A generalization of local class field theory by using K-groups. II

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Chapter II. Higher local class field theory.

Summary.

Here, we explain the main subjects and main results of Chapter II.

In § 1, we shall study the Galois cohomology of a complete discrete valuation field whose residue field is not assumed to be perfect. Next, in § 2, we shall generalize the residue homomorphism by using Quillen's K-theory. Lastly, in § 3, we shall study the local class field theory of "higher local fields" as a result of the previous two sections.

The aim of § 1 is to prove the following Th. I.

THEOREM I. (Cf. § 1.1, Th. 1 and Th. 2.) *Let K be a complete discrete valuation field with residue field F, and assume that the characteristic of K,* $\mathrm{ch}(K)$, *is zero,* $\mathrm{ch}(F)=p>0$, *and* $[F: F^p]<\infty$. *Let q be the integer such that* $[F: F^p]=p^q$. *Then:*

(1) *The cohomological dimension of K relative to p,* $\mathrm{cd}_p(K)$, *is $q+1$ or $q+2$. If F is separably closed, or more generally, if* $1-\gamma: \Omega^q_{F'/Z}\to\Omega^q_{F'/Z}$ *is surjective for any finite extension F' of F, then* $\mathrm{cd}_p(K)=q+1$. *Here, γ is the Cartier operator.*

(2) *For each $m\geqq 1$, let μ_m be the group of all m-th roots of 1 in the algebraic closure of K, and* $\mu_m^{\otimes r}$ $(r\geqq 0)$ *its r-th tensor power. Then, for each $n\geqq 1$, there is a canonical surjective homomorphism*

$$P^q_n(F)\longrightarrow H^{q+2}(K, \mu_{p^n}^{\otimes(q+1)}),$$

where $P^q_n(F)$ is a certain quotient of the group $W_n(F)\otimes\overbrace{F^*\otimes\cdots\otimes F^*}^{q\ \text{times}}$ *such that* $P^q_1(F)\cong\Omega^q_{F/Z}/(1-\gamma)\Omega^q_{F/Z}$. *This homomorphism is bijective if $q\leqq 1$ (i.e. F is perfect or* $[F: F^p]=p$), *or if* $F=E((X_1))((X_2))\cdots((X_q))$ *for some perfect field E.*

We conjecture that the canonical homomorphism in the above Th. I (2) is bijective without the assumptions in (2). The proof of its surjectivity will be completed in § 1.3. For the injectivity in the case $F=E((X_1))((X_2))\cdots((X_q))$, we shall need the "cohomological residue" defined in § 1.4, and complete the proof of the above Th. I in § 1.6.

In § 1.7, we shall prove that Milnor's K-groups ([16]) have canonical norm homomorphisms.

Next, the contents of § 2 is as follows. We shall define the generalized residue homomorphism in § 2.1. A concrete computation of this homomorphism will be stated in § 2.2 Prop. 3 and proved in § 2.5. An application of our residue homomorphism is a complement to Bloch [3]; we can eliminate the hypothesis $q \leqq p$ in [3] Ch. II (cf. § 2.2, Prop. 2).

Lastly, the main result of § 3 is the following Th. II. For any field k, let $\mathfrak{K}_*(k)$ be Milnor's K-group defined in Milnor [16] (which was denoted by K_*k in [16]), and let $K_*(k)$ be Quillen's K-group in Quillen [19].

THEOREM II. (Cf. § 3.1, Th. 1, Th. 2, and § 3.4, Prop. 3.) *Let* $N \geqq 0$, *and let* $k_0, \cdots, k_N$ *be fields satisfying the following conditions* (i) *and* (ii).

(i) k_0 *is a finite field.*

(ii) *For each* $i=1, \cdots, N$, k_i *is a complete discrete valuation field and the residue field of* k_i *is* k_{i-1}.

Denote k_N *by* K, *and* k_0 *by* k. *Then:*

(1) *There exists a canonical homomorphism*

$$\Psi_K : \mathfrak{K}_N(K) \longrightarrow \mathrm{Gal}(K^{\mathrm{ab}}/K)$$

characterized by the following properties (iii) *and* (iv).

(iii) *For each finite abelian extension* L *of* K, Ψ_K *induces an isomorphism* $\mathfrak{K}_N(K)/\mathfrak{N}_{L/K}\mathfrak{K}_N(L) \cong \mathrm{Gal}(L/K)$. *Here* $\mathfrak{N}_{L/K}$ *is the canonical norm homomorphism.*

(iv) *For each* $i=1, \cdots, N$, *let* π_i *be a lifting to* K *of a prime element of* k_i. *Then, the image of* $\Psi_K(\{\pi_1, \cdots, \pi_N\})$ *under the canonical homomorphism* $\mathrm{Gal}(K^{\mathrm{ab}}/K) \to \mathrm{Gal}(k^{\mathrm{ab}}/k)$ *coincides with the Frobenius automorphism over* k.

(2) *Let* $N \geqq 1$. *Then, there exists a canonical injective homomorphism*

$$\Phi_K : \mathrm{Br}(K) \longrightarrow \mathrm{Hom}(\mathfrak{K}_{N-1}(K), \boldsymbol{Q}/\boldsymbol{Z}),$$

where $\mathrm{Br}(K)$ *denotes the Brauer group of* K.

(3) *In the case* $\mathrm{ch}(K)>0$, *these homomorphisms* Ψ_K *and* Φ_K *are induced by canonical homomorphisms*

$$\Upsilon_K : K_N(K) \longrightarrow \mathrm{Gal}(K^{\mathrm{ab}}/K) \quad \textit{and}$$

$$\Theta_K : \mathrm{Br}(K) \longrightarrow \mathrm{Hom}(K_{N-1}(K), \boldsymbol{Q}/\boldsymbol{Z})$$

respectively, and the canonical map $\mathfrak{K}_*(K)\to K_*(K)$.

For this Th. II, in the case $\mathrm{ch}(K)=0$, we shall use the study of the Galois cohomology of § 1. In the case $\mathrm{ch}(K)>0$, we shall use the generalized residue homomorphism of § 2. The following Th. III is a consequence of Th. I and will be the key tool for Th. II (1) and (2).

THEOREM III. (Cf. § 1.1, Th. 3 or § 3.2, Prop. 1.) *Let* K *be as in Th.* II. *Then, the cohomological dimension of* K *is* $N+1$, *and there exists a canonical isomorphism*

$$H^{N+1}(K,\ \mu_m^{\otimes N})\cong\frac{1}{m}\boldsymbol{Z}/\boldsymbol{Z}$$

for any integer m *invertible in* K.

The part I (=Chapter I) of this paper was published in [26].

In the first draft of this part II, we defined the residue homomorphisms in Milnor K-theory, and it was applied in the study of abelian coverings of surfaces in Brylinski [27]. The author will publish latter that definition and related results.

Conventions.

"Ring" means commutative ring with identity and "field" means commutative field unless the word "ring" is used as "graded ring". All the graded rings in this paper are anti-commutative.

Let R be a ring. Then, we denote by R^* the multiplicative group of all invertible elements of R. For each $q\geqq 0$, the q-th exterior power over R of the absolute differential module $\Omega^1_{R/Z}$ (cf. Grothendieck [8] Ch. 0 § 20) is denoted by Ω^q_R. Furthermore, $\Omega^1_{R/Z}$ is often denoted by Ω_R.

Let k be a field. Then, we denote by $\mathrm{ch}(k)$ the characteristic of k, by k_s the separable closure of k, by k^{ab} the maximum abelian extension of k, and by X_k the group of all continuous homomorphisms $\mathrm{Gal}(k^{\mathrm{ab}}/k)\to\boldsymbol{Q}/\boldsymbol{Z}$. If m is an integer invertible in k, we denote by $\mu_{m,k}$ (or simply by μ_m) the $\mathrm{Gal}(k_s/k)$-module of all m-th roots of 1 in k_s, and by $\mu_m^{\otimes r}$ ($r\geqq 0$) the r-th tensor power of μ_m over $\boldsymbol{Z}/m\boldsymbol{Z}$ on which $\mathrm{Gal}(k_s/k)$ acts in the natural way.

Let k be a discrete valuation field. Then, we denote by v_k the normalized additive valuation of k, by O_k the valuation ring of k, and by m_k the maximal ideal of O_k. We denote by U_k and $U_k^{(n)}$ ($n\geqq 1$) the multiplicative groups $(O_k)^*$ and $\{x\in O_k \mid v_k(x-1)\geqq n\}$, respectively. For each $x\in O_k$, we denote by $\bar{x}$ the residue class in O_k/m_k of x.

For any abelian group A and for any integer n, we denote by A_n the kernel of the multiplication by $n: A\xrightarrow{\ n\ }A$.

§ 1. Galois cohomology.

§ 1.1. The main results.

The main results of the cohomological study in this Chapter II is the following Theorems.

THEOREM 1. *Let K be a complete discrete valuation field with residue field F such that* $\mathrm{ch}(K)=0$, $\mathrm{ch}(F)=p>0$, *and* $[F:F^p]=p^q$, $0\leqq q<\infty$. *Then,*

(1) *The cohomological dimension of K relative to p, $\mathrm{cd}_p(K)$, is $q+1$ or $q+2$.*

(2) *There is a canonical surjective homomorphism*

$$\Omega_F^q/(1-\gamma)\Omega_F^q \longrightarrow H^{q+2}(K, \mu_p^{\otimes(q+1)}),$$

where γ denotes the Cartier operator [5].

(3) *Assume that for any finite extension F' of F, the homomorphism $1-\gamma$: $\Omega_{F'}^q \to \Omega_{F'}^q$ is surjective. (This assumption is satisfied, for example, if F is separably closed.) Then,* $\mathrm{cd}_p(K)=q+1$.

(4) *For each $n\geqq 0$, there is a canonical surjective homomorphism*

$$h_{K,p^n}^F: P_n^q(F) \longrightarrow H^{q+2}(K, \mu_{p^n}^{\otimes(q+1)}),$$

where $P_n^q(F)$ is the group defined below.

DEFINITION 1. Let k be a field of characteristic $p>0$. Let $q\geqq 0$ and $n\geqq 0$. We define the group $P_n^q(k)$ by

$$P_n^q(k)=(W_n(k)\otimes\overbrace{k^*\otimes\cdots\otimes k^*}^{q \text{ times}})/J,$$

where $W_n(k)$ denotes the group of all p-Witt vectors of length n over k (cf. Demazure [7] Ch. III) and J denotes the subgroup of the tensor product generated by all elements of the following forms (i) (ii) (iii).

(i) $(\overbrace{0,\cdots,0}^{i \text{ times}}, a, 0, \cdots, 0)\otimes a\otimes b_1\otimes\cdots\otimes b_{q-1}$ $(0\leqq i<n,\ a, b_1, \cdots, b_{q-1}\in k^*)$.

(ii) $(\mathfrak{F}(w)-w)\otimes b_1\otimes\cdots\otimes b_q$ $(w\in W_n(k),\ b_1, \cdots, b_q\in k^*)$, where $\mathfrak{F}$ denotes the homomorphism $W_n(k)\to W_n(k)$;

$$(a_0, \cdots, a_{n-1})\mapsto(a_0^p, \cdots, a_{n-1}^p).$$

(iii) $w\otimes b_1\otimes\cdots\otimes b_q$ such that $b_i=b_j$ for some $i\neq j$.

We shall denote an element $w\otimes b_1\otimes\cdots\otimes b_q$ mod J of $P_n^q(k)$ by $\{w, b_1, \cdots, b_q\}$.

REMARK 1. Theorem 1 (2) is a special case of (4), since there is an isomorphism $P_1^q(F)\cong\Omega_F^q/(1-\gamma)\Omega_F^q$ (cf. Corollary to Lemma 5 in § 1.3).

REMARK 2. The group $P_n^q(k)$ is isomorphic to the cokernel of $F-1: C_n^q(k)\to$

$C_n^q(k)/\{C_n^{q-1}(k), T\}$ of Bloch [3] Ch. II §7. For this, cf. §2.2 Cor. 4 to Prop. 2. Hence, from the point of view of Milne [15], $P_n^q(k)$ is the group which will be denoted by the notation $H_{fl}^{q+1}(k, \mu_{p^n}^{\otimes q})$ when some cohomology theory justifies this notation in the future even in the case $q \geqq 2$ and $\mathrm{ch}(k)=p>0$.

DEFINITION 2. The definition of h_{K,p^n}^F is as follows. Let $w \in W_n(F)$ and $b_1, \cdots, b_q \in F^*$. By Witt theory [25], there is a canonical isomorphism $W_n(F)/(\mathfrak{F}-1)W_n(F) \cong (X_F)_{p^n}$. (Recall that X_k denotes the character group of the compact abelian group $\mathrm{Gal}(k^{ab}/k)$ for any field k.) Since X_F is identified with the unramified part of X_K, we have a homomorphism $i: W_n(F) \to (X_F)_{p^n} \to (X_K)_{p^n} \cong H^1(K, \boldsymbol{Z}/p^n\boldsymbol{Z})$. Now, h_{K,p^n}^F is defined by

$$\{w, b_1, \cdots, b_q\} \mapsto i(w) \cup h_{p^n,K}^1(\tilde{b}_1) \cup \cdots \cup h_{p^n,K}^1(\tilde{b}_q) \cup h_{p^n,K}^1(\pi),$$

where $\tilde{b}_i$ denotes any lifting of b_i to O_K for each i, π denotes any prime element of K, $h_{p^n,K}^1$ denotes the canonical isomorphism $K^*/(K^*)^{p^n} \cong H^1(K, \mu_{p^n})$, and $\cup$ denotes the cup product. It will be shown in §1.3 that this homomorphism h_{K,p^n}^F is well defined.

It is probable that the canonical homomorphisms in the above Th. 1 (2) and (4) are in fact bijective. For example, they are bijective in the case $q=0$, i.e. in the case where F is a perfect field, by Serre [20] Ch. XII §3 Th. 2 and by the isomorphism $P_n^0(F) \cong (X_F)_{p^n}$ (Witt theory [25]). Next, they are bijective in the case $q=1$, by Chapter I §5 Th. 1 (of this paper) and by the isomorphism $P_n^1(F) \cong \mathrm{Br}(F)_{p^n}$ (cf. §3.4 Lemma 16 for this isomorphism). They are also bijective in the following interesting case.

THEOREM 2. *Besides the hypothesis of Theorem* 1, *suppose that F is a field of the type $E((X_1))((X_2))\cdots((X_q))$, where E is a perfect field. Then, for any $n \geqq 0$, h_{K,p^n}^F is bijective. Furthermore in this case, we have $P_n^0(E) \cong P_n^q(F)$; $w \mapsto \{w, X_1, \cdots, X_q\}$. Thus,*

$$H^{q+2}(K, \mu_{p^n}^{\otimes(q+1)}) \cong P_n^0(E) \cong (X_E)_{p^n}.$$

Here $E((X_1))((X_2))\cdots((X_q))$ means the field defined inductively by the convention that $E((X_1))((X_2))\cdots((X_i))$ is the field of formal power series in the variable X_i over $E((X_1))((X_2))\cdots((X_{i-1}))$ for each i.

The following Theorem 3 is an easy consequence of Theorem 2.

THEOREM 3. *Let $N \geqq 0$, and let $k_0, \cdots, k_N$ be fields having the following properties* (i) *and* (ii).

(i) *k_0 is a finite field.*

(ii) *For each $i=1, \cdots, N$, k_i is a complete discrete valuation field and the residue field of k_i is k_{i-1}. Let $K=k_N$. Then, if m is an integer invertible in K,*

$$H^{N+1}(K, \mu_m^{\otimes N}) \cong \frac{1}{m} \boldsymbol{Z}/\boldsymbol{Z}.$$

The basic idea for our cohomological study lies in the following relation between Milnor's K-groups and Galois cohomology.

For any field k, let $\mathfrak{K}_q(k)$ $(q \geqq 0)$ be Milnor's K-groups of Milnor [16], i.e. $\mathfrak{K}_0(k) = \boldsymbol{Z}$, $\mathfrak{K}_1(k) = k^*$, and

$$\mathfrak{K}_q(k) = (\overbrace{k^* \otimes \cdots \otimes k^*}^{q \text{ times}})/J$$

for $q \geqq 2$, where J is the subgroup of the tensor product generated by all elements of the form $x_1 \otimes \cdots \otimes x_q$ satisfying $x_i + x_j = 1$ with i and j such that $i \neq j$. (Though the notation $K_* k$ was used in [16] for Milnor's K-groups, we use in this paper the notation $\mathfrak{K}_*(k)$ for them, for we use the notation $K_*(k)$ for Quillen's K-groups.) We shall denote the group structure of Milnor's K-groups additively. We shall denote the element $x_1 \otimes \cdots \otimes x_q \bmod J$ $(x_1, \cdots, x_q \in k^*)$ of $\mathfrak{K}_q(k)$ by $\{x_1, \cdots, x_q\}$, and for any $x \in \mathfrak{K}_i(k)$ and $y \in \mathfrak{K}_j(k)$ $(i, j \geqq 0)$, the image of $x \otimes y$ under the canonical homomorphism $\mathfrak{K}_i(k) \otimes \mathfrak{K}_j(k) \to \mathfrak{K}_{i+j}(k)$ by $\{x, y\}$. We have $\{x, -x\} = 0$ for any $x \in k^*$, and $\{x, y\} = (-1)^{ij}\{y, x\}$ for any $x \in \mathfrak{K}_i(k)$ and $y \in \mathfrak{K}_j(k)$.

Now, let m be an integer invertible in k. Then, the canonical isomorphism

$$h^1_{m,k}: k^*/(k^*)^m \cong H^1(k, \mu_m),$$

which comes from the exact sequence of $\mathrm{Gal}(k_s/k)$-modules

$$1 \longrightarrow \mu_m \longrightarrow k_s^* \xrightarrow{m} k_s^* \longrightarrow 1,$$

induces for each $q \geqq 0$, a homomorphism

$$h^q_{m,k}: \mathfrak{K}_q(k)/m \cdot \mathfrak{K}_q(k) \longrightarrow H^q(k, \mu_m^{\otimes q})$$
$$\{x_1, \cdots, x_q\} \mapsto h^1_{m,k}(x_1) \cup \cdots \cup h^1_{m,k}(x_q),$$

where $\cup$ denotes the cup product. Concerning this homomorphism $h^q_{m,k}$, the experts perhaps have the following Conjecture in mind (cf. [16] §6).

Conjecture 1. *The homomorphism $h^q_{m,k}$ is bijective for any field k, for any $q \geqq 0$, and for any integer m which is invertible in k.*

This Conjecture was verified in the case where k is algebraic over $\boldsymbol{Q}$, $\boldsymbol{Q}_p$, $\boldsymbol{F}_p(T)$ or $\boldsymbol{F}_p((T))$ (where p is a prime number) by Tate (cf. Tate [23] Th. 2, Bass and Tate [4] Ch. II, and Serre [21] Ch. II 6.3). In this §1, we prove the surjectivity of $h^q_{m,k}$ in the following cases.

PROPOSITION 1. *Let K be a complete discrete valuation field with residue field F such that* $\mathrm{ch}(K) = 0$, $\mathrm{ch}(F) = p > 0$, *and* $[F : F^p] = p^q < \infty$. *Then, the homo-*

morphism

$$h_{p^n,K}^{q+2}: \mathfrak{K}_{q+2}(K)/p^n\cdot\mathfrak{K}_{q+2}(K) \longrightarrow H^{q+2}(K, \mu_{p^n}^{\otimes(q+2)})$$

is surjective for any $n\geqq 0$.

Once we have Proposition 1, Theorem 1 can be reduced to a study of $\mathfrak{K}_{q+2}(K)$ (cf. § 1.3).

PROPOSITION 2. *Let k be a field, p a prime number which is invertible in k, $q\geqq 0$ an integer and K an extension of k of finite transcendental degree d. Suppose* $\mathrm{cd}_p(k)\leqq q$ *and suppose that the homomorphism $h_{p,E}^q$ is surjective for any finite separable extension E of k. Then, $h_{p^n,K}^{q+d}$ is surjective for any $n\geqq 0$.*

COROLLARY. *Let k be a field, p a prime number which is invertible in k, $q\geqq 0$ an integer and K an extension of k. Then $h_{p^n,K}^q$ is surjective for any $n\geqq 0$ if one of the following conditions is satisfied.*

(i) *k is separably closed and* $\mathrm{trans.deg}_k(K)\leqq q$.

(ii) $\mathrm{cd}_p(k)=1$ *and* $\mathrm{trans.deg}_k(K)\leqq q-1$.

(iii) *k is algebraic over $\boldsymbol{Q}_{p'}$ or $\boldsymbol{F}_{p'}((T))$ for some prime number p' and* $\mathrm{trans.deg}_k(K)\leqq q-2$.

(iv) *k is algebraic over $\boldsymbol{Q}$ and* $\mathrm{trans.deg}_k(K)\leqq q-2$. *Furthermore, if $p=2$, k is totally imaginary.*

Note that $\mathrm{cd}_p(K)\leqq q$ in each case (cf. Serre [21] Ch. II).

These Propositions will be deduced from the following

PROPOSITION 3. *Let k be a field, S a Galois extension of k of infinite degree, p a prime number which is invertible in k, and $q\geqq 0$ an integer. Suppose that* $\mathrm{cd}_p\mathrm{Gal}(S/k)\leqq q$ *and* $\mathrm{cd}_p(S)\leqq 1$, *and that for any open subgroup J of* $\mathrm{Gal}(S/k)$, *the cup product*

$$\overbrace{H^1(J, \boldsymbol{Z}/p\boldsymbol{Z})\otimes\cdots\otimes H^1(J, \boldsymbol{Z}/p\boldsymbol{Z})}^{q\ \text{times}} \longrightarrow H^q(J, \boldsymbol{Z}/p\boldsymbol{Z})$$

is surjective. Then, $h_{p^n,K}^{q+1}$ is surjective for any $n\geqq 0$.

§ 1.2. Some preliminary Lemmas.

Here, we review briefly some properties of Milnor's K-groups (cf. Bass and Tate [4]) and prove some preliminary Lemmas.

1°. The homomorphism ∂. Let K be a discrete valuation field with residue field F. Then, there is a homomorphism $\partial: \mathfrak{K}_{*+1}(K)\to\mathfrak{K}_*(F)$ having the following characterization: If $q\geqq 0$, and if $x_1, \cdots, x_q\in U_K$ and $y\in K^*$,

$$(\{x_1, \cdots, x_q, y\})=v_K(y)\cdot\{x_1, \cdots, x_q\} .$$

2°. The norm homomorphisms. Let E be a finite extension of a field k. In the case $E=k(\alpha)$ for some $\alpha\in E$, a norm homomorphism $N_{\alpha/k}: \mathfrak{K}_*(E)\to\mathfrak{K}_*(k)$ is defined in [4] Ch. I § 5 depending on the choice of α, which we review below. In fact, we can prove that for any finite extention E/k, the composite $\mathfrak{N}_{E/k}=N_{\alpha_1/E_0}\circ\cdots\circ N_{\alpha_n/E_{n-1}}: \mathfrak{K}_*(E)\to\mathfrak{K}_*(k)$ is independent of the choice of the family $(\alpha_i)_{1\leq i\leq n}$ such that $E=k(\alpha_1, \cdots, \alpha_n)$, where E_i denotes $k(\alpha_1, \cdots, \alpha_i)$ for each i. The proof of this fact is postponed until § 1.7 for the reason that we hope to prove the main results earlier, but we shall use the above notation $\mathfrak{N}_{E/k}$ freely in this § 1.

For any algebraic function field K in one variable over a field k, let $\mathfrak{P}(K/k)$ be the set of all normalized additive discrete valuations of K which are trivial on k. For each $v\in\mathfrak{P}(K/k)$, let $\kappa(v)$ be the residue field of v, and let $\partial: \mathfrak{K}_{*+1}(K)\to\bigoplus_{v\in\mathfrak{P}(K/k)}\mathfrak{K}_*(\kappa(v))$ be the homomorphism defined by the above 1°. Then, if K is the rational function field $k(X)$ and ∞ denotes the element of $\mathfrak{P}(K/k)$ such that $\infty(X)=-1$, $\bigoplus_{v\in\mathfrak{P}(K/k)}\mathfrak{K}_*(\kappa(v))$ is generated by $\partial(\mathfrak{K}_{*+1}(K))$ and the ∞-factor, and there is a homomorphism $(N_v)_{v\in\mathfrak{P}(K/k)}: \bigoplus_v\mathfrak{K}_*(\kappa(v))\to\mathfrak{K}_*(k)$ which annihilates $\partial(\mathfrak{K}_{*+1}(K))$ and induces the identity map on the ∞-factor. If $E=k(\alpha)$, $N_{\alpha/k}$ is defined to be N_v where v is the element of $\mathfrak{P}(K/k)$ corresponding to the minimum polynomial over k of α. (Cf. [4] Ch. I § 5.)

The norm homomorphism $\mathfrak{N}_{E/k}$ satisfies $\mathfrak{N}_{E/k}(\{x, y_E\})=\{\mathfrak{N}_{E/k}(x), y\}$ ($x\in\mathfrak{K}_i(E)$ and $y\in\mathfrak{K}_j(k)$), and $\mathfrak{N}_{E/k}: \mathfrak{K}_1(E)\to\mathfrak{K}_1(k)$ coincides with the usual norm homomorphism $E^*\to k^*$.

3°. These homomorphisms ∂ and $N_{\alpha/k}$ are related via the homomorphism $h^q_{m,k}$ with the following homomorphism δ and the corestriction map of Galois cohomology as in Lemma 1 and Lemma 3 below.

First, let K be a discrete valuation field with residue field F, m an integer which is invertible in F, $q\geqq 0$, and $r\in\boldsymbol{Z}$. Let $\hat{K}$ be the completion of K. Then, since $\mathrm{cd}_p(\hat{K}_{\mathrm{nr}})=1$ for any prime divisor p of m, we have a composite homomorphism δ:

$$H^{q+1}(K, \mu_m^{\otimes(r+1)})\longrightarrow H^{q+1}(\hat{K}, \mu_m^{\otimes(r+1)})\longrightarrow H^q(F, H^1(\hat{K}_{\mathrm{nr}}, \mu_m^{\otimes(r+1)}))$$
$$\xrightarrow{\cong} H^q(F, \mu_m^{\otimes r}\otimes(\hat{K}^*_{\mathrm{nr}}/\hat{K}^*_{\mathrm{nr}})^m))\xrightarrow{\cong} H^q(F, \mu_m^{\otimes r}),$$

where the last isomorphism is induced by the valuation of $\hat{K}_{\mathrm{nr}}$. Here in the case $r<0$, $\mu_m^{\otimes r}$ denotes $\mathrm{Hom}(\mu_m^{\otimes(-r)}, \boldsymbol{Q}/\boldsymbol{Z})$ on which elements σ of the Galois groups act by $f\mapsto f\circ\sigma^{-1}$.

LEMMA 1. *Let K, F, m, and q be as above. Then, the following diagram is commutative.*

$$\begin{array}{ccc} \mathfrak{K}_{q+1}(K)/m\cdot\mathfrak{K}_{q+1}(K) & \xrightarrow{h^{q+1}_{m,K}} & H^{q+1}(K,\ \mu_m^{\otimes(q+1)}) \\ \partial\downarrow & & \downarrow(-1)^q\cdot\delta \\ \mathfrak{K}_q(F)/m\cdot\mathfrak{K}_q(F) & \xrightarrow{h^q_{m,F}} & H^q(F,\ \mu_m^{\otimes q})\,. \end{array}$$

This Lemma follows from the following Corollary to Lemma 2.

LEMMA 2. *Let G be a pro-finite group, H a closed normal subgroup of G, p a prime number, A and B p-primary torsion G-modules on which G acts continuously with respect to the discrete topologies on A and B. For any i, i', j, $j'\geqq 0$ and for any $x\in H^i(G/H, H^j(H, A))$ and $y\in H^{i'}(G/H, H^{j'}(H, B))$, denote by $x\,\overline{\cup}\,y$ the element of $H^{i+i'}(G/H, H^{j+j'}(H, A\otimes B))$ obtained from the pair (x, y) by taking the cup product over G/H and then that over H. On the other hand, let $\cup$ be the cup product over G.*

(1) *Assume* $\mathrm{cd}_p(H)\leqq d<\infty$ *and let*

$$f_*:\ H^{*+d}(G,\ ?)\longrightarrow H^*(G/H,\ H^d(H,\ ?))$$

be the functorial homomorphism defined by this assumption. Let q, $i\geqq 0$, $a\in H^{i+d}(G, A)$, $b\in H^q(G/H, H^0(H, B))$. Then,

$$f_{i+q}(a\cup \mathrm{Inf}_G^{G/H}(b))=f_i(a)\,\overline{\cup}\,b$$

in $H^{i+q}(G/H, H^d(H, A\otimes B))$. Here, $\mathrm{Inf}_G^{G/H}$ *is the inflation map $H^q(G/H, H^0(H, B))\to H^q(G, B)$.*

(2) *Assume* $\mathrm{cd}_p(G/H)\leqq q<\infty$ *and let*

$$g_*:\ H^q(G/H,\ H^*(H,\ ?))\longrightarrow H^{q+*}(G,\ ?)$$

be the functorial homomorphism defined by this assumption. Let d, $j\geqq 0$, $a\in H^d(G, A)$, $b\in H^q(G/H, H^j(H, B))$. Then,

$$a\cup g_j(b)=g_{d+j}(\mathrm{Res}_H^G(a)\,\overline{\cup}\,b)$$

in $H^{d+q+j}(G, A\otimes B)$. Here, Res_H^G *is the restriction map $H^d(G, A)\to H^0(G/H, H^d(H, A))$.*

Similar result can be obtained in any cohomology theory with cup product. Since only the usual general methods in homological algebra are needed for the proof, we left it to the reader.

COROLLARY. *Let K be a complete discrete valuation field with residue field F, and m an integer invertible in F. Then, if $y\in K^*$, $q\geqq 0$, $r\in\mathbf{Z}$, and $b\in H^q(F, \mu_m^{\otimes r})$, we have*

$$\delta(h^1_{m,K}(y)\cup\mathrm{Inf}(b))=v_K(y)\cdot b\quad\text{in } H^q(F,\ \mu_m^{\otimes r})\,.$$

Here, Inf *denotes the inflation map*

$$H^q(F, \mu_m^{\otimes r}) = H^q(\mathrm{Gal}(K_{nr}/K), \quad H^0(K_{nr}, \mu_m^{\otimes r})) \longrightarrow H^q(K, \mu_m^{\otimes r}).$$

PROOF. This follows from Lemma 2 (1) if one considers the case in which $G=\mathrm{Gal}(K_s/K)$, $H=\mathrm{Gal}(K_s/K_{nr})$, p is a prime number which is invertible in F (then $\mathrm{cd}_p(H)\leqq 1$), $A=\mu_{p^n}$ $(n\geqq 0)$, $B=\mu_{p^n}^{\otimes r}$, $d=1$, $i=0$, $q=q$, $a=h_{p^n,K}^1(y)\in H^1(K, \mu_{p^n})$ and $b=b$. In this case, f_* can be identified with δ.

DEFINITION 3. Let k be a field, E a finite extension of k, and M a discrete $\mathrm{Gal}(k_s/k)$-module on which $\mathrm{Gal}(k_s/k)$ acts continuously. We define the corestriction map $\mathrm{Cor}_{E/k}: H^*(E, M)\to H^*(k, M)$ to be $[E:E_0]\circ\mathrm{Cor}_{E_0/k}\circ i^{-1}$, where E_0 is the separable closure in E of k, $\mathrm{Cor}_{E_0/k}$ is the usual corestriction map (cf. Serre [21] Ch. I § 2.4) and i denotes the isomorphism $H^*(E_0, M)\xrightarrow{\cong} H^*(E, M)$ induced by the canonical isomorphism $\mathrm{Gal}(E_s/E)\to\mathrm{Gal}((E_0)_s/E_0)$.

