博士論文（要約）

論文題目 Sharp interface limit for the stochastic Allen-Cahn equation
(確率アレン・カーン方程式に対する鋭敏な界面極限)

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Sharp Interface Limit for the Stochastic Allen-Cahn Equation

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Abstract

This paper is an abstract of Ph.D. thesis "Sharp Interface Limit for the Stochastic Allen-Cahn Equation". In this thesis, we treat our recent results about sharp interface limit for the stochastic Allen-Cahn equations in several settings. Especially, we focus on the generation and motion of interface. Finally, we show the simulation concerned with these models.

1 Introduction

The Allen-Cahn equation is a reaction-diffusion equation with a bistable reaction term f. This equation describes physical phenomena such as dynamical phase transition, and it has the form:

$$\begin{cases} \dot{u}^\varepsilon(t,x) &= \Delta u^\varepsilon(t,x) + \dfrac{1}{\varepsilon} f(u^\varepsilon(t,x)), \quad t>0,\ x\in D, \\ u^\varepsilon(0,x) &= u_0^\varepsilon(x), \quad x \in D, \end{cases} \tag{1.1}$$

where $D \subset \mathbb{R}^d$ is a domain, $\varepsilon > 0$, $\dot{u} = \frac{\partial u}{\partial t}$ and $\Delta u = \sum_{k=1}^d \frac{\partial^2 u}{\partial x_k^2}$. We assume that the reaction term f has ± 1 as stable points and satisfies $\int_{-1}^1 f(u)du = 0$. The typical example of the reaction term is $f(u) = u - u^3$. Then, it is expected that the solution u^ε tends to ± 1 as $\varepsilon \to 0$ in a very short time, and an interface appears to separate two different phases ± 1.

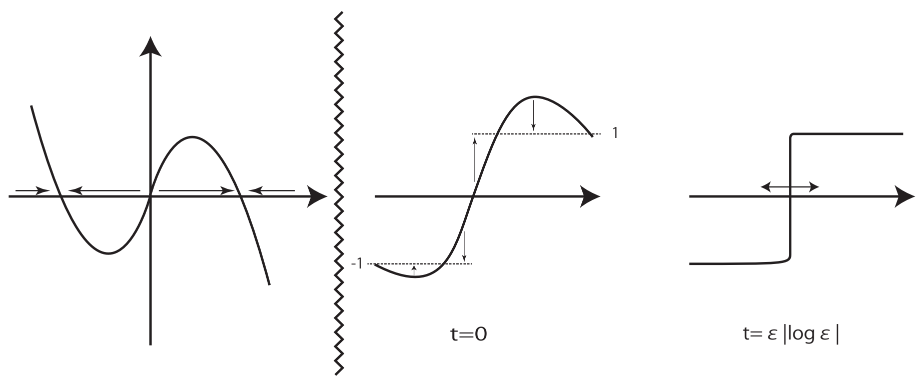


Figure 1: Reaction term and phase separation

The width of the interface is of order $O(\varepsilon^{\frac{1}{2}})$, thus the interface becomes sharp as $\varepsilon \to 0$, and we call this limit the sharp interface limit. In this thesis, we add an external random noise to PDE (1.1), and study the sharp interface limit for the stochastic Allen-Cahn equations in several settings. The stochastic Allen-Cahn equation is described as follows;

$$\dot{u}^{\varepsilon}(t,x) = \Delta u^{\varepsilon}(t,x) + \frac{1}{\varepsilon} f(u^{\varepsilon}(t,x)) + \dot{W}^{\varepsilon}(t,x), \quad t > 0,\ x \in D,$$

where $\dot{W}^{\varepsilon}(t,x)$ is an external noise term which is depend on a time variable and a space variable. We explain the meaning of this term and the solution u^{ε} in each chapter. We can also refer Chapter 4 of Funaki [8] for the sharp interface limit of SPDEs.

1.1 Related topic and motivation

In recent studies of the deterministic case, the behaviors of the solution have been investigated. For example, Chen [3] studied the initial value problem (1.1) in one space dimension, and classified the behavior of solution into four stages: (i) Phase separation: In a very short time, the solution u^{ε} tends to ± 1. In other words, interfaces are generated in a time of order $O(\varepsilon |\log \varepsilon|)$. (ii) Generation of metastable patterns: Until the time of order $O(1)$, the solution u^{ε} enters into a neighborhood of standing waves associated with f. (iii) Super-slow motion of interfaces: An approximated ODE governs the very slow interface motion for a long time of order $O(e^{\frac{C}{\varepsilon}})$ with $C > 0$ (Carr and Pego [2] also studied this stage). (iv) Annihilation of interfaces: Under the super-slow motion, when two interfaces are close enough, the interfaces are annihilated and they restore the super-slow motion. In this thesis, we are interested in the stages of generation of interface and motion of interface.

