A generalization of local class field theory by using K-groups. I.

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CONTENTS

Introduction.

Local class field theory is an example of a theory on abelian extension of fields. Let K be a field which is complete with respect to a discrete valuation and with finite residue field, and let K^{ab} be the maximum abelian extension of K. Then, local class field theory says that there exists a canonical homomorphism

$$K^* \to \mathrm{Gal}(K^{ab}/K),$$

which is "almost an isomorphism". By this homomorphism, we can translate problems on abelian extension of K, i.e. problems on the group $\mathrm{Gal}(K^{ab}/K)$, into problems on the group K^*.

There is an analogous example of a theory on abelian extension: if K is a finite field, there exists a canonical homomorphism

$$\boldsymbol{Z} \to \mathrm{Gal}(K^{ab}/K),$$

which is "almost an isomorphism". Recall that $\boldsymbol{Z}=K_0(K)$ and $K^*=K_1(K)$ for any field K, where K_0 and K_1 are notations in algebraic K-theory.

The subject of this paper is a natural generalization of these two examples.

For any ring R, let $K_*(R)$ be Quillen's K-group in Quillen [18]. On the other hand, for any field k, let $\mathfrak{K}_*(k)$ be Milnor's K-group in Milnor [12] which was denoted by K_*k in [12], i.e. $\mathfrak{K}_0(k)=\boldsymbol{Z}$, $\mathfrak{K}_1(k)=k^*$, and for any $q\geqq 2$,

$$\mathfrak{K}_q(k)=(\overbrace{k^*\otimes\cdots\otimes k^*}^{q \text{ times}})/J\,,$$

where J is the subgroup of the tensor product generated by all elements of the form $x_1\otimes\cdots\otimes x_q$ with $x_i+x_j=1$ for some i and j such that $i\neq j$. If k is a field, there is a canonical homomorphism $\mathfrak{K}_q(k)\to K_q(k)$ $(q\geqq 0)$, which is a bijection in the case $q\leqq 2$.

Our main results are the following theorems. (Cf. Kato [9] [10].)

THEOREM 1. *Let $N\geqq 0$ and let $k_0,\cdots,k_N$ be fields having the following properties* (i) *and* (ii).

(i) *k_0 is a finite field.*

(ii) *For each $i=1,\cdots,N$, k_i is complete with respect to a discrete valuation and the residue field of k_i is k_{i-1}.*

Denote k_N by K, and k_0 by k. Then:

(1) *There exists a canonical homomorphism*

$$\Psi_K\colon \mathfrak{K}_N(K)\to \mathrm{Gal}(K^{\mathrm{ab}}/K)$$

characterized by the following properties (iii) *and* (iv).

(iii) *For any finite abelian extension L of K, Ψ_K induces an isomorphism $\mathfrak{K}_N(K)/\mathfrak{N}_{L/K}\mathfrak{K}_N(L)\cong \mathrm{Gal}(L/K)$. Here, $\mathfrak{N}_{L/K}$ is a certain canonical norm homomorphism; cf. Chapter* II §3.

(iv) *For each $i=1,\cdots,N$, let π_i be a lifting to K of a prime element of k_i. Then, the image of $\Psi_K(\{\pi_1,\cdots,\pi_N\})$ under the canonical homomorphism* $\mathrm{Gal}(K^{\mathrm{ab}}/K)\to\mathrm{Gal}(k^{\mathrm{ab}}/k)$ *coincides with the Frobenius automorphism over k.*

(2) *If* $\mathrm{ch}(K)$, *the characteristic of K, is >0, there exists a canonical homomorphism*

$$\Upsilon_K\colon K_N(K)\to\mathrm{Gal}(K^{\mathrm{ab}}/K)$$

which induces the above homomorphism Ψ_K via the canonical homomorphism $\mathfrak{K}_N(K)\to K_N(K)$.

Next, in the case $N=2$ of Theorem 1, we can obtain a more satisfactory result together with a similar result on the Brauer group of K.

THEOREM 2. *Let F be a complete discrete valuation field with finite residue field and K a complete discete valuation field with residue field F. Then*:

(1) *The map $L\mapsto N_{L/K}K_2(L)$ is a bijection from the set of all finite abelian*

extensions of K in a fixed algebraic closure of K, to the set of all open subgroups of $K_2(K)$ of finite indices with respect to the topology of $K_2(K)$ defined in this paper (cf. Chapter I § 7).

(2) *There exists a canonical isomorphism*

$$\Phi_K: \mathrm{Br}(K) \xrightarrow{\cong} \mathrm{Hom}_c(K^*, \boldsymbol{Q}/\boldsymbol{Z})_{\mathrm{tor}}$$

having the following property: For each central simple algebra A over K,

$$\mathrm{Ker}(\Phi_K(\{A\}): K^* \to \boldsymbol{Q}/\boldsymbol{Z}) = \mathrm{Nrd}_{A/K}(A^*),$$

where $\mathrm{Nrd}_{A/K}: A^* \to K^*$ is the reduced norm map. Here, $\mathrm{Hom}_c(K^*, \boldsymbol{Q}/\boldsymbol{Z})_{\mathrm{tor}}$ denotes the torsion part of the group of all continuous homomorphism from K^* to $\boldsymbol{Q}/\boldsymbol{Z}$ with respect to the topology of K^* defined in this paper (cf. Chapter I § 7) and the discrete topology of $\boldsymbol{Q}/\boldsymbol{Z}$.

Lastly, let F be an algebraic function field in one variable over a finite field, and K a complete discrete valuation field with residue field F. We shall define a topological group $\mathscr{C}_K$, called the "K_2-idele class group" of K.

THEOREM 3. *Let F and K be as above.*

(1) *There exists a canonical homomorphism $\Psi_K: \mathscr{C}_K \to \mathrm{Gal}(K^{\mathrm{ab}}/K)$ having the following property: For each finite abelian extension L of K, Ψ_K induces an isomorphism $\mathscr{C}_K/N_{L/K}\mathscr{C}_L \cong \mathrm{Gal}(L/K)$.*

(2) *The map $L \mapsto N_{L/K}\mathscr{C}_L$ is a bijection from the set of all finite abelian extensions of K in a fixed algebraic closure of K, to the set of all open subgroups of $\mathscr{C}_K$ of finite indices.*

Our main tool for these studies will be the Galois cohomology in the case $\mathrm{ch}(K)=0$, and the generalized residue homomorphism (cf. Chapter II § 2) in the case $\mathrm{ch}(K)>0$.

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In Chapter I, we shall study complete discrete valuation fields K with residue fields F such that $\mathrm{ch}(F)=p>0$ and $[F: F^p]=p$. We shall study the K_2-group and the Galois cohomology of K, and prove the above Theorem 2 in the mixed characteristic case, which is the difficult case (cf. Chapter II § 3 for the equal characteristic case). Among other results, we shall prove the following theorem in Chapter I.

THEOREM 4. *Let F be a field of characteristic $p>0$ such that $[F: F^p]=p$, and K a complete discrete valuation field of characteristic zero with residue field F. Then, for each $n \geqq 0$,*

(1) $$H^2(K, \mu_{p^n}\otimes\mu_{p^n})\cong K_2(K)/K_2(K)^{p^n},$$

via the Galois symbol.

(2) $$H^3(K, \mu_{p^n}\otimes\mu_{p^n})\cong \mathrm{Br}\,(F)_{p^n}.$$

In Chapter II, we shall prove the above Theorem 1. In Chapter III, we shall prove Theorem 3.

Chapter I is contained in this Part I, and Chapter II and Chapter III will be published later.

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The subject of this paper grew up from the author's Master's thesis stimulated by Ihara [7] which suggested the study of local class field theory of the fields as in Theorem 3 together with the possibility of class field theoretic interpretation of the mapping "$j\mapsto\pi(j)$" associated with elliptic curves. A large part of the results of this paper was proved in Kato [8] and its summary was published in Kato [9] and [10].

I wish to express my sincere gratitude to Professor Y. Ihara for his suggestion of these researches and his unceasing encouragement. I also wish to express my hearty thanks to Professor Y. Kawada who guided me to the local class field theory and the theory of Galois cohomology, and to Professor T. Tasaka who carefully read the manuscript and corrected errors in it.

I was informed by Professor A. N. Parsin that he independently proved several similar results, together with their globalizations in the 2-dimensional case, which were announced in Parsin [17]. See also Parsin [15] and [16] for related results in this field.

Conventions.

"Ring" means commutative ring and "field" means commutative field, unless the contrary is explicitly stated.

If R is a ring,

R^* denotes the group of all invertible elements of R,

Ω_R denotes the absolute differential module $\Omega^1_{R/Z}$, for example, of [6] § 20.

If k is a field,

$\mathrm{ch}\,(k)$ denotes the characteristic of k,

k_s denotes the separable closure of k,

k^{ab} denotes the maximum abelian extension of k,

X_k denotes character group of $\mathrm{Gal}\,(k^{ab}/k)$; in other words,

X_k is the group of all continuous homomorphisms from $\mathrm{Gal}(k^{\mathrm{ab}}/k)$ to $\boldsymbol{Q}/\boldsymbol{Z}$ with respect to the Krull topology on $\mathrm{Gal}(k^{\mathrm{ab}}/k)$ and the discrete topology on $\boldsymbol{Q}/\boldsymbol{Z}$,

$\mathrm{Br}(k)$ denotes the Brauer group of k,

$\mu_{n,k}$ (or simply μ_n) denotes the group of all n-th roots of 1 in k_s, for each non-zero integer n.

If k is a discrete valuation field,

v_k denotes the normalized additive valuation of k ($v_k(0)=\infty$),

e_k denotes $v_k(p)$ where p is the characteristic of the residue field of k,

$O_k=\{x\in k \mid v_k(x)\geqq 0\}$,

$m_k=\{x\in k \mid v_k(x)\geqq 1\}$,

$U_k=\{x\in k \mid v_k(x)=0\}$,

$U_k^{(n)}=\{x\in k^* \mid v_k(x-1)\geqq n\}$ for each integer n such that $n\geqq 1$,

$U_k^{(0)}=U_k$ except that $U_k^{(0)}$ denotes k^* in the following two places;

$\begin{cases}\text{Ch. I §6 Proofs of the injectivity of } \Phi \text{ and } \Psi^{\vee}, \\ \text{Ch. I §8 Proof of Th. 1 Step 2-5,}\end{cases}$

$\bar{x}$ denotes the residue class of x for each $x\in O_k$.

If k is a field which is complete with respect to a discrete valuation,

k_{nr} denotes the maximum unramified extension of k, i.e. the unramified extension of k whose residue field is the separable closure of the residue field of k,

$\mathrm{Br}(k_{\mathrm{nr}}/k)$ denotes the kernel of the canonical homomorphism $\mathrm{Br}(k)\to\mathrm{Br}(k_{\mathrm{nr}})$.

If A is a commutative group whose group law is written additively,

$A\xrightarrow{n}A$ denotes the homomorphism $A\to A$; $x\mapsto nx$, and A_n denotes the kernel of the above homomorphism $A\xrightarrow{n}A$, for each integer n.

Chapter I. Complete discrete valuation fields with residue fields F such that $[F: F^p]=p$.

§ 0. Preliminaries.

In this section, we shall review some known properties of Galois cohomology and K_2 of fields, we shall fix our notations and we shall formulate the subject of this Chapter I.

§ 0.1. Preliminaries on Galois cohomology.

(Cf. Serre [20] for Galois cohomology.)

Let k be a field.

Let C_k be the following category: An object of C_k is a Gal (k_s/k)-module M such that the action

$$\mathrm{Gal}\,(k_s/k)\times M\to M$$

is continuous with respect to the Krull topology on Gal (k_s/k) and the discrete topology on M. A morphism of C_k is a Gal (k_s/k)-homomorphism.

Then C_k is an abelian category. If M is an object of C_k, we denote by $H^0(k, M)$ the group

$$\{x\in M \mid \sigma(x)=x \text{ for all } \sigma\in \mathrm{Gal}\,(k_s/k)\}\,.$$

Then, $H^0(k,\ )$ is a left-exact functor from C_k to the category of all abelian groups and all homomorphisms between them. We denote by $H^i(k,\ )$ the i-th right-derived functor of $H^0(k,\ )$. The group $H^i(k, M)$ admits an interpretation in terms of the arithmetic of k if M and i are selected as follows. ((1)-(4))

$$H^2(k, k_s^*)\cong \mathrm{Br}\,(k)\,. \tag{1}$$

$$H^1(k, \boldsymbol{Z}/n\boldsymbol{Z})\cong (X_k)_n\,, \qquad \text{for each} \quad n>0\,. \tag{2}$$

(Whenever the notation $\boldsymbol{Z}/n\boldsymbol{Z}$ is used in this paper for an object of C_k, we always assume that the action of Gal (k_s/k) on $\boldsymbol{Z}/n\boldsymbol{Z}$ is the trivial one.)

If n is any integer which is invertible in k, (1) and the exact sequence in C_k

$$1\to\mu_n\to k_s^*\xrightarrow{x\mapsto x^n}k_s^*\to 1$$

give rise to the following long exact sequence

$$k^*\xrightarrow{x\to x^n}k^*\to H^1(k, \mu_n)\to H^1(k, k_s^*)\xrightarrow{n}H^1(k, k_s^*)\to H^2(k, \mu_n)\to \mathrm{Br}\,(k)\xrightarrow{n}\mathrm{Br}\,(k)\,.$$

Since $H^1(k, k_s^*)=0$ (Hilbert's theorem 90), we have,

$$H^1(k, \mu_n)\cong k^*/(k^*)^n\,, \tag{3}$$

$$H^2(k, \mu_n)\cong \mathrm{Br}\,(k)_n\,. \tag{4}$$

§ 0.2. Preliminaries on K_2 of fields.

See Milnor [13] for the classical and conceptual definition of K_2 of rings. But the following theorem (Matsumoto [11]) also serves as a definition of K_2 of fields.

MATSUMOTO'S THEOREM. *For any field k, there is a canonical isomorphism*

$$(k^*\otimes k^*)/J\cong K_2(k)\,; \qquad x\otimes y\to\{x, y\}_k\,,$$

where J is the subgroup of the tensor product generated by all elements of the form $x\otimes(1-x)$ such that $x\in k$, $x\neq 0$ and $x\neq 1$.

We denote the group law of K_2 multiplicatively. The symbol $\{\ ,\ \}_k$ is denoted simply by $\{\ ,\ \}$ more often than not. By the above theorem and by some simple calculation, we have

$$\{x, 1-x\}=1\,, \qquad \{y, -y\}=1\,, \qquad \text{and} \qquad \{y, z\}\{z, y\}=1$$

for all $x \in k$ such that $x \neq 0$ and $x \neq 1$, and for all $y, z \in k^*$.

K_2-NORMS. If k is a field and E is a finite extension of k, there is a "natural" K_2-norm homomorphism (the transfer homomorphism of [13] § 14)

$$N_{E/k}: K_2(E) \to K_2(k)\,.$$

The most important property of K_2-norm is the following. If $a \in k^*$ and $b \in E^*$,

$$N_{E/k}(\{a, b\}_E)=\{a, N_{E/k}b\}_k\,.$$

Now, we review the tame symbol and the Galois symbol.

TAME SYMBOLES. Let K be a discrete valuation field with residue field F. Let $v: K^* \to \boldsymbol{Z}$ be the normalized additive valuation of K. The tame symbol is the homomorphism $\partial: K_2(K) \to F^*$ which is characterized by the following property: $\partial(\{x, y\})$ is equal to the residue class of $(-1)^{v(x)v(y)}x^{v(y)}/y^{v(x)}$ for all $x, y \in K^*$. The existence and the uniqueness of the tame symbol follows from Matsumoto's theorem.

TATE'S GALOIS SYMBOLS. (Cf. Bass [1] 9.) Let k be a field and let n be an integer which is invertible in k. The Galois symbol is the homomorphism

$$h_n: K_2(k)/K_2(k)^n \to H^2(k, \mu_n \otimes \mu_n) \tag{5}$$

defined as follows. (If M and N are objects of C_k (§ 0.1), we regard $M \otimes N$ as an object of C_k on which $\mathrm{Gal}\,(k_s/k)$ acts by $\sigma(x \otimes y)=\sigma(x) \otimes \sigma(y)$.)

The cup product

$$H^1(k, \mu_n) \otimes H^1(k, \mu_n) \to H^2(k, \mu_n \otimes \mu_n)$$

and the isomorphism (3) in § 0.1 induce a composite

$$g_n: k^*/(k^*)^n \otimes k^*/(k^*)^n \to H^2(k, \mu_n \otimes \mu_n)\,,$$

and this homomorphism g_n satisfies (see Bass [1])

$$g_n(x \bmod (k^*)^n \otimes (1-x) \bmod (k^*)^n)=0$$

for all $x \in k$ such that $x \neq 0$ and $x \neq 1$. Hence, by Matsumoto's theorem, we have the homomorphism h_n which is characterized by the following property:

$$h_n(\{x, y\} \bmod K_2(k)^n) = g_n(x \bmod (k^*)^n \otimes y \bmod (k^*)^n)$$

for all $x, y \in k^*$.

Finally, we add a remark about the Galois symbol. Suppose further that k contains a primitive n-th root ζ of 1. Then, the isomorphism $\boldsymbol{Z}/n\boldsymbol{Z} \cong \mu_n$; $1 \bmod n\boldsymbol{Z} \mapsto \zeta$ in C_k induces via $\otimes \mu_n$ an isomorphism $\mu_n \cong \mu_n \otimes \mu_n$ and so an isomorphism $H^2(k, \mu_n) \cong H^2(k, \mu_n \otimes \mu_n)$. Since $H^2(k, \mu_n) \cong \mathrm{Br}(k)_n$ by (4), h_n induces a homomorphism

$$h'_n : K_2(k)/K_2(k)^n \to \mathrm{Br}(k)_n .$$

It is known that for any $a, b \in k^*$, $h'_n(\{a, b\})$ coincides with the class of the central simple algebra A over k which is characterized by the following property: A has a k-basis $(x^i y^j)_{0 \leq i \leq n-1, 0 \leq j \leq n-1}$ where x and y are elements of A such that $x^n = a$, $y^n = b$ and $yxy^{-1} = \zeta x$. It is known that for any $a, b \in k^*$, the following conditions are equivalent:

$$(6) \qquad \begin{aligned} h'_n(\{a, b\}) = 0 &\Leftrightarrow a \in N_{E/K}(E^*) \qquad \text{where} \quad E = k(b^{1/n}) \\ &\Leftrightarrow \{a, b\} \in K_2(k)^n \end{aligned}$$

§ **0.3.** As is well known (cf. Serre [20] Ch. II § 5), ordinary local class field theory can be formulated in the form of Poincare's duality theorem as follows. Let K be a field which is complete with respect to a discrete valuation and with *finite* residue field. Suppose $\mathrm{ch}(K) = 0$. Then, for each $n > 0$, there exists a canonical isomorphism

$$(7) \qquad H^2(K, \mu_n) \cong \frac{1}{n}\boldsymbol{Z}/\boldsymbol{Z}$$

which arises from these special properties of K. This isomorphism (7) and the cup product induce a pairing

$$H^1(K, \mu_n) \otimes H^1(K, \boldsymbol{Z}/n\boldsymbol{Z}) \to \frac{1}{n}\boldsymbol{Z}/\boldsymbol{Z}$$

and so, by (2) and (3), a pairing

$$(8) \qquad K^*/(K^*)^n \otimes (X_K)_n \to \frac{1}{n}\boldsymbol{Z}/\boldsymbol{Z} .$$

When n varies, (8) induces a homomorphism

$$K^* \to \mathrm{Gal}(K^{ab}/K) .$$

Thus, ordinary local class field theory can be constructed on the isomorphism (7) which is analogous to the isomorphism in ordinary Poincare's duality theorem.

Now, let F be a field which is complete with respect to a discrete valuation and with *finite* residue field. Let K be a field which is complete with respect to a discrete valuation and with residue field F. Suppose $\mathrm{ch}(K)=0$. The main subject of this Chapter I is to obtain a canonical isomorphism

$$H^3(K, \mu_n\otimes\mu_n)\cong\frac{1}{n}\boldsymbol{Z}/\boldsymbol{Z} \tag{7$'$}$$

for each $n>0$. This isomorphism (7)$'$ and the cup product will induce a pairing

$$H^2(K, \mu_n\otimes\mu_n)\otimes H^1(K, \boldsymbol{Z}/n\boldsymbol{Z})\to\frac{1}{n}\boldsymbol{Z}/\boldsymbol{Z}$$

and so, by (2) and (5), a pairing

$$K_2(K)/K_2(K)^n\otimes(X_K)_n\to\frac{1}{n}\boldsymbol{Z}/\boldsymbol{Z}\,. \tag{8$'$}$$

When n varies, (8)$'$ will induce a homomorphism

$$K_2(K)\to\mathrm{Gal}\,(K^{\mathrm{ab}}/K)\,.$$

Thus, our local class field theory will be constructed on the isomorphism (7)$'$. It is easy to define this isomorphism (7)$'$ in case n is not divisible by $\mathrm{ch}(F)$, but it is highly difficult in case $\mathrm{ch}(F)=p>0$ and n is a power of p. In the latter case, for the definition of (7)$'$, we need long studies of Galois cohomology and of K_2 of fields which will continue until § 5.

§ **0.4.** Lastly, we present here an example of a field which is in the domain of our local class field theory. Let K be the field of all formal Laurent series $\sum_{i\in \boldsymbol{Z}} a_iX^i$ over $\boldsymbol{Q}_p$ such that $\lim_{i\to-\infty} a_i=0$ and such that $\{v_p(a_i)\,|\, i\in\boldsymbol{Z}\}$ is bounded below, where v_p denotes the usual normalized additive valuation of $\boldsymbol{Q}_p$. Then, K is complete with respect to the discrete valuation $\tilde{v}_p$: $\tilde{v}_p(\sum_{i\in\boldsymbol{Z}} a_iX^i)=\inf\{v_p(a_i)\,|\,i\in\boldsymbol{Z}\}$, and the residue field of K is isomorphic to $(\boldsymbol{Z}/p\boldsymbol{Z})((X))$.

§ 1. The reduced norm map of a division algebra of a certain type.

In this section, we study the reduced norm map of a division algebra of some type over a complete valuation field. Our results are analogous to those of Serre [19] Ch. V § 3 in which the norm map of a cyclic extension of a complete valuation field is studied.

NOTATIONS and ASSUMPTIONS. In this section, K is a field which is complete with respect to a discrete valuation and with residue field F such that $\mathrm{ch}(F)=$

$p>0$, and D is a division algebra with center K having the following properties (i) and (ii).

(i) $\dim_K D=p^2$.

Since K is complete, D has a canonical discrete valuation $v_K \circ \mathrm{Nrd}_{D/K}$ (cf. Serre [19] Ch. XII § 2). We define O_D, m_D, U_D and $U_D^{(n)}$ just as O_K, m_K, U_K and $U_K^{(n)}$, respectively. We denoted the residue division algebra O_D/m_D by C.

(ii) C is commutative, $[C:F]=p$ and C is inseparable over F.

The assumption (i) implies that we are concerned with the simplest case, but (ii) implies that we are concerned with the difficult part of the case. In what follows, we denote $v_K \circ \mathrm{Nrd}_{D/K}$ by v_D. From the above assumptions, it follows that v_D is a normalized valuation. If $x \in O_D$, we denote the residue class of x by $\bar{x}$.

DEFINITION of the *ramification number* of D. This is the minimum of $v_D(xyx^{-1}y^{-1}-1)$ where x and y are any elements of D^*, and denoted by t in this section.

The aim of this section is to prove the following Prop. 1, 2, 3.

PROPOSITION 1. *$1\leqq t\leqq e_K p/(p-1)$, where $e_K=v_K(p)$.*

PROPOSITION 2. (i) *If $0\leqq n\leqq t+1$, $\mathrm{Nrd}(U_D^{(n)})\subset U_K^{(n)}$.*

(ii) *If $m\geqq 0$, $\mathrm{Nrd}(U_D^{(t+mp)})\subset U_K^{(t+m)}$ and $\mathrm{Nrd}(U_D^{(t+mp+1)})\subset U_K^{(t+m+1)}$.*

(iii) *The map $\mathrm{Nrd}: U_D^{(t+1)}\to U_K^{(t+1)}$ is surjective.*

PROPOSITION 3. (On the action of Nrd on the subquotients of D^*.) *Suppose further that $[F:F^p]=p$. (In this case, we have $C=F^{1/p}$. In what follows, we denote by $\mathfrak{F}$ either the isomorphism $C\xrightarrow{\cong}F$, $C^*\xrightarrow{\cong}F^*$, or $\Omega_C\to\Omega_F$ which is induced by the isomorphism $x\mapsto x^p: C\xrightarrow{\cong}F$. Here Ω denotes the module of absolute differentials as noted at the beginning of this paper.) Then we have the following* (i)-(v).

(i) *The following diagram is commutative.*

$$\begin{array}{ccc} C^* & \xrightarrow[\mathfrak{F}]{\cong} & F^* \\ \downarrow \wr & & \downarrow \wr \\ U_D/U_D^{(1)} & \xrightarrow[\mathrm{Nrd}]{} & U_K/U_K^{(1)} \end{array}.$$

Here the vertical arrows are the canonical ones.

(ii) *Let $1\leqq n<t$ and fix an element c of D such that $v_D(c)=n$. Then the following diagram is commutative.*

$$\begin{array}{ccc} C & \xrightarrow[\mathfrak{F}]{\cong} & F \\ {\scriptstyle \bar{x} \mapsto 1+xc}\downarrow \wr & & \downarrow \wr {\scriptstyle \bar{x}\mapsto 1+x\cdot \mathrm{Nrd}(c)} \\ U_D^{(n)}/U_D^{(n+1)} & \xrightarrow[\mathrm{Nrd}]{} & U_K^{(n)}/U_K^{(n+1)} \end{array}$$

(iii) *Fix an element a of D^* such that $v_D(a)$ is prime to p. Then, the following diagram is commutative and the rows are exact.*

$$\begin{tikzcd}
0 \arrow[r] & F^* \arrow[r] \arrow[d, "\wr"] & C^* \arrow[r, "f\mapsto df/f"] \arrow[d, "\wr"] & \Omega_C \arrow[r, "(1-\gamma)\mathfrak{F}"] \arrow[d, "(1)"', "\wr"] & \Omega_F \arrow[r] \arrow[d, "(2)"', "\wr"] & \mathrm{Br}(F)_p \arrow[r] & 0 \\
0 \arrow[r] & U_K/U_K^{(1)} \arrow[r] & U_D/U_D^{(1)} \arrow[r, "g\mapsto g^{-1}aga^{-1}"'] & U_D^{(t)}/U_D^{(t+1)} \arrow[r, "\mathrm{Nrd}"'] & U_K^{(t)}/U_K^{(t+1)} . & &
\end{tikzcd}$$

Here γ denotes the Cartier operator (see below), and (1) *and* (2) *are the isomorphisms defined as follows. (Note that Ω_F is one-dimensional over F since $[F: F^p]=p$ and since $\Omega_F=\Omega^1_{F/Z}=\Omega^1_{F/F^p}$.)*

(1) $$\bar{f}\, d\bar{g}/\bar{g} \mapsto 1+f(g^{-1}aga^{-1}-1)\,, \qquad f\in O_D\,,\quad g\in U_D\,.$$

(2) $$\bar{f}\, d\bar{g}^p/\bar{g}^p \mapsto 1+f\cdot \mathrm{Nrd}\,(g^{-1}aga^{-1}-1)\,, \qquad f\in O_K\,,\quad g\in U_D\,.$$

(iv) *Let $m\geqq 1$. Fix an element c of K such that $v_K(c)=m$ and an element a of D^* such that $v_D(a)$ is prime to p. Then the following diagram is commutative and the rows are exact.*

$$\begin{tikzcd}
0 \arrow[r] & F \arrow[r] \arrow[d, "\bar{x}\mapsto 1+xc"', "\wr"] & C \arrow[r, "f\mapsto df"] \arrow[d, "\bar{x}\mapsto 1+xc"', "\wr"] & \Omega_C \arrow[r, "\gamma\circ\mathfrak{F}"] \arrow[d, "(3)"', "\wr"] & \Omega_F \arrow[r] \arrow[d, "(4)"', "\wr"] & 0 \\
0 \arrow[r] & U_K^{(m)}/U_K^{(m+1)} \arrow[r] & U_D^{(mp)}/U_D^{(mp+1)} \arrow[r, "g\mapsto g^{-1}aga^{-1}"'] & U_D^{(t+mp)}/U_D^{(t+mp+1)} \arrow[r, "\mathrm{Nrd}"'] & U_K^{(t+m)}/U_K^{(t+m+1)} &
\end{tikzcd}$$

Here (3) *and* (4) *are the isomorphisms defined as follows.*

(3) $$\bar{f}\, d\bar{g}/\bar{g} \mapsto 1+f(g^{-1}aga^{-1}-1)c\,, \qquad f\in O_D\,,\quad g\in U_D\,.$$

(4) $$\bar{f}\, d\bar{g}^p/d\bar{g}^p \mapsto 1-fc\cdot \mathrm{Nrd}\,(g^{-1}aga^{-1}-1)\,, \qquad f\in O_K\,,\quad g\in U_D\,.$$

(v) *There exists an isomorphism*

$$K^*/\mathrm{Nrd}_{D/K}\,(D^*)\cong \mathrm{Br}\,(F)_p\,.$$

REVIEW on the *Cartier operator*. (Cf. Cartier [3].) Let k be a field of characteristic $p>0$. Let $n\geqq 0$, let $\overset{n}{\underset{k}{\wedge}}\,\Omega_k$ be the n-th exterior power of the k-module Ω_k, and let $\overset{n}{\underset{k}{\wedge}}\,\Omega_{k,d=0}$ be the kernel of $d\colon \overset{n}{\underset{k}{\wedge}}\,\Omega_k \to \overset{n+1}{\underset{k}{\wedge}}\,\Omega_k$. Then, the Cartier operator γ is the unique additive homomorphism $\overset{n}{\underset{k}{\wedge}}\,\Omega_{k,d=0} \to \overset{n}{\underset{k}{\wedge}}\,\Omega_k$ such that

$$\gamma\Big(x^p\frac{dy_1}{y_1}\wedge\cdots\wedge\frac{dy_n}{y_n}\Big)=x\frac{dy_1}{y_1}\wedge\cdots\wedge\frac{dy_n}{y_n}$$
$$\gamma(dw)=0$$

for all $x\in k$, $y_1,\cdots,y_n\in k^*$ and for all $w\in \overset{n-1}{\underset{k}{\wedge}}\,\Omega_k$. In this §1, we are concerned with the case in which $n=1$ and $\overset{2}{\underset{k}{\wedge}}\,\Omega_k=0$.

