博士論文(要約)

Analysis for Viscosity Solutions with Special Emphasis on Anomalous Effects
(不規則効果を強調した粘性解析)

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Abstract

In this dissertation we consider homogenization problems and investigate the well-posedness for nonlinear evolutionary partial differential equations (PDEs for short) with substantially different effects from classical cases like a nonlocality, especially Hamilton-Jacobi equations and PDEs with second order degenerate elliptic operators in principal terms. PDEs considered in this dissertation are deeply involved with real phenomenology such as optimal control, crystal growth and anomalous diffusion. In the framework of the theory of viscosity solutions we clarify causes that lead substantial differences from classical results by means of a precise analysis for formulation of a problem or an extension of the notion of viscosity solutions.

In Chapter 1 we consider a homogenization problem for a Hamilton-Jacobi equation whose Hamiltonian is not coercive, which is motivated by a certain phenomenon of crystal growth. Homogenization is a method to estimate macroscopic aspects from microscopic aspects and it is described mathematically as a kind of singular limit problems. A homogenization result for coercive Hamiltonians is well-known. However, a similar proof does not work for non-coercive Hamiltonians and there is no unified solution.

A cell problem is a key problem for this purpose. This is like an eigenvalue problem. It reads: for a given vector P find a unique constant such that a certain (stationary) Hamilton-Jacobi equation admits a continuous viscosity solution. Then the unique constant determined for each P can be regarded as a function of P. Such a function is called an effective Hamiltonian and plays a role of a Hamiltonian of limit equation in an associated homogenization problem. It is known that cell problems are solved for every P if the Hamiltonian is coercive. Based on this classical fact, we consider an associated cell problem with a certain coercive Hamiltonians approximating the original Hamiltonian, and then get an approximate effective Hamiltonian. By introducing a generalized effective Hamiltonian as a limit of the approximate effective Hamiltonian, we characterize a set of vectors P such that cell problems are solved. Throughout several examples in one dimension, we show that the set of P can become $\mathbb{R}$, bounded open interval and empty set.

By using the solvability result of cell problems and the notion of generalized effective Hamiltonian we give a sufficient condition in order that there is a homogenized limit for an associated homogenization problem. We also show that there is a case when a homogenized limit does not exist and give a sufficient condition to occur such a situation.

In Chapter 2 we study the well-posedness for the initial-value problem of Hamilton-Jacobi equations with Caputo's time-fractional derivative (CTFD for short) whose order

is less than one. Partial differential equations with CTFDs have many applications to phenomenon in the real world. Among them, diffusion equations whose time derivatives are replaced by CTFDs which is called anomalous diffusion equations have been widely considered and a researched by using weak solutions in the sense of distribution has been begun to study in [Sakamoto-Yamamoto, '11]. In [Allen, preprint], a regularity problem for a certain equation with CTFD has been considered in terms of a viscosity solution. However, the well-posedness was not mentioned in this paper. Our aim of this research is to construct the synthetic theory of viscosity solutions so that (fully nonlinear) equations with CTFD mentioned in above papers cane be considered. Since second-order equations involves some technical issues, we consider for first-order equations in this chapter, and extend arguments to second-order cases in the next chapter. This research and results are completely new even for first-order equations.

CTFD of a function $u = u(t, x)$ is defined as a convolution of the time derivative $\partial_t u$ and the function $t^{-\alpha}$, where $\alpha \in (0, 1)$ stands for the derivative's oder. Hence the usual definition of viscosity solutions does not apply. The theory of viscosity solutions for spatial nonlocal PDEs, for example, including the fractional Laplacian has already been constructed. However, it is not clear whether or not the definition of viscosity solutions for such an equation is valid for ours since each fractional derivatives are given by different definitions. Therefore, a main issue of this research is to define a (viscosity) solution that ensures a unique existence.

A maximum principle for Caputo's derivatives was established by Luchko. Combining this with a classical maximum principle in the space direction, we get a notion of weak solutions based on the idea of the theory of viscosity solutions. However, the doubling variable method does not work for solutions in this sense. Here the doubling variable method is a standard method used for proofs of comparison principles. To overcome such a difficulty, we integrate the time derivative by parts. Then, the derived function does not have any derivatives of unknowns, so we define a viscosity solution by using it as a substitution of CTFDs. We find that the doubling variable method does work for viscosity solutions in this sense and thus prove a comparison principle (hence uniqueness). Also, an existence is proved by Perron's method similarly as usual. Totally, we establish a unique existence of the initial value problem.

In this chapter we also consider stability and regularity of a solution. For the stability section, in addition to an analogous result of the well-known one for integer-order cases, we show a stability under a limit operation when a time-derivative's order is regarded as a parameter. For the regularity, we show that a solution is Hölder continuous with the same order as time derivative's with respect to time and Lipschitz continuous with respect to space.

We emphasize that assumptions on Hamiltonians and initial data throughout this chapter are standard for an integer-order case. In particular, all results are applicable to transport equations with CTFD

$$\partial_t^\alpha u + \nu(t,x) \cdot Du = 0,$$

where Du is the spatial gradient of u.

In Chapter 3 we study the well-posedness of second order fully nonlinear PDEs with a multi-term CTFD based on considerations in Chapter 2. A multi-term CTFD is a finite sum of CTFDs with positive coefficients and it is often utilized to model more complicated phenomenon in applied fields. A single-term CTFD is also considered as a special case of the multi-term.

A viscosity solution is defined similarly as in Chapter 2. Hence an existence by Perron's method and a stability result are proved by slightly changing proofs. Here a construction of barriers required in Perron's method should be paid attention because of the multi-term CTFD. We show that they are given by a similar function as a single-term case, that is, in Chapter 2.

A main issue is to establish a comparison principle. An important step in a proof of comparison principle is to apply a maximum principle of semicontinuous functions. However, classical results (Crandall-Ishii lemma, Ishii's lemma) do not apply directly due to the nonlocality of Caputo's derivative. For this reason, we prove a relation of semijets for PDEs with CTFDs, which is called Ishii's lemma for integer-order case. The comparison principle is then proved by ideas for classical results of integer-order cases and handling the term of CTFD similar as in Chapter 2.

In Chapter 4 we investigate the well-posedness of initial-boundary value problems of Hamilton-Jacobi equations with CTFD under homogeneous Neumann boundary conditions. For integer-order case, boundary conditions are interpreted in a special sense in order to prove a unique existence of a viscosity solution for initial-boundary value problems. However, a shape of solutions changes depending on time-derivative's orders for equation with Caputo's derivatives. Thus it is not clear at all how to interpret viscosity solutions on boundaries for such equations. In this research we focus on homogeneous Neumann boundary conditions. We then show that a comparison principle and an existence results by Perron's method are proved by interpreting boundary conditions similarly as for integer-order cases.

Finally, in Chapter 5, we consider a homogenization problem of evolutionary Hamilton-Jacobi equations under state-constraint boundary conditions, which is motivated from optimal control problems. Note that Hamiltonian is coercive and time-derivative is first order. Homogenization problems for PDEs with state-constraint boundary conditions were so far considered for perforated domains and for PDEs with divergence forms.

Here the perforated domain is a domain that has holes periodically in microscopic view point. Motivated by them, we consider a similar problem for a non-perforated domain. A difficult point is what cell problems are not found by a simple asymptotic expansion. By considering carefully a procedure of convergence method, we find an associated cell problem. It is well-known that such a cell problem is solved, and thus we obtain a homogenization result. Here the argument for convergence follows the half-relaxed limit method.

Chapter 1, Chapter 2 and Chapter 5 are based on (2), (1) and (3) below, respectively. All Sections, formulas and theorems, etc., are cited only in the chapter where they appear.

(1) Y. Giga, T. Namba, *Well-posedness of Hamilton-Jacobi equations with Caputo's time-fractional derivative*, submitted.

(2) N. Hamamuki, A. Nakayasu, T. Namba, *On cell problems for Hamilton-Jacobi equations with non-coercive Hamiltonians and their application to homogenization problems*, J. Differential Equations 259 (2015) 6672–6693.

(3) T. Namba, *Homogenization problem for Hamilton-Jacobi equations with state-constraint*, Report collection of 36^{th} Young Researchers Seminar on Evolution Equations.

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Contents

Chapter 1

On cell problems for Hamilton-Jacobi equations with non-coercive Hamiltonians and their application to homogenization problems

We study a cell problem arising in homogenization for a Hamilton-Jacobi equation whose Hamiltonian is not coercive. We introduce a generalized notion of effective Hamiltonians by approximating the equation and characterize the solvability of the cell problem in terms of the generalized effective Hamiltonian. Under some sufficient conditions, the result is applied to the associated homogenization problem. We also show that homogenization for non-coercive equations fails in general.

Keywords: Cell Problem; Homogenization; Hamilton-Jacobi Equation; Non-coercive Hamiltonian; Viscosity Solution; Faceted Crystal Growth; Generalized Effective Hamiltonian; Solvability Set

1.1 Introduction

We consider a Hamilton-Jacobi equation of the form

(CP) $$H(x, Du(x) + P) = a \quad \text{in } \mathbf{T}^N$$

and study a problem to find, for a given $P \in \mathbf{R}^N$, a pair of a function $u\colon \mathbf{T}^N \to \mathbf{R}$ and a constant $a \in \mathbf{R}$ such that u is a Lipschitz continuous viscosity solution of (CP). Here, $\mathbf{T}^N := \mathbf{R}^N/\mathbf{Z}^N$ and a function u on $\mathbf{T}^N$ is regarded as a function defined on $\mathbf{R}^N$ with $\mathbf{Z}^N$-periodicity, i.e., $u(x+z) = u(x)$ for all $x \in \mathbf{R}^N$ and $z \in \mathbf{Z}^N$. Moreover, Du denotes the gradient, i.e., $Du = (\partial u/\partial x_1, \cdots, \partial u/\partial x_N)$. This kind of problem is called a *cell problem* in the theory of homogenization. The constant a satisfying (CP) is called a *critical value* if it is uniquely determined.

As a typical example in this paper, we consider the Hamiltonian $H : \mathbf{T}^N \times \mathbf{R}^N \to \mathbf{R}$ in (CP) given by

(1.1.1) $$H(x, p) = \sigma(x) m(|p|),$$

where σ and m satisfy

(H1) $\sigma\colon \mathbf{T}^N \to (0, \infty)$ is a continuous function, (H2) $m\colon [0,\infty) \to (0,1)$ is a Lipschitz continuous function,

(H3) m is strictly increasing and $m(r) \to 1$ as $r \to \infty$.

Due to the boundedness of m, our cell problem does not necessarily admit a solution (u, a), and the solvability depends on $P \in \mathbf{R}^N$. One of goals in this paper is to characterize the set of $P \in \mathbf{R}^N$ such that the cell problem admits a solution. The other goal is to apply the result to the associated homogenization problem.

A result for existence of a solution of cell problems for Hamilton-Jacobi equations was first established by Lions, Papanicolaou and Varadhan [20] under the assumption that the Hamiltonian is *coercive*, i.e.,

$$\lim_{r\to\infty} \inf\{H(x,p) \mid x \in \mathbf{T}^N, p \in \mathbf{R}^N,\ |p| \geq r\} = +\infty. \tag{1.1.2}$$

Their method begins with considering the following approximate equation with a parameter $\delta > 0$:

$$\delta u_\delta(x) + H(x, Du_\delta(x) + P) = 0 \quad \text{in } \mathbf{T}^N. \tag{1.1.3}$$

By a standard argument of viscosity solutions, it turns out that there exists a unique solution u_δ and that a family of functions $\{\delta u_\delta\}_{\delta>0}$ is uniformly bounded. Thus, (formally) $\{Du_\delta\}_{\delta>0}$ is uniformly bounded thanks to the coercivity. Therefore, by taking a subsequence if necessary, δu_δ and $u_\delta - \min u_\delta$ uniformly converge to a constant $-a$ and a function u as $\delta \to 0$, respectively. A stability argument of viscosity solutions shows that u and a solve (CP). For more details, see [20] and [13]. We point out that the paper [13] also studies second order uniformly elliptic equations by using a similar argument.

Unfortunately, our Hamiltonian (1.1.1) is not coercive because of the boundedness of the function m. When a Hamiltonian is not coercive, the method of [20] becomes very delicate. Cardaliaguet [8] shows, in fact, that δu_δ may not converge to a constant; this result does not cover our setting. We also refer the reader to [2] as a related work to [8]. Homogenization results with non-coercive Hamiltonians can be seen in [4, 5, 6, 9, 10, 18, 21, 23, 24]. Hamiltonians with some partial coercivity is studied in [4], and [5] treats equations with u/ε-term. The papers [6, 23, 18] are concerned with homogenization on spaces with a (sub-Riemannian) geometrical condition. The authors of [9] study moving interfaces with a sign changing driving force term while [10, 21, 24] considers G-equations being possibly non-coercive. Homogenization for degenerate second order equations has been developed by [1, 7]. Our Hamiltonian (1.1.1) has not been treated yet in the context of homogenization.

We now present our main results and briefly explain our approach for the non-coercive Hamilton-Jacobi equation (CP). Let us consider an approximate equation of the form

$$H_n(x, Du_n(x) + P) = \bar{H}_n(P) \quad \text{in } \mathbf{T}^N \tag{CP$_n$}$$

for each $n \in \mathbf{N}$. Here $\{H_n\}_{n\in\mathbf{N}}$ is a family of coercive Hamiltonians which approximate H. For the detailed assumptions, see (A1)–(A4) in Section 3. By the coercivity of H_n, the result of [20] ensures that, for each $n \in \mathbf{N}$, the approximate equation has a solution $(u_n, \bar{H}_n(P))$ for every $P \in \mathbf{R}^N$. The function $\bar{H}_n(\cdot)$ is called an *effective Hamiltonian*, which appears in a limit equation in homogenization problems (see [20]). Our first main result is that, for each $P \in \mathbf{R}^N$, there exists a limit $\bar{H}_\infty(P)$ of $\bar{H}_n(P)$ as $n \to \infty$ and its value is independent of approximation (Theorem 1.6). In this paper we call $\bar{H}_\infty(\cdot)$ a *generalized effective Hamiltonian*, which is defined on the whole of $\mathbf{R}^N$ even if (CP) is not solvable for some $P \in \mathbf{R}^N$. We now define the *solvability set* $\mathcal{D}$ as the set of $P \in \mathbf{R}^N$ such that (CP) admits a solution. Our second main result is a characterization of $\mathcal{D}$ in terms of the generalized effective Hamiltonian. We prove that $\mathcal{D} = \{P \in \mathbf{R}^N \mid \bar{H}_\infty(P) < \underline{\sigma}\}$, where $\underline{\sigma} := \min_{x\in\mathbf{T}^N} \sigma(x)$, and that $\bar{H}_\infty(P)$ is equal to the critical value of (CP) (Theorem 1.7). In the one-dimensional case, it turns out that $\mathcal{D}$ has a more explicit representation (Proposition 1.18).

We next present our homogenization results. Let u^ε be a viscosity solution of

$$\text{(HJ}_\varepsilon) \qquad \begin{cases} u_t^\varepsilon(x,t) + H\left(\dfrac{x}{\varepsilon}, Du^\varepsilon(x,t)\right) = 0 & \text{in } \mathbf{R}^N \times (0,T), \\ u^\varepsilon(x,0) = u_0(x) & \text{in } \mathbf{R}^N. \end{cases}$$

Here, $\varepsilon > 0$ is a parameter and $u_0 : \mathbf{R}^N \to \mathbf{R}$ is a bounded and Lipschitz continuous initial datum. In our homogenization result (Theorem 1.21) we assume either

$$(1) \quad \mathcal{D} = \mathbf{R}^N \quad \text{or} \quad (2) \quad m(\mathrm{Lip}[u_0]) < \underline{\sigma}/\overline{\sigma},$$

where $\overline{\sigma} := \max_{x\in\mathbf{T}^N} \sigma(x)$ and $\mathrm{Lip}[u_0]$ stands for the Lipschitz constant of u_0. Then, we prove that u^ε converges to the solution u of the following problem locally uniformly in $\mathbf{R}^N \times [0,T)$ as $\varepsilon \to 0$:

$$\text{(HJ)} \qquad \begin{cases} u_t(x,t) + \bar{H}_\infty(Du(x,t)) = 0 & \text{in } \mathbf{R}^N \times (0,T), \\ u(x,0) = u_0(x) & \text{in } \mathbf{R}^N. \end{cases}$$

The assumption (1) guarantees that the cell problem is solvable for every $P \in \mathbf{R}^N$. The proof is given by the half-relaxed limit method and the perturbed test function method provided by Evans [13]. The assumption (2) is a sufficient condition that $\{u^\varepsilon\}_{\varepsilon>0}$ is equi-Lipschitz continuous. Since the cell problem may not have a solution for some $P \in \mathbf{R}^N$, we are not able to apply the perturbed test function method directly. We prove the homogenization result by reducing the original equation (CP) to the approximate equation (CP$_n$) with a coercive Hamiltonian by using the equi-Lipschitz continuity of $\{u^\varepsilon\}_{\varepsilon>0}$. We also show that, under the condition $\overline{\sigma}m(0) > \underline{\sigma}$, the solutions u^ε do not converge to any function locally uniformly in $\mathbf{R}^N \times [0,T)$ as $\varepsilon \to 0$ (Theorem 1.24).