Generally, let k be a field, K an algebraic function field in one variable over k, m an integer invertible in k, $q\geqq 0$, and $r\in\boldsymbol{Z}$. For each $v\in\mathfrak{P}(K/k)$, let δ_v be the homomorphism $\delta: H^{q+1}(K, \mu_m^{\otimes(r+1)})\to H^q(\kappa(v), \mu_m^{\otimes r})$ defined by v. It is not difficult to deduce the formula

$$(*) \qquad \sum_{v\in\mathfrak{P}(K/k)} \mathrm{Cor}_{\kappa(v)/k}\circ\delta_v=0: H^{q+1}(K, \mu_m^{\otimes(r+1)}) \longrightarrow H^q(k, \mu_m^{\otimes r})$$

from $\mathrm{cd}(K\cdot k_s)=1$ and from the usual summation formula

$$\sum_{v\in\mathfrak{P}(K\cdot k_s/k_s)} \deg(v)\cdot v(x)=0 \quad \text{for all } x\in(K\cdot k_s)^*,$$

where $K\cdot k_s$ denotes the composite field of K and k_s over k. Combining Lemma 1 with the above formula $(*)$, we obtain;

LEMMA 3. *Let k be a field, E a finite extension of k, and m an integer invertible in k. Then the following diagram is commutative for any $q\geqq 0$.*

$$\begin{array}{ccc} \mathfrak{K}_q(E)/m\cdot\mathfrak{K}_q(E) & \xrightarrow{h_{m,E}^q} & H^q(E, \mu_m^{\otimes q}) \\ \mathfrak{N}_{E/k}\downarrow & & \downarrow \mathrm{Cor}_{E/k} \\ \mathfrak{K}_q(k)/m\cdot\mathfrak{K}_q(k) & \xrightarrow{h_{m,k}^q} & H^q(k, \mu_m^{\otimes q}). \end{array}$$

§ 1.3. The proofs of the Propositions and Theorem 1.

PROOF OF PROPOSITION 3. The key is the following Lemma 4.

LEMMA 4. *Let G be a pro-finite group, H a closed normal subgroup of G, and p a prime number. Let $q, d\geqq 0$ and suppose $\mathrm{cd}_p(G/H)\leqq q$ and $\mathrm{cd}_p(H)\leqq d$. Then, $H^{d+q}(G, \boldsymbol{Z}/p\boldsymbol{Z})$ is generated by elements of the form*

$$\mathrm{Cor}_G^{G'}(a\cup\mathrm{Inf}_{G'}^{G'/H}(b))$$

such that G' is an open subgroup of G containing H and $a \in H^d(G', \boldsymbol{Z}/p\boldsymbol{Z})$ and $b \in H^q(G'/H, \boldsymbol{Z}/p\boldsymbol{Z})$. Here, $\mathrm{Cor}_G^{G'}$ denotes the corestriction map $H^{d+q}(G', \boldsymbol{Z}/p\boldsymbol{Z}) \to H^{d+q}(G, \boldsymbol{Z}/p\boldsymbol{Z})$, $\mathrm{Inf}_{G'}^{G'/H}$ denotes the inflation map $H^q(G'/H, \boldsymbol{Z}/p\boldsymbol{Z}) \to H^q(G', \boldsymbol{Z}/p\boldsymbol{Z})$, and $\cup$ denotes the cup product.

PROOF. By the assumption on the cohomological dimensions,

$$H^{d+q}(G, \boldsymbol{Z}/p\boldsymbol{Z}) \cong H^q(G/H, H^d(H, \boldsymbol{Z}/p\boldsymbol{Z}))\,.$$

Since the subgroups I of H which are open in H and normal in G form a fundamental system of neighbourhoods of 1 in H, we have

$$H^q(G/H, H^d(H, \boldsymbol{Z}/p\boldsymbol{Z})) = \varinjlim_I H^q(G/H, H^d(H/I, \boldsymbol{Z}/p\boldsymbol{Z}))\,.$$

Fix a subgroup I of H which is open in H and normal in G. Then,

$$H^d(H/I, \boldsymbol{Z}/p\boldsymbol{Z}) = \varinjlim_{G'} H^d(G'/I, \boldsymbol{Z}/p\boldsymbol{Z})\,,$$

where G' ranges over all open subgroups of G containing H. Since $H^d(H/I, \boldsymbol{Z}/p\boldsymbol{Z})$ is finite, there is an open subgroup G' of G containing H such that the action of G'/H on $H^d(H/I, \boldsymbol{Z}/p\boldsymbol{Z})$ is trivial and such that the restriction map

$$\mathrm{Res}\colon H^d(G'/I, \boldsymbol{Z}/p\boldsymbol{Z}) \longrightarrow H^d(H/I, \boldsymbol{Z}/p\boldsymbol{Z})$$

is surjective. For such G', we have the following commutative diagram

$$\begin{array}{ccc}
H^d(G'/I, \boldsymbol{Z}/p\boldsymbol{Z}) \otimes H^q(G'/H, \boldsymbol{Z}/p\boldsymbol{Z}) & \xrightarrow{\mathrm{Inf}\otimes\mathrm{Inf}} & H^d(G', \boldsymbol{Z}/p\boldsymbol{Z}) \otimes H^q(G', \boldsymbol{Z}/p\boldsymbol{Z}) \\
\downarrow{\scriptstyle \mathrm{Res}\otimes 1} & & \\
H^d(H/I, \boldsymbol{Z}/p\boldsymbol{Z}) \otimes H^q(G'/H, \boldsymbol{Z}/p\boldsymbol{Z}) & \text{(i)} & \downarrow{\scriptstyle \text{cup product}} \\
\downarrow{\scriptstyle \wr} & & \\
H^q(G'/H, H^d(H/I, \boldsymbol{Z}/p\boldsymbol{Z})) & \longrightarrow & H^{d+q}(G', \boldsymbol{Z}/p\boldsymbol{Z}) \\
\downarrow{\scriptstyle \mathrm{Cor}_{G/H}^{G'/H}} & \text{(ii)} & \downarrow{\scriptstyle \mathrm{Cor}_G^{G'}} \\
H^q(G/H, H^d(H/I, \boldsymbol{Z}/p\boldsymbol{Z})) & \longrightarrow & H^{d+q}(G, \boldsymbol{Z}/p\boldsymbol{Z})\,,
\end{array}$$

in which the vertical arrows in the left side are all surjective by the assumption on G' and by $\mathrm{cd}_p(G/H) \leqq q$. Here the commutativity of the square (i) is deduced from Lemma 2 (2) by taking G'/I, H/I, q, d, 0 as G, H, q, d, j of Lemma 2 (2), respectively. The commutativity of the square (ii) is shown easily. This diagram proves Lemma 4.

Now, for the proof of Prop. 3, we present some remarks. Generally, let k

be a field, p a prime number invertible in k, and $q\geqq 0$. Then;

(1) *If* $h_{p,k}^{q}$ *is surjective,* $h_{p^n,k}^{q}$ *are surjective for all* $n\geqq 0$.

(2) *If* $\mathrm{cd}_p(k)\leqq q$ *and if* $h_{p,E}^{q}$ *is surjective for some finite extension* E *of* k, $h_{p,k}^{q}$ *is surjective.*

Indeed, (1) can be proved by induction on n, and (2) follows from Lemma 3 and from the fact that $\mathrm{Cor}_{E/k}: H^q(E, \mu_p^{\otimes q})\to H^q(k, \mu_p^{\otimes q})$ is surjective by $\mathrm{cd}_p(k)\leqq q$.

Let k, S and q be as in the hypothesis of Prop. 3. Since $\mathrm{cd}_p(k)\leqq q+1$, the above (1) and (2) show that it is sufficient to prove the surjectivity of $h_{p,k}^{q+1}$ and that we may assume k contains a primitive p-th root of 1. Now, apply Lemma 4 by taking $\mathrm{Gal}(k_s/k)$, $\mathrm{Gal}(k_s/S)$, q and 1 as G, H, q and d, respectively. Then we see that $H^{q+1}(k, \mu_p^{\otimes(q+1)})$ is generated by elements of the form

$$\mathrm{Cor}_{E/k}(h_{p,E}^{1}(x)\cup \mathrm{Inf}_{\mathrm{Gal}(k_s/E)}^{\mathrm{Gal}(S/E)}(b))$$

such that E is a finite extension of k contained in S, and such that $x\in E^*$ and $b\in H^q(\mathrm{Gal}(S/E), \mu_p^{\otimes q})$. By hypothesis, $H^q(\mathrm{Gal}(S/E), \mu_p^{\otimes q})$ is generated by the cup products of elements of $H^1(\mathrm{Gal}(S/E), \mu_p)$. It follows that $H^{q+1}(k, \mu_p^{\otimes(q+1)})$ is generated by

$$\bigcup_E \mathrm{Cor}_{E/k}\circ h_{p,E}^{q+1}(\mathfrak{K}_{q+1}(E))=\bigcup_E h_{p,k}^{q+1}\circ \mathfrak{N}_{E/k}(\mathfrak{K}_{q+1}(E))\,.$$

PROOF OF PROPOSITION 1. We prove here Prop. 1 and the fact $\mathrm{cd}_p(K)\leqq q+2$ together. Since K is not formally real, for the proof of $\mathrm{cd}_p(K)\leqq q+2$, it suffices to prove $\mathrm{cd}_p(K')\leqq q+2$ for some finite extension K' of K by Serre [22]. For each $i\geqq 0$, let ζ_{p^i} be a primitive p^i-th root of 1. As is easily seen, there is a natural number $r\geqq 2$ such that for any $i\geqq r$, $K(\zeta_{p^{i+1}})$ is a totally ramified extension of degree p of $K(\zeta_{p^i})$. By the above remark (2), we may assume $K=K(\zeta_{p^r})$.

Take elements $b_1, \cdots, b_q$ of O_K such that the residue classes $\bar{b}_1, \cdots, \bar{b}_q$ form a p-base of F (over $\boldsymbol{F}_p$) in the sense of Grothendieck [8] Ch. 0 §21. Take elements $x_{i,j}$ ($i=0, 1, 2, \cdots; 1\leqq j\leqq q$) of K_s satisfying the conditions $x_{0,j}=b_j$ and $x_{i+1,j}^p=x_{i,j}$ for any i and j. Let $M=\bigcup_i K(\zeta_{p^i})$, $N=K(\{x_{i,j}\,|\,i\geqq 0, 1\leqq j\leqq q\})$ and $S=M\cdot N$.

We claim $\mathrm{cd}_p(S)\leqq 1$. Indeed, generally, if k is a Henselian discrete valuation field as N is, $\mathrm{Gal}((\hat{k})_s/\hat{k})\xrightarrow{\cong}\mathrm{Gal}(k_s/k)$ by Artin [1] Lemma 2.2.1, where $\hat{k}$ denotes the completion of k. If $\mathrm{ch}(k)=0$, $\mathrm{Br}(k)\cong\mathrm{Br}(\hat{k})$ follows easily from this fact. If furthermore k has a perfect residue field of characteristic $p>0$ as N does, we have a commutative diagram for each finite extension k'/k,

$$\begin{array}{cccccc} w & \mathrm{Br}(k)_p & \cong & (X_{\bar{k}})_p & & \chi \\ \downarrow & \downarrow & & \downarrow & & \downarrow \\ w_{k'} & \mathrm{Br}(k')_p & \cong & (X_{\bar{k}'})_p & & e_{k'/k}\cdot\chi_{\bar{k}'} \end{array}$$

by Serre [20] Ch. XII §3, where $\bar{k}$ and $\bar{k}'$ are the residue fields of k and k'

respectively, and $e_{k'/k}$ is the ramification index of the extension k'/k. This diagram proves $\mathrm{Br}(S')_p=0$ for any finite extension S' of S (cf. the proof of Serre [21] Ch. II § 3.3 Prop. 9). The conclusion $\mathrm{cd}_p(S)\leqq 1$ follows from it by [21] Ch. II § 2.3 Prop. 4.

Now, we can apply Prop. 3 by taking K as k of Prop. 3, $q+1$ as q, and S as S. It suffices to prove $\mathrm{cd}_p(\mathrm{Gal}(S/K))\leqq q+1$ and the fact that the cup product

$$(*)\qquad \overbrace{H^1(J,\boldsymbol{Z}/p\boldsymbol{Z})\otimes\cdots\otimes H^1(J,\boldsymbol{Z}/p\boldsymbol{Z})}^{q+1\ \text{times}}\longrightarrow H^{q+1}(J,\boldsymbol{Z}/p\boldsymbol{Z})$$

is surjective for any open subgroup J of $\mathrm{Gal}(S/K)$. Regard $\mathrm{Gal}(S/M)$ as a $\mathrm{Gal}(M/K)$-module, on which an element τ of $\mathrm{Gal}(M/K)$ acts by the inner automorphism $\sigma\mapsto\tilde{\tau}\sigma\tilde{\tau}^{-1}$, where $\tilde{\tau}$ is a representative of τ. Let $\boldsymbol{Z}_p(1)$ be the inverse limit of the inverse system of $\mathrm{Gal}(M/K)$-modules $(\mu_{p^i})_{i\geqq 0}$ whose transition maps are $\mu_{p^{i+1}}\to\mu_{p^i}$; $x\mapsto x^p$. Then, as is easily seen, we have an isomorphism of $\mathrm{Gal}(M/K)$-modules

$$\mathrm{Gal}(S/M)\cong\overbrace{\boldsymbol{Z}_p(1)\times\cdots\times\boldsymbol{Z}_p(1)}^{q\ \text{times}}$$
$$\sigma\mapsto((\sigma(x_{i,1})/x_{i,1})_i,\ \cdots,\ (\sigma(x_{i,q})/x_{i,q})_i)\,.$$

Furthermore, the homomorphism $\mathrm{Gal}(M/K)\to\boldsymbol{Z}_p^*$ induced by the action of $\mathrm{Gal}(M/K)$ on $\boldsymbol{Z}_p(1)$ gives an isomorphism from $\mathrm{Gal}(M/K)$ onto the multiplicative group $1+p^r\boldsymbol{Z}_p$, which is isomorphic to the additive group $\boldsymbol{Z}_p$ because $r\geqq 2$. It is easy to deduce $\mathrm{cd}_p(\mathrm{Gal}(S/K))\leqq q+1$ and the surjectivity of the above homomorphism $(*)$ from these informations of the structure of $\mathrm{Gal}(S/K)$.

PROOF OF PROPOSITION 2. By induction on d, we may assume $d=1$. Since $\mathrm{cd}_p(K)\leqq q+1$ in this case, we may assume that K contains a primitive p-th root of 1 by the remarks (1) and (2) in the proof of Prop. 3. We can apply Prop. 3 by taking K, $K\cdot k_s$, q as k, S, q of Prop. 3, respectively.

PROOF OF COROLLARY TO PROPOSITION 2. Prop. 2 can be directly applied to the cases (i) and (ii). For the cases (iii) and (iv), it suffices to recall that $\mathrm{cd}_p(k)\leqq 2$ in these cases (cf. Serre [21] Ch. II § 4 and § 5) and the bijectivity of $h_{p,k}^2$ for these fields (Tate [23] Th. 2).

PROOF OF THEOREM 1 except the part $\mathrm{cd}_p(K)\geqq q+1$ in Th. 1 (1). (The part $\mathrm{cd}_p(K)\geqq q+1$ will be obtained in § 1.4 Cor. 2 to Prop. 4.) Since we have obtained the inequality $\mathrm{cd}_p(K)\leqq q+2$ in the proof of Prop. 1, it remains to prove the following (i) (ii) (iii).

(i) $P_1^q(F)\cong\Omega_F^q/(1-\gamma)\Omega_F^q$.

(ii) *The homomorphism* h_{K,p^n}^F *is well defined.*

(iii) h_{K,p^n}^F *is surjective.*

(The assertion (3) of Th. 1 is a consequence of (2) by the proof of Serre [21] Ch. II §2 Prop. 4 if one admits $\mathrm{cd}_p(K)\geqq q+1$.)

We need the following Lemmas.

LEMMA 5. *Let R be a ring such that the additive group R is generated by R^*. Let $q\geqq 0$. Then, there is an isomorphism*

$$(R\otimes\overbrace{R^*\otimes\cdots\otimes R^*}^{q\text{ times}})/J\cong\Omega_R^q$$

$$x\otimes y_1\otimes\cdots\otimes y_q\mapsto x\cdot\frac{dy_1}{y_1}\wedge\cdots\wedge\frac{dy_q}{y_q},$$

where J is the subgroup of the tensor product generated by all elements of the following forms (i) and (ii).

(i) $\left\{\left(\sum_{i=1}^n x_i\right)\otimes\left(\sum_{i=1}^n x_i\right)-\sum_{i=1}^n x_i\otimes x_i\right\}\otimes y_1\otimes\cdots\otimes y_{q-1}$

$$\left(n\geqq 0,\ \text{each } x_i,\ \sum_{i=1}^n x_i,\ y_1,\cdots,y_{q-1}\in R^*\right).$$

(ii) $x\otimes y_1\otimes\cdots\otimes y_q$ *satisfying* $y_i=y_j$ *with* i *and* j *such that* $i\neq j$.

PROOF. This follows easily from the fact that Ω_R is the quotient of the free R-module with basis $([a])_{a\in R}$ by the R-submodule generated by all elements of the forms $[a+b]-[a]-[b]$ and $[ab]-a[b]-b[a]$.

COROLLARY. *For any field k of characteristic $p>0$, $P_1^q(k)$ is isomorphic to the cokernel of $1-\gamma:\Omega_{k,d=0}^q\to\Omega_k^q$, where $\Omega_{k,d=0}^q$ denotes the kernel of the exterior derivation $d:\Omega_k^q\to\Omega_k^{q+1}$.*

PROOF. This follows from the fact that $\Omega_{k,d=0}^q$ is additively generated by elements of the forms dw $(w\in\Omega_k^{q-1})$ and $x^p\cdot\frac{dy_1}{y_1}\wedge\cdots\wedge\frac{dy_q}{y_q}$ $(x, y_1,\cdots,y_q\in k^*)$.

DEFINITION 4. Let k be a discrete valuation field. Let $q\geqq 1$ and $n\geqq 1$. We denote by $\mathfrak{U}_q^{(n)}(k)$ (or simply by $\mathfrak{U}_q^{(n)}$) the subgroup of $\mathfrak{K}_q(k)$ generated by all elements of the form $\{1+x, y_1,\cdots,y_{q-1}\}$ such that $x\in m_k^n$ and $y_1,\cdots,y_{q-1}\in k^*$.

LEMMA 6. *Let K be a discrete valuation field with residue field F. Let $q\geqq 1$.*

(1) (Bass and Tate [4] Ch. I §4 Prop. 4.3.) *There is an exact sequence*

$$0\longrightarrow\mathfrak{K}_q(F)\stackrel{i}{\longrightarrow}\mathfrak{K}_q(K)/\mathfrak{U}_q^{(1)}(K)\stackrel{\partial}{\longrightarrow}\mathfrak{K}_{q-1}(F)\longrightarrow 0.$$

Here i denotes the homomorphism $\{\bar{x}_1,\cdots,\bar{x}_q\}\mapsto\{x_1,\cdots,x_q\}$ $(x_1,\cdots,x_q\in U_K)$ and ∂ is as in §1.2, 1°. It splits by $\mathfrak{K}_{q-1}(F)\to\mathfrak{K}_q(K)/\mathfrak{U}_q^{(1)}$; $\{\bar{x}_1,\cdots,\bar{x}_{q-1}\}\mapsto\{x_1,\cdots,x_{q-1},\pi\}$, where π is any fixed prime element of K.

(2) *Assume* $n \geqq 1$. *Fix a prime element* π *of* K, *and elements* c *and* c' *such that* $v_K(c)=v_K(c')=n$. *Then, there is a surjective homomorphism*

$$\varphi_n^q : \Omega_F^{q-1} \oplus \Omega_F^{q-2} \longrightarrow \mathfrak{U}_q^{(n)}/\mathfrak{U}_q^{(n+1)} ;$$

$$\varphi_n^q\left(\bar{x}\cdot\frac{d\bar{y}_1}{\bar{y}_1}\wedge\cdots\wedge\frac{d\bar{y}_{q-1}}{\bar{y}_{q-1}},\ \bar{z}\cdot\frac{d\bar{u}_1}{\bar{u}_1}\wedge\cdots\wedge\frac{d\bar{u}_{q-2}}{\bar{u}_{q-2}}\right)$$

$$=\{1+xc,\ y_1,\ \cdots,\ y_{q-1}\}+\{1+zc',\ u_1,\ \cdots,\ u_{q-2},\ \pi\}$$

$$(x,\ z\in O_K,\ y_1,\ \cdots,\ y_{q-1},\ u_1,\ \cdots,\ u_{q-2}\in U_K).$$

If n *is invertible in* F, $\varphi_n^q(*, 0): \Omega_F^{q-1} \to \mathfrak{U}_q^{(n)}/\mathfrak{U}_q^{(n+1)}$ *is surjective.*

(3) *Assume* $\mathrm{ch}(F)=p>0$, *and assume* $1 \leqq n < e_K p/(p-1)$ *and* $p|n$, *where* $e_K=v_K(p)$. *Take elements* $b, b' \in K$ *such that* $v_K(b)=v_K(b')=n/p$, *and let the elements* c *and* c' *in* (2) *be* b^p *and* b'^p, *respectively. Then,* φ_n^q *induces a surjective homomorphism*

$$\bar{\varphi}_n^q : \Omega_F^{q-1}/\Omega_{F,d=0}^{q-1} \oplus \Omega_F^{q-2}/\Omega_{F,d=0}^{q-2} \longrightarrow (\mathfrak{U}_q^{(n)}+p\cdot\mathfrak{K}_q(K))/(\mathfrak{U}_q^{(n+1)}+p\cdot\mathfrak{K}_q(K)).$$

(4) *Assume* $\mathrm{ch}(F)=p>0$ *and* $\mathrm{ch}(K)=0$, *and that* K *contains a primitive* p*-th root* ζ *of* 1. *Let* $n=e_K p/(p-1)$ *and let* $c=c'=(\zeta-1)^p$. *Then,* φ_n^q *induces a surjective homomorphism*

$$\bar{\varphi}_n^q : \Omega_F^{q-1}/(1-\gamma)(\Omega_{F,d=0}^{q-1}) \oplus \Omega_F^{q-2}/(1-\gamma)(\Omega_{F,d=0}^{q-2}) \longrightarrow (\mathfrak{U}_q^{(n)}+p\cdot\mathfrak{K}_q(K))/\mathfrak{U}_q^{(n+1)}+p\cdot\mathfrak{K}_q(K)).$$

Note that $\mathfrak{U}_q^{(n)} \subset p\cdot\mathfrak{K}_q(K)$ for $n>e_K p/(p-1)$, if K is complete and $\mathrm{ch}(F)=p>0$.

REMARK 3. Let K, F and q be as in Lemma 6. Assume $\mathrm{ch}(F)=p>0$, and let $1 \leqq n < e_K p/(p-1)$. In the case that n is prime to p, let $\bar{\varphi}_n^q$ be the homomorphism $\Omega_F^{q-1} \to (\mathfrak{U}_q^{(n)}+p\cdot\mathfrak{K}_q(K))/(\mathfrak{U}_q^{(n+1)}+p\cdot\mathfrak{K}_q(K))$ induced by $\varphi_n^q(*, 0)$. In the case $p|n$, let $\bar{\varphi}_n^q$ be as in Lemma 6 (3). We shall claim in § 1.4 Prop. 4, that $\bar{\varphi}_n^q$ is bijective. Thus we can obtain exact knowledge of a large part of $\mathfrak{K}_q(K)/p\cdot\mathfrak{K}_q(K)$.

PROOF OF LEMMA 6. The homomorphisms φ_n^q are well defined by virtue of Lemma 5. All things can be checked easily, as in the proof of Chapter I § 2 Prop. 1.

LEMMA 7. *Let* k *be a field of characteristic* $p>0$. *Let* $q \geqq 1$ *and* $b_1, \cdots, b_q \in k^*$. *Then, the condition* $[k(b_i^{1/p}\ (1 \leqq i \leqq q)): k] < p^q$ *is equivalent to* $\{b_1, \cdots, b_q\} \in p\cdot\mathfrak{K}_q(k)$. *In particular,* $\mathfrak{K}_q(k)$ *is* p*-divisible if and only if* $[k: k^p] < p^q$.

PROOF. First, assume $[k(b_i^{1/p}\ (1 \leqq i \leqq q)): k] < p^q$. We can take $n<q$ and fields $E_0, \cdots, E_n$ such that $k=E_0 \subset \cdots \subset E_n$, $[E_i: E_{i-1}]=p$ and $b_i^{1/p} \in E_i$ for each $i=1, \cdots, n$, and $b_{n+1}^{1/p} \in E_n$. Then,

$$\{b_1, \cdots, b_{n+1}\} = \{\mathfrak{N}_{E_1/E_0} \circ \cdots \circ \mathfrak{N}_{E_n/E_{n-1}}\{b_1^{1/p}, \cdots, b_n^{1/p}\}, b_{n+1}\}$$
$$= p \cdot \mathfrak{N}_{E_n/k}(\{b_1^{1/p}, \cdots, b_{n+1}^{1/p}\}).$$

Conversely, assume $\{b_1, \cdots, b_q\} \in p \cdot \mathfrak{K}_q(k)$. Then, the existence of the homomorphism

$$\mathfrak{K}_q(k)/p \cdot \mathfrak{K}_q(k) \longrightarrow \Omega_k^q;\ \{x_1, \cdots, x_q\} \longmapsto \frac{dx_1}{x_1} \wedge \cdots \wedge \frac{dx_q}{x_q}$$

shows $\frac{db_1}{b_1} \wedge \cdots \wedge \frac{db_q}{b_q} = 0$. Hence $[k(b_i^{1/p}\ (1 \leqq i \leqq q)) : k] < p^q$.

Now, we start the proofs of (i), (ii) and (iii) at the beginning of this Proof of Theorem 1. We have already proved (i) in Corollary to Lemma 5. To prove (ii), let K_w be the cyclic extension of K corresponding to $i(w)$ (cf. Definition 2). Then, (ii) can be reduced to $i(w) \cup h_{p^n,K}^1(N_{K_w/K}(K_w^*)) = 0$. Indeed, $U_K^{(1)} \subset N_{K_w/K}(K_w^*)$, for K_w/K is unramified. Furthermore, if $w \in W_n(F)$ has the form

$$(\overbrace{0, \cdots, 0}^{i \text{ times}}, a, 0, \cdots, 0) \quad (0 \leqq i < n,\ a \in F^*)$$

and F_w denotes the cyclic extension of F corresponding to w via the isomorphism $W_n(F)/(\mathfrak{F}-1)W_n(F) \cong (X_F)_{p^n}$ (i.e. F_w is the residue field of K_w), then $a \in N_{F_w/F}(F_w^*)$ by Teichmüller [24] Satz 1 and hence $\tilde{a} \in N_{K_w/K}(K_w^*)$.

Lastly, we prove (iii). This is reduced to the surjectivity of $h_{K,p}^F$ by induction on n using the commutative diagram of exact sequences

$$\begin{array}{ccccccc}
P_{n-1}^q(F) & \xrightarrow{\alpha} & P_n^q(F) & \xrightarrow{\beta} & P_1^q(F) & \longrightarrow 0 \\
h_{K,p^{n-1}}^F \downarrow & & h_{K,p^n}^F \downarrow & & h_{K,p}^F \downarrow & \\
H^{q+2}(K, \mu_{p^{n-1}}^{\otimes(q+1)}) & \longrightarrow & H^{q+2}(K, \mu_{p^n}^{\otimes(q+1)}) & \longrightarrow & H^{q+2}(K, \mu_p^{\otimes(q+1)}), &
\end{array}$$

where

$$\alpha(\{(a_0, \cdots, a_{n-2}), b_1, \cdots, b_q\}) = \{(0, a_0, \cdots, a_{n-2}), b_1, \cdots, b_q\},$$
$$\beta(\{(a_0, \cdots, a_{n-1}), b_1, \cdots, b_q\}) = \{a_0, b_1, \cdots, b_q\}.$$

Thus, we are reduced to the surjectivity of the homomorphism $\Omega_F^q/(1-\gamma)\Omega_F^q \to H^{q+2}(K, \mu_p^{\otimes(q+1)})$. As is easily seen, we may assume that K contains a primitive p-th root ζ of 1. In this case, this homomorphism coincides with the composite

$$\Omega_F^q/(1-\gamma)\Omega_F^q \xrightarrow{\text{by } \bar{\varphi}_{e_K p/(p-1)}^{q+2}} \mathfrak{K}_{q+2}(K)/p \cdot \mathfrak{K}_{q+2}(K) \xrightarrow{h_{p,K}^{q+2}} H^{q+2}(K, \mu_p^{\otimes(q+2)}).$$

But the first arrow in this composite is surjective by Lemma 6, for $\mathfrak{K}_{q+1}(F)$ is p-divisible by Lemma 7, and the second is surjective by Proposition 1.

§ 1.4. The cohomological residue.

In this § 1.4, we define a homomorphism called the *cohomological residue*. This homomorphism will play a key role in the proofs of Theorem 2 and Theorem 3, and the following Proposition 4. (Cf. § 1.6.)

PROPOSITION 4. *Let K be a discrete valuation field with residue field F. Assume* $\mathrm{ch}(F)=p>0$, *and let n be an integer such that $1\leqq n<e_K p/(p-1)$. Then, the homomorphism $\bar{\varphi}_n^q$ in § 1.3 Remark 4 is bijective for any $q\geqq 1$. Assume further* $\mathrm{ch}(K)=0$ *and let $\bar{\mathfrak{U}}_q^{(n)}\subset H^q(K, \mu_p^{\otimes q})$ be the image of $\mathfrak{U}_q^{(n)}$ under the homomorphism $h_{p,K}^q$ for each $q\geqq 1$. Then, $h_{p,K}^q$ induces an isomorphism*

$$(\mathfrak{U}_q^{(n)}+p\cdot\mathfrak{K}_q(K))/(\mathfrak{U}_q^{(n+1)}+p\cdot\mathfrak{K}_q(K))\cong\bar{\mathfrak{U}}_q^{(n)}/\bar{\mathfrak{U}}_q^{(n+1)}\,.$$

COROLLARY 1. *Suppose* $\mathrm{ch}(K)=0$, *K is complete, and F is separably closed. Then, the restriction of $h_{p,K}^q$ to $(\mathfrak{U}_q^{(1)}+p\cdot\mathfrak{K}_q(K))/p\cdot\mathfrak{K}_q(K)$ is injective.*

COROLLARY 2. *Let K be any field and let $q\geqq 0$. Suppose that* $\mathrm{ch}(K)=0$ *and that there is a discrete valuation of K such that the residue field F satisfies* $\mathrm{ch}(F)=p>0$ *and $[F: F^p]\geqq p^q$. Then,* $\mathrm{cd}_p(K)\geqq q+1$.

PROOF. Since $\Omega_F^q\neq 0$, $\bar{\mathfrak{U}}_{q+1}^{(1)}/\bar{\mathfrak{U}}_{q+1}^{(2)}\neq 0$ by Prop. 4. This proves $H^{q+1}(K, \mu_p^{\otimes(q+1)})\neq 0$. This Cor. 2 completes the proof of Theorem 1.

DEFINITION of the cohomological residue. Let k and K be complete discrete valuation fields such that $k\subset K$, and E and F their residue fields, respectively. Suppose that the following conditions (i), (ii), and (iii) are satisfied.

(i) *The inclusion $k\subset K$ satisfies $O_k\subset O_K$ and $m_k\subset m_K$.*

(ii) *F is a Henselian discrete valuation field such that its valuation ring contains E and its residue field $\bar{F}$ is of finite degree over E if we regard $\bar{F}$ as an extension of E via the composite $E\to O_F\to\bar{F}$.*

(iii) *The transcendental degree of F over E is one.*

(The conditions (ii) and (iii) are satisfied, for example, if F is the algebraic closure of $E(X)$ in the field of formal power series $E((X))$.)

Let $q\geqq 0$ and $r\in \boldsymbol{Z}$, and let m be an integer invertible in k. We define now a homomorphism

$$\mathrm{Res}_m^{q,r}: H^{q+1}(K, \mu_m^{\otimes(r+1)}) \longrightarrow H^q(k, \mu_m^{\otimes r})\,,$$

called the cohomological residue. We need the following two Lemmas.

