博士論文

論文題目　Applications of Fraïssé theory to operator algebras

（Fraïssé 理論の作用素環への応用）

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Applications of Fraïssé Theory to Operator Algebras

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Introduction

Fraïssé theory was originally invented by Rolland Fraïssé in [5]. It is a topic in model theory where a bijective correspondence between certain classes consisting of finitely generated structures and countable structures with a certain homogeneity property is established. The classes and homogeneous structures in this context are called Fraïssé classes and Fraïssé limits respectively.

We shall illustrate this theory with an example. Let

$$\mathcal{A}_1 \longrightarrow \mathcal{A}_2 \longrightarrow \mathcal{A}_3 \longrightarrow \cdots$$

be an inductive system of finite totally ordered sets and order-preserving embeddings. Then its limit is a countable totally ordered set, and obviously all the countable totally ordered sets can be obtained in this way; and here, we wonder whether every countable totally ordered set appears equally often.

In order to make our question meaningful, we shall assume that the inductive system above is constructed in the following way. The first totally ordered set $\mathcal{A}_1$ is a singleton with the trivial ordering. At step n, we obtain the inclusion $\mathcal{A}_{n-1} \to \mathcal{A}_n$ by adding a point, say a_n, to $\mathcal{A}_{n-1}$ and extending the order on $\mathcal{A}_{n-1}$ to $\mathcal{A}_{n-1} \cup \{a_n\}$ at random. Then, since the ordering of the limit structure is determined by the successive random choices of extensions, the tendency of appearance of a countable totally ordered set can be measured in terms of probability. Now, one can easily verify that an inductive system which is obtained as above will almost surely satisfy the following property: For any two elements $a, b \in \mathcal{A}_n$ with $a < b$, there exist $m > n$ and $c, d, e \in \mathcal{A}_m$ with $c < a < d < b < e$. Therefore, if

$$\mathcal{B}_1 \longrightarrow \mathcal{B}_2 \longrightarrow \mathcal{B}_3 \longrightarrow \cdots$$

is another inductive system, then by passing to subsystems if necessary, one can construct a commuting diagram as follows.

$$\begin{tikzcd}
\mathcal{A}_1 \arrow[r] \arrow[d, "\varphi_1"] & \mathcal{A}_2 \arrow[r] \arrow[d, "\varphi_2"] & \mathcal{A}_3 \arrow[r] \arrow[d, "\varphi_3"] & \cdots \\
\mathcal{B}_1 \arrow[r] \arrow[ur, "\psi_1"] & \mathcal{B}_2 \arrow[r] \arrow[ur, "\psi_2"] & \mathcal{B}_3 \arrow[r] \arrow[ur, "\psi_3"] & \cdots
\end{tikzcd}$$

The order-preserving embeddings φ_n and ψ_n induce embeddings between the inductive limits, which are clearly inverses to each other and so are isomorphisms. In other words, almost all inductive systems have the same limit (up to isomorphisms). Moreover, since the first embedding φ_1 above can be chosen fairly arbitrarily, one can see that the limit has the following homogeneity property: If $\mathcal{A}$ and $\mathcal{B}$ are finite subsets of the limit, and if $\varphi\colon \mathcal{A} \to \mathcal{B}$ is an order-preserving bijection, then there exists an automorphism α of the limit which extends φ. On the other hand, since the limit is clearly a self-dense countable totally ordered set without the minimum or the maximum, it is isomorphic to the rationals $(\mathbb{Q}, <)$, and we notice that it is the unique countable totally ordered set with this homogeneity property.

Similar arguments appear here and there in mathematics. For example, instead of totally ordered sets, one can use undirected finite simple graphs in the above argument, in which case the resulting inductive limit will be almost surely what is called the Rado graph. One may also use finite dimensional extensions of a fixed countable field K, and he will obtain the algebraic closure $\overline{K}$ as the limit. Fraïssé realized that these arguments can be formulated in terms of model theory.

By definition, a *Fraïssé* class is a class $\mathcal{K}$ of finitely generated first order structures which satisfies the following axioms.

- Up to isomorphisms, $\mathcal{K}$ contains only countably many structures.
- HP: A substructure of a member of $\mathcal{K}$ is itself a member of $\mathcal{K}$.
- JEP: Any two members of $\mathcal{K}$ can be embedded into a third one.
- AP: If $\iota_1\colon \mathcal{A} \to \mathcal{B}_1$ and $\iota_2\colon \mathcal{A} \to \mathcal{B}_2$ are embeddings of members of $\mathcal{K}$, then there are embeddings η_1, η_2 of $\mathcal{B}_1, \mathcal{B}_2$ into some member C of $\mathcal{K}$ such that the diagram

$$\begin{tikzcd}
\mathcal{A} \arrow[r, "\iota_1"] \arrow[d, "\iota_2"] & \mathcal{B}_1 \arrow[d, "\eta_1"] \\
\mathcal{B}_2 \arrow[r, "\eta_2"] & C
\end{tikzcd}$$

  commutes.

In the case of totally ordered sets, the class of all finite totally ordered sets satisfies these properties, so it is a Fraïssé class. One can show that every Fraïssé class admits a "$\mathbb{Q}$-like" structure as above: If $\mathscr{K}$ is a Fraïssé class, then there exists a unique countable structure $\mathcal{M}$ such that a *generic* inductive system of members of $\mathscr{K}$ has $\mathcal{M}$ as its limit. Moreover, the structure $\mathcal{M}$ is *ultra-homogeneous* in the sense that every isomorphism between finitely generated substructures extends to an automorphism of $\mathcal{M}$. Conversely, if $\mathcal{M}$ is an ultra-homogeneous structure, then the class $\operatorname{Age} \mathcal{M}$ of all finitely generated structures embeddable into $\mathcal{M}$ is a Fraïssé class.

Fraïssé theory has been, among the rest, a target of generalization to the setting of metric structures. Roughly, a metric structure is a complete metric space together with continuous operations and relations. Typical examples are Banach spaces and Banach algebras. In [1], Itaï Ben Yaacov concisely gave a general theory for them. In his theory, the generic limit structures are the *approximately* ultra-homogeneous ones. Here, a separable metric structure $\mathcal{M}$ is said to be approximately ultra-homogeneous if every isomorphism between finitely generated substructures can be approximated by automorphisms of $\mathcal{M}$ with respect to the point-distance topology. Corresponding to this difference, the axioms for Fraïssé classes is also modified. First, the axiom AP is replaced with its approximate version, which is called NAP. Also, the countability condition is replaced with what is called PP, which claims that the class is separable and complete with respect to a variant of Gromov–Hausdorff distance. Finally, concerning with the continuity of the equipped operations and relations, an axiom called CP is added. He showed the similar results within this framework and pointed out that the Urysohn universal space, the separable infinite dimensional Hilbert space, the atomless standard probability space and the Gurariĭ space can be recognized as Fraïssé limits of suitable classes. In [3], a relaxed version of Itaï's theory has been considered and applied to several examples of operator algebras. In this version, HP is omitted and PP and CP are replaced with weaker versions called WPP and CPP respectively.

In this thesis, we consider a generalization of these theories, which can be also applied to categories of first order metric structures and embeddings between them. Then we apply it to two instances of operator algebras, i.e., UHF algebras and the Jiang–Su algebra. We note that these operator algebras are also dealt with in [3], but our approach is quite different.

A UHF algebra is defined as a C*-algebra which is obtained as an inductive limit of a sequence of the form

$$\mathbb{M}_{n_1} \longrightarrow \mathbb{M}_{n_2} \longrightarrow \mathbb{M}_{n_3} \longrightarrow \cdots,$$

where $\mathbb{M}_n$ denotes the C*-algebra of all $n \times n$ matrices and the maps are unital $*$-homomorphisms. To each such inductive system, one can assign the formal least common divisor of $\{n_i\}$, which is a formal product of the form

$$\nu = \prod_{p\text{: prime}} p^{m_p},$$

where m_p is either non-negative integer or ∞. James G. Glimm proved in [6] that this formal product is a complete invariance for the UHF algebras. We shall denote the UHF algebra corresponding to ν by $\mathbb{M}_\nu$.

The Jiang–Su algebra $\mathcal{Z}$ was introduced in [8] as the unique simple monotracial inductive limit of what we call prime dimension drop algebras. It is a nuclear infinite dimensional C*-algebra which is KK-equivalent to the complex numbers $\mathbb{C}$ and tensorially self-absorbing (i.e., $\mathcal{Z} \otimes \mathcal{Z} \simeq \mathcal{Z}$), so that it plays a key role in the Elliott's classification program of separable nuclear C*-algebras via K-theoretic invariants (for the detail, see [4] for example).

In our approach, we consider categories of C*-algebras of continuous matrix-valued functions on cubes with distinguished faithful traces and unital trace-preserving $*$-homomorphisms. Model theoretically, these C*-algebras are dealt with as unital tracial C*-algebras. For a UHF algebra $\mathbb{M}_\nu$, we consider a category $\mathscr{K}_\nu$, and for the Jiang–Su algebra, we consider a category $\mathscr{K}_{\mathcal{Z}}$. It is shown that these are Fraïssé category the generic limit of which is the corresponding C*-algebra with the unique trace.

This thesis is organized as follows. The first chapter is devoted to the Fraïssé theory for categories of metric structures. In Sections 1.1 and 1.2, we introduce approximate isometries and approximate isomorphisms. The existence and uniqueness of a Fraïssé limit is proved in Section 1.3. In the second chapter, we deal with the applications of the theory to UHF algebras and the Jiang–Su algebra. The contents of Section 1.1 are from Itaï's paper [1], while the others are from the author's papers [10, 11, 12].

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Chapter 1

Fraïssé theory for metric structures

1.1 Approximate isometries

Let X and Y be metric spaces. We denote by $\mathrm{JE}(X, Y)$ the set of all pairs (ι, η), where $\iota\colon X \to Z$ and $\eta\colon Y \to Z$ are isometries into some metric space Z. Each element of $\mathrm{JE}(X, Y)$ is called a *joint embedding* of X and Y.

Definition 1.1.1. (1) Let X be a metric space. A map $\varphi\colon X \to [0, \infty]$ is said to be *Katětov* if it satisfies the inequalities

$$\varphi(x) \leq d_X(x, x') + \varphi(x'), \quad d_X(x, x') \leq \varphi(x) + \varphi(x')$$

for all $x, x' \in X$.

(2) Suppose that X and Y are metric spaces. An *approximate isometry* from X to Y is a map $\varphi\colon X \times Y \to [0, \infty]$ which is separately Katětov:

$$\varphi(x, y) \leq d(x, x') + \varphi(x', y), \quad d(x, x') \leq \varphi(x, y) + \varphi(x', y),$$
$$\varphi(x, y) \leq d(y, y') + \varphi(x, y'), \quad d(y, y') \leq \varphi(x, y) + \varphi(x, y').$$

The class of all approximate isometries from X to Y is denoted by $\mathrm{Apx}(X, Y)$. Note that, being a closed subset of $[0, \infty]^{X \times Y}$, the space $\mathrm{Apx}(X, Y)$ is compact and Hausdorff with respect to the topology of pointwise convergence.

Intuitively, an approximate isometry is a condition to be imposed on joint embeddings. A joint embedding $(\iota, \eta) \in \mathrm{JE}(X, Y)$ is considered to satisfy an approximate isometry φ from X to Y if the inequality

$$d\big(\iota(x), \eta(y)\big) \leq \varphi(x, y)$$

holds for all $x \in X$ and $y \in Y$. We shall denote by $\mathrm{JE}^{\leq\varphi}(X, Y)$ the class of all joint embeddings satisfying φ. Clearly, the condition $\varphi \equiv \infty$ is the weakest condition. Note that if an approximate isometry φ from X to Y takes a finite value at some point, then it is real-valued, because if $\varphi(x_0, y_0) < \infty$, then

$$\varphi(x, y) \leq d_X(x, x_0) + \varphi(x_0, y_0) + d(y_0, y) < \infty.$$

Example 1.1.2. (1) If (ι, η) is a joint-embedding of X and Y, then the map

$$(x, y) \mapsto d\big(\iota(x), \eta(y)\big)$$

is an approximate isometry. We shall denote this approximate isometry by $\varphi_{\iota,\eta}$. In the case that ι is an isometry from X into Y and η is equal to id_Y, the approximate isometry $\varphi_{\iota,\eta}$ is simply written as φ_ι.

We shall show that every approximate isometry is of this form unless it is equal to ∞. To see this, let $\varphi\colon X \times Y \to [0, \infty)$ be an approximate isometry and define a symmetric function $\delta\colon (X \coprod Y)^2 \to [0, \infty)$ by

$$\delta(z, z') = \begin{cases} d_X(z, z') & \text{if } z, z' \in X, \\ \varphi(z, z') & \text{if } z \in X \text{ and } z' \in Y, \\ d_Y(z, z') & \text{if } z, z' \in Y. \end{cases}$$

Then it is easy to see that δ is a pseudo-metric. If ι and η are canonical embeddings of X and Y into the quotient metric space $X \coprod_\varphi Y$, then $d\big(\iota(x), \eta(y)\big) = \varphi(x, y)$, as desired. It follows that, for any approximate isometries φ and ψ from X to Y, the inequality $\varphi \leq \psi$ holds if and only if $\mathrm{JE}^{\leq\varphi}(X, Y)$ is included in $\mathrm{JE}^{\leq\psi}(X, Y)$, so the order $\leq$ completely reflects the strength of conditions.

Note that a net $\{\iota_\alpha\}$ of isometries from X into Y converges pointwise to an isometry ι if and only if $\{\varphi_{\iota_\alpha}\}$ converges to φ_ι. Indeed, if $\{\iota_\alpha\}$ converges to ι, then

$$\varphi_{\iota_\alpha}(x, y) = d\big(\iota_\alpha(x), y\big) \to d\big(\iota(x), y\big) = \varphi_\iota(x, y)$$

for all $x \in X$ and $y \in Y$. Conversely, if $\{\varphi_{\iota_\alpha}\}$ converges to φ_ι, then for any $x \in X$ we have

$$d\big(\iota_\alpha(x), \iota(x)\big) = \varphi_{\iota_\alpha}\big(x, \iota(x)\big) \to \varphi_\iota\big(x, \iota(x)\big) = d\big(\iota(x), \iota(x)\big) = 0.$$

(2) For an approximate isometry φ from X to Y, we set

$$\varphi^*(y, x) := \varphi(x, y).$$

Then clearly φ^* is an approximate isometry from Y to X. We call φ^* the *pseudo-inverse* of φ.

(3) Given $\varphi \in \mathrm{Apx}(X, Y)$ and $\psi \in \mathrm{Apx}(Y, Z)$, we define their *composition* by

$$\psi\varphi(x, z) := \inf_{y \in Y} \bigl(\varphi(x, y) + \psi(y, z)\bigr).$$

Here, we shall check that $\psi\varphi$ is an approximate isometry from X to Z. Indeed, if x and x' are points of X, then

$$\begin{aligned}\psi\varphi(x, z) &= \inf_{y \in Y} \bigl(\varphi(x, y) + \psi(y, z)\bigr)\\ &\le \inf_{y \in Y} \bigl(d(x, x') + \varphi(x', y) + \psi(y, z)\bigr)\\ &= d(x, x') + \psi\varphi(x'z)\end{aligned}$$

and

$$\begin{aligned}d(x, x') &\le \inf_{y, y' \in Y} \bigl(\varphi(x, y) + d(y, y') + \varphi(x', y')\bigr)\\ &\le \inf_{y, y' \in Y} \bigl(\varphi(x, y) + \psi(y, z) + \varphi(x', y') + \psi(y', z)\bigr)\\ &= \psi\varphi(x, z) + \psi\varphi(x'z),\end{aligned}$$

so $\psi\varphi(\,\cdot\,, z)$ is Katětov for all $z \in Z$. By symmetry, $\psi\varphi(x, \,\cdot\,)$ is also Katětov for all $x \in X$, so $\psi\varphi$ is an approximate isometry.

It is worth noting that if $(\iota_1, \iota_2) \in \mathrm{JE}^{\le\varphi}(X, Y)$ and $(\iota_2, \iota_3) \in \mathrm{JE}^{\le\psi}(Y, Z)$, then $(\iota_1, \iota_3) \in \mathrm{JE}^{\le\psi\varphi}(X, Z)$, and $\psi\varphi$ is the smallest approximate isometry satisfying this property. Also, it can be easily seen that the equality $\varphi_{\iota,\eta} = \varphi_\eta^*\varphi_\iota$ holds for any joint embedding (ι, η).

(4) Let $X' \subseteq X$ and $Y' \subseteq Y$ be subspaces. If φ is an approximate isometry from X to Y, then its restriction $\varphi|_{X' \times Y'}$ is an approximate isometry from X' to Y'. Note that, if $\iota \colon X' \to X$ and $\eta \colon Y' \to Y$ are the canonical embeddings, then $\varphi|_{X' \times Y'}$ is equal to $\varphi_\eta^*\varphi\varphi_\iota$. Now suppose that ψ is an approximate isometry from X' to Y'. The *trivial extension* of ψ to $X \times Y$ is defined by $\psi|^{X \times Y} := \varphi_\eta\psi\varphi_\iota^*$. It is easy to show that $\psi|^{X \times Y}$ is the largest approximate isometry such that the restriction to $X' \times Y'$ is equal to ψ. More generally, an approximate isometry θ from X to Y satisfies $\theta \le \psi|^{X \times Y}$ if and only if $\theta|_{X' \times Y'} \le \psi$.

(5) If φ is an approximate isometry from X to Y and ε is a non-negative real number, then the *relaxation* of φ by ε is defined by $(x, y) \mapsto \varphi(x, y) + \varepsilon$. We simply denote this approximate isometry by $\varphi + \varepsilon$. Note that the operation of taking relaxations commutes with that of taking compositions.

Definition 1.1.3. An approximate isometry φ from X to Y is said to be

- *ε-total* if $\varphi^*\varphi \le \varphi_{\mathrm{id}_X} + 2\varepsilon$.
- *ε-surjective* if φ^* is ε-total.
- *ε-bijective* if φ is ε-total and ε-surjective.

If φ and ψ are approximate isometries from X to Y with $\psi \le \varphi$, then clearly $\psi^*\psi \le \varphi^*\varphi$. Therefore, if φ is ε-total, then so is ψ. Similarly, if φ is ε-surjective, then so is ψ.

Proposition 1.1.4. *An approximate isometry φ from X to Y is ε-total if and only if any $(\iota, \eta) \in \mathrm{JE}^{\le\varphi}(X, Y)$ satisfies $d\big(\iota(x), \eta[Y]\big) \le \varepsilon$ for each $x \in X$. In particular, if Y is complete and φ is ε-total for any ε, then it is of the form φ_ι for a unique isometry $\iota\colon X \to Y$.*

Proof. Suppose that φ is ε-total and let (ι, η) be in $\mathrm{JE}^{\le\varphi}(X, Y)$. Then, for any $x \in X$, we have

$$\begin{aligned} 2 \inf_{y\in Y} d\big(\iota(x), \eta(y)\big) &\le \inf_{y\in Y}\big(\varphi(x, y) + \varphi^*(y, x)\big) = \varphi^*\varphi(x, x) \\ &\le \varphi_{\mathrm{id}}(x, x) + 2\varepsilon = 2\varepsilon, \end{aligned}$$

so $d\big(\iota(x), \eta[Y]\big) \le \varepsilon$.

Conversely, suppose that $d\big(\iota(x), \eta[Y]\big) \le \varepsilon$ holds for any $(\iota, \eta) \in \mathrm{JE}^{\le\varphi}(X, Y)$ and any $x \in X$. Then $\varphi \not\equiv \infty$, so it is of the form $\varphi_{\iota,\eta}$, and

$$\begin{aligned} \varphi^*\varphi(x, x') &= \inf_{y\in Y}\Big(d\big(\iota(x), \eta(y)\big) + d\big(\eta(y), \iota(x')\big)\Big) \\ &\le d(x, x') + 2 \inf_{y\in Y} d\big(\iota(x), \eta(y)\big) \\ &\le \varphi_{\mathrm{id}}(x, x') + 2\varepsilon. \end{aligned}$$

□

Let φ be an approximate isometry from X to Y. We set

$$\mathrm{Apx}^{\le\varphi}(X, Y) := \{\psi \in \mathrm{Apx}(X, Y) \mid \psi \le \varphi\}.$$

We also denote by $\mathrm{Apx}^{\lhd\varphi}(X, Y)$ the interior of the closed subset $\mathrm{Apx}^{\le\varphi}(X, Y)$ of the compact Hausdorff space $\mathrm{Apx}(X, Y)$, and write $\psi \lhd \varphi$ or $\varphi \rhd \psi$ if ψ belongs to $\mathrm{Apx}^{\lhd\varphi}(X, Y)$. If $\mathrm{Apx}^{\lhd\varphi}(X, Y)$ is nonempty, then φ is said to be *strict*. The class of all strict approximate isometries is denoted by $\mathrm{Stx}(X, Y)$.

It can be easily verified that the relation $\lhd$ is preserved under restrictions and trivial extensions. In particular, restrictions and trivial extensions of a strict approximate isometries are strict.

Proposition 1.1.5. *For $\varphi, \psi \in \mathrm{Apx}(X, Y)$, the following are equivalent.*

(i) *The relation $\psi \lhd \varphi$ holds.*

(ii) *There exist finite subsets $X_0 \subseteq X$ and $Y_0 \subseteq Y$ and a positive real number ε such that the inequality*

$$\varphi \geq (\psi|_{X_0 \times Y_0})|^{X \times Y} + \varepsilon$$

holds.

(iii) *Same as* (ii)*, with $\geq$ replaced by $\rhd$.*

Moreover, if these conditions are satisfied, then there exist finite subsets $X_0 \subseteq X$ and $Y_0 \subseteq Y$ and a rational-valued approximate isometry $\rho \in \mathrm{Apx}(X_0, Y_0)$ such that the relation $\psi \lhd \rho|^{X \times Y} \lhd \phi$ holds.

Proof. First, suppose (i) holds. Then there exist finite subsets $X_0 \subseteq X$ and $Y_0 \subseteq Y$ and a positive real number ε such that the open neighborhood

$$U := \left\{ \psi' \in \mathrm{Apx}(X, Y) \;\middle|\; |\psi'(x, y) - \psi(x, y)| < 2\varepsilon \text{ for any } x \in X_0,\ y \in Y_0 \right\}$$

is included in $\mathrm{Apx}^{\leq \varphi}(X, Y)$. Clearly $(\psi|_{X_0 \times Y_0})|^{X \times Y} + \varepsilon$ belongs to U, so (iii) follows.

