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Algebraic Study on the Quantum Calogero Model

量子カロゲロ模型の代数的な研究

平成8年12月博士(理学)申請

東京大学大学院理学系研究科物理学専攻

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論文内容の要旨

厳密な取り扱いが一般には難しい物理学に現れる様々な模型の中で、それが可能な模型は特別な存在である。例えば、量子力学に対する信用を確固たるものにする上で決定的な役割を果たした水素原子の問題は、変数分離法によって固有値問題を厳密に解くことができる量子系の典型例である。この系は、ハミルトニアンに加えて全角運動量と角運動量の一成分の都合３つ、すなわち自由度と同数の、互いに可換で同時対角化可能な保存演算子を持つ。これらの保存演算子の同時対角化基底が、水素原子の直交基底を与えるわけである。水素原子の様に、自由度と同数の互いに交換する独立な保存演算子を持つ量子系を一般に量子可積分系と呼ぶ。量子可積分系に関する問題として、互いに可換な保存演算子の構成による量子可積分性の証明、背後の対称性の解明、固有値問題の厳密解、特に全ての保存量の同時固有関数の構成などが、重要な課題である。この学位論文では、逆二乗型長距離相互作用を持つ一次元量子系の代表的な模型の一つである量子カロゲロ模型について、量子可積分系としての観点から研究した成果を発表する。

第１章では、量子可積分系に注目するきっかけとなった古典力学における完全可積分系の理論を概観し、量子化に付随して生ずる困難について述べつつ、この学位論文の問題意識を明確にする。自由度と同数の互いにポアソン括弧について交換する独立な保存量を持つ古典系を完全可積分系と呼び、その初期値問題を原理的に厳密に解くことが可能であることをリウヴィルの定理が保証する。量子可積分系の定義は、完全可積分系の定義の対応原理による読み替えなのである。量子可積分系であることは、その固有値問題について保存演算子を全て対角化することで原理的に量子数を全て求められることを意味する。さて、実際に保存量を構成し厳密に初期値問題を解くことは、完全可積分系とは言えども決して自明ではない。しかし幸い様々な完全可積分系について、これらの問題を取り扱う有力な手法である、ラックス形式や逆散乱法といった定式化が開発されてきた。古典カロゲロ模型についてもラックス形式を用いた、互いにポアソン括弧について可換な保存量の構成法と初期値問題の解法が知られている。しかしながら、一般に古典系に対して有効なラックス形式も、量子化後は正準共役な変数の間の非可換性が障害となって、保存演算子の構成ができなくなってしまうことが知られている。量子カロゲロ模型について、この困難に対する解答を与え、保存量の構成や背後にある対称性の解明、固有値問題の厳密解、特に全ての保存量の同時固有関数を明らかにするのが、この学位論文の目的、内容である。

第２章では、古典系におけるラックス形式に相当する新しい定式化を量子カロゲロ模型に対して提出し、その量子可積分性と対称性について明らかにする。古典系におけるラックス方程式を対応原理を用いて素直に量子化し、更に非可換な積を対称化したものが、結合定数に量子補正を含まないカロゲロ模型のハミルトニアンに関して演算子間の等式となることは、カロゲロやオルシャネツキーらによって指摘されており、これが量子カロゲロ模型の互いに可換な保存演算子が古典系の保存量を対応原理で読み替えたものと一致するであろうという量子可積分性に関する従来の予想の根拠となっていた。一方、非可換な積を対称化せずに古典系のラッ

クス方程式を素直に対応原理で量子化したものは、結合定数に量子補正を含む正しい量子カロゲロ模型のハミルトニアンに関する等式を与える。しかしこの量子ラックス方程式を用いて保存量を実際に構成する方法は知られておらず、まして保存量同士の可換性については全く検討されなかった。ここで注目すべき性質は、量子カロゲロ模型のラックス方程式のM行列が、片方の足について和を取るとゼロになるという性質である。この性質を用いて、古典系においてラックス形式からの保存量の構成法を与えたトレースの代りに、行列の全成分を足し合わせることで、量子カロゲロ模型の保存量を構成できることを示した。さらに、量子カロゲロ模型のラックス行列から構成され、保存量を含むより一般の複雑な演算子の族に対し、帰納的に一般化されたラックス方程式を構成し、量子カロゲロ模型の保存量の背後にW対称性のあることを示した。量子可積分系であることを示すには、互いに交換する独立な保存演算子を自由度と同数だけ構成する必要がある。そのような保存演算子の量子ラックス形式を用いた構成法を与えた。ポリクロナコスによって導入されたドゥンクル演算子を用いた定式化も量子カロゲロ模型の可積分性を考察する上で有用な方法である。量子ラックス形式とドゥンクル演算子の理論の関係を考察し、相互の対応関係を明らかにした。

量子カロゲロ模型のエネルギー固有値問題は、カロゲロによって既に解かれている。カロゲロの用いた方法は変数分離によってエネルギー固有値に効く常微分方程式を導き、これを解くというものであった。一方ペレロモフは、カロゲロ模型のエネルギー固有値がN個の調和振動子のエネルギー固有値と酷似していることに注目し、生成演算子に似た演算子をカロゲロ模型のハミルトニアンに対して構成できると予想した。N体カロゲロ模型において代数的に全ての固有関数を構成するには、縮退度の議論から独立なN個の生成演算子を構成しなければならないことがわかる。ペレロモフはその中の簡単な3個しか構成することができなかった。第3章の第一目的は、このペレロモフの予想に対する完全な答えを与えることである。第2章で導入した量子ラックス形式を利用することで、ペレロモフが構成しようとした生成演算子をN個すべて構成する事ができ、なおかつそれらの生成演算子が互いに可換であることを示すことができた。このN個の生成演算子を実ラフリン波動関数である基底状態に作用することで、すべての励起状態を構成することができた。第3章の第二の目的は、第2章で構成した量子カロゲロ模型の同時対角化可能な保存演算子の同時固有関数を実際にいくつか構成することで、その特徴を掴むことである。まず、第2章で求めた一般化されたラックス方程式を用いることで、第3章で構成したエネルギー固有関数上での第2保存量の行列表示を得る。この行列を対角化しその固有ベクトルを求めることで、第1、第2保存量の同時固有関数を得ることができる。この手順で、第2保存量の固有値の予想と、同時固有関数のうち最初の7個の表式を得た。この7個の同時固有関数については、第1、第2保存量の固有値だけで縮退が完全に解けている。これはこれらの7個の関数が全ての保存量の同時固有関数となっていて、しかも互いに直交することを意味する。さらに、対称化された単項式による展開形が、ジャック多項式と同様の三角性を示すことが分かった。この結果は、量子カロゲロ模型の保存量の同時固有関数とジャック多項式の間の何らかの関係の存在を強く示唆しており、第4章で導入する裏ジャック多項式の定義の重要なヒントとなった。

以上のような第3章の結果に基づき、第4章では、裏ジャック多項式を定義し、これが実際にカロゲロ模型の全ての保存量の同時固有関数となることを証明する。まず、量子ラックス形式及びドゥンクル演算子でカロゲロ模型と、ジャック多項式を固有関数に持つサザランド模型の代数的な構造が全く同じであることを示した。カロゲロ模型を記述するラックス形式及びドゥンクル演算子が、サザランド模型を記述するラックス形式及びドゥンクル演算子それぞれの1パラメータ変形となっていることも示し、求めるべき同時固有関数が、ジャック多項式の1パラメータ変形となることを示した。次に、第3章の結果と、カロゲロ模型とサザランド模型の対応をヒントに、裏ジャック多項式を定義した。裏ジャック多項式は、カロゲロ模型の第1、第2保存量の同時固有関数となる非斉次対称多項式で、対称化された単項式による展開形が三角

性を持つ、適当に規格化されたものとして一意的に定義できることを示した。さらに、ラポアンテとビネの方法に倣い、裏ジャック多項式のロドリーグ公式をドゥンクル演算子を用いて導入し、これを用いて裏ジャック多項式の展開係数の整数性、すべての保存量を同時対角化する同時固有関数であること、およびカロゲロ模型の基底状態のノルムの2乗を重み関数に取った内積に関する直交対称多項式であることを証明し、裏ジャック多項式が、エルミート多項式の多変数拡張の一つとなることを示した。こうして裏ジャック多項式は、ラッセルやマクドナルドによって、結合パラメータの特殊値の場合として既に知られていた直交対称多項式の変形という形で既に導入されていた多変数拡張版エルミート多項式であることが判明したのだが、カロゲロ模型の保存量の同時固有関数という観点から定義し、実際にドゥンクル演算子を用いた代数的な構成法を与えたのはこの論文が初めてのことである。

以上のように、量子カロゲロ模型について量子可積分系の観点から研究を進めた。その主な成果を列挙すると、古典ラックス形式の量子化の困難を克服する量子ラックス形式の導入、量子カロゲロ模型の同時対角化可能な保存量の構成法と量子可積分性の証明、背後にあるW対称性の導出、エネルギー固有関数の代数的な構成法の完成、全ての保存量の同時固有関数である裏ジャック多項式の導入とその代数的な構成法、及び直交性はじめとするいくつかの性質の解明、となる。これらがこの学位論文の成果である。

Thesis

Algebraic Study on the Quantum Calogero Model

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List of Papers Submitted for the Requirement of the Degree

1. H. Ujino and M. Wadati: Algebraic Approach to the Eigenstates of the Quantum Calogero Model Confined in an External Harmonic Well, Chaos, Solitons & Fractals **5** (1995) 109–118.
2. H. Ujino and M. Wadati: The Quantum Calogero Model and the W-Algebra, J. Phys. Soc. Jpn. **63** (1994) 3585–3597.
3. H. Ujino and M. Wadati: Orthogonal Symmetric Polynomials Associated with the Quantum Calogero Model, J. Phys. Soc. Jpn. **64** (1995) 2703–2706.
4. H. Ujino and M. Wadati: Correspondence between QISM and Exchange Operator Formalism, J. Phys. Soc. Jpn. **64** (1995) 4121–4128.
5. H. Ujino and M. Wadati: Algebraic Construction of the Eigenstates for the Second Conserved Operator of the Quantum Calogero Model, J. Phys. Soc. Jpn. **65** (1996) 653–656.
6. H. Ujino and M. Wadati: Rodrigues Formula for Hi-Jack Symmetric Polynomials Associated with the Quantum Calogero Model, J. Phys. Soc. Jpn. **65** (1996) 2423–2439.
7. H. Ujino and M. Wadati: Orthogonality of the Hi-Jack Polynomials Associated with the Quantum Calogero Model, to be published in J. Phys. Soc. Jpn. **66** (1997) No. 2.

Papers Added in References

1. H. Ujino, K. Hikami and M. Wadati: Integrability of the Quantum Calogero-Moser Model, J. Phys. Soc. Jpn. **61** (1992) 3425–3427.
2. M. Wadati, K. Hikami and H. Ujino: Integrability and Algebraic Structure of the Quantum Calogero-Moser Model, Chaos, Solitons & Fractals **3** (1993) 627–636.
3. H. Ujino, M. Wadati and K. Hikami: The Quantum Calogero-Moser Model: Algebraic Structures, J. Phys. Soc. Jpn. **62** (1993) 3035–3043.
4. H. Ujino and M. Wadati: The $SU(\nu)$ Quantum Calogero Model and the W_∞ Algebra with $SU(\nu)$ Symmetry, J. Phys. Soc. Jpn. **64** (1995) 39–56.
5. H. Ujino and M. Wadati: Exclusion Statistics Interpretation of the Quantum Calogero Model, J. Phys. Soc. Jpn. **64** (1995) 4064–4068.
6. H. Ujino and M. Wadati: Thermodynamics of the Exclusion Statistics Associated with the Quantum Calogero Model, J. Phys. Soc. Jpn. **65** (1996) 1203–1212.
7. H. Ujino and M. Wadati: Exclusion Statistical TBA and the Quantum Calogero Model, J. Phys. Soc. Jpn. **65** (1996) 3819–3823.
8. H. Ujino: Orthogonal Symmetric Polynomials Associated with the Calogero Model, submitted to *Extended and Quantum Algebras and their Applications in Mathematical Physics*, proceedings of the Canada-China Meeting in Mathematical Physics, Tianjin, China, August 19–24, 1996, ed. M.-L. Ge, Y. Saint-Aubin and L. Vinet (Springer-Verlag).

Abstract

Among the models in theoretical physics, exactly solvable models deserve special interest. For instance, the problem on the hydrogen atom, which played a crucial role in bringing high credit to quantum mechanics, is a typical example of the models whose eigenvalue problems are exactly solvable by separation of variables. This system has three, which is the same as the number of degrees of freedom of the system, independent and mutually commuting conserved operators, namely, the Hamiltonian, the total angular momentum and its z-axis component. We call a quantum system with N degrees of freedom that has N, independent and mutually commuting conserved operators quantum integrable system. Important problems for the quantum integrable system are, for instance, the proof of quantum integrability by construction of the commuting conserved operators, the clarification of the underlying symmetries, the exact solution for the eigenvalue problem and the identification of the simultaneous eigenfunctions of the commuting conserved operators. In this thesis, we shall study the quantum Calogero model, which is a representative model among the one-dimensional quantum particle systems with inverse-square long-range interactions, from the viewpoint of the quantum integrable systems.

In Chapter 1, we shall clarify the motive of our study by briefly summarizing the theory of completely integrable systems in classical mechanics and some difficulty in its quantization. The completely integrable system is a classical dynamical system with N degrees of freedom that has N, independent and mutually Poisson commuting conserved quantities. Exact solvability by principle of such a system is guaranteed by Liouville's theorem on integrability. The definition of the quantum integrable system is just a translation by the correspondence principle, which asserts the possibility by principle of the identification of all the quantum numbers. Even for completely integrable systems, it is not trivial to solve their initial value problems in practice. But fortunately, effective techniques for those problems such as the Lax formulation and the inverse scattering method have been developed for various completely integrable systems. The Lax formulation was also introduced for the classical Calogero model, which enables constructing mutually Poisson commuting conserved quantities and solving the initial value problem for the model. However, due to the non-commutativity between the canonical-conjugate variables, the quantized Lax formulation does not give a way to construct the commuting conserved operators. The aim and contents of the thesis are to give a solution for this difficulty, construct the conserved operators, clarify the underlying symmetries, give the exact solution for the eigenvalue problem and identify the simultaneous eigenfunctions of all the commuting conserved operators.

In Chapter 2, we shall introduce a new formulation for the quantum Calogero model that corresponds to the Lax formulation in the classical theory, and clarify its quantum integrability and the underlying symmetry. It was pointed out by Calogero and Olshanetsky et al. that the

quantization of the classical Lax equation for the classical model with symmetrizing the non-commutative products yields an equality for the "quantized classical Calogero Hamiltonian" which does not contain a quantum correction in the coupling constant. This observation had been a ground for the conjecture that the quantization of the classical conserved quantities by the correspondence principle might yield the conserved operators of the quantum Calogero model. On the other hand, a naive quantization by the correspondence principle of the classical Lax equation without symmetrization gives an equality for the correct quantum Calogero model which has a quantum correction in the coupling constant. However, there has not been a way to construct the conserved operators from the quantum Lax equation, let alone a way to prove their mutual commutativity. We pay attention to the sum-to-zero property of the M-matrix in the Lax equation of the quantum Calogero model, that is, the sum of one of the indices of the M-matrix is zero. From this property, we show that the sum of all the elements instead of the trace in the classical theory yields the conserved operators of the quantum Calogero model. Moreover, we recursively construct the generalized Lax equations for a family of operators that includes the conserved operators. We prove that the commutator algebra of the family of operators is the W-algebra that is the underlying symmetry of the quantum Calogero model. We also give a way of constructing mutually commuting conserved operators, which proves the quantum integrability. The Dunkl operator formulation, which was introduced by Polychronakos, is also a powerful method for the study of the quantum Calogero model. We study the relationships between the quantum Lax and the Dunkl operator formulations and clarify their relationship.

The energy eigenvalue problem of the quantum Calogero model was solved by Calogero. He derived and solved an ordinary differential equation by separation of variables that determines the energy eigenvalue. On the other hand, Perelomov paid attention to the close similarity of the energy eigenvalues of the quantum Calogero model and that of the N harmonic oscillators. He conjectured that there must be N independent creation-like operators for the quantum Calogero model which are proved to be sufficient for the algebraic construction of all the energy eigenfunctions from the consideration of degeneracies. And he succeeded in giving the first three of such operators. The first aim of Chapter 3 is to give the complete answer to the Perelomov's conjecture. Using the quantum Lax formulation we introduce in Chapter 2, we construct all the creation-like operators, which we call the power-sum creation operators. We also show that these power-sum creation operators mutually commute. Successive operations of the power-sum creation operators on the ground state wave function that is the real Laughlin wave function yield all the eigenfunctions of the quantum Calogero Hamiltonian. The second aim of Chapter 3 is to observe some properties of the simultaneous eigenfunctions of the commuting conserved operators by explicitly constructing some of them. From the generalized Lax equation given in Chapter 2, we get a matrix representation of the second conserved operator on the energy eigenfunction basis constructed algebraically in Chapter 3. Diagonalizing the matrix, we get the simultaneous eigenfunctions from the eigenvectors. By this method, we get the eigenvalue formula for the second conserved operators as a conjecture and the first seven of the simultaneous eigenfunctions. These seven simultaneous eigenfunctions are uniquely identified by the eigenvalues for the first and the second conserved operators because they have no degeneracy. Moreover, the expansion forms of the above seven functions with respect to the monomial symmetric polynomials show the triangularity with respect to the dominance order which is similar to the triangularity for the Jack polynomials. This observation strongly suggests a close relationship between the simultaneous eigenfunctions of the commuting conserved operators of the quantum Calogero model and the Jack polynomials, and gives an important hint for the definition of the Hi-Jack polynomials.

Following the results in Chapter 3, we define the Hi-Jack polynomials and prove that they are indeed the simultaneous eigenfunctions of all the commuting conserved operators of the quantum Calogero model in Chapter 4. First we prove that the Calogero model and the Sutherland model, whose eigenfunctions are the Jack polynomials, share the common algebraic structure. By showing that the quantum Lax and the Dunkl operator formulations for the Calogero model are one-parameter deformations of those for the Sutherland model, we prove that the Hi-Jack polynomial is a one-parameter deformation of the Jack polynomial. Next, based on the results in Chapter 3 and the correspondence between the Calogero and the Sutherland models, we define the Hi-Jack polynomials. The Hi-Jack polynomial is a normalized inhomogeneous symmetric polynomial that is the simultaneous eigenfunction of the first and the second commuting conserved operators of the Calogero model, whose expansion form with respect to the monomial symmetric polynomials shows the triangularity in the dominance order. We prove that this definition uniquely identifies the Hi-Jack polynomials. Motivated by the result by Lapointe and Vinet, we prove the Rodrigues formula for the Hi-Jack polynomials. Using the formula, we prove their integrality and the facts that the Hi-Jack polynomials are indeed the simultaneous eigenfunctions of all the commuting conserved operators of the Calogero model and hence that they are the orthogonal symmetric polynomials with the weight function given by the square of the norm of the ground state wave function. Consequently, we prove that the Hi-Jack polynomial is a multivariable generalization of the Hermite polynomial, which was introduced by Lasselle and Macdonald from the viewpoint of a deformation of an orthogonal polynomial that corresponds to the special case of the Hi-Jack polynomial. However, it is the first time to derive it from the viewpoint of the simultaneous eigenfunction of all the commuting conserved operators of the quantum integrable system, which is a natural object for physicists' interest.

As we have described above, we shall study the quantum Calogero model from the viewpoint of the quantum integrable systems. And we shall derive the results listed below:

- A new formulation named quantum Lax formulation that overcomes the difficulty in the quantization of the classical Lax formulation.
- Construction of the commuting conserved operators and the proof of the quantum integrability.
- Derivation of the underlying W-symmetry.
- Completion of the algebraic construction of the energy eigenfunction.
- Identification of the simultaneous eigenfunctions of the commuting conserved operators, their orthogonality and the Rodrigues formula.

These are the main results of the thesis.

Contents

Chapter 1

Introduction

Integrable systems have been playing a special role among many models that appear in theoretical physics. Simple and solvable models have provided not only good exercises for students, but also excellent ways for us to understand physical phenomena. For instance, the exact solution of the quantum Kepler problem gave us a convincing answer to the mystery of the series of discrete spectra of the solar ray at the beginning of this century [60, 64, 65], which brought high credit to quantum mechanics. In the analysis, the theory of the Laguerre polynomials and the spherical harmonics played an important role. These special functions are obtained by means of separation of variables of the Hamiltonian. By separation of variables, the partial differential equation that gives the eigenvalue problem of the model is decomposed into three ordinary differential equations whose solutions are the Laguerre polynomials and the spherical harmonics. From the viewpoint of the integrable systems, we should note that the model has three, independent and mutually commuting conserved operators, namely the Hamiltonian, the total angular momentum and its z-axis component. These special functions are associated with the joint wave functions for the three mutually commuting conserved operators of the model. Thus we can identify all the quantum numbers the model has. Besides the quantum Kepler problem, the theory of special functions have been helping the analysis of various kinds of important Schrödinger equations since the early days of quantum mechanics. Various physical phenomena have been explained by reducing the problems to integrable Hamiltonians as first approximations and expanding the correction terms.

Apart from the aim toward the explanation of physical phenomena, mathematical structures of integrable systems attract many researchers because of their beauty. Mathematical beauty of integrable systems motivates me to study them. In classical mechanics, the theory of the completely integrable system is well developed. The completely integrable system is a dynamical system with N degrees of freedom that has N, independent and mutually Poisson commuting conserved quantities. Liouville's theorem on integrability guarantees that we can solve the initial value problem of the completely integrable system by definite integrals and change of variables in principle. Thus construction of commuting conserved quantities has been an important problem in the theory of classical integrable systems. The Lax formulation provides a method to construct conserved quantities. The success of the inverse scattering method in the soliton theory [1, 28] benefited from the Lax formulation. Though there is no counterpart of the Liouville theorem in quantum mechanics, we define the quantum integrable system in a similar fashion to the definition of the classical completely integrable system. Namely, the quantum integrable system is a quantum system with N degrees of freedom that has N, independent and mutually commuting conserved operators. Existence of such conserved quantities asserts the possibility of identification of all the quantum numbers of the system. Thus the notion of integrability and construction of commuting conserved operators are also important in the quantum theory. However, naive quantization of the classical Lax formulation does not work because of the non-commutativity between the canonical-conjugate variables. Due to the above difficulty, development of a formulation for the quantum theory that is the counterpart of the classical Lax formulation is an attractive and challenging problem.

The main theme of my thesis is the algebraic study on the quantum Calogero model. The Calogero model was introduced by F. Calogero in 1971 [19], which describes N quantum identical particles on a line with inverse-square interactions confined in an external harmonic well. He solved the eigenvalue problem for the model by separation of variables. In the classical theory, the Calogero model is a completely integrable system and has a Lax formulation [52, 57, 58]. We shall introduce the quantum Lax formulation and the Dunkl operator formulation for the Calogero model that is the counterpart of the classical Lax formulation. Our formulations will clarify the conservation law, the algebraic structure and the special functions that are associated with the eigenvalue problem of the Calogero model. Namely, what we shall show is an old-fashioned study on the exactly solvable problem in quantum mechanics and the associated special functions by using brand-new tools.

In this chapter, we shall review what is the integrable systems and an overview history of the Calogero model so that the motive for the problems we shall study in the succeeding chapters will be clear. First, we shall summarize the Liouville theorem and the Lax formulation. We shall also present the definition of quantum integrability and difficulties in the quantization of the Lax formulation. Summarizing the overview history and the classical integrability of the Calogero model, we shall observe the effectivity of the classical Lax formulation in the study of the classical Calogero model. And last, we summarize the motivation and aims of the study and brief the outline of the thesis.

1.1 Integrable Systems

Let us first consider the notion of integrability in the classical mechanics. Generally speaking, we can not solve the initial value problem for dynamical systems with more than or equal to two degrees of freedom. We can deal with the interacting two-body problem whose interaction only depends on the distance between the particles, for we can separate variables by introducing center of mass coordinates and relative coordinates. Thus we usually say that many-body problems involving more than or equal to three particles are not generally solvable.

However, there is a class of dynamical systems whose initial value problems are, in principle, solvable. We call such a classical dynamical system completely integrable system [2]. Completely integrable systems are dynamical systems with N degrees of freedom that have N, independent and involutive, or in other words, mutually Poisson commuting conserved quantities. Liouville's theorem on integrability guarantees that completely integrable systems can be solved by quadratures. We shall verify the theorem and see how it works.

Let us state again Liouville's theorem on integrability.

Theorem 1. 1 *Consider a Hamiltonian with respect to coordinates $\boldsymbol{q}$ and momenta $\boldsymbol{p}$ that describes a dynamical system with N degrees of freedom,*

$$H(\boldsymbol{p};\boldsymbol{q};t) = H(p_1,\cdots,p_N;q_1,\cdots,q_N;t), \tag{1.1a}$$

$$\{p_i,p_j\}_{\mathrm{P}} = \{q_i,q_j\}_{\mathrm{P}} = 0, \quad \{q_i,p_j\}_{\mathrm{P}} = \delta_{ij}, \tag{1.1b}$$

where $\{a,b\}_{\mathrm{P}}$ is the Poisson bracket,

$$\{a,b\}_{\mathrm{P}} \stackrel{\mathrm{def}}{=} \sum_{i=1}^{N}\Big(\frac{\partial a}{\partial q_i}\frac{\partial b}{\partial p_i} - \frac{\partial b}{\partial q_i}\frac{\partial a}{\partial p_i}\Big). \tag{1.1c}$$

If the system has a set of N, independent and analytic conserved quantities $\{I_i|i=1,\cdots,N\}$,

$$\frac{\mathrm{d}I_i}{\mathrm{d}t} = \{I_i,H\}_{\mathrm{P}} + \frac{\partial I_i}{\partial t} = 0, \tag{1.2a}$$

$$\det\Big(\frac{\partial I_j}{\partial p_i}\Big) \neq 0, \tag{1.2b}$$

which are in involution,

$$\{I_i,I_j\}_{\mathrm{P}} = 0, \tag{1.2c}$$

then the initial value problem of the equation of motion of the system can be solved by quadratures.

Key to the proof of Theorem 1. 1 is to show the existence of a canonical transformation,

$$(\boldsymbol{p};\boldsymbol{q}) \to (\boldsymbol{P};\boldsymbol{Q}), \quad P_j = I_j, \quad j = 1,\cdots,N. \tag{1.3}$$

Since the conserved quantities are independent (1.2b), we can solve a set of equations,

$$P_j = I_j(\boldsymbol{p};\boldsymbol{q};t), \quad j=1,\cdots,N,$$

with respect to $\boldsymbol{p}$:

$$p_j = F_j(\boldsymbol{q};\boldsymbol{P};t), \quad j=1,\cdots,N. \tag{1.4}$$

Here $F_j(\boldsymbol{q};\boldsymbol{P};t)$ is an analytic function, whose existence is guaranteed by the inverse function theorem. Computing partial derivatives of both sides of eq. (1.4) with respect to p_j and q_j, we get

$$\sum_{l=1}^{N}\frac{\partial F_j}{\partial P_l}\frac{\partial P_l}{\partial q_i} = -\frac{\partial F_j}{\partial q_i}, \tag{1.5a}$$

$$\sum_{m=1}^{N}\frac{\partial F_k}{\partial P_m}\frac{\partial P_m}{\partial p_i} = \delta_{ik}. \tag{1.5b}$$

Then the product of eqs. (1.5) yields

$$\sum_{i,l,m=1}^{N}\frac{\partial F_j}{\partial P_l}\frac{\partial P_l}{\partial q_i}\frac{\partial F_k}{\partial P_m}\frac{\partial P_m}{\partial p_i} = -\frac{\partial F_j}{\partial q_k} \tag{1.6}$$

Antisymmetrizing eq. (1.6) with respect to the indices j and k, we have

$$\begin{aligned}
\frac{\partial F_k}{\partial q_j} - \frac{\partial F_j}{\partial q_k} &= \sum_{i,l,m=1}^{N}\frac{\partial F_j}{\partial P_l}\frac{\partial F_k}{\partial P_m}\Big(\frac{\partial P_l}{\partial q_i}\frac{\partial P_m}{\partial p_i} - \frac{\partial P_m}{\partial q_i}\frac{\partial P_l}{\partial p_i}\Big)\\
&= \sum_{l,m=1}^{N}\frac{\partial F_j}{\partial P_l}\frac{\partial F_k}{\partial P_m}\{I_l,I_m\}_{\mathrm{P}}\\
&= 0.
\end{aligned} \tag{1.7}$$

The last equality of eq. (1.7) comes from the mutual commutativity of the conserved quantities with respect to the Poisson bracket (1.2c).

Now we define a function $W_\Gamma(\boldsymbol{q},\boldsymbol{q}^{(\mathrm{i})};\boldsymbol{P};t)$ by the following integral:

$$W_\Gamma(\boldsymbol{q},\boldsymbol{q}^{(\mathrm{i})};\boldsymbol{P};t)=\sum_{k=1}^{N}\int_\Gamma F_k(\boldsymbol{q};\boldsymbol{P};t)\mathrm{d}q_k. \tag{1.8}$$

In the above expression (1.8), Γ means a path in N dimensional coordinate space that connects two points $\boldsymbol{q}$ and $\boldsymbol{q}^{(\mathrm{i})}$. We can see that the above integral is independent of paths Γ. Let Γ and Γ' be two different paths that connect $\boldsymbol{q}$ and $\boldsymbol{q}^{(\mathrm{i})}$. The difference between W_Γ and $W_{\Gamma'}$ is

$$W_\Gamma-W_{\Gamma'}=\sum_{k=1}^{N}\int_{\partial S}F_k\mathrm{d}q_k,$$

where S is a hypersurface whose boundary ∂S is given by the closed path, $\partial S=\Gamma-\Gamma'$. Then using the Stokes' formula, we get

$$\begin{aligned}W_\Gamma-W_{\Gamma'} &= \frac{1}{2}\sum_{j,k=1}^{N}\int_S\mathrm{d}q_j\mathrm{d}q_k\Big(\frac{\partial F_k}{\partial q_j}-\frac{\partial F_j}{\partial q_k}\Big)\\ &= 0.\end{aligned}$$

The last equality follows from eq. (1.7). Thus we see the integral (1.8) is independent of paths Γ. We denote the path-independent function by

$$W(\boldsymbol{q},\boldsymbol{q}^{(\mathrm{i})};\boldsymbol{P};t)=\sum_{k=1}^{N}\int_{\boldsymbol{q}^{(\mathrm{i})}}^{\boldsymbol{q}}F_k(\boldsymbol{q};\boldsymbol{P};t)\mathrm{d}q_k. \tag{1.9}$$

We can easily verify that this function $W(\boldsymbol{q},\boldsymbol{q}^{(\mathrm{i})};\boldsymbol{P};t)$ is a generating function of the canonical transformation (1.3) in the following way. We define the angle variable $\boldsymbol{Q}$ that is canonically conjugate to to the action variable $\boldsymbol{P}$ by

$$Q_j=\frac{\partial W}{\partial P_j},\ \ j=1,\cdots,N. \tag{1.10}$$

From the definition of the generating function (1.9), we can readily see

$$\frac{\partial W}{\partial q_j}=F_j=p_j,\ \ j=1,\cdots,N. \tag{1.11}$$

Paying attention to the independent variables of the partial derivatives,

$$\begin{aligned}\frac{\partial}{\partial q_k}\Big|_{p;q;t} &= \frac{\partial}{\partial q_k}\Big|_{q;P;t}+\sum_{l=1}^{N}\frac{\partial P_l}{\partial q_k}\Big|_{p;q;t}\frac{\partial}{\partial P_l}\Big|_{q;P;t},\\ \frac{\partial}{\partial p_k}\Big|_{p;q;t} &= \sum_{l=1}^{N}\frac{\partial P_l}{\partial p_k}\Big|_{p;q;t}\frac{\partial}{\partial P_l}\Big|_{q;P;t},\end{aligned}$$

we can calculate the Poisson brackets among the action-angle variables $(\boldsymbol{P};\boldsymbol{Q})$ as follows:

$$\begin{aligned}\big\{P_i,P_j\big\}_{\mathrm{P}} &= \big\{I_i,I_j\big\}_{\mathrm{P}}=0,\\ \big\{Q_i,Q_j\big\}_{\mathrm{P}} &= \sum_{k=1}^{N}\Big(\Big(\frac{\partial}{\partial q_k}\Big|_{p;q;t}\frac{\partial W}{\partial P_i}\Big|_{q;P;t}\Big)\Big(\frac{\partial}{\partial p_k}\Big|_{p;q;t}\frac{\partial W}{\partial P_j}\Big|_{q;P;t}\Big)\\ &\quad -\Big(\frac{\partial}{\partial p_k}\Big|_{p;q;t}\frac{\partial W}{\partial P_i}\Big|_{q;P;t}\Big)\Big(\frac{\partial}{\partial q_k}\Big|_{p;q;t}\frac{\partial W}{\partial P_j}\Big|_{q;P;t}\Big)\Big)\\ &= \sum_{l,m=1}^{N}\frac{\partial^2 W}{\partial P_i\partial P_l}\Big|_{q;P;t}\big\{P_l,P_m\big\}_{\mathrm{P}}+\sum_{l=1}^{N}\Big(\delta_{il}\frac{\partial^2 W}{\partial P_j\partial P_l}\Big|_{q;P;t}-\delta_{jl}\frac{\partial^2 W}{\partial P_i\partial P_l}\Big|_{q;P;t}\Big)\\ &= 0,\\ \big\{Q_i,P_j\big\}_{\mathrm{P}} &= \sum_{k=1}^{N}\Big(\Big(\frac{\partial}{\partial q_k}\Big|_{p;q;t}\frac{\partial W}{\partial P_i}\Big|_{q;P;t}\Big)\frac{\partial P_j}{\partial p_k}\Big|_{p;q;t}-\frac{\partial P_i}{\partial p_k}\Big|_{p;q;t}\Big(\frac{\partial}{\partial q_k}\Big|_{p;q;t}\frac{\partial W}{\partial P_j}\Big|_{q;P;t}\Big)\Big)\\ &= \frac{\partial P_j}{\partial P_i}\Big|_{q;P;t}+\sum_{l=1}^{N}\frac{\partial^2 W}{\partial P_i\partial P_l}\Big|_{q;P;t}\big\{P_l,P_j\big\}_{\mathrm{P}}\\ &= \delta_{ij}.\end{aligned}$$

Thus we have verified that the transformation (1.3) preserves the fundamental Poisson bracket (1.1b) by using properties of the generating function (1.11) and hence it is a canonical transformation.

The time evolution of the action-angle variables is described by the canonical equations of motion,

$$\frac{\mathrm{d}Q_j}{\mathrm{d}t} = \frac{\partial K}{\partial P_j}, \tag{1.12a}$$

$$\frac{\mathrm{d}P_j}{\mathrm{d}t} = -\frac{\partial K}{\partial Q_j}=0,\ \ j=1,\cdots,N, \tag{1.12b}$$

where the new Hamiltonian K is given by

$$K=H+\frac{\partial W}{\partial t}. \tag{1.12c}$$

The second equation of the canonical equation of motion (1.12) is equivalent to eq. (1.2a) because P_j is a conserved quantity I_j.

Equation (1.12b) means that the Hamiltonian K is independent of the angle variables $\boldsymbol{Q}$,

$$K(\boldsymbol{P};\boldsymbol{Q};t)=K(\boldsymbol{P};t).$$

Since the angle variables $\boldsymbol{P}$ are invariant under the time evolution, we can fix them at $\boldsymbol{P}^{(0)}$:

$$K=K(\boldsymbol{P}^{(0)};t). \tag{1.13}$$

This means that the Hamiltonian is an analytic function of time t with N initial parameters $\boldsymbol{P}^{(0)}$. Substituting eq. (1.13) into eq. (1.12a) and integrating it over t, we get

$$Q_j(t)=Q_j^{(0)}+\int_0^t\frac{\partial}{\partial P_j^{(0)}}K(P_1^{(0)},\cdots,P_N^{(0)};t')\mathrm{d}t'.$$

Thus we have solved the initial value problem in the action-angle variables. We can transform the action-angle variables and their initial values back to the original coordinates and momenta and their initial values through the canonical transformation (1.3) defined by the generating function (1.9). So dynamical systems satisfying the assumptions of the theorem can be solved by quadratures.

A short comment might be in order. Dynamical systems that satisfy the assumptions of Liouville's theorem on integrability are called completely integrable systems. Liouville's theorem says that there exists a canonical transformation that decomposes the system with N degrees of freedom into N systems with only one degree of freedom, whose time evolution equations we can integrate. However, it does not provide us an explicit form of such a canonical transformation and its inverse. This means that Liouville's theorem does not provide us a practical way to solve completely integrable systems. For practical methods to solve classical integrable systems, see, for instance, refs. [1, 28, 67, 68].

Because of Liouville's theorem, it has been an interesting problem in classical mechanics to prove the integrability of a system by showing a systematic way to construct a set of conserved quantities in involution. Historically, Lax provided such a method for the system with infinitely many degrees of freedom [47]. Suppose there is an $N \times N$ matrix L of coordinates $\boldsymbol{q}$ and momenta $\boldsymbol{p}$. Then the time evolution of the matrix L governed by a Hamiltonian H is given by

$$\frac{\mathrm{d}L}{\mathrm{d}t} = \big\{L, H\big\}_{\mathrm{P}} + \frac{\partial L}{\partial t}.$$

We suppose there is another $N \times N$ matrix M of coordinates and momenta that satisfies

$$\frac{\mathrm{d}L}{\mathrm{d}t} = \big[L, M\big], \tag{1.14}$$

where the bracket in the r.h.s. denotes the commutator between matrices,

$$\big[L, M\big] = LM - ML.$$

We call eq. (1.14) Lax equation. A pair of matrices, L and M, are called the Lax pair. The Lax equation provides us a systematic way to construct the conserved quantities of the Hamiltonian. From the Lax equation (1.14), we have the equation of motion for the n-th power of the L-matrix:

$$\frac{\mathrm{d}L^n}{\mathrm{d}t} = \big[L^n, M\big].$$

Calculating traces of both sides of the above equation, we have

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}\mathrm{Tr}L^n &= \mathrm{Tr}\big[L^n, M\big] \\ &= \mathrm{Tr}\big(L^n M - ML^n\big) \\ &= 0, \end{aligned}$$

due to the identity, $\mathrm{Tr}AB = \mathrm{Tr}BA$, for arbitrary c-number valued matrices, A and B. Thus we can obtain conserved quantities of the Hamiltonian H just by calculating traces of the powers of the L-matrix:

$$I_n \stackrel{\mathrm{def}}{=} \mathrm{Tr}L^n, \quad \frac{\mathrm{d}I_n}{\mathrm{d}t} = 0, \quad n = 1, 2, \cdots. \tag{1.15}$$

To prove the involutivity of the conserved quantities (1.2c), we usually use the classical r-matrix, which is an $N^2 \times N^2$ matrix that satisfies the following fundamental Poisson bracket [28],

$$\big\{L^{(1)}, L^{(2)}\big\}_{\mathrm{P}} = \big[r_{12}, L^{(1)}\big] - \big[r_{21}, L^{(2)}\big]. \tag{1.16}$$

The undefined symbols in the above equation are

$$L^{(1)} \stackrel{\mathrm{def}}{=} L \otimes \mathbf{1}, \quad L^{(2)} \stackrel{\mathrm{def}}{=} \mathbf{1} \otimes L$$
$$r_{21} \stackrel{\mathrm{def}}{=} P r_{12} P, \quad P \stackrel{\mathrm{def}}{=} \sum_{i,j=1}^{N} E_{ij} \otimes E_{ji}, \quad Px \otimes y = y \otimes x,$$

where $\mathbf{1}$ and E_{ij} are the unit matrix and the matrix unit, $\mathbf{1}_{kl} = \delta_{kl}$, $(E_{ij})_{kl} = \delta_{ik}\delta_{jl}$, respectively. From eq. (1.16), we can prove the mutual Poisson commutativity among the conserved quantities (1.15) as follows. We shall calculate the following Poisson bracket:

$$\begin{aligned} \big\{L, I_n\big\}_{\mathrm{P}} &= \big\{L, \mathrm{Tr}L^n\big\}_{\mathrm{P}} \\ &= \mathrm{Tr}_2\big\{L^{(1)}, (L^{(2)})^n\big\}_{\mathrm{P}}. \end{aligned}$$

Here, the symbol Tr_2 means the trace with respect to the second space of the tensor product, i.e., $\mathrm{Tr}_2 A \otimes B = A(\mathrm{Tr}B)$. Using eq. (1.16), we have

$$\begin{aligned} &\mathrm{Tr}_2\big\{L^{(1)}, (L^{(2)})^n\big\}_{\mathrm{P}} \\ &= \mathrm{Tr}_2 \sum_{k=1}^{n} (L^{(2)})^{k-1}\big\{L^{(1)}, L^{(2)}\big\}_{\mathrm{P}}(L^{(2)})^{n-k} \\ &= \mathrm{Tr}_2 \sum_{k=1}^{n} (L^{(2)})^{k-1}\Big(\big[r_{12}, L^{(1)}\big] - \big[r_{21}, L^{(2)}\big]\Big)(L^{(2)})^{n-k} \\ &= \big[L^{(1)}, -n\mathrm{Tr}_2 r_{12}(L^{(2)})^{n-1}\big]. \end{aligned}$$

Thus we obtain

$$\big\{L, I_n\big\}_{\mathrm{P}} = \big[L, M_n\big], \quad M_n \stackrel{\mathrm{def}}{=} -n\mathrm{Tr}_2 r_{12}(L^{(2)})^{n-1}. \tag{1.17}$$

This is nothing but the Lax equation for the higher conserved quantity I_n that guarantees the Poisson commutativity with other conserved quantities. Using eq. (1.17), the Poisson bracket among the conserved quantities I_n and I_m is calculated as follows:

$$\begin{aligned} \big\{I_m, I_n\big\}_{\mathrm{P}} &= \big\{\mathrm{Tr}L^m, I_n\big\}_{\mathrm{P}} \\ &= \mathrm{Tr}\big[L^m, M_n\big] \\ &= 0. \end{aligned}$$

Thus we conclude that the involutivity of the conserved quantity (1.2c) follows from the classical r-matrix and the fundamental Poisson bracket (1.16). The Lax formulation and the classical r-matrix are the standard formulation for the classical integrable systems.

Though there is no theorem in quantum theory that corresponds to Liouville's theorem in the classical theory, we often define the quantum integrable system in a similar fashion to the definition of the completely integrable system. The quantum integrable system is a quantum system with N degrees of freedom that has N, independent and mutually commuting conserved operators,

$$\frac{\mathrm{d}I_n}{\mathrm{d}t} = 0, \ \left[I_n, I_m\right] = 0, \ n, m = 1, 2, \cdots, N, \tag{1.18}$$

where the time evolution of the conserved operator is governed by the Heisenberg equation of motion,

$$-\mathrm{i}\hbar\frac{\mathrm{d}I_n}{\mathrm{d}t} = \left[H, I_n\right] - \mathrm{i}\hbar\frac{\partial I_n}{\partial t}.$$

Usually, we deal with operators that do not explicitly depend on time t. In this case, the second term in the r.h.s. of the above Heisenberg equation is zero. Existence of mutually commuting conserved operators means that we can specify, in principle, all the quantum numbers the system has by identifying the simultaneous eigenfunctions of the conserved operators. Thus, the systematic construction of the conserved operators and the identification of their simultaneous eigenfunctions are important problems in the theory of quantum integrable system. It is to be noted that the construction of the simultaneous eigenfunctions for the commuting conserved operators in practice and the proof of quantum integrability are different problems, which is somewhat similar to what the integrability in Liouville's sense means. However, the beautiful machinery for the classical integrable systems such as the Lax formulation and the classical r-matrix can not be applied to the quantum systems because of the non-commutativity between the canonical conjugate variables. For example, the trace formula for the conserved quantities (1.15) does not work in the quantum theory. Suppose that the natural quantization of the Lax equation holds:

$$-\mathrm{i}\hbar\frac{\mathrm{d}L}{\mathrm{d}t} = \left[H, L\right] - \mathrm{i}\hbar\frac{\partial L}{\partial t} = \left[L, M\right].$$

Then the time evolution of the power of the L-matrix is expressed as

$$-\mathrm{i}\hbar\frac{\mathrm{d}L^n}{\mathrm{d}t} = \left[L^n, M\right].$$

However, we cannot conclude from the above equation that the trace of the power of the L-matrix is a conserved operator because the trace identity, $\mathrm{Tr}AB = \mathrm{Tr}BA$, does not work for operator-valued matrices A and B, i.e.,

$$-\mathrm{i}\hbar\frac{\mathrm{d}}{\mathrm{d}t}\mathrm{Tr}L^n = \mathrm{Tr}\left[L^n, M\right] \stackrel{?}{=} 0.$$

Thus we have to develop some extra device for the systematic analysis of the quantum integrable systems.

1.2 Calogero Model

The Calogero model was introduced as a solvable quantum many-body model in 1971 [19]. The Calogero model describes N identical particles on a line with inverse-square long-range interactions confined in an external harmonic well,

$$H_{\mathrm{C}} = \frac{1}{2}\sum_{j=1}^{N} p_j^2 + \frac{1}{2}g\sum_{\substack{j,k=1\\ j\neq k}}^{N}\frac{1}{(x_j - x_k)^2} + \frac{1}{2}\omega^2\sum_{j=1}^{N}x_j^2, \tag{1.19}$$

where the momentum p_j is given by a partial differential operator, $p_j = -\mathrm{i}\dfrac{\partial}{\partial x_j}$. Throughout this thesis, we set the Planck constant at unity, $\hbar = 1$. Two parameters, g and ω are the coupling constant and the strength of the external harmonic well, respectively. In recent years, one-dimensional quantum systems with inverse-square interactions including the Calogero model enjoy renewed interests of theoretical physicists. Japanese researchers have also contributed toward attracting many researchers who are interested in the long-range effect of electron-electron interactions in condensed matter to the inverse-square interaction models. Among such contributions, Kawakami and Yang's study on the low-temperature critical behaviour of the Sutherland model [38] and introduction of the supersymmetric long-range t-J model by Kuramoto and Yokoyama [40] should be noted.

From the viewpoint of the classical integrable systems, the classical Calogero model is a completely integrable system in Liouville's sense that has a Lax formulation [52, 57, 58] and a classical r-matrix [6]. The idea to introduce the Lax formulation for the inverse-square interaction models is due to Moser [52]. Though Moser dealt with the Calogero-Moser model,

$$H_{\mathrm{CM}} = \frac{1}{2}\sum_{j=1}^{N} p_j^2 + \frac{1}{2}g\sum_{\substack{j,k=1\\ j\neq k}}^{N}\frac{1}{(x_j - x_k)^2}, \tag{1.20}$$

and the Sutherland model [72, 73],

$$H_{\mathrm{S}} = \frac{1}{2}\sum_{j=1}^{N} p_j^2 + \frac{1}{2}g\sum_{\substack{j,k=1\\ j\neq k}}^{N}\frac{1}{\sin^2(x_j - x_k)}, \tag{1.21}$$

the Lax formulation for the Calogero model (1.19) is straightforwardly derived from the Lax formulation for the Calogero-Moser model (1.20). The Sutherland model (1.21) is defined by a generalization of the Calogero-Moser model (1.20) that is compatible with the periodic boundary condition. In Chapter 4, we shall see a brief summary on the common algebraic structure of the Calogero and the Sutherland models. The classical Lax formulation was extended further to those for more generalized models, e.g. the elliptic Calogero-Moser model [21] and the inverse-square interaction models associated with the root lattices of Lie algebras [58]. At that time, the integrability of the models were claimed just by showing the Lax formulation, though the Lax formulation is not sufficient to prove the involutivity, or in other words, the mutual Poisson commutativity among the conserved quantities. Proof of involutivity for the classical Calogero model was completed by constructing the classical r-matrix [6]. Here we shall briefly summarize a derivation of the Lax formulation and the r-matrix method for the classical Calogero model.

Following the projection method [57, 58], we shall first introduce the classical Lax formulation for the Calogero model. The merit of the projection method is that it gives the general solution for the initial value problem of the classical Calogero model. Let us consider a time-dependent $N \times N$ Hermitian matrix $X(t)$. We shall deal with a case that each element of the matrix $X(t)$ obeys the equation of motion for the harmonic oscillator:

$$\frac{\mathrm{d}^2 X}{\mathrm{d}t^2}(t) = -\omega^2 X(t). \tag{1.22}$$

The general solution of the above equation of motion are readily obtained as

$$X(t) = X(0)\cos\omega t + \frac{1}{\omega}\frac{\mathrm{d}X}{\mathrm{d}t}(0)\sin\omega t, \tag{1.23}$$

where $X(0)$ and $\frac{\mathrm{d}X}{\mathrm{d}t}(0)$ are the initial coordinate and velocity. We note that the matrix,

$$C(t) \stackrel{\text{def}}{=} \Big[X(t), \frac{\mathrm{d}X}{\mathrm{d}t}(t)\Big], \tag{1.24a}$$

is an anti-Hermitian conserved matrix:

$$\begin{aligned} \frac{\mathrm{d}C}{\mathrm{d}t}(t) &= \Big[\frac{\mathrm{d}X}{\mathrm{d}t}(t), \frac{\mathrm{d}X}{\mathrm{d}t}(t)\Big] + \Big[X(t), \frac{\mathrm{d}^2X}{\mathrm{d}t^2}(t)\Big] = 0, & (1.24\text{b}) \\ C^\dagger(t) &= -C(t). & (1.24\text{c}) \end{aligned}$$

We consider the motion of eigenvalues of the Hermitian matrix $X(t)$ by the "projection" or an unitary transformation:

$$X(t) = U(t)D(t)U^{-1}(t), \quad D(t) = \begin{bmatrix} x_1(t) & & & \\ & x_2(t) & & \\ & & \ddots & \\ & & & x_N(t) \end{bmatrix}. \tag{1.25}$$

Calculating the time-derivative of both sides of the above equation, we have

$$\begin{aligned} \frac{\mathrm{d}X}{\mathrm{d}t}(t) &= \frac{\mathrm{d}U}{\mathrm{d}t}(t)D(t)U^{-1}(t) + U(t)\frac{\mathrm{d}D}{\mathrm{d}t}(t)U^{-1}(t) - U(t)D(t)U^{-1}(t)\frac{\mathrm{d}U}{\mathrm{d}t}(t)U^{-1}(t) & \\ &= U(t)L(t)U^{-1}(t), & (1.26\text{a}) \\ L(t) &\stackrel{\text{def}}{=} \frac{\mathrm{d}D}{\mathrm{d}t}(t) + \big[M(t), D(t)\big], & (1.26\text{b}) \\ M(t) &\stackrel{\text{def}}{=} U^{-1}(t)\frac{\mathrm{d}U}{\mathrm{d}t}(t). & (1.26\text{c}) \end{aligned}$$

Note that the L- and M-matrices are respectively Hermitian and anti-Hermitian,

$$\begin{aligned} L^\dagger(t) &= \frac{\mathrm{d}D^\dagger}{\mathrm{d}t}(t) + \Big[D^\dagger(t), \frac{\mathrm{d}U^{-1}}{\mathrm{d}t}(t)U(t)\Big] \\ &= \frac{\mathrm{d}D}{\mathrm{d}t}(t) + \Big[U^{-1}(t)\frac{\mathrm{d}U}{\mathrm{d}t}(t), D(t)\Big] \\ &= L(t), \\ M^\dagger(t) &= \frac{\mathrm{d}U^{-1}}{\mathrm{d}t}(t)U(t) \\ &= -U^{-1}(t)\frac{\mathrm{d}U}{\mathrm{d}t}(t) \\ &= -M(t). \end{aligned}$$

Using the above L-, M- and D-matrices, the equation of motion for the Hermitian matrix (1.22) is cast into the following form:

$$\frac{\mathrm{d}L}{\mathrm{d}t}(t) = \big[L(t), M(t)\big] - \omega^2 D(t). \tag{1.27}$$

Equations (1.26) and (1.27) describe the motion of the eigenvalues of the time-dependent Hermitian matrix $X(t)$. From the L- and M- matrices that satisfy eqs. (1.26) and (1.27), we can reproduce the free harmonic oscillation of the Hermitian matrix $X(t)$. For a given M-matrix, the unitary matrix $U(t)$ that satisfies eq. (1.26c) is obtained as follows:

$$U(t) = U(0)\sum_{l=0}^{\infty}\int_0^t \mathrm{d}t_l \int_0^{t_l} \mathrm{d}t_{l-1}\cdots\int_0^{t_2}\mathrm{d}t_1 M(t_1)M(t_2)\cdots M(t_l). \tag{1.28}$$

We can confirm that the above integral is indeed a solution of eq. (1.26c) by the following calculation:

$$\begin{aligned} \frac{\mathrm{d}U}{\mathrm{d}t}(t) &= \Big(U(0)\sum_{l=1}^{\infty}\int_0^t \mathrm{d}t_{l-1}\int_0^{t_{l-1}}\mathrm{d}t_{l-2}\cdots\int_0^{t_2}\mathrm{d}t_1 M(t_1)M(t_2)\cdots M(t_{l-1})\Big)M(t) \\ &= U(t)M(t). \end{aligned}$$

Thus we want to find out the Lax pair, L and M, which satisfies eqs. (1.26) and (1.27) with the time evolution defined by the classical Calogero Hamiltonian.

