数理科学実践研究レター 2018–2 July 05, 2018

# Boundary conditions for 2D shallow water equations in the half space and its application to the oceanic model

by

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# Boundary conditions for 2D shallow water equations in the half space and its application to the oceanic model

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**Abstract**

A simple discussion is given of the appropriate proposition of boundary conditions for an ocean-atmosphere coupled model, which we use to simulate the coastal phenomenon Ningaloo Niño off western Australia. Our analysis is mainly based on the basic linear algebra and characteristic theory in mathematics.

# 半空間上の２次元浅水波方程式に対する境界条件とその海洋モデルへの応用

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**概要**

西オーストラリアの沿岸部のニンガルー・ニーニョ現象をシミュレーションする為に用いられる大気海洋モデルに対して境界条件を適切な境界条件を提案した．我々の解析は数学の特性曲線論と基礎的な線形代数に基づく．

## 1 Introduction

During the austral summer of 2010-2011, an unprecedented oceanic warm event was observed off the west coast of Australia. Sea surface temperature anomaly averaged in February-March, 2011 reached about 3 °C off the west coast of Australia, which is above four times of the standard deviation of its interannual variation in recent 30 years. This coastal phenomenon was named Ningaloo Niño and has significant impacts on the precipitation over Australia (Refer to [1],[2],[4]). Because of an analogy between the equator and the coast, it will be interesting to extend the simple ocean-atmosphere coupled model of Yamagata (1985)[7], which made a large contribution to the understanding of the generation mechanism of El Niño. Here we first summarize his model. The governing equations of motion for the ocean, linearized around a state of no motion, are

$$\begin{cases} u_t - fv + gh_x = -au + \gamma U, \\ v_t + fu + gh_y = -av + \gamma V, \\ h_t + d(u_x + v_y) = -bh, \end{cases} \tag{1}$$

where $(u, v)$ are the zonal and meridional oceanic velocity components, $h$ is the surface elevation, $g$ is the acceleration due to gravity, and $d$ is the equivalent depth. Also $a$ and $b$ are Reyleigh friction and Newtonian cooling, respectively. The wind stress $(\gamma U, \gamma V)$ is assumed to enter the ocean as a body force, where $(U, V)$ satisfy the following equations as the zonal and meridional velocity of the atmosphere:

$$\begin{cases} U_t - fV + gH_x = -AU, \\ V_t + fU + gH_y = -AV, \\ H_t + D(U_x + V_y) = -BH + \alpha h, \end{cases} \tag{2}$$

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where $H$ is the depth, $A$ and $B$ are inverse time-scales for Reyleigh friction and Newtonian cooling, respectively, $D$ is the equivalent depth and $\alpha$ is the coefficient of coupling. The systems (1) and (2) of partial differential equations give a coupled model of air-sea interaction.

In this short note, we will discuss the appropriate proposition of boundary conditions for the above ocean-atmosphere model. When we apply Yamagata's model to the study of Ningaloo Niño off western Australia, the boundary coastline will be a nonnegligible factor in our case. This work is a first step toward enhancing our understanding and improving prediction skill of Ningaloo Niño and thus contribute to the mitigation of effects of abnormal weather.

## 2 Simplified oceanic model

In order to have an insight into an equatorial case, it is very useful to consider the case in which neither the atmosphere nor the ocean is rotating. That is, we take $f \equiv 0$. Also to simplify our analysis, we neglect the wind stress $(\gamma U, \gamma V)$ acting on the ocean. In our note, we assume that the ocean motion occurs in a half plane by considering the coastline as an infinite straight line. Thus we formulate the following simplified initial-boundary value problem of the two-dimensional shallow water equations:

$$\begin{cases} u_t + gh_x = 0, \\ v_t + gh_y = 0, \qquad\qquad \text{in } \ \mathbb{R}^2_{x<0} \times (0, \infty). \\ h_t + d(u_x + v_y) = 0, \end{cases} \tag{3}$$

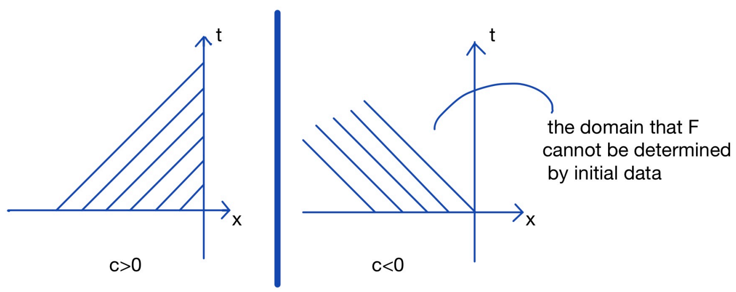
Here $\mathbb{R}^2_{x<0} := \{(x, y) | x < 0, y \in \mathbb{R}\}$ is the half plane with the boundary $\{x = 0\}$. Also, $g$ and $d$ are positive constants because of their physical meanings.

## 3 Methods and discussion

We begin with a scalar linear equation in a quarter plane

$$F_t + cF_x = 0, \quad x < 0, \quad t > 0,$$

where $c$ is a constant. We suppose that we are given the initial condition $F(x, 0)$ in $x \leq 0$, and the boundary condition $F(0, t)$ in $t \geq 0$. We ask to what extent do these values determine $F$ in the full quarter plane? It is clear that $F$ must be constant along the lines $x - ct = \text{const}$. We observe that if $c > 0$, $u$ is determined along $x = 0$ by its initial value. Thus in this case, no boundary condition can be given, while we see that the boundary condition along $x = 0$ must be given in order to determine $F$ in the entire quarter plane if $c < 0$. See the figure below.



