J. Math. Sci. Univ. Tokyo
25 (2018), 35–90.

Dehn Twists, Hypertwists, and Uniformization of Twined Singularities

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Abstract. There are two kinds of homeomorphisms of an annulus that appear as local monodromies of degenerations of Riemann surfaces: *fractional Dehn twist* and *Nielsen twist*. In this paper, they are "in a unified way" generalized to higher dimensions as a *hypertwist*, which is the monodromy of a *twined singularity* (a quotient of a multiplicative A-singularity). We moreover establish the uniformization theorem of this quotient, which generalizes the uniformization theorem in our previous paper.

Contents

2010 *Mathematics Subject Classification*. Primary 32Q30; Secondary 14D05.
Key words: Uniformization of singularity, cyclic quotient, group action, multiplicative A-singularity, monodromy, fractional Dehn twist, small group, lifting and descent.

1. Introduction

Let a and m $(0 < a < m)$ and b and n $(0 < b < n)$ be two pairs of relatively prime integers. An $\left(\frac{a}{m}, \frac{b}{n}\right)$*-fractional Dehn twist* is a self-homeomorphism of an annulus $[0,1] \times S^1$ given by $(t,\, e^{\mathrm{i}\theta}) \mapsto (t,\, e^{2\pi\mathrm{i}\{-(1-t)a/m+tb/n\}}e^{\mathrm{i}\theta})$. More generally, where κ is an integer, an $\left(\frac{a}{m},\, \frac{b}{n},\, \kappa\right)$*-fractional Dehn twist* is defined as the composite map of a κ-Dehn twist and an $\left(\frac{a}{m}, \frac{b}{n}\right)$-fractional Dehn twist (Figure 1.1). We next introduce a Nielsen twist. First let $H : [0,1] \times \mathbb{R} \to [0,1] \times \mathbb{R}$ be an affine transformation given by $H(t,y) = \left(1-t, (2t-1)\frac{a}{2m} - y\right)$. Then H and H^2 transform $[0,1] \times \mathbb{R}$ as illustrated in Figure 1.2; note that $H^2(t,y) = \left(t, (1-2t)\frac{a}{m} + y\right)$. Under the covering map $f : [0,1] \times \mathbb{R} \to [0,1] \times S^1$, $f(t,y) = (t, e^{2\pi\mathrm{i}y})$, H descends to an $\frac{a}{2m}$*-Nielsen twist* $h : [0,1] \times S^1 \to [0,1] \times S^1$, $h(t,\, e^{\mathrm{i}\theta}) = (1-t,\, e^{2\pi\mathrm{i}(2t-1)a/2m}e^{-\mathrm{i}\theta})$. Note that h^2 is a $-\left(\frac{a}{m},\, \frac{a}{m}\right)$-fractional Dehn twist.

More generally, an $\left(\frac{a}{2m},\, \kappa\right)$*-Nielsen twist* of h and a $(-\kappa)$-Dehn twist (not $(+\kappa)$-Dehn twist), explicitly given by

$$(t,\, e^{\mathrm{i}\theta}) \in [0,1] \times S^1 \longmapsto (1-t,\, e^{2\pi\mathrm{i}\{(2t-1)a/2m+t\kappa\}}e^{-\mathrm{i}\theta}) \in [0,1] \times S^1.$$

Note that its square is a $-\left(\frac{a}{m},\, \frac{a}{m},\, 2\kappa\right)$-fractional Dehn twist.

A fractional Dehn twist appears as the topological monodromy of a degeneration: Set $c := \gcd(m,n)$, $m' := m/c$, $n' := n/c$, and let $\gamma : \mathbb{C}^3 \to \mathbb{C}^3$ be an automorphism defined by

$$\gamma : (z,w,t) \longmapsto (e^{2\pi\mathrm{i}a/m}z,\, e^{2\pi\mathrm{i}b/n}w,\, e^{2\pi\mathrm{i}/m'n'c}t). \tag{1.1}$$

Suppose that γ preserves $A_{d-1} := \{(z,w,t) \in \mathbb{C}^3 \,:\, zw = t^d\}$; this is the case precisely when $e^{2\pi\mathrm{i}a/m}e^{2\pi\mathrm{i}b/n} = e^{2\pi\mathrm{i}d/m'n'c}$, that is, $\frac{a}{m} + \frac{b}{n} \equiv \frac{d}{m'n'c} \bmod \mathbb{Z}$. Write $d = m'n'c\left(\frac{a}{m} + \frac{b}{n} + \kappa\right)$ for some integer κ such that $\frac{a}{m} + \frac{b}{n} + \kappa > 0$. Let Γ the cyclic group generated by γ. Define a holomorphic map $\Phi : A_{d-1} \to \mathbb{C}$ by $\Phi(z,w,t) = t^{m'n'c}$. Then Φ is Γ-invariant, so descends to a holomorphic map $\overline{\Phi} : A_{d-1}/\Gamma \to \mathbb{C}$, which is a degeneration of annuli whose topological monodromy is a $-\left(\frac{a}{m},\, \frac{b}{n},\, \kappa\right)$-fractional Dehn twist.

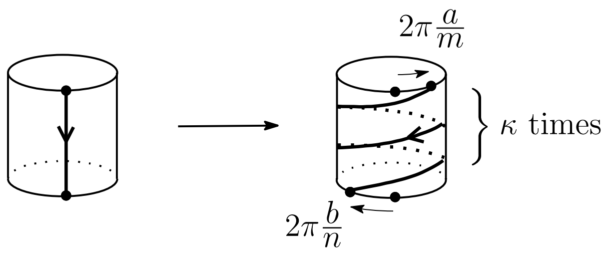

Fig. 1.1. An $\left(\frac{a}{m}, \frac{b}{n}, \kappa\right)$-fractional Dehn twist.

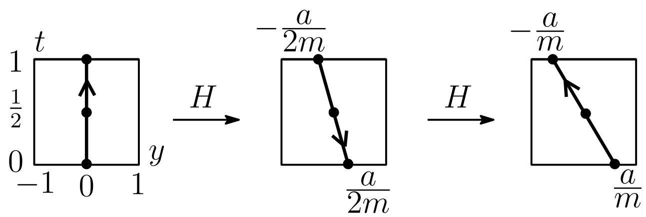

Fig. 1.2.

A Nielsen twist also appears as the topological monodromy of a degeneration: Let $\gamma' : \mathbb{C}^3 \to \mathbb{C}^3$ be an automorphism defined by

$$\gamma' : (z, w, t) \longmapsto (e^{2\pi \mathrm{i} a/2m} w, e^{2\pi \mathrm{i} a/2m} z, e^{2\pi \mathrm{i}/2m} t). \tag{1.2}$$

Suppose that γ' preserves A_{d-1}; this is the case precisely when $e^{2\pi \mathrm{i} a/m} = e^{2\pi \mathrm{i} d/2m}$, that is, $\frac{a}{m} \equiv \dfrac{d}{2m} \bmod \mathbb{Z}$. Write $d = 2a + 2m\kappa$ for some integer $\kappa \geq 0$. Let Γ' be the cyclic group generated by γ'. Define a holomorphic map $\Phi' : A_{d-1} \to \mathbb{C}$ by $\Phi'(z, w, t) = t^{2m}$. Then Φ' is Γ'-invariant, so descends to a holomorphic map $\overline{\Phi}' : A_{d-1}/\Gamma' \to \mathbb{C}$, which is a degeneration of annuli whose topological monodromy is an $\left(\frac{a}{2m}, \kappa\right)$-Nielsen twist.

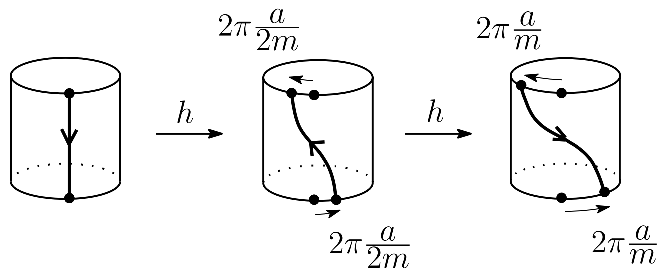

Fig. 1.3. An $\frac{a}{2m}$-Nielsen twist h.

Main results

We generalize the above notions/results to higher dimensions. Fix a positive integer d and consider a complex variety (a *multiplicative A-singularity*)

$$A_{d-1} = \{(x_1, x_2, \dots, x_n, t) \in \mathbb{C}^{n+1} \ : \ x_1 x_2 \cdots x_n = t^d\}.$$

If $n \geq 3$, the singular locus of A_{d-1} is *not* isolated — the union of ${}_nC_2$ hyperplanes $H_{ij} = \{x_i = x_j = t = 0\}$ $(1 \leq i < j \leq n)$. In contrast, the *additive A-singularity* $x_1^2 + x_2^2 + \cdots + x_n^2 = t^d$ has only an isolated singularity at the origin. In particular if $n \geq 3$, this is *not* biholomorphic to A_{d-1}. (If $n = 2$, they are biholomorphic: Via $x_1' = x_1 + \mathrm{i}x_2$ and $x_2' = x_1 - \mathrm{i}x_2$, $x_1^2 + x_2^2 = t^d$ is transformed to $x_1' x_2' = t^d$.)

Now take $\sigma \in \mathfrak{S}_n$ (a permutation of n elements) and nonzero complex numbers $\alpha_1, \dots, \alpha_n, \delta$ such that $\alpha_1 \alpha_2 \cdots \alpha_n = \delta^d$, and define an automorphism $\gamma : A_{d-1} \to A_{d-1}$ by

$$\gamma : (x_1, x_2, \dots, x_n, t) \ \longmapsto \ (\alpha_1 x_{\sigma(1)}, \alpha_2 x_{\sigma(2)}, \dots, \alpha_n x_{\sigma(n)}, \delta t).$$

Simple Case. We first consider the case that σ is cyclic of full length n. Take an (arbitrary) nth root β of $\alpha_1 \alpha_2 \cdots \alpha_n$ and define another automorphism $\gamma' : A_{d-1} \to A_{d-1}$ by

$$(*) \qquad \gamma' : (x_1, x_2, \dots, x_n, t) \mapsto (\beta x_{\sigma(1)}, \beta x_{\sigma(2)}, \dots, \beta x_{\sigma(n)}, \delta t).$$

Then irrespective of the choice of β, γ' is conjugate to γ in $\mathrm{Aut}(A_{d-1})$ (Lemma 2.3 (3)). Say $\gamma' = f^{-1} \circ \gamma \circ f$, then under a coordinate change via f

of A_{d-1}, γ' may be regarded as γ. We thus only consider an automorphism of the form $(*)$.

In what follows, suppose that $\alpha_1\alpha_2\cdots\alpha_n$ is a root of unity (this is equivalent to the finiteness of the order of γ (Corollary 2.2)). Say $\alpha_1\alpha_2\cdots\alpha_n$ is an mth root of unity, and consider an automorphism

$$(\sharp)\qquad \begin{aligned}&\gamma : (x_1, x_2, \ldots, x_n, t) \in A_{d-1} \longmapsto\\ &\qquad (e^{2\pi \mathrm{i} a/mn} x_{\sigma(1)}, e^{2\pi \mathrm{i} a/mn} x_{\sigma(2)}, \ldots, e^{2\pi \mathrm{i} a/mn} x_{\sigma(n)}, e^{2\pi \mathrm{i}/mn} t) \in A_{d-1},\end{aligned}$$

where σ is a cyclic permutation of full length n and $d = an + mn\kappa$ for some integer $\kappa \geq 0$. This generalizes the automorphism in (1.2) given by

$$\gamma :\ (z, w, t) \in A_{d-1} \longmapsto (e^{2\pi \mathrm{i} a/2m} w, e^{2\pi \mathrm{i} a/2m} z, e^{2\pi \mathrm{i}/2m} t) \in A_{d-1},$$

where $d = 2a + 2m\kappa$ for some integer $\kappa \geq 0$.

Before stating our results, we recall some terminology: A *pseudo-reflection* is a linear transformation conjugate to $(z_1, \ldots, z_i, \ldots, z_n) \mapsto (z_1, \ldots, \zeta z_i, \ldots, z_n)$, where $\zeta \neq 1$ is a root of unity. By abuse of terminology, a matrix conjugate to the diagonal matrix $\mathrm{diag}(1, \ldots, \zeta, \ldots, 1)$ is also called a pseudo-reflection. A subgroup of $GL_n(\mathbb{C})$ is *small* if it contains no pseudo-reflections.

Result 1 (Corollary 9.9) Uniformization. *Let Γ be the cyclic group generated by the automorphism γ of A_{d-1} given by $(\sharp)$. Then A_{d-1}/Γ is isomorphic to $\mathbb{C}^n/G$, where G is a small finite group generated by the automorphisms $f, g_1, g_2, \ldots, g_{n-1}$ of $\mathbb{C}^n$ given by*

$$\begin{aligned}
f &: (z_1, \ldots, z_n) \mapsto (e^{2\pi \mathrm{i} a/mnd} z_{\sigma(1)}, \ldots,\\
&\qquad\qquad e^{2\pi \mathrm{i} a/mnd} z_{\sigma(n-1)}, e^{2\pi \mathrm{i}(a+mn\kappa)/mnd} z_{\sigma(n)}),\\
g_i &: (z_1, \ldots, z_n) \mapsto (z_1, \ldots, z_{i-1}, e^{2\pi \mathrm{i}/d} z_i, z_{i+1}, \ldots, z_{n-1}, e^{-2\pi \mathrm{i}/d} z_n).
\end{aligned}$$

We remark that G is abelian only when $n = 2$ and $d = 2$ (Theorem 10.6 (2)).

Now define a holomorphic map $\Phi : A_{d-1} \to \mathbb{C}$ by $\Phi(x_1, \ldots, x_n, t) = t^{mn}$. Then Φ is Γ-invariant, so descends to a holomorphic map $\overline{\Phi} : A_{d-1}/\Gamma \to \mathbb{C}$.

Result 2 (Lemma 8.2) Correspondence of maps. *Under the isomorphism* $A_{d-1}/\Gamma \cong \mathbb{C}^n/G$, $\overline{\Phi} : A_{d-1}/\Gamma \to \mathbb{C}$ *corresponds to the holomorphic map* $\overline{\phi} : \mathbb{C}^n/G \to \mathbb{C}$ *induced by the* G*-invariant holomorphic map* $\phi : \mathbb{C}^n \to \mathbb{C}$, $\phi(v_1, v_2, \ldots v_n) = (v_1 v_2 \cdots v_n)^{mn}$.

In the case that $\sigma \in \mathfrak{S}_n$ is arbitrary, decompose it into disjoint cyclic permutations: $\sigma = \sigma_1\sigma_2 \cdots \sigma_l$, say the length of σ_i is n_i. Renumbering the indices, assume that σ_1 permutes $\{1, 2, \ldots, n_1\}$, σ_2 permutes $\{n_1+1, n_1+2, \ldots, n_1+n_2\}$, σ_3 permutes $\{n_1+n_2+1, \ldots, n_1+n_2+n_3\}$ and so on. Write $\mathbb{C}^n = \mathbb{C}^{n_1} \times \mathbb{C}^{n_2} \times \cdots \times \mathbb{C}^{n_l}$; then σ_i acts on $\mathbb{C}^{n_i}$ as $\boldsymbol{x}_i := (x_1^{(i)}, \ldots, x_{n_i}^{(i)}) \mapsto \boldsymbol{x}_i^{\sigma_i} := (x_{\sigma_i(1)}^{(i)}, \ldots, x_{\sigma_i(n_i)}^{(i)})$. As in Simple Case, the following holds (Lemma 2.6): γ is via an element of $\mathrm{Aut}(A_{d-1})$ conjugate to an automorphism $\gamma' : A_{d-1} \to A_{d-1}$ of the form

$$\gamma' : (\boldsymbol{x}_1, \ldots, \boldsymbol{x}_l, t) \longmapsto (\beta_1 \boldsymbol{x}_1^{\sigma_1}, \ldots, \beta_l \boldsymbol{x}_l^{\sigma_l}, \delta t), \quad \beta_i \in \mathbb{C}^\times.$$

It thus suffices to consider automorphisms of this form. Note that the condition that γ preserves A_{d-1} is given by

$$\beta_1^{n_1}\beta_2^{n_2}\cdots\beta_l^{n_l} = \delta^d. \tag{1.3}$$

In what follows, we consider the following automorphism of A_{d-1} generalizing ($\sharp$) in Simple Case:

$$\gamma : (\boldsymbol{x}_1, \ldots, \boldsymbol{x}_l, t) \longmapsto (e^{2\pi\mathrm{i}a_1/n_1m_1}\boldsymbol{x}_1^{\sigma_1}, \ldots, e^{2\pi\mathrm{i}a_l/n_lm_l}\boldsymbol{x}_l^{\sigma_l}, e^{2\pi\mathrm{i}/N}t), \tag{1.4}$$

where

(i) n_i is the length of σ_i, and a_i, m_i are positive integers such that a_i is relatively prime to $n_i m_i$.

(ii) $N := (m_1')^{n_1}\cdots(m_l')^{n_l}c$, where $c := \gcd(n_1m_1, \ldots, n_lm_l)$ and $m_i' := \dfrac{n_im_i}{c}$.

(iii) $(e^{2\pi\mathrm{i}a_1/n_1m_1})^{n_1}(e^{2\pi\mathrm{i}a_2/n_2m_2})^{n_2}\cdots(e^{2\pi\mathrm{i}a_l/n_lm_l})^{n_l} = e^{2\pi\mathrm{i}d/N}$ (see (1.3)), that is, $\dfrac{a_1}{m_1} + \dfrac{a_2}{m_2} + \cdots + \dfrac{a_l}{m_l} + \kappa = \dfrac{d}{N}$ for some integer κ.

We say that Γ is a *twining automorphism group*, γ is a *twining automorphism*, and the quotient A_{d-1}/Γ is a *twined singularity*. Here in case σ is the identity, Γ (and γ) is said to be *neat*. We will prove the following (if Γ is neat, this reduces to the uniformization theorem in [SaTa]):

Result 3 (Theorems 8.1, 9.6) Uniformization of twined singularity. *Let Γ be the cyclic group generated by the automorphism γ of A_{d-1} given by* (1.4). *Then there exists a small finite subgroup G of $GL_n(\mathbb{C})$ such that $A_{d-1}/\Gamma \cong \mathbb{C}^n/G$. Here $G = \langle f, g_1, g_2, \ldots, g_{n-1}\rangle$ and*
(i) *f is given as the composition $f = \varphi\psi$, where* (*below, ℓ_k is given in Remark 1.1*)

$$\begin{cases} \varphi : \ (\boldsymbol{X}_1, \ldots, \boldsymbol{X}_l) \longmapsto \left(e^{2\pi \mathrm{i} a_1 \ell_1/cd} \boldsymbol{X}_1^{\sigma_1}, \ldots, e^{2\pi \mathrm{i} a_l \ell_l/cd} \boldsymbol{X}_l^{\sigma_l}\right), \\ \psi : \ (X_1, X_2, \ldots, X_n) \longmapsto (X_1, X_2, \ldots, e^{2\pi \mathrm{i} m_l' \ell_l \kappa/d} X_{\sigma(n)}, \ldots, X_n). \end{cases}$$

(ii) *g_i is given as follows: Say $X_i \in \boldsymbol{X}_k$, then*

$$g_i : (X_1, X_2, \ldots, X_n) \longmapsto (X_1, X_2, \ldots, e^{2\pi \mathrm{i} m_k' \ell_k/d} X_i, \ldots, e^{-2\pi \mathrm{i} m_l' \ell_l/d} X_n).$$

Note: f, g_i denote $\overline{\overline{\gamma}}$, $\overline{\overline{\overline{\mathrm{id}}}}_i$ in Theorem 9.6 and φ, ψ denote $\overline{\overline{\alpha}}$, $\overline{\overline{\overline{\beta}}}_{1,\boldsymbol{q}}$ therein.

Remark 1.1. In Result 3, ℓ_k is the positive integer given in Lemma 7.4, that is, $\ell_k := Nc/n_k m_k L_k$, where $n_k = \mathrm{length}(\boldsymbol{X}_k)$ and L_k is given by (below, $\widecheck{n_k m_k}$ means the omission of $n_k m_k$)

$$L_k := \begin{cases} \mathrm{lcm}(n_1 m_1, n_2 m_2, \ldots, \widecheck{n_k m_k}, \ldots, n_l m_l) & \text{if } \mathrm{length}(\boldsymbol{X}_k) = 1, \\ \mathrm{lcm}(n_1 m_1, n_2 m_2, \ldots, n_l m_l) & \text{if } \mathrm{length}(\boldsymbol{X}_k) \geq 2. \end{cases}$$

Whether G in Result 3 is abelian depends on σ, n, d. In fact:

Result 4 (Theorem 10.6).

(1) *If $\sigma = \mathrm{id}$, then G is always abelian.* (If moreover $n = 2$, G is cyclic ([SaTa] Theorem 2.1, p.682 — originally proved in [Tak])).

(2) *If $\sigma \neq \mathrm{id}$, then G is rarely abelian — in fact only when $n = 2$ and $d = 2$* (*and in which case G is cyclic generated by f in Result 3*).

Result 3 is further enriched. Define a holomorphic map $\Phi : A_{d-1} \to \mathbb{C}$ by $\Phi(x_1, \cdots, x_n, t) = t^N$. Then Φ is Γ-invariant, so descends to a holomorphic map $\overline{\Phi} : A_{d-1}/\Gamma \to \mathbb{C}$.

Result 5 (Theorem 8.3) Correspondence of maps. *As above, let Γ be the cyclic group generated by*

$$\gamma : (\boldsymbol{x}_1, \ldots, \boldsymbol{x}_l, t) \ \longmapsto \ (e^{2\pi \mathrm{i} a_1/n_1 m_1} \boldsymbol{x}_1^{\sigma_1}, \ldots, e^{2\pi \mathrm{i} a_l/n_l m_l} \boldsymbol{x}_l^{\sigma_l}, e^{2\pi \mathrm{i}/N} t).$$

For each σ_k, let J_k be its cycle, that is, $J_k = \{i : x_i \in \boldsymbol{x}_k\}$. Then:

(1) *A holomorphic map $\phi : \mathbb{C}^n \to \mathbb{C}$ given by $\phi(x_1, \ldots, x_n) = \prod_{k=1}^{l} \Big(\prod_{i \in J_k} x_i \Big)^{L_k}$ is G-invariant.*

(2) *Under the isomorphism $A_{d-1}/\Gamma \cong \mathbb{C}^n/G$, $\overline{\Phi} : A_{d-1}/\Gamma \to \mathbb{C}$ corresponds to the descent $\overline{\phi} : \mathbb{C}^n/G \to \mathbb{C}$.*

The topological monodromy of $\overline{\Phi} : A_{d-1}/\Gamma \to \mathbb{C}$ generalizes both a fractional Dehn twist and a Nielsen twist — in a unified way! We call it a *hypertwist* (more precisely, $\Big(\frac{a_1}{n_1 m_1}, \frac{a_2}{n_2 m_2}, \cdots, \frac{a_l}{n_l m_l}, \kappa, \sigma\Big)$-hypertwist). Its action on a smooth fiber of $\overline{\Phi}$ will be described in our subsequent paper.

Acknowledgments. We would like to thank Professor Tadashi Ashikaga for useful discussions.

2. Twining Automorphisms

Let d be a positive integer and consider the multiplicative A-singularity:

$$A_{d-1} := \{(x_1, x_2, \ldots, x_n, t) \in \mathbb{C}^{n+1} : x_1 x_2 \cdots x_n = t^d\}.$$

The automorphism group $\mathrm{Aut}(A_{d-1})$ of A_{d-1} is the subgroup of $GL_{n+1}(\mathbb{C})$ consisting of elements that map A_{d-1} to itself. Now take a cyclic permutation $\sigma \in \mathfrak{S}_n$ of length n and nonzero complex numbers $\alpha_1, \alpha_2, \ldots, \alpha_n$, δ such that $\alpha_1 \alpha_2 \cdots \alpha_n = \delta^d$. Define then an automorphism γ of A_{d-1} by

$$\gamma : (x_1, x_2, \ldots, x_n, t) \ \longmapsto \ (\alpha_1 x_{\sigma(1)}, \alpha_2 x_{\sigma(2)}, \ldots, \alpha_n x_{\sigma(n)}, \delta t). \tag{2.1}$$

LEMMA 2.1. *Let k be an integer. Then $\gamma^k = 1$ if and only if k is a multiple of n and $(\alpha_1\alpha_2\cdots\alpha_n)^{k/n} = 1$ and $\delta^k = 1$.*

PROOF. Note that $\gamma^k : (x_1, \ldots, x_n, t) \mapsto (\mu_1 x_{\sigma^k(1)}, \ldots, \mu_n x_{\sigma^k(n)}, \nu t)$ for some nonzero complex numbers $\mu_1, \ldots, \mu_n, \nu$. If $\gamma^k = 1$, then it is necessary that $\sigma^k = 1$. Since σ is cyclic of length n, this implies that k is a multiple of n. Write $k = nl$, then $\gamma^{nl} = 1$. Here $\gamma^n : (x_1, \ldots, x_n, t) \mapsto (\alpha_1\alpha_2\cdots\alpha_n x_1, \ldots, \alpha_1\alpha_2\cdots\alpha_n x_n, \delta^n t)$, thus $(\alpha_1\alpha_2\cdots\alpha_n)^l = 1$ and $\delta^{nl} = 1$ (that is, $\delta^k = 1$). Conversely, if k is a multiple of n and $(\alpha_1\alpha_2\cdots\alpha_n)^{k/n} = 1$ and $\delta^k = 1$, then $\gamma^k = 1$, indeed

$$\begin{aligned}\gamma^k : (x_1, \ldots, x_n, t) \longmapsto\ & ((\alpha_1\alpha_2\cdots\alpha_n)^{k/n} x_1, \ldots, (\alpha_1\alpha_2\cdots\alpha_n)^{k/n} x_n, \delta^k t) \\ &= (x_1, \ldots, x_n, t). \ \square\end{aligned}$$

COROLLARY 2.2. *The order of γ is finite if and only if $\alpha_1\alpha_2\cdots\alpha_n$ is a root of unity.*

PROOF. $\Longrightarrow$: Say that the order of γ is k. Then from Lemma 2.1, k is a multiple of n and $(\alpha_1\alpha_2\cdots\alpha_n)^{k/n} = 1$; so $\alpha_1\alpha_2\cdots\alpha_n$ is a k/nth root of unity.