LEMMA 8. *Let k and K be as above. Then:*

(1) *There is a homomorphism $t_{K/k}: K^*\to\boldsymbol{Z}$ characterized by the properties $t_{K/k}(k^*)=0$ and*

$$t_{K/k}(x)=[\bar{F}:E]\cdot e_{K/k}\cdot v_F(\bar{x}) \quad \textit{for all } x\in U_K,$$

where $e_{K/k}$ *denotes* $v_K(\pi)$ *for prime elements* π *of* k.

(2) *There is a* $\mathrm{Gal}(k_s/k)$*-homomorphism* $T_{K/k}:(k_s\otimes_k K)^*\to \boldsymbol{Z}$, *where* $\mathrm{Gal}(k_s/k)$ *acts on* $\boldsymbol{Z}$ *trivially, characterized by the following property: If* k' *is a finite separable extension of* k *and if* $k'\otimes_k K=\prod_i K^{(i)}$ *a finite product of fields, the restriction of* $T_{K/k}$ *to* $K^{(i)*}$ *coincides with* $t_{K^{(i)}/k'}$ *of* (1).

LEMMA 9. *Let* k *and* K *be as above. Then,* $\mathrm{cd}(k_s\cdot K)=1$.

Now, we define the cohomological residue as the composite

$$\begin{aligned} H^{q+1}(K, \mu_m^{\otimes(r+1)}) &\longrightarrow H^q(k, H^1(k_s\otimes_k K, \mu_m^{\otimes(r+1)})) \\ &\cong H^q(k, \mu_m^{\otimes r}\otimes(k_s\otimes_k K)^*/((k_s\otimes_k K)^*)^m) \\ &\longrightarrow H^q(k, \mu_m^{\otimes r}), \end{aligned}$$

where the first arrow is obtained by Lemma 9 and the last arrow is the homomorphism induced by the $\mathrm{Gal}(k_s/k)$-homomorphism $T_{K/k}$ of Lemma 8 (2). (Since $k_s\otimes_k K$ is a finite product of fields, $H^*(k_s\otimes_k K, ?)$ can be defined as the direct sum of the Galois cohomology groups of the fields.)

REMARK 4. The appropriateness of the name "cohomological residue" will be explained in § 1.8 Lemma 19.

REMARK 5. It is probable that there is a canonical homomorphism $\widehat{\mathrm{Res}}_m^{q,r}$: $H^{q+1}(K, \mu_m^{\otimes(r+1)})\to H^q(k, \mu_m^{\otimes r})$ in the case that the above condition (iii) is replaced by the condition F is complete, which induces the above cohomological residue $\mathrm{Res}_m^{q,r}$. It is this homomorphism $\widehat{\mathrm{Res}}_m^{q,r}$ that is truly desired, but the author has not defined it.

PROOF OF LEMMA 8 AND LEMMA 9. The proof of Lemma 8 (1) is easy and left to the reader. Next, from the above assumptions (ii) and (iii), we can deduce the following (iv) and (v).

(iv) Let α be an element of F which is transcendental over E. Then, F is a finite extension of the separable closure in F of $E(\alpha)$.

(v) The completion $\hat{F}$ of F is a separable extension of F. It follows that $[F':F]=e_{F'/F}\cdot[\bar{F}':\bar{F}]$ for any finite extension F' of F, where $e_{F'/F}$ denotes the ramification index and $\bar{F}'$ denotes the residue field of F'.

Now, Lemma 8 (2) is a consequence of (1) and the above (v). To prove Lemma 9, let α be an element of O_K such that the residue class $\bar{\alpha}$ is transcendental over E. Then, the valuation of K induces on $k(\alpha)$ a discrete valuation with residue field $E(\bar{\alpha})$. Let M be the completion of $k(\alpha)$, N the unramified extension of M corresponding to the separable closure in F of $E(\bar{\alpha})$, and $\hat{N}$ the completion of N. Since $[K:\hat{N}]<\infty$ by the above (iv), the canonical homomorphisms

$$\mathrm{Gal}(K_s/K)\longrightarrow \mathrm{Gal}(\hat{N}_s/\hat{N})\xrightarrow{\cong}\mathrm{Gal}(N_s/N)\longrightarrow \mathrm{Gal}(M_s/M)\longrightarrow \mathrm{Gal}(k(\alpha)_s/k(\alpha))$$

are all injective. Hence $\mathrm{Gal}(K_s/(k_s\cdot K))$ is isomorphic to a closed subgroup of $\mathrm{Gal}(k(\alpha)_s/k_s(\alpha))$. It follows

$$\mathrm{cd}(k_s\cdot K)\leqq \mathrm{cd}(k_s(\alpha))=1$$

by Serre [21] Ch. I § 3 Prop. 14 and Ch. II § 4 Prop. 11.

§ 1.5. Constructions of complete discrete valuation fields.

This § 1.5 is a preliminary for the proofs of Th. 2 and Th. 3, and Prop. 4.

LEMMA 10. *Let R be a ring and $\mathfrak{p}$ a prime ideal of R. Let F be a field containing $R/\mathfrak{p}R$ as a subring. Then, there are a direct system (R_i, f_{ji}) of rings over R with directed index set, and an R-isomorphism $g:\varinjlim R_i/\mathfrak{p}R_i\xrightarrow{\cong}F$ such that*

(i) *Each R_i is finitely generated as a ring over R, and flat over R.*

(ii) *Each induced homomorphism $R_i/\mathfrak{p}R_i\to F$ is injective.*

PROOF. Endow the set F with a structure of a well-ordered set. By transfinite induction, we can construct for each $\alpha\in F$ a direct system $\Psi_\alpha=(R_i, f_{ji})$ of rings over R with directed index set I_α, and an R-homomorphism $g_\alpha:\varinjlim_{I_\alpha} R_i/\mathfrak{p}R_i\to F$, satisfying the above conditions (i) and (ii) and the following conditions (iii) and (iv).

(iii) If $\alpha\leqq\beta$, I_α is a sub-ordered set of I_β. For any $i, j\in I_\alpha$ such that $i\leqq j$, R_i and f_{ji} of Ψ_α coincide with those with respect to Ψ_β respectively, and the restriction of g_β to $\varinjlim_{I_\alpha} R_i/\mathfrak{p}R_i$ coincides with g_α.

(iv) For each α, $\{\beta\in F\mid \beta\leqq\alpha\}\subset\mathrm{Image}(g_\alpha)$.

Indeed, suppose we have defined Ψ_α and g_α for all $\alpha<\alpha_0$. Let $I=\bigcup_{\alpha<\alpha_0} I_\alpha$, $\Psi=\bigcup_{\alpha<\alpha_0}\Psi_\alpha$, $g=\varinjlim_{\alpha<\alpha_0} g_\alpha$. Here $\bigcup_{\alpha<\alpha_0}\Psi_\alpha$ means the direct system whose index set is I and whose R_i and f_{ji} are those of Ψ_α $(\alpha<\alpha_0)$. As is easily seen, there are $i_0\in I$, $b\in R_{i_0}-\mathfrak{p}R_{i_0}$ and $f\in R_{i_0}[b^{-1}][X]$ such that f is monic or zero, $f(\alpha_0)=0$ when we regard f as a polynomial over F via g, and the homomorphism $(R_i[b^{-1}]/\mathfrak{p}R_i[b^{-1}])[X]/(f(X))\to F$; $X\mapsto\alpha_0$ induced by g is injective for any $i\geqq i_0$ $(i\in I)$. Let I' be a copy of the set $\{i\in I\mid i\geqq i_0\}$ and denote by i' the element of I' corresponding to $i\in I$ such that $i\geqq i_0$. Let I_{α_0} be the disjoint union of I and I', and endow it with the following order: If $i, j \in I$, $i \leqq j$ in I_{α_0} is equivalent to $i \leqq j$ in I. If $i', j' \in I'$, $i' \leqq j'$ in I_{α_0} is equivalent to $i \leqq j$ in I. If $i \in I$ and $j' \in I'$, $i \leqq j'$ in I_{α_0} is equivalent to $i \leqq j$ in I, and $i \geqq j'$ in I_{α_0} does not occur. Now we can define the direct system Ψ_{α_0} with index set I_{α_0} as follows. For each $i \in I \subset I_{\alpha_0}$, let R_i be the R_i of Ψ. For each $i' \in I' \subset I_{\alpha_0}$ $(i \in I,\ i \geqq i_0)$, let $R_{i'} = R_i[b^{-1}][X]/(f(X))$. Let f_{ji} $(i \leqq j,\ i, j \in I_{\alpha_0})$ and g_{α_0} be the natural homomorphisms. Thus we have obtained Ψ_α and g_α for all $\alpha \in F$. Now the direct system $\bigcup_\alpha \Psi_\alpha$ and the homomorphism $g = \varinjlim g_\alpha$ satisfy all the conditions in Lemma 10.

COROLLARY 1. *Let k and K be complete discrete valuation fields such that $k \subset K$, and E, F the residue fields of k and K, respectively. Suppose that any prime element of k is still a prime element in K, and that F is separable over E. Then, there are a direct system (A_i, f_{ji}) of rings over O_k with directed index set, and an O_k-homomorphism $g: \varinjlim A_i \to O_K$ having the following properties* (i)~(iv).

(i) *Each A_i is a Noetherian ring, flat over O_k, and complete with respect to the $m_k A_i$-adic topology.*

(ii) *Each induced homomorphism $A_i/m_k A_i \to F$ is injective, and $\varinjlim A_i/m_k A_i \xrightarrow{\cong} F$.*

(iii) *Each $A_i/m_k A_i$ is finitely generated as a ring over E.*

(iv) *$\varinjlim A_i$ is a Henselian discrete valuation ring and its completion is isomorphic to O_K via g.*

PROOF. Let $R = O_k$, $\mathfrak{p} = m_k$, and $F = F$ in Lemma 10. Let A_i be the completion of R_i with respect to the $m_k R_i$-adic topology for each i. Since F is separable over E, the completion of $\varinjlim A_i$ is isomorphic to O_K over O_k by Grothendieck [8] Ch. 0 §19 Th. 19.8.2.

COROLLARY 2. *Let K be a complete discrete valuation field with residue field F. Then, there is a direct system (k_i, f_{ji}) of complete discrete valuation fields with directed index set having the following properties* (i)~(iii).

(i) *If $i \leqq j$ and π is a prime element of k_i, $f_{ji}(\pi)$ is a prime element of k_j.*

(ii) *$\varinjlim k_i$ is a Henselian discrete valuation field whose completion is isomorphic to K as a discrete valuation field.*

(iii) *The residue field of each k_i is finitely generated as a field over the prime field.*

PROOF. We may assume that $\mathrm{ch}(F) = p > 0$ and $\mathrm{ch}(K) = 0$, for in the other cases, $K \cong F((T))$ by Nagata [18] §31 Th. 31.1. By [18] §31 Th. 31.12, $O_K \cong O_{K'}[X]/(h(X))$ where K' is a complete discrete valuation field with residue field

F having p as a prime element, and $h(X)$ is an Eisenstein polynomial over $O_{K'}$. Apply Cor. 1 by taking $\boldsymbol{Q}_p$, K', $\boldsymbol{F}_p$, and F as k, K, E, and F. Let (A_i, f_{ji}) be the resulting direct system and let $A=\varinjlim A_i$. Then, for some Eisenstein polynomial $h_0(X)$ over A which is sufficiently near to $h(X)$, O_K is isomorphic to the completion of $A[X]/(h_0(X))$. Take an index i_0 such that $h_0(X)$ is defined over A_{i_0}. Then, the direct system $(A_i[X]/(h_0(X)))_{i\geq i_0}$ induces by localization and completion, the desired system $(k_i)_{i\geq i_0}$.

The above Cor. 1 has the following consequence Lemma 11. Consider the following property (P) of a field-extension K/k.

(P) Any finitely generated subring A over k of K has a k-homomorphism $A\rightarrow k$.

Example 1. *Any purely transcendental extension of an infinite field has the property* (P).

Example 2. *If k'/k and k''/k' has* (P), *so does k''/k.*

Example 3. *Let k be a Henselian discrete valuation field and $\hat{k}$ its completion. If $\hat{k}$ is separable over k, then the extension $\hat{k}/k$ has the property* (P).

This Example 3 can be easily deduced from Greenberg [9].

LEMMA 11. *Let k and K be complete discrete valuation fields such that $k\subset K$, and E, F the residue fields of k and K, respectively. Suppose that any prime element of k is still a prime element in K, and that the extension F/E has the property* (P). *Then*:

(1) *The extension K/k also has* (P).

(2) *For any discrete* $\mathrm{Gal}(k_s/k)$*-module M on which* $\mathrm{Gal}(k_s/k)$ *acts continuously, the canonical homomorphism $H^*(k, M)\rightarrow H^*(K, M)$ is injective.*

PROOF. We can easily prove that any extension having the property (P) is separable. If $\mathrm{ch}(k)=p>0$ and $k=E((T))$, take a direct system of finitely generated subrings R_i over E of F with directed index set such that $\varinjlim R_i=F$, and let $A_i=R_i[[T]]$. If $\mathrm{ch}(k)=0$, let $(A_i)_i$ be a direct system given by the above Cor. 1. Since F/E has the property (P), the set of E-rational points in $\mathrm{Spec}(A_i/m_kA_i)$ is a dense subset for each i. Hence $\mathrm{Spec}(A_i/m_kA_i)$ has a smooth E-rational point $A_i/m_kA_i\rightarrow E$. But such E-homomorphism can be lifted to an O_k-homomorphism $A_i\rightarrow O_k$. Let K° be the field of fractions of $\varinjlim A_i$. The completion of K° is isomorphic to K over k, and separable over K°. Thus, the extensions K°/k and K/K° have the property (P) (cf. Example 3).

Next, (2) follows from the following easy lemma, by taking the etale cohomology functor $H^*_{\mathrm{et}}(\ , M)$ (cf. Artin, Grothendieck and Verdier [2]) as F.

LEMMA 12. *Assume that K/k is a field-extension having the property* (P). *Let F be a functor from the category of all k-rings to the category of all sets which preserves all filtering direct limits. Then, $F(k) \to F(K)$ is injective.*

§1.6. The proofs of Theorem 2, Theorem 3, and Proposition 4.

PROOF OF THEOREM 2. Let K, F, and E be as in the hypothesis of Th. 2. We may assume that the prime number p is a prime element of K, for K is a totally ramified finite extension of a complete discrete valuation field K' in which p is a prime element and $\mathrm{Cor}_{K/K'} \circ h^F_{K,p^n} = h^F_{K',p^n}$. Since E is perfect, there is a unique subfield k of K which is complete with respect to the induced discrete valuation and whose residue field is E. Note that p is a prime element both in k and in K. Consider the following commutative diagram.

$$\begin{array}{ccc} (X_E)_{p^n} \cong P^0_n(E) & \xrightarrow{\quad\alpha\quad} & P^q_n(F) \\ \Big\downarrow h^E_{k,p^n} & & \Big\downarrow h^F_{K,p^n} \\ H^2(k, \mu_{p^n}) & \xrightarrow{\quad\beta\quad} & H^{q+2}(K, \mu_{p^n}^{\otimes(q+1)}) . \end{array}$$

Here α denotes the homomorphism $w \mapsto \{w, X_1, \cdots, X_q\}$ and β denotes the homomorphism $w \mapsto w_K \cup h^q_{p^n,K}(\{\tilde{X}_1, \cdots, \tilde{X}_q\})$, where $\tilde{X}_i$ denotes a lifting of X_i to O_K for each i. By Th. 1 (4), h^F_{K,p^n} is surjective. Since E is perfect, h^E_{k,p^n} is bijective by Serre [20] Ch. XII §3 Th. 2. Furthermore, α is surjective by the following Lemma 13 and induction on n and q. Thus, it remains to prove the injectivity of β, which follows from Lemma 14 below by induction on q.

LEMMA 13. *Let E be a field of characteristic $p>0$ such that $[E:E^p]=p^q<\infty$. Let $F=E((X))$. Then, the homomorphism*

$$\Omega^q_E/(1-\gamma)\Omega^q_E \longrightarrow \Omega^{q+1}_F/(1-\gamma)\Omega^{q+1}_F \,;\ \omega \longmapsto \omega \wedge \frac{dX}{X}$$

is bijective.

This follows from a simple computation of the action of γ.

LEMMA 14. *Let k, K, E and F be as in Cor.* 1 *to Lemma* 10. *Assume further that there is an isomorphism $i: E((X)) \cong F$ over E. Let $\tilde{X}$ be a lifting of $i(X)$ to O_K. Then, the homomorphism*

$$\beta : H^q(k, \mu_m^{\otimes r}) \longrightarrow H^{q+1}(K, \mu_m^{\otimes(r+1)}) \,;\ w \mapsto h^1_{m,K}(\tilde{X}) \cup w_K$$

is injective for any $q \geqq 0$ and $r \in \boldsymbol{Z}$ and for any integer m invertible in k.

PROOF. Let F° be the algebraic closure in F of $E(i(X))$. Then, there is a subfield K' of K, complete with respect to the restriction of the valuation of K, such that $k(\tilde{X})\subset K'\subset K$ and such that the induced inclusions of the residue fields coincide with $E(i(X))\subset F^{\circ}\subset F$. Then, β can be decomposed as

$$H^{q}(k, \mu_{m}^{\otimes r}) \xrightarrow{h^{1}_{m,K'}(\tilde{X})\cup ?_{K'}} H^{q+1}(K', \mu_{m}^{\otimes(r+1)}) \longrightarrow H^{q+1}(K, \mu_{m}^{\otimes(r+1)}).$$

The first arrow is injective, for the cohomological residue gives its left inverse as is deduced from § 1.2 Lemma 2 (1) by taking $\mathrm{Gal}(K'_s/K')$ $\mathrm{Gal}(K'_s/(k_s\cdot K'))$, q, 1, 0 as G, H, q, d, i, respectively. The second is injective by Lemma 11 (2), for the extension F/F° has the property (P) by § 1.5 Example 3.

PROOF OF THEOREM 3. Our proof is by induction on N. If $N=0$, there is a canonical isomorphism $X_{k_0}\cong \boldsymbol{Q}/\boldsymbol{Z}$; $\chi\mapsto\chi(\mathfrak{F}_{k_0})$ where $\mathfrak{F}_{k_0}$ is the Frobenius automorphism over k_0. Hence

$$H^{1}(k_0, \boldsymbol{Z}/m\boldsymbol{Z})\cong(X_{k_0})_m\cong\frac{1}{m}\boldsymbol{Z}/\boldsymbol{Z}.$$

Let $N\geqq 1$ and let $F=k_{N-1}$. We may assume $m=p^n$ for some prime number p which is invertible in K. If $\mathrm{ch}(F)=p$,

$$H^{N+1}(K, \mu_{p^n}^{\otimes N})\cong(X_{k_0})_{p^n}\cong\frac{1}{p^n}\boldsymbol{Z}/\boldsymbol{Z}$$

by Th. 2. If $\mathrm{ch}(F)\neq p$, $\mathrm{cd}_p(F)\leqq N$ by induction on N. Hence we have in this case,

$$H^{N+1}(K, \mu_{p^n}^{\otimes N})\cong H^{N}(F, H^{1}(K_{\mathrm{nr}}, \mu_{p^n}^{\otimes N}))\cong H^{N}(F, \mu_{p^n}^{\otimes(N-1)})\cong\frac{1}{p^n}\boldsymbol{Z}/\boldsymbol{Z}$$

by induction on N.

PROOF OF PROPOSITION 4. In the case $\mathrm{ch}(K)=p>0$, the first assertion can be proved by the computation using the homomorphism

$$\mathfrak{K}_q(K)/p\cdot\mathfrak{K}_q(K)\longrightarrow \Omega^q_K\,;\ \{x_1, \cdots, x_q\}\longmapsto\frac{dx_1}{x_1}\wedge\cdots\wedge\frac{dx_q}{x_q}.$$

(In this case, more is known about the structure of $\mathfrak{K}_q(K)$ by Bloch [3] Ch. II.) So, in the following, we assume $\mathrm{ch}(K)=0$. It suffices to prove that the homomorphism from Ω_F^{q-1} or $\Omega_F^{q-1}/\Omega_{F,d=0}^{q-1}\oplus\Omega_F^{q-2}/\Omega_{F,d=0}^{q-2}$ to $\bar{\mathfrak{U}}_q^{(n)}/\bar{\mathfrak{U}}_q^{(n+1)}$ induced by $\bar{\varphi}_n^q$ is injective. We may assume that K is complete and contains a primitive p-th root of 1. Furthermore, by virtue of Cor. 2 to Lemma 10, we may assume that F is finitely generated over the prime field $\boldsymbol{F}_p$. Since there is a separable extension F' of F such that $F'\cong E((X_1))\cdots((X_N))$ for some finite field E and for some natural number $N\geqq 0$, and since the homomorphism $\Omega^*_F\to\Omega^*_{F'}$ is injective, we may assume $F=E((X_1))\cdots((X_N))$ (E is finite). In this case, by virtue of Th. 2, $\bar{\varphi}^N_{e_K p/(p-1)}$ induces a non-zero isomorphism $\Omega^N_F/(1-\gamma)\Omega^N_F \xrightarrow{\cong} \bar{\mathfrak{U}}^{(e_K p/(p-1))}_{N+2}$. For brevity, let $e_1=e_K p/(p-1)$. We may assume N is sufficiently large. Assume $N\geqq q$. In the case $p\nmid n$, the diagram

$$\begin{array}{ccc}
\Omega^{q-1}_F\times\Omega^{N-q+1}_F & \xrightarrow{\text{by } \bar{\varphi}^q_n\times\bar{\varphi}^{N-q+2}_{e_1-n}} & \bar{\mathfrak{U}}^{(n)}_q/\bar{\mathfrak{U}}^{(n+1)}_q\times\bar{\mathfrak{U}}^{(e_1-n)}_{N-q+2}/\bar{\mathfrak{U}}^{(e_1-n+1)}_{N-q+2} \\
\Big\downarrow \text{exterior product} & & \text{cup product} \Big\downarrow \\
\Omega^N_F & \xrightarrow[\text{by } \bar{\varphi}^{N+2}_{e_1}]{} & \bar{\mathfrak{U}}^{(e_1)}_{N+2}\,,
\end{array}$$

which commutes for some suitable choices of elements π, c, c' of Lemma 6 (2), proves the injectivity of $\Omega^{q-1}_F\to\bar{\mathfrak{U}}^{(n)}_q/\bar{\mathfrak{U}}^{(n+1)}_q$; for the pairing $\Omega^{q-1}_F\times\Omega^{N-q+1}_F\to\Omega^N_F$ is non-degenerate and Ω^N_F is one dimensional over F. In the case $p|n$, if one replaces Ω^{q-1}_F, Ω^{N-q+1}_F and Ω^N_F in the above diagram by $\Omega^{q-1}_F/\Omega^{q-1}_{F,d=0}\oplus\Omega^{q-2}_F/\Omega^{q-2}_{F,d=0}$, $\Omega^{N-q+1}_F/\Omega^{N-q+1}_{F,d=0}\oplus\Omega^{N-q}_F/\Omega^{N-q}_{F,d=0}$ and $\Omega^N_F/d\Omega^{N-1}_F$ respectively, and the "exterior product" by the pairing

$$(\omega_1\oplus\omega_2,\ \eta_1\oplus\eta_2)\longmapsto d\omega_1\wedge\eta_2+d\omega_2\wedge\eta_1\,,$$

the rewritten diagram similarly proves the injectivity of

$$\Omega^{q-1}_F/\Omega^{q-1}_{F,d=0}\oplus\Omega^{q-2}_F/\Omega^{q-2}_{F,d=0}\longrightarrow\bar{\mathfrak{U}}^{(n)}_q/\bar{\mathfrak{U}}^{(n+1)}_q\,.$$

§ 1.7. The norm homomorphisms of Milnor's K-groups.

In this § 1.7, we prove the following Prop. 5, which implies that the norm homomorphisms of Milnor's K-groups defined in Bass and Tate [4] § 5 are canonical homomorphisms.

PROPOSITION 5. *Let k be a field and E a finite extension of k. Then, there exists a (clearly unique) homomorphism $\mathfrak{N}_{E/k}:\mathfrak{K}_*(E)\to\mathfrak{K}_*(k)$ such that; for any choice of finite family $(\alpha_i)_{1\leq i\leq n}$ of elements of E which generates E over k, $\mathfrak{N}_{E/k}$ coincides with the composite $N_{\alpha_1/E_0}\circ N_{\alpha_2/E_1}\circ\cdots\circ N_{\alpha_n/E_{n-1}}:\mathfrak{K}_*(E)\to\mathfrak{K}_*(k)$, where $E_i=k(\alpha_1,\cdots,\alpha_i)$ for $i=0,\cdots,n-1$.*

We begin with the following Lemmas.

LEMMA 15. *Prop. 5 is true in the case where E is a normal extension of k such that $[E:k]$ is a prime number.*

PROOF. The arguments in [4] Ch. I §5 show that we may assume that k has the following property $(\mathrm{E})_p$ for some prime number p.

$(\mathrm{E})_p$ *Every finite extension of k is of degree a power of p.*

In this case, by [4] Ch. I Cor. 5.3, $\mathfrak{K}_q(E)$ $(q\geqq 1)$ is generated by elements of the form $\{x, y_E\}$ $(x\in\mathfrak{K}_1(E),\ y\in\mathfrak{K}_{q-1}(k))$. But $N_{\alpha/k}(\{x, y_E\})=\{N_{E/k}(x), y\}$ for any $\alpha\in E$ such that $E=k(\alpha)$, where $N_{E/k}$ is the usual norm homomorphism $E^*\to k^*$.

By virtue of this Lemma, we may use the notation $\mathfrak{N}_{E/k}$ in the case where the extension E/k is as in Lemma 15 or trivial $(E=k)$.

LEMMA 16. *Let K be a complete discrete valuation field and L a finite normal extension of K such that $[L:K]$ is a prime number. Let F and E be the residue fields of K and L, respectively. Then, we have the following commutative diagram.*

$$\begin{array}{ccc} \mathfrak{K}_{*+1}(L) & \xrightarrow{\ \partial\ } & \mathfrak{K}_*(E) \\ {\scriptstyle \mathfrak{N}_{L/K}}\downarrow & & \downarrow{\scriptstyle \mathfrak{N}_{E/F}} \\ \mathfrak{K}_{*+1}(K) & \xrightarrow{\ \partial\ } & \mathfrak{K}_*(F)\,. \end{array}$$

PROOF. Let z be a fixed element of $\mathfrak{K}_{q+1}(L)$ $(q\geqq 0)$. By [4] Ch. I Cor. 5.3, for each prime number p, there is a finite extension K'/K such that $[K':K]$ is prime to p and $z_{L'}$ $(L'=L\cdot K')$ is generated by elements of the form $\{x, y_{L'}\}$ $(x\in\mathfrak{K}_1(L'),\ y\in\mathfrak{K}_q(K'))$. By this fact, we are reduced to the case where z itself has the form $\{x, y_L\}$ $(x\in\mathfrak{K}_1(L),\ y\in\mathfrak{K}_q(K))$, and it is easy to prove $\partial\circ\mathfrak{N}_{L/K}(z)=\mathfrak{N}_{E/F}\circ\partial(z)$ for such z.

LEMMA 17. *Let k be a field and E a finite normal extension of k such that $[E:k]$ is a prime number. Let $k'=k(\alpha)$ be a finite extension of k, and let $E'=E(\alpha)$. Then, $N_{\alpha/k}\circ\mathfrak{N}_{E'/k'}=\mathfrak{N}_{E/k}\circ N_{\alpha/E}:\ \mathfrak{K}_*(E')\to\mathfrak{K}_*(k)$.*

PROOF. By Lemma 16 and by [4] Ch. I §5 the diagram (15), we obtain the following commutative diagram for any $v\in\mathfrak{P}(k(X)/k)$.

$$\begin{array}{ccc} \mathfrak{K}_{*+1}(E(X)) & \xrightarrow{\ (\partial_w)_w\ } & \bigoplus_w \mathfrak{K}_*(\kappa(w)) \\ {\scriptstyle \mathfrak{N}_{E(X)/k(X)}}\downarrow & & \downarrow{\scriptstyle \sum_w \mathfrak{N}_{\kappa(w)/\kappa(v)}} \\ \mathfrak{K}_{*+1}(k(X)) & \xrightarrow{\ \partial_v\ } & \mathfrak{K}_*(\kappa(v))\,. \end{array}$$

Here w ranges over all elements of $\mathfrak{P}(E(X)/E)$ which lie over v. Lemma 17 follows from this diagram and the definitions of $N_{\alpha/k}$ and $N_{\alpha/E}$ (cf. §1.2).

PROOF OF PROPOSITION 5. Let k and E be as in the hypothesis of Prop. 5. By [4] Ch. I §5, we may assume that k has the property $(\mathrm{E})_p$ for some prime number p (cf. the proof of Lemma 15). In this case, the Galois group of any finite Galois extension of k is nilpotent. Hence, there is a sequence of fields $k=E'_0\subset E'_1\subset\cdots\subset E'_m=E$ such that for each j, E'_j is a normal extension of E'_{j-1} of degree p. By applying Lemma 17 to fields $E_i\cdot E'_j$, we obtain

$$N_{\alpha_1/E_0}\circ\cdots\circ N_{\alpha_n/E_{n-1}}=\mathfrak{N}_{E'_1/E'_0}\circ\cdots\circ\mathfrak{N}_{E'_m/E'_{m-1}}.$$

§ 1.8. Computation of the cohomological residue.

In this § 1.8, we compute the cohomological residues, showing the appropriateness of the name "cohomological residue". Since we shall not use later the results of this § 1.8, we omit the details of the proof.

Let k be a complete discrete valuation field with residue field E, and
A the Henselization of the two dimensional local ring $O_k[X]_{(m_k,X)}$.
M the field of fractions of A,
K the field of fractions of the completion of the local ring A_{m_kA}.
(Here (m_k, X) denotes the maximal ideal of $O_k[X]$ generated by m_k and X.) Since the local ring A_{m_kA} is a discrete valuation ring, K is a complete discrete valuation field, and the residue field F of K is isomorphic to the algebraic closure in $E((X))$ of $E(X)$. Thus, the pair (k, K) satisfies the assumptions (i), (ii) and (iii) in § 1.4, and defines the cohomological residue $\mathrm{Res}_m^{q,r}: H^{q+1}(K, \mu_m^{\otimes(r+1)})\to H^q(k, \mu_m^{\otimes r})$ for each $q\geq 0$ and $r\in\boldsymbol{Z}$, and for each integer m which is invertible in k. In the following, we compute the composite

$$\mathfrak{K}_{q+1}(K)/m\cdot\mathfrak{K}_{q+1}(K)\xrightarrow{h^{q+1}_{m,K}}H^{q+1}(K, \mu_m^{\otimes(q+1)})\xrightarrow{\mathrm{Res}_m^{q,q}}H^q(k, \mu_m^{\otimes q}).$$

Note that since M is dense in K, the homomorphism $M^*/(M^*)^m\to K^*/(K^*)^m$ and hence the homomorphism $\mathfrak{K}_{q+1}(M)/m\cdot\mathfrak{K}_{q+1}(M)\to\mathfrak{K}_{q+1}(K)/m\cdot\mathfrak{K}_{q+1}(K)$ are surjective.

First, we show that there is a kind of reciprocity law. Let $\mathfrak{S}$ be the set of all prime ideals of height one of A, and let $\mathfrak{S}'=\mathfrak{S}-\{m_kA\}$. For each $\mathfrak{p}\in\mathfrak{S}'$, let $\kappa(\mathfrak{p})$ be the residue field of A at $\mathfrak{p}$, and $\delta_\mathfrak{p}$ the homomorphism $\delta: H^{q+1}(M, \mu_m^{\otimes(r+1)})\to H^q(\kappa(\mathfrak{p}), \mu_m^{\otimes r})$ of § 1.2 defined with respect to the $\mathfrak{p}$-adic valuation. Note that for each $\mathfrak{p}\in\mathfrak{S}'$, $\kappa(\mathfrak{p})$ is a finite extension of k.

LEMMA 18. *Let k, M, K, $\mathfrak{S}'$, q, r and m as above. Then, the composite*

$$H^{q+1}(M, \mu_m^{\otimes(r+1)})\longrightarrow H^{q+1}(K, \mu_m^{\otimes(r+1)})\xrightarrow{\mathrm{Res}_m^{q,r}}H^q(k, \mu_m^{\otimes r})$$

coincides with $\sum_{\mathfrak{p}\in\mathfrak{S}'}\mathrm{Cor}_{\kappa(\mathfrak{p})/k}\circ\delta_\mathfrak{p}$.