We now adjust this idea to analyze our problem. The system can be written in matrix notation as:

$$\partial_t \vec{w} + A_x \partial_x \vec{w} + A_y \partial_y \vec{w} = 0, \qquad \text{in } \mathbb{R}^2_{x<0} \times (0, \infty),$$

where $\vec{w} = [u, v, h]^T$ is the vector of unknowns, and the matrices $A_x$ and $A_y$ are of the form

$$A_x = \begin{bmatrix} 0 & 0 & g \\ 0 & 0 & 0 \\ d & 0 & 0 \end{bmatrix} \quad \text{and} \quad A_y = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & g \\ 0 & d & 0 \end{bmatrix}.$$

Step 1: Calculation of the eigenvalues for the matrix $A_x$:

$$\lambda_1 = -\sqrt{gd}, \ \lambda_2 = 0 \text{ and } \lambda_3 = \sqrt{gd}.$$

Step 2: Calculation of the corresponding eigenvectors:
Eigenvectors $\boldsymbol{r}_i$ corresponding $\lambda_i$ are of the form

$$\boldsymbol{r}_1 = \begin{pmatrix} -\sqrt{g} \\ 0 \\ \sqrt{d} \end{pmatrix}, \quad \boldsymbol{r}_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \quad \text{and} \quad \boldsymbol{r}_3 = \begin{pmatrix} \sqrt{g} \\ 0 \\ \sqrt{d} \end{pmatrix}.$$

We thus set matrix $T := [\boldsymbol{r}_1, \boldsymbol{r}_2, \boldsymbol{r}_3]$, and $T^{-1}$ be its inverse matrix. Then it is easy to see that

$$T^{-1} A_x T = \begin{pmatrix} -\sqrt{gd} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \sqrt{gd} \end{pmatrix}.$$

Step 3: Change of variables to get a diagonalized system:
Denote

$$\vec{\omega}' = \begin{pmatrix} u' \\ v' \\ h' \end{pmatrix} = T^{-1}\vec{\omega} = T^{-1} \begin{pmatrix} u \\ v \\ h \end{pmatrix}, \quad \text{that is,} \quad \vec{\omega} = T\vec{\omega}',$$

then we obtain the equations of $\vec{\omega}' = (u', v', h')^T$ of the form

$$\partial_t \vec{\omega}' + \mathrm{Diag}(-\sqrt{gd}, 0, \sqrt{gd}) \partial_x \vec{\omega}' + \left(T^{-1} A_y T\right) \partial_y \vec{\omega}' = 0,$$

and it can be written in the form of components $\vec{\omega}' = (u', v', h')^T$:

$$\frac{\partial}{\partial t} \begin{pmatrix} u' \\ v' \\ h' \end{pmatrix} + \begin{pmatrix} -\sqrt{gd} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \sqrt{gd} \end{pmatrix} \frac{\partial}{\partial x} \begin{pmatrix} u' \\ v' \\ h' \end{pmatrix} = - \begin{pmatrix} 0 & \frac{\sqrt{d}}{2} & 0 \\ 0 & 0 & \sqrt{gd} \\ 0 & \frac{\sqrt{d}}{2} & 0 \end{pmatrix} \frac{\partial}{\partial y} \begin{pmatrix} u' \\ v' \\ h' \end{pmatrix}.$$

Note that the terms involving $y$ derivatives of $\vec{\omega}'$ do not contribute to the analysis. Refer to Thompson [6].
Step 4: Write the boundary conditions in terms of the original variables:
From the analysis at the beginning of this section, since we just have one negative eigenvalue $\lambda_1$, we need to prescribe one boundary condition at the outlet corresponding to the characteristic variable $u'$. Let us come back to the change of variables $\vec{\omega}' = T^{-1}\vec{\omega}$, that is,

$$\begin{pmatrix} u' \\ v' \\ h' \end{pmatrix} = \begin{pmatrix} -\frac{1}{2\sqrt{g}} & 0 & \frac{1}{2\sqrt{d}} \\ 0 & 1 & 0 \\ \frac{1}{2\sqrt{g}} & 0 & \frac{1}{2\sqrt{d}} \end{pmatrix} \begin{pmatrix} u \\ v \\ h \end{pmatrix}.$$

If the boundary data for the original variable $u$ is prescribed, then we can solve the problem in the following steps

- $v'$ and $h'$ are completely solvable only by using the initial data;
- Compute the boundary value of $u'$ by using those of $u, v'$ and $h'$;
- $u'$ is then solvable;
- Recover $u, v$ and $h$ from $u', v'$ and $h'$ by the above transformation.

# 4 Result

**Theorem 1** *For the two-dimensional shallow water equations (3) in the half space* $\mathbb{R}^2_{x<0}$ *, it can be solvable with the continuous initial data* $(u, v, h)|_{t=0} = (u_0, v_0, h_0)$*, provided that the boundary value* $u(x, y, t)|_{x=0} = b(y, t)$ *is given for some continuous function* $b(y, t)$*.*

This theorem says that the initial-boundary value problem is solvable provided that the normal velocity at the boundary is given as well as the initial velocity is given. We emphasize that this note only concerns with the simplified oceanic model. It will be interesting to extend the discussion to the ocean-atmosphere coupled model.

**Acknowledgement** The authors would like to express their heartfelt gratitudes to Professor Yoshikazu Giga for his useful discussion and two anonymous reviewers for their helpful comments.

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