PROOFS. First, we prove the following Lemma.

LEMMA 1. *Suppose $a \in D^*$, $b \in O_D$, $v_D(a)$ is prime to p, and the residue class in C of b does not belong to F. Then $v_D(aba^{-1}b^{-1}-1)$ is equal to the ramification number of D.*

(Roughly speaking, since the properties of a and b much differ from each other, the commutator $aba^{-1}b^{-1}$ becomes most different from 1.)

PROOF OF LEMMA 1. We can prove the following (5) and (6) easily.

(5) If $x, y \in D^*$ and $v_D(x+y)=\inf(v_D(x), v_D(y))$, we have the following inequality for any $z \in D^*$,

$$v_D(z(x+y)z^{-1}(x+y)^{-1}-1) \geqq \inf(v_D(zxz^{-1}x^{-1}-1), v_D(zyz^{-1}y^{-1}-1)) .$$

(6) For any $x, y, z \in D^*$,

$$v_D(zxyz^{-1}(xy)^{-1}-1) \geqq \inf(v_D(zxz^{-1}x^{-1}-1), v_D(zyz^{-1}y^{-1}-1)) .$$

Since each element c of D can be written as

$$c = \sum_{\substack{0 \leqq i \leqq p-1 \\ 0 \leqq j \leqq p-1}} r_{ij} a^i b^j \qquad r_{ij} \in K ,$$

and $v_D(c)=\inf\{v_D(r_{ij}a^i b^j)\}$, (5) and (6) show

$$v_D(zcz^{-1}c^{-1}-1) \geqq \inf(v_D(zaz^{-1}a^{-1}-1), v_D(zbz^{-1}b^{-1}-1)) \tag{7}$$

for any $z, c \in D^*$. From (7), by putting $z=a$ or $z=b$, we have, for any $c \in D^*$,

$$v_D(aca^{-1}c^{-1}-1) \geqq v_D(aba^{-1}b^{-1}-1) , \tag{8}$$

$$v_D(bcb^{-1}c^{-1}-1) \geqq v_D(bab^{-1}a^{-1}-1) = v_D(aba^{-1}b^{-1}-1) . \tag{9}$$

Comparing (7), (8) and (9), we have, for any $c, z \in D^*$,

$$v_D(czc^{-1}z^{-1}-1) \geqq v_D(aba^{-1}b^{-1}-1) .$$

Q.E.D.

PROOF OF PROP. 1. Since C/F is inseparable, for each $s \in D^*$, the homomorphism $x \mapsto sxs^{-1}$ induces the identity map on $O_D/m_D = C$. This shows $1 \leqq t$. It remains to prove $t \leqq e_K p/(p-1)$. Let a and b be as in Lemma 1. Let

$$a^p + r_1 a^{p-1} + \cdots + r_{p-1}a + r_p = 0 \quad (r_i \in K) \tag{10}$$

be the minimum equation of a over K. Then, the equation b^{-1} (10) b is

$$b^{-1}a^p b + r_1 b^{-1} a^{p-1} b + \cdots + r_p = 0 . \tag{10$'$}$$

We are interested in the difference (10)$'$$-$(10). As an assistance, we define a map $d_a : D \to D$ by $d_a(x) = axa^{-1} - x$. This map has the following properties.

(11) $$a^m x a^{-m} = \sum_{i=0}^{m} \binom{m}{i} (d_a)^i(x) \qquad \forall x \in D\,, \quad \forall m \geqq 0\,.$$
$$v_D(d_a(x)) \geqq v_D(x) + t \qquad \forall x \in D\,.$$
$$v_D(d_a(b)) = t\,.$$

From these properties, in case $1 \leqq m \leqq p-1$, we have

(12) $$a^m b a^{-m} = b + m d_a(b) + \cdots = b + R \qquad v_D(R) = t \qquad \textit{for some} \quad R \in D\,,$$

and so,

(13) $$b^{-1} a^m b = a^m + R \qquad v_D(R) = m v_D(a) + t \qquad \textit{for some} \quad R \in D\,.$$

Similarly, we have

(14) $$a^p b a^{-p} = b + p d_a(b) + R + (d_a)^p(b) \qquad \textit{for some} \quad R \in p d_a(b) \cdot m_D\,.$$

Suppose that $t > e_K p/(p-1)$ contrary to Prop. 1. Then,

$$v_D(p d_a(b)) = e_K p + t < pt \leqq v_D((d_a)^p(b))\,.$$

So, by (14),

$$a^p b a^{-p} = b + R \qquad v_D(R) = e_K p + t \qquad \text{for some} \quad R \in D\,.$$

This leads to

(15) $$b^{-1} a^p b = a^p + R \qquad v_D(R) = e_K p + t + p v_D(a) \qquad \textit{for some} \quad R \in D\,.$$

By (13) and (15), the difference $(10)' - (10)$ can be written as

(16) $$R_0 + r_1 R_1 + \cdots + r_{p-1} R_{p-1} = 0 \qquad (r_i \in K,\ R_i \in D)$$

where

$$v_D(R_0) = e_K p + t + p v_D(a)\,, \qquad v_D(R_i) = (p-i) v_D(a) + t \qquad \text{for} \quad 1 \leqq i \leqq p-1\,.$$

Since the class modulo p of $v_D(R_i)$ $(0 \leqq i \leqq p-1)$ are different from each other and $v_D(K^*) \subset p\boldsymbol{Z}$, this equality (16) leads to a contradiction. Q.E.D.

Proof of Prop. 2. In what follows, π denotes a fixed prime element of K. The proof consists of three steps.

Step 1. Suppose that $x \in D$, $n \geqq 1$ and $v_D(x) = n$. Then we can prove that x satisfies either of the following three conditions.

(i) $x \in K$.

(ii) $1 + x = (1+y)(1+z)$ for some $y \in K$ and $z \in D$ such that $v_K(y) \geqq n/p$, $v_D(z) \geqq n$, $p \mid v_D(z)$, and the residue class in C of $z\pi^{-v_D(z)/p}$ does not belong to F.

(iii) $1 + x = (1+y)(1+z)$ for some $y \in K$ and $z \in D$ such that $v_K(y) \geqq n/p$, $v_D(z) \geqq n$ and $v_D(z)$ is prime to p.

The proof is easy, and so we omit it.

Step 2. By step 1, for the proof of Prop. 2 (i) and (ii), it suffices to prove

"If $x \in D$ and $v_D(x)=n \geqq 1$,
$\mathrm{Nrd}(1+x)=1+\mathrm{Nrd}(x)+R$, $v_K(R) \geqq t+(n-t)p^{-1}$ for some $R \in K$",

in the following three cases:

(i) $x \in K$,

(ii) $p \mid n$ and the residue class of $x\pi^{-n/p}$ does not belong to F,

(iii) n is prime to p.

We prove the following more precise two formulae in each case.

$$\begin{aligned} &\mathrm{Nrd}(1+x)=1+\mathrm{Trd}(x)+\mathrm{Nrd}(x)+R, \quad v_K(R)>t+(n-t)p^{-1}, \\ &\qquad \textit{for some} \ R \in K. \\ &v_K(\mathrm{Trd}(x)) \geqq t+(n-t)p^{-1}. \end{aligned} \tag{17}$$

Here Trd denotes the reduced trace map.

Case (i). In this case, these formulae are easily proved because $\mathrm{Nrd}(1+x)=(1+x)^p$, $\mathrm{Trd}(x)=px$ and $\mathrm{Nrd}(x)=x^p$.

Case (ii). Let $u \in O_D$ such that $x=u\pi^{n/p}$. Let

$$u^p+r_1u^{p-1}+\cdots+r_p=0 \qquad (r_i \in K) \tag{18}$$

be the minimum equation of u over K.

It suffices to show that

$$\text{For all } 1 \leqq i \leqq p-1, \quad v_K(r_i) \geqq t-tp^{-1}. \tag{19}$$

We deduce this from the fact u is an element of O_D such that $\bar{u} \notin F$. As an assistance, let a be an element of D^* such that $v_D(a)$ is prime to p. Let $d_u\colon D \to D$ be as before. We can prove (cf. (14))

$$u^pau^{-p}=a+pd_u(a)+R+(d_u)^p(a), \quad \textit{for some} \ R \in pd_u(a)\cdot m_D. \tag{20}$$

Since $v_D(pd_u(a))=e_Kp+v_D(a)+t \geqq v_D(a)+pt$ by $t \leqq e_Kp/(p-1)$ (Prop. 1), and since $v_D((d_u)^p(a)) \geqq v_D(a)+pt$, (20) induces

$$u^pau^{-p}=a+R, \quad v_D(R) \geqq v_D(a)+pt \quad \text{for some} \ R \in D,$$

and so,

$$a^{-1}u^pa=u^p+R \quad v_D(R) \geqq pt \quad \text{for some} \ R \in D.$$

By using this, we can write the difference $a^{-1}(18)a-(18)$ as follows:

$$(21)\qquad R-\sum_{i=1}^{p-1}(ir_i a^{-1}d_u(a)u^{p-i}+R_i)=0\ ,\qquad R, R_i\in D\ ,$$
$$v_D(R)\geqq pt\ ,\qquad R_i\in r_i a^{-1}d_u(a)\cdot m_D\quad \textit{for each}\quad i\ .$$

Let $v_K(r_s)$ be the minimum of $v_K(r_i)$ $(1\leqq i\leqq p-1)$, and denote $r_s a^{-1}d_u(a)$ by z. Suppose $v_K(r_s)<t-tp^{-1}$ contrary to (19). Then, $v_D(z)<pt\leqq v_D(R)$. By sending the equation z^{-1} (21) into the residue algebra, we see that $\bar{u}$ satisfy a nontrivial equation of degree$\leqq p-1$ over F. This contradicts the fact that $[C: F]=p$. Q.E.D.

Case (iii). Let

$$(22)\qquad x^p+r_1x^{p-1}+\cdots+r_p=0\qquad (r_i\in K)$$

be the minimum equation of x over K.

It suffices to show that:

$$\text{For all}\quad 1\leqq i\leqq p-1\ ,\qquad v_K(r_i)\geqq inp^{-1}+t-tp^{-1}\ .$$

The proof of this fact is similar to those of Case (ii) and Prop. 1, so we omit it.

Step 3. It remains to prove Prop. 2 (iii). Let a and b be as in Lemma 1. Let $s=aba^{-1}b^{-1}-1$. Then, $v_D(s)=t$ by Lemma 1. By (17),

$$1=\mathrm{Nrd}\,(aba^{-1}b^{-1})=\mathrm{Nrd}\,(1+s)=1+\mathrm{Trd}\,(s)+\mathrm{Nrd}\,(s)+R\ ,$$
$$v_K(R)>t\ ,\qquad \text{for some}\quad R\in K\ .$$

It follows $v_K(s')=t$, where $s'=\mathrm{Trd}\,(s)$. By (17), we have:

$$\text{If}\quad x\in K^*\quad\text{and}\quad v_K(x)\geqq 1\ ,$$
$$\mathrm{Nrd}\,(1+xs)=1+xs'+R\ ,\qquad v_K(R)>v_K(xs')\ ,\qquad \text{for some}\quad R\in K\ .$$

This proves the inclusion $U_K^{(t+1)}\subset\mathrm{Nrd}\,(U_D^{(t+1)})$. Q.E.D.

PROOF OF PROP. 3. The assertion (i) is clear. The assertion (ii) follows from the formula (17). Since the proof of (iv) is similar to that of (iii), we omit the former. Now, we prove (iii). The first row in (iii) is exact because it is isomorphic to the exact sequence

$$0\to F^*\to F^*\to \Omega_F\to\Omega_F\to \mathrm{Br}\,(F)_p\to 0$$

which is deduced from the following well known exact sequence in C_F (§ 0.1)

$$0\to F_s^*\xrightarrow{x\mapsto x^p}F_s^*\xrightarrow{x\mapsto dx/x}\Omega_{F_s}\xrightarrow{1-\gamma}\Omega_{F_s}\to 0$$

(cf. Cartier [3]) by taking the Galois cohomology.

The homomorphisms (1) and (2) are well defined because the differential modules admit the following interpretation: by Graham [5], there is an isomorphism:

$$(k^+\underset{\mathbf{Z}}{\otimes}k^*)/J\cong\Omega_k \qquad x\otimes y\mapsto x\,dy/y$$

for any field k, where k^+ denotes the additive group of k and J denotes the subgroup of the tensor product generated by all elements of the form

$$(x+y)\otimes(x+y)-(x\otimes x)-(y\otimes y)$$

such that x, y and $x+y\in k^*$. Once the maps (1) and (2) are known to be well defined, it is clear that they are isomorphisms. For the proof of the commutativity of the diagram in (iii), it suffices to show the part

$$\begin{tikzcd} \Omega_C \arrow[r, "(1-\gamma)\cdot\mathfrak{F}"'] \arrow[d, "(1)"'] & \Omega_F \arrow[d, "(2)"'] \\ U_D^{(t)}/U_D^{(t+1)} \arrow[r, "\mathrm{Nrd}"'] & U_K^{(t)}/U_K^{(t+1)} \end{tikzcd}$$

is commutative. Since $[C:C^p]=p$, Ω_C is additively generated by elements of the forms dx and ydz/z such that $x\in C$, $y\in F$, $z\in C^*$. So, it suffices to show that if an element of Ω_C has either of these forms, its two images under $(2)\cdot(1-\gamma)\cdot\mathfrak{F}$ and $\mathrm{Nrd}\cdot(1)$ coincide. For an element $d\bar{f}$ $(f\in O_D)$, this is equivalent to

$$1+f^p\cdot\mathrm{Nrd}\,(f^{-1}afa^{-1}-1)\equiv\mathrm{Nrd}\,(1+afa^{-1}-f) \qquad \mathrm{mod}\ U_K^{(t+1)}\,,$$

which follows from the formula (17). For an element $\bar{f}d\bar{g}/\bar{g}$ $(f\in O_K,\ g\in U_D)$, this is equivalent to

$$\begin{aligned}1+f^p\cdot\mathrm{Nrd}\,(g^{-1}aga^{-1}-1)-f\cdot\mathrm{Nrd}\,(g^{-1}aga^{-1}-1)&\\ \equiv\mathrm{Nrd}\,(1+f(g^{-1}aga^{-1}-1)) \qquad \mathrm{mod}\ U_K^{(t+1)}\,.&\end{aligned} \tag{23}$$

By using the formula (17), (23) is found to be equivalent to

$$-\mathrm{Nrd}\,(g^{-1}aga^{-1}-1)\equiv\mathrm{Trd}\,(g^{-1}aga^{-1}-1) \qquad \mathrm{mod}\ m_K^{t+1}\,. \tag{24}$$

Put $h=g^{-1}aga^{-1}-1$. By the formula (17),

$$1=\mathrm{Nrd}\,(g^{-1}aga^{-1})=\mathrm{Nrd}\,(1+h)\equiv 1+\mathrm{Trd}\,(h)+\mathrm{Nrd}\,(h) \qquad \mathrm{mod}\ U_K^{(t+1)}\,,$$

which proves (24). Thus, we have proved Prop. 3 (iii).

Finally, Prop. 3 (v) follows immediately from Prop. 2 and from (i)-(iv) of Prop. 3.

The proofs of Prop. 1, 2, 3 are now complete.

REMARK 1. Let K be as at the beginning of this section. Let L be a cyclic extension of K of degree p such that the residue field C of L is an inseparable extension of F of degree p. Then, we can deduce some properties of the norm map $N_{L/K}: L^*\to K^*$ ((i)-(iv) below) from the arguments in this § 1.

Let π be a prime element of K and let σ be a generator of $\mathrm{Gal}(L/K)$. Then, the division algebra D over K defined by

$$D=\bigoplus_{i=0}^{p-1} L\cdot s^i\,,\qquad sa=\sigma(a)\cdot s \qquad \text{for all} \quad a\in L\,,\ \ s^p=\pi$$

satisfies the assumptions at the beginning of this section and its ramification number is given by $t=p\cdot v_L(\sigma(h)h^{-1}-1)$ where h is any element of O_L such that the residue class of h in C does not belong to F. Since L is a maximal commutative subalgebra of D,

$$\mathrm{Nrd}_{D/K}(x)=N_{L/K}(x) \qquad \text{for all} \quad x\in L^*\,.$$

By using this, we can easily prove the following (i)–(iii).

(i) *If* $0\leqq n\leqq tp^{-1}$, $N_{L/K}U_L^{(n)}\subset U_K^{(np)}$ *(t is as above).*

(ii) *If* $m\geqq 0$, $N_{L/K}U_L^{(tp^{-1}+m)}\subset U_K^{(t+m)}$.

(iii) $N_{L/K}\colon U_L^{(tp^{-1}+1)}\to U_K^{(t+1)}$ *is surjective.*

((i) and (ii) are deduced immediately from Prop. 2 and from the embedding $L\subset D$, and (iii) is proved by slightly changing Step 3 of Proof of Prop. 2.) We shall use (i)–(iii) in §3 and §5. We shall also use there

(iv) *results which follow immediately from Prop.* 3 *and the embedding* $L\subset D$, which we do not write down here concretely, for they are simple modifications of the assertions in Prop. 3.

§2. Computation of $K_2(K)/K_2(K)^p$.

The aim of this section is to have an exact knowledge of $K_2(K)/K_2(K)^p$ for fields K which are interesting to us.

NOTATION. Let K be a discrete valuation field. We denote by $V_K^{(n)}$ (or simply by $V^{(n)}$) the subgroup of $K_2(K)$ generated by all elements of the form $\{x,y\}$ such that $x\in U_K^{(n)}$ and $y\in K^*$. There are inclusions

$$K_2(K)=V^{(0)}\supset V^{(1)}\supset V^{(2)}\supset\cdots\,.$$

In this section, we prove the following two Propositions. In Prop. 1, we compute the subquotients $(V_K^{(n)}\cdot K_2(K)^p)/(V_K^{(n+1)}\cdot K_2(K)^p)$ of $K_2(K)/K_2(K)^p$.

PROPOSITION 1. *Suppose that K is a field which is complete with respect to a discrete valuation, F is the residue field of K,* $\mathrm{ch}(F)=p>0$ *and* $[F:F^p]=p$. *Let* $n\geqq 0$. *Then,* $(V_K^{(n)}\cdot K_2(K)^p)/(V_K^{(n+1)}\cdot K_2(K)^p)$ *is isomorphic to the following group, via the isomorphism* ρ_n *defined below.*

(i) $F^*/(F^*)^p$ *in case* $n=0$.

(ii) *F/F^p in case $0<n<e_Kp/(p-1)$ and $p\mid n$.*

(iii) *Ω_F in case $0<n<e_Kp/(p-1)$ and n is prime to p.*

(iv) *$\Omega_F/\{w-a\gamma(w)\mid w\in\Omega_F\}\oplus F/\{x^p-ax\mid x\in F\}$ in case $n=e_Kp/(p-1)$.*

(v) *0 in case $n>e_Kp/(p-1)$.*

Here, γ denotes the Cartier operator $\Omega_F\to\Omega_F$ (see the review after § 1 *Prop.* 3) *and a in* (iv) *denotes an element of F defined below. If* $\mathrm{ch}\,(K)=0$ *and K contains a primitive p-th root of* 1, *we can take* 1 *as a.*

DEFINITION of the above isomorphism. The above isomorphism is not canonical. Let n be as in the above (i) (resp. (ii), resp. (iii), resp. (iv)). We define a homomorphism ρ_n from the group in the above (i) (resp. (ii), resp. (iii), resp. (iv)) to the group $(V_K^{(n)}\cdot K_2(K)^p)/(V_K^{(n+1)}\cdot K_2(K)^p)$ as follows. This ρ_n is in fact an isomorphism as is proved later.

(i) Case $n=0$. Fix a prime element π of K. We define ρ_n (which depends on the choice of π) to be the homomorphism induced by the homomorphism $U_K\to V^{(0)}$, $x\mapsto\{x,\pi\}$.

(ii) Case "$0<n<e_Kp/(p-1)$ and $p\mid n$". Fix a prime element π of K and an element b of K such that $v_K(b)=n/p$. We define ρ_n (which depends on the choices of π and b) to be the homomorphism induced by the homomorphism $O_K\to V^{(n)}/V^{(n+1)}$, $x\mapsto\{1+xb^p,\pi\}$.

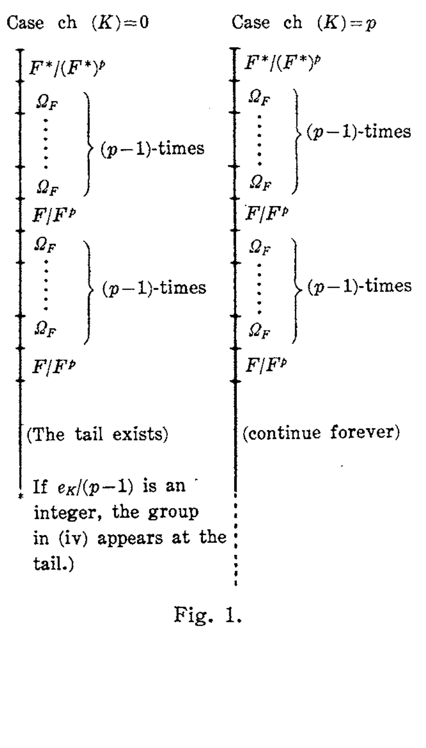

Fig. 1.

(iii) Case "$0<n<e_K p/(p-1)$ and n is prime to p". Fix an element c of K such that $v_K(c)=n$. We define ρ_n (which depends on the choice of c) to be the unique homomorphism $\Omega_F \to V^{(n)}/V^{(n+1)}$ such that

$$\bar{x}\, d\bar{y}/\bar{y} \mapsto \{1+xc, y\} \quad \text{for all} \quad x \in O_K \quad \text{and} \quad \text{all} \quad y \in U_K .$$

(iv) Case $n=e_K p/(p-1)$. Fix an element b of K such that $v_K(b)=e_K/(p-1)$. Fix a prime element π of K. We define ρ_n (which depends on the choices of b and π) to be the homomorphism induced by the following two homomorphisms:

$$\begin{gathered} \Omega_F \mapsto V^{(n)}/V^{(n+1)} \\ \bar{x}\, d\bar{y}/\bar{y} \mapsto \{1+xb^p, y\} , \\ O_K \mapsto V^{(n)}/V^{(n+1)} \\ \bar{x} \mapsto \{1+xb^p, \pi\} . \end{gathered}$$

We define $a \in F$ to be the residue class of $-pb^{1-p}$.

Roughly speaking, the structure of $K_2(K)/K_2(K)^p$ is as Figure 1.

The following Prop. 2 asserts that, roughly speaking, an element of $K_2(K)/K_2(K)^p$ is approximated by (or is equal to, if $\mathrm{ch}\,(K)=0$) some element of the form $\{\ ,\ \} \bmod K_2(K)^p$ in most cases. Note that an element of $K_2(K)$ has the form $\{\ ,\ \}\{\ ,\ \}\cdots\{\ ,\ \}$ and need not have the form $\{\ ,\ \}$, generally.

PROPOSITION 2. *Let K and F be as in Prop. 1. Let $n \geqq 0$. Let x be an element of $K_2(K)$ such that $x \in V^{(n)}K_2(K)^p$ and $x \notin V^{(n+1)}K_2(K)^p$. Then, we have $0 \leqq n \leqq e_K p/(p-1)$ and the following.*

(i) *If $n=0$, there exist a prime element π of K and an element f of U_K such that:*

$$\begin{aligned} &\bar{f} \notin F^p , \\ &x \equiv \{f, \pi\} \mod V^{(i)} \cdot K_2(K)^p \qquad \textit{for all} \quad i \geqq 0 . \end{aligned}$$

(ii) *Suppose $0<n<e_K p/(p-1)$ and $p \mid n$. Let $b \in K$ such that $v_K(b)=n/p$. Then, there exist a prime element π of K and an element f of O_K such that:*

$$\begin{aligned} &\bar{f} \notin F^p , \\ &x \equiv \{1+fb^p, \pi\} \mod V^{(i)} \cdot K_2(K)^p \qquad \textit{for all} \quad i \geqq 0 . \end{aligned}$$

(iii) *If $0<n<e_K p/(p-1)$ and n is prime to p, there exist an element c of K and an element g of U_K such that:*

$$\begin{aligned} &v_K(c)=n , \\ &\bar{g} \notin F^p , \\ &x \equiv \{1+c, g\} \mod V^{(i)} \cdot K_2(K)^p \qquad \textit{for all} \quad i \geqq 0 . \end{aligned}$$

In case $\mathrm{ch}(K)=0$, *since* $V^{(i)}\subset K_2(K)^p$ *for sufficiently large* i *in this case, the above assertions are true even if one puts* $K_2(K)^p$ *instead of* $V^{(i)}K_2(K)^p$.

The proofs of the propositions proceed as follows. First, we prove Prop. 1 except the part of the injectivity of ρ_n in Prop. 1 (ii)-(iv). Next, we prove Prop. 2. Lastly, we prove the injectivity of ρ_n in Prop. 1 (ii)-(iv).

PROOF OF PROP. 1 except the part mentioned above.

Case (i). Generally, let S be a discrete valuation field with residue field E. Let π be a fixed prime element of S. We are going to define a decomposition

$$V_S^{(0)}/V_S^{(1)}\cong K_2(E)\oplus E^* .$$

Define homomorphisms q, r, s and t as follows.

$$\begin{aligned}
&q\colon K_2(E)\to V_S^{(0)}/V_S^{(1)}\,; && \{\bar{x},\bar{y}\}\mapsto\{x,y\} \mod V_S^{(1)} \quad \text{for } x,y\in U_S .\\
&r\colon V_S^{(0)}/V_S^{(1)}\to K_2(E)\,; && \{x\pi^m, y\pi^n\} \mod V_S^{(1)}\mapsto\{\bar{x},\bar{y}\}\\
& && \qquad\qquad \text{for } x,y\in U_S \text{ and } m,n\in \boldsymbol{Z} .\\
&s\colon E^*\to V_S^{(0)}/V_S^{(1)}\,; && \bar{x}\mapsto\{x,\pi\} \mod V_S^{(1)} \quad \text{for } x\in U_S .\\
&t\colon V_S^{(0)}/V_S^{(1)}\to E^*\,; && \text{the tame symbol.}
\end{aligned}$$

By Matsumoto's theorem, these homomorphisms are well defined. We can easily show that $rq=\mathrm{id}$, $ts=\mathrm{id}$, $tq=0$, $rs=0$ and $qr+st=\mathrm{id}$ where each id denotes the identity map. Thus, we have the desired decomposition. Prop. 1 (i) follows from the decomposition and the following Lemma.

LEMMA 1. *Suppose* k *is a field such that* $\mathrm{ch}(k)=p>0$ *and* $[k:k^p]=p$. *Then,* $K_2(k)$ *is* p-*divisible.*

PROOF. Let $x,y\in k^*$. Let $E=k^{1/p}$. Then, $\{x,y\}_k=\{x,N_{E/k}y^{1/p}\}_k=N_{E/k}\{x,y^{1/p}\}_E=(N_{E/k}\{x^{1/p},y^{1/p}\}_E)^p$.

Before we proceed to Prop. 1 (ii)-(v), we prove two Lemmas.

LEMMA 2. (This Lemma is very useful throughout this paper.) *Let* S *be a discrete valuation field.*

(i) *If* $u\in U_S^{(i)}$ *and* $v\in U_S^{(j)}$, $\{u,v\}\in V_S^{(i+j)}$. *In particular, if* $u\in U_S^{(i)}$ *and* $v\in U_S^{(1)}$, $\{u,v\}\in V_S^{(i+1)}$.

(ii) *Suppose* $i\geqq 1$, $j\geqq 1$, $x\in m_S^i$, $y\in m_S^j$ *and* $x\neq 0$. *Then,*

$$\{1+x,1+y\}\equiv\{1+xy,-x\}^{-1} \mod V^{(i+j+1)} .$$

PROOF. Let x, y, and j be as in the hypothesis of (ii) above. Let N be the subgroup of $K_2(S)$ generated by all elements of $K_2(S)$ of the form $\{a,b\}$ such that $a\in U_S^{(i+j)}$ and $b\in U_S^{(j)}$. Then,

$$\begin{aligned}\{1+x, 1+y\} &\equiv \{1+x+xy, 1+y\} \mod N \\ &= \{1+x+xy, -(1+y)/(x+xy)\} \\ &= \{1+x+xy, -x\}^{-1} \\ &= \{(1+x+xy)/(1+x), -x\}^{-1} \\ &\equiv \{1+xy, -x\}^{-1} \mod V^{(i+j+1)} .\end{aligned}$$

This proves (i). Since (i) is true, $N \subset V^{(i+j+1)}$. Hence, the above calculation shows (ii).

LEMMA 3. *Let S be a discrete valuation field with residue field E. Let $n \geqq 1$, $c \in S$ and $v_S(c)=n$. Then, there exists a homomorphism $\theta: \Omega_E \to V_S^{(n)}/V_S^{(n+1)}$ which is characterized by the property*:

$$\theta(\bar{x}\, d\bar{y}/\bar{y}) = \{1+xc, y\} \mod V_S^{(n+1)} \quad \textit{for all} \quad x \in O_S \quad \textit{and} \quad \textit{all} \quad y \in U_S .$$

PROOF. By Lemma 2 and by the isomorphism $(E^+ \otimes E^*)/J \cong \Omega_E$ which was mentioned in the proof of § 1 Prop. 3, this Lemma is reduced to the following fact:

If $x, y, x+y \in U_S$,

$$\{1+(x+y)c, x+y\} \equiv \{1+xc, x\}\{1+yc, y\} \mod V_S^{(n+1)} .$$

But this can be written as

$$\{1+(x+y)c, -1/c\} \equiv \{1+xc, -1/c\}\{1+yc, -1/c\} \mod V_S^{(n+1)}$$

which is obvious.

We return to the proof of Prop. 1.