Our non-coercive Hamiltonian (1.1.1) is originally derived by Yokoyama, Giga and Rybka [25] to study the morphological stability of a faceted crystal. Two functions σ and m represent the rate of supply of molecules and the dimensionless kinetic coefficient, respectively. In [16] and [17] the authors study the large time behavior of a viscosity solution of such non-coercive Hamilton-Jacobi equations.

We conclude this section with the physical explanation of the above homogenization problem and its result. In this problem, we find an average growth of the faceted crystal with a (microscopic) heterogeneous supply of molecules. As we will mention in Subsection 3.3, the cell problem does not have a solution under the condition $\overline{\sigma}m(0) \geq \underline{\sigma}$. Thus, both the assumptions (1) and (2) imply $\overline{\sigma}m(0) < \underline{\sigma}$. This inequality means that the heterogeneity of the supply of molecules is somewhat small. In this case the growth of the faceted crystal is described by (HJ) in view of Theorem 1.21. We point out that the condition $\overline{\sigma} m(0) < \underline{\sigma}$ also appears in [17] to ensure the large time behaviour in the whole space. On the other hand, if $\overline{\sigma} m(0) > \underline{\sigma}$, i.e., the heterogeneity is somewhat large, then the growth of the faceted crystal becomes complicated since homogenization fails (Theorem 1.24).

In this paper, we show main theorems (Theorems 1.6 and 1.7) under (H1)–(H3) for simplicity, but it is possible to generalize a condition on a Hamiltonian and some generalizations are given as Remarks 1.10 and 1.14.

This paper is organized as follows. Section 2 is devoted to preparation for the viscosity solutions and the critical values. We study the cell problem in Section 3 and 4. In Section 3, we present main theorems and prove them. We also give a sufficient condition for $\mathcal{D} = \mathbf{R}^N$ and some properties of generalized effective Hamiltonians. In Section 4, we focus on the one-dimensional cell problem and give a more explicit representation of $\mathcal{D}$. Section 5 is concerned with an application to homogenization problems. In Section 6 we extend the homogenization results for more general equations.

1.2 Preliminaries

In this section let $H : \mathbf{T}^N \times \mathbf{R}^N \to \mathbf{R}$ be a general continuous Hamiltonian.

Let $\mathrm{Lip}(\mathbf{T}^N)$ denote the set of Lipschitz continuous functions on $\mathbf{T}^N$ and $\overline{B(x,r)}$ denote the closure of an open ball $B(x,r)$ of radius $r > 0$ centered at a point x.

We consider Hamilton-Jacobi equations of the form

$$H(x, Du(x)) = 0 \quad \text{in } \mathbf{T}^N. \tag{1.2.1}$$

In order to define viscosity solutions of (1.2.1), we recall notions of super- and subdifferentials. For a continuous function $u : \mathbf{T}^N \to \mathbf{R}$ and $x \in \mathbf{T}^N$, we set

$$D^+u(x) := \left\{ D\phi(x) \;\middle|\; \phi \in C^1(\mathbf{T}^N), \quad \max_{\mathbf{T}^N}(u - \phi) = (u - \phi)(x) \right\}.$$

We also define $D^-u(x)$ by replacing "max" by "min" in the above.

We call $u \in C(\mathbf{T}^N)$ a *viscosity subsolution* (resp. *supersolution*) of (1.2.1) if $H(\hat{x}, p) \le 0$ (resp. $H(\hat{x}, p) \ge 0$) for all $\hat{x} \in \mathbf{T}^N$ and $p \in D^+u(\hat{x})$ (resp. $p \in D^-u(\hat{x})$). If $u \in C(\mathbf{T}^N)$ is a viscosity sub- and supersolution of (1.2.1), we call it a *viscosity solution* of (1.2.1). The term "viscosity" is often omitted in this paper.

A pair of a function $u \in \mathrm{Lip}(\mathbf{T}^N)$ and a constant $a \in \mathbf{R}$ satisfying (CP) is called a solution of (CP). If such a constant a is unique, it is called the *critical value* of (CP). If there exists a critical value of the cell problem for every $P \in \mathbf{R}^N$, then we say that the cell problem is *fully solvable*. When the cell problem is fully solvable, we are able to define a function $\bar{H} : \mathbf{R}^N \to \mathbf{R}$ by setting $\bar{H}(P)$ as the associated critical value. We call the function $\bar{H}$ an *effective Hamiltonian* of H.

Proposition 1.1 (Comparison principle for the cell problem). *Let $P \in \mathbf{R}^N$ and let $a, b \in \mathbf{R}$. If there exist a subsolution $u \in \mathrm{Lip}(\mathbf{T}^N)$ of* (CP) *and a supersolution $v \in \mathrm{Lip}(\mathbf{T}^N)$ of $H(x, Dv(x) + P) = b$ in $\mathbf{T}^N$, then $a \ge b$. In particular, if $(u, c), (v, d) \in \mathrm{Lip}(\mathbf{T}^N) \times \mathbf{R}$ are* *solutions of the cell problem* (CP)*, then* $c = d$ *and moreover*

$$\begin{aligned} c &= \inf\{a \in \mathbf{R} \mid \textit{there exists a subsolution of } (\mathrm{CP})\} \\ &= \sup\{a \in \mathbf{R} \mid \textit{there exists a supersolution of } (\mathrm{CP})\}. \end{aligned}$$

The proof is based on the comparison principle for (1.1.3) with a small $\delta > 0$; see [20, 13]. Here we do not need an extra continuity assumption on H since u and v are now Lipschitz continuous.

Proof. Since u and v are bounded, we may assume that $u > v$ by adding a positive constant to u if necessary. Suppose by contradiction that $a < b$, i.e.,

$$H(x, Du + P) \le a < b \le H(x, Dv + P)$$

in the viscosity sense. We then see that

$$\delta u + H(x, Du + P) \le \frac{a+b}{2} \le \delta v + H(x, Dv + P).$$

The comparison principle implies that $u \le v$, which contradicts to $u > v$. Therefore, $a \ge b$.

This observation implies

$$\begin{aligned} &\inf\{a \in \mathbf{R} \mid \text{there exists a subsolution of (CP)}\} =: \overline{c} \\ &\quad \ge \sup\{a \in \mathbf{R} \mid \text{there exists a supersolution of (CP)}\} =: \underline{c}. \end{aligned}$$

We then see that $c = d = \overline{c} = \underline{c}$ since $\overline{c} \le c \le \underline{c}$ and $\overline{c} \le d \le \underline{c}$ by the definitions. □

Lemma 1.2 (Estimates of the critical value)**.** *Let* $P \in \mathbf{R}^N$ *and let* $(u, c) \in \mathrm{Lip}(\mathbf{T}^N) \times \mathbf{R}$ *be a solution of the cell problem* (CP)*. Then, we have*

$$\left.\begin{array}{l} \displaystyle\sup_{\phi \in C^1(\mathbf{T}^N)} \inf_{x \in \mathbf{T}^N} H(x, D\phi(x) + P) \\ \displaystyle\sup_{x \in \mathbf{T}^N} \sup_{p \in D^+u(x)} H(x, p + P) \end{array}\right\} \le c \le \left\{\begin{array}{l} \displaystyle\inf_{\phi \in C^1(\mathbf{T}^N)} \sup_{x \in \mathbf{T}^N} H(x, D\phi(x) + P), \\ \displaystyle\inf_{x \in \mathbf{T}^N} \inf_{p \in D^-u(x)} H(x, p + P). \end{array}\right.$$

Proof. The inequality $c \le \inf_{x \in \mathbf{T}^N} \inf_{p \in D^-u(x)} H(x, p + P)$ is trivial since it is equivalent to the definition of a viscosity supersolution of (CP). Similarly, the inequality $\sup_{x \in \mathbf{T}^N} \sup_{p \in D^+u(x)} H(x, p + P) \le c$ holds since it is equivalent to the definition of a viscosity subsolution of (CP).

For a fixed $\phi \in C^1(\mathbf{T}^N)$, since $u - \phi$ is periodic and (Lipschitz) continuous, we have $D\phi(\hat{x}) \in D^-u(\hat{x})$ at a minimum point $\hat{x} \in \mathbf{T}^N$ of $u - \phi$. Thus

$$\sup_{x \in \mathbf{T}^N} H(x, D\phi(x) + P) \ge H(\hat{x}, D\phi(\hat{x}) + P) \ge c,$$

which implies that $c \le \inf_{\phi \in C^1(\mathbf{T}^N)} \sup_{x \in \mathbf{T}^N} H(x, D\phi(x) + P)$. In a similar way, we see that $\sup_{\phi \in C^1(\mathbf{T}^N)} \inf_{x \in \mathbf{T}^N} H(x, D\phi(x) + P) \le c$ by choosing a maximum point of $u - \phi$. □

Remark 1.3. It is worth to note that if the Hamiltonian $H = H(x,p)$ is convex in p for each $x \in \mathbf{T}^N$ and satisfies the coercivity condition (1.1.2), then

$$\inf_{\phi\in \mathrm{Lip}(\mathbf{T}^N)} \sup_{x\in\mathbf{T}^N} \sup_{p\in D^+\phi(x)} H(x,p+P) = \inf_{\phi\in C^1(\mathbf{T}^N)} \sup_{x\in\mathbf{T}^N} H(x,D\phi(x)+P).$$

In particular, we have well-known formulas

$$\begin{aligned} c &= \inf_{\phi\in C^1(\mathbf{T}^N)} \sup_{x\in\mathbf{T}^N} H(x,D\phi(x)+P) \\ &= \inf_{\phi\in \mathrm{Lip}(\mathbf{T}^N)} \sup_{x\in\mathbf{T}^N} \sup_{p\in D^+\phi(x)} H(x,p+P) \\ &= \sup_{x\in\mathbf{T}^N} \sup_{p\in D^+u(x)} H(x,p+P). \end{aligned}$$

We refer the reader to [11] or [22, Subsection 4.2] for details on such a kind of representation formulas of the critical value.

We investigate the cell problem with a coercive Hamiltonian.

Proposition 1.4 ([20]). *Assume* (1.1.2)*. Then, the cell problem* (CP) *is fully solvable.*

Proposition 1.5 (Properties of the effective Hamiltonian)**.** *Assume* (1.1.2)*.*

(1) If there exists $L > 0$ such that $|H(x,p) - H(x,q)| \le L|p-q|$ for all $x \in \mathbf{T}^N$, $p,q \in \mathbf{R}^N$, then $\bar{H}$ satisfies $|\bar{H}(P) - \bar{H}(Q)| \le L|P-Q|$ for all $P,Q \in \mathbf{R}^N$.

(2) If $H(x,p) \le H(x,kp)$ for all $x \in \mathbf{T}^N$, $p \in \mathbf{R}^N$ and $k \ge 1$, then $\bar{H}(P) \le \bar{H}(kP)$ for all $P \in \mathbf{R}^N$ and $k \ge 1$.

(3) If $H(x,p) = H(x,-p)$ for all $x \in \mathbf{T}^N$ and $p \in \mathbf{R}^N$, then $\bar{H}(P) = \bar{H}(-P)$ for all $P \in \mathbf{R}^N$.

Proof. (1) Let $(u, \bar{H}(P))$ be a solution of (CP). We observe

$$H(x,Du+Q) - L|P-Q| \le H(x,Du+P) = \bar{H}(P).$$

Thus, u is a subsolution of

$$H(x,Du+Q) = \bar{H}(P) + L|P-Q|.$$

By Proposition 1.1, we obtain that $\bar{H}(Q) \le \bar{H}(P) + L|P-Q|$.

(2) Let $(u, \bar{H}(P))$ be a solution of (CP). We then see by the assumption that

$$H(x,D(ku)+kP) \ge H(x,Du+P) = \bar{H}(P),$$

which means that ku is a supersolution of $H(x,Dv+kP) = \bar{H}(P)$. Proposition 1.1 implies $\bar{H}(kP) \ge \bar{H}(P)$.

(3) Let $(u, \bar{H}(P))$ be a solution of (CP). Then, since H is even in the second variable, $(-u, \bar{H}(P))$ is a solution of

$$H(x,Dv-P) = \bar{H}(P) \quad \text{in } \mathbf{T}^N.$$

Thus, we have $\bar{H}(P) = \bar{H}(-P)$. □

1.3 The cell problem

From now on, we study a Hamiltonian H of the form (1.1.1) with (H1)–(H3). Define

$$\overline{\sigma} := \sup_{x \in \mathbf{T}^N} \sigma(x), \quad \underline{\sigma} := \inf_{x \in \mathbf{T}^N} \sigma(x), \quad m_0 := m(0).$$

We note that (H3) ensures $m_0 = \min_{r \in [0,\infty)} m(r)$.

1.3.1 Main results

For each $n \in \mathbf{N}$ let $H_n : \mathbf{T}^N \times \mathbf{R}^N \to \mathbf{R}$ be an approximating Hamiltonian of H such that

(A1) H_n is continuous on $\mathbf{T}^N \times \mathbf{R}^N$,

(A2) H_n satisfies the coercivity condition (1.1.2),

(A3) $\liminf_{n\to\infty} \inf_{\mathbf{T}^N \times \overline{B(0,R)}} (H - H_n) \geq 0$ for all $R > 0$,

(A4) $\limsup_{n\to\infty} \sup_{\mathbf{T}^N \times \mathbf{R}^N} (H - H_n) \leq 0$.

A typical approximation will be given in Remark 1.8. By (A1) and (A2), for each $n \in \mathbf{N}$, the approximation cell problem (CP$_n$) is fully solvable as noted in Proposition 1.4. Let $\bar{H}_n(P)$ be the critical value of (CP$_n$) for $P \in \mathbf{R}^N$. We define a solvability set $\mathcal{D}$ by

$$\mathcal{D} := \{P \in \mathbf{R}^N \mid \text{(CP) admits a solution } (u, c) \in \mathrm{Lip}(\mathbf{T}^N) \times \mathbf{R}\}.$$

We are now in a position to state our main theorems.

Theorem 1.6 (Convergence of $\bar{H}_n$). *There exists a unique function $\bar{H}_\infty : \mathbf{R}^N \to \mathbf{R}$ such that, for any sequence $\{H_n\}_{n\in\mathbf{N}}$ satisfying (A1)–(A4), the following conditions hold:*

$$\liminf_{n\to\infty} \inf_{\overline{B(0,R)}} (\bar{H}_\infty - \bar{H}_n) \geq 0 \quad \textit{for all } R > 0, \tag{1.3.1}$$

$$\limsup_{n\to\infty} \sup_{\mathbf{R}^N} (\bar{H}_\infty - \bar{H}_n) \leq 0. \tag{1.3.2}$$

We call the function $\bar{H}_\infty$ a *generalized effective Hamiltonian* of H.

Theorem 1.7 (Characterization of the solvability set). *We have $\mathcal{D} = \{P \in \mathbf{R}^N \mid \bar{H}_\infty(P) < \underline{\sigma}\}$. Moreover, if $P \in \mathcal{D}$, the critical value of* (CP) *is equal to $\bar{H}_\infty(P)$.*

1.3.2 The proof of Theorem 1.6

The proof of Theorem 1.6 consists of five steps. We first prove in Step 1 that $\{\bar{H}_n(P)\}_{n\in\mathbf{N}}$ is a convergent sequence for every $P \in \mathbf{R}^N$. Then it is shown in Step 2 that the limit is unique no matter how $\{H_n\}_{n\in\mathbf{N}}$ satisfying (A1)–(A4) is chosen. In Step 3 we shall derive some properties including continuity of the generalized effective Hamiltonian. This continuity will improve the convergence of $\bar{H}_n$. In Step 4 we prove that the convergence is locally uniform when $\{H_n\}_{n\in\mathbf{N}}$ is monotone, and finally, in Step 5, we derive (1.3.1) and (1.3.2) for a general approximation.

Step 1. Fix any $P \in \mathbf{R}^N$ and let $(u_n, \bar{H}_n(P)) \in \mathrm{Lip}(\mathbf{T}^N) \times \mathbf{R}$ be a solution of (CP$_n$) for each $n \in \mathbf{N}$. We first show that $\{\bar{H}_n(P)\}_{n\in\mathbf{N}}$ is bounded from below. Indeed, taking a maximum point $x_n \in \mathbf{T}^N$ of u_n, we have

$$H_n(x_n, P) \le \bar{H}_n(P).$$

Since H_n uniformly converges to H on $\mathbf{T}^N \times \overline{B(0,|P|)}$, we see that

$$H(x_n, P) - 1 \le \bar{H}_n(P)$$

for sufficiently large n. Thus

$$\left(\inf_{x\in\mathbf{T}^N} H(x,P)\right) - 1 \le \bar{H}_n(P),$$

which implies $\{\bar{H}_n(P)\}_{n\in\mathbf{N}}$ is bounded from below.

Fix $\varepsilon > 0$. By (A4) there exists some $K \in \mathbf{N}$ such that

$$H - \frac{\varepsilon}{2} \le H_n \quad \text{on } \mathbf{T}^N \times \mathbf{R}^N \tag{1.3.3}$$

for all $n \ge K$. Fix an arbitrary $n \ge K$. Recall that u_n is a Lipschitz continuous function and set $L_n = |P| + \mathrm{Lip}[u_n]$. Then, it follows from (A3) that there exists some $M \ge n$ such that

$$H_m - \frac{\varepsilon}{2} \le H \quad \text{on } \mathbf{T}^N \times \overline{B(0, L_n)} \tag{1.3.4}$$

for all $m \ge M$. Combining (1.3.3) and (1.3.4), we see that u_n is a subsolution of

$$H_m(x, Dw + P) = \bar{H}_n(P) + \varepsilon \quad \text{in } \mathbf{T}^N.$$

By Proposition 1.1, we have

$$\bar{H}_m(P) \le \bar{H}_n(P) + \varepsilon \tag{1.3.5}$$

for all $m \ge M$. This inequality implies that $\{\bar{H}_n(P)\}_{n\in\mathbf{N}}$ is bounded from above. By taking $\limsup_{m\to\infty}$ and $\liminf_{n\to\infty}$, where $\limsup_{m\to\infty}$ should be operated first since M depends on n, we have

$$\limsup_{m\to\infty} \bar{H}_m(P) \le \liminf_{n\to\infty} \bar{H}_n(P) + \varepsilon.$$

Since $\varepsilon > 0$ is arbitrary, $\bar{H}_n(P)$ converges to some value as $n \to \infty$.