PROOF. Since trans. $\deg_k(M)=1$, $\mathrm{cd}(k_s\cdot M)\leq 1$. Hence the above composite can be rewritten as

$$H^{q+1}(M, \mu_m^{\otimes(r+1)})\longrightarrow H^q(k, H^1(k_s\cdot M, \mu_m^{\otimes(r+1)}))$$

$$\xrightarrow{\cong} H^q(k,\ \mu_m^{\otimes r}\otimes(k_s\cdot M)^*/((k_s\cdot M)^*)^m)$$

$$\longrightarrow H^q(k,\ \mu_m^{\otimes r})\quad (\text{by } T_{K/k}).$$

Thus we are reduced to the fact:

For each $x\in M^*$, $t_{K/k}(x)=\sum_{\mathfrak{p}\in\mathfrak{S}'}[\kappa(\mathfrak{p}):k]\cdot v_{\mathfrak{p}}(x)$ in $\boldsymbol{Z}$, where $v_{\mathfrak{p}}$ denotes the $\mathfrak{p}$-adic normalized additive discrete valuation for each $\mathfrak{p}$, but this can be proved easily. (Here $T_{K/k}$ and $t_{K/k}$ are the homomorphisms defined in § 1.4 Lemma 8.)

COROLLARY. *The following diagram commutes.*

$$\begin{array}{ccc} \mathfrak{K}_{q+1}(M) & \xrightarrow{\text{by } h^{q+1}_{m,K}} & H^{q+1}(K,\ \mu_m^{\otimes(q+1)}) \\ {\scriptstyle \sum_{\mathfrak{p}\in\mathfrak{S}'}\mathfrak{N}_{\kappa(\mathfrak{p})/k}\circ\partial_{\mathfrak{p}}}\downarrow & & \downarrow{\scriptstyle (-1)^q\cdot \mathrm{Res}^{q,q}_m} \\ \mathfrak{K}_q(k) & \xrightarrow{\text{by } h^q_{m,k}} & H^q(k,\ \mu_m^{\otimes q}). \end{array}$$

By simple computation using this Corollary, we can obtain the following result.

LEMMA 19. *Let k, K, E, F, q and m be as above. Then,*

$$\mathrm{Res}^{q,q}_m(h^{q+1}_{m,K}(\mathfrak{U}^{(n)}_{q+1}(K)))\subset h^q_{m,k}(\mathfrak{U}^{(n)}_q(k))$$

for each $n\geqq 1$. Furthermore, the diagram

$$\begin{array}{ccc} \Omega^q_F\oplus\Omega^{q-1}_F & \xrightarrow{\text{by } \varphi^{q+1}_n} & h^{q+1}_{m,K}(\mathfrak{U}^{(n)}_{q+1}(K))/h^{q+1}_{m,K}(\mathfrak{U}^{(n+1)}_{q+1}(K)) \\ {\scriptstyle \mathrm{res}_{q-1}\oplus-\mathrm{res}_{q-2}}\downarrow & & \downarrow{\scriptstyle (-1)^q\,\mathrm{Res}^{q,q}_m} \\ \Omega^{q-1}_E\oplus\Omega^{q-2}_E & \xrightarrow{\text{by } \varphi^q_n} & h^q_{m,k}(\mathfrak{U}^{(n)}_{q+1}(k))/h^q_{m,k}(\mathfrak{U}^{(n+1)}_{q+1}(k)) \end{array}$$

commutes if one takes common π, c, and c' (§ 1.3 *Lemma* 6 (2)) *for the definitions of the above φ^{q+1}_n and φ^q_n satisfying $\pi, c, c'\in k$. Here for any $q\geqq 0$, $\mathrm{res}_q\colon \Omega^{q+1}_F\to\Omega^q_E$ denotes the unique homomorphism such that*

$$\mathrm{res}_q\left(x\frac{dy_1}{y_1}\wedge\cdots\wedge\frac{dy_{q+1}}{y_{q+1}}\right)=0$$

for any $x\in O_F$, $y_1,\cdots,y_{q+1}\in U_F$, and

$$\mathrm{res}_q\left(z\frac{du_1}{u_1}\wedge\cdots\wedge\frac{du_q}{u_q}\wedge X^i\frac{dX}{X}\right)=\begin{cases} z\dfrac{du_1}{u_1}\wedge\cdots\wedge\dfrac{du_q}{u_q} & \text{if } i=0, \\ 0 & \text{if } 0\neq i\in\boldsymbol{Z}, \end{cases}$$

for any $z\in E$, $u_1,\cdots,u_q\in E^*$. Here X is the variable in the definition of A.

Note that $\mathrm{res}_0: \Omega_F \to E$ coincides with the usual residue homomorphism $\sum_i a_i X^i \frac{dX}{X} \mapsto a_0$. This is the reason for the naming "cohomological residue".

In the following §2, we shall define residue homomorphisms in algebraic K-theory. The analogy with the cohomological residue can be illustrated as follows:

$$\begin{array}{lcccccc} \S 1.4 & E & O_F & F & k & K & H^q(k \text{ (resp. } K), \mu_m^{\otimes r}) \\ & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow \\ \S 2.1 & A & B & B_S & A[[T]][T^{-1}] & B_S[[T]][T^{-1}] & \hat{C}K_q(A \text{ (resp. } B_S)). \end{array}$$

§2. The residue homomorphisms in algebraic K-theory.

In this §2, we generalize the residue homomorphisms $\sum_i a_i X^i \frac{dX}{X} \mapsto a_0$, by using Quillen's K-groups. These generalized residue homomorphisms will be used in §3 for the construction of the canonical homomorphism $\Upsilon_K: K_N(K) \to \mathrm{Gal}(K^{ab}/K)$ for "higher" local fields K of positive characteristics. Another application is a complement to Bloch's computation of $S\hat{C}K_q(R)$ in Bloch [3]. The generalized residue homomorphisms provide a new method for the computation of $S\hat{C}K_q(R)$ and we can eliminate the assumption $q \leqq p$ of [3] (cf. §2.2 Prop. 2).

§2.1. The definition and general properties of the residue homomorphisms.

For a ring R, let $K_q(R)$ $(q \geqq 0)$ be Quillen's K-groups in Quillen [19]. We denote the group law of $K_q(R)$ additively. As in Bloch [3], let

$$\hat{C}K_q(R) = \varprojlim \mathrm{Ker}(K_q(R[T]/(T^n)) \longrightarrow K_q(R)).$$

Let $f: A \to B$ be a ring homomorphism, and S a multiplicatively closed subset of B. Assume that

(i) *B is flat as an A-module (via f).*

(ii) *Every element of S is a non-zero divisor of B.*

(iii) *For any $s \in S$, B/sB is finitely generated and projective as an A-module.*

Let $B_S = B[s^{-1} (s \in S)]$. Under these assumptions, we shall define a canonical homomorphism

$$\mathrm{Res}_{(f,S)}: \hat{C}K_{*+1}(B_S) \longrightarrow \hat{C}K_*(A),$$

which we shall call the *residue homomorphism*, having the properties as in the following Prop. 1. Recall that for any ring R, a natural $W(R)$-module structure on $\hat{C}K_q(R)$ $(q \geqq 0)$ and operators V_n and F_n $(n \geqq 1)$ on $\hat{C}K_q(R)$ are defined by Bloch [3] Ch. II §1 and §2, where $W(R)$ denotes the ring of all generalized

Witt vectors over R in Cartier [6].

Proposition 1. (1) $\mathrm{Res}_{(f,S)}: \hat{C}K_{*+1}(B_S)\to\hat{C}K_*(A)$ *is a homomorphism of $W(A)$-modules, and commutes with the actions of V_n and F_n for each $n\geqq 1$.*

(2) *Let (A', B', f', S') be another 4-ple satisfying the above assumptions* (i), (ii) *and* (iii), *and $g: A\to A'$ and $h: B\to B'$ ring homomorphisms such that $h\circ f=f'\circ g$. Then*:

(i) *Assume that the induced homomorphism $A'\otimes_A B\to B'$ is bijective and $h(S)\subset(B'_{S'})^*$. Then the following diagram is commutative.*

$$\begin{array}{ccc} \hat{C}K_{*+1}(B_S) & \xrightarrow{\mathrm{Res}_{(f,S)}} & \hat{C}K_*(A) \\ \downarrow & & \downarrow \\ \hat{C}K_{*+1}(B'_{S'}) & \xrightarrow{\mathrm{Res}_{(f',S')}} & \hat{C}K_*(A'). \end{array}$$

(ii) *Assume that A' and B' are finitely generated and projective as modules over A and B, respectively, and that $B'_{h(S)}$ is isomorphic to $B'_{S'}$ over B'. Then, the following diagram is commutative, where the notation* Tr *is as in Definition* 1 *below.*

$$\begin{array}{ccc} \hat{C}K_{*+1}(B'_{S'}) & \xrightarrow{\mathrm{Res}_{(f',S')}} & \hat{C}K_*(A') \\ \mathrm{Tr}_{B'_{S'}/B_S}\downarrow & & \downarrow\mathrm{Tr}_{A'/A} \\ \hat{C}K_{*+1}(B_S) & \xrightarrow{\mathrm{Res}_{(f,S)}} & CK_*(A). \end{array}$$

Definition 1. Let $g: R\to R'$ be a ring homomorphism such that R' is finitely generated and projective as an R-module. We denote by $\mathrm{Tr}_{R'/R}$ the homomorphism $\hat{C}K_*(R')\to\hat{C}K_*(R)$ induced by the system of transfer maps (cf. Quillen [19] § 4) $\{K_*(R'[T]/(T^m))\to K_*(R[T]/(T^m))\}$. On the other hand, we often use the notation $N_{R'/R}$ for the transfer map $K_*(R')\to K_*(R)$ itself.

Now, we define the residue homomorphisms. Let (A, B, f, S) be as above, and $H^1_{B,S}$ the category of all B-modules which admit a resolution of length one by finitely generated projective B-modules and which are annihilated by some element of S. By Grayson [11] "the localization theorem for projective modules", there is a canonical homomorphism

$$\partial: K_{*+1}(B_S)\longrightarrow K_*(H^1_{B,S}),$$

which is the boundary map in a certain long exact sequence of homotopy groups. (Cf. § 2.3 and § 2.4 to avoid the possible change of the sign of ∂ which arises from the variance of the definitions of the boundary map. In the place where we must make it clear which definition of the boundary map is adopted, we take the homomorphism $\partial_{(i,h)}$ of § 2.4 Prop. 5 as ∂.) By Lemma 1 below, we have an exact functor from $H^1_{B,S}$ to the category $P(A)$ of all finitely generated projective A-modules, which assigns to each object of $H^1_{B,S}$ its underlying A-module, and hence have a homomorphism $K_*(H^1_{B,S}) \to K_*(P(A)) = K_*(A)$ by Quillen [19]. By composing these homomorphisms, we obtain a homomorphism

$$\partial_{(f,S)} : K_{*+1}(B_S) \longrightarrow K_*(A).$$

For each $m \geqq 0$, let $f_m : A[T]/(T^m) \to B[T]/(T^m)$ be the homomorphism $\sum_i a_i T^i \mapsto \sum_i f(a_i) T^i$. By Lemma 2 (i)′ below, we obtain an inverse system of homomorphisms

$$\{\partial_{(f_m,S)} : K_{*+1}(B_S[T]/(T^m)) \longrightarrow K_*(A[T]/(T^m))\}_{m \geq 1}.$$

Hence, in the limit, we obtain the desired residue homomorphism

$$\mathrm{Res}_{(f,S)} : \hat{C}K_{*+1}(B_S) \longrightarrow \hat{C}K_*(A).$$

LEMMA 1. *Let (A, B, f, S) be as above. Then, every object of $H^1_{B,S}$ is finitely generated and projective as an A-module.*

The proof is not difficult and is left to the reader.

LEMMA 2. *The assertions in Prop. 1 (2) are valid when we replace the diagrams in* (i) *and* (ii) *by the following diagrams* (i)′ *and* (ii)′, *respectively.*

(i)′

$$\begin{array}{ccc} K_{*+1}(B_S) & \xrightarrow{\partial_{(f,S)}} & K_*(A) \\ \downarrow & & \downarrow \\ K_{*+1}(B'_{S'}) & \xrightarrow{\partial_{(f',S')}} & K_*(A') \end{array}$$

(ii)′

$$\begin{array}{ccc} K_{*+1}(B'_{S'}) & \xrightarrow{\partial_{(f',S')}} & K_*(A') \\ {\scriptstyle N_{B'_{S'}/B_S}}\downarrow & & \downarrow{\scriptstyle N_{A'/A}} \\ K_{*+1}(B_S) & \xrightarrow{\partial_{(f,S)}} & K_*(A). \end{array}$$

PROOF. For any ring R, let $P(R)$ be the category of all finitely generated projective R-modules, and $P^1(R)$ the category of all R-modules which admit a resolution of length one by finitely generated projective R-modules. If a ring homomorphism $R \to R'$ is given, let $P^1_{R'}(R)$ be the full subcategory of $P^1(R)$ consisting of all objects M such that $\mathrm{Tor}^R_1(M, R') = 0$. By Quillen [19] § 4 Th. 3, the inclusions $P(R) \subset P^1_{R'}(R) \subset P^1(R)$ induce bijections $K_*(R) = K_*(P(R)) \xrightarrow{\cong} K_*(P^1_{R'}(R)) \xrightarrow{\cong} K_*(P^1(R))$. The commutativity of the diagram (i)′ follows from that of the following diagram and from the naturality of the boundary map in a long exact sequence of homotopy groups. The commutativity of (ii)′ is proved similarly.

$$\begin{array}{ccccccc} P(A) & \longleftarrow & H^1_{B,S} & \longrightarrow & P^1_{B'}(B) & \longrightarrow & P^1_{B'_{S'}}(B_S) \\ \downarrow & & \downarrow {\scriptstyle ?\otimes_B B'} & & \downarrow & & \downarrow \\ P(A') & \longleftarrow & H^1_{B',S'} & \longrightarrow & P^1(B') & \longrightarrow & P^1(B'_{S'}) \end{array}$$

PROOF OF PROPOSITION 1. For a ring R and $a \in R$, let $[a] : \hat{C}K_*(R) \to \hat{C}K_*(R)$ be the homomorphism induced by the ring homomorphism over R; $R[T]/(T^m) \to R[T]/(T^m)$; $T \mapsto aT$ $(m \geqq 1)$. For the proof of (1), it suffices to prove that $\mathrm{Res}_{\langle f,S \rangle}$ commutes with any V_n, F_n and $[a]$ $(n \geqq 1,\ a \in A)$ (cf. [3] Ch. II § 2). Hence Prop. 1 follows from Lemma 2 by replacing $A, B, \cdots$ by $A[T]/(T^m), B[T]/(T^m), \cdots$ $(m \geqq 1)$.

§ 2.2. The computation of the residue homomorphisms.

Here we describe the restriction of the residue homomorphism to the subgroup of $\hat{\mathrm{C}}\mathrm{K}_*(B_S)$ generated by symbols (cf. Prop. 3 below). In the course of the study, we can compute $\mathrm{S}\hat{\mathrm{C}}\mathrm{K}_*$ and eliminate the assumption $q \leqq p$ in Bloch [3] (cf. Prop. 2 below). This § 2.2 contains only the statements of Prop. 2, Prop. 3 and the Corollaries to Prop. 2. The proofs will be completed in § 2.5.

We fix some notations.

By Loday [14] Ch. II, there is a natural anticommutative graded ring structure on $\bigoplus_{q \geqq 0} K_q(R)$ for any commutative ring R. We shall denote

by $\{x, y\}$ the multiplication for this structure.

Since there is a canonical injective homomorphism $R^* \to K_1(R)$, we have for any $q \geqq 0$ and $x_1, \cdots, x_q \in R^*$, an element $\{x_1, \cdots, x_q\}$ of $K_q(R)$. The following identities are known to be valid (cf. Loday [14] Ch. II Th. 2.1.12 and Milnor [17] § 9 Lemma 9.8; note that Loday's product $K_1(R) \times K_1(R) \to K_2(R)$ coincides up to sign with the product in [17] by [14] Ch. II Prop. 2.2.3).

$\{x, y\} = (-1)^{ij}\{y, x\}$ *for any* $i, j \geqq 0$ *and* $x \in K_i(R)$, $y \in K_j(R)$.

$\{u, 1-u\} = 0$ *for any* $u \in R^*$ *such that* $1-u \in R^*$.

$\{u, -u\} = 0$ *for any* $u \in R^*$.

We shall denote

by $K_q^{\mathrm{sym}}(R)$ the subgroup of $K_q(R)$ generated by all symbols $\{x_1, \cdots, x_q\}$

$(x_1, \cdots, x_q \in R^*)$.

According to Bloch [3], we shall use the notations;

$$SC_nK_q(R)=\mathrm{Ker}(K_q^{\mathrm{sym}}(R[T]/(T^{n+1})) \longrightarrow K_q^{\mathrm{sym}}(R)) \quad (n\geqq 0),$$

$$S\hat{C}K_q(R)=\varprojlim SC_nK_q(R),$$

$$SC_\infty K_q(R)=\mathrm{Ker}(K_q^{\mathrm{sym}}(R[[T]]) \longrightarrow K_q^{\mathrm{sym}}(R)),$$

$$\mathrm{filt}^n SC_\infty K_q(R)=\mathrm{Ker}(SC_\infty K_q(R) \longrightarrow SC_nK_q(R)).$$

In addition, we shall often use the following group.

DEFINITION 2. Let $\bigoplus_{q\geqq 0} S_q(R)$ be the sub-graded-ring of $\bigoplus_{q\geqq 0} K_q(R[[T]][T^{-1}])$ generated by the image of $\bigoplus_{q\geqq 0} K_q^{\mathrm{sym}}(R[[T]])$ and $T\in K_1(R[[T]][T^{-1}])$. For each $n\geqq 1$, let $\bigoplus_{q\geqq 1} S_q^{(n)}(R)$ be the graded ideal of $\bigoplus_{q\geqq 0} S_q(R)$ generated by $1+T^n\cdot R[[T]]$ $(\subset S_1(R))$.

LEMMA 3. *Suppose R is additively generated by R^*. Then:*

(1) $\{S_i^{(m)}(R), S_j^{(n)}(R)\}\subset S_{i+j}^{(m+n)}(R)$ *for any $i, j, m, n\geqq 1$.*

(2) *For each $q\geqq 1$ and $n\geqq 1$, there is a surjective homomorphism*

$$\varphi_n^q: \Omega_R^{q-1}\oplus\Omega_R^{q-2} \longrightarrow S_q^{(n)}(R)/S_q^{(n+1)}(R);$$

$$\varphi_n^q\left(x\frac{dy_1}{y_1}\wedge\cdots\wedge\frac{dy_{q-1}}{y_{q-1}},\ z\frac{du_1}{u_1}\wedge\cdots\wedge\frac{du_{q-2}}{u_{q-2}}\right)$$

$$=\{1+xT^n, y_1, \cdots, y_{q-1}\}+\{1+zT^n, u_1, \cdots, u_{q-2}, T\} \mod S_q^{(n+1)}(R).$$

PROOF. Indeed, (1) can be proved just as Chapter I § 2 Lemma 2, by replacing the field S there by $R[[T]][T^{-1}]$. Next, (2) follows from (1) and § 1.3 Lemma 5.

COROLLARY. *Suppose that R is additively generated by R^*. Then, $S_q^{(n)}(R)$ is topologically generated with respect to the topology defined by the filtration $\{S_q^{(m)}(R)\}_{m\geqq n}$, by elements of the forms*

$$\{1+fT^n, r_1, \cdots, r_{q-1}\} \text{ and } \{1+fT^n, r_1, \cdots, r_{q-2}, T\}$$

$(f\in R[[T]], r_1, \cdots, r_{q-1}\in R^*)$.

LEMMA 4. *Suppose that R is a regular Noetherian ring. Then;*

(1) *The canonical homomorphism $K_*(R[[T]])\to K_*(R[[T]][T^{-1}])$ is injective.*

(2) *If one regards $SC_\infty K_q(R)$ as a subgroup of $K_q(R[[T]][T^{-1}])$ by (1), we have*

$$S_q^{(n+1)}(R)\subset \mathrm{filt}^{n-1}SC_\infty K_q(R) \quad \textit{for any } n\geqq 1.$$

If R is additively generated by R^*, we have $S_q^{(1)}(R)=SC_\infty K_q(R)$.

PROOF. Indeed, (1) is shown in the proof of Bloch [3] Ch. II §3 Prop. 1.1 when R is a regular local ring. But the proof there can be applied to the case where R is assumed only to be regular Noetherian, for $R[[T]]$ is regular Noetherian if R is so. Next, the first assertion in (2) is proved in [3] Ch. II §3 Cor. 1.5, and the second is proved easily.

The following Prop. 2 computes the structure of $S\hat{C}K_q$, and Prop. 3 computes the restriction of the residue homomorphism to $S\hat{C}K_q(B_S)$.

PROPOSITION 2. *Let p be a prime number and R a ring over $\boldsymbol{F}_p$. Suppose that R is regular Noetherian ring having a p-base (over $\boldsymbol{F}_p$) in the sense of Grothendieck [8] Ch. 0 §21, and that R is additively generated by R^*. Then, we have, for any $q \geqq 1$;*

(1) *Assume $n=mp^r$, $p \nmid m$. Then, φ_n^q in Lemma 3 induces an isomorphism*

$$(\Omega_R^{q-1} \oplus \Omega_R^{q-2})/L_{n,q} \xrightarrow{\cong} S_q^{(n)}(R)/S_q^{(n+1)}(R)\,,$$

where $L_{n,q}$ denotes the subgroup of $\Omega_R^{q-1} \oplus \Omega_R^{q-2}$ generated by all elements of the forms

$$\left(a^{p^i}\frac{da_1}{a_1}\wedge\cdots\wedge\frac{da_{q-2}}{a_{q-2}}\wedge\frac{da}{a},\ 0\right),$$

$$\left(0,\ a^{p^j}\frac{da_1}{a_1}\wedge\cdots\wedge\frac{da_{q-3}}{a_{q-3}}\wedge\frac{da}{a}\right),$$

$$\left(a^{p^r}\frac{da_1}{a_1}\wedge\cdots\wedge\frac{da_{q-2}}{a_{q-2}}\wedge\frac{da}{a},\ m\cdot a^{p^r}\frac{da_1}{a_1}\wedge\cdots\wedge\frac{da_{q-2}}{a_{q-2}}\right)$$

such that $0 \leqq i < r$, $0 \leqq j \leqq r$, and $a, a_1, \cdots, a_{q-2} \in R^*$.

(2) $S_q^{(n+2)}(R) \subset \mathrm{filt}^n SC_\infty K_q(R) \subset S_q^{(n+1)}(R)$ *for all $n \geqq 0$.*

REMARK 1. This Prop. 2 and the following Cor. 1 and Cor. 2 were proved in Bloch [3] Ch. II under the assumptions $q \leqq p$ and the assumption that R is a local ring. The assumption "local" can be easily replaced by the assumption that R is additively generated by R^* without essential change of the argument in [3]. But it is not easy to eliminate the assumption $q \leqq p$. For this, we use the residue homomorphisms instead of the methods in [3] Ch. II §3 and §4.

REMARK 2. By Deligne and Illusie, it was shown that the results of Bloch [3] Ch. III are obtained without the hypothesis on p and $\dim(X)$ of [3] if one uses a certain complex $W\Omega_X^*$ (de Rham-Witt complex) instead of the K-theoretic groups of [3]. Cf. Illusie [13]. The complex $W\Omega_X^*$ is defined without K-theory, and so, [13] does not compute the group $S\hat{C}K_*$. The above Prop. 2 and the following Cor. 1 and Cor. 2 show that the results of [3] Ch. III can be still obtained without the hypothesis on p and $\dim(X)$ by using the K-theoretic groups.

COROLLARY 1. *Let R be as in Prop. 2. Then,*

$$S\hat{C}K_q(R) \cong \varprojlim_n S_q^{(1)}(R)/S_q^{(n)}(R).$$

PROOF. This follows from Prop. 2 (2).

By Cor. 1, if R is a ring satisfying the hypothesis of Prop. 2, $\bigoplus_{q\geqq 0} S\hat{C}K_q(R)$ has the structure of a left and right graded-$\bigoplus_{q\geqq 0} S_q(R)$-module induced by those of $\bigoplus_{q\geqq 1} S_q^{(n)}(R)$. In particular, $T\in K_1(R[[T]][T^{-1}])$ defines the multiplications $x\mapsto \{x, T\}$ and $x\mapsto \{T, x\}$ on $\bigoplus_{q\geqq 0} S\hat{C}K_q(R)$.

Assume Prop. 2 and let R be as in the hypothesis of Prop. 2. Then, $S\hat{C}K_q(R)$ is a sub-$W(R)$-module of $\hat{C}K_q(R)$. (The proof of this fact is the same as the proof of [3] Ch. II §5 Prop. 1.1.) According to [3] Ch. II §7, let $T\hat{C}K_q(R)$ be the typical part of the $W(R)$ module $S\hat{C}K_q(R)$, and for each $n\geqq 0$, let $\mathrm{filt}^n T\hat{C}K_q(R)$ be the closed subgroup of $S\hat{C}K_q(R)$ topologically generated by elements of the forms

$$\{E(aT^{p^m}), r_1, \cdots, r_{q-1}\} \text{ and } \{E(aT^{p^m}), r_1, \cdots, r_{q-2}, T\}$$

($m\geqq n$, $a\in R$, $r_1, \cdots, r_{q-1}\in R^*$), where E is the Artin-Hasse exponential $E(x)=\prod_{n\geqq 1, p\nmid n}(1-x^n)^{-\mu(n)/n}$. Since we have assumed Prop. 2, the argument in [3] Ch. II §7 shows

$$\mathrm{filt}^0 T\hat{C}K_q(R) = T\hat{C}K_q(R).$$

As in [3] Ch. II §7, let

$$C_n^q(R) = T\hat{C}K_{q+1}(R)/\mathrm{filt}^n T\hat{C}K_{q+1}(R) \quad \text{for } n\geqq 0.$$

Since we have Prop. 2 without the hypothesis $q\leqq p$, we can prove [3] Ch. II §7 Th. 2.1 without this hypothesis; i.e.,

COROLLARY 2. *Let R be as in Prop. 2. For each $n\geqq 0$ and $q\geqq 1$, let $T\Phi_n K_q(R)=\mathrm{filt}^n T\hat{C}K_q(R)/\mathrm{filt}^{n+1} T\hat{C}K_q(R)$. Then, there is an isomorphism*

$$\phi_n^q : (\Omega_R^{q-1}\oplus\Omega_R^{q-2})/L_{p^n,q} \cong T\Phi_n K_q(R);$$

$$\phi_n^q\left(x\frac{dy_1}{y_1}\wedge\cdots\wedge\frac{dy_{q-1}}{y_{q-1}},\ z\frac{du_1}{u_1}\wedge\cdots\wedge\frac{du_{q-2}}{u_{q-2}}\right)$$
$$=\{E(xT^{p^n}), y_1, \cdots, y_{q-1}\}+\{E(zT^{p^n}), u_1, \cdots, u_{q-2}, T\}$$

where $L_{p^n,q}$ is as in Prop. 2 (1). Equivalently, there is an exact sequence $0\to \Omega_R^{q-1}/D_n \to T\Phi_n K_q(R)\to \Omega_R^{q-2}/E_n\to 0$, where D_n and E_n are as in [3] Ch. II §7 Th. 2.1.

We can obtain an explicit presentation of $C_n^q(R)$. Let $W^{(p)}(R)$ be the group of all p-Witt vectors over R, i.e. the group denoted by $W(R)$ in Serre [20] Ch. II §6, and let $W_n(R)$ be the group of all p-Witt vectors of length n over R. We identify $W^{(p)}(R)$ with the typical part $T\hat{C}K_1(R)$ of $\hat{C}K_1(R)=1+T\cdot R[[T]]$ via the isomorphism $(a_0, a_1, a_2, \cdots)\mapsto \prod_{n\geq 0} E(a_nT^{p^n})$.

Corollary 3. *Let R be as in Prop. 2. Let $q\geqq 0$ and $n\geqq 0$. Then, the homomorphism*

$$(W^{(p)}(R)\otimes\overbrace{R^*\otimes\cdots\otimes R^*}^{q\text{ times}})\oplus(W^{(p)}(R)\otimes\overbrace{R^*\otimes\cdots\otimes R^*}^{q-1\text{ times}})\longrightarrow \mathrm{T\hat{C}K}_{q+1}(R)\,;$$

$$(w\otimes r_1\otimes\cdots\otimes r_q,\ w'\otimes r_1'\otimes\cdots\otimes r_{q-1}')\longmapsto\{w, r_1, \cdots, r_q\}+\{w', r_1', \cdots, r_{q-1}', T\}$$

induces an isomorphism

$$(W_n(R)\otimes\bigwedge_{\mathbf{Z}}^{q}(R^*)\oplus W_n(R)\otimes\bigwedge_{\mathbf{Z}}^{q-1}(R^*))/M_{n,q}\cong C_n^q(R)\,.$$

Here $\bigwedge_{\mathbf{Z}}^{*}(R^*)$ denotes the exterior power of the abelian group R^*, and $M_{n,q}$ denotes the subgroup generated by all elements of the forms

$$(e_i(a)\otimes(r_1\wedge\cdots\wedge r_{q-1}\wedge a),\quad p^i\cdot e_i(a)\otimes(r_1\wedge\cdots\wedge r_{q-1}))$$

and

$$(0,\quad e_i(a)\otimes(r_1\wedge\cdots\wedge r_{q-2}\wedge a))$$

such that $0\leqq i<n$, $a, r_1, \cdots, r_{q-1}\in R^*$, where $e_i(a)$ denotes the element $(\overbrace{0, \cdots, 0}^{i\text{ times}}, a, 0, \cdots, 0)$ of $W_n(R)$.

We can deduce from this Cor. 3, the following

Corollary 4. *Let k be a field of characteristic $p>0$, and let $P_n^q(k)$ be as in* §1.1 *Definition* 1 *for each n, $q\geqq 0$. Then, there is an exact sequence*

$$C_n^q(k)\xrightarrow{F-1}C_n^q(k)/\{C_n^{q-1}(k), T\}\longrightarrow P_n^q(k)\longrightarrow 0\,.$$

Here, F is the homomorphism induced by $F_p\colon S\hat{C}K_{q+1}(k)\to S\hat{C}K_{q+1}(k)$.

Proposition 3. *Let (A, B, f, S) be as in* §2.1. *Let p be a prime number, and suppose that both A and B satisfy the hypothesis on R in Prop.* 2, *and that there is an element π of S such that $A\to B/\pi B$ is bijective and $(B_S)^*$ is generated by B^* and π. Then, B_S satisfies the hypothesis on R in Prop.* 2, *and we have*:

(1) $\mathrm{Res}_{(f,S)}(S\hat{C}K_{*+1}(B_S))\subset S\hat{C}K_*(A)$,
$\mathrm{Res}_{(f,S)}(T\hat{C}K_{*+1}(B_S))\subset T\hat{C}K_*(A)$,
$\mathrm{Res}_{(f,S)}(\mathrm{filt}^n T\hat{C}K_{q+1}(B_S))\subset\mathrm{filt}^n T\hat{C}K_q(A)$ *for all n, $q\geqq 0$.*

(2) *The family of homomorphisms*

$$\{\mathrm{Res}_{(f,S)}: S\hat{C}K_{q+1}(B_S) \longrightarrow S\hat{C}K_q(A)\}_{q\geq 0}$$

is the unique family of continuous homomorphisms such that:

(i) *The induced homomorphism*

$$\bigoplus_{q\geq 1} S\hat{C}K_q(B_S) \longrightarrow \bigoplus_{q\geq 0} S\hat{C}K_q(A)$$

is a homomorphism of left-$\bigoplus_{q\geq 0} S_q(A)$*-modules (cf. the remark under Cor.* 1 *to Prop.* 2) *and annihilates the image of* $\bigoplus_{q\geq 1} S\hat{C}K_q(B)$.