The number of the initial parameters of the harmonic oscillation of the Hermitian matrix $X(t)$ is $2N^2$ whereas those of the N-body Calogero model is $2N$. Thus we have to introduce a restriction to the initial parameters of $X(t)$ and reduce its number to $2N$. As the first restriction, we restrict the initial value of matrix $X(t)$ a diagonal matrix:

$$X(0) = \begin{bmatrix} x_1(0) & & & \\ & x_2(0) & & \\ & & \ddots & \\ & & & x_N(0) \end{bmatrix} = D(0). \tag{1.29}$$

This yields the restriction to the initial value of the unitary matrix,

$$U(0) = \mathbf{1}. \tag{1.30}$$

The second one is the restriction to the conserved matrix,

$$C_{ij} = \mathrm{i}a(1-\delta_{ij}), \tag{1.31}$$

which yields the restriction to the initial velocity of the matrix $X(t)$:

$$\begin{aligned} C_{ij} &= \Big[X(0), \frac{\mathrm{d}X}{\mathrm{d}t}(0)\Big]_{ij} \\ &= (1-\delta_{ij})\Big(\frac{\mathrm{d}X}{\mathrm{d}t}(0)\Big)_{ij}\big(x_i(0) - x_j(0)\big) \\ &= \mathrm{i}a(1-\delta_{ij}) \\ \Rightarrow \; & \Big(\frac{\mathrm{d}X}{\mathrm{d}t}(0)\Big)_{ij} = p_i(0)\delta_{ij} + \mathrm{i}a(1-\delta_{ij})\frac{1}{x_i(0) - x_j(0)} = \big(L(0)\big)_{ij}. \end{aligned} \tag{1.32}$$

This observation suggests that the form of L-matrix is given by

$$\big(L(t)\big)_{ij} = p_i(t)\delta_{ij} + \mathrm{i}a(1-\delta_{ij})\frac{1}{x_i(t) - x_j(t)}. \tag{1.33}$$

Defining the time-derivative by the Poisson bracket with the classical Calogero Hamiltonian (1.19),

$$\frac{\mathrm{d}L}{\mathrm{d}t}(t) \stackrel{\mathrm{def}}{=} \left\{L(t), H_C\right\}_{\mathrm{P}},$$

we can determine the explicit form of the M-matrix by the sufficient condition of eq. (1.26),

$$\begin{aligned}\left\{L(t), H_C\right\}_{\mathrm{P}} &= \left[L(t), M(t)\right] - \omega^2 D(t) \\ \Rightarrow \left(M(t)\right)_{ij} &= \mathrm{i}a\delta_{ij}\sum_{\substack{l=1\\l\neq i}}^{N}\frac{1}{(x_i(t)-x_l(t))^2} - \mathrm{i}a(1-\delta_{ij})\frac{1}{(x_i(t)-x_j(t))^2},\end{aligned} \tag{1.34}$$

where the coupling constant is given by

$$g = a^2. \tag{1.35}$$

The above L- and M- matrices are respectively Hermitian and anti-Hermitian. They also satisfy eq. (1.26) under the identification

$$\frac{\mathrm{d}D}{\mathrm{d}t}(t) \stackrel{\mathrm{def}}{=} \left\{D(t), H_C\right\}_{\mathrm{P}}.$$

Thus we conclude that the general solution for the classical Calogero model (1.19) is obtained just by diagonalizing the time-dependent matrix (1.23) with initial values, eqs. (1.29) and (1.32).

Conserved quantities of the Calogero model are obtained by the trace formula (1.15). We introduce new matrices $L^{\pm}$ by

$$L^{\pm} \stackrel{\mathrm{def}}{=} L \pm \omega Q, \quad Q \stackrel{\mathrm{def}}{=} \mathrm{i}D. \tag{1.36}$$

Then eqs. (1.26) and (1.27) are cast into the following forms:

$$\left\{L^{\pm}, H_C\right\}_{\mathrm{P}} = \left[L^{\pm}, M\right] \pm \mathrm{i}\omega L^{\pm}. \tag{1.37}$$

We call the above equation Lax equation for the Calogero model. It is straightforward to derive the following equation from the Lax equation,

$$\left\{L^+L^-, H_C\right\}_{\mathrm{P}} = \left[L^+L^-, M\right]. \tag{1.38}$$

Thus we can obtain the conserved quantities of the Calogero model by the following trace formula,

$$I_n^{\mathrm{classical}} = \mathrm{Tr}(L^+L^-)^n, \tag{1.39}$$

and the Hamiltonian corresponds to the first conserved quantity,

$$I_1^{\mathrm{classical}} = 2H_C.$$

Involutivity of the conserved quantities is verified by the classical r-matrix [6]. The fundamental Poisson bracket of the matrix L^+L^- are expressed as

$$\left\{(L^+L^-)^{(1)}, (L^+L^-)^{(2)}\right\}_{\mathrm{P}} = \left[r_{12}, (L^+L^-)^{(1)}\right] - \left[r_{21}, (L^+L^-)^{(2)}\right], \tag{1.40a}$$

where the classical r-matrix is given by

$$\begin{aligned} r_{12} = &-\sum_{\substack{j,k=1\\j\neq k}}^{N}\frac{1}{x_j - x_k}\Big\{E_{jk}\otimes\sum_{l=1}^{N}\left(L_{lk}E_{lj} + L_{jl}E_{kl}\right) \\ &+\frac{1}{2}E_{jj}\otimes\sum_{l=1}^{N}\left(L_{lj}E_{lk} + L_{kl}E_{jl} - L_{lk}E_{lj} - L_{jl}E_{kl}\right) \\ &+\frac{\mathrm{i}\omega}{2}E_{jj}\otimes\left(E_{jk} - E_{kj}\right)\Big\}. \end{aligned} \tag{1.40b}$$

Thus we confirm the integrability of the classical Calogero model.

We have observed that the Lax formulation gives a powerful method to study the classical Calogero model. The Lax formulation gives not only a way to prove the integrability, but also a way to solve the initial value problem of the Calogero model. However, as we have mentioned in the previous section, non-commutativity between the canonical-conjugate variables in the quantum theory deprives the Lax formulation of its merits. So we have to invent a new device for the systematic study of the quantum Calogero model.

1.3 Motivation and Aims

We have briefly summarized what is the integrable systems and how effectively the Lax formulation works for the classical Calogero model. It is quite natural to expect that the quantized classical integrable systems might be the quantum integrable systems in a sense they possess mutually commuting conserved operators. It was pointed out by Calogero and Olshanetsky et al. [20, 59] that the quantization of the classical Lax equation for the classical model with symmetrizing the non-commutative products yields a equality for the "quantized classical Calogero Hamiltonian" which does not contain a quantum correction in the coupling constant. This observation had been a ground for the conjecture that the quantization of the classical conserved quantities by the correspondence principle might yield the conserved operators of the quantum Calogero model. On the other hand, a naive quantization by the correspondence principle of the classical Lax equation without symmetrization gives a equality for the correct quantum Calogero model which has a quantum correction in the coupling constant. However, the Lax formulation, which is a precision machinery for the classical integrable systems, does not work in the quantum theory because the non-commutativity between the canonical-conjugate variables breaks the trace identity. Because of this difficulty, a proof of quantum integrability does not follow from classical integrability. The fact motivates us to find out a new instrument for the quantum integrable systems. In this thesis, we shall try such a challenge for the quantum Calogero model (1.19).

The outline of the thesis is as follows. In Chapter 2, we shall study two systematic formulations for the quantum Calogero model [76, 77, 80]. One is the quantum Lax formulation, which is a natural generalization of the classical Lax formulation developed by the author and his collaborators [34, 74–78, 88, 89]. The other is the Dunkl operator formulation whose fundamental properties were studied by Dunkl [27]. This method was introduced by Polychronakos [62] into the field of the inverse-square interaction models. Using these formulations, we shall study a systematic construction of the mutually commuting conserved operators and the underlying symmetry of the quantum Calogero model. In Chapter 3, we shall present an algebraic construction of the eigenfunctions of the quantum Calogero Hamiltonian (1.19) and try a construction of the simultaneous eigenfunctions of commuting conserved operators [76, 77, 79]. As we have mentioned in Section 1.1, just presenting a set of commuting conserved operators and constructing simultaneous eigenfunctions for them in practice are completely different problems. We shall complete the algebraic construction of the energy eigenfunctions [76, 77] which was first considered by Perelomov [61]. As a first step toward the identification of the simultaneous eigenfunctions of the commuting conserved operators, we shall diagonalize the first two commuting conserved operators and construct some simultaneous eigenfunctions [79]. In Chapter 4, we shall identify the simultaneous eigenfunctions of all the commuting conserved operators [83, 84, 86, 87]. The simultaneous eigenfunctions form the orthogonal basis of the model, as is the case with the orthogonal basis of the hydrogen atom. Using the Dunkl operator formulation, we shall prove the algebraic scheme, or in other words, the Rodrigues formula for the construction of the orthogonal symmetric polynomials associated with the orthogonal basis of the Calogero model, which we call the Hi-Jack (hidden-Jack) symmetric polynomials. We shall study some properties of the Hi-Jack polynomials and prove that they indeed form the orthogonal basis of the Hilbert space of the quantum Calogero model. The final chapter is devoted to summary and concluding remarks.

Chapter 2

Algebraic Structure of the Calogero Model

Construction of the conserved currents and identification of underlying symmetry are important problems for integrable systems. For the classical integrable system, the Lax formulation [47] has played an important role to solve these problems. However for the quantum theory, the non-commutativity of the canonical conjugate variables spoils some merits of the Lax formulation. And no general method to overcome this difficulty has been developed even by now.

For quantum inverse-square-interaction models such as the Calogero-Moser (1.20), Sutherland (1.21) and Calogero models (1.19), two approaches were developed. One is the Dunkl operator formulation [27], that was first applied to the above inverse-square-interaction models by Polychronakos [62]. By the Dunkl operator formulation, or in other words, the exchange operator formalism, sets of commuting conserved operators for the three models were obtained. The other formulation is the quantum Lax formulation [34, 74–79, 88, 89], which is a natural quantum mechanics generalization of the classical Lax formulation. The quantum Lax formulation was first introduced and studied for the quantum Calogero-Moser and Sutherland models [34, 74, 75, 88, 89], and then extended to the Calogero model [76–78]. The crucial points for the formulation are the "sum-to-zero" condition for the M-matrices and the recursive construction of the generalized Lax equations. Through the generalized Lax equations, we can compute the commutators among the operators related to the three models. A set of commuting conserved operators for the Calogero-Moser model [74, 75, 88, 89] and the algebraic structures of the three models [34, 75, 77, 78, 89] were obtained through the quantum Lax formulation.

The aim of this chapter is to present the quantum Lax and Dunkl operator formulations, which will play an important role throughout the thesis, and the W-symmetry structure of the quantum Calogero model:

$$H_{\mathrm{C}} = \frac{1}{2}\sum_{j=1}^{N} p_j^2 + \frac{1}{2} g \sum_{\substack{j,k=1\\ j\neq k}}^{N} \frac{1}{(x_j - x_k)^2} + \frac{1}{2}\omega^2 \sum_{j=1}^{N} x_j^2, \tag{2.1a}$$

$$p_j = -\mathrm{i}\frac{\partial}{\partial x_j}. \tag{2.1b}$$

Motivated by the classical Lax equation for the classical Calogero Hamiltonian, which has been presented in Chapter 1, we formulate the quantum Lax equation for the quantum Calogero

Hamiltonian. Using the quantum Lax equation for the Hamiltonian, we construct three families of commuting operators, namely, commuting conserved operators and power-sum creation and annihilation operators by summing up all the elements of operator-valued matrices. By explicit calculations, we get some Lax-equation-like relations for some conserved operators and power-sum creation and annihilation operators. These relations lead to the recursion relation for the creation and annihilation operators. The recursion relation enables the recursive construction of the Lax-equation-like relations which we call generalized Lax equations. The W-algebra structure appears from the generalized Lax equations. Next, we summarize and reformulate the Dunkl operator formulation for the Calogero model and consider its relationship with the quantum Lax formulation. Integrability of the Calogero model is proved by explicitly showing the commuting conserved operators of the model.

2.1 Quantum Lax Formulation

In the classical theory, the Calogero model is solved by applying the projection method to an $N \times N$-Hermitian matrix whose time evolution is governed by the equation of motion for the harmonic oscillator [57, 58]. The projection method also gives a way to introduce the Lax equation for the classical Calogero model. Referring to the classical Lax equation (1.37), we shall construct the Lax equation for the quantum model. Let us introduce three matrices:

$$L_{jk} = p_j\delta_{jk} + \mathrm{i}a(1-\delta_{jk})\frac{1}{x_j - x_k}, \tag{2.2}$$

$$Q_{jk} = \mathrm{i}x_j\delta_{jk}, \tag{2.3}$$

$$M_{jk} = -a(1-\delta_{jk})\frac{1}{(x_j - x_k)^2} + a\delta_{jk}\sum_{\substack{l=1\\ l\neq j}}^{N}\frac{1}{(x_j - x_l)^2}. \tag{2.4}$$

The Lax equation for the quantum Calogero model is expressed as

$$-\mathrm{i}\frac{\mathrm{d}L^{\pm}}{\mathrm{d}t} = \left[H_{\mathrm{C}}, L^{\pm}\right] = \left[L^{\pm}, M\right] \pm \omega L^{\pm}, \tag{2.5}$$

where

$$L^{\pm} = L \pm \omega Q, \tag{2.6}$$

and the constant a that appears in eqs. (2.2) and (2.4) is related to the coupling constant g by

$$g = a^2 - a. \tag{2.7}$$

The second term on the r.h.s. of the Lax equation (2.5) comes from the external harmonic well. We remark that the relation (2.7) contains a quantum correction. With $\hbar$ explicitly written, the coupling constant g is expressed as $g = a^2 - \hbar a$, which reduces to the classical one (1.35) in the classical limit $\hbar \to 0$.

It is interesting to observe that the matrix M is common to the quantum Calogero-Moser model (the case $\omega = 0$) [34, 74, 75, 88, 89], and that the M-matrix satisfies the sum-to-zero condition:

$$\sum_{j=1}^{N} M_{jk} = \sum_{j=1}^{N} M_{kj} = 0. \tag{2.8}$$

The sum-to-zero condition leads to a systematic construction of the conserved operators, which is realized more easily by the trace formula in the classical case.

From the Lax equation (2.5), we see that the matrices of the following forms,

$$(L^+)^{p_1}(L^-)^{m_1}(L^+)^{p_2}(L^-)^{m_2}\cdots, \quad p_1 + p_2 + \cdots = m_1 + m_2\cdots, \tag{2.9}$$

satisfy a relation,

$$\begin{aligned}
&\left[H_{\mathrm{C}}, (L^+)^{p_1}(L^-)^{m_1}(L^+)^{p_2}(L^-)^{m_2}\cdots\right] \\
&= \left[(L^+)^{p_1}(L^-)^{m_1}(L^+)^{p_2}(L^-)^{m_2}\cdots, M\right] + \omega\Big(\sum_l p_l - \sum_k m_k\Big)(L^+)^{p_1}(L^-)^{m_1}(L^+)^{p_2}(L^-)^{m_2}\cdots \\
&= \left[(L^+)^{p_1}(L^-)^{m_1}(L^+)^{p_2}(L^-)^{m_2}\cdots, M\right].
\end{aligned} \tag{2.10}$$

Using the sum-to-zero condition on the M-matrix, we find that the conserved operators are obtained by simply summing up all the matrix elements of eq. (2.9):

$$\begin{aligned}
\Big[H_{\mathrm{C}}, \sum_{j,k=1}^{N}\big((L^+)^{p_1}(L^-)^{m_1}(L^+)^{p_2}(L^-)^{m_2}\cdots\big)_{jk}\Big] &= \sum_{j,k=1}^{N}\left[(L^+)^{p_1}(L^-)^{m_1}(L^+)^{p_2}(L^-)^{m_2}\cdots, M\right]_{jk} \\
&= 0.
\end{aligned} \tag{2.11}$$

Due to the sum-to-zero condition (2.8), the following identities hold for arbitrary operator-valued matrix A:

$$\sum_{j,k=1}^{N}(AM)_{jk} = \sum_{j,l=1}^{N} A_{jl}\Big(\sum_{k=1}^{N} M_{lk}\Big) = 0,$$

$$\sum_{j,k=1}^{N}(MA)_{jk} = \sum_{l,k=1}^{N}\Big(\sum_{j=1}^{N} M_{jl}\Big)A_{lk} = 0.$$

Then the last equality is readily verified by the above identities. Thus we can get various expressions of conserved operators made of the same number of the L^+- and L^--matrices just by changing the order of the two kinds of matrices. Among such conserved operators, we are interested in mutually commuting conserved operators, which can be simultaneously diagonalized, and the conserved operators for which we can construct the "generalized" Lax equations recursively. Hereafter, we often deal with a sum of all the matrix elements. So we denote a sum of all the elements of a matrix A by $\mathrm{T}_\Sigma A = \sum_{j,k=1}^{N} A_{jk}$.

A set of commuting conserved operators of the model is one of the important objects for quantum integrable systems because it shows the quantum integrability of the system in the analogous way for the classical completely integrable systems. Since the most simple matrix product of L^+ and L^- satisfying eq. (2.10) is L^+L^-, we study the following conserved operators as candidates of the commuting conserved operators:

$$I_n = \mathrm{T}_\Sigma(L^+L^-)^n. \tag{2.12}$$

Explicit forms of the first two conserved operators are

$$I_1 = \mathrm{T}_\Sigma(L^+L^-)$$

$$
\begin{aligned}
&= \sum_{j=1}^{N} p_j^2 + g \sum_{j,k=1}^{N}{}' \frac{1}{x_{jk}^2} + \omega^2 \sum_{j=1}^{N} x_j^2 + N\omega\big(Na + (1-a)\big) \\
&= 2H_{\mathrm{C}} + \text{constant}, \qquad (2.13a) \\
I_2 &= \mathrm{T}_\Sigma (L^+L^-)^2 \\
&= \sum_{j=1}^{N} p_j^4 + g \sum_{j,k=1}^{N}{}' \Big\{ p_j^2 \frac{1}{x_{jk}^2} + p_j \frac{1}{x_{jk}} p_k \frac{1}{x_{jk}} + p_j \frac{1}{x_{jk}^2} p_j + \frac{1}{x_{jk}} p_k^2 \frac{1}{x_{jk}} + \frac{1}{x_{jk}} p_k \frac{1}{x_{jk}} p_j + \frac{1}{x_{jk}^2} p_j^2 \Big\} \\
&\quad + 2g^2 \sum_{j,k,l=1}^{N}{}' \frac{1}{x_{jk}^2} \frac{1}{x_{kl}^2} + (g^2 - g) \sum_{j,k=1}^{N}{}' \frac{1}{x_{jk}^4} \\
&\quad + \omega^2 \Big\{ \sum_{j=1}^{N} (x_j^2 p_j^2 + p_j^2 x_j^2) + g \sum_{j,k=1}^{N}{}' \frac{x_j^2 + x_k^2}{x_{jk}^2} - a^2 \sum_{j,k,m=1}^{N}{}' \frac{x_j^2 + x_k^2}{x_{jm} x_{mk}} \Big\} + \omega^4 \sum_{j=1}^{N} x_j^4 \\
&\quad + 4\omega(a - 1 - Na) H_{\mathrm{C}} + \omega^2 \{ N(a - 1 - Na)^2 - aN(N-1) \}, \qquad (2.13b)
\end{aligned}
$$

where $x_{jk} \overset{\text{def}}{=} x_j - x_k$. The symbol $\sum'$ means that all the indices in the summand must not coincide. It is to be remarked that the total momentum is not a conserved operator. This reflects the fact that the system (2.1) is not translationally invariant. Furthermore, we see that the conserved operators generally possess the following forms:

$$
I_n = \sum_{j=1}^{N} p_j^{2n} + \cdots . \qquad (2.14)
$$

Thus we confirm that the first N conserved operators form a set of functionally independent conserved operators of the model.

In order to confirm the integrability of the model, it is not enough just to construct a set of independent conserved operators. We must verify the mutual commutativity of the conserved operators (2.12):

$$
\big[I_n, I_m \big] = 0, \quad n, m = 1, 2, \cdots . \qquad (2.15)
$$

We expect that non-trivial conserved operators also satisfy the Lax-equation-like relations with M-matrices that meet the sum-to-zero condition, as was demonstrated for the case $\omega = 0$ [75]. We generally call such Lax-equation-like relations generalized Lax equations. Using the explicit forms of the conserved operators (2.13), we obtain the generalized Lax equations for I_1 and I_2:

$$
\begin{aligned}
\big[I_1, L^\pm \big] &= \big[L^\pm, M_1 \big] \pm 2\omega L^\pm, \qquad (2.16a) \\
M_1 &= 2M, \qquad (2.16b) \\
\big[I_2, L^\pm \big] &= \big[L^\pm, M_2 \big] \pm 4\omega L^\pm L^\mp L^\pm \pm 4\omega^2 (a - 1 - Na) L^\pm, \qquad (2.16c) \\
(M_2)_{jk} &= 4\delta_{jk} \sum_{\substack{q=1 \\ q \neq j}}^{N} \Big\{ \frac{a}{x_{jq}^2} (p_j^2 + p_j p_q + p_q^2) + \mathrm{i}a \frac{1}{x_{jq}^3} (p_j - p_q) \\
&\quad + \mathrm{i}a^2 \sum_{l=1}^{N}{}' \frac{1}{x_{jl} x_{lq} x_{jq}} (p_j + p_l + p_q) + (a^3 - 3a) \frac{1}{x_{jq}^4} + a^2 \sum_{l=1}^{N}{}' \Big\{ \frac{1}{x_{jl} x_{lq} x_{jq}^2} + \frac{1}{x_{jl}^2 x_{lq}^2} \Big\} \\
&\quad + a^3 \sum_{l=1}^{N}{}' \Big\{ \frac{1}{x_{jl}^2 x_{jq}^2} - \frac{1}{x_{lq}^2 x_{jq}^2} \Big\} - \frac{1}{2} a^3 \sum_{l,m=1}^{N}{}' \frac{1}{x_{jl} x_{lq} x_{jm} x_{mq}} \Big\} \\
&\quad + 4(1 - \delta_{jk}) \Big\{ - \frac{a}{x_{jk}^2} (p_j^2 + p_j p_k + p_k^2) - \mathrm{i}a \frac{1}{x_{jk}^3} (p_j - p_k) \\
&\quad - \mathrm{i}a^2 \sum_{l=1}^{N}{}' \frac{1}{x_{jl} x_{lk} x_{jk}} (p_j + p_l + p_k) - (a^3 - 3a) \frac{1}{x_{jk}^4} - a^2 \sum_{l=1}^{N}{}' \Big\{ \frac{1}{x_{jl} x_{lk} x_{jk}^2} - \frac{1}{x_{jl}^2 x_{lk}^2} \Big\} \\
&\quad - a^3 \sum_{l=1}^{N}{}' \Big\{ \frac{1}{x_{jl}^2 x_{jk}^2} + \frac{1}{x_{lk}^2 x_{jk}^2} \Big\} + \frac{1}{2} a^3 \sum_{l,m=1}^{N}{}' \frac{1}{x_{jl} x_{lk} x_{jm} x_{mk}} \Big\} \\
&\quad + 4\omega^2 \Big\{ a\delta_{jk} \sum_{l=1}^{N}{}' \frac{x_j^2 - x_j x_l + x_l^2}{x_{jl}^2} - a(1 - \delta_{jk}) \frac{x_j^2 - x_j x_k + x_k^2}{x_{jk}^2} \Big\} \\
&\quad + 2\omega(a - 1 - Na)(M_1)_{jk}. \qquad (2.16d)
\end{aligned}
$$

Note that in the explicit forms of the generalized Lax equation and the M-matrix for the second conserved operator, eqs. (2.16c) and (2.16d), the generalized Lax equation and the M-matrix for the first conserved operator appear. This is because the second conserved operator I_2 contains the first conserved operator I_1, as is seen in eq. (2.13b). Such mixing of the lower order conserved operators generally occurs. As we see below, the L^+L^--matrix contains constant matrix terms,

$$
\begin{aligned}
L^+L^- &= (L + \omega Q)(L - \omega Q) \\
&= (L^2 - \omega^2 Q^2) + \omega [Q, L] \\
&= \frac{1}{2}(L^+L^- + L^-L^+) + \omega((a-1)\mathbf{1} - aT), \qquad (2.17)
\end{aligned}
$$

where T is a matrix whose elements are all equal to 1:

$$
T_{jk} = 1, \quad T^2 = NT. \qquad (2.18)
$$

We note that a sum of all the elements of the matrix product $A_1 T A_2$ for any operator-valued matrices A_1 and A_2 is the product of the sums of all the elements of the two matrices:

$$
\mathrm{T}_\Sigma A_1 T A_2 = (\mathrm{T}_\Sigma A_1)(\mathrm{T}_\Sigma A_2). \qquad (2.19)
$$

Thus a sum of all the elements of a power of L^+L^--matrix become a polynomial of the sums of all the elements of lower-order powers of the matrix $\frac{1}{2}(L^+L^- + L^-L^+)$:

$$
\begin{aligned}
&\mathrm{T}_\Sigma (L^+L^-)^n \\
&= \mathrm{T}_\Sigma \Big(\frac{1}{2}(L^+L^- + L^-L^+) + \omega((a-1)\mathbf{1} - aT) \Big)^n \\
&= \mathrm{T}_\Sigma \Big(\frac{1}{2}(L^+L^- + L^-L^+) \Big)^n \\
&\quad + \sum_{r=1}^{n} \mathrm{T}_\Sigma \Big(\frac{1}{2}(L^+L^- + L^-L^+) \Big)^{r-1} \omega((a-1)\mathbf{1} - aT) \Big(\frac{1}{2}(L^+L^- + L^-L^+) \Big)^{n-r} + \cdots \\
&= \mathrm{T}_\Sigma \Big(\frac{1}{2}(L^+L^- + L^-L^+) \Big)^n + n\omega(a-1) \mathrm{T}_\Sigma \Big(\frac{1}{2}(L^+L^- + L^-L^+) \Big)^{n-1} \\
&\quad - \omega a \sum_{r=1}^{n} \mathrm{T}_\Sigma \Big(\frac{1}{2}(L^+L^- + L^-L^+) \Big)^{r-1} \mathrm{T}_\Sigma \Big(\frac{1}{2}(L^+L^- + L^-L^+) \Big)^{n-r} + \cdots .
\end{aligned}
$$

The constant matrix terms generate a polynomial of lower order conserved operators in this way. That is why the conserved operators I_n (2.12) contain lower order conserved operators. Instead of the L^+L^--matrix, we adopt the matrix $\frac{1}{2}\{L^+L^- + L^-L^+\} = L^2 - \omega^2 Q^2$ and remove the constant matrix terms. Then we obtain a modified formula for conserved operators:

$$\tilde{I}_n = \mathrm{T}_\Sigma\Big(\frac{1}{2}(L^+L^- + L^-L^+)\Big)^n. \tag{2.20}$$

The first two modified conserved operators are given by

$$\begin{aligned}
\tilde{I}_1 &= \sum_{j=1}^N p_j^2 + g\sum_{j,k=1}^{N}{}' \frac{1}{x_{jk}^2} + \omega^2\sum_{j=1}^N x_j^2 = 2H_{\mathrm{C}}, &(2.21\mathrm{a})\\
\tilde{I}_2 &= \sum_{j=1}^N p_j^4 + g\sum_{j,k=1}^{N}{}' \Big\{p_j^2\frac{1}{x_{jk}^2} + p_j\frac{1}{x_{jk}}p_k\frac{1}{x_{jk}} + p_j\frac{1}{x_{jk}^2}p_j + \frac{1}{x_{jk}}p_k^2\frac{1}{x_{jk}} + \frac{1}{x_{jk}}p_k\frac{1}{x_{jk}}p_j + \frac{1}{x_{jk}^2}p_j^2\Big\}\\
&\quad +2g^2\sum_{j,k,l=1}^{N}{}' \frac{1}{x_{jk}^2}\frac{1}{x_{kl}^2} + (g^2-g)\sum_{j,k=1}^{N}{}' \frac{1}{x_{jk}^4}\\
&\quad +\omega^2\Big\{\sum_{j=1}^N (x_j^2p_j^2 + p_j^2x_j^2) + g\sum_{j,k=1}^{N}{}' \frac{x_j^2+x_k^2}{x_{jk}^2} - a^2\sum_{j,k,m=1}^{N}{}' \frac{x_j^2+x_k^2}{x_{jm}x_{mk}}\Big\} + \omega^4\sum_{j=1}^N x_j^4. &(2.21\mathrm{b})
\end{aligned}$$

Namely, $\tilde{I}_n$ corresponds to the highest order conserved operator in I_n. Note that the coefficient of the fourth term of the second conserved operator, g^2-g, contains the quantum correction. With the Planck constant $\hbar$ explicity written, the coefficient is expressed as $g^2 - \hbar g$. Moreover, the coefficient of the third term in the ω^2-order terms is not expressed as a polynomial of the coupling constant g, but as a^2. These facts show the quantum conserved operators of the Calogero model are not given by the translations of the classical conserved quantities by the correspondence principle [20, 59]. We expect that the generalized Lax equations can be obtained for the higher order conserved operators. From eq. (2.16), the generalized Lax equation for $\tilde{I}_n$ is to be expressed as

$$\begin{aligned}
&\big[\tilde{I}_n, L^\pm\big] = \big[L^\pm, \tilde{M}_n\big] \pm 2n\omega(L^\pm L^\mp)^{n-1}L^\pm, &(2.22\mathrm{a})\\
&\sum_{j=1}^N (\tilde{M}_n)_{jk} = 0, \sum_{j=1}^N (\tilde{M}_n)_{kj} = 0, &(2.22\mathrm{b})
\end{aligned}$$

where explicit forms of the first two $\tilde{M}_n$ matrices are

$$\begin{aligned}
(\tilde{M}_1)_{jk} &= 2M_{jk}, &(2.22\mathrm{c})\\
(\tilde{M}_2)_{jk} &= 4\delta_{jk}\sum_{\substack{q=1\\ q\neq j}}^N \Big\{\frac{a}{x_{jq}^2}(p_j^2 + p_jp_q + p_q^2) + \mathrm{i}a\frac{1}{x_{jq}^3}(p_j - p_q)\\
&\quad +\mathrm{i}a^2\sum_{l=1}^{N}{}' \frac{1}{x_{jl}x_{lq}x_{jq}}(p_j + p_l + p_q) + (a^3-3a)\frac{1}{x_{jq}^4} + a^2\sum_{l=1}^{N}{}'\Big\{\frac{1}{x_{jl}x_{lq}x_{jq}^2} + \frac{1}{x_{jl}^2x_{lq}^2}\Big\}\\
&\quad +a^3\sum_{l=1}^{N}{}'\Big\{\frac{1}{x_{jl}^2x_{jq}^2} - \frac{1}{x_{lq}^2x_{jq}^2}\Big\} - \frac{1}{2}a^3\sum_{l,m=1}^{N}{}' \frac{1}{x_{jl}x_{lq}x_{jm}x_{mq}}\Big\}
\end{aligned}$$

$$\begin{aligned}
&\quad +4(1-\delta_{jk})\Big\{-\frac{a}{x_{jk}^2}(p_j^2 + p_jp_k + p_k^2) - \mathrm{i}a\frac{1}{x_{jk}^3}(p_j - p_k)\\
&\quad -\mathrm{i}a^2\sum_{l=1}^{N}{}' \frac{1}{x_{jl}x_{lk}x_{jk}}(p_j + p_l + p_k) - (a^3-3a)\frac{1}{x_{jk}^4} - a^2\sum_{l=1}^{N}{}'\Big\{\frac{1}{x_{jl}x_{lk}x_{jk}^2} - \frac{1}{x_{jl}^2x_{lk}^2}\Big\}\\
&\quad -a^3\sum_{l=1}^{N}{}'\Big\{\frac{1}{x_{jl}^2x_{jk}^2} + \frac{1}{x_{lk}^2x_{jk}^2}\Big\} + \frac{1}{2}a^3\sum_{l,m=1}^{N}{}' \frac{1}{x_{jl}x_{lk}x_{jm}x_{mk}}\Big\}\\
&\quad +4\omega^2\Big\{a\delta_{jk}\sum_{l=1}^{N}{}' \frac{x_j^2 - x_jx_l + x_l^2}{x_{jl}^2} - a(1-\delta_{jk})\frac{x_j^2 - x_jx_k + x_k^2}{x_{jk}^2}\Big\}. &(2.22\mathrm{d})
\end{aligned}$$

This formula guarantees that the conserved operators $\tilde{I}_n$ (2.20) and hence I_n mutually commute. However, recursive construction of the generalized Lax equations for these conserved operators has not been successful because of the difficulty of the order problem of the L^+- and L^--matrices. Equation (2.22) for the first two cases $n = 1, 2$ are confirmed by the explicit forms of the $\tilde{M}_n$-matrices, eqs. (2.22c) and (2.22d). This proves the mutual commutativity of the conserved operators, $\big[\tilde{I}_n, \tilde{I}_m\big] = 0$ and equivalently $\big[I_n, I_m\big] = 0$ for $n = 1, 2$ and $m = 1, 2, \cdots$. But now, the cases $n \geq 3$ remain to be a conjecture. The mutual commutativity of the conserved operators for genenral cases will be verified by use of the Dunkl operator formulation in Section 2.4.

For recursive construction of the generalized Lax equations, it is convenient to introduce Weyl ordered product:

$$\begin{aligned}
\big[(L^+)^p(L^-)^m\big]_{\mathrm{W}} &= \frac{p!m!}{(p+m)!}\sum_{\text{all possible order}} (L^+)^p(L^-)^m\\
&= \frac{p!m!}{(p+m)!}\big\{(L^+)^p(L^-)^m + (L^+)^{p-1}L^-L^+(L^-)^{m-1}\\
&\quad +(L^+)^{p-1}(L^-)^2L^+(L^-)^{m-2} + \cdots + (L^-)^m(L^+)^p\big\}. &(2.23)
\end{aligned}$$

In particular, the matrix,

$$\big[(L^+)^n(L^-)^n\big]_{\mathrm{W}}, \tag{2.24}$$

satisfies a relation that is similar to eq. (2.10):

$$\Big[H, \big[(L^+)^n(L^-)^n\big]_{\mathrm{W}}\Big] = \Big[\big[(L^+)^n(L^-)^n\big]_{\mathrm{W}}, M\Big], \tag{2.25}$$

Thus we obtain a formula for a set of conserved operators:

$$O_n^n = \mathrm{T}_\Sigma\big[(L^+)^n(L^-)^n\big]_{\mathrm{W}}. \tag{2.26}$$

These conserved operators do not generally commute among themselves. However, the choice (2.26) is convenient for the investigation of the algebraic structure of the model.

Explicit forms of the first two conserved operators are

$$\begin{aligned}
O_1^1 &= \mathrm{T}_\Sigma\frac{1}{2}\big(L^+L^- + L^-L^+\big)\\
&= \sum_{j=1}^N p_j^2 + g\sum_{j,k=1}^{N}{}' \frac{1}{x_{jk}^2} + \omega^2\sum_{j=1}^N x_j^2
\end{aligned}$$

$$
\begin{aligned}
&= 2H_{\mathrm{C}}, && (2.27\mathrm{a})\\
O_2^2 &= \mathrm{T}_\Sigma \frac{1}{6}\Big\{(L^+)^2(L^-)^2 + L^+L^-L^+L^- + L^+(L^-)^2L^+ \\
&\qquad + L^-(L^+)^2L^- + L^-L^+L^-L^+ + (L^-)^2(L^+)^2\Big\} \\
&= \sum_{j=1}^N p_j^4 + g\sum_{j,k=1}^{N}{}' \Big\{p_j^2\frac{1}{x_{jk}^2} + p_j\frac{1}{x_{jk}}p_k\frac{1}{x_{jk}} + p_j\frac{1}{x_{jk}^2}p_j + \frac{1}{x_{jk}}p_k^2\frac{1}{x_{jk}} + \frac{1}{x_{jk}}p_k\frac{1}{x_{jk}}p_j + \frac{1}{x_{jk}^2}p_j^2\Big\} \\
&\quad +2g^2\sum_{j,k,l=1}^{N}{}' \frac{1}{x_{jk}^2}\frac{1}{x_{kl}^2} + (g^2-g)\sum_{j,k=1}^{N}{}'\frac{1}{x_{jk}^4} \\
&\quad +\omega^2\Big\{\sum_{j=1}^N\big(x_j^2p_j^2 + p_j^2x_j^2\big) + g\sum_{j,k=1}^{N}{}'\frac{x_j^2+x_k^2}{x_{jk}^2} - a^2\sum_{j,k,m=1}^{N}{}'\frac{x_j^2+x_k^2}{x_{jm}x_{mk}}\Big\} + \omega^4\sum_{j=1}^N x_j^4 \\
&\quad +\frac{1}{3}\omega^2\Big\{N^3a^2 - 4N^2a(a-1) + N\big(3(a-1)^2+2a\big)\Big\}. && (2.27\mathrm{b})
\end{aligned}
$$

Using the explicit forms of the conserved operators (2.27), we obtain the generalized Lax equations for O_1^1 and O_2^2:

$$
\begin{aligned}
\big[O_1^1, L^\pm\big] &= \big[L^\pm, M_1^1\big] \pm 2\omega L^\pm, && (2.28\mathrm{a})\\
M_1^1 &= 2M, && (2.28\mathrm{b})\\
\big[O_2^2, L^\pm\big] &= \big[L^\pm, M_2^2\big] \pm 4\omega\big[(L^\pm)^2L^\mp\big]_{\mathrm{W}}, && (2.28\mathrm{c})\\
(M_2^2)_{jk} &= 4\delta_{jk}\sum_{\substack{q=1\\ q\neq j}}^N\Big\{\frac{a}{x_{jq}^2}(p_j^2+p_jp_q+p_q^2) + \mathrm{i}a\frac{1}{x_{jq}^3}(p_j-p_q) \\
&\quad +\mathrm{i}a^2\sum_{l=1}^{N}{}'\frac{1}{x_{jl}x_{lq}x_{jq}}(p_j+p_l+p_q) + (a^3-3a)\frac{1}{x_{jq}^4} + a^2\sum_{l=1}^{N}{}'\Big\{\frac{1}{x_{jl}x_{lq}x_{jq}^2} + \frac{1}{x_{jl}^2x_{lq}^2}\Big\} \\
&\quad +a^3\sum_{l=1}^{N}{}'\Big\{\frac{1}{x_{jl}^2x_{jq}^2} - \frac{1}{x_{lq}^2x_{jq}^2}\Big\} - \frac{1}{2}a^3\sum_{l,m=1}^{N}{}'\frac{1}{x_{jl}x_{lq}x_{jm}x_{mq}}\Big\} \\
&\quad +4(1-\delta_{jk})\Big\{-\frac{a}{x_{jk}^2}(p_j^2+p_jp_k+p_k^2) - \mathrm{i}a\frac{1}{x_{jk}^3}(p_j-p_k) \\
&\quad -\mathrm{i}a^2\sum_{l=1}^{N}{}'\frac{1}{x_{jl}x_{lk}x_{jk}}(p_j+p_l+p_k) - (a^3-3a)\frac{1}{x_{jk}^4} - a^2\sum_{l=1}^{N}{}'\Big\{\frac{1}{x_{jl}x_{lk}x_{jk}^2} - \frac{1}{x_{jl}^2x_{lk}^2}\Big\} \\
&\quad -a^3\sum_{l=1}^{N}{}'\Big\{\frac{1}{x_{jl}^2x_{jk}^2} + \frac{1}{x_{lk}^2x_{jk}^2}\Big\} + \frac{1}{2}a^3\sum_{l,m=1}^{N}{}'\frac{1}{x_{jl}x_{lk}x_{jm}x_{mk}}\Big\} \\
&\quad +\frac{4}{3}\omega^2\Big\{a\delta_{jk}\sum_{\substack{l=1\\ l\neq j}}^N\frac{x_j^2+x_jx_l+x_l^2}{x_{jl}^2} - a(1-\delta_{jk})\frac{x_j^2+x_jx_k+x_k^2}{x_{jk}^2}\Big\}. && (2.28\mathrm{d})
\end{aligned}
$$

As we have expected, the above M-matrices (2.28b) and (2.28d) satisfy the sum-to-zero condition. The first relation (2.28a) is nothing but the Lax equation for the Hamiltonian. We expect that similar relations also hold for all the operators,

$$
O_m^p \overset{\mathrm{def}}{=} \mathrm{T}_\Sigma\big[(L^+)^p(L^-)^m\big]_{\mathrm{W}}, \tag{2.29}
$$

where p and m are non-negative integers. In the remaining part of this section, we shall recursively construct the generalized Lax equations for this family of operators.

First, we introduce two series of operators that will play an important role in an algebraic treatment of the eigenfunctions of the model. Let us consider operators defined by

$$
\begin{aligned}
O_0^n \overset{\mathrm{def}}{=} B_n^\dagger &= \mathrm{T}_\Sigma(L^+)^n, && (2.30\mathrm{a})\\
O_n^0 \overset{\mathrm{def}}{=} B_n &= \mathrm{T}_\Sigma(L^-)^n, && (2.30\mathrm{b})
\end{aligned}
$$

where n is a nonnegative integer. By use of the Lax equation (2.5), commutation relations between the Hamiltonian (3.2a) and these operators are calculated as

$$
\begin{aligned}
\big[H_{\mathrm{C}}, B_n^\dagger\big] &= \big[H_{\mathrm{C}}, \mathrm{T}_\Sigma(L^+)^n\big] \\
&= \mathrm{T}_\Sigma\Big\{\big[(L^+)^n, M\big] + n\omega(L^+)^n\Big\} \\
&= n\omega B_n^\dagger, && (2.31\mathrm{a})\\
\big[H_{\mathrm{C}}, B_n\big] &= \big[H_{\mathrm{C}}, \mathrm{T}_\Sigma(L^-)^n\big] \\
&= \mathrm{T}_\Sigma\Big\{\big[(L^-)^n, M\big] - n\omega(L^-)^n\Big\} \\
&= -n\omega B_n. && (2.31\mathrm{b})
\end{aligned}
$$

Similar to the creation and annihilation operators in the theory of quantum harmonic oscillator, these operators change energy eigenvalue by $n\omega$. We shall call the operators $B_n^\dagger$ and B_n power-sum creation and annihilation operators, respectively. The reason why we add power-sum to the name will be clear in Section 2.4 and Chapter 4.

The commutators between $B_n^\dagger$, $n = 1, 2, 3$, and $L^\pm$ are

$$
\begin{aligned}
\big[B_1^\dagger, L^\pm\big] &= \big[L^\pm, M_0^1\big] - \omega(1\mp 1)\mathbf{1}, && (2.32\mathrm{a})\\
(M_0^1)_{jk} &= 0, && (2.32\mathrm{b})\\
\big[B_2^\dagger, L^\pm\big] &= \big[L^\pm, M_0^2\big] - 2\omega(1\mp 1)L^+, && (2.32\mathrm{c})\\
(M_0^2)_{jk} &= -2a(1-\delta_{jk})\frac{1}{x_{jk}^2} + 2a\delta_{jk}\sum_{\substack{l=1\\ l\neq j}}^N\frac{1}{x_{jl}^2}, && (2.32\mathrm{d})\\
\big[B_3^\dagger, L^\pm\big] &= [L^\pm, M_0^3] - 3\omega(1\mp 1)(L^+)^2, && (2.32\mathrm{e})\\
(M_0^3)_{jk} &= -(1-\delta_{jk})\Big\{3a\frac{1}{x_{jk}^2}(p_j+p_k) + 3\mathrm{i}a^2\sum_{l=1}^{N}{}'\frac{1}{x_{jl}x_{lk}x_{jk}} + 3\mathrm{i}\omega a\frac{x_j+x_k}{x_{jk}^2}\Big\} \\
&\qquad +\delta_{jk}\sum_{\substack{m=1\\ m\neq j}}^N\Big\{3a\frac{1}{x_{jm}^2}(p_j+p_m) + 3\mathrm{i}\omega a\frac{x_j+x_m}{x_{jm}^2}\Big\}. && (2.32\mathrm{f})
\end{aligned}
$$

We remark here that three M-matrices shown above meet the sum-to-zero condition. In the same way, we can compute the commutators between B_n, $n = 1, 2, 3$, and $L^\pm$:

$$
\big[B_1, L^\pm\big] = \big[L^\pm, M_1^0\big] + \omega(1\pm 1)\mathbf{1}, \tag{2.33a}
$$

$$
\begin{aligned}
(M_1^0)_{jk} &= 0, && (2.33\text{b}) \\
\left[B_2, L^\pm\right] &= \left[L^\pm, M_2^0\right] + 2\omega(1 \pm 1)L^-, && (2.33\text{c}) \\
(M_2^0)_{jk} &= -2a(1-\delta_{jk})\frac{1}{x_{jk}^2} + 2a\delta_{jk}\sum_{\substack{l=1 \\ l\neq j}}^{N}\frac{1}{x_{jl}^2}, && (2.33\text{d}) \\
\left[B_3, L^\pm\right] &= \left[L^\pm, M_3^0\right] + 3\omega(1 \pm 1)(L^-)^2, && (2.33\text{e}) \\
(M_3^0)_{jk} &= -(1-\delta_{jk})\Big\{3a\frac{1}{x_{jk}^2}(p_j + p_k) + 3\mathrm{i}a^2{\sum_{l=1}^{N}}'\frac{1}{x_{jl}x_{lk}x_{jk}} - 3\mathrm{i}\omega a\frac{x_j + x_k}{x_{jk}^2}\Big\} \\
&\quad +\delta_{jk}\sum_{\substack{m=1 \\ m\neq j}}^{N}\Big\{3a\frac{1}{x_{jm}^2}(p_j + p_m) - 3\mathrm{i}\omega a\frac{x_j + x_m}{x_{jm}^2}\Big\}. && (2.33\text{f})
\end{aligned}
$$

These relations show that commutators $[B_n^\dagger, B_m^\dagger]$ and $[B_n, B_m]$, $n = 1, 2, 3$, $m = 1, 2, \cdots$ vanish. The explicit forms of generalized Lax equations for the power-sum creation and annihilation operators, eqs. (2.32) and (2.33), lead us to an expectation that the commutation relations $[B_n^\dagger, L^\pm]$ and $[B_n, L^\pm]$, $n = 1, 2, 3, \cdots$, take the following forms,

$$
\begin{aligned}
\left[B_n^\dagger, L^\pm\right] &= \left[L^\pm, M_0^n\right] - n\omega(1 \mp 1)(L^+)^{n-1}, && (2.34\text{a}) \\
\left[B_n, L^\pm\right] &= \left[L^\pm, M_0^n\right] + n\omega(1 \pm 1)(L^-)^{n-1}, && (2.34\text{b})
\end{aligned}
$$

with the M-matrices satisfying the sum-to-zero condition:

$$
\begin{aligned}
\sum_{j=1}^{N}(M_0^n)_{jk} &= \sum_{j=1}^{N}(M_0^n)_{kj} = 0, && (2.35\text{a}) \\
\sum_{j=1}^{N}(M_n^0)_{jk} &= \sum_{j=1}^{N}(M_n^0)_{kj} = 0. && (2.35\text{b})
\end{aligned}
$$

As a first step toward the recursive construction of the generalized Lax equations for all the operators O_m^p, we shall prove the relations (2.34) and (2.35). The proof is done by induction.

To do an inductive proof, we need a recursion formula for the power-sum creation operators:

$$
\left[B_1^\dagger, \left[O_2^2, B_n^\dagger\right]\right] = -8n\omega^2 B_{n+1}^\dagger. \tag{2.36}
$$

The proof of eq. (2.36) is carried out as follows. Commutator between O_2^2 and $B_n^\dagger$ is

$$
\begin{aligned}
\left[O_2^2, B_n^\dagger\right] &= \left[O_2^2, \mathrm{T}_\Sigma(L^+)^n\right] \\
&= \mathrm{T}_\Sigma\Big\{\left[(L^+)^n, M_2^2\right] + \frac{4}{3}\omega\sum_{l=1}^{n}(L^+)^{l-1}\big\{(L^+)^2L^- + L^+L^-L^+ + L^-(L^+)^2\big\}(L^+)^{n-l}\Big\} \\
&= \frac{4}{3}\omega\,\mathrm{T}_\Sigma\sum_{l=1}^{n}(L^+)^{l-1}\big\{(L^+)^2L^- + L^+L^-L^+ + L^-(L^+)^2\big\}(L^+)^{n-l}. && (2.37)
\end{aligned}
$$

We should remark here that we have used the sum-to-zero condition of the M_2^2-matrix. Substitution of eq. (2.37) into the l.h.s. of eq. (2.36) yields

$$
\begin{aligned}
&\left[B_1^\dagger, \left[O_2^2, B_n^\dagger\right]\right] \\
&= \Big[B_1^\dagger, \frac{4}{3}\omega\,\mathrm{T}_\Sigma\sum_{l=1}^{n}(L^+)^{l-1}\big\{(L^+)^2L^- + L^+L^-L^+ + L^-(L^+)^2\big\}(L^+)^{n-l}\Big] \\
&= \mathrm{T}_\Sigma\Big\{\Big[\frac{4}{3}\omega\sum_{l=1}^{n}\big\{(L^+)^{l-1}\big\{(L^+)^2L^- + L^+L^-L^+ + L^-(L^+)^2\big\}(L^+)^{n-l}, M_0^1\Big] - 8n\omega^2(L^+)^{n+1}\Big\} \\
&= -8n\omega^2 B_{n+1}^\dagger. && (2.38)
\end{aligned}
$$

Here we have used again the sum-to-zero condition on the M_0^1-matrix. This is nothing but the recursion formula (2.36).