It is trivial that (iii) implies (ii). Now assume (ii). Since $\lhd$ is preserved under trivial extensions, the relation $\psi|_{X_0 \times Y_0} \lhd \psi|_{X_0 \times Y_0} + \varepsilon$ implies

$$\psi \leq (\psi|_{X_0 \times Y_0})|^{X \times Y} \lhd (\psi|_{X_0 \times Y_0})|^{X \times Y} + \varepsilon \leq \varphi,$$

so (i) holds.

Finally, in order to find ρ as in the statement, suppose $\psi \lhd \varphi$. Let X_0, Y_0 be as in the proof of (i) $\Rightarrow$ (iii) above, and $F_1, \ldots, F_n$ be the partition of $X_0 \times Y_0$ induced by ψ. Without loss of generality, we may assume $\psi|_{F_1} < \cdots < \psi|_{F_n}$. Take a function $\delta \colon X_0 \times Y_0 \to (0, \varepsilon)$ so that

- δ is constant on each F_i;
- $\delta|_{F_n} < \psi|_{F_n} - \psi|_{F_{n-1}}$;
- $\delta|_{F_i} < \min\{\delta|_{F_{i+1}}, \psi|_{F_i} - \psi|_{F_{i-1}}\}$ for $i = 2, \ldots, n-1$;
- $\delta|_{F_1} < \min\{\delta|_{F_2}, \psi|_{F_1}\}$; and
- $\rho := \psi|_{X_0 \times Y_0} - \delta + \varepsilon$ is a rational valued function on $X_0 \times Y_0$.

We shall check that ρ is separately Katětov so that it is an approximate isometry. The inequality

$$d(x, x') \leq \rho(x, y) + \rho(x', y)$$

is obvious, because $\rho \geq \psi|_{X_0 \times Y_0}$. On the other hand, for $(x, y) \in F_i$ and $(x', y) \in F_j$ with $i < j$, we have

$$\begin{aligned} \rho(x, y) &= \psi|_{F_i} + \varepsilon - \delta|_{F_i} \\ &= \psi|_{F_j} + \varepsilon - \delta|_{F_j} - \left[\left((\psi|_{F_j} - \psi|_{F_i}) - \delta_{F_j} \right) + \delta|_{F_i} \right] \\ &\leq \psi|_{F_j} + \varepsilon - \delta|_{F_j} = \rho(x', y) \\ &\leq d(x, x') + \rho(x', y) \end{aligned}$$

and

$$\begin{aligned} \rho(x', y) &= \psi(x', y) + \varepsilon - \delta|_{F_j} \\ &\leq d(x', x) + \psi(x, y) + \varepsilon - \delta|_{F_i} \\ &= d(x', x) + \rho(x, y), \end{aligned}$$

so $\rho(\,\cdot\,, y)$ is Katětov for each $y \in Y_0$. By symmetry, $\rho(x, \,\cdot\,)$ is also Katětov, whence ρ is an approximate isometry. Since clearly

$$\psi|_{X_0 \times Y_0} \lhd \rho \lhd \varphi|_{X_0 \times Y_0},$$

the conclusion follows. □

1.2 Metric structures and approximate isomorphisms

By definition, a *language* is a set L such that each element of L is either a *function symbol* or a *relation symbol*. To each symbol S is associated a natural number n_S which is called the *arity* of S, and a symbol with arity n is called an n-ary symbol. A 0-ary function symbol is often called a *constant symbol*.

An L-*structure* $\mathcal{M}$ is a complete metric space M, which is called the *domain* of $\mathcal{M}$, together with an *interpretation* of symbols of L:

- to each n-ary relation symbol R is assigned a continuous map $R^{\mathcal{M}}$ from M^n to $\mathbb{R}$; and
- to each n-ary function symbol f is assigned a continuous map $f^{\mathcal{M}}$ from M^n to M.

For an L-structure $\mathcal{M}$, we shall denote its domain by $|\mathcal{M}|$.

An L-*embedding* of an L-structure $\mathcal{N}$ into another L-structure $\mathcal{M}$ is an isometry ι from $|\mathcal{N}|$ into $|\mathcal{M}|$ such that

- for any n-ary relation symbol R and any elements $a_1, \dots, a_n \in |\mathcal{N}|$, the equation
$$R^{\mathcal{N}}(a_1, \dots, a_n) = R^{\mathcal{M}}\big(\iota(a_1), \dots, \iota(a_n)\big)$$
holds, and
- for any n-ary function symbol f and any elements $a_1, \dots, a_n \in |\mathcal{N}|$, the equation
$$\iota\big(f^{\mathcal{N}}(a_1, \dots, a_n)\big) = f^{\mathcal{M}}\big(\iota(a_1), \dots, \iota(a_n)\big)$$
holds.

For an L-embedding $\iota\colon \mathcal{N} \to \mathcal{M}$ and a tuple $\bar{a} = (a_1, \dots, a_n) \in |\mathcal{N}|^n$, we shall write the tuple $\big(\iota(a_1), \dots, \iota(a_n)\big) \in |\mathcal{M}|^n$ as $\iota(\bar{a})$.

For a subset E of an L-structure $\mathcal{M}$, the L-substructure generated by E is denoted by $\langle E \rangle$. Note that the domain of an L-structure is assumed to be complete, so if D_E is the set of all elements of the form $g(a_1, \dots, a_n)$ where g is a composition of the functions equipped with $\mathcal{M}$ and $a_1, \dots, a_n$ are elements of E, then $\langle E \rangle$ is equal to the closure of D_E. The subset E is said to be a *generator* of $\mathcal{M}$ if $\langle E \rangle$ coincides with $\mathcal{M}$, and $\mathcal{M}$ is said to be *finitely generated* if there exists finite E with $\langle E \rangle = \mathcal{M}$. A tuple $\bar{a} = (a_1, \dots, a_n) \in |\mathcal{M}|^n$ is called an *ordered generator* if $\{a_i \mid i = 1, \dots, n\}$ is a generator of $\mathcal{M}$.

In the sequel, we fix a language L and a category $\mathscr{K}$ of finitely generated L-structures and L-embeddings. Embeddings and isomorphisms in $\mathrm{Mor}(\mathscr{K})$ are often referred to as *$\mathscr{K}$-embeddings* and *$\mathscr{K}$-isomorphisms* respectively. A *joint $\mathscr{K}$-embedding* is a joint embedding (ι, η) such that both ι and η are $\mathscr{K}$-embeddings. We denote by $\mathrm{JE}_{\mathscr{K}}(\mathcal{A}, \mathcal{B})$ the class of all joint $\mathscr{K}$-embeddings of $\mathcal{A}$ and $\mathcal{B}$.

In the preceding section, approximate isometries were explained as conditions to be imposed on joint embeddings. In the setting of metric structures, we are interested in approximate isometries which can be (approximately) satisfied by joint $\mathscr{K}$-embeddings, which lead us to the following definition.

Definition 1.2.1. (1) Let $\mathcal{A}, \mathcal{B}$ be objects of $\mathscr{K}$ and $\iota\colon \mathcal{A} \dashrightarrow \mathcal{B}$ be a finite partial isometry, that is, an isometry between finite subsets of $|\mathcal{A}|$ and $|\mathcal{B}|$. Then ι is called a *finite partial $\mathscr{K}$-isomorphism* if

- the L-substructures $\langle \operatorname{dom} \iota \rangle$ and $\langle \operatorname{ran} \iota \rangle$ are objects of $\mathscr{K}$;
- the canonical embeddings $\langle \operatorname{dom} \iota \rangle \to \mathcal{A}$ and $\langle \operatorname{ran} \iota \rangle \to \mathcal{B}$ are $\mathscr{K}$-embeddings; and
- ι extends to a $\mathscr{K}$-isomorphism from $\langle \operatorname{dom} \iota \rangle$ onto $\langle \operatorname{ran} \iota \rangle$.

(2) Let $\mathcal{A}, \mathcal{B}$ be objects of $\mathcal{K}$. We denote by $\mathrm{Apx}_{2,\mathcal{K}}(\mathcal{A}, \mathcal{B})$ the set of all approximate isometries from $|\mathcal{A}|$ to $|\mathcal{B}|$ which are of the form $\varphi_{\iota,\eta}|^{\mathcal{A}\times\mathcal{B}}$, where $\iota\colon \mathcal{A} \dashrightarrow C$ and $\eta\colon \mathcal{B} \dashrightarrow C$ are finite partial $\mathcal{K}$-isomorphisms into some object C of $\mathcal{K}$.
(3) A *$\mathcal{K}$-structure* is an L-structure $\mathcal{M}$ together with an inductive system of $\mathcal{K}$-embeddings

$$\mathcal{A}_1 \xrightarrow{\iota_1} \mathcal{A}_2 \xrightarrow{\iota_2} \mathcal{A}_3 \xrightarrow{\iota_3} \cdots$$

such that the inductive limit of the system is $\mathcal{M}$. We often write $\mathcal{M} = \overline{\bigcup_n \mathcal{A}_n}$, identifying each $\mathcal{A}_n$ as the corresponding L-substructure of $\mathcal{M}$. Note that $\mathcal{M}$ is not necessarily an object of $\mathcal{K}$.
(4) For $\mathcal{K}$-structures $\mathcal{M} = \overline{\bigcup_n \mathcal{A}_n}$ and $\mathcal{N} = \overline{\bigcup_m \mathcal{B}_m}$, we define

$$\mathrm{Apx}_{\mathcal{K}}(\mathcal{M}, \mathcal{N}) := \mathrm{cl}\Big(\bigcup_{n,m}\{\psi \mid \exists\varphi \in \mathrm{Apx}_{2,\mathcal{K}}(\mathcal{A}_n, \mathcal{B}_m),\ \psi \geq \varphi|^{\mathcal{M}\times\mathcal{N}}\}\Big)$$

and call its elements *approximate $\mathcal{K}$-isomorphisms*. Also, we set

$$\mathrm{Apx}^{\leq\varphi}_{\mathcal{K}}(\mathcal{M}, \mathcal{N}) := \mathrm{Apx}_{\mathcal{K}}(\mathcal{M}, \mathcal{N}) \cap \mathrm{Apx}^{\leq\varphi}(|\mathcal{M}|, |\mathcal{N}|),$$
$$\mathrm{Apx}^{\lhd\varphi}_{\mathcal{K}}(\mathcal{M}, \mathcal{N}) := \mathrm{Apx}_{\mathcal{K}}(\mathcal{M}, \mathcal{N}) \cap \mathrm{Apx}^{\lhd\varphi}(|\mathcal{M}|, |\mathcal{N}|).$$

An approximate $\mathcal{K}$-isomorphism φ from $\mathcal{M}$ to $\mathcal{N}$ is said to be *strict* if $\mathrm{Apx}^{\lhd\varphi}_{\mathcal{K}}(\mathcal{M}, \mathcal{N})$ is nonempty. We denote the set of all strict approximate $\mathcal{K}$-isomorphisms from $\mathcal{M}$ to $\mathcal{N}$ by $\mathrm{Stx}_{\mathcal{K}}(\mathcal{M}, \mathcal{N})$.
(5) An L-embedding ι of a $\mathcal{K}$-structure $\mathcal{M} = \overline{\bigcup_n \mathcal{A}_n}$ into another $\mathcal{K}$-structure $\mathcal{N} = \overline{\bigcup_m \mathcal{B}_m}$ is said to be *$\mathcal{K}$-admissible* if the corresponding approximate isometry φ_ι belongs to $\mathrm{Apx}_{\mathcal{K}}(\mathcal{M}, \mathcal{N})$. Two $\mathcal{K}$-structures are understood to be isomorphic if there exists a $\mathcal{K}$-admissible isomorphism between them.

An object $\mathcal{A}$ of $\mathcal{K}$ can be canonically identified with the $\mathcal{K}$-structure obtained from the inductive system $\mathcal{A} \xrightarrow{\mathrm{id}} \mathcal{A} \xrightarrow{\mathrm{id}} \cdots$, so that we can consider $\mathrm{Apx}_{\mathcal{K}}(\mathcal{A}, \mathcal{B})$ for objects $\mathcal{A}, \mathcal{B}$ of $\mathcal{K}$. If $\mathcal{A}, \mathcal{B}, C$ are objects of $\mathcal{K}$ and $\iota\colon \mathcal{A} \to C$ and $\eta\colon \mathcal{B} \to C$ are $\mathcal{K}$-embeddings, then $\varphi_{\iota,\eta}$ belongs to $\mathrm{Apx}_{\mathcal{K}}(\mathcal{A}, \mathcal{B})$, because it is the limit of

$$\big\{(\varphi_{\iota,\eta}|_{A_0\times B_0})|^{\mathcal{A}\times\mathcal{B}} \;\big|\; A_0 \subseteq |\mathcal{A}|, B_0 \subseteq |\mathcal{B}| \text{ are finite generators}\big\} \subseteq \mathrm{Apx}_{2,\mathcal{K}}(\mathcal{A}, \mathcal{B}).$$

In particular, every $\mathcal{K}$-embedding is $\mathcal{K}$-admissible. On the other hand, note that there might be a $\mathcal{K}$-admissible isomorphism between objects of $\mathcal{K}$ which is not a morphism of $\mathcal{K}$. There can be even a $\mathcal{K}$-admissible $\iota\colon \mathcal{A} \to \mathcal{B}$ such that no net of $\mathcal{K}$-embeddings of $\mathcal{A}$ into $\mathcal{B}$ converges to ι.

For any approximate $\mathcal{K}$-isomorphism φ from $\mathcal{M}$ to $\mathcal{N}$, the set $\mathrm{Apx}_{\mathcal{K}}^{\lhd\varphi}(\mathcal{M}, \mathcal{N})$ is obviously included in the relative interior of $\mathrm{Apx}_{\mathcal{K}}^{\leq\varphi}(\mathcal{M}, \mathcal{N})$ in $\mathrm{Apx}_{\mathcal{K}}(\mathcal{M}, \mathcal{N})$. The opposite inclusion also holds, because any relative interior point ψ in $\mathrm{Apx}_{\mathcal{K}}^{\leq\varphi}(\mathcal{M}, \mathcal{N})$ satisfies (ii) in Proposition 1.1.5.

Given a subset A of $\mathrm{Apx}(X, Y)$, we shall define

$$A^{\uparrow} := \{\psi \in \mathrm{Apx}(X, Y) \mid \exists \varphi \in A,\ \psi \geq \varphi\}.$$

Then it can be shown that $\mathrm{cl}(A^{\uparrow})$ is still upward closed, that is, $\mathrm{cl}(A^{\uparrow})^{\uparrow} = \mathrm{cl}(A^{\uparrow})$. Indeed, if $\varphi \geq \varphi'$ for $\varphi' \in \mathrm{cl}(A^{\uparrow})$, then for any $\varepsilon > 0$ and any finite subsets $X_0 \subseteq X$ and $Y_0 \subseteq Y$, we can find an approximate isometry $\psi \in A^{\uparrow}$ such that the inequality $|\psi - \varphi'| < \varepsilon$ on $X_0 \times Y_0$ holds. It follows that $(\varphi|_{X_0 \times Y_0})^{X \times Y} + \varepsilon$ is in $A^{\uparrow}$, so φ is in $\mathrm{cl}(A^{\uparrow})$. In particular, $\mathrm{Apx}_{\mathcal{K}}(\mathcal{M}, \mathcal{N})$ is upward closed for any $\mathcal{K}$-structures $\mathcal{M} = \overline{\bigcup_n \mathcal{A}_n}$ and $\mathcal{N} = \overline{\bigcup_m \mathcal{B}_m}$. This argument also implies that $\mathrm{Stx}_{\mathcal{K}}(\mathcal{M}, \mathcal{N})$ is topologically dense in $\mathrm{Apx}_{\mathcal{K}}(\mathcal{M}, \mathcal{N})$, since for any approximate $\mathcal{K}$-isomorphism φ, it automatically follows that the approximate isometries of the form $(\varphi|_{M_0 \times N_0})^{\mathcal{M} \times \mathcal{N}} + \varepsilon$ are indeed strict approximate $\mathcal{K}$-isomorphisms, where $M_0 \subseteq |\mathcal{M}|$ and $N_0 \subseteq |\mathcal{N}|$ are arbitrary finite subsets and ε is any positive real number.

Definition 1.2.2. The category $\mathcal{K}$ is said to satisfy

- the *joint embedding property* (JEP) if $\mathrm{JE}_{\mathcal{K}}(\mathcal{A}, \mathcal{B})$ is nonempty for any objects $\mathcal{A}, \mathcal{B}$ of $\mathcal{K}$.
- the *near amalgamation property* (NAP) if for any objects $\mathcal{A}, \mathcal{B}_1, \mathcal{B}_2$ in $\mathcal{K}$, any $\mathcal{K}$-embeddings $\iota_i\colon \mathcal{A} \to \mathcal{B}_i$, any finite subset $F \subseteq |\mathcal{A}|$ and any $\varepsilon > 0$, there exists a joint $\mathcal{K}$-embedding (η_1, η_2) of $\mathcal{B}_1$ and $\mathcal{B}_2$ such that the inequality

  $$d\big(\eta_1 \circ \iota_1(a),\ \eta_2 \circ \iota_2(a)\big) < \varepsilon$$

  holds for all $a \in F$.

The following two propositions are essential in proving the existence and uniqueness of Fraïssé limits in the next section. The first one claims that every strict approximate $\mathcal{K}$-isomorphism can be satisfied by joint $\mathcal{K}$-embeddings, while the second one claims that one can freely consider compositions of approximate $\mathcal{K}$-isomorphisms.

Proposition 1.2.3. *Suppose that $\mathcal{K}$ satisfies NAP. Then for any objects $\mathcal{A}, \mathcal{B}$ of $\mathcal{K}$ and any strict approximate $\mathcal{K}$-isomorphism φ from $\mathcal{A}$ to $\mathcal{B}$, there exists a joint $\mathcal{K}$-embedding (ι, η) of $\mathcal{A}$ and $\mathcal{B}$ which satisfies $\varphi_{\iota,\eta} \lhd \varphi$.*

Proof. Since φ is strict, $\mathrm{Apx}_{\mathcal{K}}^{\lhd\varphi}(\mathcal{A}, \mathcal{B})$ is an open nonempty subset of $\mathrm{Apx}_{\mathcal{K}}(\mathcal{A}, \mathcal{B})$. Therefore, it intersects with $\mathrm{Apx}_{2,\mathcal{K}}(\mathcal{A}, \mathcal{B})$, as $\mathrm{Apx}_{2,\mathcal{K}}(\mathcal{A}, \mathcal{B})^{\uparrow}$ is a dense subset. In other words, there exist an object C_0 of $\mathcal{K}$ and finite partial $\mathcal{K}$-isomorphisms $\iota_0 \colon \mathcal{A} \dashrightarrow C_0$ and $\eta_0 \colon \mathcal{B} \dashrightarrow C_0$ such that the relation

$$\varphi_{\iota_0,\eta_0}|^{\mathcal{A}\times\mathcal{B}} + \varepsilon \lhd \varphi$$

holds for some $\varepsilon > 0$. Put $\mathcal{A}_0 := \langle \operatorname{dom} \iota_0 \rangle$ and $\mathcal{B}_0 := \langle \operatorname{dom} \eta_0 \rangle$. By the definition of finite partial $\mathcal{K}$-isomorphisms, the canonical embeddings $\mathcal{A}_0 \to \mathcal{A}$ and $\mathcal{B}_0 \to \mathcal{B}$ are $\mathcal{K}$-embeddings.

Now, by NAP there exist $\mathcal{K}$-embeddings $\iota_1 \colon \mathcal{A} \to C_{\mathcal{A}}$ and $\iota_1' \colon C_0 \to C_{\mathcal{A}}$ such that the inequality

$$d\big(\iota_1(a),\ \iota_1' \circ \iota_0(a)\big) < \varepsilon/3$$

holds for all $a \in \operatorname{dom} \iota_0$. Similarly, there are $\mathcal{K}$-embeddings $\eta_1 \colon \mathcal{B} \to C_{\mathcal{B}}$ and $\eta_1' \colon C_0 \to C_{\mathcal{B}}$ with

$$d\big(\eta_1(b),\ \eta_1' \circ \eta_(b)\big) < \varepsilon/3$$

for all $b \in \operatorname{dom} \eta_0$. Then, again by NAP, there exist $\mathcal{K}$-embeddings $\iota_2 \colon C_{\mathcal{A}} \to C$ and $\eta_2 \colon C_{\mathcal{B}} \to C$ with

$$d\big(\iota_2 \circ \iota_1'(c),\ \eta_2 \circ \eta_1'(c)\big) < \varepsilon/3$$

for any $c \in \operatorname{ran} \iota_0 \cup \operatorname{ran} \eta_0$.

$$\begin{tikzcd}
 & \mathcal{A}_0 \arrow[r, hook] \arrow[d, dashed, "\iota_0"] & \mathcal{A} \arrow[d, "\iota_1"] \\
\mathcal{B}_0 \arrow[r, dashed, "\eta_0"] \arrow[d, hook] & C_0 \arrow[r, "\iota_1'"] \arrow[d, "\eta_1'"] & C_{\mathcal{A}} \arrow[d, "\iota_2"] \\
\mathcal{B} \arrow[r, "\eta_1"] & C_{\mathcal{B}} \arrow[r, "\eta_2"] & C
\end{tikzcd}$$

Set $\iota := \iota_2 \circ \iota_1$ and $\eta := \eta_2 \circ \eta_1$. Then for $a \in \operatorname{dom} \iota_0$ and $b \in \operatorname{dom} \eta_0$, we have

$$\begin{aligned}
d\big(\iota(a), \eta(b)\big) &= d\big(\iota_2 \circ \iota_1(a),\ \eta_2 \circ \eta_1(b)\big) \\
&\le d\big(\iota_2 \circ \iota_1' \circ \iota_0(a),\ \eta_2 \circ \eta_1' \circ \eta_0(b)\big) + 2\varepsilon/3 \\
&\le d\big(\iota_2 \circ \iota_1' \circ \iota_0(a),\ \iota_2 \circ \iota_1' \circ \eta_0(b)\big) + \varepsilon = d\big(\iota_0(a), \eta_0(b)\big) + \varepsilon,
\end{aligned}$$

so

$$\varphi_{\iota,\eta} \leq \varphi_{\iota_0,\eta_0}|^{\mathcal{A}\times\mathcal{B}} + \varepsilon \lhd \varphi,$$

as desired. □

Proposition 1.2.4. *Suppose that $\mathscr{K}$ satisfies NAP, and let $\mathcal{M}_1 = \overline{\bigcup_l \mathcal{A}_l}$, $\mathcal{M}_2 = \overline{\bigcup_m \mathcal{B}_m}$ and $\mathcal{M}_3 = \overline{\bigcup_n \mathcal{C}_n}$ be $\mathscr{K}$-structures. If φ and ψ belongs to* $\mathrm{Apx}_{\mathscr{K}}(\mathcal{M}_1, \mathcal{M}_2)$ *and* $\mathrm{Apx}_{\mathscr{K}}(\mathcal{M}_2, \mathcal{M}_3)$ *respectively, then the composition $\psi\varphi$ is in* $\mathrm{Apx}_{\mathscr{K}}(\mathcal{M}_1, \mathcal{M}_3)$.