(ii) $$\mathrm{Res}_{(f,S)}(\{1+a\pi^r T^n, \pi\})=\begin{cases} 1+aT^n & \text{if } r=0 \\ 0 & \text{if } r\neq 0 \end{cases}$$

for all $a\in A$, $r\in \boldsymbol{Z}$ *and* $n\geqq 1$.

(3) *The induced diagram*

$$\begin{array}{ccc} \Omega^q_{B_S}\oplus\Omega^{q-1}_{B_S} & \xrightarrow{\phi_n^{q+1}} & T\Phi_n K_{q+1}(B_S) \\ {\scriptstyle \mathrm{res}_{q-1}\oplus(-\mathrm{res}_{q-2})}\downarrow & & \downarrow{\scriptstyle \text{by } \mathrm{Res}_{(f,S)}} \\ \Omega^{q-1}_A\oplus\Omega^{q-2}_A & \xrightarrow{\phi_n^q} & T\Phi_n K_q(A) \end{array}$$

(*cf. Cor.* 2 *to Prop.* 2) *is commutative, where* $\{\mathrm{res}_q: \Omega^{q+1}_{B_S}\to\Omega^q_A\}_{q\geq 0}$ *is the unique family of homomorphisms such that:*

(i) *The induced homomorphism* $\bigoplus_{q\geq 1}\Omega^q_{B_S}\to\bigoplus_{q\geq 0}\Omega^q_A$ *is a homomorphism of left-*$\bigoplus_{q\geq 0}\Omega^q_A$*-modules and annihilates the image of* $\bigoplus_{q\geq 1}\Omega^q_B$.

(ii) $$\mathrm{res}_0\left(a\pi^r\frac{d\pi}{\pi}\right)=\begin{cases} a & \text{if } r=0 \\ 0 & \text{if } r\neq 0 \end{cases}$$

for all $a\in A$ *and* $r\in \boldsymbol{Z}$.

§ 2.3. A product structure in algebraic K-theory.

In this § 2.3, we prove the following Prop. 4.

DEFINITION 3. An *exact category over a ring* R is an exact category M in the sense of Quillen [19] § 2 whose every object X is endowed with a ring homomorphism $\theta_X: R\to\mathrm{End}_M(X)$ satisfying $h\circ\theta_X(a)=\theta_Y(a)\circ h$ for any morphism $h: X\to Y$ in M and for any $a\in R$. If M and M' are exact categories over R, an exact functor $f: M\to M'$ is called an *exact functor over* R if $f(\theta_X(a))=\theta_{f(X)}(a)$ for any object X of M and for any $a\in R$.

For example, the category $P(R)$ of all finitely generated projective R-modules is, in the natural way, an exact category over R.

PROPOSITION 4. *For any exact category M over a commutative ring R, there exists a canonical graded-left-$\boldsymbol{Z}\oplus\bigoplus_{q\geq 1}K_q(R)$-module structure on $\bigoplus_{q\geq 0}K_q(M)$ satisfying the following conditions* (1) *and* (2). *Here we regard $\boldsymbol{Z}\oplus\bigoplus_{q\geq 1}K_q(R)$ as a sub-graded ring of $\bigoplus_{q\geq 0}K_q(R)$ which is endowed with the graded ring structure of Loday* [14] *Ch.* II.

(1) *If M and M' are exact categories over R and $f: M\to M'$ is an exact functor over R, the induced homomorphism $f_*: \bigoplus_{q\geq 0}K_q(M)\to\bigoplus_{q\geq 0}K_q(M')$ (cf. Quillen* [19] §2) *is a homomorphism of left-$\boldsymbol{Z}\oplus\bigoplus_{q\geq 1}K_q(R)$-modules.*

(2) *If $M=P(R)$, this graded module structure coincides with the one induced by Loday's graded ring structure on $\bigoplus_{q\geq 0}K_q(R)$.*

We fix some notations in algebraic topology. Let X and Y be pointed topological spaces with base points x_0 and y_0, respectively. Then,

$F(X, Y)$ denotes the space of all continuous maps $X\to Y$ endowed with the compact-open topology, and

$F_0(X, Y)$ the subspace of $F(X, Y)$ consisting of all maps preserving the base points.

For $q\geqq 0$, let $\pi_q(X)$ be the q-th homotopy group, $\Omega^qX=F_0(S^q, X)$, and $\Omega X=\Omega^1X$ where S^q is the q-sphere as usual. We identify $\pi_{m+n}(X)$ with $\pi_m(\Omega^nX)$ via the general canonical map $F_0(Y\wedge Z, X)\to F_0(Y, F_0(Z, X))$.

If $h: X\to Y$ is a continuous map preserving the base points,

Γ_h denotes the homotopy fiber of h, that is, the subspace of $X\times F(I, Y)$ consisting of all elements (x, w) such that $w(0)=y_0$ and $w(1)=h(x)$, where I is the closed interval $[0, 1]$,

$\partial_h: \Omega Y\to\Gamma_h$ denotes the inclusion $w\mapsto(x_0, w)$.

Let $Z\xrightarrow{i}X\xrightarrow{h}Y$ be a sequence of pointed topological spaces and continuous maps preserving the base points. Assume

(F) *$h\circ i=0$ and the inclusion map $j: Z\to\Gamma_h$; $z\mapsto(i(z), 0)$ is a homotopy equivalence.*

Here 0 denote the constant maps with value y_0. Then, we have a long exact sequence of homotopy groups (cf. for example, Hilton [12])

$$\cdots\longrightarrow\pi_q(Z)\xrightarrow{\pi_q(i)}\pi_q(X)\xrightarrow{\pi_q(h)}\pi_q(Y)\xrightarrow{\pi_{q-1}(j)^{-1}\circ\pi_{q-1}(\partial_h)}\pi_{q-1}(Z)\longrightarrow\cdots.$$

DEFINITION 4. Let $Z\xrightarrow{i}X\xrightarrow{h}Y$ be as above. We denote by $\partial_{(i,h)}$ the above homomorphism $\pi_{*+1}(Y)\to\pi_*(Z)$; $\pi_*(j)^{-1}\circ\pi_*(\partial_h)$.

The following Lemma 5 and Lemma 6 will be useful later.

LEMMA 5. *Let* $Z \xrightarrow{i} X \xrightarrow{h} Y$ *be as above. Then, for any* $q \geqq 0$, *the sequence* $\Omega^q Z \xrightarrow{\Omega^q i} \Omega^q X \xrightarrow{\Omega^q h} \Omega^q Y$ *also satisfies the condition* (F), *and*

$$\partial_{(\Omega^q i, \Omega^q h)} = (-1)^q \partial_{(i,h)} : \pi_{m+1}(Y) \longrightarrow \pi_m(Z) \quad \textit{for all } m \geqq q.$$

Here $\partial_{(\Omega^q i, \Omega^q h)} : \pi_{m+1}(Y) \to \pi_m(Z)$ *is, in the precise form, the composite* $\pi_{m+1}(Y) = \pi_{m-q+1}(\Omega^q Y) \xrightarrow{\partial_{(\Omega^q i, \Omega^q h)}} \pi_{m-q}(\Omega^q Z) = \pi_m(Z)$.

LEMMA 6. *Suppose that* $Z \xrightarrow{i} X \xrightarrow{h} Y$ *and* $Z' \xrightarrow{i'} X' \xrightarrow{h'} Y'$ *satisfy the above condition* (F), A *is a pointed topological space, and*

$$p : A \wedge Z \longrightarrow Z', \quad q : A \wedge X \longrightarrow X', \quad r : A \wedge Y \longrightarrow Y'$$

are continuous maps preserving the base points such that the diagram

$$\begin{array}{ccccc} A \wedge Z & \xrightarrow{A \wedge i} & A \wedge X & \xrightarrow{A \wedge h} & A \wedge Y \\ p \downarrow & & q \downarrow & & r \downarrow \\ Z' & \xrightarrow{i'} & X' & \xrightarrow{h'} & Y' \end{array}$$

is commutative. Then, for any $m, n \geqq 0$ *and for any* $a \in \pi_m(A)$ *and* $b \in \pi_{n+1}(Y)$, *we have*

$$\partial_{(i', h')}(a \cdot b) = a \cdot \partial_{(i,h)}(b) \quad (\in \pi_{m+n}(Z')).$$

Here $a \cdot b \in \pi_{m+n+1}(Y')$ *and* $a \cdot \partial_{(i,h)}(b) \in \pi_{m+n}(Z')$ *denote the classes of continuous maps*

$$S^{m+n+1} = S^m \wedge S^{n+1} \xrightarrow{a \wedge b} A \wedge Y \xrightarrow{r} Y' \quad \textit{and}$$

$$S^{m+n} = S^m \wedge S^n \xrightarrow{a \wedge \partial_{(i,h)}(b)} A \wedge Z \xrightarrow{p} Z',$$

respectively.

The proofs of these lemmas are left to the reader.

PROOF OF PROPOSITION 4. Let M be an exact category over R and let QM be the category in Quillen [19] §2. Let $n \geqq 1$ and identify $\mathrm{GL}_n(R)$ with the category with unique object e such that the semigroup of all endomorphisms of e is $\mathrm{GL}_n(R)$. Let $h_n^{(0)}$ (resp. $h_n^{(1)}$) be the functor $\mathrm{GL}_n(R) \times QM \to QM$ induced by the functor $QM \to QM$; $X \mapsto X^n = X \oplus \cdots \oplus X$ (n times) and by the trivial action (resp. the action via θ_X) of $\mathrm{GL}_n(R)$ on X^n.

For a small category C, let BC be its geometric realization (cf. [19]). Then, for any small categories C and C', the canonical map $B(C\times C')\rightarrow BC\times BC'$ is a homotopy equivalence. Hence we have continuous maps up to homotopy equivalence;

$$Bh_n^{(0)},\ Bh_n^{(1)}:\ B\mathrm{GL}_n(R)\times BQM\longrightarrow BQM.$$

Let $B'h_n^{(0)}$ and $B'h_n^{(1)}: B\mathrm{GL}_n(R)\rightarrow F_0(\Omega BQM, \Omega BQM)$ be the continuous maps induced by these $Bh_n^{(0)}$ and $Bh_n^{(1)}$, respectively. Since $F_0(\Omega BQM, \Omega BQM)$ is a commutative H-group, we can define the difference $B'h_n^{(1)}-B'h_n^{(0)}$. It is easy to see that the diagram

$$\begin{array}{ccc} B\mathrm{GL}_n(R) & \longrightarrow & B\mathrm{GL}_{n+1}(R) \\ {\scriptstyle B'h_n^{(1)}-B'h_n^{(0)}}\searrow & & \swarrow{\scriptstyle B'h_{n+1}^{(1)}-B'h_{n+1}^{(0)}} \\ & F_0(\Omega BQM, \Omega BQM) & \end{array}$$

is commutative up to homotopy for any $n\geqq 1$. Thus we have a continuous map $B\mathrm{GL}(R)\rightarrow F_0(\Omega BQM, \Omega BQM)$. Since any continuous map from $B\mathrm{GL}(R)$ to an H-space H is decomposed as $B\mathrm{GL}(R)\rightarrow B\mathrm{GL}(R)^+\rightarrow H$ uniquely up to homotopy (cf. Gersten [10] § 2 Th. 2.5), we obtain a continuous map $B\mathrm{GL}(R)^+\rightarrow F_0(\Omega BQM, \Omega BQM)$. Now, by the following Lemma 7 below, this gives a continuous map

$$\theta_{R,M}:\ B\mathrm{GL}(R)^+\wedge\Omega BQM\longrightarrow \Omega BQM,$$

and consequently, a canonical pairing compatible with the graduation;

$$\boldsymbol{Z}\oplus\bigoplus_{i\geqq 1}K_i(R)\times\bigoplus_{j\geqq 0}K_j(M)\longrightarrow\bigoplus_{j\geqq 0}K_j(M),$$

which we shall denote also by $\theta_{R,M}$.

LEMMA 7. *Let X and Y be pointed topological spaces having the homotopy type of CW-complexes. Then, the homotopy classes of continuous maps $X\wedge Y\rightarrow Z$ preserving the base points and those of $X\rightarrow F_0(Y, Z)$ are in one-to-one correspondence.*

This lemma can be reduced to the well known fact that the canonical map $(X\times_w Y)/(X\vee Y)\rightarrow X\wedge Y$ is a homotopy equivalence, where $X\times_w Y$ denotes the set $X\times Y$ endowed with the compactly generated topology.

It remains to prove the following

LEMMA 8. (1) *The above pairing $\theta_{R,M}$ satisfies*

$$\theta_{R,M}(\{x, y\}, z)=\theta_{R,M}(x, \theta_{R,M}(y, z))$$

for all $x, y \in \mathbf{Z} \oplus \bigoplus_{i \geq 1} K_i(R)$ *and* $z \in \bigoplus_{j \geq 0} K_j(M)$, *where* $\{x, y\}$ *denotes the product of Loday's graded ring structure* ([14] *Ch.* II).

(2) *If* M' *is another exact category over* R *and* $f: M \to M'$ *is an exact functor over* R,

$$f_*(\theta_{R,M}(x, z)) = \theta_{R,M'}(x, f_*(z)).$$

(3) *If* $M = P(R)$, $\theta_{R,M}(x, z) = \{x, z\}$.

PROOF. Recall that Loday's product is defined as follows. For each $m, n \geq 1$, let $f^{0,0}_{m,n}$, $f^{0,1}_{m,n}$, $f^{1,0}_{m,n}$, and $f^{1,1}_{m,n}$ be the functors (i.e. group homomorphisms) $\mathrm{GL}_m(R) \times \mathrm{GL}_n(R) \to \mathrm{GL}_{mn}(R)$;

$$f^{0,0}_{m,n}(\alpha, \beta) = 1_m \otimes 1_n, \quad f^{0,1}_{m,n}(\alpha, \beta) = 1_m \otimes \beta$$
$$f^{1,0}_{m,n}(\alpha, \beta) = \alpha \otimes 1_n, \quad f^{1,1}_{m,n}(\alpha, \beta) = \alpha \otimes \beta.$$

Let $j_q: B\mathrm{GL}_q(R) \to B\mathrm{GL}(R)^+$ ($q \geq 1$) be the canonical map. Since $B\mathrm{GL}(R)^+$ is a commutative H-group, we can define a map

$$j_{mn} \circ Bf^{1,1}_{m,n} - j_{mn} \circ Bf^{1,0}_{m,n} - j_{mn} \circ Bf^{0,1}_{m,n} + j_{mn} \circ Bf^{0,0}_{m,n}:$$
$$B\mathrm{GL}_m(R) \wedge B\mathrm{GL}_n(R) \to B\mathrm{GL}(R)^+,$$

and in the limit, a map $B\mathrm{GL}(R) \wedge B\mathrm{GL}(R) \to B\mathrm{GL}(R)^+$. This map is decomposed as $B\mathrm{GL}(R) \wedge B\mathrm{GL}(R) \to B\mathrm{GL}(R)^+ \wedge B\mathrm{GL}(R)^+ \to B\mathrm{GL}(R)^+$ uniquely up to homotopy, and the resulting map

$$\gamma: B\mathrm{GL}(R)^+ \wedge B\mathrm{GL}(R)^+ \to B\mathrm{GL}(R)^+$$

induces on $\bigoplus_{q \geq 0} K_q(R)$ the structure of a graded ring.

Now, for the proof of (1), it suffices to prove the commutativity of the following diagram

$$\begin{array}{ccc}
B\mathrm{GL}(R)^+ \wedge B\mathrm{GL}(R)^+ \wedge \Omega BQM & \xrightarrow{B\mathrm{GL}(R)^+ \wedge \theta_{R,M}} & B\mathrm{GL}(R)^+ \wedge \Omega BQM \\
\downarrow{\scriptstyle \gamma \wedge \Omega BQM} & & \downarrow{\scriptstyle \theta_{R,M}} \\
B\mathrm{GL}(R)^+ \wedge \Omega BQM & \xrightarrow{\theta_{R,M}} & \Omega BQM
\end{array}$$

up to homotopy. But this can be reduced to the commutativity of the diagram of functors

$$\begin{array}{ccc}
\mathrm{GL}_m(R) \times \mathrm{GL}_n(R) \times QM & \xrightarrow{\mathrm{GL}_m(R) \times h^{(j)}_n} & \mathrm{GL}_m(R) \times QM \\
\downarrow{\scriptstyle f^{i,j}_{m,n}} & & \downarrow{\scriptstyle h^{(i)}_m} \\
\mathrm{GL}_{mn}(R) \times QM & \xrightarrow{h^{(1)}_{mn}} & QM
\end{array}$$

($0 \leqq i \leqq 1$, $0 \leqq j \leqq 1$). Next, (2) follows easily from the naturality of the construction of $\theta_{R,M}$. Lastly, for the proof of (3), it suffices to show that the canonical isomorphism

$$\bigoplus_{q \geqq 0} \pi_q(\Omega BQP(R)) \cong \bigoplus_{q=0} \pi_q(BGL(R)^+ \times K_0(R))$$

(cf. Grayson [11]) is a homomorphism of left $\boldsymbol{Z} \oplus \bigoplus_{q \geqq 1} K_q(R)$-modules when the module structure of the former group is given by $\theta_{R,M}$ and that of the latter is given as in Loday [14] Ch. II. Let S, E, $S^{-1}S$ and $S^{-1}E$ be the categories in [11] "the extension construction" defined with respect to $P=P(R)$. There is a commutative diagram of functors

$$\begin{array}{ccccc} GL_n(R)\times S^{-1}S & \longrightarrow & GL_n(R)\times S^{-1}E & \longrightarrow & GL_n(R)\times QP(R) \\ f_n^{(i)} \downarrow & & g_n^{(i)} \downarrow & & h_n^{(i)} \downarrow \\ S^{-1}S & \longrightarrow & S^{-1}E & \longrightarrow & QP(R) \end{array}$$

($i=0, 1$). In this diagram, $h_n^{(0)}$ and $h_n^{(1)}$ are as above, $f_n^{(0)}$ (resp. $f_n^{(1)}$) is the functor induced by

$$S^{-1}S \longrightarrow S^{-1}S\,;\ (X, Y) \longmapsto (X^n, Y^n)$$

and by the trivial (resp. canonical) action of $GL_n(R)$ on (X^n, Y^n), and $g_n^{(0)}$ (resp. $g_n^{(1)}$) is the functor induced by

$$S^{-1}E \longrightarrow S^{-1}E\,;\ (V, 0 \to X \to Y \to Z \to 0) \longmapsto (V^n, 0 \to X^n \to Y^n \to Z^n \to 0)$$

and by the trivial (resp. canonical) action of $GL_n(R)$ on $(V^n, 0 \to X^n \to Y^n \to Z^n \to 0)$. Here the canonical actions mean the ones defined by the fact that for any R-module M, each element σ of $GL_n(R)$ acts canonically on M^n by $\sigma \otimes 1_M$: $R^n \otimes_R M \to R^n \otimes_R M$. Since $S^{-1}S$ is a commutative H-group, the functors $f_n^{(0)}$ and $f_n^{(1)}$ ($n \geqq 1$) define a continuous map

$$BGL(R)^+ \wedge BS^{-1}S \longrightarrow BS^{-1}S$$

in the same way in which the functors $h_n^{(0)}$ and $h_n^{(1)}$ ($n \geqq 1$) defined the continuous map $\theta_{R,M}$. This map induces a graded $\boldsymbol{Z} \oplus \bigoplus_{q \geqq 1} K_q(R)$-module structure on $\bigoplus_{q \geqq 0} K_q(R)$ via the homotopy equivalence $BGL(R)^+ \times K_0(R) \cong BS^{-1}S$ of Grayson [11] "the plus construction". But it is easy to see that this structure coincides with the one defined by Loday [14] Ch. II. Now, since the sequence $BS^{-1}S \to BS^{-1}E \to BQP(R)$ satisfies the condition (F) and $BS^{-1}E$ is contractible by [11] "the extension construction", the above commutative diagram proves that the canonical isomorphisms

$$\bigoplus_{q \geqq 0} \pi_q(\Omega BQP(R)) \cong \bigoplus_{q \geqq 0} \pi_q(BS^{-1}S) \cong \bigoplus_{q \geqq 0} \pi_q(BGL(R)^+ \times K_0(R))$$

preserve the left action of $Z \oplus \bigoplus_{q \geqq 1} K_q(R)$.

§ 2.4. Properties of the boundary maps in algebraic K-theory.

In this § 2.4, we prove the following Prop. 5, which will play an important role in the computation of the residue homomorphisms.

Let R be a ring and S a multiplicatively closed subset of R consisting of non-zero-divisors in R. By Grayson [11] "the localization theorem for projective modules", the sequence

$$\Omega BQH^1_{R,S} \xrightarrow{i} \Omega BQP^1(R) \xrightarrow{h} \Omega BQP^1(R_S)$$

induced by the canonical functors $H^1_{R,S} \xrightarrow{\subset} P^1(R)$ and $P^1(R) \longrightarrow P^1(R_S)$, satisfies the condition (F) in § 2.3 (for the notations, cf. the proof of § 2.1 Lemma 2). Since $H^1_{R,S}$ is an exact category over R, the group $\bigoplus_{q\geq 0} K_q(H^1_{R,S})$ has the structure of a graded left $\bigoplus_{q\geq 0} K_q(R)$-module by § 2.3 Prop. 4 (the action of $K_0(R)$ is defined by tensor product).

PROPOSITION 5. *Let R, S, i and h be as above. Then:*

(1) *The homomorphism $\partial_{(i,h)}: \bigoplus_{q\geq 1} K_q(R_S) \to \bigoplus_{q\geq 0} K_q(H^1_{R,S})$ is a homomorphism of left $\bigoplus_{q\geq 0} K_q(R)$-modules.*

(2) *Let $s\in S$. Then, the composite*

$$\alpha: K_*(R) \longrightarrow K_*(R_S) \xrightarrow{a\mapsto \{a,\, s\}} K_{*+1}(R_S) \xrightarrow{\partial_{(i,h)}} K_*(H^1_{R,S})$$

coincides with the homomorphism $\beta: K_*(R) \to K_*(H^1_{R,S})$ induced by the exact functor $P(R) \to H^1_{R,S}$; $M \mapsto M/sM$.

This Prop. 5 has the following Corollaries.

DEFINITION 5. For any field k, let

$$\iota_k : \mathfrak{K}_*(k) \longrightarrow K_*(k)$$

be the homomorphism $\{x_1, \cdots, x_q\} \mapsto \{x_1, \cdots, x_q\}$ $(x_1, \cdots, x_q \in k^*)$.

DEFINITION 6. Let K be a discrete valuation field with residue field F, and take O_K and $O_K-\{0\}$ as the above R and S, respectively. We denote by ∂^K_F the composite $K_{*+1}(K) \xrightarrow{\partial_{(i,h)}} K_*(H^1_{R,S}) \cong K_*(F)$, where the last isomorphism is induced by the inclusion $P(F) \xrightarrow{\subset} H^1_{R,S}$. We also use the same notation ∂^K_F for the homomorphism $\partial: \mathfrak{K}_{*+1}(K) \to \mathfrak{K}_*(F)$ in § 1.2, 1°.

COROLLARY 1. *Let K and F be as above. Then, we have the following commutative diagram.*

$$\begin{array}{ccc} \mathfrak{K}_{*+1}(K) & \xrightarrow{\iota_K} & K_{*+1}(K) \\ {\scriptstyle \partial_F^K}\downarrow & & \downarrow{\scriptstyle \partial_F^K} \\ \mathfrak{K}_*(F) & \xrightarrow{\iota_F} & K_*(F)\,. \end{array}$$

COROLLARY 2. *Let k be a field and E a finite extension of k. Then we have the following commutative diagram.*

$$\begin{array}{ccc} \mathfrak{K}_*(E) & \xrightarrow{\iota_E} & K_*(E) \\ {\scriptstyle \mathfrak{N}_{E/k}}\downarrow & & \downarrow{\scriptstyle N_{E/k}} \\ \mathfrak{K}_*(k) & \xrightarrow{\iota_k} & K_*(k)\,. \end{array}$$

(Cf. §1.7 Prop. 5 for the homomorphism $\mathfrak{N}_{E/k}$ and §2.1 Definition 1 for $N_{E/k}$.)

COROLLARY 3. *Let (A, B, f, S) be as in §2.1. Then, the homomorphism $\partial_{(f,S)}: \bigoplus_{q\geqq 1} K_q(B_S) \to \bigoplus_{q\geqq 0} K_q(A)$ (cf. §2.1) is a homomorphism of left $\bigoplus_{q\geqq 0} K_q(A)$-modules.*

PROOF OF PROPOSITION 5. The fact that $\partial_{(i,h)}$ is a left-$\boldsymbol{Z}\oplus\bigoplus_{q\geqq 1} K_q(R)$-homomorphism is deduced immediately from §2.3 Lemma 6. The commutativity with the action of $K_0(R)$ can be seen easily. Next, we prove Prop. 5 (2). Since α and β are left-$\bigoplus_{q\geqq 0} K_q(R)$-homomorphisms, it suffices to prove that the image of $1 \in K_0(R)$ under α is the class of R/sR in $K_0(H^1_{R,S})$. Let φ: $\boldsymbol{Z}[X] \to R$ be the ring homomorphism such that $\varphi(X)=s$, and let $S_0=\{X^n \mid n\geqq 0\} \subset \boldsymbol{Z}[X]$. The commutative diagram

$$\begin{array}{ccccc} H^1_{\boldsymbol{Z}[X],S_0} & \longrightarrow & P^1_R(\boldsymbol{Z}[X]) & \longrightarrow & P^1_{R_S}(\boldsymbol{Z}[X, X^{-1}]) \\ {\scriptstyle ?\underset{\boldsymbol{Z}[X]}{\otimes}{}_{\varphi} R}\downarrow & & \downarrow & & \downarrow \\ H^1_{R,S} & \longrightarrow & P^1(R) & \longrightarrow & P^1(R_S) \end{array}$$

(cf. the proof of §2.1 Lemma 2 for the definition of P^1_R) shows that we may assume $R=\boldsymbol{Z}[X]$, $S=S_0$ and $s=X$. On the other hand, by Grayson [11] "the fundamental theorem", the functor $?\underset{\boldsymbol{Z}}{\otimes}\boldsymbol{Z}[X]/(X)$ induces an isomorphism $K_*(\boldsymbol{Z}) \xrightarrow{\cong} K_*(H^1_{\boldsymbol{Z}[X],S_0})$. Hence, our task becomes to prove that the sequence

$$\Omega BQP(\boldsymbol{Z}) \xrightarrow{i} \Omega BQP^1(\boldsymbol{Z}[X]) \xrightarrow{h} \Omega BQP^1(\boldsymbol{Z}[X, X^{-1}])$$

satisfies $\partial_{(i,h)}(X)=1 \in K_0(\boldsymbol{Z})$, where X is regarded as an element of $K_1(\boldsymbol{Z}[X, X^{-1}])$ and i is the map induced by $?\underset{\boldsymbol{Z}}{\otimes}\boldsymbol{Z}[X]/(X)$. Let $C\boldsymbol{Z}$ and $S\boldsymbol{Z}$ be the cone and the suspension of the ring $\boldsymbol{Z}$, respectively, as in [11] "the suspension of a ring". Let

$$\tau=\begin{pmatrix} 0 & & & \\ 1 & 0 & & \\ & 1 & 0 & \\ & & 1 & \ddots \end{pmatrix} \quad e=\begin{pmatrix} 1 & & & \\ & 0 & & \\ & & 0 & \\ & & & \ddots \end{pmatrix}$$

be elements of $C\boldsymbol{Z}$. The homomorphism $\boldsymbol{Z}[X]\to C\boldsymbol{Z}$; $X\mapsto\tau$ induces a commutative diagram

$$\begin{array}{ccccc} P(\boldsymbol{Z}) & \xrightarrow{?\otimes_{\boldsymbol{Z}}\boldsymbol{Z}[X]/(X)} & P^1_{C\boldsymbol{Z}}(\boldsymbol{Z}[X]) & \longrightarrow & P^1_{S\boldsymbol{Z}}(\boldsymbol{Z}[X, X^{-1}]) \\ \| & & \downarrow & & \downarrow \\ P(\boldsymbol{Z}) & \xrightarrow[?\otimes_{\boldsymbol{Z}}e\cdot C\boldsymbol{Z}]{} & P^1(C\boldsymbol{Z}) & \longrightarrow & P^1(S\boldsymbol{Z}). \end{array}$$

Now, $e\cdot C\boldsymbol{Z}$ is finitely generated and projective as a right $C\boldsymbol{Z}$-module, and the sequence

$$\Omega BQP(\boldsymbol{Z}) \xrightarrow{\text{by } \otimes_{\boldsymbol{Z}} e\cdot C\boldsymbol{Z}} \Omega BQP(C\boldsymbol{Z}) \longrightarrow \Omega BQP(S\boldsymbol{Z})$$

satisfies the condition (F) in §2.3. Since this sequence can be identified with

$$BS^{-1}S(\boldsymbol{Z}) \underset{(\text{by } \otimes_{\boldsymbol{Z}} e\cdot C\boldsymbol{Z})}{\xrightarrow{i'}} BS^{-1}S(C\boldsymbol{Z}) \xrightarrow{h'} BS^{-1}S(S\boldsymbol{Z})$$

by [11] "the suspension of a ring", our task finally becomes to prove

$$\partial_{(i',h')}(\bar{\tau})=1\in K_0(\boldsymbol{Z}).$$

Here $\bar{\tau}$ denotes the image of τ in $S\boldsymbol{Z}$ regarded as an element of $K_1(S\boldsymbol{Z})$. Let u (resp. v): $I\to BS^{-1}S(C\boldsymbol{Z})$ be the path from the vertex $(0, 0)$ (resp. $(e\cdot C\boldsymbol{Z}, 0)$) of $BS^{-1}S(C\boldsymbol{Z})$ to the vertex $(C\boldsymbol{Z}, \tau\cdot C\boldsymbol{Z})$ induced by the morphism in $S^{-1}S(C\boldsymbol{Z})$:

$$(0, 0)\longrightarrow(C\boldsymbol{Z}, \tau\cdot C\boldsymbol{Z});\ (C\boldsymbol{Z} \underset{\cong}{\xrightarrow{1}} C\boldsymbol{Z},\ C\boldsymbol{Z} \underset{\cong}{\xrightarrow{x\mapsto\tau x}} \tau\cdot C\boldsymbol{Z})$$

$$(\text{resp. } (e\cdot C\boldsymbol{Z}, 0)\longrightarrow(C\boldsymbol{Z}, \tau\cdot C\boldsymbol{Z});$$
$$(\tau\cdot C\boldsymbol{Z}\oplus e\cdot C\boldsymbol{Z} \underset{\cong}{\xrightarrow{1}} C\boldsymbol{Z},\ \tau\cdot C\boldsymbol{Z} \underset{\cong}{\xrightarrow{1}} \tau\cdot C\boldsymbol{Z})).$$

Let $w: I\to BS^{-1}S(C\boldsymbol{Z})$ be the path defined by

$$w(t)=u(2t) \quad \text{for } 0\leqq t\leqq 1/2,$$
$$w(t)=v(2(1-t)) \quad \text{for } 1/2\leqq t\leqq 1.$$

Then, $h'\circ w: I\to BS^{-1}S(SZ)$ satisfies $h'\circ w(0)=h'\circ w(1)$, and the induced map $S^1\to BS^{-1}S(SZ)$ represents the element $\bar{\tau}$ of $\pi_1(BS^{-1}S(SZ))=K_1(SZ)$. This proves $\partial_{(i',h')}(\bar{\tau})=1$.

REMARK 3. If one changes the definition of Γ_h in § 2.3 as

$$\Gamma_h=\{(x, w)\in X\times F(I, Y)\mid w(0)=h(x),\ w(1)=y_0\}$$

and define $\partial_{(i,h)}$ using this definition of Γ_h, the resulting equation becomes $\partial_{(i',h')}(\bar{\tau})=-1$. Thus, (and as in Lemma 5) the definition of the boundary map of the long exact sequence of homotopy groups tends to change by sign.