Preparations for the proofs of eqs. (2.34a) and (2.35a) are finished. We have already obtained explicit expressions of the first three, eq. (2.32). By the inductive assumption, we assume that the relations (2.34a) hold up to $n = p$ with M_0^p-matrices satisfying the sum-to-zero condition. We want to calculate a commutator $[B_{p+1}^\dagger, L^\pm]$. Using eq. (2.36), we have

$$
-8\omega^2 p\left[B_{p+1}^\dagger, L^\pm\right] = \left[\left[B_1^\dagger, \left[O_2^2, B_p^\dagger\right]\right], L^\pm\right]. \tag{2.39}
$$

By use of Jacobi's identity for the commutator,

$$
[A, [B, C]] + [B, [C, A]] + [C, [A, B]] = 0, \tag{2.40}
$$

the r.h.s. of eq. (2.39) is rewritten as

$$
\left[\left[B_1^\dagger, \left[O_2^2, B_p^\dagger\right]\right], L^\pm\right] = -\left[\left[\left[O_2^2, B_p^\dagger\right], L^\pm\right], B_1^\dagger\right] - \left[\left[L^\pm, B_1^\dagger\right], \left[O_2^2, B_p^\dagger\right]\right]. \tag{2.41}
$$

In order to proceed the calculation, we need a formula:

$$
\begin{aligned}
&\left[\left[O_2^2, B_p^\dagger\right], L^\pm\right] \\
&= \left[L^\pm, \left[M_2^2, M_0^p\right] + \left[M_2^2, B_p^\dagger\right] + \left[O_2^2, M_0^p\right]\right] \\
&\quad -\frac{4}{3}\omega^2 p\Big\{(1 \mp 1)\sum_{l=1}^{p-1}(L^+)^{l-1}\big\{(L^+)^2L^- + L^+L^-L^+ + L^-(L^+)^2\big\}(L^+)^{p-l-1} \\
&\qquad + (1 \mp 1)\big\{2(L^+)^pL^- + 2L^-(L^+)^p + L^+L^-(L^+)^{p-1} + (L^+)^{p-1}L^-L^+\big\} \\
&\qquad - 3(1 \pm 1)(L^+)^{p+1}\Big\}. && (2.42)
\end{aligned}
$$

This formula is proved by repeated use of Jacobi's identity (2.40) and the Lax equations, (2.28c) and (2.34a) for $n = p$. Applying Jacobi's identity to the l.h.s. of eq. (2.42), we have

$$
\left[\left[O_2^2, B_p^\dagger\right], L^\pm\right] = -\left[\left[B_p^\dagger, L^\pm\right], O_2^2\right] - \left[\left[L^\pm, O_2^2\right], B_p^\dagger\right]. \tag{2.43}
$$

Making use of the Lax equations, (2.28c) and (2.34a) for $n = p$, and applying Jacobi's identity again, we have

$$\begin{aligned}
&-\big[\big[B_p^\dagger, L^\pm\big], O_2^2\big] - \big[\big[L^\pm, O_2^2\big], B_p^\dagger\big] \\
&= -\big[\big[L^\pm, M_0^p\big] - p\omega(1\mp 1)(L^+)^{p-1}, O_2^2\big] \\
&\quad +\big[\big[L^\pm, M_2^2\big] \pm \frac{4}{3}\omega\{(L^\pm)^2L^\mp + L^\pm L^\mp L^\pm + L^\mp(L^\pm)^2\}, B_p^\dagger\big] \\
&= \big[L^\pm, \big[M_2^2, B_p^\dagger\big] + \big[O_2^2, M_0^p\big]\big] + \big[\big[O_2^2, L^\pm\big], M_0^p\big] - \big[\big[B_p^\dagger, L^\pm\big], M_2^2\big] \\
&\quad -\big[O_2^2, p\omega(1\mp 1)(L^+)^{p-1}\big] \mp \frac{4}{3}\omega\big[B_p^\dagger, (L^\pm)^2L^\mp + L^\pm L^\mp L^\pm + L^\mp(L^\pm)^2\big] \\
&= \big[L^\pm, \big[M_2^2, B_p^\dagger\big] + \big[O_2^2, M_0^p\big]\big] + \big[\big[L^\pm, M_2^2\big], M_0^p\big] + \big[\big[M_0^p, L^\pm\big], M_2^2\big] \\
&\quad -\big[O_2^2, p\omega(1\mp 1)(L^+)^{p-1}\big] + \big[p\omega(1\mp 1)(L^+)^{p-1}, M_2^2\big] \\
&\quad \mp\frac{4}{3}\omega\big[B_p^\dagger, (L^\pm)^2L^\mp + L^\pm L^\mp L^\pm + L^\mp(L^\pm)^2\big] \\
&\quad \pm\frac{4}{3}\omega\big[(L^\pm)^2L^\mp + L^\pm L^\mp L^\pm + L^\mp(L^\pm)^2, M_0^p\big] \\
&= \big[L^\pm, \big[M_2^2, M_0^p\big] + \big[M_2^2, B_p^\dagger\big] + \big[O_2^2, M_0^p\big]\big] \\
&\quad -\frac{4}{3}\omega^2p\Big\{(1\mp 1)\sum_{l=1}^{p-1}(L^+)^{l-1}\{(L^+)^2L^- + L^+L^-L^+ + L^-(L^+)^2\}(L^+)^{p-l-1} \\
&\qquad +(1\mp 1)\big\{2(L^+)^pL^- + 2L^-(L^+)^p + L^+L^-(L^+)^{p-1} + (L^+)^{p-1}L^-L^+\big\} \\
&\qquad -3(1\pm 1)(L^+)^{p+1}\Big\}.
\end{aligned} \tag{2.44}$$

Consequently, we obtain the formula (2.42). For brevity of the expressions, we adopt the following symbols:

$$\begin{aligned}
K &= \big[O_2^2, B_p^\dagger\big], \\
G &= \big[M_2^2, M_0^p\big] + \big[M_2^2, B_p^\dagger\big] + \big[O_2^2, M_0^p\big].
\end{aligned}$$

Equation (2.42) is nothing but the generalized Lax equation for the above K operator. Note that the operator-valued matrix G also satisfies the sum-to-zero condition because the matrices M_2^2 and M_0^p do.

With eqs. (2.32a), (2.32b) and (2.42), the r.h.s. of eq. (2.41) is calculated as follows:

$$\begin{aligned}
&-\big[\big[K, L^\pm\big], B_1^\dagger\big] - \big[\big[L^\pm, B_1^\dagger\big], K\big] \\
&= -\big[\big[L^\pm, G\big], B_1^\dagger\big] + \big[\big[L^\pm, M_0^1\big], K\big] - \omega(1\mp 1)\big[\mathbf{1}, K\big] \\
&\quad +\frac{4}{3}\Big[\omega^2p\Big\{(1\mp 1)\sum_{l=1}^{p-1}(L^+)^{l-1}\big\{(L^+)^2L^- + L^+L^-L^+ + L^-(L^+)^2\big\}(L^+)^{p-l-1} \\
&\qquad +(1\mp 1)\big\{2(L^+)^pL^- + 2L^-(L^+)^p + L^+L^-(L^+)^{p-1} + (L^+)^{p-1}L^-L^+\big\} \\
&\qquad -3(1\pm 1)(L^+)^{p+1}\Big\}, B_1^\dagger\Big].
\end{aligned} \tag{2.45}$$

After a calculation which is similar to that in eq. (2.44), we obtain

$$-\big[\big[K, L^\pm\big], B_1^\dagger\big] - \big[\big[L^\pm, B_1^\dagger\big], K\big] = \big[L^\pm, \big[B_1^\dagger, G\big]\big] + 8p\omega^3(p+1)(1\mp 1)(L^+)^p \tag{2.46}$$

Tools for the calculation are Jacobi's identity and eqs. (2.32a) and (2.42). We define the M_0^{p+1}-matrix as

$$M_0^{p+1} = -\frac{1}{8p\omega^2}\big[B_1^\dagger, G\big]. \tag{2.47}$$

We remark that the M_0^{p+1}-matrix satisfies the sum-to-zero condition, for the matrix G satisfies it. Then from eqs. (2.39), (2.46) and (2.47), we finally obtain the expected expression,

$$\big[B_{p+1}^\dagger, L^\pm\big] = \big[L^\pm, M_0^{p+1}\big] - (p+1)\omega(1\mp 1)(L^+)^p, \tag{2.48}$$

which is exactly the same as the formula (2.34a) for $n = p+1$. As we have confirmed before, the formulae, (2.34a) and (2.35a), hold for $n = 1, 2, 3$. Thus we complete the proof of the formula for all positive integer n.

In a similar fashion, we can deduce the relations for the annihilation operators, eqs. (2.34b) and (2.35b). Instead, we can easily check them from the generalized Lax equations for the creation operators (2.34a) and (2.35a), which has been formulated just now. Computing the Hermitian conjugate of the relation (2.34a), we have

$$\big[L^\mp, B_n\big] = \big[(M_0^n)^\dagger, L^\mp\big] - n\omega(1\mp 1)(L^-)^{n-1}. \tag{2.49}$$

Defining the M_n^0-matrix as

$$M_n^0 = (M_0^n)^\dagger, \tag{2.50}$$

we get the generalized Lax equation for the annihilation operator B_n (2.34b):

$$\begin{aligned}
\big[B_n, L^\pm\big] &= \big[L^\pm, M_n^0\big] + n\omega(1\pm 1)(L^-)^{n-1}, \\
\sum_{j=1}^{N}(M_n^0)_{jk} &= \sum_{j=1}^{N}(M_n^0)_{kj} = 0.
\end{aligned}$$

We remark here again that eqs. (2.34a) - (2.35b) assure the commutativity among the creation operators and among the annihilation operators:

$$\big[B_n^\dagger, B_m^\dagger\big] = 0, \quad \big[B_n, B_m\big] = 0. \tag{2.51}$$

The commuting power-sum creation operators will play an important role in the algebraic construction of the eigenfunctions of the Calogero Hamiltonian (2.1) in Chapter 3.

The next task is a recursive construction of the generalized Lax equations for the operators O_m^p defined by eq. (2.29). Watching closely on the relations (2.28a), (2.28c) and (2.34), we notice that the following relations might hold for O_m^p:

$$\begin{aligned}
\big[O_m^p, L^\pm\big] &= \big[L^\pm, M_m^p\big] + m\omega(1\pm 1)\big[(L^+)^p(L^-)^{m-1}\big]_{\mathrm{W}} \\
&\quad -p\omega(1\mp 1)\big[(L^+)^{p-1}(L^-)^m\big]_{\mathrm{W}},
\end{aligned} \tag{2.52a}$$

$$\sum_{j=1}^{N}(M_m^p)_{jk} = \sum_{j=1}^{N}(M_m^p)_{kj} = 0. \tag{2.52b}$$

Note that the relations hold for $m = 0$ (2.34a), $p = 0$ (2.34b) and $p = m = n$ for $n = 1, 2$ (2.28a) and (2.28c). We shall prove eq. (2.52) by induction.

First, we prove a recursion formula for O_m^{n-m}:

$$\left[B_2, O_m^{n-m}\right] = 4\omega(n-m)O_{m+1}^{n-m-1}. \tag{2.53}$$

This formula can be verified as follows. Substitution of the definition of O_m^{n-m} yields

$$\left[B_2, O_m^{n-m}\right] = \left[B_2, \mathrm{T}_\Sigma\left[(L^+)^{n-m}(L^-)^m\right]_{\mathrm{W}}\right]. \tag{2.54}$$

Using eq. (2.33c) and the sum-to-zero condition on the M_2^0-matrix (2.33d), we get the recursion formula:

$$\begin{aligned}\left[B_2, O_m^{n-m}\right] &= \mathrm{T}_\Sigma\left\{\left[\left[(L^+)^{n-m}(L^-)^m\right]_{\mathrm{W}}, M_2^0\right] + 4(n-m)\omega\left[(L^+)^{n-m-1}(L^-)^{m+1}\right]_{\mathrm{W}}\right\} \\ &= 4\omega(n-m)O_{m+1}^{n-m-1}.\end{aligned} \tag{2.55}$$

Now we shall prove eq. (2.52a). In the case of $m = 0$, the relation holds. We assume that the relations (2.52a) hold up to $m = \mu$, and that the M_m^{n-m}-matrices satisfy the sum-to-zero condition. Due to the formula (2.53), the commutator $[O_{\mu+1}^{n-\mu-1}, L^\pm]$ is expressed as

$$4\omega(n-\mu)\left[O_{\mu+1}^{n-\mu-1}, L^\pm\right] = \left[\left[B_2, O_\mu^{n-\mu}\right], L^\pm\right]. \tag{2.56}$$

By iterated use of Jacobi's identity (2.40) and the generalized Lax equations for B_2 (2.33c) and $O_\mu^{n-\mu}$ (2.52a), the r.h.s. of eq. (2.56) is calculated as follows:

$$\begin{aligned}\left[\left[B_2, O_\mu^{n-\mu}\right], L^\pm\right] &= -\left[\left[O_\mu^{n-\mu}, L^\pm\right], B_2\right] - \left[\left[L^\pm, B_2\right], O_\mu^{n-\mu}\right] \\ &= -\left[\left[L^\pm, M_\mu^{n-\mu}\right], B_2\right] + \left[\left[L^\pm, M_2^0\right], O_\mu^{n-\mu}\right] + \left[2\omega(1\pm1)L^-, O_\mu^{n-\mu}\right] \\ &\quad -\left[\mu\omega(1\pm1)\left[(L^+)^{n-\mu}(L^-)^{\mu-1}\right]_{\mathrm{W}}, B_2\right] \\ &\quad +\left[(n-\mu)\omega(1\mp1)\left[(L^+)^{n-\mu-1}(L^-)^{\mu}\right]_{\mathrm{W}}, B_2\right] \\ &= \left[L^\pm, \left[B_2, M_\mu^{n-\mu}\right] + \left[M_2^0, O_\mu^{n-\mu}\right]\right] \\ &\quad +\left[\left[B_2, L^\pm\right], M_\mu^{n-\mu}\right] - \left[\left[O_\mu^{n-\mu}, L^\pm\right], M_2^0\right] \\ &\quad +\mu\omega(1\pm1)\left[\left[(L^+)^{n-\mu}(L^-)^{\mu-1}\right]_{\mathrm{W}}, M_2^0\right] \\ &\quad -(n-\mu)\omega(1\mp1)\left[\left[(L^+)^{n-\mu-1}(L^-)^{\mu}\right]_{\mathrm{W}}, M_2^0\right] - 2\omega(1\pm1)\left[L^-, M_\mu^{n-\mu}\right] \\ &\quad +4(\mu+2)(n-\mu)\omega^2(1\pm1)\left[(L^+)^{n-\mu-1}(L^-)^{\mu}\right]_{\mathrm{W}} \\ &\quad -4(n-\mu-1)(n-\mu)\omega^2(1\mp1)\left[(L^+)^{n-\mu-2}(L^-)^{\mu+1}\right]_{\mathrm{W}} \\ &= \left[L^\pm, \left[B_2, M_\mu^{n-\mu}\right] + \left[M_2^0, O_\mu^{n-\mu}\right] + \left[M_2^0, M_\mu^{n-\mu}\right]\right] \\ &\quad +4(\mu+1)(n-\mu)\omega^2(1\pm1)\left[(L^+)^{n-\mu-1}(L^-)^{\mu}\right]_{\mathrm{W}} \\ &\quad -4(n-\mu-1)(n-\mu)\omega^2(1\mp1)\left[(L^+)^{n-\mu-2}(L^-)^{\mu+1}\right]_{\mathrm{W}}.\end{aligned} \tag{2.57}$$

We define the $M_{\mu+1}^{n-\mu-1}$-matrix by

$$M_{\mu+1}^{n-\mu-1} = \frac{1}{4\omega(n-\mu)}\left\{\left[B_2, M_\mu^{n-\mu}\right] + \left[M_2^0, O_\mu^{n-\mu}\right] + \left[M_2^0, M_\mu^{n-\mu}\right]\right\}, \tag{2.58}$$

which meets the sum-to-zero condition. Finally, we obtain the expected formula from eqs. (2.56) and (2.57):

$$\begin{aligned}\left[O_{\mu+1}^{n-\mu-1}, L^\pm\right] &= \left[L^\pm, M_{\mu+1}^{n-\mu-1}\right] \\ &\quad +(\mu+1)\omega(1\pm1)\left[(L^+)^{n-\mu-1}(L^-)^{\mu}\right]_{\mathrm{W}} \\ &\quad -(n-\mu-1)\omega(1\mp1)\left[(L^+)^{n-\mu-2}(L^-)^{\mu+1}\right]_{\mathrm{W}}.\end{aligned} \tag{2.59}$$

Thus the proof is completed. The generalized Lax equations provide an easy way to consider the algebraic structure of the Calogero model.

2.2 *W*-Symmetry of the Calogero Model

The generalized Lax equations reveal an interesting algebra among the operators O_m^p. Let us introduce a new operator by

$$W_n^{(s)} \equiv \frac{1}{4\omega}O_{s+n-1}^{s-n-1}, \quad s \geq |n| + 1. \tag{2.60}$$

In this notation, the generalized Lax equations (2.52) are written as

$$\begin{aligned}\left[W_n^{(s)}, L^\pm\right] &= \left[L^\pm, M_n^{(s)}\right] \\ &\quad +\frac{1}{4}(s+n-1)(1\pm1)\left[(L^+)^{s-n-1}(L^-)^{s+n-2}\right]_{\mathrm{W}} \\ &\quad -\frac{1}{4}(s-n-1)(1\mp1)\left[(L^+)^{s-n-2}(L^-)^{s+n-1}\right]_{\mathrm{W}},\end{aligned} \tag{2.61a}$$

$$M_n^{(s)} \equiv \frac{1}{4\omega}M_{s+n-1}^{s-n-1}, \tag{2.61b}$$

$$\sum_{j=1}^{N}(M_n^{(s)})_{jk} = \sum_{j=1}^{N}(M_n^{(s)})_{kj} = 0. \tag{2.61c}$$

From the above generalized Lax equations for the $W_n^{(s)}$-operator, the commutation relation among the $W_n^{(s)}$-operators is calculated as

$$\begin{aligned}\left[W_n^{(s)}, W_m^{(t)}\right] &= \left[W_n^{(s)}, \mathrm{T}_\Sigma\left[(L^+)^{t-m-1}(L^-)^{t+m-1}\right]_{\mathrm{W}}\right] \\ &= \mathrm{T}_\Sigma\Big\{\frac{1}{4\omega}\left[\left[(L^+)^{t-m-1}(L^-)^{t+m-1}\right]_{\mathrm{W}}, M_n^{(s)}\right] \\ &\quad +\frac{1}{2}(s+n-1)(t-m-1)\left[(L^+)^{s+t-n-m-3}(L^-)^{s+t+n+m-3}\right]_{\mathrm{W}} \\ &\quad -\frac{1}{2}(s-n-1)(t+m-1)\left[(L^+)^{s+t-n-m-3}(L^-)^{s+t+n+m-3}\right]_{\mathrm{W}} + \cdots\Big\} \\ &= \big(n(t-1) - m(s-1)\big)W_{n+m}^{(s+t-2)} + \mathcal{P}_{n,m}^{(s,t)}(W_l^{(u)}),\end{aligned} \tag{2.62}$$

where $\mathcal{P}_{n,m}^{(s,t)}(W_l^{(u)})$ is an unspecified polynomial of $W_l^{(u)}$-operators, $u \leq s+t-3$, $l \leq n+m$, which are generated by replacements of L^+ and L^-,

$$\left[L^+, L^-\right] = 2\omega\big((a-1)\mathbf{1} - aT\big). \tag{2.63}$$

Here, $\mathbf{1}$ is a unit matrix and T is a constant matrix whose matrix elements are all 1: $T_{jk} = 1$. This is the W-algebra in the sense the algebra is a higher-spin generalization of the Virasoro algebra (for review, see ref. [16]). We remark that indices s and n are integer or half-odd integer, though of course both $s-n-1$ and $s+n-1$ must be non-negative integer. The algebra possesses the operators with indices $s = 1, 3/2, 2, 5/2, \cdots$, which represent the conformal spins.

For the classical mechanics case, derivation of the W-symmetry was examined by two different methods: the collective field theory [3] and the classical r-matrix method [7]. The classical r-matrix approach was extended to the trigonometric and elliptic versions of the Calogero-Moser model [6, 8, 66]. Quantum generalization of the collective field theory [4, 5] also suggests the W-symmetry structure of the quantum Calogero model, though the relationship between the quantum Calogero model and the quantum collective field theory has not been established. Our proof is the first direct proof of the W-symmetry of the quantum Calogero model. For the spin generalizations of the quantum Calogero-Moser, Sutherland and Calogero models were also investigated by the quantum Lax formulation [34, 78]. The sum-to-zero condition was also an important key in the investigations.

At the end of this section, we give a brief summary of the correspondence of the $W_n^{(s)}$-operators to the creation operators, the annihilation operators and the conserved operators:

1. creation operator $B_n^\dagger \propto W_{-\frac{n}{2}}^{(\frac{n}{2}+1)}$
2. annihilation operator $B_n \propto W_{\frac{n}{2}}^{(\frac{n}{2}+1)}$
3. conserved operator $O_n^n \propto W_0^{(n+1)}$

In particular, the creation operators and the annihilation operators respectively form commuting subalgebras of eq. (2.62). The property of the creation operators and the annihilation operators will play an important role in the algebraic construction of the energy eigenfunctions in Chapter 3.

2.3 Dunkl Operator Formulation

In the researches on the algebraic structures of the Calogero model, two formulations were independently developed. In the last two sections, we have formulated the quantum Lax formulation [34, 74–77] that is an extension of the Lax formulation in the classical theory [52]. On the other hand, Polychronakos introduced another formulation for these models [62] that he called exchange operator formalism. His method is based on the differential operators including coordinate exchange operators. Now Polychronakos' method is usually called Dunkl operator formulation because the basic properties of this kind of differential-difference operators were first studied by Dunkl [27]. These two approaches independently reveal interesting aspects of the models. We shall study the relationship between the two, which have not been checked by serious considerations.

We summarize the Dunkl operator formulation developed by Polychronakos [62]. We introduce differential operators which he called coupled momentum operators,

$$\pi_l = p_l + \mathrm{i}a\mu^{-1}\sum_{\substack{k=1\\k\neq l}}^{N}\frac{1}{x_l - x_k}K_{lk}, \quad l = 1, 2, \cdots, N, \tag{2.64}$$

where a complex phase factor $\mu = \exp(\mathrm{i}\pi\theta)$ corresponds to the statistics of the particles in the system whose meaning will be described later. The coordinate exchange operators K_{lk} obey

$$\begin{aligned} &K_{lk} = K_{kl},\ (K_{lk})^2 = \mu^2,\ K_{lk}^\dagger = \mu^{-2}K_{lk}, \\ &K_{lk}A_l = A_k K_{lk},\ K_{lk}A_j = A_j K_{lk},\ \text{for } j \neq l, k, \end{aligned} \tag{2.65}$$

where A_j is either a momentum operator p_j, a particle coordinates x_j, or particle permutation operators K_{ji}, $i = 1, 2, \cdots, N$. These relations (2.65) guarantee that the coupled momentum operators (2.64) are Hermitian operators, $\pi_l^\dagger = \pi_l$. By linear combinations of the coupled momentum operators and the coordinate operators,

$$q_l = \mathrm{i}x_l, \tag{2.66}$$

we define creation-like and annihilation-like operators:

$$c_l = \pi_l - \omega q_l, \tag{2.67a}$$
$$c_l^\dagger = \pi_l + \omega q_l. \tag{2.67b}$$

The creation-annihilation-like operators, $\{c_l, c_l^\dagger | l = 1, 2, \cdots, N\}$, describe the algebraic structure of the Calogero model. On the other hand, the coupled-momentum operators and the coordinate operators, $\{\pi_l, q_l | l = 1, 2, \cdots, N\}$, are applied to the study on the algebraic structure of the Calogero-Moser model and the Sutherland model, which will be summarized in Chapter 4.

By a straightforward calculation, we can check the following commutation relations for the creation- and annihilation-like operators:

$$\big[c_l, c_m\big] = 0, \tag{2.68a}$$
$$\big[c_l^\dagger, c_m^\dagger\big] = 0, \tag{2.68b}$$
$$\big[c_l, c_m^\dagger\big] = 2\omega\delta_{lm}\Big(1 + a\mu^{-1}\sum_{\substack{k\neq l\\k=1}}^{N}K_{lk}\Big) - 2\omega a\mu^{-1}(1-\delta_{lm})K_{lm}. \tag{2.68c}$$

Using eq. (2.68), we can construct a set of commuting operators that correspond to the conserved operators of the Calogero model:

$$\mathsf{I}_n \equiv \sum_{l=1}^{N}\big(c_l^\dagger c_l\big)^n, \tag{2.69a}$$
$$\big[\mathsf{I}_n, \mathsf{I}_m\big] = 0,\ n, m = 1, 2, \cdots. \tag{2.69b}$$

This relation is verified as follows. We introduce the number-like operators as

$$n_j = c_j^\dagger c_j. \tag{2.70}$$

Commutators among the number-like operators are given by

$$\big[n_j, n_k\big] = 2\omega a\mu^{-1}\big(n_k - n_j\big)K_{jk}. \tag{2.71}$$

Using the above commutation relation, we can compute the commutator among the operators (2.69a) as follows:

$$
\begin{aligned}
\left[\mathsf{I}_n, \mathsf{I}_m\right] &= \sum_{j,k=1}^{N} \left[(n_j)^n, (n_k)^m\right] \\
&= \sum_{j,k=1}^{N}\sum_{p=1}^{n}\sum_{q=1}^{m} (n_j)^{p-1}(n_k)^{q-1}\left[n_j, n_k\right](n_j)^{n-p}(n_k)^{m-q} \\
&= \sum_{j,k=1}^{N}\sum_{p=1}^{n}\sum_{q=1}^{m} (n_j)^{p-1}(n_k)^{q-1} 2\omega a \mu^{-1}\left(n_k - n_j\right) K_{jk}(n_k)^{m-q}(n_j)^{n-p} \\
&= 2\omega a\mu^{-1} \sum_{j,k=1}^{N}\sum_{p=1}^{n} (n_j)^{p-1}\left((n_k)^m - (n_j)^m\right) K_{jk}(n_j)^{n-p} \\
&= \omega a \mu^{-1} \sum_{j,k=1}^{N} \left\{ \sum_{p=1}^{n} (n_j)^{p-1}\left((n_k)^m - (n_j)^m\right) K_{jk}(n_j)^{n-p} \right. \\
&\qquad \left. - \sum_{q=1}^{m} (n_j)^{q-1}\left((n_k)^n - (n_j)^n\right) K_{jk}(n_j)^{m-q} \right\} \\
&= \omega a \mu^{-1} \left(\sum_{r=0}^{n-1} - \sum_{r=m}^{n+m-1} - \sum_{r=0}^{m-1} + \sum_{r=n}^{n+m-1} \right)(n_j)^r K_{jk}(n_j)^{n+m-r-1} \\
&= 0.
\end{aligned}
$$

Thus we have confirmed eq. (2.69). Note that the above commuting operators contain coordinate exchange operators. If we restrict the operand to the states of identical and indistinguishable particles, we can obtain the differential operators that do not include particle permutation operators. These differential operators generated by the restriction must be related to the operators in the quantum Lax formulation. We shall reveal the relationship in the following section.

2.4 Relationship between the Two Formulations

In the quantum theory, identical particles are indistinguishable. This requires the wave functions to be invariant up to a phase factor under the permutations of particle indices. The action of the coordinate exchange operators to such wave functions $\Psi(x_1, x_2, \cdots, x_N)$ is described as

$$
\begin{aligned}
(K_{lk}\Psi)(x_1, \cdots, x_l, \cdots, x_k, \cdots, x_N) &= \Psi(x_1, \cdots, x_k, \cdots, x_l, \cdots, x_N) \\
&= \mu\Psi(x_1, \cdots, x_l, \cdots, x_k, \cdots, x_N), \qquad (2.72)
\end{aligned}
$$

where the phase factor μ gives the information on the statistics of the particles. For instance, the case $\mu = 1$ corresponds to the boson system and the case $\mu = -1$ corresponds to the fermion system. In order to include the fractional statistics [90], we allow μ to be a complex number, $\mu = \exp(\mathrm{i}\pi\theta)$. Existence of such wave functions is known for the Calogero model [19, 76, 77]. From now on, we shall denote by $\Big|_{\mu}$ the restriction of the operand to such wave functions. Then

we notice the following relations for arbitrary polynomials P:

$$
P(c_l^\dagger, c_l)\Big|_{\mu} = \sum_{m=1}^{N} P(L^+, L^-)_{lm}\Big|_{\mu}. \qquad (2.73)
$$

They can be proved by induction on the degree of a polynomial. First, we consider the simplest case, namely the linear case. There are only two independent elements, and we can check the relation (2.73) directly:

$$
\begin{aligned}
c_l^\dagger\Big|_{\mu} &= \Big(p_l + \mathrm{i}\omega x_l + \mathrm{i}\mu^{-1} a \sum_{\substack{k=1\\k\neq l}}^{N} \frac{1}{x_l - x_k} K_{lk}\Big)\Big|_{\mu} \\
&= \sum_{k=1}^{N} \Big(\delta_{lk}(p_l + \mathrm{i}\omega x_l) + \mathrm{i}a(1 - \delta_{lk})\frac{1}{x_l - x_k}\Big)\Big|_{\mu} \\
&= \sum_{k=1}^{N} L_{lk}^{+}\Big|_{\mu}, \qquad (2.74a) \\
c_l\Big|_{\mu} &= \Big(p_l - \mathrm{i}\omega x_l + \mathrm{i}\mu^{-1} a \sum_{\substack{k=1\\k\neq l}}^{N} \frac{1}{x_l - x_k} K_{lk}\Big)\Big|_{\mu} \\
&= \sum_{k=1}^{N} \Big(\delta_{lk}(p_l - \mathrm{i}\omega x_l) + \mathrm{i}a(1 - \delta_{lk})\frac{1}{x_l - x_k}\Big)\Big|_{\mu} \\
&= \sum_{k=1}^{N} L_{lk}^{-}\Big|_{\mu}. \qquad (2.74b)
\end{aligned}
$$

For all the independent elements of the polynomials up to degree d, we assume that the relation (2.73) holds. All we have to do is to prove the relation for $c_l^\dagger P(c_l^\dagger, c_l)$ and $c_l P(c_l^\dagger, c_l)$, where $P(c_l^\dagger, c_l)$ is an arbitrary polynomial of degree d. It can be done by a simple and straightforward calculation:

$$
\begin{aligned}
c_l^\dagger P(c_l^\dagger, c_l)\Big|_{\mu} &= \Big(p_l + \mathrm{i}\omega x_l + \mathrm{i}\mu^{-1} a \sum_{\substack{k=1\\k\neq l}}^{N} \frac{1}{x_l - x_k} K_{lk}\Big) P(c_l^\dagger, c_l)\Big|_{\mu} \\
&= \sum_{k=1}^{N} \Big(\delta_{lk}(p_l + \mathrm{i}\omega x_l) + \mathrm{i}a(1 - \delta_{lk})\frac{1}{x_l - x_k}\Big) P(\alpha_k^\dagger, \alpha_k)\Big|_{\mu} \\
&= \sum_{k=1}^{N} L_{lk}^{+} \sum_{m=1}^{N} P(L^+, L^-)_{km}\Big|_{\mu} \\
&= \sum_{k=1}^{N} \Big(L^+ P(L^+, L^-)\Big)_{lk}\Big|_{\mu}, \qquad (2.75a) \\
c_l P(c_l^\dagger, c_l)\Big|_{\mu} &= \Big(p_l - \mathrm{i}\omega x_l + \mathrm{i}\mu^{-1} a \sum_{\substack{k=1\\k\neq l}}^{N} \frac{1}{x_l - x_k} K_{lk}\Big) P(c_l^\dagger, c_l)\Big|_{\mu} \\
&= \sum_{k=1}^{N} \Big(\delta_{lk}(p_l - \mathrm{i}\omega x_l) + \mathrm{i}a(1 - \delta_{lk})\frac{1}{x_l - x_k}\Big) P(c_k^\dagger, c_k)\Big|_{\mu}
\end{aligned}
$$

$$
\begin{aligned}
&= \sum_{k=1}^{N} L^-_{lk} \sum_{m=1}^{N} P(L^+, L^-)_{km}\Big|_\mu \\
&= \sum_{k=1}^{N} \big(L^- P(L^+, L^-)\big)_{lk}\Big|_\mu . \qquad (2.75b)
\end{aligned}
$$

Thus we have proved the relation (2.73) for arbitrary polynomials.

Moreover, we can also see the correspondence of the commutator algebras,

$$
\Big[\sum_{l=1}^{N} P_1(c_l^\dagger, c_l), \sum_{m=1}^{N} P_2(c_m^\dagger, c_m)\Big]\Big|_\mu = \big[\mathrm{T}_\Sigma\, P_1(L^+, L^-), \mathrm{T}_\Sigma\, P_2(L^+, L^-)\big]\Big|_\mu , \qquad (2.76)
$$

where P_1 and P_2 are arbitrary polynomials. To prove the above relations, it is sufficient to show the following identities:

$$
P_1(c_l^\dagger, c_l) \sum_{m=1}^{N} P_2(c_m^\dagger, c_m)\Big|_\mu = \sum_{k=1}^{N} P_1(L^+, L^-)_{lk}\, \mathrm{T}_\Sigma\, P_2(L^+, L^-)\Big|_\mu . \qquad (2.77)
$$

In a similar way to the proof of eq. (2.73), we shall prove the identity (2.77) by induction on the degree of P_1. First, we get the following expression by applying eq. (2.73) on P_2:

$$
P_1(c_l^\dagger, c_l) \sum_{m=1}^{N} P_2(c_m^\dagger, c_m)\Big|_\mu = P_1(c_l^\dagger, c_l)\, \mathrm{T}_\Sigma\, P_2(L^+, L^-)\Big|_\mu . \qquad (2.78)
$$

We shall consider a simple case, namely $P_1(c_l^\dagger) = c_l^\dagger$. From the explicit form of the $L^\pm$ matrix, we can see the action of K_{lk} operator on $L^\pm$ as

$$
K_{lk} L^\pm = F_{lk} L^\pm F_{lk} K_{lk}, \qquad (2.79)
$$

where an $N \times N$ numerical matrix F_{lk} is given by

$$
F_{lk} = \begin{array}{cc} & \begin{array}{ccccccccccc} & & & l & & & & k & & & \end{array} \\ \begin{array}{c} \\ \\ \\ l \\ \\ \\ \\ k \\ \\ \\ \\ \end{array} & \begin{bmatrix} 1 & & & & & & & & & & \\ & \ddots & & & & & & & & & \\ & & 1 & & & & & & & & \\ & & & 0 & & & & 1 & & & \\ & & & & 1 & & & & & & \\ & & & & & \ddots & & & & & \\ & & & & & & 1 & & & & \\ & & & 1 & & & & 0 & & & \\ & & & & & & & & 1 & & \\ & & & & & & & & & \ddots & \\ & & & & & & & & & & 1 \end{bmatrix} \end{array} . \qquad (2.80)
$$

Note that $(F_{lk})^2$ is the identity matrix. So the action (2.79) lead us to

$$
K_{lk} P_2(L^+, L^-) = F_{lk} P_2(L^+, L^-) F_{lk} K_{lk}, \qquad (2.81)
$$

where P_2 is an arbitrary polynomial. Thus we obtain

$$
c_l^\dagger \sum_{m=1}^{N} P_2(c_m^\dagger, c_m)\Big|_\mu = L^+_{ll}\, \mathrm{T}_\Sigma\, P_2(L^+, L^-)\Big|_\mu + \sum_{\substack{k=1 \\ k\neq l}}^{N} L^+_{lk}\, \mathrm{T}_\Sigma\, F_{lk} P_2(L^+, L^-) F_{lk}\Big|_\mu . \qquad (2.82)
$$

Using an identity,

$$
\begin{aligned}
\mathrm{T}_\Sigma\, F_{lk} P_2(L^+, L^-) F_{lk} &= [1, 1, \cdots, 1] F_{lk} P_2(L^+, L^-) F_{lk} [1, 1, \cdots, 1]^T \\
&= [1, 1, \cdots, 1] P_2(L^+, L^-) [1, 1, \cdots, 1]^T \\
&= \mathrm{T}_\Sigma\, P_2(L^+, L^-), \qquad (2.83)
\end{aligned}
$$

where a superscript T means a transposition, we get

$$
c_l^\dagger \sum_{m=1}^{N} P_2(c_m^\dagger, c_m)\Big|_\mu = \sum_{k=1}^{N} L^+_{lk}\, \mathrm{T}_\Sigma\, P_2(L^+, L^-)\Big|_\mu . \qquad (2.84)
$$

This shows the validity of eq. (2.77) for the simplest case. Following the same logic, we can also confirm another simple case:

$$
c_l \sum_{m=1}^{N} P_2(c_m^\dagger, c_m)\Big|_\mu = \sum_{k=1}^{N} L^-_{lk}\, \mathrm{T}_\Sigma\, P_2(L^+, L^-)\Big|_\mu . \qquad (2.85)
$$

We assume that the relation (2.77) holds for all the polynomials P_1 up to degree d. For a polynomial of degree $d+1$, $c_l^\dagger P_1(c_l^\dagger, c_l)$, we have

$$
\begin{aligned}
c_l^\dagger P_1(c_l^\dagger, c_l) \sum_{m=1}^{N} P_2(c_m^\dagger, c_m)\Big|_\mu &= L^+_{ll} P_1(c_l^\dagger, c_l)\, \mathrm{T}_\Sigma\, P_2(L^+, L^-)\Big|_\mu \\
&\quad + \sum_{\substack{k=1 \\ k\neq l}}^{N} L^+_{lk} P_1(c_k^\dagger, c_k)\, \mathrm{T}_\Sigma\, F_{lk} P_2(L^+, L^-) F_{lk}\Big|_\mu \\
&= \sum_{k=1}^{N} L^+_{lk} P_1(c_k^\dagger, c_k)\, \mathrm{T}_\Sigma\, P_2(L^+, L^-)\Big|_\mu \\
&= \sum_{k=1}^{N} \big\{L^+ P_1(L^+, L^-)\big\}_{lk}\, \mathrm{T}_\Sigma\, P_2(L^+, L^-)\Big|_\mu . \qquad (2.86)
\end{aligned}
$$

In a similar way, we can also confirm

$$
c_l P_1(c_l^\dagger, c_l) \sum_{m=1}^{N} P_2(c_m^\dagger, c_m)\Big|_\mu = \sum_{k=1}^{N} \big\{L^- P_1(L^+, L^-)\big\}_{lk}\, \mathrm{T}_\Sigma\, P_2(L^+, L^-)\Big|_\mu . \qquad (2.87)
$$

Thus we have inductively proved the correspondence (2.77). These relations (2.73) and (2.76) assure that two series of operators,

$$
\tilde{O}^p_m \equiv \sum_{k=1}^{N} \big[(c_k^\dagger)^p (c_k)^m\big]_{\mathrm{W}}, \qquad (2.88a)
$$

$$
\tilde{W}^{(s)}_n \equiv \frac{1}{4\omega} \tilde{O}^{s-n-1}_{s+n-1}, \qquad (2.88b)
$$

respectively correspond to the operators O^p_m (2.29) and $W^{(s)}_n$ (2.60) and also satisfy the commutation relation of the quantum W-algebra under the restriction $\Big|_\mu$. As a special case, we have

$$\sum_{k=1}^{N}(c_k^\dagger)^n\Big|_\mu = B_n^\dagger\Big|_\mu, \tag{2.89a}$$

$$\sum_{k=1}^{N}(c_k)^n\Big|_\mu = B_n\Big|_\mu. \tag{2.89b}$$

That is why we called the operators $B_n^\dagger$ and B_n (2.30) power-sum creation-annihilation operators. They also guarantee the mutual commutativity of the conserved operators,

$$I_n\Big|_\mu = \mathsf{I}_n\Big|_\mu, \tag{2.90}$$

of the Calogero model (2.12).

The final task in this section is to present a correspondence with the generalized Lax equations. By a straightforward calculation, we can confirm the following commutation relations:

$$\left[\tilde{O}^p_m, c_k^\dagger\right] = 2m\omega\left[(c_k^\dagger)^p(c_k)^{m-1}\right]_{\mathrm{W}}, \tag{2.91a}$$

$$\left[\tilde{O}^p_m, c_k\right] = -2p\omega\left[(c_k^\dagger)^{p-1}(c_k)^{m}\right]_{\mathrm{W}}. \tag{2.91b}$$

We shall verify the first formula (2.91a). Using eq. (2.68), we have

$$\begin{aligned}\left[\tilde{O}^p_m, c_k^\dagger\right] &= \sum_{j=1}^{N}\left[\left[(c_j^\dagger)^p(c_j)^m\right]_{\mathrm{W}}, c_k^\dagger\right] \\ &= m\sum_{j=1}^{N}\left[(c_j^\dagger)^p(c_j)^{m-1}\left[c_j, c_k^\dagger\right]\right]_{\mathrm{W}} \\ &= 2m\omega\left[(c_k^\dagger)^p(c_k)^{m-1}\right]_{\mathrm{W}} - 2\omega a\mu^{-1}\sum_{j=1}^{N}\left[\left((c_j^\dagger)^p(c_j)^{m-1} - (c_k^\dagger)^p(c_k)^{m-1}\right)M_{jk}\right]_{\mathrm{W}}.\end{aligned}$$

From the identity,

$$\left[c_j, c_k^\dagger\right] = -\left[c_j^\dagger, c_k\right], \tag{2.92}$$

we can readily confirm the following identity,

$$\left[\left((c_j^\dagger)^p(c_j)^m - (c_k^\dagger)^p(c_k)^m\right)M_{jk}\right]_{\mathrm{W}} = 0, \text{ for all } p, m, j, k, \tag{2.93}$$

which verifies the first formula (2.91a). Equation (2.93) is proved as follows. Among the terms in the Weyl ordered product, $\left[(c_j^\dagger)^p(c_j)^m M_{jk}\right]_{\mathrm{W}}$, we pay attention to the following term:

$$(c_j^\dagger)^{p_1}(c_j)^{m_1}(c_j^\dagger)^{p_2}\cdots M_{jk}(c_j^\dagger)^{p'_1}(c_j)^{m'_1}(c_j^\dagger)^{p'_2}\cdots.$$

On the other hand, there is a term in the other Weyl ordered product, $-\left[(c_k^\dagger)^p(c_k)^m M_{jk}\right]_{\mathrm{W}}$, which uniquely corresponds to the above term:

$$(c_k^\dagger)^{p'_1}(c_k)^{m'_1}(c_k^\dagger)^{p'_2}\cdots M_{jk}(c_k^\dagger)^{p_1}(c_k)^{m_1}(c_k^\dagger)^{p_2}\cdots.$$

Summing up the above two terms, we have a commutator:

$$\left[(c_j^\dagger)^{p_1}(c_j)^{m_1}(c_j^\dagger)^{p_2}\cdots, (c_k^\dagger)^{p'_1}(c_k)^{m'_1}(c_k^\dagger)^{p'_2}\cdots\right]M_{jk}.$$

For convenience, we rewrite the above commutator as

$$\left[(j)_1 c_j (j)_2, (k)_3 c_k^\dagger (k)_4\right]M_{jk},$$

where the symbol $(j)_l$ is a sequence of $c_l^\dagger$ and c_l labeled by j. Among the terms the above commutator yields, we pick up the following term,

$$(j)_1(k)_3\left[c_j, c_k^\dagger\right](k)_4(j)_2 M_{jk}. \tag{2.94}$$

On the other hand, there is a commutator picked up form the Weyl ordered product (2.93),

$$\left[(j)_1 c_j^\dagger (j)_2, (k)_3 c_k (k)_4\right]M_{jk},$$

which yields the following term,

$$(j)_1(k)_3\left[c_j^\dagger, c_k\right](k)_4(j)_2 M_{jk}. \tag{2.95}$$

From eq. (2.92), we notice that eq. (2.95) is the unique counter term of eq. (2.94). Thus we confirm eq. (2.93). We can verify the second formula (2.91b) in the same way. The restriction of eq. (2.91) generates the following equation that is similar to the generalized Lax equation (2.52):

$$\begin{aligned}\sum_{n=1}^{N}\left(\left[O^p_m, L^\pm_{kn}\right] + \left(\Lambda^p_m L^\pm\right)_{kn}\right)\Big|_\mu &= -(1\mp 1)p\omega\sum_{n=1}^{N}\left\{\left[(L^+)^{p-1}(L^-)^m\right]_{\mathrm{W}}\right\}_{kn}\Big|_\mu \\ &\quad +(1\pm 1)m\omega\sum_{n=1}^{N}\left\{\left[(L^+)^{p}(L^-)^{m-1}\right]_{\mathrm{W}}\right\}_{kn}\Big|_\mu.\end{aligned} \tag{2.96}$$

The newly appeared operator-valued matrix, Λ^p_m, is defined as

$$\sum_{n=1}^{N}(\Lambda^p_m G)_{kn}\Big|_\mu = \sum_{n=1}^{N}(\tilde{O}^p_m - O^p_m)G_{kn}\Big|_\mu, \tag{2.97}$$

where G is an arbitrary matrix that obeys

$$K_{lk}G = F_{lk}GF_{lk}K_{lk}. \tag{2.98}$$

We can verify that the Λ^p_m matrices also satisfy the sum-to-zero condition. Substitution of the unit matrix $\mathbf{1}$ into eq. (2.97) is possible because the unit matrix satisfies eq. (2.98):

$$\sum_{n=1}^{N}(\Lambda^p_m \mathbf{1})_{kn}\Big|_\mu = \sum_{n=1}^{N}(\tilde{O}^p_m - O^p_m)\mathbf{1}_{kn}\Big|_\mu. \tag{2.99}$$

Since $\tilde{O}^p_m\Big|_\mu = O^p_m\Big|_\mu$, the r.h.s. of the above equation vanishes. Then we get

$$\sum_{n=1}^{N}(\Lambda^p_m)_{kn} = 0 \;\; k = 1, \cdots, N. \tag{2.100}$$

Computation of the sum of the other suffix k needs a preparation. From eq. (2.98), we have

$$\begin{aligned}\sum_{k,n=1}^{N} \mu^{-1} K_{lm} G_{kn}\Big|_{\mu} &= \sum_{k,n=1}^{N} F_{lm} G_{kn} F_{lm} \mu^{-1} K_{lm}\Big|_{\mu} \\ &= \sum_{k,n=1}^{N} G_{kn}\Big|_{\mu}.\end{aligned}$$

Thus the sum of all the elements of the matrix G is symmetric with respect to the exchange of the indices. Then we get from the r.h.s. of eq. (2.97),

$$\begin{aligned}\sum_{k,n=1}^{N} (\tilde{O}^p_m - O^p_m) G_{kn}\Big|_{\mu} &= \sum_{k,n=1}^{N} (O^p_m - O^p_m) G_{kn}\Big|_{\mu} \\ &= 0.\end{aligned}$$

For an arbitrary operator-valued matrix G that meets the condition (2.98), the matrix Λ^p_m must satisfy the following identity,

$$\sum_{k,l,n=1}^{N} (\Lambda^p_m)_{kl} G_{ln} = 0,$$

which means Λ^p_m meets the sum-to-zero condition:

$$\sum_{k=1}^{N} (\Lambda^p_m)_{kl} = 0, \ l = 1, \cdots, N. \tag{2.101}$$

Thus eq. (2.96) is rewritten as

$$\begin{aligned}\sum_{n=1}^{N} \Big[O^p_m, L^{\pm}_{kn}\Big]_{kn}\Big|_{\mu} &= \sum_{n=1}^{N} \Big[L^{\pm}, \Lambda^p_m\Big]_{kn}\Big|_{\mu} - (1 \mp 1) p\omega \sum_{n=1}^{N} \Big\{\Big[(L^+)^{p-1}(L^-)^m\Big]_{\mathrm{W}}\Big\}_{kn}\Big|_{\mu} \\ &\quad + (1 \pm 1) m\omega \sum_{n=1}^{N} \Big\{\Big[(L^+)^{p}(L^-)^{m-1}\Big]_{\mathrm{W}}\Big\}_{kn}\Big|_{\mu},\end{aligned} \tag{2.102}$$

and we conclude that the Λ^p_m matrices must be the M-matrices for the generalized Lax equations:

$$\Lambda^p_m = M^p_m. \tag{2.103}$$

We have clarified the correspondence between the quantum Lax and the Dunkl operator formulations for the Calogero model. We also have formulated a direct method to obtain the generalized Lax equations from the commutator among the Dunkl operators. The correspondences we have considered are based on the restriction $\Big|_{\mu}$. Thus any equality between a formula in the quantum Lax formulation and the corresponding formula in the Dunkl operator formulation is a "weak" equality. A problem whether such equalities are "strong", or independent of the restriction, is also worth considering.

2.5 Summary

We have investigated the algebraic structure of the quantum Calogero model in the framework of the quantum Lax formulation and the Dunkl operator formulation. From the Lax equation for the classical model, we have obtained the Lax equation for the quantum Calogero Hamiltonian (2.5) whose M-matrix satisfies the sum-to-zero condition (2.8). The sum-to-zero condition has enabled us to construct a set of the conserved operators of the quantum Calogero model, as was the case with the quantum Calogero-Moser model [34, 74, 75, 88, 89]. To show the quantum integrability of the Calogero model, we have considered a construction of commuting conserved operators, eqs. (2.12) and (2.20). By use of the explicit forms of the first two conserved operators, I_1 and I_2 (2.13), or $\tilde{I}_1$ and $\tilde{I}_2$ (2.21), we have obtained the first two of the generalized Lax equations for the commuting conserved operators (2.16) and have conjectured the general form (2.22). However the generalized Lax equations for the commuting conserved operators are not compatible with the recursive construction. To study the recursive construction of the generalized Lax equations, we have introduced a series of operators that are made by sum of all the elements of the Weyl ordered product of the L^+- and L^--matrices (2.29). From the explicit forms of the first few conserved operators (2.26) and power-sum creation-annihilation operators (2.30), we have obtained the corresponding generalized Lax equations, eqs. (2.28), (2.32) and (2.33). From these first few generalized Lax equations, we have found out the recursion formulae, eqs. (2.36) and (2.53). Using the recursion formulae, we have recursively constructed all the generalized Lax equations (2.52). The generalized Lax equations prove the mutual commutativity of the power-sum creation-annihilation operators, which will play an important role in the algebraic construction of the energy eigenfunctions [76, 77]. We shall study this topic in Chapter 3. Defining the $W^{(s)}_n$-operators by eq. (2.60), we have proved that the generalized Lax equations yield the W-algebra as a commutator algebra among the $W^{(s)}_n$-operators.

We have studied correspondences between the quantum Lax formulation and the Dunkl operator formulation, which are developed independently in the study of the quantum Calogero model [80]. The Dunkl operator formulation provides us a simple way of constructing the commuting conserved operators of the Calogero model (2.69). We have observed that the restriction of the operand to the wave functions of the identical particles enables us to translate the results in one of the theories into those in the other. To be concrete, we have related arbitrary operators made from the two matrices, L^+ and L^-, and their commutator algebras with those in the Dunkl operator formulation, eqs. (2.73) and (2.76). A method to directly obtain the M^p_m-matrices has been also obtained as eq. (2.97). Mutual commutativity of the conserved operators I_n has been proved.

In a sense, the quantum Lax formulation makes up for difficulties with the Dunkl operator formulation and vice versa. For example, we can readily see that the commutator algebra in the quantum Lax formulation can be written in a closed form by the operators defined by (2.29). However, this is not observed in the Dunkl operator formulation. On the other hand, the Dunkl operator formulation provides an easy way to prove the mutual commutativity of the conserved operators of the Calogero model (2.69) and a direct method to construct the generalized Lax equations, eqs. (2.102). However, in the quantum Lax formulation, a recursive construction of the generalized Lax equations for the commuting conserved operators (2.22) remains unknown because of the difficulty of the order problem of a pair of matrices, L^+ and L^-. Thus the translation between the two methods will help our deeper comprehension of the systems.

Chapter 3

Algebraic Construction of the Eigenfunctions

Historically speaking, the Calogero model appeared as a one-dimensional quantum many-body problem with square and inverse-square long-range interactions [19]:

$$H_{\text{Calogero}} = \frac{1}{2}\sum_{j=1}^{N} p_j^2 + \frac{1}{2}g\sum_{\substack{j,k=1\\ j\neq k}}^{N}\frac{1}{(x_j-x_k)^2} + \frac{1}{2}\tilde{\omega}^2\sum_{\substack{j,k=1\\ j\neq k}}^{N}(x_j-x_k)^2, \tag{3.1a}$$

$$p_j = -\mathrm{i}\frac{\partial}{\partial x_j}. \tag{3.1b}$$

By a masterful separation of variables, Calogero solved the eigenvalue problem of the above Hamiltonian in 1970. He found out a change of variables that yields an ordinary differential equation for the eigenvalue problem of the Hamiltonian (3.1), which is reducible to Laguerre's ordinary differential equation. The formula of the energy spectrum has a similar form to that for the N harmonic oscillators. The fact motivated Perelomov to try an algebraic treatment of the system (3.1), though his challenge was not completed [61]. One of the aims of this chapter is to complete Perelomov's approach using the quantum Lax formulation.

Since the Hamiltonian (3.1) is translationally invariant, we may fix the center of mass at the origin of the coordinate. This corresponds to the convention that we are not interested in the motion of the center of mass, as is often the case with the statistical mechanics problem. In this case, the Hamiltonian reduces to the following form,

$$H_{\mathrm{C}} = \frac{1}{2}\sum_{j=1}^{N} p_j^2 + \frac{1}{2}g\sum_{\substack{j,k=1\\ j\neq k}}^{N}\frac{1}{(x_j-x_k)^2} + \frac{1}{2}\omega^2\sum_{j=1}^{N}x_j^2, \tag{3.2a}$$

$$\omega^2 = 2N\tilde{\omega}^2, \tag{3.2b}$$

which contains an external harmonic well instead of mutual harmonic interactions. This modification was introduced by Sutherland [70]. He obtained the exact ground state wave function for the Hamiltonian and found out a correspondence with the theory of random matrices. Using Calogero's energy spectrum formula, he also studied the thermodynamics of the above Hamiltonian (3.2). We treat the Hamiltonian (3.2) instead of the original Hamiltonian (3.1). Note that the center of mass is not always fixed at the coordinate origin. Only when we compare our results for the Calogero Hamiltonian (3.2) with those for the original Calogero Hamiltonian (3.1), we fix the center of mass at the origin of the coordinate.

The other aim of this chapter is to consider the simultaneous eigenfunctions of the conserved operators of the Calogero model, which must form the orthogonal basis. As a similar model to the Calogero model, we know the Sutherland model [72, 73]. The Sutherland model is a quantum many-body system on a circle with inverse sine-square interactions (1.21). Among the recent works on the Sutherland model, the computation of the dynamical correlation function of the model is worth mentioning [31, 32, 48]. A key role is played by the Jack symmetric polynomials [49, 69], which form the orthogonal basis of the Sutherland model. The Jack symmetric polynomials are also related to the singular vectors of the W-algebra [9, 10, 51], which is the symmetry of the Sutherland model [34]. As we have seen in Chapter 2, the Calogero model also has the W-symmetry structure [77]. However, in contrast to the Sutherland model, we have no knowledge on the orthogonal basis of the model. It is expected to be an essential tool for deeper comprehension of the Calogero model, such as rigorous calculations of correlation functions and relationships with the representation theory of the W-algebra. Those expectations concerning the orthogonal basis of the quantum Calogero model motivate us to challenge its construction.