Case (ii). In this case, ρ_n is well defined because

$$1+f^p b^p \in (1+fb)^p \cdot U_K^{(n+1)}$$

for all $f \in O_K$. The surjectivity of ρ_n is shown as follows. It suffices to show that

$$\{1+fb^p, g\} \in V^{(n+1)} K_2(K)^p \qquad \text{for all} \quad f \in O_K \quad \text{and} \quad g \in U_K .$$

Since $[F: F^p]=p$, Ω_F is additively generated by elements of the forms $x^p \cdot dy/y$ and dz such that $x, z \in F$ and $y \in F^*$. Hence, by Lemma 3, it suffices to show that $\{1+f^p b^p, g\}$ and $\{1-gb^p, g\}$ belong to $V^{(n+1)} K_2(K)^p$ for all $f \in O_K$ and all $g \in U_K$. This follows from the facts

$$1+f^p b^p \in (1+fb)^p U_K^{(n+1)} \qquad \text{and} \qquad \{1-gb^p, g\} = \{1-gb^p, b^{-1}\}^p .$$

Case (iii). Lemma 3 shows that ρ_n is well defined. The surjectivity of ρ_n is shown as follows. Let π be a prime element of K. It suffices to show that the subgroup $\{U_K^{(n)}, \pi\}$ of $V^{(n)}$ is included in the subgroup of $V^{(n)}$ which is generated by $V^{(n+1)}$ and all elements of the form $\{u, v\}$ $u \in U_K^{(n)}$, $v \in U_K$. Take $n' \in \boldsymbol{Z}$ such that $nn' \equiv 1 \mod p$. For any $f \in U_K$, we have,

$$\{1+f\pi^n, \pi\} \equiv \{(1+n'f\pi^n)^n, \pi\} \equiv \{1+n'f\pi^n, \pi^n\} \equiv \{1+n'f\pi^n, -n'f\}^{-1} \mod V^{(n+1)} .$$

Case (iv). Assume $n=e_K p/(p-1)$. Let φ be the homomorphism

$$\begin{aligned} &\Omega_F \to V^{(n)} K_2(K)^p / V^{(n+1)} K_2(K)^p \\ &\bar{f}\, d\bar{g}/\bar{g} \mapsto \{1+fb^p, g\} \mod V^{(n+1)} K_2(K)^p . \end{aligned}$$

Then, φ annihilates the group $\{w - a\gamma(w) \mid w \in \Omega_F\}$. To see this, fix some $g \in U_K$ such that $\bar{g} \notin F^p$. Let $s_i \in O_K$ for each i such that $0 \leqq i \leqq p-1$. Then,

$$(1-a\gamma)\left(\sum_{i=0}^{p-1} \bar{s}_i^p \bar{g}^i (d\bar{g}/\bar{g})\right) \in (\bar{s}_0^p - a\bar{s}_0)\, d\bar{g}/\bar{g} + d(F) .$$

But on the one hand,

$$\begin{aligned} \varphi((\bar{s}_0^p - a\bar{s}_0)\, dg/g) &\equiv \{1+(s_0^p - \tilde{a}s_0)b^p, g\} \\ &\equiv \{1+s_0 b, g\}^p \\ &\equiv 1 \mod V^{(n+1)} K_2(K)^p . \end{aligned}$$

where $\tilde{a}$ is any representative of a. On the other hand, for all $f \in U_K$,

$$\begin{aligned} \varphi(-d\bar{f}) &\equiv \varphi(-\bar{f}\, d\bar{f}/\bar{f}) \equiv \{1-fb^p, f\} \\ &\equiv \{1-fb^p, b^{-1}\}^p \equiv 1 \mod V^{(n+1)} K_2(K)^p . \end{aligned}$$

Next, let ψ be the homomorphism

$$\begin{aligned} &F \to V^{(n)} K_2(K)^p / V^{(n+1)} K_2(K)^p \\ &\bar{f} \mapsto \{1+fb^p, \pi\} \mod V^{(n+1)} K_2(K)^p . \end{aligned}$$

Then, as is easily shown, ψ annihilates the subgroup $\{x^p - ax \mid x \in F\}$. These show that ρ_n is well defined. The surjectivity of ρ_n is obvious. Suppose K contains a primitive p-th root ζ of 1. Then, $-p(\zeta-1)^{1-p} \in U_K$ and the residue class of $-p(\zeta-1)^{1-p}$ is equal to 1. Thus, $a=1$ if one defines ρ_n using $\zeta-1$ as b.

Case (v). Assume $n > e_K p/(p-1)$. Then, $U_K^{(n)} \subset K^{*p}$. Hence $V^{(n)} \subset K_2(K)^p$.

PROOF OF PROP. 2. Prop. 2 is an easy consequence of the following Lemma.

LEMMA 4. *Let K be as in the hypothesis of Prop.* 2. *Let $\alpha \in K_2(K)$. Then, we have the following assertions* (i) (ii) (iii).

(i) *Hypothesis*: $\alpha \equiv \{f, \pi\} \mod V^{(i)} K_2(K)^p$,
$f \in U_K$, $\bar{f} \notin F^p$,
π is a prime element of K,
$i > 0$.

Conclusion: *There exist* f' *and* π' *such that*
$\alpha \equiv \{f', \pi'\} \bmod V^{(i+1)}K_2(K)^p$,
$f'/f \in U_K^{(i)}$ *and* $\pi'/\pi \in U_K^{(i)}$.

(ii) *Hypothesis*: $\alpha \equiv \{1+fb^p, \pi\} \bmod V^{(i)}K_2(K)^p$,
$f \in O_K$, $\bar{f} \notin F^p$, $v_K(b)=n/p$, $p \mid n$,
π *is a prime element of* K,
$i>n$, $0<n<e_K p/(p-1)$.

Conclusion: *There exist* f' *and* π' *such that*
$\alpha \equiv \{1+f'b^p, \pi'\} \bmod V^{(i+1)}K_2(K)^p$,
$v_K(f'-f) \geqq i-n$ *and* $\pi'/\pi \in U_K^{(i-n)}$.

(iii) *Hypothesis*: $\alpha \equiv \{1+c, g\} \bmod V^{(i)}K_2(K)^p$,
$g \in U_K$, $\bar{g} \notin F^p$, $v_K(c)=n$, n *is prime to* p,
$i>n$, $0<n<e_K p/(p-1)$.

Conclusion: *There exist* c' *and* g' *such that*
$\alpha \equiv \{1+c', g'\} \bmod V^{(i+1)}K_2(K)^p$
$v_K(c'-c) \geqq i$ *and* $g'/g \in U_K^{(i-n)}$.

PROOF OF LEMMA 4. Since (i), (ii) and (iii) of Lemma 4 are similarly proved, we present here only the proof of Lemma 4 (ii). If $0<i<e_K p/(p-1)$ and $p \mid i$, this follows from the surjectivity of ρ_i in Prop. 1 (ii). Next, suppose $0<i<e_K p/(p-1)$ and i is prime to p. Let c be an element of K such that $v_K(c)=i-n$. Lemma 2 (ii) shows, for all $s \in O_K$,

$$\begin{aligned}\{1+fb^p, 1+sc\} &\equiv \{1+sfb^p c, -fb^p\}^{-1} \\ &\equiv \{1+sfb^p c, -f\}^{-1} \quad \bmod V^{(i+1)}K_2(K)^p .\end{aligned}$$

This calculation and the surjectivity of ρ_i in Prop. 1 (iii) show that each element of $V^{(i)}K_2(K)^p/V^{(i+1)}K_2(K)^p$ has the form $\{1+fb^p, 1+sc\} \bmod V^{(i+1)}K_2(K)^p$ for some $s \in O_K$. This proves Lemma 4 (ii) in the present case. Lastly, we omit the proof in the case $i \geqq e_K p/(p-1)$ since it is similar to those in the previous cases.

Thus, we have proved Lemma 4 and hence Prop. 2.

PROOF of the injectivity of ρ_n in Prop. 1 (ii), (iii) or (iv). The proof is divided into three cases.

Case 1. Suppose $\mathrm{ch}(K)=p$.

In this case, we may suppose $K=F((T))$. Let

$$s: K_2(K)/K_2(K)^p \to \bigwedge_K^2 \Omega_K \,; \qquad s(\{x, y\} \bmod K_2(K)^p)=dx/x \wedge dy/y$$

be the well known homomorphism. Here, $\bigwedge_K^2 \Omega_K$ denotes the second exterior power of Ω_K over K. Since $[K: K^p]=p^2$, $\bigwedge_K^2 \Omega_K$ is a one dimensional linear space over the complete valuation field K. Hence $\bigwedge_K^2 \Omega_K$ has a natural topology which is characterized by the following property: For each non-zero element w of $\bigwedge_K^2 \Omega_K$, the map $K \to \bigwedge_K^2 \Omega_K$; $x \mapsto xw$ is a homeomorphism. Each element of $\bigwedge_K^2 \Omega_K$ can be uniquely written as

$$\sum_{i \in Z} w_i \wedge T^i \, dT/T$$

with respect to this natural topology, where $w_i \in \Omega_F$ for all i and $w_i=0$ if i is sufficiently near $-\infty$. For each $n \geqq 0$, let V_n be the subgroup of $\bigwedge_K^2 \Omega_K$:

$$V_n=\left\{\sum_{i \geqq n} w_i \wedge T^i \, dT/T \mid w_i \in \Omega_F\right\} .$$

Then, $s(V^{(n)} K_2(K)^p/K_2(K)^p) \subset V_n$.

Let n, b, π, ρ_n be as in Prop. 1 (ii). Let t_n be the composite

$$F/F^p \xrightarrow{\rho_n} V^{(n)} K_2(K)^p/V^{(n+1)} K_2(K)^p \xrightarrow{\text{by } s} V_n/V_{n+1} .$$

Then, we have,

$$t_n(x \bmod F^p)=dx \wedge b^p \, dT/T \mod V_{n+1} ,$$

which proves the injectivity of ρ_n in Prop. 1 (ii).

Let n, c, ρ_n be as in Prop. 1 (iii). Let $c_0 \in F$ be the residue class of c/T^n. Let t_n be the composite

$$\Omega_F \xrightarrow{\rho_n} V^{(n)} K_2(K)^p/V^{(n+1)} K_2(K)^p \xrightarrow{\text{by } s} V_n/V_{n+1} .$$

Then, we have,

$$t_n(w)=-nc_0 w \wedge T^n \, dT/T \mod V_{n+1} ,$$

which proves the injectivity of ρ_n in Prop. 1 (iii).

Case 2. Suppose $\mathrm{ch}(K)=0$ and K contains a primitive p-th root ζ of 1. We use the homomorphism

$$h_p': K_2(K)/K_2(K)^p \to \mathrm{Br}(K)_p .$$

Recall that h_p' is the composite

$$K_2(K)/K_2(K)^p \xrightarrow{h_p} H^2(K, \mu_p \otimes \mu_p) \cong H^2(K, \mu_p) \cong \mathrm{Br}(K)_p$$

where h_p denotes the Galois symbol.

First we show $\rho_{e_K p/(p-1)}$ is injective. There are many $\rho_{e_K p/(p-1)}$ as is seen from the definition, but it suffices to prove the injectivity of some one $\rho_{e_K p/(p-1)}$. Let $\mathrm{Br}(K_{nr}/K)$ be the kernel of the natural homomorphism $\mathrm{Br}(K)\to\mathrm{Br}(K_{nr})$ where K_{nr} denotes the maximum unramified extension of K. Then, we have the following Lemma.

LEMMA 5. $h'_p(V^{(e_K p/(p-1))}K_2(K)^p/K_2(K)^p)\subset \mathrm{Br}(K_{nr}/K)_p$.

PROOF. This follows from $U_{K_{\mathrm{nr}}}^{(e_K p/(p-1))}\subset(K_{\mathrm{nr}}^*)^p$. (Q.E.D.)

By Serre [19] Ch. XII,

$$\mathrm{Br}(K_{\mathrm{nr}}/K)_p\cong \mathrm{Br}(F)_p\oplus(X_F)_p\ . \tag{1}$$

But $\mathrm{Br}(F)_p\cong\Omega_F/(1-\gamma)\Omega_F$ as in §1 Prop. 3 and $(X_F)_p=F/\{x^p-x\mid x\in F\}$ by Artin-Schreier theory. Thus,

$$\mathrm{Br}(K_{\mathrm{nr}}/K)_p\cong\Omega_F/(1-\gamma)\Omega_F\oplus F/\{x^p-x\mid x\in F\}\ . \tag{2}$$

The isomorphism (1) is defined depending on the choice of a prime element of K as in Serre [19] Ch. XII. Define $\rho_{e_K p/(p-1)}$ using the prime element π which was used to define the isomorphism (1) and using $\zeta-1$ as b. Then the composite

$$\begin{aligned}&\Omega_F/(1-\gamma)\Omega_F\oplus F/\{x^p-x\mid x\in F\}\\ &\qquad\xrightarrow{\rho_{e_K p/(p-1)}} V^{(e_K p/(p-1))}K_2(K)^p/K_2(K)^p\xrightarrow{h'_p}\mathrm{Br}(K_{\mathrm{nr}}/K)_p\\ &\qquad\underset{\cong}{\xrightarrow{(2)}}\Omega_F/(1-\gamma)\Omega_F\oplus F/\{x^p-x\mid x\in F\}\end{aligned}$$

is found to be the identity map. Thus, $\rho_{e_K p/(p-1)}$ is injective.

Next, we show that ρ_n is injective in the case $0<n<e_K p/(p-1)$. We use the following Lemma 6.

LEMMA 6. *Let K and F be as in the hypothesis of Prop.* 1. *Suppose* $\mathrm{ch}(K)=0$ *and K contains a primitive p-th root of ζ of* 1. *Let*

$$h'_p\colon K_2(K)/K_2(K)^p\to\mathrm{Br}(K)_p$$

be as before. Then, we have the following (i), (ii) *and* (iii).

(i) *Suppose $f\in U_K$, $\bar f\notin F^p$ and π is a prime element of K. Then,*

$$h'_p(\{f,\pi\}\bmod K_2(K)^p)\notin\mathrm{Br}(K_{\mathrm{nr}}/K)_p\ .$$

The division algebra which corresponds to $h'_p(\{f,\pi\}\bmod K_2(K)^p)$ satisfies the assumptions at the beginning of §1 *and its ramification number is $e_K p/(p-1)$.*

(ii) *Suppose $0<n<e_K p/(p-1)$, $p\mid n$, $f\in O_K$, $\bar f\notin F^p$, $b\in K$, $v_K(b)=n/p$ and π is a prime element of K. Then,*

$$h'_p(\{1+fb^p, \pi\} \bmod K_2(K)^p) \notin \mathrm{Br}(K_{\mathrm{nr}}/K)_p .$$

The division algebra which corresponds to $h'_p(\{1+fb^p, \pi\} \bmod K_2(K)^p)$ *satisfies the assumptions at the biginning of* § 1 *and its ramification number is* $(e_K p/(p-1))-n$.

(iii) *Suppose* $0<n<e_K p/(p-1)$, n *is prime to* p, $g \in U_K$, $\bar{g} \notin F^p$, $c \in K$ *and* $v_K(c)=n$. *Then,*

$$h'_p(\{1+c, g\} \bmod K_2(K)^p) \notin \mathrm{Br}(K_{\mathrm{nr}}/K)_p .$$

The division algebra which corresponds to $h'_p(\{1+c, g\} \bmod K_2(K)^p)$ *satisfies the assumptions at the beginning of* § 1 *and its ramification number is* $(e_K p/(p-1))-n$.

PROOF. The "$\notin \mathrm{Br}(K_{\mathrm{nr}}/K)$" parts follow from § 0.2 (6) by taking K_{nr} as k. The ramification numbers can be computed using § 1 Lemma 1, for the explicit forms of the division algebras in the above Lemma are given in § 0.2.

Now, we return to the proof of the injectivity of ρ_n in the case $0<n<e_K p/(p-1)$. Since the proof in the case "$0<n<e_K p/(p-1)$ and $p \mid n$" and that in the case "$0<n<e_K p/(p-1)$ and n is prime to p" go similarly, we present here only the proof in the latter case. Let $g \in U_K$, $\bar{g} \notin F^p$, $c \in K$ and $v_K(c)=n$. It suffices to show

$$\{1+c, g\} \notin V^{(n+1)} K_2(K)^p .$$

Prop. 2 and Lemma 5 and Lemma 6 show that for any element α of $V^{(n+1)} K_2(K)^p$, $h'_p(\alpha)$ satisfies either of the following two conditions.

(i) $h'_p(\alpha) \in \mathrm{Br}(K_{\mathrm{nr}}/K)_p$.

(ii) The division algebra which corresponds to $h'_p(\alpha)$ satisfies the assumptions at the beginning of § 1 and its ramification number t satisfies

$$t<(e_K p/(p-1))-n .$$

Hence, by Lemma 6 (iii), $\{1+c, g\} \notin V^{(n+1)} K_2(K)^p$.

Case 3. Suppose $\mathrm{ch}(K)=0$. (Here, K need not contain a primitive p-th root of 1.) Let ζ be a primitive p-th root of 1 and let $L=K(\zeta)$. Since the residue field C of L is a finite Galois extension of F such that $[C: F]$ is prime to p, the canonical homomorphisms $F^*/F^{*p} \to C^*/C^{*p}$, $F/F^p \to C/C^p$, $\Omega_F \to \Omega_C$, $\Omega_F/\{w-a\gamma(w) \mid w \in \Omega_F\} \to \Omega_C/\{w-a\gamma(w) \mid w \in \Omega_C\}$, $F/\{x^p-ax \mid x \in F\} \to C/\{x^p-ax \mid x \in C\}$ are all injective. Hence the injectivity of ρ_n for K follows easily from the injectivity of ρ_n for L.

§ 3. The Galois symbol is an isomorphism.

The purpose of this section is to prove the following Theorem 1.

THEOREM 1. *Let K be a field which is complete with respect to a discrete valuation and with residue field F. Suppose that* $\mathrm{ch}(K)=0$, $\mathrm{ch}(F)=p>0$, *and* $[F:F^p]=p$. *Then for any natural number n, the Galois symbol*

$$h_{p^n}: K_2(K)/K_2(K)^{p^n} \rightarrow H^2(K, \mu_{p^n}\otimes\mu_{p^n})$$

is bijective.

PROOF. By Bass [1] 9 (11) and (12), it suffices to prove the above Theorem when $n=1$ and K contains a primitive p-th root of 1. So, in what follows, we suppose that K contains a primitive p-th root ζ of 1. It suffices to show that

$$h'_p: K_2(K)/K_2(K)^p \rightarrow \mathrm{Br}(K)_p$$

(cf. § 0.2) is bijective.

§ 3.1. We prove the injectivity of h'_p.

As in § 0.2, for any $a, b\in K^*$, the condition $h'_p(\{a, b\} \bmod K_2(K)^p)=0$ is equivalent to the condition $\{a, b\}\in K_2(K)^p$. The problem lies in the fact that an element of $K_2(K)$ has the form $\{\ ,\ \}\{\ ,\ \}\cdots\{\ ,\ \}$ in general and need not have the form $\{\ ,\ \}$. But by § 2 Prop. 2, any element of $K_2(K)/K_2(K)^p$ which does not belong to $V^{(e_K p/(p-1))}K_2(K)^p/K_2(K)^p$ has the form $\{\ ,\ \} \bmod K_2(K)^p$. Hence, it remains to show that the homomorphism

$$V^{(e_K p/(p-1))}K_2(K)^p/K_2(K)^p \xrightarrow{\text{by } h'_p} \mathrm{Br}(K)_p$$

is injective. But this follows from the proof of the injectivity of $\rho_{e_K p/(p-1)}$ (§ 2).

§ 3.2 Now, we proceed to the proof the surjectivity of h'_p. This proof is difficult because we have only little knowledge of $\mathrm{Br}(K)$. For any field k such that $\mathrm{ch}(k)\neq p$ and such that k contains a primitive p-th root of 1, let $C(k)$ be the cokernel of $h'_p: K_2(k)/K_2(k)^p \rightarrow \mathrm{Br}(k)_p$. We are going to prove $C(K)=0$.

Let x_0 be a fixed element of O_K such that $\bar{x}_0\notin F^p$. Let $x_1, x_2, x_3, \cdots$ be fixed elements of the algebraic closure of K such that $x_i=(x_{i+1})^p$ for all $i\geqq 0$. Let $K^{(i)}=K(x_i)$ and let $K^{(\infty)}=\bigcup_i K^{(i)}$. Since $K^{(\infty)}$ is a Henselian dicrete valuation field of characteristic 0, $\mathrm{Br}(K^{(\infty)})\cong\mathrm{Br}(K^{(\infty)\wedge})$ where $K^{(\infty)\wedge}$ denotes the completion of $K^{(\infty)}$. It follows that $C(K^{(\infty)})\cong C(K^{(\infty)\wedge})$. Since the residue field of $K^{(\infty)\wedge}$ is perfect, we have much knowledge of $\mathrm{Br}(K^{(\infty)\wedge})$ by Serre [19] Ch. XII, and it follows that $C(K^{(\infty)\wedge})=0$. Hence $C(K^{(\infty)})=0$. For the proof of $C(K)=0$, since $C(K^{(\infty)})=\varinjlim C(K^{(i)})$, it suffices to show that each $C(K^{(i)})\rightarrow C(K^{(i+1)})$ is injective. Now, we need the following Lemma. In what follows, if G is a group and A is a G-module, we denote by A^G the group $\{x\in A \mid \sigma x=x, \forall\, \sigma\in G\}$.

LEMMA 1. *Let k be a field and E a cyclic extension of k of finite degree m with Galois group G. Then the sequence*

$$\mathrm{Br}\,(k)_m \xrightarrow{\mathrm{Res}} \mathrm{Br}\,(E)_m^G \xrightarrow{\mathrm{Cor}} \mathrm{Br}\,(k)_m$$

is exact. Here, Res *denotes the restriction map and* Cor *denotes the corestriction map. (Cf. Serre* [20] *for the definition of these maps.)*

PROOF. This follows from the spectral sequence

$$H^*(G, H^*(E, k_s^*)) \Rightarrow H^*(k, k_s^*)\ . \qquad \text{(Q.E.D.)}$$

We return to the proof of the injectivity of $C(K^{(i)}) \to C(K^{(i+1)})$. Let $L = K^{(1)}$. It suffices to show that the homomorphism $C(K) \to C(L)$ is injective. Let $G = \mathrm{Gal}\,(L/K)$. Consider the following commutative diagram.

$$(1)\qquad \begin{array}{ccccccc} 0 \to & K_2(K)/K_2(K)^p & \xrightarrow{h'_{p,K}} & \mathrm{Br}\,(K)_p & \to & C(K) & \to 0 \\ & \downarrow{\scriptstyle \alpha} & & \downarrow{\scriptstyle \mathrm{Res}} & & \downarrow{\scriptstyle \beta} & \\ 0 \to & (K_2(L)/K_2(L)^p)^G & \xrightarrow{h'_{p,L}} & \mathrm{Br}\,(L)_p^G & \to & C(L)^G & \\ & & & \downarrow{\scriptstyle \mathrm{Cor}} & & & \\ & & & \mathrm{Br}\,(K)_p & & & \end{array}$$

Here, α and β are the canonical homomorphisms. The rows are exact by the injectivity of h'_p. The middle column is exact by Lemma 1. Furthermore, the induced homomorphism from Ker (α) to Ker (Res) is bijective. This follows from the following well known fact: "Suppose k is a field and n is an integer which is invertible in k. Suppose k contains a primitive n-th root of 1. Then, an element w of $\mathrm{Br}\,(k)_n$ belongs to the image of

$$k^* \times k^* \to \mathrm{Br}\,(k)_n\ ; \qquad (x, y) \mapsto h'_n(\{x, y\})$$

if and only if there exists a cyclic extension E of k such that the canonical homomorphism $\mathrm{Br}\,(k) \to \mathrm{Br}\,(E)$ annihilates w and such that $[E : k]$ divides n." Hence, by applying the snake lemma to the diagram (1), we find that the injectivity of β is equivalent to the exactness of the following sequence

$$(2)\qquad K_2(K)/K_2(K)^p \xrightarrow{\alpha} (K_2(L)/K_2(L)^p)^G \xrightarrow{\mathrm{Cor} \circ h'_{p,L}} \mathrm{Br}\,(K)_p\ .$$

Since the following diagram (3) is commutative as is proved in § 3.5 later and since $h'_{p,K}$ is injective,

$$(3)\qquad \begin{array}{ccc} K_2(L)/K_2(L)^p & \xrightarrow{h'_{p,L}} & \mathrm{Br}\,(L)_p \\ \downarrow{\scriptstyle N_{L/K}} & & \downarrow{\scriptstyle \mathrm{Cor}} \\ K_2(K)/K_2(K)^p & \xrightarrow{h'_{p,K}} & \mathrm{Br}\,(K)_p \end{array}$$

the exactness of (2) is equivalent to the exactness of the following sequence (4).

$$K_2(K)/K_2(K)^p \xrightarrow{\alpha} (K_2(L)/K_2(L)^p)^G \xrightarrow{N_{L/K}} K_2(K)/K_2(K)^p . \tag{4}$$

Thus all things are reduced to a property of K_2.

§ **3. 3.** It remains to prove the sequence (4) is exact. Before we proceed to the proof of this, we prove some Lemmas in § 3.3. Let K and L be as before. We denote e_K $(=e_L)$ by e. We fix a generator σ of G $(=\mathrm{Gal}(L/K))$ and a prime element π of K. Note that π is also a prime element of L. We denote $V_K^{(n)}\cdot K_2(K)^p$ by $\mathscr{V}_K^{(n)}$ and we denote $V_L^{(n)}\cdot K_2(L)^p$ by $\mathscr{V}_L^{(n)}$.

In the following, we apply Remark 1 at the end of § 1. In the present case, the division algebra D in that Remark satisfies $t=ep/(p-1)$.

LEMMA 2. *We have* $N_{L/K}\mathscr{V}_L^{(n)}\subset\mathscr{V}_K^{(n)}$ *for all* $n\geqq 0$. *Furthermore, we have,*

(i) $N_{L/K}:\mathscr{V}_L^{(0)}/\mathscr{V}_L^{(1)}\to\mathscr{V}_K^{(0)}/\mathscr{V}_K^{(1)}$ *is an isomorphism.*

(ii) *If* $0<n<ep/(p-1)$, *the following sequence is exact.*

$$0\to U_K^{(n)}/U_K^{(n+1)}\to U_L^{(n)}/U_L^{(n+1)}\xrightarrow{\{*,\pi\}}\mathscr{V}_L^{(n)}/\mathscr{V}_L^{(n+1)}\xrightarrow{N_{L/K}}\mathscr{V}_K^{(n)}/\mathscr{V}_K^{(n+1)} .$$

(iii) *The following sequence is exact.*

$$0\to U_L^{(ep/(p-1))}(L^*)^p/(L^*)^p\xrightarrow{\{*,\pi\}}\mathscr{V}_L^{(ep/(p-1))}/K_2(L)^p\xrightarrow{N_{L/K}}\mathscr{V}_K^{(ep/(p-1))}/K_2(K)^p .$$

PROOF. These are deduced from; the formula $N_{L/K}\{a,b\}_L=\{a,N_{L/K}b\}$ $(a\in K^*,$ $b\in L^*)$, § 1 Remark 1 (i) and (ii), the knowledge of $K_2/(K_2)^p$ written in § 2 Prop. 1. The details are left to the reader.

LEMMA 3. *For each* $n\geqq 1$, *let* S_n *be the cokernel of the canonical homomorphism* $U_K^{(n)}/U_K^{(n+1)}\to U_L^{(n)}/U_L^{(n+1)}$. *Then, for each* $n\geqq 1$,

$$(\sigma-1)(U_L^{(n)})\subset U_L^{(n+e')}$$

and the induced homomorphism $S_n\to S_{n+e'}$ *is an isomorphism. Here,* $\sigma-1$ *denotes the homomorphism* $x\mapsto\sigma(x)/x$ *and* $e'=e/(p-1)$.

PROOF. Consider the following commutative diagram.

$$\begin{array}{ccccccc}
 & & U_K^{(n)}/U_K^{(n+1)} & = & U_K^{(n)}/U_K^{(n+1)} & & \\
 & & \| & & \downarrow\cap & & \\
0 & \to & U_K^{(n)}/U_K^{(n+1)} & \to & U_L^{(n)}/U_L^{(n+1)} & \to & S_n \to 0\\
 & & \downarrow 0 & & \downarrow \sigma-1 & & \downarrow \\
0 & \to & U_K^{(n+e')}/U_K^{(n+e'+1)} & \to & U_L^{(n+e')}/U_L^{(n+e'+1)} & \to & S_{n+e'} \to 0\\
 & & \| & & \downarrow N_{L/K} & & \\
 & & U_K^{(n+e')}/U_K^{(n+e'+1)} & \underset{x\mapsto x^p}{\xrightarrow{\cong}} & U_K^{(n+e'p)}/U_K^{(n+e'p+1)} . & &
\end{array}$$

The middle column is exact by § 1 Remark 1 (iv) and the rows are exact. Hence the homomorphism $S_n \to S_{n+e'}$ is an isomorphism.

LEMMA 4. *If* $u \in U_L^{(e)}$ *and* $\sigma(u)/u \in (L^*)^p$, *we have* $u \in K^* \cdot (L^*)^p$.

PROOF. Generally, let k be a field and let E be a finite cyclic extension of k. Let $m=[E:k]$. Suppose that m is invertible in k and that k contains a primitive m-th root of 1. Let $G=\mathrm{Gal}(E/k)$, σ a generator of G, and $f: (E^*/(E^*)^m)^G \to \mu_m$ the homomorphism which is defined by:

If $x, y \in E^*$ and $\sigma(x)/x=y^m$, $f(x \bmod E^{*m})=N_{E/k}y$.

Then, the following sequence is known to be exact.

$$k^*/(k^*)^m \to (E^*/(E^*)^m)^G \xrightarrow{f} \mu_m .$$

Now, the proof of Lemma 4 goes as follows. Let $u \in U_L^{(e)}$, $\sigma(u)/u=v^p$ and $v \in L^*$. Then, $\sigma(u)/u \in U_L^{(ep/(p-1))}$, and so, $v \in U_L^{(e/(p-1))}$. Since $N_{L/K}U_L^{(e/(p-1))} \subset U_K^{(ep/(p-1))}$ by § 1 Remark 1 (ii), $N_{L/K}v \in U_K^{(ep/(p-1))}$. Since $N_{L/K}v$ is a p-th root of 1 and no primitive p-th root of 1 belongs to $U_K^{(ep/(p-1))}$, it follows $N_{L/K}v=1$. Hence, Lemma 4 follows from the above exact sequence.