On the other hand, Funaki [5] studied the case of stochastic Allen-Cahn equation. He consider the external noise $\dot{W}^{\varepsilon}(t,x) := \varepsilon^{\gamma} a(x) \dot{W}(t,x)$ where $a \in C_0^{\infty}(\mathbb{R})$ and $\dot{W}(t,x)$ is a space-time (Gaussian) white noise. He proved that the proper time scale for the interface motion is of order $O(\varepsilon^{-2\gamma-\frac{1}{2}})$ and the interface $\xi_t \in \mathbb{R}$ obeys the SDE;

$$d\xi_t = \alpha_1 a(\xi_t) dB_t + \alpha_2 a(\xi_t) a'(\xi_t) dt, \tag{1.2}$$

where α_1 and α_2 are the constants depending on f, and B_t is a one-dimensional Brownian motion (see Chapter 2 for the detailed conditions of α_1 and α_2). The most important point in his study is that the time scale for the motion of interface is totally different from the deterministic case, and that the dynamics of interface is also described by the SDE. These changes are from the contribution of the noise term. We proved the generation of interface in [10], in the same setting as Funaki [5]. We proved that the interfaces are formed until the time of order $O(\varepsilon |\log \varepsilon|)$. Weber [17] also investigated the annihilation of interface. In his results, he proved that the interfaces are annihilated, and restore a new phase once two interfaces reach the distance of order $O(\varepsilon^{\frac{1}{2}} |\log \varepsilon|)$.

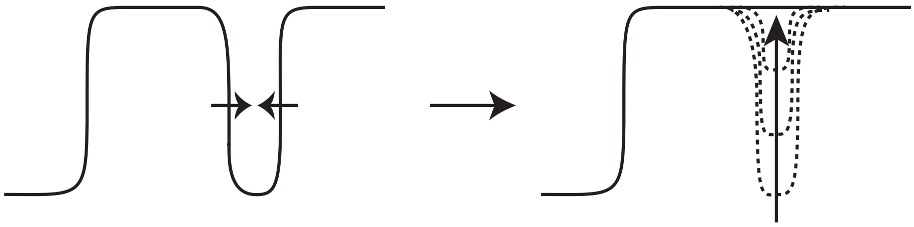


Figure 2: Annihilation of interfaces

The order of width for the annihilation is slightly wider than that of the deterministic case (see Chen [3]). He also took the time scale of order $O(\varepsilon^{-2\gamma-\frac{1}{2}})$ for the interface motion. The interfaces behave as independent Brownian motions, and they are killing each other at the limit $\varepsilon \to 0$.

The invariant measures of the stochastic Allen-Cahn equations with the boundary condition $u^\varepsilon(\pm 1) = \pm 1$ are also well-studied. For example, Weber [16] proved the concentration of the invariant measure on a minimizer of Ginzburg-Landau free energy. Allen-Cahn equation can be described as the L^2-gradient flow of this energy functional. Otto et. al. [13] also proved that the invariant measure. Roughly speaking, the invariant measure of u^ε converges weakly to the uniform distribution on the set $\{\chi_\xi | \xi \in [-1,1]\} \subset L^2[-1,1]$. Their results are deeply related to the proof in Chapter 4.

We also mention about the multi-dimensional case. In this case, the interface motion has been investigated in many articles. For example, we can refer de Mottoni and Schatzman [12]. In the multi-dimensional setting, the proper time scale is of order $O(1)$ and the interface develops by a motion by mean curvature. Alfaro et. al. [1] investigated the multi-dimensional Allen-Cahn equation with the deterministic external force term g^ε;

$$\dot{u}^\varepsilon(t,x) = \Delta u^\varepsilon(t,x) + \frac{1}{\varepsilon} f(u^\varepsilon(t,x)) + g^\varepsilon(t,x,u^\varepsilon), \quad t > 0,\ x \in D.$$

In this case, the interfaces are generated at the time of order $O(\varepsilon|\log \varepsilon|)$, and an influence of the external force remains in the mean curvature flow. Their result motivates us to consider the method of super and sub solutions for the proof of the generation in the stochastic case. We discuss about it in [11] and also in Chapter 3.

