博士論文

論文題目　Partially ordered sets, order complexes and CAT(0) properties
(半順序集合, 順序複体, 及び CAT(0) 性)

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Partially ordered sets, order complexes and CAT(0) properties

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Preface

In this thesis, we study relevance between the combinatorial structure of a poset and the geometry of its order complex. The order complex of a poset P is defined to be the abstract simplicial complex whose vertices are the elements of P, and whose faces are the finite chains of P. Its geometric realization $|P|$ is called the classifying space of P. The aim of this study is to reveal the relationship between combinatorial properties of a poset P and geometric properties of the geometric realization $|P|$ of the order complex. In this thesis, we discuss two topics. In Chapter 1, we provide some basic notions and terminology. Chapter 2 and 3 are devoted to the two topics, respectively.

In Chapter 2, we study homotopy types of Frobenius complexes. Let Λ be an additive monoid, and assume that the following two conditions hold:

(1) Λ is cancellative, that is, $\lambda+\mu=\lambda+\mu'$ implies $\mu=\mu'$ for any $\lambda,\mu,\mu'\in\Lambda$.
(2) Λ has no non-zero invertible elements, that is, $\lambda+\mu=0$ implies $\lambda=\mu=0$ for any $\lambda,\mu\in\Lambda$.

Such an Λ is said to be poset-like. Indeed, Λ can be equipped with the partial order defined by $\lambda\leq\lambda+\mu$ for $\lambda,\mu\in\Lambda$. For a non-zero element λ of Λ, the Frobenius complex $\mathcal{F}(\lambda;\Lambda)$ is defined to be the geometric realization $|(0,\lambda)_\Lambda|$ of the order complex of the open interval in Λ.

Frobenius complexes were introduced by Laudal and Sletsjøe in order to determine the torsion group $\mathrm{Tor}_*^{k[\Lambda]}(k,k)$ of the monoid algebra $k[\Lambda]$ over a field k. They proved the isomorphism

$$\mathrm{Tor}_{i,\lambda}^{k[\Lambda]}(k,k)\cong\widetilde{H}_{i-2}(\mathcal{F}(\lambda;\Lambda);k).$$

They derived a formula for the Poincaré series of the torsion group $\mathrm{Tor}_*^{k[\Lambda]}(k,k)$ in the case where Λ is a saturated rational submonoid of $\mathbb{N}^2$.

Clark and Ehrenborg focused on homotopy types of Frobenius complexes as a homotopical refinement of the Frobenius coin-exchange problem. They determined the homotopy types of the Frobenius complexes of Λ in the following two cases:

(1) Λ is generated by two relatively prime positive integers.
(2) Λ is generated by the arithmetic sequence $a, a+d, \ldots, a+(a-1)d$, where a and d are relatively prime positive integers.

Their proof is based on discrete Morse theory.

In Section 2.4, we show a broad generalization of the result about the two-generators case by Clark and Ehrenborg.

Theorem (Theorem 2.4.2). *Let Λ_1 and Λ_2 be finitely generated poset-like additive monoids. Let ρ_1 and ρ_2 be reducible elements of Λ_1 and Λ_2, respectively. Let Λ be the additive monoid obtained from the direct sum $\Lambda_1\oplus\Lambda_2$ by identifying ρ_1 with ρ_2. Let ρ denote the equivalence class of ρ_1 and ρ_2. Then there is a homotopy equivalence*

$$\mathcal{F}(\lambda;\Lambda)\simeq\bigvee_{\ell\rho+\lambda_1+\lambda_2=\lambda}S^{2\ell}*\mathcal{F}(\lambda_1;\Lambda_1)*\mathcal{F}(\lambda_2;\Lambda_2)$$

for $\lambda\in\Lambda$, where ℓ, λ_1 and λ_2 run through $\mathbb{N},\Lambda_1$ and Λ_2, respectively.

The proof is based on theory of homotopy colimits of diagrams of topological spaces over a finite poset. We provide the definition and basic properties of homotopy colimits in Section 2.3. As a corollary of the theorem, we derive the formula

$$P_k^{k[\Lambda]}(t,\mathbf{z})=\frac{P_k^{k[\Lambda_1]}(t,\mathbf{z})\cdot P_k^{k[\Lambda_2]}(t,\mathbf{z})}{1-t^2\mathbf{z}^\rho}$$

for the multi-graded Poincaré series associated to Λ. In Section 2.5, as an application of the theorem, we determine the homotopy types of Frobenius complexes and the multi-graded Poincaré series for some cases.

In Chapter 3, we discuss CAT(0) properties for orthoscheme complexes. The CAT(0) property is defined for geodesic metric spaces as a generalization of non-positive curvature property. Gromov gave a simple combinatorial characterization for cubical complexes to be a (locally) CAT(0) space. Brady and McCammond defined the orthoscheme metric on the geometric realization $|P|$ of the order complex of a graded poset P. The aim was to show that the braid groups and other Artin groups of finite types are CAT(0) groups. The geometric realization of the order complex equipped with the orthoscheme metric is called the orthoscheme complex. Orthoscheme complexes can be seen as a generalization of cubical complexes. Indeed, a cubical complex X is isometric to the orthoscheme complex of the face poset of X. Brady and McCammond gave a combinatorial characterization for a bounded graded poset of rank ≤ 4 to have CAT(0) orthoscheme complex. In the light of the above characterization, they showed that the n-strand braid group is a CAT(0) group for $n \leq 5$.

There were some sufficient conditions for a graded poset P to have CAT(0) orthoscheme complex. However, it seems that there were few necessary and sufficient conditions in a general situation. In Section 3.5, we try to give a characterization for orthoscheme complexes to be CAT(0) as a generalization of Gromov's characterization for cubical complexes. We first show the following theorem, which is equivalent to Gromov's characterization under some observations.

THEOREM (Theorem 3.5.3). *Let S be a semilattice of finite height, and assume that each principal ideal $S^{\leq x}$ is a Boolean lattice. Then the orthoscheme complex $|S|$ is a CAT(0) space if and only if S is a flag semilattice.*

A semilattice S is said to be a flag semilattice if any pairwise bounded finite subset of S is bounded. We can weaken the hypothesis of the previous theorem as follows.

THEOREM (Theorem 3.5.4). *Let S be a semilattice of finite height, and assume that each principal ideal $S^{\leq x}$ is a distributive lattice. Then the orthoscheme complex $|S|$ is a CAT(0) space if and only if S is a flag semilattice.*

One of the keys of the proof is a representation theorem for semilattices which satisfy the hypothesis of Theorem 3.5.4. This is shown in Section 3.3 as an extension of Birkhoff's representation theorem for distributive semilattices (Theorem 3.3.4). Another key is a construction of cubical cone, which is a cubical analogue of the construction of the cone. We will introduce it in Section 3.4, and show some properties. In particular, cubical cones can be seen as a partial inverse of cubical links of cubical complexes.

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Contents

CHAPTER 1

Preliminaries

In this chapter, we review the definitions and properties of some basic notions, which will be used in later chapters.

1.1. Topological spaces

In this section, we will recall some basic notions and properties on topological spaces. References for the subjects of this section are [**Bro**], [**Hat**] and [**May**].

First, we give notation which will be used in this thesis. We denote the n-dimensional sphere by S^n for $n \geq 0$, the empty space by S^{-1}, and the one-point space by $\mathbf{pt}$. Let k be a field. We denote the i-th homology group of a topological space X with coefficients in k by $H_i(X; k)$. Here we adopt the convention $H_i(X; k) = 0$ for $i < 0$. Similarly, we denote the i-th reduced homology group by $\widetilde{H}_i(X; k)$. Here $\widetilde{H}_i(X; k)$ is defined to be the kernel of the induced map $H_i(X; k) \to H_i(\mathbf{pt}; k)$ for a non-empty topological space X. For convenience, we define $\widetilde{H}_i(S^{-1}; k)$ to be k for $i = -1$ and 0 otherwise. This definition can be seen in [**Bjö**]. We denote the i-th reduced Betti number by $\widetilde{\beta}_i(X; k)$, that is, $\widetilde{\beta}_i(X; k) = \dim_k \widetilde{H}_i(X; k)$. We will omit k from the notation if no confusion can arise.

Let X be a topological space, and A a subspace of X. We say that A is a *deformation retract* of X, or the inclusion $A \hookrightarrow X$ is a *deformation retract*, if there exists a continuous map $H : X \times [0, 1] \to X$ such that

(1) $H(x, 0) = x$ for all $x \in X$.
(2) $H(a, t) = a$ for all $a \in A$ and $t \in [0, 1]$.
(3) $H(x, 1) \in A$ for all $x \in X$.

Such an H is called a *deformation retraction.* If we set $r = H(-, 1) : X \to A$, then r is a homotopy inverse of the inclusion $A \hookrightarrow X$. We also call r a deformation retraction.

Deformation retracts are closed under compositions, that is, if both $B \hookrightarrow A$ and $A \hookrightarrow X$ are deformation retracts, then so is $B \hookrightarrow X$. Let us note that deformation retracts are also closed under pushouts. More precisely, if $A \hookrightarrow X$ is a deformation retract, B is a subspace of A, and $f : B \to Y$ is a continuous map, then $Y \cup_f A \hookrightarrow Y \cup_f X$ is again a deformation retract. Here $Y \cup_f X$ denotes the adjunction space, that is, the topological space obtained from the disjoint union of X and Y by identifying $b \in B$ with $f(b) \in Y$.

Let $i : A \to X$ be a continuous map. We say i is a *Hurewicz cofibration*, or simply a *cofibration*, if i satisfies the homotopy extension property, that is, for any topological space Y and continuous maps $g : X \to Y$ and $F : A \times [0, 1] \to Y$ with $g \circ i = F(-, 0)$, there exists a continuous map $G : X \times [0, 1] \to Y$ which satisfies $g = G(-, 0)$ and $G \circ (i \times \mathrm{id}_{[0,1]}) = F$. For example, the inclusion $A \hookrightarrow X$ of a CW pair (X, A) is a cofibration. A based space (X, b) is said to be *well pointed* if the inclusion $\{b\} \hookrightarrow X$ is a cofibration. If X is a CW complex, (X, b) is well pointed for any $b \in X$.

LEMMA 1.1.1. *The following hold.*

(1) *Cofibrations are closed under compositions, that is, if* $i\colon X \to Y$ *and* $j\colon Y \to Z$ *are cofibrations, then* $j \circ i\colon X \to Z$ *is again a cofibration.*
(2) *Cofibrations are closed under pushouts, that is, if an inclusion* $i\colon A \hookrightarrow X$ *is a cofibration, and* $f\colon A \to Y$ *is a continuous map, then the induced map* $Y \to Y \cup_f X$ *is again a cofibration.*

Proof. The proof is straightforward. □

Lemma 1.1.2. *Let* $i\colon A \to X$ *and* $j\colon A \to Y$ *be cofibrations. Let* $f\colon X \to Y$ *is a homotopy equivalence satisfying* $f \circ i = j$*. Then* f *is a homotopy equivalence relative to* i *and* j*, that is, there are continuous maps* $g\colon Y \to X$*,* $F\colon X \times [0,1] \to X$ *and* $G\colon Y \times [0,1] \to Y$ *which satisfy the following:*

- $g \circ j = i$
- $F(-,0) = g \circ f$*,* $F(-,1) = \mathrm{id}_X$*,* $F(i(-),t) = i$ $(t \in [0,1])$
- $G(-,0) = f \circ g$*,* $G(-,1) = \mathrm{id}_Y$*,* $G(j(-),t) = j$ $(t \in [0,1])$

Proof. The proof is elementary; see [**Hat**, Proposition 0.19]. □

Lemma 1.1.3. *Let* X *be a topological space, and* A *be a subspace of* X*. If the inclusion* $A \hookrightarrow X$ *is a cofibration and a homotopy equivalence, then* A *is a deformation retract of* X*.*

Proof. Applying the previous lemma to the inclusion $A \hookrightarrow X$, we obtain a continuous map $G\colon X \times [0,1] \to X$ satisfying $G(x,0) = x$, $G(a,t) = a$ and $G(x,1) \in A$. □

Let $f\colon X \to Y$ be a continuous map. The *mapping cylinder* $M(f) = M(X \xrightarrow{f} Y)$ of f is obtained from the disjoint union of $X \times [0,1]$ and Y by identifying $(x,1)$ with $f(x)$ for each $x \in X$. There are canonical embeddings $i\colon X \to M(f)$ and $j\colon Y \to M(f)$, where $i(x) = [x,0]$ and $j(y) = [y]$. One can easily check that i and j are cofibrations. Moreover, Y is a deformation retract of $M(f)$. A deformation retraction is given by $H([x,t],s) = [x, \max\{t,s\}]$ and $H([y],s) = [y]$. If we set $r = H(-,1)\colon M(f) \to Y$, then we have $f = r \circ i$. Thus f is a homotopy equivalence if and only if so is i. By the previous lemma, we obtain the following:

Lemma 1.1.4. *A continuous map* $f\colon X \to Y$ *is a homotopy equivalence if and only if* $i\colon X \hookrightarrow M(f)$ *is a deformation retract.*

Let $f\colon X \to Y$ and $g\colon X \to Z$ be continuous maps. The *double mapping cylinder* $DM(f,g) = DM(Y \xleftarrow{f} X \xrightarrow{g} Z)$ is obtained from the disjoint union of $M(f)$ and $M(g)$ by identifying two canonically embedded X.

Lemma 1.1.5. *Let us consider a commutative diagram*

$$\begin{tikzcd}
X_1 \arrow[d, "h_1"'] & X_0 \arrow[l, "f_1"'] \arrow[r, "f_2"] \arrow[d, "h_0"'] & X_2 \arrow[d, "h_2"] \\
Y_1 & Y_0 \arrow[l, "g_1"] \arrow[r, "g_2"'] & Y_2
\end{tikzcd}$$

of topological spaces and continuous maps. If all vertical arrows are homotopy equivalences, then the induced map $DM(f_1,f_2) \to DM(g_1,g_2)$ *is also a homotopy equivalence.*

Proof. By Lemma 1.1.4, it is enough to show that the canonical embedding $DM(f_1,f_2) \to M(DM(f_1,f_2) \to DM(g_1,g_2))$ is a deformation retract. There is canonical homeomorphisms

$$\begin{aligned}
M(DM(f_1,f_2)\to DM(g_1,g_2)) &\approx DM(M(h_1)\leftarrow M(h_0)\to M(h_2))\\
&= M(M(h_0)\to M(h_1)) \underset{M(h_0)}{\cup} M(M(h_0)\to M(h_2))\\
&\approx M(M(f_1)\to M(g_1)) \underset{M(h_0)}{\cup} M(M(f_2)\to M(g_2)).
\end{aligned}$$

Let us set $Z_i = M(M(f_i)\to M(g_i))$ for $i=1,2$. We now show that $M(f_1)\cup_{X_0} M(h_0)\hookrightarrow Z_1$ is a deformation retract. Since h_0 is a homotopy equivalence, $X_0\hookrightarrow M(h_0)$ is a deformation retract, and thus so is $M(f_1)\hookrightarrow M(f_1)\cup_{X_0} M(h_0)$. In the commutative diagram

$$\begin{array}{ccc}
M(f_1) & \longrightarrow & X_1\\
\big\downarrow & & \big\downarrow\\
Z_1 & \longrightarrow & M(h_1),
\end{array}$$

the top, right and bottom arrows are homotopy equivalences, thus so is the left. Hence, the inclusion $M(f_1)\cup_{X_0} M(h_0)\to Z_1$ is also a homotopy equivalence. One can check that this inclusion is a cofibration. By Lemma 1.1.3, the above inclusion is a deformation retract.

Similarly, we can prove that $M(f_2)\cup_{X_0} M(h_0)\hookrightarrow Z_2$ is a deformation retract. Thus the inclusion

$$\Big(M(f_1)\underset{X_0}{\cup} M(h_0)\Big) \underset{M(h_0)}{\cup} \Big(M(f_2)\underset{X_0}{\cup} M(h_0)\Big) \hookrightarrow Z_1 \underset{M(h_0)}{\cup} Z_2$$

is a deformation retract. Since $M(h_0)$ deformation retracts to X_0, the left-hand side of this inclusion deformation retracts to $M(f_1)\cup_{X_0} M(f_2)$, which is the same as $DM(f_1,f_2)$. $\square$

Lemma 1.1.6. *Let $i\colon A\to X$ and $j\colon A\to Y$ be cofibrations. Then the canonically induced map $DM(i,j)\to X\cup_A Y$ is a homotopy equivalence, where $X\cup_A Y$ denotes the space obtained from X and Y by identifying $i(a)$ with $j(a)$ for all $a\in A$.*

Proof. Applying Lemma 1.1.2 to the obvious deformation retractions $M(i)\to X$ and $M(j)\to Y$, we obtain homotopy inverses and homotopies preserving A. By attaching them, we can construct a homotopy inverse of $DM(i,j)\to X\cup_A Y$. $\square$

Lemma 1.1.7 ([**Bro**, 7.4.1]). *Let us consider a commutative diagram*

$$\begin{array}{ccccc}
X_1 & \xleftarrow{i_1} & X_0 & \xrightarrow{i_2} & X_2\\
{\scriptstyle f_1}\big\downarrow & & {\scriptstyle f_0}\big\downarrow & & \big\downarrow{\scriptstyle f_2}\\
Y_1 & \xleftarrow[j_1]{} & Y_0 & \xrightarrow[i_2]{} & Y_2
\end{array}$$

of topological spaces and continuous maps. If all vertical arrows are homotopy equivalences, and all horizontal arrows are cofibrations, then the induced map

$$X_1\underset{X_0}{\cup} X_2\to Y_1\underset{Y_0}{\cup} Y_2$$

is a homotopy equivalence.

Proof. Applying Lemma 1.1.5 and Lemma 1.1.6, we obtain the commutative diagram

$$\begin{array}{ccc}
DM(i_1,i_2) & \longrightarrow & X_1\underset{X_0}{\cup} X_2\\
\big\downarrow & & \big\downarrow\\
DM(j_1,j_2) & \longrightarrow & Y_1\underset{Y_0}{\cup} Y_2,
\end{array}$$

whose top, left and bottom arrows are homotopy equivalences. Thus the right arrow is also a homotopy equivalence. $\square$

Let X and Y be topological spaces. The *join* $X * Y$ is defined to be the double mapping cylinder $DM(X \xleftarrow{\mathrm{pr}_1} X \times Y \xrightarrow{\mathrm{pr}_2} Y)$. Here pr_i denotes the i-th projection. By definition, we have $S^{-1} * X = X = X * S^{-1}$. The join $\mathbf{pt} * X$ with the one-point space is called the *cone* of X, which is obviously contractible. The join $S^0 * X$ with the two-point space is called the *suspension* of X, and denoted by $\operatorname{susp} X$. There is a well-known formula for the homology of the suspension

$$\widetilde{H}_i(\operatorname{susp} X) \cong \widetilde{H}_{i-1}(X).$$

In our definition, the above holds even if X is empty. Moreover, there is a formula for the homology of the join [**Mil**, Lemma 2.1]. If k is a field, it is simply given by

$$\widetilde{H}_i(X * Y; k) \cong \bigoplus_{p+q=i-1} \widetilde{H}_p(X; k) \otimes \widetilde{H}_q(Y; k). \tag{1}$$

By Lemma 1.1.5, the join is homotopy invariant, that is, homotopy equivalences $f\colon X \to X'$ and $g\colon Y \to Y'$ induces a homotopy equivalence $X * Y \to X' * Y'$. The canonical embedding $X \hookrightarrow X * Y$ is a cofibration, which is follow from the construction.

