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論文題目 Approximations of singular stochastic PDEs and their global well-posedness

（特異な確率偏微分方程式の近似とその時間大域的適切性）

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APPROXIMATIONS OF SINGULAR STOCHASTIC PDES AND THEIR GLOBAL WELL-POSEDNESS

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ABSTRACT. This paper is a summary of my Ph.D. thesis "Approximations of singular stochastic PDEs and their global well-posedness（特異な確率偏微分方程式の近似とその時間大域的適切性）".

1. INTRODUCTION

Modeling continuum phenomena in physics, biology, and economics by partial differential equations is one of the important problems in applied mathematics. In setting such dynamical systems, however, we cannot always know all parameter values precisely. In that case, it is appropriate to assume that these parameters vary in time or space depending on some unknown conditions. In addition, there are many situations where the fluctuations of microscopic parameters have essential influences in the description of the macroscopic system. In order to model such situations, it is often useful to introduce "noise" term, i.e. a random field on time and space variable. Especially, the stochastic partial differential equation (SPDE) of the form

$$\mathcal{L}u = F(u, \nabla u) + \xi,$$

where $\mathcal{L}$ is a parabolic differential operator, ξ is a random force, and F is a linear or nonlinear operator, has been actively studied from 1970s. SPDE of the above form can be seen as a generalization of stochastic differential equation (SDE) into an infinite dimensional space. A natural choice of the noise in SDE is a white noise, i.e. time derivative of a Brownian motion, so that in SPDE a natural one is a *space-time white noise*, which is a centered Gaussian random field with (formal) covariance structure

$$E[\xi(t,x)\xi(s,y)] = \delta(t-s)\delta(x-y).$$

One of the problems about SPDE is the definition of the solution space. When ξ is a space-time white noise, we can expect that the solution of SPDE behaves roughly in time and space variable. For example, it is well known that the solution u of the stochastic heat equation

$$(\partial_t - \Delta)u = \xi$$

on $(t,x) \in [0,\infty) \times \mathbb{T}^d$ belongs to the space $C([0,\infty), \mathcal{C}^{(\frac{2-d}{2})^-})$, where $\mathcal{C}^\alpha = \mathcal{B}^\alpha_{\infty,\infty}$ is the Hölder-Besov space on $\mathbb{T}^d$ and the regularity α^- means that it can be replaced by $\alpha - \kappa$ for every $\kappa > 0$. Thus when $d \geq 2$, u is not even a function-valued. This problem is related to the ill-posedness of the product of distributions $u \in \mathcal{C}^\alpha$ and $v \in \mathcal{C}^\beta$ with $\alpha + \beta \leq 0$. So nonlinear SPDEs with space-time white noise are sometimes ill-posed. Before recent breakthroughs explained below, SPDEs have been mainly studied in the case that the solution is function-valued, i.e. in a lower dimension or with smooth noise in spatial variable.

Recent theories of *regularity structures* by Hairer [10] or *paracontrolled calculus* by Gubinelli, Imkeller, and Perkowski [7] constructed general local-in-time well-posedness theories of the solutions of several singular SPDEs, e.g. KPZ equation

([11, 8]), dynamical Φ^4_d model ([10, 3]), and stochastic Navier-Stokes equation ([17]). Although the mathematical tools used in these theories are different, both of them are based on *rough path theory* of SDEs. In SDEs, the mapping from the noise term into the (strong) solution is not continuous. By rough path theory, however, we can show that the mapping from the pair of noise and its "iterated integral" to the solution is continuous. The principles in [10, 7] are as follows. Instead of considering the mapping from the noise ξ to the solution u, we construct large spaces $\mathcal{M}$ and $\mathcal{U}$ such that the "enhanced" solution map $\mathcal{M} \ni \Xi \mapsto \mathbf{u} \in \mathcal{U}$ and the "projection" $\mathbf{u} \mapsto u$ are continuous. The elements $\Xi \in \mathcal{M}$ and $\mathbf{u} \in \mathcal{U}$ are interpreted as "enhanced" noise and "enhanced" solution, respectively. When we consider a smooth approximation ξ^ϵ of ξ (as $\epsilon \downarrow 0$), we can construct a natural lift Ξ^ϵ, but Ξ^ϵ does not converge in the space $\mathcal{M}$ in general. So we introduce the "renormalization" $\hat{\Xi}^\epsilon$ of Ξ^ϵ such that $\hat{\Xi}^\epsilon$ converges to some limit $\hat{\Xi}$ and the corresponding solution $\hat{u}^\epsilon$ satisfies

$$\mathcal{L}\hat{u}^\epsilon = F^\epsilon(\hat{u}^\epsilon, \nabla\hat{u}^\epsilon) + \xi^\epsilon$$

with suitably renormalized nonlinear term F^ϵ. With the help of the continuous map $\mathcal{M} \to \mathcal{U}$, we have the convergence of $\hat{u}^\epsilon$ to some limit $\hat{u}$.