$\Longleftarrow$: Say that $\alpha_1\alpha_2\cdots\alpha_n$ is an lth root of unity: $(\alpha_1\alpha_2\cdots\alpha_n)^l = 1$. This and $\alpha_1\alpha_2\cdots\alpha_n = \delta^d$ yield $1 = \delta^{ld}$. Set $k := nld$, then k is a multiple of n and $(\alpha_1\alpha_2\cdots\alpha_n)^{k/n} = 1$ and $\delta^k = 1$, so by Lemma 2.1, $\gamma^k = 1$. $\square$

Note next the following:

LEMMA 2.3. *Let γ be the automorphism of A_{d-1} given by* (2.1). *Then*:

(1) *For an arbitrary nth root β of $\alpha_1\alpha_2\cdots\alpha_n$, $\gamma' : (x_1, \ldots, x_n, t) \mapsto (\beta x_{\sigma(1)}, \ldots, \beta x_{\sigma(n)}, \delta t)$ is an automorphism of A_{d-1}.*

(2) *Let $b_1, b_2, \ldots, b_n, c$ be nonzero complex numbers such that $b_1 b_2 \cdots b_n = c^d$. Define $f \in \operatorname{Aut}(A_{d-1})$ by $f : (x_1, \ldots, x_n, t) \mapsto (b_1 x_1, \ldots, b_n x_n, ct)$. Then*

$$f^{-1} \circ \gamma \circ f : (x_1, \ldots, x_n, t) \longmapsto \Big(\frac{\alpha_1 b_{\sigma(1)}}{b_1} x_{\sigma(1)}, \ldots, \frac{\alpha_n b_{\sigma(n)}}{b_n} x_{\sigma(n)}, \delta t\Big).$$

(3) *γ is conjugate to γ' in* $\mathrm{Aut}(A_{d-1})$.

PROOF. (1): It suffices to show that γ' preserves A_{d-1}, that is, $(\beta x_{\sigma(1)})(\beta x_{\sigma(2)})\cdots(\beta x_{\sigma(n)}) = \delta^d t^d$. This is seen as follows:

$$\begin{aligned}(\beta x_{\sigma(1)})(\beta x_{\sigma(2)})\cdots(\beta x_{\sigma(n)}) &= \beta^n x_1 x_2 \cdots x_n \\ &= \delta^d x_1 x_2 \cdots x_n \qquad \text{by } \beta^n = \alpha_1\alpha_2\cdots\alpha_n = \delta^d \\ &= \delta^d t^d \qquad\qquad\quad \text{by } x_1 x_2 \cdots x_n = t^d.\end{aligned}$$

(2): This is confirmed as follows:

$$\begin{aligned}f^{-1}\circ\gamma\circ f(x_1,\dots,x_n,t) &= f^{-1}\circ\gamma(b_1x_1,\dots,b_nx_n,ct) \\ &= f^{-1}(\alpha_1 b_{\sigma(1)}x_{\sigma(1)},\dots,\alpha_n b_{\sigma(n)}x_{\sigma(n)},\delta ct) \\ &= \Big(\frac{\alpha_1 b_{\sigma(1)}}{b_1}x_{\sigma(1)},\dots,\frac{\alpha_n b_{\sigma(n)}}{b_n}x_{\sigma(n)},\delta t\Big).\end{aligned}$$

(3): In terms of (2), it suffices to show that there exist nonzero complex numbers $b_1, b_2, \dots, b_n, c$ satisfying

(i) $b_1 b_2 \cdots b_n = c^d$,

(ii) $\beta = \dfrac{\alpha_i b_{\sigma(i)}}{b_i}$ $(i = 1, 2, \dots, n)$, that is, $b_{\sigma(i)} = \dfrac{\beta b_i}{\alpha_i}$ $(i = 1, 2, \dots, n)$.

Note that once we show the existence of $b_1, b_2, \dots, b_n$ satisfying (ii), it suffices to take c as dth root of $b_1 b_2 \cdots b_n$.

Since σ is cyclic of length n, we have $\{1, 2, \dots, n\} = \{1, \sigma(1), \dots, \sigma^{n-1}(1)\}$, so (ii) is restated as $b_{\sigma^j(1)} = \dfrac{\beta b_{\sigma^{j-1}(1)}}{\alpha_{\sigma^{j-1}(1)}}$ $(j=1, 2, \dots, n)$. Set $b_1 = 1$ and inductively define $b_{\sigma^j(1)}$ $(j = 1, 2, \dots, n-1)$ by $b_{\sigma^j(1)} := \dfrac{\beta b_{\sigma^{j-1}(1)}}{\alpha_{\sigma^{j-1}(1)}}$. It then suffices to show that $b_1 = \dfrac{\beta b_{\sigma^{n-1}(1)}}{\alpha_{\sigma^{n-1}(1)}}$. Since $\beta = \dfrac{\alpha_{\sigma^{j-1}(1)} b_{\sigma^j(1)}}{b_{\sigma^{j-1}(1)}}$ $(j = 1, 2, \dots, n-1)$, we have $\beta^{n-1} = \displaystyle\prod_{j=1}^{n-1} \frac{\alpha_{\sigma^{j-1}(1)} b_{\sigma^j(1)}}{b_{\sigma^{j-1}(1)}}$. Here $\displaystyle\prod_{j=1}^{n} \frac{\alpha_{\sigma^{j-1}(1)} b_{\sigma^j(1)}}{b_{\sigma^{j-1}(1)}} = \alpha_1\alpha_2\cdots\alpha_n = \beta^n$, so $\displaystyle\prod_{j=1}^{n-1} \frac{\alpha_{\sigma^{j-1}(1)} b_{\sigma^j(1)}}{b_{\sigma^{j-1}(1)}} = \beta^n \frac{b_{\sigma^{n-1}(1)}}{\alpha_{\sigma^{n-1}(1)} b_1}$. Thus $\beta^{n-1} = \beta^n \dfrac{b_{\sigma^{n-1}(1)}}{\alpha_{\sigma^{n-1}(1)} b_1}$, implying that $b_1 = \dfrac{\beta b_{\sigma^{n-1}(1)}}{\alpha_{\sigma^{n-1}(1)}}$. $\square$

Lemma 2.4. *If $\alpha_1\alpha_2\cdots\alpha_n$ is an mth root of unity, then* (1) *δ is a root of unity and* (2) *the order of γ' (also, of γ) is the least common multiple of nm and the order of δ.* (For a kth root of unity, k is called its *order*.)

Proof. (1): By $\alpha_1\alpha_2\cdots\alpha_n = \delta^d$. (2): Since γ' is a linear transformation, it is expressed as $\gamma' : (\boldsymbol{x}, t) \mapsto (B\boldsymbol{x}, \delta t)$, where $\boldsymbol{x} = (x_1, \ldots, x_n)$ and B is an invertible $n \times n$ matrix of order nm. Then $(\gamma')^k : (\boldsymbol{x}, t) \mapsto (B^k\boldsymbol{x}, \delta^k t)$, so the order of γ' is the least common multiple of the orders of B and δ, confirming the assertion. $\square$

General Case. We have discussed the case that $\sigma \in \mathfrak{S}_n$ is a cyclic permutation of length n. In the sequel, $\sigma \in \mathfrak{S}_n$ is *arbitrary*, for which consider the automorphism of A_{d-1} given by

$$\gamma : (x_1, x_2, \ldots, x_n, t) \longmapsto (\alpha_1 x_{\sigma(1)}, \alpha_2 x_{\sigma(2)}, \ldots, \alpha_n x_{\sigma(n)}, \delta t). \tag{2.2}$$

Decompose σ into disjoint cyclic permutations: $\sigma = \sigma_1\sigma_2\cdots\sigma_l$, say the length of σ_i is n_i. Without loss of generality, we assume that σ_1 permutes $\{1, 2, \ldots, n_1\}$, σ_2 permutes $\{n_1+1, n_1+2, \ldots, n_1+n_2\}$, σ_3 permutes $\{n_1 + n_2 + 1, \ldots, n_1+n_2+n_3\}$ and so on; these sets are *cycles* of σ. Write $\mathbb{C}^{n+1}$ as $\mathbb{C}^{n_1} \times \mathbb{C}^{n_2} \times \cdots \times \mathbb{C}^{n_l} \times \mathbb{C}$ and $(x_1, x_2, \ldots, x_n, t) \in \mathbb{C}^{n+1}$ as $(\boldsymbol{x}_1, \boldsymbol{x}_2, \ldots, \boldsymbol{x}_l, t)$, where $\boldsymbol{x}_i \in \mathbb{C}^{n_i}$. Then σ_i acts on $\mathbb{C}^{n_i}$ as a cyclic permutation, and the restriction of γ to $\mathbb{C}^{n_i}$ is of the form:

$$\gamma_i : \ \boldsymbol{x}_i = (x_{j_1}, x_{j_2}, \ldots, x_{j_{n_i}}) \longmapsto (\alpha_{j_1} x_{\sigma_i(j_1)}, \alpha_{j_2} x_{\sigma_i(j_2)}, \ldots, \alpha_{j_{n_i}} x_{\sigma_i(j_{n_i})}).$$

The order of γ is finite if and only if the orders of all γ_i are finite. As in Corollary 2.2, this is restated as follows:

Lemma 2.5. *The order of γ is finite if and only if for every i, $\prod_{j\in J_i} \alpha_j$ is a root of unity, where J_i denotes the cycle of σ_i.*

Note next the following:

Lemma 2.6. *Let γ be the automorphism of A_{d-1} given by* (2.2)*. For each i, let β_i be an arbitrary n_ith root of $\prod_{j\in J_i} \alpha_j$, where J_i denotes the cycle of σ_i. Write J_i as $\{j_1, j_2, \ldots, j_{n_i}\}$ and for $\boldsymbol{x}_i = (x_{j_1}, x_{j_2}, \ldots, x_{j_{n_i}})$, set $\boldsymbol{x}_i^{\sigma_i} := (x_{\sigma_i(j_1)}, x_{\sigma_i(j_2)}, \ldots, x_{\sigma_i(j_{n_i})})$, then:*

(1) *Irrespective of the choice of* β_i, $\beta_1^{n_1}\beta_2^{n_2}\cdots\beta_l^{n_l}$ *is constant. In fact* $\beta_1^{n_1}\beta_2^{n_2}\cdots\beta_l^{n_l}=\delta^d$.

(2) $\gamma' : (\boldsymbol{x}_1,\dots,\boldsymbol{x}_l,t) \mapsto (\beta_1\boldsymbol{x}_1^{\sigma_1},\dots,\beta_l\boldsymbol{x}_l^{\sigma_l},\delta t)$ *is an automorphism of* A_{d-1}.

(3) γ *is conjugate to* γ' *in* $\mathrm{Aut}(A_{d-1})$.

PROOF. (1): $\beta_1^{n_1}\beta_2^{n_2}\cdots\beta_l^{n_l}=\prod_{i=1}^{l}\Big(\prod_{j\in J_i}\alpha_j\Big)=\alpha_1\alpha_2\cdots\alpha_n=\delta^d$.

(2): It suffices to show that γ' preserves A_{d-1}. Temporarily write $\boldsymbol{x}_i$ as $(x_1^{(i)},x_2^{(i)},\dots,x_{n_i}^{(i)})$. By $\boldsymbol{x}_1\cdot\boldsymbol{x}_2\cdots\boldsymbol{x}_l=t^d$, we mean $(x_1^{(1)}\cdots x_{n_1}^{(1)})(x_1^{(2)}\cdots x_{n_2}^{(2)})\cdots(x_1^{(l)}\cdots x_{n_l}^{(l)})=t^d$. We then have to show that $\beta_1\boldsymbol{x}_1^{\sigma_1}\cdot\beta_2\boldsymbol{x}_2^{\sigma_2}\cdots\beta_l\boldsymbol{x}_l^{\sigma_l}=(\delta t)^d$, that is, $(\beta_1x_{\sigma_1(1)}^{(1)}\cdots\beta_1x_{\sigma_1(n_1)}^{(1)})(\beta_2x_{\sigma_2(1)}^{(2)}\cdots\ \beta_2x_{\sigma_2(n_2)}^{(2)})\cdots(\beta_lx_{\sigma_l(1)}^{(l)}\cdots\beta_lx_{\sigma_l(n_l)}^{(l)})=(\delta t)^d$, or (after reordering),

$$\beta_1^{n_1}\beta_2^{n_2}\cdots\beta_l^{n_l}(x_1^{(1)}\cdots x_{n_1}^{(1)})(x_1^{(2)}\cdots x_{n_2}^{(2)})\cdots(x_1^{(l)}\cdots x_{n_l}^{(l)})=\delta^dt^d.$$

This is equivalent to $\beta_1^{n_1}\beta_2^{n_2}\cdots\beta_l^{n_l}=\delta^d$, which is already shown in (1).

(3): The proof is similar to that of Lemma 2.3 (3). Construct first an automorphism $f_i:\mathbb{C}^{n_i}\to\mathbb{C}^{n_i}$, $f_i:\boldsymbol{x}_i=(x_1^{(i)},\dots,x_{n_i}^{(i)})\mapsto(b_1^{(i)}x_1^{(i)},\dots,b_{n_i}^{(i)}x_{n_i}^{(i)})$ such that $f_i^{-1}\circ\gamma_i\circ f_i:\boldsymbol{x}_i\mapsto\beta_i\boldsymbol{x}_i^{\sigma_i}$. Set $\boldsymbol{b}^{(i)}:=\prod_{j=1}^{n_i}b_j^{(i)}$ and take a complex number c satisfying $\boldsymbol{b}^{(1)}\boldsymbol{b}^{(2)}\cdots\boldsymbol{b}^{(l)}=c^d$. Then $f:(\boldsymbol{x}_1,\dots,\boldsymbol{x}_l,t)\mapsto\big(f_1(\boldsymbol{x}_1),\dots,f_l(\boldsymbol{x}_l),ct\big)$ is an automorphism of A_{d-1} such that $\gamma'=f^{-1}\circ\gamma\circ f$. □

LEMMA 2.7. *In Lemma 2.6, if for each* i, $\boldsymbol{\alpha}_i:=\prod_{j\in J_i}\alpha_j$ *is an* m_i*th root of unity, then*:

(1) δ *is a root of unity.*

(2) *The order of* γ' *(and so,* γ*) is finite, in fact it is the least common multiple of* $\mathrm{lcm}(n_1m_1,n_2m_2,\dots,n_lm_l)$ *and the order of* δ.

PROOF. (1) follows from $\boldsymbol{\alpha}_1\boldsymbol{\alpha}_2\cdots\boldsymbol{\alpha}_l=\delta^d$. (2):

For simplicity, express $\gamma' : (\boldsymbol{x}_1, \boldsymbol{x}_2, \ldots, \boldsymbol{x}_l, t) \mapsto (\beta_1 \boldsymbol{x}_1^{\sigma_1}, \beta_2 \boldsymbol{x}_2^{\sigma_2}, \ldots, \beta_l \boldsymbol{x}_l^{\sigma_l}, \delta t)$ as $(\boldsymbol{x}, t) \mapsto (B\boldsymbol{x}, \delta t)$, where $\boldsymbol{x} = (x_1, \ldots, x_n)$ and B is an invertible $n \times n$ matrix of the form

$$B = \begin{pmatrix} B_1 & & & O \\ & B_2 & & \\ & & \ddots & \\ O & & & B_l \end{pmatrix} \quad (B_i \text{ is an invertible } n_i \times n_i \text{ matrix}).$$

Since the order of B_i is $n_i m_i$, the order of B is $\mathrm{lcm}(n_1 m_1, n_2 m_2, \ldots, n_l m_l)$. Noting that $(\gamma')^k : (\boldsymbol{x}, t) \mapsto (B^k \boldsymbol{x}, \delta^k t)$, the order of γ' is the least common multiple of the orders of B and δ, so the assertion holds. $\square$

Corollary 2.8. *If the order of δ is a multiple of* $\mathrm{lcm}(n_1 m_1, n_2 m_2, \ldots, n_l m_l)$*, then the order of γ is that of δ.*

Definition 2.9. Let $\sigma \in \mathfrak{S}_n$ and $\alpha_1, \alpha_2, \ldots, \alpha_n, \delta$ be nonzero complex numbers such that $\alpha_1 \alpha_2 \cdots \alpha_n = \delta^d$. The automorphism of $\gamma : A_{d-1} \to A_{d-1}$ given by $(x_1, \ldots, x_n, t) \mapsto (\alpha_1 x_{\sigma(1)}, \ldots, \alpha_n x_{\sigma(n)}, \delta t)$ is called a *twining automorphism* (a *twiner*) if its order is finite.

3. Lifting and Descent

Let $p : X \to Y$ be a covering. For $f \in \mathrm{Aut}(Y)$, $g \in \mathrm{Aut}(X)$ is called a *lift* of f if the following diagram commutes:

$$\begin{array}{ccc} X & \xrightarrow{\;g\;} & X \\ {\scriptstyle p}\big\downarrow & & \big\downarrow{\scriptstyle p} \\ Y & \xrightarrow{\;f\;} & Y. \end{array}$$

In this case, f is called the *descent* of g. For a subgroup Γ of $\mathrm{Aut}(Y)$, its *lift* $\widetilde{\Gamma}$ is a subgroup of $\mathrm{Aut}(X)$ consisting of all lifts of elements of Γ. In this case, Γ is called the *descent* of $\widetilde{\Gamma}$.

We now return to twining automorphism. Let $\sigma \in \mathfrak{S}_n$ and decompose it into disjoint cyclic permutations: $\sigma = \sigma_1 \sigma_2 \cdots \sigma_l$. Say that the length of σ_i is n_i. Without loss of generality, we may assume that the cycle of σ_1 is $\{1, 2, \ldots, n_1\}$, the cycle of σ_2 is $\{n_1 + 1, \ldots, n_1 + n_2\}$, the cycle of σ_3 is

$\{n_1+n_2+1,\dots,n_1+n_2+n_3\}$ and so on. Write $\mathbb{C}^n$ as $\mathbb{C}^{n_1}\times\mathbb{C}^{n_2}\times\cdots\times\mathbb{C}^{n_l}$ and $(x_1,x_2,\dots,x_n)\in\mathbb{C}^n$ as $(\boldsymbol{x}_1,\boldsymbol{x}_2,\dots,\boldsymbol{x}_l)$. Let σ_i act on $\mathbb{C}^{n_i}$ as

$$\sigma_i : \boldsymbol{x}_i=(x_{j_1},x_{j_2},\dots,x_{j_{n_i}}) \longmapsto \boldsymbol{x}_i^{\sigma_i}:=(x_{\sigma_i(j_1)},x_{\sigma_i(j_2)},\dots,x_{\sigma_i(j_{n_i})}).$$

Consider the following automorphism of $\mathbb{C}^{n+1}$ given by

$$\gamma:\ (\boldsymbol{x}_1,\dots,\boldsymbol{x}_l,t)\longmapsto(e^{2\pi \mathrm{i}a_1/n_1m_1}\boldsymbol{x}_1^{\sigma_1},\dots,e^{2\pi \mathrm{i}a_l/n_lm_l}\boldsymbol{x}_l^{\sigma_l},e^{2\pi \mathrm{i}/N}t), \tag{3.1}$$

where

(I) a_i, m_i are positive integers such that a_i is relatively prime to n_im_i (where n_i is the length of σ_i).

(II) $N:=(m_1')^{n_1}\cdots(m_l')^{n_l}c$, where $c:=\gcd(n_1m_1,\dots,n_lm_l)$ and $m_i':=\dfrac{n_im_i}{c}$.

Note that γ preserves $A_{d-1}=\{(x_1,\dots,x_n,t)\in\mathbb{C}^{n+1}\ :\ x_1\cdots x_n=t^d\}$ precisely when $d=N\Big(\dfrac{a_1}{m_1}+\cdots+\dfrac{a_l}{m_l}+\kappa\Big)$ for some integer κ (see (iii) subsequent to (1.4)). *In what follows, we assume this.* Then:

LEMMA 3.1.

(1) *The order of γ is N.*

(2) *Let Γ be the cyclic group generated by γ. Then the holomorphic map $\Phi : A_{d-1}\to\mathbb{C}$ given by $\Phi(x_1,\cdots,x_n,t)=t^N$ is Γ-invariant. Consequently Φ descends to $\overline{\Phi}: A_{d-1}/\Gamma\to\mathbb{C}$.*

PROOF. (1): Since the order N of δ is a multiple of $\mathrm{lcm}(n_1m_1,\dots,n_lm_l)$ (see (II)), this follows from Corollary 2.8.

(2): For any $(x_1,\dots,x_n,t)\in A_{d-1}$, $\Phi\circ\gamma(x_1,\cdots,x_n,t)=(\delta t)^N=\delta^Nt^N=t^N$, so $\Phi\circ\gamma=\Phi$. $\square$

Since the order of γ is finite, γ is a twining automorphism and Γ is a twining automorphism group. If the permutation σ is the identity, Γ (and γ) is said to be *neat*, in which case $\boldsymbol{x}_i=x_i$, so γ is of the form

$$(x_1,\dots,x_n,t)\longmapsto(e^{2\pi \mathrm{i}a_1/m_1}x_1,\dots,e^{2\pi \mathrm{i}a_n/m_n}x_n,e^{2\pi \mathrm{i}/N}t).$$

For such γ, [SaTa] showed that there exists a small finite subgroup $G \subset GL_n(\mathbb{C})$ such that $A_{d-1}/\Gamma \cong \mathbb{C}^n/G$; moreover the holomorphic map $\mathbb{C}^n/G \to \mathbb{C}$ corresponding to $\overline{\Phi}$ (in Lemma 3.1) under this isomorphism is explicitly given. We will generalize these results (and more) to arbitrary γ. The construction of G is outlined as follows:

(i) Let $p : \widetilde{A}_{d-1}\ (= \mathbb{C}^n) \to A_{d-1}$ be the universal covering, and lift Γ to a group $\widetilde{\Gamma}$ acting on $\widetilde{A}_{d-1}$. Then $A_{d-1}/\Gamma \cong \widetilde{A}_{d-1}/\widetilde{\Gamma}$. If $m'_1 = m'_2 = \cdots = m'_l = 1$ (e.g. $n = 2$ and Γ is not neat), then $\widetilde{\Gamma}$ is small. Thus $\widetilde{\Gamma}$ is the desired G.

(ii) If the condition in (i) is not satisfied, let $q : \widetilde{A}_{d-1} \to \mathbb{C}^n$ be the covering map given by $q(\boldsymbol{X}_1, \boldsymbol{X}_2, \ldots, \boldsymbol{X}_l) = (\boldsymbol{X}_1^{m'_1}, \boldsymbol{X}_2^{m'_2}, \ldots, \boldsymbol{X}_l^{m'_l})$, where $\boldsymbol{X}_i^{m'_i} := (X_{j_1}^{m'_i}, \ldots, X_{j_{n_i}}^{m'_i})$, and descend $\widetilde{\Gamma}$ to a group H acting on $\mathbb{C}^n$.

Then $A_{d-1}/\Gamma \cong \widetilde{A}_{d-1}/\widetilde{\Gamma} \cong \mathbb{C}^n/H$. If $n = 2$ and Γ *is* neat, then H is a small finite group,

(iii) In (ii), if $n \geq 3$ then H is generally *not* small, in which case take the *pseudo-reflection subgroup* P of H (i.e. the subgroup generated by all pseudo-reflections in H). It is normal in H and the quotient group H/P is small and $A_{d-1}/\Gamma \cong \widetilde{A}_{d-1}/\widetilde{\Gamma} \cong \mathbb{C}^n/H \cong (\mathbb{C}^n/P)\big/(H/P) \cong \mathbb{C}^n\big/(H/P)$ (because $\mathbb{C}^n/P \cong \mathbb{C}^n$ by Chevalley-Shephard-Todd theorem). Thus H/P is the desired G.

We give some comments on the above construction:

(a) In (ii), whether H is small is *numerically* determined (Theorem 7.2).

(b) In (iii), the quotient map $H \to H/P$ is the descent of H with respect to an *explicitly-given* covering map $r : \mathbb{C}^n \to \mathbb{C}^n$ whose covering transformation group is P. See Lemma 7.1.

(c) $\widetilde{\Gamma}$ and H are generally *not* abelian, which makes the above construction much more involved than that of [SaTa].

The construction of G is systematically described in terms of lifting and descent with respect to the following diagram:

$$
\begin{tikzcd}
& & \widetilde{A}_{d-1} = \mathbb{C}^n \arrow[dl, "q"'] \arrow[dr, "p"] & \\
& \mathbb{C}^n \arrow[dl, "r"'] & & A_{d-1}. \\
\mathbb{C}^n & & &
\end{tikzcd}
\tag{3.2}
$$

4. Determination of $\widetilde{\Gamma}$ and H

Consider a twining automorphism $\gamma : A_{d-1} \to A_{d-1}$ of order N:

$$\gamma :\ (\boldsymbol{x}_1, \dots, \boldsymbol{x}_l, t) \ \longmapsto\ (e^{2\pi \mathrm{i} a_1/n_1 m_1} \boldsymbol{x}_1^{\sigma_1}, \dots, e^{2\pi \mathrm{i} a_l/n_l m_l} \boldsymbol{x}_l^{\sigma_l}, e^{2\pi \mathrm{i}/N} t),$$

where σ_i is a cyclic permutation of length n_i $(n_1 + n_2 + \cdots + n_l = n)$ and

$$(\boldsymbol{x}_1, \boldsymbol{x}_2, \dots, \boldsymbol{x}_l) \in \mathbb{C}^{n_1} \times \mathbb{C}^{n_2} \times \cdots \times \mathbb{C}^{n_l}. \tag{4.1}$$

For each γ^j $(j = 1, 2, \dots, N)$, we determine its lifts with respect to $p : \widetilde{A}_{d-1} \to A_{d-1}$, first for $j = 1$. To that end, express γ as the product of the *x-part* and the *t-part*: $\gamma = \gamma_x \gamma_t$ $(= \gamma_t \gamma_x)$, where

$$\gamma_x :\ (\boldsymbol{x}_1, \dots, \boldsymbol{x}_l, t) \ \longmapsto\ (e^{2\pi \mathrm{i} a_1/n_1 m_1} \boldsymbol{x}_1^{\sigma_1}, \dots, e^{2\pi \mathrm{i} a_l/n_l m_l} \boldsymbol{x}_l^{\sigma_l}, e^{2\pi \mathrm{i}(1/N - \kappa/d)} t),$$

$$\gamma_t :\ (\boldsymbol{x}_1, \dots, \boldsymbol{x}_l, t) \ \longmapsto\ (\boldsymbol{x}_1, \dots, \boldsymbol{x}_l, e^{2\pi \mathrm{i}\kappa/d} t).$$

The lifts of γ_x and γ_t are easy to describe. In what follows, to be consistent with the notation $(\boldsymbol{x}_1, \boldsymbol{x}_2, \dots, \boldsymbol{x}_l, t) \in A_{d-1}$, write $(X_1, X_2, \dots, X_n) \in \widetilde{A}_{d-1}$ $(= \mathbb{C}^n)$ as $(\boldsymbol{X}_1, \boldsymbol{X}_2, \dots, \boldsymbol{X}_l)$, where $\boldsymbol{X}_i \in \mathbb{C}^{n_i}$.