PROOFS OF THE COROLLARIES. Cor. 1 and Cor. 3 follow from Prop. 5 immediately. Now we prove Cor. 2. Generally, let K be an algebraic function field in one variable over a field k, and C the regular, proper and irreducible curve over k with function field K. The set of all closed points of C can be identified with $\mathfrak{P}(K/k)$ in § 1.2. For each $v\in\mathfrak{P}(K/k)$, let $\partial_v: K_{*+1}(K)\to K_*(\kappa(v))$ be the homomorphism $\partial^K_{\kappa(v)}$ in Definition 6 defined with respect to v. By Quillen [19] § 7 Prop. 3.2, we have a long exact sequence

$$\cdots\longrightarrow K_{q+1}(K)\xrightarrow{(\partial_v)_v}\bigoplus_{v\in\mathfrak{P}(K/k)}K_q(\kappa(v))\longrightarrow K_q(C)\longrightarrow K_q(K)\cdots.$$

Since C is proper over k, the structural morphism $p: C\to\mathrm{Spec}(k)$ induces a homomorphism $p_*: K_*(C)\to K_*(k)$ (cf. Quillen [19] § 7, 2.7), and for any $v\in\mathfrak{P}(K/k)$, the composite $K_*(\kappa(v))\to K_*(C)\to K_*(k)$ coincides with the transfer map $N_{\kappa(v)/k}$. Thus, we have shown

$$\sum_{v\in\mathfrak{P}(K/k)}N_{\kappa(v)/k}\circ\partial_v=0:\ K_{*+1}(K)\longrightarrow K_*(k).$$

Hence, when we take the rational function field $k(X)$ as K, the commutativity of the diagram in Cor. 2 follows from Cor. 1 and from the definition of the norm homomorphisms of Milnor's K-groups.

§ 2.5. The proofs of Proposition 2 and Proposition 3.

First, we prove the following Lemma 9.

LEMMA 9. *Let (A, B, f, S) be as in § 2.1 and π an element of S. Suppose that:*

(i) *A and B are Noetherian rings and additively generated by A^* and B^*, respectively.*

(ii) *The induced homomorphism $A\to B/\pi B$ is bijective.*

(iii) *$(B_S)^*$ is generated by B^* and π.*

Then:

(1) *The left* $\bigoplus_{q\geq 0}\Omega_A^q$*-module* $\bigoplus_{q\geq 0}\Omega_{B_S}^q$ *is generated by the image of* $\bigoplus_{q\geq 0}\Omega_B^q$ *and the elements* $\pi^r\frac{d\pi}{\pi}\in\Omega_{B_S}$ *and* $\pi^r\in\Omega_{B_S}^0=B_S$ $(r\in\mathbf{Z})$.

(2) *Let* $1\leqq i\leqq j$. *Then, the left* $\bigoplus_{q\geq 0}S_q(A)$*-module* $\bigoplus_{q\geq 1}S_q^{(i)}(B_S)/S_q^{(j)}(B_S)$ (*cf.* §2.2 *Definition* 2) *is generated by the image of* $\bigoplus_{q\geq 1}S_q^{(i)}(B)/S_q^{(j)}(B)$ *and elements of the forms* $\{1+a\pi^rT^n, \pi\}\in S_2^{(i)}(B_S)/S_2^{(j)}(B_S)$ *and* $1+a\pi^rT^n\in S_1^{(i)}(B_S)/S_1^{(j)}(B_S)$ $(a\in A,\ r\in\mathbf{Z},\ i\leqq n<j)$.

(3) *Denote by* $\tilde{f}$ *the homomorphism* $A[[T]][T^{-1}]\to B[[T]][T^{-1}]$ *induced by* f, *and let* $\tilde{S}=\{g\in B[[T]]\,|\,g \bmod (T)\in S\}$. *Then, the assumptions* (i), (ii) *and* (iii) *at the beginning of* §2.1 *are still satisfied when we replace* A, B, f *and* S *by* $A[[T]][T^{-1}]$, $B[[T]][T^{-1}]$, $\tilde{f}$ *and* $\tilde{S}$, *respectively. We have,*

$$\partial_{(\tilde{f},\tilde{S})}(\{1+a\pi^rT^n, \pi\})=\begin{cases}1+aT^n & \text{if } r=0\\ 0 & \text{if } r\neq 0\end{cases}$$

$$\partial_{(\tilde{f},\tilde{S})}(1+a\pi^rT^n)=0$$

for all $a\in A$, $r\in\mathbf{Z}$ *and* $n\geqq 1$.

(4) *Assume further that* A *and* B *are regular and the homomorphisms*

$$\varprojlim S_q^{(1)}(A)/S_q^{(n)}(A)\longrightarrow S\hat{C}K_q(A)$$

defined by §2.2 *Lemma* 4 *are bijective for all* $q\geqq 1$. *Regard* $\bigoplus_{q\geq 1}S\hat{C}K_q(A)$ *as a left* $\bigoplus_{q\geq 0}S_q(A)$*-module via these isomorphisms. Then,*

$$\mathrm{Res}_{(f,S)}(S\hat{C}K_{q+1}(B_S))\subset S\hat{C}K_q(A)\quad \text{for all } q\geqq 1,$$

$$\mathrm{Res}_{(f,S)}(\{1+a\pi^rT^n, \pi\})=\begin{cases}1+aT^n & \text{if } r=0\\ 0 & \text{if } r\neq 0\end{cases}$$

for all $a\in A$, $r\in\mathbf{Z}$ *and* $n\geqq 1$. *The induced composite*

$$\bigoplus_{q\geq 1}S_q^{(1)}(B_S)\longrightarrow\bigoplus_{q\geq 1}S\hat{C}K_q(B_S)\xrightarrow{\mathrm{Res}_{(f,S)}}\bigoplus_{q\geq 0}S\hat{C}K_q(A)$$

is a homomorphism of left $\bigoplus_{q\geq 0}S_q(A)$*-modules.*

PROOF. The proof of (1) is easy, and (2) follows from (1) and §2.2 Lemma 3 (2). Next, the proof of the fact that the 4-ple $(A[[T]][T^{-1}], B[[T]][T^{-1}], \tilde{f}, \tilde{S})$ satisfies the assumptions (i)$\sim$(iii) in §2.1 can be reduced to the following

LEMMA 10. *Let* R *be a Noetherian ring,* I *an ideal of* R, *and* M *an* R*-module such that* M/I^nM *is flat over* R/I^n *for every* $n\geqq 1$. *Then,* $\varprojlim M/I^nM$ *is flat* *over R.*

This Lemma can be proved by the standard methods in commutative algebra, and so we leave the proof to the reader.

The formula $\partial_{(\mathfrak{J},\mathfrak{S})}(\{1+aT^n, \pi\})=1+aT^n$ $(a\in A,\ n\geqq 1)$ follows from §2.4 Prop. 5 (2), and $\partial_{(\mathfrak{J},\mathfrak{S})}(1+a\pi^r T^n)=0$ $(a\in A,\ r\in \boldsymbol{Z},\ n\geqq 1)$ follows from the fact $\hat{C}K_0(R)=0$ for any ring R. To prove $\partial_{(\mathfrak{J},\mathfrak{S})}(\{1+a\pi^r T^n, \pi\})=0$ $(a\in A,\ r\neq 0,\ n\geqq 1)$, we may assume $a\in A^*$ and $r<0$. We use the following Definition 7.

DEFINITION 7. Let R be a ring. We denote by V_n $(n\geqq 1)$ (resp. F_n $(n\geqq 1)$, resp. $[a]$ $(a\in R^*)$) the covariant (resp. transfer, resp. covariant) homomorphism $K_*(R[[T]][T^{-1}])\to K_*(R[[T]][T^{-1}])$ defined by the ring homomorphism $R[[T]][T^{-1}]\to R[[T]][T^{-1}]$; $T\mapsto T^n$ (resp. $T\mapsto T^n$, resp. $T\mapsto aT$).

By §2.1 Lemma 2, $\partial_{(\mathfrak{J},\mathfrak{S})}$ commutes with the actions of V_n $(n\geqq 1)$, F_n $(n\geqq 1)$ and $[a]$ $(a\in A^*)$. It follows

$$\{1-a\pi^r T^n, \pi\}=V_n\circ[a]\circ F_{-r}(\{1-\pi^{-1}T, \pi\})=-V_n\circ[a]\circ F_{-r}(\{T, 1-\pi^{-1}T\}),$$

$$\partial_{(\mathfrak{J},\mathfrak{S})}(\{1-a\pi^r T^n, \pi\})=-V_n\circ[a]\circ F_{-r}\circ\partial_{(\mathfrak{J},\mathfrak{S})}(\{T, 1-\pi^{-1}T\})=0$$

$(a\in A^*,\ r<0,\ n\geqq 1)$. (Note that $\{T, ?\}$ commutes with $\partial_{(\mathfrak{J},\mathfrak{S})}$ by §2.4 Cor. 3 to Prop. 5.) Lastly, we prove (4). Let M be the left $\bigoplus_{q\geqq 0} S_q(A)$-submodule of $\bigoplus_{q\geqq 1} K_q(B[[T]][T^{-1}]_{\mathfrak{S}})$ generated by all elements of the form $\{1+a\pi^r T^n, \pi\}$ and $1+a\pi^r T^n$ $(a\in A,\ r\in \boldsymbol{Z},\ n\geqq 1)$. By Cor. 3 to Prop. 5, $\partial_{(\mathfrak{J},\mathfrak{S})}: \bigoplus_{q\geqq 1} K_q(B[[T]][T^{-1}]_{\mathfrak{S}}) \to \bigoplus_{q\geqq 0} K_q(A[[T]][T^{-1}])$ is a homomorphism of left $\bigoplus_{q\geqq 0} K_q(A[[T]][T^{-1}])$-modules. Hence, by (3) and §2.1 Lemma 2, we have a commutative diagram

$$\begin{array}{ccccc}
\bigoplus_{q\geqq 1} K_q(B[[T]][T^{-1}]_{\mathfrak{S}}) & \supset & M & \longrightarrow & \bigoplus_{q\geqq 1} \hat{C}K_q(B_S) \\
\downarrow \partial_{(\mathfrak{J},\mathfrak{S})} & & \downarrow \partial_{(\mathfrak{J},\mathfrak{S})} & & \downarrow \mathrm{Res}_{(f,S)} \\
\bigoplus_{q\geqq 0} K_q(A[[T]][T^{-1}]) & \supset & \bigoplus_{q\geqq 0} S_q^{(1)}(A) & \longrightarrow & \bigoplus_{q\geqq 0} \hat{C}K_q(A),
\end{array}$$

and now, the assertions in (4) follows from (2) and (3).

PROOF OF PROPOSITION 2. The proof of $\varphi_n^q(L_{n,q})=0$ is easy. To prove the rest, we need several steps (Step 1, 2, 3).

Step 1. In this Step 1, we prove that we may assume $R=k((X_1))((X_2))\cdots((X_N))$ for some perfect field k of characteristic $p>0$ and for some $N\geqq 0$. We need the following Lemma.

LEMMA 11. *Let R and R′ be rings satisfying the hypothesis of Prop. 2, and*

$R\to R'$ a ring homomorphism such that the induced homomorphism $\bigoplus_{q\geq 0}\Omega_R^q\to\bigoplus_{q\geq 0}\Omega_{R'}^q$ is injective. Then, the induced homomorphism $(\Omega_R^{q-1}\oplus\Omega_R^{q-2})/L_{n,q}(R)\to(\Omega_{R'}^{q-1}\oplus\Omega_{R'}^{q-2})/L_{n,q}(R')$ is injective for any n, $q\geqq 1$.

PROOF. Let $\Omega_{R,d=0}^q$ be the kernel of the exterior derivation $d: \Omega_R^q\to\Omega_R^{q+1}$, and let $\gamma: \Omega_{R,d=0}^q\to\Omega_R^q$ be the Cartier operator for each $q\geqq 0$. If $p\nmid n$, there is a bijection

$$\Omega_R^{q-1}\xrightarrow{\cong}(\Omega_R^{q-1}\oplus\Omega_R^{q-2})/L_{n,q}(R)\,;\ \omega\mapsto(\omega,0)\,.$$

If $p|n$, $L_{n,q}(R)\subset\Omega_{R,d=0}^{q-1}\oplus\Omega_{R,d=0}^{q-2}$ and there is an injective homomorphism

$$(\Omega_{R,d=0}^{q-1}\oplus\Omega_{R,d=0}^{q-2})/L_{n,q}(R)\xrightarrow{\gamma\oplus\gamma}(\Omega_R^{q-1}\oplus\Omega_R^{q-2})/L_{n/p,q}(R)\,.$$

Lemma 11 follows from these facts by induction on n.

By Lemma 11, if the assertions in Prop. 2 are true for R', then they are also true for R. By applying this to the case where R' is the ring of total quotients of R, which is a finite product of fields, we see that the ring R in Prop. 2 may be assumed to be a field. Since the functors SC_nK_q $(0\leqq n<\infty)$ and the functors $(\Omega_R^{q-1}\oplus\Omega_R^{q-2})/L_{n,q}(R)$ (in R) preserve all filtering direct limits, R may be assumed to be finitely generated over the prime field $\boldsymbol{F}_p$. Then, there is a separable extension R' of R of the form $R'=k((X_1))\cdots((X_N))$ where k is a finite field. Again by Lemma 11, we may assume $R=k((X_1))\cdots((X_N))$.

Step 2. Here we study the composite of residue homomorphisms.

LEMMA 12. *Let $N\geqq 0$, k_0 a perfect field, and let $k_1,\cdots,k_N$ be complete discrete valuation fields such that the residue field of k_i is k_{i-1} for each i. Denote k_N by K, and k_0 by k. Assume $\mathrm{ch}(K)=p>0$. Then, there is a homomorphism $\mathrm{Res}_{K/k}: \hat{C}K_{*+N}(K)\to\hat{C}K_*(k)$ having (and clearly characterized by) the following property: For each $i=1,\cdots,N$, take any ring homomorphism $f_i: k_{i-1}\to O_{k_i}$ such that the composite $k_{i-1}\xrightarrow{f_i}O_{k_i}\longrightarrow O_{k_i}/m_{k_i}=k_{i-1}$ is the identity map, which exists by Nagata* [18] § 31 *Th.* 31.1, *and let $S_i=O_{k_i}-\{0\}$. Then, for any choices of $f_1,\cdots,f_N$, the composite*

$$\mathrm{Res}_{(f_1,S_1)}\circ\cdots\circ\mathrm{Res}_{(f_N,S_N)}: \hat{C}K_{*+N}(K)\to\hat{C}K_*(k)$$

always coincides with $\mathrm{Res}_{K/k}$. Furthermore, this homomorphism $\mathrm{Res}_{K/k}$ has the following property: Let K' be a finite extension of K, and $k_0',\cdots,k_N'$ the fields determined by the conditions $k_N'=K'$ and k_{i-1}' is the residue field of k_i' for each $i=1,\cdots,N$. Denote k_0' by k'. Then we have the following commutative diagram.

$$\begin{array}{ccc} \hat{C}K_{*+N}(K') & \xrightarrow{\text{Res}_{K'/k'}} & \hat{C}K_*(k') \\ {\scriptstyle \text{Tr}_{K'/K}}\downarrow & & \downarrow{\scriptstyle \text{Tr}_{k'/k}} \\ \hat{C}K_{*+N}(K) & \xrightarrow{\text{Res}_{K/k}} & \hat{C}K_*(k) \end{array}$$

(*cf.* § 2.1 *Definition* 1 *for the notation* Tr).

For the proof of this Lemma, it is useful to define an operator $\gamma: \hat{C}K_* \to \hat{C}K_*$ as follows.

DEFINITION 8. Let p be a prime number, and R a ring over $\boldsymbol{F}_p$. With respect to the homomorphism $R \to R: x \mapsto x^p$, let R_1 be the former R and let R_2 be the latter R. When R_2 is finitely generated and projective as an R_1-module, we denote by γ the homomorphism $\text{Tr}_{R_2/R_1}: \hat{C}K_*(R) \to \hat{C}K_*(R)$. (Cf. Lemma 15 below for the relation between this operator γ and the Cartier operator on differential modules.)

PROOF OF LEMMA 12. For each $i=1, \cdots, N$, take a ring homomorphism $f_i': k_{i-1}' \to O_{k_i'}$ such that the composite $k_{i-1}' \to O_{k_i'} \to O_{k_i'}/m_{k_i'} = k_{i-1}'$ is the identity map, and let $S_i' = O_{k_i'} - \{0\}$. It is sufficient to prove

$$\text{Res}_{(f_1, S_1)} \circ \cdots \circ \text{Res}_{(f_N, S_N)} \circ \text{Tr}_{K'/K} = \text{Tr}_{k'/k} \circ \text{Res}_{(f_1', S_1')} \circ \cdots \circ \text{Res}_{(f_N', S_N')} .$$

(For the proof of the existence of $\text{Res}_{K/k}$, consider the case $K'=K$.) Let the operator γ be as above. Since k is perfect, the covariant homomorphism $\hat{C}K_*(k) \to \hat{C}K_*(k)$ induced by $k \to k$; $x \mapsto x^p$ is the inverse of $\gamma: \hat{C}K_*(k) \to \hat{C}K_*(k)$. By this fact and by Prop. 1 (2) (ii), we are reduced to the following

LEMMA 13. *Let A and B be rings over $\boldsymbol{F}_p$, S a multiplicatively closed subset of B, and f and f' : $A \to B$ ring homomorphisms such that the 4-ples (A, B, f, S) and (A, B, f', S) satisfy the assumptions at the beginning of* § 2.1. *Suppose that there is an element π of S such that $(B_S)^*$ is generated by B^* and π, and such that $f(x) \equiv f'(x) \bmod \pi B$ for all $x \in A$. Suppose further that via the homomorphism $A \to A$; $x \mapsto x^p$, the latter A is finitely generated and projective as a module over the former A. Then, for each $x \in \hat{C}K_{q+1}(B_S)$ $(q \geqq 0)$, the sequence $\{\gamma^n(\text{Res}_{(f,S)}(x) - \text{Res}_{(f',S)}(x))\}_{n \geqq 0}$ converges to zero in $\hat{C}K_q(A)$.*

PROOF. Let $\mathfrak{F}: A \to A$ be the homomorphism $x \mapsto x^p$. Fix $m \geqq 1$, and let $f_m, f_m': A[T]/(T^m) \to B[T]/(T^m)$ and $\mathfrak{F}_m: A[T]/(T^m) \to A[T]/(T^m)$ be the ring homomorphisms over $\boldsymbol{Z}[T]/(T^m)$ induced by f, f' and $\mathfrak{F}$ respectively. It suffices to prove the following assertion (A);

(A) *For each* $q \geqq 0$, *and for each* $x \in K_{q+1}(B_S[T]/(T^m))$, $\partial_{(f_m, S)}(x) - \partial_{(f'_m, S)}(x)$ *is annihilated by a power of* $\mathfrak{F}_{m*}$, *where* $\mathfrak{F}_{m*}$ *is the transfer map defined by* $\mathfrak{F}_m$.

Let $H = H^1_{B[T]/(T^m), S}$ and for each $n \geqq 1$, let $H(n)$ be the full subcategory of H consisting of all objects M such that $\pi^n M = 0$. As is easily seen, if $r \geqq n-1$ and if $\mathfrak{F}^r$ denotes $\mathfrak{F} \circ \cdots \circ \mathfrak{F}$ (r times), the homomorphisms $f \circ \mathfrak{F}^r$ and $f' \circ \mathfrak{F}^r$ coincide modulo $\pi^n B$. Hence $f_m \circ \mathfrak{F}^r_m$ and $f'_m \circ \mathfrak{F}^r_m$ coincide modulo $\pi^n \cdot B[T]/(T^m)$ for $r \geqq n-1$. It follows that the restriction of scalars,

$$H \xrightarrow{\text{via } f_m \circ \mathfrak{F}^r_m} P(A[T]/(T^m)) \quad \text{and} \quad H \xrightarrow{\text{via } f'_m \circ \mathfrak{F}^r_m} P(A[T]/(T^m))$$

coincide on $H(n)$. Since $H = \bigcup_n H(n)$, we have $K_q(H) = \varinjlim K_q(H(n))$ and hence the assertion (A).

Step 3. Now, we apply the homomorphism $\mathrm{Res}_{K/k}$ defined in Lemma 12 to the proof of Prop. 2. As was seen in Step 1, we may assume $R = k((X_1)) \cdots ((X_N))$ for some perfect field k of characteristic $p > 0$ and for some natural number N. We proceed by induction on N. Let $k_i = k((X_1)) \cdots ((X_i))$, f_i the canonical inclusion $k_{i-1} \subset O_{k_i}$, and $S_i = O_{k_i} - \{0\}$ for each $i = 1, \cdots, N$. By induction on N, we may assume that Prop. 2 was proved when $R = k_i$, $i \leqq N-1$. Hence we may assume $\varprojlim S_q^{(1)}(k_i)/S_q^{(n)}(k_i) \cong S\hat{C}K_q(k_i)$ for $i \leqq N-1$ (cf. Prop. 2 (2)). Let $K = k_N$ and let $j: S_*^{(1)}(K) \to S\hat{C}K_*(K)$ be the canonical homomorphism defined by § 2.2 Lemma 4. Since $\mathrm{Res}_{K/k} = \mathrm{Res}_{(f_1, S_1)} \circ \cdots \circ \mathrm{Res}_{(f_N, S_N)}$, Lemma 9 (4) shows

$$\mathrm{Res}_{K/k}(j(\{T, x\})) = 0 \quad \text{for all } x \in S_N^{(1)}(K).$$

Hence the definition of the residue homomorphisms shows

$$\mathrm{Res}_{K/k}(j(S_{N+1}^{(n)}(K)) \subset 1 + T^n \cdot k[[T]] \quad \text{for all } n \geqq 1.$$

The following Lemma 14 will be our main tool.

DEFINITION 9. Let K and k be as above. Take $n \geqq 1$. We denote by $\mathrm{res}_{K/k}$ the composite

$$\Omega_K^N \xrightarrow{\varphi_n^{N+1}(*,\,0)} S_{N+1}^{(n)}(K)/S_{N+1}^{(n+1)}(K) \quad (\text{cf. § 2.2 Lemma 3})$$

$$\xrightarrow{\text{by } \mathrm{Res}_{K/k} \circ j} (1 + T^n \cdot k[[T]])/(1 + T^{n+1} \cdot k[[T]]) \xrightarrow{1 + aT^n \mapsto a} k.$$

LEMMA 14. (1) *The homomorphism* $\mathrm{res}_{K/k} : \Omega_K^N \to k$ *is independent of the choice of* n.

(2) $\mathrm{res}_{K/k} \circ \gamma = \gamma \circ \mathrm{res}_{K/k}$ *where* γ *is the Cartier operator. (Note that* $\gamma : k \to k$ *is* $x \mapsto x^{1/p}$.)

(3) $\mathrm{res}_{K/k} \circ d(\Omega_K^{N-1}) = 0$.

(4) *Let* $n = p^r m$, $p \nmid m$, $1 \leqq q \leqq N+1$ *and* $(\omega_1, \omega_2) \in \Omega_K^{q-1} \oplus \Omega_K^{q-2}$. *Assume that for*

any $x, y_1, \cdots, y_{N+1-q} \in K^*$,

$$\mathrm{res}_{K/k}\Big(x^n\cdot\Big(\omega_1+\omega_2\wedge\frac{dx}{x}\Big)\wedge\frac{dy_1}{y_1}\wedge\cdots\wedge\frac{dy_{N+1-q}}{y_{N+1-q}}\Big)=0\,.$$

Then, $(\omega_1, \omega_2)\in L_{n,q}$, *where* $L_{n,q}$ *is the group defined in Prop.* 2 (1).

PROOF. Note that $[K:K^p]=p^N$, $\dim_K(\Omega_K)=N$, $\dim_K(\Omega_K^N)=1$, $\Omega_K^q=0$ if $q>N$, and $(\Omega_K^{q-1}\oplus\Omega_K^{q-2})/L_{n,q}=0$ if $q>N+1$.

First, (1) follows from the commutativity of $\mathrm{Res}_{K/k}$ with the operator V_n on $\hat{C}K_*$ (cf. § 2.1).

Next, (2) follows from Lemma 15 below, which describe the relation between the operator γ in Definition 8 and the Cartier operator, and from $\mathrm{Res}_{K/k}\circ\gamma=\gamma\circ\mathrm{Res}_{K/k}$ (cf. Lemma 12).

(3) is easily reduced to $\mathrm{Res}(j(\{T, \}))=0$.

Lastly, (4) is proved as follows. It suffices to prove the following (A) and (B).

(A) *If* $p\nmid n$, $n\omega_1=(-1)^q\cdot\omega_2$.

(B) *If* $p|n$, $d\omega_1=d\omega_2=0$.

Indeed, if (A) and (B) are proved, we can proceed by induction on n. If $p|n$, since we have $d\omega_1=d\omega_2=0$, we can define $\gamma(\omega_1)$ and $\gamma(\omega_2)$. By (2),

$$0=\gamma\circ\mathrm{res}_{K/k}\Big(x^n\cdot\Big(\omega_1+\omega_2\wedge\frac{dx}{x}\Big)\wedge\frac{dy_1}{y_1}\wedge\cdots\wedge\frac{dy_{N+1-q}}{y_{N+1-q}}\Big)$$

$$=\mathrm{res}_{K/k}\Big(x^{n/p}\cdot\Big(\gamma(\omega_1)+\gamma(\omega_2)\wedge\frac{dx}{x}\Big)\wedge\frac{dy_1}{y_1}\wedge\cdots\wedge\frac{dy_{N+1-q}}{y_{N+1-q}}\Big),$$

so, the induction on n can be applied (cf. the proof of Lemma 11). Next, it is easily seen that for the proofs of (A) and (B), we may assume $q=N+1$. Now we prove (A) and (B) in the case $q=N+1$. In this case, since $d(\Omega_K^N)=0$, for any n, it suffices to prove $n\omega_1=(-1)^{N+1}\cdot d\omega_2$. We may assume $N\geqq 1$. Denote k_{N-1} by F, and X_N by π. Then, $\omega_1\in\Omega_K^N$ can be written in the form $\omega_1=\sum_{i\in\mathbf{Z}}\theta_i\wedge\pi^i\frac{d\pi}{\pi}$ and $\omega_2\in\Omega_K^{N-1}$ can be written in the form $\omega_2=\sum_{i\in\mathbf{Z}}\xi_i\wedge\pi^i\frac{d\pi}{\pi}+\sum_{i\in\mathbf{Z}}\pi^i\eta_i$, where θ_i, $\eta_i\in\Omega_F^{N-1}$ and $\xi_i\in\Omega_F^{N-2}$ for all $i\in\mathbf{Z}$, and $\theta_i=\xi_i=\eta_i=0$ if i is sufficiently near to $-\infty$. The equation $n\omega_1=(-1)^{N+1}\cdot d\omega_2$ is equivalent to

(C) $n\theta_i=(-1)^{N+1}d\xi_i+i\eta_i$ *for all* $i\in\mathbf{Z}$.

For each integer i, take a sufficiently large natural number c such that $c\equiv -i \bmod n$ and such that $\theta_j=\xi_j=\eta_j=0$ for any $j\leqq i-c$. Let $e=(c+i)/n$, u any element of F, and $x=(1+u\pi^c)\cdot\pi^{-e}$. Since the residue homomorphism $\mathrm{Res}_{(f_N,s_N)}$ annihilates $\hat{C}K_*(O_K)$ and satisfies $\mathrm{Res}_{(f_N,s_N)}(\{a, \pi\})=a$ for all $a\in\hat{C}K_*(F)$ (§ 2.4 Prop. 5 (2)), we can easily deduce the following equation for the above choice of x.

$$0=\mathrm{res}_{K/k}\left(x^n\cdot\left(\omega_1+\omega_2\wedge\frac{dx}{x}\right)\right)$$
$$=\mathrm{res}_{F/k}(nu\theta_i-\xi_i\wedge du-iu\eta_i)$$
$$=\mathrm{res}_{F/k}(u\cdot(n\theta_i+(-1)^N\cdot d\xi_i-i\eta_i))\quad(\text{by } \mathrm{res}_{K/k}\circ d=0).$$

Since $\mathrm{res}_{F/k}\neq 0$, and since u is arbitrary and Ω_F^{N-1} is one-dimensional over F, we obtain $n\theta_i+(-1)^N\cdot d\xi_i-i\eta_i=0$ which proves (C).

Thus, for the proof of Lemma 14, it remains to prove the following

LEMMA 15. *Let p and R be as in the hypothesis of Prop. 2. Suppose that the rank N of the free R-module $R^{1/p}$ is finite and that there is a p-basis $(b_i)_{1\leq i\leq N}$ of R consisting of elements of R^*. For each $n\geqq 1$ and $q\geqq 1$, let $\hat{S}_q^{(n)}(R)=$ Image$(\varprojlim_i S_q^{(n)}(R)/S_q^{(i)}(R)\to S\hat{C}K_q(R))$ (cf. § 2.2 Lemma 4). Then, $\hat{S}_{N+1}^{(n)}(R)$ is stable under the action of γ in Definition 8, and the induced diagram*

$$\begin{array}{ccc} \Omega_R^N\oplus\Omega_R^{N-1} & \longrightarrow & \hat{S}_{N+1}^{(n)}(R)/\hat{S}_{N+1}^{(n+1)}(R) \\ \gamma\oplus 0\downarrow & & \downarrow \text{by } \gamma \\ \Omega_R^N\oplus\Omega_R^{N-1} & \longrightarrow & \hat{S}_{N+1}^{(n)}(R)/\hat{S}_{N+1}^{(n+1)}(R) \end{array}$$

is commutative, where the γ in the left side is the Cartier operator, the γ in the right side is the operator in Definition 8 and the horizontal arrows are the homomorphism induced by φ_n^{N+1}.

PROOF. Let j be the canonical homomorphism $S_q^{(1)}(R)\to S\hat{C}K_q(R)$. Let $R_i=R[b_1^{1/p},\cdots,b_i^{1/p}]$ for each $i=0,\cdots,N$,

$$g: R[[T]][T^{-1}]\subset R^{1/p}[[T]][T^{-1}] \text{ and } g_i: R_{i-1}[[T]][T^{-1}]\subset R_i[[T]][T^{-1}]$$

the canonical inclusions, and g_* and g_{i*} the transfer maps of the K-groups defined by g and g_i, respectively. Note that Ω_R^N is additively generated by $d(\Omega_R^{N-1})$ and elements of the form $x^p\frac{db_1}{b_1}\wedge\cdots\wedge\frac{db_N}{b_N}$ $(x\in R)$. All things are reduced to the following computations:

$$\gamma(j(\{1+x^pT^n, b_1,\cdots,b_N\}))=j(g_*(\{1+xT^n, b_1^{1/p},\cdots,b_N^{1/p}\}))$$
$$=j(g_{1*}\circ\cdots\circ g_{N*}(\{1+xT^n, b_1^{1/p},\cdots,b_N^{1/p}\}))=\{1+xT^n, b_1,\cdots,b_N\},$$
$$\gamma(j(\{1+xT^n, b_1,\cdots,b_{N-1},T\}))=j(g_*(\{1+x^{1/p}T^n, b_1^{1/p},\cdots,b_{N-1}^{1/p},T\}))$$
$$=(\text{similarly as above})=\{1+xT^{np}, b_1,\cdots,b_{N-1},T\}.$$

Here we used the projection formula (cf. Bloch [3] Ch. I § 2 Th. 4.1)

$$f_*\{s, f^*(r)\}=\{f_*(s), r\},$$

where $f: R\to S$ is any ring homomorphism between any rings R and S such that S is finitely generated and projective as an R-module, $r\in K_i(R)$, $s\in K_j(S)$ $(i, j\geqq 0)$, and f^* is the covariant homomorphism $K_i(R)\to K_i(S)$ and f_* are the transfer maps.

