_{b^\nu p}$. Then the BMO-type spaces are defined by

$$
BMO_b^{\mu,\nu}(\Omega) = \{f \in L^1_{loc}(\Omega) :\ \|f\|_{BMO_b^{\mu,\nu}} < +\infty\},
$$

$$
VMO_{b,0}^{\mu,\nu}(\Omega) = \overline{C_c^\infty}^{BMO_b^{\mu,\nu}}, VMO_{b,0,\sigma}^{\mu,\nu}(\Omega) = \overline{C_{c,\sigma}^\infty}^{BMO_b^{\mu,\nu}},
$$

where $C_c^\infty(\Omega)$ is the space of all smooth functions with the compact support in Ω and $C_{c,\sigma}^\infty(\Omega)$ is the space of all smooth solenoidal vector fields with the compact support in Ω. We also recall Miyachi's BMO space. The space $BMO^M(\Omega)$ is defined by $BMO^M(\Omega) = \{f \in L^1_{loc}(\Omega) : \ \|f\|_M = [f]_{BMO^M} + [f]_{b^M} < +\infty\}$ where

$$[f]_{BMO^M} = \sup\{\frac{1}{|B_r(x)|}\int_{B_r(x)} |f(y) - f_{B_r(x)}|dy : \ B_{2r}(x) \subset \Omega\},$$
$$[f]_{b^M} = \sup\{\frac{1}{|B_r(x)|}\int_{B_r(x)} |f(y)|dy : B_{2r}(x) \subset \Omega, \ B_{5r}(x) \cap \Omega^c \neq \emptyset\}.$$

Equivalences between our BMO spaces and Miyachi's BMO spaces are established in [Bol16], [BGST16].

Theorem 7.2.1 ([Bol16], [BGST16]). *Consider one of the following cases.*

(i) $\Omega = \mathbb{R}^n$, $\mu = \nu = \infty$,

(ii) $\Omega \subset \mathbb{R}^n$ *is a Lipschitz half space,* $\mu = \nu = \infty$,

(iii) $\Omega \subset \mathbb{R}^n$ *is a bounded Lipschitz domain,* $\mu, \nu \in (0, \infty]$,

(iv) $\Omega = \mathbb{R}^k \times G \subset \mathbb{R}^n$ *for* $1 \leq k \leq n-1$, *a bounded* C^2 *domain* G, $\mu, \nu \in (0, \infty]$,

(v) $\Omega \subset \mathbb{R}^n$ *is a Lipschitz exterior domain,* $\mu = \nu = \infty$.

Then, $\|\cdot\|_{BMO_b^{\mu,\nu}}$ *and* $\|\cdot\|_M$ *are equivalent.*

We would like to state L^q-$\mathcal{H}^1_M$ estimates of the Stokes semigroups. We will recall the definition of the Hardy space $\mathcal{H}^1_M$ on a domain defined in [Miy90] in section 7.3.1.

Theorem 7.2.2 (L^q-Hardy estimates). *Let* $\Omega \subset \mathbb{R}^n$ *be a bounded domain or an exterior domain,* $n \geq 3$, *with the* C^m *boundary* $\partial\Omega$ *for* $2m > n$. *Let* $q \in (1, n]$, $v_0 \in \mathcal{H}^1_M(\Omega)$ *be a initial data,* $S(t)$ *be the solution operator of the Stokes problem. Then there exists a constant* $C_{n,q,\Omega} > 0$ *such that for* $t > 0$

$$\|S(t)v_0\|_{L^q(\Omega)} \leq Ct^{-\frac{n}{2}(1-\frac{1}{q})}\|v_0\|_{\mathcal{H}^1_M(\Omega)},$$
$$\|\nabla S(t)v_0\|_{L^q(\Omega)} \leq Ct^{-\frac{n}{2}(1-\frac{1}{q})-\frac{1}{2}}\|v_0\|_{\mathcal{H}^1_M(\Omega)}.$$

Theorem 7.2.3 (BMO-type estimates). *Let* $\Omega \subset \mathbb{R}^n$ *be a bounded domain or an exterior domain,* $n \geq 3$, *with the* C^m *boundary* $\partial\Omega$ *for* $2m > n$. *Let* $q \in (n/(n-1), \infty)$, $\mu, \nu \in (0, \infty]$, $v_0 \in VMO^{\mu,\nu}_{b,0,\sigma}(\Omega)$ *be a initial data,* $S(t)$ *be the solution operator of the Stokes problem. Then there exists a constant* $C_{n,q,\Omega} > 0$ *such that for* $t > 0$

$$\|S(t)v_0\|_{BMO_b^{\mu,\nu}} \leq Ct^{-\frac{n}{2q}}\|v_0\|_{L^q},$$
$$\|\nabla S(t)v_0\|_{BMO_b^{\mu,\nu}} \leq Ct^{-\frac{n}{2q}-\frac{1}{2}}\|v_0\|_{L^q},$$
$$\|S(t)v_0\|_\infty \leq Ct^{-\frac{n}{2q}}\|v_0\|_{L^q},$$
$$\|\nabla S(t)v_0\|_\infty \leq Ct^{-\frac{n}{2q}-\frac{1}{2}}\|v_0\|_{L^q}.$$

Futhermore, if $\Omega \subset \mathbb{R}^n$ *is a bounded domain with the radius* R_0*, then for* $\epsilon \in (0,1)$ *there exists* $C_{n,\Omega,R_0^\epsilon} > 0$ *such that*

$$\|S(t)v_0\|_{BMO_b^{\mu,\nu}} \le C_{n,\Omega,R_0^\epsilon} t^{-\frac{\epsilon}{2}} \|v_0\|_{BMO_b^{\mu,\nu}},$$
$$\|\nabla S(t)v_0\|_{BMO_b^{\mu,\nu}} \le C_{n,\Omega,R_0^\epsilon} t^{-\frac{\epsilon+1}{2}} \|v_0\|_{BMO_b^{\mu,\nu}},$$
$$\|S(t)v_0\|_\infty \le C_{n,\Omega,R_0^\epsilon} t^{-\frac{\epsilon}{2}} \|v_0\|_{BMO_b^{\mu,\nu}},$$
$$\|\nabla S(t)v_0\|_\infty \le C_{n,\Omega,R_0^\epsilon} t^{-\frac{\epsilon+1}{2}} \|v_0\|_{BMO_b^{\mu,\nu}}.$$

For $R > 0$, we define a local space $L^1_R(\Omega) = \{f \in L^1(\Omega) : \ \mathrm{supp} f \subset B_R\}$. We state local decay estimates on some unbounded domains.

Theorem 7.2.4 (Local energy decay estimate)**.** *Let* $\Omega \subset \mathbb{R}^n$ *be any one of following cases. (i) a smooth bent half space for* $n \ge 2$*, (ii) a smooth exterior domain for* $n \ge 3$*,. Let* $R > 0$*,* $v_0 \in L^1_R(\Omega) \cap VMO^{\infty,\infty}_{b,0,\sigma}(\Omega)$*. Then, for* $0 < \epsilon < 1$ *there exists a constant* $C > 0$ *depending on* ϵ, n, R^ϵ *such that*

$$\|S(t)v_0\|_{BMO_b^{\infty,\infty}(\Omega)} \le C t^{-\frac{\epsilon}{2}} \|v_0\|_{BMO_b^{\infty,\infty}(\Omega\cap B_R)},$$
$$\|\nabla S(t)v_0\|_{BMO_b^{\infty,\infty}(\Omega)} \le C t^{-\frac{\epsilon+1}{2}} \|v_0\|_{BMO_b^{\infty,\infty}(\Omega\cap B_R)},$$
$$\|T(t)v_0\|_\infty \le C t^{-\frac{\epsilon}{2}} \|v_0\|_{BMO_b^{\infty,\infty}(\Omega\cap B_R)},$$
$$\|\nabla S(t)v_0\|_\infty \le C t^{-\frac{\epsilon+1}{2}} \|v_0\|_{BMO_b^{\infty,\infty}(\Omega\cap B_R)}.$$

Furthermore, if Ω *is a smooth aperture domain for* $n \ge 3$*, we obtain same statements by substituting* BMO^M *for* $BMO_b^{\infty,\infty}$*.*

Remark 7.2.1. *In [Mar14], The boundedness of the Stokes semigroups with non decaying initial data in* L^∞_σ *on exterior domains is established. The solution operator is decomposed into some terms. One term has a representation formula by means of the heat kernel to overcome the difficulties of a non decaying data, and the other terms are constructed by means of* L^p *theory. In [Mar14], the first term is estimated directly in* L^∞ *spaces and the other terms are estimated by a* L^1*-*L^q *duality argument. However, the part of a direct calculation seems to be difficult in BMO-type spaces. For this reason, we assume that the initial data has a compact support included uniformly in a ball.*

We prove Theorem 7.2.4 by a duality of L^q-$\mathcal{H}^1$ estimate and the embedding $BMO_b^{\infty,\infty} \to L^p$ on a bounded domain.

Estimates of the operator $S(t)P\mathrm{div}$ often imply estimates of the nonlinear term when we consider the Navier-Stokes problem in L^∞ type spaces. However, this argument would not work on BMO-type spaces because it isn't certain whether the inequality $\|v^2\|_{BMO} \le \|v\|^2_{BMO}$ is valid.

Lemma 7.2.1. *Let* $\Omega \subset \mathbb{R}^n$ *be a bounded domain,* $n \ge 3$*, with the* C^m *boundary* $\partial\Omega$ *for* $2m > n$*. Let* $\mu, \nu \in (0,\infty)$*,* $U, \nabla W \in VMO_b^{\mu,\nu}(\Omega)$*. Then for* $\epsilon \in (0,1)$ *there exists a constant* $C_{n,\Omega,\epsilon} > 0$ *such that*

$$\|S(t)P(U\cdot\nabla W)\|_{BMO_b^{\mu,\nu}} \le Ct^{-\frac{\epsilon}{2}}\|U\|_{BMO_b^{\mu,\nu}}\|\nabla W\|_{BMO_b^{\mu,\nu}},$$
$$\|\nabla S(t)P(U\cdot\nabla W)\|_{BMO_b^{\mu,\nu}} \le Ct^{-\frac{\epsilon+1}{2}}\|U\|_{BMO_b^{\mu,\nu}}\|\nabla W\|_{BMO_b^{\mu,\nu}}.$$

Theorem 7.2.5. *Let $\Omega\subset\mathbb{R}^n$ be a bounded domain, $\mu,\nu\in(0,\infty]$. Then for all $\delta\in(0,1/2)$ there exists a small constant $\gamma=\gamma(\Omega,\delta,n)>0$ such that if $V_0\in VMO_{b,0,\sigma}^{\mu,\nu}(\Omega)$ satisfies $\|V_0\|_{BMO_b^{\mu,\nu}(\Omega)}\le\gamma$ the Navier-Stokes problem admits a unique strong solution $V(t)$ on $(0,\infty)$. Moreover, for large $t>0$*

$$\|V(t)\|_{BMO_b^{\mu,\nu}} = o(t^{-\frac{\delta}{2}}), \|\nabla V(t)\|_{BMO_b^{\mu,\nu}} = o(t^{-\frac{\delta+1}{2}}).$$

7.3 Local decay estimates of the Stokes semigroups

7.3.1 Estimates on Hardy spaces

Let us define Miyachi's Hardy spaces. In order to define Miyachi's Hardy spaces, we first prepare some notations.

Definition 7.3.1 ([Miy90])**.** *Let P_m be the set of all polynomials of order not exceeding m. For $s>0$ $f\in L^1_{loc}(\mathbb{R}^n)$,*

$$\|f\|_{\Lambda(s)} = \sup_B \inf\{|B|^{-\gamma-s}\int_B |f(x)-p(x)|dx:\ p\in P_{[s]}\}.$$

Let $\Omega\subset\mathbb{R}^n$ be a open set, and $\tilde f$ be the zero extension of $f:\Omega\to\mathbb{R}$ from Ω to $\mathbb{R}$. If $\tilde f\in L^1_{loc}(\mathbb{R}^n)$, we set

$$\|f\|_{\lambda(s,\Omega)} = \|\tilde f\|_{\Lambda(s)} + \sup_{x\in\Omega}\{|f(x)|(dist(x,\Omega^c))^{-s}\},$$

$$\Lambda(s,\Omega) = \{f\in L^1_{loc}(\Omega):\ \tilde f\in L^1_{loc}(\mathbb{R}^n)\ \|f\|_{\Lambda(s,\Omega)} < +\infty\}.$$

We would like to define a good kernel. Friedrichs mollifiers are typical examples of good kernels.