§ **3.4.** Now, we prove that the sequence (4) is exact. Let I be the image of $K_2(K) \to K_2(L)$. Suppose that the sequence (4) is not exact. Then, there is an element a of $K_2(L)$ such that $a \notin K_2(L)^p \cdot I$, $\sigma(a)/a \in K_2(L)^p$ and $N_{L/K}a \in K_2(K)^p$. Let n be an integer such that $a \in \mathscr{U}_L^{(n)} \cdot I$, $a \notin \mathscr{U}_L^{(n+1)} \cdot I$ and $0 \leqq n \leqq ep/(p-1)$. Such n exists because $a \in K_2(L)=\mathscr{U}_L^{(0)} \cdot I$ and $a \notin K_2(L)^p \cdot I \supset \mathscr{U}_L^{(i)} \cdot I$ if $i > ep/(p-1)$. Let $a=bc$, $b \in \mathscr{U}_L^{(n)}$ and $c \in I$. Then, b satisfies $b \in \mathscr{U}_L^{(n)}$, $b \notin \mathscr{U}_L^{(n+1)} \cdot I$, $\sigma(b)/b \in K_2(L)^p$ and $N_{L/K}b \in K_2(K)^p$. Now, we show that these properties of b lead to a contradiction. By Lemma 2 (i), we have $n \neq 0$. Furthermore, by Lemma 2 (ii) and (iii), the class of b in $\mathscr{U}_L^{(n)}/\mathscr{U}_L^{(n+1)}$ belongs to the image of the homomorphism $\{*, \pi\}$: $U_L^{(n)}/U_L^{(n+1)} \to \mathscr{U}_L^{(n)}/\mathscr{U}_L^{(n+1)}$. In what follows, we consider three cases and we show that each case leads to a contradiction.

(i) Suppose $1 \leqq n < e$. Let S_n be as in Lemma 3. By Lemma 2 (ii), we can regard S_n as a subgroup of $\mathscr{U}_L^{(n)}/\mathscr{U}_L^{(n+1)}$ and $S_{n+e'}$ as a subgroup of $\mathscr{U}^{(n+e')}/\mathscr{U}^{(n+e'+1)}$. In the commutative diagram

$$\begin{array}{ccc} S_n & \subset & \mathscr{U}_L^{(n)}/\mathscr{U}_L^{(n+1)} \\ \downarrow{\scriptstyle \sigma-1} & & \downarrow{\scriptstyle \sigma-1} \\ S_{n+e'} & \subset & \mathscr{U}_L^{(n+e')}/\mathscr{U}_L^{(n+e'+1)} \end{array} \qquad (e'=e/(p-1)) ,$$

the left $\sigma-1$ is injective by Lemma 3. Hence $b \in \mathscr{U}_L^{(n+1)}$ and this is a contradiction.

(ii) Suppose $n=e$. Let u be an element of $U_L^{(e)}$ such that $b \equiv \{u, \pi\} \bmod \mathscr{U}_L^{(e+1)}$. In the following commutative diagram

$$\begin{array}{ccc} U_L^{(e)}/U_L^{(e+1)} & \xrightarrow{\{*,\pi\}} & \mathscr{V}_L^{(e)}/\mathscr{V}_L^{(e+1)} \\ \downarrow{\scriptstyle \sigma-1} & & \downarrow{\scriptstyle \sigma-1} \\ U_L^{(e'p)}(L^*)^p/(L^*)^p & \xrightarrow{\{*,\pi\}} & \mathscr{V}_L^{(e'p)}/K_2(L)^p \end{array} \qquad (e'=e/(p-1))\,,$$

the lower $\{*, \pi\}$ is injective by § 2 Prop. 1. Hence $\sigma(u)/u \in (L^*)^p$. By Lemma 4, $u \in K^*(L^*)^p$. Hence $b \in \mathscr{V}_L^{(n+1)}\cdot I$ and this is a contradiction.

(iii) Suppose $n>e$. Let u be an element of $U_L^{(n)}$ such that $b\equiv\{u, \pi\} \bmod \mathscr{V}_L^{(n+1)}$. Then, $\sigma(u)/u \in U_L^{(n+e')}\subset(L^*)^p$ where $e'=e/(p-1)$. By Lemma 4, $u \in K^*\cdot(L^*)^p$. Hence $b \in \mathscr{V}_L^{(n+1)}\cdot I$ and this is a contradiction. Thus, the proof of the exactness of (4) is complete if we prove the following Lemma 5.

§ **3.5.** The following Lemma is used many times in this Chapter I.

LEMMA 5. *Let k be a field and let E be a finite separable extension of k. Let n be an integer which is invertible in k. Then the following diagram is commutative.*

$$\begin{array}{ccc} K_2(E)/K_2(E)^n & \xrightarrow{h_{n,E}} & H^2(E, \mu_n\otimes\mu_n) \\ \downarrow{\scriptstyle N_{E/k}} & & \downarrow{\scriptstyle \text{Corestriction}} \\ K_2(k)/K_2(k)^n & \xrightarrow{h_{n,k}} & H^2(k, \mu_n\otimes\mu_n) \end{array}$$

where $h_{n,E}$ and $h_{n,k}$ are the Galois symbols.

PROOF. We use the technique in Bass and Tate [2] § 5. Let l be any prime number. It suffices to show that for each $\alpha \in K_2(E)$,

$$\mathrm{Cor}\circ h_{n,E}(\alpha)-h_{n,k}\circ N_{E/k}(\alpha)$$

is of finite order prime to l. If $l\neq \mathrm{ch}\,(k)$, let $\bar{k}$ be the algebraic closure of k. If $l=\mathrm{ch}\,(k)$, let k be the separable closure of k. Let L be the fixed field in $\bar{k}$ of a Sylow l-subgroup of $\mathrm{Aut}\,(\bar{k}/k)$. Then L has the following properties (i) and (ii).

(i) L is a filtered inductive limit of finite extensions of k of degrees prime to l.

(ii) Every finite extension of L is of degree a power of l. In the following, we denote $H^2(\ , \mu_n\otimes\mu_n)$ by $H(\ )$. Let

$$E\otimes_k L=L_1\times L_2\times\cdots\times L_r$$

where $L_1, \cdots, L_r$ are finite separable extensions of L. Then, since the following two diagrams commute

$$\begin{array}{ccc} H(E) & \to & \bigoplus_{i=1}^{r} H(L_i) \\ \downarrow{\scriptstyle \mathrm{Cor}} & & \downarrow{\scriptstyle \sum_{i=1}^{r}\mathrm{Cor}_{L_i/L}} \\ H(k) & \to & H(L) \end{array} \qquad \begin{array}{ccc} K_2(E) & \to & \bigoplus_{i=1}^{r} K_2(L_i) \\ \downarrow{\scriptstyle N_{E/k}} & & \downarrow{\scriptstyle \sum_{i=1}^{r}N_{L_i/L}} \\ K_2(k) & \to & K_2(L) \end{array}$$

and since, by (i), the kernel of $H(k)\to H(L)$ is a torsion group without l-torsion, we may suppose $k=L$.

Suppose $k=L$. By (ii), there exist finite extensions $E_0, \cdots, E_m$ of k such that $k=E_0\subset\cdots\subset E_m=E$ and $[E_i: E_{i-1}]=l$ for each i. Hence we may suppose $[E: k]=l$. Then, by Bass and Tate [2] § 5, $K_2(E)$ is generated by elements of the form $\{x, y\}$ such that $x\in k^*$ and $y\in E^*$. But, for such x and y, it is easy to show

$$\mathrm{Cor}\circ h_{n,E}\{x, y\}=h_{n,k}\circ N_{E/k}\{x, y\} .$$

This proves Lemma 5.

§ 4. B_i-fields.

We define a new concept "B_i-field". This is an analogue of the famous concept "C_i-field".

DEFINITION. Let $i\geqq 0$. We call a field k, a *B_i-field* if and only if for each finite extension E of k and for each finite extension F of E, the norm $N_{F/E}$: $K_i(F)\to K_i(E)$ is surjective.

EXAMPLE. B_0-field$\Leftrightarrow$algebraically closed field$\Leftrightarrow$$C_0$-field.

We propose some conjectures.

CONJECTURE 1. *A C_i-field is a B_i-field.*

CONJECTURE 2. *A field which is complete with respect to a discrete valuation is B_i if its residue field is B_{i-1}.*

These conjectures are well known to be true in the case $i=1$. The C_i-version of Conjecture 2 was called "Artin's conjecture" and is known to be false for $i\geqq 2$.

In this section, we prove the following two propositions.

PROPOSITION 1. *A C_2-field is a B_2-field.*

PROPOSITION 2. *Let K be a field which is complete with respect to a discrete valuation and with residue field F. Suppose that F is a B_1-field. Then,*

(i) *For each central simple algebra A over K, the reduced norm map* $\mathrm{Nrd}_{A/K}$: $A^*\to K^*$ *is surjective.*

(ii) *K is a B_2-field.*

Thus Conj. 1 and Conj. 2 are true also in the case $i=2$.

We shall not make use of Prop. 1 in the later part of this paper, but Prop. 2 (ii) will be used in the next section.

PPOOFS. For a while, we are concerned with general fields.

LEMMA 1. *Let k be a field and let A be a central simple algebra over k. Let $r\geqq 1$ and let $a\in k^*$. Then the following conditions* (i), (ii) *and* (iii) *are equivalent.*

(i) $a\in \mathrm{Nrd}_{A/k}A^*$.

(ii) $a\in \mathrm{Nrd}_{M_r(A)/k}M_r(A)^*$ *where $M_r(A)$ denotes the ring of all matrices of (r, r) type over A.*

(iii) *There is a finite extension E of k such that $a\in N_{E/k}E^*$ and such that A is decomposed by E.*

We omit the proof of Lemma 1. If k is a field and $w\in \mathrm{Br}(k)$, we denote by $\mathrm{Nrd}(w/k)$ the group $\mathrm{Nrd}_{A/k}A^*$ where A is any central simple algebra over k having its class w in $\mathrm{Br}(k)$. This notation makes sense by Lemma 1.

LEMMA 2. *Let k be a field, $\chi\in X_k$, E the cyclic extension of k which corresponds to χ, and $a, b\in k^*$. Let (χ, b) be the element of* $\mathrm{Br}(k)$ *which is defined in Serre* [19] *Ch.* XIV. *Then, the condition*

$$a\in \mathrm{Nrd}((\chi, b)/k)$$

implies

$$\{a, b\}\in N_{E/k}K_2(E) .$$

PROOF. Suppose $a\in \mathrm{Nrd}((\chi, b)/k)$. By Lemma 1, there are a finite extension F of k and an element c of F^* such that $a=N_{F/k}c$ and $(\chi, b)_F=0$. (For $w\in \mathrm{Br}(k)$, w_F denotes the image of w under the canonical homomorphism $\mathrm{Br}(k)\rightarrow \mathrm{Br}(F)$.) The condition $(\chi, b)_F=0$ is equivalent to the condition

$$b=N_{EF/F}d \qquad \text{for some} \quad d\in (EF)^* ,$$

by Serre [19] Ch. XIV. It follows,

$$\begin{aligned}\{a, b\}_k&=N_{F/k}\{c, b\}_F=N_{F/k}\circ N_{EF/F}\{c, d\}_{EF}=N_{EF/k}\{c, d\}_{EF}\\&=N_{E/k}\circ N_{EF/E}\{c, d\}_{EF}\in N_{E/k}K_2(E) .\end{aligned}$$

LEMMA 3. *Let k be a field having the following properties* (i) *and* (ii). *Then, k is a B_2-field.*

(i) *For each finite extension E of k and for each finite cyclic extension F of E such that $[F:E]$ is a prime number, $N_{F/E}: K_2(F)\rightarrow K_2(E)$ is surjective.*

(ii) *If* $\mathrm{ch}(k)=p>0$, $[k:k^p]\leqq p^2$.

PROOF. Suppose that k has the above properties. Let E be any finite extension of k. It suffices to prove the surjectivity of $N_{F/E}: K_2(F)\rightarrow K_2(E)$ in the following two cases.

(a) F is a finite Galois extension of E.

(b) F is a finite purely inseparable extension of E.

First, we consider the case (a). Let l be any prime number. Let L be the fixed field in F of a Sylow-l-subgroup of $\mathrm{Gal}(F/E)$. Since F/L is a Galois extension such that $\mathrm{Gal}(F/L)$ is a l-group, $N_{F/L}: K_2(F) \to K_2(L)$ is found to be surjective by (i) and by easy induction. Hence, for any $x \in K_2(E)$,

$$x^{[L:E]} = N_{L/E}(x|_L) \in N_{L/E} \circ N_{F/L} K_2(F) = N_{F/E} K_2(F) .$$

This shows that $K_2(E)/N_{F/E}K_2(F)$ is annihilated by $[L: E]$ which is prime to l. Since l is any prime number, this shows $K_2(E)/N_{F/E}K_2(F)=0$.

Next, we assume $\mathrm{ch}(k)=p>0$ and consider the case (b). It suffices to prove the surjectivity of $N_{k^{1/p}/k}: K_2(k^{1/p}) \to K_2(k)$. Let $x, y \in k^*$. For the proof of $\{x, y\} \in N_{k^{1/p}/k} K_2(k^{1/p})$, we may assume $[k: k^p]=p^2$ and $x \notin (k^*)^p$. Let $k'=k(x^{1/p})$. Then,

$$\{x, y\} = N_{k'/k}\{x^{1/p}, y\} = N_{k^{1/p}/k}\{x^{1/p}, y^{1/p}\} .$$

LEMMA 4. *Let k be a field having the following properties* (i) *and* (ii). *Then, k is a B_2-field.*

(i) *For each finite extension E of k and for each $w \in \mathrm{Br}(E)$,*

$$\mathrm{Nrd}(w/E) = E^* .$$

(ii) *If* $\mathrm{ch}(k)=p>0$, $[k: k^p] \leqq p^2$.

PROOF. It suffices to show that k has the properties (i) and (ii) in Lemma 3. Let E be a finite extension of k and let F be a finite cyclic extension of E. Let χ be an element of X_E such that F corresponds to χ. For any $a, b \in E^*$, $a \in \mathrm{Nrd}((\chi, b)/E)$ by (i). Hence $\{a, b\} \in N_{F/E}K_2(F)$ by Lemma 2. This shows $K_2(E) = N_{F/E}K_2(F)$.

PROOF OF PROP. 1. Since every finite extension of a C_2-field is C_2 (Serre [20] Ch. II §4), it is easy to show that a C_2-field satisfies (i) and (ii) in Lemma 4. (For the proof of (i), take the division algebra D corresponding to w, then $\mathrm{Nrd}_{D/E} D^* = E^*$ is easily seen.) Q.E.D.

PROOF OF PROP. 2. Prop. 2 (ii) is a consequence of Prop. 2 (i) and Lemma 4. Indeed, what we have to show is that $[K: K^p] \leqq p^2$ in the case $\mathrm{ch}(K)=p>0$. Suppose $\mathrm{ch}(K)=p>0$. Then $K \cong F((T))$. Since F is B_1, $[F: F^p] \leqq p$ and hence $[K: K^p] \leqq p^2$.

So, in what follows, we devote ourselves to the proof of Prop. 2 (i). We need three steps. Let K and F be as in the hypothesis of Prop. 2. Note that the condition "F is B_1" implies $\mathrm{Br}(F)=0$ (cf. Serre [19] Ch. X §7).

Step 1. Suppose $w \in \mathrm{Br}(K_{nr}/K)$. By Serre [19], $\mathrm{Br}(K_{nr}/K) \cong \mathrm{Br}(F) \oplus X_F$. Since $\mathrm{Br}(F)=0$, $\mathrm{Br}(K_{nr}/K) \cong X_F$. By this isomorphism, an element χ of X_F corresponds to $(\tilde{\chi}, \pi)$ where π is any prime element of K and where $\tilde{\chi}$ denotes the "unramified element" of X_K corresponding to χ. (Cf. Lemma 2 or Remark 1 below, for the notation $(\tilde{\chi}, \pi)$.) Let $w=(\tilde{\chi}, \pi)$ where χ and π are as above. Let L be the finite cyclic extension of K which corresponds to $\tilde{\chi}$, and let C be the residue field of L. Since L is unramified over K,

$$U_K^{(1)} = N_{L/K} U_L^{(1)} .$$

By this and by $F^* = N_{C/F} C^*$, we have

$$U_K = N_{L/K} U_L \subset \mathrm{Nrd}(w/K) .$$

Furthermore, by Remark 1 below, $-\pi$ belongs to $\mathrm{Nrd}(w/K)$. Thus, we have

$$K^* = \mathrm{Nrd}(w/K) .$$

REMARK 1. If $\psi \in X_K$ and $a \in K^*$, (ψ, a) coincides with the class of the central simple algebra A which is characterized by the following properties. Let L be the cyclic extension of K corresponding to ψ, and let $\bar{\psi}: \mathrm{Gal}(L/K) \to \boldsymbol{Q}/\boldsymbol{Z}$ be the induced homomorphism by $\psi: \mathrm{Gal}(K^{ab}/K) \to \boldsymbol{Q}/\boldsymbol{Z}$. Let n be the order of ψ and let σ be the generator of $\mathrm{Gal}(L/K)$ such that $\bar{\psi}(\sigma)=1/n$. Then, L is a subfield of A, and there is an element α of A such that

$$A = \bigoplus_{i=0}^{n-1} L\alpha^i ,$$
$$\alpha^n = a ,$$
$$\alpha x \alpha^{-1} = \sigma(x) \qquad \text{for all} \quad x \in L .$$

In particular, $-a = \mathrm{Nrd}_{A/K}(-\alpha) \in \mathrm{Nrd}_{A/K}(A^*)$.

Step 2. If F is perfect, $\mathrm{Br}(K) = \mathrm{Br}(K_{nr}/K)$ by Serre [19] Ch. XII. Hence, in this case, all were proved in Step 1. In Step 2 and Step 3, we suppose that F is not perfect. Let $p = \mathrm{ch}(F)$. Since F is B_1, $[F : F^p] = p$. In this Step 2, we prove that $K^* = \mathrm{Nrd}(w/K)$ if $w \in \mathrm{Br}(K)_p$ and if $w \notin \mathrm{Br}(K_{nr}/K)$. But this is clear if one combines § 1 Prop. 3 (v), the fact $\mathrm{Br}(F)=0$, and the following Lemma 5. The proof of Lemma 5 will continue until the end of Step 2.

LEMMA 5. *Let K be a field which is complete with respect to a discrete valuation and with residue field F. Suppose* $\mathrm{ch}(F) = p > 0$ and $[F : F^p] = p$. *Suppose $w \in \mathrm{Br}(K)_p$ and $w \notin \mathrm{Br}(K_{nr}/K)$. Then, the division algebra which corresponds to w satisfies the assumptions at the beginning of* § 1.

PROOF. First, suppose $\mathrm{ch}(K)=p$. Generally, let k be a field of characteristic $p>0$. Let $\Omega_{k,d=0}$ be the kernel of the exterior derivation

$$d\colon \Omega_k \to \bigwedge_k^2 \Omega_k .$$

Then the sequence in C_k (§ 0.1)

$$0 \to k_s^* \xrightarrow{x \mapsto x^p} k_s^* \xrightarrow{x \mapsto dx/x} \Omega_{k_s,d=0} \xrightarrow{1-\gamma} \Omega_{k_s} \to 0$$

is known to be exact. Here, γ denotes the Cartier operator. (Cf. [3]) Hence, by taking the Galois cohomology, we have a canonical isomorphism

$$\theta_k\colon \Omega_k/(1-\gamma)\Omega_{k,d=0} \cong \mathrm{Br}(k)_p .$$

Now, we return to the proof of Lemma 5 in the case $\mathrm{ch}(K)=p$. Since $K\cong F((T))$, we can obtain an exact knowledge of $\Omega_K/(1-\gamma)\Omega_{K,d=0}$. In what follows, for an element λ of Ω_K, we denote the class of λ in $\Omega_K/(1-\gamma)\Omega_{K,d=0}$ by $[\lambda]$. For each $n\geqq 0$, let M_n be the subgroup of $\Omega_K/(1-\gamma)\Omega_{K,d=0}$ generated by all elements of the form $[f\,dg/g]$ such that $f\in K$, $g\in K^*$ and $v_K(f)\geqq -n$. Then, $M_0\subset M_1\subset M_2\subset\cdots$, and $\bigcup_n M_n=\Omega_K/(1-\gamma)\Omega_{K,d=0}$. Put $M_{-1}=0$. By the above isomorphism θ_K, $\mathrm{Br}(K_{\mathrm{nr}}/K)_p$ corresponds to the kernel of the homomorphism

$$\Omega_K/(1-\gamma)\Omega_{K,d=0} \to \Omega_{K_{\mathrm{nr}}}/(1-\gamma)\Omega_{K_{\mathrm{nr}},d=0}$$

which is found to be M_0 by an easy calculation. As concerns M_n $n\geqq 1$, we have the following (i) and (ii).

(i) Suppose $n\geqq 1$ and $p\,|\,n$. Fix an element b of K such that $v_K(b)=n/p$, and fix a prime element π of K. Then,

$$F/F^p \cong M_n/M_{n-1} \quad \text{by}$$
$$\bar f \mapsto [fb^{-p}\,d\pi/\pi] \mod M_{n-1} \qquad (f\in O_K) .$$

(ii) Suppose $n\geqq 1$ and n is prime to p. Fix an element c of K such that $v_K(c)=n$. Then,

$$\Omega_F \cong M_n/M_{n-1} \quad \text{by}$$
$$\bar f\, d\bar g/\bar g \mapsto [fc^{-1}\,dg/g] \mod M_{n-1} \qquad (f\in O_K,\ g\in U_K) .$$

Using these (i) and (ii), we can prove the following (iii) and (iv) just as in the proof of § 2 Lemma 4.

(iii) Hypothesis: $w\in \Omega_K/(1-\gamma)\Omega_{K,d=0}$,
$w\equiv [fb^{-p}\,d\pi/\pi] \mod M_i$,
$f\in O_K$, $\bar f\notin F^p$, $b\in K^*$, $pv_K(b)>i\geqq 0$,
π is a prime element of K.

Conclusion: There exist f' and π' such that
$w \equiv [f'b^{-p}\, d\pi'/\pi'] \mod M_{i-1}$
$v_K(f'-f) \geqq pv_K(b)-i,\ \pi'/\pi \in U_K^{(pv_K(b)-i)}$.

(iv) Hypothesis: $w \in \Omega_K/(1-\gamma)\Omega_{K,d=0}$,
$w \equiv [c\, dg/g] \mod M_i$,
$g \in U_K,\ \bar{g} \notin F^p,\ c \in K^*,\ -v_K(c) > i \geqq 0$,
$v_K(c)$ is prime to p.

Conclusion: There exist c' and g' such that
$w \equiv [c'\, dg'/g'] \mod M_{i-1}$
$v_K(c'-c) \geqq -i,\ g'/g \in U_K^{(-v_K(c)-i)}$.

By (i) and (iii), or by (ii) and (iv), we have the following consequence.

Suppose $w \in \Omega_K/(1-\gamma)\Omega_{K,d=0}$ and $w \notin M_0$. Then, w has either of the following two forms.

$[fb^{-p}\, d\pi/\pi]$ where $f \in O_K$, $\bar{f} \notin F^p$, $v_K(b) > 0$, and π is a prime element of K.

$[c\, dg/g]$ where $g \in U_K$, $\bar{g} \notin F^p$, $v_K(c) < 0$. and $v_K(c)$ is prime to p.

In the former case, the element of $\mathrm{Br}(K)_p$ corresponding to w is decomposed by $K(\alpha)$ and by $K(\beta)$ where α is a root of the equation $X^p - X = fb^{-p}$ and β is a root of the equation $X^p = \pi$. In the latter case, the element of $\mathrm{Br}(K)_p$ corresponding to w is decomposed by $K(\alpha)$ and by $K(\beta)$ where α is a root of the equation $X^p - X = c$ and β is a root of the equation $X^p = g$. Thus, if $w \in \mathrm{Br}(K)_p$ and $w \notin \mathrm{Br}(K_{\mathrm{nr}}/K)$, w is decomposed by a field which is a totally ramified extension of K of degree p and by a field which is an extension of K of degree p and whose residue field is an inseparable extension of F of degree p. Hence, the division algebra which corresponds to w satisfies the assumptions at the beginning of § 1.

Next, suppose that $\mathrm{ch}(K) = 0$ and K contains a primitive p-th root ζ of 1. Then, by § 3 Theorem 1,

$$h'_p : K_2(K)/K_2(K)^p \to \mathrm{Br}(K)_p$$

is an isomorphism. Furthermore,

$$h'_p(V^{(e_K p/(p-1))} K_2(K)^p/K_2(K)^p) = \mathrm{Br}(K_{\mathrm{nr}}/K)_p\,.$$

Hence, in this case Lemma 5 follows from § 2 Prop. 2.

Lastly, suppose that $\mathrm{ch}(K) = 0$. Let ζ be a primitive p-th root of 1 and let $L = K(\zeta)$. Suppose that $w \in \mathrm{Br}(K)_p$ and $w \notin \mathrm{Br}(K_{\mathrm{nr}}/K)$. Let D be the division algebra over K which corresponds to w. Let $n = (\dim_K D)^{1/2}$. Now, we prove

$n=p$. In what follows, we use some elementary results in the general theory of central simple algebras making no references to them. Since $pw=0$, n is a power of p. On the other hand, as was proved above, w_L is decomposed by an extension M/L of degree p. Hence n divides $[M:K]$. Since $[L:K]$ is prime to p, we have $n=p$. Let C be the residue algebra of D and let d be the index of $U_K\cdot \mathrm{Nrd}_{D/K} D^*$ in K^*. By a general theory (Serre [19] Ch. XII), $d|n$ (hence $d|p$) and

$$\dim_F C=nd=pd\,.$$

Furthermore, we can show $d=1$ as follows. Since the division algebra over L corresponding to w_L satisfies the assumptions at the beginning of § 1, there is a totally ramified extension M of L of degree p such that $w_M=0$. Since $N_{M/K}M^*\subset \mathrm{Nrd}_{D/K}D^*$ and $U_L\cdot N_{M/L}M^*=L^*$, $N_{L/K}L^*\subset U_K\cdot \mathrm{Nrd}_{D/K}D^*$. Since $d|p$ and $[L:K]$ is prime to p, we have $d=1$. Hence $\dim_F C=p$. Hence, C is commutative. If C is separable over F, the unramified extension of K corresponding to C/F decomposes D and this contradicts the assumption. Hence C is inseparable over F and this completes the proof of Lemma 5.

Step 3. Now, we complete the proof of Prop. 2 (i). Suppose $w\in \mathrm{Br}(K)$. We prove $K^*=\mathrm{Nrd}(w/K)$ by induction on the order n of w. If $n=1$, this is clear. Let $n>1$. Let l be a prime divisor of n. By the hypothesis of induction, we have $K^*=\mathrm{Nrd}(lw/K)$. Let a be any element of K^*. Then, by Lemma 1, there are a finite extension L of K and an element b of L^* such that $lw_L=0$ and $a=N_{L/K}b$. First suppose $w_L\in \mathrm{Br}(L_{\mathrm{nr}}/L)$. Then, by Step 1, we have $b\in \mathrm{Nrd}(w_L/L)$. Next, suppose $w_L\notin \mathrm{Br}(L_{\mathrm{nr}}/L)$. Then, $l=p$ (It is well known that $\mathrm{Br}(K_{\mathrm{nr}})$ is a p-primary torsion group.) and hence $b\in \mathrm{Nrd}(w_L/L)$ by Step 2. In any case, by Lemma 1, there is a finite extension M of L such that $w_M=0$ and $b\in N_{M/L}M^*$. Since $w_M=0$ and $a\in N_{M/K}M^*$, by Lemma 1 we have $a\in \mathrm{Nrd}(w/K)$. This completes the proof of Prop. 2 (i).

§ 5. The cohomological dimension.

The aim of this section is to prove the following Theorem 1.

THEOREM 1. *Let K be a field which is complete with respect to a discrete valuation and with residue field F. Suppose* $\mathrm{ch}(K)=0$, $\mathrm{ch}(F)=p>0$, *and* $[F:F^p]=p$. *Let $\mathrm{cd}_p K$ be the cohomological p-dimension of K (see below). Then,*

(i) $\mathrm{cd}_p K=2$ *or* 3.

(ii) $\mathrm{cd}_p K=2$ *if and only if for each finite extension F' of F,* $\mathrm{Br}(F')$ *has no p-torsion.*

(iii) $H^3(K, \mu_{p^n}\otimes\mu_{p^n})\cong \mathrm{Br}(F)_{p^n}$, *canonically.*

COROLLARY 1. *Let K be a field which is complete with respect to a discrete valuation and with residue field F. Suppose that F is a B_1-field. Then,* $\mathrm{cd}(K)\leqq 2$, *where* $\mathrm{cd}(K)$ *denotes the cohomological dimension of K.*

COROLLARY 2. *Let F be a field which is complete with respect to a discrete valuation and with finite residue field. Let K be a field which is complete with respect to a discrete valuation and with residue field F. Suppose* $\mathrm{ch}(K)=0$. *Then for each $m>0$, there exists a canonical isomorphism*

$$\eta_{m,K}: H^3(K, \mu_m\otimes\mu_m)\xrightarrow{\cong}\frac{1}{m}\boldsymbol{Z}\Big/\boldsymbol{Z}$$

having the following properties.