We explain the stochastic case. Here, we note that we cannot take the space-time white noise as a noise term when the dimension is larger than 1, because the solution becomes ill-posed. Reader can refer Funaki's result [4] about the regularity of parabolic SPDEs. Funaki [7] assumed that $d = 2$ and the noise $\varepsilon^{-\frac{1}{2}} \dot{w}_t^\varepsilon$ where w_t^ε is smooth in time and uniform in space almost surely. It converges to a one-dimensional Brownian motion almost surely. In this case, he proved that the proper time scale is of order $O(1)$ and the interface motion is described by the mean curvature flow with the noise;

$$V_t = \kappa + \alpha \dot{w}_t,$$

where V_t is a normal velocity of the interface, κ is a mean curvature of interface, the constant α depends on f and w_t is a one-dimensional Brownian motion. This result means that the effect of noise term remains in the mean curvature flow in $O(1)$. We note that he proved it until the convexity of the interface holds. As the special case of Funaki's result, Weber [15] proved the interface motion where the noise is defined by a mollification in a time variable of the Brownian motion $w_t^\varepsilon := (\eta^\varepsilon * w)(t)$ for $d \geq 2$. We proved the generation of interface for multi-dimensional case in [11]. We take the formal derivative of Q-Brownian motion in time, which is smooth in space, as the external noise. The interface is generated until the time of order $O(\varepsilon|\log\varepsilon|)$. We prove it in Chapter 3. We can also consider a behavior of the solution after the generation as a future work. See Funaki-Yokoyama [9] for the case that the volume of u^ε is preserved.

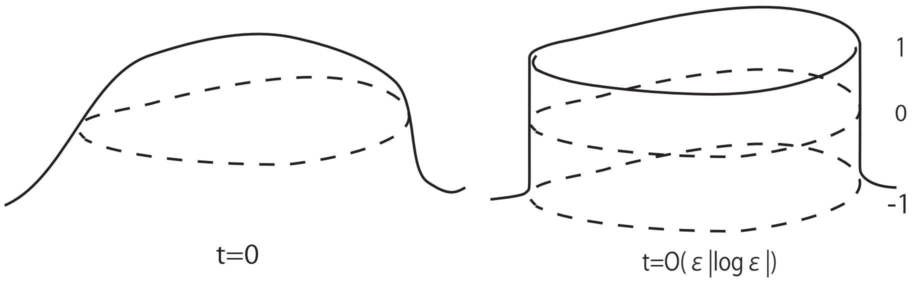


Figure 3: Generation of interface in multi dimension

Here, we summarize the contents of this thesis. This thesis consists of five chapters. Chapter 1 is an introduction. Our main results are included in Chapter 2, 3 and 4. Chapter 5 is about the simulation. We simulate the deterministic and stochastic Allen-Cahn equations in several settings in Chapter 5. We fix a filtered probability space $(\Omega, \mathcal{F}, P, \{\mathcal{F}_t\})$.

2 Generation of interface for one-dimensional case

In Chapter 2, we treat the generation of interface for the one-dimensional Allen-Cahn equation. We assume that $D = \mathbb{R}$ and $\dot{W}^\varepsilon(t,x) := \varepsilon^\gamma a(x)\dot{W}(t,x)$ where $a \in C_0^\infty(\mathbb{R})$ and $\dot{W}(t,x)$ is a space-time white noise on $\mathbb{R}$. We also assume the boundary condition $u^\varepsilon(t,\pm\infty) = \pm 1$. The initial value $u_0^\varepsilon(x)$ satisfies

$$\begin{cases} \text{(i)}\|u_0^\varepsilon\|_\infty + \|u_0^{\varepsilon\prime}\|_\infty + \|u_0^{\varepsilon\prime\prime}\|_\infty \leq C_0, \\ \text{(ii)There exists a unique } \xi_0 \in [-1,1] \text{ independent of } \varepsilon > 0 \text{ such that } u_0^\varepsilon(\xi_0) = 0, \\ \text{(iii)}|u_0^\varepsilon(x)| \geq C\varepsilon^{\frac{1}{2}} \ (|x - \xi_0| \geq C'\varepsilon^{\frac{1}{2}}), \\ \text{(iv)}|u_0^\varepsilon(x) - 1| + |u_0^{\varepsilon\prime}(x)| + |u_0^{\varepsilon\prime\prime}(x)| \leq \varepsilon^\kappa C_\mu \exp(-\frac{\sqrt{\mu}x}{2}) \ (x \geq 1), \\ \text{(v)}|u_0^\varepsilon(x) + 1| + |u_0^{\varepsilon\prime}(x)| + |u_0^{\varepsilon\prime\prime}(x)| \leq \varepsilon^\kappa C_\mu \exp(\frac{\sqrt{\mu}x}{2}) \ (x \leq -1). \end{cases}$$

See [10] for some constants C_0, C, C', κ, μ and C_μ, where $\mu := f'(0)$.