LEMMA 4.1. *A lift of γ_x is given by an automorphism $\widetilde{\gamma}_x : \widetilde{A}_{d-1} \to \widetilde{A}_{d-1}$ defined by*

$$
\begin{aligned}
&(\boldsymbol{X}_1, \boldsymbol{X}_2, \dots, \boldsymbol{X}_l) \\
&\qquad \mapsto \bigl(e^{2\pi \mathrm{i} a_1/n_1 m_1 d} \boldsymbol{X}_1^{\sigma_1}, e^{2\pi \mathrm{i} a_2/n_2 m_2 d} \boldsymbol{X}_2^{\sigma_2}, \dots, e^{2\pi \mathrm{i} a_l/n_l m_l d} \boldsymbol{X}_l^{\sigma_l}\bigr).
\end{aligned}
$$

PROOF. Since $p(X_1, X_2, \dots, X_n) = (X_1^d, X_2^d, \dots, X_n^d, X_1 X_2 \cdots X_n)$, $\widetilde{\gamma}_x$ descends to an automorphism of A_{d-1} that maps $(\boldsymbol{x}_1, \dots, \boldsymbol{x}_l, t)$ to

$$
\begin{aligned}
&\Bigl((e^{2\pi \mathrm{i} a_1/n_1 m_1 d})^d \boldsymbol{x}_1^{\sigma_1}, \dots, (e^{2\pi \mathrm{i} a_l/n_l m_l d})^d \boldsymbol{x}_l^{\sigma_l}, \\
&\qquad\qquad (e^{2\pi \mathrm{i} a_1/n_1 m_1 d})^{n_1} \cdots (e^{2\pi \mathrm{i} a_l/n_l m_l d})^{n_l} t\Bigr),
\end{aligned}
$$

that is, to $\big(e^{2\pi\mathrm{i}a_1/n_1m_1}\boldsymbol{x}_1^{\sigma_1}, \dots, e^{2\pi\mathrm{i}a_l/n_lm_l}\boldsymbol{x}_l^{\sigma_l}, e^{2\pi\mathrm{i}(a_1/m_1+\cdots+a_l/m_l)/d}t\big)$. Here since $\dfrac{a_1}{m_1}+\cdots+\dfrac{a_l}{m_l}=\dfrac{d}{N}-\kappa$, $e^{2\pi\mathrm{i}(a_1/m_1+\cdots+a_l/m_l)/d}=e^{2\pi\mathrm{i}(1/N-\kappa/d)}$. Thus $\widetilde{\gamma}_x$ descends to γ_x. □

Consider the set Λ of $(p_1, p_2, \dots, p_n)\in\mathbb{Z}^n$ satisfying $0\le p_i<d$ $(i=1,2,\dots,n)$ and

$$\frac{p_1+p_2+\cdots+p_n}{d}\equiv\frac{\kappa}{d} \bmod \mathbb{Z}. \tag{4.2}$$

Observe that the number of elements of Λ is d^{n-1}, as p_n is determined from $(p_1, p_2, \dots, p_{n-1})$ $(0\le p_i<d)$ by (4.2).

We determine the lifts of γ_t. To be consistent with the notation $(\boldsymbol{X}_1, \boldsymbol{X}_2, \dots, \boldsymbol{X}_l)\in\mathbb{C}^n$, write $(p_1, p_2, \dots, p_n)$ as $\boldsymbol{p}=(\boldsymbol{p}_1, \boldsymbol{p}_2, \dots, \boldsymbol{p}_l)$, where $\boldsymbol{p}_i\in\mathbb{Z}^{n_i}$.

Lemma 4.2. *Define an automorphism of* $\widetilde{A}_{d-1}$ *by*

$$\widetilde{\gamma}_{t,\boldsymbol{p}}:(\boldsymbol{X}_1, \boldsymbol{X}_2, \dots, \boldsymbol{X}_l)\longmapsto\big(\widetilde{\gamma}_{t,\boldsymbol{p}_1}(\boldsymbol{X}_1), \widetilde{\gamma}_{t,\boldsymbol{p}_2}(\boldsymbol{X}_2), \dots, \widetilde{\gamma}_{t,\boldsymbol{p}_l}(\boldsymbol{X}_l)\big), \tag{4.3}$$

where $\widetilde{\gamma}_{t,\boldsymbol{p}_i}:\boldsymbol{X}_i=(X_{j_1}, \dots, X_{j_{n_i}})\mapsto(e^{2\pi\mathrm{i}p_{j_1}/d}X_{j_1}, \dots, e^{2\pi\mathrm{i}p_{j_{n_i}}/d}X_{j_{n_i}})$. *Then* $\widetilde{\gamma}_{t,\boldsymbol{p}}$ *is a lift of* γ_t. *Moreover* $\{\widetilde{\gamma}_{t,\boldsymbol{p}}:\boldsymbol{p}\in\Lambda\}$ *exhausts all lifts of* γ_t.

Proof. Since $p(X_1, X_2, \dots, X_n)=(X_1^d, X_2^d, \dots, X_n^d, X_1X_2\cdots X_n)$, $\widetilde{\gamma}_{t,\boldsymbol{p}}$ descends to an automorphism of A_{d-1} that maps $(\boldsymbol{x}_1, \dots, \boldsymbol{x}_l, t)$ to

$$\Big((\widetilde{\gamma}_{t,\boldsymbol{p}_1})^d(\boldsymbol{x}_1), \dots, (\widetilde{\gamma}_{t,\boldsymbol{p}_l})^d(\boldsymbol{x}_l), (e^{2\pi\mathrm{i}p_1/d})(e^{2\pi\mathrm{i}p_2/d})\cdots(e^{2\pi\mathrm{i}p_n/d})t\Big),$$

that is, to $\big(\boldsymbol{x}_1, \dots, \boldsymbol{x}_l, e^{2\pi\mathrm{i}(p_1+p_2+\cdots+p_n)/d}t\big)$. Here by (4.2), $e^{2\pi\mathrm{i}(p_1+p_2+\cdots+p_n)/d}=e^{2\pi\mathrm{i}\kappa/d}$. Thus $\widetilde{\gamma}_{t,\boldsymbol{p}}$ descends to γ_t. We next show that $\{\widetilde{\gamma}_{t,\boldsymbol{p}}:\boldsymbol{p}\in\Lambda\}$ exhausts all lifts of γ_t. As p is d^{n-1}-fold, it suffices to show that the cardinality of this set is d^{n-1}. This is clear, as Λ consists of d^{n-1} elements and $\widetilde{\gamma}_{t,\boldsymbol{p}}\neq\widetilde{\gamma}_{t,\boldsymbol{p}'}$ for $\boldsymbol{p}\neq\boldsymbol{p}'$. □

Corollary 4.3. $\widetilde{\gamma}_x\widetilde{\gamma}_{t,\boldsymbol{p}}$ *is a lift of* γ. *Moreover* $\{\widetilde{\gamma}_x\widetilde{\gamma}_{t,\boldsymbol{p}}:\boldsymbol{p}\in\Lambda\}$ *exhausts all lifts of* γ.

Proof. $\widetilde{\gamma}_x\widetilde{\gamma}_{t,\boldsymbol{p}}$ descends to $\gamma_x\gamma_t$, i.e. γ. We show that $\{\widetilde{\gamma}_x\widetilde{\gamma}_{t,\boldsymbol{p}}:\boldsymbol{p}\in\Lambda\}$ exhausts all lifts of γ. As $p:\widetilde{A}_{d-1}\to A_{d-1}$ is d^{n-1}-fold, it suffices to show that the cardinality of this set is d^{n-1}. This is clear, as Λ consists of d^{n-1} elements and $\widetilde{\gamma}_{t,\boldsymbol{p}} \neq \widetilde{\gamma}_{t,\boldsymbol{p}'}$ for $\boldsymbol{p} \neq \boldsymbol{p}'$. □

We next determine all lifts of γ^j by replacing γ_x, γ_t with γ_x^j, γ_t^j in the above argument. First from $\gamma = \gamma_x\gamma_t$, we have $\gamma^j = \gamma_x^j\gamma_t^j$. Here since $\widetilde{\gamma}_x$ is a lift of γ_x (Lemma 4.1),

LEMMA 4.4. *$\widetilde{\gamma}_x^j$ is a lift of γ_x^j.*

We next determine lifts of γ_t^j. First for each $j = 1, 2, \ldots, N\ (= \mathrm{ord}(\gamma))$, set

$$\Lambda^{(j)} = \left\{(p_1, p_2, \ldots, p_n) \in \mathbb{Z}^n : 0 \leq p_i < d,\ \sum_{i=1}^{n} \frac{p_i}{d} \equiv \frac{j\kappa}{d} \bmod \mathbb{Z}\right\}. \tag{4.4}$$

We write $(p_1, p_2, \ldots, p_n)$ as $\boldsymbol{p} = (\boldsymbol{p}_1, \boldsymbol{p}_2 \ldots, \boldsymbol{p}_l) \in \mathbb{Z}^{n_1} \times \mathbb{Z}^{n_2} \times \cdots \times \mathbb{Z}^{n_l}$; note $n_1 + n_2 + \cdots + n_l = n$. As for Lemma 4.2, we can show:

LEMMA 4.5. *For $\boldsymbol{p} = (\boldsymbol{p}_1, \boldsymbol{p}_2 \ldots, \boldsymbol{p}_l) \in \Lambda^{(j)}$, let $\widetilde{\gamma}_{t,\boldsymbol{p}_i}$ be the automorphism of $\mathbb{C}^{n_i}$ in Lemma 4.2 and define an automorphism of $\widetilde{A}_{d-1}$ by*

$$\widetilde{\gamma}_{t,\boldsymbol{p}}^{(j)} : (\boldsymbol{X}_1, \boldsymbol{X}_2, \ldots, \boldsymbol{X}_l) \longmapsto \big(\widetilde{\gamma}_{t,\boldsymbol{p}_1}(\boldsymbol{X}_1), \widetilde{\gamma}_{t,\boldsymbol{p}_2}(\boldsymbol{X}_2), \ldots, \widetilde{\gamma}_{t,\boldsymbol{p}_l}(\boldsymbol{X}_l)\big). \tag{4.5}$$

Then $\widetilde{\gamma}_{t,\boldsymbol{p}}^{(j)}$ is a lift of γ_t^j. Moreover $\{\widetilde{\gamma}_{t,\boldsymbol{p}}^{(j)} : \boldsymbol{p} \in \Lambda^{(j)}\}$ exhausts all lifts of γ_t^j.

As for Corollary 4.3, we can show:

COROLLARY 4.6. *For $\boldsymbol{p} \in \Lambda^{(j)}$, let $\widetilde{\gamma}_{t,\boldsymbol{p}}^{(j)} : \widetilde{A}_{d-1} \to \widetilde{A}_{d-1}$ be the lift of γ_t^j given by (4.5). Then $\widetilde{\gamma}_x^j\widetilde{\gamma}_{t,\boldsymbol{p}}^{(j)}$ is a lift of γ^j. Moreover $\{\widetilde{\gamma}_x^j\widetilde{\gamma}_{t,\boldsymbol{p}}^{(j)} : \boldsymbol{p} \in \Lambda^{(j)}\}$ exhausts all lifts of γ^j.*

Let Γ be the cyclic group of order N generated by γ and $\widetilde{\Gamma}$ be the lift of Γ with respect to $p : \widetilde{A}_{d-1} \to A_{d-1}$. By Corollary 4.6, the set of lifts of $\gamma^j \in \Gamma$ is given by $\mathrm{Lift}^{(j)} := \{\widetilde{\gamma}_x^j\widetilde{\gamma}_{t,\boldsymbol{p}}^{(j)} : \boldsymbol{p} \in \Lambda^{(j)}\}$. Since $\widetilde{\Gamma} = \bigcup_{i=1}^{N} \mathrm{Lift}^{(j)}$, we obtain the following:

PROPOSITION 4.7. *The lift $\widetilde{\Gamma}$ of Γ with respect to p is given by*

$$\left\{\widetilde{\gamma}_x^j\widetilde{\gamma}_{t,\boldsymbol{p}}^{(j)} \,:\, \boldsymbol{p}\in\Lambda^{(j)},\ j=1,2,\dots,N\right\}. \tag{4.6}$$

For $\boldsymbol{p}=(\boldsymbol{p}_1,\boldsymbol{p}_2,\dots,\boldsymbol{p}_l)\in\mathbb{Z}^{n_1}\times\mathbb{Z}^{n_2}\times\cdots\times\mathbb{Z}^{n_l}$ and $\sigma=\sigma_1\sigma_2\cdots\sigma_l\in\mathfrak{S}_{n_1}\times\mathfrak{S}_{n_2}\times\cdots\times\mathfrak{S}_{n_l}$, set $\sigma(\boldsymbol{p}):=\big(\sigma_1(\boldsymbol{p}_1),\,\sigma_2(\boldsymbol{p}_2),\dots,\sigma_l(\boldsymbol{p}_l)\big)$.

LEMMA 4.8. *Let $\sigma=\sigma_1\sigma_2\cdots\sigma_l$ be the permutation appearing in the definition of γ. For $\boldsymbol{p}\in\Lambda^{(j)}$, set $\boldsymbol{q}:=\sigma^{-j}(\boldsymbol{p})$. Then $\boldsymbol{q}\in\Lambda^{(j)}$ and $\widetilde{\gamma}_{t,\boldsymbol{p}}^{(j)}\widetilde{\gamma}_x^j=\widetilde{\gamma}_x^j\widetilde{\gamma}_{t,\boldsymbol{q}}^{(j)}$.*

PROOF. Since $\boldsymbol{q}$ is a permutation of $\boldsymbol{p}$, $\{q_1,q_2,\dots,q_n\}=\{p_1,p_2,\dots,p_n\}$ as sets, so $q_1+q_2+\cdots+q_n=p_1+p_2+\cdots+p_n$. In particular

$$\begin{aligned}\frac{q_1+q_2+\cdots+q_n}{d}&=\frac{p_1+p_2+\cdots+p_n}{d}\\&\equiv\frac{j\kappa}{d}\bmod\mathbb{Z}.\end{aligned}$$

Hence $\boldsymbol{q}\in\Lambda^{(j)}$. We next show $\widetilde{\gamma}_{t,\boldsymbol{p}}^{(j)}\widetilde{\gamma}_x^j=\widetilde{\gamma}_x^j\widetilde{\gamma}_{t,\boldsymbol{q}}^{(j)}$. Note that

$$\begin{aligned}\Big((\widetilde{\gamma}_{t,\boldsymbol{q}_i})(\boldsymbol{X}_i)\Big)^{\sigma_i^j}&=(e^{2\pi\mathrm{i}q_{j_1}/d}X_{j_1},\dots,e^{2\pi\mathrm{i}q_{j_{n_i}}/d}X_{j_{n_i}})^{\sigma_i^j}\\&=(e^{2\pi\mathrm{i}p_{j_1}/d}X_{\sigma_i^j(j_1)},\dots,e^{2\pi\mathrm{i}p_{j_{n_i}}/d}X_{\sigma_i^j(j_{n_i})})\quad\text{as }\sigma_i^j(\boldsymbol{q}_i)=\boldsymbol{p}_i\\&=\widetilde{\gamma}_{t,\boldsymbol{p}_i}(X_{\sigma_i^j(j_1)},\dots X_{\sigma_i^j(j_{n_i})})=\widetilde{\gamma}_{t,\boldsymbol{p}_i}(\boldsymbol{X}_i^{\sigma_i^j}).\end{aligned}$$

Then for any $\boldsymbol{X}:=(\boldsymbol{X}_1,\dots,\boldsymbol{X}_l)\in\widetilde{A}_{d-1}$,

$$\begin{aligned}\widetilde{\gamma}_x^j\widetilde{\gamma}_{t,\boldsymbol{q}}^{(j)}(\boldsymbol{X})&=\Big(e^{2\pi\mathrm{i}ja_1/n_1m_1}\Big((\widetilde{\gamma}_{t,\boldsymbol{q}_1})(\boldsymbol{X}_1)\Big)^{\sigma_1^j},\dots,e^{2\pi\mathrm{i}ja_l/n_lm_l}\Big((\widetilde{\gamma}_{t,\boldsymbol{q}_l})(\boldsymbol{X}_1)\Big)^{\sigma_l^j}\Big)\\&=\big(e^{2\pi\mathrm{i}ja_1/n_1m_1}\widetilde{\gamma}_{t,\boldsymbol{p}_1}(\boldsymbol{X}_1^{\sigma_1^j}),\dots,e^{2\pi\mathrm{i}ja_l/n_lm_l}\widetilde{\gamma}_{t,\boldsymbol{p}_l}(\boldsymbol{X}_l^{\sigma_l^j})\big)\\&=\big(\widetilde{\gamma}_{t,\boldsymbol{p}_1}(e^{2\pi\mathrm{i}ja_1/n_1m_1}\boldsymbol{X}_1^{\sigma_1^j}),\dots,\widetilde{\gamma}_{t,\boldsymbol{p}_l}(e^{2\pi\mathrm{i}ja_l/n_lm_l}\boldsymbol{X}_l^{\sigma_l^j})\big)\\&=\widetilde{\gamma}_{t,\boldsymbol{p}}^{(j)}\widetilde{\gamma}_x^j(\boldsymbol{X}).\ \square\end{aligned}$$

We will give a necessary condition for $\widetilde{\Gamma}$ to be abelian. Recall first that for $\boldsymbol{p} = (p_1, \ldots, p_n) \in \Lambda^{(j)}$, the automorphism $\widetilde{\gamma}_{t,\boldsymbol{p}}^{(j)}$ is given by

$$\widetilde{\gamma}_{t,\boldsymbol{p}}^{(j)} : (X_1, \ldots, X_n) \longmapsto (e^{2\pi \mathrm{i} p_1/d} X_1, \ldots, e^{2\pi \mathrm{i} p_n/d} X_n).$$

Thus the following holds:

$$\begin{cases} (*) \ \ \widetilde{\gamma}_{t,\boldsymbol{p}}^{(j)} \widetilde{\gamma}_{t,\boldsymbol{p}'}^{(j')} = \widetilde{\gamma}_{t,\boldsymbol{p}'}^{(j')} \widetilde{\gamma}_{t,\boldsymbol{p}}^{(j)} \text{ for any } \boldsymbol{p} \in \Lambda^{(j)}, \ \boldsymbol{p}' \in \Lambda^{(j')}, \\ (**) \ \ \widetilde{\gamma}_{t,\boldsymbol{p}}^{(j)} = \widetilde{\gamma}_{t,\boldsymbol{p}'}^{(j')} \iff \boldsymbol{p} = \boldsymbol{p}'. \end{cases} \tag{4.7}$$

LEMMA 4.9. *If $\widetilde{\Gamma}$ is abelian, then $\sigma(\boldsymbol{p}) = \boldsymbol{p}$ for any $\boldsymbol{p} \in \Lambda^{(N)}$.* (Actually the converse holds (Proposition 10.9).)

PROOF. Taking auxiliary $\boldsymbol{q} \in \Lambda^{(1)}$, set $\eta_1 := \widetilde{\gamma}_x^N \widetilde{\gamma}_{t,\boldsymbol{p}}^{(N)}$, $\eta_2 := \widetilde{\gamma}_x \widetilde{\gamma}_{t,\boldsymbol{q}}^{(1)} \in \widetilde{\Gamma}$. If $\widetilde{\Gamma}$ is abelian, then $\eta_1\eta_2 = \eta_2\eta_1$. Here

$$\begin{cases} \eta_1\eta_2 = \widetilde{\gamma}_x^N \big(\widetilde{\gamma}_{t,\boldsymbol{p}}^{(N)} \widetilde{\gamma}_x\big) \widetilde{\gamma}_{t,\boldsymbol{q}}^{(1)} = \widetilde{\gamma}_x^N \big(\widetilde{\gamma}_x \widetilde{\gamma}_{t,\sigma^{-1}(\boldsymbol{p})}^{(N)}\big) \widetilde{\gamma}_{t,\boldsymbol{q}}^{(1)} & \text{by Lemma 4.8,} \\ \eta_2\eta_1 = \widetilde{\gamma}_x \big(\widetilde{\gamma}_{t,\boldsymbol{q}}^{(1)} \widetilde{\gamma}_x^N\big) \widetilde{\gamma}_{t,\boldsymbol{p}}^{(N)} = \widetilde{\gamma}_x \big(\widetilde{\gamma}_x^N \widetilde{\gamma}_{t,\sigma^{-N}(\boldsymbol{q})}^{(1)}\big) \widetilde{\gamma}_{t,\boldsymbol{p}}^{(N)} & \text{by Lemma 4.8.} \end{cases}$$

Thus:

$$\begin{aligned} \eta_1\eta_2 = \eta_2\eta_1 &\iff \widetilde{\gamma}_x^{N+1} \widetilde{\gamma}_{t,\sigma^{-1}(\boldsymbol{p})}^{(N)} \widetilde{\gamma}_{t,\boldsymbol{q}}^{(1)} = \widetilde{\gamma}_x^{N+1} \widetilde{\gamma}_{t,\sigma^{-N}(\boldsymbol{q})}^{(1)} \widetilde{\gamma}_{t,\boldsymbol{p}}^{(N)} \\ &\iff \widetilde{\gamma}_{t,\sigma^{-1}(\boldsymbol{p})}^{(N)} \widetilde{\gamma}_{t,\boldsymbol{q}}^{(1)} = \widetilde{\gamma}_{t,\sigma^{-N}(\boldsymbol{q})}^{(1)} \widetilde{\gamma}_{t,\boldsymbol{p}}^{(N)} \\ &\iff \widetilde{\gamma}_{t,\sigma^{-1}(\boldsymbol{p})}^{(N)} \widetilde{\gamma}_{t,\boldsymbol{q}}^{(1)} = \widetilde{\gamma}_{t,\boldsymbol{q}}^{(1)} \widetilde{\gamma}_{t,\boldsymbol{p}}^{(N)} \quad \text{as } \sigma^{-N} = \mathrm{id} \\ &\iff \widetilde{\gamma}_{t,\sigma^{-1}(\boldsymbol{p})}^{(N)} \widetilde{\gamma}_{t,\boldsymbol{q}}^{(1)} = \widetilde{\gamma}_{t,\boldsymbol{p}}^{(N)} \widetilde{\gamma}_{t,\boldsymbol{q}}^{(1)} \quad \text{by } (*) \text{ of } (4.7) \\ &\iff \widetilde{\gamma}_{t,\sigma^{-1}(\boldsymbol{p})}^{(N)} = \widetilde{\gamma}_{t,\boldsymbol{p}}^{(N)} \\ &\iff \sigma^{-1}(\boldsymbol{p}) = \boldsymbol{p} \quad \text{by } (**) \text{ of } (4.7). \ \square \end{aligned}$$

We next determine the descent H of $\widetilde{\Gamma}$ with respect to the covering $q : \widetilde{A}_{d-1} \to \mathbb{C}^n$ given by $q(\boldsymbol{X}_1, \boldsymbol{X}_2, \ldots, \boldsymbol{X}_l) = (\boldsymbol{X}_1^{m_1'}, \boldsymbol{X}_2^{m_2'}, \ldots, \boldsymbol{X}_l^{m_l'})$. For simplicity, set $\alpha := \gamma_x$, $\beta := \gamma_t$ and $\widetilde{\alpha} := \widetilde{\gamma}_x$, $\widetilde{\beta}_{j,\boldsymbol{p}} := \widetilde{\gamma}_{t,\boldsymbol{p}}^{(j)}$, where $\boldsymbol{p} =$

$(\boldsymbol{p}_1, \boldsymbol{p}_2, \ldots, \boldsymbol{p}_l)$. The latter pair is explicitly given by (see Lemma 4.1 and (4.5)):

$$
\begin{aligned}
&\widetilde{\alpha} : (\boldsymbol{X}_1, \ldots, \boldsymbol{X}_l) \longmapsto \bigl(e^{2\pi \mathrm{i} a_1/n_1 m_1 d} \boldsymbol{X}_1^{\sigma_1}, \ldots, e^{2\pi \mathrm{i} a_l/n_l m_l d} \boldsymbol{X}_l^{\sigma_l}\bigr), \\
&\widetilde{\beta}_{j,\boldsymbol{p}} : (\boldsymbol{X}_1, \boldsymbol{X}_2, \ldots, \boldsymbol{X}_l) \\
&\qquad\qquad \longmapsto \bigl(\widetilde{\beta}_{j,\boldsymbol{p}_1}(\boldsymbol{X}_1), \widetilde{\beta}_{j,\boldsymbol{p}_2}(\boldsymbol{X}_2), \ldots, \widetilde{\beta}_{j,\boldsymbol{p}_l}(\boldsymbol{X}_l)\bigr),
\end{aligned}
\tag{4.8}
$$

where we set $\widetilde{\beta}_{j,\boldsymbol{p}_k} := \widetilde{\gamma}_{t,\boldsymbol{p}_k}$. Since $q(\boldsymbol{X}_1, \boldsymbol{X}_2, \ldots, \boldsymbol{X}_l) = (\boldsymbol{X}_1^{m'_1}, \boldsymbol{X}_2^{m'_2}, \ldots, \boldsymbol{X}_l^{m'_l})$, the following holds:

Lemma 4.10. *The descents $\overline{\alpha}$, $\overline{\beta}_{j,\boldsymbol{p}}$ of $\widetilde{\alpha}$, $\widetilde{\beta}_{j,\boldsymbol{p}}$ with respect to q are explicitly given by*

$$
\begin{aligned}
&\overline{\alpha} : (\boldsymbol{u}_1, \boldsymbol{u}_2, \ldots, \boldsymbol{u}_l) \longmapsto \bigl(e^{2\pi \mathrm{i} a_1/cd} \boldsymbol{u}_1^{\sigma_1}, e^{2\pi \mathrm{i} a_2/cd} \boldsymbol{u}_2^{\sigma_2}, \ldots, e^{2\pi \mathrm{i} a_l/cd} \boldsymbol{u}_l^{\sigma_l}\bigr), \\
&\overline{\beta}_{j,\boldsymbol{p}} : (\boldsymbol{u}_1, \boldsymbol{u}_2, \ldots, \boldsymbol{u}_l) \\
&\qquad\qquad \longmapsto \bigl((\widetilde{\beta}_{j,\boldsymbol{p}_1})^{m'_1}(\boldsymbol{u}_1), (\widetilde{\beta}_{j,\boldsymbol{p}_2})^{m'_2}(\boldsymbol{u}_2), \ldots, (\widetilde{\beta}_{j,\boldsymbol{p}_l})^{m'_l}(\boldsymbol{u}_l)\bigr).
\end{aligned}
$$

Lemma 4.11.

(1) $\widetilde{\Gamma} = \bigl\{\alpha^j \beta_{j,\boldsymbol{p}} : \boldsymbol{p} \in \Lambda^{(j)},\ j = 1, 2, \ldots, N\bigr\}$.

(2) $H = \bigl\{\overline{\alpha}^j \overline{\beta}_{j,\boldsymbol{p}} : \boldsymbol{p} \in \Lambda^{(j)},\ j = 1, 2, \ldots, N\bigr\}$.

Proof. (1): Proposition 4.7. (2) follows from (1) as the induced homomorphism $q_* : \widetilde{\Gamma} \to H$ from $q : \widetilde{A}_{d-1} \to \mathbb{C}^n$ is surjective, $\Box$

Remark 4.12. If $\sigma \neq \mathrm{id}$, $\widetilde{\Gamma}$ is generally not abelian (see Lemma 4.9). Accordingly H is generally not abelian.

Lemma 4.11 (2) implies the following:

Lemma 4.13. *Each element of H is of the form*

$$
(u_1, u_2, \ldots, u_n) \longmapsto (\zeta_1 u_{\sigma^j(1)}, \zeta_2 u_{\sigma^j(2)}, \ldots, \zeta_n u_{\sigma^j(n)}),
$$

where $\zeta_1, \zeta_2, \ldots, \zeta_n$ are roots of unity, σ is the permutation appearing in the definition of γ, and $j \in \mathbb{Z}$.