As we have mentioned, we shall complete the algebraic construction of the energy eigenfunctions *à la* Perelomov and study the simultaneous eigenfunctions of the conserved operators of the Calogero model. Factorizing the Hamiltonian into a pair of Hermitian conjugate operators, we get the ground state energy and the ground state wave function. By successive operations of the power-sum creation operators obtained in Chapter 2 on the ground state wave function, we construct all the energy eigenfunctions of the Calogero model (3.2). Fixing the center of mass at the coordinate origin, we formulate the algebraic construction of the eigenfunctions of the original Calogero Hamiltonian (3.1). On the algebraically constructed basis of the Hilbert space of the Calogero model, we get a matrix representation of the second commuting conserved operator $\hat{I}_2$ (2.21b). By a straightforward diagonalization of the matrix, we conjecture an eigenvalue formula for the second commuting conserved operator and present the first seven of the simultaneous eigenfunctions.

3.1 Perelomov's Program

We demonstrate an algebraic treatment of the eigenfunctions of the quantum Calogero model (3.2a). First, we shall identify the ground state wave function. We decompose the Hamiltonian (3.2a) into the sum of the non-negative operators. From eq. (2.21a), we see that the Hamiltonian is expressed as follows:

$$H_{\mathrm{C}} = \frac{1}{4}\mathrm{T}_\Sigma\big(L^+L^- + L^-L^+\big). \tag{3.3}$$

Changing the order of L^+ and L^- in the second term of the r.h.s. of eq. (3.3), we get

$$\begin{aligned} H_{\mathrm{C}} &= \mathrm{T}_\Sigma\Big(\frac{1}{2}L^+L^- + \frac{1}{4}[L^+, L^-]\Big) \\ &= \frac{1}{2}\mathrm{T}_\Sigma(L^+L^-) + \frac{1}{2}N\omega\big(Na + (1-a)\big). \end{aligned} \tag{3.4}$$

We define a pair of Hermitian conjugate operators as

$$h_j^\dagger = \sum_{k=1}^{N} L^+_{kj} = p_j + \mathrm{i}\omega x_j + \mathrm{i}a\sum_{\substack{k=1\\k\neq j}}^{N}\frac{1}{x_k - x_j}, \tag{3.5a}$$

$$h_j = \sum_{k=1}^{N} L^-_{jk} = p_j - \mathrm{i}\omega x_j + \mathrm{i}a\sum_{\substack{k=1\\k\neq j}}^{N}\frac{1}{x_j - x_k}. \tag{3.5b}$$

Using the above operators, a new expression for the Hamiltonian (3.2a) is obtained:

$$H_{\mathrm{C}} = \frac{1}{2}\sum_{j=1}^{N} h_j^\dagger h_j + \frac{1}{2}N\omega\big(Na + (1-a)\big). \tag{3.6}$$

Note that the operator $\sum_{j=1}^{N} h_j^\dagger h_j$ is a nonnegative Hermitian operator. Consequently, all the eigenvalue of the Hamiltonian (3.6) must be larger than or equal to the ground state energy, $E_{\mathrm{g}} = \frac{1}{2}N\omega\big(Na + (1-a)\big)$. It is interesting to observe that the ground-state energy E_{g} reduces to that of N bosonic (fermionic) harmonic oscillators when we take the harmonic oscillator limit $a \to 0$ ($a \to 1$). In these limits, the coupling constant g vanishes, as is easily recognized from eq. (2.7). In order to obtain the ground state wave function, we have only to solve the following equations:

$$h_j|0\rangle = 0, \quad \text{for } j = 1, 2, \cdots, N. \tag{3.7}$$

Namely we seek a state that is annihilated by the operator h_j. The solution of eq. (3.7) is expressed as the real wave function of Laughlin's type:

$$\langle x|0\rangle = \prod_{1\le j<k\le N}(x_j - x_k)^a \exp\Big(-\sum_{j=1}^{N}\frac{1}{2}\omega x_j^2\Big). \tag{3.8}$$

Excited states are obtained by applications of the power-sum creation operators $B_n^\dagger$, $n = 1, 2, \cdots, N$,

$$\begin{aligned} B_n^\dagger &= \mathrm{T}_\Sigma(L^+)^n, \\ \big[H_{\mathrm{C}}, B_n^\dagger\big] &= n\omega B_n^\dagger, \end{aligned}$$

on the ground state. The above relations are given in eqs. (2.30a) and (2.31a). Let λ' denote a set of positive integers less than or equal to N,

$$\begin{aligned} \lambda' &= \{N^{n_N}, (N-1)^{n_{N-1}}, \cdots, 1^{n_1}\} \\ &= \{\underbrace{N, \cdots, N}_{n_N}, \underbrace{N-1, \cdots, N-1}_{n_{N-1}}, \cdots, \underbrace{1, \cdots, 1}_{n_1}\} \\ &= \{\lambda'_1, \lambda'_2, \cdots, \lambda'_{l(\lambda')}\}. \end{aligned} \tag{3.9}$$

We call λ' dual Young tableau, which is the dual of the Young tableau. Later we shall introduce a definition of the Young tableau. For details about the Young tableau and the dual Young tableau, consult, for instance, ref. [50]. Here a function of the dual Young tableau $l(\lambda')$ is the length of the dual Young tableau,

$$l(\lambda') = \sum_{k=1}^{N} n_k, \tag{3.10}$$

which is the number of non-zero elements in the dual Young tableau. The excited state labeled by a dual Young tableau λ' and its energy eigenvalue $E(\lambda')$ are expressed as follows:

$$|\lambda'\rangle = \prod_{k=1}^{N} (B_k^\dagger)^{n_k} |0\rangle, \tag{3.11a}$$

$$H_C|\lambda'\rangle = (E_g + |\lambda'|\omega)|\lambda'\rangle \stackrel{\text{def}}{=} E(\lambda')|\lambda'\rangle, \tag{3.11b}$$

where the symbol $|\lambda'|$ denotes the weight of the dual Young tableau,

$$|\lambda'| = \sum_{k=1}^{N} k n_k = \sum_{k=1}^{l(\lambda')} \lambda'_k. \tag{3.12}$$

Multiplicity of the n-th level for the system whose energy eigenvalue is $E_g + n\omega$ is equal to the number of partitions $p(n; N)$, i.e., the number of the different dual Young tableau λ' whose weight is equal to n, $\#\{\lambda' \big| |\lambda'| = n\}$. This is exactly the same result as is well known for the N harmonic oscillators.

Independence of the states, which correspond to two different dual Young tableaux, λ' and μ' respectively, can be confirmed as follows. First, we remark that the order of the creation operators dose not matter because of the mutual commutativity of the creation operators, $[B_n^\dagger, B_m^\dagger] = 0$, $n, m = 1, 2, \cdots$, as has been confirmed in Section 2.1. In the harmonic oscillator limit, $a \to 0$, $B_n^\dagger$ is expressed as

$$\begin{aligned} B_n^\dagger &\stackrel{a\to 0}{=} \mathrm{T}_\Sigma\Big(\mathrm{diag}(p_1 + \mathrm{i}\omega x_1, p_2 + \mathrm{i}\omega x_2, \cdots, p_N + \mathrm{i}\omega x_N)\Big)^n \\ &= \sum_{j=1}^{N} (a_j^\dagger)^n, \quad n = 1, 2, \cdots. \end{aligned} \tag{3.13}$$

Hence two different indices λ' and μ' generate two different polynomials of the conventional creation operators for the quantum harmonic oscillators, $\{a_j^\dagger | j = 1, 2, \cdots, N\}$. Homogeneous polynomials with degree n have independent elements whose number agrees with the number of partitions, $p(n; N)$. Thus we can get all the energy eigenfunctions algebraically by eq. (3.11) and hence they form a basis of the Hilbert space of the Calogero model.

When we consider the original Calogero Hamiltonian (3.1), we must take it into account that the center of mass is fixed at the origin of the coordinate. In this case, the operator $B_1^\dagger$ (and also B_1) reduces to zero:

$$\begin{aligned} B_1^\dagger &= \mathrm{T}_\Sigma L^+ = \sum_{k=1}^{N} h_k^\dagger \\ &= \sum_{k=1}^{N} \Big(-\mathrm{i}\frac{\partial}{\partial x_k} + \mathrm{i}\omega x_k\Big) = 0. \end{aligned} \tag{3.14}$$

This formula means that the indices of the eigenfunctions of the original Calogero Hamiltonian (3.1) should be the dual Young tableaux λ' that meet the following condition,

$$\lambda' = \{N^{n_N}, \cdots, 2^{n_2}\}, \tag{3.15}$$

or in other words, n_1 must be zero, $n_1 = 0$. Then the eigenfunctions are expressed as

$$|\lambda'\rangle_{\text{Calogero}} = \prod_{k=2}^{N} (B^\dagger)^{n_k} |0\rangle, \tag{3.16}$$

whose energy eigenvalue is $E(\lambda') = E_g + |\lambda'|\omega$. This agrees with the result of Calogero [19]. Perelomov investigated an algebraic approach to the Calogero's system (3.1a) and succeeded in obtaining three kinds of "creation (annihilation)"operators, namely $B_2^\dagger$,$B_3^\dagger$ and $B_4^\dagger$ (and their Hermitian conjugates) by direct calculation. However, he did not give higher operators $B_n^\dagger, n \geq 5$. The correct formula for these operators was first given in eq. (2.30a) [76, 77].

3.2 Simultaneous Eigenfunctions

We have obtained an algebraic formula for the energy eigenfunctions of the quantum Calogero model [76, 77]. Since the series of the eigenfunctions contains the same number of independent elements as that of the non-interacting quantum harmonic oscillators, they span the Hilbert space of the quantum Calogero model. However, they do not form an orthogonal basis of the Hilbert space because of the large degeneracy of its Hamiltonian. A conventional approach to the construction of the orthogonal basis is the Gram-Schmidt method. The Gram-Schmidt method compels us to compute many complicated multiple integrals and seems hopeless. Instead, we shall consider the diagonalization of nontrivial higher conserved operators of the model and remove the degeneracy, as has been done for the wave functions of the hydrogen atom. The eigenvalues corresponding to the conserved operators are expected to uniquely identify the elements of the orthogonal basis obtained as simultaneous eigenfunctions.

From the study in Chapter 2, we know the commuting conserved operators of the Calogero model [76, 80]. They take the forms

$$\tilde{I}_n = \mathrm{T}_\Sigma\Big(\frac{1}{2}(L^+L^- + L^-L^+)\Big)^n, \quad n = 1, 2, \cdots, N, \tag{3.17}$$

where the $N \times N$ operator-valued matrices L^+ and L^- are defined by eq. (2.6) in Chapter 2. The Hamiltonian corresponds to the first conserved operator $\tilde{I}_1$, i.e., $2H_C = \tilde{I}_1$. Thus we have a set of independent operators that are simultaneously diagonalizable. The basis that diagonalizes them all simultaneously is expected to be the orthogonal basis of the quantum Calogero model. As a first step of the research of the orthogonal basis of the model, we shall diagonalize the first nontrivial conserved operator $\tilde{I}_2$, and observe how the degeneracy disappears.

First, we consider how to obtain the matrix representation of the operator $\tilde{I}_2$ on the basis given by the algebraic formula in the previous section [76, 77]. We denote the ground state of the model by $|0\rangle_1$. The eigenfunction of the quantum Calogero model is labeled by a dual Young tableau λ' (3.9). The corresponding eigenfunction is expressed as

$$|\lambda'\rangle_1 = \prod_{k=1}^{N} (B_k^\dagger)^{n_k} |0\rangle_1. \tag{3.18}$$

The suffix 1 of the ket in the above expression means that the state $|\lambda'\rangle_1$ diagonalize the first conserved operator $\tilde{I}_1$, or equivalently, the Hamiltonian of the Calogero model. The operator, $B_n^\dagger = \mathrm{T}_\Sigma (L^+)^n$, is the power-sum creation operator. We recall that the eigenvalue of the first conserved operator, or equivalently twice the energy eigenvalue, of the state $|\lambda'\rangle_1$ is given by [19, 76, 77]

$$\tilde{E}_1(\lambda') = 2|\lambda'|\omega + N\omega\Big(Na + (1-a)\Big), \tag{3.19}$$

where the symbol $|\lambda'|$ is called the weight of the dual Young tableau (3.12). Any two states with different weights are orthogonal. This means that a higher conserved operator $\tilde{I}_2$ is block-diagonalized by the energy eigenfunction basis and each block consists of the states of the same weight.

In the calculation of the matrix elements on the basis (3.18), we shall utilize the fact that the operator $O_A = \mathrm{T}_\Sigma AL^-$, where A is an arbitrary $N \times N$ operator-valued matrix, annihilates the ground state, i.e., $O_A|0\rangle_1 = 0$. We use the generalized Lax equations obtained in Chapter 2 [76, 77] to derive the following formulae:

$$\begin{aligned}
\tilde{I}_2|0\rangle_1 &= \omega^2 N\Big((a-1) - Na\Big)^2 |0\rangle_1 \stackrel{\text{def}}{=} \tilde{E}_2(0)|0\rangle_1, && (3.20)\\
\Big[\tilde{I}_2, B_n^\dagger\Big] &= n\omega \sum_{j,k=1}^{N} (L^+)^{n+1}L^- + n\omega^2 a \sum_{k=1}^{n-1} B_k^\dagger B_{n-k}^\dagger \\
&\quad + n\omega^2\Big(2Na - (n+1)(a-1)\Big)B_n^\dagger, && (3.21)\\
\Big[\Big[\tilde{I}_2, B_n^\dagger\Big], B_m^\dagger\Big] &= 2nm\omega^2 B_{n+m}^\dagger. && (3.22)
\end{aligned}$$

By using the above formulae, we can express the operator $\tilde{I}_2$ in the form of a c-number-valued matrix. For instance, the block with weight 2 is expressed as

$$\tilde{I}_2 \begin{bmatrix} |2\rangle_1 \\ |1^2\rangle_1 \end{bmatrix} = \begin{bmatrix} \tilde{E}_2(0) - 6\omega^2(a-1) + 4\omega^2 Na & 2\omega^2 a \\ 2 & \tilde{E}_2(0) - 4\omega^2(a-1) + 4\omega^2 Na \end{bmatrix} \begin{bmatrix} |2\rangle_1 \\ |1^2\rangle_1 \end{bmatrix}. \tag{3.23}$$

We have implicitly introduced above a lexicographic order among dual Young tableaux. That is, two different tableaux λ' and μ' with the same weight are denoted by $\lambda' > \mu'$ if the first non-vanishing difference $\lambda'_k - \mu'_k$ is positive.

Similar calculations lead us to the $(n+1)$-th block of the operator $\tilde{I}_2$, $M_2(n)$, on the weight-n basis in the lexicographic order denoted by $\boldsymbol{D}_1(n) = [|n\rangle_1, |n-1,1\rangle_1, \cdots, |1^n\rangle_1]^T$:

$$\tilde{I}_2 \boldsymbol{D}_1(n) = M_2(n)\boldsymbol{D}_1(n). \tag{3.24}$$

Diagonalization of the matrix $M_2(n)$ gives the eigenvalue $\tilde{E}_2(\lambda')$ and the corresponding eigenvector $\boldsymbol{E}_2(\lambda')$. Then the following relation must hold among $\tilde{E}_2(\lambda')$, $\boldsymbol{E}_2(\lambda')$, $M_2(n)$ and $\tilde{I}_2$:

$$\begin{aligned}
\tilde{I}_2 T_n^{-1}\boldsymbol{D}_1(n) &= T_n^{-1}M_2(n)T_nT_n^{-1}\boldsymbol{D}_1(n) \\
&= \mathrm{diag}[\tilde{E}_2(n), \tilde{E}_2(n-1,1), \cdots, \tilde{E}_2(1^n)]T_n^{-1}\boldsymbol{D}_1(n), && (3.25)\\
T_n &= [\boldsymbol{E}_2(n), \boldsymbol{E}_2(n, n-1), \cdots, \boldsymbol{E}_2(1^n)]. && (3.26)
\end{aligned}$$

We denote by $|\lambda'\rangle_2$ the weight-n states that diagonalize the operator $\tilde{I}_2$. The equality (3.25) shows us their forms as

$$\boldsymbol{D}_2(n) \stackrel{\text{def}}{=} \Big[|n\rangle_2, |n-1,1\rangle_2, \cdots, |1^n\rangle_2\Big]^T = T_n^{-1}\boldsymbol{D}_1(n), \tag{3.27}$$

where the suffices 2 of the kets in the above expression mean that the state $|\lambda'\rangle_2$ diagonalizes the first two commuting conserved operators, $\tilde{I}_1$ and $\tilde{I}_2$. In particular, the elements of $D_2(n)$ that have no degeneracy in $\tilde{E}_2(\lambda')$ turn out to be the elements of the orthogonal basis of the model.

After a straightforward but lengthy calculation, we can obtain the eigenvalues of $\tilde{I}_2$ for up to the seventh block (corresponding to weight 6). The eigenvalue $\tilde{E}_2(\lambda')$ is expressed as

$$\tilde{E}_2(\lambda') = 4\omega^2\left(\frac{1}{4}N\Big((a-1) - Na\Big)^2 + (1 - a + 2Na)|\lambda'| - a\sum_{k=1}^{l(\lambda')}(\lambda'_k)^2 + \sum_{k=1}^{N}(\lambda_k)^2\right). \tag{3.28}$$

Here, λ_k is the k-th element of the Young tableau λ, which is the dual of the dual Young tableau λ' defined by

$$\begin{aligned}
\lambda_k &= \sum_{j=k}^{N} n_j, && (3.29a)\\
n_k &= \lambda_k - \lambda_{k+1}, \quad \lambda_{N+1} \stackrel{\text{def}}{=} 0. && (3.29b)
\end{aligned}$$

Namely, the dual Young tableau λ' is obtained by exchanging the rows and the columns of the Young tableau λ and vice versa. The eigenvalue $\tilde{E}_2(\lambda')$ removes most of the degeneracy. Indeed, there is no degeneracy up to the states with weight 5. We can obtain the expressions for the orthogonal eigenfunctions from the simultaneous eigenvectors corresponding to the nondegenerate eigenvalues $\tilde{E}_1(\lambda')$ and $\tilde{E}_2(\lambda')$. However, there still remains degeneracy. For example, two pairs of different tableaux, $\{4, 1^2\}$, $\{3^2\}$ and $\{3, 1^3\}$, $\{2^3\}$ respectively give the same eigenvalues of $\tilde{I}_1$ and $\tilde{I}_2$. Since eqs. (3.19) and (3.28) suggest that the eigenvalues $\tilde{E}_m(\lambda')$ of the higher conserved operators $\tilde{I}_m$, $m \geq 3$, involve the higher-order power sums of the Young tableau, $\mathsf{p}_m(\lambda) = \sum_{k=1}^{l}(\lambda_k)^m$, remaining degeneracy must be completely removed by their diagonalizations.

Generally speaking, the eigenfunctions of the quantum Calogero model (3.18) are given by products of symmetric polynomials and the ground state wave function. We denote the symmetric-polynomial parts of the energy eigenfunctions by

$$[\lambda']_1 \stackrel{\text{def}}{=} \frac{\langle \boldsymbol{x}|\lambda'\rangle_1}{\langle \boldsymbol{x}|0\rangle_1}.$$

The meaning of the suffix 1 in the above expression is the same as that of the suffix for the ket $|\lambda'\rangle_1$. The coordinate representation of eigenfunction (3.18) is expressed in terms of power sums $\mathsf{p}_n(x_1, \cdots, x_N) = \sum_{k=1}^{N} x_k^n$. The first seven eigenfunctions are

$$\begin{aligned}
&[0]_1 = 1, && (3.30a)\\
&[1]_1 = 2\mathrm{i}\omega\mathsf{p}_1, && (3.30b)\\
&[2]_1 = -4\omega^2\mathsf{p}_2 + 2\omega N + 2\omega N(N-1)a, && (3.30c)\\
&[1^2]_1 = -4\omega^2(\mathsf{p}_1)^2 + 2\omega N, && (3.30d)\\
&[3]_1 = -8\mathrm{i}\omega^3\mathsf{p}_3 + 12\mathrm{i}\omega^2\Big(1 + a(N-1)\Big)\mathsf{p}_1, && (3.30e)
\end{aligned}$$

$$[2,1]_1 = -8\mathrm{i}\omega^3 \mathsf{p}_2\mathsf{p}_1 + 4\mathrm{i}\omega^2\Big((N+2) + aN(N-1)\Big)\mathsf{p}_1, \tag{3.30f}$$
$$[1^3]_1 = -8\mathrm{i}\omega^3(\mathsf{p}_1)^3 + 12\mathrm{i}\omega^2 N\mathsf{p}_1. \tag{3.30g}$$

Note that coefficients of the power sums depend on the particle number N and the parameter a that determines the coupling constant. We denote polynomial parts of the simultaneous eigenfunctions for $\tilde{I}_1$ and $\tilde{I}_2$ in a similar way by

$$[\lambda']_2 = \frac{\langle \boldsymbol{x}|\lambda^2\rangle_2}{\langle \boldsymbol{x}|0\rangle_1}.$$

Then the first seven of them are expressed by the basis $[\lambda']_1$ as

$$[0]_2 = [0]_1 = 1, \tag{3.31a}$$
$$[1]_2 = [1]_1 = 2\mathrm{i}\omega\mathsf{p}_1, \tag{3.31b}$$
$$[2]_2 = [2]_1 - [1^2]_1 = -4\omega^2\big(\mathsf{p}_2 - (\mathsf{p}_1)^2\big) + 2\omega aN(N-1), \tag{3.31c}$$
$$[1^2]_2 = [2]_1 + a[1^2]_1 = -4\omega^2\big(\mathsf{p}_2 + a(\mathsf{p}_1)^2\big) + 2\omega N(Na+1), \tag{3.31d}$$
$$\begin{aligned}[3]_2 &= 2[3]_1 - 3[2,1]_1 + [1^3]_1 \\ &= -8\mathrm{i}\omega^3\big(2\mathsf{p}_3 - 3\mathsf{p}_2\mathsf{p}_1 + (\mathsf{p}_1)^3\big) - 12\mathrm{i}\omega^2 a(N-1)(N-2)\mathsf{p}_1,\end{aligned} \tag{3.31e}$$
$$\begin{aligned}[2,1]_2 &= [3]_1 - (1-a)[2,1]_1 - a[1^3]_1 \\ &= -8\mathrm{i}\omega^3\big(\mathsf{p}_3 - (1-a)\mathsf{p}_2\mathsf{p}_1 - a(\mathsf{p}_1)^3\big) - 4\mathrm{i}\omega^2(1-a)(N-1)(Na+1)\mathsf{p}_1,\end{aligned} \tag{3.31f}$$
$$\begin{aligned}[1^3]_2 &= 2[3]_1 + 3a[2,1]_1 + a^2[1^3]_1 \\ &= -8\mathrm{i}\omega^3\big(2\mathsf{p}_3 + 3a\mathsf{p}_2\mathsf{p}_1 + a^2(\mathsf{p}_1)^3\big) + 12\mathrm{i}\omega^2(a^2N^2 + 3aN + 2)\mathsf{p}_1.\end{aligned} \tag{3.31g}$$

Since the eigenvalues $\tilde{E}_1(\lambda')$ and $\tilde{E}_2(\lambda')$ for the above seven symmetric polynomials have no degeneracy, they are indeed the simultaneous eigenfunctions of all the commuting conserved operators, $\tilde{I}_n$ and hence I_n, $n = 1, 2, \cdots$. In this sense, they should be denoted by $[\lambda']_\infty$. These are the first seven of a new series of orthogonal symmetric polynomials with respect to the inner product:

$$\begin{aligned}\langle [\lambda']_\infty, [\mu']_\infty\rangle &= \int_{-\infty}^{\infty}\prod_{k=1}^{N} dx_k \prod_{1\le k<l\le N}|x_k - x_l|^{2a}\exp(-\omega\boldsymbol{x}^2)[\lambda']_\infty[\mu']_\infty \\ &= \mathcal{N}_{\lambda'}\delta_{\lambda'\mu'}.\end{aligned} \tag{3.32}$$

Thus they should be regarded as a deformed multivariable generalization of the Hermite polynomials.

We notice that the eigenvalue formula for the commuting conserved operator I_2 (2.13b),

$$E_2(\lambda) = 4\omega^2\sum_{k=1}^{N}\Big((\lambda_k)^2 + a(N+1-2k)\lambda_k\Big), \tag{3.33}$$

which can be derived from eqs. (2.13), (2.21), (3.19) and (3.28), has the same form as that of the eigenvalue of the Sutherland Hamiltonian on the Jack polynomial [69, 73]. Moreover, the first seven orthogonal symmetric polynomials (3.31) are cast into the following forms,

$$[0]_\infty = 1, \tag{3.34a}$$
$$[1]_\infty = 2\mathrm{i}\omega m_1, \tag{3.34b}$$
$$[2]_\infty = 8\omega^2\Big(m_{1^2} + \frac{a}{2\omega}\frac{N(N-1)}{2}\Big), \tag{3.34c}$$
$$[1^2]_\infty = -4\omega^2\Big((1+a)m_2 + 2am_{1^2} - \frac{1}{2\omega}N(Na+1)\Big), \tag{3.34d}$$
$$[3]_\infty = -48\mathrm{i}\omega^3\Big(m_{1^3} + \frac{1}{2\omega}a\frac{(N-1)(N-2)}{2}m_1\Big), \tag{3.34e}$$
$$[2,1]_\infty = 8\mathrm{i}\omega^3\Big((2a+1)m_{2,1} + 6am_{1^3} - \frac{1}{2\omega}(1-a)(N-1)(Na+1)m_1\Big), \tag{3.34f}$$
$$[1^3]_\infty = -8\mathrm{i}\omega^3\Big((a^2+3a+2)m_3 + 3a(a+1)m_{2,1} + 6am_{1^3} - \frac{3}{2\omega}(a^2N^2 + 3aN + 2)m_1\Big), \tag{3.34g}$$

where m_λ denotes the symmetrized monomial, or in other words, the monomial symmetric polynomial whose definition is

$$m_\lambda(x_1, \cdots, x_N) = \sum_{\substack{\sigma\in S_N,\ \text{distinct} \\ \text{permutations}}} (x_{\sigma(1)})^{\lambda_1}\cdots(x_{\sigma(N)})^{\lambda_N}.$$

The above expansions show the triangularity, which is also observed for the Jack polynomials. The meaning of this observation will be discussed in more detail in the next chapter. These observation strongly suggests that there must be a similarity between the Jack polynomials and the unidentified orthogonal symmetric polynomials associated with the Calogero model. To be concrete, the observation of the eigenvalue formula (3.33) and the triangularity (3.34) show the essential part of the definition of the simultaneous eigenfunctions of all the commuting conserved operators of the Calogero model. Based on the results of this section, we shall introduce the definition and investigate the properties of the simultaneous eigenfunctions in the next chapter.

3.3 Summary

We have studied an algebraic construction of all the eigenfunctions of the Calogero Hamiltonian with the help of the quantum Lax formulation. By the factorization of the Hamiltonian, we have obtained the ground state wave function. Using the power-sum creation operators which has been obtained in Chapter 2, we have formulated an algebraic method to construct the eigenfunctions of the Calogero model. From the number of independent eigenfunctions, we have confirmed that the eigenfunctions form the basis of the Hilbert space of the Calogero model. We have also reproduced the result for the original Calogero model by fixing the center of mass at the coordinate origin. Thus we have completed Perelomov's dream of the algebraic construction of the eigenfunctions of the Calogero model. We have also considered a construction of the orthogonal basis of the Calogero model by diagonalizing mutually commuting conserved operators. We have directly diagonalized the first nontrivial conserved operator $\tilde{I}_2$ using the energy eigenfunctions with weights up to 6. The results indicate a general formula for the eigenvalue of $\tilde{I}_2$. In addition, we have presented explicit expressions of the first seven of the unidentified orthogonal symmetric polynomials associated with the Calogero model. From the explicit form of the weight function of the inner-product, we have concluded that the seven unidentified orthogonal symmetric polynomials should be regarded as multivariable generalizations of the Hermite polynomials. From the eigenvalue formula for the commuting conserved operator I_2 and the expansion of the explicit forms with respect to the monomial symmetric function, we have observed a similarity between the Jack polynomials and the unidentified orthogonal symmetric polynomials. This observation give us a crucial hint toward the identification of the orthogonal symmetric polynomials associated with the Calogero model. Detailed mathematical properties of the polynomials, such as their definition in a compact form and their formulation, are to be investigated in the next chapter.

Chapter 4

Orthogonal Basis

Exact solutions for the Schrödinger equations have provided important problems in physics and mathematical physics. Most of us have studied the Laguerre polynomials and the spherical harmonics in the theory of the hydrogen atom, and the Hermite polynomials and their Rodrigues formula in the theory of the quantum harmonic oscillator. The former is also a good example that shows the role of conserved operators in quantum mechanics. The hydrogen atom has three, independent and mutually commuting conserved operators, namely, the Hamiltonian, the total angular momentum and its z-axis component. The simultaneous eigenfunctions for the three conserved operators give the orthogonal basis of the hydrogen atom. A classical system with a set of independent and mutually Poisson commuting (involutive) conserved quantities whose number of elements is the same as the degrees of freedom of the system can be integrated by quadrature. This is guaranteed by the Liouville theorem. Such a system is called the completely integrable system. Quantum systems with enough number of such conserved operators are analogously called quantum integrable systems. The hydrogen atom is a simple example of the quantum integrable system.

Among the various quantum integrable systems, one-dimensional quantum many-body systems with inverse-square long-range interactions are now attracting much interests of theoretical physicists. Of the various integrable inverse-square-interaction models, the quantum Calogero model [19] has the longest history. Its Hamiltonian is expressed as

$$\hat{H}_{\mathrm{C}} = \frac{1}{2}\sum_{j=1}^{N}\left(p_j^2 + \omega^2 x_j^2\right) + \frac{1}{2}\sum_{\substack{j,k=1\\ j\neq k}}^{N} \frac{a^2 - a}{(x_j - x_k)^2}, \tag{4.1}$$

where the constants N, a and ω are the particle number, the coupling parameter and the strength of the external harmonic well respectively. The momentum operator p_j is given by a partial differential operator, $p_j = -\mathrm{i}\dfrac{\partial}{\partial x_j}$. In Chapter 2, we have confirmed that the model is a quantum integrable system in the sense that it has sufficient number of independent and mutually commuting conserved operators [62, 76–78, 80]. On the other hand, the Sutherland model [72, 73], which is a one-dimensional quantum integrable system with inverse-sine-square interactions,

$$\hat{H}_{\mathrm{S}} = \frac{1}{2}\sum_{j=1}^{N} p_j^2 + \frac{1}{2}\sum_{\substack{j,k=1\\ j\neq k}}^{N} \frac{a^2 - a}{\sin^2(x_j - x_k)}, \tag{4.2}$$

has been thoroughly investigated since its orthogonal basis was known to consist of the Jack symmetric polynomials [36, 50, 69]. The quantum Lax formulation and the Dunkl operator formulation (exchange operator formalism) showed that these two models share the same algebraic structure [13, 14, 34, 35, 62, 76–78, 80]. We have also observed in Chapter 3 that the first seven simultaneous eigenfunctions of the commuting conserved operators of the Calogero model and the Jack polynomials have a common property, namely the triangularity with respect to the dominance order [69]. These facts strongly suggest some similarities in the structures of the Hilbert spaces of the Calogero and the Sutherland models. In order to clarify this problem, we shall apply a naive approach that we use in the study of the hydrogen atom to the quantum Calogero model and study the deformed multivariable extension of the Hermite polynomials, namely, the Hi-Jack (hidden-Jack) symmetric polynomials [79, 83, 84, 86, 87].

The Jack symmetric polynomials are uniquely determined by three properties [50, 69]. First, they are the eigenfunctions of the differential operator that is derived from the Hamiltonian of the Sutherland model. Second, they possess triangular expansions in monomial symmetric polynomials with respect to the dominance order. And last, they are properly normalized. For detail, see eq. (4.38). Quite recently, Lapointe and Vinet discovered the Rodrigues formula for the Jack symmetric polynomials using the Dunkl operator for the Sutherland model [42]. Here we shall extend their results to the quantum Calogero model and give the Rodrigues formula for the Hi-Jack symmetric polynomials [83, 84]. Through the Rodrigues formula, we shall also investigate the properties of the Hi-Jack symmetric polynomials and compare them with the basis of the model that was given by the quantum Lax formulation [76, 77] in Chapter 3 and by the Dunkl operators [17, 18]. The algebraic construction of the eigenfunctions for the Hamiltonian (the first conserved operator) of the quantum Calogero model has already been given. Thus the Hi-Jack symmetric polynomials must be linear combinations of them. We shall specify the linear combinations that relate the Hi-Jack polynomials and the eigenfunctions of the Hamiltonian given before [17, 18, 76, 77]. We also want to see the relationships and similarities between the Jack polynomials and the Hi-Jack polynomials. A proof of the orthogonality of the Hi-Jack polynomials is also an important problem. We shall present clear answers to these questions.

The outline of this chapter is as follows. In Section 4.1, we summarize and reformulate the results of the quantum Lax and the Dunkl operator formulations for the Calogero and Sutherland models. We shall clarify the common algebraic structure of the models. The Jack symmetric polynomials are also introduced. In Section 4.2, we present the Rodrigues formula for the Hi-Jack symmetric polynomials and introduce some propositions that guarantee the results. Some properties of the Hi-Jack polynomials such as integrality, triangularity, orthogonality and relationships with the Jack polynomials are also presented. In Section 4.3, we prove the propositions. And in the final section, we give a brief summary.

4.1 Models and Formulations

In our derivation of the Rodrigues formula for the Hi-Jack symmetric polynomials, we do many computations involving the Dunkl operators for the Calogero model. We also compare the Hi-Jack polynomials with the Jack polynomials and with the basis of the Calogero model given by the quantum Lax formulation in Chapter 3. Thus we need a summary of the Calogero model, the Sutherland model, and the quantum Lax and the Dunkl operator formulations.

First, we reformulate the quantum Lax formulation and the Dunkl operators for the Calogero model. The Lax matrices for the N-body system are given by $N \times N$ operator-valued matrices. To express them, we have to introduce two operator-valued matrices:

$$\check{L}_{jk} = \mathrm{i}p_j\delta_{jk} - a(1-\delta_{jk})\frac{1}{x_j - x_k}, \tag{4.3a}$$

$$\check{Q}_{jk} = x_j\delta_{jk}, \tag{4.3b}$$

where $j, k = 1, 2, \cdots, N$. The above $\check{L}$-matrix is for the (rational) Calogero-Moser model (1.20) whose Hamiltonian is obtained by taking $\omega = 0$ of the Calogero model (4.1) or the rational limit of the Sutherland model (4.2). The Lax matrices for the Calogero model are

$$\hat{L}^- = \check{L} + \omega\check{Q}, \tag{4.4a}$$

$$\hat{L}^+ = -\frac{1}{2\omega}(\check{L} - \omega\check{Q}). \tag{4.4b}$$

In eqs. (4.3) and (4.4) above, we have introduced different normalizations from those matrices in Chapter 2. We have also introduced accent marks. They are just for convenience of later discussions in this chapter. Then the Hamiltonian (4.1) is expressed by the above Lax matrices as

$$\begin{aligned}\hat{H}_{\mathrm{C}} &= \omega\,\mathrm{T}_{\Sigma}\,\hat{L}^+\hat{L}^- + \frac{1}{2}N\omega\big(Na + (1-a)\big)\\ &= \omega\,\mathrm{T}_{\Sigma}\,\hat{L}^+\hat{L}^- + E_{\mathrm{g}},\end{aligned} \tag{4.5}$$

where T_{Σ} denotes a sum of all the matrix elements, $\mathrm{T}_{\Sigma}A = \sum_{i,j=1}^{N} A_{ij}$, and E_{g} is the ground state energy. Note that the first term of the r.h.s. of eq. (4.5) is a nonnegative Hermitian operator. Thus the ground state is the solution of the following equations,

$$\sum_{k=1}^{N}\hat{L}^-_{jk}\hat{\phi}_{\mathrm{g}} = 0, \text{ for } j = 1, 2, \cdots, N \Rightarrow \hat{H}_{\mathrm{C}}\hat{\phi}_{\mathrm{g}} = E_{\mathrm{g}}\hat{\phi}_{\mathrm{g}}. \tag{4.6}$$

The ground state wave function is the real Laughlin wave function:

$$\hat{\phi}_{\mathrm{g}} = \prod_{1\le j<k\le N}|x_j - x_k|^a \exp\Big(-\frac{1}{2}\omega\sum_{j=1}^{N}x_j^2\Big). \tag{4.7}$$

A short note might be in order. The phase of the difference product of the above real Laughlin wave function, which determines the statistics of the particles, or in other words, the symmetry of all the eigenfunctions, can be arbitrary. We can assign any phase factor to all the exchanges of particles. However, we must introduce a phase factor to the definition of the Dunkl operators [80], as has been discussed in Chapter 2. To avoid unnecessary complexity, we fix the phase at unity.

The eigenfunction of the Calogero model is fatorized into an inhomogeneous symmetric polynomial and the ground state wave function. For convenience of investigations on the inhomogeneous symmetric polynomials, we redefine the Lax matrices (4.4) by the following similarity transformation:

$$L^- = \hat{\phi}_{\mathrm{g}}^{-1}\hat{L}^-\hat{\phi}_{\mathrm{g}}, \tag{4.8a}$$

$$L^+ = \hat{\phi}_{\mathrm{g}}^{-1}\hat{L}^+\hat{\phi}_{\mathrm{g}}. \tag{4.8b}$$

Any operator with a hat, $\hat{\mathcal{O}}$, is related to an operator $\mathcal{O}$ by the similarity transformation using the ground state wave function $\hat{\phi}_g$,

$$\mathcal{O} = \hat{\phi}_g^{-1}\hat{\mathcal{O}}\hat{\phi}_g, \tag{4.9a}$$
$$\hat{\mathcal{O}} = \hat{\phi}_g\mathcal{O}\hat{\phi}_g^{-1}. \tag{4.9b}$$

A set of mutually commuting conserved operators of the Calogero model $\{I_n|n = 1, 2, \cdots, N\}$ is given by

$$I_n = \mathrm{T}_\Sigma(L^+L^-)^n. \tag{4.10}$$

The Hamiltonian H_C is equal to $\omega I_1 + E_g$. We regard the first conserved operator I_1 as the Hamiltonian of the Calogero model. The Heisenberg equations for the L^- and L^+ matrices are expressed in the forms of the Lax equation [76]. Moreover, we have more general relations for a class of operators,

$$V_p^m = \mathrm{T}_\Sigma[(L^-)^m(L^+)^p]_\mathrm{w}, \tag{4.11}$$

where the subscript w means the Weyl ordered product. The class of operators naturally includes the Hamiltonian, $H_C = \omega V_1^1$. The generalized Lax equations are

$$\left[V_p^m, L^-\right] = \left[L^-, Z_p^m\right] - p[(L^-)^m(L^+)^{p-1}]_\mathrm{w}, \tag{4.12a}$$
$$\left[V_p^m, L^+\right] = \left[L^+, Z_p^m\right] + m[(L^-)^{m-1}(L^+)^p]_\mathrm{w}, \tag{4.12b}$$

where the symbol Z_p^m is an $N \times N$ operator-valued matrix that satisfies the sum-to-zero condition:

$$\sum_{j=1}^N (Z_p^m)_{jk} = \sum_{j=1}^N (Z_p^m)_{kj} = 0, \quad \text{for } k = 1, 2, \cdots, N. \tag{4.13}$$

As we have confirmed in Chapter 2, the generalized Lax equations exhibit that the operators (4.11) satisfy the commutation relations of the W-algebra [77, 78].

The operators with $m = 0$ are important in the construction of the eigenfunctions of the Hamiltonian, because they satisfy

$$\left[I_1, V_p^0\right] = pV_p^0, \quad \text{for } p = 1, 2, \cdots. \tag{4.14}$$

Thus the operators V_p^0 play the same role as the creation operator in the theory of the quantum harmonic oscillator. We call these mutually commuting operators V_p^0 power-sum creation operators, whose meaning has been explained in Chapters 2 and 3.

Successive operations of the power sum creation operators generate all the eigenfunctions of the Hamiltonian, which are labeled by the Young tableaux. The Young tableau λ is a non-increasing sequence of N nonnegative integers:

$$\lambda = \{\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_N \geq 0\}. \tag{4.15}$$

Then the polynomial part of the excited state ϕ_λ is given by [76, 77]

$$\begin{aligned}\phi_\lambda &= (V_N^0)^{\lambda_N}(V_{N-1}^0)^{\lambda_{N-1}-\lambda_N}\cdots(V_1^0)^{\lambda_1-\lambda_2}\cdot 1\\ &= \prod_{p=1}^N (V_p^0)^{\lambda_k-\lambda_{k+1}}\cdot 1,\end{aligned} \tag{4.16}$$

where $\lambda_{N+1} \equiv 0$ and the eigenvalue for the first conserved operator is

$$\begin{aligned}I_1\phi_\lambda &= \sum_{k=1}^N \lambda_k\phi_\lambda\\ &= E_1(\lambda)\phi_\lambda.\end{aligned} \tag{4.17}$$

It is nontrivial that ϕ_λ is indeed an inhomogeneous symmetric polynomial. We shall prove it in Section 4.3. Note that the eigenfunction of the original Hamiltonian $\hat{H}_C$ (4.1) and its eigenvalue are $\hat{\phi}_\lambda = \hat{\phi}_g\phi_\lambda$ and $\omega E_1(\lambda) + E_g$. These eigenfunctions give the complete set of the eigenfunctions. However, they are not orthogonal because of the remaining large degeneracy.

Using the Dunkl operator formalism, we can do an analogous investigation on the Calogero model. The Dunkl operators for the model are

$$\alpha_l = \mathrm{i}\Big(p_l + \mathrm{i}a\sum_{\substack{k=1\\k\neq l}}^N \frac{1}{x_l - x_k}(K_{lk} - 1)\Big), \tag{4.18a}$$
$$\alpha_l^\dagger = -\frac{\mathrm{i}}{2\omega}\Big(p_l + \mathrm{i}a\sum_{\substack{k=1\\k\neq l}}^N \frac{1}{x_l - x_k}(K_{lk} - 1) + 2\mathrm{i}\omega x_l\Big), \tag{4.18b}$$
$$d_l = \alpha_l^\dagger\alpha_l, \tag{4.18c}$$

where K_{lk} is the coordinate exchange operator. The operator K_{lk} has the properties

$$\begin{aligned}&K_{lk} = K_{kl}, \ (K_{lk})^2 = 1, \ K_{lk}^\dagger = K_{lk}, \ K_{lk}\cdot 1 = 1,\\ &K_{lk}A_l = A_kK_{lk}, \ K_{lk}A_j = A_jK_{lk}, \ \text{for } j \neq l, k,\end{aligned} \tag{4.19}$$

where A_j is either a partial differential operator $\dfrac{\partial}{\partial x_j}$ (or equivalently, a momentum operator p_j), a particle coordinate x_j or a coordinate exchange operator K_{jk}, $k = 1, 2, \cdots, N$, $k \neq j$. The above properties of the coordinate exchange operator are also expressed as the action on a multivariable function:

$$(K_{lk}f)(x_1, \cdots, x_l, \cdots, x_k, \cdots, x_N) = f(x_1, \cdots, x_k, \cdots, x_l, \cdots, x_N). \tag{4.20}$$

Note that the action on the ground state of the above Dunkl operators has already been removed by the similarity transformation (4.9). The Dunkl operators satisfy the relations,

$$[\alpha_l, \alpha_m] = 0, \quad [\alpha_l^\dagger, \alpha_m^\dagger] = 0, \tag{4.21a}$$
$$[\alpha_l, \alpha_m^\dagger] = \delta_{lm}\Big(1 + a\sum_{\substack{k=1\\k\neq l}}^N K_{lk}\Big) - a(1 - \delta_{lm})K_{lm}, \tag{4.21b}$$
$$[d_l, d_m] = a(d_m - d_l)K_{lm}, \tag{4.21c}$$
$$\alpha_l\cdot 1 = 0. \tag{4.21d}$$

As we have mentioned, the phase factor of the difference product part of the ground state wave function can be arbitrary. This phase factor affects the definition of the Dunkl operators and coordinate exchange operators with hat, i.e., $\hat{\alpha}_l$, $\hat{\alpha}_l^\dagger$, $\hat{d}_l$ and $\hat{K}_{lk}$. We have to introduce a phase factor in the defining relations of the coordinate exchange operators (4.19), the action on the multivariable functions (4.20) and the commutation relations of the Dunkl operators (4.21), as we have seen in Chapter 2 [80]. This modification is naturally introduced by the inverse of the similarity transformation of the Dunkl operators (4.9b). Using the above relations, we can confirm that the eigenfunctions of the Hamiltonian I_1 are given by [17, 18]

$$\begin{aligned} \varphi_\lambda &= \sum_{\substack{\sigma:\in S_N \text{ distinct} \\ \text{permutation}}} (\alpha^\dagger_{\sigma(1)})^{\lambda_1}(\alpha^\dagger_{\sigma(2)})^{\lambda_2}\cdots(\alpha^\dagger_{\sigma(N)})^{\lambda_N}\cdot 1 \\ &= m_\lambda(\alpha_1^\dagger,\alpha_2^\dagger,\cdots,\alpha_N^\dagger)\cdot 1, && (4.22a) \\ I_1\varphi_\lambda &= E_1(\lambda)\varphi_\lambda, && (4.22b) \end{aligned}$$

where $m_\lambda(x_1,x_2,\cdots,x_N)$ is a monomial symmetric polynomial [50, 69] defined by

$$m_\lambda(x_1,x_2,\cdots,x_N) = \sum_{\substack{\sigma\in S_N:\text{ distinct} \\ \text{permutation}}} (x_{\sigma(1)})^{\lambda_1}(x_{\sigma(2)})^{\lambda_2}\cdots(x_{\sigma(N)})^{\lambda_N}. \tag{4.23}$$

Note that the summation over S_N is performed so that any monomial in the summand appears only once. In terms of the Dunkl operators, we can express the commuting conserved operators I_n (4.10) and the operators V_p^m (4.11) as

$$I_n = \sum_{l=1}^N (d_l)^n\Big|_{\text{Sym}}, \quad n=1,2,\cdots,N, \tag{4.24}$$

$$V_p^m = \sum_{l=1}^N \Big[(\alpha_l)^m(\alpha_l^\dagger)^p\Big]_{\text{W}}\Big|_{\text{Sym}}, \tag{4.25}$$

where the symbol $\Big|_{\text{Sym}}$ means that the operand is restricted to symmetric functions [80]. Then the power sum creation operators are expressed by

$$V_n^0 = \sum_{l=1}^N(\alpha_l^\dagger)^n\Big|_{\text{Sym}} = \mathsf{p}_n(\alpha_1^\dagger,\alpha_2^\dagger,\cdots,\alpha_N^\dagger)\Big|_{\text{Sym}}, \tag{4.26}$$

where $\mathsf{p}_n(x_1,x_2,\cdots,x_N)$ is the power sum symmetric polynomial of degree n [50, 69]. This shows that two kinds of eigenfunctions (4.16) and (4.22) are related by the transformation between the power sum symmetric polynomials and the monomial symmetric polynomials.

Next, we consider the quantum Lax and the Dunkl operator formulations for the Sutherland model. The Lax matrix for the Sutherland model is

$$\tilde{\mathcal{L}}_{jk} = p_j\delta_{jk} + \mathrm{i}a(1-\delta_{jk})\cot(x_j - x_k). \tag{4.27}$$

The above Lax matrix gives the Hamiltonian of the Sutherland model by

$$\begin{aligned} \tilde{H}_{\text{S}} &= \frac{1}{2}\mathrm{T}_\Sigma\tilde{\mathcal{L}}^2 + \frac{1}{6}a^2N(N-1)(N+1) \\ &= \frac{1}{2}\mathrm{T}_\Sigma\tilde{\mathcal{L}}^2 + \epsilon_{\text{g}}, && (4.28) \end{aligned}$$

where ϵ_{g} is the ground state energy. As is similar to eq. (4.6), the ground state of the Sutherland model satisfies the following equations,

$$\sum_{k=1}^N \tilde{\mathcal{L}}_{jk}\tilde{\psi}_{\text{g}} = 0, \text{ for } j=1,2,\cdots,N \Rightarrow \tilde{H}_{\text{S}}\tilde{\psi}_{\text{g}} = \epsilon_{\text{g}}\tilde{\psi}_{\text{g}}, \tag{4.29}$$

because the first term of the r.h.s. of eq. (4.28) is a nonnegative operator. The ground state is given by the trigonometric Jastraw wave function:

$$\tilde{\psi}_{\text{g}} = \prod_{1\le j<k\le N}|\sin(x_j-x_k)|^a. \tag{4.30}$$

The phase factor of the above trigonometric Jastraw wave function can be arbitrary. By the change of the variables,

$$\exp 2\mathrm{i}x_j = z_j, \quad j=1,2,\cdots,N, \tag{4.31}$$

the Hamiltonian of the Sutherland model (4.2) is transformed to

$$\tilde{H}_{\text{S}} = -2\Big(\sum_{j=1}^N(z_jp_{z_j})^2 + (a^2-a)\sum_{\substack{j,k=1\\ j\ne k}}^N\frac{z_jz_k}{(z_j-z_k)^2}\Big), \tag{4.32}$$

where $p_{z_j} = -\mathrm{i}\dfrac{\partial}{\partial z_j}$. The ground state wave function (4.30) is transformed to

$$\tilde{\psi}_{\text{g}} = \prod_{1\le j<k\le N}|z_j-z_k|^a\prod_{j=1}^N z_j^{-\frac{1}{2}a(N-1)}. \tag{4.33}$$

Here we do not mind the difference of the scalar factor of the ground state wave function. The similarity transformation of the above Hamiltonian yields

$$\begin{aligned} H_{\text{S}} - \epsilon_{\text{g}} &= \tilde{\psi}_{\text{g}}^{-1}(\tilde{H}_{\text{S}}-\epsilon_{\text{g}})\tilde{\psi}_{\text{g}} \\ &= -2\sum_{j=1}^N(z_jp_{z_j})^2 + \mathrm{i}a\sum_{\substack{j,k=1\\ j\ne k}}^N\frac{z_j+z_k}{z_j-z_k}(z_jp_{z_j} - z_kp_{z_k}). && (4.34) \end{aligned}$$

The above projected Hamiltonian can be derived from the Lax matrices (4.3). We define the "ground state" for the $\hat{L}$-matrix (4.3a) by the solution of the equations,

$$\sum_{k=1}^N \hat{L}_{jk}\Delta^a = 0, \text{ for } j=1,2,\cdots,N, \tag{4.35a}$$

and their solution is

$$\Delta^a = \prod_{1\le j<k\le N}|x_j-x_k|^a. \tag{4.35b}$$

The phase of the above Jastraw wave function also can be arbitrary. The effect of the phase to the Dunkl operators for the Sutherland model, which can be made explicit by the similarity transformation, is also the same as that of the Calogero model. By the similarity transformation using the above Jastraw function, we define L and Q as

$$\begin{aligned} L &= \Delta^{-a}\check{L}\Delta^{a}, && (4.36a)\\ Q &= \Delta^{-a}\check{Q}\Delta^{a} = \check{Q}. && (4.36b)\end{aligned}$$

Then we get the projected Hamiltonian (4.34), whose variables are not $\{z_j\}$ but $\{x_j\}$ by

$$\begin{aligned} H_{\mathrm{S}} - \epsilon_{\mathrm{g}} &= 2\,\mathrm{T}_{\Sigma}(QL)^2 \\ &\stackrel{\text{def}}{=} 2\mathcal{I}_2. && (4.37)\end{aligned}$$

From now on, we take $\mathcal{I}_2$ as the Hamiltonian of the Sutherland model. The Jack symmetric polynomials $J_\lambda(\boldsymbol{x};1/a)$ are uniquely defined by [69]

$$\mathcal{I}_2 J_\lambda(\boldsymbol{x};1/a) = \sum_{k=1}^{N}\Big(\lambda_k^2 + a(N+1-2k)\lambda_k\Big)J_\lambda(\boldsymbol{x};1/a), \tag{4.38a}$$

$$J_\lambda(\boldsymbol{x};1/a) = \sum_{\mu\overset{\mathrm{D}}{\leq}\lambda} v_{\lambda\mu}(a)m_\mu(\boldsymbol{x}), \tag{4.38b}$$

$$v_{\lambda\lambda}(a) = 1, \tag{4.38c}$$

where $\boldsymbol{x} = (x_1, x_2, \cdots, x_N)$ and λ and μ are the Young tableaux (4.15). The symbol $\overset{\mathrm{D}}{\leq}$ is the dominance order among the Young tableaux [50, 69]:

$$\mu\overset{\mathrm{D}}{\leq}\lambda \Leftrightarrow \sum_{k=1}^{N}\mu_k = \sum_{k=1}^{N}\lambda_k \text{ and } \sum_{k=1}^{l}\mu_k \leq \sum_{k=1}^{l}\lambda_k \text{ for all } l. \tag{4.39}$$

Note that the dominance order is not a total order but a partial order. A total order among the Young tableaux is given by the lexicographic order:

$$\mu \leq \lambda \Leftrightarrow \sum_{k=1}^{N}\mu_k = \sum_{k=1}^{N}\lambda_k \text{ and the first non-vanishing difference } \lambda_l - \mu_l > 0. \tag{4.40}$$

Commuting conserved operators of the Sutherland model are given by

$$\mathcal{I}_n = \mathrm{T}_{\Sigma}(QL)^n. \tag{4.41}$$

We have similar relations to the generalized Lax equations for the Calogero model (4.12),

$$\begin{aligned} \big[U_q^l, L\big] &= \big[L, Y_q^l\big] - q[L^lQ^{q-1}]_{\mathrm{W}}, && (4.42a)\\ \big[U_q^l, Q\big] &= \big[Q, Y_q^l\big] + m[L^lQ^q]_{\mathrm{W}}, && (4.42b)\end{aligned}$$

where the operator U_q^l is defined by

$$U_q^l = \mathrm{T}_{\Sigma}[L^lQ^q]_{\mathrm{W}}. \tag{4.43}$$

The operator-valued matrix Y_q^l also satisfies the sum-to-zero condition,

$$\sum_{j=1}^{N}(Y_q^l)_{jk} = \sum_{j=1}^{N}(Y_q^l)_{kj} = 0, \quad \text{for } k = 1, 2, \cdots, N. \tag{4.44}$$

Since eqs. (4.12) and (4.42) has the same form, we notice the correspondence between the quantum Lax formulation of the Calogero model and that of the Sutherland model:

$$L^- = L, L^+ \leftrightarrow Q, I_k \leftrightarrow \mathcal{I}_k. \tag{4.45}$$

This means that the two quantum Lax formulations for the Calogero and Sutherland models give two different representations of the same commutator algebra.