Definition 7.3.2 ([Miy90])**.** $\Phi:\mathbb{R}^n\times\mathbb{R}^n\times(0,\infty)\to\mathbb{R}$ *is said to be a good kernel if for some functions K_s with respect to the valuable $s\in(0,\infty)$ Φ satisfies the following properties*

(i) $\Phi(x,\cdot,t)\in C_0^\infty(B_t(x))$ *for $x\in\mathbb{R}^n$, $t>0$,*

(ii) For $x\in\mathbb{R}^n$, $t>0$, $s>0$,

$$\|\Phi(x,\cdot,t)\|_{\Lambda(s)} \le K_s t^{-\gamma-s},$$

(iii) *For $y \in \mathbb{R}^n$, $t > 0$,*

$$\Phi(\cdot, y, t) \in L^1(\mathbb{R}^n), \int \Phi(x, y, t) dx = 1,$$

(iv) *For $g \in L^\infty(\mathbb{R}^n)$ with the compact support supp g, $t > 0$, we define $g\#\Phi(t)$ by $(g\#\Phi(t))(y) = \int g(x)\Phi(x, y, t)dx$. Then, for $s > 0$*

$$\|g\#\Phi(t)\|_{\Lambda(s)} \leq K_s \|g\|_{\Lambda(s)},$$

(v) *If $g \in L^\infty(\mathbb{R}^n)$ with the compact support supp g, $t > 0$, then $g\#\Phi(t) \in C_0^\infty(\mathbb{R}^n)$,*

(vi) *For $f \in D'(\mathbb{R}^n)$, $t > 0$, a function $x \to < f, \Phi(x, \cdot, t) >$ is in $L^1_{loc}(\mathbb{R}^n)$,*

(vii) *If $f \in D'(\mathbb{R}^n)$, $g \in L^\infty(\mathbb{R}^n)$ with the compact support supp g, $t > 0$, then*

$$< f, g\#\Phi(t) >= \int g(x) < f, \Phi(x, \cdot, t) > dx,$$

(viii) *For a open set $U \subset \mathbb{R}^n$, $f \in D'(\Omega)$, let $M^+_{\Phi,\Omega}(f)(x)$ be a maximal function $M^+_{\Phi,\Omega}(f)(x) = \sup\{| < f, \Phi(x, \cdot, t) > | : \ 0 < t < dist(x, \Omega^c)\}$. Then, $M^+_{\Phi,\Omega}(f)(x)$ is measurable.*

We shall use the notation $\|f\|_{p,\Phi,\Omega} = \|M^+_{\Phi,\Omega}(f)\|_{L^p(\Omega)}$ as seen in [Miy90]. We define the grand maximal function by

$$f^*_{s,\Omega} = \sup_\Psi | < f, \Psi > |$$

where sup is taken over Ψ for which there exists $t = t_\Psi > 0$ such that $\Psi \in C_0^\infty(B_t(x) \cap \Omega)$ and $\|\Psi\|_{\Lambda(s,B_t(x)\cap\Omega)} \leq t^{-\gamma-s}$. It is shown by [Miy90] that the maximal function $M^+_{\Phi,\Omega}$ is well-defined with respect to Φ.

Theorem 7.3.1 ([Miy90]). *Let Φ, Φ' be good kernels, $s > 0$, $0 < p \leq \infty$. Then, there exists $C_{p,\Phi,\Phi'} > 0$ such that for all open subsets $\Omega \subset \mathbb{R}^n$ and $f \in D'(\Omega)$*

$$\|f\|_{p,\Phi',\Omega} \leq C_{p,\Phi,\Phi'} \|f\|_{p,\Phi,\Omega}.$$

A proof of this equivalence theorem is obtained by comparing each maximal function with the grand maximal function. We would like to recall Miyachi's Hardy spaces.

Definition 7.3.3 ([Miy90]). *Let $\Omega \subset \mathbb{R}^n$, Φ be a good kernel.*

$$\mathcal{H}^1_M(\Omega) = \{f \in L^1_{loc}(\Omega) : \ \|f\|_{1,\Phi,\Omega} < \infty\}.$$

The density property of $\mathcal{H}^1_M(\Omega)$ is shown in [Miy90].

Lemma 7.3.1 (Density property of $\mathcal{H}^1_M(\Omega)$ [Miy90])**.** *$C_0^\infty(\Omega)$ is densely included in $\mathcal{H}^1_M(\Omega)$.*

$$C_0^\infty(\Omega) \subset \mathcal{H}^1_M(\Omega).$$

It is also shown by [Miy90] that if Ω satisfies some extension property then there exists an extension operator from $\mathcal{H}^1_M(\Omega)$ to $\mathcal{H}^1(\mathbb{R}^n)$.

Definition 7.3.4. *$\Omega \subset \mathbb{R}^n$ is said to have $\mathcal{H}^p_M$ extension properties if there exists a constant $A > 0$ such that for all $x \in \Omega$ there exists $x' \in \Omega^c$ which satisfies*

$$dist(x, x') < A dist(x, \Omega^c),$$
$$dist(x', \Omega) > A^{-1} dist(x, \Omega^c).$$

Remark 7.3.1. *There are some examples of domains which have $\mathcal{H}^p_M$ extension properties. If we consider one of the following cases,*

(i) $\Omega \subset \mathbb{R}^n$ *is a bounded domain,*

(ii) $\Omega \subset \mathbb{R}^n$ *is a sector-like domain,*

(iii) $\Omega \subset \mathbb{R}^n$ *is a bent half space,*

(iv) $\Omega = \mathbb{R}^k \times G \subset \mathbb{R}^n$ *for* $1 \le k \le n-1$*, a bounded domain* G*,*

then Ω has $\mathcal{H}^p_M$ extension properties.

Lemma 7.3.2 (Miyachi's extension [Miy90])**.** *Let $\Omega \subset \mathbb{R}^n$ have $\mathcal{H}^p_M$ extension properties. Then, there exists an extension operator $T : \mathcal{H}^1(\Omega) \to \mathcal{H}^1(\mathbb{R}^n)$ such that*

$$Tf|_\Omega = f, \ \|Tf\|_{\mathcal{H}^1(\mathbb{R}^n)} \le C\|f\|_{\mathcal{H}^1(\Omega)}.$$

We remark that when the domain $\Omega \subset \mathbb{R}^n$ is the half space $\mathbb{R}^n_+$ there is an another definition of Hardy spaces $\mathcal{H}^1_{GMS}(\mathbb{R}^n_+)$ defined in [GMS99] :

$$\mathcal{H}^1_{GMS}(\mathbb{R}^n_+) = \{f \in L^1_{loc}(\mathbb{R}^n_+) : \ \|f\|_{\mathcal{H}^1_{GMS}(\mathbb{R}^n_+)} < \infty\}$$

where $\|f\|_{\mathcal{H}^1_{GMS}(\mathbb{R}^n_+)} = \inf\{\|\tilde{f}\|_{\mathcal{H}^1(\mathbb{R}^n)} : \ \tilde{f} \in \ \mathcal{H}^1(\mathbb{R}^n), \ \tilde{f}|_{\mathbb{R}^n_+} = f\}$. We remark that $\mathcal{H}^1_M(\Omega)$ is included in $L^1(\Omega)$.

Lemma 7.3.3.

$$\|f\|_{L^1(\Omega)} \le C\|f\|_{\mathcal{H}^1_M(\Omega)}.$$

Proof. A proof of Lemma 7.3.3 is obtained samely as the case of $\Omega = \mathbb{R}^n$, for example, as seen in [Oga13, Proposition 9.1.1, p150]. $\square$

Remark 7.3.2. *The Miyachi's Hardy space is originally defined as a subspace of $D'(\Omega)$, but it turns out that the original Miyachi's Hardy space is the same space as defined a subspace of $L^1_{loc}(\Omega)$.*

By the extension lemma (Lemma 7.3.2) we can also compare $\mathcal{H}^1_M(\mathbb{R}^n_+)$ with $\mathcal{H}^1_{GMS}(\mathbb{R}^n_+)$.

Corollary 7.3.1.

$$\|u\|_{\mathcal{H}^1_{GMS}(\mathbb{R}^n_+)} \leq C\|u\|_{\mathcal{H}^1_M(\mathbb{R}^n_+)}.$$

Proof. Corollary 7.3.1 is obtained from Lemma 7.3.2 and the definitions of norms of Hardy spaces. □

We also recall the duality theorem of $\mathcal{H}^1_M(\Omega)$ in [Miy90].

Theorem 7.3.2 (Duality of $\mathcal{H}^1_M(\Omega)$ [Miy90])**.** *Let $\Omega \subset \mathbb{R}^n$ be a proper open set. Then, the dual space $\mathcal{H}^1_M(\Omega)^*$ is $BMO^M(\Omega)$, i.e.,*

$$\mathcal{H}^1_M(\Omega)^* = BMO^M(\Omega).$$

7.3.2 Decay estimates

We consider decay estimates of the initial value problem of the Stokes equation with the Dirichlet boundary conditions.

$$\begin{cases} \partial_t v - \Delta v + \nabla\pi = 0 & \text{in } \Omega \times (0,\infty) \\ \mathrm{div} v = 0 & \text{in } \Omega \times (0,\infty) \\ v = 0 & \text{on } \partial\Omega \times (0,\infty) \\ v(0) = v_0 & \text{in } \Omega, \end{cases}$$

Let $S(t)(v_0) = v$ be the solution operator of the Stokes problem. This section is mainly due to the following L^1-L^q estimates. Let us recall L^1-L^q on bounded domains in [Mar11].

Theorem 7.3.3 ([Mar11])**.** *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain or an exterior domain, $n \geq 3$, with the C^m boundary $\partial\Omega$ for $2m > n$. Let $v_0 \in L^1(\Omega)$. There exists a unique solution (v, π) of the Stokes problem such that*

1. *$\eta > 0$, $q > 1$, $v = S(t)v_0 \in C(\eta, T; L^q_\sigma(\Omega)) \cap L^\infty(\eta, T : W^{1,q}(\Omega))$, $D^2 v(t,x)$, $\nabla\pi(t,x) \in L^\infty(\eta, T; L^q(\Omega))$,*
2. *there exists a constant $C > 0$ such that*

$$\|S(t)v_0\|_{L^q} \leq Ct^{-\frac{n}{2}(1-\frac{1}{q})}\|v_0\|_{L^1} \text{ for } t > 0, \tag{7.3.1}$$

$$\|\nabla S(t)v_0\|_{L^q} \leq Ct^{-\frac{1}{2}-\frac{n}{2}(1-\frac{1}{q})}\|v_0\|_{L^1} \text{ for } t > 0,\ q \in (1, n], \tag{7.3.2}$$

$$\|\nabla S(t)v_0\|_{L^q} \leq Ct^{-\frac{1}{2}-\frac{n}{2}(1-\frac{1}{q})}\|v_0\|_{L^1} \text{ for } t \in (0, 1],\ q \in (n, \infty], \tag{7.3.3}$$

$$\|\nabla S(t)v_0\|_{L^q} \leq Ct^{-\frac{n}{2}}\|v_0\|_{L^1} \text{ for } t > 1,\ q \in (n, \infty], \tag{7.3.4}$$

$$\|\partial_t S(t)v_0\|_{L^q} \leq Ct^{-1-\frac{n}{2}(1-\frac{1}{q})}\|v_0\|_{L^1} \text{ for } t > 0. \tag{7.3.5}$$

In order to obtain BMO-type estimates by a duality argument, we need estimates in Hardy spaces $\mathcal{H}^1_M(\Omega)$ on domains introduced in [Miy90].