5. Simple Pseudo-Reflections

To determine the pseudo-reflection subgroup of H, some technical preparation is needed. A pseudo-reflection is *simple* if it is of the following form (and a general pseudo-reflection is conjugate to such):

$$(u_1, \dots, u_n) \longmapsto (u_1, \dots, \zeta u_i, \dots, u_n) \quad (\zeta \neq 1 \text{ is a root of unity}).$$

This is denoted by $h_{i,\zeta}$. In the particular case $\zeta = -1$, it is a *simple reflection*. Note that the order of a pseudo-reflection is finite (if ζ is a kth root of unity, its order is k) and its fixed point set is an $(n-1)$-dimensional subspace (for $h_{i,\zeta}$, this is defined by $u_i = 0$).

An example of a non-simple pseudo-reflection is

$$k_{ij,\alpha} : (u_1, \dots, u_i, \dots, u_j, \dots, u_n) \longmapsto (u_1, \dots, \alpha u_j, \dots, \alpha^{-1} u_i, \dots, u_n),$$

where $\alpha \neq 0$. This is called an (i,j)*-switching*. Note $k_{ij,\alpha}$ is conjugate to $h_{i,-1}$, for instance if $n = 3$ and $(i,j) = (1,2)$, then via $A = \begin{pmatrix} -\alpha & \alpha & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$:

$$A^{-1} \begin{pmatrix} 0 & \alpha & 0 \\ \alpha^{-1} & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} A = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

LEMMA 5.1. *A linear automorphism of $\mathbb{C}^n$ is a pseudo-reflection if and only if its order is finite and the dimension of its fixed point set is $n-1$.*

PROOF. It suffices to show "if". Suppose that a linear automorphism $f(\boldsymbol{z}) = A\boldsymbol{z}$ satisfies the condition. Then $A^k = E$ for some positive integer k. The minimal polynomial of A thus divides $x^k - 1$, so its roots are distinct kth roots of unity. Hence A is diagonalizable to a matrix of the form $\begin{pmatrix} \zeta_1 & & & O \\ & \zeta_2 & & \\ & & \ddots & \\ O & & & \zeta_n \end{pmatrix}$, where ζ_i is a kth root of unity. Here by assumption the dimension of the fixed point set of f is $n-1$, so only one of $\zeta_1, \zeta_2, \dots, \zeta_n$ is not 1 and the others are 1, implying that f is a pseudo-reflection. $\square$

LEMMA 5.2. *Let $h : (u_1, \dots, u_n) \mapsto (\zeta_1 u_{\tau(1)}, \dots, \zeta_n u_{\tau(n)})$ be an automorphism of $\mathbb{C}^n$ $(n \geq 2)$, where $\zeta_1, \dots, \zeta_n$ are roots of unity and $\tau \in \mathfrak{S}_n$ is a cyclic permutation of length n.*

(1) *Let $\mathrm{Fix}(h)$ be the fixed point set of h, then*

$$\dim \mathrm{Fix}(h) = \begin{cases} 1 & \text{if } \zeta_1\zeta_2\cdots\zeta_n = 1, \\ 0 & \text{otherwise.} \end{cases}$$

(2) *If h is a pseudo-reflection, n must be 2 (so τ is necessarily a transposition) and $h : (u_1, u_2) \mapsto (\zeta_1 u_2, \zeta_1^{-1} u_1)$ (a $(1,2)$-switching).*

PROOF. (1): First

$$\mathrm{Fix}(h) = \bigl\{(u_1, \dots, u_n) \in \mathbb{C}^n : u_1 = \zeta_1 u_{\tau(1)}, u_2 = \zeta_2 u_{\tau(2)}, \dots, u_n = \zeta_n u_{\tau(n)}\bigr\}.$$

Without loss of generality, we assume $\tau = (1\ 2\ \cdots\ n)$. Then $\mathrm{Fix}(h)$ is defined by $u_1 = \zeta_1 u_2$, $u_2 = \zeta_2 u_3$, $\dots$, $u_n = \zeta_n u_1$; this is equivalent to

$$(*) \qquad u_1 = \zeta_1 u_2 = \zeta_1\zeta_2 u_3 = \cdots = \zeta_1\zeta_2\cdots\zeta_{n-1} u_n = \zeta_1\zeta_2\cdots\zeta_n u_1.$$

We claim that setting $\boldsymbol{v} := (1, \zeta_1^{-1}, \zeta_1^{-1}\zeta_2^{-1}, \dots, \zeta_1^{-1}\zeta_2^{-1}\cdots\zeta_{n-1}^{-1}) \in \mathbb{C}^n$, then $\mathrm{Fix}(h)$ is $\{c\boldsymbol{v} : c \in \mathbb{C}\}$ if $\zeta_1\zeta_2\cdots\zeta_n = 1$, and $\{0\}$ otherwise. Note that from $(*)$, in particular $u_1 = \zeta_1\zeta_2\cdots\zeta_n u_1$, whose solution is, if $\zeta_1\zeta_2\cdots\zeta_n \neq 1$, unique $u_1 = 0$, accordingly the solution of $(*)$ is unique $u_1 = u_2 = u_3 = \cdots = u_n = 0$, so $\mathrm{Fix}(h) = \{0\}$. If $\zeta_1\zeta_2\cdots\zeta_n = 1$, solving $(*)$ with respect to u_1 yields $u_2 = \zeta_1^{-1} u_1$, $u_3 = \zeta_1^{-1}\zeta_2^{-1} u_1$, $\dots$, $u_n = \zeta_1^{-1}\zeta_2^{-1}\cdots\zeta_{n-1}^{-1} u_1$. Thus setting $c := u_1$, then $(u_1, u_2, \dots, u_n) = c(1, \zeta_1^{-1}, \zeta_1^{-1}\zeta_2^{-1}, \dots, \zeta_1^{-1}\zeta_2^{-1}\cdots\zeta_{n-1}^{-1})$, hence $\mathrm{Fix}(h) = \bigl\{c\boldsymbol{v} : c \in \mathbb{C}\bigr\}$.

(2): If h is a pseudo-reflection of $\mathbb{C}^n$ $(n \geq 2)$, then by Lemma 5.1, $\dim \mathrm{Fix}(h) = n - 1 \geq 1$. This combined with (1) implies $n - 1 = 1$ and $\zeta_1\zeta_2\cdots\zeta_n = 1$, that is, $n = 2$ and $\zeta_1\zeta_2 = 1$. Thus $h : (u_1, u_2) \mapsto (\zeta_1 u_2, \zeta_1^{-1} u_1)$. $\square$

Lemma 5.2 (2) is generalized to:

LEMMA 5.3. *Let $h : (u_1, \dots, u_n) \mapsto (\zeta_1 u_{\tau(1)}, \dots, \zeta_n u_{\tau(n)})$ be an automorphism of $\mathbb{C}^n$ $(n \geq 2)$, where $\zeta_1, \dots, \zeta_n$ are roots of unity and $\tau \in \mathfrak{S}_n$. If h is a pseudo-reflection, then it is either simple or switching.*

PROOF. Decompose τ into disjoint cyclic permutations: $\tau = \tau_1\tau_2\cdots\tau_l$. Without loss of generality, we assume that τ_1 permutes $\{1, 2, \dots, n_1\}$, τ_2 permutes $\{n_1+1, n_1+2, \dots, n_1+n_2\}$ and so on. Write $\mathbb{C}^n$ as $\mathbb{C}^{n_1} \times \mathbb{C}^{n_2} \times \cdots \times \mathbb{C}^{n_l}$ and $(u_1, u_2, \dots, u_n) \in \mathbb{C}^n$ as $(\boldsymbol{u}_1, \boldsymbol{u}_2, \dots, \boldsymbol{u}_l)$, where $\boldsymbol{u}_i \in \mathbb{C}^{n_i}$. Express then h as

$$h : (\boldsymbol{u}_1, \boldsymbol{u}_2, \dots, \boldsymbol{u}_l) \longmapsto \big(h_1(\boldsymbol{u}_1), h_2(\boldsymbol{u}_2), \dots, h_l(\boldsymbol{u}_l)\big),$$

where $h_i : \mathbb{C}^{n_i} \to \mathbb{C}^{n_i}$ is a linear automorphism of finite order (as h is). Then $\mathrm{Fix}(h)$ is expressed as $\mathrm{Fix}(h_1) \times \mathrm{Fix}(h_2) \times \cdots \times \mathrm{Fix}(h_l)$, so

$$\dim \mathrm{Fix}(h) = \dim \mathrm{Fix}(h_1) + \dim \mathrm{Fix}(h_2) + \cdots + \dim \mathrm{Fix}(h_l).$$

Here if h is a pseudo-reflection, then by Lemma 5.1, $\dim \mathrm{Fix}(h) = n - 1 = n_1 + n_2 + \cdots + n_l - 1$, thus

$$\dim \mathrm{Fix}(h_1) + \dim \mathrm{Fix}(h_2) + \cdots + \dim \mathrm{Fix}(h_l) = n_1 + n_2 + \cdots + n_l - 1.$$

Noting $\dim \mathrm{Fix}(h_i) \le n_i$, we have: For some h_k, $\dim \mathrm{Fix}(h_k) = n_k - 1$ (so h_k is a pseudo-reflection by Lemma 5.1) and for any other h_i, $\dim \mathrm{Fix}(h_i) = n_i$ (so h_i is the identity). Thus $h(\boldsymbol{u}_1, \boldsymbol{u}_2, \dots, \boldsymbol{u}_l) = \big(\boldsymbol{u}_1, \boldsymbol{u}_2, \dots, h_k(\boldsymbol{u}_k), \dots, \boldsymbol{u}_l\big)$ such that h_k is a pseudo-reflection. Here if $n_k \ge 2$, h is switching and if $n_k = 1$, simple, because: in the former case, by Lemma 5.2 (2), n_k must be 2 and h_k is switching and in the latter case, $h_k : \mathbb{C} \to \mathbb{C}$ is of the form $u \mapsto \zeta u$ ($\zeta \neq 1$ is a root of unity). $\square$

6. The Pseudo-Reflection Subgroup of H

LEMMA 6.1. *Let G be a finite subgroup of $GL_n(\mathbb{C})$ and Q be the pseudo-reflection subgroup of G* (i.e. *the subgroup generated by all pseudo-reflections of G*). *Then Q is normal in G.*

PROOF. By definition, any element conjugate to a pseudo-reflection is also a pseudo-reflection, so Q is normal in G. $\square$

The G-action on $\mathbb{C}^n$ naturally descends to a G/Q-action on $\mathbb{C}^n/Q$. Here:

THEOREM 6.2 (Chevalley-Shephard-Todd). *$\mathbb{C}^n/Q \cong \mathbb{C}^n$ and under this isomorphism, G/Q acts on $\mathbb{C}^n$ linearly. So G/Q may be regarded as a* *subgroup of* $GL_n(\mathbb{C})$. (Note G/Q is a small group, as the pseudo-reflection subgroup of G/Q is trivial.)

We return to the cyclic group Γ generated by a twining automorphism $\gamma : A_{d-1} \to A_{d-1}$ given by

$$(6.1) \qquad (\boldsymbol{x}_1, \dots, \boldsymbol{x}_l, t) \mapsto (e^{2\pi \mathrm{i} a_1/n_1 m_1} \boldsymbol{x}_1^{\sigma_1}, \dots, e^{2\pi \mathrm{i} a_l/n_l m_l} \boldsymbol{x}_l^{\sigma_l}, e^{2\pi \mathrm{i}/N} t).$$

Recall that $\widetilde{\Gamma}$ is the lift of Γ with respect to the universal covering $p : \widetilde{A}_{d-1}(=\mathbb{C}^n) \to A_{d-1}$ and H is the descent of $\widetilde{\Gamma}$ with respect to the covering $q : \widetilde{A}_{d-1} \to \mathbb{C}^n$. We apply to H the above results, to determine its *pseudo-reflection subgroup* — the subgroup generated by all pseudo-reflections in H. Note first that:

Lemma 6.3.

(1) *The cyclic group* Γ *contains no switching that leaves* t *fixed.*

(2) *Any pseudo-reflection in* H *is simple.*

Proof. (1): We only show that Γ contains no $(1,2)$-switching (other cases are similarly shown). Note first that from (6.1), $\gamma^k \in \Gamma$ maps t to $e^{2\pi \mathrm{i} k/N} t$. If γ^k is a $(1,2)$-switching, then $e^{2\pi \mathrm{i} k/N}$ must be 1; so k is a multiple of N. Since the order of γ is N, this implies that γ^k is the identity, which contradicts that γ^k is a $(1,2)$-switching.

(2): Let $h \in H$ be a pseudo-reflection. By Lemma 4.13, h is of the form:

$$(6.2) \qquad h : (u_1, u_2, \dots, u_n) \mapsto (\zeta_1 u_{\sigma^j(1)}, \zeta_2 u_{\sigma^j(2)}, \dots, \zeta_n u_{\sigma^j(n)})$$

for some j and some roots $\zeta_1, \zeta_2, \dots, \zeta_n$ of unity. Then by Lemma 5.3, h is either simple or switching. The assertion is thus confirmed by showing the latter does *not* occur. We only show that h cannot be a $(1,2)$-switching (other cases are similarly shown). Otherwise

$$h : (u_1, u_2, u_3, \dots, u_n) \longmapsto (\alpha u_2, \alpha^{-1} u_1, u_3, \dots, u_n) \quad (\alpha\text{: a root of unity}).$$

Comparing this with (6.2) yields $\sigma^j = (1\ 2)$.

Recall that $\sigma = \sigma_1 \sigma_2 \cdots \sigma_l$, where $\sigma_1, \sigma_2, \dots, \sigma_l$ are the cyclic permutations appearing in (6.1) and n_i is the length of σ_i. From $\sigma^j = (1\ 2)$, we have $\sigma_1^j = (1\ 2)$ and $\sigma_2^j = \sigma_3^j = \cdots = \sigma_l^j = \mathrm{id}$. Note that $\sigma_1^j = (1\ 2)$ implies $\sigma_1 = (1\ 2)$ and $n_1 = 2$ (see Remark 6.4 (2) below); from the latter, $\boldsymbol{X}_1 = (X_1, X_2)$, so the covering $q : \widetilde{A}_{d-1} \to \mathbb{C}^n$ is given by

$$q : (X_1, X_2, X_3, \ldots, X_n) \longmapsto (X_1^{m_1'}, X_2^{m_1'}, X_3^{m_2'}, \ldots, X_n^{m_l'}).$$

Define a lift $\widetilde{h} \in \widetilde{\Gamma}$ of h with respect to q by

$$\widetilde{h} : (X_1, X_2, X_3, \ldots, X_n) \longmapsto (\alpha^{1/m_1'} X_2, \alpha^{-1/m_1'} X_1, X_3, \ldots, X_n).$$

The descent $\overline{h} \in \Gamma$ of $\widetilde{h}$ with respect to $p : \widetilde{A}_{d-1} \to A_{d-1}$ is then

$$\overline{h} : (x_1, x_2, x_3, \ldots, x_n, t) \longmapsto (\alpha^{d/m_1'} x_2, \alpha^{-d/m_1'} x_1, x_3, \ldots, x_n, t).$$

This is a $(1,2)$-switching, which contradicts that Γ contains no switching (as shown in (1)). $\square$

REMARK 6.4. For a cyclic permutation τ, τ^j is generally *decomposable*: Say the length of τ is l and set $k := \gcd(j, l)$, then τ^j is a product of k cyclic permutations of the *same* length l/k (note k divides l).
(1) In case $k = 1$, τ^j is indecomposable, and the length $l/1$ of τ^j is the same as that of τ.
(2) If $l = 2$ (i.e. τ is a transposition), then necessarily $k = 1$ or 2. In the former case, by (1) the length of τ^j is also 2, so τ^j is a transposition — necessarily $\tau^j = \tau$ and j is odd.

We turn to determine the pseudo-reflection subgroup of H.

PROPOSITION 6.5. *The pseudo-reflection subgroup P of H is a direct product $P_1 \times P_2 \times \cdots \times P_n$, where P_i is the subgroup of H generated by ith simple pseudo-reflections, that is, of the form*

$$(u_1, u_2, \ldots, u_n) \longmapsto (u_1, u_2, \ldots, \zeta u_i, \ldots, u_n), \quad \zeta \text{ is a root of unity.}$$

PROOF. Clearly $P_1 P_2 \cdots P_n \subset P$. Since any pseudo-reflection in H is contained in some P_i (from Lemma 6.3 (2)), $P = P_1 P_2 \cdots P_n$. Here by definition, $P_i \cap P_j = \{1\}$ $(i \neq j)$, thus $P = P_1 \times P_2 \times \cdots \times P_n$. $\square$

We next determine P_i explicitly. Recall first the following diagram with group actions:

$$\begin{tikzcd}[column sep=small] & \widetilde{A}_{d-1} = \mathbb{C}^n \curvearrowleft \widetilde{\Gamma} \arrow[dl, "q"'] \arrow[dr, "p"] & \\ H \curvearrowright \mathbb{C}^n & & A_{d-1} \curvearrowleft \Gamma. \end{tikzcd} \tag{6.3}$$

Here Γ is the cyclic group generated by a twining automorphism

$$\gamma : (\boldsymbol{x}_1, \dots, \boldsymbol{x}_l, t) \longmapsto (e^{2\pi \mathrm{i} a_1/n_1 m_1} \boldsymbol{x}_1^{\sigma_1}, \dots, e^{2\pi \mathrm{i} a_l/n_l m_l} \boldsymbol{x}_l^{\sigma_l}, e^{2\pi \mathrm{i}/N} t),$$

and $\widetilde{\Gamma}$ is the lift of Γ with respect to p, and H is the descent of $\widetilde{\Gamma}$ with respect to q.

Notation 6.6. The subsequent discussion involves the following groups:

- $\widetilde{\Gamma}_i \subset \widetilde{\Gamma}$: the subgroup generated by ith simple pseudo-reflections, that is, of the form $(X_1, X_2, \dots, X_n) \mapsto (X_1, X_2, \dots, \zeta X_i, \dots, X_n)$, where ζ is a root of unity.
- $\Gamma_i \subset \Gamma$: the subgroup generated by automorphisms of the form $(x_1, \dots, x_n, t) \mapsto (x_1, \dots, \mu^d x_i, \dots, x_n, \mu t)$, where μ is a root of unity.
- $P_i \subset H$: the subgroup generated by ith simple pseudo-reflections.

Definition 6.7. The surjective homomorphism $p_* : \widetilde{\Gamma} \to \Gamma$ (resp. $q_* : \widetilde{\Gamma} \to H$) induced by p (resp. q) is called a *descent homomorphism*.

Lemma 6.8.

(1) Γ_i *is the descent of* $\widetilde{\Gamma}_i$ *with respect to* p*, that is,* $p_*(\widetilde{\Gamma}_i) = \Gamma_i$*. In fact* $p_* : \widetilde{\Gamma}_i \to \Gamma_i$ *is an isomorphism.*

(2) P_i *is the descent of* $\widetilde{\Gamma}_i$ *with respect to* q*, that is,* $q_*(\widetilde{\Gamma}_i) = P_i$*.*

Proof. (1): Since

$$p : (X_1, X_2, \dots, X_n) \longmapsto (X_1^d, X_2^d, \dots, X_n^d, X_1 X_2 \cdots X_n),$$

an ith pseudo-reflection $(X_1, \dots, X_n) \mapsto (X_1, \dots, \zeta X_i, \dots, X_n)$ descends to $(x_1, \dots, x_n, t) \mapsto (x_1, \dots, \zeta^d x_i, \dots, x_n, \zeta t)$. This correspondence is clearly surjective, so $p_*(\widetilde{\Gamma}_i) = \Gamma_i$. Moreover this is injective: Distinct automorphisms $\begin{cases} (X_1,\dots,X_n) \mapsto (X_1,\dots,\zeta X_i,\dots,X_n) \\ (X_1,\dots,X_n) \mapsto (X_1,\dots,\zeta' X_i,\dots,X_n) \end{cases}$ descend to distinct automorphisms $\begin{cases} (x_1,\dots,x_n,t) \mapsto (x_1,\dots,\zeta^d x_i,\dots,x_n,\zeta t) \\ (x_1,\dots,x_n,t) \mapsto (x_1,\dots,(\zeta')^d x_i,\dots,x_n,\zeta' t). \end{cases}$

(2): Write $(X_1,\dots,X_n) \in \mathbb{C}^n$ as $(\boldsymbol{X}_1,\boldsymbol{X}_2,\dots,\boldsymbol{X}_l) \in \mathbb{C}^{n_1}\times\mathbb{C}^{n_2}\times\cdots\times\mathbb{C}^{n_l}$ $(n = n_1+n_2+\cdots+n_l)$, then

$$q:\ (\boldsymbol{X}_1,\boldsymbol{X}_2,\dots,\boldsymbol{X}_l) \longmapsto (\boldsymbol{X}_1^{m'_1},\boldsymbol{X}_2^{m'_2},\dots,\boldsymbol{X}_l^{m'_l}). \tag{6.4}$$

Say $X_i \in \boldsymbol{X}_k$, then under q, $(X_1,\dots,X_n) \mapsto (X_1,\dots,\zeta X_i,\dots,X_n)$ descends to $(u_1,\dots,u_n) \mapsto (u_1,\dots,\ \zeta^{m'_k} u_i,\dots,u_n)$. This correspondence is clearly surjective. $\square$

Recall that Γ is the cyclic group of order N generated by

$$\gamma : (\boldsymbol{x}_1,\dots,\boldsymbol{x}_l,t) \longmapsto (e^{2\pi \mathrm{i} a_1/n_1m_1}\boldsymbol{x}_1^{\sigma_1},\dots,e^{2\pi \mathrm{i} a_l/n_lm_l}\boldsymbol{x}_l^{\sigma_l},e^{2\pi \mathrm{i}/N}t). \tag{6.5}$$

Thus

$$\gamma^j:\ (\boldsymbol{x}_1,\dots,\boldsymbol{x}_l,t) \longmapsto (e^{2\pi \mathrm{i} j a_1/n_1m_1}\boldsymbol{x}_1^{\sigma_1^j},\dots,e^{2\pi \mathrm{i} j a_l/n_lm_l}\boldsymbol{x}_l^{\sigma_l^j},e^{2\pi \mathrm{i} j/N}t). \tag{6.6}$$

We investigate when $\gamma^j \in \Gamma_i$, that is, γ^j is of the form $(x_1,\dots,x_n,t) \mapsto (x_1,\dots,\zeta^d x_i,\dots,x_n,\zeta t)$ for some root ζ of unity. Say $x_i \in \boldsymbol{x}_k$, then

$$\gamma^j:\ (x_1,\dots,x_n,t) \longmapsto \big(\underbrace{x_1,\dots}_{\boldsymbol{x}_1}\cdots\underbrace{\dots,\zeta^d x_i,\dots}_{\boldsymbol{x}_k}\cdots\underbrace{\dots,x_n}_{\boldsymbol{x}_l},\zeta t\big). \tag{6.7}$$

Comparing (6.6) and (6.7) yields $\sigma_1^j = 1$, $\sigma_2^j = 1$, $\dots$, $\sigma_l^j = 1$, accordingly (6.6) reduces to

$$\gamma^j:\ (\boldsymbol{x}_1,\dots,\boldsymbol{x}_l,t) \longmapsto (e^{2\pi \mathrm{i} j a_1/n_1m_1}\boldsymbol{x}_1,\dots,e^{2\pi \mathrm{i} j a_l/n_lm_l}\boldsymbol{x}_l,e^{2\pi \mathrm{i} j/N}t). \tag{6.8}$$

We then compare the coefficients in (6.7) and (6.8):

- Comparison for $\boldsymbol{x}_s$ $(s = 1,\dots,\check{k},\dots,l)$ gives $e^{2\pi \mathrm{i} j a_s/n_sm_s} = 1$, where $\check{k}$ means the omission of k.

- Comparison for $\boldsymbol{x}_k$ gives $e^{2\pi \mathrm{i} j a_k/n_k m_k}\boldsymbol{x}_k = (\underbrace{\ldots, x_{i-1}, \zeta^d x_i, \ldots}_{\boldsymbol{x}_k})$, that is,
$$(\ldots, e^{2\pi \mathrm{i} j a_k/n_k m_k} x_{i-1}, e^{2\pi \mathrm{i} j a_k/n_k m_k} x_i, \ldots) = (\ldots, x_{i-1}, \zeta^d x_i, \ldots).$$
If $\mathrm{length}(\boldsymbol{x}_k) = 1$, this reduces to $(e^{2\pi \mathrm{i} j a_k/n_k m_k} x_i) = (\zeta^d x_i)$, so $e^{2\pi \mathrm{i} j a_k/n_k m_k} = \zeta^d$. If $\mathrm{length}(\boldsymbol{x}_k) \geq 2$, then $e^{2\pi \mathrm{i} j a_k/n_k m_k} = 1$ and $\zeta^d = 1$.
- Comparison for t gives $e^{2\pi \mathrm{i} j/N} = \zeta$.

Note. If $\mathrm{length}(\boldsymbol{x}_k) = 1$ (resp. ≥ 2), then $(\zeta, \zeta^d) = (e^{2\pi \mathrm{i} j/N}, e^{2\pi \mathrm{i} j a_k/n_k m_k})$ (resp. $(\zeta, \zeta^d) = (e^{2\pi \mathrm{i} j/N}, 1)$). Accordingly $(e^{2\pi \mathrm{i} j/N})^d = e^{2\pi \mathrm{i} j a_k/n_k m_k}$ (resp. $(e^{2\pi \mathrm{i} j/N})^d = 1$), which also follows from the fact that γ^j preserves A_{d-1}, that is, $x_1 x_2 \cdots x_n = t$.

We summarize the above results as follows:

Lemma 6.9. *Let Γ_i be the subgroup of Γ defined in Notation 6.6. Then $\gamma^j \in \Gamma_i$ if and only if γ^j is of the form (say $x_i \in \boldsymbol{x}_k$):*

$$\begin{cases} (\boldsymbol{x}_1, \ldots, \boldsymbol{x}_l, t) \mapsto (\boldsymbol{x}_1, \ldots, e^{2\pi \mathrm{i} d j/N}\boldsymbol{x}_k, \ldots, \boldsymbol{x}_l, e^{2\pi \mathrm{i} j/N} t) & \textit{if } \mathrm{length}(\boldsymbol{x}_k) = 1, \\ (\boldsymbol{x}_1, \ldots, \boldsymbol{x}_l, t) \mapsto (\boldsymbol{x}_1 \ldots, \boldsymbol{x}_l, e^{2\pi \mathrm{i} j/N} t) & \textit{if } \mathrm{length}(\boldsymbol{x}_k) \geq 2. \end{cases}$$

This condition is 'more explicitly' given by: $\sigma_1^j = 1$, $\sigma_2^j = 1$, $\ldots$, $\sigma_l^j = 1$ and (below, $\check{k}$ is the omission of k)

$$(*) \quad \begin{cases} e^{2\pi \mathrm{i} j a_s/n_s m_s} = 1 \textit{ for } s = 1, 2, \ldots, \check{k}, \ldots, l & \textit{if } \mathrm{length}(\boldsymbol{x}_k) = 1, \\ e^{2\pi \mathrm{i} j a_s/n_s m_s} = 1 \textit{ for } s = 1, 2, \ldots, l & \textit{if } \mathrm{length}(\boldsymbol{x}_k) \geq 2. \end{cases}$$

Here a_s and $n_s m_s$ $(s = 1, 2, \ldots, l)$ are relatively prime, so $(*)$ is restated as: *j is a multiple of L_k, where* (below, $\widecheck{n_k m_k}$ is the omission of $n_k m_k$)

$$(6.9) \quad L_k := \begin{cases} \mathrm{lcm}(n_1 m_1, n_2 m_2, \ldots, \widecheck{n_k m_k}, \ldots, n_l m_l) & \textit{if } \mathrm{length}(\boldsymbol{x}_k) = 1, \\ \mathrm{lcm}(n_1 m_1, n_2 m_2, \ldots, n_l m_l) & \textit{if } \mathrm{length}(\boldsymbol{x}_k) \geq 2, \end{cases}$$

Here $n_s = \mathrm{length}(\boldsymbol{x}_s)$ (the order of σ_s). Hence $\gamma^j \in \Gamma_i$ if and only if j is a common multiple of L_k and the orders of $\sigma_1, \sigma_2, \dots, \sigma_l$, that is, j is a multiple of $\mathrm{lcm}(L_k, n_1, n_2, \dots, n_l) = L_k$. The following is thus obtained:

COROLLARY 6.10. *In Lemma 6.9, $\gamma^j \in \Gamma_i$ if and only if j is a multiple of L_k given by* (6.9).