In this thesis, we apply Gubinelli-Imkeller-Perkowski's theory to some singular SPDEs and obtain suitable renormalizations of these equations. In addition, we consider the global-in-time well-posedness of the solution. Since the general theories ignore the concrete form of the nonlinear term F, global existence of the solution u is non-trivial in general. We need to use technical properties of F in order to get the global well-posedness.

2. Paracontrolled calculus and Funaki-Quastel approximation for the KPZ equation

In Chapter 2, we consider the approximation of the KPZ equation

$$\partial_t h = \tfrac{1}{2}\partial_x^2 h + \tfrac{1}{2}(\partial_x h)^2 + \xi$$

on $(t, x) \in [0, \infty) \times \mathbb{T}$, where ξ is a space-time white noise. The KPZ equation was introduced by Kardar, Parisi and Zhang [14] as a model for a growing interface represented by a height function h with fluctuations. When we replace ξ by a smeared noise $\xi^\epsilon(t, x) = (\xi(t) * \eta^\epsilon)(x)$ in x with even mollifier η^ϵ and consider the approximation

$$\partial_t h^\epsilon = \tfrac{1}{2}\partial_x^2 h^\epsilon + \tfrac{1}{2}\{(\partial_x h^\epsilon)^2 - c^\epsilon\} + \xi^\epsilon$$

with a constant $c^\epsilon \sim \frac{1}{\epsilon}$ depending on the choice of mollifier η^ϵ, we can show that h^ϵ converges to the so-called *Cole-Hopf solution* h_{CH} as $\epsilon \downarrow 0$. However, this approximation is not useful in considering the invariant measure. Instead, here we consider the Funaki-Quastel's approximation [6]

$$\partial_t \widetilde{h}^\epsilon = \tfrac{1}{2}\partial_x^2 \widetilde{h}^\epsilon + \tfrac{1}{2}\{(\partial_x \widetilde{h}^\epsilon)^2 - c^\epsilon\} * \eta_2^\epsilon + \xi^\epsilon, \tag{2.1}$$

where $\eta_2^\epsilon = \eta^\epsilon * \eta^\epsilon$. It is a common fact that the invariant measures are essentially unchanged if we apply an operator A to the noise term and A^2 to the drift term at the same time. Indeed the Cole-Hopf solution h_{CH} admits the distribution of the Brownian bridge $(B(x))_{x\in\mathbb{T}}$ as an invariant measure (in the sense of the tilt process $\partial_x h_{\mathrm{CH}}$), and the approximation $\widetilde{h}^\epsilon$ admits the distribution of $(B * \eta^\epsilon(x))_{x\in\mathbb{T}}$ as an invariant measure. In addition, the convergence of $\widetilde{h}^\epsilon$ to the process $h_{\mathrm{CH}}(t) + \frac{1}{24}t$ was shown in [6] but only in the equilibrium case. In this chapter, we use the paracontrolled calculus and show that the convergence of the approximation (2.1) holds even for non-equilibrium case. As a consequence, the derivation of the difference $\frac{1}{24}t$ is easier than the method in [6].

Let $\varphi \in C_0^\infty(\mathbb{R})$ satisfy $\varphi(0) = 1$ and $\varphi(x) = \varphi(-x)$. Let $\eta = \mathcal{F}^{-1}\varphi$ and consider the mollifier $\eta^\epsilon = \epsilon^{-1}\eta(\epsilon^{-1}\cdot)$. Denote by $\mathcal{C}^\delta = \mathcal{B}^\delta_{\infty,\infty}(\mathbb{T})$ the inhomogeneous Besov space on $\mathbb{T}$. Our main result is formulated as follows.