Theorem 7.3.4 ($\mathcal{H}^1_M$-L^q estimates of the Stokes semigroups for large time)**.** *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain or an exterior domain, $n \geq 3$, with the C^m boundary $\partial\Omega$ for $2m > n$. Let $v_0 \in \mathcal{H}^1_M(\Omega)$. Then, there exists a constant $C > 0$ such that*

$$\|S(t)v_0\|_{L^q} \leq Ct^{-\frac{n}{2}(1-\frac{1}{q})}\|v_0\|_{\mathcal{H}^1_M(\Omega)} \text{ for } t > 0, \tag{7.3.6}$$

$$\|\nabla S(t)v_0\|_{L^q} \leq Ct^{-\frac{1}{2}-\frac{n}{2}(1-\frac{1}{q})}\|v_0\|_{\mathcal{H}^1_M(\Omega)} \text{ for } t > 0,\ q \in (1,n], \tag{7.3.7}$$

$$\|\nabla S(t)v_0\|_{L^q} \leq Ct^{-\frac{1}{2}-\frac{n}{2}(1-\frac{1}{q})}\|v_0\|_{\mathcal{H}^1_M(\Omega)} \text{ for } t \in (0,1],\ q \in (n,\infty], \tag{7.3.8}$$

$$\|\nabla S(t)v_0\|_{L^q} \leq Ct^{-\frac{n}{2}}\|v_0\|_{\mathcal{H}^1_M(\Omega)} \text{ for } t > 1,\ q \in (n,\infty], \tag{7.3.9}$$

$$\|\partial_t S(t)v_0\|_{L^q} \leq Ct^{-1-\frac{n}{2}(1-\frac{1}{q})}\|v_0\|_{\mathcal{H}^1_M(\Omega)} \text{ for } t > 0. \tag{7.3.10}$$

Proof. Theorem 7.3.3 and Lemma 7.3.3 imply this lemma. □

Remark 7.3.3. *We remark that when we consider other cases,*

(i) Ω is the whole space $\mathbb{R}^n$,

(ii) Ω is a smooth bent half space,

(iii) Ω is a smooth aperture domain,

we can also obtain $\mathcal{H}^1_M$-L^q estimates of the Stokes semigroups for large time.

7.4 Duality arguments

In this section, we obtain BMO-type estimates from Hardy type estimates by a duality argument. Let u_λ be a solution of the Stokes resolvent equation with the Dirichlet boundary condition

$$\begin{cases} (\lambda - \Delta)u_\lambda + \nabla p_\lambda = f & \text{in } \Omega \\ \mathrm{div} u_\lambda = 0 & \text{in } \Omega \\ u_\lambda = 0 & \text{on } \partial\Omega \end{cases}$$

where f is a given external force term and p_λ is an unknown pressure term.

Lemma 7.4.1. *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain or an exterior domain, $n \geq 3$, with the C^m boundary $\partial\Omega$ for $2m > n$, $q \in (1,n]$. Let $f \in \mathcal{H}^1_M(\Omega)$. Then, there exists a constant $C > 0$ such that u_λ satisfies the Stokes resolvent estimate for $Re\lambda \geq 0$ $(\lambda \neq 0)$*

$$|\lambda|^{-\frac{n}{2}(1-\frac{1}{q})+1}\|u_\lambda\|_{L^q} \leq C\|f\|_{\mathcal{H}^1_M}. \tag{7.4.1}$$

Proof. This resolvent estimate (7.4.1) is obtained from Theorem 7.3.4 by arguments with the Laplace transform, for example, as in [His04, Lemma 3.3]. □

Lemma 7.4.2. *Consider one of the following cases.*

(i) $\Omega = \mathbb{R}^n$, $\mu = \nu = \infty$,

(ii) $\Omega \subset \mathbb{R}^n$ *is a Lipschitz half space,* $\mu = \nu = \infty$,

(iii) $\Omega \subset \mathbb{R}^n$ *is a bounded Lipschitz domain,* $\mu, \nu \in (0, \infty]$,

(iv) $\Omega = \mathbb{R}^k \times G \subset \mathbb{R}^n$ *for* $1 \leq k \leq n-1$*, a bounded* C^2 *domain* G, $\mu, \nu \in (0, \infty]$,

(v) $\Omega \subset \mathbb{R}^n$ *is a Lipschitz exterior domain,* $\mu = \nu = \infty$.

Then,

$$\|u_\lambda\|_{BMO_b^{\mu,\nu}} = \sup\{|(u_\lambda, g)| : \ g \in \mathcal{H}^1_M(\Omega), \ \|g\|_{\mathcal{H}^1_M(\Omega)} \leq 1\}. \tag{7.4.2}$$

Proof. Since $BMO_b^{\mu,\nu}(\Omega) = BMO_b^M(\Omega) = (\mathcal{H}^1_M(\Omega))^*$ by Theorem 7.2.1, and Theorem 7.3.2, we obtain (7.4.2). □

Lemma 7.4.3. *Let* $\Omega \subset \mathbb{R}^n$ *be a bounded domain or an exterior domain,* $n \geq 3$, *with the* C^m *boundary* $\partial\Omega$ *for* $2m > n$*. Let* $\mu, \nu \in (0, \infty]$, $q \in [n/(n-1), \infty)$, $f \in VMO^{\mu,\nu}_{b,0,\sigma}(\Omega)$*. Then, there exists a constant* $C > 0$ *such that*

$$\|u_\lambda\|_{BMO_b^{\mu,\nu}} \leq C|\lambda|^{\frac{n}{2q}-1}\|f\|_q. \tag{7.4.3}$$

$$\|\nabla u_\lambda\|_{BMO_b^{\mu,\nu}} \leq C|\lambda|^{-\frac{1}{2}+\frac{n}{2q}}\|f\|_q \tag{7.4.4}$$

Furthermore, if Ω *is bounded, then there exists a constant* $C_{n,\epsilon,diam(\Omega)} > 0$ *such that for* $\epsilon \in (0,1)$

$$\|u_\lambda\|_{BMO_b^{\mu,\nu}} \leq C_{n,\epsilon,diam(\Omega)}|\lambda|^{\frac{\epsilon}{2}-1}\|f\|_{BMO_b^{\mu,\nu}(\Omega)}. \tag{7.4.5}$$

Proof. Let ψ_λ be a solution of the Stokes problem with the external force g,

$$\begin{cases} (\lambda - \Delta)\psi_\lambda + \nabla q_\lambda = g & \text{in } \Omega \\ \mathrm{div}\psi_\lambda = 0 & \text{in } \Omega \\ \psi_\lambda = 0 & \text{on } \partial\Omega. \end{cases}$$

Then we can calculate as

$$\begin{aligned} (u_\lambda, g) &= (u_\lambda, (\lambda - \Delta)\psi_\lambda + \nabla q_\lambda) \\ &= ((\lambda - \Delta)u_\lambda, \psi_\lambda) - (\mathrm{div} u_\lambda, q_\lambda) \\ &= ((\lambda - \Delta)u_\lambda, \psi_\lambda) \\ &= ((\lambda - \Delta)u_\lambda, \psi_\lambda) - (p_\lambda, \mathrm{div}\psi_\lambda) \\ &= ((\lambda - \Delta)u_\lambda + \nabla p_\lambda, \psi_\lambda) \\ &= (f, \psi_\lambda). \end{aligned}$$

Therefore, we obtain the following estimate by [Bol16, Theorem 5.44], and (7.4.1)

$$
\begin{aligned}
|(u_\lambda, g)| &\leq \|f\|_{L^p(\Omega)}\|\psi_\lambda\|_{L^q(\Omega)} \\
&\leq C_{n,p,diam(\Omega)}\|f\|_{BMO_b^{\mu,\nu}(\Omega)}\|\psi_\lambda\|_{L^q(\Omega)} \\
&\leq C_{n,p,q,diam(\Omega)}|\lambda|^{\frac{n}{2}(1-\frac{1}{q})-1}\|f\|_{BMO_b^{\mu,\nu}(\Omega)}\|g\|_{\mathcal{H}^1_M(\Omega)}
\end{aligned}
$$

By (7.4.2), this implies that

$$
\|u_\lambda\|_{BMO_b^{\mu,\nu}} \leq C_{n,q,diam(\Omega)}|\lambda|^{\frac{n}{2}(1-\frac{1}{q})-1}\|f\|_{BMO_b^{\mu,\nu}(\Omega)}. \tag{7.4.6}
$$

We can also obtain gradient estimates. □

We recall the inversion Laplace transform formula for large time given by [Shi83], [His04].

Lemma 7.4.4 (Inversion Laplace transform formula for large time [Shi83], [His04])**.** *Let X be a Banach space with norm $\|\cdot\|$ and $g \in L^1(\mathbb{R}; X)$. If there are constants $\theta \in (0,1)$ and $M > 0$ such that*

$$
\int_\infty^\infty \|g(s)\|ds + \sup_{h\neq 0}\frac{1}{|h|^\theta}\int_\infty^\infty \|g(s+h) - g(s)\|ds \leq M,
$$

then the inverse image $G(t) = \dfrac{1}{2\pi}\displaystyle\int_\infty^\infty e^{ist}g(s)ds$ enjoys

$$
\|G(t)\| \leq CM(1+|t|)^{-\theta}
$$

for some $C > 0$ indpendent of $t \in \mathbb{R}$.

Lemma 7.4.5. *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain or an exterior domain, $n \geq 3$, with the C^m boundary $\partial\Omega$ for $2m > n$. Let $\mu, \nu \in (0, \infty]$, $q \in [n/(n-1), \infty)$, $v_0 \in VMO^{\mu,\nu}_{b,0,\sigma}(\Omega)$. Then, there exists a constant $C > 0$ such that*

$$
\begin{aligned}
\|S(t)v_0\|_{BMO_b^{\mu,\nu}} &\leq Ct^{-\frac{n}{2q}}\|v_0\|_{L^q} \\
\|\nabla S(t)v_0\|_{BMO_b^{\mu,\nu}} &\leq Ct^{-\frac{n}{2q}-\frac{1}{2}}\|v_0\|_{L^q}
\end{aligned} \tag{7.4.7}
$$

Furthermore, if Ω is bounded, then there exists a constant $C_{n,\epsilon,diam(\Omega)} > 0$ such that for $\epsilon \in (0,1)$

$$
\|S(t)v_0\|_{BMO_b^{\mu,\nu}} \leq C_{n,\epsilon,diam(\Omega)}t^{-\frac{\epsilon}{2}}\|v_0\|_{BMO_b^{\mu,\nu}(\Omega)}. \tag{7.4.8}
$$

$$
\|\nabla S(t)v_0\|_{BMO_b^{\mu,\nu}} \leq C_{n,\epsilon,diam(\Omega)}t^{-\frac{\epsilon}{2}-\frac{1}{2}}\|v_0\|_{BMO_b^{\mu,\nu}(\Omega)}. \tag{7.4.9}
$$

Proof. We also need decay estimates of $u_{s+h} - u_s$ to finish our duality argument by the inverse Laplace transform formula as in Lemma 7.4.4. This decay estimates of $u_{s+h} - u_s$ in BMO is obtained from a duality argument and the following deformations of equality

$$\begin{aligned}(u_{s+h} - u_s, g) &= (u_{s+h}, g) - (u_s, g) \\ &= (u_{s+h}, (s+h-\Delta)\psi_{s+h} + \nabla q_{s+h}) - (u_s, (s-\Delta)\psi_s + \nabla q_s) \\ &= (f, \psi_{s+h}) - (f, \psi_s) = (f, \psi_{s+h} - \psi_s).\end{aligned}$$

Decay estimates of $\psi_{s+h} - \psi_s$ are obtained together with the case of ψ_λ by an argument, for example , seen in [His04].Therefore, these resolvent estimates imply estimates of the Stokes semigroups by Lemma 7.4.4. □