We explicitly determine Γ_i and $\widetilde{\Gamma}_i$:

LEMMA 6.11.

(1) *The group Γ_i (in Notation 6.6) is cyclic: Say $x_i \in \boldsymbol{x}_k$, then Γ_i is generated by the following automorphism:*

$$\gamma^{L_k} : (\boldsymbol{x}_1, \dots, \boldsymbol{x}_l, t) \longmapsto (\boldsymbol{x}_1, \dots, e^{2\pi \mathrm{i} L_k d/N} \boldsymbol{x}_k, \dots, \boldsymbol{x}_l, e^{2\pi \mathrm{i} L_k/N} t),$$

(Note: If $n_k \geq 2$, then $e^{2\pi \mathrm{i} L_k d/N} = 1$.)

(2) *The subgroup $\widetilde{\Gamma}_i$ of $\widetilde{\Gamma}$ (in Notation 6.6) is cyclic: Say $X_i \in \boldsymbol{X}_k$, then $\widetilde{\Gamma}_i$ is generated by the following automorphism*

$$\xi_i : (X_1, \dots, X_n) \longmapsto (X_1, \dots, e^{2\pi \mathrm{i} L_k/N} X_i, \dots, X_n). \tag{6.10}$$

PROOF. (1): Γ_i is cyclic, because it is a subgroup of the cyclic group Γ. Say now $x_i \in \boldsymbol{x}_k$, then since $\gamma^j \in \Gamma_i$ if and only if j is a multiple of L_k (Corollary 6.10), Γ_i is generated by γ^{L_k}.

(2): $\widetilde{\Gamma}_i$ is cyclic, because $\widetilde{\Gamma}_i$ is isomorphic to the cyclic group Γ_i (Lemma 6.8 (1)). Say $X_i \in \boldsymbol{X}_k$. We then show that $\widetilde{\Gamma}_i$ is generated by the ξ_i given by (6.10). Since $X_i \in \boldsymbol{X}_k$, $x_i \in \boldsymbol{x}_k$, and thus by (1), Γ_i is generated by γ^{L_k}. Since $p_* : \widetilde{\Gamma}_i \to \Gamma_i$ is isomorphic (Lemma 6.8 (1)) and $p_*(\xi_i) = \gamma^{L_k}$, $\widetilde{\Gamma}_i$ is generated by $p_*^{-1}(\gamma^{L_k}) = \xi_i$. $\square$

Recall that H is the descent of $\widetilde{\Gamma}$ with respect to q.

COROLLARY 6.12. *The subgroup P_i of H generated by ith pseudo-reflections is actually cyclic: Say $u_i \in \boldsymbol{u}_k$, when we write $(u_1, \dots, u_n) \in \mathbb{C}^n$ as $(\boldsymbol{u}_1, \dots, \boldsymbol{u}_l) \in \mathbb{C}^{n_1} \times \cdots \times \mathbb{C}^{n_l}$. Then P_i is generated by*

$$h_i : (u_1, \dots, u_n) \longmapsto (u_1, \dots, e^{2\pi \mathrm{i} n_k m_k L_k/Nc} u_i, \dots, u_n). \tag{6.11}$$

PROOF. Since $q_*(\widetilde{\Gamma}_i) = P_i$ (Lemma 6.8 (2)) and $\widetilde{\Gamma}_i$ is generated by ξ_i (Lemma 6.11 (2)), P_i is generated by $q_*(\xi_i)$. Here $q_*(\xi_i) = h_i$, confirming the assertion. □

Let P be the pseudo-reflection subgroup of H. Then $P = P_1 \times P_2 \times \cdots \times P_n$ (Lemma 6.5), thus from Corollary 6.12 the following holds:

PROPOSITION 6.13. *The pseudo-reflection subgroup P of H is generated by the automorphisms $h_1, h_2, \ldots, h_n$ in Corollary 6.12.*

7. Numerical Criterion of Smallness

That is, its pseudo-reflection subgroup P is nontrivial. Consider the quotient map $r : \mathbb{C}^n \to \mathbb{C}^n/P$. By Chevalley-Shephard-Todd theorem, $\mathbb{C}^n/P \cong \mathbb{C}^n$ and under this isomorphism, H/P acts on $\mathbb{C}^n$ linearly. So H/P may be regarded as a subgroup of $GL_n(\mathbb{C})$ and r as a map $r : \mathbb{C}^n \to \mathbb{C}^n$. Since the covering transformation group of r is P, the following is obvious:

$$
\begin{aligned}
r : \mathbb{C}^n \to \mathbb{C}^n \text{ is the identity map} &\iff P = \{1\} \\
&\iff H \text{ is small.}
\end{aligned}
\tag{7.1}
$$

We explicitly give r. We begin with observation. Let $\mathbb{Z}_\ell := \langle e^{2\pi \mathrm{i}/\ell} \rangle$ act on $\mathbb{C}$ by multiplication, then the quotient map $\mathbb{C} \to \mathbb{C}/\mathbb{Z}_\ell \cong \mathbb{C}$ is given by $z \mapsto z^\ell$. More generally let $\mathbb{Z}_{\ell_1} \times \cdots \times \mathbb{Z}_{\ell_n} = \langle e^{2\pi \mathrm{i}/\ell_1} \rangle \times \cdots \times \langle e^{2\pi \mathrm{i}/\ell_n} \rangle$ act on $\mathbb{C}^n = \mathbb{C} \times \cdots \times \mathbb{C}$ by multiplication, then the quotient map $\mathbb{C}^n \to \mathbb{C}^n/(\mathbb{Z}_{\ell_1} \times \cdots \times \mathbb{Z}_{\ell_n}) \cong \mathbb{C}^n$ is given by

$$
(z_1, \ldots, z_n) \longmapsto (z_1^{\ell_1}, \ldots, z_n^{\ell_n}).
\tag{7.2}
$$

Similarly the quotient map $r : \mathbb{C}^n \to \mathbb{C}^n/P \cong \mathbb{C}^n$ may be explicitly given. Recall first that $P = \langle h_1 \rangle \times \langle h_2 \rangle \times \cdots \times \langle h_n \rangle$ (Proposition 6.13), where h_i is an automorphism of $\mathbb{C}^n$ given by (6.11): Set $\ell_k := Nc/n_k m_k L_k$, where L_k is the positive integer given by (6.9) and $N := (m_1')^{n_1} \cdots (m_l')^{n_l} c$ and $c := \gcd(n_1 m_1, \ldots, n_l m_l)$ and $m_k' = \frac{n_k m_k}{c}$ (ℓ_k is an integer by Lemma 7.4 below), then explicitly

$$
h_i : (u_1, \ldots, u_n) \longmapsto (u_1, \ldots, e^{2\pi \mathrm{i}/\ell_k} u_i, \ldots, u_n),
$$

As for (7.2), $r : \mathbb{C}^n \to \mathbb{C}^n/P \cong \mathbb{C}^n$ is then given by

$$(u_1, \dots, u_n) \mapsto \big(\underbrace{u_1^{\ell_1}, \dots}_{\boldsymbol{u}_1} \cdots \underbrace{\dots, u_i^{\ell_k}, \dots}_{\boldsymbol{u}_k} \cdots \underbrace{\dots, u_n^{\ell_l}}_{\boldsymbol{u}_l}\big).$$

We formalize this result as follows:

LEMMA 7.1. *Write* $(u_1, u_2, \dots, u_n) \in \mathbb{C}^n$ *as* $(\boldsymbol{u}_1, \boldsymbol{u}_2, \dots, \boldsymbol{u}_l) \in \mathbb{C}^{n_1} \times \mathbb{C}^{n_2} \times \cdots \times \mathbb{C}^{n_l}$*, where* $\boldsymbol{u}_k := (u_{j_1}, \dots, u_{j_{n_k}})$*. Then the covering map* $r : \mathbb{C}^n \to \mathbb{C}^n$ *is explicitly given by* $r(\boldsymbol{u}_1, \boldsymbol{u}_2, \dots, \boldsymbol{u}_l) = (\boldsymbol{u}_1^{\ell_1}, \boldsymbol{u}_2^{\ell_2}, \dots, \boldsymbol{u}_l^{\ell_l})$*, where* $\boldsymbol{u}_k^{\ell_k} := (u_{j_1}^{\ell_k}, \dots, u_{j_{n_k}}^{\ell_k})$.

The following is immediate from Lemma 7.1:

$$\begin{aligned} r \text{ is the identity map} \iff\; & \ell_1 = \ell_2 = \cdots = \ell_l = 1 \\ & (\text{i.e. } Nc/n_1 m_1 L_1 = \cdots = Nc/n_l m_l L_l = 1) \\ \iff\; & m_1' L_1 = \cdots = m_l' L_l = N. \end{aligned}$$

This combined with (7.1) yields the following:

THEOREM 7.2. *The following are equivalent:*
(1) H *is small.*
(2) *The covering* $r : \mathbb{C}^n \to \mathbb{C}^n$ *is the identity map.*
(3) $m_1' L_1 = m_2' L_2 = \cdots = m_l' L_l = N$.

COROLLARY 7.3. *If* $n = 2$*, then* H *is small.*

PROOF. From Theorem 7.2, it suffices to show $m_1' L_1 = m_2' L_2 = \cdots = m_l' L_l = 1$. Note first that the permutation $\sigma \in \mathfrak{S}_n$ appearing in the definition of γ is, if $n = 2$, either the identity or a transposition $(1\,2)$. We separate into two cases:
(i) If σ is the identity, then $n_1 = n_2 = 1$, $c = \gcd(m_1, m_2)$, $m_1' = \frac{m_1}{c}$, $m_2' = \frac{m_2}{c}$, $N = m_1' m_2' c$, $L_1 = m_2' c$, and $L_2 = m_1' c$. Thus $m_1' L_1 = m_2' L_2 = N$.
(ii) If σ is the transposition $(1\,2)$, then $n_1 = 2$, $c = 2m_1$, $m_1' = \dfrac{2m_1}{c} = 1$, $N = (m_1')^2 c = 2m_1$, and $L_1 = n_1 m_1 = 2m_1$. Thus $m_1' L_1 = N$. □

Supplement. We show that $\ell_k := Nc/n_k m_k L_k$ is an integer. Recall that $N := (m_1')^{n_1} \cdots (m_l')^{n_l} c$, where $c := \gcd(n_1 m_1, \dots, n_l m_l)$ and $m_k' =$

$\frac{n_k m_k}{c}$ and L_k is given by (6.9):

$$L_k = \begin{cases} \mathrm{lcm}(n_1m_1, n_2m_2, \ldots, \widecheck{n_k m_k}, \ldots, n_l m_l) & \text{if } n_k = 1, \\ \mathrm{lcm}(n_1m_1, n_2m_2, \ldots, n_l m_l) & \text{if } n_k \geq 2. \end{cases}$$

LEMMA 7.4. *$\ell_k := Nc/n_k m_k L_k$ is an integer.*

PROOF. Rewrite L_k as

$$L_k = \begin{cases} \mathrm{lcm}(m_1', m_2' \ldots, \check{m}_k', \ldots, m_l')c & \text{if } n_k = 1, \\ \mathrm{lcm}(m_1', m_2', \ldots, m_l')c & \text{if } n_k \geq 2. \end{cases}$$

Here

$$\begin{cases} \mathrm{lcm}(m_1', m_2' \ldots, \check{m}_k', \ldots, m_l') \text{ divides } m_1'm_2' \cdots \check{m}_k' \cdots m_l', \\ \mathrm{lcm}(m_1', m_2', \ldots, m_l') \text{ divides } m_1'm_2' \cdots m_l'. \end{cases}$$

In either case L_k divides $m_1' \cdots (m_k')^{n_k - 1} \cdots m_l' c$, so $n_k m_k L_k$ $(= m_k' L_k c)$ divides $m_1' \cdots (m_k')^{n_k} \cdots m_l' c^2$, in particular, divides $Nc = (m_1')^{n_1} \cdots (m_l')^{n_l} c^2$. $\square$

8. Uniformization of Twined Singularities

8.1. Uniformization theorem

In what follows, set $G := H/P$. Consider the diagram expanding (6.3):

(8.1)
$$\begin{tikzcd}[column sep=small] & & \widetilde{A}_{d-1} = \mathbb{C}^n \curvearrowleft \widetilde{\Gamma} \arrow[dl, "q"'] \arrow[dr, "p"] & \\ & H \curvearrowright \mathbb{C}^n \arrow[dl, "r"'] & & A_{d-1} \curvearrowleft \Gamma. \\ G := H/P \curvearrowright \mathbb{C}^n & & & \end{tikzcd}$$

Then

(8.2)
$$A_{d-1}/\Gamma \cong \widetilde{A}_{d-1}/\widetilde{\Gamma} \cong \mathbb{C}^n/H \cong \mathbb{C}^n/G.$$

Here G is a small finite subgroup of $GL_n(\mathbb{C})$ (Theorem 6.2). We thus proved (1) of the following:

THEOREM 8.1 (Uniformization theorem). *Let Γ be the cyclic group generated by a twining automorphism $\gamma : A_{d-1} \to A_{d-1}$ given by*

$$\gamma : (\boldsymbol{x}_1, \dots, \boldsymbol{x}_l, t) \longmapsto (e^{2\pi \mathrm{i} a_1/n_1 m_1} \boldsymbol{x}_1^{\sigma_1}, \dots, e^{2\pi \mathrm{i} a_l/n_l m_l} \boldsymbol{x}_l^{\sigma_l}, e^{2\pi \mathrm{i}/N} t).$$

Then:

(1) *There exists a small finite group $G \subset GL_n(\mathbb{C})$ such that $A_{d-1}/\Gamma \cong \mathbb{C}^n/G$; this isomorphism is the composition $\overline{r} \circ \overline{q} \circ \overline{p}^{-1}$, where $\overline{p} : \widetilde{A}_{d-1}/\widetilde{\Gamma} \xrightarrow{\cong} A_{d-1}/\Gamma$, $\overline{q} : \widetilde{A}_{d-1}/\widetilde{\Gamma} \xrightarrow{\cong} \mathbb{C}^n/H$, $\overline{r} : \mathbb{C}^n/H \xrightarrow{\cong} \mathbb{C}^n/G$ are induced from p, q, r.*

(2) *The isomorphism $\Psi := \overline{r} \circ \overline{q} \circ \overline{p}^{-1} : A_{d-1}/\Gamma \xrightarrow{\cong} \mathbb{C}^n/G$ in (1) is explicitly given by*

$$\Psi\big([\boldsymbol{x}_1, \dots, \boldsymbol{x}_l, t]\big) = \big[\boldsymbol{x}_1^{\ell_1 m_1'/d}, \dots, \boldsymbol{x}_l^{\ell_l m_l'/d}\big],$$

where $[\boldsymbol{x}_1, \dots, \boldsymbol{x}_l, t] \in A_{d-1}/\Gamma$ and $\big[\boldsymbol{x}_1^{\ell_1 m_1'/d}, \dots, \boldsymbol{x}_l^{\ell_l m_l'/d}\big] \in \mathbb{C}^n/G$ denote the images of $(\boldsymbol{x}_1, \dots, \boldsymbol{x}_l, t) \in A_{d-1}$ and $\big(\boldsymbol{x}_1^{\ell_1 m_1'/d}, \dots, \boldsymbol{x}_l^{\ell_l m_l'/d}\big) \in \mathbb{C}^n$ respectively.

PROOF. It remains to show (2). Since

$$\overline{p}\big([\boldsymbol{X}_1, \boldsymbol{X}_2, \dots, \boldsymbol{X}_l]\big) = [\boldsymbol{X}_1^d, \boldsymbol{X}_2^d, \dots, \boldsymbol{X}_l^d, \boldsymbol{X}_1 \boldsymbol{X}_2 \cdots \boldsymbol{X}_l],$$

we have $\overline{p}^{-1}\big([\boldsymbol{x}_1, \boldsymbol{x}_2, \dots, \boldsymbol{x}_l, t]\big) = \big[\boldsymbol{x}_1^{1/d}, \boldsymbol{x}_2^{1/d}, \dots, \boldsymbol{x}_l^{1/d}\big]$. Thus

$$\begin{aligned}\Psi\big([\boldsymbol{x}_1, \boldsymbol{x}_2, \dots, \boldsymbol{x}_l, t]\big) &= \overline{r} \circ \overline{q} \circ \overline{p}^{-1}\big([\boldsymbol{x}_1, \boldsymbol{x}_2, \dots, \boldsymbol{x}_l, t]\big) \\ &= \overline{r} \circ \overline{q}\big([\boldsymbol{x}_1^{1/d}, \boldsymbol{x}_2^{1/d}, \dots, \boldsymbol{x}_l^{1/d}]\big) = \overline{r}\big([\boldsymbol{x}_1^{m_1'/d}, \boldsymbol{x}_2^{m_2'/d}, \dots, \boldsymbol{x}_l^{m_l'/d}]\big) \\ &= \big[\boldsymbol{x}_1^{\ell_1 m_1'/d}, \boldsymbol{x}_2^{\ell_2 m_2'/d}, \dots, \boldsymbol{x}_l^{\ell_l m_l'/d}\big]. \ \square\end{aligned}$$

Correspondence between maps

We keep the notation above: Γ is the cyclic group of order N generated by the automorphism of A_{d-1} given by

$$\gamma : \ (\boldsymbol{x}_1, \dots, \boldsymbol{x}_l, t) \longmapsto (e^{2\pi \mathrm{i} a_1/n_1 m_1} \boldsymbol{x}_1^{\sigma_1}, \dots, e^{2\pi \mathrm{i} a_l/n_l m_l} \boldsymbol{x}_l^{\sigma_l}, e^{2\pi \mathrm{i}/N} t).$$

Define a holomorphic map $\Phi : A_{d-1} \to \mathbb{C}$ by

$$\Phi(x_1, x_2, \ldots, x_n, t) = t^N. \tag{8.3}$$

This, being Γ-invariant, descends to a holomorphic map $\overline{\Phi} : A_{d-1}/\Gamma \to \mathbb{C}$ (which is a local model of a degeneration of compact complex manifolds). We shall explicitly give the corresponding map $\mathbb{C}^n/G \to \mathbb{C}$ under the isomorphism $A_{d-1}/\Gamma \cong \mathbb{C}^n/G$ in Theorem 8.1.

Consider first the case $l = 1$, that is, $\gamma : (\boldsymbol{x}_1, t) \mapsto (e^{2\pi \mathrm{i} a_1/nm_1}\boldsymbol{x}_1, e^{2\pi\mathrm{i}/N}t)$. Explicitly γ is of the form (below, write a_1, m_1, L_1 as a, m, L etc):

$$\gamma : (x_1, \ldots, x_n, t) \longmapsto (e^{2\pi\mathrm{i}a/nm}x_{\sigma(1)}, \ldots, e^{2\pi\mathrm{i}a/nm}x_{\sigma(n)}, e^{2\pi\mathrm{i}/N}t),$$

where $\sigma \in \mathfrak{S}_n$ is a cyclic permutation of length n. In this case, $c = nm$, $m' = 1$, $L = nm$, $N = (m')^n c = nm$. Accordingly $\ell := Nc/nmL = 1$ and $d = N(\frac{a}{m} + \kappa) = na + nm\kappa$. The following then hold:

Lemma 8.2.

(i) *Let $G \subset GL_n(\mathbb{C})$ be the small finite group in Theorem 8.1. Then the holomorphic map $\phi : \mathbb{C}^n \to \mathbb{C}$ given by $\phi(v_1, \ldots, v_n) = (v_1 \cdots v_n)^{nm}$ is G-invariant.* (So ϕ descends to a holomorphic map $\overline{\phi} : \mathbb{C}^n/G \to \mathbb{C}$.)

(ii) *Let $\Phi : A_{d-1} \to \mathbb{C}$ be the Γ-invariant map given by* (8.3). *Under the isomorphism $\Psi : A_{d-1}/\Gamma \xrightarrow{\cong} \mathbb{C}^n/G$ in Theorem 8.1, $\overline{\Phi} : A_{d-1}/\Gamma \to \mathbb{C}$ corresponds to $\overline{\phi}$, that is, $\overline{\Phi} = \overline{\phi} \circ \Psi$.*

Proof. (i): As seen in Theorem 9.1 (3) below, $G = \{g_{j,\boldsymbol{p}} : \boldsymbol{p} \in \Lambda^{(j)},\ j = 1, 2, \ldots, N\}$, where $g_{j,\boldsymbol{p}}$ is $\overline{\overline{\alpha}}^j \overline{\overline{\beta}}_{j,\boldsymbol{p}}$ therein. Explicitly

$$g_{j,\boldsymbol{p}} : (v_1, \ldots, v_n) \longmapsto (e^{2\pi\mathrm{i}(ja+nmp_1)/nmd}v_{\sigma(1)}, \ldots, e^{2\pi\mathrm{i}(ja+nmp_n)/nmd}v_{\sigma(n)}).$$

For simplicity, set $\zeta_i := e^{2\pi\mathrm{i}(ja+nmp_i)/nmd}$, then

$$g_{j,\boldsymbol{p}} : (v_1, v_2, \ldots, v_n) \longmapsto (\zeta_1 v_{\sigma(1)}, \zeta_2 v_{\sigma(2)}, \ldots, \zeta_n v_{\sigma(n)}).$$

It suffices to show $\phi \circ g_{j,\boldsymbol{p}}(v_1, v_2, \ldots, v_n) = \phi(v_1, v_2, \ldots, v_n)$. Note first that $(\zeta_1\zeta_2 \cdots \zeta_n)^{nm} = 1$, indeed

$$\begin{aligned}(\zeta_1\zeta_2\cdots\zeta_n)^{nm} &= e^{2\pi\mathrm{i}\{jna+nm(p_1+\cdots+p_n)\}/d} \\ &= e^{2\pi\mathrm{i}(jna+nmj\kappa)/d} \qquad \text{as } (p_1, \ldots, p_n) \in \Lambda^{(j)} \\ &= e^{2\pi\mathrm{i}j} = 1.\end{aligned}$$

Then

$$\begin{aligned}\phi \circ g_{j,\boldsymbol{p}}(v_1, v_2, \ldots, v_n) &= \phi(\zeta_1 v_{\sigma(1)}, \zeta_2 v_{\sigma(2)}, \ldots, \zeta_n v_{\sigma(n)})\\ &= (\zeta_1\zeta_2\cdots\zeta_n)^{nm}(v_{\sigma(1)}v_{\sigma(2)}\cdots v_{\sigma(n)})^{nm}\\ &= (v_{\sigma(1)}v_{\sigma(2)}\cdots v_{\sigma(n)})^{nm} \quad \text{as } (\zeta_1\zeta_2\cdots\zeta_n)^{nm} = 1\\ &= (v_1v_2\cdots v_n)^{nm} = \phi(v_1, v_2, \ldots, v_n).\end{aligned}$$

(ii): From Theorem 8.1 (2), $\Psi\big([x_1, \ldots, x_n, t]\big) = [x_1^{1/d}, \ldots, x_n^{1/d}]$. Thus

$$\begin{aligned}\overline{\phi} \circ \Psi\big([x_1, \ldots, x_n, t]\big) &= \overline{\phi}\big([x_1^{1/d}, \ldots, x_n^{1/d}]\big) = (x_1\cdots x_n)^{nm/d}\\ &= t^{nm} \quad \text{as } x_1\cdots x_n = t^d\\ &= t^N \quad \text{as } N = nm\\ &= \overline{\Phi}\big([x_1, \ldots, x_n, t]\big) \quad \text{by definition.} \ \square\end{aligned}$$

We turn to the general case:

$$(*) \quad \gamma : (\boldsymbol{x}_1, \ldots, \boldsymbol{x}_l, t) \longmapsto (e^{2\pi \mathrm{i} a_1/n_1m_1}\boldsymbol{x}_1^{\sigma_1}, \ldots, e^{2\pi \mathrm{i} a_l/n_lm_l}\boldsymbol{x}_l^{\sigma_l}, e^{2\pi\mathrm{i}/N}t).$$

As for Lemma 8.2, we can show the following:

THEOREM 8.3. *Write $(v_1, \ldots, v_n) \in \mathbb{C}^n$ as $(\boldsymbol{v}_1, \ldots, \boldsymbol{v}_l) \in \mathbb{C}^n = \mathbb{C}^{n_1} \times \cdots \times \mathbb{C}^{n_l}$. For each permutation σ_k appearing in $(*)$, let J_k be its cycle, that is, $J_k = \{i : v_i \in \boldsymbol{v}_k\}$. Then:*

(1) *Let $G \subset GL_n(\mathbb{C})$ be the small finite group in Theorem 8.1 and $\phi : \mathbb{C}^n \to \mathbb{C}$ be a holomorphic map given by $\phi(v_1, \ldots, v_n) = \prod_{k=1}^{l}\Big(\prod_{i \in J_k} v_i\Big)^{L_k}$, where L_k is the integer given by (6.9). Then ϕ is G-invariant.*

(2) *Let $\Phi : A_{d-1} \to \mathbb{C}$ be the Γ-invariant map given by (8.3). Under the isomorphism $\Psi : A_{d-1}/\Gamma \xrightarrow{\cong} \mathbb{C}^n/G$ in Theorem 8.1, $\overline{\Phi} : A_{d-1}/\Gamma \to \mathbb{C}$ corresponds to the descent $\overline{\phi} : \mathbb{C}^n/G \to \mathbb{C}$.*

9. Explicit Forms of Elements of $\widetilde{\Gamma}$, H, G

We subsequently deal with many notations — to reduce the burden of memorizing them, H, G are denoted by $\overline{\Gamma}$, $\overline{\overline{\Gamma}}$. Recall:

- Express $\gamma = \alpha\beta$, where

$$\begin{cases} \alpha : (\boldsymbol{x}_1, \dots, \boldsymbol{x}_l, t) \\ \qquad\qquad \longmapsto (e^{2\pi\mathrm{i}a_1/n_1m_1}\boldsymbol{x}_1^{\sigma_1}, \dots, e^{2\pi\mathrm{i}a_l/n_lm_l}\boldsymbol{x}_l^{\sigma_l}, e^{2\pi\mathrm{i}(1/N-\kappa/d)}t), \\ \beta : (\boldsymbol{x}_1, \dots, \boldsymbol{x}_l, t) \longmapsto (\boldsymbol{x}_1, \dots, \boldsymbol{x}_l, e^{2\pi\mathrm{i}\kappa/d}t). \end{cases}$$

- Set $\Lambda^{(j)} := \left\{\boldsymbol{p} = (p_1, \dots, p_n) \in \mathbb{Z}^n \ : \ 0 \le p_i \le d, \ \sum_{i=1}^{n} \frac{p_i}{d} \equiv \frac{j\kappa}{d} \bmod \mathbb{Z}\right\}$ (see (4.4)).