Step 2. We next prove that the limit of $\bar{H}_n(P)$ is independent of a choice of $\{H_n\}_{n\in\mathbf{N}}$ satisfying (A1)–(A4). Let $\{H_n\}_{n\in\mathbf{N}}$ and $\{H'_n\}_{n\in\mathbf{N}}$ be two sequences of Hamiltonians satisfying (A1)–(A4). For each $P \in \mathbf{R}^N$, let $(u_n, \bar{H}_n(P))$ and $(u'_n, \bar{H}'_n(P))$ be, respectively, solutions of (CP$_n$) and

$$H'_n(x, Du'_n + P) = \bar{H}'_n(P) \quad \text{in } \mathbf{T}^N.$$

Consider a new sequence

$$H_1, H_1', H_2, H_2', H_3, H_3', \cdots .$$

This satisfies (A3) and (A4), so that

$$\bar{H}_1(P), \bar{H}_1'(P), \bar{H}_2(P), \bar{H}_2'(P), \bar{H}_3(P), \bar{H}_3'(P), \cdots$$

has a limit $a \in \mathbf{R}$. Therefore

$$a = \lim_{n\to\infty} \bar{H}_n(P) = \lim_{n\to\infty} \bar{H}_n'(P)$$

since both $\{\bar{H}_n(P)\}_{n\in\mathbf{N}}$ and $\{\bar{H}_n'(P)\}_{n\in\mathbf{N}}$ are subsequences. We denote this common limit by $\bar{H}_\infty(P)$.

Remark 1.8. Thanks to the uniqueness of the pointwise limit we are able to take a specific approximation. Let us take an approximating Hamiltonian H_n of the form

$$H_n(x, p) = \sigma(x) M_n(|p|),$$

where $M_n : [0, \infty) \to [m_0, \infty)$ is an approximating function of m such that

(B1) M_n is Lipschitz continuous,

(B2) $M_n(r) \to \infty$ as $r \to \infty$,

(B3) there exists $\alpha_n \in \mathbf{R}$ such that

$$M_n(r) = m(r) \quad \text{for } r \in [0, \alpha_n], \quad M_n(r) > m(r) \quad \text{for } r \in (\alpha_n, \infty),$$

for each $n \in \mathbf{N}$ and $\alpha_n \to \infty$ as $n \to \infty$,

(B4) $M_n(r) \geq M_{n'}(r)$ for all $n' \geq n$ and $r \in [0, \infty)$.

For instance,

$$M_n(r) = \max\{m(r), Lr - n\} \tag{1.3.6}$$

satisfies (B1)–(B4), where L is the Lipschitz constant of m.

Step 3. We next derive some properties of the generalized effective Hamiltonian. Among them the continuity given in (1) will be required in order to improve the convergence of $\bar{H}_n$.

Proposition 1.9 (Properties of the generalized effective Hamiltonian). *We have*

(1) $|\bar{H}_\infty(P) - \bar{H}_\infty(Q)| \leq \overline{\sigma} L |P - Q|$ *for all* $P, Q \in \mathbf{R}^N$*, where* L *is the Lipschitz constant of* m*,*

(2) $\bar{H}_\infty(kP) \geq \bar{H}_\infty(P)$ *for all* $P \in \mathbf{R}^N$ *and* $k \geq 1$*,*

(3) $\bar{H}_\infty(P) = \bar{H}_\infty(-P)$ *for all* $P \in \mathbf{R}^N$*,*

(4) $\max\{\underline{\sigma} m(|P|), \overline{\sigma} m_0\} \leq \bar{H}_\infty(P) \leq \overline{\sigma} m(|P|) < \overline{\sigma}$ *for all* $P \in \mathbf{R}^N$*.*

Proof. Take H_n as in Remark 1.8, where we set M_n by (1.3.6). Let $\bar{H}_n$ be the effective Hamiltonian of H_n. We then have

$$|H_n(x,p) - H_n(x,q)| \leq \overline{\sigma}L|p-q| \quad \text{for all } x \in \mathbf{T}^N,\, p,q \in \mathbf{R}^N.$$

Hence, Proposition 1.5 (1) shows

$$|\bar{H}_n(P) - \bar{H}_n(Q)| \leq \overline{\sigma}L|P-Q| \quad \text{for all } P,Q \in \mathbf{R}^N.$$

Sending $n \to \infty$ yields the conclusion (1).

By a similar argument the properties (2)–(3) are verified from Proposition 1.5 since our coercive Hamiltonians H_n satisfy the assumptions of Proposition 1.5 (2)–(3). The property (4) is a consequence of Lemma 1.2. □

Step 4. Assume that $\{H_n\}_{n\in\mathbf{N}}$ is monotone, i.e., $H_n \geq H_{n'}$ on $\mathbf{T}^N \times \mathbf{R}^N$ for all $n \leq n'$. By this monotonicity we see that $\bar{H}_n \geq \bar{H}_{n'}$ if $n \leq n'$. Indeed, a solution u_n of (CP$_n$) is always a subsolution of

$$H_{n'}(x, Du_n + P) = \bar{H}_n(P)$$

for $n' \geq n$. Thus Proposition 1.1 yields $\bar{H}_n(P) \geq \bar{H}_{n'}(P)$. Since $\bar{H}_\infty$ is continuous in view of Proposition 1.9 (1), Dini's lemma implies that $\bar{H}_n$ converges to $\bar{H}_\infty$ locally uniformly in $\mathbf{R}^N$ as $n \to \infty$. (For the proof of Proposition 1.9 (1) we only need a pointwise convergence of $\bar{H}_n$ to $\bar{H}_\infty$ and the uniqueness of $\bar{H}_\infty$.)

Step 5. We shall show (1.3.1) and (1.3.2) for a general $\{H_n\}_{n\in\mathbf{N}}$. Sending $m \to \infty$ in (1.3.5) of Step 1, we obtain

$$\bar{H}_\infty(P) \leq \bar{H}_n(P) + \varepsilon.$$

This inequality holds for all $\varepsilon > 0$, $n \geq K$ and $P \in \mathbf{R}^N$, where K does not depend on P. Accordingly we have $\sup_{\mathbf{R}^N}(\bar{H}_\infty - \bar{H}_n) \leq \varepsilon$, and thus taking $\limsup_{n\to\infty}$ yields (1.3.2) since $\varepsilon > 0$ is arbitrary.

To prove (1.3.1) we define $\{H'_n\}_{n\in\mathbf{N}}$ by $H'_n(x,p) := \sup_{m\geq n} H_m(x,p)$. Then $\{H'_n\}_{n\in\mathbf{N}}$ is monotone and $H'_n \geq H_n$ on $\mathbf{T}^N \times \mathbf{R}^N$ for all n. Also, $\{H'_n\}_{n\in\mathbf{N}}$ satisfies (A1)–(A4); it is easy to see that (A2)–(A4) hold while the continuity condition (A1) is due to Ascoli-Arzelà theorem, which asserts that, for a compact set $K \subset \mathbf{R}^N$, a sequence $\{f_n\}_{n\in\mathbf{N}} \subset C(K)$ is uniformly bounded and equicontinuous if and only if every subsequence of $\{f_n\}$ has a uniformly convergent subsequence. We apply the if-part of this theorem to see that $\{H_n\}$ is equi-continuous on each compact set of $\mathbf{T}^N \times \mathbf{R}^N$ since $H_n \to H$ uniformly on the set. Therefore the supremum H'_n is continuous. From Step 4 it follows that $\bar{H}'_n$ converges to $\bar{H}_\infty$ locally uniformly. Therefore, using $\bar{H}'_n \geq \bar{H}_n$, we observe

$$\liminf_{n\to\infty} \inf_{B(0,R)} (\bar{H}_\infty - \bar{H}_n) \geq \liminf_{n\to\infty} \inf_{B(0,R)} (\bar{H}_\infty - \bar{H}'_n) = 0.$$

The proof is now complete.

Remark 1.10. Theorem 1.6 still holds for more general, continuous Hamiltonians which are not necessarily of the form (1.1.1). Indeed, the above proof works if we require H

to satisfy

$$
|H(x,p) - H(x,q)| \leq L|p-q| \quad \text{for some } L > 0, \tag{1.3.7}
$$

which is used to guarantee Proposition 1.9 (1).

1.3.3 The proof of Theorem 1.7

We first prepare

Proposition 1.11. *Let $P \in \mathcal{D}$ and let $c \in \mathbf{R}$ be the critical value of* (CP). *Then,*

$$
\left.\begin{array}{r} \underline{\sigma} m(|P|) \\ \overline{\sigma} m_0 \end{array}\right\} \leq c \leq \overline{\sigma} m(|P|), \quad c < \underline{\sigma}.
$$

In particular, we have $\mathcal{D} = \emptyset$ if $\overline{\sigma} m_0 \geq \underline{\sigma}$.

Proof. Taking $\phi \equiv 0$ in Lemma 1.2 implies

$$
\begin{aligned}
c &\leq \inf_{\phi \in C^1(\mathbf{T}^N)} \sup_{x \in \mathbf{T}^N} H(x, D\phi(x) + P) \leq \sup_{x \in \mathbf{T}^N} H(x,P) = \overline{\sigma} m(|P|), \\
c &\geq \sup_{\phi \in C^1(\mathbf{T}^N)} \inf_{x \in \mathbf{T}^N} H(x, D\phi(x) + P) \geq \inf_{x \in \mathbf{T}^N} H(x,P) = \underline{\sigma} m(|P|).
\end{aligned}
$$

We next show $c < \underline{\sigma}$. Take a solution $u \in \mathrm{Lip}(\mathbf{T}^N)$ of (CP). For every $x \in A^- := \{x \in \mathbf{T}^N \mid D^- u(x) \neq \emptyset\}$, take $p \in D^- u(x)$. Since $|p| \leq \mathrm{Lip}[u]$, we have

$$
\inf_{p \in D^- u(x)} H(x, p+P) \leq H(x, p+P) \leq \sigma(x) m(\mathrm{Lip}[u] + |P|).
$$

Therefore, by Lemma 1.2,

$$
\begin{aligned}
c &\leq \inf_{x \in \mathbf{T}^N} \inf_{p \in D^- u(x)} H(x, p+P) \leq \inf_{x \in A^-} \inf_{p \in D^- u(x)} H(x, p+P) \\
&\leq \inf_{x \in A^-} \sigma(x) m(\mathrm{Lip}[u] + |P|).
\end{aligned}
$$

According to [3, Lemma 1.8 (d)], the set A^- is dense in $\mathbf{T}^N$. Thus, we obtain $c \leq \underline{\sigma} m(\mathrm{Lip}[u] + |P|) < \underline{\sigma}$. The proof of the inequality $\overline{\sigma} m_0 \leq c$ is easier. □

Let $\widehat{\mathcal{D}} := \{P \in \mathbf{R}^N \mid \bar{H}_\infty(P) < \underline{\sigma}\}$. We note that $\mathcal{D} = \widehat{\mathcal{D}} = \emptyset$ when $\overline{\sigma} m_0 \geq \underline{\sigma}$. Indeed, Proposition 1.11 implies $\mathcal{D} = \emptyset$ and it is easy to check $\widehat{\mathcal{D}} = \emptyset$ since Proposition 3.4 (4) yields $\bar{H}_\infty(P) \geq \underline{\sigma} m(|P|) \geq \underline{\sigma} m_0 \geq \overline{\sigma}$ for all $P \in \mathbf{R}^N$. We may hereafter assume that $\overline{\sigma} m_0 < \underline{\sigma}$. Let us take the special approximating Hamiltonian $H_n(x,p) = \sigma(x) M_n(|p|)$ with the conditions (B1)–(B4) in Remark 1.8.
<u>**Proof of $\mathcal{D} \supset \widehat{\mathcal{D}}$**</u>. We define

$$
\mathcal{D}_\ell := \{P \in \mathbf{R}^N \mid \bar{H}_\ell(P) \leq \underline{\sigma} m(\alpha_\ell)\}
$$

for $\ell \in \mathbf{N}$. Note that $\cup_{\ell=0}^\infty \mathcal{D}_\ell = \widehat{\mathcal{D}}$. It is easy to check this equation since $\bar{H}_\ell(P) \to \bar{H}_\infty(P)$ as $\ell \to \infty$ with $\bar{H}_\ell(P) \geq \bar{H}_\infty(P)$ and $\underline{\sigma} m(\alpha_\ell) \to \underline{\sigma}$ as $\ell \to \infty$ with $\underline{\sigma} m(\alpha_\ell) < \underline{\sigma}$.

Therefore, if $\mathcal{D} \supset \mathcal{D}_\ell$ for every $\ell \in \mathbf{N}$, then we will have $\mathcal{D} \supset \cup_{\ell=0}^\infty \mathcal{D}_\ell = \widehat{\mathcal{D}}$.

Fix any $P \in \mathcal{D}_\ell$ and let $(u_n, \bar{H}_n(P))$ be a solution of (CP$_n$). Note that $\bar{H}_n(P)$ is monotone decreasing with respect to n by (B4) and Proposition 1.1. For each $n \in \mathbf{N}$ such that $n \geq \ell$,

$$M_n(|Du_n + P|) = \frac{\bar{H}_n(P)}{\sigma(x)} \leq \frac{\bar{H}_\ell(P)}{\underline{\sigma}} \leq m(\alpha_\ell) \quad \text{in } \mathbf{T}^N$$

in the viscosity sense. Note that the last inequality follows from $P \in \mathcal{D}_\ell$. Since $m \leq M_n$ on $[0, \infty)$ and m is strictly increasing, we have

$$|Du_n(x)| \leq \alpha_\ell + |P| \quad \text{in } \mathbf{T}^N$$

in the viscosity sense. Thus,

$$\sup_{n \geq \ell} \mathrm{Lip}[u_n] \leq \alpha_\ell + |P| < \infty.$$

Set $v_n(y) := u_n(y) - \min u_n$. Then, $\{v_n\}_{n \in \mathbf{N}}$ is uniformly bounded and equi-Lipschitz continuous in $\mathbf{T}^N$. Thus, by taking a subsequence if necessary, Ascoli-Arzelà theorem implies that v_n uniformly converges to some Lipschitz continuous function u in $\mathbf{T}^N$ as $n \to \infty$. Since M_n converges to m locally uniformly in $[0, \infty)$ by (B3) of M_n, the stability of viscosity solutions (see [12]) implies that $(u, \bar{H}_\infty(P))$ is a solution of (CP), which means that $P \in \mathcal{D}$. We get the desired inclusion $\mathcal{D} \supset \widehat{\mathcal{D}}$.
Proof of $\mathcal{D} \subset \widehat{\mathcal{D}}$. Fix any $P \in \mathcal{D}$ and let $(u, c) \in$Lip$(\mathbf{T}^N) \times \mathbf{R}$ be a solution of (CP). The condition (B3) of M_n implies

$$M_n(r) = m(r) \quad \text{for all } r \leq \mathrm{Lip}[u] + |P|$$

for sufficiently large n. Hence, (u, c) is a solution of

$$\sigma(x) M_n(|Du + P|) = c \quad \text{in } \mathbf{T}^N.$$

Since $\bar{H}_n(P)$ is the critical value of the above problem, we have $\bar{H}_n(P) = c$. Sending $n \to \infty$ yields $\bar{H}_\infty(P) = c$. Since $c < \underline{\sigma}$ by Proposition 1.11, we have $\bar{H}_\infty(P) < \underline{\sigma}$. Thus, $\mathcal{D} \subset \widehat{\mathcal{D}}$. The proof of Theorem 1.7 is complete.

Remark 1.12. By the last part of the proof, we see that for every $R > 0$ there exists $N_R \in \mathbf{N}$ such that $\bar{H}_\infty(P) = \bar{H}_n(P)$ for all $P \in \overline{B(0, R)}$ and $n \geq N_R$. This is thanks to the conditions (B1)–(B4).

Remark 1.13. Applying a method in the proof of Proposition 1.11 to the approximating Hamiltonians H_n above, we see $\max\{\underline{\sigma} M_n(|P|), \overline{\sigma} M_n(0)\} \leq \bar{H}_n(P) \leq \overline{\sigma} M_n(|P|)$. Since $M_n(0) = m(0)$, letting $P = 0$ gives $\bar{H}_n(0) = \overline{\sigma} m_0$, so that $\bar{H}_\infty(0) = \overline{\sigma} m_0$. Thus Theorem 1.7 implies that $0 \in \mathcal{D}$ if $\overline{\sigma} m_0 < \underline{\sigma}$. Moreover, from the Lipschitz continuity of $\bar{H}_\infty$ (Proposition 1.9 (1)) it follows that $B(0, (\underline{\sigma} - \overline{\sigma} m_0)/\overline{\sigma} L) \subset \mathcal{D}$, where L is the Lipschitz constant of m.