7.5 Application to the Navier-Stokes problem

7.5.1 Estimates of the nonlinear term

In this section, we consider the estimate of the nonlinear term of the integral equation

$$V(t) = S(t)V_0 - \int_0^t T(t-s)P(V \cdot \nabla V)(s)ds. \tag{7.5.1}$$

Lemma 7.5.1. *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain, $n \geq 3$, with the C^m boundary $\partial\Omega$ for $2m > n$. Let $\mu, \nu \in (0, \infty)$, $U, \nabla W \in VMO_b^{\mu,\nu}(\Omega)$. Then for $\epsilon \in (0,1)$ there exists a constant $C_{n,\Omega,\epsilon} > 0$ such that*

$$\begin{aligned}&\|S(t)P(U \cdot \nabla W)\|_{BMO_b^{\mu,\nu}} \leq Ct^{-\frac{\epsilon}{2}} \|U\|_{BMO_b^{\mu,\nu}} \|\nabla W\|_{BMO_b^{\mu,\nu}}, \\ &\|\nabla S(t)P(U \cdot \nabla W)\|_{BMO_b^{\mu,\nu}} \leq Ct^{-\frac{\epsilon+1}{2}} \|U\|_{BMO_b^{\mu,\nu}} \|\nabla W\|_{BMO_b^{\mu,\nu}}.\end{aligned} \tag{7.5.2}$$

Proof. By Lemma 7.4.5 and L^q boundedness of the Helmholtz projection on bounded domains, and [Bol16, Theorem 5.44],

$$\begin{aligned}\|S(t)P(U \cdot \nabla W)\|_{BMO_b^{\mu,\nu}} &\leq C_{n,p,diam(\Omega)} t^{-\frac{n}{2q}} \|P(U \cdot \nabla W)\|_{L^q} \\ &\leq C_{n,p,diam(\Omega)} t^{-\frac{n}{2q}} \|U \cdot \nabla W)\|_{L^q} \\ &\leq C_{n,p,diam(\Omega)} t^{-\frac{n}{2q}} \|U\|_{L^{2q}} \|\nabla W\|_{L^{2q}} \\ &\leq Ct^{-\frac{\epsilon}{2}} \|U\|_{BMO_b^{\mu,\nu}} \|\nabla W\|_{BMO_b^{\mu,\nu}}.\end{aligned}$$

As a consequence, we conclude that

$$\begin{aligned}&\|S(t)P(U \cdot \nabla W)\|_{BMO_b^{\mu,\nu}} \leq Ct^{-\frac{\epsilon}{2}} \|U\|_{BMO_b^{\mu,\nu}} \|\nabla W\|_{BMO_b^{\mu,\nu}}, \\ &\|\nabla S(t)P(U \cdot \nabla W)\|_{BMO_b^{\mu,\nu}} \leq Ct^{-\frac{\epsilon+1}{2}} \|U\|_{BMO_b^{\mu,\nu}} \|\nabla W\|_{BMO_b^{\mu,\nu}},\end{aligned}$$

□

7.5.2 Kato's arguments

In this section, we consider the initial value problem of the Navier-Stokes equation with the Dirichlet boundary condition:

$$\begin{cases} \partial_t V - \Delta V + (V\cdot\nabla)V + \nabla\Pi = 0 & \text{in } \Omega\times(0,\infty) \\ \mathrm{div}V = 0 & \text{in } \Omega\times(0,\infty) \\ V = 0 & \text{on } \partial\Omega\times(0,\infty) \\ V(0) = V_0 & \text{in } \Omega, \end{cases}$$

We would like to start from the existence and uniqueness of a solution of the integral equation (7.5.1). This section is based on Kato's arguments, for example, as in [Kat84] in which the Navier-Stokes problem on L^p spaces are considered.

Lemma 7.5.2. *Let $\Omega\subset\mathbb{R}^n$ be a bounded domain, $n\geq 3$, with the C^m boundary $\partial\Omega$ for $2m>n$, $\mu,\nu\in(0,\infty]$. Then for all $\delta\in(0,1/2)$ there exists a small constant $\gamma=\gamma(\Omega,\delta,n)>0$ such that if $V_0\in VMO^{\mu,\nu}_{b,0,\sigma}(\Omega)$ satisfies $\|V_0\|_{BMO^{\mu,\nu}_b(\Omega)}\leq\gamma$ the Navier-Stokes problem admits a solution $V(t)$ of the integral equation (7.5.1) on $(0,\infty)$. Moreover, for large $t>0$*

$$\|V(t)\|_{BMO^{\mu,\nu}_b} = o(t^{-\frac{\delta}{2}}), \|\nabla V(t)\|_{BMO^{\mu,\nu}_b} = o(t^{-\frac{\delta+1}{2}}).$$

Lemma 7.5.3. *Let $\Omega\subset\mathbb{R}^n$ be a bounded domain, $n\geq 3$, with the C^m boundary $\partial\Omega$ for $2m>n$, $\mu,\nu\in(0,\infty]$. Then for all $\delta\in(0,1/2)$ there exists a small constant $\gamma=\gamma(\Omega,\delta,n)>0$ such that if $V_0\in VMO^{\mu,\nu}_{b,0,\sigma}(\Omega)$ satisfies $\|V_0\|_{BMO^{\mu,\nu}_b(\Omega)}\leq\gamma$ the Navier-Stokes problem admits a unique strong solution $V(t)$ on $(0,\infty)$. Moreover, for large $t>0$*

$$\|V(t)\|_{BMO^{\mu,\nu}_b} = o(t^{-\frac{\delta}{2}}), \|\nabla V(t)\|_{BMO^{\mu,\nu}_b} = o(t^{-\frac{\delta+1}{2}}).$$

Proof. We construct a solution of (7.5.1) by a successive approximation. Let $X_0 = S(t)V_0$, and let us consider

$$X_{m+1} = X_0 + GX_m,$$

where $GU(t) = -\int_0^t T(t-s)P(U\cdot\nabla U)(s)ds$. By (7.5.2),

$$\begin{aligned} \|GU(t)\|_{BMO^{\mu,\nu}_b} &\leq C\int_0^t (t-s)^{-\frac{\epsilon}{2}}\|U\|_{BMO^{\mu,\nu}_b}\|\nabla U\|_{BMO^{\mu,\nu}_b}ds, \\ \|\nabla GU(t)\|_{BMO^{\mu,\nu}_b} &\leq C\int_0^t (t-s)^{-\frac{\epsilon+1}{2}}\|U\|_{BMO^{\mu,\nu}_b}\|\nabla U\|_{BMO^{\mu,\nu}_b}ds \end{aligned} \quad (7.5.3)$$

We argue by induction. We take $K_0 = K_0' = c\|V_0\|_{BMO^{\mu,\nu}_b}$, and we fix an arbitrary $\delta\in(0,\frac{1}{2})$. We assume that there exist small constant K_m, K_m'

$$\|X_m\|_{BMO^{\mu,\nu}_b} \leq ct^{-\frac{\delta}{2}}K_m,\ \|\nabla X_m\|_{BMO^{\mu,\nu}_b} \leq ct^{-\frac{\delta+1}{2}}K_m'. \quad (7.5.4)$$

The continuity of $t^{\frac{\delta}{2}}X_0$ at $t=0$, and the decay estimates of X_0 are corollaries of the boundedness of the Stokes semigroups on $BMO^{\mu,\nu}_{b,\sigma}(\Omega)$ in [Bol16, Theorem 6.50]. By (7.4.8), we can also obtain the decay estimates for large time of X_0. By assumption and the estimates (7.5.2), we obtain the estimate of the nonlinear terms

$$\begin{aligned}
\|GX_m\|_{BMO^{\mu,\nu}_b} &\le C\int_0^t (t-s)^{-\frac{\epsilon}{2}} s^{-\frac{2\delta+1}{2}} K_m K'_m ds,\\
\|\nabla GX_m\|_{BMO^{\mu,\nu}_b} &\le C\int_0^t (t-s)^{-\frac{\epsilon+1}{2}} s^{-\frac{2\delta+1}{2}} K_m K'_m ds
\end{aligned} \tag{7.5.5}$$

The two integrals $\int_0^t (t-s)^{-\frac{\epsilon}{2}} s^{-\frac{2\delta+1}{2}} ds = t^{-\frac{\epsilon+2\delta-1}{2}} \int_0^1 (1-z)^{-\frac{\epsilon}{2}} z^{-\frac{2\delta+1}{2}} dz$ and $\int_0^t (t-s)^{-\frac{\epsilon+1}{2}} s^{-\frac{2\delta+1}{2}} ds$ converge for $\epsilon\in(0,1),\ \delta\in(0,\frac{1}{2})$. Since $\epsilon\in(0,1)$ is arbitrary, we choose $\epsilon=\delta$ so that we can compare the local decays with respect to t, i.e., $t^{-\frac{\epsilon}{2}}+t^{-\frac{\epsilon+2\delta-1}{2}}\le 2t^{-\frac{\delta}{2}}$ for $0<t\le 1$. For the case of $1\le t$, we choose $\epsilon=1-\delta$ so that $t^{-\frac{\epsilon}{2}}+t^{-\frac{\epsilon+2\delta-1}{2}}\le 2t^{-\frac{\delta}{2}}$ for $1\le t$. As a consequence, we obtain the decay of $t^{\frac{\delta}{2}}X_{m+1}$ and $t^{\frac{\delta+1}{2}}X'_{m+1}$

$$\begin{aligned}
t^{-\frac{\delta}{2}}K_{m+1} &\le Ct^{-\frac{\delta}{2}}K_0 + \|GX_m\|_{BMO^{\mu,\nu}_b}\\
&\le C_{n,\delta,diam(\Omega)}t^{-\frac{\delta}{2}}K_0 + C_{n,\delta,diam(\Omega)}t^{-\frac{\epsilon+2\delta-1}{2}}K_mK'_m\\
&\le C_{n,\delta,diam(\Omega)}t^{-\frac{\delta}{2}}(K_0+K_mK'_m),\\
t^{-\frac{\delta+1}{2}}K'_{m+1} &\le t^{-\frac{\delta+1}{2}}K'_0 + \|\nabla GX_m\|_{BMO^{\mu,\nu}_b} \le C_{n,\delta,diam(\Omega)}t^{-\frac{\delta+1}{2}}(K_0+K_mK'_m).
\end{aligned} \tag{7.5.6}$$

These imply the continuity of $t^{\frac{\delta}{2}}X_{m+1}$ and $t^{\frac{\delta+1}{2}}X'_{m+1}$ at $t=0$.

As was shown in [Kat84], We can take small $\gamma>0$ depending on n, δ, $diam(\Omega)$ so that if $\|V_0\|_{BMO^{\mu,\nu}_{b,\sigma}}\le\gamma$ then $K_m,\ K'_m$ are uniformly bounded by a fixed constant K.

Therefore, we conclude that $\{X_m\},\ \{\nabla X_m\}$ are uniformly bounded in $BMO^{\mu,\nu}_b$.

$$\|X_m\|_{BMO^{\mu,\nu}_b}\le ct^{-\frac{\delta}{2}}K,\ \|\nabla X_m\|_{BMO^{\mu,\nu}_b}\le ct^{-\frac{\delta+1}{2}}K'. \tag{7.5.7}$$

By carrying on with an argument based on [Kat84], we obtain a unique global solution $V(t)$ of the integral equation $V(t)=S(t)V_0-\int_0^t T(t-s)P(V\cdot\nabla V)(s)ds$. By a Hölder estimate, it turns out that $V(t)$ is a strong solution of the Navier-Stokes problem. □

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Chapter 8

Gaussian bounds for the semigroups generated by higher order elliptic operators and application to L^p maximal regularity.

8.1 Introduction

In this chapter, we consider the Gaussian bounds for the semigroups generated by higher order divergence type operators. It is claimed in [AQ00], [Dav97], [KW04] that semigroups generated by divergence type operators fails to have Gaussian bounds when p is larger than the Sobolev exponent, and it is shown in [Dav97] that there is a divergence type operator generating the semigroup which fails to have L^p regularity for large p. However, it turns out that the semigroup seems to fail to have L^p regularity because the divergence operator in [Dav97] has coefficients with a singularity. We would like to establish the Gaussian bounds for the semigroups generated by higher order divergence type operators for large p provided that the coefficients of the leading terms of the operator has C_{bu} regularity and the boundary of the domain is Lipschitz provided that the Lipschitz constant is small enough where $C_{bu}(\overline{\Omega})$ is the spaces of bounded uniformly continuous functions in $\overline{\Omega}$.

Our argument is based on a Davies perturbation. Instead of stating results for general operators we first discuss the bi-Laplace operator Δ^2 as the simplest example.