- For $\boldsymbol{p} \in \Lambda^{(j)}$, let $\widetilde{\alpha}$, $\widetilde{\beta}_{j,\boldsymbol{p}}$ be the lifts of α, β given by (4.8), and $\overline{\alpha}$, $\overline{\beta}_{j,\boldsymbol{p}}$ be their descents with respect to q. Let $\overline{\overline{\alpha}}$, $\overline{\overline{\beta}}_{j,\boldsymbol{p}}$ be the descents of $\overline{\alpha}$, $\overline{\beta}_{j,\boldsymbol{p}}$ with respect to r.

The following then holds:

Theorem 9.1.

(1) $\widetilde{\Gamma} = \left\{\widetilde{\alpha}^j\widetilde{\beta}_{j,\boldsymbol{p}} \ : \ \boldsymbol{p} \in \Lambda^{(j)}, \ j = 1, 2, \dots, N\right\}$, *where*

$$\begin{cases} \widetilde{\alpha} : (\boldsymbol{X}_1, \dots, \boldsymbol{X}_l) \longmapsto \left(e^{2\pi\mathrm{i}a_1/n_1m_1d}\boldsymbol{X}_1^{\sigma_1}, \dots, e^{2\pi\mathrm{i}a_l/n_lm_ld}\boldsymbol{X}_l^{\sigma_l}\right), \\ \widetilde{\beta}_{j,\boldsymbol{p}} : (\boldsymbol{X}_1, \dots, \boldsymbol{X}_l) \longmapsto \left(\widetilde{\beta}_{j,\boldsymbol{p}_1}(\boldsymbol{X}_1), \dots, \widetilde{\beta}_{j,\boldsymbol{p}_l}(\boldsymbol{X}_l)\right). \end{cases}$$

(2) $\overline{\Gamma} = \left\{\overline{\alpha}^j\overline{\beta}_{j,\boldsymbol{p}} \ : \ \boldsymbol{p} \in \Lambda^{(j)}, \ j = 1, 2, \dots, N\right\}$, *where*

$$\begin{cases} \overline{\alpha} : (\boldsymbol{u}_1, \dots, \boldsymbol{u}_l) \longmapsto \left(e^{2\pi\mathrm{i}a_1/cd}\boldsymbol{u}_1^{\sigma_1}, \dots, e^{2\pi\mathrm{i}a_l/cd}\boldsymbol{u}_l^{\sigma_l}\right), \\ \overline{\beta}_{j,\boldsymbol{p}} : (\boldsymbol{u}_1, \dots, \boldsymbol{u}_l) \longmapsto \left((\widetilde{\beta}_{j,\boldsymbol{p}_1})^{m_1'}(\boldsymbol{u}_1), \dots, (\widetilde{\beta}_{j,\boldsymbol{p}_l})^{m_l'}(\boldsymbol{u}_l)\right). \end{cases}$$

(3) $\overline{\overline{\Gamma}} = \left\{\overline{\overline{\alpha}}^j\overline{\overline{\beta}}_{j,\boldsymbol{p}} \ : \ \boldsymbol{p} \in \Lambda^{(j)}, \ j = 1, 2, \dots, N\right\}$, *where*

$$\begin{cases} \overline{\overline{\alpha}} : (\boldsymbol{v}_1, \dots, \boldsymbol{v}_l) \longmapsto \left(e^{2\pi\mathrm{i}a_1\ell_1/cd}\boldsymbol{v}_1^{\sigma_1}, \dots, e^{2\pi\mathrm{i}a_l\ell_l/cd}\boldsymbol{v}_l^{\sigma_l}\right), \\ \overline{\overline{\beta}}_{j,\boldsymbol{p}} : (\boldsymbol{v}_1, \dots, \boldsymbol{v}_l) \longmapsto \left((\widetilde{\beta}_{j,\boldsymbol{p}_1})^{m_1'\ell_1}(\boldsymbol{v}_1), \dots, (\widetilde{\beta}_{j,\boldsymbol{p}_l})^{m_l'\ell_l}(\boldsymbol{v}_l)\right). \end{cases}$$

Namely

$$
\begin{tikzcd}[column sep=small]
 & & \widetilde{\Gamma}=\{\widetilde{\alpha}^j\widetilde{\beta}_{j,\boldsymbol{p}}\} \arrow[dl, "q_*"'] \arrow[dr, "p_*"] & \\
 & \overline{\Gamma}=\{\overline{\alpha}^j\overline{\beta}_{j,\boldsymbol{p}}\} \arrow[dl, "r_*"'] & & \Gamma=\{\gamma^j=\alpha^j\beta^j\}. \\
\overline{\overline{\Gamma}}=\{\overline{\overline{\alpha}}^j\overline{\overline{\beta}}_{j,\boldsymbol{p}}\} & & &
\end{tikzcd}
\tag{9.1}
$$

PROOF. (1) and (2) are already shown in Lemma 4.11. (3) follows from (2), as the descent homomorphism $r_* : \overline{\Gamma} \to \overline{\overline{\Gamma}}$ is surjective and the covering $r : \mathbb{C}^n \to \mathbb{C}^n$ is given by $r : (\boldsymbol{u}_1, \boldsymbol{u}_2, \dots, \boldsymbol{u}_l) \mapsto (\boldsymbol{u}_1^{\ell_1}, \boldsymbol{u}_2^{\ell_2}, \dots, \boldsymbol{u}_l^{\ell_l})$ (see Lemma 7.1). $\square$

Note:

$\alpha,\ \beta \notin \Gamma$	$\widetilde{\alpha},\ \widetilde{\beta}_{j,\boldsymbol{p}} \notin \widetilde{\Gamma}$	$\overline{\alpha},\ \overline{\beta}_{j,\boldsymbol{p}} \notin \overline{\Gamma}$	$\overline{\overline{\alpha}},\ \overline{\overline{\beta}}_{j,\boldsymbol{p}} \notin \overline{\overline{\Gamma}}$
$\alpha\beta \in \Gamma$	$\widetilde{\alpha}^j\widetilde{\beta}_{j,\boldsymbol{p}} \in \widetilde{\Gamma}$	$\overline{\alpha}^j\overline{\beta}_{j,\boldsymbol{p}} \in \overline{\Gamma}$	$\overline{\overline{\alpha}}^j\overline{\overline{\beta}}_{j,\boldsymbol{p}} \in \overline{\overline{\Gamma}}$

Here explicitly:

LEMMA 9.2. *Setting* $\zeta_k := e^{2\pi \mathrm{i} m'_k/d}$, *then for* $\boldsymbol{p} = (p_1, p_2, \dots, p_n) \in \Lambda^{(j)}$,

(1) $\widetilde{\beta}_{j,\boldsymbol{p}} : (X_1, X_2, \dots, X_n) \longmapsto (e^{2\pi \mathrm{i} p_1/d} X_1, e^{2\pi \mathrm{i} p_2/d} X_2, \dots, e^{2\pi \mathrm{i} p_n/d} X_n)$.

(2) $\overline{\beta}_{j,\boldsymbol{p}} : (X_1, X_2, \dots, X_n) \longmapsto (Y_1, Y_2, \dots, Y_l)$, *where*

$$
\begin{aligned}
Y_1 &= (\underbrace{\zeta_1^{p_1} X_1, \zeta_1^{p_2} X_2, \dots, \zeta_1^{p_{n_1}} X_{n_1}}_{n_1}) \\
Y_2 &= (\underbrace{\zeta_2^{p_{n_1+1}} X_{n_1+1}, \zeta_2^{p_{n_1+2}} X_{n_1+2}, \dots, \zeta_2^{p_{n_1+n_2}} X_{n_1+n_2}}_{n_2}) \\
Y_3 &= (\underbrace{\zeta_3^{p_{n_1+n_2+1}} X_{n_1+n_2+1}, \zeta_3^{p_{n_1+n_2+2}} X_{n_1+n_2+2}, \dots, \zeta_3^{p_{n_1+n_2+n_3}} X_{n_1+n_2+n_3}}_{n_3}) \\
&\cdots .
\end{aligned}
$$

(3) $\overline{\overline{\beta}}_{j,\boldsymbol{p}} : (X_1, X_2, \ldots, X_n) \longmapsto (Z_1, Z_2, \ldots, Z_l)$, *where*

$$Z_1 = (\underbrace{\zeta_1^{\ell_1 p_1} X_1, \zeta_1^{\ell_1 p_2} X_2, \ldots, \zeta_1^{\ell_1 p_{n_1}} X_{n_1}}_{n_1})$$

$$Z_2 = (\underbrace{\zeta_2^{\ell_2 p_{n_1+1}} X_{n_1+1}, \zeta_2^{\ell_2 p_{n_1+2}} X_{n_1+2}, \ldots, \zeta_2^{\ell_2 p_{n_1+n_2}} X_{n_1+n_2}}_{n_2})$$

$$Z_3 = (\underbrace{\zeta_3^{\ell_3 p_{n_1+n_2+1}} X_{n_1+n_2+1}, \zeta_3^{\ell_3 p_{n_1+n_2+2}} X_{n_1+n_2+2}, \ldots, \zeta_3^{\ell_3 p_{n_1+n_2+n_3}} X_{n_1+n_2+n_3}}_{n_3})$$

$$\cdots.$$

PROOF. (1): Write $\boldsymbol{p} = (p_1, p_2, \ldots, p_n)$ as $(\boldsymbol{p}_1, \boldsymbol{p}_2, \ldots, \boldsymbol{p}_l) \in \mathbb{Z}^{n_1} \times \mathbb{Z}^{n_2} \times \cdots \times \mathbb{Z}^{n_l}$. Note that (see Theorem 9.1 (1))

$\widetilde{\beta}_{j,\boldsymbol{p}} : (\boldsymbol{X}_1, \boldsymbol{X}_2, \ldots, \boldsymbol{X}_l) \longmapsto \big(\widetilde{\beta}_{j,\boldsymbol{p}_1}(\boldsymbol{X}_1), \widetilde{\beta}_{j,\boldsymbol{p}_2}(\boldsymbol{X}_2), \ldots, \widetilde{\beta}_{j,\boldsymbol{p}_l}(\boldsymbol{X}_l)\big)$,

where $\widetilde{\beta}_{j,\boldsymbol{p}_i} : \boldsymbol{X}_i = (X_{j_1}, \ldots, X_{j_{n_i}}) \mapsto (e^{2\pi \mathrm{i} p_{j_1}/d} X_{j_1}, \ldots, e^{2\pi \mathrm{i} p_{j_{n_i}}/d} X_{j_{n_i}})$. In the cooridinates $(X_1, X_2, \ldots, X_n)$,

$\widetilde{\beta}_{j,\boldsymbol{p}} : (X_1, X_2, \ldots, X_n) \longmapsto (e^{2\pi \mathrm{i} p_1/d} X_1, e^{2\pi \mathrm{i} p_2/d} X_2, \ldots, e^{2\pi \mathrm{i} p_n/d} X_n)$.

(2): Note that (see Theorem 9.1 (2))

$\overline{\beta}_{j,\boldsymbol{p}} : (\boldsymbol{X}_1, \ldots, \boldsymbol{X}_l) \longmapsto \big((\widetilde{\beta}_{j,\boldsymbol{p}_1})^{m'_1}(\boldsymbol{X}_1), \ldots, (\widetilde{\beta}_{j,\boldsymbol{p}_l})^{m'_l}(\boldsymbol{X}_l)\big)$.

Writing this in the cooridinates $(X_1, X_2, \ldots, X_n)$ yields the assertion.

(3): Note that (see Theorem 9.1 (3))

$\overline{\overline{\beta}}_{j,\boldsymbol{p}} : (\boldsymbol{X}_1, \ldots, \boldsymbol{X}_l) \longmapsto \big((\widetilde{\beta}_{j,\boldsymbol{p}_1})^{m'_1 \ell_1}(\boldsymbol{X}_1), \ldots, (\widetilde{\beta}_{j,\boldsymbol{p}_l})^{m'_l \ell_l}(\boldsymbol{X}_l)\big)$.

Writing this in the cooridinates $(X_1, X_2, \ldots, X_n)$ yields the assertion. $\square$

REMARK 9.3. If $\sigma \neq \mathrm{id}$, $\overline{\Gamma}\,(= H)$ is generally not abelian — this is also the case for $\overline{\overline{\Gamma}}\,(= G)$. We will determine when $\widetilde{\Gamma}$ (and G) is abelian. See Theorem 10.11.

9.1. Generators of $\widetilde{\Gamma}$, $\overline{\Gamma}\,(= H)$ and $\overline{\overline{\Gamma}}\,(= G)$

The covering maps p, q, r induce surjective homomorphisms (*descent homomorphisms*) $p_* : \widetilde{\Gamma} \to \Gamma$, $q_* : \widetilde{\Gamma} \to \overline{\Gamma}$, $r_* : \overline{\Gamma} \to \overline{\overline{\Gamma}}$ (see (9.1)). As q_* and r_* are surjective, generators of $\widetilde{\Gamma}$ descend to those of $\overline{\Gamma}$, and then, to those of $\overline{\overline{\Gamma}}$. Subsequently we will explicitly give generators of $\widetilde{\Gamma}$ and descend them to $\overline{\Gamma}$, and then to $\overline{\overline{\Gamma}}$.

First take a lift $\widetilde{\gamma} := \widetilde{\alpha}\widetilde{\beta}_{1,\boldsymbol{p}}$ of γ (recall $\widetilde{\alpha}^j\widetilde{\beta}_{j,\boldsymbol{p}}$ is a lift of γ^j; Corollary 4.6). To simplify discussion, for $\boldsymbol{p}$ we take $\boldsymbol{q} := (0,\dots,0,\overset{\sigma(n)}{\check{\kappa}},0\dots,0)$:

$$\widetilde{\alpha}:\ (\boldsymbol{X}_1,\dots,\boldsymbol{X}_l)\longmapsto\bigl(e^{2\pi\mathrm{i}a_1/n_1m_1d}\boldsymbol{X}_1^{\sigma_1},\dots,e^{2\pi\mathrm{i}a_l/n_lm_ld}\boldsymbol{X}_l^{\sigma_l}\bigr),\tag{9.2}$$

$$\widetilde{\beta}_{1,\boldsymbol{q}}:\ (X_1,X_2,\dots,X_n)\longmapsto(X_1,X_2,\dots,e^{2\pi\mathrm{i}\kappa/d}X_{\sigma(n)},\dots,X_n).\tag{9.3}$$

We next take lifts $\widetilde{\mathrm{id}}_1,\widetilde{\mathrm{id}}_2,\dots,\widetilde{\mathrm{id}}_{n-1}$ of $\mathrm{id}\in\Gamma$ as follows:

$$\widetilde{\mathrm{id}}_i:(X_1,\dots,X_n)\longmapsto(X_1,\dots,X_{i-1},e^{2\pi\mathrm{i}/d}X_i,X_{i+1},\dots,e^{-2\pi\mathrm{i}/d}X_n).\tag{9.4}$$

Recall that $\widetilde{\Gamma}=\bigl\{\widetilde{\alpha}^j\widetilde{\beta}_{j,\boldsymbol{p}}\ :\ \boldsymbol{p}\in\Lambda^{(j)},\ j=1,2,\dots,N\bigr\}$ (Theorem 9.1 (1)).

LEMMA 9.4. *Set* $\delta:=(\widetilde{\beta}_{1,\sigma(\boldsymbol{q})})^j$ (*note in general* $\delta\notin\widetilde{\Gamma}$), *and for simplicity write* $\widetilde{\gamma}$, $\widetilde{\mathrm{id}}_i$ *as* φ, ψ_i. *Then*:

(1) $\varphi^j\bigl(\psi_{\sigma(n)}\psi_{\sigma^2(n)}\cdots\psi_{\sigma^j(n)}\bigr)^{-\kappa}=\widetilde{\alpha}^j\delta$.

(2) *For* $\boldsymbol{p}=(p_1,\dots,p_n)\in\Lambda^{(j)}$, $(\psi_1)^{p_1}(\psi_2)^{p_2}\cdots(\psi_{n-1})^{p_{n-1}}=\delta^{-1}\widetilde{\beta}_{j,\boldsymbol{p}}$.

(3) *For* $\boldsymbol{p}=(p_1,\dots,p_n)\in\Lambda^{(j)}$,

$$\varphi^j\bigl(\psi_{\sigma(n)}\psi_{\sigma^2(n)}\cdots\psi_{\sigma^j(n)}\bigr)^{-\kappa}(\psi_1)^{p_1}(\psi_2)^{p_2}\cdots(\psi_{n-1})^{p_{n-1}}=\widetilde{\alpha}^j\widetilde{\beta}_{j,\boldsymbol{p}}.$$

PROOF. (1): Note first that

$$\begin{aligned}\varphi^j&=(\widetilde{\alpha}\widetilde{\beta}_{1,\boldsymbol{q}})^j\\&=\widetilde{\alpha}^j\widetilde{\beta}_{1,\sigma^{-j+1}(\boldsymbol{q})}\cdots\widetilde{\beta}_{1,\sigma^{-1}(\boldsymbol{q})}\widetilde{\beta}_{1,\boldsymbol{q}}\qquad\text{as }\widetilde{\beta}_{1,\boldsymbol{q}}\widetilde{\alpha}=\widetilde{\alpha}\widetilde{\beta}_{1,\sigma^{-1}(\boldsymbol{q})}\text{ (Lemma 4.8).}\end{aligned}$$

Here $(\psi_{\sigma^i(n)})^{-\kappa}=(\widetilde{\beta}_{1,\sigma^{-i+1}(\boldsymbol{q})})^{-1}\widetilde{\beta}_{1,\sigma(\boldsymbol{q})}$ and $\delta=(\widetilde{\beta}_{1,\sigma(\boldsymbol{q})})^j$, so

$$\bigl(\psi_{\sigma(n)}\psi_{\sigma^2(n)}\cdots\psi_{\sigma^j(n)}\bigr)^{-\kappa}=(\widetilde{\beta}_{1,\sigma^{-j+1}(\boldsymbol{q})}\cdots\widetilde{\beta}_{1,\sigma^{-1}(\boldsymbol{q})}\widetilde{\beta}_{1,\boldsymbol{q}})^{-1}\delta.$$

Thus $\varphi^j\bigl(\psi_{\sigma(n)}\psi_{\sigma^2(n)}\cdots\psi_{\sigma^j(n)}\bigr)^{-\kappa}=\widetilde{\alpha}^j\delta$.

(2): Since $\boldsymbol{p}\in\Lambda^{(j)}$, we have

$$(*)\qquad-(p_1+\cdots+p_{n-1})/d\equiv(p_n-j\kappa)/d\bmod\mathbb{Z}.$$

Now

$$\begin{aligned}
&(\psi_1)^{p_1}(\psi_2)^{p_2}\cdots(\psi_{n-1})^{p_{n-1}}(X_1,\dots,X_n)\\
&\qquad = (e^{2\pi \mathrm{i} p_1/d}X_1,\dots,e^{2\pi \mathrm{i} p_{n-1}/d}X_{n-1},e^{-2\pi \mathrm{i}(p_1+\cdots+p_{n-1})/d}X_n)\\
&\qquad = (e^{2\pi \mathrm{i} p_1/d}X_1,\dots,e^{2\pi \mathrm{i} p_{n-1}/d}X_{n-1},e^{2\pi \mathrm{i}(p_n-j\kappa)/d}X_n) \quad \text{by } (*)\\
&\qquad = \delta^{-1}\widetilde{\beta}_{j,\boldsymbol{p}}(X_1,\dots,X_n).
\end{aligned}$$

The equation of (3) is the product of (1) and (2). □

From Lemma 9.4 (3), any element of $\widetilde{\Gamma}$ is written as a product of $\widetilde{\gamma}$, $\widetilde{\mathrm{id}}_i$ $(i=1,2,\dots,n-1)$, so they generate $\widetilde{\Gamma}$, therefore:

Corollary 9.5. *Set* $\overline{\gamma} := q_*(\widetilde{\gamma})$, $\overline{\mathrm{id}}_i := q_*(\widetilde{\mathrm{id}}_i)$ *and* $\overline{\overline{\gamma}} := r_*(\overline{\gamma})$, $\overline{\overline{\mathrm{id}}}_i := r_*(\overline{\mathrm{id}}_i)$, *then*:

(1) $\widetilde{\gamma}$, $\widetilde{\mathrm{id}}_1,\widetilde{\mathrm{id}}_2,\dots,\widetilde{\mathrm{id}}_{n-1}$ *generate* $\widetilde{\Gamma}$.

(2) $\overline{\gamma}$, $\overline{\mathrm{id}}_1,\overline{\mathrm{id}}_2,\dots,\overline{\mathrm{id}}_{n-1}$ *generate* $\overline{\Gamma}\,(=H)$.

(3) $\overline{\overline{\gamma}}$, $\overline{\overline{\mathrm{id}}}_1,\overline{\overline{\mathrm{id}}}_2,\dots,\overline{\overline{\mathrm{id}}}_{n-1}$ *generate* $\overline{\overline{\Gamma}}\,(=G)$.

(9.5)
$$\begin{tikzcd}
 & & \widetilde{\Gamma}\ni\widetilde{\gamma},\widetilde{\mathrm{id}}_1,\widetilde{\mathrm{id}}_2,\dots,\widetilde{\mathrm{id}}_{n-1} \arrow[dl, "q_*"'] \arrow[dr, "p_*"] & \\
 & \overline{\gamma},\overline{\mathrm{id}}_1,\overline{\mathrm{id}}_2,\dots,\overline{\mathrm{id}}_{n-1}\in\overline{\Gamma}\,(=H) \arrow[dl, "r_*"'] & & \Gamma\ni\gamma,\ \mathrm{id}. \\
\overline{\overline{\gamma}},\overline{\overline{\mathrm{id}}}_1,\overline{\overline{\mathrm{id}}}_2,\dots,\overline{\overline{\mathrm{id}}}_{n-1}\in\overline{\overline{\Gamma}}\,(=G) & & &
\end{tikzcd}$$

We summarize the explicit forms of relevant automorphisms. Set $\ell_k := Nc/n_k m_k L_k$ $(k=1,2,\dots,l)$, where L_k is the integer given by (6.9). Then:

THEOREM 9.6.

(1) $\widetilde{\gamma} = \widetilde{\alpha}\widetilde{\beta}_{1,\boldsymbol{q}}$, *where*

$$\begin{cases} \widetilde{\alpha} : \ (\boldsymbol{X}_1, \ldots, \boldsymbol{X}_l) \longmapsto \big(e^{2\pi \mathrm{i} a_1/n_1 m_1 d} \boldsymbol{X}_1^{\sigma_1}, \ldots, e^{2\pi \mathrm{i} a_l/n_l m_l d} \boldsymbol{X}_l^{\sigma_l}\big), \\ \widetilde{\beta}_{1,\boldsymbol{q}} : (X_1, X_2, \ldots, X_n) \longmapsto (X_1, X_2, \ldots, e^{2\pi \mathrm{i}\kappa/d} X_{\sigma(n)}, \ldots, X_n). \end{cases}$$

$$\widetilde{\mathrm{id}}_i : (X_1, X_2, \ldots, X_n) \longmapsto (X_1, X_2, \ldots, e^{2\pi \mathrm{i}/d} X_i, \ldots, e^{-2\pi \mathrm{i}/d} X_n).$$

(2) $\overline{\gamma} = \overline{\alpha}\overline{\beta}_{1,\boldsymbol{q}}$, *where*

$$\begin{cases} \overline{\alpha} : \ (\boldsymbol{X}_1, \ldots, \boldsymbol{X}_l) \longmapsto \big(e^{2\pi \mathrm{i} a_1/cd} \boldsymbol{X}_1^{\sigma_1}, \ldots, e^{2\pi \mathrm{i} a_l/cd} \boldsymbol{X}_l^{\sigma_l}\big), \\ \overline{\beta}_{1,\boldsymbol{q}} : (X_1, X_2, \ldots, X_n) \longmapsto \\ \qquad\qquad (X_1, X_2, \ldots, e^{2\pi \mathrm{i} m_l' \kappa/d} X_{\sigma(n)}, \ldots, X_n). \end{cases}$$

$$\overline{\mathrm{id}}_i : (X_1, X_2, \ldots, X_n) \longmapsto$$
$$(X_1, X_2, \ldots, e^{2\pi \mathrm{i} m_k'/d} X_i, \ldots, e^{-2\pi \mathrm{i} m_l'/d} X_n) \quad (\textit{say } X_i \in \boldsymbol{X}_k).$$

(3) $\overline{\overline{\gamma}} = \overline{\overline{\alpha}}\overline{\overline{\beta}}_{1,\boldsymbol{q}}$, *where*

$$\begin{cases} \overline{\overline{\alpha}} : \ (\boldsymbol{X}_1, \ldots, \boldsymbol{X}_l) \longmapsto \big(e^{2\pi \mathrm{i} a_1 \ell_1/cd} \boldsymbol{X}_1^{\sigma_1}, \ldots, e^{2\pi \mathrm{i} a_l \ell_l/cd} \boldsymbol{X}_l^{\sigma_l}\big), \\ \overline{\overline{\beta}}_{1,\boldsymbol{q}} : (X_1, X_2, \ldots, X_n) \longmapsto \\ \qquad\qquad (X_1, X_2, \ldots, e^{2\pi \mathrm{i} m_l' \ell_l \kappa/d} X_{\sigma(n)}, \ldots, X_n). \end{cases}$$

$$\overline{\overline{\mathrm{id}}}_i : (X_1, X_2, \ldots, X_n) \longmapsto$$
$$(X_1, X_2, \ldots, e^{2\pi \mathrm{i} m_k' \ell_k/d} X_i, \ldots, e^{-2\pi \mathrm{i} m_l' \ell_l/d} X_n) \quad (\textit{say } X_i \in \boldsymbol{X}_k).$$

PROOF. (1): $\widetilde{\gamma} = \widetilde{\alpha}\widetilde{\beta}_{1,\boldsymbol{q}}$ is the definition of $\widetilde{\gamma}$, and the explicit forms of $\widetilde{\alpha}$, $\widetilde{\beta}_{1,\boldsymbol{q}}$, $\widetilde{\mathrm{id}}_i$ are respectively given by (9.2), (9.3), and (9.4), confirming (1). (2) is the descent of (1) with respect to q: Writing $(X_1, \ldots, X_n) \in \mathbb{C}^n$ as $(\boldsymbol{X}_1, \ldots, \boldsymbol{X}_l) \in \mathbb{C}^{n_1} \times \mathbb{C}^{n_l}$, then by (6.4),

$$q : (\boldsymbol{X}_1, \boldsymbol{X}_2, \ldots, \boldsymbol{X}_l) \longmapsto (\boldsymbol{X}_1^{m_1'}, \boldsymbol{X}_2^{m_2'}, \ldots, \boldsymbol{X}_l^{m_l'}). \tag{9.6}$$

Similarly (3) is the descent of (2) with respect to $r : (\boldsymbol{u}_1, \boldsymbol{u}_2, \ldots, \boldsymbol{u}_l) \mapsto (\boldsymbol{u}_1^{\ell_1}, \boldsymbol{u}_2^{\ell_2}, \ldots, \boldsymbol{u}_l^{\ell_l})$ (this explicit form of r is given in Lemma 7.1). $\square$

Note that while $\widetilde{\mathrm{id}}_i \in \widetilde{\Gamma}$ is a lift of id $\in \Gamma$, $\widetilde{\mathrm{id}}_i$ itself is *not* the identity map; neither are its descents $\overline{\mathrm{id}}_i$, $\overline{\overline{\mathrm{id}}}_i$.