Remark 1.14. A similar proof applies to more general Hamiltonians. Let H be a Hamiltonian satisfying (1.3.7). We define

$$h(\rho) := \inf_{x \in \mathbf{T}^N} \inf_{|p| \geq \rho} H(x, p), \quad h_\infty := \sup_{\rho \geq 0} h(\rho),$$

and assume

(H4) $\inf_{x\in\mathbf{T}^N}\sup_{|p|\le\rho} H(x,p) < h_\infty$ for all $\rho \ge 0$.

Then it turns out that $\mathcal{D} = \{P \in \mathbf{R}^N \mid \bar{H}_\infty(P) < h_\infty\}$. We shall give a sketch of the proof of this generalization.

We first show that the critical value c of (CP) satisfies $c < h_\infty$. In a similar way to the proof of Proposition 1.11, we see

$$c \le \inf_{x\in A^-}\inf_{p\in D^-u(x)} H(x,p+P) \le \inf_{x\in A^-}\sup_{|p|\le\rho} H(x,p)$$

with $\rho := \mathrm{Lip}[u] + |P|$, where $u \in \mathrm{Lip}(\mathbf{T}^N)$ is a solution of (CP) and $A^- = \{x \in \mathbf{T}^N \mid D^-u(x) \ne \emptyset\}$. Since $\sup_{|p|\le\rho} H(\cdot,p)$ is continuous by the compactness of $\overline{B(0,\rho)}$ and since A^- is dense in $\mathbf{T}^N$, using (H4), we estimate

$$c \le \inf_{x\in\mathbf{T}^N}\sup_{|p|\le\rho} H(x,p) < h_\infty.$$

Next, we see that $\{p \in \mathbf{R}^N \mid H(x,p) \le \tau \text{ for some } x \in \mathbf{T}^N\}$ is bounded for every $\tau < h_\infty$. Indeed, if there were some sequence $\{(x_j,p_j)\}_{j\in\mathbf{N}}$ such that $|p_j| \to \infty$ as $j \to \infty$, we would have $h(|p_j|) \le H(x_j,p_j) \le \tau < h_\infty$, which is a contradiction since $\sup_{j\in\mathbf{N}} h(|p_j|) < h_\infty$.

Define $\widehat{\mathcal{D}} := \{P \in \mathbf{R}^N \mid \bar{H}_\infty(P) < h_\infty\}$, and take an approximate Hamiltonian H_n as $H_n(x,p) = \max\{H(x,p), |p| - n\}$. To prove $\mathcal{D} \supset \widehat{\mathcal{D}}$ we set $\mathcal{D}_\ell := \{P \in \mathbf{R}^N \mid \bar{H}_\ell(P) \le \tau_\ell\}$, where $\{\tau_\ell\}_{\ell\in\mathbf{N}}$ is a sequence such that $\tau_\ell < h_\infty$ and $\tau_\ell \to h_\infty$ as $\ell \to \infty$. Then $\bigcup_{\ell=1}^\infty \mathcal{D}_\ell = \widehat{\mathcal{D}}$. Fix $\ell \in \mathbf{N}$. For every $P \in \mathcal{D}_\ell$ and $n \ge l$, a solution $(u_n, \bar{H}_n(P))$ of (CP$_n$) satisfies

$$H(x, Du_n + P) \le H_n(x, Du_n + P) = \bar{H}_n(P) \le \bar{H}_\ell(P) \le \tau_\ell.$$

Since $\tau_\ell < h_\infty$, we have $\sup_{n\ge\ell}\mathrm{Lip}[u_n] < \infty$. Ascoli-Arzelà theorem ensures that $u_n - \min u$ subsequently converges to some u, and thus $(u, \bar{H}_\infty(P))$ solves (CP). The proof of $\mathcal{D} \subset \widehat{\mathcal{D}}$ is easier. Indeed, by the choice of H_n, a solution (u,c) of (CP) is also a solution of (CP$_n$) for n sufficiently large, and therefore $\bar{H}_\infty(P) = \bar{H}_n(P) = c < h_\infty$.

1.3.4 A sufficient condition for the fully solvability

Applying the result in Theorem 1.7, we give a sufficient condition which guarantees that (CP) is fully solvable, i.e., $\mathcal{D} = \mathbf{R}^N$.

Theorem 1.15. *Assume $\overline{\sigma}m_0 < \underline{\sigma}$. Let $P \in \mathbf{R}^N$ and assume that there exists $\psi \in C^1(\mathbf{T}^N)$ such that $D\psi = -P$ on $\{\sigma \ne \underline{\sigma}\}$. Then $P \in \mathcal{D}$.*

If there exists such a ψ for every $P \in \mathbf{R}^N$, then (CP) is fully solvable. A simple condition for the existence of ψ will be given after the proof; see Remark 1.16.

Proof. We take H_n as in Remark 1.8. By the representation of $\mathcal{D}$ obtained in Theorem 1.7, the proof is completed by showing that $\bar{H}_n(P) < \underline{\sigma}$ for $n \in \mathbf{N}$ sufficiently large. To this end, we use the estimate

$$\bar{H}_n(P) \le \inf_{\phi\in C^1(\mathbf{T}^N)}\sup_{x\in\mathbf{T}^N} H_n(x, D\phi(x) + P)$$

in Lemma 1.2. Choosing $\phi = \psi$, where ψ is the function in our assumption, we see

$$\bar{H}_n(P) \leq \sup_{x \in \mathbf{T}^N} H_n(x, D\psi(x) + P). \tag{1.3.8}$$

On $\{\sigma \neq \underline{\sigma}\}$ we compute

$$H_n(x, D\psi(x) + P) = H_n(x, 0) = \sigma(x) m_0 \leq \overline{\sigma} m_0 < \underline{\sigma}.$$

For $x \in \mathbf{T}^N$ such that $\sigma(x) = \underline{\sigma}$, we have

$$H_n(x, D\psi(x) + P) = \underline{\sigma} M_n(|D\psi(x) + P|).$$

We now set $r_0 := \max_{x \in \mathbf{T}^N} |D\psi(x) + P| < \infty$ and choose n large so that $M_n(r) = m(r)$ for all $r \leq r_0$. Then

$$H_n(x, D\psi(x) + P) \leq \underline{\sigma} m(r_0) < \underline{\sigma}.$$

Consequently, (1.3.8) implies $\bar{H}_n(P) < \underline{\sigma}$. □

Remark 1.16. If $\overline{\{\sigma \neq \underline{\sigma}\}} \subset (0,1)^N$, then there exists ψ in Theorem 1.15 for every $P \in \mathbf{R}^N$. Indeed, letting $A \subset (0,1)^N$ be an open set such that $\overline{\{\sigma \neq \underline{\sigma}\}} \subset A$ and $\overline{A} \subset (0,1)^N$, we are able to construct a function $\psi \in C^1(\mathbf{T}^N)$ so that $\psi(x) = -\langle P, x \rangle$ for $x \in \{\sigma \neq \underline{\sigma}\}$ and $\psi(x) = 0$ for $x \notin A$. The existence of such a ψ is due to Whitney's extension theorem; see, e.g., [14, Section 6.5, Theorem 1]. Let us briefly check the assumption in [14]. Let $f : [0,1]^N \to \mathbf{R}$ and $d : [0,1]^N \to \mathbf{R}^N$ be continuous functions such that $f(x) = -\langle P, x \rangle$, $d(x) = -P$ on $K_1 := \overline{\{\sigma \neq \underline{\sigma}\}}$ and $f(x) = d(x) = 0$ on $K_2 := [0,1]^N \setminus A$. Also, define $K := K_1 \cup K_2$ and $\delta_0 := \operatorname{dist}(K_1, K_2) > 0$. If $x, y \in K$ satisfy $|x - y| < \delta_0$, we have $x, y \in K_1$ or $x, y \in K_2$, and hence $R(y, x) := (f(y) - f(x) - \langle d(x), y - x \rangle)/|x - y| = 0$. The theorem is thus applicable.

Remark 1.17. The existence of ψ in Theorem 1.15 is not a necessary condition for $P \in \mathcal{D}$. In Example 1.2 (1), where we consider the one-dimensional case, the cell problem is fully solvable, but there is no such periodic ψ for $P \neq 0$ because σ attains a minimum at one point.

1.4 One-dimensional cell problem

In this section we investigate the cell problem in one dimension. In this case the solvability set $\mathcal{D}$ has a more explicit representation. We first rewrite (CP) as

$$|u'(x) + P| = f_a(x) \quad \text{in } \mathbf{T}, \tag{1.4.1}$$

where

$$f_a(x) := m^{-1}\left(\frac{a}{\sigma(x)}\right).$$

Here, $m^{-1} : [m_0, 1) \to [0, \infty)$ is the inverse function of m, and f_a is well-defined as a $[0,\infty)$-valued function if $\overline{\sigma} m_0 \leq a < \underline{\sigma}$. We now set $m^{-1}(1) = \infty$. Then, $f_{\underline{\sigma}}$ is a $[0,\infty]$-valued function. Note that $a \mapsto f_a(x)$ is increasing for every $x \in \mathbf{T}$.

The authors of [20] consider

$$(1.4.2) \qquad |u'(x)+P|^2 - V(x) = a \quad \text{in } \mathbf{T},$$

as an example of the cell problem in one dimension. Here, V is a continuous function on $\mathbf{T}$ such that $\min_{\mathbf{T}} V = 0$. According to [20], for each $P \in \mathbf{R}$, the critical value c of (1.4.2) is given by

$$(1.4.3) \qquad c = \begin{cases} 0 & \text{if } |P| \le \int_0^1 \sqrt{V(z)}dz, \\ a \quad \text{such that } |P| = \int_0^1 \sqrt{V(z)+a}dz,\ a \ge 0, & \text{otherwise.} \end{cases}$$

As an analogue of this formula, we establish

Proposition 1.18. *(1) If $\overline{\sigma} m_0 \ge \underline{\sigma}$, then $\mathcal{D} = \emptyset$.*

(2) If $\overline{\sigma} m_0 < \underline{\sigma}$, then

$$\mathcal{D} = \begin{cases} (-\int_0^1 f_{\underline{\sigma}}(z)dz, \int_0^1 f_{\underline{\sigma}}(z)dz) & \textit{if } f_{\underline{\sigma}} \in L^1(0,1), \\ \mathbf{R} & \textit{otherwise}. \end{cases}$$

Moreover, the critical value c is given by

$$(1.4.4) \qquad c = \begin{cases} \overline{\sigma} m_0 & \textit{if } |P| \le \int_0^1 f_{\overline{\sigma} m_0}(z)dz, \\ a \quad \textit{such that } |P| = \int_0^1 f_a(z)dz & \textit{otherwise.} \end{cases}$$

Proof. (1) This is obvious by Proposition 1.11.

(2) We set $\widetilde{\mathcal{D}} = (-\int_0^1 f_{\underline{\sigma}}(z)dz, \int_0^1 f_{\underline{\sigma}}(z)dz)$. When $f_{\underline{\sigma}} \notin L^1(0,1)$, we read $\widetilde{\mathcal{D}} = \mathbf{R}$. We first prove $\mathcal{D} \supset \widetilde{\mathcal{D}}$. To do this, take $P \in \widetilde{\mathcal{D}}$. What we have to do is to find $u \in \mathrm{Lip}(\mathbf{T})$ such that (u,c) is a solution of (1.4.1), where c is the constant in (1.4.4).

When $|P| \le \int_0^1 f_{\overline{\sigma} m_0}(z)dz$, we set

$$u(x) = \begin{cases} \displaystyle\int_{x_0}^x f_{\overline{\sigma} m_0}(z)dz - Px & \text{for } x \in [x_0, x_1], \\ \displaystyle\int_x^{x_0+1} f_{\overline{\sigma} m_0}(z)dz + P(1-x) & \text{for } x \in [x_1, x_0+1]. \end{cases}$$

Here, $x_0 \in [0,1]$ and $x_1 \in [x_0, x_0+1]$ are points such that

$$f_{\overline{\sigma} m_0}(x_0) = 0, \qquad \int_{x_0}^{x_1} f_{\overline{\sigma} m_0}(z)dz = \int_{x_1}^{x_0+1} f_{\overline{\sigma} m_0}(z)dz + P.$$

We regard u as a function on $\mathbf{T}$ by extending it periodically. Then, it is easy to see that u is a solution of (1.4.1).

When $|P| \ge \int_0^1 f_{\overline{\sigma} m_0}(z)dz$, for c chosen by (1.4.4), we set

$$u(x) = \mathrm{sign}(P) \int_0^x f_c(z)dz - Px \quad \text{for } x \in \mathbf{R}.$$

Note that u is a $\mathbf{Z}$-periodic function since, by the definition of c,

$$\operatorname{sign}(P)\int_0^1 f_c(z)dz - P = 0.$$

Then, it is easy to see that u is a solution of (1.4.1). Therefore, we have obtained $\mathcal{D} \supset \widetilde{\mathcal{D}}$.

We next show the reverse inclusion $\mathcal{D} \subset \widetilde{\mathcal{D}}$. Let $P \in \mathcal{D}$ and take a solution (u, c) of (1.4.1), then

$$|P| = \left|\int_0^1 (u'(z) + P)dz\right| \le \int_0^1 |u'(z) + P|dz \le \int_0^1 f_c(z)dz < \int_0^1 f_{\underline{\sigma}}(z)dz.$$

The first equality follows from the periodicity of u. Thus, $P \in \widetilde{\mathcal{D}}$ and so the proof is complete. □

Remark 1.19. The representation of the critical value (1.4.4) is also obtained via the formula (1.4.3) given in [20]. In fact, a is a critical value of (CP) if and only if the critical value c_a of

$$|u'(x) + P| = f_a(x) + c_a \quad \text{in } \mathbf{T}$$

is equal to 0. It is easily seen that the condition $c_a = 0$ yields (1.4.4).

When σ attains a minimum on some interval $[a, b]$ with $a < b$, it is easily seen that $f_{\underline{\sigma}}$ is not integrable since $f_{\underline{\sigma}} = +\infty$ on $[a, b]$. Consequently, (1.4.1) is fully solvable by Proposition 1.18. If $\sigma(x) = \underline{\sigma}$ at only one point $x \in \mathbf{T}$, the integrability of $f_{\underline{\sigma}}$ depends on σ and m as the next examples indicate.

Example 1.1. *Let us consider* (1.4.1) *with*

$$m(r) = \frac{1}{2}\frac{r}{1+r} + \frac{1}{2} \quad (r \in [0, \infty)), \quad \sigma(x) = x^\alpha(1-x)^\alpha + \beta \quad (x \in [0, 1]),$$

where $\alpha, \beta > 0$. We note that $\overline{\sigma} m_0 < \underline{\sigma}$ holds when $\beta > 1/4^\alpha$. Since

$$f_{\underline{\sigma}}(x) = \frac{\underline{\sigma}}{2}\frac{1}{\sigma(x) - \underline{\sigma}} - 1 = \frac{\underline{\sigma}}{2}\frac{1}{x^\alpha(1-x)^\alpha} - 1,$$

the integrability of $f_{\underline{\sigma}}$ is determined by the choice of $\alpha > 0$.

Example 1.2. *Set*

$$\sigma(x) = \begin{cases} x + \frac{3}{2} & (0 \le x < \frac{1}{2}), \\ -x + \frac{5}{2} & (\frac{1}{2} \le x < 1). \end{cases}$$

(1) We let

$$m(r) = \frac{1}{2}\frac{r}{1+r} + \frac{1}{2}.$$

We extend σ periodically to $\mathbf{R}$ and still denote it by σ. Note that $\overline{\sigma} m_0 < \underline{\sigma}$ holds. Since

$$f_{\underline{\sigma}}(x) = \frac{\underline{\sigma}}{2}\frac{1}{\sigma(x) - \underline{\sigma}} - \frac{1}{2},$$

we observe that

$$\int_0^1 f_{\underline{\sigma}}(z)dz = 2\int_0^{1/2} f_{\underline{\sigma}}(z)dz = \underline{\sigma}\int_0^{1/2} \frac{1}{z}dz - \frac{1}{2} = \infty.$$

Thus, $\mathcal{D} = \mathbf{R}$, i.e., the cell problem is fully solvable.

(2) We next study

$$m(r) = \frac{1}{2}\tanh r + \frac{1}{2} \quad (r \in [0,\infty)).$$

Note that $\overline{\sigma} m_0 < \underline{\sigma}$. Since

$$f_{\underline{\sigma}}(x) = \frac{1}{2}\log\left(\frac{\underline{\sigma}}{\sigma(x) - \underline{\sigma}}\right) = \frac{1}{2}\{\log\underline{\sigma} - \log(\sigma(x) - \underline{\sigma})\},$$

we observe that

$$\int_0^1 f_{\underline{\sigma}}(z)dz = 2\int_0^{1/2} f_{\underline{\sigma}}(z)dz = 1 + \frac{1}{2}\log 6.$$

Therefore, $\mathcal{D} = \left(-1 - \frac{1}{2}\log 6, 1 + \frac{1}{2}\log 6\right)$.