The same situation can also be observed in the Dunkl operator formulation. We introduce the Dunkl operators for the Sutherland model whose action on the ground state is removed in a similar way to deal with the Dunkl operators for the Calogero model:

$$\begin{aligned} \nabla_l &= \mathrm{i}\Big(p_l + \mathrm{i}a\sum_{\substack{k=1\\k\neq l}}^{N}\frac{1}{x_l - x_k}(K_{lk} - 1)\Big), && (4.46a)\\ &x_l, && (4.46b)\\ D_l &= x_l\nabla_l. && (4.46c)\end{aligned}$$

These Dunkl operators satisfies the following relations,

$$\begin{aligned} &[\nabla_l, \nabla_m] = 0, \quad [x_l, x_m] = 0, && (4.47a)\\ &[\nabla_l, x_m] = \delta_{lm}\Big(1 + a\sum_{\substack{k=1\\k\neq l}}^{N}K_{lk}\Big) - a(1-\delta_{lm})K_{lm}, && (4.47b)\\ &[D_l, D_m] = a(D_m - D_l)K_{lm}, && (4.47c)\\ &\nabla_l\cdot 1 = 0, && (4.47d)\end{aligned}$$

which are completely the same as those of Dunkl operators for the Calogero model (4.21). Commuting conserved operators (4.41) are expressed by the Dunkl operator as

$$\mathcal{I}_n = \sum_{l=1}^{N}(D_l)^n\Big|_{\mathrm{Sym}}, \quad n = 1, 2, \cdots, N. \tag{4.48}$$

Thus we notice the correspondence between the two sets of Dunkl operators:

$$\alpha_l = \nabla_l, \alpha_l^\dagger \leftrightarrow x_l, d_l \leftrightarrow D_l. \tag{4.49}$$

Moreover, in the limit $\omega \to \infty$, the Lax matrices and the Dunkl operators for the Calogero model reduce to those for the Sutherland model. Thus our theory for the Hi-Jack symmetric polynomials described by the Dunkl operators for the Calogero model contains the results for the Jack symmetric polynomials written by the Dunkl operators for the Sutherland model.

We have summarized the quantum Lax and the Dunkl operator formulations for the Calogero and Sutherland models. They give two different representations of the same commutator algebra. The quantum Lax formulation and the Dunkl operators for the Calogero model include those for the Sutherland model as a special case $\omega \to \infty$. Thus we can say that the theory of the Calogero model and the Hi-Jack polynomials is a one-parameter deformation of that of the Sutherland model and the Jack polynomials. In the following section, we investigate the Hi-Jack polynomials using the Dunkl operators.

4.2 Hi-Jack Symmetric Polynomials

Following the definition of the Jack symmetric polynomials (4.38), we define the Hi-Jack symmetric polynomials $j_\lambda(\boldsymbol{x};\omega,1/a)$.

Definition 4. 1 *The Hi-Jack polynomials $j_\lambda(\boldsymbol{x};\omega,1/a)$ are uniquely specified by the following four conditions:*

$$\begin{aligned}
I_1 j_\lambda(\boldsymbol{x};\omega,1/a) &= \sum_{k=1}^{N}\lambda_k j_\lambda(\boldsymbol{x};\omega,1/a)\\
&= E_1(\lambda) j_\lambda(\boldsymbol{x};\omega,1/a), \qquad (4.50a)\\
I_2 j_\lambda(\boldsymbol{x};\omega,1/a) &= \sum_{k=1}^{N}\Big(\lambda_k^2 + a(N+1-2k)\lambda_k\Big) j_\lambda(\boldsymbol{x};\omega,1/a)\\
&= E_2(\lambda) j_\lambda(\boldsymbol{x};\omega,1/a), \qquad (4.50b)\\
j_\lambda(\boldsymbol{x};\omega,1/a) &= \sum_{\substack{\mu \overset{\mathrm{D}}{\leq} \lambda \\ \text{or } |\mu|<|\lambda|}} w_{\lambda\mu}(a,1/2\omega) m_\mu(\boldsymbol{x}), \qquad (4.50c)\\
w_{\lambda\lambda}(a,1/2\omega) &= 1. \qquad (4.50d)
\end{aligned}$$

Here $\lambda = \{\lambda_1 \geq \lambda_2 \geq \cdots \lambda_N \geq 0\} \in Y_N$ is a Young tableau and Y_N means the set of all the Young tableaux. The symbol $|\lambda|$ is the weight of the Young tableau, $|\lambda| = \sum_{k=1}^{N}\lambda_k$.

The first two formulae (4.50a) and (4.50b) indicate that the Hi-Jack polynomials are the simultaneous eigenfunctions of the first two conserved operators of the Calogero model with specifying their eigenvalue formulae. The third property of the Hi-Jack polynomials (4.50c) is called triangularity. The last one is normalization. The first three conditions are based on the observation in Chapter 3. Triangularity of the Hi-Jack polynomials with respect to the dominance order plays an essential role in the unique identification of the Hi-Jack polynomials. Because of remaining degeneracy, an eigenfunction cannot be uniquely identified only by the eigenvalues for the first two out of the N commuting conserved operators. However, combining the eigenvalues and triangularity, we can uniquely identify the Hi-Jack polynomials.

Proposition 4. 2 *Let λ and μ be distinct Young tableaux. If the two eigenvalues for the Young tableaux are equal,*

$$E_1(\lambda) = E_1(\mu), \quad E_2(\lambda) = E_2(\mu), \qquad (4.51)$$

then we can not define the dominance order between the two Young tableaux λ and μ.

The proposition asserts that we can distinguish any two eigenfunctions which share common first two eigenvalues E_1 and E_2 by triangularity of the Hi-Jack polynomials.

In order to write down the Rodrigues formula for the Hi-Jack polynomials, it is convenient to introduce the following symbols (cf. eq. (4.18)):

$$\begin{aligned}
a_J^\dagger &= \prod_{j\in J} a_j^\dagger, \qquad (4.52a)\\
d_{m,J} &= (d_{j_1}+ma)(d_{j_2}+(m+1)a)\cdots(d_{j_k}+(m+k-1)a), \qquad (4.52b)
\end{aligned}$$

where J is a subset of a set $\{1,2,\cdots,N\}$ whose number of elements $|J|$ is equal to k, $J \subseteq \{1,2,\cdots,N\}$, $|J| = k$. From eq. (4.21c), we can verify an identity,

$$(d_i+ma)(d_j+(m+1)a)\Big|_{\mathrm{Sym}}^{\{i,j\}} = (d_j+ma)(d_i+(m+1)a)\Big|_{\mathrm{Sym}}^{\{i,j\}}, \qquad (4.53)$$

where m is an integer. The symbol $\Big|_{\mathrm{Sym}}^{J}$ where J is some set of integers means that the operand is restricted to the space of the multivariable functions that are symmetric with respect to the exchanges of any indices in the set J. This identity guarantees that the operator $d_{m,J}$ does not depend on the order of the elements of a set J. The raising operators of the Hi-Jack polynomials are expressed as

$$\begin{aligned}
b_k^+ &= \sum_{\substack{J\subseteq\{1,2,\cdots,N\}\\ |J|=k}} a_J^\dagger d_{1,J}, \text{ for } k = 1,2,\cdots,N-1, \qquad (4.54a)\\
b_N^+ &= a_1^\dagger a_2^\dagger \cdots a_N^\dagger. \qquad (4.54b)
\end{aligned}$$

Using the raising operators (4.54), we can write down the Rodrigues formula for the Hi-Jack polynomials $j_\lambda(\boldsymbol{x};\omega,1/a)$ as

$$j_\lambda(\boldsymbol{x};\omega,1/a) = C_\lambda^{-1}(b_N^+)^{\lambda_N}(b_{N-1}^+)^{\lambda_{N-1}-\lambda_N}\cdots(b_1^+)^{\lambda_1-\lambda_2}\cdot 1, \qquad (4.55)$$

with the constant C_λ given by

$$C_\lambda = \prod_{k=1}^{N-1} C_k(\lambda_1,\lambda_2,\cdots,\lambda_{k+1};a), \qquad (4.56)$$

where

$$C_k(\lambda_1,\lambda_2,\cdots,\lambda_{k+1};a) = (a)_{\lambda_k-\lambda_{k+1}}(2a+\lambda_{k-1}-\lambda_k)_{\lambda_k-\lambda_{k+1}}\cdots(ka+\lambda_1-\lambda_k)_{\lambda_k-\lambda_{k+1}}. \qquad (4.57)$$

In the above expression, the symbol $(\beta)_n$ is the Pochhammer symbol, that is, $(\beta)_n = \beta(\beta+1)\cdots(\beta+n-1)$, $(\beta)_0 \overset{\mathrm{def}}{=} 1$. What we want to prove is summarized as the following proposition.

Proposition 4. 3 *The symmetric polynomials generated by the Rodrigues formula (4.55) satisfy the definition of the Hi-Jack symmetric polynomials (4.50).*

The first two out of four requirements (4.50) are derived from the following propositions.

Proposition 4. 4

$$\left[I_1, b_k^+\right]\Big|_{\mathrm{Sym}} = k b_k^+\Big|_{\mathrm{Sym}}. \qquad (4.58)$$

Proposition 4. 5 *The null operators $n_{i+1,J}$, which are defined by*

$$n_{k+1,J} = d_{0,J},\ J \subseteq \{1,2,\cdots,N\},\ |J| = k+1, \qquad (4.59)$$

satisfy

$$n_{k+1,J}(b_k^+)^{\lambda_k}(b_{k-1}^+)^{\lambda_{k-1}-\lambda_k}\cdots(b_1^+)^{\lambda_1-\lambda_2}\cdot 1 = 0. \qquad (4.60)$$

Proposition 4. 6

$$\left[I_2, b_k^+\right]\Big|_{\mathsf{Sym}} = \Big\{b_k^+\Big(2I_1 + k + ak(N-k)\Big) + \sum_{\substack{J\subseteq\{1,2,\cdots,N\}\\ |J|=k+1}} g_{k+1,J}n_{k+1,J}\Big\}\Big|_{\mathsf{Sym}}. \tag{4.61}$$

Here $g_{k+1,J}$ is an unspecified nonsingular operator that satisfies $g_{N+1,J}=0$.

The first requirement directly follows from Proposition 4. 4 . For a while, we forget about the normalization constant. From the l.h.s. of eq. (4.50a), we get

$$\begin{aligned} &I_1(b_N^+)^{\lambda_N}(b_{N-1}^+)^{\lambda_{N-1}-\lambda_N}\cdots(b_1^+)^{\lambda_1-\lambda_2}\cdot 1\\ &= \Big(\left[I_1, (b_N^+)^{\lambda_N}(b_{N-1}^+)^{\lambda_{N-1}-\lambda_N}\cdots(b_1^+)^{\lambda_1-\lambda_2}\right] + (b_N^+)^{\lambda_N}(b_{N-1}^+)^{\lambda_{N-1}-\lambda_N}\cdots(b_1^+)^{\lambda_1-\lambda_2}I_1\Big)\cdot 1. \end{aligned} \tag{4.62}$$

Because of eq. (4.21d), the second term of the above equation vanishes:

$$I_1\cdot 1 = \sum_{k=1}^{N}\alpha_k^\dagger\alpha_k\cdot 1 = 0. \tag{4.63}$$

Then using Proposition 4. 4 , we get the expected result:

$$\begin{aligned} &I_1(b_N^+)^{\lambda_N}(b_{N-1}^+)^{\lambda_{N-1}-\lambda_N}\cdots(b_1^+)^{\lambda_1-\lambda_2}\cdot 1\\ &= \Big(\sum_{k=1}^{N-1}k(\lambda_k-\lambda_{k+1}) + N\lambda_N\Big)(b_N^+)^{\lambda_N}(b_{N-1}^+)^{\lambda_{N-1}-\lambda_N}\cdots(b_1^+)^{\lambda_1-\lambda_2}\cdot 1\\ &= \sum_{k=1}^{N}\lambda_k(b_N^+)^{\lambda_N}(b_{N-1}^+)^{\lambda_{N-1}-\lambda_N}\cdots(b_1^+)^{\lambda_1-\lambda_2}\cdot 1. \end{aligned} \tag{4.64}$$

The second requirement (4.50b) is shown by induction. It is easy to show it for $\lambda=0$,

$$I_2\cdot j_0(\boldsymbol{x};\omega,1/a) = E_2(0)j_0(\boldsymbol{x};\omega,1/a) = 0, \tag{4.65}$$

by using eq. (4.21d) because $j_0(\boldsymbol{x};\omega,1/a)$ is equal to 1 as a polynomial. By inductive assumption, eq. (4.50b) holds up to $\lambda=\{\lambda_1,\lambda_2,\cdots,\lambda_k,\underbrace{0,\cdots,0}_{N-k}\}$. Then for $\lambda=\{\lambda_1+1,\lambda_2+1,\cdots,\lambda_k+1,\underbrace{0,\cdots,0}_{N-k}\}$, we have

$$I_2b_k^+(b_k^+)^{\lambda_k}(b_{k-1}^+)^{\lambda_{k-1}-\lambda_k}\cdots(b_1^+)^{\lambda_1-\lambda_2}\cdot 1 = \Big(\left[I_2,b_k^+\right] + b_k^+I_2\Big)(b_k^+)^{\lambda_k}(b_{k-1}^+)^{\lambda_{k-1}-\lambda_k}\cdots(b_1^+)^{\lambda_1-\lambda_2}\cdot 1. \tag{4.66}$$

From the inductive assumption and Proposition 4. 6 , we get

$$\begin{aligned} &I_2b_k^+(b_k^+)^{\lambda_k}(b_{k-1}^+)^{\lambda_{k-1}-\lambda_k}\cdot(b_1^+)^{\lambda_1-\lambda_2}\cdot 1\\ &= b_k^+\Big(E_2(\{\lambda_1,\lambda_2,\cdots,\lambda_k,0,\cdots,0\}) + 2\sum_{j=1}^{k}\lambda_j + k + ak(N-k)\Big)\\ &\quad\times(b_k^+)^{\lambda_k}(b_{k-1}^+)^{\lambda_{k-1}-\lambda_k}\cdots(b_1^+)^{\lambda_1-\lambda_2}\cdot 1\\ &= E_2(\{\lambda_1+1,\lambda_2+1,\cdots,\lambda_k+1,0,\cdots,0\})b_k^+(b_k^+)^{\lambda_k}(b_{k-1}^+)^{\lambda_{k-1}-\lambda_k}\cdots(b_1^+)^{\lambda_1-\lambda_2}\cdot 1. \end{aligned} \tag{4.67}$$

which completes the proof.

As byproducts of the proof of the last two requirements of Proposition 4. 3 , we notice the following results.

Proposition 4. 7 *The expansion coefficients of the Hi-Jack polynomials $C_\lambda w_{\lambda\mu}(a,1/2\omega)$ are polynomials of a and $1/2\omega$ with integer coefficients. This property is analogous to that stated by the Macdonald-Stanley conjecture for the Jack polynomials [41, 50, 69].*

Proposition 4. 8

$$J_\lambda(\alpha_1^\dagger,\alpha_2^\dagger,\cdots,\alpha_N^\dagger;1/a)\cdot 1 = j_\lambda(\boldsymbol{x};\omega,1/a). \tag{4.68}$$

We also notice the stronger form of the triangularity of the Hi-Jack polynomials (4.50c). We define the "weak" dominance order among the Young tableaux by

$$\mu\overset{\mathrm{d}}{\leq}\lambda \Leftrightarrow \sum_{k=1}^{l}\mu_k\leq\sum_{k=1}^{l}\lambda_k \text{ for all } l. \tag{4.69}$$

Namely, the definition of the dominance order (4.39) is given by adding another condition, $|\lambda|=|\mu|$ to the above weak dominance order. The stronger form of the triangularity (4.50c) is given by the following formula.

Proposition 4. 9

$$j_\lambda(\boldsymbol{x};\omega,1/a) = \sum_{\substack{\mu\overset{\mathrm{d}}{\leq}\lambda\\ \text{and } |\mu|\equiv|\lambda|\ (\mathrm{mod}2)}}\left(\frac{1}{2\omega}\right)^{(|\lambda|-|\mu|)/2}w_{\lambda\mu}(a)m_\mu(\boldsymbol{x}). \tag{4.70a}$$

$$w_{\lambda\lambda}(a) = 1. \tag{4.70b}$$

Then the following formula straightforwardly follows from the above.

Proposition 4. 10

$$j_\lambda(\boldsymbol{x};\omega,1/a) = J_\lambda(\boldsymbol{x};\omega,1/a) + \sum_{\substack{\mu\overset{\mathrm{d}}{\leq}\lambda \text{ and } |\mu|<|\lambda|\\ \text{and } |\mu|\equiv|\lambda|\ (\mathrm{mod}2)}}\left(\frac{1}{2\omega}\right)^{(|\lambda|-|\mu|)/2}\mathrm{w}_{\lambda\mu}(a)J_\mu(\boldsymbol{x};1/a). \tag{4.71}$$

During the discussion in Section 4.1, we have noticed that the Hi-Jack polynomials should reduce to the Jack polynomials in the limit $\omega\to\infty$:

$$j_\lambda(\boldsymbol{x};\omega=\infty,1/a) = J_\lambda(\boldsymbol{x};1/a). \tag{4.72}$$

In Proposition 4. 10 , the above relation is expressed in more detail. Proposition 4. 8 gives another relationship between the Jack polynomials and the Hi-Jack polynomials. The relationship between the eigenfunctions of the Hamiltonian H_C given by the QISM (4.16) or the eigenfunctions (4.22a) and the Hi-Jack polynomials is now clear. Several bases for the ring of homogeneous symmetric polynomials are known [50, 69]. The power sums, the monomial symmetric polynomials and the Jack polynomials are examples of such bases. Thus, the transformation between the Hi-Jack polynomials and two kinds of the eigenfunctions of the Hamiltonian H_C is the transformation between the bases of homogeneous symmetric polynomials. Defining the transformations by

$$f_\lambda^{pJ}(\{p_\lambda\}) = J_\lambda, \tag{4.73a}$$
$$f_\lambda^{mJ}(\{m_\lambda\}) = J_\lambda, \tag{4.73b}$$

we have

$$f_\lambda^{pJ}(\{\phi_\lambda\}) = j_\lambda, \tag{4.74a}$$
$$f_\lambda^{mJ}(\{\varphi_\lambda\}) = j_\lambda. \tag{4.74b}$$

Note that the transformation (4.73b) is nothing but the expansion of the Jack polynomials in the monomial symmetric polynomials (4.38b). Propositions 4. 7 , 4. 9 and 4. 10 and the triangularity of the Hi-Jack polynomials (4.50c) are observed in the explicit forms. The explicit forms of the first seven Hi-Jack polynomials are

$$\begin{aligned}
j_0(\boldsymbol{x};\omega,1/a) &= J_0(\boldsymbol{x};1/a) = m_0(\boldsymbol{x}) = 1, && (4.75a)\\
j_1(\boldsymbol{x};\omega,1/a) &= J_1(\boldsymbol{x};1/a) = m_1(\boldsymbol{x}), && (4.75b)\\
j_{1^2}(\boldsymbol{x};\omega,1/a) &= J_{1^2}(\boldsymbol{x};1/a) + \frac{a}{2\omega}\frac{N(N-1)}{2}J_0(\boldsymbol{x};1/a)\\
&= m_{1^2}(\boldsymbol{x}) + \frac{a}{2\omega}\frac{N(N-1)}{2}m_0(\boldsymbol{x}), && (4.75c)\\
(a+1)j_2(\boldsymbol{x};\omega,1/a) &= (a+1)J_2(\boldsymbol{x};1/a) - \frac{1}{2\omega}N(Na+1)J_0(\boldsymbol{x};1/a)\\
&= (a+1)m_2(\boldsymbol{x}) + 2am_{1^2}(\boldsymbol{x}) - \frac{1}{2\omega}N(Na+1)m_0(\boldsymbol{x}), && (4.75d)\\
j_{1^3}(\boldsymbol{x};\omega,1/a) &= J_{1^3}(\boldsymbol{x};1/a) + \frac{1}{2\omega}a\frac{(N-1)(N-2)}{2}J_1(\boldsymbol{x};1/a)\\
&= m_{1^3}(\boldsymbol{x}) + \frac{1}{2\omega}a\frac{(N-1)(N-2)}{2}m_1(\boldsymbol{x}), && (4.75e)\\
(2a+1)j_{2,1}(\boldsymbol{x};\omega,1/a) &= (2a+1)J_{2,1}(\boldsymbol{x};1/a)\\
&\quad -\frac{1}{2\omega}(1-a)(N-1)(Na+1)J_1(\boldsymbol{x};1/a)\\
&= (2a+1)m_{2,1}(\boldsymbol{x}) + 6am_{1^3}(\boldsymbol{x})\\
&\quad -\frac{1}{2\omega}(1-a)(N-1)(Na+1)m_1(\boldsymbol{x}), && (4.75f)\\
(a^2+3a+2)j_3(\boldsymbol{x};\omega,1/a) &= (a^2+3a+2)J_3(\boldsymbol{x};1/a) - \frac{3}{2\omega}(a^2N^2+3aN+2)J_1(\boldsymbol{x};1/a)\\
&= (a^2+3a+2)m_3(\boldsymbol{x}) + 3a(a+1)m_{2,1}(\boldsymbol{x}) + 6a^2m_{1^3}(\boldsymbol{x})\\
&\quad -\frac{3}{2\omega}(a^2N^2+3aN+2)m_1(\boldsymbol{x}). && (4.75g)
\end{aligned}$$

The explicit forms show the fact that the Hi-Jack polynomial is a one-parameter deformation of the Jack polynomial,

$$j_\lambda(\boldsymbol{x};\omega=\infty,1/a) = J_\lambda(\boldsymbol{x};1/a).$$

This has been clarified as eq. (4.72) by the discussion of the common algebraic structure of the Calogero and the Sutherland models in Section 4.1. The above Hi-Jack polynomials coincide with the seven orthogonal symmetric polynomials (3.34) given in Chapter 3 up to normalization.

Orthogonality of the Hi-Jack polynomials follows from the following proposition.

Proposition 4. 11 *The Hi-Jack polynomials are the simultaneous eigenfunctions of all the commuting conserved operators $\{I_n|n=1,\cdots,N\}$ with no degeneracy in the eigenvalues. Hence they are the orthogonal symmetric polynomials with respect to the following inner product:*

$$\begin{aligned}
\langle j_\lambda, j_\mu\rangle &= \int_{-\infty}^{\infty}\prod_{k=1}^{N}\mathrm{d}x_k|\hat{\phi}_g|^2 j_\lambda(\boldsymbol{x};\omega,1/a)j_\mu(\boldsymbol{x};\omega,1/a)\\
&\propto \delta_{\lambda,\mu}.
\end{aligned} \tag{4.76}$$

In the next section, we shall prove Propositions 4. 2 – 4. 11 .

4.3 Proofs

4.3.1 Unique Identification of the Hi-Jack Polynomials

In order to prove Proposition 4. 2 , we need a preparation. We introduce an elementary deformation which will be applied to the Young tableaux later. Let $\chi = \{\chi_1,\cdots,\chi_N\}$ be a set of N nonnegative integers and X_N be the set of all χ's, which includes Y_N, $Y_N \subset X_N$. Then we can introduce a deformation $R_l : X_N \to X_N$ for $1 \le l \le N-1$,

$$R_l(\{\chi_1,\cdots,\chi_l,\chi_{l+1},\cdots,\chi_N\}) = \{\chi_1,\cdots,\chi_l-1,\chi_{l+1}+1,\cdots,\chi_N\}, \tag{4.77}$$

when χ satisfies the condition $\chi_l > \chi_{l+1}$. Defining the dominance order in X_N by eq. (4.39), we notice

$$R_l(\chi)\overset{\mathrm{D}}{\le}\chi. \tag{4.78}$$

This means that the elementary deformation R_l generates a smaller set of N nonnegative integers from a set of N nonnegative integers. We want to show that successive applications of the elementary deformations to a Young tableau λ generate all the Young tableaux μ satisfying $\mu\overset{\mathrm{D}}{\le}\lambda$. Necessary properties of the elementary deformation are summarized in two lemmas, Lemmas 4. 12 and 4. 13 . The latter one will be used in the proof of Proposition 4. 2 .

The elementary deformation is not generally a deformation within the Young tableaux Y_N because a deformed tableau $R_l(\lambda)$ does not belong to Y_N when the Young tableau λ satisfies $\lambda_l = \lambda_{l+1}+1$. But we can show that successive applications of the elementary deformations give us an deformation among the Young tableaux Y_N.

Lemma 4. 12 *By iterative applications of the elementary deformations to a Young tableau* λ*, we can make* $\lambda' \in Y_N$ *that is the largest in the dominance order of all the Young tableaux* μ *that meet*

$$\mu \overset{\mathrm{D}}{\leq} \lambda, \ \mu_k = \lambda_k, \ \textit{for } 1 \leq k \leq l-1, \ \mu_l = \lambda_l - 1, \tag{4.79}$$

when λ *satisfies an inequality,*

$$\sum_{k=l}^{N} \lambda_k \leq (N-l+1)(\lambda_l - 1). \tag{4.80}$$

A proof of the above lemma is as follows. Since the Young tableau λ satisfies the condition (4.80), there is some suffix $k \geq l+1$ such that $\lambda_k \leq \lambda_l - 1$. We denote the minimum number of such suffices by $m+1$. This means that the Young tableau $\lambda \in Y_N$ satisfies

$$\lambda_{l+1} = \cdots = \lambda_m = \lambda_l > \lambda_l - 1 \geq \lambda_{m+1}.$$

Thus we can apply R_m to λ since $\lambda_m > \lambda_{m+1}$. Generally speaking, when you can apply R_k to a Young tableau μ, $R_k(\mu)$ satisfies

$$R_k(\mu)_{k+1} = \mu_{k+1} + 1 > \mu_{k+2} = R_k(\mu)_{k+2},$$

for $\mu_{k+1} \geq \mu_{k+2}$. Therefore we can define a successive operation of the elementary deformation, $R_{k+1} \circ R_k(\mu)$. By further successive operations, we have

$$R_n \circ \cdots R_{m+1} \circ R_m(\lambda), \ m \leq n \leq N-1,$$

for the Young tableau λ now we are dealing with. Due to the condition (4.80), there exists an integer n such that

$$R_n \circ \cdots \circ R_m(\lambda) \in Y_N, \ m \leq n \leq N-1. \tag{4.81}$$

If not, then the condition (4.81) is not satisfied for all n, $m \leq n \leq N-1$. This means

$$\lambda_{m+1} = \cdots = \lambda_N = \lambda_l - 1,$$

and the Young tableau λ breaks the condition (4.80). We denote the minimum of such integers by n_m. Then we get a Young tableau:

$$R_{n_m} \circ \cdots \circ R_m(\lambda) = \{\cdots, \lambda_m - 1, \cdots, \lambda_{n_m} + 1, \cdots\} \in Y_N, \tag{4.82a}$$

$$R_{n_m} \circ \cdots \circ R_m(\lambda)_k = \begin{cases} \lambda_k, & 1 \leq k \leq l-1 \\ \lambda_l, & l \leq k \leq m-1 \\ \lambda_l - 1, & m \leq k \leq n_m \\ \lambda_{n_m} + 1, & k = n_m + 1 \\ \lambda_k, & n_m + 2 \leq k \leq N \end{cases}. \tag{4.82b}$$

This Young tableau (4.82) satisfies the condition (4.80) because of the following relation:

$$\sum_{k=l}^{N} R_{n_m} \circ \cdots \circ R_m(\lambda)_k = \sum_{k=l}^{N} \lambda_k \leq (N-l+1)(\lambda_l - 1).$$

Thus following the discussions to find out the minimum suffices m and n_m for the Young tableau λ, we can find out minimum suffices m' and $n_{m'}$ for the Young tableau (4.82). From the explicit form (4.82b), we can identify the minimum suffices as

$$m' = m - 1 \geq l, \tag{4.83a}$$

$$\begin{aligned} n_{m'} &\overset{\mathrm{def}}{=} n_{m-1} \\ &= \begin{cases} n_m, & R_{n_m} \circ \cdots \circ R_m(\lambda)_{n_m} \\ & \quad > R_{n_m} \circ \cdots \circ R_m(\lambda)_{n_m+1} \\ n_m + 1, & R_{n_m} \circ \cdots \circ R_m(\lambda)_{n_m} \\ & \quad = R_{n_m} \circ \cdots \circ R_m(\lambda)_{n_m+1} \end{cases} \\ &\geq n_m, \end{aligned} \tag{4.83b}$$

when $m \geq l+1$. Then we get

$$R_{n_{m-1}} \circ \cdots \circ R_{m-1} \circ R_{n_m} \circ \cdots \circ R_m(\lambda) \in Y_N, \tag{4.84a}$$

$$\begin{aligned} & R_{n_{m-1}} \circ \cdots \circ R_{m-1} \circ R_{n_m} \circ \cdots \circ R_m(\lambda)_k \\ & = \begin{cases} \lambda_k, & 1 \leq k \leq l-1 \\ \lambda_l, & l \leq k \leq m-2 \\ \lambda_l - 1, & m-1 \leq k \leq n_{m-1} \\ R_{n_m} \circ \cdots \circ R_m(\lambda)_{n_{m-1}+1} + 1, & k = n_{m-1} + 1 \\ \lambda_k, & n_{m-1} + 2 \leq k \leq N \end{cases}. \end{aligned} \tag{4.84b}$$

Iteration of the same procedure until m' becomes l finally yields a Young tableau,

$$\lambda' = R_{n_l} \circ \cdots R_l \circ \cdots \circ R_{n_m} \circ \cdots \circ R_m(\lambda) \in Y_N, \tag{4.85}$$

where $l \leq m \leq n_m \leq \cdots \leq n_l$. By construction, we can readily see that the Young tableau λ' satisfies the condition (4.79). The explicit form of the Young tableau is

$$\lambda'_k = \begin{cases} \lambda_k, & 1 \leq k \leq l-1 \\ \lambda_l - 1, & l \leq k \leq n_l \\ |\lambda| - \displaystyle\sum_{\substack{k=1 \\ k \neq n_l+1}}^{N} \lambda'_k, & k = n_l + 1 \\ \lambda_k, & n_l + 2 \leq k \leq N-1 \end{cases}. \tag{4.86}$$

Let μ be an arbitrary Young tableau satisfying the condition (4.79). We shall compare the partial sums of the two Young tableaux, λ' and μ. From the condition (4.79), we see

$$\sum_{k=1}^{m} \lambda'_k = \sum_{k=1}^{m} \mu, \text{ for } 1 \leq m \leq l. \tag{4.87}$$

Due to the definition of the Young tableau and the explicit form of λ' (4.86), we get the following inequality:

$$\lambda'_k = \lambda_l - 1 = \mu_l \geq \mu_k, \text{ for } l+1 \leq k \leq n_l.$$

The explicit form of λ' (4.86) also shows

$$\sum_{k=1}^{m}\lambda'_k=\sum_{k=1}^{m}\lambda_k,\ \text{for } n_l+1\le m\le N.$$

The above two relations are combined to give

$$\sum_{k=1}^{m}\lambda'_k\ge\sum_{k=1}^{m}\mu,\ \text{for } l+1\le m\le N. \tag{4.88}$$

Thus eqs. (4.87) and (4.88) prove $\mu\overset{\mathrm{D}}{\le}\lambda'$, which means that the Young tableau λ' is the largest of all the Young tableaux satisfying the condition (4.79).

Next we show that we can make any Young tableau μ such that $\mu\overset{\mathrm{D}}{\le}\lambda$ by iterative applications of the elementary deformations (4.77) to a Young tableau λ.

Lemma 4. 13 *Let λ and μ be two Young tableaux, $\lambda,\mu\in Y_N$. If $\mu\overset{\mathrm{D}}{\le}\lambda$, then we can obtain the smaller Young tableau μ by successive applications of the elementary deformations (4.77) to the larger Young tableau λ.*

We assume μ is different from λ, because the case $\mu=\lambda$ is trivial. Then there is a minimum number of l such that the l-th partial sums of the two Young tableaux satisfy

$$\sum_{k=1}^{l}\lambda_k-\sum_{k=1}^{l}\mu_k=\Delta_l>0,$$

where Δ_l is some positive integer. For convenience, we add a superscript $(l-1)$ to λ to indicate that the elements of two tableaux $\lambda^{(l-1)}$ and μ are the same up to the $(l-1)$-th element. Since the two Young tableaux satisfy $\mu\overset{\mathrm{D}}{\le}\lambda^{(l-1)}$, we have

$$\begin{aligned}\sum_{k=l}^{N}\lambda_k &= |\lambda|-\sum_{k=1}^{l-1}\lambda_k\\ &\le |\mu|-\sum_{k=1}^{l-1}\mu_k\\ &= \sum_{k=l}^{N}\mu_k\\ &\le (N-l+1)\mu_l=(N-l+1)(\lambda_l-\Delta_l).\end{aligned}$$

The above inequality guarantees that we can apply the deformation shown in Lemma 4. 12 Δ_l times to the larger Young tableau λ. The deformation yields a Young tableau $\lambda^{(l)}$, which is the largest in the dominance order of all the Young tableaux that satisfy

$$\nu\overset{\mathrm{D}}{\le}\lambda,\ \nu_k=\lambda_k,\ 1\le k\le l-1, \nu_l=\lambda_l-\Delta_l. \tag{4.89}$$

The Young tableau μ also satisfies the condition (4.89). Thus we conclude that $\mu\overset{\mathrm{D}}{\le}\lambda^{(l)}$. This means that we can apply the same procedure to $\lambda^{(l)}$ again. Thus iteration of the procedure finally yields the Young tableau $\lambda^{(N)}$ that is equal to μ.

Now we are in position to prove Proposition 4. 2 . We use the reductive absurdity. From the assumption of the proposition, eq. (4.51), two different Young tableaux, $\lambda,\mu\in Y_N$, give the same first two eigenvalues. Assume we can define the dominance order between the two Young tableaux. Without loss of generality, we can assume $\mu\overset{\mathrm{D}}{\le}\lambda$. Then from Lemma 4. 13 , there must be a way of applying iteratively the elementary deformations to the larger Young tableau λ that yields the smaller Young tableau μ, say $R_{l'}\circ\cdots\circ R_l(\lambda)=\mu$. The iterative applications consist of the elementary deformations (4.77). Since the elementary deformation does not change the weight, the first eigenvalues E_1 for any $\chi\in X_N$ and well-defined $R_l(\chi)$ are the same. Comparing the second eigenvalues for χ and $R_l(\chi)$, we have

$$E_2(\chi)-E_2(R_l(\chi))=2(\chi_l-\chi_{l+1}-1)+2a>0,$$

because the inequality $\chi_l>\chi_{l+1}$ holds to make $R_l(\chi)$ well-defined. This inequality shows that iterative applications of the elementary deformations R_k to χ monotonically decrease the second eigenvalue E_2 for the deformed χ. Thus the second eigenvalues for any pairs of distinct Young tableaux λ and μ that meet $\mu\overset{\mathrm{D}}{\le}\lambda$ must satisfy the inequality,

$$E_2(\lambda)>E_2\big(R_{l'}\circ\cdots\circ R_l(\lambda)\big)=E_2(\mu).$$

This is contradictory to the assumption of the proposition, eq. (4.51). Thus we have proved the proposition. Proposition 4. 2 is essential to the unique identification of the Hi-Jack polynomials just by diagonalizing the first two commuting conserved operators. As a result, it will implicitly play an important role to show that the Hi-Jack polynomials are the simultaneous eigenfunctions of all the commuting conserved operators.

4.3.2 Hamiltonian

We shall prove Proposition 4. 4 . It is easy to prove the case $k=N$. From the definition of I_1 and b_N^+, we have

$$\begin{aligned}\big[I_1,b_N^+\big] &= \sum_{j=1}^{N}\big[d_j,\alpha_1^\dagger\alpha_2^\dagger\cdots\alpha_N^\dagger\big]\\ &= \sum_{j=1}^{N}\sum_{l=1}^{N}\alpha_1^\dagger\cdots\alpha_{l-1}^\dagger\big[d_j,\alpha_l^\dagger\big]\alpha_{l+1}^\dagger\cdots\alpha_N^\dagger.\end{aligned} \tag{4.90}$$

Using eq. (4.21c), we get

$$\begin{aligned}\big[I_1,b_N^+\big] &= \sum_{j=1}^{N}\sum_{l=1}^{N}\alpha_1^\dagger\cdots\alpha_{l-1}^\dagger\alpha_j^\dagger\big(\delta_{jl}(1+a\sum_{\substack{i=1\\ i\ne j}}^{N}K_{ji})-a(1-\delta_{jl})K_{jl}\big)\alpha_{l+1}^\dagger\cdots\alpha_N^\dagger\\ &= \big(Nb_N^++a\alpha_1^\dagger\cdots\alpha_{l-1}^\dagger\sum_{\substack{j,l=1\\ j\ne l}}^{N}\alpha_j^\dagger(K_{jl}-K_{jl})\alpha_{l+1}^\dagger\cdots\alpha_N^\dagger\big)\\ &= Nb_N^+,\end{aligned} \tag{4.91}$$

which says the validity of eq. (4.58) for the case $k=N$. Note that we do not have to restrict the operand in the above calculation.

For the case $k \neq N$, we need more computation. First, we decompose the Hamiltonian I_1 into two parts:

$$\left[I_1, b_k^+\right] = \sum_{\substack{J\subseteq\{1,2,\cdots,N\} \\ |J|=k}} \left[\sum_{i\in J} d_i + \sum_{i\notin J} d_i, \alpha_J^\dagger d_{1,J}\right]. \tag{4.92}$$

The first part of the r.h.s. of eq. (4.92) is calculated as

$$\begin{aligned}
&\left[\sum_{i\in J} d_i, \alpha_J^\dagger d_{1,J}\right] \\
&= \sum_{i\in J}\sum_{l=1}^{k}\Big\{\alpha_{j_1}^\dagger\cdots\alpha_{j_{l-1}}^\dagger\left[d_i,\alpha_{j_l}^\dagger\right]\alpha_{j_{l+1}}^\dagger\cdots\alpha_{l_k}^\dagger d_{1,J} \\
&\quad +\alpha_J^\dagger(d_{j_1}+a)\cdots(d_{j_{l-1}}+(l-1)a)\left[d_i, d_{j_l}+la\right](d_{j_{l+1}}+(l+1)a)\cdots(d_{l_k}+ka)\Big\} \\
&= \sum_{i\in J}\sum_{l=1}^{k}\Big\{\alpha_{j_1}^\dagger\cdots\alpha_{j_{l-1}}^\dagger\Big(\delta_{ij_l}\alpha_{j_l}^\dagger(1+a\sum_{\substack{j\in J\\ j\neq j_l}}K_{j_lj}+a\sum_{j\notin J}K_{j_lj})-\alpha_i^\dagger(1-\delta_{ij_l})aK_{ij_l}\Big)\alpha_{j_{l+1}}^\dagger\cdots\alpha_{l_k}^\dagger d_{1,J} \\
&\quad +\alpha_J^\dagger(d_{j_1}+a)\cdots(d_{j_{l-1}}+(l-1)a)\Big((d_{j_l}+la)-(d_i+la)\Big)K_{ij_l} \\
&\quad (d_{j_{l+1}}+(l+1)a)\cdots(d_{l_k}+ka)\Big\}.
\end{aligned} \tag{4.93}$$

We move exchange operators to the rightmost and utilize the restriction $\Big|_{\mathrm{Sym}}$. Using eqs. (4.19) and (4.53), we get

$$\begin{aligned}
&\left[\sum_{i\in J} d_i, \alpha_J^\dagger d_{1,J}\right]\Big|_{\mathrm{Sym}} \\
&= k\alpha_J d_{1,J}\Big|_{\mathrm{Sym}} + a\sum_{i\notin J}\sum_{l=1}^{k}\alpha_J^\dagger d_{1,J\setminus\{j_l\}}(d_i+ka)\Big|_{\mathrm{Sym}} \\
&\quad +a\sum_{l=1}^{k}\sum_{\substack{i\in J\\ i\neq j_l}}\Big\{\Big(\alpha_{J\setminus\{i\}}^\dagger\alpha_{j_l}^\dagger-\alpha_{J\setminus\{j_l\}}^\dagger\alpha_i^\dagger\Big)d_{1,J}+\alpha_J^\dagger\Big(d_{1,J\setminus\{i\}}(d_{j_l}+ka)-d_{1,J\setminus\{j_l\}}(d_i+ka)\Big)\Big\}\Big|_{\mathrm{Sym}} \\
&= k\alpha_J d_{1,J}\Big|_{\mathrm{Sym}} + a\sum_{i\notin J}\sum_{l=1}^{k}\alpha_J^\dagger d_{1,J\setminus\{j_l\}}(d_i+ka)\Big|_{\mathrm{Sym}}.
\end{aligned} \tag{4.94}$$

The second part of the r.h.s. of eq. (4.92) is calculated as

$$\begin{aligned}
&\left[\sum_{i\notin J} d_i, \alpha_J^\dagger d_{1,J}\right]\Big|_{\mathrm{Sym}} \\
&= \sum_{i\notin J}\sum_{l=1}^{k}\Big\{\alpha_{j_1}^\dagger\cdots\alpha_{j_{l-1}}^\dagger\left[d_i,\alpha_{j_l}^\dagger\right]\alpha_{j_{l+1}}^\dagger\cdots\alpha_{j_k}^\dagger d_{1,J} \\
&\quad +\alpha_J^\dagger(d_{j_1}+a)\cdots(d_{j_{l-1}}+(l-1)a)\left[d_i, d_{j_l}+la\right](d_{j_{l+1}}+(l+1)a)\cdots(d_{l_k}+ka)\Big\}\Big|_{\mathrm{Sym}} \\
&= \sum_{i\notin J}\sum_{l=1}^{k}\Big\{-a\alpha_{j_1}^\dagger\cdots\alpha_{j_{l-1}}^\dagger\alpha_i^\dagger K_{ij_l}\alpha_{j_{l+1}}^\dagger\cdots\alpha_{j_k}^\dagger d_{1,J} \\
&\quad +\alpha_J^\dagger(d_{j_1}+a)\cdots(d_{j_{l-1}}+(l-1)a)\Big((d_{j_l}+la)-(d_i+la)\Big)K_{ij_l} \\
&\qquad \times(d_{j_{l+1}}+(l+1)a)\cdots(d_{l_k}+ka)\Big\}\Big|_{\mathrm{Sym}} \\
&= \sum_{i\notin J}\sum_{l=1}^{k}\Big\{-a\alpha_{J\setminus\{j_l\}}^\dagger\alpha_i^\dagger d_{1,J\setminus\{j_l\}}(d_i+ka)+\alpha_J^\dagger\Big(d_{1,J}-d_{1,J\setminus\{j_l\}}(d_i+ka)\Big)\Big\}\Big|_{\mathrm{Sym}}.
\end{aligned} \tag{4.95}$$

Substitution of eqs. (4.94) and (4.95) into eq. (4.92) yields

$$\begin{aligned}
&\left[I_1, b_k^+\right]\Big|_{\mathrm{Sym}} \\
&= \sum_{\substack{J\subseteq\{1,2,\cdots,N\} \\ |J|=k}}\Big\{k\alpha_J d_{1,J} \\
&\quad +a\sum_{i\notin J}\sum_{l=1}^{k}\Big\{(\alpha_J^\dagger-\alpha_{J\setminus\{j_l\}}^\dagger\alpha_i^\dagger)d_{1,J\setminus\{j_l\}}(d_i+ka)+\alpha_J^\dagger\Big(d_{1,J}-d_{1,J\setminus\{j_l\}}(d_i+ka)\Big)\Big\}\Big\}\Big|_{\mathrm{Sym}} \\
&= kb_k^+\Big|_{\mathrm{Sym}}.
\end{aligned} \tag{4.96}$$

Thus we have proved Proposition 4. 4 .

4.3.3 Null Operators

We shall prove Proposition 4. 5 . Since the function $(b_k^+)^{\lambda_k}(b_{k-1}^+)^{\lambda_{k-1}-\lambda_k}\cdots(b_1^+)^{\lambda_1-\lambda_2}\cdot 1$ is a symmetric function of N variables $\{x_1,x_2,\cdots,x_N\}$, it is sufficient to prove the case $J=\{1,2,\cdots,k+1\}$. For brevity, we use the symbol $n_{k+1}=n_{k+1,\{1,\cdots,k+1\}}$ hereafter. Then the expression to be proved is

$$n_{k+1}(b_k^+)^{\lambda_k}(b_{k-1}^+)^{\lambda_{k-1}-\lambda_k}\cdots(b_1^+)^{\lambda_1-\lambda_2}\cdot 1 = 0. \tag{4.97}$$

This follows from

$$\left[n_{i+1}, b_k^+\right]\Big|_{\mathrm{Sym}} \sim n_{k+1}\Big|_{\mathrm{Sym}}, \text{ for } i \geq k, \tag{4.98}$$

where the symbol $\sim$ means that the term on the r.h.s. can be multiplied on the left by some nonsingular operator $\mathcal{O}$, $\mathcal{O}\cdot 0 = 0$. We can easily verify

$$\begin{aligned}
n_{k+1}(b_k^+)^{\lambda_k}(b_{k-1}^+)^{\lambda_{k-1}-\lambda_k}\cdots(b_1^+)^{\lambda_1-\lambda_2}\cdot 1 &\sim n_{k+1}(b_{k-1}^+)^{\lambda_{k-1}-\lambda_k}\cdots(b_1^+)^{\lambda_1-\lambda_2}\cdot 1 \\
&\ \vdots \\
&\sim n_2\cdot 1 \\
&= 0,
\end{aligned} \tag{4.99}$$

using eqs. (4.21d) and (4.98). For convenience of explanation, we introduce a symbol $[m]$ for a set $\{1,2,\cdots,m\}$ with an integer m. We also introduce $\Big|_{\mathrm{Sym}}^{J}$ with a set of integers J that indicates the operand is a symmetric function of x_j, $j\in J$. From the identity (4.53), we have

$$n_{k+1}\Big|_{\mathrm{Sym}}^{[k+1]} = n_k(d_{k+1}+ka)\Big|_{\mathrm{Sym}}^{[k+1]}$$

$$
\begin{aligned}
&= \left\{ kan_k + n_{k-1}d_{k+1}(d_k + (k-1)a) + an_{k-1}(d_{k+1} - d_k) \right\}\Big|_{\mathrm{Sym}}^{[k+1]} \\
&= \left\{ kan_k + (aK_{k\,k+1} - a)n_k + n_{k-1}d_{k+1}(d_k - (k-1)a) \right\}\Big|_{\mathrm{Sym}}^{[k+1]} \\
&\ \vdots \\
&= \Big\{ \Big((k - (k-1))a + a\sum_{j=2}^{k} K_{j\,k+1} \Big) n_k \\
&\quad + d_1 K_{1\,k+1} K_{1\,k+1} d_{k+1}(d_2 + a) \cdots (d_k + (k-1)a) \Big\}\Big|_{\mathrm{Sym}}^{[k+1]} \\
&= \Big\{ d_1 K_{1\,k+1} + a + a\sum_{j=2}^{k} K_{j\,k+1} \Big\} n_k\Big|_{\mathrm{Sym}}^{[k+1]}, \qquad (4.100)
\end{aligned}
$$

which means

$$
n_{k+1}\Big|_{\mathrm{Sym}}^{[k+1]} \sim n_k\Big|_{\mathrm{Sym}}^{[k+1]}. \qquad (4.101)
$$

Thanks to the above relation, eq. (4.98) for $i > k$ follows from that for $i = k$, i.e.,

$$
\left[n_{k+1}, b_k^+\right]\Big|_{\mathrm{Sym}} \sim n_{k+1}\Big|_{\mathrm{Sym}}. \qquad (4.102)
$$

We shall prove a stronger statement, namely,

Proposition 4. 14

$$
\left[n_{k+1}^{[M]}, b_k^{+[M]}\right]\Big|_{\mathrm{Sym}}^{[N]} \sim n_{k+1}^{[M]}\Big|_{\mathrm{Sym}}^{[N]}, \quad M \geq N. \qquad (4.103)
$$

Here, the superscript $[M]$ over Dunkl operators indicates that they are made from the Dunkl operators (4.18) that depend not only on the variables $x_1, x_2, \cdots, x_N$ but also on $x_{N+1}, \cdots, x_M$. Note that $n_{k+1}^{[M]}$ and $b_k^{+[M]}$ are symmetric under S_N but not under S_M. We just changed the number of variables of Dunkl operators (4.18) but do not change the definition of operators made from them, such as conserved operators (4.24) and raising operators (4.52) and (4.54). Namely, the indices and subsets in the summand of their definitions are included in the set $[N]$.

All the Dunkl operators in the remainder of this Section 4.3.3 will always depend on $x_1, \cdots, x_M$. We shall omit the superscript $[M]$ in the following.

To prove Proposition 4. 14 , we need several lemmas. We define the restricted raising operator by

$$
b_{k,J}^+ = \sum_{\substack{J' \subseteq J \\ |J'| = k}} \alpha_{J'}^\dagger d_{1,J'}, \qquad (4.104)
$$

where J is a set of integers. From the definition of the raising operator (4.54), we can easily verify

$$
b_k^+ \overset{\mathrm{def}}{=} \sum_{\substack{J \subseteq [N] \\ |J| = k}} \alpha_J^\dagger d_{1,J}
$$

$$
\begin{aligned}
&= \sum_{l=0}^{k} \sum_{\substack{J' \subseteq [k+1] \\ |J'| = k-l}} \sum_{\substack{J \subseteq \{k+2,\cdots,N\} \\ |J| = l}} \alpha_{J'}^\dagger \alpha_J^\dagger d_{1,J'} d_{k-l+1,J} \\
&= \sum_{l=0}^{k} \sum_{\substack{J \subseteq \{k+2,\cdots,N\} \\ |J| = l}} \alpha_J^\dagger b_{k-l,[k+1]}^+ d_{k-l+1,J}.
\end{aligned}
$$

Thus we have the following formula.

Lemma 4. 15

$$
b_k^+ = \sum_{l=0}^{k} \sum_{\substack{J \subseteq \{k+2,\cdots,N\} \\ |J| = l}} \alpha_J^\dagger b_{k-l,[k+1]}^+ d_{k-l+1,J}, \qquad (4.105)
$$

with $b_{0,J}^+ = 1$ *and when* $|J| = 0$, $\alpha_J^\dagger = 1$ *and* $d_{k+1,J} = 1$.