LEMMA 9.7. *The set of lifts of* id $\in \Gamma$ *is given by*

$$\left\{(\widetilde{\mathrm{id}}_1)^{k_1}(\widetilde{\mathrm{id}}_2)^{k_2}\cdots(\widetilde{\mathrm{id}}_{n-1})^{k_{n-1}} \ : \ k_i \in \mathbb{Z},\ 0 \le k_i < d\right\}.$$

PROOF. For simplicity, set $\widetilde{\mathrm{id}}_{\boldsymbol{k}} := (\widetilde{\mathrm{id}}_1)^{k_1}(\widetilde{\mathrm{id}}_2)^{k_2}\cdots(\widetilde{\mathrm{id}}_{n-1})^{k_{n-1}}$, where $\boldsymbol{k} = (k_1, k_2, \ldots, k_{n-1})$. Note that $\widetilde{\mathrm{id}}_{\boldsymbol{k}}$ is a lift of id $\in \Gamma$ as $\widetilde{\mathrm{id}}_1, \widetilde{\mathrm{id}}_2, \ldots, \widetilde{\mathrm{id}}_{n-1}$ are lifts of id $\in \Gamma$. Note next that explicitly

$$\begin{aligned}\widetilde{\mathrm{id}}_{\boldsymbol{k}} : (X_1, \ldots, X_n)& \\ &\mapsto (e^{2\pi \mathrm{i} k_1/d}X_1, \ldots e^{2\pi \mathrm{i} k_{n-1}/d}X_{n-1}, e^{-2\pi\mathrm{i}(k_1+\cdots+k_{n-1})/d}X_n).\end{aligned}$$

So $\widetilde{\mathrm{id}}_{\boldsymbol{k}} \ne \widetilde{\mathrm{id}}_{\boldsymbol{l}}$ if $\boldsymbol{k} \ne \boldsymbol{l}$, and the elements of $S := \{\widetilde{\mathrm{id}}_{\boldsymbol{k}} \ : \ \boldsymbol{k} \in \mathbb{Z}^{n-1}, 0 \le k_i < d\}$ are all distinct. Thus S consists of d^{n-1} elements. Since $p : \widetilde{A}_{d-1} \to A_{d-1}$ is d^{n-1}-fold, this implies that S exhausts all lifts of id $\in \Gamma$. □

From the explicit forms of $\widetilde{\mathrm{id}}_i$, $\overline{\mathrm{id}}_i$, $\overline{\overline{\mathrm{id}}}_i$ in Theorem 9.6, the following is clear:

COROLLARY 9.8. $\widetilde{\mathrm{id}}_i \ne \widetilde{\mathrm{id}}_j$, $\overline{\mathrm{id}}_i \ne \overline{\mathrm{id}}_j$, $\overline{\overline{\mathrm{id}}}_i \ne \overline{\overline{\mathrm{id}}}_j$ *for* $i \ne j$.

Consider the special case that $\sigma \in \mathfrak{S}_n$ is cyclic of length n. Then γ is of the following form (a_1, m_1 are for simplicity denoted as a, m):

$$\gamma : (x_1, \ldots, x_n, t) \longmapsto (e^{2\pi \mathrm{i} a/nm}x_{\sigma(1)}, \ldots, e^{2\pi \mathrm{i} a/nm}x_{\sigma(n)}, e^{2\pi\mathrm{i}/nm}t). \tag{9.7}$$

COROLLARY 9.9. *For the cyclic group* Γ *generated by* (9.7)*, the small finite subgroup* $G \subset GL_n(\mathbb{C})$ *such that* $A_{d-1}/\Gamma \cong \mathbb{C}^n/G$ *(see Theorem 8.1) satisfies*:

(1) $\widetilde{\Gamma} = H = G$*, that is, the covering maps* q *and* r *in* (8.1) *are the identity maps.*

(2) G *is generated by the automorphisms* $f, g_1, g_2, \ldots, g_{n-1}$ *given by*

$$\begin{aligned}f : (x_1, \ldots, x_n)& \\ &\mapsto (e^{2\pi\mathrm{i}a/nmd}x_{\sigma(1)}, \ldots, e^{2\pi\mathrm{i}a/nmd}x_{\sigma(n-1)}, e^{2\pi\mathrm{i}(a+nm\kappa)/nmd}x_{\sigma(n)}),\\ g_i : (x_1, \ldots, x_n) &\mapsto (x_1, x_2, \ldots, e^{2\pi\mathrm{i}/d}x_i, \ldots, e^{-2\pi\mathrm{i}/d}x_n).\end{aligned}$$

PROOF. (1): In the present case, σ is cyclic of length n, so $l = 1$ in (9.6) and Lemma 7.1, and thus $q : \boldsymbol{X} \mapsto \boldsymbol{X}^{m_1'}$, $r : \boldsymbol{u} \mapsto \boldsymbol{u}^{\ell_1}$. We claim that $m_1' = \ell_1 = 1$ (so q and r are the identity maps). First since $c = \gcd(n_1 m_1) = n_1 m_1$, we have $m_1' := n_1 m_1 / c = 1$. Next $N = (m_1')^{n_1} c = n_1 m_1$ and $L_1 = \mathrm{lcm}(n_1 m_1) = n_1 m_1$, thus $\ell_1 := Nc/n_1 m_1 L_1 = 1$, confirming (1).

(2): Since $\widetilde{\Gamma} = G$, this follows from Theorem 9.6 (1) (note n_1, m_1, a_1 are denoted by n, m, a in the assertion). □

9.2. Preparation to deduce relations

Recall that $\widetilde{\gamma}$, $\widetilde{\mathrm{id}}_i \in \widetilde{\Gamma}$ are lifts of γ, $\mathrm{id} \in \Gamma$, and their descents are $\overline{\gamma}$, $\overline{\mathrm{id}}_i \in \overline{\Gamma}$, whose descents are $\overline{\overline{\gamma}}$, $\overline{\overline{\mathrm{id}}}_i \in \overline{\overline{\Gamma}}$. None of them are identity maps (see Theorem 9.6 for their explicit forms). Note that $i = 1, 2, \ldots, n-1$. *Convention*: *Define* $\widetilde{\mathrm{id}}_n$, $\overline{\mathrm{id}}_n$, $\overline{\overline{\mathrm{id}}}_n$ *as identity maps.*

Recall that $\widetilde{\Gamma}$ is generated by $\widetilde{\gamma}$, $\widetilde{\mathrm{id}}_i$ $(i = 1, 2, \ldots, n-1)$, and $\overline{\Gamma}$ by $\overline{\gamma}$, $\overline{\mathrm{id}}_i$, and $\overline{\overline{\Gamma}}$ by $\overline{\overline{\gamma}}$, $\overline{\overline{\mathrm{id}}}_i$ (Corollary 9.5). We deduce relations among $\widetilde{\gamma}$, $\widetilde{\mathrm{id}}_i$ (which descend to relations among $\overline{\gamma}$, $\overline{\mathrm{id}}_i$ and then those among $\overline{\overline{\gamma}}$, $\overline{\overline{\mathrm{id}}}_i$). We begin with preparation. By Theorem 9.6 (1), $\widetilde{\gamma} = \widetilde{\alpha}\widetilde{\beta}_{1,\boldsymbol{q}}$, where $\boldsymbol{q} := (0, \ldots, 0, \kappa, 0, \ldots, 0)$ (κ lies in the $\sigma(n)$th place) and

$$\begin{cases} \widetilde{\alpha} : \ (\boldsymbol{X}_1, \ldots, \boldsymbol{X}_l) \longmapsto \left(e^{2\pi \mathrm{i} a_1/n_1 m_1 d} \boldsymbol{X}_1^{\sigma_1}, \ldots, e^{2\pi \mathrm{i} a_l/n_l m_l d} \boldsymbol{X}_l^{\sigma_l}\right), \\ \widetilde{\beta}_{1,\boldsymbol{q}} : (X_1, X_2, \ldots, X_n) \longmapsto (X_1, X_2, \ldots, e^{2\pi \mathrm{i}\kappa/d} X_{\sigma(n)}, \ldots, X_n). \end{cases}$$

REMARK 9.10. $\widetilde{\beta}_{1,\boldsymbol{p}}$ (for general $\boldsymbol{p} = (p_1, p_2, \ldots, p_n)$) is given as follows (see Lemma 9.2 (1)):

$$\widetilde{\beta}_{1,\boldsymbol{p}} : (X_1, X_2, \ldots, X_n) \longmapsto (e^{2\pi \mathrm{i} p_1/d} X_1, e^{2\pi \mathrm{i} p_2/d} X_2, \ldots, e^{2\pi \mathrm{i} p_n/d} X_n).$$

Using the relation $\widetilde{\beta}_{1,\boldsymbol{p}}\widetilde{\alpha} = \widetilde{\alpha}\widetilde{\beta}_{1,\sigma^{-1}(\boldsymbol{p})}$ (Lemma 4.8), we may rewrite $\widetilde{\gamma}^N = \underbrace{(\widetilde{\alpha}\widetilde{\beta}_{1,\boldsymbol{q}}) \cdots (\widetilde{\alpha}\widetilde{\beta}_{1,\boldsymbol{q}})}_{N}$ as $\widetilde{\gamma}^N = \widetilde{\alpha}^N (\widetilde{\beta}_{1,\sigma^{-N+1}(\boldsymbol{q})} \cdots \widetilde{\beta}_{1,\sigma^{-1}(\boldsymbol{q})} \widetilde{\beta}_{1,\boldsymbol{q}})$; for instance if $N = 3$,

$$\begin{aligned} \widetilde{\gamma}^3 &= (\widetilde{\alpha}\widetilde{\beta}_{1,\boldsymbol{q}})(\widetilde{\alpha}\widetilde{\beta}_{1,\boldsymbol{q}})(\widetilde{\alpha}\widetilde{\beta}_{1,\boldsymbol{q}}) = (\widetilde{\alpha}\widetilde{\beta}_{1,\boldsymbol{q}})\widetilde{\alpha}\widetilde{\alpha}\widetilde{\beta}_{1,\sigma^{-1}\boldsymbol{q}}\widetilde{\beta}_{1,\boldsymbol{q}} \\ &= \widetilde{\alpha}\widetilde{\alpha}\widetilde{\alpha}\widetilde{\beta}_{1,\sigma^{-2}(\boldsymbol{q})}\widetilde{\beta}_{1,\sigma^{-1}\boldsymbol{q}}\widetilde{\beta}_{1,\boldsymbol{q}}. \end{aligned}$$

From the explicit form of $\widetilde{\beta}_{1,\boldsymbol{p}}$ (see Remark 9.10), $\widetilde{\beta}_{1,\boldsymbol{p}}\widetilde{\beta}_{1,\boldsymbol{p}'} = \widetilde{\beta}_{1,\boldsymbol{p}'}\widetilde{\beta}_{1,\boldsymbol{p}}$ for any $\boldsymbol{p}$, $\boldsymbol{p}'$. Thus

$$\widetilde{\gamma}^N = \widetilde{\alpha}^N \prod_{i=0}^{N-1} \widetilde{\beta}_{1,\sigma^{-i}(\boldsymbol{q})}. \tag{9.8}$$

To rewrite this, recall that $\sigma = \sigma_1\sigma_2\cdots\sigma_l$ (cycle decomposition) and the length of σ_j is n_j.

Lemma 9.11.

(i) $\sigma_j^{n_j} = \mathrm{id}$.

(ii) $\sigma^{n_l}(\boldsymbol{q}) = \boldsymbol{q}$. *Consequently* $\sigma^i(\boldsymbol{q}) = \sigma^{i'}(\boldsymbol{q})$ *if* $i \equiv i' \bmod n_l$.

(iii) n_l *divides* N.

(iv) $\sigma^{-i}(\boldsymbol{q}) = \sigma^{N-i}(\boldsymbol{q})$.

Proof. (i) is clear as σ_j is a cyclic permutation of length n_j.

(ii): Since $\boldsymbol{q} := (0,\dots,0,\kappa,0,\dots,0)$ (κ lies in the $\sigma(n)$th place), we have $\sigma^{n_l}(\boldsymbol{q}) = (0,\dots,0,\kappa,0,\dots,0)$ (κ lies in the $\sigma^{-n_l+1}(n)$th place). To show $\sigma^{n_l}(\boldsymbol{q}) = \boldsymbol{q}$, it thus suffices to show $\sigma^{-n_l+1}(n) = \sigma(n)$, that is, $\sigma^{n_l}(n) = n$. Note that n is contained in the cycle J_l of σ_l (indeed $J_l = \{n-n_l+1,\dots,n-1,n\}$), so $\sigma_1,\sigma_2,\dots,\sigma_{l-1}$ are 'irrelevant' to the transformation of n. Hence $\sigma(n) = \sigma_l(n)$, so $\sigma^{n_l}(n) = \sigma_l^{n_l}(n) = n$ (as $\sigma_l^{n_l} = \mathrm{id}$ by (i)).

(iii): Note that

$$\begin{aligned} N &= (m_1')^{n_1}\cdots(m_l')^{n_l}c = (m_1')^{n_1}\cdots(m_l')^{n_l-1}m_l'c \\ &= (m_1')^{n_1}\cdots(m_l')^{n_l-1}n_lm_l \quad \text{as } m_l'c = n_lm_l. \end{aligned}$$

Thus n_l divides N.

(iv): Since n_l divides N, we have $N-i \equiv -i \bmod n_l$. Thus $\sigma^{N-i}(\boldsymbol{q}) = \sigma^{-i}(\boldsymbol{q})$ by (ii). □

Using (iv), rewrite (9.8) as $\widetilde{\gamma}^N = \widetilde{\alpha}^N \prod_{i=0}^{N-1} \widetilde{\beta}_{1,\sigma^i(\boldsymbol{q})}$. This is further rewritten. For instance if $N = 6$ and $n_l = 2$,

$$\begin{aligned} \widetilde{\gamma}^6 &= \widetilde{\alpha}^6(\widetilde{\beta}_{1,\boldsymbol{q}}\widetilde{\beta}_{1,\sigma^1(\boldsymbol{q})})(\widetilde{\beta}_{1,\sigma^2(\boldsymbol{q})}\widetilde{\beta}_{1,\sigma^3(\boldsymbol{q})})(\widetilde{\beta}_{1,\sigma^4(\boldsymbol{q})}\widetilde{\beta}_{1,\sigma^5(\boldsymbol{q})}) \\ &= \widetilde{\alpha}^6(\widetilde{\beta}_{1,\boldsymbol{q}}\widetilde{\beta}_{1,\sigma^1(\boldsymbol{q})})^3 \quad \text{as } \sigma^2(\boldsymbol{q}) = \boldsymbol{q}. \end{aligned}$$

In general, the following holds:

$$\widetilde{\gamma}^N = \widetilde{\alpha}^N \Big(\prod_{i=0}^{n_l-1} \widetilde{\beta}_{1,\sigma^i(\boldsymbol{q})} \Big)^{N/n_l}. \tag{9.9}$$

Here

$$\begin{cases} \widetilde{\alpha}^N : (\boldsymbol{X}_1, \dots, \boldsymbol{X}_l) \mapsto \big(e^{2\pi \mathrm{i} a_1 N/n_1 m_1 d} \boldsymbol{X}_1 \dots, e^{2\pi \mathrm{i} a_l N/n_l m_l d} \boldsymbol{X}_l \big), \\ \widetilde{\beta}_{1,\sigma^i(\boldsymbol{q})} : (X_1, \dots, X_n) \mapsto (X_1, \dots, e^{2\pi \mathrm{i} \kappa/d} X_{\sigma_l^{-i+1}(n)}, \dots, X_n). \end{cases} \tag{9.10}$$

We claim that

$$\prod_{i=0}^{n_l-1} \widetilde{\beta}_{1,\sigma^i(\boldsymbol{q})} : (X_1, \dots, X_n) \mapsto (X_1, X_2, \dots, \underbrace{e^{2\pi \mathrm{i} \kappa/d} X_{n-n_l+1}, \dots, e^{2\pi \mathrm{i} \kappa/d} X_n}_{n_l}),$$

that is,

$$\prod_{i=0}^{n_l-1} \widetilde{\beta}_{1,\sigma^i(\boldsymbol{q})} : (\boldsymbol{X}_1, \dots, \boldsymbol{X}_l) \mapsto (\boldsymbol{X}_1, \dots, \boldsymbol{X}_{l-1}, e^{2\pi \mathrm{i} \kappa/d} \boldsymbol{X}_l). \tag{9.11}$$

Since $\widetilde{\beta}_{1,\sigma^i(\boldsymbol{q})} : (X_1, \dots, X_n) \mapsto (X_1, \dots, e^{2\pi \mathrm{i} \kappa/d} X_{\sigma_l^{-i+1}(n)}, \dots, X_n)$, the composition $\prod_{i=0}^{n_l-1} \widetilde{\beta}_{1,\sigma^i(\boldsymbol{q})}$ is the multiplication of each coordinate $X_{\sigma_l^{-i+1}(n)}$ $(i = 0, 1, \dots, n_l - 1)$ by $e^{2\pi \mathrm{i} \kappa/d}$. Here

$$\begin{aligned} \{\sigma_l^{-i+1}(n) : i = 0, 1, \dots, n_l - 1\} &= \{n - n_l + 1, \dots, n - 1, n\} \\ &= \{j : X_j \in \boldsymbol{X}_l\}. \end{aligned}$$

So $\prod_{i=0}^{n_l-1} \widetilde{\beta}_{1,\sigma^i(\boldsymbol{q})}$ is given by the multiplication of every $X_j \in \boldsymbol{X}_l$ by $e^{2\pi \mathrm{i} \kappa/d}$, that is, of the form (9.11). Consequently

$$\begin{aligned} \Big(\prod_{i=0}^{n_l-1} \widetilde{\beta}_{1,\sigma^i(\boldsymbol{q})} \Big)^{N/n_l} &: (\boldsymbol{X}_1, \dots, \boldsymbol{X}_l) \\ &\mapsto (\boldsymbol{X}_1, \dots, \boldsymbol{X}_{l-1}, e^{2\pi \mathrm{i} \kappa N/n_l d} \boldsymbol{X}_l), \end{aligned} \tag{9.12}$$

where recall that n_l divides N (Lemma 9.11 (iii)).

LEMMA 9.12. *Set* $\xi_k := \begin{cases} e^{2\pi \mathrm{i} a_k N/n_k m_k d} & (k \neq l) \\ e^{2\pi \mathrm{i} (a_l + m_l \kappa) N/n_l m_l d} & (k = l). \end{cases}$ *Then:*

(1) $\widetilde{\gamma}^N : (\boldsymbol{X}_1, \boldsymbol{X}_2, \ldots, \boldsymbol{X}_l) \longmapsto (\xi_1 \boldsymbol{X}_1, \xi_2 \boldsymbol{X}_2, \ldots, \xi_l \boldsymbol{X}_l)$.

(2) $\overline{\gamma}^N : (\boldsymbol{X}_1, \boldsymbol{X}_2, \ldots, \boldsymbol{X}_l) \longmapsto (\xi_1^{m'_1} \boldsymbol{X}_1, \xi_2^{m'_2} \boldsymbol{X}_2, \ldots, \xi_l^{m'_l} \boldsymbol{X}_l)$.

(3) $\overline{\overline{\gamma}}^N : (\boldsymbol{X}_1, \boldsymbol{X}_2, \ldots, \boldsymbol{X}_l) \longmapsto (\xi_1^{m'_1 \ell_1} \boldsymbol{X}_1, \xi_2^{m'_2 \ell_2} \boldsymbol{X}_2, \ldots, \xi_l^{m'_l \ell_l} \boldsymbol{X}_l)$.

Proof. It suffices to show (1), as (2) and (3) are descents of (1). First $\widetilde{\gamma}^N = \widetilde{\alpha}^N \Big(\prod_{i=0}^{n_l - 1} \widetilde{\beta}_{1, \sigma^i(\boldsymbol{q})} \Big)^{N/n_l}$ (see (9.9)). By (9.10) and (9.12), setting $\alpha := e^{2\pi \mathrm{i} a_l N / n_l m_l d}$ and $\beta := e^{2\pi \mathrm{i} \kappa N / n_l d}$, then

$$\widetilde{\gamma}^N : (\boldsymbol{X}_1, \ldots, \boldsymbol{X}_l) \longmapsto (\xi_1 \boldsymbol{X}_1, \ldots, \xi_{l-1} \boldsymbol{X}_{l-1}, \alpha\beta \boldsymbol{X}_l).$$

Here $\alpha\beta = e^{2\pi \mathrm{i} a_l N / n_l m_l d} e^{2\pi \mathrm{i} \kappa N / n_l d} = e^{2\pi \mathrm{i} (a_l + m_l \kappa) N / n_l m_l d} = \xi_l$, so

$$\widetilde{\gamma}^N : (\boldsymbol{X}_1, \ldots, \boldsymbol{X}_l) \longmapsto (\xi_1 \boldsymbol{X}_1, \ldots, \xi_l \boldsymbol{X}_l). \ \Box$$

9.3. Relations between generators

We keep the notation above. We claim that the following relation holds:

$$\widetilde{\gamma}^N = \widetilde{\mathbf{id}}_1 \widetilde{\mathbf{id}}_2 \cdots \widetilde{\mathbf{id}}_l, \tag{9.13}$$

where $\widetilde{\mathbf{id}}_k$ is defined as follows: Write $\{1, 2, \ldots, n\} = J_1 \amalg J_2 \amalg \cdots \amalg J_l$ (the cycle decomposition, where J_k is the cycle of σ_k), then

$$\widetilde{\mathbf{id}}_k := \begin{cases} \prod_{i \in J_k} (\widetilde{\mathrm{id}}_i)^{a_k N / n_k m_k} & (k \neq l), \\ \prod_{i \in J_l} (\widetilde{\mathrm{id}}_i)^{(a_l + m_l \kappa) N / n_l m_l} & (k = l). \end{cases}$$

More explicitly, letting $f_k : \mathbb{C}^{n_l} \to \mathbb{C}^{n_l}$ $(k = 1, 2, \ldots, l)$ be the automorphism given by $\boldsymbol{X}_l = (X_{j_1}, \ldots, X_{j_{n_l}}) \mapsto (X_{j_1}, \ldots, X_{j_{n_l - 1}}, \xi_k^{-n_k} X_{j_{n_l}})$, then

$$\widetilde{\mathbf{id}}_k : \begin{cases} (\boldsymbol{X}_1, \ldots, \boldsymbol{X}_l) \mapsto (\boldsymbol{X}_1, \ldots, \xi_k \boldsymbol{X}_k \ldots, \boldsymbol{X}_{l-1}, f_k(\boldsymbol{X}_l)) & \text{if } k \neq l, \\ (\boldsymbol{X}_1, \ldots, \boldsymbol{X}_l) \mapsto (\boldsymbol{X}_1 \ldots, \boldsymbol{X}_{l-1}, \xi_l \, f_l(\boldsymbol{X}_l)) & \text{if } k = l. \end{cases} \tag{9.14}$$

So

$$\widetilde{\mathbf{id}}_1 \widetilde{\mathbf{id}}_2 \cdots \widetilde{\mathbf{id}}_l : (\boldsymbol{X}_1, \ldots, \boldsymbol{X}_l) \mapsto (\xi_1 \boldsymbol{X}_1 \ldots, \xi_{l-1} \boldsymbol{X}_{l-1}, \xi_l \, f_1 f_2 \cdots f_l(\boldsymbol{X}_l)).$$

Here $f_1 f_2 \cdots f_l = 1$, indeed $\xi_1^{-n_1} \xi_2^{-n_2} \cdots \xi_l^{-n_l} = e^{-2\pi \mathrm{i} N (a_1/m_1 + \cdots + a_l/m_l + \kappa)/d} = e^{-2\pi \mathrm{i} d/d} = 1$. Thus $\widetilde{\mathbf{id}}_1 \widetilde{\mathbf{id}}_2 \cdots \widetilde{\mathbf{id}}_l = \widetilde{\gamma}^N$.

LEMMA 9.13.

(1.a) *For any* k, $\widetilde{\mathbf{id}}_k = 1 \iff \xi_k = 1$.

(1.b) $\widetilde{\mathbf{id}}_1 = \widetilde{\mathbf{id}}_2 = \cdots = \widetilde{\mathbf{id}}_l = 1 \iff \widetilde{\gamma}^N = 1$.

PROOF. (1.a) is immediate from (9.14).

(1.b): From Lemma 9.12 (1), $\widetilde{\gamma}^N = 1 \iff \xi_1 = \xi_2 = \cdots = \xi_l = 1$. This and (1.a) gives (1.b). $\square$

Corresponding to the relation $\widetilde{\gamma}^N = \widetilde{\mathbf{id}}_1\widetilde{\mathbf{id}}_2\cdots\widetilde{\mathbf{id}}_l$, $\overline{\gamma}^N = \overline{\mathbf{id}}_1\overline{\mathbf{id}}_2\cdots\overline{\mathbf{id}}_l$ and $\overline{\overline{\gamma}}^N = \overline{\overline{\mathbf{id}}}_1\overline{\overline{\mathbf{id}}}_2\cdots\overline{\overline{\mathbf{id}}}_l$, where explicitly

$$\overline{\mathbf{id}}_k = \begin{cases} \prod_{i\in J_k} (\overline{\mathrm{id}}_i)^{a_kN/n_km_k}, \\ \prod_{i\in J_l} (\overline{\mathrm{id}}_i)^{(a_l+m_l\kappa)N/n_lm_l}, \end{cases} \qquad \overline{\overline{\mathbf{id}}}_k = \begin{cases} \prod_{i\in J_k} (\overline{\overline{\mathrm{id}}}_i)^{a_kN/n_km_k} & (k\neq l), \\ \prod_{i\in J_l} (\overline{\overline{\mathrm{id}}}_i)^{(a_l+m_l\kappa)N/n_lm_l} & (k=l). \end{cases}$$

LEMMA 9.14.

(2.a) *For any* k, $\overline{\mathbf{id}}_k=1 \iff \xi_k^{m'_k}=1$ *and* $\xi_k^{-n_km'_l}=1$.

(2.b) $\overline{\mathbf{id}}_1 = \overline{\mathbf{id}}_2 = \cdots = \overline{\mathbf{id}}_l = 1 \implies \overline{\gamma}^N = 1$.

PROOF. (2.a): From (9.14), $\overline{\mathbf{id}}_k : (\boldsymbol{X}_1,\dots,\boldsymbol{X}_l) \mapsto (\boldsymbol{X}_1,\dots,\xi_k^{m'_k}\boldsymbol{X}_k, \dots,\boldsymbol{X}_{l-1}, f_k^{m'_l}(\boldsymbol{X}_l))$. Here $f_k^{m'_l} = 1 \iff \xi_k^{-n_km'_l} = 1$, so the assertion holds.