1.2. Simplicial complexes

In this section, we briefly review some basic definitions and properties on simplicial complexes and their geometric realizations. Reference for the subjects of this section are [**Bjö**] and [**Wal**].

An *abstract simplicial complex* K is a family of finite sets such that any subset of any element of K is again an element of K. An element of K is said to be a *face* of K, and an element of a face of K is said to be a *vertex* of K. In our definition, the empty set is a face of K unless $K = \emptyset$. The set of the vertices of K is denoted by $V(K)$. The *dimension* of a face σ of K is defined to be $\#\sigma - 1$, and denoted by $\dim \sigma$. A face of dimensions d is simply called a *d-face*. If the dimensions of the faces of K are bounded above, then K is said to be *finite dimensional*. In this case, the *dimension* of K is defined by

$$\dim K = \max_{\sigma \in K} \dim \sigma.$$

Simplicial maps. Let K and L be abstract simplicial complexes. A *simplicial map* from K to L is a map $f\colon V(K) \to V(L)$ such that the image $f(\sigma)$ of any face σ of K is a face of L. A simplicial map f is an *isomorphism* if f is bijective and the inverse f^{-1} is a simplicial map from L to K. If an isomorphism between K and L exists, then K and L are said to be *isomorphic*, and we write $K \cong L$.

Geometric realizations. For a finite set σ, the *standard simplex* of vertex set σ is defined by

$$\Delta^\sigma = \Big\{ \sum_{v \in \sigma} t_v v \;\Big|\; t_v \geq 0,\ \sum_{v \in \sigma} t_v = 1 \Big\} \subset \mathbb{R}^{(\sigma)},$$

where $\mathbb{R}^{(\sigma)}$ denote the free linear space $\bigoplus_{v \in \sigma} \mathbb{R}\sigma$ with basis σ. Geometrically, Δ^σ is a point if $\dim \sigma = 0$, a segment if $\dim \sigma = 1$, and a triangle if $\dim \sigma = 2$. For an abstract simplicial complex K, the *geometric realization* of K is defined by

$$|K| = \bigcup_{\sigma \in K} \Delta^\sigma \subset \mathbb{R}^{(V(K))}.$$

Equivalently, $|K|$ can be defined as

$$|K| = \Big\{ x = \sum_{v\in V(K)} t_v v \;\Big|\; t_v \geq 0,\ \sum_{v\in V(K)} t_v = 1,\ \operatorname{supp} x \in K \Big\},$$

where $\operatorname{supp} x = \{\, v \in V(K) \mid t_v \neq 0 \,\}$. Usually, we consider $|K|$ as a topological space with the weak topology with respect to Δ^σ for $\sigma \in K$, that is, the coarsest topology on $|K|$ such that the inclusion $\Delta^\sigma \hookrightarrow |K|$ is a continuous map for each $\sigma \in K$. In Chapter 2, we focus on homotopy types of geometric realizations with this topology. In Chapter 3, we consider piecewise Euclidean metrics on geometric realizations, and study their curvature properties. Such a metric defines another topology on the geometric realization. This topology coincides with the weak topology if and only if K is locally finite, that is, each vertex of K belongs to only finitely many faces of K. In the rest of this section, we consider the weak topology on the geometric realization.

Let $f\colon K \to L$ be a simplicial map. The *geometric realization* $|f|\colon |K| \to |L|$ is defined by

$$|f|\Big(\sum_{v\in V(K)} t_v v\Big) = \sum_{v\in V(K)} t_v f(v).$$

Then $|f|$ is a continuous map. It is easily checked that the geometric realization defines a functor from the category of abstract simplicial complexes with simplicial maps to the category of topological spaces with continuous maps.

Simplices. Let σ be a finite set. The abstract simplicial complex consisting of all subsets of σ is called the *simplex* of vertex set σ, which will be denoted by $\widetilde{\sigma}$. Then the geometric realization $|\widetilde{\sigma}|$ is the same as the standard simplex Δ^σ of vertex set σ.

Joins. Let K and L be abstract simplicial complexes. For simplicity, we assume that $V(K)$ and $V(L)$ are disjoint. Otherwise, we replace $v \in V(K)$ with $(1, v)$, and $w \in V(L)$ with $(2, w)$. The *join* of K and L is defined by

$$K * L = \{\, \sigma \cup \tau \mid \sigma \in K,\ \tau \in L \,\}.$$

The vertex set $V(K*L)$ is given by the disjoint union $V(K) \sqcup V(L)$. The inclusions induce simplicial maps $K \hookrightarrow K * L$ and $L \hookrightarrow K * L$. If K and L are finite dimensional, then so is $K * L$, and $\dim(K * L) = \dim K + \dim L + 1$ holds.

For disjoint finite sets σ and τ, the obvious map

$$|\widetilde{\sigma}| * |\widetilde{\tau}| = \Delta^\sigma * \Delta^\tau \to \Delta^{\sigma\cup\tau} = |\widetilde{\sigma} * \widetilde{\tau}|$$

is a homeomorphism. Using the inverse of the above map, we obtain a continuous bijection

$$|K * L| \to |K| * |L|.$$

This map is a homeomorphism if we consider the compactly generated topology on the right-hand side [**Wal**, Section 2].

Links. Let K be an abstract simplicial complexes, and σ a face of K. The *link* of σ in K is defined by

$$\operatorname{lk}(\sigma; K) = \{\, \tau \in K \mid \sigma \cap \tau = \emptyset,\ \sigma \cup \tau \in K \,\}.$$

The link $\operatorname{lk}(\emptyset, K)$ of the empty face is the same as K itself. The link $\operatorname{lk}(\{v\}; K)$ of a 0-face is simply denoted by $\operatorname{lk}(v; K)$. If τ is a face of $\operatorname{lk}(\sigma; K)$, then the iterated link $\operatorname{lk}(\tau; \operatorname{lk}(\sigma; K))$ coincides with $\operatorname{lk}(\sigma \cup \tau; K)$.

1.3. Partially ordered sets

Let X be a set. A *partial order* $\leq$ on X is a binary relation on X which satisfies the following three conditions.

reflexivity: $x \leq x$ for any $x \in X$.
transitivity: $x \leq y$ and $y \leq z$ imply $x \leq z$ for any $x, y, z \in X$.
anti-symmetry: $x \leq y$ and $y \leq x$ imply $x = y$ for any $x, y \in X$.

We write $y \geq x$ if $x \leq y$ holds, and $x < y$ if both $x \leq y$ and $x \neq y$ hold. Similarly, we write $x > y$ if $y < x$ holds. A *partially ordered set* (*poset* for short) is a pair of a set P and a partial order $\leq$ on P. We denote a poset $(P, \leq)$ simply by P if no confusion can arise. Let S be a subset of a poset P. Then S can be seen as a poset by the restriction of the partial order on P. In this case, S is said to be a *induced subposet* of P. For a poset $P = (P, \leq)$, the poset $(P, \geq)$ with the opposite order is called the *opposite poset* of P, and denoted by P^{op}.

Let a and b be elements of P. We define intervals of P by

$$\begin{aligned} P_{\geq a} &= \{\, x \in P \mid x \geq a \,\}, \\ [a, b]_P &= \{\, x \in P \mid a \leq x \leq b \,\}, \\ (a, b)_P &= \{\, x \in P \mid a < x < b \,\}. \end{aligned}$$

Similarly, we define $P_{>a}, P^{\leq b}, P^{<b}, [a, b)_P$ and $(a, b]_P$. Usually, we see intervals of P as induced subposets of P.

Let $P = (P, \leq_P)$ and $Q = (Q, \leq_Q)$ be posets. A map $f\colon P \to Q$ is *order preserving* if $x \leq_P y$ implies $f(x) \leq_Q f(y)$ for any $x, y \in P$. We say f is *strictly order preserving* if $x <_P y$ implies $f(x) <_Q f(y)$ for any $x, y \in P$.

Order complexes. Let P be a poset. A *chain* of P is a totally ordered subset of P, that is, a subset σ of P such that for any $x, y \in \sigma$, either $x \leq y$ or $x \geq y$ holds. The *order complex* $\Delta(P)$ of P is defined to be the abstract simplicial complex whose faces are the finite chains of P. Then $V(\Delta(P)) = P$ holds. We denote the geometric realization $|\Delta(P)|$ of the order complex simply by $|P|$, and we sometimes refer to the geometric realization of the order complex of P simply as the order complex of P. An order-preserving map $f\colon P \to Q$ induces a simplicial map between the order complexes. Its geometric realization $|P| \to |Q|$ will be denoted by $|f|$.

Face posets. Let K be an abstract simplicial complex. The inclusion defines a partial order on K. The poset $(K, \subset)$ is called the *face poset* of K, and denoted by $F(K)$. Unless $K = \emptyset$, $F(K)$ has the minimum $\emptyset$. The induced subposet $F(K) \setminus \{\emptyset\}$ is also called the face poset of K, and denoted by $F_+(K)$.

Barycentric subdivisions. Let K be an abstract simplicial complex. The order complex $\Delta(F_+(K))$ of the face poset of K is called the *barycentric subdivision* of K, and denoted by $\operatorname{Sd} K$. The affine map from $|\operatorname{Sd} k|$ to $|K|$ which sends the vertex σ of $\operatorname{Sd} K$ to its barycenter $\frac{1}{\#\sigma}\sum_{v\in\sigma} v$ gives a well-known homeomorphism $|\operatorname{Sd} K| \approx |K|$. Similarly, the face poset $F_+(\Delta(P))$ of the order complex of a poset P is also called the barycentric subdivision of P, and denoted by $\operatorname{Sd} P$. Then we have

$$|\operatorname{Sd} P| = |\Delta(F_+(\Delta(P)))| \approx |\Delta(P)| = |P|.$$

Products. Let $P = (P, \leq_P)$ and $Q = (Q, \leq_Q)$ be posets. The product $P \times Q$ can be equipped with the partial ordered $\leq_{P\times Q}$ such that $(p, q) \leq_{P\times Q} (p', q')$ holds if and only if both $p \leq_P p'$ and $q \leq_Q q'$ hold. Equivalently, $\leq_{P\times Q}$ is the strongest partial order on $P \times Q$ such that both projections $P \times Q \to P$ and $P \times Q \to Q$ are order preserving. The geometric realizations of these projections induces a continuous bijection $|P \times Q| \to |P| \times |Q|$. This map is a homeomorphism if we give the compactly generated topology on the product $|P| \times |Q|$ [**Qui**, (1.2)]. In particular, if either $|P|$ or $|Q|$ is locally compact, there is a homeomorphism $|P \times Q| \approx |P| \times |Q|$. The following lemma plays an important role for a combinatorial approach to homotopy theory.

Proposition 1.3.1 (Quillen [**Qui**, 1.3]). *Let P and Q be posets, and f and g be order-preserving maps. Let us assume $f \leq g$, that is, $f(x) \leq g(x)$ holds for each $x \in P$. Then the induced maps $|f|$ and $|g|$ are homotopic.*

Proof. Let $H\colon P \times \{0,1\} \to Q$ be the map defined by $H(x,0) = f(x)$ and $H(x,1) = g(x)$ for $x \in P$. Then H is order preserving, where the order on $\{0,1\}$ is given by $0 < 1$. The geometric realization of $\{0,1\}$ is homeomorphic to the closed interval $[0,1] \subset \mathbb{R}$, which is compact. Thus we obtain a homotopy

$$|P| \times [0,1] \approx |P \times \{0,1\}| \xrightarrow{|H|} |Q|$$

from $|f|$ to $|g|$. $\square$

Using the above proposition, we obtain some useful lemma to study the homotopy types of order complexes.

Lemma 1.3.2. *If a poset P has either a minimum or a maximum, then $|P|$ is contractible.*

Proof. The constant map to the minimum (or the maximum) is homotopic to the identity map on P. $\square$

Lemma 1.3.3. *Let S be an induced subposet of a poset P. If $S^{\leq x} = S \cap P^{\leq x}$ has a maximum for each $x \in P$, then $|S|$ is homotopy equivalent to $|P|$.*

Proof. Let $i\colon S \hookrightarrow P$ be the inclusion, and $r\colon P \to S$ the map defined by $r(x) = \max S^{\leq x}$ for $x \in P$. Then r is an order-preserving map satisfying $r \circ i = \mathrm{id}_S$ and $i \circ r \leq \mathrm{id}_P$. Thus $|r|$ is a homotopy inverse of $|i|$. $\square$

Walker showed the following:

Proposition 1.3.4 ([**Wal**, Theorem 5.1(d)]). *Let x and x' be elements of P with $x < x'$. Let y and y' be elements of Q with $y < y'$. Then there is a homeomorphism*

$$\left| \big((x,y),(x',y')\big)_{P \times Q} \right| \approx \operatorname{susp}\big(|(x,x')_P| * |(y,y')_Q|\big),$$

where the join has the compactly generated topology.

CHAPTER 2

Frobenius complexes and diagrams over a finite poset

2.1. Introduction

Let $\alpha_1, \dots, \alpha_g$ be positive integers with $\gcd(\alpha_1, \dots, \alpha_g) = 1$. Let Λ be the submonoid of the additive monoid $\mathbb{N}$ generated by $\alpha_1, \dots, \alpha_g$. The *Frobenius coin-exchange problem* is to determine the largest integer which does not belong to Λ [**BR**]. It is a homotopical refinement of this problem to determine the homotopy types of Frobenius complexes of Λ [**CE**].

We now define a Frobenius complex in a more general situation (a more precise definition is given in Section 2.2). Let Λ be an additive monoid. We say that Λ is *poset-like* if Λ is cancellative and has no non-zero invertible elements. A poset-like additive monoid Λ can be equipped with the partial order defined by $\lambda \leq_\Lambda \lambda + \mu$ for $\lambda, \mu \in \Lambda$. For a non-zero element λ of Λ, the *Frobenius complex* $\mathcal{F}(\lambda; \Lambda)$ is defined to be the order complex $|(0, \lambda)_\Lambda|$ of the open interval of Λ.

Let us fix a field k. Laudal and Sletsjøe [**LS**] proved that there is an isomorphism between the graded component $\operatorname{Tor}_{i,\lambda}^{k[\Lambda]}(k, k)$ of the torsion group over the monoid algebra $k[\Lambda]$ and the reduced homology group $\widetilde{H}_{i-2}(\mathcal{F}(\lambda; \Lambda); k)$ of the Frobenius complex with coefficients in k. As an application, they showed that if Λ is a saturated rational submonoid of $\mathbb{N}^2$, then its Poincaré series is given by a rational function. Moreover, they gave an explicit formula for its denominator. The multi-graded Poincaré series $P_k^{k[\Lambda]}(t, \mathbf{z})$ is defined by

$$P_k^{k[\Lambda]}(t, \mathbf{z}) = \sum_{i \in \mathbb{N},\, \lambda \in \Lambda} \dim_k \operatorname{Tor}_{i,\lambda}^{k[\Lambda]}(k, k) \cdot t^i \mathbf{z}^\lambda.$$

The rationality of the multi-graded Poincaré series was discussed in [**PRS**].

Clark and Ehrenborg [**CE**] studied homotopy types of Frobenius complexes, and they showed the following:

- If Λ is generated by two relatively prime integers, then any Frobenius complex of Λ is either contractible or homotopy equivalent to a sphere.
- If Λ is generated by the arithmetic sequence $a, a + d, \dots, a + (a - 1)d$ for relatively prime integers a and d, then any Frobenius complex of Λ is homotopy equivalent to a wedge of spheres.

In both cases the multi-graded Poincaré series are determined and proved to be rational. They showed the above statements by discrete Morse theory.

In this chapter, we establish a new method to determine the homotopy types of Frobenius complexes. Let Λ_1 and Λ_2 be finitely generated poset-like additive monoids. Let ρ_1 and ρ_2 be reducible elements of Λ_1 and Λ_2, respectively. Let Λ be the additive monoid obtained from the direct sum $\Lambda_1 \oplus \Lambda_2$ by identifying ρ_1 with ρ_2. Let ρ denote the equivalence class of ρ_1 and ρ_2 in Λ. We show the following homotopy equivalence (Theorem 2.4.2):

$$\mathcal{F}(\lambda; \Lambda) \simeq \bigvee_{\ell\rho + \lambda_1 + \lambda_2 = \lambda} S^{2\ell} * \mathcal{F}(\lambda_1; \Lambda_1) * \mathcal{F}(\lambda_2; \Lambda_2)$$

The proof is done by using homotopy colimits of diagrams over a finite poset. As a consequence, we derive the formula

$$P_k^{k[\Lambda]}(t,\mathbf{z}) = \frac{P_k^{k[\Lambda_1]}(t,\mathbf{z}) \cdot P_k^{k[\Lambda_2]}(t,\mathbf{z})}{1 - t^2\mathbf{z}^\rho}$$

for the multi-graded Poincaré series (Corollary 2.4.5).

As an application, we determine the homotopy types of the Frobenius complexes and the multi-graded Poincaré series for some Λ. For example, we show that if Λ is generated by a finite geometric sequence, then any Frobenius complex of Λ is homotopy equivalent to a wedge of spheres, and give a formula for the multi-graded Poincaré series (Proposition 2.5.5). This gives an answer to a question raised by Clark and Ehrenborg [**CE**, Question 6.4].

The rest of this chapter is organized as follows. In Section 2.2, we give a precise definition of Frobenius complexes, and show some basic properties. In Section 2.3, we review homotopy colimits of diagrams of topological spaces over a finite poset, and give self-contained proofs for the basic properties of homotopy colimits. In Section 2.4, we show Theorem 2.4.2 and derive its corollaries. In Section 2.5, we determine homotopy types of Frobenius complexes of some additive monoids by using the theorem.

2.2. Preliminaries

2.2.1. Notation. For convenience, we introduce a formal symbol S^{-2}, which is not a topological space. We define its reduced homology $\widetilde{H}_i(S^{-2};k)$ to be k for $i=-2$ and 0 otherwise. The reduced Betti numbers are similarly defined by $\widetilde{\beta}_i(S^{-2}) = \delta_{i,-2}$. Here $\delta_{i,j}$ denotes the Kronecker delta.

Let $X \circledast Y$ denote the suspension $\operatorname{susp}(X * Y)$ of the join of topological spaces X and Y. We also define

$$S^{-2} \circledast X = X = X \circledast S^{-2}.$$

In the light of (1), we have

$$\widetilde{\beta}_{i-2}(X \circledast Y) = \sum_{p+q=i} \widetilde{\beta}_{p-2}(X) \cdot \widetilde{\beta}_{q-2}(Y). \tag{2}$$

In our definition, the above holds even in the case $X = S^{-2}$ or $Y = S^{-2}$.

2.2.2. Frobenius complexes. An *additive monoid* is a triple $\Lambda = (\Lambda, +, 0)$ of a set Λ, a binary operator $+$ on Λ, and an element 0 of Λ which satisfies the following three conditions:

associativity: $(\lambda+\mu)+\nu = \lambda+(\mu+\nu)$ holds for any $\lambda,\mu,\nu \in \Lambda$.
commutativity: $\lambda+\mu = \mu+\lambda$ holds for any $\lambda,\mu \in \Lambda$.
identity element: $\lambda + 0 = \lambda$ holds for any $\lambda \in \Lambda$.

For example, the non-negative integers form an additive monoid $\mathbb{N}$. The Cartesian product $\mathbb{N}^g$ is also an additive monoid with the coordinatewise addition.

There is a canonical action of $\mathbb{N}$ on an additive monoid Λ. The action is inductively defined by $0\cdot\lambda = 0$ and $(n+1)\cdot\lambda = n\cdot\lambda+\lambda$ for $n \in \mathbb{N}$ and $\lambda \in \Lambda$. Then this action is distributive, that is, $(n+m)\cdot\lambda = n\cdot\lambda+m\cdot\lambda$ and $n\cdot(\lambda+\mu) = n\cdot\lambda+n\cdot\mu$ hold. We will denote $n\cdot\lambda$ simply by $n\lambda$.