Let $\Omega \subset \mathbb{R}^n$ be a Lipschitz domain with small Lipschitz constant. We remark that Ω can be unbounded and the assumption of regularity is owing to the assumption of $W^{m,p} - W^{-m,p}$ resolvent estimates. Let $1 < p < \infty$, and $a(u,v) = \int_\Omega \Delta u \Delta v dx$ for $u \in W_0^{2,p}(\Omega)$ and $v \in W_0^{2,q}(\Omega)$. In this section, we consider the generalized Gaussian bounds for the semigroup generated by bi-Laplacian Δ^2. Δ^2 can be defined in the weak sense as follows:

$$\Delta^2 : W_0^{2,p}(\Omega) \to W^{-2,p}(\Omega) \ \Delta^2(u)(v) = a(u,v).$$

By this interpretation in this way, we can regard Δ^2 as a divergence type differential operator.

Let $W = \{\phi \in C_0^\infty(\mathbb{R}^n) : \|\partial^\alpha \phi\|_\infty \le 1 \text{ for all } 1 \le |\alpha| \le 4\}$. Then it is known that $d(x,y) = \sup\{|\phi(x) - \phi(y)| : \phi \in W\}$ defines a metric on $\mathbb{R}^n$ equivalent to the Euclidean distance. We define the Davis perturbation $e^{-\lambda\phi}\Delta^2 e^{\lambda\phi}$ of Δ^2 by

$$e^{-\lambda\phi}\Delta^2 e^{\lambda\phi}(u)(v) = a_{\lambda\phi}(u,v) = \int_\Omega \Delta e^{\lambda\phi} u \Delta e^{-\lambda\phi} v dx.$$

Let $\theta_0 = \sup\{\theta : e^{-t\Delta^2}$ has an analytic extension to $\Sigma_\theta\}$. θ_0 will be obtained by Theorem 8.2.1 later. Let χ denote a characteristic function defined by $\chi_A(x) = 1$ if $x \in A$, and $\chi_A(x) = 0$ if $x \notin A$.

Theorem 8.1.1. *Let $m = 4$, $1 < p < \infty$, $p \le q \le \infty$, $\theta \in (0, \theta_0)$, and let Ω be a Lipschitz domain with a small Lipschitz constant. Then there exist $A, B, C > 0$ such that*

$$\|\chi_{B_r(x)} e^{-t\Delta^2} \chi_{B_r(y)}\|_{p\to q} \le C|t|^{-\frac{n}{m}(\frac{1}{p}-\frac{1}{q})} e^{-A\frac{|x-y|^{m/m-1}}{\mathrm{Re}t^{1/m-1}}}$$

for $r > 0, x, y \in \Omega, |x-y| \ge 2\sqrt{n}r$,

$$\|\chi_{B_r(x)} e^{-t\Delta^2} \chi_{B_{(k+1)r}(x)\setminus B_{kr}(x)}\|_{p\to q} \le C|t|^{-\frac{n}{m}(\frac{1}{p}-\frac{1}{q})} e^{-B\frac{(kr)^m}{\mathrm{Re}t^{1/m-1}}}$$

for $r > 0, x \in \Omega, k \ge 2\sqrt{n}$.

In order to obtain generalized Gaussian bounds, we need estimates of Davis perturbation.

Theorem 8.1.2. *Let $1 < p < \infty$, $\theta \in (0, \theta_0)$, and let Ω be a Lipschitz domain with a small Lipschitz constant. Then there exist C, ω such that for $|\arg t| \le \theta$, $\lambda \in \mathbb{R}^n$, and $\phi \in W$*

$$\|e^{-\lambda\phi} e^{-t\Delta^2} e^{\lambda\phi}\|_{L^p \to L^p} \le C e^{\omega(1+|\lambda|^4)\mathrm{Re}t}, \tag{8.1.1}$$

$$\|\nabla^2 e^{-\lambda\phi} e^{-t\Delta^2} e^{\lambda\phi}\|_{L^p \to L^p} \le C|t|^{-\frac{1}{2}} e^{\omega(1+|\lambda|^4)\mathrm{Re}t}, \tag{8.1.2}$$

$$\|e^{-\lambda\phi} e^{-t\Delta^2} e^{\lambda\phi}\|_{L^p \to L^{p^*}} \le C|t|^{-\frac{n}{4}(\frac{1}{p}-\frac{1}{p^*})} e^{\omega(1+|\lambda|^4)\mathrm{Re}t}. \tag{8.1.3}$$

Interpolation arguments imply estimates from p to q, and estimates from q to p^* for $q \le p^*$, and semigroup arguments implies estimates from p to q for general $p \le q$. Let us illustrate our proof to establish Theorem 8.1.2. Our method is a Davies perturbation which has been first introduced by E. B. Davies.

Outline of the proof of Theorem 8.1.2.

First of all, we consider the difference between the operator and its Davies perturbation.

Lemma 8.1.1. *For arbitrary* $0 < \epsilon < 1$

$$\sup_{\|v\|_{2,q}=1} |a_{\lambda\phi}(u,v) - a(u,v)| \leq C(\epsilon\|\nabla^2 u\|_p + C_\epsilon(1+|\lambda|^4)\|u\|_p).$$

Proof. Since $\Delta(FG) = \Delta(F)G + 2\nabla F \cdot \nabla G + F\Delta G$ by the Leibniz' rule, we can calculate

$$|\Delta e^{\lambda\phi} u \Delta e^{-\lambda\phi} v - \Delta u \Delta v| \leq C \sum_{0\leq\gamma,\delta\leq 2, |\gamma+\delta|\leq 3} (1+|\lambda|)^{4-|\gamma|-|\delta|} |\partial^\gamma u||\partial^\delta v|.$$

This implies by Hölder inequality

$$|a_{\lambda\phi}(u,v) - a(u,v)| \leq C \sum_{0\leq\gamma,\delta\leq 2, |\gamma+\delta|\leq 3} (1+|\lambda|)^{4-|\gamma|-|\delta|} \|\partial^\gamma u\|_p \|\partial^\delta v\|_q.$$

By applying interpolation inequality with respect to the order of derivatives and taking sup over v we obtain the desired estimate

$$\sup_{\|v\|_{2,q}=1} |a_{\lambda\phi}(u,v) - a(u,v)| \leq C(\epsilon\|\nabla^2 u\|_p + C_\epsilon(1+|\lambda|)^4\|u\|_p).$$

□

Let us define $B_{\lambda\phi} = \Delta^2_{\lambda\phi} - \Delta^2$ in the weak sense. Then $B_{\lambda\phi}$ corresponds to the bilinear form $a_{\lambda\phi}(\cdot,\cdot) - a(\cdot,\cdot)$.

By Theorem 8.2.1 the operator $(\mu+\Delta^2)^{-1} : W^{-2,p} \to W^{2,p}_0$ is well defined. For $f \in L^p(\Omega)$ let u be $(\mu+\Delta^2)^{-1} f$. Then $u \in W^{2,p}_0(\Omega)$ satisfies

$$\begin{cases} (\mu+\Delta^2)u = f & \text{in } \Omega \\ u = \partial_\nu u = 0 & \text{on } \partial\Omega, \end{cases}$$

where ∂_ν is the normal derivative. The L^p resolvent estimate (Theorem 8.2.1) implies

$$|\mu|^{-\frac{1}{2}}\|u\|_p + |\mu|^{-\frac{1}{4}}\|\nabla u\|_p + \|\nabla^2 u\|_p \leq C\|f\|_{W^{-2,p}},$$
$$|\mu|^{-1}\|u\|_p + |\mu|^{-\frac{3}{4}}\|\nabla u\|_p + |\mu|^{-\frac{1}{2}}\|\nabla^2 u\|_p \leq C\|f\|_p.$$

Lemma 8.1.2. *(estimates of lower order terms)*
There exists $M' > 0$ *such that for* $\mu \in \Sigma_{\pi-\theta_0}$ *and* $|\mu| > M'(1+|\lambda|^4)$

$$\|B_{\lambda\phi}(\mu+\Delta^2)^{-1}\|_{L^p\to W^{-2,p}} < \frac{|\mu|^{-\frac{1}{2}}}{2}, \tag{8.1.4}$$

$$\|B_{\lambda\phi}(\mu+\Delta^2)^{-1}\|_{W^{-2,p}\to W^{-2,p}} < \frac{1}{2}. \tag{8.1.5}$$

Proof. First we prove the inequality (8.1.4). Let $f \in L^p(\Omega)$, and let $u = (\mu + \Delta^2)^{-1} f$. By Lemma 8.1.1 for arbitrary $0 < \epsilon < 1$

$$\sup_{\|v\|_{2,q}=1} |a_{\lambda\phi}(u,v) - a(u,v)| \leq C(\epsilon \|\nabla^2 u\|_p + C_\epsilon (1 + |\lambda|^4)\|u\|_p).$$

By Theorem 8.2.1 we can apply $W^{2,p} - L^p$ estimates to obtaining the inequality

$$\sup_{\|v\|_{2,q}=1} |a_{\lambda\phi}(u,v) - a(u,v)| \leq C'|\mu|^{-\frac{1}{2}}(\epsilon + C_\epsilon(1 + |\lambda|^4)|\mu|^{-\frac{1}{2}})\|f\|_p).$$

We take a small ϵ so that $C'\epsilon < \frac{1}{4}$, and then we can also take large $M'_{C',C_\epsilon} > 0$ so that if $|\mu| > M'(1 + |\lambda|^4)$ then $C'C_\epsilon(1 + |\lambda|^4)|\mu|^{-\frac{1}{2}} < \frac{1}{4}$. This implies the desired estimate (8.1.4)

$$\sup_{\|v\|_{2,q}=1} |a_{\lambda\phi}((\mu + \Delta^2)^{-1} f, v) - a((\mu + \Delta^2)^{-1} f, v)| \leq \frac{|\mu|^{-\frac{1}{2}}}{2} \|f\|_p.$$

For a proof of the inequality (8.1.5), we can prove similarly by using $W^{2,p} - W^{-2,p}$ estimates in Theorem 8.2.1. □

Lemma 8.1.3. *(resolvent estimate for $\Delta^2_{\lambda\phi}$) Let $\mu \in \Sigma_{\pi-\theta_0} \cap \{|\mu| > M'(1 + |\lambda|^4)\}$. Then*

$$\|(\mu + \Delta^2 + B_{\lambda\phi})^{-1}\|_{L^p \to L^p} \leq C|\mu|^{-1}.$$

Proof. We can obtain the following inequality as seen in [Kun08]

$$\begin{aligned}
\|(\mu + \Delta^2 + B_{\lambda\phi})^{-1}\|_{L^p \to L^p} &= \|(\mu + \Delta^2)^{-1} \sum_{k=0}^{\infty} (-B_{\lambda\phi}(\mu + \Delta^2)^{-1})^k\|_{L^p \to L^p} \\
&\leq \|(\mu + \Delta^2)^{-1}\|_{L^p \to L^p} + \|(\mu + \Delta^2)^{-1}(-B_{\lambda\phi}(\mu + \Delta^2)^{-1})\|_{L^p \to L^p} \\
&\quad + \|(\mu + \Delta^2)^{-1}(\sum_{k=1}^{\infty} (-B_{\lambda\phi}(\mu + \Delta^2)^{-1})^k)(-B_{\lambda\phi}(\mu + \Delta^2)^{-1})\|_{L^p \to L^p}.
\end{aligned}$$

By Theorem 8.2.1, the first term is estimated by $\frac{C}{|\mu|}$. By Theorem 8.2.1 and Lemma 8.1.2 (8.1.4), the second term is estimated by

$$\begin{aligned}
\|(\mu + \Delta^2)^{-1}(-B_{\lambda\phi}(\mu + \Delta^2)^{-1})\|_{L^p \to L^p} &\leq \|(\mu + \Delta^2)^{-1}\|_{W^{-2,p} \to L^p} \|B_{\lambda\phi}(\mu + \Delta^2)^{-1}\|_{L^p \to W^{-2,p}} \\
&\leq \frac{C}{|\mu|^{\frac{1}{2}}} \frac{1}{2|\mu|^{\frac{1}{2}}} \leq \frac{C}{2|\mu|}.
\end{aligned}$$