Theorem 2.1. *We set* $\bar{u}^\varepsilon(t,x) := u^\varepsilon(\varepsilon^{-2\gamma-\frac{1}{2}}t, x)$ *and* γ *is a constant such that*

there exist constants $\kappa > \kappa' > 1$ which satisfy

$$\begin{cases} (\kappa' + \frac{21}{40} + \frac{\gamma}{10}) \vee 2\kappa' < \kappa < \gamma - \frac{C_f}{\mu}, \\ 1 < \kappa' < \frac{1}{20} + \frac{\gamma}{5}. \end{cases}$$

Then there exist an a.s. positive random variable $C(\omega) \in L^\infty(\Omega)$ *and stochastic processes* ξ_t^ε *such that*

$$P(\|\bar{u}^\varepsilon(t,\cdot) - \chi_{\xi_t^\varepsilon}(\cdot)\|_{L^2(\mathbb{R})} \le \delta \textit{ for all } t \in [C(\omega)\varepsilon^{2\gamma+\frac{3}{2}}|\log\varepsilon|, T]) \to 1 \quad (\varepsilon \to 0),$$

for all $\delta > 0$ *and* $T > 0$*. Moreover, the distribution of the process* ξ_t^ε *on* $C([0,T],\mathbb{R})$ *weakly converges to that of* ξ_t *and* ξ_t *obeys the SDE (1.2).*

We prove that the interface is formed until the time of order $O(\varepsilon|\log\varepsilon|)$. We prove it by estimating the energy inequality for the solution u^ε, and the convergence of u^ε to the minimizer of Ginzburg-Landau free energy which corresponds to the Allen-Cahn equation.

3 Generation of interface for multi-dimensional case

We also show the generation of interface in the multi-dimensional setting by assuming that a noise term has a regularity in a space variable. We show it in Chapter 3 and in [11]. In this chapter, we set $D \subset \mathbb{R}^d$ as a domain, and $\dot{W}^\varepsilon(t,x) := \varepsilon^\gamma \dot{W}^Q(t,x)$ where $W^Q(t,x)$ is a Q-Brownian motion which is smooth in space. We also assume the Neumann boundary condition, and the initial value $u_0 \in C^2(\mathbb{R})$ satisfies $\|u_0\|_\infty + \|u_0'\|_\infty + \|u_0''\|_\infty \le C_0$.

Theorem 3.1. *If there exist constants* $C_1 > 0$*,* κ *and* α *satisfying* $\kappa > \alpha > \frac{1}{2}$*,* $\kappa > 1$ *and* $\frac{\alpha}{\mu} + \frac{\kappa}{p} < C_1 < \frac{1}{\mu}$*, then there exists a positive constants* $\tilde{\gamma}_d > 0$ *which depends on a dimension* d*, and, for all* $\gamma \ge \tilde{\gamma}_d$*, we have that*

(i) $\lim_{\varepsilon\to 0} P(-1 - \varepsilon^\kappa \le u^\varepsilon(x, C_1\varepsilon|\log\varepsilon|) \le 1 + \varepsilon^\kappa \textit{ for all } x \in D) = 1$

(ii) $\lim_{\varepsilon\to 0} P(u^\varepsilon(x, C_1\varepsilon|\log\varepsilon|) \ge 1 - \varepsilon^\kappa \textit{ for } x \in D \textit{ such that } u_0(x) \ge \varepsilon^{1-C_1\mu}) = 1$

(iii) $\lim_{\varepsilon\to 0} P(u^\varepsilon(x, C_1\varepsilon|\log\varepsilon|) \le -1 + \varepsilon^\kappa \textit{ for } x \in D \textit{ such that } u_0(x) \le -\varepsilon^{1-C_1\mu}) = 1$

Also in this case, the generation of interface occurs until the time of order $O(\varepsilon|\log\varepsilon|)$. We prove the comparison theorem for SPDEs by applying the maximal principle for PDEs, in the process of proof for this theorem. We also apply these methods to a one-dimensional setting where the motion of interface is studied by Funaki [6] with the special noise. And thus, we connect it to his result.

4 Reflected Brownian motion derived by a sharp interface limit

In Chapter 4, we study the case that the equation on the interval $[-1,1]$ has Dirichlet boundary conditions $u^\varepsilon(\pm 1) = \pm 1$. We take the noise $\dot{W}^\varepsilon(t,x) := \varepsilon^\gamma \dot{W}(t,x)$ where $\dot{W}(t,x)$

is a space-time white noise on $[-1,1]$. In this case, we can expect that interface can be reflected at the boundary.

Theorem 4.1. *Let a probability measure P^ε on $C([0,T],L^2[-1,1])$ be the distribution of $\bar{u}^\varepsilon(t,x) := u^\varepsilon(\varepsilon^{-2\gamma-\frac{1}{2}}t,x)$ with the initial value u_0^ε, and let P on $C([0,T],L^2[-1,1])$ be that of the Markov process $\chi_{\sqrt{2}B(\alpha_1^2 t)}$ where $B(t)$ is a reflected Brownian motion on $[-1,1]$ which start at $\xi_0 \in [-1,1]$ and $\alpha_1 := \|\nabla m\|_{L^2}^{-1}$. If $\gamma > \frac{19}{4}$, then P^ε converges to P weakly on $C([0,T],L^2[-1,1])$ as $\varepsilon \to 0$.*

As we can see, the proper time scale is of order $O(\varepsilon^{-2\gamma-\frac{1}{2}})$, and we get a reflected Brownian motion as the interface motion. However, analyzing the behavior like a reflection at the boundary is not simple because of its singularity. We regard the solution u^ε as the $L^2[-1,1]$-valued Markov process, consider the Dirichlet form which corresponds to u^ε and specify the interface motion through Mosco convergence of the Dirichlet form.