A map φ from an additive monoid Λ to another Λ' is a *homomorphism* if φ preserves the additions and the zero elements, that is, $\varphi(\lambda+\mu) = \varphi(\lambda)+\varphi(\mu)$ and $\varphi(0) = 0$ hold. An *isomorphism* is a bijective homomorphism. We say a homomorphism $\varphi\colon \Lambda \to \Lambda'$ is *proper* if $\varphi(\lambda) = 0$ implies $\lambda = 0$ for any $\lambda \in \Lambda$. Let us note that a proper homomorphism is not necessarily injective.

An additive monoid Λ is *finitely generated* if there exist $\alpha_1, \dots, \alpha_g \in \Lambda$ such that the homomorphism $\pi\colon \mathbb{N}^g \to \Lambda$ defined by

$$\pi(x) = x_1\alpha_1 + \cdots + x_g\alpha_g \quad \Big(x = (x_1, \dots, x_g) \in \mathbb{N}^g\Big)$$

is surjective.

Let Λ be an additive monoid. We can associate Λ with the small category whose objects are the elements of Λ, and whose morphisms from λ to ν are the elements μ of Λ satisfying $\lambda + \mu = \nu$. The associated category forms a poset if and only if the following hold.

- Λ is cancellative, that is, $\lambda + \mu = \lambda + \mu'$ implies $\mu = \mu'$ for any $\lambda, \mu, \mu' \in \Lambda$.
- Λ has no non-zero invertible elements, that is, $\lambda + \mu = 0$ implies $\lambda = \mu = 0$ for any $\lambda, \mu \in \Lambda$.

In this case, we say that Λ is *poset-like*.

Let Λ be a poset-like additive monoid. The partial order $\leq_\Lambda$ on Λ is characterized by $\lambda \leq_\Lambda \lambda + \mu$ for $\lambda, \mu \in \Lambda$. In the case $\lambda \leq_\Lambda \nu$, the unique element μ satisfying $\lambda + \mu = \nu$ will be denoted by $\nu - \lambda$. Let Λ_+ denote the set of the non-zero elements of Λ. For $\lambda \in \Lambda_+$, the *Frobenius complex* $\mathcal{F}(\lambda; \Lambda)$ is defined to be the order complex $|(0, \lambda)_\Lambda|$ of the open interval in Λ. In the case $\lambda = 0$, we define $\mathcal{F}(0; \Lambda) = S^{-2}$ for convenience. Let us note that any open interval $(\mu, \nu)_\Lambda$ of Λ is isomorphic to $(0, \nu - \mu)_\Lambda$. Thus its order complex is homeomorphic to the Frobenius complex $\mathcal{F}(\nu - \mu; \Lambda)$.

A non-zero element λ of Λ is said to be *reducible* if there exist non-zero elements σ and τ of Λ satisfying $\sigma + \tau = \lambda$. Otherwise, λ is *irreducible*, which is equivalent to that λ is a minimal element of Λ_+. Thus $\mathcal{F}(\lambda; \Lambda)$ is empty if and only if λ is irreducible.

Frobenius complexes were introduced by Laudal and Sletsjøe [**LS**], and they showed the following.

Theorem 2.2.1 (Laudal-Sletsjøe [**LS**, Proposition 1.3]). *Let Λ be a poset-like additive monoid, and k a field. Then there is an isomorphism*

$$\mathrm{Tor}^{k[\Lambda]}_{i,\lambda}(k, k) \cong \widetilde{H}_{i-2}(\mathcal{F}(\lambda; \Lambda); k),$$

where $k[\Lambda]$ denotes the monoid algebra of Λ over k. In particular, the multi-graded Poincaré series is given by

$$P^{k[\Lambda]}_k(t, \mathbf{z}) = \sum_{i \in \mathbb{N}, \lambda \in \Lambda} \widetilde{\beta}_{i-2}(\mathcal{F}(\lambda; \Lambda); k) \cdot t^i \mathbf{z}^\lambda.$$

Proof. The following proof is based on [**PRS**]. The monoid algebra $k[\Lambda]$ is defined to be the free k-linear space with basis $\{\, \mathbf{z}^\lambda \mid \lambda \in \Lambda \,\}$. The multiplication is given by $\mathbf{z}^\lambda \cdot \mathbf{z}^\mu = \mathbf{z}^{\lambda+\mu}$ for $\lambda, \mu \in \Lambda$. The subspace spanned by $\mathbf{z}^\lambda$ for $\lambda \in \Lambda_+$ forms an ideal of $k[\Lambda]$, which will be denoted by $k[\Lambda_+]$. Moreover, $k[\Lambda_+]$ is a maximal ideal with quotient $k[\Lambda]/k[\Lambda_+] \cong k$. In this proof, we denote $k[\Lambda]$ and $k[\Lambda_+]$ by R and R_+, respectively. We see k as an R-module by the isomorphism $k \cong R/R_+$, that is, $\mathbf{z}^\lambda \cdot 1$ is defined to be 1 for $\lambda = 0$ and 0 otherwise. We also see k as a k-subalgebra of R by the obvious inclusion $k \cong k\mathbf{z}^0 \hookrightarrow R$.

We now construct a left R-free resolution of k, which is known as the bar resolution. We denote the tensor over R simply by $\otimes$, and the tensor over k by the vertical bar $|$. Let us define

$$B_n = R \mid \underbrace{R_+ \mid \cdots \mid R_+}_{n} \mid R$$

for $n \geq 0$. We see B_n as a left-right R-module by

$$s \cdot (r_0 \mid r_1 \mid \cdots \mid r_n \mid r_{n+1}) \cdot t = (sr_0) \mid r_1 \mid \cdots \mid r_n \mid (r_{n+1}t)$$

for $r_0 \mid \cdots \mid r_{n+1} \in B_n$ and $s, t \in R$. The boundary operator $\partial_n\colon B_n \to B_{n-1}$ is defined by

$$\partial_n(r_0 \mid \cdots \mid r_{n+1}) = \sum_{i=0}^{n} (-1)^i r_0 \mid \cdots \mid r_i r_{i+1} \mid \cdots \mid r_{n+1}$$

for $n \geq 1$ and $r_0 \mid \cdots \mid r_{n+1} \in B_n$. The augmentation map $\varepsilon\colon B_0 \to R$ is defined by

$$\varepsilon(r_0 \mid r_1) = r_0 r_1.$$

We now show that the augmented left-right R-chain complex $\{B_n, \partial_n, \varepsilon\}$ is right R-contractible, that is, there exist right R-homomorphisms $h_n\colon B_n \to B_{n+1},\ n \geq 0$ and $\eta\colon R \to B_0$ such that the following equations hold:

(1) $\varepsilon \circ \eta = \mathrm{id}_R$
(2) $\eta \circ \varepsilon + \partial_1 \circ h_0 = \mathrm{id}_{B_0}$
(3) $h_{n-1} \circ \partial_n + \partial_{n+1} \circ h_n = \mathrm{id}_{B_n}$ $(n \geq 1)$

We define h_n and η by

$$\begin{aligned} h_n(r_0 \mid \cdots \mid r_{n+1}) &= 1 \mid \pi(r_0) \mid r_1 \mid \cdots \mid r_{n+1} \\ \eta(r_0) &= 1 \mid r_0, \end{aligned}$$

where π denotes the projection $R = k \oplus R_+ \to R_+$. Then h_n and η are right R-homomorphisms. We can derive the above equations from definition. For example,

$$\begin{aligned} (\eta \circ \varepsilon + \partial_1 \circ h_0)(r_0 \mid r_1) &= 1 \mid r_0 r_1 + \pi(r_0) \mid r_1 - 1 \mid \pi(r_0) r_1 \\ &= 1 \mid (r_0 - \pi(r_0)) r_1 + \pi(r_0) \mid r_1 \\ &= (r_0 - \pi(r_0)) \mid r_1 + \pi(r_0) \mid r_1 \\ &= r_0 \mid r_1, \end{aligned}$$

where we use the fact that $r_0 - \pi(r_0)$ belongs to k.

Thus we have a left R-resolution $\{B_n \otimes k, \partial_n \otimes \mathrm{id}_k, \varepsilon \otimes \mathrm{id}_k\}$ of $R \otimes k \cong k$. Each $B_n \otimes k$ is left R-free since a left R-basis given by

$$\left\{ 1 \mid \mathbf{z}^{\lambda_1} \mid \cdots \mid \mathbf{z}^{\lambda_n} \mid 1 \otimes 1 \;\middle|\; \lambda_1, \ldots, \lambda_n \in \Lambda_+ \right\}.$$

Hence, the torsion group is given by

$$\mathrm{Tor}_i^R(k, k) \cong H_i(k \otimes B_* \otimes k, \mathrm{id}_k \otimes \partial_* \otimes \mathrm{id}_k).$$

Let us focus on the chain complex $\{B'_n, \partial'_n\} = \{k \otimes B_n \otimes k, \mathrm{id}_k \otimes \partial_n \otimes \mathrm{id}_k\}$. Then we have

$$\begin{aligned} B'_n &= k \otimes R \mid R_+ \mid \cdots \mid R + | R \otimes k \\ &\cong k \mid R_+ \mid \cdots \mid R_+ \mid k \\ &\cong \underbrace{R_+ \mid \cdots \mid R_+}_{n}. \end{aligned}$$

Under this identification, B'_n is a free k-linear space generated by $\mathbf{z}^{\lambda_1} \mid \cdots \mid \mathbf{z}^{\lambda_n}$ for $\lambda_1, \ldots, \lambda_n \in \Lambda_+$. The boundary operator ∂'_n is given by

$$\partial'_n(r_1 \mid \cdots \mid r_n) = \sum_{i=1}^{n-1} (-1)^i r_1 \mid \cdots \mid r_i r_{i+1} \mid \cdots \mid r_n.$$

Each B'_n has an obvious decomposition $B'_n = \bigoplus_{\lambda \in \Lambda} B'_{n,\lambda}$, where $B'_{n,\lambda}$ denotes the k-subspace generated by $\mathbf{z}^{\lambda_1} \mid \cdots \mid \mathbf{z}^{\lambda_n}$ with $\lambda_1 + \cdots + \lambda_n = \lambda$. This Λ-grading comes from the natural Λ-grading on B_n. The boundary operators ∂'_n preserve the Λ-gradings, that is, $\partial'_n(B'_{n,\lambda}) \subset B'_{n-1,\lambda}$ holds for $n \geq 1$ and $\lambda \in \Lambda$. Thus the homology groups also can be decomposed as

$$H_i(B'_*, \partial'_*) = \bigoplus_{\lambda \in \Lambda} H_i(B'_{*,\lambda}, \partial'_{*,\lambda}).$$

Thus we have $\mathrm{Tor}^R_{i,\lambda}(k,k) \cong H_i(B'_{*,\lambda}, \partial'_{*,\lambda})$.

In the case $\lambda = 0$, we have $B'_{0,0} = k$ and $B'_{n,0} = 0$ for $n > 0$, which implies the assertion. In the case where λ is irreducible, we have $B'_{0,\lambda} = 0$, $B'_{1,\lambda} = k\mathbf{z}^\lambda$ and $B'_{n,\lambda} = 0$ for $n \geq 2$, thus the assertion similarly follows.

Let us assume that λ is reducible, and show that $H_i(B'_{*,\lambda}, \partial'_{*,\lambda})$ is isomorphic to $\widetilde{H}_{i-2}(\mathcal{F}(\lambda;\Lambda);k)$. We have $B'_{0,\lambda} = 0$ and $B'_{1,\lambda} = k\mathbf{z}^\lambda$. Let us consider the simplicial chain complex $C_* = \{C_*, \partial_*\}$ associated to $\mathcal{F}(\lambda;\Lambda)$ with coefficients in k. Then $H_i(\mathcal{F}(\lambda;\Lambda);k)$ is isomorphic to $H_n(C_*)$. Here, C_n is a free k-linear space generated by the chains of $(0,\lambda)_\Lambda$ of length n. The boundary operator $\partial_n \colon C_n \to C_{n+1}$ is given by

$$\partial_n(\{\mu_0 < \cdots < \mu_n\}) = \sum_{i=0}^{n} (-1)^i \{\mu_0 < \cdots < \mu_{i-1} < \mu_{i+1} < \cdots < \mu_n\}.$$

For $n \geq 2$, there is an isomorphism between $B'_{n,\lambda}$ and C_{n-2}, which sends $z^{\lambda_1}|\cdots|z^{\lambda_n}$ to $\{\lambda_1 < \lambda_1 + \lambda_2 < \cdots < \lambda_1 + \cdots + \lambda_{n-1}\}$. Moreover, these isomorphisms commute with the boundary operators up to sign. In the case $n = 2$, $\partial'_{2,\lambda} \colon B'_{2,\lambda} \to B'_{1,\lambda} \cong k$ corresponds to the augmentation map $\varepsilon \colon C_0 \to k$. Hence, we have the desired isomorphism $H_i(B'_{*,\Lambda}, \partial'_{*,\lambda}) \cong \widetilde{H}_{i-2}(\mathcal{F}(\lambda;\Lambda);k)$. $\square$

PROPOSITION 2.2.2. *There is a homotopy equivalence*

$$\mathcal{F}(n;\mathbb{N}) \simeq \begin{cases} S^{-2} & (n = 0) \\ S^{-1} & (n = 1) \\ \mathbf{pt} & (n \geq 2), \end{cases}$$

and the multi-graded Poincaré series is given by

$$P_k^{k[\mathbb{N}]}(t, \mathbf{z}) = 1 + t\mathbf{z}.$$

PROOF. In the case $n \geq 2$, we have $\mathcal{F}(n;\mathbb{N}) = |[1,n)_\mathbb{N}| \simeq \mathbf{pt}$. $\square$

2.2.3. Local finiteness. Let Λ be a finitely generated poset-like additive monoid.

PROPOSITION 2.2.3. *As a poset, Λ is locally finite, that is, $\Lambda^{\leq \lambda}$ is finite for each $\lambda \in \Lambda$.*

PROOF. Let us take a finite generating system $\{\alpha_1, \ldots, \alpha_g\}$ of Λ. We can assume that each α_i is non-zero. Let $\pi \colon \mathbb{N}^g \to \Lambda$ be the homomorphism defined by

$$\pi(x) = \sum_{i=1}^{g} x_i \alpha_i \qquad \big(x = (x_1, \ldots, x_g) \in \mathbb{N}^g\big).$$

Then π is a proper surjective homomorphism.

For $\lambda \in \Lambda$, we have

$$\Lambda^{\leq \lambda} = \bigcup_{x \in \pi^{-1}(\lambda)} \pi\big((\mathbb{N}^g)^{\leq x}\big).$$

We can check that $\pi^{-1}(\lambda)$ is an antichain of $\mathbb{N}^g$, that is, $x \leq_{\mathbb{N}^g} y$ implies $x = y$ for any $x, y \in \pi^{-1}(\lambda)$. By Dickson's lemma, which will be shown below, the antichain $\pi^{-1}(\lambda)$ of $\mathbb{N}^g$ is finite. Since each $(\mathbb{N}^g)^{\leq x}$ is finite, $\Lambda^{\leq \lambda}$ is also finite. $\square$

Lemma 2.2.4 (Dickson's lemma [**Dic**, Lemma A]). *Any antichain of $\mathbb{N}^g$ is finite.*

Proof. The proof is done by induction on g. The case $g \leq 1$ is trivial. Let us assume $g \geq 2$, and let A be a non-empty antichain of $\mathbb{N}^g$. Set

$$X_{i,n} = \{\, x = (x_1, \ldots, x_g) \in \mathbb{N}^g \mid x_i = n \,\}.$$

Then $A \cap X_{i,n}$ is finite since it is an antichain of $X_{i,n} \cong \mathbb{N}^{g-1}$. Fix an element $a = (a_1, \ldots, a_g)$ of A. Then we have $A = \bigcup_{i=1}^{g} \bigcup_{n=0}^{a_i} A \cap X_{i,n}$. Thus A is finite. □

Corollary 2.2.5. *The Frobenius complex $\mathcal{F}(\lambda; \Lambda)$ is a finite simplicial complex for any $\lambda \in \Lambda_+$.*

Lemma 2.2.6. *For any $\rho \in \Lambda_+$ and $\lambda \in \Lambda$, there uniquely exist $\ell \in \mathbb{N}$ and $\check{\lambda} \in \Lambda$ satisfying $\lambda = \ell\rho + \check{\lambda}$ and $\check{\lambda} \not\geq_\Lambda \rho$. In this case, ℓ is the largest integer satisfying $\ell\rho \leq \lambda$.*

Proof. Let A be the set of all $\ell \in \mathbb{N}$ satisfying $\ell\rho \leq_\Lambda \lambda$, and define $f\colon \mathbb{N} \to \Lambda$ by $f(\ell) = \ell\rho$. Then f is injective and the image of A under f is contained in the finite subset $\Lambda^{\leq \lambda}$. Hence A is finite. It is enough to take as $\ell = \max A$ and $\check{\lambda} = \lambda - \ell\rho$. □

2.2.4. Frobenius complexes of a direct sum. Let Λ_1 and Λ_2 be finitely generated poset-like additive monoids.

Proposition 2.2.7. *There is a homeomorphism*

$$\mathcal{F}(\lambda_1 \oplus \lambda_2; \Lambda_1 \oplus \Lambda_2) \approx \mathcal{F}(\lambda_1; \Lambda_1) \circledast \mathcal{F}(\lambda_2; \Lambda_2)$$

for $\lambda_1 \in \Lambda_1$ and $\lambda_2 \in \Lambda_2$.

Proof. As a poset, $\Lambda_1 \oplus \Lambda_2$ is canonically isomorphic to the product $\Lambda_1 \times \Lambda_2$ of posets. The assertion follows by Proposition 1.3.4 if both λ_1 and λ_2 are non-zero. The other cases follow by definition. □

Corollary 2.2.8. *The multi-graded Poincaré series is given by*

$$P_k^{k[\Lambda_1 \oplus \Lambda_2]}(t, \mathbf{z}) = P_k^{k[\Lambda_1]}(t, \mathbf{z}) \cdot P_k^{k[\Lambda_2]}(t, \mathbf{z}).$$

Proof. By the previous proposition and the equation (2), we have

$$\widetilde{\beta}_{i-2}(\mathcal{F}(\lambda_1 \oplus \lambda_2; \Lambda_1 \oplus \Lambda_2)) = \sum_{j_1 + j_2 = i} \widetilde{\beta}_{j_1-2}(\mathcal{F}(\lambda_1; \Lambda_1)) \cdot \widetilde{\beta}_{j_2-2}(\mathcal{F}(\lambda_2; \Lambda_2)),$$

which implies the assertion. □

2.2.5. Barycentric subdivisions of Frobenius complexes. Let Λ be a finitely generated poset-like additive monoid, and λ a non-zero element of Λ. In this subsection, we consider the barycentric subdivision of the Frobenius complex $\mathcal{F}(\lambda; \Lambda)$. A *composition* of λ in Λ is a tuple $\xi = (\xi^{(1)}, \ldots, \xi^{(s)})$ of non-zero elements of Λ satisfying $\sum_{i=1}^{s} \xi^{(i)} = \lambda$. We say ξ is *non-trivial* if $s \geq 2$. Let us define $C(\lambda; \Lambda)$ to be the set of the non-trivial compositions of λ in Λ. Then there is a bijection $\Phi\colon C(\lambda; \Lambda) \cong \mathrm{Sd}(0, \lambda)_\Lambda$, which is given by

$$\Phi\Big((\xi^{(1)}, \ldots, \xi^{(s)})\Big) = \{\xi^{(1)}, \xi^{(1)} + \xi^{(2)}, \ldots, \xi^{(1)} + \cdots + \xi^{(s-1)}\}.$$

We see $C(\lambda; \Lambda)$ as a poset by using this bijection. Thus we have

$$\mathcal{F}(\lambda; \Lambda) = |(0, \lambda)_\Lambda| \approx |\mathrm{Sd}(0, \lambda)_\Lambda| \cong |C(\lambda; \Lambda)|.$$

Let Λ' be another finitely generated poset-like additive monoid, and $\varphi\colon \Lambda \to \Lambda'$ a proper homomorphism. Then φ induces an order-preserving map from $C(\lambda; \Lambda)$ to $C(\varphi(\lambda); \Lambda')$, which will be denoted by φ_*.