Proposition 1.20. *We have*

$$\bar{H}_\infty(P) > \bar{H}_\infty(Q) \quad \textit{for all } P, Q \in \mathcal{D} \textit{ such that } |P| > |Q| \geq \int_0^1 f_{\overline{\sigma} m_0}(z)dz.$$

Proof. By (1.4.4), we observe

$$\int_0^1 \{f_{\bar{H}_\infty(P)}(z) - f_{\bar{H}_\infty(Q)}(z)\}dz = |P| - |Q| > 0,$$

which implies that $\bar{H}_\infty(P) > \bar{H}_\infty(Q)$. □

1.5 Application to homogenization problems

We present our homogenization result for the equation (HJ$_\varepsilon$) with the Hamiltonian (1.1.1) satisfying (H1)–(H3). Here, $u_0 : \mathbf{R}^N \to \mathbf{R}$ is a bounded and Lipschitz continuous initial datum. We remark that there exists a unique bounded solution $u^\varepsilon \in C(\mathbf{R}^N \times [0,T))$ of (HJ$_\varepsilon$). Similarly, there exists a unique bounded solution $u \in C(\mathbf{R}^N \times [0,T))$ of (HJ). Indeed, the comparison principle holds for a viscosity sub- and supersolution (see [12]). This yields uniqueness of solutions. Existence is a consequence of Perron's method (see [19]).

Theorem 1.21 (Homogenization result)**.** *Assume either*

$$(1)\ \mathcal{D} = \mathbf{R}^N \quad \text{or} \quad (2)\ m(\mathrm{Lip}[u_0]) < \underline{\sigma}/\overline{\sigma}.$$

Then the solution u^ε of (HJ$_\varepsilon$) *converges to the solution u of* (HJ) *locally uniformly in $\mathbf{R}^N \times [0,T)$ as $\varepsilon \to 0$.*

Proof of Theorem 1.21 under the assumption (1). Recall that, for each $P \in \mathcal{D}$, $\bar{H}_\infty(P)$ is the critical value of (CP) from Theorem 1.7. As we mentioned in Introduction the assumption (1) means that the cell problem is fully solvable, and so the conclusion follows from the same argument as in [13] involving the perturbed test function method. Here we do not need equi-Lipschitz continuity of $\{u_\varepsilon\}$ since the half-relaxed limit method works for our equation; see [13, Proof of Theorem 4.4]. To be more precise, it turns out that the upper- and lower half-relaxed limits

$$\overline{u}(x,t) := \lim_{\delta\to 0} \sup\{u^\varepsilon(y,s) \mid (y,s) \in B(x,\delta) \times (t-\delta, t+\delta),\ \varepsilon < \delta\},$$
$$\underline{u}(x,t) := \lim_{\delta\to 0} \inf\{u^\varepsilon(y,s) \mid (y,s) \in B(x,\delta) \times (t-\delta, t+\delta),\ \varepsilon < \delta\}$$

are, respectively, a sub- and supersolution of (HJ), so that the comparison principle ensures that these two limits are equal to the solution u. This implies locally uniform convergence to u. □

We shall hereafter prove Theorem 1.21 under the assumption (2).

Proposition 1.22 (Regularity of the solution of (HJ$_\varepsilon$))**.** *Assume (2) in Theorem 1.21. Then, the solutions u^ε of* (HJ$_\varepsilon$) *satisfy*

$$|u^\varepsilon(x,t) - u^\varepsilon(x,s)| \le L|t-s|, \quad |u^\varepsilon(x,t) - u^\varepsilon(y,t)| \le K|x-y|$$

for all $x, y \in \mathbf{R}^N, t, s \in [0,T)$ with the constants

$$L := \overline{\sigma} m(\mathrm{Lip}[u_0]) < \infty, \quad K := m^{-1}\left(\frac{\overline{\sigma}}{\underline{\sigma}} m(\mathrm{Lip}[u_0])\right) < \infty.$$

We omit the proof since this proposition is verified by the same argument as in [15, Appendix A]. We point out that [15, Proposition 3.17] holds under the assumption $R_+(m) < \infty$ even if the Hamiltonian does not satisfy the coercivity condition (H$_{R+}$).

We give two different proofs of Theorem 1.21 under the assumption (2).

Proof I of Theorem 1.21 under the assumption (2). By Proposition 1.22, Ascoli-Arzelà theorem implies that u^ε subsequently converges to some Lipschitz continuous function u locally uniformly in $\mathbf{R}^N \times [0,T)$ as $\varepsilon \to 0$.

We prove that u is a supersolution of (HJ). The proof is based on the perturbed test function method (see [13]). Let $(x_0, t_0) \in \mathbf{R}^N \times (0,T)$ and $\phi \in C^1(\mathbf{R}^N \times (0,T))$ such that $u - \phi$ has a strict local minimum at (x_0,t_0). Suppose that

$$\phi_t(x_0,t_0) + \bar{H}_\infty(D\phi(x_0,t_0)) =: -\theta < 0.$$

We take $\{H_n\}_{n\in\mathbf{N}}$ as in Remark 1.8 and let $\bar{H}_n$ be the effective Hamiltonian of (CP$_n$). Since $\bar{H}_n$ converges to $\bar{H}_\infty$, we have

$$\phi_t(x_0,t_0) + \bar{H}_n(D\phi(x_0,t_0)) \le -\frac{\theta}{2} \tag{1.5.1}$$

for sufficiently large n. On the other hand, by the Lipschitz continuity of u^ε and (B3), we see that it is a solution of

$$w_t(x,t) + H_n\left(\frac{x}{\varepsilon}, Dw(x,t)\right) = 0 \tag{1.5.2}$$

in $\mathbf{R}^N \times (0,T)$ for sufficiently large n. We hereafter fix n satisfying the above two conditions.

Set

$$\phi_n^\varepsilon(x,t) := \phi(x,t) + \varepsilon v_n\left(\frac{x}{\varepsilon}\right),$$

where v_n is a solution of (CP$_n$). By the same argument as in [13], we see that ϕ_n^ε is a subsolution of (1.5.2) in $B(x_0,r) \times (t_0 - r, t_0 + r)$ for sufficiently small $r > 0$. The comparison principle for (HJ$_\varepsilon$) implies a contradiction (see [13]) and so u is a supersolution of (HJ).

Similarly, it is proved that u is a subsolution of (HJ), and therefore, u is a unique solution of (HJ). Consequently, u^ε converges to u locally uniformly in $\mathbf{R}^N \times [0,T)$ as $\varepsilon \to 0$ without taking subsequences. □

Proof II of Theorem 1.21 under the assumption (2). Recall that $\{u^\varepsilon\}_{\varepsilon>0}$ is equi-Lipschitz continuous in view of Proposition 1.22 and therefore subsequently converges to some u locally uniformly in $\mathbf{R}^N \times [0,T)$ as $\varepsilon \to 0$. Take $\{H_n\}_{n\in\mathbf{N}}$ as in Remark 1.8. By (B3) and the equi-Lipschitz continuity of $\{u^\varepsilon\}_{\varepsilon>0}$, we have

$$u_t^\varepsilon(x,t) + H_n\left(\frac{x}{\varepsilon}, Du^\varepsilon(x,t)\right) = 0 \quad \text{in } \mathbf{R}^N \times (0,T)$$

for all $n \in \mathbf{N}$ large enough and all $\varepsilon > 0$. We now apply the homogenization result for coercive Hamiltonians [13] to see that u^ε converges to the solution w_n of

$$\begin{cases} w_t(x,t) + \bar{H}_n\left(Dw(x,t)\right) = 0 & \text{in } \mathbf{R}^N \times (0,T), \\ w(x,0) = u_0(x) & \text{in } \mathbf{R}^N \end{cases}$$

locally uniformly in $\mathbf{R}^N \times [0,T)$. Since u is a limit of a subsequence, it turns out that $w_n \equiv u$. Since $\bar{H}_n$ converges to $\bar{H}_\infty$ locally uniformly, the stability result for viscosity solutions yields the conclusion that u is a viscosity solution of (HJ). □

Remark 1.23. The main difference between two proofs is the order of limits of ε and n. We point out that Proof I does not require the locally uniform convergence of $\bar{H}_n$. However, we need the equi-Lipschitz continuity of $\{u^\varepsilon\}_{\varepsilon>0}$ in both proofs in order to ensure that u^ε is a solution of the approximate equation.

Theorem 1.24 (Non-homogenization result)**.** *Assume that $\overline{\sigma} m_0 > \underline{\sigma}$. Let u^ε be the solutions of* (HJ$_\varepsilon$)*. Then, u^ε does not have a locally uniformly convergent limit in $\mathbf{R}^N \times [0,T)$ as $\varepsilon \to 0$.*

Proof. Set

$$u_-^\varepsilon(x,t) := u_0(x) - \sigma\left(\frac{x}{\varepsilon}\right)t, \quad u_+^\varepsilon(x,t) := u_0(x) - \sigma\left(\frac{x}{\varepsilon}\right)m_0 t$$

for $(x,t) \in \mathbf{R}^N \times [0,T)$. Then, we see that u_-^ε and u_+^ε are a subsolution and a supersolution of (HJ$_\varepsilon$) respectively. By the comparison principle, the solution u^ε satisfies

$$u_-^\varepsilon(x,t) \le u^\varepsilon(x,t) \le u_+^\varepsilon(x,t) \tag{1.5.3}$$

for all $(x,t) \in \mathbf{R}^N \times [0,T)$ and $\varepsilon > 0$.

Let $\overline{u}$ and $\underline{u}$ be the upper half-relaxed limit and the lower half-relaxed limit of u^ε, respectively. Moreover, let $\overline{u}_-$ and $\underline{u}_+$ be the upper half-relaxed limit of u^ε_- and the lower half-relaxed limit of u^ε_+, respectively. Then, we have

$$\overline{u}_-(x,t) = u_0(x) - \underline{\sigma} t \quad \text{and} \quad \underline{u}_+(x,t) = u_0(x) - \overline{\sigma} m_0 t.$$

Thus, by (1.5.3) and the assumption $\overline{\sigma} m_0 > \underline{\sigma}$, we have

$$\underline{u}(x,t) \le \underline{u}_+(x,t) < \overline{u}_-(x,t) \le \overline{u}(x,t) \tag{1.5.4}$$

for all $(x,t) \in \mathbf{R}^N \times [0,T)$. Therefore, $\overline{u}$ and $\underline{u}$ are different and so we conclude that u^ε does not converge to any functions locally uniformly as $\varepsilon \to 0$. $\square$

Remark 1.25. When $\overline{\sigma} m_0 = \underline{\sigma}$, we do not know whether or not u^ε has a limit as $\varepsilon \to 0$. However, by (1.5.4), we see that the limit of u^ε should be $u_0(x) - \underline{\sigma} t (= u_0(x) - \overline{\sigma} m_0 t)$ if it exists.

1.6 Generalization

Our homogenization results can be extended for more general equations of the form

$$u^\varepsilon_t(x,t) + H\left(x, \frac{x}{\varepsilon}, u^\varepsilon(x,t), Du^\varepsilon(x,t)\right) = 0 \quad \text{in } \mathbf{R}^N \times (0,T). \tag{1.6.1}$$

Here $H = H(x,y,u,p) : \mathbf{R}^N \times \mathbf{T}^N \times \mathbf{R} \times \mathbf{R}^N \to \mathbf{R}$ is Lipschitz continuous in $\mathbf{R}^N \times \mathbf{T}^N \times (-L, L) \times B(0,L)$ for every $L > 0$ and non-decreasing in u. They guarantee the comparison principle; similar assumptions can be seen in [13]. The corresponding cell problem is

$$H\left(x, y, u, Dv(y) + P\right) = a \quad \text{in } \mathbf{T}^N, \tag{1.6.2}$$

where the unknown is $(v,a) \in \mathrm{Lip}(\mathbf{T}^N) \times \mathbf{R}$ and $(x,u) \in \mathbf{R}^N \times \mathbf{R}$ is fixed. Define $\mathcal{D}_{x,u}$ as the set of $P \in \mathbf{R}^N$ such that (1.6.2) admits a solution (v,a) for a given (x,u). For homogenization of (1.6.1) we assume either

$$(1)\ \ \mathcal{D}_{x,u} = \mathbf{R}^N \text{ for all } (x,u) \in \mathbf{R}^N \times \mathbf{R} \quad \text{or} \quad (2)\ \ \sup_{\varepsilon > 0} \mathrm{Lip}[u^\varepsilon] < \infty.$$

Choose $H_n(x,y,u,p) = \max\{H(x,y,u,p), |p| - n\}$, which is a coercive Hamiltonian approximating H. Since (x,u) is fixed in cell problems, a similar method in this paper gives a generalized effective Hamiltonian $\bar{H}_\infty(x,u,P)$ as the limit of $\bar{H}_n(x,u,P)$. (Here we do not pursue generalization of approximation to H and study only a homogenization problem. Also, in this case $\bar{H}_\infty$ is just the infimum of $\bar{H}_n$.) According to [13, Lemma 2.2], $\bar{H}_n$ possesses the same regularity and monotonicity properties as H_n, and thus so is $\bar{H}_\infty$. Moreover, since $\bar{H}_n$ is monotone in n, Dini's lemma ensures that $\bar{H}_n$ converges to $\bar{H}_\infty$ locally uniformly. One is now able to show homogenization results for (1.6.1) with the same argument as in three proofs of Theorem 1.21 above.

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Chapter 2

Well-posedness of Hamilton-Jacobi equations with Caputo's time-fractional derivative

In this chapter a notion of a viscosity solution for the initial value problem of a Hamilton-Jacobi equation with Caputo's time-fractional derivative of order less than one under periodic boundary conditions is introduced. By using the notion of a solution, a unique existence is proved. For this purpose the comparison principle as well as Perron's method is established. Stability with respect to the order of derivative as well as the standard one is also studied and a regularity of a solution is discussed. Our results does not require the coercivity for Hamiltonians and thus apply to a linear transport equation with time-fractional derivatives with variable coefficients.

Chapter 3

Second order fully nonlinear multi-term time-fractional PDEs with positive constant coefficients

In this chapter second order fully nonlinear PDEs with multi-term Caputo's time-fractional derivative (CTFD) whose orders are less than one are considered. A notion of a weak solution introduced in Chapter 2 for (first order) Hamilton-Jacobi equations, which is based on the idea of viscosity solutions, is extended. A unique existence and some stabilities including the vanishing viscosity are proved. In order to establish a comparison principle, a relation of semijets is proved for PDEs with CTFD, which is called Ishii's lemma for an integer-order case. The existence is proved via the Perron's method. In particular, all results apply to anomalous diffusion equations which is allowed to be degenerate.

Chapter 4

Hamilton-Jacobi equations with Caputo's time-fractional derivative under homogeneous Neumann boundary conditions

The inital-boundary-value problem of a Hamilton-Jacobi equation with Caputo's time-fractional derivative whose order is less than one is considered under homogeneous Neumann boundary conditions. A notion of viscosity solutions is introduced based on the idea for the initial-value problem of a Hamilton-Jacobi equation with Caputo's time-fractional derivative, which was given by Giga and the author, and for integer-order cases. In particular, boundary conditions are interpreted similarly as for integer-order cases. In order to prove a unique existence, a comparison principle as well as Perron's method are established.

Keywords: Homogeneous Neumann boundary condition; Caputo's time-fractional derivatives; Hamilton-Jacobi equations; Viscosity solutions

4.1 Introduction

Let $T > 0$ and $\alpha \in (0,1]$ be given constants, Ω be a bounded C^1-domain in $\mathbb{R}^d$ and $\nu(\cdot)$ be an outer unit normal vector on $\partial\Omega$. We study the well-posedness of the initial-boundary value problem

$$\begin{cases} \partial_t^\alpha u + H(x, Du) = 0 & \text{in } (0,T)\times\Omega =: \Omega_T, \quad (4.1.1) \\ Du\cdot\nu(x) = 0 & \text{on } [0,T]\times\partial\Omega, \quad (4.1.2) \\ u|_{t=0} = u_0 & \text{in } \Omega.. \quad (4.1.3) \end{cases}$$

Here Du is the spatial gradient and ∂_t^α denotes Caputo's time-fractional derivative (CTFD for short), which is defined by

$$(\partial_t^\alpha f)(t) = \begin{cases} \dfrac{1}{\Gamma(1-\alpha)}\displaystyle\int_0^t \frac{f'(s)}{(t-s)^\alpha}ds & \text{for } \alpha \in (0,1), \\ f'(t) & \text{for } \alpha = 1. \end{cases}$$

Here $\Gamma(\cdot)$ is the usual Gamma function. The function H is continuous on $\overline{\Omega}\times\mathbb{R}$ and the initial data u_0 is continuous on $\overline{\Omega}$.

Partial differential equations (PDEs for short) with fractional derivatives or nonlocal terms have been attracted by motivations from various physical phenomena. Among them, the number of literatures for PDEs with time-fractional derivatives is growing rapidly in recent years. A typical example of such PDEs is an *anomalous diffusion equation*, which is a diffusion equation whose time-derivative is replaced by CTFDs. For a diffusion phenomenon in a complex region like a fractal, say, for example a rock with many fractures, anomalous diffusion equations rather fit observational data by putting an appropriate α ([9], [10] and [27]). Luchko ([20] and [21]) and Sakamoto and Yamamoto ([25]) proved, respectively, a unique existence of classical solutions and weak solutions in the sense of distribution for the initial-boundary problem of

$$\partial_t^\alpha u - \operatorname{div}(p(x)Du) + q(x) = f.$$

Here typically p is smooth and uniformly positive with nonnegative continuous q. We refer the reader to [8], [14], [15], [24], [26], [28] and [29] for further topics of the above works and other examples of PDEs with CTFDs.