We shall also use the following formulae:

Lemma 4. 16

$$
\begin{aligned}
\left[d_i, \alpha_J^\dagger\right] &= -a\alpha_i^\dagger \sum_{j\in J} \alpha_{J\setminus\{j\}}^\dagger K_{ij}, \quad i \notin J, && (4.106\mathrm{a}) \\
\left[d_i, \alpha_J^\dagger\right] &= \alpha_J^\dagger\Big(1 + a\sum_{j\in[M]\setminus J} K_{ij}\Big), \quad i \in J, && (4.106\mathrm{b}) \\
\left[n_{k+1}, \alpha_l^\dagger\right]\Big|_{\mathrm{Sym}}^{[k+1]} &= -a\Big(\alpha_1^\dagger K_{12}K_{23}\cdots K_{k\,k+1}K_{k+1\,l} + \cdots + \alpha_{k+1}^\dagger K_{k+1\,l}\Big) n_k\Big|_{\mathrm{Sym}}^{[k+1]} \\
&\sim n_k\Big|_{\mathrm{Sym}}^{[k+1]}, \ l \notin [k+1]. && (4.106\mathrm{c})
\end{aligned}
$$

The first two formulae can be checked from the definition of d_i and $\alpha_J^\dagger$ and commutation relations (4.21):

$$
\begin{aligned}
\left[d_i, \alpha_J^\dagger\right] &= \alpha_i^\dagger \sum_{j\in J} \prod_{j'\in J\setminus\{j\}} \alpha_{j'}^\dagger \left[\alpha_i, \alpha_j^\dagger\right] \\
&= -a\alpha_i^\dagger \sum_{j\in J} \alpha_{J\setminus\{j\}}^\dagger K_{ij}, \quad i \notin J, \\
\left[d_i, \alpha_J^\dagger\right] &= \Big[\alpha_i^\dagger\alpha_i, \alpha_i^\dagger \prod_{j\in J\setminus\{i\}} \alpha_j^\dagger\Big] \\
&= \alpha_i^\dagger\left[\alpha_i, \alpha_i^\dagger\right]\alpha_{J\setminus\{i\}}^\dagger + \alpha_i^\dagger\left[\alpha_i^\dagger, \alpha_{J\setminus\{i\}}^\dagger\right] \\
&= \alpha_i^\dagger\Big(1 + a\sum_{\substack{k=1 \\ k\neq i}}^{M} K_{ik}\Big)\alpha_{J\setminus\{i\}}^\dagger - a\alpha_i^\dagger \sum_{j\in J\setminus\{i\}} \alpha_{J\setminus\{j\}}^\dagger K_{ij} \\
&= \alpha_J^\dagger + a\alpha_i^\dagger \sum_{k\in J\setminus\{i\}} \alpha_{J\setminus\{k\}}^\dagger K_{ik} + a\alpha_J^\dagger \sum_{k\in[M]\setminus J} K_{ik} - a\alpha_i^\dagger \sum_{j\in J\setminus\{i\}} \alpha_{J\setminus\{j\}}^\dagger K_{ij} \\
&= \alpha_J^\dagger\Big(1 + a\sum_{j\in[M]\setminus J} K_{ij}\Big), \quad i \in J.
\end{aligned}
$$

The last formula (4.106c) is proved by induction on k. By a straightforward calculation using the commutation relations (4.21) and the definition of the null operator (4.59), we can confirm the case $k=1$:

$$
\begin{aligned}
\Big[n_2, \alpha_l^\dagger\Big]\Big|_{\mathrm{Sym}}^{[2]} &= \Big(d_1\alpha_2^\dagger\big[\alpha_2, \alpha_l^\dagger\big] + \alpha_1^\dagger\big[\alpha_1, \alpha_l^\dagger\big](d_2+a)\Big)\Big|_{\mathrm{Sym}}^{[2]} \\
&= -a\Big(\alpha_1^\dagger\big[\alpha_1, \alpha_2^\dagger\big]K_{2l} + \alpha_2^\dagger d_1 K_{2l} + \alpha_1^\dagger K_{1l}(d_2+a)\Big)\Big|_{\mathrm{Sym}}^{[2]} \\
&= -a\Big(-a\alpha_1^\dagger K_{12}K_{2l} + \alpha_2^\dagger K_{2l}d_1 + \alpha_1^\dagger K_{1l}(d_2+a)K_{12}\Big)\Big|_{\mathrm{Sym}}^{[2]} \\
&= -a\Big(\big(\alpha_1^\dagger K_{12}K_{2l} + \alpha_2^\dagger K_{2l}\big)n_1 - a\alpha_1^\dagger K_{12}K_{2l} + a\alpha_1^\dagger K_{12}K_{2l}\Big)\Big|_{\mathrm{Sym}}^{[2]} \\
&= -a\Big(\alpha_1^\dagger K_{12}K_{2l} + \alpha_2^\dagger K_{2l}\Big)n_1\Big|_{\mathrm{Sym}}^{[2]} \sim n_1\Big|_{\mathrm{Sym}}^{[2]}, \quad l \notin [2].
\end{aligned}
$$

Note that we have used the relation $K_{12}\Big|_{\mathrm{Sym}}^{[2]} = 1\Big|_{\mathrm{Sym}}^{[2]}$ in the above calculation. Assuming the validity of eq. (4.106c) up to the case $k-1$, we can calculate the case k:

$$
\begin{aligned}
\Big[n_{k+1}, \alpha_l^\dagger\Big]\Big|_{\mathrm{Sym}}^{[k+1]} &= \Big(\big[n_k, \alpha_l^\dagger\big]\Big|_{\mathrm{Sym}}^{[k]}(d_{k+1}+ka) + n_k\big[d_{k+1}, \alpha_l^\dagger\big]\Big)\Big|_{\mathrm{Sym}}^{[k+1]} \\
&= -a\Big(\big(\alpha_1^\dagger K_{12}K_{23}\cdots K_{kl} + \cdots + \alpha_k^\dagger K_{kl}\big)n_{k-1}(d_{k+1}+ka) \\
&\quad + \big[n_k, \alpha_{k+1}^\dagger\big]\Big|_{\mathrm{Sym}}^{[k]} K_{k+1\,l} + \alpha_{k+1}^\dagger n_k K_{k+1\,l}\Big)\Big|_{\mathrm{Sym}}^{[k+1]} \\
&= -a\Big(\big(\alpha_1^\dagger K_{12}K_{23}\cdots K_{kl} + \cdots + \alpha_k^\dagger K_{kl}\big)n_{k-1}(d_{k+1}+ka) \\
&\quad -a\big(\alpha_1^\dagger K_{12}K_{23}\cdots K_{k\,k+1} + \cdots + \alpha_k^\dagger K_{k\,k+1}\big)K_{k+1\,l}n_{k-1} \\
&\quad + \alpha_{k+1}^\dagger K_{k+1\,l}n_k\Big)\Big|_{\mathrm{Sym}}^{[k+1]}, \quad l \notin [k+1].
\end{aligned}
$$

Using the merit of the restriction $K_{k\,k+1}\Big|_{\mathrm{Sym}}^{[k+1]} = 1\Big|_{\mathrm{Sym}}^{[k+1]}$, we have

$$
\begin{aligned}
&\Big[n_{k+1}, \alpha_l^\dagger\Big]\Big|_{\mathrm{Sym}}^{[k+1]} \\
&= -a\Big(\big(\alpha_1^\dagger K_{12}K_{23}\cdots K_{kl} + \cdots + \alpha_k^\dagger K_{kl}\big)n_{k-1}(d_{k+1}+ka)K_{k\,k+1} \\
&\quad -a\big(\alpha_1^\dagger K_{12}K_{23}\cdots K_{k\,k+1} + \cdots + \alpha_k^\dagger K_{k\,k+1}\big)K_{k+1\,l}n_{k-1} + \alpha_{k+1}^\dagger K_{k+1\,l}n_k\Big)\Big|_{\mathrm{Sym}}^{[k+1]} \\
&= -a\Big(\big(\alpha_1^\dagger K_{12}K_{23}\cdots K_{k\,k+1}K_{k+1\,l} + \cdots + \alpha_k^\dagger K_{k\,k+1}K_{k+1\,l}\big)(n_k + an_{k-1}) \\
&\quad -a\big(\alpha_1^\dagger K_{12}K_{23}\cdots K_{k\,k+1}K_{k+1\,l} + \cdots + \alpha_k^\dagger K_{k\,k+1}K_{k+1\,l}\big)n_{k-1} + \alpha_{k+1}^\dagger K_{k+1\,l}n_k\Big)\Big|_{\mathrm{Sym}}^{[k+1]} \\
&= -a\Big(\alpha_1^\dagger K_{12}K_{23}\cdots K_{k\,k+1}K_{k+1\,l} + \cdots + \alpha_{k+1}^\dagger K_{k+1\,l}\Big)n_k\Big|_{\mathrm{Sym}}^{[k+1]}
\end{aligned}
$$

$$
\sim \; n_k\Big|_{\mathrm{Sym}}, \quad l \notin [k+1],
$$

which proves the formula (4.106c).

In the following, we often need to deal with the terms that do not have $\alpha_1^\dagger$ appearing as an explicit factor on the left of the terms. When $\mathcal{O}$ represents a Dunkl operator of our interest, such terms are denoted by $\mathcal{O}\Big|_{\alpha_1^\dagger\sim 0}$.

Lemma 4. 17 *For $M \geq n \geq k+1$, we have*

$$
\Big(\big[n_{k+1}, \alpha_2^\dagger\cdots\alpha_n^\dagger\big] + \alpha_2^\dagger\cdots\alpha_n^\dagger\big[d_1, d_{\{2,\cdots,k+1\}}\big]\Big)\Big|_{\alpha_1^\dagger\sim 0} \sim d_1. \tag{4.107}
$$

This formula is also proved by induction on k. The case $k=1$ is verified as follows. From the definitions of the operators and the first two formulae of Lemma 4. 16 , we have

$$
\begin{aligned}
&\Big(\big[n_2, \alpha_2^\dagger\cdots\alpha_n^\dagger\big] + \alpha_2^\dagger\cdots\alpha_n^\dagger\big[d_1, d_2+a\big]\Big)\Big|_{\alpha_1^\dagger\sim 0} \\
&= \Big(\big[d_1, \alpha_2^\dagger\cdots\alpha_n^\dagger\big](d_2+a) + d_1\big[d_2, \alpha_2^\dagger\cdots\alpha_n^\dagger\big] + a\alpha_2^\dagger\cdots\alpha_n^\dagger(d_2-d_1)K_{12}\Big)\Big|_{\alpha_1^\dagger\sim 0} \\
&= \Big(d_1\alpha_2^\dagger\cdots\alpha_n^\dagger\big(1 + a\sum_{j\in[M]\setminus\{2,\cdots,n\}}K_{2j}\big) + a\alpha_2^\dagger\cdots\alpha_n^\dagger(d_2-d_1)K_{12}\Big)\Big|_{\alpha_1^\dagger\sim 0} \\
&= \alpha_2^\dagger\cdots\alpha_n^\dagger\Big(d_1\big(1 + aK_{21} + a\sum_{j=n+1}^{M}K_{2j}\big)\Big)\Big|_{\alpha_1^\dagger\sim 0} \\
&= \alpha_2^\dagger\cdots\alpha_n^\dagger\big(1 + aK_{12} + a\sum_{j=n+1}^{M}K_{2j}\big)d_1\Big|_{\alpha_1^\dagger\sim 0} \sim d_1.
\end{aligned}
$$

As the inductive assumption, we assume that the lemma is valid up to $k-1$. Then the case k is expressed as

$$
\begin{aligned}
&\Big(\big[n_{k+1}, \alpha_2^\dagger\cdots\alpha_n^\dagger\big] + \alpha_2^\dagger\cdots\alpha_n^\dagger\big[d_1, d_{\{2,\cdots,k+1\}}\big]\Big)\Big|_{\alpha_1^\dagger\sim 0} \\
&\quad = \Big(\big[n_k, \alpha_2^\dagger\cdots\alpha_n^\dagger\big] + \alpha_2^\dagger\cdots\alpha_n^\dagger\big[d_1, d_{1,\{2,\cdots,k\}}\big]\Big)(d_{k+1}+ka)\Big|_{\alpha_1^\dagger\sim 0} \\
&\qquad + \Big(n_k\big[d_{k+1}, \alpha_2^\dagger\cdots\alpha_n^\dagger\big] + \alpha_2^\dagger\cdots\alpha_n^\dagger d_{1,\{2,\cdots,k\}}\big[d_1, d_{k+1}\big]\Big)\Big|_{\alpha_1^\dagger\sim 0}.
\end{aligned} \tag{4.108}
$$

Using the second formula of Lemma 4. 16 and eq. (4.21), we can readily calculate the second term in the r.h.s. of the above expression:

$$
\begin{aligned}
&\Big(n_k\big[d_{k+1}, \alpha_2^\dagger\cdots\alpha_n^\dagger\big] + \alpha_2^\dagger\cdots\alpha_n^\dagger d_{1,\{2,\cdots,k\}}\big[d_1, d_{k+1}\big]\Big)\Big|_{\alpha_1^\dagger\sim 0} \\
&= \Big(\big(\alpha_2^\dagger\cdots\alpha_n^\dagger n_k + \big[n_k, \alpha_2^\dagger\cdots\alpha_n^\dagger\big]\big)\big(1 + aK_{1\,k+1} + a\sum_{j=n+1}^{M}K_{k+1\,j}\big)
\end{aligned}
$$

$$
\begin{aligned}
&+a\alpha_2^\dagger\cdots\alpha_n^\dagger d_{1,\{2,\cdots,k\}}(d_{k+1}-d_1)K_{1\,k+1}\Big)\Big|_{\alpha_1^\dagger\sim 0}\\
=\;&\bigg(\Big(\big[n_k,\alpha_2^\dagger\cdots\alpha_n^\dagger\big]+\alpha_2^\dagger\cdots\alpha_n^\dagger\big[d_1,d_{1,\{2,\cdots,k\}}\big]+\alpha_2^\dagger\cdots\alpha_n^\dagger d_{1,\{2,\cdots,k\}}d_1\Big)\\
&\times\Big(1+aK_{1\,k+1}+a\sum_{j=n+1}^{M}K_{k+1\,j}\Big)\\
&+a\alpha_2^\dagger\cdots\alpha_n^\dagger\Big(d_{1,\{2,\cdots,k\}}K_{1\,k+1}d_1-d_{1,\{2,\cdots,k\}}d_1K_{1\,k+1}\Big)\bigg)\bigg|_{\alpha_1^\dagger\sim 0}\\
=\;&\bigg(\Big(\big[n_k,\alpha_2^\dagger\cdots\alpha_n^\dagger\big]+\alpha_2^\dagger\cdots\alpha_n^\dagger\big[d_1,d_{1,\{2,\cdots,k\}}\big]\Big)\Big(1+aK_{1\,k+1}+a\sum_{j=n+1}^{M}K_{k+1\,j}\Big)\\
&+\alpha_2^\dagger\cdots\alpha_n^\dagger d_{1,\{2,\cdots,k\}}\Big(1+aK_{1\,k+1}+a\sum_{j=n+1}^{M}K_{k+1\,j}\Big)d_1\bigg)\bigg|_{\alpha_1^\dagger\sim 0}.
\end{aligned}
\tag{4.109}
$$

Substitution of the above equation into eq. (4.108) yields

$$
\begin{aligned}
&\Big(\big[n_{k+1},\alpha_2^\dagger\cdots\alpha_n^\dagger\big]+\alpha_2^\dagger\cdots\alpha_n^\dagger\big[d_1,d_{\{2,\cdots,k+1\}}\big]\Big)\Big|_{\alpha_1^\dagger\sim 0}\\
&=\bigg(\Big(\big[n_k,\alpha_2^\dagger\cdots\alpha_n^\dagger\big]+\alpha_2^\dagger\cdots\alpha_n^\dagger\big[d_1,d_{1,\{2,\cdots,k\}}\big]\Big)\Big(d_{k+1}+ka+1+aK_{1\,k+1}+a\sum_{j=n+1}^{M}K_{k+1\,j}\Big)\\
&\quad+\alpha_2^\dagger\cdots\alpha_n^\dagger d_{1,\{2,\cdots,k\}}\Big(1+aK_{1\,k+1}+a\sum_{j=n+1}^{M}K_{k+1\,j}\Big)d_1\bigg)\bigg|_{\alpha_1^\dagger\sim 0}.
\end{aligned}
\tag{4.110}
$$

The second term of the r.h.s. of the above expression has d_1 at its rightmost. Thus the proof of eq. (4.107) reduces to the proof of the following relation for the first term of the r.h.s. of eq. (4.110):

$$
\Big(\big[n_k,\alpha_2^\dagger\cdots\alpha_n^\dagger\big]+\alpha_2^\dagger\cdots\alpha_n^\dagger\big[d_1,d_{1,\{2,\cdots,k\}}\big]\Big)\Big(d_{k+1}+ka+1+aK_{1\,k+1}+a\sum_{j=n+1}^{M}K_{k+1\,j}\Big)\Big|_{\alpha_1^\dagger\sim 0}\sim d_1.
\tag{4.111}
$$

This formula can be readily verified by the inductive assumption:

$$
\begin{aligned}
&\Big(\big[n_k,\alpha_2^\dagger\cdots\alpha_n^\dagger\big]+\alpha_2^\dagger\cdots\alpha_n^\dagger\big[d_1,d_{1,\{2,\cdots,k\}}\big]\Big)\Big(d_{k+1}+ka+1+aK_{1\,k+1}+a\sum_{j=n+1}^{M}K_{k+1\,j}\Big)\Big|_{\alpha_1^\dagger\sim 0}\\
&\qquad\sim\; d_1\Big(d_{k+1}+ka+1+aK_{1\,k+1}+a\sum_{j=n+1}^{M}K_{k+1\,j}\Big)\\
&\qquad=\;\Big(d_{k+1}+ka+1+aK_{1\,k+1}+a\sum_{j=n+1}^{M}K_{k+1\,j}\Big)d_1+\big[d_1,d_{k+1}+aK_{1\,k+1}\big]\\
&\qquad=\;\Big(d_{k+1}+ka+1+aK_{1\,k+1}+a\sum_{j=n+1}^{M}K_{k+1\,j}\Big)d_1\\
&\qquad\quad+a(d_{k+1}-d_1)K_{1\,k+1}+a(d_1-d_{k+1})K_{1\,k+1}\\
&\qquad=\;\Big(d_{k+1}+ka+1+aK_{1\,k+1}+a\sum_{j=n+1}^{M}K_{k+1\,j}\Big)d_1\sim d_1.
\end{aligned}
$$

Equations (4.110) and (4.111) prove Lemma 4. 17 .

As a step toward proving Proposition 4. 14 , we prove the case $N=k+1$.

Proposition 4. 18

$$
\Big[n_{k+1},b^+_{k,[k+1]}\Big]\Big|_{\mathrm{Sym}}^{[k+1]}\sim n_{k+1}\Big|_{\mathrm{Sym}}^{[k+1]}.
\tag{4.112}
$$

Both sides of the above equation are symmetric under permutations of indices $1,\cdots,k+1$. From the first two equations of Lemma 4. 16 , we have

$$
\Big[n_{k+1},\alpha_1^\dagger\cdots\overset{i}{\vee}\cdots\alpha_{k+1}^\dagger\Big]=\sum_{j=1}^{N}\alpha_1^\dagger\cdots\overset{j}{\vee}\cdots\alpha_{k+1}^\dagger\mathcal{O}_{ij},
\tag{4.113}
$$

where $\mathcal{O}_{ij}$ is some unspecified operator that can be written by d_l and K_{lm} with $1\le l,m\le k+1$ and $\alpha_1^\dagger\cdots\overset{i}{\vee}\cdots\alpha_{k+1}^\dagger=\prod_{\substack{j=1\\ j\neq i}}^{k+1}\alpha_j^\dagger$. Commutators among the operators made of d_l operators are also operators made of d_l and K_{lm} with $1\le l,m\le k+1$. Thus we can say that the raising operator $b^+_{k,[k+1]}$ and the l.h.s. of eq. (4.112) have factors $\alpha_1^\dagger\cdots\overset{i}{\vee}\cdots\alpha_{k+1}^\dagger$ on the left. Therefore, in order to prove Proposition 4. 18 , it is sufficient for us to prove the following expression:

$$
\Big[n_{k+1},b^+_{k,[k+1]}\Big]\Big|_{\mathrm{Sym}}^{[k+1]}\Big|_{\alpha_1^\dagger\sim 0}\sim n_{k+1}\Big|_{\mathrm{Sym}}^{[k+1]}\Big|_{\alpha_1^\dagger\sim 0}.
\tag{4.114}
$$

This is proved by induction on k. The case $k=1$ can be verified by a straightforward calculation:

$$
\begin{aligned}
\Big[n_2,b^+_{1,[2]}\Big]\Big|_{\mathrm{Sym}}^{[2]}\Big|_{\alpha_1^\dagger\sim 0}
&=\Big[d_1(d_2+a),\alpha_1^\dagger(d_1+a)+\alpha_2^\dagger(d_2+a)\Big]\Big|_{\mathrm{Sym}}^{[2]}\Big|_{\alpha_1^\dagger\sim 0}\\
&=\Big(\alpha_1^\dagger\Big(1+a\sum_{j=3}^{M}K_{1j}\Big)+\alpha_2^\dagger\Big(1+a\sum_{j=3}^{M}K_{2j}\Big)\Big)n_2\Big|_{\mathrm{Sym}}^{[2]}\Big|_{\alpha_1^\dagger\sim 0}\\
&=\alpha_2^\dagger\Big(1+a\sum_{j=3}^{M}K_{2j}\Big)n_2\Big|_{\mathrm{Sym}}^{[2]}\Big|_{\alpha_1^\dagger\sim 0}\sim n_2.
\end{aligned}
$$

We assume that eq. (4.114) is valid up to $k-1$. From the definitions of the null operator (4.59) and the restricted raising operator (4.104), we can decompose them as

$$
\begin{aligned}
b^+_{k,[k+1]}&=\alpha_1^\dagger b^+_{k-1,\{2,\cdots,k+1\}}(d_1+ka)+\alpha_2^\dagger\cdots\alpha_{k+1}^\dagger d_{1,\{2,\cdots,k+1\}},\\
n_{k+1}&=d_1d_{1,\{2,\cdots,k+1\}}.
\end{aligned}
$$

Then the l.h.s. of eq. (4.114) of the case k is cast into

$$
\begin{aligned}
&\Big[n_{k+1},b^+_{k,[k+1]}\Big]\Big|_{\mathrm{Sym}}^{[k+1]}\Big|_{\alpha_1^\dagger\sim 0}\\
&=\Big[d_1d_{1,\{2,\cdots,k+1\}},\alpha_1^\dagger b^+_{k-1,\{2,\cdots,k+1\}}(d_1+ka)+\alpha_2^\dagger\cdots\alpha_{k+1}^\dagger d_{1,\{2,\cdots,k+1\}}\Big]\Big|_{\mathrm{Sym}}^{[k+1]}\Big|_{\alpha_1^\dagger\sim 0}\\
&=\bigg(d_1\Big[d_{1,\{2,\cdots,k+1\}},\alpha_1^\dagger\Big]b^+_{k-1,\{2,\cdots,k+1\}}(d_1+ka)\\
&\quad+\Big(\big[n_{k+1},\alpha_2^\dagger\cdots\alpha_{k+1}^\dagger\big]+\alpha_2^\dagger\cdots\alpha_{k+1}^\dagger\big[d_1,d_{1,\{2,\cdots,k+1\}}\big]\Big)d_{1,\{2,\cdots,k+1\}}\bigg)\bigg|_{\mathrm{Sym}}^{[k+1]}\bigg|_{\alpha_1^\dagger\sim 0}.
\end{aligned}
\tag{4.115}
$$

From Lemma 4. 17 , we can easily confirm that the second term of the r.h.s. of the above equation satisfies the following relation:

$$
\begin{aligned}
&\Big(\big[n_{k+1}, \alpha_2^\dagger \cdots \alpha_{k+1}^\dagger\big] + \alpha_2^\dagger \cdots \alpha_{k+1}^\dagger \big[d_1, d_{1,\{2,\cdots,k+1\}}\big]\Big) d_{1,\{2,\cdots,k+1\}}\Big|_{\mathrm{Sym}}^{[k+1]}\Big|_{\alpha_1^\dagger\sim 0} \\
&\sim d_1 d_{1,\{2,\cdots,k+1\}}\Big|_{\mathrm{Sym}}^{[k+1]}\Big|_{\alpha_1^\dagger\sim 0} \\
&= n_{k+1}\Big|_{\mathrm{Sym}}^{[k+1]}\Big|_{\alpha_1^\dagger\sim 0}.
\end{aligned}
$$

Thus we have to show the following relation so as to prove eq. (4.114):

$$
d_1\big[d_{1,\{2,\cdots,k+1\}}, \alpha_1^\dagger\big] b_{k-1,\{2,\cdots,k+1\}}^{+}(d_1 + ka)\Big|_{\mathrm{Sym}}^{[k+1]}\Big|_{\alpha_1^\dagger\sim 0} \sim n_{k+1}\Big|_{\mathrm{Sym}}^{[k+1]}\Big|_{\alpha_1^\dagger\sim 0}. \tag{4.116}
$$

We need a formula that is similar to the third formula of Lemma 4. 16 :

Lemma 4. 19

$$
\big[d_{1,\{2,\cdots,k+1\}}, \alpha_1^\dagger\big]\Big|_{\mathrm{Sym}}^{\{2,\cdots,k+1\}} = -a\Big(\alpha_2^\dagger K_{23}\cdots K_{k+1\,1} + \cdots + \alpha_{k+1}^\dagger K_{k+1\,1}\Big) d_{1,\{2,\cdots,k\}}\Big|_{\mathrm{Sym}}^{\{2,\cdots,k+1\}}.
$$

This lemma is also proved by induction. Verification of the case $k = 1$ is easily done:

$$
\big[d_2 + a, \alpha_1^\dagger\big] = -a\alpha_2^\dagger K_{12}.
$$

We assume the lemma holds up to the case $k-1$. Using the inductive assumption, we can calculate the l.h.s. of the lemma of the case k as

$$
\begin{aligned}
&\big[d_{1,\{2,\cdots,k+1\}}, \alpha_1^\dagger\big]\Big|_{\mathrm{Sym}}^{\{2,\cdots,k+1\}} \\
&= \Big(\big[d_{1,\{2,\cdots,k\}}, \alpha_1^\dagger\big](d_{k+1} + ka) + d_{1,\{2,\cdots,k\}}\big[d_{k+1}, \alpha_1^\dagger\big]\Big)\Big|_{\mathrm{Sym}}^{\{2,\cdots,k+1\}} \\
&= \Big(-a\Big(\alpha_2^\dagger K_{23}\cdots K_{k1} + \cdots + \alpha_k^\dagger K_{k1}\Big) d_{1,\{2,\cdots,k-1\}}(d_{k+1} + ka) K_{k\,k+1} \\
&\qquad -a d_{1,\{2,\cdots,k\}} \alpha_{k+1}^\dagger K_{k+1\,1}\Big)\Big|_{\mathrm{Sym}}^{\{2,\cdots,k+1\}} \\
&= -a\Big(\Big(\alpha_2^\dagger K_{23}\cdots K_{k\,k+1}K_{k+1\,1} + \cdots + \alpha_k^\dagger K_{k\,k+1}K_{k+1\,1}\Big)\big(d_{1,\{2,\cdots,k\}} + a d_{1,\{2,\cdots,k-1\}}\big) \\
&\qquad + \alpha_{k+1}^\dagger K_{k+1\,1} d_{1,\{2,\cdots,k\}} + \big[d_{1,\{2,\cdots,k\}}, \alpha_{k+1}^\dagger\big] K_{k+1\,1}\Big)\Big|_{\mathrm{Sym}}^{\{2,\cdots,k+1\}} \\
&= -a\Big(\Big(\alpha_2^\dagger K_{23}\cdots K_{k+1\,1} + \cdots + \alpha_{k+1}^\dagger K_{k+1\,1}\Big) d_{1,\{2,\cdots,k\}} \\
&\qquad + a\Big(\alpha_2^\dagger K_{23}\cdots K_{k\,k+1}K_{k+1\,1} + \cdots + \alpha_k^\dagger K_{k\,k+1}K_{k+1\,1}\Big) d_{1,\{2,\cdots,k-1\}} \\
&\qquad - a\Big(\alpha_2^\dagger K_{23}\cdots K_{k\,k+1} + \cdots + \alpha_k^\dagger K_{k\,k+1}\Big) d_{1,\{2,\cdots,k-1\}} K_{k+1\,1}\Big)\Big|_{\mathrm{Sym}}^{\{2,\cdots,k+1\}} \\
&= -a\Big(\alpha_2^\dagger K_{23}\cdots K_{k+1\,1} + \cdots + \alpha_{k+1}^\dagger K_{k+1\,1}\Big) d_{1,\{2,\cdots,k\}}\Big|_{\mathrm{Sym}}^{\{2,\cdots,k+1\}},
\end{aligned}
$$

which complete the proof of Lemma 4. 19 . Using Lemma 4. 19 , we can calculate the l.h.s. of eq. (4.116) as

$$
\begin{aligned}
&d_1\big[d_{1,\{2,\cdots,k+1\}}, \alpha_1^\dagger\big] b_{k-1,\{2,\cdots,k+1\}}^{+}(d_1 + ka)\Big|_{\mathrm{Sym}}^{[k+1]}\Big|_{\alpha_1^\dagger\sim 0} \\
&\quad\sim\ -a d_1\Big(\alpha_2^\dagger K_{23}\cdots K_{k+1\,1} + \cdots + \alpha_{k+1}^\dagger K_{k+1\,1}\Big) d_{1,\{2,\cdots,k\}} b_{k-1,\{2,\cdots,k+1\}}^{+}(d_1 + ka)\Big|_{\mathrm{Sym}}^{[k+1]}\Big|_{\alpha_1^\dagger\sim 0}.
\end{aligned}
\tag{4.117}
$$

Paying attention to the relations,

$$
\big[d_1, \alpha_j^\dagger\big]\Big|_{\alpha_1^\dagger\sim 0} = 0, \quad d_{k+1} d_{1,\{2,\cdots,k\}} = n_{k,\{2,\cdots,k+1\}}, \quad n_{k,\{2,\cdots,k+1\}}(d_1 + ka) = n_{k+1},
$$

we can further calculate eq. (4.117):

$$
\begin{aligned}
&- a d_1\Big(\alpha_2^\dagger K_{23}\cdots K_{k+1\,1} + \cdots + \alpha_{k+1}^\dagger K_{k+1\,1}\Big) d_{1,\{2,\cdots,k\}} b_{k-1,\{2,\cdots,k+1\}}^{+}(d_1 + ka)\Big|_{\mathrm{Sym}}^{[k+1]}\Big|_{\alpha_1^\dagger\sim 0} \\
&\quad= -a\Big(\alpha_2^\dagger K_{23}\cdots K_{k+1\,1} + \cdots + \alpha_{k+1}^\dagger K_{k+1\,1}\Big) d_{k+1} d_{1,\{2,\cdots,k\}} b_{k-1,\{2,\cdots,k+1\}}^{+}(d_1 + ka)\Big|_{\mathrm{Sym}}^{[k+1]}\Big|_{\alpha_1^\dagger\sim 0} \\
&\quad\sim \Big(\big[n_{k,\{2,\cdots,k+1\}}, b_{k-1,\{2,\cdots,k+1\}}^{+}\big] + b_{k-1,\{2,\cdots,k+1\}}^{+} n_{k,\{2,\cdots,k+1\}}\Big)(d_1 + ka)\Big|_{\mathrm{Sym}}^{[k+1]}\Big|_{\alpha_1^\dagger\sim 0} \\
&\quad= \Big(\big[n_{k,\{2,\cdots,k+1\}}, b_{k-1,\{2,\cdots,k+1\}}^{+}\big](d_1 + ka) + b_{k-1,\{2,\cdots,k+1\}}^{+} n_{k+1}\Big)\Big|_{\mathrm{Sym}}^{[k+1]}\Big|_{\alpha_1^\dagger\sim 0}.
\end{aligned}
$$

Note that the second term in the last expression is the expected null operator n_{k+1}. Due to the inductive assumption, the first term can be cast into the following form:

$$
\big[n_{k,\{2,\cdots,k+1\}}, b_{k-1,\{2,\cdots,k+1\}}^{+}\big](d_1 + ka)\Big|_{\mathrm{Sym}}^{[k+1]}\Big|_{\alpha_1^\dagger\sim 0} \sim n_{k,\{2,\cdots,k+1\}}(d_1 + ka)\Big|_{\mathrm{Sym}}^{[k+1]}\Big|_{\alpha_1^\dagger\sim 0} = n_{k+1}\Big|_{\mathrm{Sym}}^{[k+1]}\Big|_{\alpha_1^\dagger\sim 0}.
$$

Thus we have proved eq. (4.114) and hence Proposition 4. 18 .

We need a few more formulae to prove Proposition 4. 14 .

Lemma 4. 20 *For all sets of positive integers* $J = \{j_1, j_2, \cdots, j_l\} \subseteq [N]$ *such that* $J \cap \{1, 2, \cdots, k+1\} = \emptyset$, *we have*

$$
\big[n_{k+1}, \alpha_J^\dagger\big]\Big|_{\mathrm{Sym}}^{[k+1]} \sim n_{k+1-l}\Big|_{\mathrm{Sym}}^{[k+1]}, \tag{4.118a}
$$

$$
\Big(n_{k+1-l} d_{k+1-l,J}\Big|_{\mathrm{Sym}}^{[k+1]}\Big)\Big|_{\mathrm{Sym}}^{[N]} \sim n_{k+1}\Big|_{\mathrm{Sym}}^{[N]}, \tag{4.118b}
$$

$$
\big[n_{k+1}, d_{k+1-l,J}\big]\Big|_{\mathrm{Sym}}^{[N]} \sim n_{k+1}\Big|_{\mathrm{Sym}}^{[N]}. \tag{4.118c}
$$

These formulae can be straightforwardly verified by using the definitions of operators, the third equation of Lemma 4. 16 and the fundamental commutation relations among Dunkl operators (4.21). For convenience of the proof of the above formulae, we introduce a symbol $\clubsuit$ that means an unspecified non-singular-operator-valued coefficient. This symbol enables us to compactify the expressions in the following way:

$$\begin{aligned}&(\text{some involved expression}) \sim (\text{an operator}) \Leftrightarrow (\text{some involved expression}) = \clubsuit(\text{an operator}),\\ &(\text{some involved expression}) \times (\text{an operator}) + \clubsuit(\text{an operator}) = \clubsuit(\text{an operator}).\end{aligned}$$

Using this $\clubsuit$-symbol, we shall prove Lemma 4. 20 . By iterated use of eq. (4.101) and the third formula of Lemma 4. 16 , we have

$$\begin{aligned}
&\left[n_{k+1}, \alpha_J^\dagger\right]\Big|_{\mathrm{Sym}}^{[k+1]}\\
&= \left(\clubsuit n_k \alpha_{j_2}^\dagger \cdots \alpha_{j_l}^\dagger + \clubsuit n_k \alpha_{j_3}^\dagger \cdots \alpha_{j_l}^\dagger + \cdots + \clubsuit n_k\right)\Big|_{\mathrm{Sym}}^{[k+1]}\\
&= \left(\clubsuit\left[n_k, \alpha_{j_2}^\dagger \cdots \alpha_{j_l}^\dagger\right] + \clubsuit\left[n_k, \alpha_{j_3}^\dagger \cdots \alpha_{j_l}^\dagger\right] + \cdots + \clubsuit n_k\right)\Big|_{\mathrm{Sym}}^{[k+1]}\\
&\ \vdots\\
&= \left(\clubsuit n_{k+1-l} + \clubsuit n_{k+2-l} + \cdots + \clubsuit n_k\right)\Big|_{\mathrm{Sym}}^{[k+1]}\\
&= \clubsuit n_{k+1-l}\Big|_{\mathrm{Sym}}^{[k+1]},
\end{aligned}$$

which proves eq. (4.118a). Verification of the second formula is simple. From the definitions of the operators n_{k+1-l} and $d_{k+1-l,J}$, we have

$$n_{k+1-l} d_{k+1-l,J}\Big|_{\mathrm{Sym}}^{[k+1]}\Big|_{\mathrm{Sym}}^{[N]} = d_1\left(d_2+a\right)\cdots\left(d_{k+1-l}+(k-l)a\right)\left(d_{j_1}+(k-l+1)a\right)\cdots\left(d_{j_l}+ka\right)\Big|_{\mathrm{Sym}}^{[n]}.$$

We denote a sequence of coordinate exchange operators that transform the indices $\{1, \cdots, k-l+1, j_1, \cdots, j_l\}$ to $\{1, \cdots, k+1\}$ by $K_\clubsuit$. Then we have

$$\begin{aligned}
&d_1\left(d_2+a\right)\cdots\left(d_{k+1-l}+(k-l)a\right)\left(d_{j_1}+(k-l+1)a\right)\cdots\left(d_{j_l}+ka\right)K_\clubsuit K_\clubsuit^{-1}\Big|_{\mathrm{Sym}}^{[N]}\\
&\quad = K_\clubsuit d_1\left(d_2+a\right)\cdots\left(d_{k+1}+ka\right)\Big|_{\mathrm{Sym}}^{[N]}\\
&\quad = \clubsuit n_{k+1},
\end{aligned}$$

which proves eq. (4.118b) The following commutation relation is convenient for the proof of eq. (4.118c):

$$\begin{aligned}
\left[n_{k+1}, d_i\right]\Big|_{\mathrm{Sym}}^{[k+1]\cup\{i\}} &= \left[d_1\left(d_2+a\right)\cdots\left(d_{k+1}+ka\right), d_i\right]\Big|_{\mathrm{Sym}}^{[k+1]\cup\{i\}}\\
&= \sum_{j=1}^{k} a d_1 \cdots \left(d_{j-1}+(j-2)a\right)K_{ij}\left(\left(d_j+(j-1)a\right)-\left(d_i+(j-1)a\right)\right)
\end{aligned}$$

$$\begin{aligned}
&\quad\times\left(d_{j+1}+ja\right)\cdots\left(d_{k+1}+ka\right)\Big|_{\mathrm{Sym}}^{[k+1]\cup\{i\}}\\
&= \sum_{j=1}^{k} a(K_{ij}-1)n_{k+1} \sim n_{k+1}\Big|_{\mathrm{Sym}}^{[k+1]\cup\{i\}} \sim n_{k+1}\Big|_{\mathrm{Sym}}^{[k+1]\cup\{i\}}, \quad i \notin [k+1].
\end{aligned}$$

By iterative use of the above formula, we have

$$\begin{aligned}
\left[n_{k+1}, d_{k+1-l,J}\right]\Big|_{\mathrm{Sym}}^{[N]} &= \left(n_{k+1}\left(d_{j_1}+(k+1-l)a\right)\cdots\left(d_{j_l}+ka\right) - d_{k+1-l,J}n_{k+1}\right)\Big|_{\mathrm{Sym}}^{[N]}\\
&= \left(\left[n_{k+1}, \left(d_{j_1}+(k+1-l)a\right)\right] + \left(d_{j_1}+(k+1-l)a\right)n_{k+1}\right)\\
&\quad \times\left(d_{j_2}+(k+2-l)a\right)\cdots\left(d_{j_l}+ka\right)\Big|_{\mathrm{Sym}}^{[N]} + \clubsuit n_{k+1}\Big|_{\mathrm{Sym}}^{[N]}\\
&= \clubsuit n_{k+1}\left(d_{j_2}+(k+2-l)a\right)\cdots\left(d_{j_l}+ka\right)\Big|_{\mathrm{Sym}}^{[N]} + \clubsuit n_{k+1}\Big|_{\mathrm{Sym}}^{[N]}\\
&\ \vdots\\
&= \clubsuit n_{k+1}\Big|_{\mathrm{Sym}}^{[N]}.
\end{aligned}$$

Thus we have proved Lemma 4. 20 .

Now we are ready to prove Proposition 4. 14 . We shall prove it by induction on k. First, we shall check the case $k = 1$. From the definitions of the raising operators, eqs. (4.54) and (4.104), we have

$$\left[n_2, b_1^+\right]\Big|_{\mathrm{Sym}}^{[N]} = \left[n_2, b_{1,[2]}^+\right]\Big|_{\mathrm{Sym}}^{[N]} + \sum_{i=3}^{N}\left[n_2, \alpha_i^\dagger(d_i+a)\right]\Big|_{\mathrm{Sym}}^{[N]}. \tag{4.119}$$

According to Proposition 4. 18 , the first term of the r.h.s. of the above equation is confirmed to be

$$\left[n_2, b_{1,[2]}^+\right]\Big|_{\mathrm{Sym}}^{[N]} \sim n_2\Big|_{\mathrm{Sym}}^{[N]}. \tag{4.120}$$

The second term is calculated as

$$\left[n_2, \alpha_i^\dagger(d_i+a)\right]\Big|_{\mathrm{Sym}}^{[N]} = \left[n_2, \alpha_i^\dagger\right](d_i+a)\Big|_{\mathrm{Sym}}^{[N]} + \alpha_i^\dagger\left[n_2, (d_i+a)\right]\Big|_{\mathrm{Sym}}^{[N]}. \tag{4.121}$$

From the third formula of Lemma 4. 20 , we notice that the second term of the r.h.s. of eq. (4.121) satisfies

$$\alpha_i^\dagger\left[n_2, (d_i+a)\right]\Big|_{\mathrm{Sym}}^{[N]} \sim n_2\Big|_{\mathrm{Sym}}^{[N]}. \tag{4.122}$$

Using eq. (4.106c), we can verify that the first term of the r.h.s. of eq. (4.121) reduces to

$$\begin{aligned}
\left[n_2, \alpha_i^\dagger\right](d_i+a)\Big|_{\mathrm{Sym}}^{[N]} &\sim n_1(d_i+a)\Big|_{\mathrm{Sym}}^{[N]}\\
&\sim K_{2i}d_1(d_2+a)\Big|_{\mathrm{Sym}}^{[N]}\\
&\sim n_2\Big|_{\mathrm{Sym}}^{[N]}.
\end{aligned} \tag{4.123}$$

Summarizing the results, we have

$$\left[n_2, b_1^+\right]\Big|_{\mathrm{Sym}}^{[N]} \sim n_2\Big|_{\mathrm{Sym}}^{[N]}. \tag{4.124}$$

Thus we have confirmed that the proposition holds for the case $k=1$.

By inductive assumption, the proposition holds up to $k-1$. What we like to show is that the commutator between n_{k+1} and each term of the decomposition of the raising operator (4.105) is similar to n_{k+1}, i.e.,

$$\left[n_{k+1}, \alpha_J^\dagger b^+_{k-l,[k+1]} d_{k-l+1,J}\right]\Big|_{\mathrm{Sym}}^{[N]} \sim n_{k+1}\Big|_{\mathrm{Sym}}^{[N]}. \tag{4.125}$$

Using the Leibniz rule, we decompose the l.h.s. into two parts:

$$\begin{aligned}&\left[n_{k+1}, \alpha_J^\dagger b^+_{k-l,[k+1]} d_{k-l+1,J}\right]\Big|_{\mathrm{Sym}}^{[N]}\\ &= \left[n_{k+1}, \alpha_J^\dagger b^+_{k-l,[k+1]}\right] d_{k-l+1,J}\Big|_{\mathrm{Sym}}^{[N]} + \alpha_J^\dagger b^+_{k-l,[k+1]}\left[n_{k+1}, d_{k-l+1,J}\right]\Big|_{\mathrm{Sym}}^{[N]}.\end{aligned} \tag{4.126}$$

Then from the third formula of Lemma 4. 20 , we notice that the second term is similar to n_{k+1}:

$$\alpha_J^\dagger b^+_{k-l,[k+1]}\left[n_{k+1}, d_{k-l+1,J}\right]\Big|_{\mathrm{Sym}}^{[N]} \sim n_{k+1}\Big|_{\mathrm{Sym}}^{[N]}. \tag{4.127}$$

Our remaining task is to check the first term. When $l \stackrel{\mathrm{def}}{=} |J| = 0$, the first term is similar to n_{k+1} because of Proposition 4. 18 . When $l \neq 0$, we have to do some calculation:

$$\begin{aligned}&\left[n_{k+1}, \alpha_J^\dagger b^+_{k-l,[k+1]}\right] d_{k-l+1,J}\Big|_{\mathrm{Sym}}^{[N]}\\ &= \left[n_{k+1}, \alpha_J^\dagger\right] b^+_{k-l,[k+1]} d_{k-l+1,J}\Big|_{\mathrm{Sym}}^{[N]} + \alpha_J^\dagger\left[n_{k+1}, b^+_{k-l,[k+1]}\right] d_{k-l+1,J}\Big|_{\mathrm{Sym}}^{[N]}.\end{aligned} \tag{4.128}$$

From the first formula of Lemma 4. 20 , the first term of the r.h.s. of eq. (4.128) is calculated as

$$\begin{aligned}&\left[n_{k+1}, \alpha_J^\dagger\right] b^+_{k-l,[k+1]} d_{k-l+1,J}\Big|_{\mathrm{Sym}}^{[N]}\\ &= \left[n_{k+1}, \alpha_J^\dagger\right]\Big|_{\mathrm{Sym}}^{[k+1]} b^+_{k-l,[k+1]} d_{k-l+1,J}\Big|_{\mathrm{Sym}}^{[N]}\\ &\sim n_{k+1-l} b^+_{k-l,[k+1]} d_{k-l+1,J}\Big|_{\mathrm{Sym}}^{[N]}\\ &= b^+_{k-l,[k+1]} n_{k+1-l} d_{k-l+1,J}\Big|_{\mathrm{Sym}}^{[N]} + \left[n_{k+1-l}, b^+_{k-l,[k+1]}\right] d_{k-l+1,J}\Big|_{\mathrm{Sym}}^{[N]}.\end{aligned} \tag{4.129}$$

Note that the operators $b^+_{k-l,[k+1]}$ and $d_{k-l+1,J}$ are invariant under the permutations of indices $1, \cdots, k+1$. Using the second formula of Lemma 4. 20 , we can verify that the first term of the r.h.s. of eq. (4.129) reduces to

$$b^+_{k-l,[k+1]} n_{k+1-l} d_{k-l+1,J}\Big|_{\mathrm{Sym}}^{[N]} \sim n_{k+1}\Big|_{\mathrm{Sym}}^{[N]}. \tag{4.130}$$

From the inductive hypothesis, we have

$$\left[n_{k-l+1}, b^+_{k-l,[k+1]}\right]\Big|_{\mathrm{Sym}}^{[k+1]} \sim n_{k-l+1}\Big|_{\mathrm{Sym}}^{[k+1]}. \tag{4.131}$$

Then the second term of the r.h.s. of eq. (4.129) is calculated as

$$\begin{aligned}\left[n_{k+1-l}, b^+_{k-l,[k+1]}\right]\Big|_{\mathrm{Sym}}^{[k+1]} d_{k-l+1,J}\Big|_{\mathrm{Sym}}^{[N]} &\sim n_{k-l+1} d_{k-l+1,J}\Big|_{\mathrm{Sym}}^{[N]}\\ &\sim n_{k+1}\Big|_{\mathrm{Sym}}^{[N]}.\end{aligned} \tag{4.132}$$

On the other hand, the second term of the r.h.s. of eq. (4.128) is separated into two parts as

$$\alpha_J^\dagger\left[n_{k+1}, b^+_{k-l,[k+1]}\right] d_{k-l+1,J}\Big|_{\mathrm{Sym}}^{[N]} = \alpha_J^\dagger n_{k+1} b^+_{k-l,[k+1]} d_{k-l+1,J}\Big|_{\mathrm{Sym}}^{[N]} - \alpha_J^\dagger b^+_{k-l,[k+1]} n_{k+1} d_{k-l+1,J}\Big|_{\mathrm{Sym}}^{[N]}. \tag{4.133}$$

In an analogous way to the verification of the first and the second terms in the r.h.s. of eq. (4.129), we can confirm both the second and the first terms in the r.h.s. of eq. (4.133) are respectively similar to n_{k+1}. Then we obtain

$$\alpha_J^\dagger\left[n_{k+1}, b^+_{k-l,[k+1]}\right] d_{k-l+1,J}\Big|_{\mathrm{Sym}}^{[N]} \sim n_{k+1}\Big|_{\mathrm{Sym}}^{[N]}. \tag{4.134}$$

Equations (4.130), (4.132) and (4.134) yield

$$\left[n_{k+1}, \alpha_J^\dagger b^+_{k-l,[k+1]}\right] d_{k-l+1,J}\Big|_{\mathrm{Sym}}^{[N]} \sim n_{k+1}\Big|_{\mathrm{Sym}}^{[N]}. \tag{4.135}$$

Thus we have proved Proposition 4. 14 and hence Proposition 4. 5 .

4.3.4 Raising Operators

Proposition 4. 6 is given in a form whose operators depend on N variables $x_1, \cdots, x_N$. It is convenient for us to explicitly indicate the number of variables, for we shall change the number of variables during the induction procedure.

Proposition 4. 21

$$\left[I_2^{[N]}(N), b^{+[N]}_{k,[N]}\right]\Big|_{\mathrm{Sym}}^{[N]} = \left\{b^{+[N]}_{k,[N]}\left(2I_1^{[N]}(N) + k + ak(N-k)\right) + \sum_{\substack{J\subseteq[N]\\ |J|=k+1}} g^{[N]}_{k+1,J} n^{[N]}_{k+1,J}\right\}\Big|_{\mathrm{Sym}}^{[N]},\ N \geq k. \tag{4.136}$$

Here $g^{[N]}_{k+1,J}$ is an unspecified nonsingular operator that satisfies $g^{[N]}_{N+1,J} = 0$.

Note that we have introduced the notation

$$I_n^{[M]}(N) = \sum_{i=1}^{N} (d_i^{[M]})^n. \tag{4.137}$$

It is obvious that Proposition 4. 6 is tantamount to the above proposition. We shall prove Proposition 4. 21 by induction on k and N. Precisely speaking, we use the induction on l, which relates k and N by $k = l+1$ and $N = l+M$ with arbitrary integer M. The plan requires several lemmas.

Lemma 4. 22

$$
\begin{aligned}
&\Big[(d_j^{[M]})^2, \alpha_1^{\dagger[M]} \cdots \overset{i}{\vee} \cdots \alpha_N^{\dagger[M]}\Big] \\
&= \alpha_1^{\dagger[M]} \cdots \overset{i}{\vee} \cdots \alpha_N^{\dagger[M]}\Big((1 + aK_{ij} + a\sum_{k=N+1}^{M} K_{jk})^2 + (1 + aK_{ij} + a\sum_{k=N+1}^{M} K_{jk})d_j^{[M]} \\
&\quad + d_j^{[M]}(1 + aK_{ij} + a\sum_{k=N+1}^{M} K_{jk})\Big),\ 1 \le i,j \le N,\ i \neq j,
\end{aligned} \tag{4.138a}
$$

$$
\begin{aligned}
&\Big[(d_j^{[M]})^2, \alpha_1^{\dagger[M]} \cdots \overset{i}{\vee} \cdots \alpha_N^{\dagger[M]}\Big]\Big|_{\alpha_1^\dagger \sim 0} \\
&= -a\alpha_2^{\dagger[M]} \cdots \overset{i}{\vee} \cdots \alpha_N^{\dagger[M]}\alpha_j^{\dagger[M]}\Big(d_j^{[M]}K_{1j} + K_{1j}d_j^{[M]} + K_{1j} + aK_{1j}(K_{1i} + \sum_{k=N+1}^{M} K_{1k})\Big), \\
&\quad j = i \text{ or } N+1 \le j \le M.
\end{aligned} \tag{4.138b}
$$

Both formulae are readily verified from the first two formulae of Lemma 4. 16 . Using the second formula of Lemma 4. 16 , we can prove eq. (4.138a) as follows:

$$
\begin{aligned}
&\Big[(d_j^{[M]})^2, \alpha_1^{\dagger[M]} \cdots \overset{i}{\vee} \cdots \alpha_N^{\dagger[M]}\Big] \\
&= d_j^{[M]}\alpha_1^{\dagger[M]} \cdots \overset{i}{\vee} \cdots \alpha_N^{\dagger[M]}(1 + aK_{ij} + a\sum_{k=N+1}^{M} K_{jk}) \\
&\quad + \alpha_1^{\dagger[M]} \cdots \overset{i}{\vee} \cdots \alpha_N^{\dagger[M]}(1 + aK_{ij} + a\sum_{k=N+1}^{M} K_{jk})d_j^{[M]} \\
&= \alpha_1^{\dagger[M]} \cdots \overset{i}{\vee} \cdots \alpha_N^{\dagger[M]}\Big((1 + aK_{ij} + a\sum_{k=N+1}^{M} K_{jk})^2 + (1 + aK_{ij} + a\sum_{k=N+1}^{N} K_{jk})d_j^{[M]} \\
&\quad + d_j^{[M]}(1 + aK_{ij} + a\sum_{k=N+1}^{N} K_{jk})\Big),\ 1 \le i,j \le N,\ i \neq j.
\end{aligned}
$$

Verification of the second formula (4.138b) is done with the help of eqs. (4.106a) and (4.106b):

$$
\begin{aligned}
&\Big[(d_j^{[M]})^2, \alpha_1^{\dagger[M]} \cdots \overset{i}{\vee} \cdots \alpha_N^{\dagger[M]}\Big]\Big|_{\alpha_1^\dagger \sim 0} \\
&= -ad_j^{[M]}\alpha_2^{\dagger[M]} \cdots \overset{i}{\vee} \cdots \alpha_N^{\dagger[M]}\alpha_j^\dagger K_{1j} - a\alpha_2^{\dagger[M]} \cdots \overset{i}{\vee} \cdots \alpha_N^{\dagger[M]}\alpha_j^\dagger K_{1j}d_j^{[M]}\Big)\Big|_{\alpha_1^\dagger \sim 0} \\
&= -a\alpha_2^{\dagger[M]} \cdots \overset{i}{\vee} \cdots \alpha_N^{\dagger[M]}\alpha_j^\dagger\Big(d_j^{[M]}K_{1j} + K_{1j}d_j^{[M]} + (1 + a\sum_{k \in ([M]\setminus([N]\cup\{j\}))\cup\{1\}\cup\{i\}} K_{jk})K_{1j}\Big) \\
&= -a\alpha_2^{\dagger[M]} \cdots \overset{i}{\vee} \cdots \alpha_N^{\dagger[M]}\alpha_j^\dagger\Big(d_j^{[M]}K_{ij} + K_{ij}d_j^{[M]} + K_{1j} + aK_{1j}(K_{1i} + \sum_{k=N+1}^{M} K_{1k})\Big) \\
&\quad j = i \text{ or } N+1 \le j \le M.
\end{aligned}
$$

Thus Lemma 4. 22 is proved. Lemma 4. 22 leads to the following formula.