Theorem 2.1. *For every periodic function $h_0 \in \mathcal{C}^{0^+} = \cup_{\delta>0}\mathcal{C}^\delta$, there exists a survival time $T^\epsilon \in (0,\infty]$ such that,* (2.1) *has a unique solution $\widetilde{h}^\epsilon$ on $[0,T]$ for every $T < T^\epsilon$, and $\lim_{\epsilon\downarrow 0} T^\epsilon = \infty$ in probability. Furthermore, $\widetilde{h}^\epsilon$ converges to the process $h(t) = h_{\mathrm{CH}}(t) + \frac{1}{24}t$ in $C((0,T],\mathcal{C}^{\frac{1}{2}-\delta})$ in probability for every $\delta > 0$ and $T < \infty$, where h_{CH} is the Cole-Hopf solution starting at $\widetilde{h}_0$.*

Remark 2.2. *Precisely, the convergence $\tilde{h}^\epsilon \to h$ in probability considered here means that*

$$P(\|\tilde{h}^\epsilon - h\|_{C([t,T],\mathcal{C}^{\frac{1}{2}-\delta})} > \lambda,\ T < T^\epsilon) + P(T \geq T^\epsilon) \to 0$$

for every $0 < t < T$ and $\lambda > 0$.

This result is an extension of [6] to non-stationary solutions and furthermore shows the convergence in probabilistically strong sense instead of law sense.

3. A coupled KPZ equation, its two types of approximations and existence of global solutions

In Chapter 3, we consider the coupled $\mathbb{R}^d$-valued KPZ equation

$$\partial_t h^\alpha = \tfrac{1}{2}\partial_x^2 h^\alpha + \tfrac{1}{2}\Gamma^\alpha_{\beta\gamma}\partial_x h^\beta \partial_x h^\gamma + \sigma^\alpha_\beta \xi^\beta, \quad 1 \leq \alpha \leq d$$

on $(t,x) \in [0,\infty) \times \mathbb{T}$, with given constants $\Gamma^\alpha_{\beta\gamma}$ and σ^α_β. Here (ξ^α) is a d-tuple of independent space-time white noises. At least heuristically, if a microscopic system with d (local) conserved quantities involves a weak asymmetry and if we expand a macroscopic equation to second order, we can expect to obtain the coupled KPZ equations in the space-time scaling limit of the microscopic system. See e.g. [5]. As in the scalar valued case, we can consider two types of approximations

$$\partial_t h^{\epsilon,\alpha} = \tfrac{1}{2}\partial_x^2 h^{\epsilon,\alpha} + \tfrac{1}{2}\Gamma^\alpha_{\beta\gamma}(\partial_x h^{\epsilon,\beta}\partial_x h^{\epsilon,\gamma} - c^\epsilon A^{\beta\gamma} - B^{\epsilon,\beta\gamma}) + \sigma^\alpha_\beta \xi^{\epsilon,\beta}, \tag{3.1}$$

$$\partial_t \tilde{h}^{\epsilon,\alpha} = \tfrac{1}{2}\partial_x^2 \tilde{h}^{\epsilon,\alpha} + \tfrac{1}{2}\Gamma^\alpha_{\beta\gamma}(\partial_x \tilde{h}^{\epsilon,\beta}\partial_x \tilde{h}^{\epsilon,\gamma} - c^\epsilon A^{\beta\gamma} - \tilde{B}^{\epsilon,\beta\gamma}) * \eta_2^\epsilon + \sigma^\alpha_\beta \xi^{\epsilon,\beta}, \tag{3.2}$$

where c^ϵ is the same constant as above, $A^{\beta\gamma} = \sum_\delta \sigma^\beta_\delta \sigma^\gamma_\delta$, and $B^{\epsilon,\beta\gamma}$ and $\tilde{B}^{\epsilon,\beta\gamma}$ are constants behaving as $O(|\log\epsilon|)$ in general. We can show that these approximations have the same limit h under a well-adjusted choice of $B^{\epsilon,\beta\gamma}$ and $\tilde{B}^{\epsilon,\beta\gamma}$.

For $\kappa \in \mathbb{R}$ and $r \in \mathbb{N}$, $(\mathcal{C}^\kappa)^r := \mathcal{B}^\kappa_{\infty,\infty}(\mathbb{T};\mathbb{R}^r)$ denotes the $\mathbb{R}^r$-valued Besov space on $\mathbb{T}$. Our first two main theorems are formulated as follows.