5 Simulations

In Chapter 5, we simulate the deterministic and stochastic Allen-Cahn equations, and discuss about the difference between both settings visually. For the simulation in one space dimension, we used C language, and for multi dimension, we used FreeFem++-cs.

5.1 One-dimensional settings

At first, we simulate the one-dimensional stochastic Allen-Cahn equation;

$$\dot{u}(t,x) = \Delta u(t,x) + af(u(t,x)) + b\dot{W}_t(x), \quad t > 0,\ x \in [-1,1],$$

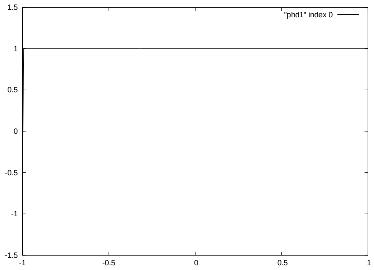
where $a > 0$, $b \in \mathbb{R}$, $f(u) = u - u^3$ and $\dot{W}_t(x)$ is a space-time white noise. We impose Dirichlet boundary conditions $u(\pm 1) = \pm 1$. The methods of this simulation is based on Saito [14]. First we take $N \in \mathbb{N}$ and set $h := \frac{1}{N+1}$ and $\tau := h^2$. We also set a lattice $Q_h := \{(x_i,t_n) | x_i = -1 + ih\ (0 \le i \le 2N+1),\ t_n = n\tau\}$, and define a discretized solution u_i^n on $(x_i,t_n) \in Q_h$. By discretizing this equation, we can describe this as following;

$$\frac{u_i^{n+1} - u_i^n}{\tau} = \frac{u_{i+1}^n - 2u_i^n + u_{i-1}^n}{h^2} + af(u_i^n) + b\frac{W_i^{n+1} - W_i^n}{\tau}.$$

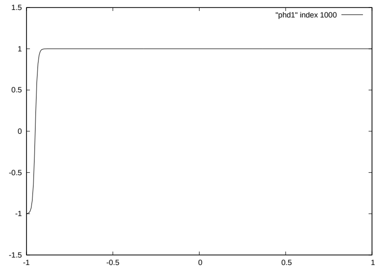
Here, the external noise term is defined by $W_i^{n+1} - W_i^n = \sqrt{\tau}G_i^n$ where $\{G_i^n\}$ are random variables which are parametrized by i and n, they are independent and obey the normal distribution $N(0,1)$. We note that $\sqrt{\tau}G_i^n$ obeys $N(0,\tau)$, and this represents the difference of Brownian motion on each time step. This term is an approximation of a space-time white noise. Finally, we calculate u_i^n inductively.

5.1.1 Reflection of interface at the boundary

Before the stochastic case, we consider the deterministic case ($b = 0$). We change the time as $\bar{u}(t) := u(ct)$. The initial value takes value -1 on $x_i = -1$ and takes 1 on $x_i \neq -1$. Now we simulate the case that $a = 10^3$, $b = 0$, $c = 10^4$ and $N = 150$.

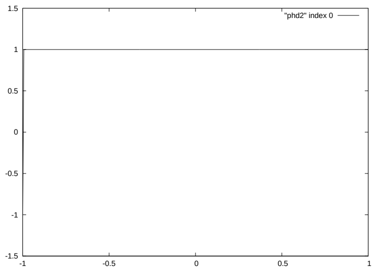


(1) $t = 0$

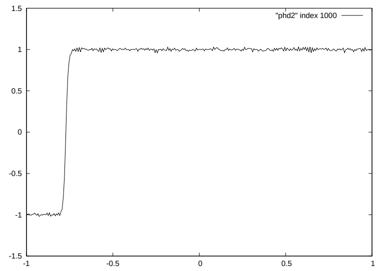


(2) $t = 1000$

We can see that the interface almost stops immediately although we take very long time scale. Actually, this moves, however, the speed of interface is extremely slow. This is the super slow motion. On the other hand, the motion of interface becomes totally different if we take $b = 2$.