Proof. First, assume that both φ and ψ are strict and $\mathcal{M}_1$, $\mathcal{M}_2$, $\mathcal{M}_3$ are objects of $\mathscr{K}$. Then, by Proposition 1.2.3, there exist objects $\mathcal{D}$ and $\mathcal{E}$ of $\mathscr{K}$ and $\mathscr{K}$-embeddings $\iota_i\colon \mathcal{M}_i \to \mathcal{D}$ $(i = 1, 2)$ and $\eta_j\colon \mathcal{M}_j \to \mathcal{E}$ $(j = 2, 3)$ such that $\varphi_{\iota_1,\iota_2} \lhd \varphi$ and $\varphi_{\eta_2,\eta_3} \lhd \psi$. It follows from Proposition 1.1.5 that there exist a finite subset $F_0 \subseteq |\mathcal{M}_2|$ and a positive real number $\varepsilon > 0$ with

$$(\varphi_{\iota_1,\iota_2}|_{\mathcal{M}_1\times F_0})|^{\mathcal{M}_1\times\mathcal{M}_2} + \varepsilon \lhd \varphi, \quad (\varphi_{\eta_2,\eta_3}|_{F_0\times\mathcal{M}_3})|^{\mathcal{M}_2\times\mathcal{M}_3} + \varepsilon \lhd \psi.$$

By NAP, we can find $\mathscr{K}$-embeddings $\theta_1\colon \mathcal{D} \to \mathcal{F}$ and $\theta_2\colon \mathcal{E} \to \mathcal{F}$ such that the inequality

$$d\big(\theta_1 \circ \iota_2(b),\ \theta_2 \circ \eta_2(b)\big) < 2\varepsilon$$

holds for all $b \in F_0$.

$$\begin{tikzcd} & & \mathcal{M}_1 \arrow[d, "\iota_1"] \\ & \mathcal{M}_2 \arrow[r, "\iota_2"] \arrow[d, "\eta_2"] & \mathcal{D} \arrow[d, "\theta_1"] \\ \mathcal{M}_3 \arrow[r, "\eta_3"] & \mathcal{E} \arrow[r, "\theta_2"] & \mathcal{F} \end{tikzcd}$$

For $a \in |\mathcal{M}_1|$ and $c \in |\mathcal{M}_3|$, we have

$$\begin{aligned} & d\big(\theta_1 \circ \iota_1(a),\ \theta_2 \circ \eta_3(c)\big) \\ \leq & \inf_{b\in F_0}\Big[d\big(\theta_1 \circ \iota_1(a),\ \theta_1 \circ \iota_2(b)\big) + d\big(\theta_1 \circ \iota_2(b),\ \theta_2 \circ \eta_3(c)\big)\Big] \\ < & \inf_{b\in F_0}\Big[d\big(\theta_1 \circ \iota_1(a),\ \theta_1 \circ \iota_2(b)\big) + d\big(\theta_2 \circ \eta_2(b),\ \theta_2 \circ \eta_3(c)\big) + 2\varepsilon\Big] \\ = & \big(\varphi_{\eta_2,\eta_3}|_{F_0\times\mathcal{M}_3} + \varepsilon\big)\big(\varphi_{\iota_1,\iota_2}|_{\mathcal{M}_1\times F_0} + \varepsilon\big)(a, c), \end{aligned}$$

so

$$\varphi_{\theta_1\circ\iota_1,\theta_2\circ\eta_3} \leq \Big[\big(\varphi_{\eta_2,\eta_3}|_{F_0\times\mathcal{M}_3} + \varepsilon\big)|^{\mathcal{M}_2\times\mathcal{M}_3}\Big]\Big[\big(\varphi_{\iota_1,\iota_2}|_{\mathcal{M}_1\times F_0} + \varepsilon\big)|^{\mathcal{M}_1\times\mathcal{M}_2}\Big] \leq \psi\varphi.$$

Since $\varphi_{\theta_1\circ\iota_1,\theta_2\circ\eta_3}$ is in $\mathrm{Apx}_{\mathcal{K}}(\mathcal{M}_1,\mathcal{M}_3)$, so is $\psi\varphi$.

Next, assume that both φ and ψ are still strict, but $\mathcal{M}_1$, $\mathcal{M}_2$ and $\mathcal{M}_3$ are general $\mathcal{K}$-structures. Then there exist sufficiently large $l,m,m',n\in\mathbb{N}$ and approximate $\mathcal{K}$-isomorphisms φ' from $\mathcal{A}_l$ to $\mathcal{B}_m$ and ψ' from $\mathcal{B}_{m'}$ to C_n with $\varphi'|^{\mathcal{M}_1\times\mathcal{M}_2}\lhd\varphi$ and $\psi'|^{\mathcal{M}_2\times\mathcal{M}_3}\lhd\psi$. We may assume without loss of generality that m is equal to m', since in general, if $\iota\colon\mathcal{A}\to\mathcal{A}'$ and $\eta\colon\mathcal{B}\to\mathcal{B}'$ are $\mathcal{K}$-embeddings, then one can directly check from the definition that the trivial extension of an approximate $\mathcal{K}$-isomorphism in $\mathrm{Apx}_{2,\mathcal{K}}(\mathcal{A},\mathcal{B})$ via these $\mathcal{K}$-embeddings belong to $\mathrm{Apx}_{2,\mathcal{K}}(\mathcal{A}',\mathcal{B}')$. Also, we may assume that both φ' and ψ' are strict by Proposition 1.1.5. By what we proved in the preceding paragraph, $\psi'\varphi'$ is in $\mathrm{Apx}_{\mathcal{K}}(\mathcal{A}_l,C_n)$. By direct computation, one can check that $(\psi'|^{\mathcal{M}_2\times\mathcal{M}_3})(\varphi'|^{\mathcal{M}_1\times\mathcal{M}_2})$ is equal to $(\psi'\varphi')|^{\mathcal{M}_1\times\mathcal{M}_3}$, so $\psi\varphi$ is in $\mathrm{Apx}_{\mathcal{K}}(\mathcal{M}_1,\mathcal{M}_3)$.

Finally, let φ and ψ be general approximate $\mathcal{K}$-isomorphisms between general $\mathcal{K}$-structures. Then there exist nets $\{\varphi_\alpha\}$ and $\{\psi_\beta\}$ of strict approximate $\mathcal{K}$-isomorphisms which converge to φ and ψ respectively, and

$$\psi\varphi=(\overline{\lim_{\beta}}\,\psi_\beta)(\overline{\lim_{\alpha}}\,\varphi_\alpha)\geq\overline{\lim_{\alpha,\beta}}(\psi_\beta\varphi_\alpha)\in\mathrm{Apx}_{\mathcal{K}}(\mathcal{M}_1,\mathcal{M}_3),$$

so $\psi\varphi$ belongs to $\mathrm{Apx}_{\mathcal{K}}(\mathcal{M}_1,\mathcal{M}_3)$. □

Corollary 1.2.5. *Trivial extensions and restrictions of approximate $\mathcal{K}$-isomorphisms via $\mathcal{K}$-admissible embeddings are approximate $\mathcal{K}$-isomorphisms.*

1.3 Fraïssé categories and their limits

Let L be a language and $\mathcal{K}$ be a category of finitely generated L-structures and L-embeddings which satisfies JEP and NAP. For each $n\in\mathbb{N}$, we denote by $\mathcal{K}_n$ the class of all pairs $\langle\mathcal{A},\bar{a}\rangle$, where $\mathcal{A}$ is an object of $\mathcal{K}$ and $\bar{a}$ is an ordered generator of $\mathcal{A}$. We simply write $\langle\bar{a}\rangle$ instead of $\langle\mathcal{A},\bar{a}\rangle$ when there is no danger of confusion.

We shall consider a variant of the Gromov–Hausdorff distance on $\mathcal{K}_n$. Define a pseudo-metric $d^{\mathcal{K}}$ on $\mathcal{K}_n$ by

$$\begin{aligned}d^{\mathcal{K}}\left(\langle\bar{a}\rangle,\langle\bar{b}\rangle\right)&:=\inf\left\{\max_i\varphi(a_i,b_i)\;\middle|\;\varphi\in\mathrm{Apx}_{\mathcal{K}}\left(\langle\bar{a}\rangle,\langle\bar{b}\rangle\right)\right\}\\&=\inf\left\{\max_i\varphi(a_i,b_i)\;\middle|\;\varphi\in\mathrm{Stx}_{\mathcal{K}}\left(\langle\bar{a}\rangle,\langle\bar{b}\rangle\right)\right\}\\&=\inf\left\{\max_i d\left(\iota(a_i),\eta(b_i)\right)\;\middle|\;(\iota,\eta)\in\mathrm{JE}_{\mathcal{K}}\left(\langle\bar{a}\rangle,\langle\bar{b}\rangle\right)\right\},\end{aligned}$$

where a_i and b_i denotes the i-th component of $\bar{a}$ and $\bar{b}$ respectively. The fact that $d^{\mathscr{K}}$ is indeed a pseudo-metric easily follows from JEP and NAP.

Definition 1.3.1. The category $\mathscr{K}$ is said to satisfy

- the *weak Polish property* (WPP) if $\mathscr{K}_n$ is separable with respect to the pseudo-metric $d^{\mathscr{K}}$ for each n.
- the *Cauchy continuity property* (CCP) if
  (i) for any n-ary predicate symbol P in L, the map

$$\langle \mathcal{A}, (\bar{a}, \bar{b}) \rangle \mapsto P^{\mathcal{A}}(\bar{a})$$

  from $\mathscr{K}_{n+m}$ into $\mathbb{R}$ sends Cauchy sequences into Cauchy sequences; and
  (ii) for any n-ary function symbol f in L, the map

$$\langle \mathcal{A}, (\bar{a}, \bar{b}) \rangle \mapsto \langle \mathcal{A}, \left(\bar{a}, \bar{b}, f^{\mathcal{A}}(\bar{a})\right) \rangle$$

  from $\mathscr{K}_{n+m}$ into $\mathscr{K}_{n+m+1}$ sends Cauchy sequences into Cauchy sequences.

Remark 1.3.2. If $\mathscr{K}$ satisfies CCP, then $d^{\mathscr{K}}(\langle \bar{a} \rangle, \langle \bar{b} \rangle)$ is equal to zero if and only if there exists a $\mathscr{K}$-admissible isomorphism from $\langle \bar{a} \rangle$ onto $\langle \bar{b} \rangle$ which sends a_i to b_i. To see this, first suppose that the map $a_i \mapsto b_i$ extends to a $\mathscr{K}$-admissible isomorphism ι. Then $(\varphi_\iota|_{\bar{a}\times\bar{b}})|^{\langle \bar{a} \rangle \times \langle \bar{b} \rangle} + \varepsilon$ belongs to $\mathrm{Stx}_{\mathscr{K}}\left(\langle \bar{a} \rangle, \langle \bar{b} \rangle\right)$, and

$$d^{\mathscr{K}}\left(\langle \bar{a} \rangle, \langle \bar{b} \rangle\right) \leq (\varphi_\iota|_{\bar{a}\times\bar{b}})|^{\langle \bar{a} \rangle \times \langle \bar{b} \rangle}(a_i, b_i) + \varepsilon = \varepsilon$$

for arbitrary $\varepsilon > 0$. Conversely, suppose $d^{\mathscr{K}}\left(\langle \bar{a} \rangle, \langle \bar{b} \rangle\right) = 0$. Let $D_{\bar{a}}$ be the set of all elements of $\langle \bar{a} \rangle$ of the form $g(\bar{a})$, where g is a composition of functions equipped with $\langle \bar{a} \rangle$, and $D_{\bar{b}}$ be the set obtained from $\langle \bar{b} \rangle$ by the same way. Then it follows from CCP that the map $a_i \mapsto b_i$ extends to an isometry from $D_{\bar{a}}$ onto $D_{\bar{b}}$ and the interpretations of the symbols can be identified via this isometry, so that it extends to an L-isomorphism ι from $\langle \bar{a} \rangle$ onto $\langle \bar{b} \rangle$. If $\bar{c} = \left(g_1(\bar{a}), \ldots, g_n(\bar{a})\right)$ and $\bar{d} = \left(g_1(\bar{b}), \ldots, g_n(\bar{b})\right)$, then $d^{\mathscr{K}}\left(\langle \bar{a}, \bar{c} \rangle, \langle \bar{b}, \bar{d} \rangle\right) = 0$ by CCP, so there exists a joint $\mathscr{K}$-embedding (η_1, η_2) of $\langle \bar{a}, \bar{c} \rangle$ and $\langle \bar{b}, \bar{d} \rangle$ such that the $\eta_1(\bar{a}, \bar{c})$ and $\eta_2(\bar{b}, \bar{d})$ are arbitrarily close to each other, whence ι is $\mathscr{K}$-admissible.

Definition 1.3.3. (1) A category $\mathscr{K}$ of finitely generated separable L-structures is called a *Fraïssé category* if it satisfies JEP, NAP, WPP and CCP.

(2) Let $\mathscr{K}$ be a Fraïssé category. A $\mathscr{K}$-structure $\mathcal{M}$ is called a *Fraïssé limit* of $\mathscr{K}$ if for any $\mathscr{K}$-structure $\mathcal{N}$ and any strict approximate $\mathscr{K}$-isomorphism $\varphi \in \mathrm{Stx}_{\mathscr{K}}(\mathcal{N}, \mathcal{M})$, there exists a $\mathscr{K}$-admissible embedding $\iota \colon \mathcal{N} \to \mathcal{M}$ with $\varphi_\iota \lhd \varphi$.

We shall begin with characterizing Fraïssé limits. Fix a Fraïssé category $\mathscr{K}$.

Definition 1.3.4. A $\mathscr{K}$-structure $\mathcal{M}$ is said to be

- *$\mathscr{K}$-universal* if for any object $\mathcal{A}$ of $\mathscr{K}$, there exists a $\mathscr{K}$-admissible embedding of $\mathcal{A}$ into $\mathcal{M}$.
- *approximately $\mathscr{K}$-ultra-homogeneous* if for any $\langle \bar{a} \rangle \in \mathscr{K}_n$, any $\varepsilon > 0$ and any $\mathscr{K}$-admissible embeddings $\iota, \eta \colon \langle \bar{a} \rangle \to \mathcal{M}$, there exists a $\mathscr{K}$-admissible automorphism α of $\mathcal{M}$ with
$$\max_i d\big(\alpha \circ \iota(a_i),\ \eta(a_i)\big) \le \varepsilon.$$

Theorem 1.3.5. *For a $\mathscr{K}$-structure $\mathcal{M}$, the following are equivalent.*

(i) *The structure $\mathcal{M}$ is a Fraïssé limit of $\mathscr{K}$.*
(ii) *For any object $\mathcal{A}$ of $\mathscr{K}$ and any $\varphi \in \mathrm{Stx}_{\mathscr{K}}(\mathcal{A}, \mathcal{M})$, there exists a $\mathscr{K}$-admissible embedding $\iota \colon \mathcal{A} \to \mathcal{M}$ with $\varphi_\iota \lhd \varphi$.*
(iii) *If $\langle \bar{a} \rangle$ is in $\mathscr{K}_n$ and φ is a strict approximate $\mathscr{K}$-isomorphism from $\langle \bar{a} \rangle$ to $\mathcal{M}$, then for any $\varepsilon > 0$ there is an approximate $\mathscr{K}$-isomorphism $\psi \in \mathrm{Stx}_{\mathscr{K}}^{\lhd\varphi}\big(\langle \bar{a} \rangle, \mathcal{M}\big)$ such that ψ is ε-total on $\bar{a}$, that is, $\psi|_{\bar{a}\times\mathcal{M}}$ is ε-total.*
(iv) *The structure $\mathcal{M}$ is $\mathscr{K}$-universal and approximately $\mathscr{K}$-ultra-homogeneous.*

Moreover, if a Fraïssé limit exists, then it is unique up to $\mathscr{K}$-admissible isomorphisms.

Proof. First, assume that $\mathcal{M} = \overline{\bigcup_n \mathcal{A}_n}$ and $\mathcal{N} = \overline{\bigcup_m \mathcal{B}_m}$ are $\mathscr{K}$-structures satisfying (iii). We shall show that if φ is a strict approximate $\mathscr{K}$-isomorphism from $\mathcal{M}$ to $\mathcal{N}$, then there exists a $\mathscr{K}$-admissible isomorphism α from $\mathcal{M}$ onto $\mathcal{N}$ with $\varphi_\alpha \lhd \varphi$. Since φ is strict, there exist an approximate $\mathscr{K}$-isomorphism ψ from $\mathcal{M}$ to $\mathcal{N}$, finite subsets $E \subseteq \bigcup_n |\mathcal{A}_n|$ and $F \subseteq \bigcup_m |\mathcal{B}_m|$, and a positive real number $\varepsilon \le 1$ with
$$(\psi|_{E\times F})|^{\mathcal{M}\times\mathcal{N}} + \varepsilon \lhd \varphi.$$

Take increasing sequences $\{X_i\}$ and $\{Y_j\}$ of finite sets such that

- $X_0 = E$ and $Y_0 = F$;
- $X_i \subseteq \bigcup_n |\mathcal{A}_n|$ and $Y_j \subseteq \bigcup_m |\mathcal{B}_m|$ for all i, j; and
- $\bigcup_i X_i$ and $\bigcup_j Y_j$ are dense in $|\mathcal{M}|$ and $|\mathcal{N}|$ respectively.

We claim the existence of a sequence $\{\psi_l\}$ of strict approximate $\mathscr{K}$-isomorphisms from $\mathcal{M}$ to $\mathcal{N}$ with the following properties.

- each ψ_l is of the form $(\theta|_{X_{i(l)}\times Y_{j(l)}})|^{\mathcal{M}\times\mathcal{N}} + \delta_l$ for some $\theta \in \mathrm{Apx}_{\mathscr{K}}(\mathcal{M},\mathcal{N})$, where $\delta_l \le 2^{-l}$ and $i(l), j(l) \uparrow \infty$;
- $\psi_{l+1} \lhd \psi_l$; and
- $\psi_{l+1}|_{X_{i(l)}\times\mathcal{N}}$ is δ_l-total if l is even, while $\psi_{l+1}|_{\mathcal{M}\times Y_{i(l)}}$ is δ_l-surjective if l is odd.

The construction of such a sequence proceeds as following. Set

$$\psi_0 := (\psi|_{X_0\times Y_0})|^{\mathcal{M}\times\mathcal{N}} + \varepsilon.$$

Assume l is even and ψ_l is given. Then, by assumption on $\mathcal{N}$, one can find $\theta \lhd \psi_l$ such that $\theta|_{X_{i(l)}\times\mathcal{N}}$ is $\delta_l/2$-total. Since $(\theta|_{X_i\times Y_j})|^{\mathcal{M}\times\mathcal{N}} + \delta$ converges to θ as $i, j \to \infty$ and $\delta \to 0$, for sufficiently large $i(l+1) > i(l)$ and $j(l+1) > j(l)$ and sufficiently small $\delta_{l+1} < \delta_l/2$, we have

$$\psi_l \rhd (\theta|_{X_{i(l+1)}\times Y_{j(l+1)}})|^{\mathcal{M}\times\mathcal{N}} + \delta_{l+1}.$$

We let ψ_{l+1} be the right-hand side. Then it is clear that $\psi_{l+1}|_{X_{i(l)}\times\mathcal{N}}$ is δ_l-total. The case l is odd is similar, and the description of the construction of $\{\psi_l\}$ is completed. Now the sequence being decreasing, there exists the limit $\psi_\infty \in \mathrm{Apx}^{\lhd\varphi}_{\mathscr{K}}(\mathcal{M},\mathcal{N})$, which is clearly of the form φ_α for some isomorphism $\alpha\colon \mathcal{M} \to \mathcal{N}$ by Proposition 1.1.4, as desired.

(iii) $\Rightarrow$ (i) can be verified by similar arguments. Also, (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) is trivial.

(iii) $\Rightarrow$ (iv). It follows from (iii) $\Rightarrow$ (ii) that $\mathcal{M}$ is $\mathscr{K}$-universal. Let $\iota, \eta\colon \langle \bar{a}\rangle \to \mathcal{M}$ be $\mathscr{K}$-admissible embeddings and ε be a positive real number. Then

$$\varphi := \big((\varphi_\eta\varphi_\iota^*)|_{\iota(\bar{a})\times\eta(\bar{a})}\big)|^{\mathcal{M}\times\mathcal{M}} + \varepsilon$$

is in $\mathrm{Stx}_{\mathscr{K}}(\mathcal{M},\mathcal{M})$, so by what we proved in the first paragraph, one can find a $\mathscr{K}$-admissible automorphism α of $\mathcal{M}$ with $\varphi_\alpha \lhd \varphi$. Since $\varphi\big(\iota(a_i), \eta(a_i)\big) = \varepsilon$, the inequality

$$d\big(\alpha\circ\iota(a_i),\ \eta(a_i)\big) \le \varepsilon$$

follows.