Lemma 2.2.9. *For any $\xi \in C(\lambda;\Lambda)$ and any $\eta \in C(\varphi(\lambda);\Lambda')$ with $\varphi_*(\xi) \geq \eta$, there uniquely exists $\xi' \in C(\lambda;\Lambda)$ satisfying $\xi' \leq \xi$ and $\varphi_*(\xi') = \eta$.*

Proof. Since φ is proper, φ induces a strictly order-preserving map from $(0,\lambda)_\Lambda$ to $(0,\varphi(\lambda))_{\Lambda'}$. It is enough to show that the induced map $\mathrm{Sd}(0,\lambda)_\Lambda \to \mathrm{Sd}(0,\varphi(\lambda))_{\Lambda'}$ satisfies the above property. Let σ be a non-empty finite chain of $(0,\lambda)_\Lambda$, and τ a subset of $\varphi(\sigma)$. Note that the restriction of φ on σ is injective. Thus $\sigma' = (\varphi|_\sigma)^{-1}(\tau)$ is the desired element. $\square$

2.3. Diagrams over a finite poset

In this section, we will review the definition of homotopy colimits of diagrams of topological spaces over a finite poset, and give self-contained proofs for their basic properties. References for the subjects of this section are [**BK**] and [**ZZ**].

Let Q be a finite poset. A *diagram* over Q (Q-diagram for short) in a category $\mathcal{C}$ is a functor $D\colon Q \to \mathcal{C}$. Here we see Q as a category by the following way. The objects are the elements of Q. There is a unique morphism $q \to q'$ if $q \leq q'$, and no morphism otherwise. We denote the induced morphism from $D(q)$ to $D(q')$ by $D_{qq'}$ for $q, q' \in Q$ with $q \leq q'$.

Let D be a Q-diagram of topological spaces, that is, D is a Q-diagram in the category of topological spaces and continuous maps. The *homotopy colimit* of D is defined by

$$\operatorname*{hocolim}_Q D = \frac{\coprod_{q\in Q} D(q) \times |Q_{\geq q}|}{\sim},$$

where $\sim$ denotes the equivalence relation generated by

$$(x,y) \sim (D_{qq'}(x), y) \qquad \Big(q, q' \in Q,\ q \leq q',\ x \in D(q),\ y \in |Q_{\geq q'}|\Big).$$

The construction of the homotopy colimit is a generalization of that of the mapping cylinder. In fact, the following holds.

Lemma 2.3.1. *If Q has a maximum m, then the obvious map $r\colon \operatorname{hocolim}_Q D \to D(m)$ is a homotopy equivalence.*

Proof. Here $r\colon \operatorname{hocolim}_Q D \to D(m)$ is defined by

$$r([x,y]) = D_{qm}(x) \quad (q \in Q,\ x \in D(q),\ y \in |Q_{\geq q}|).$$

Let $i\colon D(m) \to \operatorname{hocolim}_Q D$ be the canonical embedding, that is,

$$i(x) = [x,m] \quad (x \in D(m)).$$

Then r is a retraction, and a homotopy $H\colon \operatorname{hocolim}_Q D \times [0,1] \to \operatorname{hocolim}_Q D$ from $i \circ r$ to the identify is given by

$$H([x,y],t) = [x, ty + (1-t)m] \quad (q \in Q,\ x \in D(q),\ y \in |Q_{\geq q}|). \qquad \square$$

The construction of the homotopy colimits is functorial. Let E be another Q-diagram of topological spaces, and α a natural transformation from D to E. Then α induces a continuous map from $\operatorname{hocolim}_Q D$ to $\operatorname{hocolim}_Q E$. The induced map will be denoted by $\operatorname{hocolim}_Q \alpha$. Let us note that if α is a natural homeomorphism then the induced map is a homeomorphism. Moreover, the following holds.

Lemma 2.3.2 (Homotopy Lemma [**BK**, XII.4.2], [**ZZ**, 1.7]). *Let us assume that α is a natural homotopy equivalence, that is, α is a natural transformation and each $\alpha_q\colon D(q) \to E(q)$ is a homotopy equivalence. Then* $\operatorname{hocolim}_Q \alpha\colon \operatorname{hocolim}_Q D \to \operatorname{hocolim}_Q E$ *is a homotopy equivalence.*

Proof. We first show the case where Q has a maximum m. Let us consider the commutative diagram

$$\begin{array}{ccc} \operatorname{hocolim}_Q D & \longrightarrow & D(m) \\ {\scriptstyle \operatorname{hocolim}_Q \alpha}\downarrow & & \downarrow{\scriptstyle \alpha_q} \\ \operatorname{hocolim}_Q E & \longrightarrow & E(m). \end{array}$$

By assumption, the right arrow is a homotopy equivalence. By Lemma 2.3.1, the horizontal arrows are homotopy equivalences. Thus $\operatorname{hocolim}_Q \alpha$ is also a homotopy equivalence.

We next show the general cases by induction on the size of Q. The case $Q = \emptyset$ is trivial, so we assume that Q is non-empty. Let us fix a maximal element m of Q, and consider the following commutative diagram:

$$\begin{array}{ccccc} \operatorname{hocolim}_{Q^{\leq m}} D & \longleftarrow & \operatorname{hocolim}_{Q^{<m}} D & \longrightarrow & \operatorname{hocolim}_{Q\setminus\{m\}} D \\ {\scriptstyle \operatorname{hocolim}_{Q^{\leq m}} \alpha}\downarrow & & {\scriptstyle \operatorname{hocolim}_{Q^{<m}} \alpha}\downarrow & & \downarrow{\scriptstyle \operatorname{hocolim}_{Q\setminus\{m\}} \alpha} \\ \operatorname{hocolim}_{Q^{\leq m}} E & \longleftarrow & \operatorname{hocolim}_{Q^{<m}} E & \longrightarrow & \operatorname{hocolim}_{Q\setminus\{m\}} E \end{array}$$

The left vertical arrow is a homotopy equivalence as we show above. The other vertical arrows are homotopy equivalences by the induction hypotheses. Applying Lemma 1.1.7, we obtain a homotopy equivalence

$$\operatorname*{hocolim}_{Q^{\leq m}} D \underset{\operatorname*{hocolim}_{Q^{<m}} D}{\cup} \operatorname*{hocolim}_{Q\setminus\{m\}} D \to \operatorname*{hocolim}_{Q^{\leq m}} E \underset{\operatorname*{hocolim}_{Q^{<m}} E}{\cup} \operatorname*{hocolim}_{Q\setminus\{m\}} E,$$

which is the same as $\operatorname{hocolim}_Q \alpha \colon \operatorname{hocolim}_Q D \to \operatorname{hocolim}_Q E$ up to obvious homeomorphisms. ☐

We say a Q-diagram D is *cofibrant* if the obvious map $\operatorname{colim}_{Q^{<q}} D \to D(q)$ is a cofibration for each $q \in Q$. In this case, we can replace the homotopy colimit with the colimit up to homotopy equivalence.

Lemma 2.3.3 (Projection Lemma [**BK**, XII.3.1(iv)], [**ZZ**, 1.6]). *If D is a cofibrant Q-diagram, then the canonical map*

$$\operatorname*{hocolim}_Q D \to \operatorname*{colim}_Q D$$

is a homotopy equivalence.

Proof. The proof is similar to that of the Homotopy Lemma. If Q has a maximum m, then the canonical map

$$\operatorname*{hocolim}_Q D \to \operatorname*{colim}_Q D \approx D(m)$$

is a homotopy equivalence by Lemma 2.3.1.

We now show the general cases by induction on the size of Q. The case $Q = \emptyset$ is trivial, so we assume that Q is non-empty. Let us fix a maximal element m of Q, and consider the following commutative diagram:

$$\begin{array}{ccccc} \operatorname{hocolim}_{Q^{\leq m}} D & \longleftarrow & \operatorname{hocolim}_{Q^{<m}} D & \longrightarrow & \operatorname{hocolim}_{Q\setminus\{m\}} D \\ \downarrow & & \downarrow & & \downarrow \\ \operatorname{colim}_{Q^{\leq m}} D & \longleftarrow & \operatorname{colim}_{Q^{<m}} D & \longrightarrow & \operatorname{colim}_{Q\setminus\{m\}} D \end{array}$$

The left vertical arrow is a homotopy equivalence as we show above. The other vertical arrows are homotopy equivalences by the induction hypotheses. Since D is cofibrant, the lower horizontal arrows are cofibrations. Applying Lemma 1.1.7, we obtain a homotopy equivalence

$$\operatorname*{hocolim}_{Q^{\le m}} D \underset{\operatorname*{hocolim}_{Q^{<m}} D}{\cup} \operatorname*{hocolim}_{Q\setminus\{m\}} D \to \operatorname*{colim}_{Q^{\le m}} D \underset{\operatorname*{colim}_{Q^{<m}} D}{\cup} \operatorname*{colim}_{Q\setminus\{m\}} D,$$

which is the same as the canonical map $\operatorname{hocolim}_Q D \to \operatorname{colim}_Q D$ up to obvious homeomorphisms. □

In a certain situation, the homotopy colimit is homotopically decomposed into a wedge.

Lemma 2.3.4 (Wedge Lemma [**ZZ**, 1.8]). *Let us assume that for each $q \in Q$ there exists a point c_q in $D(q)$ which satisfies the following:*

- *$(D(q), c_q)$ is well pointed.*
- *the obvious map $\operatorname{hocolim}_{Q^{<q}} D \to D(q)$ is homotopic to the constant map to c_q.*

Then there is a homotopy equivalence

$$\operatorname*{hocolim}_{Q} D \simeq |Q| \vee \left\{ D(q) * |Q_{>q}| \;\middle|\; q \in Q \right\},$$

where the wedge is formed by identifying $q \in |Q|$ with $c_q \in D(q) \subset D(q) * |Q_{>q}|$ for each $q \in Q$.

Proof. Let W be the topological space of the right-hand side of the required homotopy equivalence. Let us define two Q-diagrams $\widetilde{D}$ and E by

$$\widetilde{D}(q) = \operatorname*{hocolim}_{Q^{\le q}} D \subset \operatorname*{hocolim}_{Q} D$$
$$E(q) = \left|Q^{\le q}\right| \vee \left\{ D(p) * |(p,q]_Q| \;\middle|\; p \in Q^{\le q} \right\} \subset W.$$

The induced maps $\widetilde{D}(q) \to \widetilde{D}(q')$ and $E(q) \to E(q')$ are inclusions. By the Projection Lemma, we have

$$\operatorname*{hocolim}_{Q} \widetilde{D} \simeq \operatorname*{colim}_{Q} \widetilde{D} = \operatorname*{hocolim}_{Q} D$$
$$\operatorname*{hocolim}_{Q} E \simeq \operatorname*{colim}_{Q} E = W.$$

By the Homotopy Lemma, it is enough to construct a natural homotopy equivalence $\alpha\colon \widetilde{D} \to E$.

By Lemma 2.3.1, the obvious map $\widetilde{D}(q) \to D(q)$, say γ_q, is a homotopy equivalence. We now show that the obvious embedding $D(q) \hookrightarrow E(q)$ is a deformation retract. For $p \in Q^{<q}$, we have

$$D(p) * |(p,q]_Q| \simeq D(p) * \mathbf{pt} \simeq \mathbf{pt}.$$

Since the composition $\{c_p\} \hookrightarrow D(p) \hookrightarrow D(p) * |(p,q]_Q|$ is a cofibration, the inclusion $\{c_p\} \hookrightarrow D(p) * |(p,q]_Q|$ is a deformation retract. Similarly, $\{q\} \hookrightarrow \left|Q^{\le q}\right|$ is a deformation retract. Thus the composition

$$D(q) \hookrightarrow \left|Q^{\le q}\right| \vee D(q) \hookrightarrow \left|Q^{\le q}\right| \vee \left\{ D(p) * |(p,q]_Q| \;\middle|\; p \in Q^{\le q} \right\} = E(q)$$

is a deformation retract. Let $r_q\colon E(q) \to D(q)$ be the deformation retraction obtained by the above argument. Let us note that the composition $E(p) \hookrightarrow E(q) \xrightarrow{r_q} D(q)$ is the constant map to $c_q \in D(Q)$ for each $p < q$.

Let $q_1, \dots, q_n$ be a linear extension of a finite poset Q, that is, $q_1, \dots, q_n$ are distinct elements of Q such that

- $Q = \{q_1, \dots, q_n\}$.
- $q_i < q_j$ implies $i < j$ for each $1 \le i, j \le n$.

We now inductively construct homotopy equivalences $\{\alpha_{q_i} \colon \widetilde{D}(q_i) \to E(q_i)\}_{1\le i\le n}$ such that the diagram

$$\begin{tikzcd} \widetilde{D}(q_i) \arrow[r, "\alpha_{q_i}"] \arrow[d, hook] & E(q_i) \arrow[d, hook] \\ \widetilde{D}(q_j) \arrow[r, "\alpha_{q_j}"'] & E(q_j) \end{tikzcd}$$

commutes for each $q_i < q_j$. Let us assume that we could construct $\{\alpha_{q_i}\}_{1\le i<j}$ satisfying the commutativity described above. Let us consider the diagram

$$\begin{tikzcd} \bigcup\limits_{q_i<q_j} \widetilde{D}(q_i) \arrow[rr, "\bigcup_{q_i<q_j}\alpha_{q_i}"] \arrow[d, hook] & & \bigcup\limits_{q_i<q_j} E(q_i) \arrow[d, hook] \\ \widetilde{D}(q_j) \arrow[r, "\sim", "\gamma_{q_j}"'] & D(q_j) \arrow[r, hook, "\sim"] & E(q_j). \end{tikzcd}$$

Let us note that the composition

$$\bigcup_{q_i<q_j} \widetilde{D}(q_i) \hookrightarrow \widetilde{D}(q_j) \xrightarrow{\gamma_{q_j}} D(q_j)$$

is the same as the canonical map $\operatorname{hocolim}_{Q^{<q_j}} D \to D(q_j)$ in the assumption, which is homotopic to the constant map to c_q. As we see above, the composition

$$\bigcup_{q_i<q_j} E(q_i) \hookrightarrow E(q_j) \xrightarrow[\sim]{r_{q_j}} D(q_j)$$

is the constant map to c_q, and r_{q_j} is a deformation retraction. Thus the inclusion $\bigcup_{q_i<q_j} E(q_i) \hookrightarrow E(q_j)$ is homotopic to the constant map to c_q. In particular, the rectangle in the above diagram commutes up to homotopy. Since the inclusion $\bigcup_{q_i<q_j} \widetilde{D}(q_i) \hookrightarrow \widetilde{D}(q_j)$ is a cofibration, we can take $\alpha_{q_j} \colon D(q_j) \to E(q_j)$ such that in the diagram

$$\begin{tikzcd} \bigcup\limits_{q_i<q_j} \widetilde{D}(q_i) \arrow[rr, "\bigcup_{q_i<q_j}\alpha_{q_i}"] \arrow[d, hook] & & \bigcup\limits_{q_i<q_j} E(q_i) \arrow[d, hook] \\ \widetilde{D}(q_j) \arrow[rr, "\alpha_{q_j}"] \arrow[dr, "\sim", "\gamma_{q_j}"'] & & E(q_j) \\ & D(q_j) \arrow[ur, hook, "\sim"'] & \end{tikzcd},$$

the upper rectangle commutes, and the lower triangle commutes up to homotopy. Thus α_{q_j} is a homotopy equivalence, and satisfies the required commutativity. $\square$

Let X be a Q-diagram of posets, that is, X is a Q-diagram in the category of posets and order-preserving maps. Taking the order complexes, we obtain a

Q-diagram of topological spaces, which will be denoted by $|X|$. The *Grothendieck construction* $Q \int X$ is a poset defined by

$$Q \textstyle\int X = \{\, (q, x) \mid q \in Q,\ x \in X(q) \,\}$$
$$(q, x) \leq (q', x') \iff q \leq q' \text{ and } X_{qq'}(x) \leq x'.$$

THEOREM 2.3.5 (Thomason [**Tho**, Theorem 1.2]). *There is a homotopy equivalence*

$$\operatorname*{hocolim}_Q |X| \simeq \left|Q \textstyle\int X\right| .$$

PROOF. Let Y be the Q-diagram defined by $Y(q) = Q^{\leq q} \int X$. The map $Y_{qq'}$ is defined to be the inclusion. Let us define the natural transformation α from Y to X by $\alpha_q(q', x) = X_{q'q}(x)$. Then α_q induces a homotopy equivalence, since the obvious section $x \mapsto (q, x)$ gives a homotopy inverse. By the Homotopy Lemma, we have $\operatorname{hocolim}_Q |X| \simeq \operatorname{hocolim}_Q |Y|$. By the Projection Lemma, we have $\operatorname{hocolim}_Q |Y| \simeq \operatorname{colim}_Q |Y| \approx \left|Q \int X\right|$. □

THEOREM 2.3.6 (Quillen Fiber Lemma [**Qui**, Proposition 1.6]). *Let P and Q be finite posets, and $f\colon P \to Q$ be an order-preserving map. If the fiber $\left|f^{-1}(Q^{\leq q})\right|$ is contractible for each $q \in Q$, then $|f| : |P| \to |Q|$ is a homotopy equivalence.*

PROOF. Let X and Y be the Q-diagrams defined by $X(q) = f^{-1}(Q^{\leq q})$ and $Y(q) = Q^{\leq q}$. The maps $X_{qq'}$ and $Y_{qq'}$ are defined to be the inclusions. Let $\alpha\colon X \to Y$ be the natural transformation defined by $\alpha_q(p) = f(p)$. Since both $|X(q)|$ and $|Y(q)|$ are contractible for each $q \in Q$, $|\alpha| : |X| \to |Y|$ is a natural homotopy equivalence. By the Homotopy Lemma, $\operatorname{hocolim}_Q |\alpha|$ is a homotopy equivalence. By the Projection Lemma, the canonical maps $\operatorname{hocolim}_Q |X| \to \operatorname{colim}_Q |X| \approx |P|$ and $\operatorname{hocolim}_Q |Y| \to \operatorname{colim}_Q |Y| \approx |Q|$ are homotopy equivalences. Then we have the commutative diagram

$$\begin{tikzcd}
\operatorname{hocolim}_Q |X| \arrow[r, "\operatorname{hocolim}_Q|\alpha|"] \arrow[d] & \operatorname{hocolim}_Q |Y| \arrow[d] \\
|P| \arrow[r, "|f|"'] & |Q| ,
\end{tikzcd}$$

whose top, left and right arrows are homotopy equivalences. Thus $|f|$ is also a homotopy equivalence. □

REMARK. *Let us note that the order complex $|P^{\mathrm{op}}|$ of the opposite poset is isomorphic to $|P|$. Thus the dual statement of the Quillen Fiber Lemma is also true, that is, if $\left|f^{-1}(Q_{\geq q})\right|$ is contractible for each $q \in Q$, then $|f|$ is a homotopy equivalence.*

As a generalization of the Quillen Fiber Lemma, Björner, Wachs and Welker showed the following.