Theorem 3.1. (1) *Let $\delta \in (0,\frac{1}{2})$. For every $h(0) \in (\mathcal{C}^\delta)^d$, there exists a unique solution h^ϵ of the KPZ approximating equation* (3.1) *up to the survival time $T^\epsilon_{\mathrm{sur}} \in (0,\infty]$ (i.e. $T^\epsilon_{\mathrm{sur}} = \infty$ or $\lim_{t\uparrow T^\epsilon_{\mathrm{sur}}} \|h^\epsilon\|_{C([0,t],(\mathcal{C}^\delta)^d)} = \infty$). With a proper choice of $B^{\epsilon,\beta\gamma}$, there exists a random time $T_{\mathrm{sur}} \in (0,\infty]$ such that $T_{\mathrm{sur}} \leq \liminf_{\epsilon\downarrow 0} T^\epsilon_{\mathrm{sur}}$ in probability and h^ϵ converges to some h in $C((0,T],(\mathcal{C}^{1/2-\delta'})^d)$ in probability for every $\delta' > 0$ and $0 < T < T_{\mathrm{sur}}$. This T_{sur} can be chosen maximal in the sense that $T_{\mathrm{sur}} = \infty$ or $\lim_{T\uparrow T_{\mathrm{sur}}} \|h\|_{C([0,T],(\mathcal{C}^\delta)^d)} = \infty$. The survival time T_{sur} depends on the initial value $h(0)$ and driving processes.*
(2) *A similar result holds for the solution $\tilde{h}^\epsilon$ of the KPZ approximating equation* (3.2) *with some limit $\tilde{h}$ under a proper choice of $\tilde{B}^{\epsilon,\beta\gamma}$. Moreover, under a well-adjusted choice of the renormalization factors $B^{\epsilon,\beta\gamma}$ and $\tilde{B}^{\epsilon,\beta\gamma}$, we can make $h = \tilde{h}$.*

Remark 3.2. *Precisely, the convergence $h^\epsilon \to h$ considered here means that*

$$P(\|h^\epsilon - h\|_{C([t,T],(\mathcal{C}^{\frac{1}{2}-\delta'})^d)} > \lambda,\ T < T_{\mathrm{sur}} \wedge T^\epsilon_{\mathrm{sur}})$$

$$+P(T^\epsilon_{\rm sur} \le T_{\rm sur} - \lambda,\ T_{\rm sur} < \infty) + P(T^\epsilon_{\rm sur} \le T,\ T_{\rm sur} = \infty) \to 0$$

for every $0 < t < T$ and $\lambda > 0$. The convergence $\tilde{h}^\epsilon \to h$ also.

Theorem 3.3. *All components of the renormalization matrices B^ϵ and $\tilde{B}^\epsilon$ behave as $O(1)$ if and only if the trilinear condition*

$$\hat\Gamma^\alpha_{\beta\gamma} = \hat\Gamma^\alpha_{\gamma\beta} = \hat\Gamma^\beta_{\alpha\gamma}, \quad \forall \alpha, \beta, \gamma \tag{3.3}$$

holds, where $\hat\Gamma^\alpha_{\beta\gamma} = (\sigma^{-1})^\alpha_{\alpha'}\Gamma^{\alpha'}_{\beta'\gamma'}\sigma^{\beta'}_\beta\sigma^{\gamma'}_\gamma$. In particular, when (3.3) *holds, we can choose $B^\epsilon = \tilde{B}^\epsilon = 0$ in the approximations* (3.1) *and* (3.2), *and the corresponding solutions $h_{B=0}$ and $\tilde{h}_{\tilde{B}=0}$ respectively converge as $\epsilon \downarrow 0$. In the limit, we have*

$$\tilde{h}^\alpha_{\tilde{B}=0}(t,x) = h^\alpha_{B=0}(t,x) + c^\alpha t, \quad 1 \le \alpha \le d,$$

where

$$c^\alpha = \frac{1}{24}\sum_{\gamma,\gamma'} \sigma^\alpha_\beta \hat\Gamma^\beta_{\alpha'\alpha''}\hat\Gamma^{\alpha'}_{\gamma\gamma'}\hat\Gamma^{\alpha''}_{\gamma\gamma'}.$$

In order to obtain the global well-posedness, we assume the trilinear condition (3.3). Under this condition, we will show that the process h admits the distribution of $((\sigma^\alpha_\beta B^\beta)(x))_{x\in\mathbb{T}}$ as an invariant measure (in the sense of the tilt process $\partial_x h$), where (B^α) is a d-tuple of independent Brownian bridges. As a consequence, we will show that the limit h exists on whole $[0,\infty)$ almost surely, when the initial value is sampled from this invariant measure. In addition, since Hairer and Mattingly [12] showed that the solutions of several singular SPDEs in the sense of Hairer's theory are strong Feller processes, so we can indeed show the global well-posedness of the coupled KPZ equation for every initial value.