(1) $t = 0$

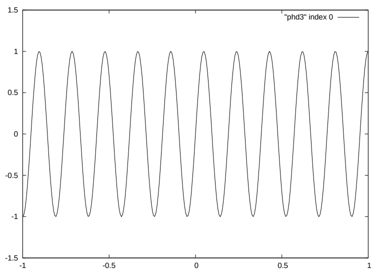


(2) $t = 1000$

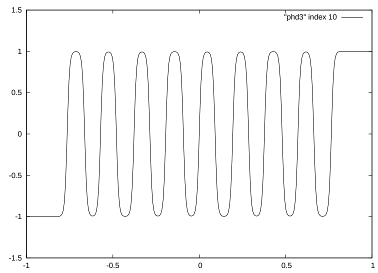
In this case, the solution becomes singular, and the interface perturbs randomly and fast. We can observe a reflected Brownian motion as an interface motion. Moreover, we can expect that we can take the value γ to be smaller than $\frac{19}{4}$ which is lower bound of γ in Theorem 4.1.

5.1.2 Annihilation of interfaces

We also consider the annihilation of interface. First, we simulate the deterministic case. We set the initial value $u_0(x) := \sin \frac{21\pi x}{2}$.

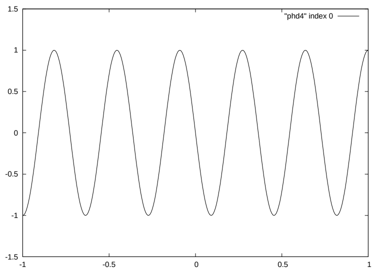


(1) $t = 0$

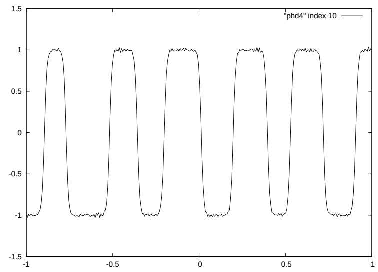


(2) $t = 10$

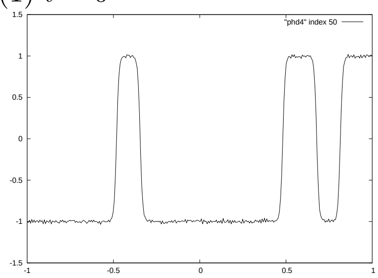
The annihilation occurs symmetrically because of the boundary conditions and the definition of the reaction term f. Next we consider the stochastic case, which is investigated by Weber [17]. We set the initial value $u_0(x) := -\sin\frac{11\pi x}{2}$. We change the initial value because the annihilation occurs too fast if we take $u_0(x) := \sin\frac{21\pi x}{2}$. However, we can observe essentially the same phenomena as the deterministic case.



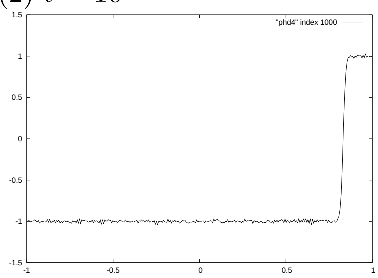
(1) $t = 0$



(2) $t = 10$



(3) $t = 50$



(4) $t = 1000$

As we can see, the interfaces move like the independent Brownian motions, and the annihilation randomly occurs. This result is involved with that of Weber [17].

5.2 Multi-dimensional settings

Now, we consider the two-dimensional Allen-Cahn equation;

$$\begin{cases} \dot{u}(t,x) = \Delta u(t,x) + af(u(t,x)) + \sqrt{a}\dot{W}_t, & t > 0,\ x \in D, \\ u(0,x) = u_0(x), & x \in D, \\ \frac{\partial u}{\partial \nu}(t,x) = 0, & t > 0,\ x \in \partial D \end{cases}$$

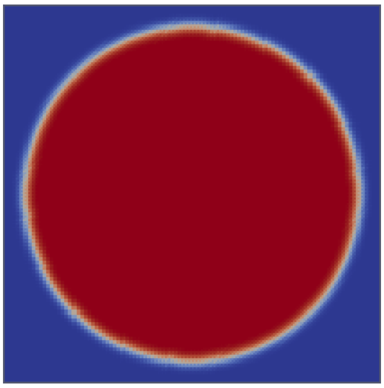
The domain $D \subset \mathbb{R}^2$ is a square $[0,1] \times [0,1]$, the reaction term is defined by $f(u) := \frac{1}{2}(u - u^3)$ and we impose Neumann boundary condition. We take a white noise only in time as the external noise, and $a := 10^{-2}$. We use the finite element method for the simulation. If we set a time change $\bar{u}(t,x) = u(a^{-1}t, x)$, then we have the weak form;

$$\langle \bar{u}(t) - \bar{u}(0), \varphi \rangle = \int_0^t \{a^{-1}\langle \nabla \bar{u}(s), \nabla \varphi \rangle + \langle f(\bar{u}(s)), \varphi \rangle\} ds + \langle W_t, \varphi \rangle$$