(i) *If L is a finite extension of K, the following diagrams are commutative.*

$$\begin{array}{ccc} H^3(K, \mu_m\otimes\mu_m) & \xrightarrow[\cong]{\eta_{m,K}} & \frac{1}{m}\boldsymbol{Z}\Big/\boldsymbol{Z} \\ \downarrow\mathrm{Res} & & \downarrow[L:K] \\ H^3(L, \mu_m\otimes\mu_m) & \xrightarrow[\cong]{\eta_{m,L}} & \frac{1}{m}\boldsymbol{Z}\Big/\boldsymbol{Z} \end{array} \qquad \begin{array}{ccc} H^3(L, \mu_m\otimes\mu_m) & \xrightarrow[\cong]{\eta_{m,L}} & \frac{1}{m}\boldsymbol{Z}\Big/\boldsymbol{Z} \\ \downarrow\mathrm{Cor} & & \| \\ H^3(K, \mu_m\otimes\mu_m) & \xrightarrow[\cong]{\eta_{m,K}} & \frac{1}{m}\boldsymbol{Z}\Big/\boldsymbol{Z} \end{array}$$

(ii) *Suppose $m|n$. Let $s: \mu_m\otimes\mu_m\to\mu_n\otimes\mu_n$ be the injective* $\mathrm{Gal}(K_s/K)$*-homomorphism which is characterized by the following propety:*

$$s(x\otimes y^{n/m})=x\otimes y \quad \text{for all} \quad x\in\mu_m \quad \text{and} \quad \text{for all} \quad y\in\mu_n\,.$$

Then, the following diagram is commutative.

$$\begin{array}{ccc} H^3(K, \mu_m\otimes\mu_m) & \xrightarrow[\cong]{\eta_{m,K}} & \frac{1}{m}\boldsymbol{Z}\Big/\boldsymbol{Z} \\ \downarrow\text{by } s & & \cap \\ H^3(K, \mu_n\otimes\mu_n) & \xrightarrow[\cong]{\eta_{n,K}} & \frac{1}{n}\boldsymbol{Z}\Big/\boldsymbol{Z} \end{array}$$

Besides the above results, we prove the following Proposition 1 in this section. This is an analogue of the known fact:

"Let K be a field which is complete with respect to a discrete valuation and with residue field F. Suppose $\mathrm{ch}(K)=0$, $\mathrm{ch}(F)=p>0$, F is perfect, and $e_K/(p-1)$ is not an integer. Then X_K is p-divisible."

PROPOSITION 1. *Let K be a field which is complete with respect to a discrete valuation and with residue field F. Suppose* $\mathrm{ch}(K)=0$, $\mathrm{ch}(F)=p>0$, $[F: F^p]=p$, *and $e_K/(p-1)$ is not an integer. Then* $\mathrm{Br}(K)$ *is p-divisible.*

REVIEW on the *cohomological dimension.* (Cf. Serre [20])

Before we begin the proof of Th. 1, we review the definitions of the cohomological p-dimension $\mathrm{cd}_p k$ and the cohomological dimension $\mathrm{cd}(k)$ of a field k. Let k be a field and let p be a prime number. Then, by definition, $\mathrm{cd}_p k$ is an element of the set $\{0, 1, 2, \cdots, \infty\}$ characterized by the following property:

For each integer $i \geqq 0$, the following conditions (i) and (ii) are equivalent.

(i) $\mathrm{cd}_p k \leqq i$.

(ii) If M is an object of C_k (§ 0.1), and if M is a p-primary torsion group, $H^n(k, M)=0$ for all n such that $n>i$.

Next, $\mathrm{cd}(k)$ is defined by

$$\mathrm{cd}(k)=\mathrm{Sup}\cdot \mathrm{cd}_p k\ .$$

PROOF OF THEOREM 1. The outline of the proof is as follows. In Step 1 and Step 2, we prove the following assertion:

"Besides the hypothesis of Th. 1, suppose that F is separably closed. Then $\mathrm{cd}_p K \leqq 2$."

This assertion induces (i) of Th. 1 (Step 3). By using the results in § 3, we prove (iii) (Step 4). Finally, (ii) is deduced from (iii) (Step 5).

Step 1. Besides the hypothesis of Th. 1, suppose that F is separably closed. For the proof of $\mathrm{cd}_p K \leqq 2$, by Serre [20] Ch. II § 2 Prop. 4, it suffices to show that $H^2(K, K_s^*)$ ($\cong \mathrm{Br}(K)$) is p-divisible and that $H^3(K, K_s^*)$ has no p-torsion. So, we prove that $\mathrm{Br}(K)$ is p-divisible in Step 1, and that $H^3(K, K_s^*)$ has no p-torsion in Step 2. We may suppose that K contains a primitive p-th root ζ of 1 in the proof of $\mathrm{cd}_p K \leqq 2$, because $\mathrm{cd}_p K = \mathrm{cd}_p K(\zeta)$, for $[K(\zeta): K]$ is prime to p. So, we suppose that K contains a primitive p-th root ζ of 1 in Step 1 and in Step 2.

For the proof of the p-divisibility of $\mathrm{Br}(K)$, since $\mathrm{Br}(K)$ is a torsion group, it suffices to show the following assertion (A).

(A) If $x \in \mathrm{Br}(K)_p$ and n is a natural number, there exists $y \in \mathrm{Br}(K)$ such that $x = p^n y$.

As an assistant to the proof of (A), let $L = K(\zeta')$ where ζ' is a primitive p^{n+1}-th root of 1. We have the following commutative diagram (cf. § 3 Lemma 5).

$$\begin{array}{ccc} K_2(L)/K_2(L)^p & \xrightarrow{h'_{p,L}} & \mathrm{Br}(L)_p \\ \downarrow{\scriptstyle N_{L/K}} & & \downarrow{\scriptstyle \mathrm{Cor}} \\ K_2(K)/K_2(K)^p & \xrightarrow{h'_{p,K}} & \mathrm{Br}(K)_p \end{array}$$

By § 3 Th. 1, $h'_{p,K}$ and $h'_{p,L}$ are isomorphisms. Since F is a B_1-field, K is a B_2-field by § 4 Prop. 2. Hence $N_{L/K}: K_2(L)/K_2(L)^p \to K_2(K)/K_2(K)^p$ is surjective. Thus, $\mathrm{Cor}: \mathrm{Br}(L)_p \to \mathrm{Br}(K)_p$ is surjective. So, if $x \in \mathrm{Br}(K)_p$, there exists $u \in \mathrm{Br}(L)_p$ such that $x=\mathrm{Cor}\,(u)$. On the other hand, we have the following commutative diagram in virtue of the fact that L contains a primitive p^{n+1}-th root of 1.

$$\begin{array}{ccc} K_2(L)/K_2(L)^{p^{n+1}} & \xrightarrow{h'_{p^{n+1},L}} & \mathrm{Br}\,(L)_{p^{n+1}} \\ \downarrow\alpha & & \downarrow\beta \\ K_2(L)/K_2(L)^p & \xrightarrow{h'_{p,L}} & \mathrm{Br}\,(L)_p \end{array}$$

Here, α is the natural projection and β is the homomorphism $w\mapsto p^n w$. Since $h'_{p,L}$ and $h'_{p^{n+1},L}$ are isomorphisms by § 3 Th. 1 and since α is surjective, β is surjective. So, $u=p^n v$ for some $v\in\mathrm{Br}\,(L)$. Hence,

$$x=\mathrm{Cor}\,(u)=p^n\,\mathrm{Cor}\,(v)\ .$$

Thus, we have proved (A).

Step 2. As in Step 1, besides the hypothesis of Th. 1, we suppose that F is separably closed and that K contains a primitive p-th root ζ of 1. We are going to prove that $H^3(K, K_s^*)$ has no p-torsion. Let $K^{(i)}$ $(i\geqq 0)$ and $K^{(\infty)}$ be as in § 3.2. Since $K^{(\infty)}$ is a Henselian discrete valuation field of characteristic 0, $H^3(K^{(\infty)}, K_s^*)\cong H^3(K^{(\infty)\wedge},(K^{(\infty)\wedge})_s^*)$ where $K^{(\infty)\wedge}$ denotes the completion of $K^{(\infty)}$. Since the residue field of $K^{(\infty)\wedge}$ is algebraically closed, $H^i(K^{(\infty)\wedge},)$ is the zero functor for $i\geqq 3$ as is well known. Thus, $H^3(K^{(\infty)}, K_s^*)=0$. Hence it suffices to show that each $H^3(K^{(i)}, K_s^*)\to H^3(K^{(i+1)}, K_s^*)$ is injective. Let $L=K^{(1)}$. It suffices to show that $H^3(K, K_s^*)\to H^3(L, K_s^*)$ is injetive. Now, we need the following Lemma.

Lemma 1. *Let k be a field and let E be a finite cyclic extension of k. Let $G=\mathrm{Gal}\,(E/k)$ and let σ be a generator of G. Then the kernel of $H^3(k, k_s^*)\to H^3(E, k_s^*)$ is isomorphic to the group A/B where A is the kernel of* $\mathrm{Cor}:\mathrm{Br}\,(E)\to\mathrm{Br}\,(k)$ *and B is the image of* $\sigma-1:\mathrm{Br}\,(E)\to\mathrm{Br}\,(E)$.

Proof. This follows from the spectral sequence

$$H^*(G, H^*(E, k_s^*))\Rightarrow H^*(k, k_s^*)\ . \qquad \text{(Q.E.D.)}$$

We return to the proof of the injectivity of $H^3(K, K_s^*)\to H^3(L, K_s^*)$. Let $G=\mathrm{Gal}\,(L/K)$ and let σ be a generator of G. By Lemma 1, it suffices to show that the sequence

$$\mathrm{Br}\,(L)\xrightarrow{\sigma-1}\mathrm{Br}\,(L)\xrightarrow{\mathrm{Cor}}\mathrm{Br}\,(K)$$

is exact. Since $[L:K]=p$, the "prime to p" part of this sequence is exact. Since $\mathrm{Br}\,(L)$ is p-divisible (Step 1), it suffices to show that the sequence

$$\mathrm{Br}\,(L)_p\xrightarrow{\sigma-1}\mathrm{Br}\,(L)_p\xrightarrow{\mathrm{Cor}}\mathrm{Br}\,(K)_p$$

is exact. Hence, by § 3 Theorem 1 and by § 3 Lemma 5, it suffices to show the exactness of the sequence

(1) $$K_2(L)/K_2(L)^p \xrightarrow{\sigma-1} K_2(L)/K_2(L)^p \xrightarrow{N_{L/K}} K_2(K)/K_2(K)^p .$$

Thus all things were reduced to a property of K_2.

Now, we prove the exactness of (1). Let π, e, $\mathscr{V}_K^{(n)}$ and $\mathscr{V}_L^{(n)}$ be as in § 3.3. Let $e'=e/(p-1)$. Since F is separably closed, $U_K^{(e'p)}\subset(K^*)^p$ and $U_L^{(e'p)}\subset(L^*)^p$. For each n such that $0\leqq n<e'p$, let A_n be the subset of $K_2(L)$ defined by

$$A_n=\{\{f, \pi\} \mid f\in U_L \text{ and } \bar{f}\notin F\} \quad \text{if} \quad n=0 ,$$
$$A_n=\{\{1+f\pi^{n/p}, \pi\} \mid f\in U_L \text{ and } \bar{f}\notin F\} \quad \text{if} \quad p\mid n \quad \text{and} \quad n>0 ,$$
$$A_n=\{\{1+c, g\} \mid c\in K,\ v_K(c)=n,\ g\in U_L,\ \bar{g}\notin F\} \quad \text{if} \quad n \quad \text{is prime to} \quad p .$$

Then, we have

(2) *If* $\theta\in A_n$, $N_{L/K}\theta\in\mathscr{V}_K^{(n)}$ *and* $N_{L/K}\theta\notin\mathscr{V}_K^{(n+1)}$.

This follows from § 2 Prop. 1, from the formula

$$N_{L/K}\{x, y\}_L=\{x, N_{L/K}y\} \qquad (x\in K^*,\ y\in L^*) ,$$

and from the calculation of $N_{L/K}: L^*\to K^*$ given by § 1 Remark 1 (iv) (with $t=ep/(p-1)$). By § 2 Prop. 1, each element α of $K_2(L)$ satisfies

$$\alpha\equiv\{u, \pi\}\prod_{0\leqq n<e'p}\theta_n \mod K_2(L)^p$$

for some $u\in U_L^{(e')}$ and for some $(\theta_n)_{0\leqq n<e'p}$ where $\theta_n=1$ or $\theta_n\in A_n$ for each n. Suppose $N_{L/K}\alpha\in K_2(K)^p$. Then, $\theta_n=1$ for all n. Indeed, if $\theta_n\neq1$ for some n, let $n_0=\min\{n\mid\theta_n\neq1\}$. Since $N_{L/K}u\in(K^*)^p$ by § 1 Remark 1 (i), (2) shows

$$N_{L/K}\alpha\equiv N_{L/K}\theta_{n_0}\not\equiv1 \mod \mathscr{V}_K^{(n_0+1)} .$$

This is a contradiction. Hence $\theta_n=1$ for all n and we have

$$\alpha\equiv\{u, \pi\} \mod K_2(L)^p .$$

Since $N_{L/K}u\in(K^*)^p$, $u\in(L^*)^{\sigma-1}\cdot K^*$ by a general field theory. Hence, for the proof of $\alpha\in K_2(L)^{\sigma-1}\cdot K_2(L)^p$ (the exactness of (1) follows from this), it suffices to show that the image of $K_2(K)\to K_2(L)$ is contained in $K_2(L)^{\sigma-1}\cdot K_2(L)^p$. Let $\beta\in K_2(K)$. Since K is a B_2-field (see Step 1), $\beta=N_{L/K}\beta'$ for some $\beta'\in K_2(L)$. Hence, in $K_2(L)$,

$$\beta=\prod_{\tau\in G}\tau(\beta')=(\beta')^p\prod_{\tau\in G}(\tau(\beta')/\beta')\in K_2(L)^{\sigma-1}\cdot K_2(L)^p .$$

Thus, we have proved the exactness of (1). And hence we have proved $\mathrm{cd}_p K\leqq2$ in the case F is separably closed.

Step 3. Let K_{nr} be the maximum unramified extension of K and let $\hat{K}_{\mathrm{nr}}$ be its completion. By Step 1 and Step 2, $\mathrm{cd}_p \hat{K}_{\mathrm{nr}} \leqq 2$. Generally, for any Henselian discrete valuation field S and for any prime number l, $\mathrm{cd}_l S = \mathrm{cd}_l \hat{S}$ where $\hat{S}$ is the completion of S. So, $\mathrm{cd}_p K_{\mathrm{nr}} \leqq 2$. Next, since $\mathrm{ch}(F)=p>0$, $\mathrm{cd}_p F \leqq 1$ (cf. Serre [20] Ch. II §2). So, (cf. [20] Ch. I §3)

$$\mathrm{cd}_p K \leqq \mathrm{cd}_p F + \mathrm{cd}_p K_{\mathrm{nr}} \leqq 1+2=3 .$$

It remains to show $\mathrm{cd}_p K>1$. By §3 Th. 1,

$$K_2(K)/K_2(K)^p \cong H^2(K, \mu_p \otimes \mu_p) .$$

Since $K_2(K)/K_2(K)^p \neq 0$ (§2 Prop. 1), we have $H^2(K, \mu_p \otimes \mu_p) \neq 0$ and so, $\mathrm{cd}_p K>1$.

Step 4. Let n be any natural number. We identify F_s with the residue field of K_{nr}. Consider the following diagram in C_F (here, C_F is the category defined in §0.1).

$$\begin{array}{ll} K_2(K_{\mathrm{nr}})/K_2(K_{\mathrm{nr}})^{p^n} \xrightarrow{h_{p^n}} H^2(K_{\mathrm{nr}}, \mu_{p^n} \otimes \mu_{p^n}) \\ \quad \big\downarrow \text{tame symbol} \\ F_s^*/(F_s^*)^{p^n} \end{array} \tag{3}$$

By §3 Theorem 1 and by running to the inductive limit, the above h_{p^n} is an isomorphism. Hence, (3) induces a homomorphism

$$H^1(F, H^2(K_{\mathrm{nr}}, \mu_{p^n} \otimes \mu_{p^n})) \to H^1(F, F_s^*/(F_s^*)^{p^n}) . \tag{4}$$

On the one hand, $H^1(F, F_s^*/(F_s^*)^{p^n}) \cong \mathrm{Br}(F)_{p^n}$ follows from the exact sequence

$$0 \to F_s^* \xrightarrow{x \mapsto x^{p^n}} F_s^* \to F_s^*/(F_s^*)^{p^n} \to 0 .$$

On the other hand,

$$H^3(K, \mu_{p^n} \otimes \mu_{p^n}) \cong H^1(F, H^2(K_{\mathrm{nr}}, \mu_{p^n} \otimes \mu_{p^n}))$$

follows from the spectral sequence

$$H^*(F, H^*(K_{\mathrm{nr}}, \mu_{p^n} \otimes \mu_{p^n})) \Rightarrow H^*(K, \mu_{p^n} \otimes \mu_{p^n}) ,$$

since $\mathrm{cd}_p K_{\mathrm{nr}} \leqq 2$ (Step 3) and $\mathrm{cd}_p F \leqq 1$. So, the homomorphism (4) can be rewritten as

$$H^3(K, \mu_{p^n} \otimes \mu_{p^n}) \to \mathrm{Br}(F)_{p^n} . \tag{5}$$

It remains to show that (5) is an isomorphism. By induction on n using the following commutative diagram of exact sequences

$$\begin{array}{ccccccc} H^3(K, \mu_p \otimes \mu_p) & \to & H^3(K, \mu_{p^n} \otimes \mu_{p^n}) & \to & H^3(K, \mu_{p^{n-1}} \otimes \mu_{p^{n-1}}) & \to & 0 \\ \downarrow & & \downarrow & & \downarrow & & \\ 0 \to \mathrm{Br}(F)_p & \to & \mathrm{Br}(F)_{p^n} & \to & \mathrm{Br}(F)_{p^{n-1}} & & \end{array}$$

(the upper row is exact since $H^4(K, \mu_p\otimes\mu_p)=0$ by $\mathrm{cd}_p K\leqq 3$), it is found that we may suppose $n=1$. Let N be the kernel of $K_2(K_{\mathrm{nr}})/K_2(K_{\mathrm{nr}})^p\to F_s^*/(F_s^*)^p$ which is induced by the tame symbol. By § 2 Prop. 1 and by running to the inductive limit, $\mathrm{Gal}(F_s/F)$-module N is built up by a finite number of extensions from $\mathrm{Gal}(F_s/F)$-modules of the forms Ω_{F_s} and F_s/F_s^p. Hence, for $i=1, 2$, $H^i(F, N)=0$ by $H^i(F, \Omega_{F_s})=0$ and by $H^i(F, F_s/F_s^p)=0$. This shows that (4) is, and hence (5) is an isomorphism.

Step 5. (The proof of (ii) of Th. 1) By (iii) of Th. 1, the condition

"For each finite extension F' of F, $\mathrm{Br}(F')_p=0$"

is equivalent to the following condition

"For each finite extension L of K, $H^3(L, \mu_p\otimes\mu_p)=0$".

But the latter condition is equivalent to $\mathrm{cd}_p K\leqq 2$, by the argument in the proof of Serre [20] Ch. II § 2 Prop. 4. This completes the proof of Th. 1.

PROOF OF COROLLARY 1. Let K and F be as in the hypothesis of Corollary 1. Suppose l is any prime number. Since F is a B_1-field, $\mathrm{cd}_l F\leqq 1$ (cf. Serre [20] Ch. II § 3).

If $l\neq \mathrm{ch}(F)$, $\mathrm{ch}_l K_{\mathrm{nr}}=1$ as is well known, and hence we have

$$\mathrm{cd}_l K\leqq \mathrm{cd}_l F+\mathrm{cd}_l K_{\mathrm{nr}}\leqq 2 .$$

If $l=\mathrm{ch}(F)$ and $\mathrm{ch}(K)=0$, we have $\mathrm{cd}_l K\leqq 2$ by (ii) of Theorem.

If $l=\mathrm{ch}(K)$, we have $\mathrm{cd}_l K\leqq 1$ by Serre [20] Ch. II § 2 Prop. 3.

Hence we have $\mathrm{cd}_l K\leqq 2$ in all cases.

PROOF OF COROLLARY 2. Generally, let K be a field which is complete with respect to a discrete valuation and with arbitrary residue field F.

If $\mathrm{ch}(F)=p>0$, the "prime to p" part of the Galois cohomology theory over K is easy. If $\mathrm{ch}(F)=0$, the whole of the Galois cohomology theory over K is easy. For a while, we will study this easy part. If $\mathrm{ch}(F)=p>0$, we will combine this study with the study of the p-primary part given in Th. 1.

Let m be an integer which is invertible in F. Let K_{nr} be the maximum unramified extension of K. Identify F_s with the residue field of K_{nr}. As is well known, there is a canonical isomorphism in C_F (§ 0.1),

$$H^i(K_{\mathrm{nr}}, \mu_m\otimes\mu_m)\cong\begin{cases}\mu_m\otimes\mu_m & \text{if } i=0 ,\\ \mu_m & \text{if } i=1 ,\\ 0 & \text{if } i>1 .\end{cases}$$

Hence, using the spectral sequence

$$H^*(F, H^*(K_{\mathrm{nr}}, \mu_m\otimes\mu_m))\Rightarrow H^*(K, \mu_m\otimes\mu_m)\ ,$$

we have an exact sequence

$$H^3(F, \mu_m\otimes\mu_m)\to H^3(K, \mu_m\otimes\mu_m)\to H^2(F, \mu_m)\to H^4(F, \mu_m\otimes\mu_m)\ .$$

Suppose that F is complete with respect to a discrete valuation and with finite residue field. Then, $\mathrm{cd}\,(F)=2$ by Serre [20] Ch. II § 5 and by its analogue in the characteristic >0 case. Hence we have

$$H^3(K, \mu_m\otimes\mu_m)\cong H^2(F, \mu_m)\cong \mathrm{Br}\,(F)_m\ . \tag{6}$$

Suppose further $\mathrm{ch}\,(K)=0$. For any non-zero integer m, by (6) and by (iii) of Th. 1, we have,

$$H^3(K, \mu_m\otimes\mu_m)\cong \mathrm{Br}\,(F)_m\ .$$

Since $\mathrm{Br}\,(F)\cong \boldsymbol{Q}/\boldsymbol{Z}$ canonically by ordinary local class field theory, we have the canonical isomorphism $\eta_{m,K}$.

It remains to prove the commutativity of the three diagrams in Corollary 2. The commutativity of the left diagram in (i) and that of the diagram in (ii) is deduced easily and left to the reader. The commutativity of the right diagram in (i) is proved as follows.

If $m|n$, let $s_{m,n}$ be the homomorphism s defined in (ii). Let A be the inductive limit of the system $(\mu_n\otimes\mu_n, s_{m,n})$. Then, by the commutativity of the diagram in (ii), we have a canonical isomorphism

$$\eta_K: H^3(K, A)\cong \boldsymbol{Q}/\boldsymbol{Z}\ .$$

By the commutativity of the left diagram in (i), we have a commutative diagram

$$\begin{array}{ccc} H^3(K, A) & \xrightarrow{\eta_K} & \boldsymbol{Q}/\boldsymbol{Z} \\ \downarrow{\scriptstyle \mathrm{Res}} & & \downarrow{\scriptstyle [L:K]} \\ H^3(L, A) & \xrightarrow{\eta_L} & \boldsymbol{Q}/\boldsymbol{Z} \end{array}$$

for each finite extension L of K. Since $\mathrm{Cor}_{L/K}\circ\mathrm{Res}_{L/K}=[L:K]$ and since $[L:K]: \boldsymbol{Q}/\boldsymbol{Z}\to\boldsymbol{Q}/\boldsymbol{Z}$ is surjective, it follows that the diagram

$$\begin{array}{ccc} H^3(L, A) & \xrightarrow{\eta_L} & \boldsymbol{Q}/\boldsymbol{Z} \\ \downarrow{\scriptstyle \mathrm{Cor}} & & \| \\ H^3(K, A) & \xrightarrow{\eta_K} & \boldsymbol{Q}/\boldsymbol{Z} \end{array}$$

is commutative. This proves the commutativity of the right diagram in (i).

PROOF OF PROP. 1. Since there is an exact sequence

$$\mathrm{Br}\,(K)\xrightarrow{p}\mathrm{Br}\,(K)\to H^3(K, \mu_p)\ ,$$

it suffices to show $H^3(K, \mu_p)=0$. Let $L=K(\zeta)$ where ζ denotes a primitive p-th root of 1, let $G=\mathrm{Gal}(L/K)$ and let H be the inertial group of L/K. The condition that $e_K/(p-1)$ is not an integer implies that $H\neq\{1\}$. Let C be the residue field of L. Let $\varphi: G\to(\boldsymbol{Z}/p\boldsymbol{Z})^*$ be the injective homomorphism induced by the action of G on the p-th roots of 1. For any $\boldsymbol{Z}/p\boldsymbol{Z}[G]$-module A, let $A^{\Sigma}=\{x\in A \mid \sigma(x)=\varphi(\sigma)x, \ \forall \sigma\in G\}$. Since the order of G is prime to p, $H^3(K, \mu_p)$ is isomorphic to $H^3(L, \mu_p)^G$ and hence is isomorphic to $H^3(L, \mu_p\otimes\mu_p)^{\Sigma}\cong \mathrm{Br}(C)_p^{\Sigma}$. Let σ be an element of H such that $\sigma\neq 1$. Since H acts on $\mathrm{Br}(C)$ trivially, any $x\in \mathrm{Br}(C)_p^{\Sigma}$ satisfies $x=\varphi(\sigma)x$. But $1-\varphi(\sigma)\neq 0$, and hence $x=0$. Q.E.D.

§ 6. Local class field theory.

In this section, we are concerned with the main problem of this Chapter I. In this section, F denotes a field of characteristic $p>0$ which is complete with respect to a discrete valuation and with *finite* residue field. And K deontes a field of characteristic 0 which is complete with respect to a discrete valuation and with residue field F. Our aim is to prove the following Theorem 1 and Theorem 2.

THEOREM 1. *There exists a canonical injective homomorphism*

$$\Phi: \mathrm{Br}(K)\to \mathrm{Hom}(K^*, \boldsymbol{Q}/\boldsymbol{Z})$$

having the following property: For each $w\in \mathrm{Br}(K)$, $\mathrm{Ker}(\Phi(w))=\mathrm{Nrd}(w/K)$.

THEOREM 2. *There exists a canonical homomorphism*

$$\Psi: K_2(K)\to \mathrm{Gal}(K^{\mathrm{ab}}/K)$$

having the following property: For each finite abelian extension L of K, Ψ induces an isomorphism

$$K_2(K)/N_{L/K}K_2(L)\xrightarrow{\cong}\mathrm{Gal}(L/K).$$

DEFINITIONS OF Φ AND Ψ. We have constructed a canonical isomorphism

$$\eta_{m,K}: H^3(K, \mu_m\otimes\mu_m)\cong \frac{1}{m}\boldsymbol{Z}\Big/\boldsymbol{Z}$$

for each $m>0$ (§ 5 Corollary 2). Hence we have the following composites

$$K^*/(K^*)^m\otimes \mathrm{Br}(K)_m\cong H^1(K, \mu_m)\otimes H^2(K, \mu_m) \tag{1}$$
$$\xrightarrow{\text{cup product}} H^3(K, \mu_m\otimes\mu_m)\cong \frac{1}{m}\boldsymbol{Z}\Big/\boldsymbol{Z},$$

$$K_2(K)/K_2(K)^m \otimes (X_K)_m \xrightarrow{\text{by } h_m} H^2(K, \mu_m \otimes \mu_m) \otimes H^1(K, \boldsymbol{Z}/m\boldsymbol{Z}) \tag{2}$$

$$\xrightarrow{\text{cup product}} H^3(K, \mu_m \otimes \mu_m) \cong \frac{1}{m}\boldsymbol{Z}\Big/\boldsymbol{Z}\,.$$

By the commutativity of the diagram in §5 Corollary 2 (ii), (1) induces a homomorphism

$$K^* \otimes \mathrm{Br}\,(K) \to \boldsymbol{Q}/\boldsymbol{Z} \tag{3}$$

and (2) induces a homomorphism

$$K_2(K) \otimes X_K \to \boldsymbol{Q}/\boldsymbol{Z}\,. \tag{4}$$

The homomorphism Φ is defined by (3) and the homomorphism Ψ is defined by (4).

Some *formulae*. We denote the pairings (3) and (4) both by $\langle\ ,\ \rangle_K$. Let L be a finite extension of K. Then, we have the following formulae.

$$\langle a, w_L\rangle_L = \langle N_{L/K}a, w\rangle_K\,, \qquad \forall\, a \in L^*\,, \quad \forall\, w \in \mathrm{Br}\,(K)\,. \tag{5}$$

$$\langle a, w\rangle_L = \langle a, \mathrm{Cor}_{L/K}w\rangle_K\,, \qquad \forall\, a \in K^*\,, \quad \forall\, w \in \mathrm{Br}\,(L)\,. \tag{6}$$

$$\langle \alpha, \chi_L\rangle_L = \langle N_{L/K}\alpha, \chi\rangle_K\,, \qquad \forall\, \alpha \in K_2(L)\,, \quad \forall\, \chi \in X_K\,. \tag{7}$$

$$\langle \alpha, \chi\rangle_L = \langle \alpha, \mathrm{Cor}_{L/K}\chi\rangle_K\,, \qquad \forall\, \alpha \in K_2(K)\,, \quad \forall\, \chi \in X_L\,. \tag{8}$$

Here, the corestriction maps are defined by the following identifications:

$$\mathrm{Br}\,(k) = H^2(k, k_s^*) \qquad \text{and} \qquad X_k = H^1(k, \boldsymbol{Q}/\boldsymbol{Z}) \qquad \text{where } k \text{ is any field.}$$

These formulae are proved easily by using

§3 Lemma 5,
the formula $\mathrm{Cor}\,(x \cup \mathrm{Res}\,(y)) = \mathrm{Cor}\,(x) \cup y$, ($\cup$: cup product)
the commutativity of the right diagram in §5 Corollary 2 (i).

Proofs of the injectivity of Φ: $\mathrm{Br}\,(K) \to \mathrm{Hom}\,(K^*, \boldsymbol{Q}/\boldsymbol{Z})$ and $\Psi^\vee$: $X_K \to \mathrm{Hom}\,(K_2(K), \boldsymbol{Q}/\boldsymbol{Z})$ where $\Psi^\vee$ denotes the dual of Ψ.