THEOREM 2.3.7 (Poset fiber theorem [**BWW**, Theorem 2.5]). *Let P and Q be finite posets, and $f\colon P \to Q$ be an order-preserving map. Let us assume that for each $q \in Q$ the inclusion $\left|f^{-1}(Q^{<q})\right| \hookrightarrow \left|f^{-1}(Q^{\leq q})\right|$ is homotopic to the constant map to some point c_q in $\left|f^{-1}(Q^{\leq q})\right|$. Then there is a homotopy equivalence*

$$|P| \simeq |Q| \vee \Big\{ \left|f^{-1}(Q^{\leq q})\right| * |Q_{>q}| \ \Big|\ q \in Q \Big\},$$

where the wedge is formed by identifying $q \in |Q|$ with $c_q \in \left|f^{-1}(Q^{\leq q})\right|$ for each $q \in Q$.

PROOF. Let us consider the Q-diagram D of topological spaces defined by $D(q) = \left|f^{-1}(Q^{\leq q})\right|$. The induced maps $D(q) \to D(q')$ are inclusions. Since D is cofibrant, we have

$$\operatorname*{hocolim}_{Q} D \simeq \operatorname*{colim}_{Q} D = |P| .$$

We now check that the assumption of the Wedge Lemma for D. Since each $D(q)$ is a CW complex, $(D(q), c_q)$ is well pointed. The obvious map $\operatorname{hocolim}_{Q^{<q}} D \to D(q)$ is factored as

$$\operatorname*{hocolim}_{Q^{<q}} D \to \operatorname*{colim}_{Q^{<q}} D = \left|f^{-1}(Q^{<q})\right| \hookrightarrow \left|f^{-1}(Q^{\leq q})\right| = D(q).$$

By the assumption, this map is homotopic to the constant map to c_q. Thus we obtain the required homotopy equivalence. □

2.4. Frobenius complexes of a sum with one relation

Let Λ_1 and Λ_2 be finitely generated poset-like additive monoids. Let ρ_1 and ρ_2 be reducible elements of Λ_1 and Λ_2, respectively. Let Λ be the quotient additive monoid of $\Lambda_1 \oplus \Lambda_2$ by the equivalence relation generated by $\rho_1 \sim \rho_2$. The equivalence class of ρ_1 and ρ_2 will be denoted by ρ. We define

$$\check{\Lambda}_i = \Lambda_i \setminus (\Lambda_i)_{\geq \rho_i}$$

for $i = 1, 2$.

PROPOSITION 2.4.1. *The following hold.*

(1) *Λ is a finitely generated poset-like additive monoid.*
(2) *Any element λ of Λ can be uniquely written as $n\rho + \check{\lambda}_1 + \check{\lambda}_2$ for $n \in \mathbb{N}$, $\check{\lambda}_1 \in \check{\Lambda}_1$ and $\check{\lambda}_2 \in \check{\Lambda}_2$.*
(3) *Any element of Λ can be uniquely written as $\check{\lambda}_1 + \lambda_2$ for $\check{\lambda}_1 \in \check{\Lambda}_1$ and $\lambda_2 \in \Lambda_2$. Equivalently, for any $\lambda \in \Lambda$ there uniquely exists $\widetilde{\lambda} \in \Lambda_1 \oplus \Lambda_2$ satisfying $\pi(\widetilde{\lambda}) = \lambda$ and $\widetilde{\lambda} \not\geq \rho_1$, where π denotes the canonical projection $\Lambda_1 \oplus \Lambda_2 \to \Lambda$.*

PROOF. The proof is straightforward. □

THEOREM 2.4.2. *There is a homotopy equivalence*

$$\mathcal{F}(\lambda; \Lambda) \simeq \bigvee_{\ell\rho + \lambda_1 + \lambda_2 = \lambda} S^{2\ell-2} \circledast \mathcal{F}(\lambda_1; \Lambda_1) \circledast \mathcal{F}(\lambda_2; \Lambda_2)$$

for $\lambda \in \Lambda \setminus \{\rho\}$, where ℓ, λ_1 and λ_2 run through $\mathbb{N}$, Λ_1 and Λ_2, respectively. In the case $\lambda = \rho$, there is a homeomorphisms

$$\mathcal{F}(\lambda; \Lambda) \approx \mathcal{F}(\rho_1; \Lambda_1) \sqcup \mathcal{F}(\rho_2; \Lambda_2).$$

We show this theorem in the rest of this section. Let λ be an element of Λ. Then λ can be uniquely written as $\lambda = n\rho + \check{\lambda}_1 + \check{\lambda}_2$ for $n \in \mathbb{N}$, $\check{\lambda}_1 \in \check{\Lambda}_1$ and $\check{\lambda}_2 \in \check{\Lambda}_2$. In the case $n = 0$, the obvious map $(0, \check{\lambda}_1 \oplus \check{\lambda}_2)_{\Lambda_1 \oplus \Lambda_2} \to (0, \lambda)_\Lambda$ is a poset isomorphism, which implies the assertion. In the case $\lambda = \rho$, the open interval $(0, \rho)_\Lambda$ is isomorphic to the disjoint union of $(0, \rho_1)_{\Lambda_1}$ and $(0, \rho_2)_{\Lambda_2}$, which implies the assertion.

In the rest of proof, we assume $\lambda >_\Lambda \rho$. It is enough to show

$$\mathcal{F}(\lambda; \Lambda) \simeq \bigvee_{\ell + \ell_1 + \ell_2 = n} S^{2\ell-2} \circledast \mathcal{F}(\ell_1\rho_1 + \check{\lambda}_1; \Lambda_1) \circledast \mathcal{F}(\ell_2\rho_2 + \check{\lambda}_2; \Lambda_2).$$

Let $\hat{Q}$ be the set of all subsets of $[n] = \{0, 1, \ldots, n\}$. We see $\hat{Q}$ as a poset by reverse inclusion, that is, $q \leq_{\hat{Q}} q'$ means $q \supset q'$. We denote the minimum $[n]$ and the maximum $\emptyset$ of $\hat{Q}$ by $\hat{0}$ and $\hat{1}$, respectively. Let Q be the induced subposet $Q \setminus \{\hat{1}\}$. We will construct two $\hat{Q}$-diagrams of posets. Let us set

$$\begin{aligned}\widetilde{\Lambda} &= \Lambda_1 \oplus \mathbb{N}\alpha_1 \oplus \cdots \oplus \mathbb{N}\alpha_n \oplus \Lambda_2 \\ \widetilde{\lambda} &= \check{\lambda}_1 + \alpha_1 + \cdots + \alpha_n + \check{\lambda}_2.\end{aligned}$$

For $q \in \hat{Q}$, let Λ_q be the quotient additive monoid of $\widetilde{\Lambda}$ by the equivalence relation $\sim_q$ generated by $\alpha_i \sim_q \alpha_{i+1}$ for $i \in [n] \setminus q$, where α_0 and α_{n+1} denote ρ_1 and ρ_2, respectively. There are canonical isomorphisms $\Lambda_{\hat{0}} \cong \widetilde{\Lambda}$ and $\Lambda_{\hat{1}} \cong \Lambda$. For $q, q' \in \hat{Q}$ with $q \leq q'$, there is a canonical projection from Λ_q onto $\Lambda_{q'}$, which will be denoted by $\varphi_{qq'}$. Then $\varphi_{qq'}$ is a proper homomorphism. For $q \in \hat{Q}$, let us define $\lambda_q = \varphi_{\hat{0}q}(\widetilde{\lambda}) \in \Lambda_q$. For each $q = \{q_0 < \cdots < q_s\} \in Q$, the obvious inclusion $\Lambda_1 \oplus \mathbb{N}\alpha_{q_1} \oplus \cdots \oplus \mathbb{N}\alpha_{q_s} \oplus \Lambda_2 \hookrightarrow \widetilde{\Lambda}$ induces an isomorphism

$$\Lambda_1 \oplus \mathbb{N}\alpha_{q_1} \oplus \cdots \oplus \mathbb{N}\alpha_{q_s} \oplus \Lambda_2 \cong \Lambda_q. \tag{3}$$

Using Proposition 2.2.7, we have

$$\begin{aligned}\mathcal{F}(\lambda_q;\Lambda_q) \simeq \mathcal{F}(q_0\rho_1 + \check{\lambda}_1;\Lambda_1) \circledast \mathcal{F}(q_1 - q_0;\mathbb{N}) \circledast \cdots \\ \cdots \circledast \mathcal{F}(q_s - q_{s-1};\mathbb{N}) \circledast \mathcal{F}((n-q_s)\rho_2 + \check{\lambda}_2;\Lambda_2).\end{aligned}$$

Thus $\mathcal{F}(\lambda_q;\Lambda_q)$ is contractible if $q_i - q_{i-1} \geq 2$ for some $i = 1, \ldots, s$. Otherwise, q can be written as $\{\ell_1, \ell_1 + 1, \ldots, n - \ell_2 - 1, n - \ell_2\}$. In this case, we have

$$\mathcal{F}(\lambda_q;\Lambda_q) \simeq S^{n-\ell_1-\ell_2-2} \circledast \mathcal{F}(\ell_1\rho_1 + \check{\lambda}_1) \circledast \mathcal{F}(\ell_2\rho_2 + \check{\lambda}_2). \tag{4}$$

Let X and Y be the Q-diagrams of poset defined by

$$\begin{aligned}&X(q) = (0, \lambda_q)_{\Lambda_q} \\ &X_{qq'} = \varphi_{qq'} \colon X(q) \to X(q') \\ &Y(q) = C(\lambda_q;\Lambda_q) \\ &Y_{qq'} = (\varphi_{qq'})_* \colon Y(q) \to Y(q').\end{aligned}$$

Then $|X(q)|$ and $|Y(q)|$ are homeomorphic to $\mathcal{F}(\lambda_q;\Lambda_q)$. Moreover, two Q-diagrams $|X|$ and $|Y|$ of spaces are naturally homeomorphic.

The rest of the proof goes as follows. First, we show that $\left|Q \int Y\right|$ is homotopy equivalent to $\mathcal{F}(\lambda;\Lambda)$ by using the Quillen Fiber Lemma. Second, we show that $\left|Q \int X\right|$ is homotopy equivalent to the right-hand side of the theorem by using the Poset Fiber Theorem. Therefore we can conclude

$$\mathcal{F}(\lambda;\Lambda) \simeq \left|Q \textstyle\int Y\right| \simeq \operatorname*{hocolim}_Q |Y| \approx \operatorname*{hocolim}_Q |X| \simeq \left|Q \textstyle\int X\right| \simeq (\text{RHS}).$$

Lemma 2.4.3. *There is a homotopy equivalence* $\left|Q \int Y\right| \simeq \mathcal{F}(\lambda;\Lambda)$.

Proof. We use the Quillen Fiber Lemma to the map $f \colon Q \int Y \to C(\lambda;\Lambda)$ which sends (q,ξ) to $(\varphi_{q\hat{1}})_*(\xi)$. Let us fix an element $\eta = (\eta^{(1)}, \ldots, \eta^{(s)})$ of $C(\lambda;\Lambda)$. We now show that the fiber $\left|f^{-1}(C(\lambda;\Lambda)_{\geq\eta})\right|$ is contractible. By Lemma 1.3.3 and Lemma 2.2.9, the fiber is homotopy equivalent to $\left|f^{-1}(\eta)\right|$.

Let $\mathrm{rem}_1 \colon \Lambda \to \check{\Lambda}_1$ be the map defined by $\mathrm{rem}_1(\check{\lambda}_1 + \lambda_2) = \check{\lambda}_1$ for $\check{\lambda}_1 \in \check{\Lambda}_1$ and $\lambda_2 \in \Lambda_2$. For $q \in Q$, let $\mathrm{pr}_q \colon \Lambda_q \to \Lambda_1$ be the projection via the isomorphism (3). We also denote the composition of rem_1 after $\varphi_{q\hat{1}}$ by rem_1. Then we have $\mathrm{rem}_1(\mathrm{pr}_q(\mu_q)) = \mathrm{rem}_1(\mu_q)$, and thus $\mu_q \geq \mathrm{rem}_1(\mu_q)$ for $q \in Q$ and $\mu_q \in \Lambda_q$. By Lemma 2.2.6, we can take the largest integer ℓ_1 satisfying

$$\ell_1\rho_1 \leq_{\Lambda_1} \sum_{i=1}^{s} \mathrm{rem}_1(\eta^{(i)}).$$

Let us set

$$S = \{\, (q, \xi) \in f^{-1}(\eta) \mid \ell_1 \in q \,\}.$$

We now show that the inclusion $S \subset f^{-1}(\eta)$ satisfies the hypothesis of Lemma 1.3.3. Let $(q, \xi = (\xi^{(1)}, \dots, \xi^{(s)}))$ be an element of $f^{-1}(\eta)$. Then we have

$$\lambda_q = \sum_{i=1}^{s} \xi^{(i)} \geq \sum_{i=1}^{s} \mathrm{rem}_1(\xi^{(i)}) = \sum_{i=1}^{s} \mathrm{rem}_1(\eta^{(i)}) \geq \ell_1 \rho_1.$$

Thus at least ℓ_1 of $\alpha_1, \dots, \alpha_n$ are identified with ρ_1 in Λ_q, which implies $\min q \geq \ell_1$. In the case $\min q = \ell_1$, (q, ξ) itself is the maximum of $S^{\leq (q,\xi)}$. Let us assume $\min q > \ell_1$ and set $\bar{q} = \{\ell_1\} \cup q$. By the same manner of Proposition 2.4.1(3), there uniquely exists $\bar{\xi}^{(i)} \in \Lambda_{\bar{q}}$ satisfying $\varphi_{\bar{q}q}(\bar{\xi}^{(i)}) = \xi^{(i)}$ and $\bar{\xi}^{(i)} \not\geq \rho_1$, which means $\mathrm{pr}_{\bar{q}}(\bar{\xi}^{(i)}) = \mathrm{rem}_1(\eta^{(i)})$. Then we have

$$\mathrm{pr}_{\bar{q}} \left(\sum_{i=1}^{s} \bar{\xi}^{(i)} \right) = \sum_{i=1}^{s} \mathrm{rem}_1(\eta^{(i)}) = \ell_1 \rho_1 + \check{\lambda}_1 = \mathrm{pr}_{\bar{q}}(\lambda_q).$$

Combining this with

$$\varphi_{\bar{q}q} \left(\sum_{i=1}^{s} \bar{\xi}^{(i)} \right) = \sum_{i=1}^{s} \xi^{(i)} = \lambda_q,$$

we have $\sum_{i=1}^{s} \bar{\xi}^{(i)} = \lambda_{\bar{q}}$, which implies $\bar{\xi} = (\bar{\xi}^{(1)}, \dots, \bar{\xi}^{(s)}) \in C(\lambda_{\bar{q}}; \Lambda_{\bar{q}})$. Moreover, $(\bar{q}, \bar{\xi})$ is the maximum of $S^{\leq (q,\xi)}$. Thus $\left| f^{-1}(\eta) \right|$ is homotopy equivalent to $|S|$.

Similarly, there uniquely exists $\bar{\eta} \in C(\lambda_{\{\ell_1\}}; \Lambda_{\{\ell_1\}})$ satisfying $(\varphi_{\{\ell_1\}\hat{1}})_*(\bar{\eta}) = \eta$. Then $(\{\ell_1\}, \bar{\eta})$ is the maximum of S. Thus we have

$$\left| f^{-1}(C(\lambda; \Lambda)_{\geq \eta}) \right| \simeq \left| f^{-1}(\eta) \right| \simeq |S| \simeq \mathbf{pt}. \qquad \square$$

Lemma 2.4.4. *There is a homotopy equivalence*

$$\left| Q \textstyle\int X \right| \simeq \bigvee_{\ell + \ell_1 + \ell_2 = n} S^{2\ell - 2} \circledast \mathcal{F}(\ell_1 \rho_1 + \check{\lambda}_1; \Lambda_1) \circledast \mathcal{F}(\ell_2 \rho_2 + \check{\lambda}_2; \Lambda_2).$$

Proof. We use the Poset Fiber Theorem to the map $g \colon Q \int X \to Q$ which sends (q, μ) to q. We first show that g satisfies the hypothesis of the Poset Fiber Theorem. By definition, we have $g^{-1}(Q^{<q}) = Q^{<q} \int X$ and $g^{-1}(Q^{\leq q}) = Q^{\leq q} \int X$. By the assumption $\lambda >_\Lambda \rho$, $X(\hat{0})$ is non-empty. The canonical map

$$Q^{\leq q} \textstyle\int X \to X(q) \; ; \; (p, \mu) \mapsto \varphi_{pq}(\mu)$$

induces a homotopy equivalence, since the obvious section $\mu \mapsto (q, \mu)$ gives a homotopy inverse. Thus it is enough to show that the restriction

$$h \colon Q^{<q} \textstyle\int X \to X(q) \; ; \; (p, \mu) \mapsto \varphi_{pq}(\mu)$$

induces a null-homotopic map. We can assume that q can be written as $\{\ell_1, \ell_1 + 1, \dots, n - \ell_2\}$, since otherwise $|X(q)|$ is contractible. Since each ρ_i is reducible, we can take $\sigma_i, \tau_i \in (\Lambda_i)_+$ satisfying $\sigma_i + \tau_i = \rho_i$. For $p \in Q^{<q}$ we define the homomorphism $\psi_{pq} \colon \Lambda_p \to \Lambda_q$ by

$$\begin{aligned} \psi_{pq}(\mu_1) &= \mu_1 \qquad (\mu_1 \in \Lambda_1) \\ \psi_{pq}(\mu_2) &= \mu_2 \qquad (\mu_2 \in \Lambda_2) \\ \psi_{pq}(\alpha_i) &= \begin{cases} \sigma_1 & (\min p < i \leq \ell_1) \\ \sigma_2 & (n - \ell_2 < i \leq \max p) \\ \alpha_i & (\text{otherwise}). \end{cases} \end{aligned}$$

Then the following hold.

- ψ_{pq} is well-defined.

- ψ_{pq} is a proper homomorphism.
- $\psi_{pq} \leq \varphi_{pq}$ holds.
- $\psi_{pq} \leq \psi_{p'q} \circ \varphi_{pp'}$ holds for any $p, p' \in Q^{<q}$ with $p \leq p'$.

Let us define $h' \colon Q^{<q} \int X \to X(q)$ by $h'(p, \mu) = \psi_{pq}(\mu)$. Then h' is order preserving and satisfies $h' \leq h$, which implies $|h'| \simeq |h|$. Next, we show that the image of $|h'|$ is contained in a contractible subset. The case $\ell_1 = \ell_2 = 0$ is trivial, since q is the minimum of Q. In the case $\ell_1 = 0$ and $\ell_2 > 0$, we have $n - \ell_2 < \max p$ holds for any $p \in Q^{<q}$. Thus the image of h' is contained in $(0, \lambda_q - \tau_2]_{\Lambda_q}$, whose order complex is contractible. Similarly we can show the case $\ell_1 > 0$ and $\ell_2 = 0$. We assume that both ℓ_1 and ℓ_2 are positive. Then the image of h' is contained in the union $(0, \lambda_q - \tau_1]_{\Lambda_q} \cup (0, \lambda_q - \tau_2]_{\Lambda_q}$. Then the order complex of the union is contractible since all order complexes of $(0, \lambda_q - \tau_1]_{\Lambda_q}$, $(0, \lambda_q - \tau_2]_{\Lambda_q}$ and the intersection $(0, \lambda_q - \tau_1]_{\Lambda_q} \cap (0, \lambda_q - \tau_2]_{\Lambda_q} = (0, \lambda_q - \tau_1 - \tau_2]_{\Lambda_q}$ are contractible.