(iv) $\Rightarrow$ (ii). Suppose that $\mathcal{A}$ is an object of $\mathscr{K}$ and φ is a strict approximate $\mathscr{K}$-isomorphism from $\mathcal{A}$ to $\mathcal{M}$. By assumption, there exists a $\mathscr{K}$-admissible embedding $\iota\colon \mathcal{A} \to \mathcal{M} = \overline{\bigcup_n \mathcal{A}_n}$, so it suffices to show that there is a $\mathscr{K}$-admissible automorphism α of $\mathcal{M}$ with $\varphi_{\alpha\circ\iota} \lhd \varphi$, or equivalently, $\varphi_\alpha \lhd \varphi\varphi_\iota^*$. To see this, find sufficiently large n and finite partial $\mathscr{K}$-isomorphisms ι_1, ι_2 from $\mathcal{A}_n$ into some object C of $\mathscr{K}$ with

$$(\varphi_{\iota_1,\iota_2})|^{\mathcal{M}\times\mathcal{M}} + \varepsilon \lhd \varphi\varphi_\iota^*.$$

Since there exists a $\mathscr{K}$-admissible embedding of C into $\mathcal{M}$, and since $\mathcal{M}$ is approximately $\mathscr{K}$-ultra-homogeneous, there exists a $\mathscr{K}$-admissible embedding $\eta\colon C \to \mathcal{M}$ with

$$d\big(b, \eta \circ \iota_2(b)\big) < \varepsilon/2$$

for $b \in \operatorname{dom} \iota_2$. Again by the $\mathscr{K}$-ultra-homogeneity of $\mathcal{M}$, one can find a $\mathscr{K}$-admissible automorphism α of $\mathcal{M}$ such that the inequality

$$d\big(\alpha(a),\ \eta \circ \iota_1(a)\big) < \varepsilon/2$$

holds for all $a \in \operatorname{dom} \iota_1$. Then

$$\begin{aligned} d\big(\alpha(a), b\big) &\le d\big(\alpha(a),\ \eta \circ \iota_1(a)\big) + d\big(\eta \circ \iota_1(a),\ \eta \circ \iota_2(b)\big) + d\big(\eta \circ \iota_2(b),\ b\big) \\ &\le d\big(\iota_1(a), \iota_2(b)\big) + \varepsilon = \varphi_{\iota_1,\iota_2}(a, b) + \varepsilon, \end{aligned}$$

which completes the proof. □

Next, we shall prove the existence of the Fraïssé limit. For this, we need the following lemma which claims that, in order to see (iii) in Theorem 1.3.5, we only have to check a countable dense part.

Lemma 1.3.6. *Let $\mathcal{M}$ be a $\mathscr{K}$-structure and M_0 be a countable dense subset of $|\mathcal{M}|$. Suppose that, for each $n \in \mathbb{N}$, a countable dense subset $\mathscr{K}_{n,0}$ of $\mathscr{K}_n$ is given. Then, in order for $\mathcal{M}$ to be the Fraïssé limit of $\mathscr{K}$, it is sufficient that, for any $n \in \mathbb{N}$, any $\langle \bar{a} \rangle \in \mathscr{K}_{n,0}$, any finite subset $F \subseteq M_0$ and any $\varphi \in \operatorname{Stx}_{\mathscr{K}}\big(\langle \bar{a} \rangle, \mathcal{M}\big)$ which is rational-valued on $\bar{a} \times F$, there exists $\psi \in \operatorname{Apx}^{\le\varphi}_{\mathscr{K}}\big(\langle \bar{a} \rangle, \mathcal{M}\big)$ such that ψ is ε-total on $\bar{a}$.*

Proof. Let $\mathcal{B}$ be an object of $\mathscr{K}$ and φ be a strict approximate $\mathscr{K}$-isomorphism from $\mathcal{B}$ to $\mathcal{M}$, and take $\varphi' \in \operatorname{Stx}^{\lhd\varphi}_{\mathscr{K}}(\mathcal{B}, \mathcal{M})$. Then there exists an arbitrarily large finite subsets $F \in |\mathcal{B}|^n$ and $G \subseteq M_0$ and arbitrarily small $\varepsilon > 0$ with

$$\varphi'' := (\varphi'|_{F\times G})|^{\mathcal{B}\times\mathcal{M}} + \varepsilon \lhd \varphi.$$

Without loss of generality, we may assume that F is a generator of $\mathcal{B}$. Let $\bar{b} = (b_1, \ldots, b_n)$ be an enumeration of F. Take $\langle \bar{a} \rangle \in \mathscr{K}_{n,0}$ with $d^{\mathscr{K}}\big(\langle \bar{a} \rangle, \langle \bar{b} \rangle\big) < \varepsilon/4$ and find a joint $\mathscr{K}$-embedding (ι, η) of $\langle \bar{a} \rangle$ and $\langle \bar{b} \rangle$ satisfying $\max_i \varphi_{\iota,\eta}(a_i, b_i) < \varepsilon/4$. Then, being a restriction of an extension of a strict approximate $\mathscr{K}$-isomorphism, $\varphi''\varphi_{\iota,\eta}$ is strict, so there exists $\psi' \in \operatorname{Stx}_{\mathscr{K}}\big(\langle \bar{a} \rangle, \mathcal{M}\big)$ which is rational-valued on $\bar{a} \times F$ and satisfies

$\psi' \lhd \varphi''\varphi_{\iota,\eta}$, by Proposition 1.1.5. By assumption, we can take $\psi'' \in \mathrm{Apx}_{\mathscr{K}}^{\leq\psi'}(\langle\bar{a}\rangle, \mathcal{M})$ such that $\psi''|_{\bar{a}\times\mathcal{M}}$ is $\varepsilon/4$-total. Then, $(\psi''\varphi_{\eta,\iota})|_{\bar{b}\times\mathcal{M}}$ is $\varepsilon/2$-total, and

$$(\psi''\varphi_{\eta,\iota})|_{\bar{b}\times\mathcal{M}} \leq (\varphi''\varphi_{\iota,\eta}\varphi_{\eta,\iota})|_{\bar{b}\times\mathcal{M}} \leq \varphi''|_{\bar{b}\times\mathcal{M}} + \varepsilon/2,$$

since $\varphi_{\eta,\iota}|_{\bar{b}\times\langle\bar{a}\rangle}$ is $\varepsilon/4$-total. Now take a finite subset $H \subseteq |\mathcal{M}|$ such that G is included in H and $(\psi''\varphi_{\eta,\iota})|_{\bar{b}\times H}$ is $3\varepsilon/4$-total, and put

$$\psi := \left((\psi''\varphi_{\eta,\iota})|_{\bar{b}\times H}\right)|^{\mathcal{B}\times\mathcal{M}} + \varepsilon/4.$$

Then $\psi|_{\bar{b}\times|\mathcal{M}|}$ is ε-total, and

$$\psi \leq (\varphi''|_{\bar{b}\times H})|^{\mathcal{B}\times\mathcal{M}} + 3\varepsilon/4 \leq (\varphi'|_{F\times G})|^{\mathcal{B}\times\mathcal{M}} + \varepsilon \lhd \varphi.$$

Since this shows that (iii) in Theorem 1.3.5 holds, it follows that $\mathcal{M}$ is the Fraïssé limit of $\mathscr{K}$. □

Theorem 1.3.7. *Every Fraïssé category has its limit.*

Proof. Take a countable dense subset $\mathscr{K}_{n,0} \subseteq \mathscr{K}_n$ for each n. In view of Proposition 1.2.3, we can inductively find a $\mathscr{K}$-structure $\mathcal{A}_k$, a $\mathscr{K}$-embedding $\iota_{k-1}\colon \mathcal{A}_{k-1} \to \mathcal{A}_k$ and a countable dense subset $A_{k,0} \subseteq |\mathcal{A}_k|$ so that, if $\langle\bar{a}\rangle$ is in $\mathscr{K}_{n,0}$, if F is a finite subset of $A_{k,0}$, and if φ is a strict approximate $\mathscr{K}$-isomorphism from $\langle\bar{a}\rangle$ to $\mathcal{A}_k$ which is rational-valued on $\bar{a}\times F$, then there exists a $\mathscr{K}$-embedding $\iota\colon \langle\bar{a}\rangle \to \mathcal{A}_l$ for some $l > k$ with $\varphi_\iota \lhd \varphi_{\iota_{l,k}}\varphi$, where $\iota_{l,k}$ denotes the composition of $\iota_k, \ldots, \iota_{l-1}$. Then the $\mathscr{K}$-structure obtained from the inductive system satisfies the assumption in the previous lemma, so we are done. □

Chapter 2

Applications to operator algebras

In this chapter, we give two applications of the theory presented in the preceding chapter. The first section is devoted to the application to UHF algebras: it is shown that a category of C*-algebras of matrix-valued continuous functions with distinguished traces and diagonalizable $*$-homomorphisms is a Fraïssé category the limit of which is a UHF algebra. In the second section, we prove a similar result for a category of dimension drop algebras with distinguished traces and $*$-homomorphisms, in which case the limit is the Jiang–Su algebra. In both cases, the structures under consideration are unital tracial C*-algebras, so the appropriate language for them is L_{TC^*} which consists of the following symbols:

- two constant symbols 0 and 1;
- an unary function symbol λ for each $\lambda \in \mathbb{C}$, which are to be interpreted as multiplication by λ;
- an unary function symbol $*$ for involution;
- a binary function symbol $+$ and $\cdot$;
- an unary predicate symbol tr.

Every unital C*-algebra with a distinguished trace can be canonically considered as a metric L_{TC^*}-structure. Note that the distance we adopt is the norm distance, and that a map between unital C*-algebras with fixed traces are L-embeddings if and only if it is a trace-preserving injective $*$-homomorphism.

2.1 UHF algebras

In the sequel, we denote by $\mathbb{M}_n$ the C*-algebra of all $n\times n$ matrices. Also, for non-negative integer p and positive integer n, we shall denote by $\mathcal{A}_{p,n}$ the C*-algebra $C([0,1]^p, \mathbb{M}_n)$ of all $n \times n$ matrix-valued continuous functions on $[0,1]^p$. We note that there are canonical isomorphisms $C([0,1]^p, \mathbb{M}_n) \simeq C([0,1]^p) \otimes \mathbb{M}_n$, $C([0,1]^p) \otimes C([0,1]^q) \simeq C([0,1]^{p+q})$ and $\mathbb{M}_n \otimes \mathbb{M}_m \simeq \mathbb{M}_{mn}$, so that $\mathcal{A}_{p,n} \otimes \mathcal{A}_{q,m}$ is canonically isomorphic to $\mathcal{A}_{p+q,mn}$. Now, for a probability Radon measure μ on $[0,1]^p$, we can define a trace (which is still denoted by μ) on $\mathcal{A}_{p,n}$ by

$$\mu(f) := \int \operatorname{tr}\big(f(t)\big)\, d\mu(t),$$

where tr is the normalized trace on $\mathbb{M}_n$. It can be easily verified that every trace on $\mathcal{A}_{p,n}$ is of this form, so that we can identify the traces on $\mathcal{A}_{p,n}$ with the probability Radon measures on $[0,1]^p$. Since the group $\mathrm{Homeo}([0,1]^p)$ of all homeomorphisms of $[0,1]^p$ acts on the set of probability Radon measures on $[0,1]^p$, it also acts on the traces of $\mathcal{A}_{p,n}$. In this section, we only consider the traces which are in the $\mathrm{Homeo}([0,1]^p)$-orbit of λ, where λ is the Lebesgue measure. Note that such traces are faithful.

By definition, a *supernatural number* is a formal product

$$\nu = \prod_{p:\text{ prime}} p^{n_p},$$

where n_p is either a non-negative integer or ∞ for each p such that $\sum_p n_p = \infty$. Given a supernatural number ν, we shall define a category $\mathcal{K}_\nu$ as following. Let $\mathbb{N}_\nu$ be the set of all natural numbers which formally divides ν.

- $\mathrm{Obj}(\mathcal{K}_\nu)$ is the class of all the pairs $\langle \mathcal{A}_{p,n}, \tau\rangle$, where n is in $\mathbb{N}_\nu$ and τ is in the $\mathrm{Homeo}([0,1]^p)$-orbit of λ.
- $\mathrm{Mor}_{\mathcal{K}_\nu}\big(\langle \mathcal{A}_{p,n}, \tau\rangle, \langle \mathcal{A}_{p',n'}, \tau'\rangle\big)$ is the set of all (unital trace-preserving injective) diagonalizable $*$-homomorphisms from $\langle \mathcal{A}_{p,n}, \tau\rangle$ to $\langle \mathcal{A}_{p',n'}, \tau'\rangle$.

Here, a $*$-homomorphism ι from $\mathcal{A}_{p,n}$ to $\mathcal{A}_{p',n'}$ is said to be *diagonalizable* if there exist a unitary $v \in \mathcal{A}_{p',n'}$ and continuous functions $t_1, \ldots, t_k \colon [0,1]^{p'} \to [0,1]^p$ such that

$$\iota\big(f\big)(s) = \mathrm{Ad}(v_s)\Big(\mathrm{diag}\big[f(t_1(s)), \ldots, f(t_k(s))\big]\Big)$$

for $f \in \mathcal{A}_{p,n}$ and $s \in [0,1]^{p'}$, where $\mathrm{Ad}(v)$ denotes the inner automorphism of $\mathcal{A}_{p',n'}$ associated to v, and $\mathrm{diag}[a_1, \ldots, a_n]$ is the block diagonal matrix with a_i as its i-th block. Note that for each diagonalizable ι, the choice of the unitary v and the continuous functions $t_1, \dots, t_k$ in the above expression is *not* unique in general. Nevertheless, we often have to consider the diameters of the ranges of the functions $t_1, \dots, t_k$. For the sake of convenience, we shall set

$$V(t_1, \dots, t_k) := \max_l \operatorname{diam} \operatorname{Im} t_l.$$

Remark 2.1.1. Here, we shall give an example of L_{TC^*}-morphisms between objects of $\mathcal{K}_\nu$ which cannot be approximated by diagonalizable ones with respect to point-norm topology. We assume that 2 is in $\mathbb{N}_\nu$ and use $\mathbb{D} := \{z \in \mathbb{C} \mid |z| \le 1\}$ instead of $[0,1]^2$. Define $t_1, t_2 \colon \mathbb{D} \to \mathbb{D}$ by

$$t_1(re^{i\theta}) := re^{i\theta/2}, \quad t_2(re^{i\theta}) := -re^{i\theta/2} \quad \bigl(r \in [0,1],\ \theta \in [0, 2\pi)\bigr).$$

Also, let $u \colon \mathbb{D} \to \mathbb{M}_2$ be a unitary-valued function defined by

$$u(re^{i\theta}) := \begin{cases} \begin{pmatrix} e^{i\theta/4}\cos\frac{\theta}{4} & \sin\frac{\theta}{4} \\ -e^{i\theta/4}\sin\frac{\theta}{4} & \cos\frac{\theta}{4} \end{pmatrix} & \bigl(r \ne 0,\ \theta \in [0, 2\pi)\bigr) \\ 1_{\mathbb{M}_2} & (r = 0), \end{cases}$$

and note that $u(r) = 1_{\mathbb{M}_2}$ while

$$\lim_{\theta \to 2\pi - 0} u(re^{i\theta}) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

if $r \ne 0$. Now, given $f \in C(\mathbb{D})$, consider a matrix-valued function

$$\varphi(f) \colon z \mapsto \operatorname{Ad}(u(z))\bigl(\operatorname{diag}[f(t_1(z)), f(t_2(z))]\bigr).$$

Clearly this is continuous on the complement of the non-negative part of the real axis. It is also continuous on the positive part of the real axis, as the switch of the eigenvalues is offset by the unitary. Finally, it is continuous at the origin, because it converges to the scalar matrix $f(0)1_2$. Therefore, this matrix-valued function belongs to $C(\mathbb{D}, \mathbb{M}_2)$, so that φ defines a $*$-homomorphism from $C(\mathbb{D})$ into $C(\mathbb{D}, \mathbb{M}_2)$. We can also easily verify that this is unital, injective, and trace-preserving with respect to μ, where μ corresponds to the normalized Lebesgue measure on $\mathbb{D}$.

We shall show that the map φ is not approximately diagonalizable. Indeed, if φ is approximately diagonalizable, then there is a $*$-homomorphism $\psi \colon C(\mathbb{D}) \to C(\mathbb{D}, \mathbb{M}_2)$ of the form

$$\psi(f) = \operatorname{Ad}(v)\bigl(\operatorname{diag}[f \circ t_1', f \circ t_2']\bigr)$$

for some continuous maps $t'_1, t'_2 \colon \mathbb{D} \to \mathbb{D}$ and a continuous unitary-valued map v which satisfies $\|\varphi(\mathrm{id}_{\mathbb{D}}) - \psi(\mathrm{id}_{\mathbb{D}})\| < 1/2$. It follows that the Hausdorff distance between $\{t_1(z), t_2(z)\}$ and $\{t'_1(z), t'_2(z)\}$ is less than $1/2$ for all z with $|z| = 1$, but this is impossible. We note that this is also a counterexample of [9, Theorem 6.3] claiming that any unital $*$-homomorphism from $C(X)$ to $C(Y, \mathbb{M}_n)$ is approximately diagonalizable if X is a compact metrizable locally absolute retract and Y is a compact metric space with $\dim Y \leq 2$.

Lemma 2.1.2. (i) *For any object $\langle \mathcal{A}_{p,n}, \tau \rangle$ of $\mathcal{K}_\nu$, there exists a $\mathcal{K}_\nu$-isomorphism from $\langle \mathcal{A}_{p,n}, \tau \rangle$ onto $\langle \mathcal{A}_{p,n}, \lambda \rangle$.*
(ii) *For any p, there exists a $\mathcal{K}_\nu$-embedding from $\langle \mathcal{A}_{p,n}, \tau \rangle$ into $\langle \mathcal{A}_{1,n}, \lambda \rangle$.*

Proof. (i) Let α be the homeomorphism of $[0,1]^p$ with $\alpha_*(\lambda) = \tau$. Then the induced $*$-homomorphism $\alpha^* \colon f \mapsto f \circ \alpha$ is the desired one.

(ii) We may assume $\tau = \lambda$ by (i). It suffices to give a proof for the case $p = 2$. Let $\beta \colon [0,1] \to [0,1]^2$ be the Hilbert curve [7], which is a surjective continuous map such that any interval of the form $[k/4^l, (k+1)/4^l]$ is sent to a square of the form $[k_1/2^l, (k_1+1)/2^l] \times [k_2/2^l, (k_2+1)/2^l]$, so that $\beta_*(\lambda) = \lambda$. Then the $*$-homomorphism $\beta^* \colon f \mapsto f \circ \beta$ is the desired one. $\square$

Lemma 2.1.3. *Suppose that $\iota_1, \iota_2 \colon \langle \mathcal{A}_{p,n}, \tau \rangle \to \langle \mathcal{A}_{p',n'}, \tau' \rangle$ are $\mathcal{K}_\nu$-embeddings of the form*

$$\iota_i(f) = \mathrm{diag}[f \circ t_{1,i}, \ldots, f \circ t_{k,i}].$$

If $V(t_{1,i}, \ldots, t_{k,i})$ is less than δ for $i = 1, 2$, then there exists a permutation $\sigma \in \mathfrak{S}_k$ such that the inequality

$$\|t_{l,1} - t_{\sigma(l),2}\| < 2\delta$$

holds for all l.

Proof. For each l, let S_l be the set of all l' with $\mathrm{Im}\, t_{l,1} \cap \mathrm{Im}\, t_{l',2} \neq \emptyset$. Then, for any $F \subseteq \{1, \ldots, k\}$, we have

$$\Big|\bigcup_{l \in F} S_l\Big| = k \sum_{l' \in \bigcup_{l \in F} S_l} \tau\big(\mathrm{Im}\, t_{l',2}\big) \geq k\tau\Big(\bigcup_{l \in F} \mathrm{Im}\, t_{l,1}\Big) \geq |F|,$$

since both ι_1 and ι_2 are trace-preserving. By Hall's marriage theorem there exists a permutation $\sigma \in \mathfrak{S}_k$ with $t_{\sigma(l),2} \in S_l$ for all l. Now the inequality $\|t_{l,1} - t_{\sigma(l),2}\| < 2\delta$ is clear. $\square$

Theorem 2.1.4. *The category $\mathcal{K}_\nu$ is a Fraïssé category.*

Proof. JEP is a direct consequence of Lemma 2.1.2 and the fact that if n divides n', then there exists a $\mathcal{K}_\nu$-embedding from $\langle \mathcal{A}_{1,n}, \lambda \rangle$ to $\langle \mathcal{A}_{1,n'}, \lambda \rangle$ defined by

$$f \mapsto \mathrm{diag}[f, \dots, f].$$

For NAP, let ι_i be a $\mathcal{K}_\nu$-embedding from $\langle \mathcal{A}_{p_0,n_0}, \tau_0 \rangle$ into $\langle \mathcal{A}_{p_i,n_i}, \tau_i \rangle$ for $i = 1, 2$, and suppose that a finite subset F of $\mathcal{A}_{p_0,n_0}$ and a positive real number $\varepsilon > 0$ are given. Our goal is to find $\mathcal{K}_\nu$-embeddings η_i from $\langle \mathcal{A}_{p_i,n_i}, \tau_i \rangle$ into some object $\langle \mathcal{A}_{p_3,n_3}, \tau_3 \rangle$ such that the inequality

$$\|\eta_1 \circ \iota_1(f) - \eta_2 \circ \iota_2(f)\| < \varepsilon$$

holds for all $f \in F$. To see this, take $\delta > 0$ so that $|t - t'| < \delta$ implies $\|f(t) - f(t')\| < \varepsilon$ for all $f \in F$. Apply JEP to find $\mathcal{K}_\nu$-embeddings η'_i from $\langle \mathcal{A}_{p_i,n_i}, \tau_i \rangle$ into some object $\langle \mathcal{A}_{p',n'}, \tau' \rangle$. By Proposition 2.1.2, we may assume without loss of generality that $\tau' = \lambda$ and $p' = 1$. Now, Since $\eta'_i \circ \iota_i$ is a $\mathcal{K}_\nu$-isomorphism, it is of the form

$$\eta'_i \circ \iota_i(f) = \mathrm{Ad}(v'_i)\big(\mathrm{diag}[f \circ t'_{1,i}, \dots, f \circ t'_{k',i}]\big).$$

Take sufficiently large natural number m such that $n'm$ is in $\mathbb{N}_\nu$ and $|s - s'| < 1/m$ implies $|t'_{l,i}(s) - t'_{l,i}(s')| < \delta/2$ for all l and i. Define $r_c \colon [0,1] \to [0,1]$ by

$$r_c(x) := \frac{x + c - 1}{m} \quad (c = 1, \dots, m),$$

and let ρ be a $\mathcal{K}_\nu$-embedding from $\langle \mathcal{A}_{1,n'}, \lambda \rangle$ into $\langle \mathcal{A}_{1,n'm}, \lambda \rangle$ of the form

$$\rho(f) = \mathrm{diag}[f \circ r_1, \dots, f \circ r_m].$$

Then $\rho \circ \eta'_i \circ \iota_i$ is of the form

$$\rho \circ \eta'_i \circ \iota_i(f) = \mathrm{Ad}(v_i)\big(\mathrm{diag}[f \circ t_{1,i}, \dots, f \circ t_{k,i}]\big),$$

where $V(t_{1,i}, \dots, t_{k,i})$ is less than $\delta/2$ for $i = 1, 2$. By Lemma 2.1.3, we may assume without loss of generality that the inequality $\|t_{l,1} - t_{l,2}\| < \delta$ holds for all l. It can be easily verified that $\eta_1 := \rho \circ \eta'_1$ and $\eta_2 := \mathrm{Ad}(v_1 v_2^*) \circ \rho \circ \eta'_2$ are the desired $\mathcal{K}_\nu$-embeddings.