The injectivity of the "prime to p" parts of these homomorphisms are easily proved as follows. If n is prime to p, there are induced commutative diagrams (9) and (10) with exact rows

$$\begin{array}{ccccccccc} 0 \to & \mathrm{Br}\,(F)_n & \to & \mathrm{Br}\,(K)_n & \to & (X_F)_n & \to 0 \\ & \downarrow\wr & & \downarrow \text{by } \Phi & & \downarrow \alpha & \\ 0 \to & \frac{1}{n}\boldsymbol{Z}\Big/\boldsymbol{Z} & \to & \mathrm{Hom}\left(K^*, \frac{1}{n}\boldsymbol{Z}\Big/\boldsymbol{Z}\right) & \to & \mathrm{Hom}\left(F^*, \frac{1}{n}\boldsymbol{Z}\Big/\boldsymbol{Z}\right) & \to 0 \end{array} \tag{9}$$

$$(10)\qquad \begin{array}{ccccccc} 0\to & (X_F)_n & \to & (X_K)_n & \to & \mathrm{Hom}\left(k^*, \frac{1}{n}\boldsymbol{Z}\big/\boldsymbol{Z}\right)\to 0 \\ & \downarrow\alpha & & \downarrow\text{by } \Psi^\vee & & \downarrow\beta \\ 0\to\mathrm{Hom}\left(F^*, \frac{1}{n}\boldsymbol{Z}\big/\boldsymbol{Z}\right) & & \to\mathrm{Hom}\left(K_2(K), \frac{1}{n}\boldsymbol{Z}\big/\boldsymbol{Z}\right) & & \to\mathrm{Hom}\left(K_2(F), \frac{1}{n}\boldsymbol{Z}\big/\boldsymbol{Z}\right)\to 0\,, \end{array}$$

where α is the homomorphism defined by ordinary local class field theory, k is the residue field of F, and β is the homomorphism induced by the tame symbol. The upper exact row of the first (resp. second) diagram is obtained from the spectral sequence

$$H^*(F, H^*(K_{\mathrm{nr}},\quad))\Rightarrow H^*(K,\quad)$$

(cf. the proof of §5 Corollary 2). The homomorphism α is bijective by ordinary local class field theory, and β is bijective as is easily seen. Hence, if n is prime to p, the homomorphisms

$$\mathrm{Br}\,(K)_n\to\mathrm{Hom}\left(K^*, \frac{1}{n}\boldsymbol{Z}\big/\boldsymbol{Z}\right)$$
$$(X_K)_n\to\mathrm{Hom}\left(K_2(K), \frac{1}{n}\boldsymbol{Z}\big/\boldsymbol{Z}\right),$$

which are induced by Φ and $\Psi^\vee$, are bijective.

Next, we prove the injectivity of the p-primary parts of Φ and $\Psi^\vee$. It suffices to show the induced homomorphisms

$$\mathrm{Br}\,(K)_p\to\mathrm{Hom}\left(K^*/(K^*)^p, \frac{1}{p}\boldsymbol{Z}\big/\boldsymbol{Z}\right)$$
$$(X_K)_p\to\mathrm{Hom}\left(K_2(K)/K_2(K)^p, \frac{1}{p}\boldsymbol{Z}\big/\boldsymbol{Z}\right)$$

are injective. We may suppose that K contains a primitive p-th root ζ of 1. Indeed, if we can prove our assertions for $L=K(\zeta)$, we can deduce $w=0$ from the assumptions $w\in\mathrm{Br}\,(K)_p$ and $\langle K^*, w\rangle_K=0$, as follows:

$$\langle K^*, w\rangle_K=0\Rightarrow\langle N_{L/K}L^*, w\rangle_K=0\overset{\text{by (5)}}{\Longrightarrow}\langle L^*, w_L\rangle_L=0\Rightarrow w_L=0$$
$$\Rightarrow[L:K]w=\mathrm{Cor}_{L/K}w_L=0\Rightarrow w=0\,.$$

(The last $\Rightarrow$ follows from the fact that $[L:K]$ is prime to p and $pw=0$.) The similar argument goes for $(X_K)_p\to\mathrm{Hom}\,(K_2(K)/K_2(K)^p, (1/p)\boldsymbol{Z}/\boldsymbol{Z})$. Hence we suppose that K contains a primitive p-th root ζ of 1. In this case, we have isomorphisms

$$\mathrm{Br}\,(K)_p\cong K_2(K)/K_2(K)^p \qquad (\S 3\ \text{Th. } 1)$$
$$(X_K)_p\cong H^1(K, \boldsymbol{Z}/p\boldsymbol{Z})\cong(\text{by } \boldsymbol{Z}/p\boldsymbol{Z}\cong\mu_p;\ 1\mapsto\zeta)H^1(K, \mu_p)$$
$$\cong K^*/(K^*)^p \qquad (\text{Kummer theory})\,.$$

Hence it suffices to show that the induced homomorphisms

$$K_2(K)/K_2(K)^p \to \mathrm{Hom}\left(K^*/(K^*)^p, \frac{1}{p}\boldsymbol{Z}\Big/\boldsymbol{Z}\right) \tag{11}$$

$$K^*/(K^*)^p \to \mathrm{Hom}\left(K_2(K)/K_2(K)^p, \frac{1}{p}\boldsymbol{Z}\Big/\boldsymbol{Z}\right) \tag{12}$$

are injective. The group $K^*/(K^*)^p$ has the structure illustrated in the following Figure 2. If one compares Figure 2 with Figure 1 in § 2, one finds that each pair of the groups which are combined by $\leftrightarrow$ in Figure 3 consists of a locally compact abelian group and its dual group (with respect to the usual topologies on these groups).

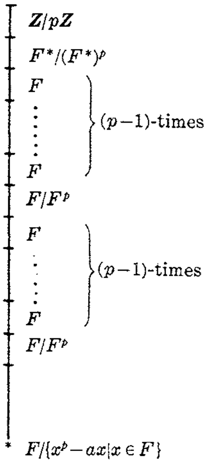

Fig. 2.

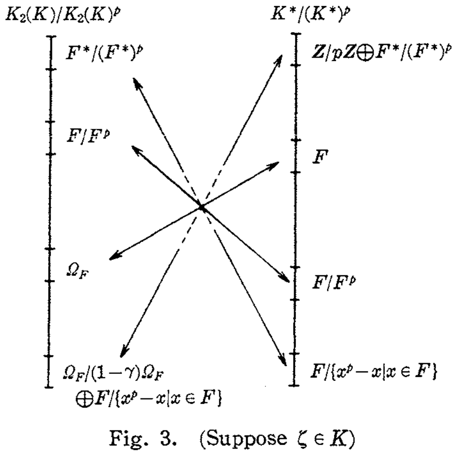

Fig. 3. (Suppose $\zeta\in K$)

Hence, it remains to show that the homomorphisms (11) and (12) induce the duality map of each pair (the pair (Ω_F, F), the pair $(F/F^p, F/F^p)$, etc.) of the groups which are combined by $\leftrightarrow$ in Figure 3. But the homomorphisms (11) and (12) induce the same homomorphism

$$\varepsilon\colon K_2(K)/K_2(K)^p\otimes K^*/(K^*)^p \to \frac{1}{p}\boldsymbol{Z}\Big/\boldsymbol{Z}\,,$$

and the composite ε_0:

$$K^*/(K^*)^p\otimes K^*/(K^*)^p\otimes K^*/(K^*)^p \xrightarrow{x\otimes y\otimes z\mapsto\{x,y\}\otimes z} K_2(K)/K_2(K)^p\otimes K^*/(K^*)^p \xrightarrow{\varepsilon} \frac{1}{p}\boldsymbol{Z}\Big/\boldsymbol{Z}$$

coincides with the following composite

$$K^*/(K^*)^p\otimes K^*/(K^*)^p\otimes K^*/(K^*)^p\xrightarrow{\cong}H^1(K,\mu_p)\otimes H^1(K,\mu_p)\otimes H^1(K,\mu_p)$$
$$\xrightarrow{\text{cup product}}H^3(K,\mu_p\otimes\mu_p\otimes\mu_p)\underset{\cong}{\xrightarrow{\alpha}}H^3(K,\mu_p\otimes\mu_p)\underset{\cong}{\xrightarrow{\S 5\text{ Th. }1}}\frac{1}{p}\boldsymbol{Z}\Big/\boldsymbol{Z}$$

where α is the isomorphism induced by $\boldsymbol{Z}/p\boldsymbol{Z}\cong\mu_p$; $1\mapsto\zeta$. Since ε has the following properties (13) and (14), the homomorphisms (11) and (12) realize the duality between two groups which are combined by $\leftrightarrow$ in Figure 3 and we arrive at the goal.

In the following (13) and (14), we use the notation δ for the composite

$$\Omega_F\xrightarrow{\text{residue map}}k\xrightarrow{\text{trace}}\boldsymbol{Z}/p\boldsymbol{Z}\underset{\cong}{\xrightarrow{1\mapsto 1/p}}\frac{1}{p}\boldsymbol{Z}\Big/\boldsymbol{Z}\,,$$

where k is the residue field of F. This homomorphism δ has the following properties: As is easily seen, the pairings $\Omega_F\otimes F\to(1/p)\boldsymbol{Z}/\boldsymbol{Z}$; $w\otimes x\mapsto\delta(xw)$ and $F/F^p\otimes F/F^p\to(1/p)\boldsymbol{Z}/\boldsymbol{Z}$; $x\otimes y\mapsto\delta(x\,dy)$ give the perfect dualities of the pairs of locally compact abelian groups (Ω_F, F) and $(F/F^p, F/F^p)$. Furthermore, as in Serre [19] Ch. XIV § 5 Cor. to Prop. 15, the canonical isomorphism $\mathrm{Br}(F)_p\cong(1/p)\boldsymbol{Z}/\boldsymbol{Z}$ and the canonical pairing $(X_F)_p\otimes F^*/(F^*)^p\to(1/p)\boldsymbol{Z}/\boldsymbol{Z}$ defined by ordinary local class field theory are given by

$$\mathrm{Br}(F)_p\cong\Omega_F/(1-\gamma)\Omega_F(\text{cf. the proof of } \S 4 \text{ Lemma } 5)\xrightarrow{\delta}\frac{1}{p}\boldsymbol{Z}\Big/\boldsymbol{Z}\,,$$

$$(X_F)_p\otimes F^*/(F^*)^p\cong F/\{x^p-x\}\otimes F^*/(F^*)^p\to\frac{1}{p}\boldsymbol{Z}\Big/\boldsymbol{Z}\,;\qquad x\otimes y\mapsto\delta(x\,dy/y)\,,$$

respectively. We fix a prime element π of K, and we denote $\zeta-1$ by τ.

(13) *If* $0\leqq n\leqq e_Kp/(p-1)$ *and* $i>(e_Kp/(p-1))-n$,

$$\varepsilon(V_K^{(n)}\cdot K_2(K)^p/K_2(K)^p\otimes U_K^{(i)}\cdot(K^*)^p/(K^*)^p)=0\,.$$

(14) *Suppose* $0\leqq n\leqq e_Kp/(p-1)$. *Let* $i=(e_Kp/(p-1))-n$. *Then, the induced homomorphism by* (13):

$$V_K^{(n)}K_2(K)^p/V_K^{(n+1)}K_2(K)^p\otimes U_K^{(i)}(K^*)^p/U_K^{(i+1)}(K^*)^p\to\frac{1}{p}\boldsymbol{Z}\Big/\boldsymbol{Z}$$

coincides with the following, provided that the notation $U_K^{(0)}$ *stands for* K^* *(not for* U_K*) only for a moment.*

If $n=0$,

$$F^*/(F^*)^p\otimes F/\{x^p-x\mid x\in F\}\to\frac{1}{p}\boldsymbol{Z}\Big/\boldsymbol{Z}\,;\qquad x\otimes y\mapsto\delta(y\,dx/x)\,,$$

if $0<n<e_K p/(p-1)$ *and* $p\,|\,n$,

$$F/F^p \otimes F/F^p \to \frac{1}{p}\boldsymbol{Z}\Big/\boldsymbol{Z}\,; \qquad x\otimes y \mapsto \delta(x\,dy)\,,$$

if $0<n<e_K p/(p-1)$ *and* n *is prime to* p,

$$\Omega_F \otimes F \to \frac{1}{p}\boldsymbol{Z}\Big/\boldsymbol{Z}\,; \qquad w\otimes x \mapsto -n\delta(xw)\,,$$

if $n=e_K p/(p-1)$,

$$\text{the sum of } \begin{cases} \Omega_F/(1-\gamma)\Omega_F \otimes \boldsymbol{Z}/p\boldsymbol{Z} \to \dfrac{1}{p}\boldsymbol{Z}\Big/\boldsymbol{Z}\,; & w\otimes 1 \mapsto \delta(w) \\ F/\{x^p-x\} \otimes F^*/(F^*)^p \to \dfrac{1}{p}\boldsymbol{Z}\Big/\boldsymbol{Z}\,; & x\otimes y \mapsto -\delta(x\,dy/y)\,. \end{cases}$$

Here, we identify each subquotient of $K_2(K)/K_2(K)^p$ with $F^*/(F^*)^p$, F/F^p, Ω_F, or $\Omega_F/(1-\gamma)\Omega_F \oplus F/\{x^p-x\,|\,x\in F\}$ via the isomorphism ρ_n (§2 Prop. 1) which we define using the above fixed π in the case of §2 Prop. 1 (i), (ii), or (iv), and choosing b or c arbitrarily in the case of §2 Prop. 1 (ii) or (iii), and taking τ $(=\zeta-1)$ as b in the case of §2 Prop. 1 (iv). We identify each subquotient of $K^*/(K^*)^p$ with $\boldsymbol{Z}/p\boldsymbol{Z}\oplus F^*/(F^*)^p$, F, F/F^p, or $F/\{x^p-x\,|\,x\in F\}$ via the following isomorphisms:

$\boldsymbol{Z}/p\boldsymbol{Z}\oplus F^*/(F^*)^p \cong K^*/U_K^{(1)}(K^*)^p\,;$ the sum of $1\mapsto\pi$ and $\bar f\mapsto f$ $(f\in U_K)$,

if $0<i<e_K p/(p-1)$ and $p\,|\,i$,

$$F/F^p \cong U_K^{(i)}(K^*)^p/U_K^{(i+1)}(K^*)^p\,; \qquad \bar f \mapsto 1+f(\tau/b)^p \quad (f\in O_K)\,,$$

if $0<i<e_K p/(p-1)$ and i is prime to p,

$$F \cong U_K^{(i)}(K^*)^p/U_K^{(i+1)}(K^*)^p\,; \qquad \bar f \mapsto 1+f\tau^p c^{-1} \quad (f\in O_K)$$

(b and c are the elements which were used to define the above ρ_n),

$$F/\{x^p-x\,|\,x\in F\} \cong U_K^{(e_K p/(p-1))}(K^*)^p/(K^*)^p\,; \qquad \bar f \mapsto 1+f\tau^p \quad (f\in O_K)\,.$$

The proofs of (13) and (14) are as follows. Since ε_0 is anti-symmetric, (13) follows from the fact:

If $i+j>e_K p/(p-1)$, $\{U_K^{(i)}, U_K^{(j)}\}\subset V_K^{(i+j)}\subset K_2(K)^p$ (§2 Lemma 2).

Next, the part "if $n=0, \cdots$" of (14) is deduced directly from the definition of ε. (We omit the details.) The rest of (14) can be reduced to the above part by using the fact ε_0 is anti-symmetric. For example, suppose $0<n<e_K p/(p-1)$ and n is prime to p. (The proofs of the other parts go similarly and are left to the reader.) Let $f\in U_K$, $g\in U_K$, $h\in O_K$, $c\in K$, and $v_K(c)=n$. Then,

$$\varepsilon(\{1+fc, g\}\otimes(1+h\tau^p c^{-1})) = -\varepsilon(\{1+fc, 1+h\tau^p c^{-1}\}\otimes g)$$
$$= \varepsilon(\{1+fch\tau^p c^{-1}, -fc\}\otimes g)$$
(here, we used § 2 Lemma 2 (ii))
$$= \varepsilon(\{-fc, g\}\otimes(1+fh\tau^p)) .$$

But
$$\varepsilon(\{-fc, g\}\otimes(1+fh\tau^p)) = -n\delta(\bar{f}\bar{h}\, d\bar{g}/\bar{g})$$
by the part "if $n=0, \cdots$" of (14).

Proof of $\mathrm{Ker}(\Phi(w))=\mathrm{Nrd}(w/K)$. The following proof is applicable to the case $\mathrm{ch}(K)=p$ in Ch. II with no change. Let $w\in \mathrm{Br}(K)$. If $a\in \mathrm{Nrd}(w/K)$, there are a finite extension L of K and $b\in L^*$ such that $w_L=0$ and $a=N_{L/K}b$. Hence $\langle a, w\rangle_K= \langle b, w_L\rangle_L=0$ by the formula (5). This proves $\mathrm{Ker}(\Phi(w))\supset \mathrm{Nrd}(w/K)$.

Next, we prove $\mathrm{Ker}(\Phi(w))=\mathrm{Nrd}(w/K)$. We need two steps.

Step 1. In Step 1, by induction on the order of w, we prove that we may assume $w\in \mathrm{Br}(K)_l$ for some prime number l. Indeed, suppose $w\in \mathrm{Br}(K)_n$, $n>1$, $a\in K^*$, and $\langle a, w\rangle_K=0$. Let l be a prime divisor of n. Since $\langle a, lw\rangle_K=0$, we have $a\in \mathrm{Nrd}(lw/K)$ by the hypothesis of our induction. Hence, by § 4 Lemma 1, there are a finite extension L of K and $b\in L^*$ such that $lw_L=0$ and $a=N_{L/K}b$. By the formula (5), $\langle b, w_L\rangle_L=\langle a, w\rangle_K=0$. Suppose we have proved the fact $\mathrm{Ker}(\Phi(w'))=\mathrm{Nrd}(w'/K)$ for all $w'\in \mathrm{Br}(K)_l$. Since L is a field which and whose residue field satisfy the assumptions at the beginning of this section, too, it follows that $b\in \mathrm{Nrd}(w_L/L)$. Hence, there are a finite extension M of L and $c\in M^*$ such that $w_M=0$ and $b=N_{M/L}c$. Thus, $w_M=0$ and $a=N_{M/K}c$. This shows $a\in \mathrm{Nrd}(w/K)$ by § 4 Lemma 1. And we have arrived at the goal of Step 1.

Step 2. In Step 2, we suppose $w\in \mathrm{Br}(K)_l$, $w\neq 0$, and that l is a prime number. Since $\Phi(w): K^*\to \boldsymbol{Q}/\boldsymbol{Z}$ is non-zero by the injectivity of Φ, the image of $\Phi(w)$ is of order l. Hence, for the proof of $\mathrm{Ker}(\Phi(w))=\mathrm{Nrd}(w/K)$, it suffices to show $\sharp(K^*/\mathrm{Nrd}(w/K))\leqq l$. (For a set S, we denote the order of S by $\sharp(S)$.)

First, suppose $w\notin \mathrm{Br}(K_{\mathrm{nr}}/K)$. Then, we have $l=p$, and the division algebra D which corresponds to w satisfies the conditions at the beginning of § 1 by § 4 Lemma 5. Hence, by § 1 Prop. 3 (v), there is an isomorphism
$$K^*/\mathrm{Nrd}_{D/K}D^*\cong \mathrm{Br}(F)_p .$$
Since $\sharp(\mathrm{Br}(F)_p)=p$ by ordinary local class field theory, we have $\sharp(K^*/\mathrm{Nrd}(w/K))=p$.

Next, suppose $w\in \mathrm{Br}(K_{\mathrm{nr}}/K)_l$ and $w\neq 0$. Let D be the division algebra over K which corresponds to w. By Serre [19] Ch. XII, there is a natural splitting exact sequence

$$0 \to \mathrm{Br}\,(F) \xrightarrow{\alpha} \mathrm{Br}\,(K_{\mathrm{nr}}/K) \xrightarrow{\beta} X_F \to 0\,.$$

First, suppose that $w=\alpha(w_0)$ for some $w_0 \in \mathrm{Br}\,(F)$. Then the residue algebra C of D is nothing but the division algebra with center F which corresponds to w_0. Since the order of w_0 is l, we have $\dim_F C=l^2$ by ordinary local class field theory. Let F' be a maximal commutative subfield of C such that the extension F'/F is separable. Let L be the unramified extension of K which corresponds to F'. Then, since w is the image of w_0, the following commutative diagram

$$\begin{array}{ccc} \mathrm{Br}\,(F) & \to & \mathrm{Br}\,(K_{\mathrm{nr}}/K) \\ \downarrow & & \downarrow \\ \mathrm{Br}\,(F') & \to & \mathrm{Br}\,(L_{\mathrm{nr}}/L) \end{array} \tag{15}$$

shows that L decomposes D. Hence we have $\dim_K D=l^2$. Since L is unramified over K,

$$U_K^{(1)}=N_{L/K}U_L^{(1)}\subset \mathrm{Nrd}_{D/K}D^*\,.$$

The map $\mathrm{Nrd}_{D/K}\colon U_D/U_D^{(1)} \to U_K/U_K^{(1)}$ is surjective because it is equal to the map $\mathrm{Nrd}_{C/F}\colon C^* \to F^*$ which is well known (and is known from §4 Prop. 2 (i), for a finite field is B_1) to be surjective. Thus, we have $U_K \subset \mathrm{Nrd}_{D/K} D^*$. Since $\dim_K D=l^2$, the map $\mathrm{Nrd}_{D/K}\colon D^*/U_D \to K^*/U_K$ can be rewritten as $\boldsymbol{Z}\to\boldsymbol{Z}$; $1 \mapsto l$. Hence we have $\sharp(K^*/\mathrm{Nrd}_{D/K} D^*)=l$. Next, suppose that $w \in \mathrm{Br}\,(K_{\mathrm{nr}}/K)_l$ and that w does not belong to the image of α. Let $\chi=\beta(w)$ and let F' be the cyclic extension of F corresponding to χ. Let $\tilde{\chi}$ be the "unramified" element of X_K corresponding to χ and let L be the unramified cyclic extension of K corresponding to $\tilde{\chi}$. Then, we can show $w_L=0$ as follows. Let π be a prime element of K. Then, by Serre [19] Ch. XII, $\beta((\tilde{\chi}, \pi))=\chi=\beta(w)$. (Cf. §4 Remark 1 after Step 1 of the proof of Prop. 2, or Serre [19] Ch. XIV, for the above notation (,).) Hence $w-(\tilde{\chi}, \pi)$ comes from $\mathrm{Br}\,(F)$. By ordinary local class field theory, any element in $\mathrm{Br}\,(F)_l$ is decomposed by any extension of F of degree l, and so, in particular, by F'. Hence, the commutative diagram (15) shows that $w-(\tilde{\chi}, \pi)$ is decomposed by L. Thus, $w_L=0$. By Serre [19] Ch. XIV §1, we have $w=(\tilde{\chi}, a)$ for some $a \in K^*$. Since $\beta((\tilde{\chi}, a))=v_K(a)\chi$, we have $w=(\tilde{\chi}, \pi')$ for some prime element π' of K. It follows that D has the following properties.

$\dim_K D=l^2$.
$U_K^{(1)}=N_{L/K}U_L^{(1)}\subset \mathrm{Nrd}_{D/K} D^*$.
The map $\mathrm{Nrd}_{D/K}\colon U_D/U_D^{(1)} \to U_K/U_K^{(1)}$ is nothing but the map $N_{F'/F}\colon F'^* \to F^*$.
$\mathrm{Nrd}_{D/K} D^*$ contains $-\pi'$.

Hence $\sharp(K^*/\mathrm{Nrd}_{D/K} D^*)=\sharp(F^*/N_{F'/F}F'^*)$ and this is equal to l by ordinary local class field theory.

PROOF of the fact that for each finite abelian extension L of K, Ψ induces an isomorphism

$$K_2(K)/N_{L/K}K_2(L) \cong \mathrm{Gal}(L/K) .$$

Let L be a finite abelian extension of K. Let θ denote the homomorphism $K_2(K) \to \mathrm{Gal}(L/K)$ induced by Ψ. Tne surjectivity of θ follows immediately from the injectivity of $\Psi^\vee : X_K \to \mathrm{Hom}(K_2(K), \boldsymbol{Q}/\boldsymbol{Z})$. Next, we show $N_{L/K}K_2(L) \subset \mathrm{Ker}(\theta)$. For each character χ of $\mathrm{Gal}(L/K)$,

$$\langle N_{L/K}K_2(L), \chi \rangle_K = \langle K_2(L), \chi_L \rangle_L = 0$$

by the formula (7), where we regard χ as an element of X_K. Hence $N_{L/K}K_2(L) \subset \mathrm{Ker}(\theta)$.

It remains to show that $N_{L/K}K_2(L)$ coincides with $\mathrm{Ker}(\theta)$. It suffices to show

$$\sharp(K_2(K)/N_{L/K}K_2(L)) \leqq [L : K] . \tag{16}$$

Generally, if $f : A \to B$ and $g : B \to C$ are homomorphisms of abelian groups, there is an exact sequence

$$\mathrm{Coker}(f) \to \mathrm{Coker}(gf) \to \mathrm{Coker}(g) \to 0 ,$$

and so, we have an inequality

$$\sharp(\mathrm{Coker}(gf)) \leqq \sharp(\mathrm{Coker}(f)) \cdot \sharp(\mathrm{Coker}(g)) .$$

Hence, to prove (16), we may assume that $[L : K]$ is a prime number l. But in this case, we can show that $N_{L/K}K_2(L)$ coincides with $\mathrm{Ker}(\theta)$, from which (16) follows immediately. Suppose L is a cyclic extension of K such that $[L : K]$ is a prime number l. Let χ be an element of X_K to which L corresponds. Suppose $\alpha \in K_2(K)$ and $\theta(\alpha) = 1$. This implies $\langle \alpha, \chi \rangle_K = 0$. By Lemma 1 below, there are $a, b \in K^*$ such that

$$\alpha \equiv \{a, b\} \mod K_2(K)^l .$$

We have $\langle \{a, b\}, \chi \rangle_K = 0$. Hence $\langle a, (\chi, b) \rangle_K = 0$. Since $\mathrm{Nrd}(w/K) = \mathrm{Ker}(\Phi(w))$ as has been shown, we have, $a \in \mathrm{Nrd}((\chi, b)/K)$. Hence $\{a, b\} \in N_{L/K}K_2(L)$ by §4 Lemma 2. Hence $\alpha \in N_{L/K}K_2(L)$ and this is our goal.

LEMMA 1. *Let K be as at the beginning of this section. Let l be a prime number. Then, for each element α of $K_2(K)$, there are $a, b \in K^*$ such that $\alpha \equiv \{a, b\} \bmod K_2(K)^l$.*

PROOF. First, suppose $l = p$. By §2 Prop. 2, we may assume $\alpha \in$

$V_K^{(e_K p/(p-1))} K_2(K)^p$. If K contains a primitive p-th root of 1, there is an isomorphism

$$V_K^{(e_K p/(p-1))} K_2(K)^p / K_2(K)^p \cong \mathrm{Br}\,(K_{\mathrm{nr}}/K)_p$$

induced by h'_p. Hence, in this case, it suffices to show that each element of $\mathrm{Br}\,(K_{\mathrm{nr}}/K)_p$ is decomposed by a cyclic extension of K of degree p. But this has been proved in Step 2 in the proof of "$\mathrm{Ker}\,(\Phi(w)) = \mathrm{Nrd}\,(w/K)$". If K contains no primitive p-th root of 1, our assertion follows from § 2 Prop. 2, from the existence of the isomorphism $\rho_{e_K p/(p-1)}$ (§ 2 Prop. 1) and (assume $e_K/(p-1) \in \boldsymbol{Z}$) from

$$\Omega_F / \{w - a\gamma(w) \mid w \in \Omega_F\} = 0 \quad \text{in this case}\,, \tag{17}$$

where a is the element of F given in § 2 Prop. 1. The proof of (17) is as follows. Let $K' = K(\zeta)$ where ζ denotes a primitive p-th root of 1, let $G = \mathrm{Gal}\,(K'/K)$, and let C be the residue field of K'. Let $\varphi : G \to (\boldsymbol{Z}/p\boldsymbol{Z})^*$ be the injective homomorphism induced by the action of G on the p-th roots of 1. For any $\boldsymbol{Z}/p\boldsymbol{Z}[G]$-module A, let $A^{\Pi} = \{x \in A \mid \sigma(x) = \varphi(\sigma)^{-1} x,\ \forall\, \sigma \in G\}$. Then, we have isomorphisms

$$\Omega_F / \{w - a\gamma(w) \mid w \in \Omega_F\} \cong (\Omega_C / (1-\gamma)\Omega_C)^{\Pi} \cong (\mathrm{Br}\,(C)_p)^{\Pi} \cong (\boldsymbol{Z}/p\boldsymbol{Z})^{\Pi} = 0\,,$$

where G acts on $\boldsymbol{Z}/p\boldsymbol{Z}$ trivially.

Next, suppose $l \neq p$. Since $U_K^{(1)} \subset (K^*)^l$, we have $V_K^{(1)} \subset K_2(K)^l$. Since (a) $K_2(K)/V_K^{(1)} \cong K_2(F) \oplus F^*$ by the proof of § 2 Prop. 1 (i), (b) $V_F^{(1)} \subset K_2(F)^l$, (c) $K_2(F)/V_F^{(1)} \cong K_2(k) \oplus k^*$, (d) $K_2(k) = 0$ (K_2 of finite fields are well known to vanish), where k denotes the residue field of F, it follows

$$K_2(K)/K_2(K)^l \cong \boldsymbol{Z}/l\boldsymbol{Z} \oplus k^*/(k^*)^l \oplus k^*/(k^*)^l\,.$$

By writing down this isomorphism explicitly, our assertion is proved without any difficulty.

REMARK 1. Lastly, we describe the relation between the local class field theory of K and that of F. We have the following commutative diagram (18), where Ψ' denotes the homomorphism in ordinary local class field theory and α denotes the canonical restriction to the unramified part.

$$\begin{array}{ccc} K_2(K) & \xrightarrow{\Psi} & \mathrm{Gal}\,(K^{\mathrm{ab}}/K) \\ \Big\downarrow{\scriptstyle \text{tame symbol}} & & \Big\downarrow{\scriptstyle \alpha} \\ F^* & \xrightarrow{\Psi'} & \mathrm{Gal}\,(F^{\mathrm{ab}}/F)\,. \end{array} \tag{18}$$

It is not difficult to deduce the commutativity of (18) from the definition of Ψ, and its proof is left to the reader.

§ 7. The topologies of K^* and $K_2(K)$.

In this section, F denotes a field of characteristic $p>0$ which is complete with respect to a discrete valuation and with *finite* residue field, and K denotes a field which is complete with respect to a discrete valuation and with residue field F. We always regard F as a topological field with respect to the ordinary topology, i.e. the valuation topology. In this section, we define natural topologies on K^* and $K_2(K)$ which take the topology of the residue field F into account. (We do not adopt the topology on K^* defined by the valuation of K since it induces on the residue field the discrete topology. Our topology is weaker than it.)

§ 7.1. The definition of the topologies of K^* and $K_2(K)$.