In the very recent past, PDEs with CTFDs have started to be researched in the framework of the theory of viscosity solutions, which is a theory for generalized solutions that ensures a unique existence of continuous solutions introduced by Crandall and Lions in [7]. To the best of our knowledge, Allen ([2]) first introduced a notion of a viscosity solution for such PDEs. More precisely, he introduced a notion of a viscosity solution for

$$\partial_t^\alpha u - \sup_i \inf_j \left(\int_{\mathbb{R}^d} \frac{u(x+y)-u(x)}{|y|^{d+2\sigma}} a^{ij}(t,x,y)dy \right) = f \tag{4.1.4}$$

and studied regularity problems of solution. Here a_{ij} is positive, bounded function that is symmetric with respect to the third variable and f is a given function. It was not clear whether or not his viscosity solution exactly exists since the well-posedness was not mentioned in [2]. The well-posedness was studied by Giga and the author ([12]) for first order equations and by the author ([23]) for second order equations although they do not cover rigorously the equation (4.1.4) Allen considered. In [12] they extended the notion of viscosity solutions to the Hamilton-Jacobi equation with CTFD of the form

$$\partial_t^\alpha u + H(t,x,u,Du) = 0.$$

They proved a unique existence for the initial-value problem under natural assumptions on H and initial data for $\alpha = 1$ and also studied stability and regularity problems. In [23] this notion of a viscosity solution was extended to the second order equation of the form

$$\partial_t^\alpha u + F(t,x,u,Du,D^2u) = 0.$$

Here D^2u is the Hesse matrix of u and the function F is continuous and degenerate elliptic. He proved a unique existence for the initial-value problem also with no special assumptions on F and initial data. See also [17] and [18] for a significance to develop the theory of viscosity solutions to PDEs with CTFDs.

In this chapter we extend the notion of a viscosity solution introduced in [12] to an initial-boundary value problem. For viscosity solutions for $\alpha = 1$, boundary conditions are interpreted in a special sense which is often said "in the viscosity sense". As an example, let us consider the problem

$$\begin{cases} \partial_t u + H(x, Du) = 0 & \text{in } (0, T] \times \Omega, \\ \nu(x) \cdot Du = g & \text{on } [0, T] \times \partial\Omega. \end{cases} \tag{4.1.5}$$

Here the function $g = g(x)$ is continuous. We now ignore the initial condition. Then a (upper semicontinuous) viscosity subsolution u of (4.1.5) is defined as follows: for any point $(\hat{t}, \hat{x}) \in (0, T] \times \overline{\Omega}$ and any test function $\phi \in C^1((0, T] \times \overline{\Omega})$ such that $\max_{\overline{\Omega_T}}(u - \phi) = (u - \phi)(\hat{t}, \hat{x})$,

$$\partial_t \phi(\hat{t}, \hat{x}) + H(\hat{x}, D\phi(\hat{t}, \hat{x})) \le 0$$

if $\hat{x} \in \Omega$ and

$$\min\{\partial_t \phi(\hat{t}, \hat{x}) + H(\hat{x}, D\phi(\hat{t}, \hat{x})), \nu(\hat{x}) \cdot D\phi(\hat{t}, \hat{x}) - g(\hat{x})\} \le 0$$

if $\hat{x} \in \partial\Omega$. A (lower semicontinuous) viscosity supersolution of (4.1.5) is similarly defined and a viscosity solution is defined as being a viscosity sub- and supersolution of (4.1.5). A reason why such a definition is useful is made clear for example by seeking a problem that a (classical) solution u^ε of

$$\begin{cases} -\varepsilon u'' + u' + u = x + 1 & \text{in } (0, 1), \\ u'(0) = u'(1) = 0 \end{cases}$$

satisfies as $\varepsilon \to 0$. See [6] for more detail. For the above definition, the well-posedness is well studied ([1], [3], [4], [5], [6], [11], [13], [16] and [19]).

In an extension of a definition of viscosity solutions to the initial-boundary value problem of (4.1.1), we should pay attention to what a shape of solution may change depending on α. In fact, Mainardi, Mura and Pagnini ([22]) showed that a solution of the initial-value problem of the transport equation of the form

$$\partial_t^\alpha u + \partial_x u = 0 \quad \text{in } (0, \infty) \times \mathbb{R}$$

is given as

$$u(t, x) = \frac{1}{t^\alpha} \int_0^\infty W_{-\alpha, 1-\alpha}\left(-\frac{z}{t^\alpha}\right) u_0(x - z) dz \tag{4.1.6}$$

with different motivations. Here $W_{-\alpha,1-\alpha}$ is Wright function and u_0 is a continuous initial data. The function u given by (4.1.6) is obviously different from the case $\alpha = 1$. For this reason it is not clear whether or not we may interpret boundary conditions for (4.1.1) similarly as the case $\alpha = 1$.

We will show that the same interpretation can be applied to (4.1.1)-(4.1.3). Then a comparison principle and an existence by Perron's method are proved by handling CTFDs similarly to [12] and [23] and dealing with terms of Hamiltonian similarly to classical way. We note that Hamiltonians may depend on t and u although we restrict ourselves to the case $H = H(x, p)$ for simplicity of arguments. In fact, all results are proved with appropriate modifications. In addition, Caputo's derivatives are allowed to be multi-term, that is, a finite sum of Caputo's derivative with positive coefficients. A treatment of such derivatives is similar as in [23].

This chapter is organized as follows: In Section 2, we give a definition of a viscosity solution. In Section 3 we prove a comparison principle and, in Section 4, an existence result through Perron's method.

4.2 Definition of solutions and properties

In this section we assume that Hamiltonians H are merely continuous on $\overline{\Omega} \times \mathbb{R}^d$ and Ω is a general domain with boundary in $\mathbb{R}^d$. For constants $a, b \in \mathbb{R}$ such that $a < b$ and a locally compact set O in $\mathbb{R}^\ell$ ($\ell \geq 1$) we define

$$\begin{aligned}&\mathcal{C}^1([a,b] \times O)\\ &= \{\phi \in C^1((a,b] \times O) \cap C([a,b] \times O) \mid \partial_t \phi(\cdot, x) \in L^1(a,b) \text{ for every } x \in O\}.\end{aligned}$$

Here $L^1(a,b)$ is a set of Lebesugue integrable functions in (a,b). We also define two functions $J_r[f], K_r[f] : (0,T] \to \mathbb{R}$ for a measurable function $f : [0,T] \to \mathbb{R}$ by

$$J_r^\alpha[f](t) = \frac{\alpha}{\Gamma(1-\alpha)} \int_0^r (f(t) - f(t-\tau)) \frac{d\tau}{\tau^{\alpha+1}}$$

and

$$K_r^\alpha[f](t) = \frac{f(t) - f(0)}{t^\alpha \Gamma(1-\alpha)} + \frac{\alpha}{\Gamma(1-\alpha)} \int_r^t (f(t) - f(t-\tau)) \frac{d\tau}{\tau^{\alpha+1}},$$

where $r > 0$ is a parameter such that $0 < r < t$. Note that $\partial_t^\alpha f = J_r[f] + K_r[f]$ for all $r \in (0,t)$ if $f \in C^1((0,T]) \cap C([0,T])$ and $f' \in L^1(0,T)$. This is easily derived by applying integration by parts and changing a variable of integration; see [12]. For a locally compact set $E \subset \mathbb{R}^l$ with $l \geq 1$, let $USC(E)$ and $LSC(E)$ be sets of real-valued upper and lower semicontinuous functions on E, respectively. Note that semicontinuous functions are (Borel-)measurable.

Definition 4.1 (Viscosity solutions)**.** A function $u \in USC(\overline{\Omega_T})$ (resp. $LSC(\overline{\Omega_T})$) is called a *viscosity subsolution* (resp. *viscosity supersolution*) of (4.1.1)-(4.1.2) if, for any $a, b \in [0,T]$ with $a < b$ and any open ball $B(z)$ centered at $z \in \overline{\Omega}$ in $\mathbb{R}^d$, the following holds: for any $(\hat{t}, \hat{x}) \in (a,b] \times (\overline{\Omega} \cap B(z))$ and $\phi \in \mathcal{C}^1(\overline{\Omega_T})$ such that $\max_{[a,b] \times \overline{\Omega \cap B(z)}}(u - \phi) = (u - \phi)(\hat{t}, \hat{x})$ (resp. $\min_{[a,b] \times \overline{\Omega \cap B(z)}}(u - \phi) = (u - \phi)(\hat{t}, \hat{x})$),

- if $\hat{x} \in \Omega$,

$$J_{\hat{t}-a}[\phi](\hat{t}, \hat{x}) + K_{t-a}[u](\hat{t}, \hat{x}) + H(\hat{x}, D\phi(\hat{t}, \hat{x})) \leq 0 \quad (\text{resp. } \geq 0)$$

- if $\hat{x} \in \partial\Omega$,

$$\begin{aligned}\min\{J_{\hat{t}-a}[\phi](\hat{t}, \hat{x}) + K_{t-a}[u](\hat{t}, \hat{x}) + H(\hat{x}, D\phi(\hat{t}, \hat{x})), \nu(\hat{x}) \cdot D\phi(\hat{t}, \hat{x})\} \leq 0,\\ (\text{resp. } \max\{J_{\hat{t}-a}[\phi](\hat{t}, \hat{x}) + K_{t-a}[u](\hat{t}, \hat{x}) + H(\hat{x}, D\phi(\hat{t}, \hat{x})), \nu(\hat{x}) \cdot D\phi(\hat{t}, \hat{x})\} \geq 0.)\end{aligned}$$

If u is both a viscosity sub- and supersolution of (4.1.1), then u is called a *viscosity solution* of (4.1.1).

Remark 4.2. In Definition 4.1, $J_{\hat{t}-a}[\phi](\hat{t},\hat{x})$ exists and $K_{\hat{t}-a}[u](\hat{t},\hat{x})$ makes sense, where both are in the sense given in [12].

If a viscosity subsolution (resp. viscosity supersolution) u of (4.1.1)-(4.1.2) satisfies $u(0,\cdot) \le u_0$ (resp. $u(0,\cdot) \ge u_0$) on $\overline{\Omega}$, then u is called a viscosity subsolution (resp. viscosity supersolution) of (4.1.1)-(4.1.3). We suppress the word "viscosity" since viscosity solutions are only considered in this chapter.

The following equivalence for the definition of solutions was utilized in [12] when there is no boundary conditions in the viscosity sense.

Lemma 4.3 (Equivalence). *A function $u \in USC(\overline{\Omega_T})$ (resp. $LSC(\overline{\Omega_T})$) is a subsolution (resp. supersolution) of* (4.1.1)-(4.1.2) *if and only if, for any $(\hat{t},\hat{x}) \in (0,T]\times\overline{\Omega}$ and $\phi \in \mathcal{C}^1(\overline{\Omega_T})$ such that $\max_{\overline{\Omega_T}}(u-\phi) = (u-\phi)(\hat{t},\hat{x})$, $K_0[u](\hat{t},\hat{x})$ exists and*

- *if $\hat{x} \in \Omega$,*
$$K_0[u](\hat{t},\hat{x}) + H(\hat{x}, D\phi(\hat{t},\hat{x})) \le 0$$
- *if $\hat{x} \in \partial\Omega$,*
$$\min\{K_0[u](\hat{t},\hat{x}) + H(\hat{x}, D\phi(\hat{t},\hat{x})), \nu(\hat{x})\cdot D\phi(\hat{t},\hat{x})\} \le 0,$$
$$(\textit{resp. } \max\{K_0[u](\hat{t},\hat{x}) + H(\hat{x}, D\phi(\hat{t},\hat{x})), \nu(\hat{x})\cdot D\phi(\hat{t},\hat{x})\} \ge 0.)$$

Proof. We first prove the 'only if' part. We may assume that $\hat{x} \in \partial\Omega$ since the case of $\hat{x} \in \Omega$ was proved already in [12]. Then, for each small $r>0$, $(\hat{t},\hat{x}) \in (\hat{t}-r,\hat{t}+r]\times(\overline{\Omega}\cap B(\hat{x};r))$ and it is a maximum point of $u-\phi$ over $[\hat{t}-r,\hat{t}+r]\times(\overline{\Omega\cap B(\hat{x};r)})$. Here $B(\hat{x};r)$ and $\overline{B(\hat{x};r)}$ are a open ball with radius $r>0$ centered at $\hat{x}$ and its closure, respectively. By the definition of viscosity subsolution, either

$$J_r[\phi](\hat{t},\hat{x}) + K_r[u](\hat{t},\hat{x}) + H(\hat{x}, D\phi(\hat{t},\hat{x})) \le 0 \tag{4.2.1}$$

or $\nu(\hat{x})\cdot D\phi(\hat{t},\hat{x}) \le 0$ holds. Handling (4.2.1) as $r\to 0$ similarly as in the case that $\hat{x}\in\Omega$, we see that $K_0[u](\hat{t},\hat{x})$ exists and that

$$K_0[u](\hat{t},\hat{x}) + H(\hat{x}, D\phi(\hat{t},\hat{x})) \le 0.$$

Consequently, we have

$$\min\{K_0[u](\hat{t},\hat{x}) + H(\hat{x}, D\phi(\hat{t},\hat{x})) \le 0, \nu(\hat{x})\cdot D\phi(\hat{t},\hat{x})\} \le 0,$$

which is a conclusion.

We next prove the 'if' part. Fix any $a,b \in [0,T]$ with $a<b$ and any open ball $B(z)$ centered at $z\in\overline{\Omega}$ in $\mathbb{R}^d$. Assume that $u-\phi$ attains a maximum at $(\hat{t},\hat{x}) \in (a,b]\times(\overline{\Omega}\cap B(z))$ over $[a,b]\times\overline{\Omega\cap B(z)}$ for $\phi\in\mathcal{C}^1(\overline{\Omega_T})$. We may assume that $\hat{x}\in\partial\Omega$. By re-defining ϕ appropriately, there is a function $\psi\in\mathcal{C}^1(\overline{\Omega_T})$ such that $\psi=\phi$ near $(\hat{t},\hat{x})$ and $u-\psi$ attains a maximum at $(\hat{t},\hat{x})$ over $\overline{\Omega_T}$; see [12,]. Thus $K_0[u](\hat{t},\hat{x})$ exists and it holds that

$$\min\{K_0[u](\hat{t},\hat{x}) + H(\hat{x}, D\phi(\hat{t},\hat{x})) \le 0, \nu(\hat{x})\cdot D\phi(\hat{t},\hat{x})\} \le 0. \tag{4.2.2}$$

Since $K_0[u](\hat{t},\hat{x}) = J_{\hat{t}-a}[u](\hat{t},\hat{x}) + K_{\hat{t}-a}[u](\hat{t},\hat{x})$ and since $(u-\phi)(\hat{t},\hat{x}) \geq (u-\phi)(\hat{t}-\tau,\hat{x})$ for all $\tau \in [0,\hat{t}-a]$, we see that $K_0[u](\hat{t},\hat{x}) \geq J_{\hat{t}-a}[\phi](\hat{t},\hat{x}) + K_{\hat{t}-a}[u](\hat{t},\hat{x})$. Combining this with (4.2.2) we get the desired inequality. $\square$

4.3 Comparison principle

In order to get a comparison principle we impose the following assumptions:

(A1) Ω is bounded domain in $\mathbb{R}^d$ and of class C^1,

(A2) $H \in C(\overline{\Omega} \times \mathbb{R}^d)$,

(A3) there is a modulus $\omega_1 : [0,\infty) \to [0,\infty)$ such that

$$|H(x,p) - H(y,p)| \leq \omega_1(|x-y|(1+|p|))$$

for all $x, y \in \overline{\Omega}$ and $p \in \mathbb{R}^d$,

(A4) there is a modulus $\omega_2 : [0,\infty) \to [0,\infty)$ such that

$$|H(x,p) - H(x,q)| \leq \omega_2(|p-q|)$$

for all $x \in \overline{\Omega}$ and $p, q \in \mathbb{R}^d$.

Here we say for Ω to be of class C^1 if there is a function $\rho \in C^1(\mathbb{R}^d)$ which satisfies

$$\Omega = \{x \in \mathbb{R}^d \mid \rho(x) < 0\}, \quad D\rho(x) \neq 0 \quad \text{for all } x \in \partial\Omega.$$

Such a function ρ is called a defining function of Ω. Note that $\nu(x) = D\rho(x)/|D\rho(x)|$ at $x \in \partial\Omega$, where ν is an outer unit normal vector on $\partial\Omega$.

Theorem 4.4 (Comparison principle). *Assume (A1)-(A4). Let u and v be a subsolution and supersolution of* (4.1.1)-(4.1.2). *If $u(0,\cdot) \leq v(0,\cdot)$ on $\overline{\Omega}$, then $u \leq v$ on $\overline{\Omega_T}$.*

Proof. Suppose that the conclusion were false: $\sup_{\overline{\Omega_T}}(u-v) = (u-v)(\hat{t},\hat{x}) =: \theta > 0$. When $\hat{x} \in \Omega$, the argument is quite similar as in [12,], so we may assume that $\hat{x} \in \partial\Omega$.