Corollary 4. 23 *For $1 \le i < N \le M$, we have*

$$
\begin{aligned}
&\Big[I_2^{[M]}(N-1), \alpha_1^{\dagger[M]} \cdots \overset{i}{\vee} \cdots \alpha_{N-1}^{\dagger[M]}\alpha_M^{\dagger\,[M]}\Big]\Big|_{\alpha_1^\dagger \sim 0} \\
&\quad = \alpha_M^{\dagger\,[M]}\Big[I_2^{[M-1]}(N-1), \alpha_1^{\dagger[M-1]} \cdots \overset{i}{\vee} \cdots \alpha_{N-1}^{\dagger[M-1]}\Big]\Big|_{\alpha_1^\dagger \sim 0}\Big|_{\mathcal{O}^{[M-1]} \sim \mathcal{O}^{[M]}}.
\end{aligned} \tag{4.139}
$$

The restriction $\Big|_{\mathcal{O}^{[M-1]} \sim \mathcal{O}^{[M]}}$ means that we respectively identify the Dunkl operators $\alpha_i^{\dagger[M]}, \alpha_i^{[M]}$ and $d_i^{[M]}$ with $\alpha_i^{\dagger[M-1]}, \alpha_i^{[M-1]}$ and $d_i^{[M-1]}$, where $i \le M-1$. The l.h.s. of the above formula is calculated as

$$
\begin{aligned}
&\Big[I_2^{[M]}(N-1), \alpha_1^{\dagger[M]} \cdots \overset{i}{\vee} \cdots \alpha_{N-1}^{\dagger[M]}\alpha_M^{\dagger\,[M]}\Big]\Big|_{\alpha_1^\dagger \sim 0} \\
&\quad = \Big[\sum_{j=1}^{N-1}(d_j^{[M]})^2, \alpha_1^{\dagger[M]} \cdots \overset{i}{\vee} \cdots \alpha_{N-1}^{\dagger[M]}\alpha_M^{\dagger\,[M]}\Big]\Big|_{\alpha_1^\dagger \sim 0} \\
&\quad = \Big[(d_i^{[M]})^2, \alpha_1^{\dagger[M]} \cdots \overset{i}{\vee} \cdots \alpha_{N-1}^{\dagger[M]}\alpha_M^{\dagger\,[M]}\Big]\Big|_{\alpha_1^\dagger \sim 0} \\
&\quad = -a\alpha_2^{\dagger[M]} \cdots \alpha_{N-1}^{\dagger\,[M]}\alpha_M^{\dagger\,[M]}\Big(d_i^{[M]}K_{1i} + K_{1i}d_i^{[M]} + K_{1i} + aK_{1i}(K_{1i} + \sum_{k=N}^{M-1} K_{1k})\Big).
\end{aligned} \tag{4.140}
$$

In a similar way, the r.h.s. of eq. (4.139) is calculated as

$$
\begin{aligned}
&\alpha_M^{\dagger\,[M]}\Big[I_2^{[M-1]}(N-1), \alpha_1^{\dagger[M-1]} \cdots \overset{i}{\vee} \cdots \alpha_{N-1}^{\dagger[M-1]}\Big]\Big|_{\alpha_1^\dagger \sim 0}\Big|_{\mathcal{O}^{[M-1]} \sim \mathcal{O}^{[M]}} \\
&\quad = \alpha_M^{\dagger\,[M]}\Big(-a\alpha_2^{\dagger[M-1]} \cdots \alpha_{N-1}^{\dagger\,[M-1]} \\
&\qquad \times\Big(d_i^{[M-1]}K_{1i} + K_{1i}d_i^{[M-1]} + K_{1i} + aK_{1i}(K_{1i} + \sum_{k=N}^{M-1} K_{1k})\Big)\Big)\Big|_{\mathcal{O}^{[M-1]} \sim \mathcal{O}^{[M]}} \\
&\quad = -a\alpha_2^{\dagger[M]} \cdots \alpha_{N-1}^{\dagger\,[M]}\alpha_M^{\dagger\,[M]}\Big(d_i^{[M]}K_{1i} + K_{1i}d_i^{[M]} + K_{1i} + aK_{1i}(K_{1i} + \sum_{k=N}^{M-1} K_{1k})\Big),
\end{aligned} \tag{4.141}
$$

which coincides with eq. (4.140). Thus we have confirmed Corollary 4. 23 .

The following formulae are valid for arbitrary number of variables.

Lemma 4. 24 *For $M \ge N$, we have*

$$
\Big[I_n^{[M]}(N-1), d_N^{[M]}\Big]\Big|_{\mathrm{Sym}}^{[N]} = a\Big((N-1)(d_N^{[M]})^n - I_n^{[M]}(N-1)\Big)\Big|_{\mathrm{Sym}}^{[N]}, \tag{4.142a}
$$

$$
\Big[(d_N^{[M]})^n, d_{1,[N]}^{[M]}\Big]\Big|_{\mathrm{Sym}}^{[N]} = ad_{1,[N-1]}^{[M]}\Big(I_n^{[M]}(N-1) - (N-1)(d_N^{[M]})^n\Big)\Big|_{\mathrm{Sym}}^{[N]},\ \forall N \ge 2. \tag{4.142b}
$$

The first identity can be checked from eq. (4.21c):

$$
\begin{aligned}
&\Big[I_n^{[M]}(N-1), d_N^{[M]}\Big]\Big|_{\mathrm{Sym}}^{[N]} \\
&= \sum_{j=1}^{N-1}\Big[(d_j^{[M]})^n, d_N^{[M]}\Big]\Big|_{\mathrm{Sym}}^{[N]} \\
&= a\sum_{j=1}^{N-1}\sum_{k=1}^{n}(d_j^{[M]})^{k-1}(d_N^{[M]}-d_j^{[M]})K_{jN}(d_j^{[M]})^{n-k}\Big|_{\mathrm{Sym}}^{[N]} \\
&= a\sum_{j=1}^{N-1}\sum_{k=1}^{n}\Big(-(d_j^{[M]})^{k}(d_N^{[M]})^{n-k}+(d_j^{[M]})^{k-1}(d_N^{[M]})^{n-k+1}\Big)K_{jN}\Big|_{\mathrm{Sym}}^{[N]} \\
&= \sum_{j=1}^{N-1}\Big(-(d_j^{[M]})^{n}+(d_N^{[M]})^{n}\Big)K_{jN}\Big|_{\mathrm{Sym}}^{[N]} \\
&= a\Big((N-1)(d_N^{[M]})^n - I_n^{[M]}(N-1)\Big)\Big|_{\mathrm{Sym}}^{[N]}.
\end{aligned}
$$

The second one is proved by induction on N from again eq. (4.21c). The case $N=2$ is straightforwardly calculated as follows:

$$
\begin{aligned}
&\Big[(d_2^{[M]})^n, d_{1,[2]}^{[M]}\Big]\Big|_{\mathrm{Sym}}^{[2]} \\
&= \Big[(d_2^{[M]})^n, (d_1^{[M]}+a)(d_2^{[M]}+2a)\Big]\Big|_{\mathrm{Sym}}^{[2]} \\
&= a\Big((d_1^{[M]})^n-(d_2^{[M]})^n\Big)K_{12}(d_2^{[M]}+2a)\Big|_{\mathrm{Sym}}^{[2]} \\
&= a\Big((d_1^{[M]})^n-(d_2^{[M]})^n\Big)(d_1^{[M]}+2a)\Big|_{\mathrm{Sym}}^{[2]} \\
&= a\Big((d_1^{[M]}+2a)\Big((d_1^{[M]})^n-(d_2^{[M]})^n\Big)-a\Big((d_1^{[M]})^n-(d_2^{[M]})^n\Big)\Big)\Big|_{\mathrm{Sym}}^{[2]} \\
&= a(d_1^{[M]}+a)\Big((d_1^{[M]})^n-(d_2^{[M]})^n\Big)\Big|_{\mathrm{Sym}}^{[2]} = ad_{1,[1]}^{[M]}\Big(I_n^{[M]}(1)-(d_2^{[M]})^n\Big)\Big|_{\mathrm{Sym}}^{[2]}.
\end{aligned}
$$

By inductive assumption, we assume the formula for the case $N-1$, i.e.,

$$
\Big[(d_N^{[M]})^n, d_{1,\{1,\cdots,N-2,N\}}^{[M]}\Big]\Big|_{\mathrm{Sym}}^{\{1,\cdots,N-2,N\}} = ad_{1,[N-2]}^{[M]}\Big(I_n^{[M]}(N-2)-(N-2)(d_N^{[M]})^n\Big)\Big|_{\mathrm{Sym}}^{\{1,\cdots,N-2,N\}}. \tag{4.143}
$$

We decompose the l.h.s. of eq. (4.142b) into two terms as

$$
\begin{aligned}
&\Big[(d_N^{[M]})^n, d_{1,[N]}^{[M]}\Big]\Big|_{\mathrm{Sym}}^{[N]} \\
&= \Big(\Big[(d_N^{[M]})^n, d_{1,\{1,\cdots,N-2,N\}}^{[M]}(d_{N-1}^{[M]}+Na) + d_{1,\{1,\cdots,N-2\}}^{[M]} \\
&\qquad \times\Big((d_{N-1}^{[M]}+(N-1)a)(d_N^{[M]}+Na)-(d_N^{[M]}+(N-1)a)(d_{N-1}^{[M]}+Na)\Big)\Big]\Big)\Big|_{\mathrm{Sym}}^{[N]}
\end{aligned}
$$

$$
\begin{aligned}
&= \Big[(d_N^{[M]})^n, d_{1,\{1,\cdots,N-2,N\}}^{[M]}(d_{N-1}^{[M]}+Na)\Big]\Big|_{\mathrm{Sym}}^{[N]} \\
&\quad - \Big[(d_N^{[M]})^n, ad_{1,\{1,\cdots,N-2\}}^{[M]}(d_{N-1}^{[M]}-d_N^{[M]})(K_{N-1\,N}-1)\Big]\Big|_{\mathrm{Sym}}^{[N]}.
\end{aligned} \tag{4.144}
$$

Since the operand is restricted to the symmetric functions of the indices $\{1,\cdots,N\}$, the second term in the r.h.s. of eq. (4.144) is simplified as

$$
\begin{aligned}
&-\Big[(d_N^{[M]})^n, ad_{1,\{1,\cdots,N-2\}}^{[M]}(d_{N-1}^{[M]}-d_N^{[M]})(K_{N-1\,N}-1)\Big]\Big|_{\mathrm{Sym}}^{[N]} \\
&= ad_{1,\{1,\cdots,N-2\}}^{[M]}(d_{N-1}^{[M]}-d_N^{[M]})(K_{N-1\,N}-1)(d_N^{[M]})^n\Big|_{\mathrm{Sym}}^{[N]} \\
&\quad - ad_{1,\{1,\cdots,N-2\}}^{[M]}(d_N^{[M]})^n(d_{N-1}^{[M]}-d_N^{[M]})(K_{N-1\,N}-1)\Big|_{\mathrm{Sym}}^{[N]} \\
&= ad_{1,\{1,\cdots,N-2\}}^{[M]}(d_{N-1}^{[M]}-d_N^{[M]})\Big((d_{N-1}^{[M]})^nK_{N-1\,N}-(d_N^{[M]})^n\Big)\Big|_{\mathrm{Sym}}^{[N]} \\
&= ad_{1,\{1,\cdots,N-2\}}^{[M]}(d_{N-1}^{[M]}-d_N^{[M]})\Big((d_{N-1}^{[M]})^n-(d_N^{[M]})^n\Big)\Big|_{\mathrm{Sym}}^{[N]} \\
&= a(d_{1,\{1,\cdots,N-1\}}^{[M]}-d_{1,\{1,\cdots,N-2,N\}}^{[M]})\Big((d_{N-1}^{[M]})^n-(d_N^{[M]})^n\Big)\Big|_{\mathrm{Sym}}^{[N]}.
\end{aligned} \tag{4.145}
$$

Using the inductive assumption (4.143), we simplify the first term in the r.h.s. of eq. (4.144) as

$$
\begin{aligned}
&\Big[(d_N^{[M]})^n, d_{1,\{1,\cdots,N-2,N\}}^{[M]}(d_{N-1}^{[M]}+Na)\Big]\Big|_{\mathrm{Sym}}^{[N]} \\
&= \Big[(d_N^{[M]})^n, d_{1,\{1,\cdots,N-2,N\}}^{[M]}\Big]\Big|_{\mathrm{Sym}}^{\{1,\cdots,N-2,N\}}(d_{N-1}^{[M]}+Na)\Big|_{\mathrm{Sym}}^{[N]} + d_{1,\{1,\cdots,N-2,N\}}^{[M]}\Big[(d_N^{[M]})^n, d_{N-1}^{[M]}\Big]\Big|_{\mathrm{Sym}}^{[N]} \\
&= ad_{1,\{1,\cdots,N-2\}}^{[M]}\Big(\sum_{j=1}^{N-2}(d_j^{[M]})^n-(N-2)(d_N^{[M]})^n\Big)(d_{N-1}^{[M]}+Na)\Big|_{\mathrm{Sym}}^{[N]} \\
&\quad + ad_{1,\{1,\cdots,N-2,N\}}\Big((d_{N-1}^{[M]})^n-(d_N^{[M]})^n\Big)\Big|_{\mathrm{Sym}}^{[N]} \\
&= ad_{1,\{1,\cdots,N-2\}}^{[M]}\Big((d_{N-1}^{[M]}+Na)\Big(\sum_{j=1}^{N-2}(d_j^{[M]})^n-(N-2)(d_N^{[M]})^n\Big) \\
&\quad + a\sum_{j=1}^{N-2}\Big((d_{N-1}^{[M]})^n-(d_j^{[M]})^n\Big)-a(N-2)\Big((d_{N-1}^{[M]})^n-(d_N^{[M]})^n\Big)\Big)\Big|_{\mathrm{Sym}}^{[N]} \\
&\quad + ad_{1,\{1,\cdots,N-2,N\}}\Big((d_{N-1}^{[M]})^n-(d_N^{[M]})^n\Big)\Big|_{\mathrm{Sym}}^{[N]} \\
&= ad_{1,\{1,\cdots,N-1\}}^{[M]}\Big(\sum_{j=1}^{N-2}(d_j^{[M]})^n-(N-2)(d_N^{[M]})^n\Big)\Big|_{\mathrm{Sym}}^{[N]} \\
&\quad + ad_{1,\{1,\cdots,N-2,N\}}\Big((d_{N-1}^{[M]})^n-(d_N^{[M]})^n\Big)\Big|_{\mathrm{Sym}}^{[N]}.
\end{aligned} \tag{4.146}
$$

Note that we have used the identity (4.142a) several times in the calculation. From eqs. (4.144) - (4.146), we finally obtain

$$\Big[(d_N^{[M]})^n, d_{1,[N]}^{[M]}\Big]\Big|_{\mathrm{Sym}}^{[N]} = a d_{1,[N-1]}^{[M]}\Big(I_n^{[M]}(N-1) - (N-1)(d_N^{[M]})^n\Big)\Big|_{\mathrm{Sym}}^{[N]},$$

which is nothing but the second formula of Lemma 4. 24 .

Now we shall start the proof of Proposition 4. 21 . We have to separately prove the case for $k = N$ because of the difference of the definition of the raising operator (4.54):

Proposition 4. 25

$$\Big[I_2^{[N]}(N), b_{N,[N]}^{+[N]}\Big] = b_{N,[N]}^{+[N]}\Big(2I_1^{[N]}(N) + N\Big). \tag{4.147}$$

This formula can be straightforwardly verified from the definition of $b_{N,[N]}^{+[N]}$ (4.54b) and the second formula of Lemma 4. 16 :

$$\begin{aligned}
\Big[I_2^{[N]}(N), b_{N,[N]}^{+[N]}\Big] &= \sum_{j=1}^{N}\Big[(d_j^{[N]})^2, \alpha_1^{\dagger[N]}\cdots\alpha_N^{\dagger\,[N]}\Big] \\
&= \sum_{j=1}^{N}\Big(d_j^{[N]}\alpha_1^{\dagger[N]}\cdots\alpha_N^{\dagger\,[N]} + \alpha_1^{\dagger[N]}\cdots\alpha_N^{\dagger\,[N]}d_j^{[N]}\Big) \\
&= \sum_{j=1}^{N}\alpha_1^{\dagger[N]}\cdots\alpha_N^{\dagger\,[N]}\Big(2d_j^{[N]} + 1\Big) \\
&= b_{N,[N]}^{+[N]}\Big(2I_1^{[N]}(N) + N\Big).
\end{aligned}$$

As a ground for inductive assumption, we need a proposition:

Proposition 4. 26

$$\Big[I_2^{[M]}(M), b_{1,[M]}^{+[M]}\Big]\Big|_{\mathrm{Sym}}^{[M]} = \Big(\Big(b_{1,[M]}^{+[M]}I_1^{[M]}(M)+1+a(M-1)\Big)+\sum_{\substack{J\subseteq[M]\\|J|=2}} g_{2,J}^{[M]}n_{2,J}^{[M]}\Big)\Big|_{\mathrm{Sym}}^{[M]},\ \forall M\geq 1. \tag{4.148}$$

This is nothing but Proposition 4. 21 for $l = 0$, i.e., $k = 1$ and $N = M \geq 1$. The proof is as follows. Using eq. (4.21c), we can rewrite the second conserved operator $I_2^{[M]}(M)$ as

$$\begin{aligned}
&I_2^{[M]}(M)\Big|_{\mathrm{Sym}}^{[M]} \\
&= \Big((d_1^{[M]}+\cdots+d_M^{[M]})^2 - \sum_{1\leq i<j\leq M}(d_i^{[M]}d_j^{[M]} + d_j^{[M]}d_i^{[M]})\Big)\Big|_{\mathrm{Sym}}^{[M]} \\
&= \Big(\Big(d_1^{[M]}+\cdots+d_M^{[M]}\Big)\Big(d_1^{[M]}+\cdots+d_M^{[M]} + a(M-1)\Big) - 2\sum_{1\leq i<j\leq M} d_i^{[M]}(d_j^{[M]} + a)\Big)\Big|_{\mathrm{Sym}}^{[M]} \\
&= \Big(I_1^{[M]}(M)\Big(I_1^{[M]}(M) + a(M-1)\Big) - 2\sum_{\substack{J\subseteq[M]\\|J|=2}} n_{2,J}^{[M]}\Big)\Big|_{\mathrm{Sym}}^{[M]}.
\end{aligned} \tag{4.149}$$

Then from Proposition 4. 4 and Proposition 4. 14 for $k = 1$, i.e., eq. (4.124), we obtain the results.

We assume that Proposition 4. 21 is valid up to l, namely, $k = l + 1$ and $N = l + M$. Assuming the validity of Proposition 4. 21 for the case of k and N, we shall verify the case of $k + 1$ and $N + 1$:

$$\begin{aligned}
&\Big[I_2^{[N+1]}(N+1), b_{k+1,[N+1]}^{+[N+1]}\Big]\Big|_{\mathrm{Sym}}^{[N+1]} \\
&\quad = \Big\{b_{k+1,[N+1]}^{+[N+1]}\Big(2I_1^{[N+1]}(N+1) + (k+1) + a(k+1)(N-k)\Big) + \sum_{\substack{J\subseteq[N+1]\\|J|=k+2}} g_{k+2,J}^{[N+1]}n_{k+2,J}^{[N+1]}\Big\}\Big|_{\mathrm{Sym}}^{[N+1]}.
\end{aligned} \tag{4.150}$$

Since both sides of the above expression are invariant under the permutations of indices $1, 2, \cdots, N + 1$, it is sufficient to check the term with the factor $\alpha_2^{\dagger[N+1]}\cdots\alpha_{k+1}^{\dagger[N+1]}\alpha_{N+1}^{\dagger[N+1]}$ on the left. Thus we introduce the restriction $\alpha_1^{\dagger[N+1]}, \alpha_{k+2}^{\dagger[N+1]}, \cdots, \alpha_N^{\dagger[N+1]} \sim 0$, which we denote by $\Big|_{\alpha_{1,k+2,\cdots,N}^{\dagger[N+1]}\sim 0}$. From the definition of the conserved operator $I_2^{[N]}(N)$ (4.137) and Lemma 4. 15 , we have

$$I_2^{[N+1]}(N+1) = I_2^{[N+1]}(N) + (d_{N+1}^{[N+1]})^2, \tag{4.151}$$

$$b_{k+1,[N+1]}^{+[N+1]} = \alpha_{N+1}^{\dagger[N+1]}b_{k,[N]}^{+[N+1]}\Big(d_{N+1}^{[N+1]} + (k+1)a\Big) + b_{k+1,[N]}^{+[N+1]}. \tag{4.152}$$

Then the l.h.s. of eq. (4.150) is decomposed as

$$\begin{aligned}
&\Big[I_2^{[N+1]}(N+1), b_{k+1,[N+1]}^{+[N+1]}\Big]\Big|_{\alpha_{1,k+2,\cdots,N}^{\dagger[N+1]}\sim 0}\Big|_{\mathrm{Sym}}^{[N+1]} \\
&= \Big\{\Big[I_2^{[N+1]}(N), \alpha_{N+1}^{\dagger[N+1]}b_{k,[N]}^{+[N+1]}\Big(d_{N+1}^{[N+1]} + (k+1)a\Big)\Big] + \Big[I_2^{[N+1]}(N), b_{k+1,[N]}^{+[N+1]}\Big] \\
&\quad + \Big[(d_{N+1}^{[N+1]})^2, \alpha_{N+1}^{\dagger[N+1]}b_{k,[N]}^{+[N+1]}\Big(d_{N+1}^{[N+1]} + (k+1)a\Big)\Big] + \Big[(d_{N+1}^{[N+1]})^2, b_{k+1,[N]}^{+[N+1]}\Big]\Big\}\Big|_{\alpha_{1,k+2,\cdots,N}^{\dagger[N+1]}\sim 0}\Big|_{\mathrm{Sym}}^{[N+1]}.
\end{aligned} \tag{4.153}$$

Due to Lemma 4. 22 , we can easily confirm

$$\Big[I_2^{[N+1]}(N), b_{k+1,[N]}^{+[N+1]}\Big]\Big|_{\alpha_{1,k+2,\cdots,N}^{\dagger[N+1]}\sim 0}\Big|_{\mathrm{Sym}}^{[N+1]} = 0, \tag{4.154a}$$

$$\begin{aligned}
&\Big[(d_{N+1}^{[N+1]})^2, \alpha_{N+1}^{\dagger[N+1]}b_{k,[N]}^{+[N+1]}\Big]\Big|_{\alpha_{1,k+2,\cdots,N}^{\dagger[N+1]}\sim 0}\Big|_{\mathrm{Sym}}^{[N+1]} \\
&= \Big[(d_{N+1}^{[N+1]})^2, \alpha_{N+1}^{\dagger[N+1]}\alpha_2^{\dagger[N+1]}\cdots\alpha_{k+1}^{\dagger[N+1]}d_{1,\{2,\cdots,k+1\}}^{[N+1]}\Big]\Big|_{\alpha_{1,k+2,\cdots,N}^{\dagger[N+1]}\sim 0}\Big|_{\mathrm{Sym}}^{[N+1]},
\end{aligned} \tag{4.154b}$$

$$\begin{aligned}
&\Big[(d_{N+1}^{[N+1]})^2, b_{k+1,[N]}^{+[N+1]}\Big]\Big|_{\alpha_{1,k+2,\cdots,N}^{\dagger[N+1]}\sim 0}\Big|_{\mathrm{Sym}}^{[N+1]} \\
&= \sum_{i\in\{1,k+2,\cdots,N\}}\Big[(d_{N+1}^{[N+1]})^2, \alpha_i^{\dagger[N+1]}\alpha_2^{\dagger}\cdots\alpha_{k+1}^{\dagger[N+1]}\Big]d_{1,\{2,\cdots,k+1\}}^{[N+1]}\Big(d_i^{[N+1]} + (k+1)a\Big) \\
&\quad \Big|_{\alpha_{1,k+2,\cdots,N}^{\dagger[N+1]}\sim 0}\Big|_{\mathrm{Sym}}^{[N+1]}.
\end{aligned} \tag{4.154c}$$

Thus the r.h.s. of eq. (4.153) becomes

$$
\begin{aligned}
&\Big[I_2^{[N+1]}(N+1), b^{+[N+1]}_{k+1,[N+1]}\Big] \\
&= \Big\{\Big[I_2^{[N+1]}(N), \alpha^{\dagger[N+1]}_{N+1} b^{+[N+1]}_{k,[N]}\Big]\Big(d^{[N+1]}_{N+1} + (k+1)a\Big) \\
&\quad + \alpha^{\dagger[N+1]}_{N+1}\alpha^{\dagger[N+1]}_{2}\cdots\alpha^{\dagger[N+1]}_{k+1} d^{[N+1]}_{1,\{2,\cdots,k+1\}}\Big[I_2^{[N+1]}(N), d^{[N+1]}_{N+1}\Big] \\
&\quad + \Big[(d^{[N+1]}_{N+1})^2, \alpha^{\dagger[N+1]}_{N+1}\alpha^{\dagger[N+1]}_{2}\cdots\alpha^{\dagger[N+1]}_{k+1} d^{[N+1]}_{1,\{2,\cdots,k+1,N+1\}}\Big] \\
&\quad + \sum_{i\in\{1,k+2,\cdots,N\}}\Big[(d^{[N+1]}_{N+1})^2, \alpha^{\dagger[N+1]}_{i}\alpha^{\dagger}_{2}\cdots\alpha^{\dagger[N+1]}_{k+1}\Big] d^{[N+1]}_{1,\{2,\cdots,k+1\}}\Big(d^{[N+1]}_{i} + (k+1)a\Big)\Big\} \\
&\quad \Big|_{\alpha^{\dagger[N+1]}_{1,k+2,\cdots,N}\sim 0}\Big|^{[N+1]}_{\mathrm{Sym}}.
\end{aligned}
\tag{4.155}
$$

We shall calculate the above equation term by term. Using Corollary 4. 23 , we can rewrite the commutator of the first term of the r.h.s. of eq. (4.155) as

$$
\begin{aligned}
&\Big[I_2^{[N+1]}(N), \alpha^{\dagger[N+1]}_{N+1} b^{+[N+1]}_{k,[N]}\Big]\Big(d^{[N+1]}_{N+1} + (k+1)a\Big)\Big|_{\alpha^{\dagger[N+1]}_{1,k+2,\cdots,N}\sim 0}\Big|^{[N+1]}_{\mathrm{Sym}} \\
&= \alpha^{\dagger[N+1]}_{N+1}\Big[I_2^{[N]}(N), b^{+[N]}_{k,[N]}\Big]\Big|^{[N]}_{\mathrm{Sym}}\Big|_{\mathcal{O}^{[N]}\sim\mathcal{O}^{[N+1]}}\Big(d^{[N+1]}_{N+1} + (k+1)a\Big)\Big|_{\alpha^{\dagger[N+1]}_{1,k+2,\cdots,N}\sim 0}\Big|^{[N+1]}_{\mathrm{Sym}}.
\end{aligned}
\tag{4.156}
$$

The above commutator allows the use of inductive assumption. Thus we have

$$
\begin{aligned}
&\Big[I_2^{[N+1]}(N), \alpha^{\dagger[N+1]}_{N+1} b^{+[N+1]}_{k,[N]}\Big]\Big(d^{[N+1]}_{N+1} + (k+1)a\Big)\Big|_{\alpha^{\dagger[N+1]}_{1,k+2,\cdots,N}\sim 0}\Big|^{[N+1]}_{\mathrm{Sym}} \\
&= \Big\{\alpha^{\dagger[N+1]}_{2}\cdots\alpha^{\dagger[N+1]}_{k+1}\alpha^{\dagger[N+1]}_{N+1} d^{[N+1]}_{1,\{2,\cdots,k+1,N+1\}}\Big(2I_1^{[N+1]}(N) + k + ak(N-k)\Big) \\
&\quad + 2a\alpha^{\dagger[N+1]}_{2}\cdots\alpha^{\dagger[N+1]}_{k+1}\alpha^{\dagger[N+1]}_{N+1} d^{[N+1]}_{1,\{2,\cdots,k+1\}}\Big(Nd^{[N+1]}_{N+1} - I_1^{[N+1]}(N)\Big) \\
&\quad + \alpha^{\dagger[N+1]}_{N+1}\sum_{\substack{J\subseteq[k]\cup\{N+1\}\\ N+1\in J,|J|=k+2}} g^{[N+1]}_{N-1,J\setminus\{N+1\}} n^{[N+1]}_{k+2,J}\Big\}\Big|_{\alpha^{\dagger[N+1]}_{1,k+2,\cdots,N}\sim 0}\Big|^{[N+1]}_{\mathrm{Sym}}.
\end{aligned}
\tag{4.157}
$$

The second term is straightforwardly calculated from eq. (4.142a):

$$
\begin{aligned}
&\alpha^{\dagger[N+1]}_{N+1}\alpha^{\dagger[N+1]}_{2}\cdots\alpha^{\dagger[N+1]}_{k+1} d^{[N+1]}_{1,\{2,\cdots,k+1\}}\Big[I_2^{[N+1]}(N), d^{[N+1]}_{N+1}\Big]\Big|_{\alpha^{\dagger[N+1]}_{1,k+2,\cdots,N}\sim 0}\Big|^{[N+1]}_{\mathrm{Sym}} \\
&= a\alpha^{\dagger[N+1]}_{N+1}\alpha^{\dagger[N+1]}_{2}\cdots\alpha^{\dagger[N+1]}_{k+1} d^{[N+1]}_{1,\{2,\cdots,k+1\}}\Big(N(d^{[N+1]}_{N+1})^2 - I_2^{[N+1]}(N)\Big)\Big|_{\alpha^{\dagger[N+1]}_{1,k+2,\cdots,N}\sim 0}\Big|^{[N+1]}_{\mathrm{Sym}}.
\end{aligned}
\tag{4.158}
$$

By using eqs. (4.138a) and (4.142b), the third term is cast into

$$
\begin{aligned}
&\Big[(d^{[N+1]}_{N+1})^2, \alpha^{\dagger[N+1]}_{2}\cdots\alpha^{\dagger[N+1]}_{k+1}\alpha^{\dagger[N+1]}_{N+1} d^{[N+1]}_{1,\{2,\cdots,k+1,N+1\}}\Big]\Big|_{\alpha^{\dagger[N+1]}_{1,k+2,\cdots,N}\sim 0}\Big|^{[N+1]}_{\mathrm{Sym}} \\
&= \Big\{\Big[(d^{[N+1]}_{N+1})^2, \alpha^{\dagger[N+1]}_{2}\cdots\alpha^{\dagger[N+1]}_{k+1}\alpha^{\dagger[N+1]}_{N+1}\Big] d^{[N+1]}_{1,\{2,\cdots,k+1,N+1\}} \\
&\quad + \alpha^{\dagger[N+1]}_{2}\cdots\alpha^{\dagger[N+1]}_{k+1}\alpha^{\dagger[N+1]}_{N+1}\Big[(d^{[N+1]}_{N+1})^2, d^{[N+1]}_{1,\{2,\cdots,k+1,N+1\}}\Big]\Big\}\Big|_{\alpha^{\dagger[N+1]}_{1,k+2,\cdots,N}\sim 0}\Big|^{[N+1]}_{\mathrm{Sym}} \\
&= \alpha^{\dagger[N+1]}_{2}\cdots\alpha^{\dagger[N+1]}_{k+1}\alpha^{\dagger[N+1]}_{N+1}\Big\{\Big(2d^{[N+1]}_{N+1} + 1 + a^2(N-k) \\
&\quad + a\sum_{i\in\{1,k+2,\cdots,N\}}\Big(2K_{iN+1} + d^{[N+1]}_{N+1}K_{iN+1} + K_{iN+1}d^{[N+1]}_{N+1}\Big) \\
&\quad + a^2\sum_{\substack{i,j\in\{1,k+2,\cdots,N\}\\ i\neq j}} K_{iN+1}K_{jN+1}\Big) d^{[N+1]}_{1,\{2,\cdots,k+1,N+1\}} \\
&\quad + ad^{[N+1]}_{1,\{2,\cdots,k+1\}}\Big(\sum_{i=2}^{k+1}(d^{[N+1]}_{i})^2 - k(d^{[N+1]}_{N+1})^2\Big)\Big\}\Big|_{\alpha^{\dagger[N+1]}_{1,k+2,\cdots,N}\sim 0}\Big|^{[N+1]}_{\mathrm{Sym}}.
\end{aligned}
\tag{4.159}
$$

Using eq. (4.138b), we get the following expression from the fourth term:

$$
\begin{aligned}
&\sum_{i\in\{1,k+2,\cdots,N\}}\Big[(d^{[N+1]}_{N+1})^2, \alpha^{\dagger[N+1]}_{i}\alpha^{\dagger[N+1]}_{2}\cdots\alpha^{\dagger[N+1]}_{k+1}\Big] d^{[N+1]}_{1,\{2,\cdots,k+1\}}\Big(d^{[N+1]}_{i} + (k+1)a\Big) \\
&\quad \Big|_{\alpha^{\dagger[N+1]}_{1,k+2,\cdots,N}\sim 0}\Big|^{[N+1]}_{\mathrm{Sym}} \\
&= -a\sum_{i\in\{1,k+2,\cdots,N\}}\alpha^{\dagger[N+1]}_{2}\cdots\alpha^{\dagger[N+1]}_{k+1}\alpha^{\dagger[N+1]}_{N+1} \\
&\quad \Big(K_{iN+1}d^{[N+1]}_{N+1} + d^{[N+1]}_{N+1}K_{iN+1} + K_{iN+1} + a + a\sum_{\substack{j\in\{1,k+2,\cdots,N\}\\ j\neq i}} K_{jN+1}K_{iN+1}\Big) \\
&\quad d^{[N+1]}_{1,\{2,\cdots,k+1\}}\Big(d^{[N+1]}_{i} + (k+1)a\Big)\Big|_{\alpha^{\dagger[N+1]}_{1,k+2,\cdots,N}\sim 0}\Big|^{[N+1]}_{\mathrm{Sym}}.
\end{aligned}
\tag{4.160}
$$

Assembling eqs. (4.157) - (4.160) and doing some calculation with the help of Lemma 4. 24 and the definition of null operators (4.59), we obtain

$$
\begin{aligned}
&\Big[I_2^{[N+1]}(N+1), b^{+[N+1]}_{k+1,[N+1]}\Big]\Big|_{\alpha^{\dagger[N+1]}_{1,k+2,\cdots,N}\sim 0}\Big|^{[N+1]}_{\mathrm{Sym}} \\
&= \Big\{\alpha^{\dagger[N+1]}_{2}\cdots\alpha^{\dagger[N+1]}_{k+1}\alpha^{\dagger[N+1]}_{N+1} d^{[N+1]}_{1,\{2,\cdots,k+1,N+1\}}\Big(2I_1^{[N+1]}(N+1) + (k+1) + a(k+1)(N-k)\Big) \\
&\quad + \sum_{\substack{J\subseteq[N+1]\\ N+1\in J,|J|=k+2}} g'^{[N+1]}_{k+2,J} n^{[N+1]}_{k+2,J}\Big\}\Big|_{\alpha^{\dagger[N+1]}_{1,k+2,\cdots,N}\sim 0}\Big|^{[N+1]}_{\mathrm{Sym}} \\
&= \Big\{b^{+[N+1]}_{k+1}\Big(2I_1^{[N+1]}(N+1) + (k+1) + a(k+1)(N-k)\Big) \\
&\quad + \sum_{\substack{J\subseteq[N+1]\\ |J|=k+2}} g^{[N+1]}_{k+2,J} n^{[N+1]}_{k+2,J}\Big\}\Big|_{\alpha^{\dagger[N+1]}_{1,k+2,\cdots,N}\sim 0}\Big|^{[N+1]}_{\mathrm{Sym}},
\end{aligned}
\tag{4.161}
$$

which proves Proposition 4. 21 .

4.3.5 Normalization, Triangularity and Expansion

We shall prove the last two properties of the Hi-Jack polynomials (4.50c) and (4.50d) and Propositions 4. 7 , 4. 8 and 4. 10 . Here again we implicitly assume that the Dunkl operators depend only on N variables $x_1, \cdots, x_N$. The restriction symbol without explicit indication of indices also means the restriction of the operand to symmetric functions of indices $1, 2, \cdots, N$.

First we shall prove the following proposition.

Proposition 4. 27 *Operation of symmetric polynomials of Dunkl operators* $\{\alpha_1^\dagger, \alpha_2^\dagger, \cdots, \alpha_N^\dagger\}$ *on any symmetric polynomials of* N *variables* $\{x_1, x_2, \cdots, x_N\}$ *yields symmetric polynomials of the* N *variables* $\{x_1, x_2, \cdots, x_N\}$.

Let us denote an arbitrary polynomial of $x_1, x_2, \cdots, x_N$ by $P(x_1, x_2, \cdots, x_N)$. Acting the Dunkl operator $\alpha_j^\dagger$ on $P(x_1, x_2, \cdots, x_N)$, we have

$$\begin{aligned} \alpha_j^\dagger P(x_1, x_2, \cdots, x_N) &= x_j P(x_1, x_2, \cdots, x_N) - \frac{1}{2\omega}\frac{\partial}{\partial x_j} P(x_1, x_2, \cdots, x_N) \\ &+ \frac{a}{2\omega}\sum_{\substack{i=1 \\ i \neq j}}^{N} \frac{1}{x_i - x_j}\big(P(x_1, x_2, \cdots, x_N) - P(x_i \leftrightarrow x_j)\big). \end{aligned} \tag{4.162}$$

It is obvious that the first and the second terms of the r.h.s. of eq. (4.162) are polynomials of $x_1, x_2, \cdots, x_N$. Since the difference of polynomials $P(x_1, x_2, \cdots, x_N) - P(x_i \leftrightarrow x_j)$ has a zero at $x_i = x_j$, the third term is also a polynomial of $x_1, x_2, \cdots, x_N$. Proposition 4. 27 follows from this property. We note that the orders of the second and the third terms, which are the terms with the coefficient $1/2\omega$, are less than that of the first term by two in the r.h.s. of the above expression.

As a basis of symmetric polynomials, we employ the monomial symmetric polynomials [50, 69] defined by

$$m_\lambda(x_1, x_2, \cdots, x_N) = \sum_{\substack{\sigma:\ \text{distinct} \\ \text{permutation}}} x_{\sigma(1)}^{\lambda_1} x_{\sigma(2)}^{\lambda_2} \cdots x_{\sigma(N)}^{\lambda_N}, \tag{4.163}$$

where λ is a Young tableau (4.15). By definition, monomial symmetric polynomials are symmetric polynomials with respect to any exchange of indices $1, 2, \cdots, N$. Calculating the action of the Dunkl operator $\alpha_{\sigma(j)}^\dagger$ on the monomial $x_{\sigma(1)}^{\lambda_1} x_{\sigma(2)}^{\lambda_2} \cdots x_{\sigma(N)}^{\lambda_N}$, we have

$$\begin{aligned} \alpha_{\sigma(j)}^\dagger x_{\sigma(1)}^{\lambda_1} x_{\sigma(2)}^{\lambda_2} \cdots x_{\sigma(N)}^{\lambda_N} &= x_{\sigma(1)}^{\lambda_1} \cdots x_{\sigma(j)}^{\lambda_j+1} \cdots x_{\sigma(N)}^{\lambda_N} - \frac{1}{2\omega}\lambda_j x_{\sigma(1)}^{\lambda_1} \cdots x_{\sigma(j)}^{\lambda_j-1} \cdots x_{\sigma(N)}^{\lambda_N} \\ &+ \frac{a}{2\omega}\sum_{\substack{i=1 \\ i \neq j}}^{N} x_{\sigma(1)}^{\lambda_1} \cdots \overset{\sigma(i)}{\vee} \cdots \overset{\sigma(j)}{\vee} \cdots x_{\sigma(N)}^{\lambda^N} \\ &\times \begin{cases} \displaystyle\sum_{k=1}^{\lambda_i - \lambda_j} x_{\sigma(i)}^{\lambda_i - k} x_{\sigma(j)}^{\lambda_j + k - 1} & \lambda_i > \lambda_j \\ \displaystyle\Big(-\sum_{k=1}^{\lambda_j - \lambda_i} x_{\sigma(i)}^{\lambda_j - k} x_{\sigma(j)}^{\lambda_i + k - 1}\Big) & \lambda_i < \lambda_j \\ 0 & \lambda_i = \lambda_j \end{cases}. \end{aligned}$$

Defining the Young tableaux from the sets of integers that appear in the r.h.s. of the above expression as exponents by

$$\begin{aligned} \lambda^{(j\pm1)} &= \{\lambda_1, \cdots, \lambda_j \pm 1, \cdots, \lambda_N\}, \\ \lambda^{(i-k,j+k-1)} &= \{\lambda_1, \cdots, \lambda_i - k, \cdots, \lambda_j + k - 1, \cdots, \lambda_N\}, \\ \lambda^{(j-k,i+k-1)} &= \{\lambda_1, \cdots, \lambda_j - k, \cdots, \lambda_i + k - 1, \cdots, \lambda_N\}, \end{aligned}$$

where the integers in the r.h.s. are regarded to be arranged in the non-increasing order, we notice the following relations:

$$\lambda^{(j-1)} \overset{\mathrm{d}}{\leq} \lambda^{(j+1)}, \ \lambda^{(i-k,j+k-1)} \overset{\mathrm{d}}{\leq} \lambda^{(j+1)}, \ \lambda^{(j-k,i+k-1)} \overset{\mathrm{d}}{\leq} \lambda^{(j+1)}. \tag{4.164}$$

From eqs. (4.162), (4.163) and (4.164), we have the following result as a special case of Proposition 4. 27 :

Corollary 4. 28 *Operation of the monomial symmetric polynomial whose arguments are Dunkl operators on 1 yields a symmetric polynomial of the following expansion,*

$$\begin{aligned} m_\lambda(\alpha_1^\dagger, \alpha_2^\dagger, \cdots, \alpha_N^\dagger) \cdot 1 &= m_\lambda(x_1, x_2, \cdots, x_N) \\ &+ \sum_{\substack{\mu \overset{\mathrm{d}}{\leq} \lambda \text{ and } |\mu| < |\lambda| \\ \text{and } |\mu| \equiv |\lambda| \ (\mathrm{mod} 2)}} \Big(\frac{1}{2\omega}\Big)^{(|\lambda| - |\mu|)/2} u_{\lambda\mu}(a) m_\mu(x_1, x_2, \cdots, x_N), \end{aligned} \tag{4.165}$$

where an unspecified coefficient $(1/2\omega)^{(|\lambda|-|\mu|)/2} u_{\lambda\mu}(a)$ *is an integer coefficient polynomial of* a *and* $1/2\omega$. *In the above expansion, increasing the order of* $1/2\omega$ *by one causes decreasing of the weight of the symmetrized monomial by two.*

Next we shall consider the action of d_i operator on the monomial symmetric polynomials of $\alpha_1^\dagger, \alpha_2^\dagger, \cdots, \alpha_N^\dagger$. We shall consider the case where the length of the Young tableau λ, l is less than or equal to N, $l \leq N$, i.e., $\lambda = \{\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_l > 0\}$. From the monomial symmetric polynomial $m_\lambda(\alpha_1^\dagger, \alpha_2^\dagger, \cdots, \alpha_N^\dagger)$, we single out a monomial $(\alpha_{\sigma(1)}^\dagger)^{\lambda_1}(\alpha_{\sigma(2)}^\dagger)^{\lambda_2} \cdots (\alpha_{\sigma(l)}^\dagger)^{\lambda_l}$. Because of eq. (4.21d), we have

$$d_i (\alpha_{\sigma(1)}^\dagger)^{\lambda_1}(\alpha_{\sigma(2)}^\dagger)^{\lambda_2} \cdots (\alpha_{\sigma(l)}^\dagger)^{\lambda_l} \cdot 1 = \Big[d_i, (\alpha_{\sigma(1)}^\dagger)^{\lambda_1}(\alpha_{\sigma(2)}^\dagger)^{\lambda_2} \cdots (\alpha_{\sigma(l)}^\dagger)^{\lambda_l}\Big]\Big|_{\mathrm{Sym}} \cdot 1. \tag{4.166}$$

From eq. (4.21c), we can easily verify

$$\big[d_i, \alpha_j^\dagger\big] = \alpha_i^\dagger\Big\{\delta_{ij}\Big(1 + a\sum_{\substack{k=1 \\ k \neq i}}^{N} K_{ik}\Big) - a(1 - \delta_{ij})K_{ij}\Big\}. \tag{4.167}$$

Using the above formula, we get the following expressions:

$$\begin{aligned} &\Big[d_i, (\alpha_{\sigma(1)}^\dagger)^{\lambda_1}(\alpha_{\sigma(2)}^\dagger)^{\lambda_2} \cdots (\alpha_{\sigma(l)}^\dagger)^{\lambda_l}\Big]\Big|_{\mathrm{Sym}} \\ &= \Big\{\Big(\lambda_h + (N - l)a\Big)(\alpha_{\sigma(1)}^\dagger)^{\lambda_1}(\alpha_{\sigma(2)}^\dagger)^{\lambda_2} \cdots (\alpha_{\sigma(l)}^\dagger)^{\lambda_l} \end{aligned}$$

$$
\begin{aligned}
&+a\sum_{k=1}^{\lambda_h-1}\sum_{j=l+1}^{N}(\alpha^\dagger_{\sigma(1)})^{\lambda_1}\cdots\overset{h}{\vee}\cdots(\alpha^\dagger_{\sigma(l)})^{\lambda_l}(\alpha^\dagger_{\sigma(h)})^{k}(\alpha^\dagger_{\sigma(j)})^{\lambda_h-k}\\
&+a\sum_{j=h+1}^{l}(\alpha^\dagger_{\sigma(1)})^{\lambda_1}\cdots\overset{h}{\vee}\cdots\overset{j}{\vee}\cdots(\alpha^\dagger_{\sigma(l)})^{\lambda_l}\sum_{k=1}^{\lambda_h-\lambda_j}(\alpha^\dagger_{\sigma(h)})^{\lambda_h-k+1}(\alpha^\dagger_{\sigma(j)})^{\lambda_j+k-1}\\
&-a\sum_{j=1}^{h-1}(\alpha^\dagger_{\sigma(1)})^{\lambda_1}\cdots\overset{j}{\vee}\cdots\overset{h}{\vee}\cdots(\alpha^\dagger_{\sigma(l)})^{\lambda_l}\sum_{k=1}^{\lambda_j-\lambda_h}(\alpha^\dagger_{\sigma(h)})^{\lambda_j-k+1}(\alpha^\dagger_{\sigma(j)})^{\lambda_h+k-1}\Big\}\Big|_{\mathrm{Sym}},\\
&i=\sigma(h),\ 1\le h\le l,
\end{aligned}
\tag{4.168a}
$$

$$
\begin{aligned}
&\Big[d_i,(\alpha^\dagger_{\sigma(1)})^{\lambda_1}(\alpha^\dagger_{\sigma(2)})^{\lambda_2}\cdots(\alpha^\dagger_{\sigma(l)})^{\lambda_l}\Big]\Big|_{\mathrm{Sym}}\\
&=\Big\{-a\sum_{k=1}^{l}(\alpha^\dagger_{\sigma(1)})^{\lambda_1}\cdots\overset{k}{\vee}\cdots(\alpha^\dagger_{\sigma(l)})^{\lambda_l}(\alpha^\dagger_{\sigma(h)})^{\lambda_k}\\
&\quad-a\sum_{k=1}^{l}\sum_{m=1}^{\lambda_k-1}(\alpha^\dagger_{\sigma(1)})^{\lambda_1}\cdots\overset{k}{\vee}\cdots(\alpha^\dagger_{\sigma(l)})^{\lambda_l}(\alpha^\dagger_{\sigma(h)})^{\lambda_k-m}(\alpha^\dagger_{\sigma(k)})^{m}\Big\}\Big|_{\mathrm{Sym}},\\
&\quad i=\sigma(h),\ l+1\le h\le N.
\end{aligned}
\tag{4.168b}
$$

Note that the summation $\sum_{k=s}^{e}$, $e<s$ means zero, which occurs in the third and fourth term of eq. (4.168a) when $\lambda_h=\lambda_j$ or $h=1,l$. From the above calculation, we get the following result.

Proposition 4. 29 *No higher order monomial in the dominance order is generated by the action of d_i operator on a monomial $(\alpha_1^\dagger)^{\lambda_{\sigma(1)}}(\alpha_2^\dagger)^{\lambda_{\sigma(2)}}\cdots(\alpha_N^\dagger)^{\lambda_{\sigma(N)}}\cdot 1$, where $\lambda=\{\lambda_1\ge\lambda_2\ge\cdots\ge\lambda_N\ge 0\}$ and $\sigma\in S_N$, and the coefficients of the monomials are integer coefficient polynomials of a. The weight of the Young tableaux of the monomials are the same as that of the original monomial.*

This property causes the triangularity of the Hi-Jack polynomials (4.50c). From the definition of the raising operator of the Hi-Jack polynomial and the above proposition, we notice the weak form of the third requirement of Proposition 4. 3 .

Proposition 4. 30

$$
\begin{aligned}
C_\lambda j_\lambda(\boldsymbol{x};\omega,1/a) &= (b_N^+)^{\lambda_N}(b_{N-1}^+)^{\lambda_{N-1}-\lambda_N}\cdots(b_1^+)^{\lambda_1-\lambda_2}\cdot 1\\
&= \sum_{\mu\overset{\mathrm{D}}{\le}\lambda}\mathrm{v}_{\lambda\mu}(a)m_\mu(\alpha_1^\dagger,\cdots,\alpha_N^\dagger)\cdot 1,
\end{aligned}
\tag{4.169}
$$

where $\mathrm{v}_{\lambda\mu}(a)$ is an unspecified integer coefficient polynomial of a.

Combining Corollary 4. 28 and Proposition 4. 30 , we can easily confirm the first formula of Proposition 4. 9 symmetrized and hence the third requirement of Proposition 4. 3 , i.e., eq. (4.50c). To verify the second formula of Proposition 4. 9 , or equivalently, the fourth requirement (4.50d), we have to show

$$
\mathrm{v}_{\lambda\lambda}(a)=C_\lambda. \tag{4.170}
$$

Computation of $\mathrm{v}_{\lambda\lambda}(a)$ needs consideration on the cancellation among the monomials in eq. (4.168).