Let μ_A be the Gaussian measure on the space $(\mathcal{C}_0^{-1/2-\delta})^d := \{u \in (\mathcal{C}^{1/2-\delta})^d\,;\int_{\mathbb{T}} u = 0\}$, $\delta > 0$, under which $u = (u^\alpha)_{\alpha=1}^d \in (\mathcal{C}_0^{-1/2-\delta})^d$ has the covariance

$$E[u^\alpha(x)u^\beta(y)] = A^{\alpha\beta}\delta(x-y).$$

Note that μ_A is the distribution of $(\partial_x\sigma B)_{x\in\mathbb{T}}$, which is the limit in law of that of $\big(\partial_x(\sigma B * \eta^\epsilon)\big)_{x\in\mathbb{T}}$ as $\epsilon \downarrow 0$.

Theorem 3.4. *Let $\delta \in (0, \frac{1}{2})$ and assume the trilinear condition* (3.3). *Then there exists a subset $H \subset (\mathcal{C}_0^{-1/2-\delta})^d$ such that $\mu_A(H) = 1$, and if $\partial_x h(0) \in H$, the convergence to the limiting process h as above holds on whole $[0,\infty)$ (i.e. $T_{\rm sur} = \infty$ almost surely). Moreover, the spatial derivative $u = \partial_x h$ of the limit process h is a Markov process on $(\mathcal{C}_0^{-1/2-\delta})^d$ which admits μ_A as an invariant measure.*

Remark 3.5. *Proposition 5.4 of Hairer and Mattingly* [12] *(combined with Theorem 3.4) shows that the limit process h exists on $[0,\infty)$ almost surely for all initial values $h(0) \in (\mathcal{C}^{1/2-\delta})^d$, since the measure μ_A has a dense support in $(\mathcal{C}_0^{-1/2-\delta})^d$.*

4. Global well-posedness of complex Ginzburg-Landau equation with space-time white noise

In Chapter 4, we consider the stochastic complex Ginzburg-Landau (CGL) equation

$$\partial_t u = (i+\mu)\Delta u - \nu|u|^2u + \lambda u + \xi$$

on $(t,x) \in [0,\infty)\times\mathbb{T}^3$, where $\mu > 0$, $\nu \in \{z \in \mathbb{C}\,;\Re z > 0\}$, $\lambda \in \mathbb{C}$, and ξ is a complex space-time white noise, which is a complex-valued centered Gaussian random field with covariance structure

$$E[\xi(t,x)\xi(s,y)] = 0, \quad E[\xi(t,x)\overline{\xi(s,y)}] = \delta(t-s)\delta(x-y).$$

The CGL equation appears as a generic amplitude equation near the threshold for an instability in fluid mechanics, as well as in the theory of phase transition in superconductivity. Randomly forced CGL equation in spatial dimension $d \geq 1$ has been mainly studied in the case that ξ is a smeared noise in x, or that $d = 1$ and ξ is a space-time white noise, e.g. [1, 2, 9, 15]. In our case ($d = 3$ and ξ is a space-time white noise), as an application of the paracontrolled calculus, we can show that the solution u^ϵ of the approximating equation

$$\partial_t u^\epsilon = (i+\mu)\Delta u^\epsilon - \nu|u^\epsilon|^2 u^\epsilon + C^\epsilon u^\epsilon + \xi^\epsilon \tag{4.1}$$

with suitable $C^\epsilon \sim \frac{1}{\epsilon}$ converges to a process u locally in time. See [13] for details.

In order to obtain the global well-posedness, we use the method in [16], where Mourrat and Weber showed the global well-posedness of the dynamical Φ^4 model on the 3-dimensional torus. In the deterministic setting [4], the a priori L^{2p}-inequality of the solution u of the CGL equation holds with $p > \frac{3}{2}$ and

$$p < 1 + \mu(\mu + \sqrt{1+\mu^2}).$$

In this chapter, we improve the method in [16] in order that it can be applied to the a priori L^{2p}-inequality of the CGL equation for every p close to $\frac{3}{2}$, and we show the a priori estimate of the solution of the stochastic CGL equation when $\mu > \frac{1}{2\sqrt{2}}$.

Let $\mathcal{C}^\alpha = \mathcal{B}^\alpha_{\infty,\infty}$ be the complex-valued Besov space on $\mathbb{T}^3$.

Theorem 4.1. *Let $\mu > \frac{1}{2\sqrt{2}}$. Choose sufficiently small $\kappa > 0$ depending on μ. For every initial value $u_0 \in \mathcal{C}^{-\frac{2}{3}+\kappa}$, the solution u^ϵ of* (4.1) *has a limit u, which is independent of the choice of η, such that for every $T > 0$*

$$\lim_{\epsilon \downarrow 0} \|u^\epsilon - u\|_{C([0,T],\mathcal{C}^{-\frac{2}{3}+\kappa})} = 0$$

in probability.

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