WPP is clear, because up to $\mathcal{K}_\nu$-isomorphisms, there are only countably many objects in $\mathcal{K}_\nu$, by Lemma 2.1.2. Also, CCP automatically follows from the fact that all the relevant functions are 1-Lipschitz on the unit ball. □

We shall find a concrete description of the limit of $\mathcal{K}_\nu$. For this, the following proposition is useful.

Proposition 2.1.5. *Let $\mathcal{K}$ be a Fraïssé category and $\mathcal{M} = \overline{\bigcup_n \mathcal{A}_n}$ be a $\mathcal{K}$-structure. Denote by $\iota_{k,j}$ the canonical $\mathcal{K}$-embedding from $\mathcal{A}_j$ into $\mathcal{A}_k$. Suppose that the following two conditions hold:*

(a) *Any object C of $\mathcal{K}$ is $\mathcal{K}$-embeddable into $\mathcal{A}_n$ for some n.*
(b) *Given a finite subset $F \subseteq |\mathcal{A}_i|$, a positive real number ε and a $\mathcal{K}$-embedding $\eta \colon \mathcal{A}_i \to \mathcal{A}_j$ for some $j > i$, one can find $k > j$ and a $\mathcal{K}$-automorphism $\alpha \in \mathrm{Aut}(\mathcal{A}_k)$ such that the inequality*

$$d\big(\alpha \circ \iota_{k,j} \circ \eta(a),\ \iota_{k,i}(a)\big) < \varepsilon$$

holds for all $a \in F$.

Then $\mathcal{M} = \overline{\bigcup_n \mathcal{A}_n}$ is the Fraïssé limit of $\mathcal{K}$.

Proof. We shall check (iii) in Theorem 1.3.5. Let ε be a positive real number, $\mathcal{B}$ be an object of $\mathcal{K}$ and φ be in $\mathrm{Stx}_{\mathcal{K}}(\mathcal{B}, \mathcal{M})$. Then one can find finite subsets $F_1 \subseteq |\mathcal{B}|$ and $F_2 \subseteq |\mathcal{A}_i|$, an object C of $\mathcal{K}$, and $\mathcal{K}$-embeddings $\iota \colon \mathcal{B} \to C$ and $\eta \colon \mathcal{A}_i \to C$ such that the relation

$$\big(\varphi_{\iota,\eta}|_{F_1 \times F_2}\big)|^{\mathcal{B} \times \mathcal{M}} \lhd \varphi$$

holds. By assumption (a), there exists a $\mathcal{K}$-embedding θ of C into some $\mathcal{A}_j$ with $j > i$. Then one can find a $\mathcal{K}$-automorphism $\alpha \in \mathrm{Aut}(\mathcal{A}_k)$ for some $k > j$ such that the inequality

$$d\big(\alpha \circ \iota_{k,j} \circ \theta \circ \eta(a),\ \iota_{k,i}(a)\big) < \varepsilon$$

holds for all $a \in F_2$, by assumption (b). Now, for $b \in F_1$ and $a \in F_2$, we have

$$\begin{aligned} &d\big(\alpha \circ \iota_{k,j} \circ \theta \circ \iota(b),\ \iota_{k,i}(a)\big) \\ &< d\big(\alpha \circ \iota_{k,j} \circ \theta \circ \iota(b),\ \alpha \circ \iota_{k,j} \circ \theta \circ \eta(a)\big) + \varepsilon \\ &= d\big(\iota(b), \eta(a)\big) + \varepsilon, \end{aligned}$$

whence

$$\varphi_{\alpha \circ \iota_{k,j} \circ \theta \circ \iota} \leq \big(\varphi_{\iota,\eta}|_{F_1 \times F_2}\big)|^{\mathcal{B} \times \mathcal{M}} \lhd \varphi,$$

which completes the proof. □

Corollary 2.1.6. *Let $\mathcal{M} = \overline{\bigcup_j \langle \mathcal{A}_{p_j,n_j}, \tau_j \rangle}$ be a $\mathcal{K}_\nu$-structure and $\iota_{k,j}$ denote the canonical $\mathcal{K}_\nu$-embedding from $\langle \mathcal{A}_{p_j,n_j}, \tau_j \rangle$ into $\langle \mathcal{A}_{p_k,n_k}, \tau_k \rangle$. Suppose the following conditions hold:*

(a) $p_j \geq 1$.
(b) *For any $n \in \mathbb{N}_\nu$, there exists $j \in \mathbb{N}$ such that n divides n_j.*
(c) *For any $j \in \mathbb{N}$ and $\varepsilon > 0$ there exists $k > j$ such that $\iota_{k,j}$ is of the form*

$$\iota_{k,j}(f) = \mathrm{Ad}(v)\bigl(\mathrm{diag}[f \circ t_1, \dots, f \circ t_m]\bigr),$$

where $V(t_1, \dots, t_m)$ is less than ε.

Then $\mathcal{M} = \overline{\bigcup_j \langle \mathcal{A}_{p_j,n_j}, \tau_j \rangle}$ is the Fraïssé limit of $\mathcal{K}_\nu$.

Proof. This is immediate from Lemmas 2.1.2 and 2.1.3 and Proposition 2.1.5. □

Take an increasing sequence $\{n_j\} \subseteq \mathbb{N}_\nu$ so that (b) in Corollary 2.1.6 is satisfied. Define $\iota_i \colon \mathcal{A}_{1,n_j} \to \mathcal{A}_{1,n_{j+1}}$ by

$$\iota_i(f) = \mathrm{diag}[f \circ r_1, \dots, f \circ r_m],$$

where m is equal to n_{j+1}/n_j and $r_1, \dots, r_m$ are as in the proof of Theorem 2.1.4. Then the diagram

$$\begin{array}{ccccccc}
\langle \mathcal{A}_{1,n_1}, \lambda \rangle & \xrightarrow{\iota_1} & \langle \mathcal{A}_{1,n_2}, \lambda \rangle & \xrightarrow{\iota_2} & \langle \mathcal{A}_{1,n_3}, \lambda \rangle & \xrightarrow{\iota_3} & \cdots \\
\uparrow & & \uparrow & & \uparrow & & \\
\langle \mathbb{M}_{n_1}, \mathrm{tr} \rangle & \longrightarrow & \langle \mathbb{M}_{n_2}, \mathrm{tr} \rangle & \longrightarrow & \langle \mathbb{M}_{n_3}, \mathrm{tr} \rangle & \longrightarrow & \cdots
\end{array}$$

commutes, where $\mathbb{M}_{n_j}$ is canonically identified with the C*-subalgebra of constant functions on the interval $[0, 1]$. Since the upper inductive system satisfies the assumption of Corollary 2.1.6 and the limit of the lower inductive system is clearly dense in that of the upper one, it follows that the Fraïssé limit of $\mathcal{K}_\nu$ is isomorphic to what is called the UHF algebra of type ν as C*-algebras.

We conclude this section by showing that all L_{TC^*}-embeddings into the Fraïssé limit of $\mathcal{K}_\nu$ is indeed $\mathcal{K}_\nu$-admissible. To see this, we use the following lemmas [2, Exercise II.8 and Lemma III.3.2].

Lemma 2.1.7. *Let f be a continuous function on a compact subset X of $\mathbb{C}$. Then, for any $\varepsilon > 0$, there exists $\delta > 0$ such that if a and b are normal elements of a C*-algebra $\mathcal{A}$ with $\|a - b\| < \delta$, then $\|f(a) - f(b)\| < \varepsilon$.*

Lemma 2.1.8. *For any $\varepsilon > 0$ and $n \in \mathbb{N}$, there exists $\delta > 0$ such that if $\mathcal{A}$ and $\mathcal{B}$ are C*-subalgebras of a unital C*-algebra $\mathcal{D}$, if $\dim \mathcal{A}$ is less than n, and if $\{e_{ij}^{(k)}\}$ is a system of matrix units which spans $\mathcal{A}$ and satisfies $d(e_{ij}^{(k)}, \mathcal{B}) < \delta$, then there exists a unitary u in $\mathcal{D}$ with $\|u - 1\| < \varepsilon$ and $\mathrm{Ad}(u)[\mathcal{A}] \subseteq \mathcal{B}$.*

Lemma 2.1.9. *Let $\{e_{ij}\}$ be the system of standard matrix units of $\mathbb{M}_n$ and a be an element of $\mathbb{M}_m \otimes \mathbb{M}_n$ satisfying*

$$\|a(1 \otimes e_{ij}) - (1 \otimes e_{ij})a\| < \varepsilon.$$

Then the inequality

$$\|a - \big(1 \otimes \mathrm{tr}\big)(a)\| < n^2\varepsilon$$

holds.

Proof. If a is represented as $\sum a_{ij} \otimes e_{ij}$, then one can easily verify the inequality

$$\left\| a_{ij} \otimes e_{ij} - \frac{\delta_{ij}}{n} \sum_k a_{kk} \otimes e_{kk} \right\| < \varepsilon,$$

from which the conclusion follows. □

Theorem 2.1.10. *Every L-embedding from an object of $\mathcal{K}_\nu$ into the Fraïssé limit of $\mathcal{K}_\nu$ is $\mathcal{K}_\nu$-admissible.*

Proof. Let $\mathcal{M}$ be the Fraïssé limit of $\mathcal{K}_\nu$ and $\iota\colon \langle \mathcal{A}_{p,n}, \tau\rangle \to \mathcal{M}$ be an L-embedding. Our goal is to show that ι can be approximated by $\mathcal{K}$-admissible embeddings with respect to the topology of pointwise convergence. For simplicity, we only show the case $p = 1$ and $\tau = \lambda$. Set

$$G := \{1 \otimes e_{ij} \mid i, j = 1, \dots n\} \cup \{\mathrm{id}_{[0,1]} \otimes 1\} \subseteq C[0,1] \otimes \mathbb{M}_n \simeq \mathcal{A}_{1,n},$$

where $\{e_{ij}\}$ is the system of standard matrix units of $\mathbb{M}_n$, and note that G is a generator of $\mathcal{A}_{1,n}$. Given $\varepsilon > 0$, it suffices to find a $\mathcal{K}$-admissible embedding η of $\langle \mathcal{A}_{1,n}, \lambda\rangle$ into $\mathcal{M}$ satisfying $\|\iota(g) - \eta(g)\| < \varepsilon$ for all $g \in G$. For this, take $N \in \mathbb{N}$ with $1/N < \varepsilon/6$ and $nN \in \mathbb{N}_\nu$. For $c, d \in \mathbb{N}$ with $0 \le c < d \le N$, define a continuous function $f_{c,d}$ on $[0,1]$ by

$$f_{c,d}(t) := \begin{cases} 0 & (t \notin [(c-1)/N, (d+1)/N]) \\ 1 & (t \in [c/N, d/N]) \\ Nt - c + 1 & (t \in [(c-1)/N, c/N]) \\ -Nt + d - 1 & (t \in [d/N, (d+1)/N]). \end{cases}$$

Then by Lemma 2.1.7, there exists positive $\delta < \varepsilon/2$ such that if a is a normal element of $\mathcal{M}$ with $\|a - \iota(\mathrm{id}_{[0,1]} \otimes 1)\| < \delta$, then the inequality

$$\|f_{c,d}(a) - \iota(f_{c,d} \otimes 1)\| < \frac{1}{N}$$

holds for all $c, d \in \mathbb{N}$ with $0 \leq c < d \leq N$. Take such δ and set $\delta' := \delta/(6n^2 + 1)$.

Let

$$\begin{array}{ccccccc}
\langle \mathcal{A}_{1,n_1}, \lambda \rangle & \xrightarrow{\iota_1} & \langle \mathcal{A}_{1,n_2}, \lambda \rangle & \xrightarrow{\iota_2} & \langle \mathcal{A}_{1,n_3}, \lambda \rangle & \xrightarrow{\iota_3} & \cdots \\
\uparrow & & \uparrow & & \uparrow & & \\
\langle \mathbb{M}_{n_1}, \mathrm{tr} \rangle & \longrightarrow & \langle \mathbb{M}_{n_2}, \mathrm{tr} \rangle & \longrightarrow & \langle \mathbb{M}_{n_3}, \mathrm{tr} \rangle & \longrightarrow & \cdots
\end{array}$$

be the inductive system we saw before Lemma 2.1.7. Then, by Lemma 2.1.8, there exists a unitary u in $\mathcal{M}$ with $\|u - 1\| < \delta'$ and $e'_{ij} := u\big[\iota(1 \otimes e_{ij})\big]u^* \in \bigcup_k \mathbb{M}_{n_k}$. We shall denote by $\mathcal{B}$ the finite dimensional simple C*-subalgebra generated by $\{e'_{ij}\}$. Note that the inequality

$$\|\iota(1 \otimes e_{ij}) - e'_{ij}\| < 2\delta' \leq \varepsilon$$

holds for all i, j. Also, if $\mathcal{B}$ is included in $\mathbb{M}_{n_k}$, then $\mathbb{M}_{n_k}$ is canonically isomorphic to $\mathcal{B} \otimes \mathbb{M}_{n_k/n}$. Now, take $a \in \bigcup_k \mathbb{M}_{n_k}$ with $\|a - \iota(\mathrm{id}_{[0,1]} \otimes 1)\| < \delta'$. By Lemma 2.1.7, we may assume without loss of generality that a is a positive element with $\|a\| \leq 1$. Then

$$\|ae'_{ij} - e'_{ij}a\| < 6\delta,$$

so by Lemma 2.1.9, there exists a positive element $a' \in \bigcup_k \mathbb{M}_{n_k}$ which commutes with every element of $\mathcal{B}$ and satisfies the inequalities

$$\|a' - \iota(\mathrm{id}_{[0,1]} \otimes 1)\| < (6n^2 + 1)\delta' \leq \delta, \quad \|a'\| \leq 1.$$

By definition of δ, we have

$$\|f_{c,d}(a') - \iota(f_{c,d} \otimes 1)\| < \frac{1}{N}$$

for $0 \leq c < d \leq N$.

Let k_0 be sufficiently large so that both $\mathcal{B}$ and a' is included in $\mathbb{M}_{n_{k_0}}$ and $m := n_{k_0}/n$ is a multiple of N. Since the commutant $\mathcal{B}' \cap \mathbb{M}_{n_{k_0}}$ is canonically isomorphic to $\mathbb{M}_m$, the positive element a' can be identified with a diagonal matrix of $\mathbb{M}_m$, say $\mathrm{diag}[t_1, \ldots, t_m]$. Without loss of generality, we may assume $t_1 \leq \cdots \leq t_m$. Then we have

$$\begin{aligned}
\mathrm{tr}\big(\mathrm{diag}\big[f_{c,d}(t_1), \ldots, f_{c,d}(t_m)\big]\big) &= \mathrm{tr}^{\mathcal{M}}\big(f_{c,d}(a')\big) \\
&\geq \lambda(f_{c,d} \otimes 1) - \frac{1}{N} = \frac{d - c}{N}.
\end{aligned}$$

This inequality together with Hall's marriage theorem implies that the real numbers $t_{mc/N+1}, \ldots, t_{m(c+1)/N}$ are included in $[(c-1)/N, (c+2)/N]$. Consequently, the element

$$a'' := \operatorname{diag}[r_1, \ldots, r_m] \otimes 1 \in C([0,1], \mathbb{M}_m) \otimes \mathcal{B} \simeq \mathcal{A}_{1,n_{k_0}}$$

satisfies

$$\|a'' - a'\| \le \frac{3}{N} < \frac{\varepsilon}{2},$$

where $r_1, \ldots, r_m$ are as in the proof of Theorem 2.1.4. One can easily check that the $\mathcal{K}$-embedding $\eta\colon \mathcal{A}_{1,n} \to \mathcal{A}_{1,n_{k_0}}$ defined by

$$\eta(1 \otimes e_{ij}) := e'_{ij}, \quad \eta(\mathrm{id}_{[0,1]} \otimes 1) = a''$$

has the desired property. □

2.2 The Jiang–Su algebra

For natural numbers p and q, the *dimension drop algebra* $\mathcal{Z}_{p,q}$ is defined by

$$\mathcal{Z}_{p,q} := \{f \in \mathcal{A}_{1,pq} \mid f(0) \in \mathbb{M}_p \otimes 1_q \ \&\ f(1) \in 1_p \otimes \mathbb{M}_q\},$$

where we took over the notation $\mathcal{A}_{p,n} := C([0,1]^p, \mathbb{M}_n)$ from the preceding section. It is said to be *prime* if p and q are coprime. In the sequel, we frequently use the fact that every ideal of $\mathcal{Z}_{p,q}$ is of the form

$$\mathcal{I} = \{f \in \mathcal{Z}_{p,q} \mid f|_\Sigma \equiv 0\}$$

for a unique closed subset Σ of $[0,1]$.

We denote by $c_p^{p,q}$ the map from $\mathbb{M}_p \otimes 1_q$ to $\mathbb{M}_p$ defined by $a \otimes 1 \mapsto a$. The map $c_q^{p,q}\colon 1_p \otimes \mathbb{M}_q \to \mathbb{M}_q$ is defined similarly. When no confusion arises, these maps are simply denoted by c. Also, for $t \in [0,1]$, we denote by ev_t the evaluation map at t.

We shall define the category $\mathcal{K}_\mathcal{Z}$ as following.

- $\mathrm{Obj}(\mathcal{K}_\mathcal{Z})$ is the class of all the pairs $\langle \mathcal{Z}_{p,q}, \tau\rangle$, where p, q are coprime and τ is a faithful tracial state on $\mathcal{Z}_{p,q}$.
- Every L_{TC^*}-embedding between objects of $\mathcal{K}_\mathcal{Z}$ is a morphism of $\mathcal{K}_\mathcal{Z}$.

We note that every trace on a dimension drop algebra bijectively corresponds to a probability Radon measure on $[0,1]$, as in the case of $\mathcal{A}_{p,n}$.

In [8], Xinhui Jiang and Hongbing Su constructed the so-called Jiang–Su algebra as an inductive limit of prime dimension drop algebras, and proved that it is the unique monotracial simple C*-algebra among such inductive limits. Our goals here are to show that $\mathscr{K}_{\mathcal{Z}}$ is a Fraïssé category the limit of which is a simple C*-algebra with a unique tracial state (so it is the Jiang–Su algebra), and to give a Fraïssé theoretic proof of the uniqueness result above by Jiang and Su.

The first step is to show that every unital $*$-homomorphism between dimension drop algebras are approximately diagonalizable in the following sense.

Proposition 2.2.1. *Let $\iota\colon \mathcal{Z}_{p,q} \to \mathcal{Z}_{p',q'}$ be a unital $*$-homomorphism. Then the following statements hold.*
(1) *There exist integers a, b with $0 \le a < q$ and $0 \le b < p$, continuous maps $t_1, \dots, t_k$ from $[0,1]$ into $[0,1]$ and a family $\{v_s\}_{s\in[0,1]}$ of unitary matrices of size $p'q'$ such that ι is of the form*

$$\begin{aligned}\iota(f)(s) = \mathrm{Ad}(v_s)\Big(\mathrm{diag}\big[&\overbrace{c(f(0)),\dots,c(f(0))}^{a},\\ &f(t_1(s)),\dots,f(t_k(s)),\underbrace{c(f(1)),\dots,c(f(1))}_{b}\big]\Big)\end{aligned}$$

for $f \in \mathcal{Z}_{p,q}$ and $s \in [0,1]$.
(2) *Suppose that $t_1, \dots, t_k$ are as in (1). Then for any finite $G \subseteq \mathcal{Z}_{p,q}$ and any $\varepsilon > 0$, there exists a continuous path $u\colon [0,1] \to \mathbb{M}_{p'q'}$ of unitaries such that the $*$-homomorphism $\iota'\colon \mathcal{Z}_{p,q} \to \mathcal{Z}_{p',q'}$ defined by*

$$\begin{aligned}\iota'(f)(s) = \mathrm{Ad}(u(s))\Big(\mathrm{diag}\big[&\overbrace{c(f(0)),\dots,c(f(0))}^{a},\\ &f(t_1(s)),\dots,f(t_k(s)),\underbrace{c(f(1)),\dots,c(f(1))}_{b}\big]\Big)\end{aligned}$$

satisfies $\|\iota(g) - \iota'(g)\| < \varepsilon$ for all $g \in G$.

Proof. (1) For each $s \in [0,1]$, the map $\mathrm{ev}_s \circ \iota$ is a unital $*$-homomorphism from $\mathcal{Z}_{p,q}$ into the finite dimensional C*-algebra $\mathbb{M}_{p'q'}$, so there exist non-negative integers a^s, b^s, a unitary v_s and real numbers $t_1^s, \dots, t_{k^s}^s \in [0,1]$ such that the equation

$$\begin{aligned}\mathrm{ev}_s \circ \iota(f) = \mathrm{Ad}(v_s)\Big(\mathrm{diag}\big[&\overbrace{c(f(0)), \dots, c(f(0))}^{a^s},\\ &f(t_1^s), \dots, f(t_{k^s}^s), \underbrace{c(f(1)), \dots, c(f(1))}_{b^s}\big]\Big)\end{aligned}$$

holds for all $f \in \mathcal{Z}_{p,q}$. Without loss of generality, we may assume $t_1^s \le \dots \le t_{k^s}^s$.

Let T^s be the multiset

$$\{\underbrace{0, \dots, 0}_{pa^s}, \underbrace{t_1^s, \dots, t_1^s}_{pq}, \dots, \underbrace{t_{k^s}^s, \dots, t_{k^s}^s}_{pq}, \underbrace{1, \dots, 1}_{qb^s}\},$$

and note that T^s is equal to the spectrum of $\mathrm{ev}_s \circ \iota(\mathrm{id}_{[0,1]} \otimes 1)$. By continuity, if s_1 and s_2 are close to each other, then so are T_{s_1} and T_{s_2} with respect to the Hausdorff distance, whence in particular pa^s and qb^s must be constant modulo pq. Since

$$\begin{aligned}\mathrm{diag}\big[\overbrace{c(f(0)), \dots, c(f(0))}^{q}\big] &\sim f(0),\\ \mathrm{diag}\big[\underbrace{c(f(1)), \dots, c(f(1))}_{p}\big] &\sim f(1),\end{aligned}$$

we may assume from the outset that a^s, b^s, k^s are independent of s, so we shall simply write a, b, k for them. Also, by the same reason, we may assume $a < q$ and $b < p$.