For a parameter $\varepsilon > 0$ we consider the function

$$\Phi(t,x,s,y) = u(t,x) - v(s,y) - \frac{|t-s|^2 + |x-y|^2}{\varepsilon} - \delta(\rho(x) + \rho(y) + 2 + |x-\hat{x}|^2)$$

on $([0,T] \times \overline{\Omega})^2$, where $\delta > 0$ is a small constant. Let $(t_\varepsilon, x_\varepsilon, s_\varepsilon, y_\varepsilon)$ be a maximum point of Φ. From the inequality $\Phi(t_\varepsilon, x_\varepsilon, s_\varepsilon, y_\varepsilon) \geq \Phi(\hat{t},\hat{x},\hat{t},\hat{x}) = \theta$, that is,

$$\frac{|t_\varepsilon - s_\varepsilon|^2 + |x_\varepsilon - y_\varepsilon|^2}{\varepsilon} + \delta|x_\varepsilon - \hat{x}|^2 \leq u(t_\varepsilon, x_\varepsilon) - v(s_\varepsilon, y_\varepsilon) - \theta,$$

it follows that

$$\begin{cases} (t_\varepsilon, x_\varepsilon, s_\varepsilon, y_\varepsilon) \to (\tilde{t}, \hat{x}, \tilde{t}, \hat{x}), \\ |x_\varepsilon - y_\varepsilon|^2/\varepsilon \to 0, \\ u(t_\varepsilon, x_\varepsilon) \to u(\tilde{t}, \hat{x}), \quad v(s_\varepsilon, y_\varepsilon) \to v(\tilde{t}, \hat{x}) \end{cases}$$

as $\varepsilon \to 0$ (by taking a subsequence if necessary) for some $\tilde{t} \in [0,T]$ such that $(u - v)(\tilde{t},\hat{x}) = \theta$. Notice that $\tilde{t} > 0$.

By the viscosity property of u, it holds either

$$K_0[u](t_\varepsilon, x_\varepsilon) + H(x_\varepsilon, p_\varepsilon + \delta(D\rho(x_\varepsilon) + 2(x_\varepsilon - \hat{x}))) \le 0 \tag{4.3.1}$$

if $x_\varepsilon \in \Omega$ or

(4.3.2)

$$\min\{K_0[u](t_\varepsilon, x_\varepsilon)+H(x_\varepsilon, p_\varepsilon+\delta(D\rho(x_\varepsilon)+2(x_\varepsilon-\hat{x}))), \nu(x_\varepsilon)\cdot(p_\varepsilon+\delta(D\rho(x_\varepsilon)+2(x_\varepsilon-\hat{x})))\} \le 0$$

if $x_\varepsilon \in \partial\Omega$, where $p_\varepsilon = (x_\varepsilon - y_\varepsilon)/\varepsilon$. If $x_\varepsilon \in \partial\Omega$, since $\nu(x_\varepsilon) \cdot D\rho(x_\varepsilon) = |D\rho(x_\varepsilon)| > 0$,

$$\nu(x_\varepsilon) \cdot (p_\varepsilon + \delta(D\rho(x_\varepsilon) + 2(x_\varepsilon - \hat{x}))) \ge -\frac{|x_\varepsilon - y_\varepsilon|^2}{\hat{r}\varepsilon} + \delta(1 - 2|x_\varepsilon - \hat{x}|) \ge \frac{\delta}{2} > 0$$

for suitable small ε. Thus (4.3.2) is rewritten by the same form as (4.3.1). Similarly, by the viscosity property of v, we have

$$K_0[v](x_\varepsilon, y_\varepsilon) + H(y_\varepsilon, p_\varepsilon - \delta D\rho(y_\varepsilon)) \ge 0.$$

Applying (A4), we obtain

$$K_0[u](t_\varepsilon, x_\varepsilon) - K_0[v](s_\varepsilon, y_\varepsilon) + H(x_\varepsilon, p_\varepsilon) - H(y_\varepsilon, p_\varepsilon) \le \omega_2(\delta C_1) - \omega_2(\delta C_2), \tag{4.3.3}$$

where $C_1 =: \sup_{x\in\overline{\Omega}} |D\rho(x) + 2(x - \hat{x})|$ and $C_2 := \sup_{x\in\overline{\Omega}} |D\rho(x)|$.

Let us divide the term $K_0[u](t_\varepsilon, x_\varepsilon) - K_0[v](s_\varepsilon, y_\varepsilon)$ into three parts as follows:

$$I_{1,\varepsilon} = \frac{u(t_\varepsilon, x_\varepsilon) - u(0, x_\varepsilon)}{t_\varepsilon^\alpha} - \frac{v(s_\varepsilon, y_\varepsilon) - v(0, y_\varepsilon)}{s_\varepsilon^\alpha},$$

$$I_{2,\varepsilon} = \int_0^r (u(t_\varepsilon, x_\varepsilon) - u(t_\varepsilon - \tau, x_\varepsilon) - v(s_\varepsilon, y_\varepsilon) + v(s_\varepsilon - \tau, y_\varepsilon))\frac{d\tau}{\tau^{\alpha+1}}$$

and

$$I_{3,\varepsilon} = \int_r^{t_\varepsilon} (u(t_\varepsilon, x_\varepsilon) - u(t_\varepsilon - \tau, x_\varepsilon))\frac{d\tau}{\tau^{\alpha+1}} - \int_r^{s_\varepsilon} (v(s_\varepsilon, y_\varepsilon) - v(s_\varepsilon - \tau, y_\varepsilon))\frac{d\tau}{\tau^{\alpha+1}}.$$

Here $r > 0$ is a constant such that $0 < r < \min\{\tilde{t}, t_\varepsilon\}$, which exists by considering only small ε if necessary since $\bar{t} \to \tilde{t} > 0$ as $\varepsilon \to 0$. It is not hard to see that

$$\liminf_{\varepsilon\to 0} I_{1,\varepsilon} \ge \frac{u(\tilde{t},\hat{x}) - v(\tilde{t},\hat{x}) - (u(0,\hat{x}) - v(0,\hat{x}))}{\tilde{t}^\alpha}.$$

It follows immediately from the inequality $\Phi(t_\varepsilon, x_\varepsilon, s_\varepsilon, y_\varepsilon) \ge \theta > 0$ that the integrand of $I_{2,\varepsilon}$ is nonnegative. Hence $\liminf_{\varepsilon\to 0} I_{2,\varepsilon} \ge 0$. Since $\lim_{\varepsilon\to 0} u(t_\varepsilon, x_\varepsilon) = u(\tilde{t},\hat{x})$, for some constant $C > 0$, it holds that $u(t_\varepsilon, x_\varepsilon) \ge u(\tilde{t},\hat{x}) - C$ for suitably small ε. On the another hand, $u(t_\varepsilon - \tau, x_\varepsilon) \le \max_{[0,T]\times\overline{\Omega}} u$ by the upper semicontinuity of u. Totally we have

$$\begin{aligned}(u(t_\varepsilon, x_\varepsilon) - u(t_\varepsilon - \tau, x_\varepsilon))\mathbb{1}_{(r,t_\varepsilon)}(\tau) &\ge (u(\tilde{t}, \hat{x}) - C - \max_{[0,T]\times\overline{\Omega}} u)\mathbb{1}_{(r,\tilde{t})}(\tau) \\ &\ge -|u(\tilde{t}, \hat{x}) - C - \max_{[0,T]\times\overline{\Omega}} u|\mathbb{1}_{(r,T)}(\tau),\end{aligned}$$

where $\mathbb{1}_I$ is the indicator funcdtion on an interval I. The right-hand side multiplied by $\tau^{-\alpha-1}$ is integrable on $(0,T)$, so Fatou's lemma yields

$$\liminf_{\varepsilon\to 0} \int_r^{t_\varepsilon} (u(t_\varepsilon, x_\varepsilon) - u(t_\varepsilon - \tau, x_\varepsilon))\frac{d\tau}{\tau^{\alpha+1}} \ge \int_r^{\tilde{t}} (u(\tilde{t}, \hat{x}) - u(\tilde{t} - \tau, \hat{x}))\frac{d\tau}{\tau^{\alpha+1}}.$$

Similarly,

$$\limsup_{\varepsilon\to 0} \int_r^{s_\varepsilon} (v(s_\varepsilon, y_\varepsilon) - v(s_\varepsilon - \tau, y_\varepsilon))\frac{d\tau}{\tau^{\alpha+1}} \le \int_r^{\tilde{t}} (v(\tilde{t}, \hat{x}) - v(\tilde{t} - \tau, \hat{x}))\frac{d\tau}{\tau^{\alpha+1}}.$$

Recalling that $(u-v)(\tilde{t}, \hat{x}) = \theta = \sup_{[0,T]\times\overline{\Omega}}(u-v)$, we find that $(u-v)(\tilde{t}, \hat{x}) \ge (u-v)(\tilde{t} - \tau, \hat{x})$ for all $\tau \in (r, \tilde{t})$. This implies that $\liminf_{\varepsilon\to 0} I_{3,\varepsilon} \ge 0$.

Applying (A3) to (4.3.3) and then taking a limit inferior $\varepsilon \to 0$ yields

$$\frac{(u-v)(\tilde{t}, \hat{x}) - (u-v)(0, \hat{x})}{\tilde{t}^\alpha \Gamma(1-\alpha)} \le \omega_2(\delta C_1) - \omega_2(\delta C_2).$$

This is a contradiction for sufficiently small δ since $(u-v)(\tilde{t}, \hat{x}) = \theta > 0$ and $(u-v)(0, \hat{x}) \le 0$. □

4.4 Existence

Proposition 4.5 (Barriers). *There are a subsolution $u^- \in USC(\overline{\Omega_T})$ and a supersolution $u^+ \in LSC(\overline{\Omega_T})$ of* (4.1.1)-(4.1.3) *such that $u^-(t,x) \le u_0(x) \le u^+(t,x)$ for all $(t,x) \in \overline{\Omega_T}$ and*

$$\lim_{t\to 0} u^\pm(t,x) = u^\pm(0,x) = u_0(x)$$

for all $x \in \overline{\Omega}$.

Proof. For each $\varepsilon \in (0,1)$, we choose $u_{0,\varepsilon} \in C^1(\overline{\Omega})$ so that $|u_0(x) - u_{0,\varepsilon}(x)| < \varepsilon$ for all $x \in \overline{\Omega}$.

Let ρ be a defining function of Ω. Since $D\rho(z) = |D\rho(z)|\nu(z)$ for $z \in \partial\Omega$, there is a large constant $C_\varepsilon > 0$ such that

$$C_\varepsilon \nu(z) \cdot D\rho(z) \ge \max_{\partial\Omega} |\nu \cdot Du_{0,\varepsilon}|$$

for all $z \in \partial\Omega$.

Define the function

$$u_\varepsilon^\pm(t,x) = u_{0,\varepsilon}(x) \pm \left(\varepsilon \arctan\left(\frac{C_\varepsilon \rho(x)}{\varepsilon}\right) + 2\varepsilon\right) \pm \frac{M_\varepsilon t^\alpha}{\Gamma(1+\alpha)},$$

where $M_\varepsilon > 0$ is a large constant. Then u_ε^- and u_ε^+ are a viscosity subsolution and supersolution of (4.1.1)-(4.1.2), respectively. To see this we first note that $u_\varepsilon^\pm \in \mathcal{C}^1(\overline{\Omega_T})$. It is easy to see that $\nu(x)\cdot Du_\varepsilon^+(x) \geq 0$ and $\nu(x)\cdot Du_\varepsilon^-(x) \leq 0$ for all $x \in \partial\Omega$. We note that $|Du_\varepsilon^\pm|$ is bounded on $\overline{\Omega_T}$ and $\partial_t^\alpha u_\varepsilon^\pm = \pm M_\varepsilon$. For latter equality we used the well-known formula [24, (2.56)]. Hence if M_ε is taken sufficiently large, then we see that $u_\varepsilon^\pm$ satisfy the desired inequalities. Thus the assertion is immediately made clear.

Set $u^-(t,x) = (\sup\{u_\varepsilon^-(t,x) \mid \varepsilon \in (0,1)\})^*$ and $u^+(t,x) = (\inf\{u_\varepsilon^+(t,x) \mid \varepsilon \in (0,1)\})_*$. Here, for a real-valued function h defined on a set L in $\mathbf{R}^\ell$ ($\ell \in \mathbf{N}$), h^* and h_* denote, respectively, the upper semicontinuous envelope and the lower semicontinuous envelope, which are defined by

$$h^*(z) = \lim_{r \searrow 0} \sup\{h(\xi) \mid \xi \in L \cap \overline{B(z;r)}\}$$

and

$$h_*(z) = \lim_{r \searrow 0} \inf\{h(\xi) \mid \xi \in L \cap \overline{B(z;r)}\}$$

for $z \in \overline{L}$. Then u^- and u^+ are respectively a subolution and supersolution of (4.1.1)-(4.1.2). In fact, the closedness under supremum operator proved in [12] is easily extended to the current situation.

Since $-1 \leq \arctan r \leq 0$ for all $r \geq 0$, we see that

$$u_\varepsilon^-(t,x) \leq u_{0,\varepsilon}(x) - \varepsilon \leq u_0(x)$$

for all $x \in \overline{\Omega}$. Hence $u^- \leq u_0$ on $\overline{\Omega_T}$. Moreover, from the definition, we observe that $u^-(0,x) \geq \sup\{u_{0,\varepsilon}(x) - 2\varepsilon \mid \varepsilon \in (0,1)\} \geq u_0(x)$ for all $x \in \overline{\Omega}$. Thus $u^-(0,\cdot) = u_0$ on $\overline{\Omega}$. Simiarly, it follows that $u^+ \geq u_0$ on $\overline{\Omega_T}$ and that $u^+(0,x) \leq u_0$ on $\overline{\Omega}$, so that $u^+(0,x) = u_0$ on $\overline{\Omega}$. □

Theorem 4.6. *Assume (A1)-(A4) and that $u_0 \in C(\overline{\Omega})$. Then there exists at most one solution of* (4.1.1)-(4.1.3).

Proof. Let $u^\pm$ be functions constructed in Proposition 4.5. We denote by X a set of subsolutions v of (4.1.1)-(4.1.3) such that $v \leq u^+$ on $\overline{\Omega_T}$. Notice that $X \neq \emptyset$ since $u^- \in X$. We define

$$u(t,x) = \sup\{v(t,x) \mid v \in X\}.$$

Then it turns out by the closedness under supremum (cf; [12]) that u is a subsolution of (4.1.1)-(4.1.2). Also, if u were not a supersolution of (4.1.1)-(4.1.2), then there would exist a subsolution $U \in X$ of (4.1.1)-(4.1.2) such that $U > u$ at some point, which is a contradiction to the maximality of u. An existence of such U is easily checked from an analogous result of [12]. Accordingly, u is a supersolution of (4.1.1)-(4.1.2). Since $u^- \leq u \leq u^+$ on $\overline{\Omega_T}$ and $u^-(0,\cdot) = u^+(0,\cdot) = u_0$ on $\overline{\Omega}$, we see that $u(0,\cdot) = u_0$ on $\overline{\Omega}$. Therefore u is a solution of (4.1.1)-(4.1.3). □

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Chapter 5

Homogenization for Hamilton-Jacobi equations with State-Constraint Boundary Conditions

A homogenization problem for a Hamilton-Jacobi equation with state-constraint boundary condition on non-perforated domain is considered. To find a cell problem, the procedure of convergence is analyzed carefully.

Keywords: Hamilton-Jacobi equations; Homogenization; Non-perforated domain; State-constraint boundary conditions; Viscosity solutions

5.1 Introduction

In this chapter we consider the initial boundary value problem for Hamilton-Jacobi equations of the form

$$
\text{(SC)}_\varepsilon \quad
\begin{cases}
\partial_t u^\varepsilon + H\left(t, x, \frac{x}{\varepsilon}, Du^\varepsilon\right) = 0 & \text{in } (0,T)\times\Omega, & (5.1.1)\\
\partial_t u^\varepsilon + H\left(t, x, \frac{x}{\varepsilon}, Du^\varepsilon\right) \geq 0 & \text{on } (0,T)\times\partial\Omega, & (5.1.2)\\
u|_{t=0} = u_0 & \text{in } \overline{\Omega}. & (5.1.3)
\end{cases}
$$

Here $\varepsilon > 0, T > 0$ are given constants, Ω is a bounded domain in $\mathbb{R}^d$, $H : [0,T) \times \overline{\Omega} \times \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}$ is a given function called a *Hamiltonian* and $u_0 : \overline{\Omega} \to \mathbb{R}$ is a given initial data. Also, Du denotes a spatial gradient of an unknown function $u : (0,T] \times \overline{\Omega} \to \mathbb{R}$, i.e., $Du = (\partial u/\partial x_1, \cdots, \partial u/\partial x_d)$. The goal in this chapter is to clarify that a viscosity solution of (SC)$_\varepsilon$ converges to a viscosity solution of a proper limit equation as $\varepsilon \to 0$.

A boundary condition of the type of (5.1.2) is called a *state-constraint boundary condition*. Soner ([12]) first gave a definition of viscosity solutions for Hamilton-Jacobi equations with state-constraint boundary conditions and researched their properties; see also [**?**]. Such solutions are sometimes called a *constrained viscosity solutions*; see [2, Section IV.5]. Here let us recall the definition of constrained viscosity solutions for (SC)$_\varepsilon$ without the third variable of H, say (SC)$_*$, for the reader's convenience.

Definition 5.1 (Constrained viscosity solutions)**.** 1. An upper semicontinuous function $u : [0,T) \times \Omega \to \mathbb{R}$ is called a *viscosity subsolution* of (SC)$_*$ if $u(0,\cdot) \leq u_0$ on $\overline{\Omega}$ and it holds that

$$\partial_t \phi(\hat{t}, \hat{x}) + H(\hat{t}, \hat{x}, D\phi(\hat{t}, \hat{x})) \leq 0$$

whenever $u - \phi$ attains a maximum at $(\hat{t}, \hat{x}) \in (0,T) \times \Omega$ for $\phi \in C^1((0,T) \times \Omega)$.