Let $I=\boldsymbol{Z}_p[[X]]$ (resp. $I=\boldsymbol{F}_p[[T, X]]$) and J the completion of the local ring of I at the prime ideal pI (resp. TI). Then, J is a complete discrete valuation ring with a prime element p (resp. T) and with residue field $\boldsymbol{F}_p((X))$. The ring J can be described as the set of all Laurent power series $\sum_{n\in\boldsymbol{Z}} a_nX^n$ over $\boldsymbol{Z}_p$ (resp. $\boldsymbol{F}_p[[T]]$) such that $\lim a_n=0$ when $n\to-\infty$. We regard J as a topological ring by taking the set $\{U_{i,j}\,|\,i\geqq 0, j\geqq 0\}$ of subsets $U_{i,j}=X^iI+p^jJ$ (resp. $U_{i,j}=X^iI+T^jJ$) as a basis of the neighbourhoods of zero in J.

Now, let K and F be as above. Suppose that $\mathrm{ch}(K)=0$ (resp. $\mathrm{ch}(K)=p$). Then K can be regarded as a finite extension of the field of fractions of J. Precisely, there is a ring homomorphism $\varphi: J\to O_K$ such that;

(i) O_K is a free module of finite rank over J via φ.

(ii) Via the induced homomorphism $\boldsymbol{F}_p((X))=J/pJ\to F$ (resp. $\boldsymbol{F}_p((X))=J/TJ\to F$), F is finite dimensional over $\boldsymbol{F}_p((X))$ and the given valuation of F is the one induced by the natural valuation of $\boldsymbol{F}_p((X))$.

(This fact follows easily from Nagata [14] 31.12.) Take a basis $(e_i)_{1\leqq i\leqq n}$ of O_K over J and endow O_K with the topology for which the bijection $J^n\to O_K$; $(x_i)_{1\leqq i\leqq n}\to\sum_i x_ie_i$ is a homeomorphism. It is clear that this topology of O_K is independent of the choice of the basis $(e_i)_i$, and is compatible with the ring structure of O_K. Furthermore,

LEMMA 1. *This topology of O_K is independent of the choice of the above homomorphism φ.*

This Lemma will be proved in the next § 7.2. We endow U_K with the topology induced by the topology of O_K. Then,

LEMMA 2. *This topology of U_K is compatible with the* (*multiplicative*) *group structure of U_K.*

(For the proof, see § 7.2.) We endow K^* with the unique topology which is compatible with the group structure of K^* and for which U_K is open in K^*, and which induces on U_K the above topology of U_K. Lastly, we endow $K_2(K)$ with the strongest topology which is compatible with the group structure and for which the map

$$K^* \times K^* \to K_2(K)\ ; \qquad (x, y) \mapsto \{x, y\}$$

is continuous. There is a natural bijection between the following two sets (i) and (ii) for any commutative topological group H.

(i) The set of all continuous homomorphisms $K_2(K) \to H$.

(ii) The set of all continuous maps $h: K^* \times K^* \to H$ such that

$$h(xy, z) = h(x, z)h(y, z)$$
$$h(x, yz) = h(x, y)h(x, z)$$
$$h(1-u, u) = 1$$

for all $x, y, z \in K^*$ and for all $u \in K$ such that $u \neq 0$ and $u \neq 1$.

REMARK 1. Suppose $\mathrm{ch}\,(K) = p$. Choose a ring homomorphism $\theta: F \to O_K$ such that the composite

$$F \xrightarrow{\theta} O_K \xrightarrow{x \to \bar{x}} F$$

is the identity map. Choose a prime element π of K. Then, these choices determine a ring isomorphism

$$\varphi: F[[T]] \to O_K$$

such that $\varphi(T) = \pi$ and such that the restriction of φ to F is θ. If one regards $F[[T]]$ as a topological ring with the product topology of the valuation topology of F, and if one endows O_K with the topology defined above, φ becomes a homeomorphism.

§ 7.2. The standard topology of a module over a Noetherian local ring.

For a while, we review a certain topology on a module over a Noetherian local ring, which will be used for the proofs of the above Lemma 1 and Lemma 2.

Let A be any Noetherian local ring and m_A the maximal ideal of A. We call an A-module M a discrete A-module if and only if:

(1) For each $x \in M$, there is a natural number n such that $m_A^n \cdot x = 0$.

We regard any A-module M as a topological group in the following way: As a base of the neighbourhoods of 0 in M, take all A-submodules N of M such that

M/N are discrete A-modules. We call this topology on M, the *standard topology* on M over A. This topology has the following properties (a)-(e).

(a) If M is a finitely generated A-module, this topology coincides with the m_A-adic topology.

(b) If M is an A-module, M is a discrete A-module in the sense of the above (1) if and only if the standard topology on M over A is discrete.

(c) Any A-homomorphism is continuous.

(d) If N is an A-submodule of M, the standard topology on N (resp. on M/N) over A coincides with the restriction (resp. quotient) of the standard topology on M over A.

(e) Let B be another Noetherian local ring and $f: A\to B$ a ring homomorphism such that B is finitely generated as an A-module. Then, for each B-module M, the standard topology on M over B coincides with the standard topology on M over A which is defined by regarding M as an A-module by f.

Now, let I, J and φ be as in §7.1. The above argument can be applied to the I-module O_K/m_K^n $(n\geqq 0)$ because;

LEMMA 3. *Let $n\geqq 0$ and endow O_K/m_K^n with the quotient topology of the topology of O_K defined in §7.1. Then, this topology of O_K/m_K^n coincides with the standard topology over I.*

The proof is easy and left to the reader.

PROOF OF LEMMA 1. We can easily prove that any neighbourhood of zero in O_K contains m_K^n for some $n\geqq 0$. Hence, it is enough to prove that the quotient topology on O_K/m_K^n is independent of the choice of φ. Now, for any ring R of characteristic p, and for any natural number r, denote by $W_r(R)$ the ring of all Witt vectors of length r relative to p over R as in Demazure [4] Ch. III. 1. Note that if $\mathfrak{A}$ is an ideal of a ring A such that $p\in\mathfrak{A}$, and if $r\geqq n-1$, there is a well defined ring homomorphism

$$(1)\qquad W_r(A/\mathfrak{A})\to A/\mathfrak{A}^n\,;\qquad (x_0, x_1, \cdots, x_{r-1})\mapsto\sum_{i=0}^{r-1}p^i\cdot x_i^{p^{r-i}}\,.$$

Take a natural number $r\geqq n-1$, and let α: $W_r(\boldsymbol{F}_p[[X]])\to I/p^nI$ (resp. I/T^nI) and β: $W_r(F)\to O_K/m_K^n$ be the homomorphisms of the above type (1). We have a commutative diagram

$$\begin{array}{ccccc} W_r(\boldsymbol{F}_p[[X]]) & & \xrightarrow{\alpha} & & I/p^nI \text{ (resp. } I/T^nI) \\ \downarrow{\scriptstyle\varphi'} & & & & \downarrow{\scriptstyle\text{by }\varphi} \\ W_r(O_F) & \overset{\subset}{\to} & W_r(F) & \xrightarrow{\beta} & O_K/m_K^n\,, \end{array}$$

where φ' denotes the homomorphism induced by φ. Since α and φ' are finite ring-homomorphisms, the topology of O_K/m_K^n coincides with the standard topology over $W_r(O_F)$ (Lemma 3 and the property (e) of the standard topologies), which is independent of φ.

PROOF OF LEMMA 2. It suffices to prove that for each $n \geqq 0$, the topology on $(O_K/m_K^n)^*$ induced by the topology of O_K/m_K^n is compatible with the group structure. We can take an I-subalgebra R of O_K/m_K^n such that R is finitely generated as an I-module and such that O_K/m_K^n is generated by R as an $I[X^{-1}]$-module. Then, by the definition of the standard topology and by Lemma 3, R is open in O_K/m_K^n. On the other hand, the topology on R induced by the topology of O_K/m_K^n coincides with the standard topology over I (property (d)), and hence with the m_R-adic topology. Since R^* is open in R for the m_R-adic topology of R, it is also open in O_K/m_K^n. Since the m_R-adic topology of R^* is compatible with the group structure of R^*, the map $(O_K/m_K^n)^* \to (O_K/m_K^n)^*$; $x \mapsto x^{-1}$ is continuous on the neighbourhood $R^* \cdot a$ of each element a of $(O_K/m_K^n)^*$ and hence is continuous on $(O_K/m_K^n)^*$. This proves Lemma 2.

§ 7.3. Some properties of the topologies of K^* and $K_2(K)$.

Let F and K be as at the beginning of § 7. The following Lemma is proved easily.

LEMMA 4. *Let L be a finite extension of K. Then, both the inclusion map $K^* \to L^*$ and the norm map $N_{L/K}: L^* \to K^*$ are continuous.*

The following Proposition is important to the study of the topology of $K_2(K)$, but its proof is rather long and continues until the end of § 7.

PROPOSITION 1. *Let $m \geqq 0$ and $n \geqq 0$. Fix a prime element π of K and an element g of O_K such that $\bar{g} \notin F^p$. Then, the homomorphism ν*

$$\nu: U_K \times U_K^{(1)} \to K_2(K)/(K_2(K)^{p^m} \cdot V_K^{(n)})$$
$$(x, y) \mapsto \{x, \pi\}\{y, g\}$$

is surjective and the topology of $K_2(K)/(K_2(K)^{p^m} \cdot V_K^{(n)})$, i.e. the quotient of the topology of $K_2(K)$, coincides with the quotient of the topology of $U_K \times U_K^{(1)}$ with respect to the surjection ν.

COROLLARY. *Let H be a discrete abelian group on which p is nilpotent. Then, for a homomorphism $h: K_2(K) \to H$, the following conditions* (i) *and* (ii) *are equivalent.*

(i) *h is continuous.*

(ii) *For each $a \in K^*$, the map $K^* \to H$; $x \mapsto h(\{x, a\})$ is continuous. Furthermore, $h(V_K^{(n)}) = \{1\}$ for some n.*

This Corollary is deduced from the above Proposition as follows. What we must show is that the condition (i) implies that $h(V_K^{(n)}) = \{1\}$ for some n. But this is shown easily by the method in the proof of Graham [5] 2 Lemma 1.

PROOF OF PROPOSITION 1. To prove that ν is surjective, we may assume $m=1$. Then, this follows from § 2 Prop. 1.

Now, we prove the rest of Prop. 1. It suffices to show that there is a continuous map φ which makes the following diagram commutative.

$$\begin{tikzcd}
K^*/U_K^{(n)} \times K^*/U_K^{(n)} \arrow[rr, "\varphi"] \arrow[dr] & & U_K/U_K^{(n)} \times U_K^{(1)}/U_K^{(n)} \arrow[dl, "\nu"'] \\
& K_2(K)/(K_2(K)^{p^m} \cdot V_K^{(n)}) &
\end{tikzcd}$$

$$(x, y) \mapsto \{x, y\} \qquad\qquad (x, y) \mapsto \{x, \pi\}\{y, g\}$$

The rest of this section is occupied by the proof of the existence of such a map φ. This is easily reduced to the fact: for each $i = 0, 1, \cdots, n-1$, there is a continuous map φ_i which makes the following diagram commutative

$$\begin{tikzcd}
U_K^{(i)}/U_K^{(n)} \times U_K/U_K^{(n)} \arrow[rr, "\varphi_i"] \arrow[dr] & & U_K/U_K^{(n)} \times U_K^{(1)}/U_K^{(n)} \times (U_K^{(i+1)}/U_K^{(n)} \times U_K/U_K^{(n)})^N \arrow[dl] \\
& K_2(K)/(K_2(K)^{p^m} \cdot V_K^{(n)}) &
\end{tikzcd}$$

$$(x, y) \mapsto \{x, y\} \qquad\qquad (u, v, (a_j, b_j)_j) \mapsto \{u, \pi\}\{v, g\} \prod_j \{a_j, b_j\}$$

where N is a suitable natural number which we may take anyway for each i.

Step 1. In this step, we prove that there is such a map φ_i in case $i \neq 0$. It suffices to show that: for each $i' = 0, 1, \cdots, n-1$, there is a continuous map $\varphi_{i,i'}$ which makes the following diagram commutative

$$\begin{tikzcd}
U_K^{(i)}/U_K^{(n)} \times U_K^{(i')}/U_K^{(n)} \arrow[rr, "\varphi_{i,i'}"] \arrow[dr] & & \begin{array}{l} U_K/U_K^{(n)} \times U_K^{(1)}/U_K^{(n)} \times (U_K^{(i+1)}/U_K^{(n)} \times U_K/U_K^{(n)})^M \\ \times (U_K^{(i)}/U_K^{(n)} \times U_K^{(i'+1)}/U_K^{(n)})^N \end{array} \arrow[dl] \\
& K_2(K)/(K_2(K)^{p^m} \cdot V_K^{(n)}) &
\end{tikzcd}$$

$$(x, y) \mapsto \{x, y\} \qquad\qquad (u, v, (a_j, b_j)_j, (a'_j, b'_j)_j) \mapsto \{u, \pi\}\{v, g\} \prod_j \{a_j, b_j\} \cdot \prod_j \{a'_j, b'_j\}$$

where M and N are suitable numbers which we may take anyway for each i'.

The proof of the existence of $\varphi_{i,i'}$ consists of the "case $i'=0$" part and the "case $i' \neq 0$" part. Since these two parts go similarly, we present here only the "$i'=0$" part.

Let r be a natural number such that $r \geqq n-1$ and $r \geqq m$. Let η be the continuous map

$$\eta: U_K^{(i)}/U_K^{(n)} \to F\,; \qquad 1+a\pi^i \mod U_K^{(n)} \mapsto \bar{a} \qquad (a \in O_K)\,.$$

For each $k=0, 1, \cdots, p^r-1$, let ψ_k be the continuous map

$$\psi_k: F \to O_K/m_K^n\,; \qquad \sum_{0 \leq s < p^r} \bar{a}_s^{p^r} \bar{g}^s \mapsto a_k^{p^r} \mod m_K^n \qquad (a_s \in O_K)\,.$$

Let θ be the continuous map

$$\theta: U_K/U_K^{(n)} \to U_K/U_K^{(n)}\,; \qquad x \mod U_K^{(n)} \mapsto \sum_{0 \leq s < p^r} \psi_s(\bar{x}) g^s \mod U_K^{(n)}\,.$$

In what follows, we use the fact that the map $(O_K/m_K^n)^* \to (O_K/m_K^n)^*$; $x \mapsto x^{-1}$ is continuous, making no reference to it. In what follows, the notation $\equiv$ means the congruence modulo $K_2(K)^{p^m} \cdot V^{(n)}$. It is easily seen that there are continuous maps

$$\begin{aligned}
f_1&: U_K^{(i)}/U_K^{(n)} \to U_K^{(i+1)}/U_K^{(n)}\,,\\
f_2&: U_K/U_K^{(n)} \to U_K^{(1)}/U_K^{(n)}\,,\\
f_3&: U_K^{(i)}/U_K^{(n)} \times U_K/U_K^{(n)} \to U_K^{(i+1)}/U_K^{(n)}\,,\\
f_4&: U_K^{(i)}/U_K^{(n)} \times U_K/U_K^{(n)} \to U_K/U_K^{(n)}\,,\\
f_5&: U_K^{(i)}/U_K^{(n)} \times U_K/U_K^{(n)} \to U_K^{(1)}/U_K^{(n)}
\end{aligned}$$

which satisfy the following congruences (i)-(iv) for all $x \in U_K^{(i)}/U_K^{(n)}$ and $y \in U_K/U_K^{(n)}$.

$$\text{(i)} \qquad \{x, y\} \equiv \Big(\prod_{0 \leq s < p^r} \{1+\psi_s(\eta(x))g^s\pi^i, y\}\Big)\{f_1(x), y\}$$

$$\text{(ii)} \qquad \{1+\psi_s(\eta(x))g^s\pi^i, y\} \equiv \{1+\psi_s(\eta(x))g^s\pi^i, \theta(y)\} \cdot \{1+\psi_s(\eta(x))g^s\pi^i, f_2(y)\}$$

$$\begin{aligned}
\text{(iii)} \qquad \{1+\psi_s(\eta(x))g^s\pi^i, \theta(y)\} &\equiv \{1+\psi_s(\eta(x))g^s\pi^i, -g^{-s}\pi^{-i}\theta(y)\}\\
&\equiv \{1+\psi_s(\eta(x))g^s\pi^i\theta(y)^{-1}\theta(y), -g^{-s}\pi^{-i}\theta(y)\}\\
&\equiv \Big(\prod_{0 \leq k < p^r} \{1+\psi_s(\eta(x))g^s\pi^i\theta(y)^{-1}\psi_k(\bar{y})g^k, -g^{-s}\pi^{-i}\theta(y)\}\Big)\\
&\quad \cdot \{f_3(x, y), -g^{-s}\theta(y)\} \cdot \{f_4(x, y), \pi\}
\end{aligned}$$

$$\begin{aligned}
\text{(iv)} \qquad \{1+\psi_s(\eta(x))g^s\pi^i\theta(y)^{-1}\psi_k(\bar{y})g^k, -g^{-s}\pi^{-i}\theta(y)\}&\\
\equiv \{1+\psi_s(\eta(x))g^s\pi^i\theta(y)^{-1}\psi_k(\bar{y})g^k, g^k\}&\\
\equiv \{f_5(x, y), g\}\,.&
\end{aligned}$$

This shows the existence of $\varphi_{i,0}$.

Step 2. In this step, we prove the existence of φ_0. In the case $\mathrm{ch}\,(K)=p$, the proof is easy. In this case, we may assume $K=F((T))$. For each $x \in U_K/U_K^{(n)}$, let $f(x) \in F^*$ and $g(x) \in U_K^{(1)}/U_K^{(n)}$ be the elements such that $x=f(x)g(x)$. Since $K_2(F)$ is p-divisible, $K_2(F) \subset K_2(K)^{p^m}$. Hence, for all $x, y \in U_K/U_K^{(n)}$,

$$\{x, y\} \equiv \{g(x), y\}\{g(y), f(x)^{-1}\} \mod K_2(K)^{p^m} \cdot V_K^{(n)}\,,$$

which proves the existence of φ_0.

Next, suppose $\mathrm{ch}(K)=0$. Since we have proved the existence of φ_i for $i\neq 0$, we can define a continuous map φ_+ which makes the following diagram commutative.

$$\begin{tikzcd}[row sep=small]
U_K^{(1)}/U_K^{(n)}\times U_K/U_K^{(n)} \arrow[rr, "\varphi_+"] \arrow[ddr] & & U_K/U_K^{(n)}\times U_K^{(1)}/U_K^{(n)} \arrow[ddl, "\nu"'] \\
(x, y) \arrow[d, mapsto] & & (x, y) \arrow[d, mapsto] \\
\{x, y\} & K_2(K)/(K_2(K)^{p^m}\cdot V_K^{(n)}) & \{x, \pi\}\{y, g\}
\end{tikzcd}$$

Now, we need the following Lemma.

LEMMA 5. *Let K and F be as at the beginning of this §7. Suppose L is an extension of K of degree p such that the residue extension is inseparable and of degree p. Suppose $m\geqq 0$, $n\geqq 0$, $n'\geqq n$, $N_{L/K}U_L^{(n')}\subset U_K^{(n)}\cdot(K^*)^{p^m}$, and $N_{L/K}V_L^{(n')}\subset V_K^{(n)}K_2(K)^{p^m}$. Fix a prime element π of K. Then, for a suitable number N, there is a continuous map ξ which makes the following diagram commutative.*

$$\begin{tikzcd}[row sep=small]
U_L^{(1)}/U_L^{(n')}\times U_L/U_L^{(n')} \arrow[r, "\xi"] \arrow[ddd] & U_K/U_K^{(n)}\times(U_K^{(1)}/U_K^{(n)}\times U_K/U_K^{(n)})^N \arrow[ddd] \\
(x, y) \arrow[d, mapsto] & (u, (x_i, y_i)_i) \arrow[d, mapsto] \\
\{x, y\} & \{u, \pi\}\prod_i\{x_i, y_i\} \\
K_2(L)/(K_2(L)^{p^m}\cdot V_L^{(n')}) \arrow[r, "N_{L/K}"] & K_2(K)/(K_2(K)^{p^m}\cdot V_K^{(n)})\,.
\end{tikzcd}$$

PROOF. By applying the existence of φ_+ to L, and by Lemma 4, it suffices to prove the following assertion:

"Fix an element h of O_L such that $\bar{h}\notin F$. Then, for a suitable number N, there is a continuous map ξ which makes the following diagram

$$\begin{tikzcd}[row sep=small]
U_L^{(1)}/U_L^{(n')} \arrow[rr, "\xi"] \arrow[ddr] & & U_K/U_K^{(n)}\times(U_K^{(1)}/U_K^{(n)}\times U_K/U_K^{(n)})^N \arrow[ddl] \\
x \arrow[d, mapsto] & & (u, (x_i, y_i)_i) \arrow[d, mapsto] \\
N_{L/K}\{x, h\} & K_2(K)/(K_2(K)^{p^m}\cdot V_K^{(n)}) & \{u, \pi\}\prod_i\{x_i, y_i\}
\end{tikzcd}$$

commutative."

This assertion is proved as follows. Let r be a number such that $r\geqq m$ and $r\geqq n'-1$. Let

$$I=\{(i, j)\mid 1\leqq i<n',\ 0\leqq j<p^{r+1},\ \text{and } j \text{ is prime to } p\}\,.$$

As is easily seen, the map

$$U_K^{(1)}/U_K^{(n')}\times F^I\to U_L^{(1)}/U_L^{(n')}$$
$$(x \bmod U_K^{(n')},\ (\bar{a}_{ij})_{(i,j)\in I})\mapsto x\cdot\prod_{(i,j)\in I}(1-a_{ij}^{p^r}\pi^i h^j) \bmod U_L^{(n')} \qquad (a_{ij}\in O_K)$$

is a homeomorphism. But if $x\in U_K^{(1)}/U_K^{(n')}$ and $a\in O_K$, and if $(i, j)\in I$,

$$N_{L/K}\{x, h\}=\{x, N_{L/K}h\}\,, \qquad \text{and}$$
$$\begin{aligned}N_{L/K}\{1-a^{p^r}\pi^i h^j, h\} &\equiv N_{L/K}\{1-a^{p^r}\pi^i h^j, h^j\}^{1/j}\\ &\equiv N_{L/K}\{1-a^{p^r}\pi^i h^j, \pi\}^{-i/j}\\ &\equiv \{N_{L/K}(1-a^{p^r}\pi^i h^j)^{-i/j}, \pi\} \mod K_2(K)^{p^m}\cdot V_K^{(n)}\,.\end{aligned}$$

(Here, the action of $1/j$ is well defined, for the group $K_2(K)/(K_2(K)^{p^m}\cdot V_K^{(n)})$ is annihilated by p^m, and j is prime to p.) Hence, by Lemma 4, we have the above assertion. Q.E.D.

Now, we return to the proof of the existence of φ_0 in the case $\mathrm{ch}(K)=0$. Take extensions L_i $0\leqq i\leqq m$ of K such that $K=L_0\subset L_1\subset\cdots\subset L_m$, and such that $[L_i: L_{i-1}]=p$ and the residue extension of L_i/L_{i-1} is inseparable and of degree p for each $i=1, \cdots, m$. Take numbers $n_0, n_1, \cdots, n_m$ such that $n=n_0\leqq n_1\leqq\cdots\leqq n_m$, $N_{L_i/L_{i-1}}U_{L_i}^{(n_i)}\subset U_{L_{i-1}}^{(n_{i-1})}$ and $N_{L_i/L_{i-1}}V_{L_i}^{(n_i)}\subset K_2(L_{i-1})^{p^m}\cdot V_{L_{i-1}}^{(n_{i-1})}$ for each i. Such numbers exist since the assumption $\mathrm{ch}(K)=0$ implies $V_{L_i}^{(N)}\subset K_2(L_i)^{p^m}$ for sufficiently large N. Fix an element h of O_{L_m} such that the residue field of L_m is $F(\bar{h})$. Let r be a number such that $r-m\geqq n_m-1$ and $r\geqq m$. Let ψ be the continuous map $F\to O_{L_m}/(m_{L_m})^{n_m}$;

$$\psi\colon \sum_{0\leqq s<p^r}\bar{a}_s^{p^r}\cdot\bar{h}^{p^m s}\mapsto\sum_{0\leqq s<p^r}a_s^{p^{r-m}}\cdot h^s \mod (m_{L_m})^{n_m} \qquad (a_s\in O_K)\,.$$

Then, there are a natural number M and continuous maps

$$\begin{aligned}&f_1\colon U_K/U_K^{(n)}\to U_K^{(1)}/U_K^{(n)}\\ &f_2\colon U_K/U_K^{(n)}\to U_{L_m}^{(1)}/U_{L_m}^{(n_m)}\\ &f_3 \text{ and } f_{4,j}\ \ 1\leqq j\leqq M\colon\ U_K/U_K^{(n)}\times U_K/U_K^{(n)}\to U_K/U_K^{(n)}\\ &f_{5,j}\ \ 1\leqq j\leqq M\colon\ U_K/U_K^{(n)}\times U_K/U_K^{(n)}\to U_K^{(1)}/U_K^{(n)}\end{aligned}$$

such that for all $x, y\in U_K/U_K^{(n)}$,

$$\begin{aligned}&\{x, y\}\equiv\{N_{L_m/K}\psi(\bar{x}), y\}\{f_1(x), y\}\,, \qquad \text{and}\\ &N_{L_m/K}\{\psi(\bar{x}), y\}\equiv N_{L_m/K}\{\psi(\bar{x}), \psi(\bar{y})^{p^m}\}\cdot N_{L_m/K}\{\psi(\bar{x}), f_2(y)\} \quad \text{(the first term vanishes)}\\ &\qquad\equiv\{f_3(x, y), \pi\}\prod_{j=1}^{M}\{f_{4,j}(x, y), f_{5,j}(x, y)\}\,.\end{aligned}$$

Indeed, f_3, $f_{4,j}$ and $f_{5,j}$ exist by Lemma 5 and by induction on m. This shows the existence of φ_0. Q.E.D.

§ 8. The existence theorem.

In this section, F denotes a field of characteristic $p>0$ which is complete with respect to a discrete valuation and with *finite* residue field, and K denotes a field of characteristic 0 which is complete with respect to a discrete valuation and with residue field F.

Let $\mathrm{Hom}_c(K^*, \boldsymbol{Q}/\boldsymbol{Z})$ be the group of all continuous homomorphisms from K^* to $\boldsymbol{Q}/\boldsymbol{Z}$ with respect to the topology on K^* defined in §7 and with respect to the discrete topology on $\boldsymbol{Q}/\boldsymbol{Z}$. Define $\mathrm{Hom}_c(K_2(K), \boldsymbol{Q}/\boldsymbol{Z})$ similarly.

In §6, we defined canonical injective homomorphisms

$$\Phi: \mathrm{Br}(K) \to \mathrm{Hom}(K^*, \boldsymbol{Q}/\boldsymbol{Z}), \qquad \text{and} \qquad \Psi^\vee: X_K \to \mathrm{Hom}(K_2(K), \boldsymbol{Q}/\boldsymbol{Z}).$$

(We denote by $\Psi^\vee$ the dual homomorphism of Ψ.) The aim of this section is to prove the following Th. 1 and Th. 2 which determine the images of Φ and $\Psi^\vee$.

THEOREM 1. $\mathrm{Im}(\Phi) = \mathrm{Hom}_c(K^*, \boldsymbol{Q}/\boldsymbol{Z})$.

THEOREM 2. $\mathrm{Im}(\Psi^\vee) = \mathrm{Hom}_c(K_2(K), \boldsymbol{Q}/\boldsymbol{Z})$.

COROLLARY 1. (i) $\mathrm{Br}(K) \cong \mathrm{Hom}_c(K^*, \boldsymbol{Q}/\boldsymbol{Z})$.

(ii) $X_K \cong \mathrm{Hom}_c(K_2(K), \boldsymbol{Q}/\boldsymbol{Z})$.

(iii) *The group X_K is isomorphic to the group of all continuous maps $h: K^* \times K^* \to \boldsymbol{Q}/\boldsymbol{Z}$ such that*:

$$h(xy, z) = h(x, z)h(y, z)$$
$$h(x, yz) = h(x, y)h(x, z)$$
$$h(1-u, u) = 0$$

or all $x, y, z \in K^*$ and for all $u \in K$ such that $u \neq 0$ and $u \neq 1$.

COROLLARY 2. *The map $L \mapsto N_{L/K}K_2(L)$ is a bijection from the set of all finite abelian extensions of K in a fixed algebraic closure of K to the set of all open subgroups of $K_2(K)$ of finite indices.*

This Corollary 2 follows from Th. 2 and from §6 Th. 2 immediately.

PROOF OF THEOREM 1. (Step 1 and Step 2.)

Step 1. In Step 1, we prove that $\Phi(w): K^* \to \boldsymbol{Q}/\boldsymbol{Z}$ is continuous for all $w \in \mathrm{Br}(K)$. The following proof is applicable with no change to the case $\mathrm{ch}(K) = p$ in Ch. II.

Step 1.1. First, we prove this in the case $w \in \mathrm{Br}(K_{\mathrm{nr}}/K)$. Then, $w = w' + (\chi, \pi)$ for some w', χ, and π where w' belongs to the image of the canonical homomorphism $\mathrm{Br}(F) \to \mathrm{Br}(K_{\mathrm{nr}}/K)$, χ is an unramified element of X_K and π is a prime element of K. Hence, it suffices to prove that $\Phi(w')$ and $\Phi((\chi, \pi))$ are continuous. By §6 Th. 1, it suffices to prove that $\mathrm{Nrd}(w'/K)$ and $\mathrm{Nrd}((\chi, \pi)/K)$ are open in K^*. But (cf. §6 Step 2 of Proof of $\mathrm{Ker}(\Phi(w)) = \mathrm{Nrd}(w/K)$) the former group contains U_K and the latter contains all elements x of U_K such that $\bar{x}\in N_{F'/F}F'^*$ where F' denotes the cyclic extension of F corresponding to χ. Hence the former is open immediately, and the latter is open since $N_{F'/F}F'^*$ is open in F^* by the ordinary local class field theory.