Now set

$$t_i(s) := t_i^s \quad (i = 1, \dots, k).$$

Then, since the inequality $t_1^s \le \dots \le t_k$ holds and the map $s \mapsto T^s$ is continuous, the functions $t_1, \dots, t_k$ are continuous. Clearly, we have

$$\begin{aligned}\iota\big(f\big)(s) = \mathrm{Ad}(v_s)\Big(\mathrm{diag}\big[&\overbrace{c(f(0)), \dots, c(f(0))}^{a},\\ &f(t_1(s)), \dots, f(t_k(s)), \underbrace{c(f(1)), \dots, c(f(1))}_{b}\big]\Big),\end{aligned}$$

as desired.

(2) Without loss of generality, we may assume that G is included in the unit ball. Fix $s_0 \in [0,1]$. We first claim that there exists $\delta(s_0) > 0$ with the following property: if

$|s - s_0| < \delta(s_0)$, then there exists a unitary $w_{s_0} \in \mathbb{M}_{p'q'}$ with $\|v_s - w_{s_0}\| < \varepsilon$ such that the equation

$$\begin{aligned}\iota(f)(s_0) = \mathrm{Ad}(w_{s_0})\Big(\mathrm{diag}\big[&\overbrace{c(f(0)), \dots, c(f(0))}^{a},\\ &f(t_1(s_0)), \dots, f(t_k(s_0)), \underbrace{c(f(1)), \dots, c(f(1))}_{b}\big]\Big)\end{aligned}$$

holds for all $f \in \mathcal{Z}_{p,q}$. To see this, let $\{x_1, \dots, x_l\}$ be the set of *distinct* eigenvalues of $\mathrm{ev}_{s_0} \circ\iota(\mathrm{id}_{[0,1]} \otimes 1)$, and take mutually orthogonal non-negative continuous functions $f_1, \dots, f_l$ such that f_i is constantly equal to 1 on some neighborhood of x_i for each i. Note that if $\{e^i_{lm}\}$ is the system of standard matrix units of $\mathrm{ev}_{t_i}[\mathcal{Z}_{p,q}]$, then $\{\mathrm{ev}_{s_0} \circ\iota(f_i \otimes e^i_{lm})\}_{i,l,m}$ forms a system of matrix units which spans $\mathrm{Im}(\mathrm{ev}_{s_0} \circ\iota)$, and if s is sufficiently close to s_0, then $\{\mathrm{ev}_s \circ\iota(f_i \otimes e^i_{lm})\}_{i,l,m}$ is a system of matrix units in $\mathrm{Im}(\mathrm{ev}_s \circ\iota)$ which is close to $\{\mathrm{ev}_{s_0} \circ\iota(f_i \otimes e^i_{lm})\}_{i,l,m}$. Hence, as in the proof of [2, Lemma III.3.2], we can find a unitary w with $\|w - 1\| < \varepsilon$ such that

$$w\big(\mathrm{ev}_{s_0} \circ\iota(f_i \otimes e^i_{lm})\big)\, w^* = \mathrm{ev}_s \circ\iota(f_i \otimes e^i_{lm}),$$

and $w_{s_0} := v_s w$ has the desired property.

Now take $\delta_0 > 0$ sufficiently small so that the inequalities

$$\big\|g(t_i(s)) - g(t_i(s'))\big\| < \varepsilon/4, \quad \big\|\iota(g)(s) - \iota(g)(s')\big\| < \varepsilon/4$$

hold for all $g \in G$ whenever $|s - s'| < \delta_0$, and consider an open covering

$$\mathcal{U} := \big\{U_\delta(s) \bigm| s \in [0, 1] \ \&\ \delta < \min\{\delta(s), \delta_0\}\big\}$$

of $[0, 1]$, where $U_\delta(s)$ denotes the open ball of radius δ and center s. Since $[0, 1]$ is compact, there exists a finite subcovering, say $\{I_1, \dots, I_r\}$. We denote the center of I_j by c_j, and without loss of generality, we may assume $c_1 < \cdots < c_r$ and $I_j \cap I_{j+1} \neq \emptyset$ for all j. Take small $\gamma > 0$ and $b_j \in I_j \cap I_{j+1} \cap (c_j + \gamma, c_{j+1} - \gamma)$ for each j, and find a unitary $u \in \mathcal{A}_{p'q'}$ such that

- $u(b_j)$ is equal to v_{b_j} for all j;
- $u(0)$ and $u(1)$ are equal to v_0 and v_1 respectively;
- the image of u on $[c_j + \gamma, c_{j+1} - \gamma]$ is included in the $\varepsilon/4$-ball of center $u(b_j) = v_{b_j}$;

- the images of u on $[0, c_1 - \gamma]$ and $[c_r + \gamma, 1]$ are included in the $\varepsilon/4$-balls of center $u(0) = v_0$ and $u(1) = v_1$ respectively; and
- the image of u on $[c_j - \gamma, c_j + \gamma]$ is included in the path-connected subset
$$\{w \mid \iota(f)(c_j) = \mathrm{Ad}(wv_{c_j}^*)(\iota(f)(c_j))\}$$
  of unitaries,

which is possible by the claim we proved in the previous paragraph.

We shall set
$$\begin{aligned}\iota'(f)(s) = \mathrm{Ad}(u(s))\Big(\mathrm{diag}\big[&\overbrace{c(f(0)), \ldots, c(f(0))}^{a},\\ &f(t_1(s)), \ldots, f(t_k(s)), \underbrace{c(f(1)), \ldots, c(f(1))}_{b}\big]\Big)\end{aligned}$$

and show that this ι' has the desired property. For $g \in G$ and $t \in [c_j + \gamma, c_{j+1} - \gamma]$, we have
$$\begin{aligned}\big\|\iota'(g)(s) - \iota(g)(s)\big\| &\leq \big\|\iota'(g)(s) - \iota(g)(b_j)\big\| + \big\|\iota(g)(b_j) - \iota(g)(s)\big\|\\ &< \frac{3\varepsilon}{4} + \frac{\varepsilon}{4} = \varepsilon.\end{aligned}$$
The same inequality holds if t is in $[0, c_1 - \gamma]$ or $[c_r + \gamma, 1]$. On the other hand, if $t \in [c_j - \gamma, c_j + \gamma]$, then
$$\begin{aligned}\big\|\iota'(g)(s) - \iota(g)(s)\big\| &\leq \big\|\iota'(g)(s) - \iota(g)(c_j)\big\| + \big\|\iota(g)(c_j) - \iota(g)(s)\big\|\\ &< \frac{\varepsilon}{4} + \frac{\varepsilon}{4} < \varepsilon.\end{aligned}$$
Consequently, it follows that
$$\|\iota'(g) - \iota(g)\| < \varepsilon$$
for all $g \in G$, which completes the proof. □

The family $t_1, \ldots, t_k$ of continuous maps and the integers a and b that appeared in Proposition 2.2.1 are called an *eigenvalue pattern* and the *remainder indices* of the $*$-homomorphism ι. An eigenvalue pattern $t_1, \ldots, t_k$ is said to be *normalized* if it satisfies the inequality $t_1 \leq \cdots \leq t_k$. Note that the normalized eigenvalue pattern is unique for each $*$-homomorphism. Also, if $\mathcal{Z}_{p,q}$ is prime, then the remainder indices depend only on the integers p, q, p' and q'. Indeed, if η is another $*$-homomorphism from $\mathcal{Z}_{p,q}$ into $\mathcal{Z}_{p',q'}$, and if a_η and b_η are the remainder indices of η, then the congruence equation

$$pa + qb \equiv p'q' \equiv pa_\eta + qb_\eta \pmod{pq}$$

holds, so that

$$a \equiv a_\eta \pmod{q}, \quad b \equiv b_\eta \pmod{p},$$

as p and q are coprime.

Let ι be a $*$-homomorphism between dimension drop algebras with an eigenvalue pattern $t_1, \ldots, t_k$. As in the preceding section, we shall denote by $V(t_1, \ldots, t_k)$ the maximum of the diameters of the images of $t_1, \ldots, t_k$, and call it the *variation* of the eigenvalue pattern. The infimum of the variations of all the eigenvalue patterns of ι is clearly equal to the variation of the normalized eigenvalue pattern, which is called the variation of ι and denoted by $V(\iota)$.

Proposition 2.2.2. *Let $p, q \in \mathbb{N}$ be coprime and ε be a positive real number. Then there exists $M \in \mathbb{N}$ such that if p', q' are larger than M, then there exists a unital embedding of $\mathcal{Z}_{p,q}$ into $\mathcal{Z}_{p',q'}$ with its variation less than ε.*

Proof. Since p and q are coprime, there exists $M \in \mathbb{N}$ with $M \geq pq(1/\varepsilon + 2)$ such that if $p', q' > M$, then

$$pa + pqk + qb = p'q'$$

for some $a, b, k \in \mathbb{N}$. We shall show that this M has the desired property. For this, suppose $p', q' > M$ and $a, b, k \in \mathbb{N}$ satisfy the equality above. Without loss of generality, we may assume $0 \leq a < q$ and $0 \leq b < p$. Also, one can find $l^0, m^0 \in [0, q)$ and $l^1, m^1 \in [0, p)$ such that

$$\begin{aligned} pl^0 &\equiv p' \pmod{q}, & pm^0 &\equiv q' \pmod{q}, \\ ql^1 &\equiv p' \pmod{p}, & qm^1 &\equiv q' \pmod{p}. \end{aligned}$$

Then,

$$\begin{aligned} pq'l^0 &\equiv pp'm^0 \equiv p'q' \equiv pa \pmod{q}, \\ qq'l^1 &\equiv qp'm^1 \equiv p'q' \equiv qb \pmod{p}, \end{aligned}$$

so

$$q'l^0 \equiv p'm^0 \equiv a \pmod{q}, \quad q'l^1 \equiv p'm^1 \equiv b \pmod{p}.$$

We set

$$n_0^0 := \frac{q'l^0 - a}{q}, \quad n_0^1 := \frac{q'l^1 - b}{p},$$
$$n_0^0 := \frac{p'm^0 - a}{q}, \quad n_0^1 := \frac{p'm^1 - b}{p}.$$

so that

$$a + qn_0^0 \equiv b + pn_0^1 \equiv 0 \pmod{q'},$$
$$a + qn_1^0 \equiv b + pn_1^1 \equiv 0 \pmod{p'}.$$

We claim that

- $n_0^0 + n_0^1$ and $n_1^0 + n_1^1$ are smaller than k;
- $k - n_0^0 - n_0^1$ and $k - n_1^0 - n_1^1$ are multiples of q' and p' respectively; and
- $(k - n_0^0 - n_0^1)/q'$ and $(k - n_1^0 - n_1^1)/p'$ are larger than $1/\varepsilon$.

Indeed, we have

$$\begin{aligned} n_0^0 + n_0^1 &= \frac{q'l^0 - a}{q} + \frac{q'l^1 - b}{p} \\ &= \frac{q'(pl^0 + ql^1) - pa - qb}{pq} \\ &< \frac{2q'pq - p'q' + pqk}{pq} < k. \end{aligned}$$

Also, note that

$$pq(k - n_0^0 - n_0^1) = p'q' - pq'l^0 - qq'l^1 = q'(p' - pl^0 - ql^1).$$

Since p and q divide $p' - ql^1$ and $p' - pl^0$ respectively, and since p and q are coprime, it follows that pq divides $p' - pl^0 - ql^1$, so q' divides $k - n_0^0 - n_0^1$; and

$$\frac{k - n_0^0 - n_0^1}{q'} = \frac{p' - pl^0 - ql^1}{pq} > \frac{p' - 2pq}{pq} > \frac{1}{\varepsilon}.$$

Similarly, it follows that $n_1^0 + n_1^1$ is smaller than k, that p' divides $k - n_1^0 + n_1^1$, and that $(k - n_1^0 - n_1^1)/p'$ is larger than $1/\varepsilon$.

From the claim in the previous paragraph, one can easily construct a family $t_1, \ldots t_k$ of continuous maps from $[0, 1]$ into $[0, 1]$ such that

- the union of the images of $t_1, \ldots, t_k$ is equal to $[0, 1]$;
- the diameter of the image of t_i is smaller than ε for all i;
- $\#\{i \mid t_i(x) = y\} = n_x^y$ for $x, y = 0, 1$; and

- for each y with $0 < y < 1$, the integers q' and p' divide $\#\{i \mid t_i(0) = y\}$ and $\#\{i \mid t_i(1) = y\}$ respectively.

If we define a $*$-homomorphism η from $\mathcal{Z}_{p,q}$ into $\mathcal{A}_{1,p'q'}$ by

$$\begin{aligned}\eta(f)(s) = \Big(\mathrm{diag}\big[&\overbrace{c(f(0)), \dots, c(f(0))}^{a},\\ &f(t_1(s)), \dots, f(t_k(s)), \underbrace{c(f(1)), \dots, c(f(1))}_{b}\big]\Big),\end{aligned}$$

then one can easily verify from the construction of $t_1, \dots, t_k$ that the images of $\mathrm{ev}_0 \circ \eta$ and $\mathrm{ev}_1 \circ \eta$ are included in isomorphic copies of $\mathbb{M}_{p'} \otimes 1_{q'}$ and $1_{p'} \otimes \mathbb{M}_{q'}$ respectively, so there is a unitary $u \in \mathcal{A}_{1,p'q'}$ with $\mathrm{Im}\big(\mathrm{Ad}(u) \circ \eta\big) \subseteq \mathcal{Z}_{p',q'}$. $\square$

Note that the integers a and b in the proof of the previous proposition is the reminder indices of the embedding that is constructed. In particular, both of the indices are equal to 0 if pq divides $p'q'$.

For the next proposition, we shall introduce a terminology. By definition, a *modulus of uniform continuity* of a function f on $[0, 1]$ is a map $\Delta_f\colon (0, 1] \to (0, 1]$ such that $|s - s'| < \Delta_f(\varepsilon)$ implies $\|f(s) - f(s')\| \leq \varepsilon$.

Proposition 2.2.3. *Let p, q be coprime positive integers, $\iota_1, \iota_2\colon \mathcal{Z}_{p,q} \to \mathcal{Z}_{p',q'}$ be unital $*$-homomorphisms with eigenvalue patterns $t^1_1, \dots t^1_k$ and $t^2_1, \dots, t^2_k$ respectively, G be a finite subset of $\mathcal{Z}_{p,q}$, and ε be a positive real number. If the inequality*

$$\max_i \|t^1_i - t^2_i\|_\infty < \min_{g \in G} \Delta_g(\varepsilon)$$

holds, where Δ_g is a modulus of uniform continuity of g, then there exists a unitary $w \in \mathcal{Z}_{p',q'}$ with

$$\big\|\big(\mathrm{Ad}(w) \circ \iota_1\big)(g) - \iota_2(g)\big\| < 5\varepsilon$$

for all $g \in G$.

Proof. By Proposition 2.2.1, we may assume without loss of generality that ι_j is of the form

$$\begin{aligned}\iota_j(f)(s) = \mathrm{Ad}(u^j(s))\Big(\mathrm{diag}\big[&\overbrace{c(f(0)), \dots, c(f(0))}^{a},\\ &f(t^j_1(s)), \dots, f(t^j_k(s)), \underbrace{c(f(1)), \dots, c(f(1))}_{b}\big]\Big)\end{aligned}$$

for $f \in \mathcal{Z}_{p,q}$, where $u^j \in \mathcal{A}_{1,p'q'}$ is a unitary and $t_1^j \leq \cdots \leq t_k^j$. Also, we may assume that $\|g\| \leq 1$ for all $g \in G$.

Let n_0^0 and n_0^1 be the least non-negative integers such that

$$a + qn_0^0 \equiv b + pn_0^1 \equiv 0 \pmod{q'}.$$

Then, from the condition $\iota_j(f) \in \mathbb{M}_{p'} \otimes 1_{q'}$, it easily follows that

$$\begin{aligned} 0 = t^j_{n_0^0}(0) &\leq t^j_{n_0^0+1}(0) = \cdots = t^j_{n_0^0+q'}(0) \\ &\leq t^j_{n_0^0+q'+1}(0) = \cdots = t^j_{n_0^0+2q'}(0) \\ &\leq \cdots \leq t^j_{k-n_0^1+1}(0) = 1, \end{aligned}$$

so there exists a unitary $v_0^j \in \mathbb{M}_{p'}$ such that

$$\begin{aligned} c\big(\iota_j\big(f\big)(0)\big) = \operatorname{Ad}(v_0^j)\Big(\operatorname{diag}\big[&\overbrace{c(f(0)), \ldots, c(f(0))}^{a'}, \\ &f\big(t^j_{n_0^0+q'}(0)\big), f\big(t^j_{n_0^0+2q'}(0)\big), \ldots, f\big(t^j_{k-n_0^1}(0)\big), \\ &\underbrace{c(f(1)), \ldots, c(f(1))}_{b'}\big]\Big) \end{aligned}$$

for some non-negative integers a' and b'. Similarly, for suitable non-negative integers n_1^0, n_1^1, a'' and b'' and a unitary $v_1^j \in \mathbb{M}_{q'}$, we have

$$\begin{aligned} c\big(\iota_j\big(f\big)(1)\big) = \operatorname{Ad}(v_1^j)\Big(\operatorname{diag}\big[&\overbrace{c(f(0)), \ldots, c(f(0))}^{a''}, \\ &f\big(t^j_{n_1^0+p'}(1)\big), f\big(t^j_{n_1^0+2p'}(1)\big), \ldots, f\big(t^j_{k-n_1^1}(1)\big), \\ &\underbrace{c(f(1)), \ldots, c(f(1))}_{b''}\big]\Big). \end{aligned}$$

Thus, if v is a path of unitaries connecting $\big[v_0^2(v_0^1)^*\big] \otimes 1_{q'}$ to $1_{p'} \otimes \big[v_1^2(v_1^1)^*\big]$, then v is in $\mathcal{Z}_{p',q'}$ and $\big(\operatorname{Ad}(v) \circ \iota_1\big)\big(f\big)(s) = \big(\operatorname{Ad}(u^2(u^1)^*) \circ \iota_1\big)\big(f\big)(s)$ for $s = 0, 1$. Therefore, considering $\operatorname{Ad}(v) \circ \iota_1$ instead of ι_1 if necessary, we may assume from the outset that $u^2(0)u^1(0)^*$ and $u^2(1)u^1(1)^*$ commutes with every matrix in the image of $\mathrm{ev}_0 \circ \iota_1$ and $\mathrm{ev}_1 \circ \iota_1$.

Now, take $\delta > 0$ so that $|s - s'| < \delta$ implies

$$|t_i^j(s) - t_i^j(s')| < \min_{g \in G} \Delta_g(\varepsilon), \quad \|u^j(s) - u^j(s')\| < \varepsilon.$$

Let $w\colon [0,1] \to \mathbb{M}_{p'q'}$ be a path of unitaries such that

- $w|_{[0,\delta/2]}$ connects $1_{p'q'}$ to $u^2(0)u^1(0)^*$ within the commutant of the image of $\mathrm{ev}_0 \circ \iota_1$;
- $w(s) = u^2(2s - \delta')u^1(2s - \delta)^*$ for $s \in [\delta/2, \delta]$;
- $w(s) = u^2(s)u^1(s)^*$ for $s \in [\delta, 1 - \delta]$;
- $w(s) = u^2(2s - 1 + \delta)u^1(2s - 1 + \delta)^*$ for $s \in [1 - \delta, 1 - \delta/2]$; and
- $w|_{[1-\delta/2,1]}$ connects $u^2(1)u^1(1)^*$ to $1_{p'q'}$ within the commutant of the image of $\mathrm{ev}_0 \circ \iota_1$.

Then it is not difficult to see that this w has the desired property. $\square$

Lemma 2.2.4. *If μ is an atomless faithful probability measure on $[0,1]$, then for any faithful probability measure ν on $[0,1]$, there exists a non-decreasing continuous surjection β from $[0,1]$ onto $[0,1]$ with $\beta_*(\mu) = \nu$, so that β^* is a trace-preserving embedding of $\langle \mathcal{Z}_{p,q}, \nu\rangle$ into $\langle \mathcal{Z}_{p,q}, \mu\rangle$.*

Proof. We first assume that the measure μ is equal to the Lebesgue measure λ and set $\alpha(t) := \nu([0,t))$. Note that α is a strictly increasing lower semi-continuous function from $[0,1]$ into $[0,1]$. Let β be the unique non-decreasing function extending α^{-1}. Then,

$$\beta_*(\lambda)\big([0,t)\big) = \lambda\big(\beta^{-1}\big[[0,t)\big]\big) = \lambda\big([0,\alpha(t))\big) = \alpha(t) = \nu\big([0,t)\big),$$

so $\beta_*(\lambda)$ is equal to ν. Also, if ν is atomless, then α is continuous, whence $\beta = \alpha^{-1}$ is a homeomorphism.

For the general case, let β_μ, β_ν be the non-decreasing continuous functions such that $(\beta_\mu)_*(\lambda) = \mu$ and $(\beta_\nu)_*(\lambda) = \nu$. Then β_μ is a homeomorphism and $\beta := \beta_\nu \circ \beta_\mu^{-1}$ satisfies $\beta_*(\mu) = \nu$, which completes the proof. $\square$

Theorem 2.2.5. *The category $\mathcal{K}_{\mathcal{Z}}$ is a Fraïssé category.*

Proof. In view of Lemma 2.2.4, one can easily modify the proof of Proposition 2.2.2 to show the following claim: For any coprime integers p and q and any faithful tracial state τ on $\mathcal{Z}_{p,q}$, there exists a natural number M such that if p' and q' are larger than M and pq divides $p'q'$, then we can construct a *trace-preserving* $*$-homomorphism from $\langle \mathcal{Z}_{p,q}, \tau\rangle$ into $\langle \mathcal{Z}_{p',q'}, \lambda\rangle$, where λ is the Lebesgue measure. So $\mathcal{K}_{\mathcal{Z}}$ satisfies JEP. Also, a combination of Propositions 2.2.1, 2.2.2 and 2.2.3 immediately yields a proof of NAP:

The proof is essentially the same as that of Theorem 2.1.4. Since any object $\langle \mathcal{Z}_{p,q}, \tau\rangle$ of $\mathcal{K}_{\mathcal{Z}}$ can be embedded into $\langle \mathcal{Z}_{p,q}, \lambda\rangle$ by Lemma 2.2.4, WPP holds. Finally, CCP is automatic, since all the relevant functions and relations are 1-Lipschitz on the unit ball. □

Henceforth, we shall denote by $\langle \mathcal{Z}, \mathrm{tr}\rangle$ the Fraïssé limit of $\mathcal{K}_{\mathcal{Z}}$. From Theorem 2.2.13 below, it follows that the C*-algebra $\mathcal{Z}$ is the same as the one constructed in [8, Section 2], the so-called Jiang–Su algebra.