Now, we can complete the proof of Prop. 2. Let K and k be as at the beginning of this Step 3. Assume $1\leqq q\leqq N+1$ and $(\omega_1, \omega_2)\in\Omega_K^{q-1}\oplus\Omega_K^{q-2}$, $n=p^r m$, $p\nmid m$, and that the image of $\varphi_n^q(\omega_1, \omega_2)$ in $S_q^{(1)}(K)/(S_q^{(n+1)}(K)+\mathrm{filt}^n SC_\infty K_q(K))$ is zero. For the proof of Prop. 2, it is sufficient to prove $(\omega_1, \omega_2)\in L_{n,q}$. The groups $S_q^{(n)}(K)$ and $\mathrm{filt}^n SC_\infty K_q(K)$ are clearly stable under the action of $[x]$ (cf. Definition 7) for any $x\in K^*$, and the induced diagram

$$\begin{array}{ccccc} (\omega_1, \omega_2) & & \Omega_K^{q-1}\oplus\Omega_K^{q-2} & \xrightarrow{\varphi_n^q} & S_q^{(n)}(K)/S_q^{(n+1)}(K) \\ \downarrow & & \downarrow & & \downarrow [x] \\ \left(x^n\cdot\left(\omega_1+\omega_2\wedge\dfrac{dx}{x}\right),\ x^n\omega_2\right) & & \Omega_K^{q-1}\oplus\Omega_K^{q-2} & \xrightarrow{\varphi_n^q} & S_q^{(n)}(K)/S_q^{(n+1)}(K) \end{array}$$

is commutative. Hence, it follows that for any $x, y_1, \cdots, y_{N+1-q}\in K^*$, the image of

$$\varphi_n^{N+1}\left(x^n\cdot\left(\omega_1+\omega_2\wedge\frac{dx}{x}\right)\wedge\frac{dy_1}{y_1}\wedge\cdots\wedge\frac{dy_{N+1-q}}{y_{N+1-q}},\ x^n\omega_2\wedge\frac{dy_1}{y_1}\wedge\cdots\wedge\frac{dy_{N+1-q}}{y_{N+1-q}}\right)$$

in $S_{N+1}^{(1)}(K)/(S_{N+1}^{(n+1)}(K)+\mathrm{filt}^n SC_\infty K_{N+1}(K))$ is zero. Thus we have

$$\mathrm{res}_{K/k}\left(x^n\cdot\left(\omega_1+\omega_2\wedge\frac{dx}{x}\right)\wedge\frac{dy_1}{y_1}\wedge\cdots\wedge\frac{dy_{N+1-q}}{y_{N+1-q}}\right)=0\,.$$

Hence, by Lemma 14 (4), we have $(\omega_1, \omega_2)\in L_{n,q}$.

Proof of the Corollaries of Prop. 2. Cor. 1 and Cor. 2 are immediate. For the proof of Cor. 3, let

$$C_n'^q(R)=(W_n(R)\otimes\bigwedge_{\mathbf{Z}}^{q}(R^*)\oplus W_n(R)\otimes\bigwedge_{\mathbf{Z}}^{q-1}(R^*))/M_{n,q}\,,$$

where $M_{n,q}$ is the group defined in Cor. 3. Our task is to prove $C_n'^q(R)\cong C_n^q(R)$. Let $\Psi_n^q(R)=\mathrm{Ker}(C_{n+1}'^{q+1}(R)\to C_n'^{q+1}(R))$. It suffices to prove $\Psi_n^q(R)\cong T\Phi_n K_q(R)$. But this follows from the fact that the homomorphism

$$(\Omega_R^{q-1}\oplus\Omega_R^{q-2})/L_{n,q}\to\Psi_n^q(R)\,;\ \left(x\frac{dy_1}{y_1}\wedge\cdots\wedge\frac{dy_{q-1}}{y_{q-1}},\ z\frac{du_1}{u_1}\wedge\cdots\wedge\frac{du_{q-2}}{u_{q-2}}\right)$$
$$\mapsto(e_n(x)\otimes y_1\wedge\cdots\wedge y_{q-1},\ e_n(z)\otimes u_1\wedge\cdots\wedge u_{q-2})$$

is surjective and the composite

$$(\Omega_R^{q-1}\oplus\Omega_R^{q-2})/L_{n,q}\longrightarrow\Psi_n^q(R)\longrightarrow T\Phi_n K_q(R)$$

is bijective by Cor. 2. Lastly, Cor. 4 is a consequence of Cor. 3, for the action of F_p on $T\hat{C}K_1(k)=W^{(p)}(k)$ coincides with $(a_0, a_1, a_2, \cdots)\mapsto(a_0^p, a_1^p, a_2^p, \cdots)$.

PROOF OF PROPOSITION 3. Since Prop. 2 has been proved, all assertions in Prop. 3 follows from Lemma 9, and from the fact that the residue homomorphism commutes with the action of $W(A)$ and hence preserve the typical parts.

§ 3. Local class field theory.

§ 3.1. The results.

The aim of § 3 is to prove the following Theorems.

Recall that we denote by $\mathfrak{K}_q(k)$ ($q\geqq 0$) the Milnor's K-groups of k, whereas we denote by $K_q(k)$ the Quillen's K-groups of k.

THEOREM 1. *Let $N\geqq 0$, and let $k_0, \cdots, k_N$ be fields satisfying the following conditions.*

(i) *k_0 is a finite field.*

(ii) *For each $i=1, \cdots, N$, k_i is a complete discrete valuation field with residue field k_{i-1}.*

Denote k_N by K, and k_0 by k. Then, there exists a unique homomorphism

$$\Psi_K : \mathfrak{K}_N(K) \longrightarrow \mathrm{Gal}(K^{ab}/K)$$

having the following properties (1) *and* (2).

(1) *For each finite abelian extension L of K, Ψ_K induces an isomorphism $\mathfrak{K}_N(K)/\mathfrak{N}_{L/K}\mathfrak{K}_N(L)\cong\mathrm{Gal}(L/K)$, where $\mathfrak{N}_{L/K}$ is the norm homomorphism of* § 1.7 *Prop.* 5.

(2) *For each $i=1, \cdots, N$, let π_i be any lifting to K of any prime element of k_i. Then, the image of $\Psi_K(\{\pi_1, \cdots, \pi_N\})$ under the canonical homomorphism $\mathrm{Gal}(K^{ab}/K)\to\mathrm{Gal}(k^{ab}/k)$ coincides with the Frobenius automorphism over k.*

THEOREM 2. *Besides the hypothesis of Th.* 1, *assume* $\mathrm{ch}(K)=p>0$. *Then, there is a canonical homomorphism*

$$\Upsilon_K : K_N(K) \longrightarrow \mathrm{Gal}(K^{ab}/K)$$

such that the composite with the canonical map $\iota_K : \mathfrak{K}_N(K)\to K_N(K)$ (§ 3.4 *Definition* 5) *coincides with Ψ_K.*

The definitions of these homomorphisms Ψ_K and Υ_K are contained in § 3.2 with their functorial properties. In § 3.3, we compute the norm group $\mathfrak{N}_{L/K}\mathfrak{K}_N(L)$ and complete the proofs of Th. 1 and Th. 2. In § 3.4, we study the Brauer group of K.

In § 3.5, we shall study the case $N=2$. This case was closely studied in Chapter I in the mixed characteristic case. Here we study the equal characteristic case and prove the following Theorem 3. Recall that in Chapter I §7, we have defined natural topologies of K^* and $K_2(K)$ in the case $\mathrm{ch}(k_1)=p>0$. In the case $\mathrm{ch}(k_1)=0$, we can take the discrete topologies of K^* and $K_2(K)$ in the following Theorem 3 (cf. §3.5 Remark 3).

THEOREM 3. *Besides the hypothesis of Th.* 1, *assume* $N=2$. *Then*;

(1) *The map* $L\mapsto N_{L/K}K_2(L)$ *is a bijection from the set of all finite abelian extensions of* K *in a fixed algebraic closure of* K *to the set of all open subgroups of* $K_2(K)$ *of finite indices.*

(2) *There is a canonical isomorphism*

$$\Phi_K : \mathrm{Br}(K)\cong \mathrm{Hom}_c(K^*, \mathbf{Q}/\boldsymbol{Z})_{\mathrm{tor}}$$

such that for any central simple algebra A *over* K,

$$\mathrm{Ker}(\Phi_K(\{A\}) : K^* \longrightarrow \boldsymbol{Q}/\boldsymbol{Z})=\mathrm{Nrd}_{A/K}(A^*)\,.$$

Here $\mathrm{Hom}_c(K^*, \boldsymbol{Q}/\boldsymbol{Z})_{\mathrm{tor}}$ *denotes the torsion part of the group of all continuous homomorphisms* $K^*\to\boldsymbol{Q}/\boldsymbol{Z}$.

§3.2. The definitions of Ψ_K and Υ_K.

The definition of Ψ_K will be given by a kind of duality. For the description of this duality, it is useful to pay attention to the following groups and denote them by simple notations $H^*(k)$.

DEFINITION 1. Let k be a field and let $q\geqq 0$.

(1) If $\mathrm{ch}(k)=0$, let

$$H^q(k)=\varinjlim H^q(k, \mu_m^{\otimes(q-1)})\,,$$

where m ranges over all non-zero integers.

(2) If $\mathrm{ch}(k)=p>0$, let

$$H^q(k)=\varinjlim H^q(k, \mu_m^{\otimes(q-1)}) \qquad \text{in the case } q=0, \text{ and}$$

$$H^q(k)=\varinjlim H^q(k, \mu_m^{\otimes(q-1)}) \oplus \varinjlim P_n^{q-1}(k) \quad \text{in the case } q\geqq 1\,,$$

where m ranges over all integers which are invertible in k, and n ranges over all natural numbers.

Here the transition map of the system $\{H^q(k, \mu_m^{\otimes(q-1)})\}_m$ is given for each pair (m, n) such that $m|n$, as follows. Let ζ be a primitive n-th root of 1. Then, the isomorphisms as abelian groups $\boldsymbol{Z}/n\boldsymbol{Z}\cong\mu_n$; $1\mapsto\zeta$ and $\boldsymbol{Z}/m\boldsymbol{Z}\cong\mu_m$; $1\mapsto\zeta^{n/m}$ naturally induce isomorphisms $\boldsymbol{Z}/n\boldsymbol{Z}\cong\mu_n^{\otimes(q-1)}$ and $\boldsymbol{Z}/m\boldsymbol{Z}\cong\mu_m^{\otimes(q-1)}$. The transition map is induced by the injection $\mu_m^{\otimes(q-1)}\to\mu_n^{\otimes(q-1)}$ which corresponds to $\boldsymbol{Z}/m\boldsymbol{Z}\to\boldsymbol{Z}/n\boldsymbol{Z}$;

$1 \mapsto \frac{n}{m}$ via the above isomorphisms, and which is in fact independent of the choice of ζ. The transition map of the system $\{P_n^{q-1}(k)\}_n$ is given by

$$P_n^{q-1}(k) \longrightarrow P_{n+1}^{q-1}(k);$$

$$\{(a_0, \cdots, a_{n-1}), b_1, \cdots, b_{q-1}\} \longmapsto \{(0, a_0, \cdots, a_{n-1}), b_1, \cdots, b_{q-1}\}$$

$$(a_0, \cdots, a_{n-1} \in k,\ b_1, \cdots, b_{q-1} \in k^*).$$

Example. $H^1(k) \cong X_k$ *(via* $H^1(k, \boldsymbol{Z}/m\boldsymbol{Z}) \cong (X_k)_m$ *and Witt theory).*
$H^2(k) \cong \mathrm{Br}(k)$ *(cf.* § 3.4 *Cor. to Lemma* 16).

We shall often identify $H^1(k)$ with X_k via the canonical isomorphism.

These groups $H^*(k)$ have the following elementary properties.

First, $\bigoplus_{q \geq 0} H^q(k)$ have the structure of a right $\bigoplus_{q \geq 0} \mathfrak{K}_q(k)$-module which is induced by the pairings

$$H^i(k, \mu_m^{\otimes(i-1)}) \otimes \mathfrak{K}_j(k) \longrightarrow H^{i+j}(k, \mu_m^{\otimes(i+j-1)});\ x \otimes y \longmapsto x \cup h_{m,k}^j(y),$$

$$P_n^i(k) \otimes \mathfrak{K}_j(k) \longrightarrow P_n^{i+j}(k);$$

$$\{w, a_1, \cdots, a_i\} \otimes \{b_1, \cdots, b_j\} \longmapsto \{w, a_1, \cdots, a_i, b_1, \cdots, b_j\}.$$

We denote the product of this module structure by $\{w, a\}$ ($w \in \bigoplus_{q \geq 0} H^q(k)$, $a \in \bigoplus_{q \geq 0} \mathfrak{K}_q(k)$).

Next, let E be an extension of k. Then, the covariant homomorphism $H^*(k) \to H^*(E)$; $w \mapsto w_E$ is defined in the obvious way. If $[E : k] < \infty$, we can define a natural corestriction map $H^*(E) \to H^*(k)$ as follows. On the part $\varinjlim H^q(\ , \mu_m^{\otimes(q-1)})$, this is the corestriction map in § 1.2 Definition 3. Assume $\mathrm{ch}(k) = p > 0$. Let $\mathrm{Tr}_{E/k} : \hat{C}K_*(E) \to \hat{C}K_*(k)$ be the homomorphism in § 2.1 Definition 1. We can prove as follows that $\mathrm{Tr}_{E/k}$ induces a homomorphism $P_n^{q-1}(E) \to P_n^{q-1}(k)$ ($n, q \geq 1$), whose limit ($n \to \infty$) we shall adopt as the corestriction map for the p-primary part. By § 2.2 Cor. 4 to Prop. 2, it is sufficient to prove

$$\mathrm{Tr}_{E/k}(\mathrm{filt}^n \mathrm{T\hat{C}K}_*(E)) \subset \mathrm{filt}^n \mathrm{T\hat{C}K}_*(k) \quad \text{for all } n \geq 0.$$

But this can be proved using the projection formula $f_*\{s, f^*(r)\} = \{f_*(s), r\}$ (cf. the proof of § 2.5 Lemma 15), for the subquotients $\mathrm{filt}^n \mathrm{T\hat{C}K}/\mathrm{filt}^{n+1}\mathrm{T\hat{C}K}$ can be described in terms of differential modules as in § 2.2 Cor. 2 to Prop. 2.

LEMMA 1. *Let k be a field and E a finite extension of k. Then:*

(1) $\mathrm{Cor}_{E/k}(\{w, a_E\}) = \{\mathrm{Cor}_{E/k}(w), a\}$ *for all* $w \in \bigoplus_{q \geq 0} H^q(E)$ *and* $a \in \bigoplus_{q \geq 0} \mathfrak{K}_q(k)$.

(2) $\mathrm{Cor}_{E/k}(\{w_E, a\}) = \{w, \mathfrak{N}_{E/k}(a)\}$ *for all* $w \in \bigoplus_{q \geq 0} H^q(k)$ *and* $a \in \bigoplus_{q \geq 0} \mathfrak{K}_q(E)$, *where*

$\mathfrak{N}_{E/k}$ is the norm homomorphism of § 1.7 Prop. 5.

PROOF. We can relate $\mathfrak{N}_{E/k}$ with $\mathrm{Cor}_{E/k}$ and $N_{E/k}$ by § 1.2 Lemma 3 and § 2.4 Cor. 2 to Prop. 5.

COROLLARY. *Let k be a field, $\chi \in H^1(k)=X_k$, and $a \in \mathfrak{K}_q(k)$, $q \geqq 0$. Let E be the cyclic extension of k corresponding to χ. Then, if $a \in \mathfrak{N}_{E/k}(\mathfrak{K}_q(E))$, we have $\{\chi, a\}=0$ in $H^{q+1}(k)$.*

Lastly, let K be a complete discrete valuation field with residue field F. We consider the relation between $H^*(K)$ and $H^*(F)$ in the following Lemma 2 and Lemma 3.

DEFINITION 2. Let K and F be as above. Let $i_K^F: H^*(F) \to H^*(K)$ be the homomorphism induced by

$$H^q(F, \mu_m^{\otimes(q-1)})=H^q(\mathrm{Gal}(K_{\mathrm{nr}}/K), H^0(K_{\mathrm{nr}}, \mu_m^{\otimes(q-1)})) \xrightarrow{\mathrm{Inf}} H^q(K, \mu_m^{\otimes(q-1)}),$$

where m ranges over all integers which are invertible in F, and (in the case $\mathrm{ch}(F)>0$)

$$P_n^{q-1}(F) \to H^q(K); \ \{w, b_1, \cdots, b_{q-1}\} \longrightarrow \{i(w), \tilde{b}, \cdots, \tilde{b}_{q-1}\},$$

where i is the canonical homomorphism $P_n^0(F) \cong (X_F)_{p^n} \to (X_K)_{p^n} \subset H^1(K)$ and $\tilde{b}_j$ denotes a lifting of b_j to U_K.

LEMMA 2. *Let K and F be as above. For each $q \geqq 0$, let $H_l^q(F)$ and $H_l^q(K)$ be the subgroups of $H^q(F)$ and $H^q(K)$, respectively, consisting of all elements whose orders are invertible in F. Then, we have a splitting exact sequence*

$$0 \longrightarrow H_l^{q+1}(F) \xrightarrow{i_K^F} H_l^{q+1}(K) \xrightarrow{\delta} H_l^q(F) \longrightarrow 0,$$

where δ is the direct limit of the homomorphism δ in § 1.2. For any prime element π of K,

$$H_l^q(F) \longrightarrow H_l^{q+1}(K); \ w \longmapsto (-1)^q \cdot \{i_K^F(w), \pi\}$$

is a right inverse of δ.

PROOF. This follows from the spectral sequence

$$H^*(F, H^*(K_{\mathrm{nr}}, \ )) \Rightarrow H^*(K, \ )$$

and § 1.2 Corollary to Lemma 2.

LEMMA 3. *Let K and F be as above. Let $q \geqq 0$, and assume $H^{q+1}(F)=0$. Then, the homomorphism*

$$h_K^F : H^q(F) \longrightarrow H^{q+1}(K)\,;\ w \longmapsto \{i_K^F(w),\ \pi\},$$

where π *is a prime element of* K, *does not depend on the choice of* π. *Assume further* $[F: F^p]=p^{q-1}$ *if* $\mathrm{ch}(F)=p>0$. *Then,* h_K^F *is bijective in the case* $\mathrm{ch}(K)=\mathrm{ch}(F)$. *In the case* $\mathrm{ch}(K)=0$ *and* $\mathrm{ch}(F)=p>0$, *the non-p-part of* h_K^F *is bijective and its p-primary part coincides with the direct limit of the homomorphisms*

$$h_{K,p^n}^F : P_n^{q-1}(F) \longrightarrow H^{q+1}(K,\ \mu_{p^n}^{\otimes q}) \quad (n \geqq 0)$$

(*cf.* §1.1 *Definition* 2).

PROOF. In the case $\mathrm{ch}(K)=p>0$, we may assume $K=F((\pi))$. If $[F: F^p]=p^{q-1}$, the homomorphism $P_n^{q-1}(F)\to P_n^q(K)\,;\ w\mapsto\{w,\ \pi\}$ is surjective by §1.6 Lemma 13, but the residue homomorphism $\hat{C}K_{q+1}(K)\to\hat{C}K_q(F)$ induces its left inverse by §2.4 Prop. 5 (2).

Now we can prove the following Proposition. For any finite field k, let $h_k : X_k \xrightarrow{\cong} \boldsymbol{Q}/\boldsymbol{Z}$ be the isomorphism $\chi\mapsto\chi(\mathfrak{F}_k)$ where $\mathfrak{F}_k$ is the Frobenious automorphism over k.

PROPOSITION 1. *Let* N, $k_0, \cdots, k_N$, K *and* k *be as in the hypothesis of Theorem* 1. *Then*:

(1) $H^q(K)=0$ *for all* $q>N+1$, *and there is a unique isomorphism*

$$h_K : H^{N+1}(K) \xrightarrow{\cong} \boldsymbol{Q}/\boldsymbol{Z}$$

which satisfies $h_K(\{\chi,\ \pi_1, \cdots, \pi_N\})=h_k(\chi)$ *for any* $\chi\in X_k$ *and for any* $\pi_1, \cdots, \pi_N$ *such that* π_i *is a lifting of a prime element of* k_i *to* K *for each* i. (*Here* h_k *is as above, and we identify* X_k *with its canonical image in* X_K.)

(2) *Let* L *be a finite extension of* K. *Since* L *is of the same type as* K, *we obtain the isomorphism* h_L *by* (1). *Then, the following diagrams are commutative.*

$$\text{(i)}\quad \begin{array}{ccc} H^{N+1}(K) & \overset{h_K}{\cong} & \boldsymbol{Q}/\boldsymbol{Z} \\ \downarrow & & \downarrow [L:K] \\ H^{N+1}(L) & \underset{h_L}{\cong} & \boldsymbol{Q}/\boldsymbol{Z} \end{array} \qquad \text{(ii)}\quad \begin{array}{ccc} H^{N+1}(L) & \overset{h_L}{\cong} & \boldsymbol{Q}/\boldsymbol{Z} \\ \mathrm{Cor}_{L/K}\downarrow & & \| \\ H^{N+1}(K) & \underset{h_K}{\cong} & \boldsymbol{Q}/\boldsymbol{Z}. \end{array}$$

(3) *Let* $N\geqq 1$, *and let* $F=k_{N-1}$. *Then, the following diagram is commutative.*

$$\begin{array}{ccc} H^{N}(F) & \overset{h_F}{\cong} & \boldsymbol{Q}/\boldsymbol{Z} \\ h_K^F\downarrow & & \| \\ H^{N+1}(K) & \underset{h_K}{\cong} & \boldsymbol{Q}/\boldsymbol{Z}. \end{array}$$

Here h_K^F *is the homomorphism in Lemma* 3.

PROOF. The first assertion in (1) follows from § 1.6 Proof of § 1 Th. 3. The rest of (1) follows from Lemma 3 and § 1 Th. 2. Indeed, we can define h_K by the diagram

$$H^{N+1}(K) \xleftarrow[\cong]{h_{k_N}^{k_{N-1}} \circ \cdots \circ h_{k_1}^{k_0}} H^1(k) \xrightarrow[\cong]{h_k} \boldsymbol{Q}/\boldsymbol{Z}.$$

The commutativity of the diagram (i) in (2) is proved easily by the characterizations of h_K and h_L in (1), and the commutativity of (ii) is deduced from that of (i). The assertion (3) follows from the above definition of h_K.

This proposition enables us to define the desired homomorphism Ψ_K. Let K be as in Prop. 1. Then, we have a pairing

$$X_K \otimes \mathfrak{K}_N(K) = H^1(K) \otimes \mathfrak{K}_N(K) \xrightarrow{\chi \otimes a \mapsto \{\chi, a\}} H^{N+1}(K) \xrightarrow[\cong]{h_K} \boldsymbol{Q}/\boldsymbol{Z}.$$

We define Ψ_K as the homomorphism $\mathfrak{K}_N(K) \to \mathrm{Gal}(K^{\mathrm{ab}}/K)$ induced by this pairing.

COROLLARY 1. *Let* $N \geqq 0$, *and let* K *and* L *be as in Prop.* 1 (2). *Then, the diagrams*

$$\text{(i)}\quad \begin{array}{ccc} \mathfrak{K}_N(L) & \xrightarrow{\Psi_L} & \mathrm{Gal}(L^{\mathrm{ab}}/L) \\ {\scriptstyle \mathfrak{N}_{L/K}}\downarrow & & \downarrow \\ \mathfrak{K}_N(K) & \xrightarrow{\Psi_K} & \mathrm{Gal}(K^{\mathrm{ab}}/K) \end{array} \qquad \text{(ii)}\quad \begin{array}{ccc} \mathfrak{K}_N(K) & \xrightarrow{\Psi_K} & \mathrm{Gal}(K^{\mathrm{ab}}/K) \\ \downarrow & & \downarrow{\scriptstyle \text{transfer}} \\ \mathfrak{K}_N(L) & \xrightarrow{\Psi_L} & \mathrm{Gal}(L^{\mathrm{ab}}/L) \end{array}$$

are commutative. Here $\mathrm{Gal}(L^{\mathrm{ab}}/L) \to \mathrm{Gal}(K^{\mathrm{ab}}/K)$ *is the canonical restriction, and the "transfer" is the dual map of the corestriction map* $\mathrm{Cor}_{L/K}: H^1(L) \to H^1(K)$.

COROLLARY 2. *Let* $N \geqq 1$, *and let* K *and* F *be as in Prop.* 1 (3). *Then, the diagram*

$$\begin{array}{ccc} \mathfrak{K}_N(K) & \xrightarrow{\Psi_K} & \mathrm{Gal}(K^{\mathrm{ab}}/K) \\ {\scriptstyle \partial_F^K}\downarrow & & \downarrow \\ \mathfrak{K}_{N-1}(F) & \xrightarrow{\Psi_F} & \mathrm{Gal}(F^{\mathrm{ab}}/F) \end{array}$$

is commutative, where ∂_F^K *is as in* § 2.4 *Definition* 6 *and* $\mathrm{Gal}(K^{\mathrm{ab}}/K) \to \mathrm{Gal}(F^{\mathrm{ab}}/F)$ *is the restriction to the residue field.*

Our next task is to define the homomorphism Υ_K in the case $\mathrm{ch}(K) = p > 0$. Let $N \geqq 0$, and let K and k be as in the hypothesis of Theorem 2. In this case, the pro-p part $K_N(K) \to \mathrm{Gal}(K^{\mathrm{ab}}/K)(p)$ of Υ_K and the non-p part $K_N(K) \to$

Gal(K^{ab}/K)(non-p) of Υ_K will be defined by rather different methods. Here and in the following, for any pro-finite (resp. torsion) abelian group G, $G(p)$ denotes the pro-p (resp. p-primary) part of G, and G(non-p) denotes the non-p part of G.

Recall that we have defined in §2.5 Lemma 12 a canonical homomorphism $\mathrm{Res}_{K/k}: \hat{C}K_{*+N}(K) \to \hat{C}K_*(k)$. Let Res_K be the composite

$$\mathrm{Res}_K: \hat{C}K_{N+1}(K) \xrightarrow{\mathrm{Res}_{K/k}} \hat{C}K_1(k) \xrightarrow{\mathrm{Tr}_{k/\boldsymbol{F}_p}} \hat{C}K_1(\boldsymbol{F}_p).$$

Then, Res_K is a homomorphism of $W(\boldsymbol{F}_p)$-modules and commutes with the action of the operators V_n and F_n ($n \geqq 1$) by §2 Prop. 1. By these properties, Res_K induces a pairing

$$W^{(p)}(K) \otimes K_N(K) \longrightarrow W^{(p)}(\boldsymbol{F}_p)\,;\ w \otimes a \longmapsto \mathrm{Res}_K\{w, a\},$$

and induces for each $n \geqq 1$,

$$W_n(K) \otimes K_N(K) \longrightarrow W_n(\boldsymbol{F}_p) = \boldsymbol{Z}/p^n\boldsymbol{Z}$$

(cf. §2.2 for the notations $W^{(p)}$ and W_n). For any $w \in W^{(p)}(K)$ and $a \in K_N(K)$,

$$\mathrm{Res}_K\{F_p(w), a\} = \mathrm{Res}_K \circ F_p\{w, a\} = F_p \circ \mathrm{Res}_K\{w, a\} = \mathrm{Res}_K\{w, a\}.$$

Hence we have a pairing

$$W_n(K)/(\mathfrak{F}-1)W_n(K) \otimes K_N(K) \longrightarrow \boldsymbol{Z}/p^n\boldsymbol{Z},$$

where $\mathfrak{F}$ is the homomorphism $(a_0, \cdots, a_{n-1}) \mapsto (a_0^p, \cdots, a_{n-1}^p)$. Since

$$W_n(K)/(\mathfrak{F}-1)W_n(K) \cong (X_K)_{p^n},$$

we have a pairing

$$(X_K)_{p^n} \otimes K_N(K) \longrightarrow \boldsymbol{Z}/p^n\boldsymbol{Z} \xrightarrow{1 \mapsto p^{-n}} \boldsymbol{Q}/\boldsymbol{Z}.$$

When n varies, as is easily seen, this pairing induces a pairing $X_K(p) \otimes K_N(K) \to \boldsymbol{Q}/\boldsymbol{Z}$ and equivalently, induces a homomorphism

$$K_N(K) \longrightarrow \mathrm{Gal}(K^{ab}/K)(p),$$

which is the definition of the pro-p part of Υ_K.

Next, we are concerned with the non-p part of Υ_K. We define it as the homomorphism $K_N(K) \to \mathrm{Gal}(K^{ab}/K)$(non-$p$) induced by the following pairing τ_K in Lemma 4.

DEFINITION 2. Let K be as in the hypothesis of Th. 1. We denote by ∂_k^K the composite

$$K_{*+N}(K) \xrightarrow{\partial_{k_{N-1}}^K} K_{*+N-1}(k_{N-1}) \xrightarrow{\partial_{k_{N-2}}^{k_{N-1}}} \cdots \xrightarrow{\partial_{k_0}^{k_1}} K_*(k),$$

where the homomorphisms $\partial_{k_{i-1}}^{k_i}$ ($i=1, \cdots, N$) are as in §2.4 Definition 6.

LEMMA 4. *Let K be as in the hypothesis of Th. 2. Then, there exists a unique homomorphism*

$$\tau_K : X_K(\text{non-}p)\otimes K_N(K) \longrightarrow \boldsymbol{Q}/\boldsymbol{Z}$$

having the following properties (1) *and* (2).

(1) *For any $\chi\in X_k$ and $a\in K_N(K)$,*

$$\tau_K(\chi\otimes a)=h_k(\partial_k^K(a)\cdot\chi) .$$

Here h_k is the canonical isomorphism $X_k\cong\boldsymbol{Q}/\boldsymbol{Z}$ (cf. Prop. 1) *and we regard $\partial_k^K(a)\in K_0(k)$ as an element of $\boldsymbol{Z}$.*

(2) *For any $\theta\in H^0(K)=\mathrm{Hom}(k^*, \boldsymbol{Q}/\boldsymbol{Z})$, $u\in K^*$, and $a\in K_N(K)$,*

$$\tau_K(\{\theta, u\}\otimes a)=\theta\circ\partial_k^K(\{u, a\}) .$$

Here we regard $\partial_k^K(\{u, a\})\in K_1(k)$ as an element of k^*.

For the proof of Lemma 4, we need the following Lemma.

LEMMA 5. *Let k be a field, and let $\theta\in H^0(k)$ and $u\in k^*$. Take a natural number m invertible in k such that $\mu_m\subset k$ and $\theta\in\mathrm{Hom}(\mu_m, \boldsymbol{Q}/\boldsymbol{Z})$, and take an element v of k_s such that $v^m=u$. Then, $\{\theta, u\}\in H^1(k)=X_k$ coincides with*

$$\mathrm{Gal}(k^{\mathrm{ab}}/k) \longrightarrow \boldsymbol{Q}/\boldsymbol{Z} ; \ \sigma \longmapsto \theta(\sigma(v)v^{-1}) .$$

PROOF OF LEMMA 4. We prove Lemma 4 by induction on N. In the case $N=0$, we must prove that $h_k(\{\theta, u\})=\theta(u)$ for any $\theta\in H^0(k)=\mathrm{Hom}(k^*, \boldsymbol{Q}/\boldsymbol{Z})$ and for any $u\in k^*$, but this is deduced from Lemma 5. Next, assume $N\geqq 1$. Denote k_{N-1} by F and fix a prime element π of K. Then, by Lemma 2, we have an isomorphism

$$H_t^1(F)\oplus H_t^0(F) \xrightarrow{\cong} H_t^1(K) ; \ (\chi, \theta) \longmapsto i_K^F(\chi)+\{i_K^F(\theta), \pi\} .$$

Hence, we can define a homomorphism $\tau_K : X_K(\text{non-}p)\otimes\mathfrak{K}_N(K)\to\boldsymbol{Q}/\boldsymbol{Z}$ by

$$(i_K^F(\chi)+\{i_K^F(\theta), \pi\})\otimes a \longmapsto \tau_F(\chi\otimes\partial_F^K(a))+\theta\circ\partial_k^K(\{\pi, a\}) .$$

(Note that $H_t^0(F)=H_t^0(K)=\mathrm{Hom}(k^*, \boldsymbol{Q}/\boldsymbol{Z})$, and $H_t^1(?)=X_?(\text{non-}p)$.) It is easy to see that τ_K has the properties (1) and (2).

Thus we have defined the homomorphism Υ_K. The relation $\Upsilon_K\circ\iota_K=\Psi_K$ will be proved in § 3.3. In the rest of this § 3.2, we prove the following Lemma 6.