By Theorem 8.2.1 and Lemma 8.1.2, we can estimate the last term

$$\|(\mu + \Delta^2)^{-1}(\sum_{k=1}^{\infty} (-B_{\lambda\phi}(\mu + \Delta^2)^{-1})^k)(-B_{\lambda\phi}(\mu + \Delta^2)^{-1})\|_{L^p \to L^p}$$

$$\leq \|(\mu + \Delta^2)^{-1}\|_{W^{-2,p} \to L^p} \|\sum_{k=1}^{\infty} (-B_{\lambda\phi}(\mu + \Delta^2)^{-1})^k\|_{W^{-2,p} \to W^{-2,p}} \|B_{\lambda\phi}(\mu + \Delta^2)^{-1}\|_{L^p \to W^{-2,p}}$$

$$\leq \frac{C}{|\mu|^{\frac{1}{2}}} (\sum_{k=1}^{\infty} \frac{1}{2^k}) \frac{1}{2|\mu|^{\frac{1}{2}}} \leq \frac{C}{|\mu|}.$$

Thus, we have completed the proof of Lemma 8.1.3. □

Proof of Theorem 8.1.2 . We first prove (8.1.1). Lemma 8.1.3 implies $\Sigma_{\pi-\theta_0} \cap \{|\mu| > M'(1+|\lambda|^4)\} \subset \rho(-\Delta^2_{\lambda\phi})$. Let $\theta \in (0,\theta_0)$. We set $\theta' = \pi/2 - (\theta_0+\theta)/2$, then $\Sigma_{\pi-\theta'} + r/\sin\theta' \subset \Sigma_{\pi-\theta'} \cap \{|\mu| \geq r\}$ for $r > 0$. We take $r = M'(1+|\lambda|^4)$ and we define $\tilde{\Delta}^2{}_{\lambda\phi} = \Delta^2_{\lambda\phi} + M'(1+|\lambda|^4)/\sin\theta'$. In this setting, we conclude that $\Sigma_{\pi-\theta'} \subset \rho(-\tilde{\Delta}^2{}_{\lambda\phi})$, and for $\mu \in \Sigma_{\pi-\theta'}$ we have

$$\begin{aligned}\|\mu(\mu+\tilde{\Delta}^2{}_{\lambda\phi})^{-1}\|_{L^p\to L^p} &= \|(\mu + \frac{M'(1+|\lambda|^4)}{\sin\theta'} + \Delta^2 + B_{\lambda\phi})^{-1}\|_{L^p\to L^p} \\ &\leq C\frac{|\mu|}{|\mu + \frac{M'(1+|\lambda|^4)}{\sin\theta'}|}.\end{aligned}$$

By calculating the right hand side as (16) in [Kun08, pp2742], we obtain the following resolvent estimates for $\mu \in \Sigma_{\pi-\theta'}$, $\lambda \in \mathbb{R}^n$, and $\phi \in W$

$$\|\mu(\mu+\tilde{\Delta}^2{}_{\lambda\phi})^{-1}\|_{L^p\to L^p} \leq \frac{C}{\sin\theta'}.$$

Therefore, $\tilde{\Delta}^2{}_{\lambda\phi}$ generates a bounded analytic semigroup $e^{-t\tilde{\Delta}^2{}_{\lambda\phi}}$ with the estimate

$$\|e^{-t\tilde{\Delta}^2{}_{\lambda\phi}}\|_{L^p\to L^p} \leq C.$$

This implies (8.1.1) of Theorem 8.1.2.

We next prove (8.1.2). For $\mu \in \Sigma_{\pi-\theta'} \subset \rho(-\tilde{\Delta}^2{}_{\lambda\phi})$ we can argue similarly in Lemma 8.1.3

$$\begin{aligned}&\|\mu^{\frac{1}{2}}\nabla^2(\mu+\tilde{\Delta}^2{}_{\lambda\phi})^{-1}\|_{L^p\to L^p} \\ &= \|\mu^{\frac{1}{2}}\nabla^2(\mu + \frac{M'(1+|\lambda|^4)}{\sin\theta'} + \Delta^2 + B_{\lambda\phi})^{-1}\|_{L^p\to L^p} \\ &= \|\mu^{\frac{1}{2}}\{\nabla^2(\mu + \frac{M'(1+|\lambda|^4)}{\sin\theta'} + \Delta^2)^{-1}\}\sum_{k=0}^{\infty}(-B_{\lambda\phi}(\mu + \frac{M'(1+|\lambda|^4)}{\sin\theta'} + \Delta^2)^{-1})^k\|_{L^p\to L^p} \\ &\leq C\frac{|\mu|^{\frac{1}{2}}}{|\mu + \frac{M'(1+|\lambda|^4)}{\sin\theta'}|^{\frac{1}{2}}}.\end{aligned}$$

By the same argument in (8.1.1) of Theorem 8.1.2, we obtain (8.1.2). We finally prove (8.1.3). Let $v = e^{-t\tilde{\Delta}^2{}_{\lambda\phi}}f$. We consider the three cases, $n > 2p$, $n < 2p$, $n = 2p$. When $n > 2p$, by $v(t,\cdot) \in W_0^{2,p}(\Omega)$ for each t, the zero extension, and the Sobolev inequality [AF03, Theorem 4.31] we obtain for $p^* = np/(n-2p)$

$$\|v\|_{p^*} \leq C_{p,n}\|\nabla^2 v\|_p.$$

By (8.1.2), we obtain

$$\|v\|_{p\to p^*} \leq C_{p,n}\|\nabla^2 e^{-\lambda\phi}e^{-t\Delta^2}e^{\lambda\phi}\|_{L^p\to L^p} \leq C|t|^{-\frac{1}{2}}e^{\omega(1+|\lambda|^4)\mathrm{Re}t},$$

Since $-1/2 = -n/4(1/p - 1/p^*)$, (8.1.3) is proved if $n > 2p$. When $n < 2p$, by $v(t,\cdot) \in W_0^{2,p}(\Omega)$ for each t, the zero extension, and the Gagliardo-Nirenberg inequality we obtain for $p^* = \infty$

$$\|v\|_\infty \le C_{n,p}\|\nabla^2 v\|_p^{\frac{n}{2p}}\|v\|_p^{1-\frac{n}{2p}}.$$

(8.1.1), (8.1.2) imply (8.1.3) when $n < 2p$, In the case $n = 2p$, estimates are obtained from $p - \epsilon'$ case by Sneiberg's lemma [Kun05, Theorem 8]. ☐

Proof of Theorem 8.1.1. Theorem 8.1.2 implies Gaussian bounds by [Kun08, Lemma 3.4]. The proof of [Kun08, Lemma 3.4] is based on a direct calculation of semigroups. ☐

8.2 Property (R_p)

We would like to define a property (R_p).

Definition 8.2.1. *Let $\Omega \subset \mathbb{R}^n$ be a domain. Let L be a divergence type operator $L = -\sum_{|\alpha|,|\beta|\le m}(-1)^m\partial^\beta a_{\alpha,\beta}\partial^\alpha$. Then, the pair (L,Ω) is said to satisfy a property (R_p) for $p \in (1,\infty)$ if for some $\theta \in (0,\frac{\pi}{2})$ there exists a constant there exist $M > 0,\ C > 0$ such that for all $\mu \in \Sigma_{\pi-\theta} \cap \{|z| \ge M\}$ the resolvent $(\mu + L)^{-1}$ exists with the estimate for $0 \le i,j \le m$*

$$\|(\mu + L)^{-1}\|_{L(W^{-i,p}(\Omega),W^{j,p}(\Omega))} \le C|\mu|^{-1+\frac{i+j}{2m}}.$$

We recall $W^{m,p} - W^{-m,p}$ resolvent estimates. Let L be a divergence type operator $L = -\sum_{|\alpha|,|\beta|=m}(-1)^m\partial^\beta a_{\alpha,\beta}\partial^\alpha$. We give assumptions in order to state $W^{m,p} - W^{-m,p}$ resolvent estimates obtained by [Miy06]. Let $b(x,\xi) = \sum_{|\alpha|,|\beta|=m} a_{\alpha,\beta}(x)\xi^{\alpha+\beta}$ denote the principal symbol of L, and let $C_{bu}(\overline{\Omega})$ be the spaces of bounded uniformly continuous functions in $\overline{\Omega}$. We define Miyazaki's conditions (M1), (M2) as

$$\text{(M1)}\ \ a_{\alpha,\beta} \in \begin{cases} C_{bu}(\overline{\Omega}) & \text{if } |\alpha| = |\beta| = m \\ L^\infty(\Omega) & \text{if } |\alpha| + |\beta| \le 2m-1, \end{cases}$$

(M2) L is uniformly strongly elliptic, i.e., there exists $\delta_L > 0$ such that

$$\mathrm{Re}\, b(x,\xi) \ge \delta_L|\xi|^{2m} \text{ for } x \in \Omega,\ \xi \in \mathbb{R}^n.$$

Let us denote $\kappa_L = \sup\limits_{x\in\Omega}\ \sup\limits_{\xi\in\mathbb{R}^n,\ \xi\ne 0} |\arg b(x,\xi)|$.

Theorem 8.2.1. ([Miy06]) *Let $\Omega \subset \mathbb{R}^n$ be a uniformly Lipschitz domain with a small constant L, $p \in (1,\infty)$, and $\theta_0 \in (\kappa_L, \frac{\pi}{2})$. Assume L satisfies (M1) and (M2). Then, there exist $M > 0,\ C > 0$ such that for all $\mu \in \Sigma_{\pi-\theta_0} \cap \{|z| \ge M\}$ the resolvent $(\mu + L)^{-1}$ exists with the estimate for $0 \le i,j \le m$*

$$\|(\mu + L)^{-1}\|_{L(W^{-i,p},W^{j,p})} \le C|\mu|^{-1+\frac{i+j}{2m}}.$$

Corollary 8.2.1. *Let $\Omega \subset \mathbb{R}^n$ be a uniformly Lipschitz domain with a small constant L, $p \in (1,\infty)$, and $\theta_0 \in (\kappa_L, \frac{\pi}{2})$. Assume L satisfies (M1) and (M2). Then, the pair (L,Ω) has property (R_p) for $p \in (1,\infty)$.*

Remark 8.2.1. *The regularity assumption of the boundary and coefficients can be improved when we consider stationary problems. In the paper of S-S. Byun and S. Ryu [BR11], estimates for higher order stationary problems in the setting of Orlicz space are established. $W^{m,p}$ estimates are obtained as a corollary of the results of [BR11] together with uniqueness results. In the paper of V. Maz'ya, M. Mitrea, and T. Shaposhnikova [MMS10], stationary problems of higher order elliptic systems with bounded coefficients in bounded Lipschitz domains are considered.*

8.3 Higher order case

Let $\Omega \subset \mathbb{R}^n$ be a domain, $1 < p < \infty$. Let m be a positive integer, and $L = L_0 + L_1$ be a divergence type differential operator of order $2m$ with the leading term $L_0 = -\sum_{|\alpha|,|\beta|=m}(-1)^m \partial^\beta a_{\alpha,\beta}\partial^\alpha$ of L and the lower order term $L_1 = -\sum_{|\alpha|+|\beta|\le 2m-1}(-1)^{|\beta|}\partial^\beta a_{\alpha,\beta}\partial^\alpha$ of L. We assume that the pair (L,Ω) satisfies a property (R_p) for $p \in (1,\infty)$.

Let $a(u,v) = \displaystyle\sum_{|\alpha|,|\beta|\le m} (a_{\alpha,\beta}\partial^\alpha u, \partial^\beta v)_{L^2(\Omega)}$ for $u \in W_0^{m,p}(\Omega)$ and $v \in W_0^{m,q}(\Omega)$.
In this section, we consider the generalized Gaussian bounds for the semigroup generated by L. The divergence type operator L can be defined in the weak sense as follows:

$$L : W_0^{m,p}(\Omega) \to W^{-m,p}(\Omega)\ L(u)(v) = a(u,v).$$

Let $W_{2m} = \{\phi \in C_0^\infty(\mathbb{R}^n) : \|\partial^\alpha \phi\|_\infty \le 1 \text{ for all } 1 \le |\alpha| \le 2m\}$. Then it is known that $d(x,y) = \sup\{|\phi(x) - \phi(y)| : \phi \in W_{2m}\}$ defines a metric on $\mathbb{R}^n$ equivalent to the Euclidean distance. We define the Davis perturbation $e^{-\lambda\phi}Le^{\lambda\phi}$ of Δ^2 by

$$a_{\lambda\phi}(u,v) = \sum_{|\alpha|,|\beta|\le m} (a_{\alpha,\beta}\partial^\alpha e^{\lambda\phi}u, \partial^\beta e^{-\lambda\phi}v)_{L^2(\Omega)}.$$

Let $\theta_0 = \sup\{\theta : e^{-t\Delta^2}$ has an analytic extension to $\Sigma_\theta\}$. θ_0 can be obtained, for example, by Theorem 8.2.1.