Let us note that $\Delta(Q_{>q})$ is isomorphic to the barycentric subdivision of the boundary of $(n - \ell_1 - \ell_2)$-simplex, whose geometric realization is homeomorphic to $S^{n-\ell_1-\ell_2-1}$. By the Poset Fiber Theorem, we have

$$\begin{aligned} \left|Q \textstyle\int X\right| &\simeq |Q| \vee \left\{ \left|Q^{\leq q} \textstyle\int X\right| * |Q_{>q}| \;\middle|\; q \in Q \right\} \\ &\simeq \bigvee_{\ell+\ell_1+\ell_2=n} S^{2\ell-2} \circledast \mathcal{F}(\ell_1\rho_1 + \check{\lambda}_1; \Lambda_1) \circledast \mathcal{F}(\ell_2\rho_2 + \check{\lambda}_2; \Lambda_2). \qquad \square \end{aligned}$$

Corollary 2.4.5. *The multi-graded Poincaré series is given by*

$$P_k^{k[\Lambda]}(t, \mathbf{z}) = \frac{P_k^{k[\Lambda_1]}(t, z) \cdot P_k^{k[\Lambda_2]}(t, z)}{1 - t^2 z^\rho}.$$

Proof. By the previous theorem and equation (2), we have

$$\widetilde{\beta}_{i-2}(\mathcal{F}(\lambda; \Lambda)) = \sum_{\substack{\ell\rho+\lambda_1+\lambda_2=\lambda \\ 2\ell+j_1+j_2=i}} \widetilde{\beta}_{j_1-2}(\mathcal{F}(\lambda_1; \Lambda_1)) \cdot \widetilde{\beta}_{j_2-2}(\mathcal{F}(\lambda_2; \Lambda_2)).$$

Combining this with Theorem 2.2.1, we obtain the assertion. $\square$

We say that X has *homotopy type of a wedge of spheres* if X is S^{-2} or a topological space which is homotopy equivalent to

$$\coprod_{i=1}^{c} \bigvee_{j=1}^{w_i} S^{d_{ij}}$$

for some non-negative integers $c, \{w_i\}_{1 \leq i \leq c}$ and $\{d_{ij}\}_{1 \leq j \leq w_i}^{1 \leq i \leq c}$. In this case, the reduced Betti numbers of X is independent of the choice of a field. Moreover, the homotopy type of X is determined only by the reduced Betti numbers of X if $\widetilde{\beta}_0(X) = 0$.

Corollary 2.4.6. *Let us assume that any Frobenius complex of Λ_i has homotopy type of a wedge of spheres for $i = 1, 2$. Then any Frobenius complex of Λ has homotopy types of a wedge of spheres.*

2.5. Examples

Let A be an additive group, and $\alpha_1, \ldots, \alpha_g$ elements of A. Let Λ be the submonoid of A generated by $\alpha_1, \ldots, \alpha_g$, that is,

$$\Lambda = \langle \alpha_1, \ldots, \alpha_g \rangle_{\mathbb{N}} = \{ n_1\alpha_1 + \cdots + n_g\alpha_g \mid n_1, \ldots, n_g \in \mathbb{N} \}.$$

In this case, Λ is poset-like if and only if $\Lambda \cap (-\Lambda) = \{0\}$ holds. We denote the smallest additive subgroup of A containing Λ by $\overline{\Lambda}$, that is,

$$\overline{\Lambda} = \{ \lambda - \mu \mid \lambda, \mu \in \Lambda \}.$$

The following is a convenient form of Theorem 2.4.2.

THEOREM 2.5.1. *Let Λ_1 and Λ_2 be finitely generated poset-like submonoids of an additive group A, and ρ a non-zero element of $\Lambda_1 \cap \Lambda_2$. Let us assume the following.*

(1) *$\overline{\Lambda}_1 \cap \overline{\Lambda}_2 = \mathbb{Z}\rho$ holds.*
(2) *ρ is an reducible element of Λ_i for $i = 1, 2$.*
(3) *Any Frobenius complex of Λ_i has homotopy type of a wedge of spheres for $i = 1, 2$.*

Then any Frobenius complex of the sum $\Lambda_1 + \Lambda_2$ in A has homotopy type of a wedge of spheres, and the multi-graded Poincaré series is given by

$$P_k^{k[\Lambda_1+\Lambda_2]}(t, \mathbf{z}) = \frac{P_k^{k[\Lambda_1]}(t, z) \cdot P_k^{k[\Lambda_2]}(t, z)}{1 - t^2 z^\rho}.$$

PROOF. Let Λ be the additive monoid obtained from $\Lambda_1 \oplus \Lambda_2$ by identifying $\rho \oplus 0$ with $0 \oplus \rho$. Clearly, the map

$$\widetilde{\varphi}\colon \Lambda_1 \oplus \Lambda_2 \to \Lambda_1 + \Lambda_2 \ ; \ \lambda_1 \oplus \lambda_2 \mapsto \lambda_1 + \lambda_2$$

induces a surjective homomorphism $\varphi\colon \Lambda \to \Lambda_1 + \Lambda_2$. We now show that φ is injective. If $\lambda_1 + \lambda_2 = \lambda_1' + \lambda_2'$ holds in A for $\lambda_1, \lambda_1' \in \Lambda_1$ and $\lambda_2, \lambda_2' \in \Lambda_2$, then we have $\lambda_1 - \lambda_1' = \lambda_2' - \lambda_2 \in \overline{\Lambda}_1 \cap \overline{\Lambda}_2 = \mathbb{Z}\rho$. Let us assume $\lambda_1 - \lambda_1' = n\rho$ for $n \geq 0$. Then we have

$$\lambda_1 \oplus \lambda_2 = (\lambda_1' + n\rho) \oplus \lambda_2 \sim \lambda_1' \oplus (\lambda_2 + n\rho) = \lambda_1' \oplus \lambda_2'.$$

The case $n \leq 0$ is similarly proved. Thus we can apply Theorem 2.4.2 to $\Lambda_1 + \Lambda_2$. $\square$

PROPOSITION 2.5.2. *Let A be a torsion-free additive group, and Λ a finitely generated poset-like submonoid of A. Let us fix a positive integer p. Then there is a canonical homeomorphism $\mathcal{F}(p\lambda; p\Lambda) \approx F(\lambda; \Lambda)$ for $\lambda \in \Lambda_+$, and the multi-graded Poincaré series is given by*

$$P_k^{k[p\Lambda]}(t, \mathbf{z}) = P_k^{k[\Lambda]}(t, \mathbf{z}^p).$$

PROOF. The proof is straightforward. $\square$

The following is an alternative proof for [**CE**, Theorem 4.1]

PROPOSITION 2.5.3 (Clark-Ehrenborg). *Let a and b be integers with $1 < a < b$, and assume that b is not a multiple of a. Then any Frobenius complex of $\langle a, b\rangle_{\mathbb{N}} \subset \mathbb{N}$ has homotopy type of a wedge of spheres, and the multi-graded Poincaré series is given by*

$$P_k^{k[\langle a,b\rangle_{\mathbb{N}}]}(t, \mathbf{z}) = \frac{(1 + t\mathbf{z}^a)(1 + t\mathbf{z}^b)}{1 - t^2\mathbf{z}^m},$$

where m is the least common multiple of a and b.

PROOF. We can easily check that $\langle a, b\rangle_{\mathbb{N}} = \langle a\rangle_{\mathbb{N}} + \langle b\rangle_{\mathbb{N}}$ satisfies all hypotheses of Theorem 2.5.1 with $\rho = m$. Thus we have

$$P_k^{k[\langle a,b\rangle_{\mathbb{N}}]}(t, \mathbf{z}) = \frac{P_k^{k[\langle a\rangle_{\mathbb{N}}]}(t, \mathbf{z}) \cdot P_k^{k[\langle b\rangle_{\mathbb{N}}]}(t, \mathbf{z})}{1 - t^2\mathbf{z}^m} = \frac{(1 + t\mathbf{z}^a)(1 + t\mathbf{z}^b)}{1 - t^2\mathbf{z}^m}. \qquad \square$$

PROPOSITION 2.5.4. *Let a be a positive even number, and d a positive odd number. Let us assume that $a+2d$ is not a multiple of a. Let Λ be the submonoid of $\mathbb{N}$ generated by the arithmetic sequence $a, a+d, a+2d$. Then each Frobenius complex of Λ has homotopy types of a wedge of spheres, and the multi-graded Poincaré series is given by*

$$P_k^{k[\Lambda]}(t, \mathbf{z}) = \frac{(1 + t\mathbf{z}^a)(1 + t\mathbf{z}^{a+2d})}{(1 - t^2\mathbf{z}^m)(1 - t\mathbf{z}^{a+d})},$$

where m is the least common multiple of a and $a+2d$.

PROOF. We can easily check that $\Lambda = \langle a, a+2d\rangle_{\mathbb{N}} + \langle a+d\rangle_{\mathbb{N}}$ satisfies all hypotheses of Theorem 2.5.1 with $\rho = a + (a+2d) = 2(a+d)$. Thus we have

$$P_k^{k[\Lambda]}(t, \mathbf{z}) = \frac{P_k^{k[\langle a,a+2d\rangle_{\mathbb{N}}]}(t, \mathbf{z}) \cdot P_k^{k[\langle a+d\rangle_{\mathbb{N}}]}(t, \mathbf{z})}{1 - t^2\mathbf{z}^{2(a+d)}} = \frac{(1+t\mathbf{z}^a)(1+t\mathbf{z}^{a+2d})}{(1-t^2\mathbf{z}^m)(1-t\mathbf{z}^{a+d})}. \qquad \square$$

The following gives an answer to a question raised by Clark and Ehrenborg [**CE**, Question 6.4].

PROPOSITION 2.5.5. *Let p and q be relatively prime integers with $1 < p < q$, and n a positive integer. Let Λ_n be the submonoid of $\mathbb{N}$ generated by the geometric sequence $p^n, p^{n-1}q, \ldots, pq^{n-1}, q^n$. Then each Frobenius complex of Λ_n has homotopy type of a wedge of spheres, and the multi-graded Poincaré series is given by*

$$P_k^{k[\Lambda_n]}(t, \mathbf{z}) = \frac{\prod_{i=0}^{n}(1 + t\mathbf{z}^{p^{n-i}q^i})}{\prod_{i=1}^{n}(1 - t^2\mathbf{z}^{p^{n-i+1}q^i})}.$$

In particular, in the case $p = 2$,

$$P_k^{k[\Lambda_n]}(t, \mathbf{z}) = \frac{1 + t\mathbf{z}^{2^n}}{\prod_{i=1}^{n}(1 - t\mathbf{z}^{2^{n-i}q^i})}.$$

PROOF. We show the assertion by induction on n. The case $n = 1$ follows from Proposition 2.5.3. Let us assume $n \geq 2$. By the induction hypothesis, $\Lambda_n = p\Lambda_{n-1} + \langle q^n\rangle_{\mathbb{N}}$ satisfies all hypotheses of Theorem 2.5.1 with $\rho = q \cdot pq^{n-1} = p \cdot q^n$. Thus we have

$$\begin{aligned} P_k^{k[\Lambda_n]}(t, \mathbf{z}) &= \frac{P_k^{k[p\Lambda_{n-1}]}(t, \mathbf{z}) \cdot P_k^{k[\langle q^n\rangle_{\mathbb{N}}]}(t, \mathbf{z})}{1 - t^2\mathbf{z}^{pq^n}} \\ &= \frac{\prod_{i=0}^{n}(1 + t\mathbf{z}^{p^{n-i}q^i})}{\prod_{i=1}^{n}(1 - t^2\mathbf{z}^{p^{n-i+1}q^i})}. \qquad \square \end{aligned}$$

PROPOSITION 2.5.6. *Let $p_1, \ldots, p_n$ ($n \geq 1$) be mutually relatively prime integers with $1 < p_1 < \cdots < p_n$. Let Λ_n be the submonoid of $\mathbb{Q}$ generated by $1/p_1, \ldots, 1/p_n$. Then any Frobenius complex of Λ_n has homotopy type of a wedge of spheres, and the multi-graded Poincaré series is given by*

$$P_k^{k[\Lambda_n]}(t, \mathbf{z}) = \frac{\prod_{i=1}^{n}(1 + t\mathbf{z}^{1/p_i})}{(1 - t^2\mathbf{z})^{n-1}}.$$

PROOF. We show the assertion by induction on n. The case $n = 1$ follows from Proposition 2.2.2. Let us assume $n \geq 2$. By the induction hypothesis, $\Lambda_n = \Lambda_{n-1} + \langle 1/p_n\rangle_{\mathbb{N}}$ satisfies all hypotheses of Theorem 2.5.1 with $\rho = 1$. Thus we have

$$P_k^{k[\Lambda_n]}(t, \mathbf{z}) = \frac{P_k^{k[\Lambda_{n-1}]}(t, \mathbf{z}) \cdot P_k^{k[\langle 1/p_n\rangle_{\mathbb{N}}]}(t, \mathbf{z})}{1 - t^2\mathbf{z}} = \frac{\prod_{i=1}^{n}(1 + t\mathbf{z}^{1/p_i})}{(1 - t^2\mathbf{z})^{n-1}}. \qquad \square$$

PROPOSITION 2.5.7. *Let a, b, c and d be integers greater than 1, and assume $\gcd(a, b) = \gcd(c, d) = \gcd(a+b, c+d) = 1$. Let Λ be the submonoid of $\mathbb{N}$ generated by $a(c+d)$, $b(c+d)$, $(a+b)c$ and $(a+b)d$. Then each Frobenius complex of Λ has* *homotopy type of a wedge of spheres, and the multi-graded Poincaré series is given by*

$$P_k^{k[\Lambda]}(t,\mathbf{z}) = \frac{(1+t\mathbf{z}^{a(c+d)})(1+t\mathbf{z}^{b(c+d)})(1+t\mathbf{z}^{(a+b)c})(1+t\mathbf{z}^{(a+b)d})}{(1-t^2\mathbf{z}^{ab(c+d)})(1-t^2\mathbf{z}^{(a+b)cd})(1-t^2\mathbf{z}^{(a+b)(c+d)})}.$$

Proof. We can easily check that $\Lambda = (c+d)\,\langle a,b\rangle_{\mathbb{N}} + (a+b)\,\langle c,d\rangle_{\mathbb{N}}$ satisfies all hypotheses of Theorem 2.5.1 with $\rho = (a+b)(c+d)$. Thus we have

$$\begin{aligned} P_k^{k[\Lambda]}(t,\mathbf{z}) &= \frac{P_k^{k[(c+d)\langle a,b\rangle_{\mathbb{N}}]}(t,\mathbf{z}) \cdot P_k^{k[(a+b)\langle c,d\rangle_{\mathbb{N}}]}(t,\mathbf{z})}{1-t^2\mathbf{z}^{\rho}} \\ &= \frac{(1+t\mathbf{z}^{a(c+d)})(1+t\mathbf{z}^{b(c+d)})(1+t\mathbf{z}^{(a+b)c})(1+t\mathbf{z}^{(a+b)d})}{(1-t^2\mathbf{z}^{ab(c+d)})(1-t^2\mathbf{z}^{(a+b)cd})(1-t^2\mathbf{z}^{(a+b)(c+d)})}. \qquad \square \end{aligned}$$

CHAPTER 3

An extension of Gromov's characterization for orthoscheme complexes

3.1. Introduction

Gromov [**Gro**] showed that a cubical complex has non-positive curvature if and only if the link of each vertex is a flag complex. This theorem has a lot of applications. A typical example is the proof that any right-angled Artin group is a CAT(0) group, which goes as follows (see [**CD**] for more details). For a right-angled Artin group A_Γ, one can construct a cubical complex $\mathcal{S}_\Gamma$ with fundamental group A_Γ, which is called the Salvetti complex associated to A_Γ. Using Gromov's characterization, one can check that $\mathcal{S}_\Gamma$ have non-positive curvature. Thus A_Γ acts properly, cocompactly by isometries on the universal cover of $\mathcal{S}_\Gamma$, which is a CAT(0) geodesic space.

It is, however, still open whether all Artin groups are CAT(0) groups. Brady and McCammond [**BM**] introduced orthoscheme complexes as a generalization of cubical complexes. Orthoschemes are Euclidean simplices which appear in the barycentric subdivision of the cube $[-1, 1]^d$. The orthoscheme complex of a graded poset P is a piecewise Euclidean complex obtained by gluing orthoschemes along the chains of P. A precise definition will be given in Section 3.5. Brady and McCammond showed the following.

(1) If the orthoscheme complex of the poset NPC_n of the non-crossing partitions is a CAT(0) space, then the n-string braid group is a CAT(0) group.
(2) For $n \leq 5$, the orthoscheme complex of NPC_n is a CAT(0) space.

Thus the n-string braid group is a CAT(0) group for $n \leq 5$. They conjectured that (2) holds for arbitrary n. Haettel, Kielak and Schwer showed that (2) holds for $n \leq 6$ [**HKS**].

Now, it seems to be important to develop criteria for a graded poset to have CAT(0) orthoscheme complex. Chalopin et al. [**CCHO**] established some sufficient conditions. For example, they showed the following.

(1) The orthoscheme complex of a modular lattice is a CAT(0) space.
(2) The orthoscheme complex of a locally distributive flag semilattice is a CAT(0) space.

Relevance between the CAT(0) properties of orthoscheme complexes and the computational complexity of the 0-extension problem was pointed out (see [**CCHO**] for more details).

It seems, however, that there were few necessary and sufficient conditions for a graded poset to have CAT(0) orthoscheme complex. In this chapter, we discuss a translation and an extension of Gromov's characterization for orthoscheme complexes. We say a semilattice S is a *flag semilattice* if any pairwise bounded finite subset of S is again bounded. As a translation, we show that the orthoscheme complex of a locally Boolean semilattice S is a CAT(0) space if and only if S is a flag semilattice (Theorem 3.5.3). As an extension, we show that the orthoscheme complex of a locally distributive semilattice S is a CAT(0) space if and only if S is a flag semilattice (Theorem 3.5.4). We also show that the orthoscheme complex of any locally distributive semilattice can be embedded in that of some locally Boolean semilattice as a convex subset.

The rest of this chapter is organized as follows. In Section 3.2, we introduce some notions and terminology. In Section 3.3, we establish a representation theorem for locally distributive semilattices. In Section 3.4, we review some notions concerning CAT(0) geodesic spaces and Euclidean polyhedral complexes. In Section 3.5, we discuss an extension of Gromov's characterization for orthoscheme complexes.

3.2. Preliminaries

An abstract simplicial complex K is said to be a *flag complex* if the following condition holds for any finite subset σ of vertices: if any two-element subset of σ forms a face of K, then σ itself is also a face of K.