We shall say an inductive system of prime dimension drop algebras with distinguished traces is *regular* if its inductive limit is isomorphic to $\langle \mathcal{Z}, \mathrm{tr}\rangle$. In the sequel, we shall establish a method of recognizing regular systems.

Lemma 2.2.6. *Suppose that p and q are coprime and $\mathcal{Z}_{p,q}$ is embeddable into $\mathcal{Z}_{p',q'}$. Then there exists a tracial state $\lambda_{p',q'}$ on $\mathcal{Z}_{p,q}$ with the following properties.*
(1) *There exists a trace-preserving embedding from $\langle \mathcal{Z}_{p,q}, \lambda_{p',q'}\rangle$ into $\langle \mathcal{Z}_{p',q'}, \lambda\rangle$.*
(2) *If τ is a tracial state on $\mathcal{Z}_{p,q}$ of the form $\iota^*(\tau')$ for some embedding ι of $\mathcal{Z}_{p,q}$ into $\mathcal{Z}_{p',q'}$ and some faithful tracial state τ' on $\mathcal{Z}_{p'q'}$, then there exists a non-decreasing continuous map β from $[0, 1]$ onto $[0, 1]$ with $\beta_*(\lambda_{p,q}) = \tau$.*

Proof. Since $\mathcal{Z}_{p,q}$ is embeddable into $\mathcal{Z}_{p',q'}$, there is an embedding ρ of $\mathcal{Z}_{p,q}$ into $\mathcal{Z}_{p',q'}$ of the form

$$\rho(f)(s) = \mathrm{Ad}(u(s))\Big(\mathrm{diag}\big[\overbrace{c(f(0)), \dots, c(f(0))}^{a},$$
$$f(t_1(s)), \dots, f(t_k(s)), \underbrace{c(f(1)), \dots, c(f(1))}_{b}\big]\Big),$$

where $t_1, \dots, t_k$ are piecewise strictly monotone functions such that the union of the images is equal to $[0, 1]$. We shall set $\lambda_{p',q'} := \rho^*(\lambda)$. Note that if $\lambda_{p',q'} = \lambda^d_{p',q'} + \lambda^c_{p',q'}$ is the Lebesgue decomposition, then the discrete measure $\lambda^d_{p',q'}$ is equal to $(ap\delta_0 + bq\delta_1)/p'q'$, where δ_0 and δ_1 are the Dirac measures supported on $\{0\}$ and $\{1\}$ respectively, and the support of the atomless measure $\lambda^c_{p,q}$ is $[0, 1]$.

If τ is of the form $\iota^*(\tau')$ for some embedding ι of $\mathcal{Z}_{p,q}$ into $\mathcal{Z}_{p',q'}$ and some faithful tracial state τ' on $\mathcal{Z}_{p',q'}$, then necessarily $\tau = \lambda^d_{p',q'} + \mu$ for a suitable measure μ on $[0, 1]$, and $\|\lambda^c_{p',q'}\| = \|\mu\|$. Since $\lambda^c_{p',q'}$ is continuous, there exists a non-decreasing continuous map β from $[0, 1]$ onto $[0, 1]$ with $\beta_*(\lambda^c_{p',q'}) = \mu$, so $\beta_*(\lambda_{p',q'}) = \tau$, as desired. □

Lemma 2.2.7. *Let $\iota\colon \mathcal{Z}_{p,q} \to \mathcal{Z}_{p',q'}$ be a unital $*$-homomorphism, $\beta\colon [0,1] \to [0,1]$ be a non-decreasing continuous surjection, G be a finite subset of $\mathcal{Z}_{p,q}$, and ε be a positive real number. Suppose that the inequality*

$$V(\iota) < \min_{g\in G} \Delta_g(\varepsilon)$$

holds, where Δ_g denotes a modulus of uniform continuity of g. Then there exists a unitary $w \in \mathcal{Z}_{p,q}$ with

$$\left\| \big(\mathrm{Ad}(w) \circ \beta^* \circ \iota\big)(g) - \iota(g) \right\| < 5\varepsilon$$

for all $g \in G$.

Proof. Note that if $t_1 \leq \cdots \leq t_k$ is the normalized eigenvalue pattern of ι, then

$$\|t_i - t_i \circ \beta\|_\infty < \min_{g\in G} \Delta_g(\varepsilon).$$

Thus, the claim is immediate from Proposition 2.2.3. □

At first sight, the proof of the following proposition might seem to be complicated. However, the underlying idea is very simple; see Remark 2.2.9.

Proposition 2.2.8. *An inductive system $\{\langle \mathcal{Z}_{p_n,q_n}, \tau_n\rangle, \iota_{n,m}\}$ of prime dimension drop algebras with distinguished traces is regular if*

$$\lim_{n\to\infty} V(\iota_{n,m}) = 0$$

for all m.

Proof. We shall apply Theorem 1.3.5(iii), or more precisely, a weaker version of Lemma 1.3.6. Let $\langle \mathcal{Z}_{p,q}, \tau\rangle$ be a prime dimension drop algebra with a fixed faithful trace, F be a finite subset of $\mathcal{Z}_{p,q}$, and φ be a strict approximate $\mathscr{K}_{\mathcal{Z}}$-isomorphism from $\langle \mathcal{Z}_{p,q}, \tau\rangle$ to $\langle \mathcal{Z}_{p_n,q_n}, \tau_n\rangle$. Our goal is to find an approximate $\mathscr{K}_{\mathcal{Z}}$-isomorphism ψ from $\langle \mathcal{Z}_{p,q}, \tau\rangle$ to $\langle \mathcal{Z}_{p_N,q_N}, \tau_N\rangle$ for some $N > n$ such that

- $\psi(f, g) \leq \varphi(f, g)$ for $f \in \mathcal{Z}_{p,q}$ and $g \in \mathcal{Z}_{p_n,q_n}$, and
- ψ is ε-total on F for a given $\varepsilon > 0$.

By the definition of strict approximate isomorphisms, there exist finite subsets $G_1 \subseteq \mathcal{Z}_{p,q}$ and $G_2 \subseteq \mathcal{Z}_{p_n,q_n}$, morphisms θ_1, θ_2 from $\langle \mathcal{Z}_{p,q}, \tau\rangle$ and $\langle \mathcal{Z}_{p_n,q_n}, \tau_n\rangle$ into some $\langle \mathcal{Z}_{r,s}, \sigma\rangle$, and a positive real number δ such that

$$\varphi \geq (\varphi_{\theta_1,\theta_2}|_{G_1\times G_2})|^{\mathcal{Z}_{p,q}\times\mathcal{Z}_{p_n,q_n}} + \delta.$$

Here, we may assume the following. Fix an arbitrary positive real number γ.

(1) The subset G_1 includes F. This is because we may replace G_1 with a larger subset.
(2) There exist $m < n$ and a finite subset $G_2' \subseteq \mathcal{Z}_{p_m,q_m}$ such that $\iota_{n,m}[G_2'] = G_2$ and $V(\iota_{n,m}) < \Delta_g(\gamma)$ for all $g \in G_2'$, where Δ_g is a modulus of uniform continuity for g. This is because, taking our goal into account, we may replace φ with $\varphi|^{\mathcal{Z}_{p,q}\times\mathcal{Z}_{p_l,q_l}}$ for $l > n$, and we have

$$\begin{aligned}\varphi|^{\mathcal{Z}_{p,q}\times\mathcal{Z}_{p_l,q_l}} &\geq \left[(\varphi_{\theta_1,\theta_2}|_{G_1\times G_2})|^{\mathcal{Z}_{p,q}\times\mathcal{Z}_{p_n,q_n}} + \delta\right]|^{\mathcal{Z}_{p,q}\times\mathcal{Z}_{p_l,q_l}}\\ &= (\varphi_{\theta_1,\theta_2}|_{G_1\times\iota_{l,n}[G_2]})|^{\mathcal{Z}_{p,q}\times\mathcal{Z}_{p_n,q_n}} + \delta.\end{aligned}$$

(3) The embedding θ_1 satisfies $V(\theta_1) < \Delta_f(\gamma)$ for all $f \in G_1$, by Proposition 2.2.2.
(4) The tracial state σ is atomless, by Lemma 2.2.4.

Now, take sufficiently large N so that there exists an embedding ζ of $\mathcal{Z}_{r,s}$ into $\mathcal{Z}_{p_N,q_N}$ with $V(\zeta) < \Delta_g(\gamma)$ for all $g \in \theta_i[G_i]$. Let λ be the tracial state on $\mathcal{Z}_{p_N,q_N}$ corresponding to the Lebesgue measure, α be the nondecreasing surjective continuous map from $[0,1]$ to $[0,1]$ with $\alpha_*(\lambda) = \tau_N$, and Σ_α be the closed subset of $[0,1]$ such that $f \in \mathcal{Z}_{p_N,q_N}$ is in the image of α^* if and only if f is constant on Σ_α. Also, let λ_{p_N,q_N} be the tracial state on $\mathcal{Z}_{p_n,q_n}$ as in Lemma 2.2.6, and set

$$\sigma' := \zeta^*(\lambda), \quad \tau' := \theta_1^*(\sigma'), \quad \tau_n' := \theta_2^*(\sigma').$$

By Lemmas 2.2.6 and 2.2.7 and assumption (2) in the first paragraph, there exists a morphism η from $\langle \mathcal{Z}_{p_n,q_n}, \tau_n\rangle$ to $\langle \mathcal{Z}_{p_n,q_n}, \lambda_{p_N,q_N}\rangle$ with

$$\|\eta(g) - g\| < 5\gamma \quad (g \in G_2).$$

Similarly, there exists a morphism η' from $\langle \mathcal{Z}_{p_n,q_n}, \tau_n'\rangle$ to $\langle \mathcal{Z}_{p_n,q_n}, \lambda_{p_N,q_N}\rangle$ with

$$\|\eta'(g) - g\| < 5\gamma \quad (g \in G_2).$$

Also, by Lemmas 2.2.4 and 2.2.7 and assumption (3) in the first paragraph, there exists a morphism ρ from $\langle \mathcal{Z}_{r,s}, \sigma'\rangle$ to $\langle \mathcal{Z}_{r,s}, \sigma\rangle$ with

$$\|\rho(f) - f\| < 5\gamma \quad (f \in \theta_1[G_1]).$$

Finally, by Lemma 2.2.6, one can find a morphism ι from $\langle \mathcal{Z}_{p_n,q_n}, \lambda_{p_N,q_N}\rangle$ to $\langle \mathcal{Z}_{p_N,q_N}, \lambda\rangle$. Here, by Proposition 2.2.3 and assumption (2) in the first paragraph, we can modify ι and ζ by inner automorphisms so that

$$\|\alpha^* \circ \iota_{N,n}(g) - \iota \circ \eta(g)\| < 5\gamma, \quad \|\zeta \circ \theta_2(g) - \iota \circ \eta'(g)\| < 5\gamma \quad (g \in G_2).$$

Furthermore, by Proposition 2.2.1, we may assume that ζ is of the form

$$\zeta(f)(s) = \mathrm{Ad}(u(s))\Big(\mathrm{diag}\big[c(f(0)),\dots,c(f(0)),\\ f(t_1(s)),\dots,f(t_k(s)),c(f(1)),\dots,c(f(1))\big]\Big),$$

where $t_1,\dots,t_k$ is the normalized eigenvalue pattern of ζ. Since $V(\zeta) < \Delta_g(\gamma)$ for all $g \in \theta_i[G_i]$, and since

$$\begin{aligned} &\|\zeta\circ\theta_2(g) - \alpha^*\circ\iota_{N,n}(g)\| \\ &< \|\zeta\circ\theta_2(g) - \iota\circ\eta'(g)\| + \|\iota\circ\eta'(g) - \iota\circ\eta(g)\| + \|\iota\circ\eta(g) - \alpha^*\circ\iota_{N,n}(g)\| \\ &< 20\gamma, \end{aligned}$$

for all $g \in G_2$, one can easily modify the unitary u as in the last paragraph of the proof of Proposition 2.2.3 so that u is constant on Σ_α while ζ still satisfies the inequality

$$\|\zeta\circ\theta_2(g) - \alpha^*\circ\iota_{N,n}(g)\| < 100\gamma \quad (g \in G_2).$$

Then, since u is constant on Σ_α and $V(\zeta) < \Delta_f(\gamma)$ for all $f \in \theta_1[G_1]$, the inequality

$$\inf_{g\in\mathcal{Z}_{p_N,q_N}} \|\zeta\circ\theta_1(f) - \alpha^*(g)\| < \gamma$$

holds for all $f \in G_1$.

$$\begin{tikzcd}
 & \langle\mathcal{Z}_{p_n,q_n},\tau_n'\rangle \arrow[dr, "\theta_2"] \arrow[dd, "\eta'"] & & \\
\langle\mathcal{Z}_{p,q},\tau'\rangle \arrow[rr, "\theta_1"] & & \langle\mathcal{Z}_{r,s},\sigma'\rangle \arrow[dr, "\zeta"] \arrow[dd, "\rho"] & \\
 & \langle\mathcal{Z}_{p_n,q_n},\lambda_{p_N,q_N}\rangle \arrow[rr, "\iota"] & & \langle\mathcal{Z}_{p_N,q_N},\lambda\rangle \\
\langle\mathcal{Z}_{p,q},\tau\rangle \arrow[rr, "\theta_1"] & & \langle\mathcal{Z}_{r,s},\sigma\rangle & \\
 & \langle\mathcal{Z}_{p_n,q_n},\tau_n\rangle \arrow[uu, "\eta"] \arrow[ur, "\theta_2"] \arrow[rr, "\iota_{N,n}"] & & \langle\mathcal{Z}_{p_N,q_N},\tau_N\rangle \arrow[uu, "\alpha^*"']
\end{tikzcd}$$

Set $\psi := \varphi_{\zeta,\alpha^*}\varphi_{\theta_1,\rho}$. Then, for $f \in G_1$ and $g \in G_2$, we have

$$\begin{aligned} \psi\big(f,\iota_{N,n}(g)\big) &\le \varphi_{\theta_1,\rho}\big(f,\theta_1(f)\big) + \varphi_{\zeta,\alpha^*}\big(\theta_1(f),\iota_{N,n}(g)\big) \\ &= \|\theta_1(f) - \rho\circ\theta_1(f)\| + \|\zeta\circ\theta_1(f) - \alpha^*\circ\iota_{N,n}(g)\| \\ &\le \|\zeta\circ\theta_1(f) - \zeta\circ\theta_2(g)\| + 105\gamma \\ &= \varphi_{\theta_1,\theta_2}(f,g) + 105\gamma. \end{aligned}$$

Also, since $\|\theta_1(f)-\rho\circ\theta_1(f)\| < 5\gamma$ and $\inf_g \|\zeta\circ\theta_1(f)-\alpha^*(g)\| < \gamma$, one can easily see that ψ is 6γ-total on G_1. Since γ was arbitrary, we may assume $\gamma < \min\{\varepsilon/6, \delta/105\}$ so that ψ has the desired property. □

Remark 2.2.9. Here, for the reader's better understanding, we shall present a simpler version of the proof above in a certain special case. Let φ be a strict approximate $\mathscr{K}_{\mathcal{Z}}$-isomorphism from an object $\langle \mathcal{Z}_{p,q}, \tau\rangle$ of $\mathscr{K}_{\mathcal{Z}}$ to $\langle \mathcal{Z}_{p_n,q_n}, \tau_n\rangle$ and F be a finite subset of $\mathcal{Z}_{p,q}$. Then there exist finite subsets $G_1 \subseteq \mathcal{Z}_{p,q}$ and $G_2 \subseteq \mathcal{Z}_{p_n,q_n}$, a joint $\mathscr{K}$-embedding (θ_1, θ_2) of $\langle \mathcal{Z}_{p,q}, \tau\rangle$ and $\langle \mathcal{Z}_{p_n,q_n}, \tau_n\rangle$ into some $\langle \mathcal{Z}_{r,s}, \sigma\rangle$ and $\delta > 0$ such that

$$\varphi \geq (\varphi_{\theta_1,\theta_2}|_{G_1\times G_2})|^{\mathcal{Z}_{p,q}\times\mathcal{Z}_{p_n,q_n}} + \delta.$$

Without loss of generality, we may assume that G_1 includes F.

Now, assume that there happens to be a trace-preserving $*$-homomorphism ζ' from $\langle \mathcal{Z}_{r,s}, \sigma\rangle$ to $\langle \mathcal{Z}_{p_N,q_N}, \tau_N\rangle$ for sufficiently large N. Since $V(\iota_{N,m}) \to 0$ as $N \to \infty$ for each m, we may assume that both $V(\iota_{N,n})$ and $V(\zeta' \circ \theta_2)$ are smaller than $\delta/5$, whence there is a unitary u in $\mathcal{Z}_{p_N,q_N}$ with

$$\left\|\left(\mathrm{Ad}(u)\circ\zeta'\circ\theta_2\right)(g) - \iota_{N,n}(g)\right\| < \delta$$

for all $g \in G_2$. Now, set $\psi := \varphi_{\zeta'\circ\theta_1}$. Then we have

$$\begin{aligned}
\psi\big(f, \iota_{N,n}(g)\big) &= \|\zeta'\circ\theta_1(f) - \iota_{N,n}(g)\| \\
&\leq \|\zeta'\circ\theta_1(f) - \zeta'\circ\theta_2(g)\| + \|\zeta'\circ\theta_2(g) - \iota_{N,n}(g)\| \\
&< \varphi_{\theta_1,\theta_2}(f,g) + \delta,
\end{aligned}$$

so $\psi \leq \varphi|^{\mathcal{Z}_{p,q}\times\mathcal{Z}_{p_N,q_N}}$. Clearly, ψ is ε-total for any $\varepsilon > 0$, since

$$\inf_{g\in\mathcal{Z}_{p_N,q_N}} \psi(f,g) = 0$$

for all f. This was what we would like to show, in view of Theorem 1.3.5.

In general, $\langle \mathcal{Z}_{r,s}, \sigma\rangle$ is not necessarily embeddable into some $\langle \mathcal{Z}_{p_N,q_N}, \tau_N\rangle$, however. This is why we need to approximate the measures τ_N and σ by λ and σ' in the original proof above, which causes all the other additional steps.

In the sequel, we fix an inductive system $\{\iota_{n,m} \colon \mathcal{Z}_m \to \mathcal{Z}_n\}$ of prime dimension drop algebras and write its limit by $\mathcal{Z}_0$. Note that every $*$-homomorphism between prime dimension drop algebras is automatically unital and injective, and $\mathcal{Z}_0$ admits a tracial state. We also let $t_1^{m,n} \leq \cdots \leq t_{k(m,n)}^{m,n}$ be the normalized eigenvalue pattern of $\iota_{n,m}$.

Lemma 2.2.10. *The following two conditions are equivalent.*
(1) *The limit $\mathcal{Z}_0$ is simple.*
(2) *For any $\varepsilon > 0$, any $y \in [0,1]$ and any $m \in \mathbb{N}$, there exists $n > m$ such that if $x \in [0,1]$ satisfies $t_i^{m,n}(x) = y$ for some i, then the Hausdorff distance between $\{t_1^{m,n}(x), \ldots, t_{k(m,n)}^{m,n}(x)\}$ and $[0,1]$ is less than ε.*

Proof. (1) $\Rightarrow$ (2). Suppose that (2) does not hold. Then there exist $\varepsilon > 0$, $y_0 \in [0,1]$ and $m_0 \in \mathbb{N}$ such that for any $n > m_0$ there is $x_n \in [0,1]$ with $t_i^{m_0,n}(x_n) = y_0$ for some i and

$$d\big(\{t_1^{m_0,n}(x_n), \ldots, t_{k(m_0,n)}^{m_0,n}(x_n)\}, [0,1]\big) \geq \varepsilon.$$

Take $N \in \mathbb{N}$ so that $1/N < \varepsilon/2$. For each $n > m_0$ there is $a(n) \in \{0, \ldots, N\}$ with

$$U_{1/N}(a(n)/N) \cap \{t_1^{m_0,n}(x_n), \ldots, t_{k(m_0,n)}^{m_0,n}(x_n)\} = \varnothing,$$

where $U_\delta(z)$ denotes the open ball of center z and radius δ. Passing to a subsystem if necessary, we may assume that $a(n)$ is constant, say a. Put $U := U_{1/N}(a/N)$.

For each $m, n \in \mathbb{N}$ with $m_0 \leq m \leq n$, set

$$\begin{aligned}
C_{m_0,n} &:= \{x \in [0,1] \mid t_i^{m_0,n}(x) \notin U \text{ for any } i\},\\
C_{m,n} &:= \{t_i^{m,n}(x) \mid x \in C_{m_0,n}\},\\
C_m &:= \bigcap_{n \geq m} C_{m,n}.
\end{aligned}$$

Note that $C_{m_0,n}$ is nonempty, since x_n is in $C_{m_0,n}$. Also, if y is in $C_{m,n+1}$, then there exists x in $C_{m_0,n+1}$ with $t_i^{m,n+1}(x) = y$ for some i. Now, since $\iota_{n+1,m} = \iota_{n+1,n} \circ \iota_{n,m}$, there are some j, j' with $t_i^{m,n+1}(x) = t_j^{m,n}\big(t_{j'}^{n,n+1}(x)\big)$. On the other hand, $t_l^{m_0,n}\big(t_{j'}^{n,n+1}(x)\big)$ is not in U for any l, because x is in $C_{m_0,n+1}$. Therefore, $t_{j'}^{n,n+1}(x)$ is in $C_{m_0,n}$, whence $y = t_j^{m,n}\big(t_{j'}^{n,n+1}(x)\big)$ is in $C_{m,n}$. Consequently, $\{C_{m,n}\}_n$ is a decreasing sequence of nonempty closed subsets, so C_m is a nonempty closed subset of $[0,1]$.