Theorem 8.3.1. *Let $\Omega \subset \mathbb{R}^n$ be a domain, $1 < p < \infty$. Let m be a positive integer, and L be a divergence type differential operator of order $2m$. We assume that the pair (L,Ω) satisfies a property (R_p) for $p \in (1,\infty)$. Let $p \le q \le \infty$, $\theta \in (0,\theta_0)$. Then there exist $A, B, C > 0$ such that*

$$\|\chi_{B_r(x)} e^{tL} \chi_{B_r(y)}\|_{p\to q} \le C|t|^{-\frac{n}{m}(\frac{1}{p}-\frac{1}{q})} e^{-A\frac{|x-y|^{m/m-1}}{\mathrm{Re}t^{1/m-1}}}$$

for $r > 0, x, y \in \Omega, |x-y| \geq 2\sqrt{n}r$,

$$\|\chi_{B_r(x)} e^{tL} \chi_{B_{(k+1)r}(x)\setminus B_{kr}(x)}\|_{p\to q} \leq C|t|^{-\frac{n}{m}(\frac{1}{p}-\frac{1}{q})} e^{-B\frac{(kr)^m}{\mathrm{Re}t^{1/m-1}}}$$

for $r > 0, x \in \Omega, k \geq 2\sqrt{n}$.

In order to obtain generalized Gaussian bounds, we need estimates of Davis perturbation.

Theorem 8.3.2. *Let $\Omega \subset \mathbb{R}^n$ be a domain, $1 < p < \infty$. Let m be a positive integer, and L be a divergence type differential operator of order $2m$. We assume that the pair (L, Ω) satisfies a property (R_p) for $p \in (1, \infty)$. Let $\theta \in (0, \theta_0)$.*

Then there exist C, ω such that for $|\arg t| \leq \theta$, $\lambda \in \mathbb{R}^n$, and $\phi \in W$

$$\|e^{-\lambda\phi} e^{tL} e^{\lambda\phi}\|_{L^p \to L^p} \leq C e^{\omega(1+|\lambda|^{2m})\mathrm{Re}t}, \tag{8.3.1}$$

$$\|\nabla^m e^{-\lambda\phi} e^{tL} e^{\lambda\phi}\|_{L^p \to L^p} \leq C|t|^{-\frac{1}{2}} e^{\omega(1+|\lambda|^{2m})\mathrm{Re}t}, \tag{8.3.2}$$

$$\|e^{-\lambda\phi} e^{tL} e^{\lambda\phi}\|_{L^p \to L^{p^*}} \leq C|t|^{-\frac{n}{2m}(\frac{1}{p}-\frac{1}{p^*})} e^{\omega(1+|\lambda|^{2m})\mathrm{Re}t}. \tag{8.3.3}$$

We will establish Theorem 8.3.2 by a Davies perturbation which was first introduced by E. B. Davies. We recall calculations of $\partial^\beta e^{\lambda\phi}$ in [KW04] and interpolation inequalities with respect to the order of derivatives in [AF03].

Lemma 8.3.1 ([KW04])**.**

$$|\partial^\beta e^{\lambda\phi}| \leq \sum_{k=1}^{|\beta|} |\lambda|^k C_{\beta,k} e^{\lambda\phi}.$$

Lemma 8.3.2 ([AF03])**.** *Let $\Omega \subset \mathbb{R}^n$ be a domain, $f \in W_0^{m,p}(\Omega)$. Then there exists a constant $C > 0$ such that*

$$\|\nabla^k f\|_p \leq \frac{1}{C}\epsilon\|\nabla^m f\|_p + C^{c_{m,k}} C_\epsilon \|f\|_p.$$

We will calculate the difference between $L_{\lambda\phi}$ and L.

Lemma 8.3.3. *For arbitrary $0 < \epsilon < 1$*

$$\sup_{\|v\|_{2,q}=1} |a_{\lambda\phi}(u,v) - a(u,v)| \leq C(\epsilon\|\nabla^m u\|_p + C_\epsilon(1+|\lambda|^{2m})\|u\|_p).$$

Proof. By Lemma 8.3.1, we can calculate

$$\begin{aligned} &|\sum_{|\alpha|,|\beta|\leq m} a_{\alpha,\beta}\partial^\alpha e^{\lambda\phi} u \partial^\beta e^{-\lambda\phi} v - \sum_{|\alpha|,|\beta|\leq m} a_{\alpha,\beta}\partial^\alpha u \partial^\beta v| \\ &\leq C \sum_{0\leq\gamma,\delta\leq m, |\gamma+\delta|\leq 2m-1} (1+|\lambda|)^{2m-|\gamma|-|\delta|} |\partial^\gamma u||\partial^\delta v|. \end{aligned}$$

This implies by Hölder inequality

$$|a_{\lambda\phi}(u,v) - a(u,v)| \le C \sum_{0\le\gamma,\delta\le m, |\gamma+\delta|\le 2m-1} (1+|\lambda|)^{2m-|\gamma|-|\delta|} \|\partial^\gamma u\|_p \|\partial^\delta v\|_q.$$

By applying Lemma 8.3.2 and taking sup over v we obtain the desired estimate

$$\sup_{\|v\|_{m,q}=1} |a_{\lambda\phi}(u,v) - a(u,v)| \le C(\epsilon\|\nabla^m u\|_p + C_\epsilon(1+|\lambda|)^{2m}\|u\|_p).$$

□

Let us define $B_{\lambda\phi} = L_{\lambda\phi} - L$ in the weak sense. Then $B_{\lambda\phi}$ corresponds to the bilinear form $a_{\lambda\phi}(\cdot,\cdot) - a(\cdot,\cdot)$.

By the property (R_p) the operator $(\mu - L)^{-1} : W^{-m,p} \to W_0^{m,p}$ is well defined. For $f \in L^p(\Omega)$ let u be $(\mu - L)^{-1} f$. Then $u \in W_0^{m,p}(\Omega)$ satisfies

$$\begin{cases} (\mu - L)u = f & \text{in } \Omega \\ u = \partial_\nu u = \cdots = \partial_\nu^{m-1} u = 0 & \text{on } \partial\Omega, \end{cases}$$

where ∂_ν is the normal derivative. Let us denote

$$N(u,\mu) = \sum_{|\alpha|\le m} |\mu|^{1-\frac{|\alpha|}{2m}} \|\partial^\alpha u\|_p.$$

The property (R_p) implies

$$N(u,\mu) \le C|\mu|^{\frac{1}{2}} \|f\|_{W^{-m,p}}, \ N(u,\mu) \le C\|f\|_p.$$

Lemma 8.3.4. *(Estimates of lower order terms)*
There exists $M' > 0$ such that for $\mu \in \Sigma_{\pi-\theta_0}$ and $|\mu| > M'(1+|\lambda|^{2m})$

$$\|B_{\lambda\phi}(\mu - L)^{-1}\|_{L^p \to W^{-m,p}} < \frac{|\mu|^{-\frac{1}{2}}}{2}, \tag{8.3.4}$$

$$\|B_{\lambda\phi}(\mu - L)^{-1}\|_{W^{-m,p} \to W^{-m,p}} < \frac{1}{2}. \tag{8.3.5}$$

Proof. First we prove the inequality (8.3.4). Let $f \in L^p(\Omega)$, and let $u = (\mu - L)^{-1} f$. By Lemma 8.3.3 for arbitrary $0 < \epsilon < 1$

$$\sup_{\|v\|_{m,q}=1} |a_{\lambda\phi}(u,v) - a(u,v)| \le C(\epsilon\|\nabla^m u\|_p + C_\epsilon(1+|\lambda|)^{2m}\|u\|_p).$$

By the property (R_p) we can apply $W^{m,p} - L^p$ estimates to obtaining the inequality

$$\sup_{\|v\|_{m,q}=1} |a_{\lambda\phi}(u,v) - a(u,v)| \le C'|\mu|^{-\frac{1}{2}}(\epsilon + C_\epsilon(1+|\lambda|^{2m})|\mu|^{-\frac{1}{2}})\|f\|_p).$$

We take a small ϵ so that $C'\epsilon < \frac{1}{4}$, and then we can also take large $M'_{C',C_\epsilon} > 0$ so that if $|\mu| > M'(1+|\lambda|^{2m})$ then $C'C_\epsilon(1+|\lambda|^{2m})|\mu|^{-\frac{1}{2}} < \frac{1}{4}$. This implies the desired estimate (8.3.4)

$$\sup_{\|v\|_{m,q}=1} |a_{\lambda\phi}((\mu-L)^{-1}f,v) - a((\mu-L)^{-1}f,v)| \leq \frac{|\mu|^{-\frac{1}{2}}}{2}\|f\|_p.$$

For a proof of the inequality (8.3.5), we can prove similarly by using $W^{m,p} - W^{-m,p}$ estimates of the property (R^p). $\square$

Lemma 8.3.5. *(Resolvent estimate for $L_{\lambda\phi}$) Let $\mu \in \Sigma_{\pi-\theta_0} \cap \{|\mu| > M'(1+|\lambda|^4)\}$. Then*

$$\|(\mu - L - B_{\lambda\phi})^{-1}\|_{L^p\to L^p} \leq C|\mu|^{-1}.$$

Proof. We can formally obtain the following inequality by considering von Neumann series

$$\begin{aligned}
&\|(\mu - L - B_{\lambda\phi})^{-1}\|_{L^p\to L^p} \\
&= \|(\mu-L)^{-1}\sum_{k=0}^{\infty}(B_{\lambda\phi}(\mu-L)^{-1})^k\|_{L^p\to L^p} \\
&\leq \|(\mu-L)^{-1}\|_{L^p\to L^p} + \|(\mu-L)^{-1}(B_{\lambda\phi}(\mu-L)^{-1})\|_{L^p\to L^p} \\
&+ \|(\mu-L)^{-1}(\sum_{k=1}^{\infty}(B_{\lambda\phi}(\mu-L)^{-1})^k)(B_{\lambda\phi}(\mu-L)^{-1})\|_{L^p\to L^p}.
\end{aligned}$$

By the property (R_p), the first term is estimated by $\frac{C}{|\mu|}$. By the property (R_p) and Lemma 8.3.4 (8.3.4), the second term is estimated by

$$\begin{aligned}
&\|(\mu-L)^{-1}(B_{\lambda\phi}(\mu-L)^{-1})\|_{L^p\to L^p} \\
&\leq \|(\mu-L)^{-1}\|_{W^{-m,p}\to L^p}\|B_{\lambda\phi}(\mu-L)^{-1}\|_{L^p\to W^{-m,p}} \\
&\leq \frac{C}{|\mu|^{\frac{1}{2}}}\frac{1}{2|\mu|^{\frac{1}{2}}} \leq \frac{C}{2|\mu|}.
\end{aligned}$$

By the property (R_p) and Lemma 8.3.4, we can estimate the last term

$$\begin{aligned}
&\|(\mu-L)^{-1}(\sum_{k=1}^{\infty}(B_{\lambda\phi}(\mu-L)^{-1})^k)(B_{\lambda\phi}(\mu-L)^{-1})\|_{L^p\to L^p} \\
&\leq \|(\mu-L)^{-1}\|_{W^{-m,p}\to L^p}\|\sum_{k=1}^{\infty}(B_{\lambda\phi}(\mu-L)^{-1})^k\|_{W^{-m,p}\to W^{-m,p}}\|B_{\lambda\phi}(\mu-L)^{-1}\|_{L^p\to W^{-m,p}} \\
&\leq \frac{C}{|\mu|^{\frac{1}{2}}}(\sum_{k=1}^{\infty}\frac{1}{2^k})\frac{1}{2|\mu|^{\frac{1}{2}}} \leq \frac{C}{|\mu|}.
\end{aligned}$$