2. An lower semicontinuous function $u : [0,T) \times \Omega \to \mathbb{R}$ is called a *viscosity supersolution* of (SC)$_*$ if $u(0,\cdot) \geq u_0$ on $\overline{\Omega}$ and it holds that

$$\partial_t \phi(\hat{t},\hat{x}) + H(\hat{t},\hat{x},D\phi(\hat{t},\hat{x})) \geq 0$$

whenever $u - \phi$ attains a minimum at $(\hat{t},\hat{x}) \in (0,T) \times \overline{\Omega}$ for $\phi \in C^1((0,T) \times \overline{\Omega})$.

3. If $u \in C([0,T) \times \overline{\Omega})$ is a both viscosity sub- and supersolution of (SC)$_*$, then u is called a *viscosity solution* of (SC)$_*$.

Throughout this chapter we only deal with constrained or usual viscosity solutions as a notion of solutions, so words "constrained" and "viscosity" are hereafter suppressed.

Hamilton-Jacobi equations with state-constraint boundary conditions arise in control problems under a special situation. More precisely, they are derived as a dynamic programming equation for an optimization problem with a control as a parameter of a given cost functional which states/dynamics are restricted in a given region. In more realistic situation there are much possibilities that dynamics can oscillate frequently by some external factors such as a heterogeneity of (restricted) region for instance. Our equation (SC)$_\varepsilon$ reflects such effects by including a small parameter $\varepsilon > 0$ or an oscillating variable x/ε. An interesting problem is what an optimal control for dynamics when $\varepsilon \to 0$ is. The problem to find an optimal control unfortunately may be difficult in general, so we focus on behavior of solutions of (SC)$_\varepsilon$ as $\varepsilon \to 0$. Such a problem is called a *homogenization problem*.

Homogenization problems for Hamilton-Jacobi equations was considered firstly by Lions, Papanicolaou and Varadhan ([11]). Since they established a homogenization result under quite general assumptions, their results have been referred in many literatures although it is unpublished paper. After that, Evans ([6]) improved it partially by using a *perturbed test function method* he suggested. There are few studies for pdes with state-constraints as far as we know. Kesavan and Muthukumar ([**?**]) considered the homogenization problem for equations of divergence form with state-constraint on both of a perforated domain and a non-perforated domain. Here the perforated domain means a domain with holes periodically. Although Horie and Ishii ([7]) considered for Hamilton-Jacobi equations, their result is only on the perforated domain.

In contract with these works our equation (SC)$_\varepsilon$ is considered on non-perforated domain. As with usual we need to consider a stationary equation called a *cell problem* but there occurs a difficulty to find it. Such a difficulty is similar for boundary value problems except for the Dirichlet problem, so let us share it now. For simplicity and explanation of a difficulty let us consider stationry Hamilton-Jacobi equations with Dirichlet boundary conditions

$$(\text{DP})_\varepsilon \quad \begin{cases} u^\varepsilon + H\left(x, \dfrac{x}{\varepsilon}, Du^\varepsilon\right) = 0 & \text{in } \Omega, \qquad (5.1.4) \\ u^\varepsilon = 0 & \text{on } \partial\Omega. \qquad (5.1.5) \end{cases}$$

To derive a cell problem let u^ε expand with respect to ε as follows:

$$u^\varepsilon(x) = u(x) + \varepsilon v\left(x, \frac{x}{\varepsilon}\right) + o(\varepsilon^2). \qquad (5.1.6)$$

Here v is a certain function mentioned later such that $y \mapsto v(\cdot, y)$ is $\mathbb{Z}^d$-periodic, i.e., $v(\cdot, y+z) = v(\cdot, y)$ for all $y \in \mathbb{R}^d$ and $z \in \mathbb{Z}^d$. Plugging (5.1.4) to (5.1.6) and identifying the terms in front of powers of ε, it follows formally that

$$u(x) + H(x, y, D_x u(x) + D_y v(x,y)) = 0 \quad \text{for } (t, y) \in \Omega \times \mathbb{R}^d.$$

Therefore we are led to the cell problem for (DP)$_\varepsilon$

$$H(x, y, Dv(y) + P) = \bar{H}(x, P) \quad \text{in } \mathbb{T}^d.$$

Here $\mathbb{T}^d := \mathbb{R}^d/\mathbb{Z}^d$ is a d-dimensional torus and a function v on $\mathbb{T}^d$ is regarded as a function defined on $\mathbb{R}^d$ with $\mathbb{Z}^d$-periodicity. The cell problem precisely consists with problems that is, for given $(x, P) \in \overline{\Omega} \times \mathbb{R}^d$, to find a unique constant $\bar{H}(x, P) \in \mathbb{R}$ such that the above equation admits a (viscosity) solution $v \in C(\mathbb{T}^d)$. Such a solution v is often called a *corrector*. Also, if we can solve the cell problem, then $\bar{H}(x, P)$ is regarded as a function corresponding to a variable (x, P) and such a function is often called an *effective Hamiltonian*. Under our assumptions, it is well-known that (CP) is solvable; see [**?**]. Moreover, the perturbed test function works well and hence a solution u^ε of (DP)$_\varepsilon$ converges to a solution u of the following equation uniformly in $\overline{\Omega}$ as $\varepsilon \to 0$:

$$\begin{cases} u(x) + \bar{H}(x, Du(x)) = 0 & \text{in } \Omega, \\ u(x) = 0 & \text{on } \partial\Omega. \end{cases}$$

A situation for (SC)$_\varepsilon$ is different a little. Arguing such as above, we have to handle also the implicit boundary condition (5.1.2). This means that the cell problem for (SC)$_\varepsilon$ may not does is not have a simpler form as (CP). However, in fact, such a special form for the cell problem is not needed. By focusing on this point, we will establish a homogenization result for (SC)$_\varepsilon$ under the following standard assumptions:

(A1) $H : [0, T) \times \overline{\Omega} \times \mathbb{T}^d \times \mathbb{R}^d$ is a continuous function,

(A2) H is coercive in p, i.e.,

$$\lim_{r \to \infty} \inf\{H(x, y, p) \mid x \in \overline{\Omega}, y \in \mathbb{R}^d, |p| \geq r\} = +\infty,$$

(A3) for some constant $L > 0$

$$|H(x, y, p) - H(x', y', p')| \leq L(|x - x'| + |y - y'| + |p - p'|)$$

for all $(x, x', y, y', p, p') \in \overline{\Omega}^2 \times \mathbb{R}^{4d}$,

(A4) Ω is a bounded, connected open set in $\mathbb{R}^d$ and, for $z \in \partial\Omega$, there exists a unit vector $\eta \in \mathbb{R}^d$ and a constant $c \in (0, 1)$ such that

$$B(x + \delta\eta, \delta c) \subset \Omega \quad \text{for all } x \in \overline{\Omega} \cap B(z, 2c) \text{ and } 0 < \delta \leq c,$$

(A4) $u_0 : \overline{\Omega} \to \mathbb{R}$ is a Lipschitz continuous function,

(A5) there exists a unique solution $u^\varepsilon \in C([0, T) \times \overline{\Omega})$ of (SC)$_\varepsilon$.

This chapter is organized as follows: In Section 6.2 we state main result and prove it.

5.2 Homogenization result

A main theorem in this chapter is

Theorem 5.2 (Homogenization result for (SC)$_\varepsilon$)**.** *Assume that (A1)-(A5). Then the solution u^ε of (SC)$_\varepsilon$ converges to the solution u of the following equation (SC) uniformly in $\overline{\Omega}$ as $\varepsilon \to 0$.*

$$\text{(SC)}\quad \begin{cases} \partial_t u + \bar{H}(t,x,Du) = 0 & \text{in } (0,T)\times\Omega, \quad (5.2.1)\\ \partial_t u + \bar{H}(t,x,Du) \geq 0 & \text{on } (0,T)\times\partial\Omega, \quad (5.2.2)\\ u|_{t=0} = u_0 & \text{in } \overline{\Omega}. \quad (5.2.3) \end{cases}$$

Let us here gather fundamental facts without proofs in order to use in a proof of Theorem 5.2.

Lemma 5.3 (Cell problems)**.** *Assume (A1)-(A3). Then for each $(t,x,P) \in (0,T)\times\overline{\Omega}\times\mathbb{R}^d$ there exists a unique constant $\bar{H}(t,x,P) \in \mathbb{R}$ such that*

$$H(t,x,D_y v(y) + P) = \bar{H}(t,x,P) \quad \text{in } \mathbb{T}^d \tag{5.2.4}$$

admits a solution $v \in C(\mathbb{T}^d)$.

This is a well-known fact; see [11].

Lemma 5.4 (Comparison principle for (SC))**.** *Assume (A1),(A2) and (A4). Let u and v be a subsolution and a supersolution of (SC), respectively. If $u(0,\cdot) \leq v(0,\cdot)$ on $\overline{\Omega}$, then $u \leq v$ on $[0,T)\times\overline{\Omega}$.*

This lemma can proved similarly as [8, Theorem 7.3]. Note that there is no difficulty to state the similar for (SC)$_\varepsilon$. The following proposition is easily established from Lemma 5.4.

Proposition 5.5 (Uniformly boundedness)**.** *Let u^ε be a solution of (SC)$_\varepsilon$. Then $\{u^\varepsilon\}_\varepsilon$ is uniformly bounded in $[0,T)\times\overline{\Omega}$.*

Proof of Theorem 2.1.1. Let u^+ and u^- be the *upper half-relaxed limit* and the *lower half-relaxed limit* for the family $\{u^\varepsilon\}_\varepsilon$ of solutions of (SC)$_\varepsilon$. More precisely, they are defined by

$$u^+(t,x) = \lim_{\delta\searrow 0} \sup\{u^\varepsilon(s,y) \mid (s,y) \in B_\delta(t,x) \cap ([0,T)\times\overline{\Omega}), \varepsilon < \delta\}$$

and $u^- = -(-u)^+$. Here $B_\delta(t,x)$ is an open ball with radius δ centered (t,x). Note that both of them are real-valued functions on $[0,T)\times\overline{\Omega}$ thanks to Proposition 5.5. Moreover, u^+ is an upper semicontinuous function and so u^- is a lower semicontinuous function; see, e.g., [**?**]. By the definition of u^+ and u^-, we see $u^- \leq u^+$ on $[0,T)\times\overline{\Omega}$. Thus, if u^+ and u^- are respectively a subsolution and a supersolution of (5.2.1)-(5.2.2), Theorem 5.2 concludes. That is why $u^+ = u^- =: u$ on $[0,T)\times\overline{\Omega}$ by Lemma 5.4 (comparison principle for (SC)) and hence u is a solution of (SC). It is proved by the almost same argument as [6, Theorem 3] that u^+ and u^- are a subsolution and a supersolution of (5.2.1), respectively. Therefore it is enough to prove that u^- satisfies the boundary condition (5.2.2). A proof is a slightly modification of [7, Proof of Proposition 1.7 and Theorem 1.2].

Assume that $u^- - \phi$ attains a strict local minimum at $(t_0, x_0) \in (0,T) \times \partial\Omega$ for $\phi \in C^1((0,T) \times \overline{\Omega})$. Then we shall show that

$$\partial_t \phi(t_0, x_0) + \bar{H}(t_0, x_0, D\phi(t_0, x_0)) \geq 0.$$

Let v be a solution of (5.2.4) with $(t, x, P) = (t_0, x_0, D\phi(t_0, x_0))$; see Lemma 5.3. By the definition of u^-, there are subsequences $\{\varepsilon_n\}_{n=1,2,\cdots}$ and $\{(t_n, x_n)\}_{n=1,2,\cdots}$ such that

(5.2.5)
$$\begin{cases} (\varepsilon_n, t_n, x_n, u^{\varepsilon_n}(t_n, x_n)) \to (0, t_0, x_0, u^-(t_0, x_0)) \text{ as } n \to \infty, \\ (t, x) \mapsto u^{\varepsilon_n}(t,x) - \phi(t,x) - \varepsilon_n v(x/\varepsilon_n) \text{ attains a local minimum at } (t_n, x_n). \end{cases}$$

From (A4), we get a unit vector $\eta_{\varepsilon_n} \in \mathbb{R}^d$ and a constant $c_{\varepsilon_n} \in (0,1)$ such that

(5.2.6)
$$B(x + \tau\eta_{\varepsilon_n}, \tau c_{\varepsilon_n}) \subset \Omega \quad \text{for } x \in \overline{\Omega} \cap B(x_n, 2c_{\varepsilon_n}) \text{ and } 0 < \tau \leq c_{\varepsilon_n}.$$

Fix any $n = 1, 2, \cdots$. Consider the auxiliary function

(5.2.7)
$$\Phi(t,x,y) := u^{\varepsilon_n}(t,x) - \phi(t,x) - \varepsilon_n v\left(\frac{y}{\varepsilon_n}\right) + \left|\frac{y-x}{\delta} - \eta_{\varepsilon_n}\right|^2 + |t-t_n|^2 + |x-x_n|^2$$

on $[0,T] \times \overline{\Omega} \times \overline{\Omega}$, where $\delta > 0$. Let $(t^\delta, x^\delta, y^\delta)$ be a minimum point of Φ over $[0,T] \times (\overline{\Omega} \cap B(x_n, c_{\varepsilon_n})) \times (\overline{\Omega} \cap B(x_n, c_{\varepsilon_n}))$. We hereafter assume that $\delta \leq c_\varepsilon$ so that $x_n + \delta\eta_{\varepsilon_n} \in \overline{\Omega} \cap B(x_n, c_{\varepsilon_n})$ from (5.2.6).

The inequality $\Phi(t^\delta, x^\delta, y^\delta) \leq \Phi(t_n, x_n, x_n + \delta\eta_{\varepsilon_n})$ implies that $(t^\delta, x^\delta, y^\delta) \to (t_n, x_n, x_n)$ and $(y^\delta - x^\delta)/\delta \to \eta_{\varepsilon_n}$ as $\delta \to 0$. Hence we may choose δ so small that

$$\max\left\{|t^\delta - t_n|, |x^\delta - y^\delta|, |x^\delta - x_n|, \left|\frac{y^\delta - x^\delta}{\delta} - \eta_{\varepsilon_n}\right|\right\} < \frac{c_{\varepsilon_n}}{2}.$$

This implies that $y^\delta \in B(x^\delta + \delta\eta_{\varepsilon_n}, \delta c_{\varepsilon_n})$ from the fourth value in the left-hand side, whence $x^\delta + \delta\eta_\varepsilon, y^\delta \in \Omega$ by (5.2.6). From the above Φ attains a local minimum at the interior point $(t^\delta, x^\delta, y^\delta) \in (0,T) \times \Omega \times \Omega$ over $[0,T] \times \overline{\Omega} \times \overline{\Omega}$.

Since u^{ε_n} is a supersolution of (SC)$_\varepsilon$, it holds that

$$\partial_t \phi(t^\delta, x^\delta) - 2(t^\delta - t_n) + H\left(t^\delta, x^\delta, \frac{x^\delta}{\varepsilon_n}, D\phi(t^\delta, x^\delta) - \frac{2}{\delta}\left(\frac{y^\delta - x^\delta}{\delta} - \eta_{\varepsilon_n}\right) - 2(x^\delta - x_n)\right) \geq 0.$$

Also, since v is a subsolution of (5.2.4),

$$H\left(x_0, \frac{y^\delta}{\varepsilon_n}, D\phi(t_0, x_0) - \frac{2}{\delta}\left(\frac{y^\delta - x^\delta}{\delta} - \eta_{\varepsilon_n}\right)\right) \leq \bar{H}(t_0, x_0, D\phi(t_0, x_0)).$$

Thus, from (A4), we obtain that

$$
\begin{aligned}
0 \leq & \partial_t \phi(t^\delta, x^\delta) - 2(t^\delta - t_n) + H\left(t^\delta, x^\delta, \frac{x^\delta}{\varepsilon_n}, D\phi(t^\delta, x^\delta) - \frac{2}{\delta}\left(\frac{y^\delta - x^\delta}{\delta} - \eta_{\varepsilon_n}\right) - 2(x^\delta - x_n)\right) \\
\leq & \partial_t \phi(t^\delta, x^\delta) - 2(t^\delta - t_n) + H\left(t_0, x_0, \frac{y^\delta}{\varepsilon_n}, D\phi(t_0, x_0) - \frac{2}{\delta}\left(\frac{y^\delta - x^\delta}{\delta} - \eta_{\varepsilon_n}\right)\right) \\
& + L\left(|x^\delta - x_0| + \frac{|y^\delta - x^\delta|}{\varepsilon_n} + 2|x^\delta - x_n| + |D\phi(t^\delta, x^\delta) - D\phi(t_0, x_0)|\right) \\
\leq & \partial_t \phi(t^\delta, x^\delta) - 2(t^\delta - t_n) + \bar{H}(t_0, x_0, D\phi(t_0, x_0)) \\
& + L\left(|x^\delta - x_0| + \frac{|y^\delta - x^\delta|}{\varepsilon_n} + 2|x^\delta - x_n| + |D\phi(t^\delta, x^\delta) - D\phi(t_0, x_0)|\right).
\end{aligned}
$$

Passing to the limit $\delta \to 0$ implies that

$$
\partial_t \phi(t_n, x_n) + \bar{H}(t_0, x_0, D\phi(t_0, x_0)) \geq L(|x_n - x_0| + |D\phi(t_n, x_n) - D\phi(t_0, x_0)|).
$$

Since this inequality holds for all $n = 1, 2, \cdots$, we get the desired inequality when sending $n \to \infty$. $\square$

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