We shall show

$$C_m = \bigcup_{i=1}^{k(m,n)} t_i^{m,n}[C_n]. \qquad (*)$$

Clearly, the right-hand side is included in the left-hand side. To see the opposite inclusion, let y be in C_m. Then, for each $l \geq n$, there is z_l in $C_{n,l}$ with $t_i^{m,n}(z_l) = y$ for some i. By the pigeonhole principle, there is i_0 with $t_{i_0}^{m,n}(z_l) = y$ for infinitely many l. Let z be a limit point of such z_l's. Then clearly z is in C_n and $t_{i_0}^{m,n}(z) = y$.

For each $m \geq m_0$, set

$$\mathcal{I}_m := \{f \in \mathcal{Z}_{p_m,q_m} \mid f|_{C_m} \equiv 0\} \subsetneq \mathcal{Z}_{p_m,q_m}.$$

Then, by $(*)$, we have $\iota_{m+1,m}[\mathcal{Z}_{p_m,q_m}] \cap \mathcal{I}_{m+1} = \iota_{m+1,m}[\mathcal{I}_m]$, so the sequence $\{\mathcal{I}_m\}$ defines a closed ideal $\mathcal{I}$ of $\mathcal{Z}_0$. Since $\mathcal{I}_{m_0}$ includes $\{f \mid \operatorname{supp} f \subseteq U\}$, the ideal $\mathcal{I}$ is nontrivial, so $\mathcal{Z}_0$ is not simple.

(2) $\Rightarrow$ (1). Let $\mathcal{I}$ be a proper ideal of $\mathcal{Z}_0$, and set

$$\begin{aligned} \mathcal{I}_m &:= \mathcal{I} \cap \mathcal{Z}_{p_m,q_m}, \\ C_m &:= \{x \mid f(x) = 0 \text{ for all } f \in \mathcal{I}_m\} \end{aligned}$$

It suffices to show that C_m coincides with $[0,1]$. For this, let y be in C_m. By assumption, for any $\varepsilon > 0$ there is $n_0 > m$ such that if $t_i^{m,n_0}(x) = y$, then

$$d\big(\{t_1^{m,n_0}(x), \ldots, t_{k(m,n_0)}^{m,n_0}(x)\}, [0,1]\big) < \varepsilon.$$

However, since $C_m = \bigcup_{i=1}^{k(m,n_0)} t_i^{m,n_0}[C_{n_0}]$ by construction, we can find $x \in C_{n_0}$ with $t_i^{m,n_0}(x) = y$ for some i, and

$$\{t_1^{m,n_0}(x), \ldots, t_{k(m,n_0)}^{m,n_0}(x)\} \subseteq C_m.$$

Consequently, it follows that the Hausdorff distance between C_m and $[0,1]$ is less than arbitrary ε, so $C_m = [0,1]$. □

For $y \in [0,1]$ and $\varepsilon > 0$, we set

$$\begin{aligned} a_{m,n}(y,\varepsilon) &:= \max\{i \mid \max t_i^{m,n} \leq y + \varepsilon\}, \\ b_{m,n}(y,\varepsilon) &:= \max\{i \mid \min t_i^{m,n} < y - \varepsilon\} \\ c_{m,n}(y,\varepsilon) &:= \max\{b_{m,n}(y,\varepsilon) - a_{m,n}(y,\varepsilon), 0\} \\ &= \#\{i \mid \min t_i^{m,n} < y - \varepsilon \;\&\; \max t_i^{m,n} > y + \varepsilon\}. \end{aligned}$$

Lemma 2.2.11. *The following are equivalent.*

(1) *The limit $\mathcal{Z}_0$ is monotracial.*

(2) *For any y, any ε and any m,*

$$\lim_{n\to\infty} \frac{c_{m,n}(y,\varepsilon)}{k(m,n)} = 0.$$

Proof. (1) $\Rightarrow$ (2). Suppose (2) does not hold. Then, passing to a subsystem if necessary, we may assume that there exist $y \in [0,1]$, $\varepsilon > 0$ and $\delta > 0$ with $c_{m,n}(y,\varepsilon)/k(m,n) \geq \delta$ for all $n > m$. Let $x^1_{m,n}, x^2_{m,n} \in [0,1]$ be such that

$$t^{m,n}_{a_{m,n}(y,\varepsilon)+1}(x^1_{m,n}) > y + \varepsilon, \quad t^{m,n}_{b_{m,n}(y,\varepsilon)+1}(x^2_{m,n}) < y - \varepsilon,$$

and τ_1, τ_2 be limit points of the tracial states $\iota^*_{n,m}(\delta_{x^1_{m,n}}), \iota^*_{n,m}(\delta_{x^2_{m,n}})$ respectively. We note that these are restrictions of some tracial states on $\mathcal{Z}_0$. Now, if $f \in C[0,1]$ is taken so that

$$f|_{[0,y-\varepsilon]} \equiv 0, \quad f|_{[y+\varepsilon,1]} \equiv 1, \quad 0 \leq f \leq 1,$$

then

$$\begin{aligned} \tau_1(f) &\geq \overline{\lim}\left[1 - \frac{a_{m,n}(y,\varepsilon)}{k(m,n)}\right], \\ \tau_2(f) &\leq \underline{\lim}\left[1 - \frac{b_{m,n}(y,\varepsilon)}{k(m,n)}\right], \end{aligned}$$

whence

$$\tau_1(f) - \tau_2(f) \geq \overline{\lim}\, \frac{c_{m,n}(y,\varepsilon)}{k_{m,n}(y,\varepsilon)} \geq \delta.$$

Consequently, $\mathcal{Z}_0$ is multitracial.

(2) $\Rightarrow$ (1). Suppose (2) holds. We shall first show that, given $m \in \mathbb{N}$, $\delta > 0$ and $\varepsilon > 0$, one can find $n > m$ with

$$\frac{\#\{i \mid \operatorname{diam} \operatorname{Im} t^{m,n}_i > \delta\}}{k(m,n)} < \varepsilon.$$

Indeed, take $N \in \mathbb{N}$ with $1/N < \delta/3$, and let $n(j)$ be sufficiently large so that

$$\frac{c_{m,n(j)}(j/N, 1/N)}{k(m,n(j))} < \frac{\varepsilon}{N} \quad (j = 1, \ldots, N-1).$$

Set $n := \max_j n(j)$. If $\operatorname{diam} \operatorname{Im} t^{m,n}_i > \delta$, then

$$\min t^{m,n}_i < \frac{j-1}{N}, \quad \max t^{m,n}_i > \frac{j+1}{N}$$

for some j, so the desired inequality follows.

We shall next show that, for $f \in C[0,1]$, $m \in \mathbb{N}$ and $\varepsilon > 0$, there exists $n > m$ with

$$\sup_{x,x' \in [0,1]} \left| \left[\iota^*_{n,m}(\delta_x) - \iota^*_{n,m}(\delta_{x'})\right](f) \right| \leq \varepsilon.$$

For this, we may assume $\|f\| \le 1$. Take $\delta > 0$ so that $|y - y'| < \delta$ implies $\|f(y) - f(y')\| \le \varepsilon/3$, and put $J := \{i \mid \operatorname{diam} \operatorname{Im} t_i^{m,n} > \delta\}$. By what we proved in the preceding paragraph, there exists $n > m$ with $\# J/k(m,n) < \varepsilon/3$. Then, for $x, x' \in [0,1]$, we have

$$\begin{aligned}
&\left|\left[\iota_{n,m}^*(\delta_x) - \iota_{n,m}^*(\delta_{x'})\right](f)\right| \\
&\le \frac{1}{k(m,n)}\Big|\sum_i f\big(t_i^{m,n}(x)\big) - f\big(t_i^{m,n}(x')\big)\Big| \\
&\le \frac{1}{k(m,n)}\Big(\sum_{i\in J} + \sum_{i\notin J}\Big)\big|f\big(t_i^{m,n}(x)\big) - f\big(t_i^{m,n}(x)\big)\big| \\
&\le \varepsilon,
\end{aligned}$$

as desired.

Finally, we shall show that $\mathcal{Z}_0$ is monotracial. Let τ, τ' be tracial states on $\mathcal{Z}_0$. Fix an element f in the center of $\mathcal{Z}_{p_m,q_m}$, which is canonically identified with an element of $C[0,1]$, and take sufficiently large $m > n$ so that

$$\sup_{x,x'\in[0,1]}\left|\left[\iota_{n,m}^*(\delta_x) - \iota_{n,m}^*(\delta_{x'})\right](f)\right| \le \frac{\varepsilon}{3}.$$

Since the convex combinations of the Dirac measures are weakly* dense in the set of probability measures, we can find $x_1, \ldots, x_l, x_1', \ldots, x_l'$ in $[0,1]$ with

$$\begin{aligned}
\Big|\Big(\tau|_{\mathcal{Z}_{p_n,q_n}} - \frac{1}{l}\sum_j \delta_{x_j}\Big)(f \circ t_i^{m,n})\Big| &< \frac{\varepsilon}{3}, \\
\Big|\Big(\tau'|_{\mathcal{Z}_{p_n,q_n}} - \frac{1}{l}\sum_j \delta_{x_j'}\Big)(f \circ t_i^{m,n})\Big| &< \frac{\varepsilon}{3}
\end{aligned}$$

for all i. Consequently,

$$\begin{aligned}
&|\tau(f) - \tau'(f)| \\
&\le \frac{1}{k(m,n)}\Big|\sum_i \tau|_{\mathcal{Z}_{p_n,q_n}}(f \circ t_i^{m,n}) - \tau'|_{\mathcal{Z}_{p_n,q_n}}(f \circ t_i^{m,n})\Big| \\
&\le \frac{2}{3}\varepsilon + \frac{1}{k(m,n)\cdot l}\Big|\sum_{i,j} \delta_{x_j}(f \circ t_i^{m,n}) - \delta_{x_j'}(f \circ t_i^{m,n})\Big| \\
&= \frac{2}{3}\varepsilon + \frac{1}{l}\Big|\sum_j \left[\iota_{n,m}^*(\delta_{x_j}) - \iota_{n,m}^*(\delta_{x_j'})\right](f)\Big| \\
&\le \varepsilon.
\end{aligned}$$

Since ε was arbitrary, $\tau(f) = \tau'(f)$, and so $\tau = \tau'$. $\square$

Proposition 2.2.12. *The inductive limit C*-algebra $\mathcal{Z}_0$ is simple and monotracial if and only if* $\lim_n V(\iota_{n,m}) = 0$ *for each m.*

Proof. It is clear from Lemmas 2.2.10 and 2.2.11 that if $\lim_n V(\iota_{n,m}) = 0$ for all m, then $\mathcal{Z}_0$ is simple and monotracial. For the opposite implication, first note that if $\mathcal{Z}_0$ is simple, then

$$\lim_n \operatorname{diam} \operatorname{Im} t_1^{m,n} = 0$$

for each m. Indeed, for any $\varepsilon > 0$, there exists sufficiently large n such that if $t_i^{m,n}(x) = \varepsilon$ for some i and $x \in [0, 1]$, then

$$d\big(\{t_1^{m,n}(x), \dots, t_{k(m,n)}^{m,n}(x)\}, [0, 1]\big) < \varepsilon,$$

by Lemma 2.2.10. This implies that $\varepsilon \notin \operatorname{Im} t_1^{m,n}$, and since $0 \in \operatorname{Im} t_1^{m,n}$, it follows that $\mathrm{t}_1^{m,n} \subseteq [0, \varepsilon)$.

Next, for each $m, n \in \mathbb{N}$, let $\Delta_{m,n}$ be a map from $(0, 1]$ to $(0, 1]$ such that $|x - x'| \leq \Delta_{m,n}(\varepsilon)$ implies $|t_i^{m,n}(x) - t_i^{m,n}(x')| \leq \varepsilon$ for all i. Passing to a subsystem if necessary, we may assume that $\operatorname{Im} t_1^{n,n+1}$ is included in $[0, \Delta_{n-1,n}(\varepsilon/2)]$. For a fixed $m \in \mathbb{N}$, set

$$F_n := \operatorname{Im} t_1^{m,m+1} \circ \cdots \circ t_1^{n,n+1}$$

and take $y_0 \in \bigcap_n F_n$. By Lemma 2.2.10, there is $n > m$ such that if $x \in [0, 1]$ satisfies $t_i^{m,n}(x) = y_0$ for some i, then the distance between $\{t_1^{m,n}(x), \dots, t_{k(m,n)}^{m,n}(x)\}$ and $[0, 1]$ is less than $\varepsilon/2$. On the other hand, by definition of F_n, we can find $x \in \operatorname{Im} t_1^{n,n+1} \subseteq [0, \Delta_{n,n+1}(\varepsilon/2)]$ with $t_i^{m,n}(x) = y_0$ for some i. Consequently, it follows that for any $y \in [0, 1]$ there exists i with $\operatorname{Im} t_i^{m,n} \circ t_1^{n,n+1} \subseteq [y - \varepsilon, y + \varepsilon]$.

Now, let $f\colon [0, 1] \to [0, 1]$ be a continuous map such that the image of f includes $[y - \varepsilon, y + \varepsilon]$ for some $y \in [0, 1]$. We shall show the existence of $\delta > 0$ such that if a continuous map $g\colon [0, 1] \to [0, 1]$ satisfies $\operatorname{Im} f \circ g \supseteq [y - \varepsilon, y + \varepsilon]$, then $\operatorname{diam} \operatorname{Im} g \geq \delta$. Indeed, let $(a_n)_n$ and $(b_n)_n$ be enumerations of the boundaries of $f^{-1}(y - \varepsilon)$ and $f^{-1}(y + \varepsilon)$ respectively. If the image of $f \circ g$ includes $[y - \varepsilon, y + \varepsilon]$, then the image of g must contain $a_{n'}$ and $b_{n''}$ such that there is no a_n or b_n between $a_{n'}$ and $b_{n''}$. However, there can be only finitely many such pairs (n', n''), because otherwise f cannot be uniformly continuous. Thus,

$$\delta := \min\big\{|a_{n'} - b_{n''}| \;\big|\; \nexists n,\ a_{n'} \lesseqgtr a_n \lesseqgtr b_{n''} \text{ or } a_{n'} \lesseqgtr b_n \lesseqgtr b_{n''}\big\}$$

has the desired property.

Finally, suppose that there is $m \in \mathbb{N}$ with $\lim_n V(\iota_{n,m}) > 0$. Without loss of generality, we may assume $m = 1$. Also, by passing to a subsystem if necessary, we may assume that there is $y \in [0,1]$ and $\varepsilon > 0$ with the following property: For any n, there exists i such that the image of $t_i^{1,n}$ includes $[y-\varepsilon, y+\varepsilon]$. By what we proved in the second paragraph, it is easy to find $n_0, i_1, i_2 \in \mathbb{N}$ with

$$\operatorname{Im} t_{i_1}^{1,2} \circ t_{i_2}^{2,n_0} \circ t_1^{n_0,n_0+1} \subseteq [y-\varepsilon/2, y+\varepsilon/2].$$

We may assume $n_0 = 3$. Set

$$F := \left\{ t_h^{1,2} \circ t_i^{2,3} \circ t_j^{3,4} \;\middle|\; \operatorname{Im} t_h^{1,2} \circ t_i^{2,3} \circ t_j^{3,4} \supseteq [y-\varepsilon, y+\varepsilon] \right\}$$

and take $\delta > 0$ so that if f is in F and if the image of $f \circ g$ includes $[y-\varepsilon, y+\varepsilon]$, then $\operatorname{diam} \operatorname{Im} g \geq \delta$. Since $\mathcal{Z}_0$ is monotracial, we may assume

$$\frac{\#\left\{ t_k^{4,5} \;\middle|\; \operatorname{diam} \operatorname{Im} t_k^{4,5} \geq \delta \right\}}{k(4,5)} < \frac{1}{\#F},$$

by Lemma 2.2.11. Then

$$\begin{aligned}
&\#\left\{ t_h^{1,2} \circ t_i^{2,3} \circ t_j^{3,4} \circ t_k^{4,5} \;\middle|\; \operatorname{Im} t_h^{1,2} \circ t_i^{2,3} \circ t_j^{3,4} \circ t_k^{4,5} \supseteq [y-\varepsilon, y+\varepsilon] \right\} \\
&\leq \#F \times \#\{t_k^{4,5} \mid \operatorname{diam} t_k^{4,5}\} \leq k(4,5) \\
&\leq \#\left\{ t_h^{1,2} \circ t_i^{2,3} \circ t_j^{3,4} \circ t_k^{4,5} \;\middle|\; \operatorname{Im} t_h^{1,2} \circ t_i^{2,3} \circ t_j^{3,4} \circ t_k^{4,5} \subseteq [y-\varepsilon/2, y+\varepsilon/2] \right\}.
\end{aligned}$$

However, this implies that there is no i with $\operatorname{Im} t_i^{1,5} \supseteq [y-\varepsilon, y+\varepsilon]$, which is a contradiction. □

Combining Propositions 2.2.8 and 2.2.12, we obtain the following result.

Theorem 2.2.13. *For an inductive system $\{\iota_{n,m} \colon \mathcal{Z}_m \to \mathcal{Z}_n\}$ of prime dimension drop algebras, the following are all equivalent.*

(1) *The inductive limit of $\{\iota_{n,m} \colon \mathcal{Z}_m \to \mathcal{Z}_n\}$ is isomorphic to $\mathcal{Z}$.*
(2) *The equality $\lim_n V(\iota_{n,m}) = 0$ holds for all m.*
(3) *The inductive limit of $\{\iota_{n,m} \colon \mathcal{Z}_m \to \mathcal{Z}_n\}$ is simple and monotracial.*

It was shown by Jiang and Su that every unital $*$-endomorphism of $\mathcal{Z}$ is approximately inner. We shall conclude this section by partially recovering this result.

Proposition 2.2.14. *Every $\mathcal{K}_{\mathcal{Z}}$-admissible endomorphism of $\langle \mathcal{Z}, \mathrm{tr} \rangle$ is approximately inner.*

Proof. Let

$$\langle \mathcal{Z}_{p_1,q_1}, \tau_1\rangle \xrightarrow{\iota_{2,1}} \langle \mathcal{Z}_{p_2,q_2}, \tau_2\rangle \xrightarrow{\iota_{3,2}} \langle \mathcal{Z}_{p_3,q_3}, \tau_3\rangle \xrightarrow{\iota_{4,3}} \cdots$$

be a regular sequence with the following property, the existence of which follows from Theorem 2.2.13:

(1) $p_n q_n$ divides $p_{n+1}q_{n+1}$ and τ_n is atomless for all n.

(2) For any natural number a, there exists sufficiently large n such that a divides $p_n q_n$.

We shall first show that if ρ is a $\mathcal{K}_{\mathcal{Z}}$-admissible endomorphism of $\langle \mathcal{Z}, \mathrm{tr}\rangle$, then for any finite subset $F \subseteq \mathcal{Z}_{p_n,q_n}$ and any $\varepsilon > 0$, there exists a morphism ι from $\langle \mathcal{Z}_{p_n,q_n}, \tau_n\rangle$ to $\langle \mathcal{Z}_{p_N,q_N}, \tau_N\rangle$ with

$$\|\rho(f) - \iota(f)\| < \varepsilon \quad (f \in F).$$

Take sufficiently large m and so that for any $f \in F$, there exists $f' \in \mathcal{Z}_{p_m,q_m}$ with $\|\rho(f) - f'\| < \varepsilon/4$. We shall fix such f' for each $f \in F$ and set $F' := \{f' \mid f \in F\}$. Put

$$\psi := (\varphi_\rho|_{F\times F'})|^{\mathcal{Z}_{p_n,q_n}\times\mathcal{Z}_{p_m,q_m}} + \frac{\varepsilon}{4}$$

and note that this is a strict approximate $\mathcal{K}_{\mathcal{Z}}$-isomorphism, as ρ is $\mathcal{K}$-admissible. Since ψ is strict, there exists a joint $\mathcal{K}$-embedding (θ_1, θ_2) of $\langle \mathcal{Z}_{p_n,q_n}, \tau_n\rangle$ and $\langle \mathcal{Z}_{p_m,q_m}, \tau_m\rangle$ into some object $\langle \mathcal{Z}_{r,s}, \sigma\rangle$ with $\varphi_{\theta_1,\theta_2} \le \psi$, whence

$$\|\theta_1(f) - \theta_2(f')\| \le \frac{\varepsilon}{2}.$$

Now by Proposition 2.2.2, one can embed $\mathcal{Z}_{r,s}$ into $\mathcal{Z}_{p_{m'},q_{m'}}$ for some $m' > m$. By assumption (2), we may assume that rs divides $p_{m'}q_{m'}$, so the remainder indices vanish. Consequently, since $\tau_{m'}$ is atomless by assumption (1), one can easily find a morphism η from $\langle \mathcal{Z}_{r,s}, \sigma\rangle$ to $\langle \mathcal{Z}_{p_{m'},q_{m'}}, \tau_{m'}\rangle$. Since $V(\iota_{N,m''}) \to 0$ as $N \to \infty$ by Theorem 2.2.13, one can find $N > m''$ and a unitary u in $\mathcal{Z}_{p_N,q_N}$ with

$$\left\|\left(\mathrm{Ad}(u)\circ\iota_{N,m'}\circ\zeta\circ\theta_2\right)(f') - \iota_{N,m}(f')\right\| < \frac{\varepsilon}{4}$$

for all $f' \in F'$, by Proposition 2.2.3. We set

$$\iota := \mathrm{Ad}(u)\circ\iota_{N,m'}\circ\zeta\circ\theta_1.$$

Then, for $f \in F$, we have

$$\begin{aligned}
\|\rho(f) - \iota(f)\| \le\ & \|\rho(f) - f'\| \\
& + \left\|\iota_{N,m}(f') - \left(\mathrm{Ad}(u)\circ\iota_{N,m'}\circ\zeta\circ\theta_2\right)(f')\right\| \\
& + \left\|\left(\mathrm{Ad}(u)\circ\iota_{N,m'}\circ\zeta\circ\theta_2\right)(f') - \left(\mathrm{Ad}(u)\circ\iota_{N,m'}\circ\zeta\circ\theta_1\right)(f)\right\| \\
& < \varepsilon,
\end{aligned}$$

as desired.

Now, since $V(\iota_{M,N}) \to 0$ as $M \to \infty$, there exists sufficiently large M and a unitary v in $\mathcal{Z}_{p_M,q_M}$ with

$$\left\|\left(\mathrm{Ad}(v) \circ \iota\right)(f) - f\right\| < \varepsilon$$

for all $f \in F$, by Proposition 2.2.3 This implies

$$\|\rho(f) - \mathrm{Ad}(v^*)(f)\| < 2\varepsilon,$$

so ρ is approximately inner, which completes the proof. □

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