Thus, we finish the proof of Lemma 8.3.5. $\square$

proof of Theorem 8.3.2 . We first prove (8.3.1). Lemma 8.3.5 implies $\Sigma_{\pi-\theta_0} \cap \{|\mu| > M'(1+|\lambda|^{2m})\} \subset \rho(L_{\lambda\phi})$. Let $\theta \in (0, \theta_0)$. We set $\theta' = \pi/2 - (\theta_0 + \theta)/2$, then $\Sigma_{\pi-\theta'} + r/\sin\theta' \subset \Sigma_{\pi-\theta'} \cap \{|\mu| \geq r\}$ for $r > 0$. We take $r = M'(1+|\lambda|^{2m})$ and we define $\tilde{L}_{\lambda\phi} = L_{\lambda\phi} + M'(1+|\lambda|^{2m})/\sin\theta'$. In this setting, we conclude that $\Sigma_{\pi-\theta'} \subset \rho(\tilde{L}_{\lambda\phi})$, and for $\mu \in \Sigma_{\pi-\theta'}$ we have

$$\begin{aligned}\|\mu(\mu - \tilde{L}_{\lambda\phi})^{-1}\|_{L^p\to L^p} &= \|(\mu + \frac{M'(1+|\lambda|^{2m})}{\sin\theta'} - L - B_{\lambda\phi})^{-1}\|_{L^p\to L^p} \\ &\leq C\frac{|\mu|}{|\mu + \frac{M'(1+|\lambda|^{2m})}{\sin\theta'}|}.\end{aligned}$$

By calculating the right hand side as seen in [Kun08, pp2742], we obtain the following resolvent estimates for $\mu \in \Sigma_{\pi-\theta'}$, $\lambda \in \mathbb{R}^n$, and $\phi \in W_{2m}$

$$\|\mu(\mu + \tilde{\Delta}^2{}_{\lambda\phi})^{-1}\|_{L^p\to L^p} \leq \frac{C}{\sin\theta'}.$$

Therefore, $\tilde{L}_{\lambda\phi}$ generates a bounded analytic semigroup $e^{t\tilde{L}_{\lambda\phi}}$ with the estimate

$$\|e^{t\tilde{L}_{\lambda\phi}}\|_{L^p\to L^p} \leq C.$$

This implies (8.3.1) of Theorem 8.3.2.

We next prove (8.3.2). For $\mu \in \Sigma_{\pi-\theta'} \subset \rho(\tilde{L}_{\lambda\phi})$ we can argue similarly in Lemma 8.3.5

$$\begin{aligned}&\|\mu^{\frac{1}{2}}\nabla^m(\mu - \tilde{L}_{\lambda\phi})^{-1}\|_{L^p\to L^p} \\ &= \|\mu^{\frac{1}{2}}\nabla^m(\mu + \frac{M'(1+|\lambda|^{2m})}{\sin\theta'} - L - B_{\lambda\phi})^{-1}\|_{L^p\to L^p} \\ &= \|\mu^{\frac{1}{2}}\{\nabla^m(\mu + \frac{M'(1+|\lambda|^{2m})}{\sin\theta'} - L)^{-1}\}\sum_{k=0}^{\infty}(B_{\lambda\phi}(\mu + \frac{M'(1+|\lambda|^{2m})}{\sin\theta'} - L)^{-1})^k\|_{L^p\to L^p} \\ &\leq C\frac{|\mu|^{\frac{1}{2}}}{|\mu + \frac{M'(1+|\lambda|^{2m})}{\sin\theta'}|^{\frac{1}{2}}}.\end{aligned}$$

By the same argument in (8.3.1) of Theorem 8.3.2, we obtain (8.3.2). We finally prove (8.3.3). Let $v = e^{t\tilde{L}_{\lambda\phi}}f$. We consider the three cases, $n > mp$, $n < mp$, $n = mp$. When $n > mp$, by $v(t,\cdot) \in W_0^{m,p}(\Omega)$ for each t, the zero extension, and the Sobolev inequality [AF03, Theorem 4.31] we obtain for $p^* = np/(n-mp)$

$$\|v\|_{p^*} \leq C_{p,n}\|\nabla^m v\|_p.$$

By (8.3.2), we obtain

$$\|v\|_{p\to p^*} \leq C_{p,n}\|\nabla^m e^{-\lambda\phi}e^{tL}e^{\lambda\phi}\|_{L^p\to L^p} \leq C|t|^{-\frac{1}{2}}e^{\omega(1+|\lambda|^{2m})\mathrm{Re}t},$$

Since $-1/2 = -n/2m(1/p - 1/p^*)$, (8.3.3) is proved if $n > mp$. When $n < mp$, by $v(t,\cdot) \in W_0^{m,p}(\Omega)$ for each t, the zero extension, and the Gagliardo-Nirenberg inequality we obtain for $p^* = \infty$

$$\|v\|_\infty \leq C_{n,p}\|\nabla^m v\|_p^{\frac{n}{mp}}\|v\|_p^{1-\frac{n}{mp}}.$$

(8.3.1), (8.3.2) imply (8.3.3) when $n < mp$, In the case $n = mp$, estimates are obtained from $p - \epsilon'$ case by Sneiberg's lemma [Kun05, Theorem 8]. □

We would to apply the following lemma to obtain Theorem 8.3.1 from Theorem 8.3.2.

Lemma 8.3.6 ([Kun08])**.** *Let $G \subset \mathbb{C}$ and $S(t)$ be a semigroup. Assume that there exist $C(t)$, ω such that for $\lambda \in \mathbb{R}$, $\phi \in W$ and $t \in G$*

$$\|e^{-\lambda\phi} S(t) e^{\lambda\phi}\|_{L^p \to L^q} \leq C(t) e^{\omega|\lambda|^4 \mathrm{Re} t}.$$

Then there exist $A, B > 0$ such that

$$\|\chi_{B_r(x)\cap\Omega} S(t) \chi_{B_r(y)\cap\Omega}\|_{p\to q} \leq C(t) e^{-A\frac{|x-y|^{m/m-1}}{\mathrm{Re} t^{1/m-1}}} \text{ for } r > 0, x, y \in \Omega, |x-y| \geq 2\sqrt{n}r,$$

$$\|\chi_{B_r(x)\cap\Omega} S(t) \chi_{B_{(k+1)r}(x)\setminus B_{kr}(x)\cap\Omega}\|_{p\to q} \leq C(t) e^{-B\frac{(kr)^m}{\mathrm{Re} t^{1/m-1}}} \text{ for } r > 0, x \in \Omega, k \geq 2\sqrt{n}.$$

Proof. By using the zero extension, we obtain this lemma from original statement in [Kun08]. □

This lemma and Theorem 8.3.2 imply Theorem 8.3.1 for $q = p^*$.

Lemma 8.3.7. *Let $\Omega \subset \mathbb{R}^n$ be a domain, $1 < p < \infty$. Let m be a positive integer, and L be a divergence type differential operator of order $2m$. We assume that the pair (L, Ω) satisfies a property (R_p) for $p \in (1, \infty)$. Let $p \leq q \leq p^*$, $\theta \in (0, \theta_0)$. Then there exist $A, B, C > 0$ such that*

$$\|\chi_{B_r(x)} e^{tL} \chi_{B_r(y)}\|_{p\to q} \leq C|t|^{-\frac{n}{m}(\frac{1}{p}-\frac{1}{q})} e^{-A\frac{|x-y|^{m/m-1}}{\mathrm{Re} t^{1/m-1}}} \text{ for } r > 0, x, y \in \Omega, |x-y| \geq 2\sqrt{n}r$$

$$\|\chi_{B_r(x)} e^{tL} \chi_{B_{(k+1)r}(x)\setminus B_{kr}(x)}\|_{p\to q} \leq C|t|^{-\frac{n}{m}(\frac{1}{p}-\frac{1}{q})} e^{-B\frac{(kr)^m}{\mathrm{Re} t^{1/m-1}}} \text{ for } r > 0, x \in \Omega, k \geq 2\sqrt{n}$$

$$\|\chi_{B_r(x)} e^{tL} \chi_{B_r(y)}\|_{q\to p^*} \leq C|t|^{-\frac{n}{m}(\frac{1}{q}-\frac{1}{p^*})} e^{-A\frac{|x-y|^{m/m-1}}{\mathrm{Re} t^{1/m-1}}} \text{ for } r > 0, x, y \in \Omega, |x-y| \geq 2\sqrt{n}r$$

$$\|\chi_{B_r(x)} e^{tL} \chi_{B_{(k+1)r}(x)\setminus B_{kr}(x)}\|_{q\to p^*} \leq C|t|^{-\frac{n}{m}(\frac{1}{q}-\frac{1}{p^*})} e^{-B\frac{(kr)^m}{\mathrm{Re} t^{1/m-1}}} \text{ for } r > 0, x \in \Omega, k \geq 2\sqrt{n}$$

Proof. By interpolating (8.3.1), (8.3.3), we obtain this lemma from Lemma 8.3.6. □

Proof of Theorem 8.3.1. By a similar argument in [Kun08, pp2746], we can extend the results in Lemma 8.3.7 to general $q \in [p, \infty)$. □

8.4 Application to L^p maximal regularity

We would like to consider L^p maximal regularity as an application of Gaussian bounds. This section is due to [Kun08]. We recall homogeneous type spaces.

Definition 8.4.1 ([Kun08])**.** *Let $X = (\Omega, d, |\cdot|)$ be a measured metric space. X is said to be of homogeneous type if there exist $C_\Omega, n > 0$ such that for $\rho > 0$, $\lambda \geq 1$, $x \in \Omega$,*

$$|B_{\lambda\rho}(x)| \leq C_\Omega \lambda^n |B_\rho(x)|.$$

Theorem 8.4.1 ([Kun08])**.** *Let $(\Omega, d, |\cdot|)$ be a measured metric space of homogeneous type. Let $1 \leq q_0 \leq q_1 \leq \infty$, and assume that $\{S(t)\}_{t\in\tau}$ is a family of linear operators on $L^{q_0} \cap L^{q_1}$ such that for $t \in \tau$, $x \in \Omega$, $k \in \mathbb{N}$*

$$\|\chi_{B_{\rho(t)}(x)} S(t) \chi_{B_{(k+1)\rho(t)}(x) \setminus B_{k\rho(t)}(x)}\|_{q_0 \to q_1} \leq |B_{\rho(t)}(x)|^{-(\frac{1}{q_0} - \frac{1}{q_1})} h(k)$$

where $\rho : \tau \to (0, \infty)$ is a function and the sequence $\{h(k)\}_k$ satisfies $h(k) \leq C_\delta (k+1)^{-\delta}$ for some $\delta > \frac{n}{q_0} + \frac{1}{q_0'}$. Then $\{S(t) : t \in \tau\}$ is R_s-bounded in $L^q(\Omega)$ for all $(q, s) \in (q_0, q_1) \times [q_0, q_1] \cup \{(q_0, q_0), (q_1, q_1)\}$.

Theorem 8.4.2 ([Wei01], [Wei201])**.** *Let $q \in (1, \infty)$ and let L be the generator of a bounded analytic semigroup in $L^q(\Omega)$. Then $-L$ has maximal regularity if and only if, for some $\delta > 0$, the set $\{e^{zL} : |\arg z| \leq \delta\}$ is R_2-bounded in $L^q(\Omega)$, which is the case if and only if $\{e^{e^{\pm i\delta} tL} : t \geq 0\}$ is R_2-bounded in $L^q(\Omega)$.*

These imply the following maximal regularity from generalized Gaussian bounds.

Corollary 8.4.1. *Let $\Omega \subset \mathbb{R}^n$ be a domain, $1 < p < \infty$. Let m be a positive integer, and L be a divergence type differential operator of order $2m$. We assume that the pair (L, Ω) satisfies a property (R_p) for $p \in (1, \infty)$. Then $-L$ has L^p maximal regularity.*

Corollary 8.4.2. *Let $\Omega \subset \mathbb{R}^n$ be a uniformly Lipschitz domain with a small constant L, $p \in (1, \infty)$. Assume that L satisfies (M1) and (M2). Then, $-L$ has L^p maximal regularity.*

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