PROPOSITION 3.2.1. *The following hold.*

(1) *An abstract simplicial complex K is a flag complex if and only if the following condition holds for any faces $\sigma_1, \sigma_2, \sigma_3$ of K: if all of $\sigma_1 \cup \sigma_2$, $\sigma_1 \cup \sigma_3$ and $\sigma_2 \cup \sigma_3$ are faces of K, then $\sigma_1 \cup \sigma_2 \cup \sigma$ is also a face of K.*
(2) *The simplex $\widetilde{\sigma}$ is a flag complex for any finite set σ.*
(3) *If K is a flag complex, then the link $\mathrm{lk}(\sigma; K)$ is a flag complex for any face σ of K.*
(4) *For abstract simplicial complexes K and L, the join $K * L$ is a flag complex if and only if both K and L are flag complexes.*

PROOF. The proof is straightforward. □

Let P be a poset. A *chain* of P is a totally ordered subset of P. The *length* of a chain C is defined to be $\#C - 1$. The *height* $\mathrm{ht}(P)$ of P is defined to be the least upper bound of the lengths of all chains of P, which might be ∞. The *height* $\mathrm{ht}_P(x)$ of an element x of P is defined to be $\mathrm{ht}(P^{\leq x})$. We say that P has *locally finite height* if the height of any elements of P is finite. Let us note that P has finite height if and only if the order complex of P is finite dimensional. In this case, $\mathrm{ht}(P) = \dim \Delta(P)$ holds. A subset A of P is said to be *bounded above*, or simply *bounded*, if there exists $u \in P$ satisfying $A \subset P^{\leq u}$.

A *lattice* is a poset L such that any pair $x, y \in L$ has the greatest lower bound and the least upper bound, which will be denoted by $x \wedge y$ and $x \vee y$, respectively. We say L is *modular* if the modular law

$$(x \vee y) \wedge z = x \vee (y \wedge z)$$

holds for any $x, y, z \in L$ with $x \leq z$. We say L is *distributive* if the distributive law

$$x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$$

holds for any $x, y, z \in L$. We say L is *bounded* if L has a minimum and a maximum, which will be denoted by 0 and 1, respectively. A bounded lattice L is said to be *complemented* if for any $x \in L$ there exists $y \in L$ satisfying $x \wedge y = 0$ and $x \vee y = 1$. A complemented distributive lattice is called a *Boolean* lattice.

A *meet-semilattices*, or simply a *semilattice*, is a poset S such that any pair $x, y \in S$ has the greatest lower bound, which will be denoted by $x \wedge y$.

LEMMA 3.2.2. *Let S be a semilattice of locally finite height, and A a non-empty subset of S closed under $\wedge$, that is, $x, y \in A$ implies $x \wedge y \in A$. Then A has a minimum.*

PROOF. First, we show that A has a minimal element. Otherwise, we can take an infinite strictly decreasing sequence $a_0 > a_1 > \cdots$ of A. Thus we obtain

$$\infty > \mathrm{ht}_S(a_0) > \mathrm{ht}_S(a_1) > \cdots,$$

but $\mathrm{ht}_S(x)$ is non-negative by definition, which is a contradiction.

Thus A has a minimal element m. Then m is the minimum of A, since we have $x \geq x \wedge m = m$ for any $x \in A$. □

PROPOSITION 3.2.3. *Let S be a non-empty semilattice of locally finite height. Then the following hold.*

(1) *S has a minimum, which will be denoted by* 0.
(2) *Any bounded pair of elements of S has the least upper bound.*
(3) *$S^{\leq x}$ is a bounded lattice for any $x \in S$.*

PROOF. To show (1), apply the previous lemma to S itself. To show (2), similarly consider $S^{\leq x} \cap S^{\leq y}$. (3) follows from (1) and (2). □

Let S be a non-empty semilattice of locally finite height. For a bounded pair $x, y \in S$, we denote their least upper bound by $x \vee y$. We can see $\vee$ as a partial binary operator on S. For a property (Φ) for bounded lattices, we say S is *locally* (Φ) if $S^{\leq x}$ satisfies (Φ) for any $x \in S$. For example, a locally distributive semilattice is a semilattice S such that $S^{\leq x}$ is a distributive lattice for any $x \in S$. We say that S is a *flag semilattice* if any pairwise bounded finite subset of S is bounded. By the similar way in Proposition 3.2.1, a semilattice S is a flag semilattice if and only if any pairwise bounded triple of elements of S is bounded.

3.3. A representation theorem for locally distributive semilattices

It is well-known that any distributive lattice of finite height is isomorphic to the poset of the down sets of a finite poset, which is known as Birkhoff's representation theorem for distributive lattices (see [**Grä**, Theorem 107]). In this section, we discuss its extension for locally distributive semilattices. The basic idea of this extension can be seen in [**CCHO**, Proposition 7.6].

Let S be a non-empty locally distributive semilattice of locally finite height. An element x of S is said to be *join-reducible*, or simply *reducible*, if there exist $y, z \in S^{<x}$ satisfying $x = y \vee z$. An element x of S is said to be *join-irreducible*, or simply *irreducible*, if x is neither reducible nor equal to 0. Let $\mathrm{Irr}\, S$ denote the induced subposet of all irreducible elements of S.

PROPOSITION 3.3.1. *Let x be an element of S. Then the following are equivalent.*

- *x is irreducible.*
- *For any finite subset F of $S^{\leq x}$, $\bigvee F = x$ implies $x \in F$.*

PROOF. The proof is done by induction on $\#F$. □

LEMMA 3.3.2. *If a subset A of $\mathrm{Irr}\, S$ is bounded in S, then A is finite.*

PROOF. Let us take $u \in S$ satisfying $A \subset S^{\leq u}$. We now show $\#A \leq \mathrm{ht}_S(u)$. Otherwise, we can take $n > \mathrm{ht}_S(u)$ and distinct n elements $a_1, \ldots, a_n$ of A. By permutation, we can assume that $a_i \leq a_j$ implies $i \leq j$. Set $b_j = \bigvee_{i \leq j} a_i$ for $j = 0, \ldots, n$. Clearly, the sequence $b_0, \ldots, b_n$ is weakly increasing. If the equation $b_{j-1} = b_j$ holds, then we have

$$a_j = b_j \wedge a_j = b_{j-1} \wedge a_j = \Big(\bigvee_{i<j} a_i\Big) \wedge a_j = \bigvee_{i<j} (a_i \wedge a_j).$$

Since a_j is irreducible, there exists $i < j$ satisfying $a_i \wedge a_j = a_j$, which implies $a_i \geq a_j$. This contradicts the assumption for $a_1, \ldots, a_n$. Hence the sequence $b_0, \ldots, b_n$ is strictly increasing, and thus forms a chain in $S^{\leq u}$ of length n, which contradicts the assumption $n > \mathrm{ht}_S(u)$. $\square$

A *down set* of a poset P is a subset I of P such that $x \leq y$ and $y \in I$ imply $x \in I$ for any $x, y \in P$. Let $\operatorname{Down} P$ denote the set of all down sets of P. For a subset σ of P, we define the closure of σ by

$$\overline{\sigma} = \bigcup_{x \in \sigma} P^{\leq x}.$$

Then $\overline{\sigma}$ is the smallest down set of P which contains σ.

Let K be an abstract simplicial complex, and fix a partial order $\leq$ on $V(K)$. A face of K is said to be a *down face* if it is a down set of $V(K)$ with respect to this order. Let $DF(K)$ denote the set of all down faces of K. We see $DF(K)$ as an induced subposet of $F(K)$. A partial order $\leq$ on $V(K)$ is said to be a *compatible order* on K if any face of K is contained in some down face of K. Equivalently, $\overline{\sigma}$ is a face of K for any face σ of K.

Proposition 3.3.3. *Let K be an abstract simplicial complex, and fix a partial order on $V(K)$. Then $F(K)$ is a locally Boolean semilattice of locally finite height, and $DF(K)$ is a locally distributive semilattice of locally finite height. Moreover, the meets, the joins and the heights in $DF(K)$ coincide with the restrictions of those in $F(K)$.*

Proof. The meet is given by the intersection, the join by the union if exists, and the height by the size of a face, which is finite. $\square$

Theorem 3.3.4. *Let S be a locally distributive semilattice of locally finite height. Then there exist an abstract simplicial complex K and a compatible order on K such that $DF(K)$ is isomorphic to S.*

Proof. Let K be the abstract simplicial complex whose faces are the subsets σ of $\operatorname{Irr} S$ bounded in S. The finiteness of faces of K follows from Lemma 3.3.2. The induced order on $V(K) = \operatorname{Irr} S$ is a compatible order on K, since $\sigma \subset S^{\leq u}$ implies $\overline{\sigma} \subset S^{\leq u}$.

Let $\varphi\colon S \to DF(K)$ and $\psi\colon DF(K) \to S$ be the maps defined by

$$\varphi(x) = (\operatorname{Irr} S)^{\leq x} \qquad (x \in S)$$
$$\psi(\sigma) = \bigvee \sigma \qquad (\sigma \in DF(K)).$$

Clearly, φ and ψ are well-defined and order-preserving. We will show these maps are inverses of each other.

It is clear that $\psi \circ \varphi(x) \leq x$ holds for any $x \in S$. We now show the other direction by induction on $\mathrm{ht}_S(x)$. The case either $x = 0$ or $x \in \operatorname{Irr} S$ is trivial. Let us assume that x is reducible, that is, $x = y \vee z$ for some $y, z \in S^{<x}$. By the induction hypothesis, we have

$$\psi \circ \varphi(x) = \psi \circ \varphi(y \vee z) \geq \psi \circ \varphi(y) \vee \psi \circ \varphi(z) = y \vee z = x.$$

It is clear that $\varphi \circ \psi(\sigma) \supset \sigma$ holds for any $\sigma \in DF(K)$. Let x be an element of $\varphi \circ \psi(\sigma) = (\operatorname{Irr} S)^{\leq \bigvee \sigma}$. Then we have

$$x = x \wedge \Bigl(\bigvee \sigma\Bigr) = \bigvee_{y \in \sigma} (x \wedge y).$$

Since x is irreducible, there exists $y \in \sigma$ satisfying $x = x \wedge y$, which implies $x \leq y$. Since σ is a down set, x belongs to σ. $\square$

Corollary 3.3.5. *Let S be a locally Boolean semilattice of locally finite height. Then there exists an abstract simplicial complex K such that $F(K)$ is isomorphic to S.*

Proof. It is enough to show that $\operatorname{Irr} S$ is an antichain, that is, there is no non-trivial ordering in S. Otherwise, there exist $x, y \in \operatorname{Irr} S$ satisfying $x > y$. Since $S^{\leq x}$ is a Boolean lattice, there exists $z \in S^{\leq x}$ satisfying $y \wedge z = 0$ and $y \vee z = x$. Since x is irreducible and $y < x$, we have $z = x$. Thus we have

$$0 = y \wedge z = y \wedge x = y,$$

which contradicts to the assumption that y is irreducible. $\square$

Proposition 3.3.6. *Let K be an abstract simplicial complex, and fix a compatible order on K. Then the following are equivalent.*

(1) *K is a flag complex.*
(2) *$F(K)$ is a flag semilattice.*
(3) *$DF(K)$ is a flag semilattice.*

Proof. (1) $\Leftrightarrow$ (2) and (2) $\Rightarrow$ (3) are trivial. We now show (3) $\Rightarrow$ (2). Let us assume that $DF(K)$ is a flag semilattice. Let $\sigma_1, \sigma_2, \sigma_3$ be pairwise bounded elements of $F(K)$. Then $\overline{\sigma}_1, \overline{\sigma}_2, \overline{\sigma}_3$ are pairwise bounded in $DF(K)$, since an upper bound of $\overline{\sigma}_i$ and $\overline{\sigma}_j$ is given by $\overline{\sigma_i \cup \sigma_j}$. Thus there exists an upper bound of $\{\overline{\sigma}_1, \overline{\sigma}_2, \overline{\sigma}_3\}$ in $DF(K)$, which is also an upper bound of $\{\sigma_1, \sigma_2, \sigma_3\}$ in $F(K)$. $\square$

3.4. Metric spaces

In this section, we first review some notions concerning CAT(0) geodesic spaces and Euclidean polyhedral complexes. References for these subjects are [**BH**] and [**Gro**]. Second, we construct the cubical cone from an abstract simplicial complex, and show some properties of cubical cones.

A *pseudo-metric* on a set X is a binary function $d\colon X \times X \to [0, \infty]$ which satisfies the following conditions:

- $d(x, x) = 0$ holds for any $x \in X$.
- $d(x, y) = d(y, x)$ holds for any $x, y \in X$.
- $d(x, z) \leq d(x, y) + d(y, z)$ holds for any $x, y, z \in X$.

A *metric* on a set X is a pseudo-metric d on X which satisfy

- $d(x, y) = 0$ implies $x = y$ for any $x, y \in X$.

A *metric space* is a set X equipped with a metric d_X on X. Similarly, we define a *pseudo-metric space*. The metric on $\mathbb{R}^n$ defined by $d(x, y) = \sqrt{\sum_{i=1}^{n} (x_i - y_i)^2}$ is called the *Euclidean metric*, and $\mathbb{R}^n$ with the Euclidean metric is called the *Euclidean space*, which will be denoted by $\mathbb{E}^n$. A metric space X is said to be *complete* if any Cauchy sequence in X converges. For a metric space (X, d_X) and a subset A of X, the restriction of d_X on $A \times A$ is called the *induced metric* on A. For two pseudo-metric spaces X and Y, a map $f\colon X \to Y$ is *non-expanding* if $d_Y(f(x), f(x')) \leq d_X(x, x')$ holds for any $x, x' \in X$, and f is *distance preserving* if $d_Y(f(x), f(x')) = d_X(x, x')$ holds for any $x, x' \in X$. Clearly, a distance-preserving map from a metric space is injective. A bijective distance-preserving map is called an *isometry*. Two metric spaces are called *isometric* if an isometry between them exists. For a metric space X and $x, y \in X$, a *geodesic path* from x to y in X is a distance-preserving map $\gamma\colon [0, \ell] \to X$ which sends the endpoints 0 and ℓ to x and y, respectively. Here $[0, \ell]$ denotes the closed interval of $\mathbb{R}$ with the standard metric, that is, $d_{[0,\ell]}(s, t) = |s - t|$. In this case, we have $d_X(x, y) = \ell$. A metric space X is said to be *geodesic* if for any $x, y \in X$ there exists a geodesic path from x to y in X.

CAT(0) properties. A geodesic metric space X is said to be *CAT(0)* if for any $x, y, z \in X$ and any geodesic path $\gamma\colon [0, \ell] \to X$ from x to y in X, the inequality

$$d_X(\gamma(t\ell), z)^2 \leq t \cdot d_X(y, z)^2 + (1-t) \cdot d_X(x, z)^2 - t(1-t) \cdot d_X(x, y)^2$$

holds for all $t \in [0, 1]$. Roughly speaking, this inequality means that the any triangle in X whose edges are geodesic paths is at least as thin as the comparison triangle of the same side lengths in the Euclidean space. We say a metric space X has *non-positive curvature*, or is *locally CAT(0)*, if for any $x \in X$ there exists $r > 0$ such that the open ball $\{ y \in X \mid d_X(x, y) < r \}$ of radius r centered at x is a CAT(0) geodesic space with the induced metric.

Let us note that if a geodesic space X is CAT(0), then X is uniquely geodesic, that is, for any pair of points in X there uniquely exists a geodesic path between them. Since the unique geodesic path can be taken continuously with respect to the end points, any non-empty CAT(0) geodesic space must be contractible.

Theorem 3.4.1 (The Cartan-Hadamard Theorem [**BH**, II.4.1(2)]). *Let X be a complete metric space. Then the following are equivalent.*

(1) *X has non-positive curvature, and is simply connected.*
(2) *X is a CAT(0) geodesic space.*

Euclidean polyhedral complexes. In this subsection, we review the definition and basic properties of Euclidean polyhedral complexes. Roughly speaking, Euclidean polyhedral complexes are obtained from Euclidean polytopes by gluing them along isometric faces. We are interested in conditions for Euclidean polyhedral complexes to have (locally) CAT(0) metric.

A *Euclidean polytope* is a polytope in a Euclidean space with the induced metric. A *Euclidean polyhedral complex* is a set X equipped with a family $\{(P_\lambda, i_\lambda)\}_{\lambda \in \Lambda}$ of pairs of a Euclidean polytope P_λ and an injection $i_\lambda\colon P_\lambda \to X$ which satisfies the following conditions:

- The images of i_λ cover X, that is, $X = \bigcup_{\lambda \in \Lambda} i_\lambda(P_\lambda)$.
- If $i_\lambda(P_\lambda)$ and $i_\mu(P_\mu)$ have non-empty intersection, then the inverse image of the intersection under i_λ is a face of P_λ, similarly the inverse image under i_μ is a face of P_μ, and the induced bijection

  $$i_\mu^{-1} \circ i_\lambda\colon i_\lambda^{-1}(i_\lambda(P_\lambda) \cap i_\mu(P_\mu)) \to i_\mu^{-1}(i_\lambda(P_\lambda) \cap i_\mu(P_\mu))$$

  is an isometry with respect to the induced metrics.

The maps $\{i_\lambda\}_{\lambda \in \Lambda}$ are called *face maps* of X, and their images are called *faces* of X. The restriction of i_λ on a face of P_λ is also called a face map of X, and its image is also called a face of X.

By definition, our Euclidean polyhedral complexes are regular, that is, all face maps are injective. Moreover, our Euclidean polyhedral complexes are simple, that is, any two faces intersect in at most one face of them.

For $x, y \in X$, a *string* from x to y in X is a finite sequence $\Sigma = \{(\lambda_i, x_i, y_i)\}_{i=1}^m$ of triples which satisfies the following conditions:

- $\lambda_i \in \Lambda_i$ for $i = 1, \ldots, m$
- $x_i, y_i \in P_{\lambda_i}$ for $i = 1, \ldots, m$
- $x = i_{\lambda_1}(x_1)$
- $i_{\lambda_i}(y_i) = i_{\lambda_{i+1}}(x_{i+1})$ for $i = 1, \ldots, m-1$
- $i_{\lambda_m}(y_m) = y$

The *length* of a string $\Sigma = \{(\lambda_i, x_i, y_i)\}_{i=1}^m$ is defined by

$$\ell(\Sigma) = \sum_{i=1}^{m} d_{P_{\lambda_i}}(x_i, y_i).$$

The *intrinsic pseudo-metric* on X is defined by

$$d_X(x,y) = \inf\{\, \ell(\Sigma) \mid \Sigma \text{ is a string from } x \text{ to } y \text{ in } X\,\}.$$

If there is no string from x to y in X, we define $d_X(x,y) = \infty$. The intrinsic pseudo-metric can be characterized as follows: for any pseudo-metric space Z and any map $f\colon X \to Z$, f is non-expanding if and only if $f \circ i_\lambda\colon P_\lambda \to Z$ is non-expanding for each $\lambda \in \Lambda$. Equivalently, the intrinsic pseudo-metric is the largest pseudo-metric such that all face maps $i_\lambda\colon P_\lambda \to X$ are non-expanding. Let us note that a string $\Sigma = \{(\lambda_i, x_i, y_i)\}_{i=1}^m$ induces a path in X by concatenating the line segments $[x_i, y_i]$ in P_{λ_i}. We say X is *connected* if any pair of points in X can be connected by a string in X. We say that X has *finite shapes* if the number of isometry types of $\{\, P_\lambda \mid \lambda \in \Lambda\,\}$ is finite. Bridson showed the following.

Theorem 3.4.2 ([**BH**, I.7]). *If X is a connected Euclidean polyhedral complex of finite shapes, then the intrinsic pseudo-metric is a metric, and X is a complete geodesic metric space. Moreover, any geodesic path in X is obtained from a string.*

Cubical complexes. A *cubical complex* is a Euclidean polyhedral complex X such that each face of X is isometric to a unit cube $I^d = [0,1]^d \subset \mathbb{E}^d$ for some $d \geq 0$. Let us note that a cubical complex has finite shapes if and only if it is finite dimensional. A face of X isometric to I^0 is called a *vertex* of X. Since a vertex v of X is a one-point subspace of X, we identify v as a point in X. A face of X isometric to I^1 is called an *edge* of X. Two distinct vertices v and w of X is said to be *adjacent* if there exists an edge of X which contains both v and w. Let v be a vertex of X. The *cubical link*, or simply the *link*, $\operatorname{lk}_\square(v;X)$ of v in X is an abstract simplicial complex whose vertices are the vertices of X adjacent to v. A finite subset σ of such vertices forms a face of $\operatorname{lk}_\square(v;X)$ if and only if there exists a face of X containing both v and σ.