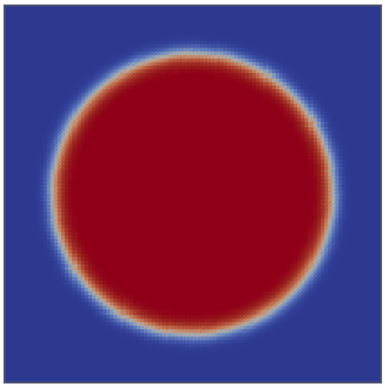
where $\langle \cdot, \cdot \rangle$ is an inner product on $L^2(D)$ and $\varphi \in C_0^\infty(D)$ is a test function. Then, we can apply the finite element method. We reset the time scale when we visualize the simulation. In this case, the interface motion appears as the motion by mean curvature.

5.2.1 Motion by mean curvature

We simulate the deterministic case $\dot{u}(t,x) = \Delta u(t,x) + af(u(t,x))$ at first. We take the initial value $u_0 := 1_A(x) - 1_{D\setminus A}(x)$ where $A := \{(x,y) \in \mathbb{R}^2 | (x-0.5)^2 + (y-0.5)^2 \leq 0.2\}$ and $a = 100$. The red part is the phase 1, and the blue part is the phase -1.

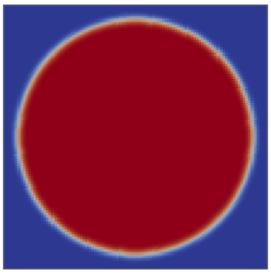


(1) $t = 0$

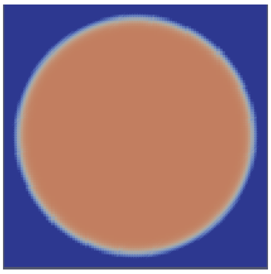


(2) $t = 300$

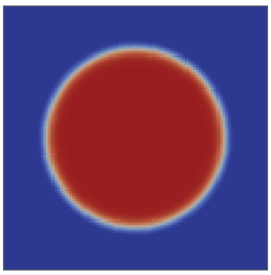
The interface shrinks because the interface motion obeys the motion by mean curvature. Next, we consider the stochastic case $\dot{u}(t,x) = \Delta u(t,x) + af(u(t,x)) + \sqrt{a}\dot{W}_t$.



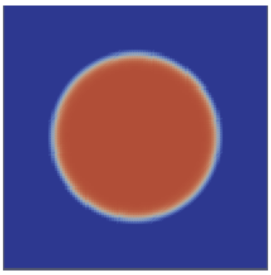
(1) $t = 0$



(2) $t = 100$



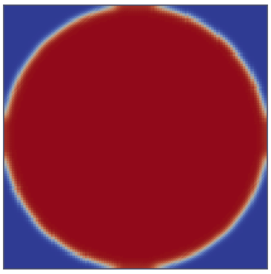
(3) $t = 200$



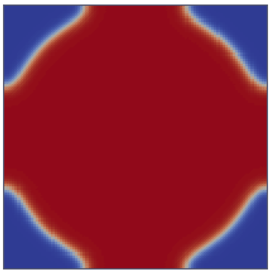
(4) $t = 300$

The interface perturbs, and the radius of the circle moves randomly. The color of each phase also perturbs uniformly because we take the noise which is uniform in a space variable. Next, we consider the case that interface touches the boundary at the initial time. In this case, we have the initial value $u_0 := 1_B(x) - 1_{D\setminus B}(x)$ where $B := \{(x,y) \in$

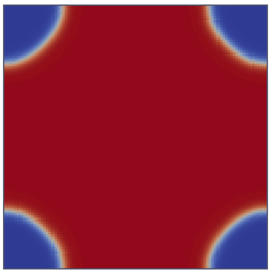
$\mathbb{R}^2 | (x - 0.5)^2 + (y - 0.5)^2 \leq 0.25\}$. The interface touches at $(0.5, 0)$, $(1, 0.5)$, $(0.5, 1)$ and $(0, 0.5)$ at the initial time. The interface motion becomes totally different from the previous case. Here, we consider the deterministic case.



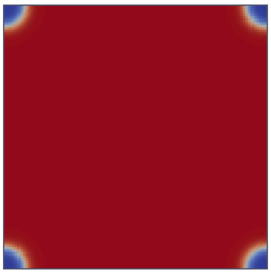
(1) $t = 0$



(2) $t = 20$

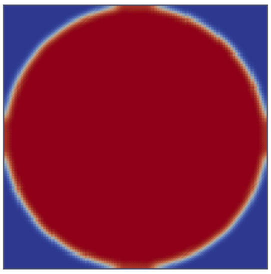


(3) $t = 100$

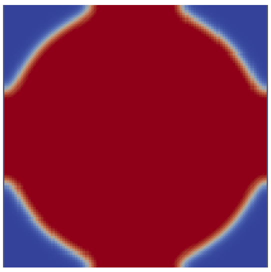


(4) $t = 300$

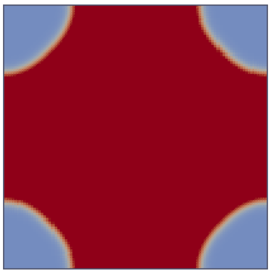
In this case, the interface sticks to the boundary, and it does not shrink. The phase which takes value 1 becomes wide. Finally, we can observe one quarter of a circle at each corner, and this shrinks to the corner slowly. We consider the stochastic case at last.