Step 1.2. Here, we prove that $\mathrm{Br}(K_{\mathrm{nr}}/K)$ is a direct summand of $\mathrm{Br}(K)$. Since $\mathrm{Br}(F)$ and X_F are p-divisible, and since there is an exact sequence

$$0\to\mathrm{Br}(F)\to\mathrm{Br}(K_{\mathrm{nr}}/K)\to X_F\to 0\ ,$$

it follows that $\mathrm{Br}(K_{\mathrm{nr}}/K)$ is p-divisible. By this and by the fact that the "prime to p" part of the torsion group $\mathrm{Br}(K)$ is contained in $\mathrm{Br}(K_{\mathrm{nr}}/K)$, $\mathrm{Br}(K_{\mathrm{nr}}/K)$ is a direct summand of $\mathrm{Br}(K)$.

Step 1.3. (The main part of Step 1.) By Step 1.2, there is a subgroup M of $\mathrm{Br}(K)$ such that $\mathrm{Br}(K)=\mathrm{Br}(K_{\mathrm{nr}}/K)\oplus M$. Suppose $w\in M$. In what follows, we prove that $\Phi(w)$ is continuous.

Let p^m be the order of w. We denote by A the set

$$\{0, 1, \cdots, p-1\}\times\{0, 1, \cdots, p-1\}\ .$$

For each $i=0, 1, \cdots, m$, and for each $\alpha\in A^i=\overset{(i\text{ times})}{A\times\cdots\times A}$, take a finite extension K_α of K inductively, as follows. If $i=0$ and if e is the unique element of A^0, let $K_e=K$. Suppose $0\leqq i<m$ and $(a_1, \cdots, a_i)\in A^i$. Suppose that $L=K_{(a_1,\cdots,a_i)}$ is taken already, such that $[L: K]=p^i$ and the order of w_L is p^{m-i}. Let $b=(j, k)\in A$. We take $K_{(a_1,\cdots,a_i,b)}$, which we denote by L_b, as follows.

By the assumption $w\in M$, $p^{m-i-1}w\notin\mathrm{Br}(L_{\mathrm{nr}}/L)$. Hence, by § 4 Lemma 5, the division algebra D over L which corresponds to $p^{m-i-1}w_L$ satisfies the assumptions at the beginning of § 1. Fix a prime element π of D and an element g of O_D such that $\bar{g}$ does not belong to the residue field of L. Then, we take L_b to be a (commutative) subfield of D such that $L\subset L_b$, $\pi^j g^k\in L_b$, and $[L_b: L]=p$.

Then, $[L_b: K]=p^{i+1}$ and the order of w_{L_b} is p^{m-i-1}. In this way, we can take K_α for each $\alpha\in A^i$ such that $0\leqq i\leqq m$.

Next, for each $\alpha\in A^i$ such that $0\leqq i\leqq m$, we take a natural number n_α inductively, as follows. If $i=0$ and if e is the unique element of A^0, let n_e be a natural number which satisfies $U_K^{(n_e)}\subset\mathrm{Nrd}(w/K)$. (The existence of n_e follows from the existence of a separable extension of K which decomposes w. We omit the details.) If $0\leqq i\leqq m$ and $\alpha\in A^i$, and if $b\in A$, let $n_{\alpha b}$ be a natural number which satisfies

$$N_{K_{\alpha b}/K_\alpha}(U_{K_{\alpha b}}^{(n_{\alpha b})})\subset U_{K_\alpha}^{(n_\alpha)}\ .$$

Here, if $\alpha=(a_1, \cdots, a_i)$, αb denotes $(a_1, \cdots, a_i, b)$. For each $i=0, \cdots, m-1$, let

$$N_i: \prod_{\alpha\in A^{i+1}} (K_\alpha^*/U_{K_\alpha}^{(n_\alpha)}) \to \prod_{\alpha\in A^i} (K_\alpha^*/U_{K_\alpha}^{(n_\alpha)})$$

be the sum of the homomorphisms:

$$(K_{\alpha b}^*/U_{K_{\alpha b}}^{(n_{\alpha b})} \xrightarrow{\text{norm map}} K_\alpha^*/U_{K_\alpha}^{(n_\alpha)})_{\alpha\in A^i, b\in A} .$$

Now, our task is to prove that each N_i $0\leqq i\leqq m-1$ is an open map. If we can prove this, the image of the composite $N_0\circ N_1\circ\cdots\circ N_{m-1}$ is an open subgroup of $K^*/U_K^{(n_e)}$, where e denotes the unique element of A^0. Since $U_K^{(n_e)}\subset \mathrm{Nrd}\,(w/K)$ and since $\mathrm{Im}\,(N_0\circ N_1\circ\cdots\circ N_{m-1})$ is contained in $\mathrm{Nrd}\,(w/K)/U_K^{(n_e)}$, it follows that $\mathrm{Nrd}\,(w/K)$ is open in K^*. Since $\mathrm{Ker}\,(\Phi(w))=\mathrm{Nrd}\,(w/K)$, we have the consequence that $\Phi(w)$ is continuous. Hence, it suffices to prove the following Lemma.

LEMMA 1. *Suppose $w\in \mathrm{Br}\,(K)_p$ and $w\notin \mathrm{Br}\,(K_{\mathrm{nr}}/K)$. Let D be the division algebra over K which corresponds to w. (Then, by § 4 Lemma 5, D satisfies the assumptions at the beginning of § 1.) Let π be a prime element of D and let g be an element of O_D such that $\bar{g}\notin F$. For each*

$$b=(j, k)\in A=\{0, 1, \cdots, p-1\}\times\{0, 1, \cdots, p-1\} ,$$

let K_b a (commutative) subfield of D such that $\pi^j g^k\in K_b$ and such that $[K_b: K]=p$. Let $n\geqq 0$. For each $b\in A$, let n_b be a natural number such that $N_{K_b/K}U_{K_b}^{(n_b)}\subset U_K^{(n)}$. Then, there are a neighbourhood V of 1 in $K^*/U_K^{(n)}$ and a continuous map $\varphi=(\varphi_b)_{b\in A}: V\to \prod_{b\in A} K_b^*/U_{K_b}^{(n_b)}$ such that $x\equiv \prod_{b\in A} N_{K_b/K}\varphi_b(x) \bmod U_K^{(n)}$ for all $x\in V$.

PROOF. We denote the group $\prod_{b\in A} K_b^*/U_{K_b}^{(n_b)}$ by G. Let N be the homomorphism

$$N: G\to K^*/U_K^{(n)} ; \qquad (x_b)_{b\in A}\mapsto \prod_{b\in A} N_{K_b/K}x_b .$$

It suffices to show that for each $i=0, 1, \cdots, n-1$, there are a neighbourhood V of 1 in $U_K^{(i)}/U_K^{(n)}$ and a continuous map φ which make the following diagram commutative.

$$\begin{array}{ccc} V & \xrightarrow{\varphi} & G\times(U_K^{(i+1)}/U_K^{(n)}) \\ & \searrow \quad \swarrow & (x, y) \\ & & \downarrow \\ & K^*/U_K^{(n)} & N(x)\cdot y . \end{array}$$

The proof of this fact will continue until the end of Step 1.

First, let t be the ramification number of D (§ 1). Let i be an integer such that $0\leqq i\leqq n-1$. Define i' as follows. If $0\leqq i\leqq t$, let $i'=i$. If $i\geqq t$, let $i'=t+p(i-t)$.

As was shown in § 1, $\mathrm{Nrd}_{D/K}\, U_D^{(i')}\subset U_K^{(i)}$ and $\mathrm{Nrd}_{D/K}\, U_D^{(i'+1)}\subset U_K^{(i+1)}$. Let

$$\mathrm{Nrd}_i\colon U_D^{(i')}/U_D^{(i'+1)}\to U_K^{(i)}/U_K^{(i+1)}$$

be the homomorphism induced by $\mathrm{Nrd}_{D/K}$. By § 1 Prop. 3, there are a neighbourhood W of 1 in $U_K^{(i)}/U_K^{(i+1)}$ and a continuous map $\theta\colon W\to U_D^{(i')}/U_D^{(i'+1)}$ such that the composite $\mathrm{Nrd}_i\circ\theta$ coincides with the inclusion map $W\overset{\subset}{\to}U_K^{(i)}/U_K^{(i+1)}$. Here the topology of $U_K^{(i)}/U_K^{(i+1)}$ is the one induced by the topology of K^* defined in § 7; in other words, $U_K^{(i)}/U_K^{(i+1)}$ is homeomorphic to F^* (when $i=0$), or to F (when $i>0$) in the usual way. And the topology of $U_D^{(i')}/U_D^{(i'+1)}$ is the similar one; in other words, if C denotes the residue field of D, $U_D^{(i')}/U_D^{(i'+1)}$ is homeomorphic to C^* or C in the usual way. (The existence of W and that of θ is not clear only when $i=t$. But in this case Nrd_t is isomorphic to

$$(1-\gamma)\circ\mathfrak{F}\colon \Omega_C\to\Omega_F$$

as a continuous homomorphism between topological groups. Fix $h\in F$ such that $h\notin F^p$. Let $W=m_F\cdot dh/h\subset\Omega_F$ and let θ be the map

$$\theta\colon W\to\Omega_C\,;\qquad x\,dh/h\mapsto(-x-x^p-x^{p^2}-\cdots)\,d(h^{1/p})/(h^{1/p})\,.$$

Then, $(1-\gamma)\circ\mathfrak{F}\circ\theta$ coincides with the inclusion map $W\overset{\subset}{\to}\Omega_F$. This proves the existence of W and θ.)

Next, let r be a sufficiently large number. We define a continuous map λ: $U_D^{(i')}/U_D^{(i'+1)}\to G$ as follows. If $i=0$, define λ by:

the b-component of $\lambda\Big(\sum_{0\leqq s<p^{r+1}}\bar{a}_s^{p^r}\cdot\bar{g}^s\Big)$, where $b\in A$ and each $a_s\in O_K$,

is equal to $\begin{cases}\sum_{0\leqq s<p^{r+1}}a_s^{p^r}\cdot g^s \mod U_{K_b}^{(n_b)} & \text{if } b=(0,1)\,.\\ 1 & \text{otherwise}\,.\end{cases}$

If $i>0$ and $p\,|\,i'$, fix $c\in K$ such that $v_K(c)=i'/p$ and define λ by:

the b-component of $\lambda\Big(1+c\sum_{0\leqq s<p^{r+1}}a_s^{p^r}\cdot g^s \mod U_D^{(i'+1)}\Big)$, where $b\in A$ and each $a_s\in O_K$, is equal to $\begin{cases}1+c\sum_{0\leqq s<p^{r+1}}a_s^{p^r}\cdot g^s \mod U_{K_b}^{(n_b)} & \text{if } b=(0,1)\,.\\ 1 & \text{otherwise}\,.\end{cases}$

Suppose $i>0$ and i' is prime to p. Let d be the integer such that $0\leqq d<p$ and $d\equiv i' \mod p$. Fix $c\in K$ such that $v_K(c)=(i'-d)/p$ and fix $h\in O_K$ such that $\bar{h}=\bar{g}^p$. Define λ by:

the (j, k)-component of $\lambda\left(1+c\cdot\pi^d \sum_{0\leqq s<p^{r+1}} a_s^{p^r}\cdot g^s \mod U_D^{(i'+1)}\right)$,

where $(j, k)\in A$ and each $a_s\in O_K$, is equal to

$$\begin{cases} 1+c\cdot\pi^d\cdot g^k \sum\limits_{\substack{0\leqq s<p^{r+1} \\ s\equiv k \bmod p}} a_s^{p^r}\cdot h^{(s-k)/p} \mod U_K^{(n_{(d,k)})} & \text{if } j=d . \\ 1 & \text{otherwise} . \end{cases}$$

Lastly, we define V to be the inverse image of W under the canonical projection $\tau: U_K^{(i)}/U_K^{(n)}\to U_K^{(i)}/U_K^{(i+1)}$. We can define $\varphi: V\to G\times(U_K^{(i+1)}/U_K^{(n)})$ by

$$\varphi(x)=(\lambda\theta\tau(x), x/N(\lambda\theta\tau(x))) .$$

This completes the proof of Lemma 1.

Step 2. Conversely, we prove that each continuous homomorphism $\varphi: K^*\to \boldsymbol{Q}/\boldsymbol{Z}$ has the form $\Phi(w)$ for some $w\in \mathrm{Br}(K)$. (Step 2.1–Step 2.5.)

Step 2.1. First, we prove that φ is of finite order. By the definition of the topology of K^*, $\varphi(U_K^{(n)})=0$ for some n. Hence, if m is sufficiently large, $p^m\varphi(U_K^{(1)})\subset\varphi(U_K^{(n)})=0$. Thus, $p^m\varphi$ is factorized as

$$K^*\to K^*/U_K^{(1)}\to \boldsymbol{Q}/\boldsymbol{Z} .$$

Since $K^*/U_K^{(1)}\cong \boldsymbol{Z}\times\boldsymbol{Z}\times U_F$ as a topological group and since U_F is compact, the image of $p^m\varphi$ is finite. Hence φ is of finite order.

Step 2.2. By Step 2.1, it suffices to study the following two cases.

Case 1. The order of φ is prime to p.

Case 2. The order of φ is a power of p.

The study of Case 1 is easy as follows. If n is prime to p, Φ induces an isomorphism

$$\mathrm{Br}(K)_n \xrightarrow{\cong} \mathrm{Hom}\left(K^*, \frac{1}{n}\boldsymbol{Z}\middle/\boldsymbol{Z}\right)$$

as was shown in §6 Proof of the injectivity of Φ and $\Psi^\vee$.

Hence it remains to study Case 2, which will be done in Step 2.3, 2.4, and 2.5.

Step 2.3. We prove here that we may assume that K contains a primitive p-th root ζ of 1. Let $L=K(\zeta)$. Let $\varphi: K^*\to\boldsymbol{Q}/\boldsymbol{Z}$ be a continuous homomorphism and suppose that the order of φ is a power of p. Then, the composite

$$\varphi\circ N_{L/K}: L^*\xrightarrow{N_{L/K}} K^*\xrightarrow{\varphi}\boldsymbol{Q}/\boldsymbol{Z}$$

is continuous by §7 Lemma 4. Assume that our problem is solved for L. Then, there is $w\in \mathrm{Br}(L)$ such that $\varphi\circ N_{L/K}=\Phi_L(w)$. By the formula §6(6), it follows $[L:K]\varphi=\Phi_K(\mathrm{Cor}_{L/K}(w))$. Since $[L:K]$ is prime to p, $\varphi=\Phi_K(w')$ for some $w'\in \mathrm{Br}(K)$. (Φ_K or Φ_L denotes Φ of each field.)

Step 2.4. In Step 2.4 and in Step 2.5, we assume that K contains a primitive p-th root ζ of 1. In Step 2.4, we prove that we may assume $p\varphi=0$. We use induction on the order of φ. Assume that our problem is solved in case $p\varphi=0$. Let $\varphi: K^*\to \boldsymbol{Q}/\boldsymbol{Z}$ be a continuous homomorphism and suppose that the order of φ is p^r, $r\geqq 1$. By the assumption, $p^{r-1}\varphi$ has the form $\Phi_K(w)$ for some $w\in \mathrm{Br}(K)$. Since $\Phi_K(pw)=p^r\varphi=0$, $pw=0$ by the injectivity of Φ_K (§6 Th. 1). Hence, the isomorphism

$$h'_p: K_2(K)/K_2(K)^p\cong \mathrm{Br}(K)_p \qquad (\S 3 \text{ Theorem } 1)$$

and §6 Lemma 1 show that there is a cyclic extension L of K of degree p such that $w_L=0$. By the formula §6(5),

$$p^{r-1}\varphi\circ N_{L/K}=\Phi_L(w_L)=0: L^*\to \boldsymbol{Q}/\boldsymbol{Z}\,.$$

Hence, by applying the hypothesis of induction to L, there is $w'\in \mathrm{Br}(L)$ such that $\varphi\circ N_{L/K}=\Phi_L(w')$.

Now, let $G=\mathrm{Gal}(L/K)$. For any G-module A, let

$$A^G=\{x\in A\,|\,\sigma(x)=x \text{ for all } \sigma\in G\}\,.$$

Clearly, $\mathrm{Br}(L)$ has a natural G-module structure. For all $\sigma\in G$, $\sigma(w')=w'$ follows from

$$\begin{aligned}\Phi_L(\sigma(w'))&=\Phi_L(w')\circ\sigma^{-1} \qquad (\text{here, we used the formula } \S 6\,(5))\\ &=\varphi\circ N_{L/K}\circ\sigma^{-1}=\varphi\circ N_{L/K}=\Phi_L(w')\end{aligned}$$

and from the injectivity of Φ_L. Thus, $w'\in \mathrm{Br}(L)^G$. Since G is cyclic, we have an exact sequence

$$0\to H^2(G, L^*)\to \mathrm{Br}(K)\to \mathrm{Br}(L)^G\to 0\,.$$

It follows that there is $w''\in \mathrm{Br}(K)$ such that $w'=(w'')_L$. Hence $\varphi-\Phi_K(w'')$ annihilates $N_{L/K}L^*$, and so, $(K^*)^p$. Hence, by the assumption, $\varphi-\Phi_K(w'')=\Phi_K(w''')$ for some $w'''\in \mathrm{Br}(K)$.

Step 2.5. Suppose that K contains a primitive p-th root ζ of 1 and that $\varphi: K^*/(K^*)^p\to(1/p)\boldsymbol{Z}/\boldsymbol{Z}$ is a continuous homomorphism. We prove that φ has the form $\Phi(w)$ for some $w\in \mathrm{Br}(K)_p$. Look at Figure 3 in §6 once more. In this Step 2.5, let the notation $U_K^{(0)}$ stand for K^* (not for U_K) contrary to the custom. By induction on i starting with $i=(e_Kp/(p-1))+1$ (until $i=0$), we prove that for each i, each continuous homomorphism

$$\psi : U_K^{(i)}(K^*)^p/(K^*)^p \to \frac{1}{p}\boldsymbol{Z}/\boldsymbol{Z}$$

is the restriction $\Phi(w) \mid U_K^{(i)}(K^*)^p/(K^*)^p$ for some $w \in \mathrm{Br}(K)_p$.

If $i=(e_K p/(p-1))+1$, this assertion is clear. Suppose that $e_K p/(p-1) \geqq i \geqq 0$. By our induction, we may assume that the restriction $\psi \mid U_K^{(i+1)}(K^*)^p/(K^*)^p$ of ψ is the restriction of $\Phi(w)$ for some $w \in \mathrm{Br}(K)_p$. Since $\psi-\Phi(w)$ annihilates $U_K^{(i+1)}(K^*)^p/(K^*)^p$, $\psi-\Phi(w)$ induces a continuous homomorphism: $U_K^{(i)}(K^*)^p/U_K^{(i+1)}(K^*)^p \to (1/p)\boldsymbol{Z}/\boldsymbol{Z}$. For each i, let A_i be the topological group

$$\begin{array}{ll} \boldsymbol{Z}/p\boldsymbol{Z} \times F^*/(F^*)^p & \text{if } i=0\,, \\ F/F^p & \text{if } 0<i<e_K p/(p-1) \text{ and } p \mid i\,, \\ F & \text{if } 0<i<e_K p/(p-1) \text{ and } i \text{ is prime to } p\,, \\ F/\{x^p-x \mid x \in F\} & \text{if } i=e_K p/(p-1)\,. \end{array}$$

Then, for each i, the homomorphism $A_i \to U_K^{(i)}(K^*)^p/U_K^{(i+1)}(K^*)^p$ which is given in §6 Proof of the injectivity of Φ and $\Psi^\vee$, is continuous. Hence the composite

$$A_i \to U_K^{(i)}(K^*)^p/U_K^{(i+1)}(K^*)^p \xrightarrow{\text{by } \psi-\Phi(w)} \frac{1}{p}\boldsymbol{Z}/\boldsymbol{Z}$$

is continuous. By duality, this composite is induced by an element of B_i where B_i is the group

$$\begin{array}{ll} \boldsymbol{Z}/p\boldsymbol{Z} \times F/\{x^p-x \mid x \in F\} & \text{if } i=0\,, \\ F/F^p & \text{if } 0<i<e_K p/(p-1) \text{ and } p \mid i\,, \\ \Omega_F & \text{if } 0<i<e_K p/(p-1) \text{ and } i \text{ is prime to } p\,, \\ F^*/(F^*)^p & \text{if } i=e_K p/(p-1)\,. \end{array}$$

Hence, the isomorphism $h_p' : K_2(K)/K_2(K)^p \to \mathrm{Br}(K)_p$ and the explicit calculation in §6 of the homomorphism

$$V_K^{(n)}K_2(K)^p/V_K^{(n+1)}K_2(K)^p \otimes U_K^{(i)}(K^*)^p/U_K^{(i+1)}(K^*)^p \to \frac{1}{p}\boldsymbol{Z}/\boldsymbol{Z} \qquad (n+i=e_K p/(p-1))$$

show that there is some $w' \in \mathrm{Br}(K)_p$ such that $\psi-\Phi(w)=\Phi(w')$ on $U_K^{(i)}(K^*)^p/(K^*)^p$.

Q.E.D.

PROOF OF THEOREM 2. (Step 1 and Step 2.)

Step 1. First, we prove that $\Psi^\vee(\chi)$ is continuous for each $\chi \in X_K$. It suffices to prove this in the following two cases.

Case 1. Suppose the order n of χ is prime to p. It suffices to show that

$K_2(K)/K_2(K)^n$ is a discrete group. Let k be the residue field of F. Let

π be a fixed prime element of K,
$s: K_2(K)\to K_2(F)$ be the unique homomorphism such that
$s(\{x\pi^i, y\pi^j\})=\{\bar{x}, \bar{y}\}$ for all $x, y\in U_K$ and for all i, j,
$t: K_2(K)\to F^*$, and $t': K_2(F)\to k^*$ be the tame symbols.

By the proof of § 6 Lemma 1, and by the definition of the topology of $K_2(K)$, the homomorphism

$$K_2(K)/K_2(K)^n \xrightarrow{\text{by } (t'\circ s, t)} k^*/(k^*)^n \times F^*/(F^*)^n$$

is a continuous bijection with respect to the ordinary topologies of k^* and F^*. Since $k^*/(k^*)^n$ and $F^*/(F^*)^n$ are discrete, it follows that $K_2(K)/K_2(K)^n$ is discrete.

Case 2. Suppose that the order of χ is a power of p. By § 7.3 Corollary, it suffices to prove the following (i) and (ii).

(i) For sufficiently large n, $\langle V_K^{(n)}, \chi\rangle_K=0$.

(ii) For each $a\in K^*$, the map $K^*\to \boldsymbol{Q}/\boldsymbol{Z}$; $x\mapsto\langle\{x, a\}, \chi\rangle_K$ is continuous.

Let L be the cyclic extension of K which corresponds to χ. For sufficiently large n, $U_K^{(n)}\subset N_{L/K}L^*$, and so, $V_K^{(n)}\subset N_{L/K}K_2(L)$. This proves (i). It remains to prove (ii). Let $w=(\chi, a)\in \mathrm{Br}(K)$. Then, the homomorphism $K^*\to\boldsymbol{Q}/\boldsymbol{Z}$; $x\mapsto \langle\{x, a\}, \chi\rangle_K$ coincides with $-\Phi(w)$, which is continuous by Th. 1.

Step 2. Conversely, we prove that each continuous homomorphism $\varphi: K_2(K)\to \boldsymbol{Q}/\boldsymbol{Z}$ has the form $\Psi^\vee(\chi)$ for some $\chi\in X_K$. (Step 2.1-2.5.)

Step 2.1. First, we prove that φ is of finite order. Just as in the proof of Graham [5] Lemma 1, it is easily seen that $\varphi(V_K^{(n)})=0$ for sufficiently large n. Hence, if m is sufficiently large, $p^m\varphi(V^{(1)})\subset\varphi(V_K^{(n)})=0$. Thus, the composite

$$K^*\times K^*\to K_2(K)\xrightarrow{p^m\varphi}\boldsymbol{Q}/\boldsymbol{Z}$$

is factorized as

$$K^*\times K^*\to K^*/U_K^{(1)}\times K^*/U_K^{(1)}\xrightarrow{\varphi'}\boldsymbol{Q}/\boldsymbol{Z}\,.$$

where φ' is a continuous map. Let A be the topological group

$$\{x\in U_K \mid \bar{x}\in U_F\}/U_K^{(1)} \quad (\cong U_F)\,.$$

Fix a prime element π of K and an element τ of O_K such that $\bar{\tau}$ is a prime element of F. Then, the image of $p^m\varphi$ is generated by $\varphi'(A\times A)$, $\varphi'(A\times\{\pi \bmod U_K^{(1)}\})$, $\varphi'(A\times\{\tau \bmod U_K^{(1)}\})$, and $\varphi'(\pi, \tau)$. Since A is compact and $\boldsymbol{Q}/\boldsymbol{Z}$ is discrete, these four sets are finite. Thus, the image of $p^m\varphi$ is finite and hence φ is of finite order.

Step 2.2. The "prime to p" part is easy. If n is prime to p, the homomorphism $(X_K)_n \to \mathrm{Hom}(K_2(K), (1/n)\boldsymbol{Z}/\boldsymbol{Z})$ induced by $\Psi^\vee$ is bijective as was shown in §6 Proof of the injectivity of Φ and $\Psi^\vee$. Hence we may assume that the order of φ is a power of p.

Step 2.3. Here, we prove that the homomorphism

$$N_{L/K}: K_2(L)/K_2(L)^{p^m} \to K_2(K)/K_2(K)^{p^m}$$

is continuous for each $m \geqq 0$ and for each finite cyclic extension L of K. We use this result in Step 2.4. Clearly we may assume that $[L: K]$ is a prime number. Since $\mathrm{ch}(K)=0$, $V_K^{(n)} \subset K_2(K)^{p^m}$ and $V_L^{(n)} \subset K_2(L)^{p^m}$ for sufficiently large n. Hence, we can apply §7 Proposition 1. We consider the following three case (i), (ii), (iii).

(i) If L/K is unramified, or if $[L: K]$ is prime to p, the proof goes as follows. Let π be a prime element of K and let g be an element of O_K such that $\bar{g} \notin F^p$. Then, there are an integer i and a prime element π' of L such that $\pi' \equiv \pi^i \bmod (L^*)^{p^m}$. Hence, by §7 Proposition 1, it suffices to show that the homomorphism

$$U_L \times U_L^{(1)} \to K_2(K)/K_2(K)^{p^m}$$
$$(x, y) \mapsto N_{L/K}\{x, \pi\} \cdot N_{L/K}\{y, g\} = \{N_{L/K}(x), \pi\} \cdot \{N_{L/K}(y), g\}$$

is continuous. But this is clear since the homomorphism $N_{L/K}: L^* \to K^*$ is continuous by §7 Lemma 4.

(ii) If L/K is totally ramified and $[L: K]=p$, the proof goes as follows. Let π be a prime element of L. By §7 Proposition 1, it suffices to show that the homomorphism

$$f: U_L \to K_2(K)/K_2(K)^{p^m}; \qquad x \mapsto N_{L/K}\{x, \pi\}$$

is continuous. Let g be an element of O_K such that $\bar{g} \notin F^p$. Fix a number n such that f annihilates $U_L^{(np)}$, and then fix a sufficiently large number r. Let I be the set of all integers i such that $0 < i < np$ and such that i is prime to p. Then, the map

$$U_K/U_K^{(n)} \times F^I \to U_L/U_L^{(np)}$$
$$\left(x, \left(\sum_{0 \leqq j < p^r} \bar{a}_{ij}^{p^r} \cdot \bar{g}^j\right)_{i \in I}\right) \mapsto x \cdot \prod_{\substack{0 \leqq j < p^r \\ i \in I}} (1 - a_{ij}^{p^r} \cdot g^j \cdot \pi^i) \mod U_L^{(np)} \qquad (a_{ij} \in O_K)$$

is a homeomorphism. Since

$$N_{L/K}\{x, \pi\} = \{x, N_{L/K}(\pi)\} \qquad \text{for all} \quad x \in U_K$$

and since

$$N_{L/K}\{1-a^{p^r}g^j\pi^i,\pi\} \qquad (a\in O_K,\ i\in I)$$
$$\equiv N_{L/K}\{1-a^{p^r}g^j\pi^i,g\}^{-j/i}$$
$$\equiv\{N_{L/K}(1-a^{p^r}g^j\pi^i),g\}^{-j/i} \mod K_2(K)^{p^m},$$

it follows that f is continuous.

(iii) Suppose $[L: K]=p$, and suppose the residue extension is inseparable and of degree p. Let h be an element of O_L such that $\bar{h}\notin F$. By §7 Proposition 1, it suffices to show that the homomorphism

$$U_L^{(1)}\to K_2(K)/K_2(K)^{p^m}\ ; \qquad x\mapsto N_{L/K}\{x,h\}$$

is continuous. But this follows immediately from §7 Lemma 5, since that Lemma claims that the map

$$U_L^{(1)}\times U_L\to K_2(K)/K_2(K)^{p^m}\ ; \qquad (x,y)\mapsto N_{L/K}\{x,y\}$$

is continuous.

Step 2.4. We claim here that we may assume K contains a primitive p-th root of 1, and that we may assume $p\varphi=0$. By virtue of Step 2.3, the proofs of these claims proceed just as Proof of Th. 1 Step 2.3, and 2.4, using the exact sequence

$$0\to H^2(G,\boldsymbol{Z})\to X_K\to(X_L)^G\to 0$$

where G acts on $\boldsymbol{Z}$ trivially. The details are left to the reader.

Step 2.5. Suppose K contains a primitive p-th root of 1. Let $\varphi: K_2(K)/K_2(K)^p\to(1/p)\boldsymbol{Z}/\boldsymbol{Z}$ be a continuous homomorphism. We claim that $\varphi=\Phi^\vee(\chi)$ for some $\chi\in(X_K)_p$. Indeed, by induction starting with $i=(e_Kp/(p-1))+1$ (until $i=0$), we can prove that for each i, each continuous homomorphism $V^{(i)}K_2(K)^p/K_2(K)^p\to(1/p)\boldsymbol{Z}/\boldsymbol{Z}$ coincides with the restriction of $\Phi^\vee(\chi)$ for some $\chi\in(X_K)_p$; for we can proceed just as in Step 2.5 of the proof of Th. 1 and we can use the fact that the homomorphism ρ_i (§2 Prop. 1) is continuous (this is shown easily) instead of the fact that the homomorphism $A_i\to U_K^{(i)}(K^*)^p/U_K^{(i+1)}(K^*)^p$ is continuous.

Thus, we have proved Th. 1 and Th. 2.

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