Gromov characterized the condition for cubical complexes to have non-positive curvature.

Theorem 3.4.3 (Gromov [**Gro**, 4.2.C], [**BH**, Theorem II.5.20]). *Let X be a finite-dimensional cubical complex. Then X has non-positive curvature if and only if $\operatorname{lk}_\square(v;X)$ is a flag complex for any vertex v of X.*

Cubical cones. We will discuss a translation and an extension of this characterization by Gromov in the next section. In order to do this, we now introduce cubical cones, which behave as a partial inverse of cubical links. Let K be a finite-dimensional abstract simplicial complex. The *cubical cone* $C_\square(K)$ of K is the cubical complex defined by

$$C_\square(K) = \bigcup_{\sigma \in K} I^\sigma \subset \mathbb{E}^{(V(K))},$$

where

$$I^\sigma = \Big\{ \sum_{v \in \sigma} t_v v \;\Big|\; t_v \in [0,1] \Big\} \subset \mathbb{E}^{(\sigma)} \subset \mathbb{E}^{(V(K))}.$$

Here $\mathbb{E}^{(A)}$ denotes the direct sum $\bigoplus_{a\in A} \mathbb{R}a$ with the Euclidean metric with respect to A, that is, $d_{\mathbb{E}^{(A)}}(\sum_a t_a a, \sum_a s_a a) = \sqrt{\sum_a (t_a - s_a)^2}$. Equivalently, $C_\square(K)$ can be defined as

$$C_\square(K) = \Big\{ x = \sum_{v \in V(K)} t_v v \;\Big|\; t_v \in [0,1],\ \operatorname{supp} x \in K \Big\},$$

where $\operatorname{supp} x = \{\, v \in V(K) \mid t_v \neq 0\,\}$. Face maps of $C_\square(K)$ are the inclusions $I^\sigma \hookrightarrow C_\square(K)$ for $\sigma \in K$. Here we see I^σ as a Euclidean polytope in $\mathbb{E}^{(\sigma)}$. By definition, I^σ is isometric to the unit cube $I^{\#\sigma}$.

Proposition 3.4.4. *Any vertex of $C_\square(K)$ has form $\chi_\sigma = \sum_{v\in\sigma} v$ for some $\sigma \in F(K)$. Moreover, the link $\mathrm{lk}_\square(\chi_\sigma; C_\square(K))$ is isomorphic to $\widetilde{\sigma} * \mathrm{lk}(\sigma; K)$. In particular, the link $\mathrm{lk}_\square(0; C_\square(K))$ at the origin $0 = \chi_\emptyset$ is isomorphic to K itself.*

Proof. The first assertion is obvious. We now show the second. A vertex adjacent to χ_σ has form either $\chi_{\sigma\setminus\{v\}}$ for $v \in \sigma$ or $\chi_{\sigma\cup\{w\}}$ for $w \in V(\mathrm{lk}(\sigma; K))$. The obvious bijection

$$V(\widetilde{\sigma} * \mathrm{lk}(\sigma; K)) = V(\widetilde{\sigma}) \sqcup V(\mathrm{lk}(\sigma; K)) \to V(\mathrm{lk}_\square(\chi_\sigma; C_\square(K)))$$

gives an isomorphism between abstract simplicial complexes. $\square$

Proposition 3.4.5. *The cubical cone $C_\square(K)$ is a CAT(0) space if and only if K is a flag complex.*

Proof. By Theorem 3.4.2, $C_\square(K)$ is a complete metric space. Since $C_\square(K)$ is star-shaped at the origin, $C_\square(K)$ is contractible, and thus simply connected. By Theorem 3.4.1, $C_\square(K)$ is CAT(0) if and only if it has non-positive curvature. By Theorem 3.4.3, this is equivalent to that the link of each vertex in $C_\square(K)$ is a flag complex. Combining the previous proposition and Proposition 3.2.1, we have the assertion. $\square$

Proposition 3.4.6. *The inclusion $i\colon C_\square(K) \hookrightarrow \mathbb{E}^{(V(K))}$ is a non-expanding map. Moreover, if a pair $\xi, \eta \in C_\square(K)$ satisfies the equation*

$$d_{C_\square(K)}(\xi, \eta) = d_{\mathbb{E}^{(V(K))}}(\xi, \eta),$$

then the line segment between ξ and η in $E^{(V(K))}$ is contained in $C_\square(K)$.

Proof. The first assertion follows from that the composition

$$I^\sigma \hookrightarrow C_\square(K) \hookrightarrow \mathbb{E}^{(V(K))}$$

is distance preserving, and thus non-expanding for $\sigma \in F(K)$. We now show the second. Let us assume $\xi, \eta \in C_\square(K)$ satisfies $d_{C_\square(K)}(\xi, \eta) = d_{\mathbb{E}^{(V(K))}}(\xi, \eta)$. Let us take a geodesic path $\gamma\colon [0, \ell] \to C_\square(K)$ from ξ to η in $C_\square(K)$. Then $i \circ \gamma$ is a geodesic path from ξ to η in the Euclidean space $\mathbb{E}^{(V(K))}$, which implies the assertion. $\square$

3.5. Orthoscheme complexes

In this section, we consider the orthoscheme complex of a poset, which is the order complex equipped with a certain Euclidean polyhedral complex structure.

For positive real numbers $\ell_1, \dots, \ell_d$, the *orthoscheme* $O(\ell_1, \dots, \ell_d)$ is defined to be the Euclidean polytope in $\mathbb{E}^d$ spanned by $v_i = \sum_{j=1}^{i} \ell_j e_j$ for $i = 0, \dots, d$. Here $e_1, \dots, e_d$ denote the standard orthonormal basis of $\mathbb{E}^d$. Then the orthoscheme $O(\ell_1, \dots, \ell_d)$ is a d-dimensional Euclidean simplex satisfying the following properties:

- The edge $v_i v_j$ is orthogonal to $v_j v_k$ for $0 \le i \le j \le k \le d$.
- The edge $v_{i-1} v_i$ has length ℓ_i for $1 \le i \le d$.
- The edge $v_i v_j$ has length $\sqrt{\sum_{k=i+1}^{j} \ell_k^2}$ for $0 \le i \le j \le d$.

Let us note that the unit d-orthoscheme $O(1, \dots, 1)$ is isometric to the facet of the barycentric subdivision of the cube $[-1, 1]^d$.

Let P be a poset, and $r\colon P \to \mathbb{R}$ be a strictly order-preserving map. We now construct a Euclidean polyhedral complex structure on the order complex $|P|$ by using r. For a finite chain $\sigma = \{x_0 < \cdots < x_d\}$ of P, Let us define

$$O^\sigma = O(\sqrt{r(x_1) - r(x_0)}, \sqrt{r(x_2) - r(x_1)}, \dots, \sqrt{r(x_d) - r(x_{d-1})}),$$

and define $i_\sigma \colon O_\sigma \to |P|$ to be the affine map which sends v_i to x_i for $i = 0, \dots, d$. Then i_σ is an injection onto Δ^σ. We can see that i_σ gives a Euclidean metric on Δ^σ satisfying $d_{\Delta^\sigma}(x_i, x_j) = \sqrt{r(x_j) - r(x_i)}$ for $0 \le i \le j \le d$. The orthoscheme complex of P with respect to r is defined to be the Euclidean polyhedral complex on the geometric realization $|P|$ whose face maps are $i_\sigma \colon O^\sigma \to |P|$ for $\sigma \in \Delta(P)$. We say that a poset P is *connected* if for any $x, y \in P$ there exists a finite sequence $x_0, \dots, x_{2n}$ in P satisfying

$$x = x_0 \le x_1 \ge x_2 \le \cdots \ge x_{2n} = y.$$

Let us note that P is connected if and only if the orthoscheme complex $|P|$ is connected. By using Theorem 3.4.2 we have the following.

PROPOSITION 3.5.1. *If P is connected and the image of $r \colon P \to \mathbb{R}$ is finite, then the orthoscheme complex $|P|$ of P with respect to r is a complete geodesic metric space.*

A poset P is said to be *graded* if there exists a strictly order-preserving map $r \colon P \to \mathbb{Z}$ such that the equation $\mathrm{ht}[x, y]_P = r(y) - r(x)$ holds for any pair $x, y \in P$ with $x \le y$. Such an r is called a *rank function* on P. If P is a poset of finite height with a minimum, a rank function on P is given by the height function $\mathrm{ht}_P \colon P \to \{0, 1, \dots, \mathrm{ht}\, P\}$ if exists. In the rest of this chapter, we treat only graded posets of finite height, and discuss their orthoscheme complexes with respect to rank functions. Clearly, the orthoscheme complex of a graded poset P with respect to a rank function is independent of the choice of a rank function. Let us note that any locally distributive semilattice of locally finite height is graded by Theorem 3.3.4.

LEMMA 3.5.2. *Let K be a finite-dimensional abstract simplicial complex. Then the orthoscheme complex $|F(K)|$ is isometric to the cubical cone $C_\Box(K)$.*

PROOF. Let us define $\varphi \colon |F(K)| \to C_\Box(K)$ by

$$\varphi\Big(\sum_{i=0}^{d} t_i \sigma_i\Big) = \sum_{i=0}^{d} t_i \chi_{\sigma_i}.$$

We now construct the inverse ψ of φ. Let $\xi = \sum_{v \in V(K)} t_v v$ be an element of $C_\Box(K)$. Let us note that $\{\, v \in V(K) \mid t_v > 0 \,\}$ is finite and forms a face of K. Let us take a descending sequence $1 = s_0 > s_1 > \cdots > s_{d+1} = 0$ such that

$$\{\, t_v \mid v \in V(K) \,\} \cup \{0, 1\} = \{s_0, s_1, \dots, s_{d+1}\},$$

and let

$$\sigma_i = \{\, v \in V(K) \mid t_v \ge s_i \}$$

for $i = 0, \dots, d$. Then $\sigma_0 \subsetneq \sigma_1 \subsetneq \cdots \subsetneq \sigma_d$ is a finite chain of $F(K)$. We define

$$\psi(\xi) = \sum_{i=0}^{d} (s_i - s_{i+1}) \sigma_i.$$

We can easily check that ψ is the inverse of φ.

We next show that φ is distance preserving. By definition we can check that the restriction of φ on Δ^C is distance preserving for any finite chain C of $F(K)$. Using the characterization of the intrinsic metric on $|F(K)|$, it follows that φ is non-expanding. Moreover, a string in $C_\Box(K)$ can be decomposed via ψ into a string in $F(K)$ of the same length, which implies that φ is distance preserving. $\Box$

The following gives a translation of Theorem 3.4.3 for orthoscheme complex.

THEOREM 3.5.3. *Let S be a locally Boolean semilattice of finite height. Then the orthoscheme complex $|S|$ is a CAT(0) space if and only if S is a flag semilattice.*

Proof. By Corollary 3.3.5, it is enough to show the case $S = F(K)$ for a finite-dimensional abstract simplicial complex K. Let us note that $F(K)$ is a flag semilattice if and only if K is a flag complex. The assertion follows from Proposition 3.4.5 and the previous lemma. $\square$

The following is an extension of the previous theorem.

Theorem 3.5.4. *Let S be a locally distributive semilattice of finite height. Then the orthoscheme complex $|S|$ is a CAT(0) space if and only if S is a flag semilattice.*

To show this theorem, we first show the following.

Lemma 3.5.5. *Let K be a finite-dimensional abstract simplicial complex, and fix a compatible order on K. Then $|DF(K)|$ is a convex subset of $|F(K)|$, that is, any geodesic path in $|F(K)|$ between points in $|DF(K)|$ is contained in $|DF(K)|$. In particular, the induced metric on $|DF(K)|$ from $|F(K)|$ coincides with the intrinsic metric of its own.*

Proof. Let $\varphi\colon |F(K)| \to C_\square(K)$ be the isometry defined in the proof of Lemma 3.5.2. Let X be the image of $|DF(K)|$ under φ. By the definition of φ and the construction of its inverse, we have

$$X = \Big\{ \sum_{u \in V(K)} t_u u \in C_\square(K) \;\Big|\; t_v \geq t_w \text{ for } v \leq w \text{ in } V(K) \Big\}.$$

It is enough to show that X is convex in $C_\square(K)$. Set

$$\widetilde{Y}_{vw} = \Big\{ \sum_{u \in V(K)} t_u u \in \mathbb{E}^{(V(K))} \;\Big|\; t_v \geq t_w \Big\}$$

and $Y_{vw} = C_\square(K) \cap \widetilde{Y}_{vw}$ for $v < w$ in $V(K)$. Then we have $X = \bigcap_{v<w} Y_{vw}$. Thus it is enough to show that Y_{vw} is convex in $C_\square(K)$ for $v < w$. We now define $\widetilde{\psi}_{vw}\colon \mathbb{E}^{(V(K))} \to \widetilde{Y}_{vw}$ as follows. Let ξ be an element of $\mathbb{E}^{(V(K))}$. Let us define $\widetilde{\psi}_{vw}(\xi) \in \widetilde{Y}_{vw}$ to be the unique point satisfying

$$d_{\mathbb{E}^{(V(K))}}(\xi, \widetilde{\psi}_{vw}(\xi)) = \inf_{\eta \in \widetilde{Y}_{vw}} d_{\mathbb{E}^{(V(K))}}(\xi, \eta).$$

Indeed, $\widetilde{\psi}_{vw}$ is given by

$$\widetilde{\psi}_{vw}\Big(\sum_{u \in V(K)} t_u u \Big) = \max\Big\{ t_v, \frac{t_v + t_w}{2} \Big\} v + \min\Big\{ t_w, \frac{t_v + t_w}{2} \Big\} w + \sum_{u \neq v,w} t_u u$$

Let us note that if σ is a face of K and $w \in \sigma$, then $\sigma \cup \{v\} \subset \overline{\sigma}$ is also a face of K. Thus the image of $C_\square(K)$ under $\widetilde{\psi}_{vw}$ is contained in $C_\square(K)$. We define $\psi_{vw}\colon C_\square(K) \to Y_{vw}$ to be the restriction of $\widetilde{\psi}_{vw}$. Since $\widetilde{\psi}_{vw}$ is non-expanding, ψ_{vw} is non-expanding on each face, and thus entirely on $C_\square(K)$. Moreover, ψ_{vw} is a retraction. Let us assume that Y_{vw} is not a convex subset of $C_\square(K)$. Then there exist $\xi, \eta, \zeta \in C_\square(K)$ satisfying $\xi, \zeta \in Y_{vw}$, $\eta \notin Y_{vw}$, and $d_{C_\square(K)}(\xi, \eta) + d_{C_\square(K)}(\eta, \zeta) = d_{C_\square(K)}(\xi, \zeta)$. Let us take a shortest string $\Sigma = \{(\sigma_i, x_i, y_i)\}_{i=1}^m$ from ξ to η. Then there exists $i = 1, \ldots, m$ such that $x_i \in Y_{vw}$ but $y_i \notin Y_{vw}$. For such x_i and y_i, ψ_{vw} strictly shortens their distance. Thus the resulting string $\Sigma' = \{(\overline{\sigma}_i, \psi_{vw}(x_i), \psi_{vw}(y_i))\}_{i=1}^m$ from ξ to $\psi_{vw}(\eta)$ has length less than that of Σ. Hence we have

$$\begin{aligned} d_{C_\square(K)}(\xi, \zeta) &\leq d_{C_\square(K)}(\xi, \psi_{vw}(\eta)) + d_{C_\square(K)}(\psi_{vw}(\eta), \zeta) \\ &< d_{C_\square(K)}(\xi, \eta) + d_{C_\square(K)}(\eta, \zeta) \\ &= d_{C_\square(K)}(\xi, \zeta), \end{aligned}$$

which is a contradiction. $\square$

Proof of Theorem 3.5.4. By Theorem 3.3.4, we can assume $S = DF(K)$ for some finite-dimensional abstract simplicial complex K with a fixed compatible order on K. If $DF(K)$ is a flag semilattice, then K is a flag complex, and thus $|F(K)| \cong C_{\square}(K)$ is a CAT(0) space. Hence its convex subset $|DF(K)|$ is also a CAT(0) space.

We now show the converse. Let us assume that $|DF(K)|$ is a CAT(0) space. Let σ_i $(i = 1, 2, 3)$ be pairwise bounded elements of $DF(K)$, that is, $\sigma_i \cup \sigma_j \in DF(K)$ holds for $i, j \in \{1, 2, 3\}$. Let X be the image of $|DF(K)|$ under the isometry $\varphi\colon |F(K)| \to C_{\square}(K)$ in the proof of Lemma 3.5.2. Then X is isometric to $|DF(K)|$ with the induced metric from $C_{\square}(K)$, and thus X is a CAT(0) space. Since the line segments $[\chi_{\sigma_i}, \chi_{\sigma_j}]$ in $\mathbb{E}^{(V(K))}$ are contained in X, we have

$$d_X(\chi_{\sigma_i}, \chi_{\sigma_j}) = d_{\mathbb{E}^{(V(K))}}(\chi_{\sigma_i}, \chi_{\sigma_j})$$

for $i, j \in \{1, 2, 3\}$. By the CAT(0) inequality, we have

$$d_X(\frac{\chi_{\sigma_1} + \chi_{\sigma_2}}{2}, \chi_{\sigma_3}) \leq d_{\mathbb{E}^{(V(K))}}(\frac{\chi_{\sigma_1} + \chi_{\sigma_2}}{2}, \chi_{\sigma_3}).$$

This is possible only if the line segment $[\frac{\chi_{\sigma_1}+\chi_{\sigma_2}}{2}, \chi_{\sigma_3}]$ is contained in X, which implies $\sigma_1 \cup \sigma_2 \cup \sigma_3 \in F(K)$, and thus $\sigma_1 \cup \sigma_2 \cup \sigma_3 \in DF(K)$. □

Chalopin et al. conjectured that a locally modular flag semilattice of finite height has CAT(0) orthoscheme complex [**CCHO**, Conjecture 7.3]. We suggest the following stronger form.

Conjecture 3.5.6. *Let S be a locally modular semilattice of finite height. Then the orthoscheme complex $|S|$ is a CAT(0) space if and only if S is a flag semilattice.*

The following gives a criterion for a graded poset of finite height to have locally CAT(0) orthoscheme complex.

Proposition 3.5.7. *Let P be a graded poset of finite height. Then the orthoscheme complex $|P|$ has non-positive curvature if and only if the intervals $P^{\leq x}$ and $P_{\geq x}$ have CAT(0) orthoscheme complexes for each $x \in P$.*

Proof. The assertion follows from [**BM**, Proposition 3.11, Proposition 5.7] and [**BH**, Theorem II.5.2]. □

Corollary 3.5.8. *Let P be a graded poset of finite height, and assume that the intervals $(P^{\leq x})^{\mathrm{op}}$ and $P_{\geq x}$ are locally distributive semilattices for each $x \in P$. Then the orthoscheme complex $|P|$ is a CAT(0) space if and only if the intervals $(P^{\leq x})^{\mathrm{op}}$ and $P_{\geq x}$ are flag semilattices.*

Proof. The assertion immediately follows from Theorem 3.5.4 and the previous proposition. □

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