After summing over the distinct permutation $\sigma\in S_N$, the terms that come from the third and the fourth terms of (4.168a) cancel out and vanish, for the fourth term with the permutation σ replaced by $\sigma'=\sigma(jh)$ yields

$$
\begin{aligned}
&-a(\alpha^\dagger_{\sigma'(1)})^{\lambda_1}\cdots\overset{h}{\vee}\cdots\overset{j}{\vee}\cdots(\alpha^\dagger_{\sigma'(l)})^{\lambda_l}\sum_{k=1}^{\lambda_h-\lambda_j}(\alpha^\dagger_{\sigma'(j)})^{\lambda_h-k+1}(\alpha^\dagger_{\sigma'(h)})^{\lambda_j+k-1}\\
&\quad=-a(\alpha^\dagger_{\sigma(1)})^{\lambda_1}\cdots\overset{h}{\vee}\cdots\overset{j}{\vee}\cdots(\alpha^\dagger_{\sigma(l)})^{\lambda_l}\sum_{k=1}^{\lambda_h-\lambda_j}(\alpha^\dagger_{\sigma(h)})^{\lambda_h-k+1}(\alpha^\dagger_{\sigma(j)})^{\lambda_j+k-1},
\end{aligned}
\tag{4.171}
$$

which is the summand in the third term of eq. (4.168a) with negative sign. Similar cancellation also occurs between the second term in the bracket of the first term of eq. (4.168a) and the first term of eq. (4.168b) with σ replaced by $\sigma'=\sigma(kh)$, $l+1\le k\le N$, where σ in the r.h.s. is that of eq. (4.168a):

$$
-a(\alpha^\dagger_{\sigma'(1)})^{\lambda_1}\cdots\overset{h}{\vee}\cdots(\alpha^\dagger_{\sigma'(l)})^{\lambda_l}(\alpha^\dagger_{\sigma'(k)})^{\lambda_h}=-a(\alpha^\dagger_{\sigma(1)})^{\lambda_1}(\alpha^\dagger_{\sigma(2)})^{\lambda_2}\cdots(\alpha^\dagger_{\sigma(l)})^{\lambda_l}. \tag{4.172}
$$

Since there are $N-l$ permutations σ' that yield the monomial $-a(\alpha^\dagger_{\sigma(1)})^{\lambda_1}(\alpha^\dagger_{\sigma(2)})^{\lambda_2}\cdots(\alpha^\dagger_{\sigma(l)})^{\lambda_l}$ through the commutator (4.168b), we can cancel the second term of the first bracket in the first term of eq. (4.168a). Thus the coefficient of the term $(\alpha^\dagger_{\sigma(1)})^{\lambda_1}\cdots(\alpha^\dagger_{\sigma(l)})^{\lambda_l}$ coming from the commutator $\Big[d_i,m_\lambda(\alpha_1^\dagger,\alpha_2^\dagger,\cdots,\alpha_N^\dagger)\Big]\Big|_{\mathrm{Sym}}$ is λ_h where $i=\sigma(h)$.

Since the Hi-Jack polynomial is symmetric with respect to the exchange of indices $1,\cdots,N$, we have only to calculate the coefficient of $(\alpha_1^\dagger)^{\lambda_1}\cdots(\alpha_l^\dagger)^{\lambda_l}$. In the following calculation, we shall omit all the monomials except for $(\alpha_1^\dagger)^{\lambda_1}\cdots(\alpha_l^\dagger)^{\lambda_l}$, namely the monomial with identity permutation. Any lower order monomial and the same order monomial with different permutation are omitted in the expression. However, we implicitly sum up over the distinct permutations to use the above cancellation. To know the coefficient of the monomial of interest that is yielded from $b_l^+m_\lambda(\alpha_1^\dagger,\cdots,\alpha_N^\dagger)\cdot 1$, we have only to do the following calculation using eq. (4.168):

$$
\begin{aligned}
&d_{1,\{l,\cdots,1\}}(\alpha_1^\dagger)^{\lambda_1}\cdots(\alpha_l^\dagger)^{\lambda_l}\cdot 1\\
&=d_{1,\{l,\cdots,2\}}\Big\{\Big[d_1,(\alpha_1^\dagger)^{\lambda_1}\cdots(\alpha_l^\dagger)^{\lambda_l}\Big]\Big|_{\mathrm{Sym}}+la(\alpha_1^\dagger)^{\lambda_1}\cdots(\alpha_l^\dagger)^{\lambda_l}\Big\}\cdot 1\\
&=d_{1,\{l,\cdots,2\}}\Big\{(\lambda_1+la)(\alpha_1^\dagger)^{\lambda_1}\cdots(\alpha_l^\dagger)^{\lambda_l}+a(N-l)(\alpha_1^\dagger)^{\lambda_1}\cdots(\alpha_l^\dagger)^{\lambda_l}+a\sum_{\substack{i=2\\ \lambda_i<\lambda_1}}^{N}(\alpha_1^\dagger)^{\lambda_1}\cdots(\alpha_l^\dagger)^{\lambda_l}\Big\}\cdot 1.
\end{aligned}
\tag{4.173}
$$

Summing over the distinct permutation, we can cancel out the second and third term as has been explained in eqs. (4.171) and (4.172). Next, we operate d_2 on the operand:

$$
\begin{aligned}
&d_{1,\{l,\cdots,1\}}(\alpha_1^\dagger)^{\lambda_1}\cdots(\alpha_l^\dagger)^{\lambda_l}\cdot 1\\
&=(\lambda_1+la)d_{1,\{l,\cdots,3\}}\Big\{(\lambda_2+(l-1)a)(\alpha_1^\dagger)^{\lambda_1}\cdots(\alpha_l^\dagger)^{\lambda_l}+(N-l)(\alpha_1^\dagger)^{\lambda_1}\cdots(\alpha_l^\dagger)^{\lambda_l}\\
&\quad-a\sum_{\substack{i=1\\ \lambda_i>\lambda_2}}^{1}(\alpha_2^\dagger)^{\lambda_i}(\alpha_i^\dagger)^{\lambda_2}(\alpha_3^\dagger)^{\lambda_3}\cdots(\alpha_l^\dagger)^{\lambda_l}+a\sum_{\substack{i=3\\ \lambda_i<\lambda_2}}^{N}(\alpha_1^\dagger)^{\lambda_1}\cdots(\alpha_l^\dagger)^{\lambda_l}\Big\}\cdot 1.
\end{aligned}
\tag{4.174}
$$

Since we have already operated d_1 on the monomial symmetric polynomial $m_\lambda(\alpha_1^\dagger, \cdots, \alpha_N^\dagger)$, summation over the distinct permutation is not anymore invariant under the permutation of the indices $1, \cdots, N$, but invariant under the permutation of indices $2, \cdots, N$. To cancel out the second and fourth terms, we only need transpositions $(2j)$, $3 \leq j \leq N$. The third term, which is a monomial with the same Young tableau and a permutation (12), can not be canceled out because of the break of invariance under the permutation involving the index 1. However, this monomial can not be changed to the monomial with identity permutation by operating d_i, $i \geq 3$. Thus, we have

$$d_{1,\{l,\cdots,1\}}(\alpha_1^\dagger)^{\lambda_1} \cdots (\alpha_l^\dagger)^{\lambda_l} \cdot 1 = (\lambda_1 + la)(\lambda_2 + (l-1)a) d_{1,\{l,\cdots,3\}}(\alpha_1^\dagger)^{\lambda_1} \cdots (\alpha_l^\dagger)^{\lambda_l} \cdot 1. \tag{4.175}$$

Repeating analogous calculations, we get

$$d_{1,\{l,\cdots,1\}}(\alpha_1^\dagger)^{\lambda_1} \cdots (\alpha_l^\dagger)^{\lambda_l} \cdot 1 = (\lambda_1 + la)(\lambda_2 + (l-1)a) \cdots (\lambda_l + a)(\alpha_1^\dagger)^{\lambda_1} \cdots (\alpha_l^\dagger)^{\lambda_l} \cdot 1. \tag{4.176}$$

Then the following expansion follows from the above formula,

$$\begin{aligned} & b_l^+ m_\lambda(\alpha_1^\dagger, \cdots, \alpha_N^\dagger) \cdot 1 \\ = & \Big\{(\lambda_1 + la)(\lambda_2 + (l-1)a) \cdots (\lambda_l + a) m_{\lambda+1^l}(\alpha_1^\dagger, \cdots, \alpha_N^\dagger) \\ & + \sum_{\substack{\mu \overset{\mathrm{D}}{\leq} \lambda + 1^l \\ \mu \neq \lambda}} y_{\lambda+1^l \mu}(a) m_\mu(\alpha_1^\dagger, \cdots, \alpha_N^\dagger)\Big\} \cdot 1, \end{aligned} \tag{4.177}$$

where $\lambda + 1^l = \{\lambda_1 + 1, \cdots, \lambda_l + 1\}$ and $y_{\lambda+1^l\mu}(a)$ is an unspecified integer coefficient polynomial of a. We remark that 1 is a monomial symmetric polynomial with $\lambda = 0$. Then from the definition of the raising operator (4.54) and repeated use of eq. (4.177), we finally verify eq. (4.170):

Lemma 4. 31

$$\mathrm{v}_{\lambda\lambda}(a) = C_\lambda.$$

Combining Corollary 4. 28 , Proposition 4. 30 and Lemma 4. 31 , we can confirm the third and fourth requirements of Proposition 4. 3 , eqs. (4.50c) and (4.50d), and Propositions 4. 7 and 4. 9 .

In the limit $\omega \to \infty$, eq. (4.169) reduces to

$$\begin{aligned} C_\lambda j_\lambda(\boldsymbol{x}; \omega \to \infty, 1/a) &= (b_N^+)^{\lambda_N} (b_{N-1}^+)^{\lambda_{N-1} - \lambda_N} \cdots (b_1^+)^{\lambda_1 - \lambda_2} \cdot 1 \Big|_{\omega \to \infty} \\ &= \sum_{\mu \overset{\mathrm{D}}{\leq} \lambda} \mathrm{v}_{\lambda\mu}(a) m_\mu(x_1, \cdots, x_N). \end{aligned} \tag{4.178}$$

As has been remarked in Section 4.1, the Dunkl operators for the Calogero model reduce to corresponding Dunkl operators (4.49) for the Sutherland model in the limit $\omega \to \infty$. Then it

is obvious that the Hi-Jack polynomials in the limit $\omega \to \infty$ are the Jack polynomials. Thus we have

$$C_\lambda J_\lambda(\boldsymbol{x}; 1/a) = \sum_{\mu \overset{\mathrm{D}}{\leq} \lambda} \mathrm{v}_{\lambda\mu}(a) m_\mu(x_1, \cdots, x_N), \tag{4.179}$$

which means the coefficients $v_{\lambda\mu}(a)$ in eq. (4.38b) and $\mathrm{v}_{\lambda\mu}(a)$ in (4.169) are essentially same:

$$\mathrm{v}_{\lambda\mu}(a) = C_\lambda v_{\lambda\mu}(a). \tag{4.180}$$

This proves Proposition 4. 8 . Calculating the $\omega \to \infty$ limit of Proposition 4. 9 , we have

$$\begin{aligned} j_\lambda(\boldsymbol{x}; \omega \to \infty, 1/a) &= \sum_{\mu \overset{\mathrm{D}}{\leq} \lambda} w_{\lambda\mu}(a) m_\mu(\boldsymbol{x}) \\ &= J_\lambda(\boldsymbol{x}; 1/a), \end{aligned}$$

which shows the top weight monomial symmetric functions in the expansion of the Hi-Jack polynomial (4.70a), or equivalently, eq. (4.50c) form the Jack polynomial of the same Young tableau. Since the monomial symmetric polynomials $m_\lambda(\boldsymbol{x})$ and the Jack polynomials $J_\lambda(\boldsymbol{x}; 1/a)$ respectively form the bases of the symmetric polynomials, the lower weight monomial symmetric polynomials can be rewritten by the lower weight Jack polynomials. Thus we confirm Proposition 4. 10 .

4.3.6 Orthogonality

In our proof of Proposition 4. 11 , it is convenient to use some results in previous sections.

Proposition 4. 32 *Operation of d_i operator on a monomial of $\alpha_k^\dagger$'s, $(\alpha_1^\dagger)^{\lambda_{\sigma(1)}} \cdots (\alpha_N^\dagger)^{\lambda_{\sigma(N)}} \cdot 1$, where $\lambda \in Y_N$ and $\sigma \in S_N$ yields the monomials of $\alpha_k^\dagger$'s of the form, $(\alpha_1^\dagger)^{\mu_{\tau(1)}} \cdots (\alpha_N^\dagger)^{\mu_{\tau(N)}} \cdot 1$, where $\mu \in Y_N$, $\mu \overset{\mathrm{D}}{\leq} \lambda$ and $\tau = (i, h)\sigma \in S_N$, $1 \leq h \leq N$.*

The above assertion is a part of Proposition 4. 29 . We also use the following property.

Proposition 4. 33 *The Hi-Jack polynomials are expanded into the monomial symmetric polynomials of $\alpha_k^\dagger$'s as*

$$\begin{aligned} j_\lambda(\boldsymbol{x}; \omega, 1/a) &= \sum_{\mu \overset{\mathrm{D}}{\leq} \lambda} v_{\lambda\mu}(a) m_\mu(\alpha_1^\dagger, \cdots, \alpha_N^\dagger) \cdot 1 && (4.181\mathrm{a}) \\ &= J_\lambda(\alpha_1^\dagger, \cdots, \alpha_N^\dagger; 1/a) \cdot 1, && (4.181\mathrm{b}) \end{aligned}$$

where $v_{\lambda\lambda}(a) = 1 \neq 0$.

This result is shown in Propositions 4. 8 and 4. 30 . We can readily see that the above proposition and Corollary 4. 28 lead to the triangularity of the Hi-Jack polynomial (4.50c). Here we only use the triangularity (4.181a) and do not mind if the r.h.s. corresponds with the Jack polynomial.

First, we shall prove that the Hi-Jack polynomials are the simultaneous eigenfunctions of all the commuting conserved operators $\{I_n | 1 \leq n \leq N\}$. By definition, the Hi-Jack polynomials are the simultaneous eigenfunctions of the first two conserved operators, I_1 and I_2. Due to

Corollary 4. 28 and Propositions 4. 32 and 4. 33 , operation of I_n on a Hi-Jack polynomial j_λ yields

$$\begin{aligned} I_n \cdot j_\lambda(\boldsymbol{x};\omega,1/a) &= \sum_{l=1}^{N}(d_l)^n \sum_{\mu \overset{\mathrm{D}}{\leq} \lambda} v_{\lambda\mu} m_\mu(\alpha_1^\dagger,\cdots,\alpha_N^\dagger)\cdot 1 \\ &= \sum_{\mu \overset{\mathrm{D}}{\leq} \lambda} v'_{\lambda\mu} m_\mu(\alpha_1^\dagger,\cdots,\alpha_N^\dagger)\cdot 1 \\ &= \sum_{\substack{\mu \overset{\mathrm{D}}{\leq} \lambda \\ \text{or } |\mu|<|\lambda|}} w'_{\lambda\mu} m_\lambda(x_1,\cdots,x_N). \end{aligned} \tag{4.182}$$

Since the n-th conserved operator commutes with the first and second conserved operator, $[I_1, I_n] = [I_2, I_n] = 0$, we have

$$I_1 I_n j_\lambda(\boldsymbol{x};\omega,1/a) = E_1(\lambda) I_n j_\lambda(\boldsymbol{x};\omega,1/a), \tag{4.183a}$$
$$I_2 I_n j_\lambda(\boldsymbol{x};\omega,1/a) = E_2(\lambda) I_n j_\lambda(\boldsymbol{x};\omega,1/a). \tag{4.183b}$$

Equations (4.183a), (4.183b) and (4.182) for $I_n j_\lambda$ are respectively the same as eqs. (4.50a), (4.50b) and (4.50c) for the Hi-Jack polynomial j_λ. We have proved in Section 4.3.1 the uniqueness of the Hi-Jack polynomial j_λ determined by the definition (4.50). Therefore we conclude that $I_n j_\lambda$ must coincide with j_λ up to normalization. Thus we confirm that the Hi-Jack polynomials j_λ simultaneously diagonalize all the commuting conserved operators I_n, $n = 1,\cdots,N$.

Next, we shall verify that there is no degeneracy in the eigenvalues. The eigenvalue of the n-th conserved operator for a Hi-Jack polynomial j_λ is denoted by $E_n(\lambda)$:

$$I_n j_\lambda(\boldsymbol{x};\omega,1/a) = E_n(\lambda) j_\lambda(\boldsymbol{x};\omega,1/a),\ n = 1,\cdots,N. \tag{4.184}$$

Both the conserved operator I_n and the Hi-Jack polynomial j_λ are polynomials of the coupling parameter a, which can be readily recognized from the definition of the conserved operators (4.24) and Proposition 4. 7 . Thus the eigenvalue $E_n(\lambda)$ is also a polynomial of a,

$$E_n(a) = e_n^{(0)}(\lambda) + e_n^{(1)}(\lambda)a + \cdots. \tag{4.185}$$

Considering the case $a = 0$, which corresponds to N free bosons confined in an external harmonic well, we can easily obtain the constant term $e_n^{(0)}(\lambda)$ as

$$e_n^{(0)}(\lambda) = \sum_{k=1}^{N}(\lambda_k)^n. \tag{4.186}$$

It is clear that there is no degeneracy in the constant terms of the eigenvalues $\{e_n^{(0)}(\lambda)|n = 1,\cdots,N\}$. We shall confirm it by showing uniqueness of the solution of the following algebraic equations:

$$e_n^{(0)}(\lambda) = \mathsf{p}_n,\ \ n = 1,2,\cdots,N. \tag{4.187}$$

We define the elementary symmetric polynomial by

$$\mathsf{e}_n(\lambda_1,\lambda_2,\cdots,\lambda_N) = \sum_{\substack{\sigma:\text{distinct}\\ \text{permutations}}} \lambda_{\sigma(1)}\lambda_{\sigma(2)}\cdots\lambda_{\sigma(n)}. \tag{4.188}$$

The generating function of the elementary symmetric polynomials is given by

$$\mathsf{E}(t) \overset{\text{def}}{=} \sum_{n=0}^{N}\mathsf{e}_n(\lambda)t^n = \prod_{k=1}^{N}(1+\lambda_k t). \tag{4.189}$$

From the above definition of the generating function, it is obvious that the solution of the algebraic equations,

$$\mathsf{e}_n(\lambda) = \mathsf{e}_n,\ \ n = 1,2,\cdots,N, \tag{4.190}$$

are uniquely determined by the solution of the algebraic equation,

$$\mathsf{E}(t) \overset{\text{def}}{=} \sum_{n=1}^{N}\mathsf{e}_n t^n = 0 \Rightarrow \lambda_k = -\frac{1}{t_k},\ k = 1,2,\cdots,N,\ t_1 \geq t_2 \geq \cdots \geq t_N, \tag{4.191}$$

where t_k is the k-th zero of the generating function in non-increasing order. For convenience, we take the maximum number of the non-zero elements of the Young tableau as infinity, $N = \infty$. The logarithmic derivative of the generating function (4.189) yields

$$\frac{\mathrm{d}}{\mathrm{d}t}\log\mathsf{E}(t) = \sum_{k=1}^{\infty}\frac{\lambda_k}{1+\lambda_k t} = \sum_{n=1}^{\infty}\mathsf{p}_n(\lambda)(-t)^{n-1},$$

where $\mathsf{p}_n(\lambda)$ is the power sum of degree n. Paying attention to the fact that $\mathsf{E}(0) = 1$, we can calculate the definite integral of the above expression over t:

$$\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n}\mathsf{p}_n(\lambda)t^n = \log\Big(\sum_{n=0}^{\infty}\mathsf{e}_n(\lambda)t^n\Big), \tag{4.192a}$$
$$\sum_{n=0}^{\infty}\mathsf{e}_n(\lambda)t^n = \exp\Big(\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n}\mathsf{p}_n(\lambda)t^n\Big). \tag{4.192b}$$

Comparing the terms with the same order of t, we can transform the power sums to the elementary symmetric polynomials and vice versa. We should note that no higher order polynomial appears in the transformation given by the above relations:

$$\begin{aligned} \mathsf{p}_n &= \mathsf{p}_n(\mathsf{e}_1,\mathsf{e}_2,\cdots,\mathsf{e}_n), \\ \mathsf{e}_n &= \mathsf{e}_n(\mathsf{p}_1,\mathsf{p}_2,\cdots,\mathsf{p}_n). \end{aligned}$$

For given constant parts of the eigenvalues, p_n, $n = 1,2,\cdots,N$, we can uniquely identify the corresponding parameters, e_n, $n = 1,2,\cdots,N$, by the above transformation (4.192b). Then the algebraic equations (4.187) are cast into another forms (4.190). Thus the solution of the algebraic equations (4.187) are uniquely given by the zeroes of the algebraic equation (4.191) and hence we have proved that there is no degeneracy in the constant terms of the eigenvalues. Since the conserved operators I_n are Hermitian operators concerning the inner product (4.76), this proves that the Hi-Jack polynomials are the orthogonal symmetric polynomials with respect to the inner product.

From the explicit form of the weight function,

$$|\hat{\phi}_{\mathrm{g}}|^2 = \prod_{1\leq j<k\leq N}|x_j - x_k|^{2a}\exp\Big(-\omega\sum_{l=1}^{N}x_l^2\Big), \tag{4.193}$$

we conclude that the Hi-Jack polynomials are a multivariable generalization of the Hermite polynomials [46]. We note that a proof based on the isomorphism between the Dunkl operators for the Hi-Jack polynomials and those for the Jack polynomials [62, 83] has been reported [37, 84]. Our proof is a direct proof that does not rely on the isomorphism. As has been pointed out before, all the Dunkl operators for the Hi-Jack polynomials reduce to those for the Jack polynomials in the limit, $\omega \to \infty$. Thus our proof contains a proof of orthogonality of the Jack polynomials as a special case.

4.4 Summary

We have studied the orthogonal basis of the Calogero model. Two crucial observations are the reasons why we have introduced the Hi-Jack polynomials by Definition 4. 1 in a similar way to a definition of the Jack polynomials, eqs. (4.38), which form the orthogonal basis of the Sutherland model. One is the observation of the explicit forms of the first seven orthogonal symmetric polynomials (3.34) in Chapter 3 that indicates a similarity between these polynomials and the Jack polynomials. The other is the common algebraic structure of the Calogero and Sutherland models, which has been explicitly shown in Section 4.1. We have introduced the elementary deformation of the Young tableau which generates all the Young tableaux μ that meet $\mu \overset{\mathrm{D}}{\leq} \lambda$ from a Young tableau λ. Using the property of the deformation, we have confirmed that our definition of the Hi-Jack polynomials, Definition 4. 1 , uniquely specifies the Hi-Jack polynomials. We have proved that the functions generated by the Rodrigues formula [83] that is an extension of the Rodrigues formula for the Jack symmetric polynomials discovered by Lapointe and Vinet [42] satisfy the definition of the Hi-Jack polynomials. Our proof is based on the algebraic relations among the Dunkl operators. In the consideration of their normalizations, we have clarified that expansions of the Hi-Jack symmetric polynomials in terms of the monomial symmetric polynomials have triangular forms, as is similar to the Jack symmetric polynomials. We have also confirmed that the Hi-Jack symmetric polynomials exhibits the integrality corresponding to the weak form of the Macdonald-Stanley conjecture for the Jack symmetric polynomials [41]. The Hi-Jack symmetric polynomials and the eigenfunctions for the Hamiltonian that were algebraically constructed through the quantum Lax formulation in Chapter 3 are related by the transformation between the Jack symmetric polynomials and the power sum symmetric polynomials. We have studied on the orthogonality of the Hi-Jack polynomials. The orthogonal basis provides a very useful tool for the study of physical quantities in quantum theory. The orthogonality of the Hi-Jack symmetric polynomials is expected to be important in the exact calculation of the thermodynamic quantities such as the Green functions and the correlation functions, as has been done for the Sutherland model using the properties of the Jack polynomials [29, 30, 32, 48]. Orthogonality of the Jack polynomials is proved by showing that all the commuting conserved operators of the Sutherland model $\{\mathcal{I}_k | k = 1, 2, \cdots, N\}$ are simultaneously diagonalized by the Jack polynomials [49, 50]. Considering the correspondence between the Calogero model and the Sutherland model, we can expect that all the conserved operators of the Calogero model $\{I_k | k = 1, 2, \cdots, N\}$ are also diagonalized by the Hi-Jack polynomials. This expectation has been verified. We have verified that the Hi-Jack polynomial on which the n-th conserved operator operates, $I_n j_\lambda$, also satisfies the definition of the Hi-Jack polynomial except for the normalization condition. This means that $I_n j_\lambda$ coincides with the Hi-Jack polynomial j_λ up to a scalar factor. Therefore the Hi-Jack polynomials are the simultaneous eigenfunctions of all the conserved operators. We have calculated the eigenvalues of all the conserved operators for the case when the coupling parameter a is zero and have found out there is no degeneracy. Thus we have proved the orthogonality of the Hi-Jack polynomials with respect to the inner product (4.76). From the explicit form of the weight function of the inner product, we have confirmed that the Hi-Jack polynomials which we have found should be regarded as a multivariable generalization of the Hermite polynomials.

Chapter 5

Summary and Concluding Remarks

In this thesis, we have studied the quantum Calogero model in an algebraic fashion. In the classical theory, the Calogero model is known to be a completely integrable system. The model has a Lax formulation, which has been developed as a powerful tool for various completely integrable systems. For the classical Calogero model, the Lax formulation not only shows its integrability, but also gives a way to solve the initial value problem of the model. In this sense, the Calogero model is a special model even in the classical integrable systems, because the notion of integrability just asserts the existence of the canonical transformation to the action-angle variables. However, because of the non-commutativity between the canonical conjugate variables, the merits of the Lax formulation for the model seemed to be completely destroyed in the quantum theory. This difficulty has motivated us to find a powerful method to study the integrability and the underlying symmetry of the quantum Calogero model. We have also wanted to find out an elegant way to construct the eigenfunctions. Though the energy eigenvalue problem was solved by Calogero in 1971 [19], an algebraic construction of the energy eigenfunctions has not been completed. We have pursued more detailed information on the eigenfunctions of the quantum Calogero model. Quantum integrability means that we can identify, in principle, all the quantum number of the system by diagonalizing all the mutually commuting conserved operators. But just proving it possible is far different from doing it in practice. Thus we have aimed at giving a method to construct the simultaneous eigenfunctions of the commuting conserved operators and hence to identify the orthogonal basis of the quantum Calogero model.

In Chapter 2, we have developed the quantum Lax formulation of the Calogero model, which is a natural generalization of the Lax formulation for the classical one, introduced the Dunkl operator formulation and investigated the algebraic structure of the quantum Calogero model in the framework of the quantum Lax formulation and the Dunkl operator formulation. From the Lax equation for the classical model, we have obtained the Lax equation for the quantum Calogero Hamiltonian whose M-matrix satisfies the sum-to-zero condition. The fact enables us to construct a set of the conserved operators of the quantum model, as was the case with the quantum Calogero-Moser model [34, 74, 75, 88, 89]. To show the quantum integrability of the Calogero model, we have considered a construction of commuting conserved operators, eqs. (2.12) and (2.20). By use of the explicit forms of the first two conserved operators, I_1 and I_2 (2.13), or $\tilde{I}_1$ and $\tilde{I}_2$ (2.21), we have obtained the first two of the generalized Lax equations for the commuting conserved operators (2.16) and have conjectured the general form (2.22). However the generalized Lax equations for the commuting conserved operators are not compatible with the recursive construction. To study the recursive construction of the generalized Lax equations, we have introduced a series of operators that are made by summation of all the elements of the Weyl ordered product of L^+- and L^-- matrices (2.29). From the explicit forms of the first few conserved operators (2.26) and power-sum creation-annihilation operators (2.30), we have obtained the corresponding generalized Lax equations, eqs. (2.28), (2.32) and (2.33). From these first few generalized Lax equations, we have found out the recursion formulae, eqs. (2.36) and (2.53). Using the recursion formulae, we have recursively constructed the generalized Lax equations (2.52). The generalized Lax equations prove the mutual commutativity of the power-sum creation-annihilation operators, which have played an important role in the algebraic construction of the energy eigenfunctions [76, 77] in Chapter 3. Defining the $W_n^{(s)}$-operators by eq. (2.60), we have proved that the generalized Lax equations yields the W-algebra as a commutator algebra among the $W_n^{(s)}$-operators. We have studied correspondences between the quantum Lax formulation [76–78, 80] and the Dunkl operator formulation [27] that was introduced by Polychronakos into the problems on the inverse-square interaction models [62]. The Dunkl operator formulation provides us a simple way of constructing the commuting conserved operators of the Calogero model (2.69). We have observed that the restriction of the operand to the wave functions of the identical particles enables us to translate the results in one of the theories into those in the other. To be concrete, we have related arbitrary operators made from the two matrices, L^+ and L^-, and their commutator algebras with those in the Dunkl operator formulation, eqs. (2.73) and (2.76). A method to directly obtain the M_m^p-matrices has also been given as eq. (2.97). Mutual commutativity of the conserved operators I_n has been proved. The simultaneous eigenfunctions of these mutually commuting conserved operators have been studied in Chapters 3 and 4.

In Chapter 3, we have studied an algebraic construction of all the eigenfunctions of the Calogero Hamiltonian with the help of the quantum Lax formulation [76–78]. Our approach is based on Perelomov's idea on an algebraic treatment of the eigenfunctions of the Calogero model. By the factorization of the Hamiltonian, we have obtained the ground state wave function. Using the power-sum creation operators which has been obtained in Chapter 2, we have obtained an algebraic method to construct the eigenfunctions of the Calogero model. From the number of independent eigenfunctions, we have confirmed that the eigenfunctions form the basis of the Hilbert space of the Calogero model. We have also reproduced the result for the original Calogero model by fixing the center of mass at the coordinate origin. Thus we have completed Perelomov's dream of the algebraic construction of the eigenfunctions of the Calogero model. Another approach to the algebraic construction of the eigenfunction of the Calogero Hamiltonian that uses the Dunkl operators was also reported [17, 18], which has been briefly summarized in Chapter 4. We have also considered a construction of the orthogonal basis of the Calogero model by diagonalizing mutually commuting conserved operators. We have directly diagonalized the first nontrivial conserved operator $\tilde{I}_2$ using the eigenfunctions with weights up to 6. The results indicate a general formula for the eigenvalue of $\tilde{I}_2$. In addition, we have presented explicit expressions of the first seven of the unidentified orthogonal symmetric polynomials associated with the Calogero model, which have been identified as the Hi-Jack polynomials in Chapter 4. From the explicit form of the weight function of the inner-product, we have concluded that the orthogonal symmetric polynomial should be regarded as a multivariable generalization of the Hermite polynomial. From the eigenvalue formula for the commuting conserved operator I_2 and the expansion of the explicit forms with respect to the monomial symmetric function, we have observed a similarity between the Jack polynomials and

the seven orthogonal symmetric polynomials. The results have given an important suggestion toward the definition of the Hi-Jack polynomials.

In Chapter 4, we have studied the orthogonal basis of the Calogero model further and also have completed the construction of the orthogonal basis in an algebraic fashion [79, 83, 84, 86]. Based on the fact that the Calogero model has a set of mutually commuting conserved operators [62, 76–78, 80], we have tried to construct the simultaneous eigenfunctions for all the conserved operators of the Calogero model that must form the orthogonal basis of the model. Since the Calogero and the Sutherland models share the common algebraic structure, it is natural to introduce the Hi-Jack symmetric polynomials in a similar way to a definition of the Jack polynomials [83, 84]. The results in Chapter 3 also support this expectation. We have introduced the elementary deformation of the Young tableau which generates all the Young tableaux μ that meet $\mu \overset{\mathrm{D}}{\leq} \lambda$ from a Young tableau λ. Using the property of the deformation, we have confirmed that our definition of the Hi-Jack polynomials, Definition 4. 1 , uniquely specifies the Hi-Jack polynomials. We have proved that the functions generated by the Rodrigues formula [83] satisfy the definition of the Hi-Jack polynomials. This result is an extension of the Rodrigues formula for the Jack symmetric polynomials discovered by Lapointe and Vinet [42]. Our proof is based on the algebraic relations among the Dunkl operators. In the consideration of their normalizations, we have clarified that expansions of the Hi-Jack symmetric polynomials in terms of the monomial symmetric polynomials have triangularity, as is similar to the Jack symmetric polynomials. We have also confirmed that the Hi-Jack symmetric polynomials exhibits the integrality corresponding to the weak form of the Macdonald-Stanley conjecture for the Jack symmetric polynomials [41]. The Hi-Jack symmetric polynomials and the eigenfunctions for the Hamiltonian that was algebraically constructed through the quantum Lax formulation in Chapter 3 are related by the transformation between the Jack symmetric polynomials and the power sum symmetric polynomials. We have confirmed the orthogonality of the Hi-Jack polynomials. The orthogonal basis provides a very useful tool for the study of physical quantities in quantum theory. The orthogonality of the Hi-Jack symmetric polynomials is expected to be important in the exact calculation of the thermodynamic quantities such as the Green functions and the correlation functions, as has been done for the Sutherland model using the properties of the Jack polynomials [29–32, 48]. Orthogonality of the Jack polynomials is proved by showing that all the commuting conserved operators of the Sutherland model, $\mathcal{I}_k$, $k = 1, 2, \cdots, N$, are simultaneously diagonalized by the Jack polynomials [49, 50]. Considering the correspondence between the Calogero model and the Sutherland model, we can expect that all the conserved operators of the Calogero model, I_k, $k = 1, 2, \cdots, N$, are also diagonalized by the Hi-Jack polynomials. This expectation has been verified. We have proved that the Hi-Jack polynomial on which the n-th conserved operator operates, $I_n j_\lambda$, also satisfies the definition of the Hi-Jack polynomial except for the normalization condition. This means that $I_n j_\lambda$ coincides with the Hi-Jack polynomial j_λ up to a scalar factor and therefore the Hi-Jack polynomials are the simultaneous eigenfunctions of all the conserved operators. We have calculated the eigenvalues of all the conserved operators for the case when the coupling parameter a is zero and have found out there is no degeneracy. Thus we have proved the orthogonality of the Hi-Jack polynomials with respect to the inner product (4.76). From the explicit form of the weight function of the inner product, we have concluded that the Hi-Jack polynomial is a multivariable generalization of the Hermite polynomial. According to recent preprints [11, 12, 26], the generalized Hermite polynomials were also introduced by Lasselle [46] and by Macdonald in an unpublished manuscript. They defined the generalized Hermite polynomials as a deformation of known orthogonal symmetric polynomials with respect to the inner product (4.76) with the parameter a fixed at 1/2. On the other hand, our definition specifies the Hi-Jack polynomials, or in other words, the generalized Hermite polynomials, as the simultaneous eigenfunctions for the commuting conserved operators of the Calogero model, which are natural objects for physicists' interest.

Compactly summarizing, we have developed the formulation for an algebraic study of the quantum Calogero model. We have presented a way to construct the commuting conserved operators, and have identified the underlying symmetry. We have given a simple method for the algebraic construction of the energy eigenfunctions. We have also identified the simultaneous eigenfunctions of all the commuting conserved operators and have presented their Rodrigues formula. The simultaneous eigenfunction is a one-parameter deformation of the well-known Jack polynomial which we call Hi-Jack polynomial. Proving their orthogonality, we have identified that the Hi-Jack polynomial as a multivariable generalization of the Hermite polynomial. Thus we have succeeded in clarifying the fundamental properties on the quantum Calogero model from the viewpoint of the quantum integrable systems.

We believe that the thesis has opened the gate toward the exact calculations of various correlation functions of the Calogero model. As we have mentioned before, the knowledge on the Jack polynomials [36, 50, 69] enabled the exact calculations of correlation functions of the Sutherland model [29–32, 48]. The correlation functions exhibit an interesting connection with the exclusion statistics [33], which was first recognized by the asymptotic Bethe ansatz [71] of the Sutherland model [15, 91, 92]. Similar connections between the exclusion statistics and the quantum Calogero model were also reported [53, 54, 81, 82, 85]. Thus we expect similar relationships between the exclusion statistics and the correlation functions of the Calogero model. Some progresses related to our results were also reported recently [11, 12, 26]. Multivariable generalizations of the classical Laguerre and Jacobi polynomials, which form the orthogonal basis of Calogero models associated with root lattices other than A_{N-1} [59], were studied. Their non-symmetric extensions, which describe the spin or multi-component generalizations of Calogero models, were also reported. Further investigations on these orthogonal polynomials must be important for the study of Green functions and correlation functions of Calogero models, which was done for the Sutherland model with the help of the properties of the Jack polynomials [29, 30, 32, 48]. The quantum Lax formulation and the Dunkl operator formulation for the Calogero and Sutherland models revealed their W-symmetry and the Yangian symmetry structures [13, 14, 34, 35, 77, 78]. It is interesting to study the orthogonal polynomials from the viewpoint of representation theory of such symmetries. As examples of such studies, we should note that the Jack polynomials were identified with the singular vectors of the Virasoro and W_N-algebras [10, 51]. The Macdonald polynomials [50] and their generalizations, which are q-deformations of the orthogonal symmetric polynomials, are also interesting topics. The Macdonald polynomials are associated with the discretization, or in other words, the relativistic generalization of the Sutherland model [63]. The Rodrigues formula for the Macdonald polynomials was given [43–45, 55, 56]. Further studies on the continuous Hahn polynomials [23] which are associated with the relativistic or discretized Calogero model [22] and more generalized q-deformed orthogonal polynomials such as BC_N-Askey-Wilson polynomials [39] related to the discretized Calogero model associated with root lattices of BC_N-type [24, 25] are interesting. Thus we believe that the field of the quantum integrable systems with inverse-square interactions will continue to produce lots of attractive open problems.

Bibliography

[1] M. J. Ablowitz and H. Segur: Solitons and the Inverse Scattering Transform (SIAM, 1981).

[2] V. I. Arnold: Mathematical Methods of Classical Mechanics (Springer-Verlag, New-York, 1989) 2nd ed.

[3] J. Avan and A. Jevicki: Classical Integrability and Higher Symmetries of Collective String Field Theory, Phys. Lett. **B266** (1991) 35–41.

[4] J. Avan and A. Jevicki: Quantum Integrability and Exact Eigenstates of the Collective String Field Theory, Phys. Lett. **B272** (1991) 17–24.

[5] J. Avan and A. Jevicki: Algebraic Structures and Eigenstates for Integrable Collective Field Theories, Commun. Math. Phys. **150** (1992) 149–166.

[6] J. Avan and M. Talon: Classical R-Matrix Structure for the Calogero Model, Phys. Lett. **B303** (1993) 33–37.

[7] J. Avan: Integrable Extensions of the Rational and Trigonometric A_N Calogero Moser Potentials, Phys. Lett. **A185** (1994) 293–303.

[8] J. Avan, O. Babelon and M. Talon: Construction of the Classical R-Matrices for the Toda and Calogero Models, J. Geom. Phys. (1994)

[9] H. Awata, Y. Matsuo, S. Odake and J. Shiraishi: Collective Field Theory, Calogero-Sutherland Model and Generalized Matrix Model, Phys. Lett. B **347** (1995) 49–55.

[10] H. Awata, Y. Matsuo, S. Odake and J. Shiraishi: Excited States of the Calogero-Sutherland Model and Singular Vectors of the W_N Algebra, Nucl. Phys. B **449** (1995) 347–374.

[11] T. H. Baker and P. J. Forrester: The Calogero-Sutherland Model and Generalized Classical Polynomials, preprint, solv-int/9608004.

[12] T. H. Baker and P. J. Forrester: The Calogero-Sutherland Models and Polynomials with Prescribed Symmetry, preprint, solv-int/9609010.

[13] D. Bernard, M. Gaudin, F. D. M. Haldane and V. Pasquier: Yang-Baxter Equation in Long-Range Interacting Systems, J. Phys. A: Math. Gen. **26** (1993) 5219–5236.

[14] D. Bernard, K. Hikami and M. Wadati: The Yangian Deformation of the W-Algebras and the Calogero-Sutherland System, New Developments of Integrable Systems and Long-Ranged Interaction Models, ed. M.-L. Ge and Y.-S. Wu (World Scientific Singapore, 1995) pp.1–9.

[15] D. Bernard and Y.-S. Wu: A Note on Statistical Interactions and the Thermodynamic Bethe Ansatz, New Developments of Integrable Systems and Long-Ranged Interaction Models, ed. M.-L. Ge and Y.-S. Wu (World Scientific Singapore, 1995) pp.10–20.

[16] P. Bouwknegt and K. Schoutens: $\mathcal{W}$ Symmetry in Conformal Field Theory, Phys. Rep. **223** (1993) 183–276.

[17] L. Brink, T. H. Hansson and M. A. Vasiliev: Explicit Solution to the N-Body Calogero Problem, Phys. Lett. B **286** (1992) 109–111.

[18] L. Brink, T. H. Hansson, S. Konstein and M. A. Vasiliev: The Calogero Model – Anyonic Representation, Fermionic Extension and Supersymmetry, Nucl. Phys. B **401** (1993) 591–612.

[19] F. Calogero: Solution of the One-Dimensional N-Body Problems with Quadratic and/or Inversely Quadratic Pair Potentials, J. Math. Phys. **12** (1971) 419–436.

[20] F. Calogero, O. Ragnisco and C. Marchioro: Exact Solution of the Classical and Quantal One-Dimensional Many-Body Problems with the Two-Body Potential $V_a(x) = g^2a^2/\sinh^2(ax)$, Lett. Nuovo. Cim. **13** (1975) 383–387.

[21] F. Calogero: A Sequence of Lax Matrices for Certain Integrable Hamiltonian Systems, Lett. Nuovo Cim. **16** (1976) 22–24.

[22] J. F. van Diejen: Difference Calogero-Moser Systems and Finite Toda Chains, J. Math. Phys. **36** (1995) 1299–1323.

[23] J. F. van Diejen: Multivariable Continuous Hahn and Wilson Polynomials Related to Integrable Difference Systems, J. Phys. A: Math. Gen. **28** (1995) L369–L374.

[24] J. F. van Diejen: Commuting Difference Operators with Polynomial Eigenfunctions, Composite Math. **95** (1995) 183–233.

[25] J. F. van Diejen: Properties of Some Families of Hypergeometric Orthogonal Polynomials in Several Variables, preprint, UTMS 96–10.

[26] J. F. van Diejen: Confluent Hypergeometric Orthogonal Polynomials Related to the Rational Quantum Calogero System with Harmonic Confinement, preprint, q-alg/9609032.

[27] C. F. Dunkl: Differential-Difference Operators Associated to Reflection Group, Trans. Amer. Math. Soc. **311** (1989) 167–183.

[28] L. D. Faddeev and L. A. Takhtajan: Hamiltonian Methods in the Theory of Solitons (Springer-Verlag, Berlin, Heidelberg, 1987)

[29] P. J. Forrester: Selberg Correlation Integrals and the $1/r^2$ Quantum Many-Body System, Nucl. Phys. B **388** (1992) 671–699.

[30] P. J. Forrester: Addendum to 'Selberg Correlation Integrals and the $1/r^2$ Quantum Many Body System', Nucl. Phys. B **416** (1994) 377–385.

[31] Z. N. C. Ha: Exact Dynamical Correlation Functions of Calogero-Sutherland Model and One-Dimensional Fractional Statistics, Phys. Rev. Lett. **73** (1994) 1574–1577.

[32] Z. N. C. Ha: Fractional Statistics in One-Dimension: View from an Exactly Solvable Model, Nucl. Phys. B **435** (1995) 604–636.

[33] F. D. M. Haldane: "Fractional Statistics" in Arbitrary Dimensions: A Generalization of Pauli Principle, Phys. Rev. Lett. **67** (1991) 937–940.

[34] K. Hikami and M. Wadati: Infinite Symmetry of the Spin Systems with Inverse Square Interactions, J. Phys. Soc. Jpn. **62** (1993) 4203–4217.

[35] K. Hikami: Symmetry of the Calogero Model Confined in the Harmonic Potential – Yangian and W Algebra, J. Phys. A: Math. Gen. **28** (1995) L131–L136.

[36] H. Jack: A Class of Symmetric Polynomials with a Parameter, Proc. R. Soc. Edinburgh (A) **69** (1970) 1–18.

[37] S. Kakei: Common Algebraic Structure for the Calogero-Sutherland Models, preprint, solv-int/9608009.

[38] N. Kawakami and S.-K. Yang: Finite-Size Scaling in One-Dimensional Quantum Liquid with Long-Range Interaction, Phys. Rev. Lett. **67** (1991) 2493–2496.

[39] T. H. Koornwinder: Askey-Wilson Polynomials for Root Systems of type BC, Hypergeometric Functions on Domains of Positivity, Jack Polynomials, and Applications, Contemporary Math. **138**, ed. D. St. P. Richards (Amer. Math. Soc., Providence, R. I., 1992) pp.189–204.

[40] Y. Kuramoto and H. Yokoyama: Exactly Soluble Supersymmetric t-J Type Model with Long-Range Exchange and Transfer, Phys. Rev. Lett. **67** (1991) 1338–1341.

[41] L. Lapointe and L. Vinet: A Rodrigues Formula for the Jack Polynomials and the Macdonald-Stanley Conjecture, IMRN **9** (1995) 419–424.

[42] L. Lapointe and L. Vinet: Exact Operator Solution of the Calogero-Sutherland Model, Commun. Math. Phys. **178** (1996) 425–452.

[43] L. Lapointe and L. Vinet: Creation Operators for the Macdonald and Jack Polynomials, preprint, q-alg/9607024.

[44] L. Lapointe and L. Vinet: Rodrigues Formulas for the Macdonald Polynomials, preprint, q-alg/9607025.

[45] L. Lapointe and L. Vinet: A Short Proof of the Integrality of the Macdonald (q, t)-Kostka Coefficients, preprint, q-alg/9607026.

[46] M. Lasselle: Polynômes de Hermite Généralisés, C. R. Acad. Sci. Paris. t. Séries I **313** (1991) 579–582.

[47] P. Lax: Integrals of Nonlinear Equations of Evolution and Solitary Waves, Comm. Pure Appl. Math. **21** (1968) 467–490.

[48] F. Lesage, V. Pasquier and D. Serban: Dynamical Correlation Functions in the Calogero-Sutherland Model, Nucl. Phys. B **435** (1995) 585–603.

[49] I. G. Macdonald: Commuting Differential Operators and Zonal Spherical Functions, Algebraic Groups, Utrecht, 1986, Proceedings, Lecture Notes in Mathematics **1271** ed. A. M. Cohen, W. H. Hesselink, W. L. J. van der Kallen and J. R. Strooker (Springer-Verlag,Berlin/Heidelberg/New York, 1987) pp.189–200.

[50] I. G. Macdonald: Symmetric Functions and Hall Polynomials (Clarendon Press, Oxford, 1995) 2nd ed.

[51] K. Mimachi and Y. Yamada: Singular Vectors of the Virasoro Algebra in Terms of Jack Symmetric Polynomials, Commun. Math. Phys. **174** (1995) 447–455.

[52] J. Moser: Three Integrable Hamiltonian Systems Connected with Isospectral Deformations, Adv. Math. **16** (1975) 197–220.

[53] M. V. N. Murthy and R. Shankar: Haldane Exclusion Statistics and Second Virial Coefficient, Phys. Rev. Lett. **72** (1994) 3629–3633.

[54] M. V. N. Murthy and R. Shankar: Thermodynamics of a One-Dimensional Ideal Gas with Fractional Exclusion Statistics, Phys. Rev. Lett. **73** (1994) 3331–3334.

[55] A. N. Kirillov and M. Noumi: Affine Hecke Algebras and Raising Operators for Macdonald Polynomials, preprint, q-alg/9605004.

[56] A. N. Kirillov and M. Noumi: q-Difference Raising Operators for Macdonald Polynomials and the Integrality of Transition Coefficients, preprint, q-alg/9605005.

[57] M. A. Olshanetsky and A. M. Perelomov: Explicit Solution of the Calogero Model in the Classical Case and Geodesic Flows on Symmetric Spaces of Zero Curvature, Lett. Nuovo. Cim. **16** (1976) 333–339.

[58] M. A. Olshanetsky and A. M. Perelomov: Classical Integrable Finite-Dimensional Systems Related to Lie Algebras, Phys. Rep. **71** (1981) 313–400.

[59] M. A. Olshanetsky and A. M. Perelomov: Quantum Integrable Systems Related to Lie Algebras, Phys. Rep. **94** (1983) 313–404.

[60] W. Pauli: Über das Wasserstoffspektrum vom Standpunkt der neuen Quantenmechanik, Z. Physik **36** (1926) 336–363.

[61] A. M. Perelomov: Algebraic Approach to the Solution of a One-Dimensional Model of N Interacting Particles, Theor. Math. Phys. **6** (1971) 263–282.

[62] A. P. Polychronakos: Exchange Operator Formalism for Integrable Systems of Particles Phys. Rev. Lett. **69** (1992) 703–705.

[63] S. N. M. Ruijsenaars: Complete Integrability of Relativistic Calogero-Moser Systems and Elliptic Function Identities, Commun. Math. Phys. **110** (1987) 191–213.

[64] L. I. Schiff: Quantum Mechanics (McGrawhill Book Company, New York, St. Louis, San Francisco, Toronto, London, Sydney, 1968) 3rd ed.

[65] E. Schrödinger: Quantisierung als Eigenwertproblem, Ann. Physik **79** (1926) 361–376.

[66] E. K. Sklyanin: Dynamical r-matrices for the Elliptic Calogero-Moser Model, Alg. and Anal. **6** (1994) 227–237.

[67] E. K. Sklyanin: Separation of Variables. New Trends, Prog. Theor. Phys. Suppl. **118** (1995) 35–60.

[68] E. K. Sklyanin: Lecture at School of Mathematical Science, University of Tokyo, 1994.

[69] R. P. Stanley: Some Combinatorial Properties of Jack Symmetric Functions, Adv. Math. **77** (1989) 76–115.

[70] B. Sutherland: Quantum Many-Body Problem in One Dimension: Ground State, J. Math. Phys. **12** (1971) 246–250.

[71] B. Sutherland: Quantum Many-Body Problem in One Dimension: Thermodynamics, J. Math. Phys. **12** (1971) 251–256.

[72] B. Sutherland: Exact Results for a Quantum Many-Body Problem in One Dimension, Phys. Rev. A **4** (1971) 2019–2021.

[73] B. Sutherland: Exact Results for a Quantum Many-Body Problem in One Dimension. II, Phys. Rev. A **5** (1972) 1372–1376.

[74] H. Ujino, K. Hikami and M. Wadati: Integrability of the Quantum Calogero-Moser Model, J. Phys. Soc. Jpn. **61** (1992) 3425–3427.

[75] H. Ujino, M. Wadati and K. Hikami: The Quantum Calogero-Moser Model: Algebraic Structures, J. Phys. Soc. Jpn. **62** (1993) 3035–3043.

[76] H. Ujino and M. Wadati: Algebraic Approach to the Eigenstates of the Quantum Calogero Model Confined in an External Harmonic Well, Chaos, Solitons & Fractals **5** (1995) 109–118.

[77] H. Ujino and M. Wadati: The Quantum Calogero Model and the W-Algebra, J. Phys. Soc. Jpn. **63** (1994) 3585–3597.

[78] H. Ujino and M. Wadati: The $SU(\nu)$ Quantum Calogero Model and the W_∞ Algebra with $SU(\nu)$ Symmetry, J. Phys. Soc. Jpn. **64** (1995) 39–56.

[79] H. Ujino and M. Wadati: Orthogonal Symmetric Polynomials Associated with the Quantum Calogero Model, J. Phys. Soc. Jpn. **64** (1995) 2703–2706.

[80] H. Ujino and M. Wadati: Correspondence between QISM and Exchange Operator Formalism, J. Phys. Soc. Jpn. **64** (1995) 4121–4128.

[81] H. Ujino and M. Wadati: Exclusion Statistics Interpretation of the Quantum Calogero Model, J. Phys. Soc. Jpn. **64** (1995) 4064–4068.

[82] H. Ujino and M. Wadati: Thermodynamics of the Exclusion Statistics Associated with the Quantum Calogero Model, J. Phys. Soc. Jpn. **65** (1996) 1203–1212.

[83] H. Ujino and M. Wadati: Algebraic Construction of the Eigenstates for the Second Conserved Operator of the Quantum Calogero Model, J. Phys. Soc. Jpn. **65** (1996) 653–656.

[84] H. Ujino and M. Wadati: Rodrigues Formula for Hi-Jack Symmetric Polynomials Associated with the Quantum Calogero Model, J. Phys. Soc. Jpn. **65** (1996) 2423–2439.

[85] H. Ujino and M. Wadati: Exclusion Statistical TBA and the Quantum Calogero Model, J. Phys. Soc. Jpn. **65** (1996) 3819–3823.

[86] H. Ujino and M. Wadati: Orthogonality of the Hi-Jack Polynomials Associated with the Calogero Model, to be published in J. Phys. Soc. Jpn. **66** (1997) No. 2.

[87] H. Ujino: Orthogonal Symmetric Polynomials Associated with the Calogero Model, submitted to Extended and Quantum Algebras and their Applications in Mathematical Physics, proceedings of the Canada-China Meeting in Mathematical Physics, Tianjin, China, August 19–24, 1996, ed. M.-L. Ge, Y. Saint-Aubin and L. Vinet (Springer-Verlag).

[88] M. Wadati, T. Nagao and K. Hikami: Quantum Integrable Systems, Physica **D68** (1993) 162–168.

[89] M. Wadati, K. Hikami and H. Ujino: Integrability and Algebraic Structure of the Quantum Calogero-Moser Model, Chaos, Solitons & Fractals **3** (1993) 627–636.

[90] F. Wilczek: *Fractional Statistics and Anyon Superconductivity*, (World Scientific, Singapore, 1989).

[91] Y.-S. Wu: Statistical Distribution for Generalized Ideal Gas of Fractional-Statistics Particles, Phys. Rev. Lett. **73** (1994) 922–925.

[92] Y.-S. Wu: Fractional Exclusion Statistics and Generalized Ideal Gases, New Developments of Integrable Systems and Long-Ranged Interaction Models, ed. M.-L. Ge and Y.-S. Wu (World Scientific Singapore, 1995) pp.159–174.