(2.b): From Lemma 9.12 (2), $\overline{\gamma}^N = 1 \iff \xi_1^{m'_1} = \xi_2^{m'_2} = \cdots = \xi_l^{m'_l} = 1$. This and (2.a) gives (2.b). $\square$

REMARK 9.15. In (2.b), "$\Longleftarrow$" does *not* hold: Since m'_k $(k\neq l)$ does not divide $n_km'_l$, even if $\xi_k^{m'_k} = 1$, in general $\xi_k^{-n_km'_l} \neq 1$ (that is, $\overline{\mathbf{id}}_k \neq 1$).

From (9.14),

$$\overline{\overline{\mathbf{id}}}_k : (\boldsymbol{X}_1,\dots,\boldsymbol{X}_l) \longmapsto (\boldsymbol{X}_1,\dots,\xi_k^{m'_k\ell_k}\boldsymbol{X}_k \dots,\boldsymbol{X}_{l-1}, f_k^{m'_l\ell_l}(\boldsymbol{X}_l)),$$

where $f_k^{m'_l\ell_l} : \boldsymbol{X}_l = (X_{j_1},\dots,X_{j_{n_l}}) \mapsto (X_{j_1},\dots,X_{j_{n_l-1}}, \xi_k^{-n_km'_l\ell_l}X_{j_{n_l}})$. Here if $\xi_k^{m'_k}=1$, then $\xi_k^{-n_km'_l\ell_l}=1$; otherwise $\overline{\overline{\mathbf{id}}}_k \in \overline{\overline{\Gamma}}$ is a pseudo-reflection, but this contradicts the fact that $\overline{\overline{\Gamma}}\,(= G)$ is a small group (Theorem 8.1 (1)). This proves (1) of the following ((2) is immediate from (1)):

Lemma 9.16. *For any k,*

(1) *If $\xi_k^{m'_k} = 1$, then $\xi_k^{-n_k m'_l \ell_l} = 1$.*

(2) $\overline{\overline{\mathbf{id}}}_k = 1 \iff \xi_k^{m'_k \ell_k} = 1$.

From Lemma 9.12 (3), $\overline{\overline{\gamma}}^N = 1 \iff \xi_1^{m'_1\ell_1} = \xi_2^{m'_2\ell_2} = \cdots = \xi_l^{m'_l\ell_l} = 1$. This combined with Lemma 9.16 (2) gives:

Lemma 9.17. $\overline{\overline{\mathbf{id}}}_1 = \overline{\overline{\mathbf{id}}}_2 = \cdots = \overline{\overline{\mathbf{id}}}_l = 1 \iff \overline{\overline{\gamma}}^N = 1$.

We summarize the above results as follows:

Proposition 9.18.

(1) $\widetilde{\gamma}^N = \widetilde{\mathbf{id}}_1\widetilde{\mathbf{id}}_2\cdots\widetilde{\mathbf{id}}_l$. *Here* $\widetilde{\mathbf{id}}_1 = \widetilde{\mathbf{id}}_2 = \cdots = \widetilde{\mathbf{id}}_l = 1 \iff \widetilde{\gamma}^N = 1$.

(2) $\overline{\gamma}^N = \overline{\mathbf{id}}_1\overline{\mathbf{id}}_2\cdots\overline{\mathbf{id}}_l$.

(3) $\overline{\overline{\gamma}}^N = \overline{\overline{\mathbf{id}}}_1\overline{\overline{\mathbf{id}}}_2\cdots\overline{\overline{\mathbf{id}}}_l$. *Here* $\overline{\overline{\mathbf{id}}}_1 = \overline{\overline{\mathbf{id}}}_2 = \cdots = \overline{\overline{\mathbf{id}}}_l = 1 \iff \overline{\overline{\gamma}}^N = 1$.

For (2), *we merely have*: $\overline{\mathbf{id}}_1 = \overline{\mathbf{id}}_2 = \cdots = \overline{\mathbf{id}}_l = 1 \implies \overline{\gamma}^N = 1$.

Another relation. There is another relation among $\widetilde{\gamma}$, $\widetilde{\mathrm{id}}_i$ (and also among $\overline{\gamma}$, $\overline{\mathrm{id}}_i$ and among $\overline{\overline{\gamma}}$, $\overline{\overline{\mathrm{id}}}_i$):

Lemma 9.19. *For each $i = 1, 2, \ldots, n-1$,*

(1) $\widetilde{\mathrm{id}}_i\widetilde{\gamma} = \widetilde{\gamma}\,\widetilde{\mathrm{id}}_{\sigma(i)}(\widetilde{\mathrm{id}}_{\sigma(n)})^{-1}$.

(2) $\overline{\mathrm{id}}_i\overline{\gamma} = \overline{\gamma}\,\overline{\mathrm{id}}_{\sigma(i)}(\overline{\mathrm{id}}_{\sigma(n)})^{-1}$.

(3) $\overline{\overline{\mathrm{id}}}_i\overline{\overline{\gamma}} = \overline{\overline{\gamma}}\,\overline{\overline{\mathrm{id}}}_{\sigma(i)}(\overline{\overline{\mathrm{id}}}_{\sigma(n)})^{-1}$.

In particular if $\sigma(i) = i$, *then* $\widetilde{\mathrm{id}}_i\widetilde{\gamma} = \widetilde{\gamma}\,\widetilde{\mathrm{id}}_i(\widetilde{\mathrm{id}}_{\sigma(n)})^{-1}$, $\overline{\mathrm{id}}_i\overline{\gamma} = \overline{\gamma}\,\overline{\mathrm{id}}_i(\overline{\mathrm{id}}_{\sigma(n)})^{-1}$, *and* $\overline{\overline{\mathrm{id}}}_i\overline{\overline{\gamma}} = \overline{\overline{\gamma}}\,\overline{\overline{\mathrm{id}}}_i(\overline{\overline{\mathrm{id}}}_{\sigma(n)})^{-1}$ (these indicate that $\widetilde{\Gamma}, \overline{\Gamma}$ and $\overline{\overline{\Gamma}}$ are not abelian. Indeed they are not except for $\sigma = \mathrm{id}$ or $n = d = 2$ (Theorem 10.11)).

PROOF. (1) can be shown as in the proof of Lemma 4.8. (2) and (3) are the descents of (1). □

REMARK 9.20. If $\sigma(n) = n$, then $\widetilde{\mathrm{id}}_{\sigma(n)}$ is the identity map (as $\widetilde{\mathrm{id}}_{\sigma(n)} = \widetilde{\mathrm{id}}_n$ is the identity map), thus (1) becomes $\widetilde{\mathrm{id}}_i\widetilde{\gamma} = \widetilde{\gamma}\,\widetilde{\mathrm{id}}_{\sigma(i)}$. In particular if σ is the identity, then $\widetilde{\mathrm{id}}_i\widetilde{\gamma} = \widetilde{\gamma}\,\widetilde{\mathrm{id}}_i$. This implies that $\widetilde{\Gamma}$ is abelian. Accordingly $\overline{\Gamma}$ and $\overline{\overline{\Gamma}}$ are abelian.

10. When G is Abelian?

We will determine when $G\,(=\overline{\overline{\Gamma}})$ is abelian. We begin with preparation. Recall that G is generated by $\overline{\overline{\gamma}}$, $\overline{\overline{\mathrm{id}}}_i$ $(i = 1, 2, \ldots, n-1)$ (Corollary 9.5 (3)).

LEMMA 10.1. *Set* $f := \overline{\overline{\gamma}}$ *and* $g_i := \overline{\overline{\mathrm{id}}}_i$ $(i = 1, 2, \ldots, n-1)$. *Then*:

(1) *G is abelian if and only if* $(g_i)^{-1}g_{\sigma(i)} = g_{\sigma(n)}$ *for every* i.

(2) *Suppose that G is abelian. If* $\sigma = \mathrm{id}$, *then* $g_{\sigma(n)} = \mathrm{id}$ (*so* $\sigma(n) = n$). *Otherwise* $g_{\sigma(n)} \neq \mathrm{id}$ (*so* $\sigma(n) \neq n$).

PROOF. (1): As G is generated by f, g_i $(i = 1, 2, \ldots, n-1)$ it is abelian precisely when $g_i f = f g_i$ for every i. By Lemma 9.19 (3), this is equivalent to $g_i = g_{\sigma(i)}\,(g_{\sigma(n)})^{-1}$ for every i.

(2): If $\sigma = \mathrm{id}$, then $g_{\sigma(n)} = g_n = \mathrm{id}$. We next show that if $\sigma \neq \mathrm{id}$, then $g_{\sigma(n)} \neq \mathrm{id}$. Since G is abelian, $(g_i)^{-1}g_{\sigma(i)} = g_{\sigma(n)}$ by (1). Thus if $g_{\sigma(n)} = \mathrm{id}$, then $(g_i)^{-1}g_{\sigma(i)} = \mathrm{id}$, so $g_i = g_{\sigma(i)}$. This implies $i = \sigma(i)$ (note: $g_i = g_j \Leftrightarrow i = j$ by Corollary 9.8). Hence $\sigma = \mathrm{id}$, contradicting the assumption. □

LEMMA 10.2. *If* $\sigma \neq \mathrm{id}$ *and G is abelian, then* $\{1, \sigma(1)\} = \{2, \sigma(2)\} = \cdots = \{n, \sigma(n)\}$ (*as sets*).

PROOF. Since G is abelian, $(g_i)^{-1}g_{\sigma(i)} = g_{\sigma(n)}$ for every i (Lemma 10.1 (1)). We explicitly give both sides. First from Theorem 9.6 (3), g_i and $g_{\sigma(i)}$

are given by (say $x_i \in \boldsymbol{x}_k$, so $x_{\sigma(i)} \in \boldsymbol{x}_k$):

$$\begin{aligned} g_i &: (x_1, x_2, \ldots, x_n) \longmapsto (x_1, x_2, \ldots, e^{2\pi \mathrm{i} m'_k \ell_k/d} x_i, \ldots, e^{-2\pi \mathrm{i} m'_l \ell_l/d} x_n), \\ g_{\sigma(i)} &: (x_1, x_2, \ldots, x_n) \longmapsto (x_1, x_2, \ldots, e^{2\pi \mathrm{i} m'_k \ell_k/d} x_{\sigma(i)}, \ldots, e^{-2\pi \mathrm{i} m'_l \ell_l/d} x_n). \end{aligned}$$

Accordingly

$$\begin{aligned} (g_i)^{-1} g_{\sigma(i)} &: (x_1, \ldots, x_n) \\ &\qquad\qquad \mapsto (x_1, \ldots, e^{-2\pi \mathrm{i} m'_k \ell_k/d} x_i, \ldots, e^{2\pi \mathrm{i} m'_k \ell_k/d} x_{\sigma(i)}, \ldots, x_n). \end{aligned}$$

Note next that as $\sigma \neq \mathrm{id}$, we have $\sigma(n) \neq n$ (Lemma 10.1 (2)). From Theorem 9.6 (3),

$$g_{\sigma(n)} : (x_1, x_2, \ldots, x_n) \longmapsto (x_1, x_2, \ldots, e^{2\pi \mathrm{i} m'_l \ell_l/d} x_{\sigma(n)}, \ldots, e^{-2\pi \mathrm{i} m'_l \ell_l/d} x_n).$$

As $(g_i)^{-1} g_{\sigma(i)} = g_{\sigma(n)}$, we have $\{i,\, \sigma(i)\} = \{n,\, \sigma(n)\}$ for every i. □

Corollary 10.3. *If $\sigma \neq \mathrm{id}$ and G is abelian, then $n = 2$ and $\sigma = (1\,2)$.*

Proof. By Lemma 10.2, $\{1,\, \sigma(1)\} = \{2,\, \sigma(2)\} = \cdots = \{n, \sigma(n)\}$. This equation indeed holds for $n = 2$, $\sigma = (1\,2)$, as $\{1,\, 2\} = \{2,\, 1\}$. In contrast, this fails for $n \geq 3$. For instance, if $n = 3$ and $\sigma = (1\,2\,3)$, then $\{1,\, 2\} = \{2,\, 3\} = \{3,\, 1\}$, which is absurd. The general case is similarly confirmed. □

We revive the notation $\overline{\overline{\gamma}}$, $\overline{\overline{\mathrm{id}}}_i$ for f, g_i. Recall that $A_{d-1}/\Gamma \cong \mathbb{C}^n/G$ as well as the following diagram:

$$\begin{tikzcd}[column sep=small] & & \widetilde{A}_{d-1} = \mathbb{C}^n \curvearrowleft \widetilde{\gamma},\, \widetilde{\mathrm{id}}_i \arrow[dl, "q"'] \arrow[dr, "p"] & \\ & \overline{\gamma},\, \overline{\mathrm{id}}_i \curvearrowright \mathbb{C}^n \arrow[dl, "r"'] & & A_{d-1} \curvearrowleft \gamma,\, \mathrm{id}. \\ \overline{\overline{\gamma}},\, \overline{\overline{\mathrm{id}}}_i \curvearrowright \mathbb{C}^n & & & \end{tikzcd} \tag{10.1}$$

Lemma 10.4. *Suppose $n = 2$ and $\sigma = (1\,2)$. Then:*

(A) *The covering maps* q, r *in* (10.1) *are the identity maps. Accordingly* $\widetilde{\Gamma} = \overline{\Gamma} = G$ *and* $\widetilde{\gamma} = \overline{\gamma} = \overline{\overline{\gamma}}$, $\widetilde{\mathrm{id}}_i = \overline{\mathrm{id}}_i = \overline{\overline{\mathrm{id}}}_i$.

(B) G *is abelian if and only if* $d = 2$.

PROOF. Since $\sigma = (1\,2)$ is cyclic, (A) follows from Corollary 9.9 (1). We next show (B). For simplicity, set $\psi_i := \widetilde{\mathrm{id}}_i$ and $g_i := \overline{\overline{\mathrm{id}}}_i$. By (A) in the present case, $\psi_i = g_i$. By Lemma 10.1 (1), G is abelian if and only if $(g_i)^{-1} g_{\sigma(i)} = g_{\sigma(n)}$. Substituting $n = 2$, $\sigma = (1\,2)$ and $\psi_i = g_i$ into this equation yields $(\psi_1)^{-1}\psi_2 = \psi_1$, so $(\psi_1)^2 = \mathrm{id}$. By Theorem 9.6 (1), this is equivalent to $(e^{2\pi \mathrm{i}/d})^2 = 1$, that is, $d = 2$. □

Hence:

PROPOSITION 10.5. $\sigma \neq \mathrm{id}$ *and* G *is abelian if and only if* $n = 2$, $\sigma = (1\,2)$ *and* $d = 2$.

In this case G is actually *cyclic*. To see this, note first that when $n = 2$ and $\sigma = (1\,2)$, G is generated by $\overline{\overline{\gamma}}$, $\overline{\overline{\mathrm{id}}}_1$ (Corollary 9.5 (3)) and $\widetilde{\gamma} = \overline{\overline{\gamma}}$, $\widetilde{\mathrm{id}}_i = \overline{\overline{\mathrm{id}}}_i$ (Lemma 10.4 (A)) and $2 = d = 2a + 2m\kappa$, so $a = 1$ and $\kappa = 0$. Then from Theorem 9.6 (1),

$$\begin{aligned} \overline{\overline{\gamma}}\ (= \widetilde{\gamma}) &: (x_1, x_2) \longmapsto (e^{2\pi \mathrm{i}/4m} x_2, e^{2\pi \mathrm{i}/4m} x_1), \\ \overline{\overline{\mathrm{id}}}_1\ (= \widetilde{\mathrm{id}}_1) &: (x_1, x_2) \longmapsto (e^{2\pi \mathrm{i}/2} x_1, e^{2\pi \mathrm{i}/2} x_2). \end{aligned}$$

Hence $\overline{\overline{\mathrm{id}}}_1 = (\overline{\overline{\gamma}})^{2m}$, so G is generated by $\overline{\overline{\gamma}}$. This confirms (2) of the following; (1) is already shown in Remark 9.20.

THEOREM 10.6. *Whether* G *is abelian depends on* σ, n, *and* d. *More precisely*:

(1) *If* $\sigma = \mathrm{id}$, *then* G *is always abelian.* (If moreover $n = 2$, G is cyclic ([SaTa] Theorem 2.1, p.682 — originally proved in [Tak])).

(2) *If* $\sigma \neq \mathrm{id}$, *then* G *is rarely abelian — in fact only when* $n = 2$ *and* $d = 2$ (*and in which case* G *is cyclic generated by* $\overline{\overline{\gamma}}$).

For (2), we will determine when $\widetilde{\Gamma}$ is abelian. The following is needed.

LEMMA 10.7. *For each* $i = 1, 2, \ldots, n-1$,

$$\widetilde{\mathrm{id}}_i = \widetilde{\alpha}^N \widetilde{\beta}_{N,\boldsymbol{p}_i} \quad \textit{for some } \boldsymbol{p}_i \in \Lambda^{(N)},$$

where as in (4.4),

$$\Lambda^{(N)} = \Big\{ \boldsymbol{p} = (p_1, \ldots, p_n) \in \mathbb{Z}^n \ : \ 0 \le p_i < d, \quad \sum_{i=1}^{n} \frac{p_i}{d} \equiv \frac{N\kappa}{d} \bmod \mathbb{Z} \Big\}. \tag{10.2}$$

PROOF. Since $\widetilde{\mathrm{id}}_i$ is a lift of $1\,(=\gamma^N) \in \Gamma$, this follows from Corollary 4.6. □

For $\boldsymbol{p} = (p_1, \ldots, p_n) \in \Lambda^{(j)}$, the automorphism $\widetilde{\beta}_{j,\boldsymbol{p}}$ is given by

$$\widetilde{\beta}_{j,\boldsymbol{p}} : (X_1, \ldots, X_n) \longmapsto (e^{2\pi \mathrm{i} p_1/d} X_1, \ldots, e^{2\pi \mathrm{i} p_n/d} X_n) \quad \text{(Lemma 9.2 (1))}.$$

Thus

$$\begin{cases} (*) & \widetilde{\beta}_{j,\boldsymbol{p}} \widetilde{\beta}_{j',\boldsymbol{p}'} = \widetilde{\beta}_{j',\boldsymbol{p}'} \widetilde{\beta}_{j,\boldsymbol{p}} \ \text{ for any } \boldsymbol{p} \in \Lambda^{(j)},\ \boldsymbol{p}' \in \Lambda^{(j')}, \\ (**) & \widetilde{\beta}_{j,\boldsymbol{p}} = \widetilde{\beta}_{j',\boldsymbol{p}'} \iff \boldsymbol{p} = \boldsymbol{p}'. \end{cases} \tag{10.3}$$

Actually: $\widetilde{\Gamma}$ is abelian $\iff \sigma = \mathrm{id}$ or $n = d = 2$. The following is the first step to show this.

LEMMA 10.8. $\widetilde{\Gamma}$ *is abelian* $\iff \sigma(\boldsymbol{p}_i) = \boldsymbol{p}_i$ *for every* i.
(Notation: For $\boldsymbol{x} = (x_1, \ldots, x_n)$, set $\sigma(\boldsymbol{x}) := (x_{\sigma(1)}, \ldots, x_{\sigma(n)})$. So $\sigma(\boldsymbol{x}) = \boldsymbol{x}$ means $x_{\sigma(1)} = x_1 \ldots, x_{\sigma(n)} = x_n$, i.e. σ fixes all elements of $\boldsymbol{x}$.)

PROOF. Since $\widetilde{\Gamma}$ is generated by $\widetilde{\gamma}$ and $\widetilde{\mathrm{id}}_i$ $(i = 1, 2, \ldots, n-1)$ (Corollary 9.5 (1)), we have

$$\widetilde{\Gamma} \text{ is abelian} \iff \widetilde{\gamma}\,\widetilde{\mathrm{id}}_i = \widetilde{\mathrm{id}}_i \widetilde{\gamma} \ \text{ for every } i.$$

Since $\widetilde{\gamma} = \widetilde{\alpha}\widetilde{\beta}_{1,\boldsymbol{q}}$ (Theorem 9.6 (1)) and $\widetilde{\mathrm{id}}_i = \widetilde{\alpha}^N \widetilde{\beta}_{N,\boldsymbol{p}_i}$ for some $\boldsymbol{p}_i \in \Lambda^{(N)}$ (Lemma 10.7), the condition on R.H.S. is rewritten as

$$\widetilde{\alpha}\widetilde{\beta}_{1,\boldsymbol{q}} \widetilde{\alpha}^N \widetilde{\beta}_{N,\boldsymbol{p}_i} = \widetilde{\alpha}^N \widetilde{\beta}_{N,\boldsymbol{p}_i} \widetilde{\alpha}\widetilde{\beta}_{1,\boldsymbol{q}} \ \text{ for every } i.$$

By Lemma 4.8, $\widetilde{\beta}_{N,\boldsymbol{p}_i}\widetilde{\alpha} = \widetilde{\alpha}\widetilde{\beta}_{N,\sigma^{-1}(\boldsymbol{p}_i)}$ and $\widetilde{\beta}_{1,\boldsymbol{q}}\widetilde{\alpha}^N = \widetilde{\alpha}^N\widetilde{\beta}_{1,\sigma^{-N}(\boldsymbol{q})}$. Here $\widetilde{\beta}_{1,\sigma^{-N}(\boldsymbol{q})} = \widetilde{\beta}_{1,\boldsymbol{q}}$ (as $\sigma^{-N} = \mathrm{id}$), thus

$$\begin{aligned}
\widetilde{\Gamma} \text{ is abelian } &\iff \widetilde{\alpha}^{N+1}\widetilde{\beta}_{1,\boldsymbol{q}}\widetilde{\beta}_{N,\boldsymbol{p}_i} = \widetilde{\alpha}^{N+1}\widetilde{\beta}_{N,\sigma^{-1}(\boldsymbol{p}_i)}\widetilde{\beta}_{1,\boldsymbol{q}}, \ {}^{\forall}i \\
&\iff \widetilde{\beta}_{1,\boldsymbol{q}}\widetilde{\beta}_{N,\boldsymbol{p}_i} = \widetilde{\beta}_{N,\sigma^{-1}(\boldsymbol{p}_i)}\widetilde{\beta}_{1,\boldsymbol{q}}, \ {}^{\forall}i \\
&\iff \widetilde{\beta}_{N,\boldsymbol{p}_i}\widetilde{\beta}_{1,\boldsymbol{q}} = \widetilde{\beta}_{N,\sigma^{-1}(\boldsymbol{p}_i)}\widetilde{\beta}_{1,\boldsymbol{q}}, \ {}^{\forall}i \qquad \text{by } (*) \text{ of } (10.3) \\
&\iff \widetilde{\beta}_{N,\boldsymbol{p}_i} = \widetilde{\beta}_{N,\sigma^{-1}(\boldsymbol{p}_i)}, \ {}^{\forall}i \\
&\iff \boldsymbol{p}_i = \sigma^{-1}(\boldsymbol{p}_i), \ {}^{\forall}i \qquad \text{by } (**) \text{ of } (10.3). \ \square
\end{aligned}$$

Furthermore:

PROPOSITION 10.9. *The following are equivalent*:

(1) $\widetilde{\Gamma}$ *is abelian.*

(2) $\sigma(\boldsymbol{p}) = \boldsymbol{p}$ *for any* $\boldsymbol{p} \in \Lambda^{(N)}$.

(3) $\sigma = \mathrm{id}$ *or* $n = d = 2$.

(From the equivalence of (1) and (3), in most cases $\widetilde{\Gamma}$ is not abelian.)

PROOF. "(1) $\Longrightarrow$ (2)" was shown as Lemma 4.9.

(2) $\Longrightarrow$ (1): If $\sigma(\boldsymbol{p}) = \boldsymbol{p}$ for every $\boldsymbol{p} \in \Lambda^{(N)}$, then in particular $\sigma(\boldsymbol{p}_i) = \boldsymbol{p}_i$ for every i. The assertion thus follows from Lemma 10.8.

(3) $\Longrightarrow$ (2): First if $\sigma = \mathrm{id}$, (2) is obvious. Next if $n = d = 2$, then either $\sigma = \mathrm{id}$ or $\sigma = (1\,2)$. It suffices to consider the latter case — for which $2 = d = 2a + 2m\kappa$, so $a = 1$ and $\kappa = 0$, accordingly (10.2) is

$$\begin{aligned}
\Lambda^{(N)} &= \left\{ (p_1, p_2) \in \mathbb{Z}^2 \,:\, 0 \le p_i < 2, \ \frac{p_1 + p_2}{2} \equiv 0 \bmod \mathbb{Z} \right\} \\
&= \{(0,0),\,(1,1)\}.
\end{aligned}$$

Then for $\boldsymbol{p} \in \Lambda^{(N)}$, clearly $\sigma(\boldsymbol{p}) = \boldsymbol{p}$ (note: for $\boldsymbol{p} = (p_1, p_2)$, $\sigma(\boldsymbol{p}) = \boldsymbol{p}$ precisely when $p_{\sigma(1)} = p_1$, $p_{\sigma(2)} = p_2$).

(1) $\Longrightarrow$ (3): If $\widetilde{\Gamma}$ is abelian, its descent G is necessarily abelian, thus $\sigma = \mathrm{id}$ or $n = d = 2$ by Theorem 10.6. $\square$

LEMMA 10.10. *The following are equivalent*:

(A) $\widetilde{\Gamma}$ *is abelian.* (B) H *is abelian.* (C) G *is abelian.*

PROOF. "(A) $\Longrightarrow$ (B)" and "(B) $\Longrightarrow$ (C)" follow from the facts that H is the descent of $\widetilde{\Gamma}$ and G is the descent of H. "(C) $\Longrightarrow$ (A)": If G is abelian, then $\sigma = \mathrm{id}$ or $n = d = 2$ by Theorem 10.6, so $\widetilde{\Gamma}$ is abelian by Proposition 10.9. $\square$

Lemma 10.10 combined with Proposition 10.9 yields:

THEOREM 10.11. *The following are equivalent*:

(1) $\sigma = \mathrm{id}$ *or* $n = d = 2$.

(2) $\widetilde{\Gamma}$ *is abelian.*

(3) H *is abelian.*

(4) G *is abelian.*

Supplement. For each $\sigma \in \mathfrak{S}_n$, define an automorphism f_σ of $\mathbb{C}^n$ by $f_\sigma(x_1, x_2, \ldots, x_n) = (x_{\sigma(1)}, x_{\sigma(2)}, \ldots, x_{\sigma(n)})$. This does "not" define a *group* action of $\mathfrak{S}_n$ on $\mathbb{C}^n$. Indeed $f_\tau\big(f_\sigma(x_1, \ldots, x_n)\big) = f_\tau(x_{\sigma(1)}, \ldots, x_{\sigma(n)}) = (x_{\sigma\tau(1)}, \ldots, x_{\sigma\tau(n)}) = f_{\sigma\tau}(x_1, \ldots, x_n)$, so $f_\tau \circ f_\sigma = f_{\sigma\tau} \neq f_{\tau\sigma}$. In contrast, $f_\sigma(x_1, x_2, \ldots, x_n) = (x_{\sigma^{-1}(1)}, x_{\sigma^{-1}(2)}, \ldots, x_{\sigma^{-1}(n)})$ defines a group action of $\mathfrak{S}_n$, as $f_\tau \circ f_\sigma = f_{\tau\sigma}$.

References

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(Received March 17, 2017)

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