(1) $t = 0$



(2) $t = 20$



(3) $t = 100$

(4) $t = 300$

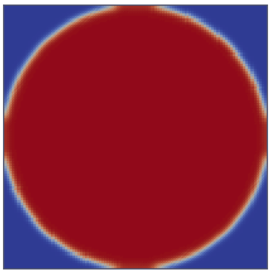
The figure of interface develops similarly as the deterministic case, however it perturbs randomly. The effect of noise remains in $O(1)$ as in the result of Funaki [7].

5.2.2 Volume preserving case

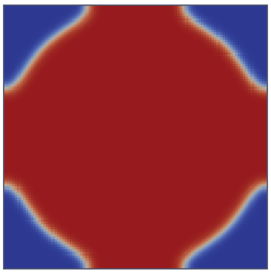
Finally, we consider the volume preserving case;

$$\begin{cases} \dot{u}(t,x) = \Delta u(t,x) + a\left\{ f(u(t,x)) - \fint_D f(u(t,x))dx \right\} + \alpha \dot{w}_t^{\varepsilon}, \quad t > 0,\ x \in D, \\ u(0,x) = u_0(x), \quad x \in D, \\ \frac{\partial u}{\partial \nu}(t,x) = 0, \quad t > 0,\ x \in \partial D \end{cases}$$

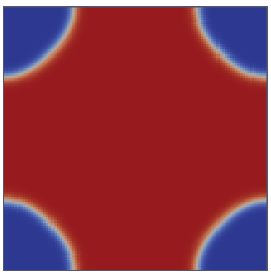
which is discussed in Funaki and Yokoyama [9]. The constant α depends on f. The term $\fint_D f(u)dx := \frac{1}{|D|}\int_D f(u)dx$ is an average of $f(u)$. In the deterministic case ($\dot{w}_t^{\varepsilon} \equiv 0$), the volume of u is preserved because of Neumann boundary condition.



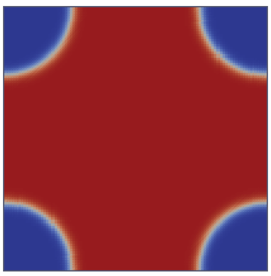
(1) $t = 0$



(2) $t = 20$

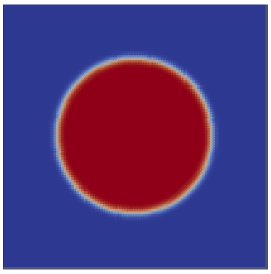


(3) $t = 100$

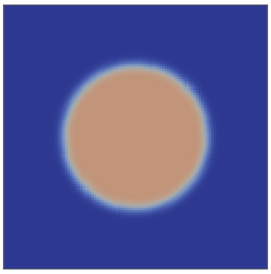


(4) $t = 300$

We take the constant $\alpha = 1$ in this section. We observe that the volume of two phases is preserved. Thus, the interface motion is similar to that of the non-preserving case, and we can see that one quarter of a circle appears at each corner, however, the interface motion stops. In the stochastic case, we can get $\int_D u(t,x)dx = w_t^\varepsilon$ from an easy calculation. The process w_t^ε, in Funaki and Yokoyama [9], is smooth and depends only on time t. This process converges to a Brownian motion, and the convergence is very slow (the speed of convergence is of order $O(\log\log|\log\varepsilon|)$). However, we take a white noise in our simulation directly. We also change the initial value $u_0 := 1_C(x) - 1_{D\setminus C}(x)$ where $C := \{(x,y) \in \mathbb{R}^2 | (x-0.5)^2 + (y-0.5)^2 \leq 0.09\}$.



(1) $t = 0$



(2) $t = 2$

(3) $t = 3$

(4) $t = 8$

In this case, the interface perturbs, however, it disappears suddenly. The non-local term $⨍_D f(u)dx := \frac{1}{|D|}\int_D f(u)dx$ perturbs because of the noise. Then, the reaction term does not become odd, and the solution becomes a traveling wave temporally. This causes the results such that the interface expands or shrinks suddenly. Also in the article [9], they mentioned that this dynamics is sensitive to the noise, and we can expect that this is the reason of this result.

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