LEMMA 6. *Let K be as in the hypothesis of Th.* 2.

(1) *If L is a finite extension of K, the following diagrams are commutative.*

$$\text{(i)}\quad \begin{array}{ccc} K_N(L) & \xrightarrow{\gamma_L} & \mathrm{Gal}(L^{\mathrm{ab}}/L) \\ {\scriptstyle N_{L/K}}\downarrow & & \downarrow \\ K_N(K) & \xrightarrow{\gamma_K} & \mathrm{Gal}(K^{\mathrm{ab}}/K) \end{array} \qquad \text{(ii)}\quad \begin{array}{ccc} K_N(K) & \xrightarrow{\gamma_K} & \mathrm{Gal}(K^{\mathrm{ab}}/K) \\ \downarrow & & \downarrow{\scriptstyle \text{transfer}} \\ K_N(L) & \xrightarrow{\gamma_L} & \mathrm{Gal}(L^{\mathrm{ab}}/L). \end{array}$$

(2) *Let* $N\geqq 1$ *and denote* k_{N-1} *by* F. *Then, the following diagram is commutative.*

$$\begin{array}{ccc} K_N(K) & \xrightarrow{\gamma_K} & \mathrm{Gal}(K^{\mathrm{ab}}/K) \\ {\scriptstyle \partial_F^K}\downarrow & & \downarrow \\ K_{N-1}(F) & \xrightarrow{\gamma_F} & \mathrm{Gal}(F^{\mathrm{ab}}/F). \end{array}$$

PROOF. First, we prove (1). The pro-p part follows from the projection formula (cf. the proof of §2.5 Lemma 15) and the relation

$$\mathrm{Res}_L=\mathrm{Res}_K\circ\mathrm{Tr}_{L/K}: \hat{C}K_{N+1}(L)\longrightarrow \hat{C}K_1(\boldsymbol{F}_p)$$

(cf. §2.5 Lemma 12). For the non-p part, we need the following Lemmas.

LEMMA 7. *Let K be a complete discrete valuation field with residue field F. Let L be a finite extension of K and E the residue field of L. Then, the following diagrams* (1) *and* (2) *are commutative.*

$$\text{(1)}\quad \begin{array}{ccc} K_{*+1}(L) & \xrightarrow{\partial_E^L} & K_*(E) \\ {\scriptstyle N_{L/K}}\downarrow & & \downarrow{\scriptstyle N_{E/F}} \\ K_{*+1}(K) & \xrightarrow{\partial_F^K} & K_*(F) \end{array} \qquad \text{(2)}\quad \begin{array}{ccc} K_{*+1}(K) & \xrightarrow{\partial_F^K} & K_*(F) \\ {\scriptstyle ?_L}\downarrow & & \downarrow{\scriptstyle e_{L/K}\cdot ?_E} \\ K_{*+1}(L) & \xrightarrow{\partial_E^L} & K_*(E) \end{array}$$

Here $e_{L/K}$ denotes the ramification index of the extension L/K.

PROOF. By §2.1 Lemma 2, it remains to prove the following fact; the functors

$$f=O_L\otimes_{O_K}?,\ g=E\otimes_F?: P(F)\longrightarrow H^1_{O_L,O_L-\{0\}}$$

satisfy $f_*=e_{L/K}\cdot g_*: K_*(F)\to K_*(H^1_{O_L,O_L-\{0\}})\cong K_*(E)$. Let π_L be a prime element of L. Then, all the functors $M\mapsto \pi_L^i f(M)/\pi_L^{i+1}f(M)$ $(0\leqq i<e_{L/K})$ are exact functors isomorphic to g, and $\pi_L^{e_{L/K}}\cdot f=0$. Hence by Quillen [19] §3 Cor. 2 to Th. 2, we have $f_*=e_{L/K}\cdot g_*$.

COROLLARY. *Let K and k be as in the hypothesis of Th.* 1, *L a finite extension of K, and $k'_0,\cdots,k'_N$ the fields determined by the conditions $k'_N=L$ and k'_{i-1} is the residue field of k'_i for each $i=1,\cdots,N$. Denote k'_0 by k'. Then, when we replace F, E, K_{*+1} and $e_{L/K}$ in the diagrams of Lemma* 7 *by k, k', K_{*+N}, and*

$[L:K]\cdot[k':k]^{-1}$ *respectively, the resulting diagrams are commutative.*

LEMMA 8. *Let k be a finite field and E a finite extension of k.*

(1) *For any $\theta\in H^0(k)=\mathrm{Hom}(k^*, \boldsymbol{Q}/\boldsymbol{Z})$, θ_E $(\in H^0(E))$ coincides with the composite* $E^*\xrightarrow{N_{E/K}}k^*\xrightarrow{\theta}\boldsymbol{Q}/\boldsymbol{Z}$.

(2) *For any $\theta\in H^0(E)=\mathrm{Hom}(E^*, \boldsymbol{Q}/\boldsymbol{Z})$, $\mathrm{Cor}_{E/k}(\theta)$ coincides with the composite* $k^*\longrightarrow E^*\xrightarrow{\theta}\boldsymbol{Q}/\boldsymbol{Z}$.

The proof is easy and left to the reader.

Now, we return to the proof of the non-p part of Lemma 6 (1). It suffices to prove the formulae

(i)′ $\tau_K(\chi\otimes N_{L/K}a)=\tau_L(\chi_L\otimes a)$ $(\chi\in X_K(\text{non-}p),\ a\in K_N(L))$,

(ii)′ $\tau_K(\mathrm{Cor}_{L/K}(\chi)\otimes a)=\tau_L(\chi\otimes a_L)$ $(\chi\in X_L(\text{non-}p),\ a\in K_N(K))$.

These formulae can be proved using the following remarks. By Lemma 2 and by induction on N, $X_K(\text{non-}p)$ is generated by $X_k(\text{non-}p)$ and elements of the form $\{\theta, u\}$ $(\theta\in H^0(K)=\mathrm{Hom}(k^*, \boldsymbol{Q}/\boldsymbol{Z}),\ u\in K^*)$. For the proof of (ii)′, it suffices to consider the case $k'=k$ and the case $[k':k]=[L:K]$, where k' is as in Corollary to Lemma 7. In the latter case, as is easily seen, $X_L(\text{non-}p)$ is generated by $X_{k'}(\text{non-}p)$ and elements of the form $\{\theta, u\}$ $(\theta\in H^0(L)=\mathrm{Hom}((k')^*, \boldsymbol{Q}/\boldsymbol{Z}),$ $u\in K^*$ (not only $\in L^*$)). For example, in (ii)′, assume $[k':k]=[L:K]$, $\theta\in H^0(L)$ $=\mathrm{Hom}((k')^*, \boldsymbol{Q}/\boldsymbol{Z})$, $u\in K^*$ and $a\in K_N(K)$. Then,

$$\tau_K(\mathrm{Cor}_{L/K}(\{\theta, u\})\otimes a)=\tau_K(\{\mathrm{Cor}_{L/K}(\theta), u\}\otimes a)=\theta((\partial_k^K(\{u, a\}))_{k'})\quad\text{(Lemma 8)}$$

$$=\theta(\partial_{k'}^L(\{u, a\}_L))\quad\text{(Cor. to Lemma 7)}\qquad=\tau_L(\{\theta, u\}\otimes a_L).$$

The proofs of the other cases are easier and left to the reader.

Next, we prove Lemma 6 (2). The non-p part is clear from the definition of τ_K in the proof of Lemma 4. The pro-p part follows from § 2.4 Cor. 3 to Prop. 5 applied to the case in which $A=F[T]/(T^n)$, $B=O_K[T]/(T^n)$ $(n\geqq 1)$, $S=O_K-\{0\}$ and f is the homomorphism $F[T]/(T^n)\rightarrow O_K[T]/(T^n)$ induced by a ring homomorphism $F\rightarrow O_K$ such that the composite $F\rightarrow O_K\rightarrow O_K/m_K=F$ is the identity map.

§ 3.3. The norm groups.

In this § 3.3, we compute the norm homomorphisms of the Milnor's K-groups of complete discrete valuation fields, and prove Th. 1 as a consequence.

Recall that for any field k and for any finite cyclic extension E of k, there is an exact sequence

$$E^*\xrightarrow{N_{E/k}}k^*\longrightarrow \mathrm{Br}(k)\longrightarrow \mathrm{Br}(E)$$

(cf. Serre [20] Ch. XIV § 1). In the usual local class field theory, this exact sequence enables us to prove $k^*/N_{E/k}(E^*)\cong \mathrm{Gal}(E/k)$ if we assume the well known results on the Brauer groups of the local fields k and E. Similarly, in the local class field theory of this section, it is natural to expect that a sequence of the type

$$\mathfrak{K}_*(E) \xrightarrow{\mathfrak{N}_{E/k}} \mathfrak{K}_*(k) \longrightarrow \cdots$$

will play the same role.

Conjecture 1. *Let k be a field, $\chi\in X_k$, and E the cyclic extension of k corresponding to χ. Then the sequence*

$$\mathfrak{K}_q(E) \xrightarrow{\mathfrak{N}_{E/k}} \mathfrak{K}_q(k) \xrightarrow{a\mapsto\{\chi,\, a\}} H^{q+1}(k) \longrightarrow H^{q+1}(E)$$

is exact for any $q\geqq 0$.

Though we can not prove this Conjecture, we take it as the basic idea of the study in this § 3.3. In the following, we shall prove that the assertion in Conjecture 1 is true in the case where we take K and N in Th. 1 as k and q respectively, and we shall prove Th. 1 by this fact. In the course of the proofs, we obtain the following result.

DEFINITION 3. (Cf. Chapter I § 4.) Let $i\geqq 0$ and let p be a prime number. We call a field k, a *$\mathfrak{B}_i$-field* (resp. *$\mathfrak{B}_i(p)$-field*) if and only if for any finite extension E of k and for any finite extension F of E, the group (resp. the p-primary part of the torsion group) $\mathfrak{K}_i(E)/\mathfrak{N}_{F/E}\mathfrak{K}_i(F)$ is zero. Of course, a field k is a $\mathfrak{B}_i$-field if and only if it is $\mathfrak{B}_i(p)$ for all p.

PROPOSITION 2. *Let K be a complete discrete valuation field with residue field F, p a prime number, and $i\geqq 0$. Then the following conditions* (i) *and* (ii) *are equivalent.*

(i) *F is a $\mathfrak{B}_i(p)$-field.*

(ii) *K is a $\mathfrak{B}_{i+1}(p)$-field.*

Furthermore, if these equivalent conditions are satisfied, and if $\mathrm{ch}(F)=p$ *and* $\mathrm{ch}(K)=0$, *we have* $\mathrm{cd}_p(K)\leqq i+1$.

REMARK 1. It is probable that the property $\mathfrak{B}_i(p)$ is closely related to the cohomological p-dimension cd_p. In fact, if the above Conjecture 1 is true for any field k and for a fixed q, it is easily deduced that a field k such that $\mathrm{ch}(k)\neq p$ is $\mathfrak{B}_q(p)$ if and only if $H^{q+1}(k')(p)=0$ for any finite extension k' of k. If § 1.1 Conjecture 1 is true for any field k, for q and $q+1$ and for all $m=p^n$ $(n\geqq 0)$, it is deduced that a field k such that $\mathrm{ch}(k)\neq p$ is $\mathfrak{B}_q(p)$ if and only if $\mathrm{cd}_p(k)\leqq q$.

Before we start the proofs of Prop. 2 and Th. 1, we prove some preliminary Lemmas.

LEMMA 9. *A field k is $\mathfrak{B}_i(p)$ if and only if for any finite extension E of k and for any finite normal extension F of E of degree p, $\mathfrak{N}_{F/E}: \mathfrak{K}_i(F)\to\mathfrak{K}_i(E)$ is surjective.*

LEMMA 10. *If k is a $\mathfrak{B}_i(p)$-field, it is $\mathfrak{B}_q(p)$ for all $q\geqq i$ and $\mathfrak{K}_q(k)$ is p-divisible for all $q>i$.*

LEMMA 11. *Let k be a $\mathfrak{B}_i(p)$-field and assume* $\mathrm{ch}(k)=p$. *Then,*

(1) $[k: k^p]\leqq p^i$.

(2) $P_n^i(k)=0$ *for all n.*

In particular, $1-\gamma: \Omega_k^i\to\Omega_k^i$ is surjective, where γ is the Cartier operator.

LEMMA 12. *Let k be a field and assume* $\mathrm{ch}(k)=p>0$, $[k: k^p]=p^q<\infty$. *Then, $\mathfrak{N}_{k^{1/p}/k}: \mathfrak{K}_q(k^{1/p})\to\mathfrak{K}_q(k)$ coincides with the isomorphism $\{x_1, \cdots, x_q\}\mapsto\{x_1^p, \cdots, x_q^p\}$.*

LEMMA 13. *For a discrete valuation field K with residue field F, and for a prime element π of K, let $\partial_\pi: \mathfrak{K}_{q+1}(K)\to\mathfrak{K}_{q+1}(F)\oplus\mathfrak{K}_q(F)$ be the surjective homomorphism with kernel $\mathfrak{U}_{q+1}^{(1)}(K)$;*

$$\partial_\pi(\{x_1, \cdots, x_{q+1}\}+\{y_1, \cdots, y_q, \pi\})=(\{\bar{x}_1, \cdots, \bar{x}_{q+1}\}, \{\bar{y}_1, \cdots, \bar{y}_q\})$$

$(x_1, \cdots, x_{q+1}, y_1, \cdots, y_q\in U_K, q\geqq 0)$ (*cf.* Bass and Tate [4] Ch. I *Prop.* 4.3). *Let K be a complete discrete valuation field, L a finite extension of K, F and E the residue fields of K and L, and π_K and π_L prime elements of K and L, respectively. Let $\pi_K=u\cdot\pi_L^e$ $(u\in U_L, e\in \boldsymbol{Z})$. Then, the following diagrams* (i) *and* (ii) *are commutative.*

$$\text{(i)}\quad \begin{array}{ccc} \mathfrak{K}_{q+1}(K) & \xrightarrow{\partial_{\pi_K}} & \mathfrak{K}_{q+1}(F)\oplus\mathfrak{K}_q(F) \\ \downarrow & & \downarrow j_{\pi_L,\pi_K} \\ \mathfrak{K}_{q+1}(L) & \xrightarrow{\partial_{\pi_L}} & \mathfrak{K}_{q+1}(E)\oplus\mathfrak{K}_q(E) \end{array} \qquad \text{(ii)}\quad \begin{array}{ccc} \mathfrak{K}_{q+1}(L) & \xrightarrow{\partial_{\pi_L}} & \mathfrak{K}_{q+1}(E)\oplus\mathfrak{K}_q(E) \\ \mathfrak{N}_{L/K}\downarrow & & \downarrow \mathfrak{N}_{\pi_K,\pi_L} \\ \mathfrak{K}_{q+1}(K) & \xrightarrow{\partial_{\pi_K}} & \mathfrak{K}_{q+1}(F)\oplus\mathfrak{K}_q(F) \end{array}$$

Here j_{π_L,π_K} and $\mathfrak{N}_{\pi_K,\pi_L}$ are the homomorphisms defined by:

$$j_{\pi_L,\pi_K}(a, b)=(a_E+\{b_E, \bar{u}\}, e\cdot b_E),$$

$$\mathfrak{N}_{\pi_K,\pi_L}(a, b)=(e\cdot\mathfrak{N}_{E/F}(a)-\mathfrak{N}_{E/F}(\{b, \bar{u}\}), \mathfrak{N}_{E/F}(b)).$$

PROOF OF LEMMA 9. This is proved easily using the Sylow subgroups of Galois groups.

PROOF OF LEMMA 10. The first assertion is obvious. For the rest, let x be any element of k^*, y a solution of $y^p=x$, and $E=k(y)$. For $q>i$, $\mathfrak{K}_{q-1}(k)=p\cdot\mathfrak{K}_{q-1}(k)+\mathfrak{N}_{E/k}\mathfrak{K}_{q-1}(E)$, and $\{\mathfrak{N}_{E/k}\mathfrak{K}_{q-1}(E), x\}\subset p\cdot\mathfrak{N}_{E/k}\{\mathfrak{K}_{q-1}(E), y\}$.

PROOF OF LEMMA 11. By Lemma 10, $\mathfrak{K}_{i+1}(k)$ is p-divisible. Hence, by § 1.3 Lemma 7, we have $[k:k^p]\leqq p^i$. On the other hand, it is easily seen that in § 3.2 Corollary to Lemma 1, we may replace $H^1(k)$ by $P_n^0(k)$ and $H^{q+1}(k)$ by $P_n^q(k)$. Since k is $\mathfrak{B}_i(p)$, we have $\{\chi, a\}=0$ in $P_n^i(k)$ for all $\chi\in P_n^0(k)$ and $a\in\mathfrak{K}_i(k)$. Thus $P_n^i(k)=0$.

PROOF OF LEMMA 12. Let $x_1, \cdots, x_q\in(k^{1/p})^*$. We can take a sequence $k=E_0\subset E_1\subset\cdots\subset E_q=k^{1/p}$ such that $[E_i: E_{i-1}]=p$ and $x_i\in E_i$ for all $i=1, \cdots, q$. We have

$$\mathfrak{N}_{k^{1/p}/k}\{x_1, \cdots, x_q\}=\mathfrak{N}_{E_1/E_0}\circ\cdots\circ\mathfrak{N}_{E_q/E_{q-1}}\{x_1, \cdots, x_q\}=\{x_1^p, \cdots, x_q^p\}.$$

PROOF OF LEMMA 13. The commutativity of (i) is easily seen. Next, we prove (ii). By $\mathfrak{N}_{\pi_K,\pi_L}\circ j_{\pi_L,\pi_K}=[L:K]$, and by [4] Ch. I § 5 diagram (15) and [4] Ch. I Cor. 5.3, it is sufficient to prove

$$\partial_{\pi_K}\circ\mathfrak{N}_{L/K}(z)=\mathfrak{N}_{\pi_K,\pi_L}\circ\partial_{\pi_L}(z)$$

for elements $z\in\mathfrak{K}_{q+1}(L)$ of the form $\{x, y_L\}$ $(x\in\mathfrak{K}_1(L),\ y\in\mathfrak{K}_q(K))$ assuming that $[L:K]$ is a prime number. But this can be proved by easy computation.

PROOFS OF PROPOSITION 2 AND THEOREM 1. Now, we are ready to prove Prop. 2 and Th. 1. Generally, let K be a complete discrete valuation field with residue field F, p a prime number, L a cyclic extension of K of degree p, and E the residue field of L. We study the group $\mathfrak{K}_q(K)/\mathfrak{N}_{L/K}\mathfrak{K}_q(L)$ in the following cases (A), (B), (C) and (D).

(A) *Assume that L is unramified over K. We claim*

$$\mathfrak{K}_q(K)/\mathfrak{N}_{L/K}\mathfrak{K}_q(L)\cong\mathfrak{K}_q(F)/\mathfrak{N}_{E/F}\mathfrak{K}_q(E)\oplus\mathfrak{K}_{q-1}(F)/\mathfrak{N}_{E/F}\mathfrak{K}_{q-1}(E).$$

Indeed, since $U_K^{(1)}\subset N_{L/K}(L^*)$, $\mathfrak{U}_q^{(1)}(K)\subset\mathfrak{N}_{L/K}\mathfrak{K}_q(L)$ for all $q\geqq 1$. So, our claim follows from Lemma 13.

(B) *Assume that L is totally ramified over K, and that $p\neq\mathrm{ch}(F)$. We claim*

$$\mathfrak{K}_q(K)/\mathfrak{N}_{L/K}\mathfrak{K}_q(L)\cong\mathfrak{K}_q(F)/p\cdot\mathfrak{K}_q(F).$$

Indeed, as above, $\mathfrak{U}_q^{(1)}(K)\subset\mathfrak{N}_{L/K}\mathfrak{K}_q(L)$ for all $q\geqq 1$, and our claim follows from Lemma 13.

(C) *Assume $p=\mathrm{ch}(F)$, $[F:F^p]\leqq p^{q-1}$ and that L is totally ramified over K. Let π_L be a prime element of L, $\pi_K=N_{L/K}\pi_L$, σ a generator of* Gal(L/K), *$a=\sigma(\pi_L)\pi_L^{-1}-1$, and $b=N_{L/K}(a)$. We claim:*

There is a unique surjective homomorphism

$$f\colon \Omega_F^{q-1}/(1-\gamma)\Omega_F^{q-1} \longrightarrow \mathfrak{K}_q(K)/\mathfrak{N}_{L/K}\mathfrak{K}_q(L)$$

such that $f\left(\bar{x}\frac{d\bar{y}_1}{\bar{y}_1}\wedge\cdots\wedge\frac{d\bar{y}_{q-1}}{\bar{y}_{q-1}}\right)=\{1+xb,\ y_1,\ \cdots,\ y_{q-1}\} \bmod \mathfrak{N}_{L/K}\mathfrak{K}_q(L)$ *for all* $x\in O_K$, *and* $y_1,\cdots,y_{q-1}\in U_K$.

Indeed, since $[F\colon F^p]\leqq p^{q-1}$ by the assumption, $\mathfrak{K}_q(F)$ is p-divisible by § 1.3 Lemma 7. Hence we have

$$\mathfrak{K}_q(K)\subset\mathfrak{U}_q^{(1)}(K)+\mathfrak{N}_{L/K}\mathfrak{K}_q(L)$$

by Lemma 13. Next, since $[F\colon F^p]\leqq p^{q-1}$, Ω_F^{q-1} is additively generated by $d(\Omega_F^{q-2})$ and elements of the form $x^p\frac{dy_1}{y_1}\wedge\cdots\wedge\frac{dy_{q-1}}{y_{q-1}}$ $(x\in F,\ y_1,\cdots,y_{q-1}\in F^*)$. Hence, it follows from § 1.3 Lemma 6 (2) that for any $n\geqq 1$, $\mathfrak{U}_q^{(n)}(K)/\mathfrak{U}_q^{(n+1)}(K)$ is generated by elements of the forms $\{1+x^p\pi_K^n,\ y_1,\cdots,y_{q-1}\}$ and $\{1+x\pi_K^n,\ y_1,\cdots,y_{q-2},\ \pi_K\}$ $(x\in O_K,\ y_1,\cdots,y_{q-1}\in U_K)$. Let t be the ramification number of L/K, i.e. $t=v_K(b)$, which is independent of the choices of π_L and σ. If $0<n<t$, by Serre [20] Ch. V § 3, we have

$$1+x^p\pi_K^n\equiv N_{L/K}(1+x\pi_L^n) \mod U_K^{(n+1)} \quad \text{for all } x\in O_K,$$

which proves

$$\mathfrak{U}_q^{(n)}(K)\subset\mathfrak{N}_q^{(n+1)}(K)+\mathfrak{N}_{L/K}\mathfrak{K}_q(L) \quad \text{for } 0<n<t.$$

On the other hand, $U_K^{(t+1)}\subset N_{L/K}(L^*)$ by [20] Ch. V § 3, and hence $\mathfrak{U}_q^{(t+1)}(K)\subset\mathfrak{N}_{L/K}\mathfrak{K}_q(L)$. Furthermore, $(1-\gamma)\Omega_F^{q-1}$ is generated by $d(\Omega_F^{q-2})$ and elements of the form $(x^p-x)\frac{dy_1}{y_1}\wedge\cdots\wedge\frac{dy_{q-1}}{y_{q-1}}$. Now, our claim follows from;

$$N_{L/K}(1+xa)\equiv 1+(x^p-x)b \mod U_K^{(t+1)} \quad \text{for all } x\in O_K$$

([20] Ch. V § 3).

(D) *Assume* $p=\mathrm{ch}(F)$, $[F\colon F^p]\leqq p^{q-1}$, $q\geqq 1$ *and that E is purely inseparable over F. Let h be an element of O_L such that $E=F(\bar{h})$, $g=N_{L/K}(h)$, σ a generator of* $\mathrm{Gal}(L/K)$, $a=\sigma(h)h^{-1}-1$, *and* $b=N_{L/K}(a)$. *We claim:*

There is a unique surjective homomorphism

$$f\colon \Omega_F^{q-1}/(1-\gamma)\Omega_F^{q-1} \longrightarrow \mathfrak{K}_q(K)/\mathfrak{N}_{L/K}\mathfrak{K}_q(L)$$

such that $f\left(\bar{x}\frac{d\bar{y}_1}{\bar{y}_1}\wedge\cdots\wedge\frac{d\bar{y}_{q-2}}{\bar{y}_{q-2}}\wedge\frac{d\bar{g}}{\bar{g}}\right)=\{1+xb,\ y_1,\cdots,y_{q-2},\ \pi\} \bmod \mathfrak{N}_{L/K}\mathfrak{K}_q(L)$ *for all* $x\in O_K$, $y_1,\cdots,y_{q-2}\in U_K$ *and for all prime elements π of K.*

Indeed, since $[F\colon F^p]\leqq p^{q-1}$, the sequence

$$\Omega_F^{q-3}\xrightarrow{?\wedge\frac{d\bar{g}}{\bar{g}}}\Omega_F^{q-2}\xrightarrow{?\wedge\frac{d\bar{g}}{\bar{g}}}\Omega_F^{q-1}\longrightarrow 0$$

is exact and we can deduce from this fact that Ω_F^{q-2} is additively generated by $d(\Omega_F^{q-3})$, $\Omega_F^{q-3}\wedge\frac{d\bar{g}}{\bar{g}}$, and elements of the form $x^p\bar{g}^i\frac{dy_1}{y_1}\wedge\cdots\wedge\frac{dy_{q-2}}{y_{q-2}}$ $(0\leqq i<p$, $x\in F$, $y_1,\cdots,y_{q-2}\in F^*)$. It follows that for any $n\geqq 1$, $\mathfrak{U}_q^{(n)}(K)/\mathfrak{U}_q^{(n+1)}(K)$ is generated by $\{\mathfrak{U}_{q-1}^{(n)}(K), g\}$ and elements of the form $\{1+x^pg^i\pi^n, y_1,\cdots,y_{q-2},\pi\}$ where π is a prime element of K, and if $p\nmid n$, it is generated by $\{\mathfrak{U}_{q-1}^{(n)}(K), g\}$. Let $t=v_K(b)$, which is in fact independent of the choices of h and σ (cf. Remark 1 at the end of Chapter I § 1). Then, since $g\in N_{L/K}(L^*)$, and since

$$1+x^pg^i\pi^n\equiv N_{L/K}(1+xh^i\pi^{n/p}) \mod U_K^{(n+1)}$$

in the case $p|n$ and $0<n<t$ (cf. Chapter I § 1 (17) and Remark 1 of that section), we have

$$\mathfrak{U}_q^{(n)}(K)\subset\mathfrak{U}_q^{(n+1)}(K)+\mathfrak{N}_{L/K}\mathfrak{K}_q(L) \quad \text{if } 0<n<t.$$

On the other hand, $U_K^{(t+1)}\subset N_{L/K}(L^*)$ (cf. Ch. I § 1 Remark 1 (iii)) and hence $\mathfrak{U}_q^{(t+1)}(K)\subset\mathfrak{N}_{L/K}\mathfrak{K}_q(L)$. By these facts, our claim follows from;

$$N_{L/K}\Big(1+\sum_{i=0}^{p-1}x_ih^ia\Big)\equiv 1+\Big(x_0^p-x_0+\sum_{i=1}^{p-1}x_i^pg^i\Big)b \mod U_K^{(t+1)}$$

for all $x_0,\cdots,x_{p-1}\in O_K$, which is a consequence of Chapter I § 1 (17), and of $N_{L/K}(\sigma(h)h^{-1})=1$ and $\mathrm{Tr}_{L/K}(\sigma(h^i)-h^i)=0$.

These claims in (A)~(D) are sufficient for the proof of Prop. 2. Indeed, first, assume that F is a $\mathfrak{B}_i(p)$-field. For the proof of the fact K is $\mathfrak{B}_{i+1}(p)$, by Lemma 9, it suffices to prove $\mathfrak{K}_{i+1}(K)=\mathfrak{N}_{L/K}\mathfrak{K}_{i+1}(L)$ in the case where L is a cyclic extension of K of degree p, and in the case where $\mathrm{ch}(K)=p$ and L is a purely inseparable finite extension of K. By Lemma 10, we have $\mathfrak{K}_{i+1}(F)/p\cdot\mathfrak{K}_{i+1}(F)=0$. By Lemma 11, if $\mathrm{ch}(F)=p$, we have $[F:F^p]\leqq p^i$ and $\Omega_F^i/(1-\gamma)\Omega_F^i=0$. When we apply the above (A)~(D) by taking $q=i+1$, these facts show $\mathfrak{K}_{i+1}(K)=\mathfrak{N}_{L/K}\mathfrak{K}_{i+1}(L)$ in the case where L is a cyclic extension of K of degree p. On the other hand, if $\mathrm{ch}(K)=p$, $[K:K^p]\leqq p^{i+1}$ by $[F:F^p]\leqq p^i$. Hence Lemma 12 shows $\mathfrak{K}_{i+1}(K)=\mathfrak{N}_{L/K}\mathfrak{K}_{i+1}(L)$ in the case where L is a purely inseparable finite extension of K. Conversely, assume that K is $\mathfrak{B}_{i+1}(p)$. Then, by (A), $\mathfrak{K}_i(F)=\mathfrak{N}_{E/F}\mathfrak{K}_i(E)$ for any cyclic extension E/F of degree p. Furthermore, if $\mathrm{ch}(F)=p$, we can deduce $\mathfrak{K}_{i+1}(F)=p\cdot\mathfrak{K}_{i+1}(F)$ from Lemma 13 by taking a totally ramified cyclic extension L of K of degree p. Hence $[F:F^p]\leqq p^i$, and by Lemma 12, $\mathfrak{K}_i(F)=\mathfrak{N}_{E/F}\mathfrak{K}_i(E)$ for any purely inseparable finite extension E of F. The assertion concerning the cohomological dimension follows from § 1.1 Th. 1 (3) with the help of Lemma 11.

COROLLARY (to Prop. 2). *Let N and K be as in the hypothesis of Th.* 1. *Then, K is a $\mathfrak{B}_{N+1}$-field. For any $q\geqq N+2$, $\mathfrak{K}_q(K)$ is a divisible group.*

Now we proceed to the proof of Th. 1. Let N and K be as in the hypothesis of Th. 1. Our tasks are to prove the bijectivity of the homomorphism $\mathfrak{K}_N(K)/\mathfrak{N}_{L/K}\mathfrak{K}_N(L)\to \mathrm{Gal}(L/K)$ induced by Ψ_K and by Cor. 1 to §3.2 Prop. 1 for any finite abelian extension L of K, and to prove the uniqueness of Ψ_K. The former can be reduced to the case where L is a cyclic extension of a prime degree by the induction using the commutative diagram of exact sequences

$$\begin{array}{ccccccc} \mathfrak{K}_N(K')/\mathfrak{N}_{L/K'}\mathfrak{K}_N(L) & \xrightarrow{\mathfrak{N}_{K'/K}} & \mathfrak{K}_N(K)/\mathfrak{N}_{L/K}\mathfrak{K}_N(L) & \longrightarrow & \mathfrak{K}_N(K)/\mathfrak{N}_{K'/K}\mathfrak{K}_N(K') & \longrightarrow & 0 \\ \text{by } \Psi_{K'}\downarrow & & \text{by } \Psi_K\downarrow & & \text{by } \Psi_K\downarrow & & \\ 0 \longrightarrow \mathrm{Gal}(L/K') & \longrightarrow & \mathrm{Gal}(L/K) & \longrightarrow & \mathrm{Gal}(K'/K) & \longrightarrow & 0 \end{array}$$

$(K\subset K'\subset L)$. Furthermore, in the case where L is cyclic over K, the former is equivalent (modulo Prop. 1 (2) (ii)) to the exactness of the sequence

$$\mathfrak{K}_N(L)\xrightarrow{\mathfrak{N}_{L/K}}\mathfrak{K}_N(K)\xrightarrow{\{\chi,\ ?\}}H^{N+1}(K)\longrightarrow H^{N+1}(L)\,,$$

where χ is an element of X_K to which L corresponds. Thus, for the former, it is sufficient to prove the following assertion (H).

(H) *Let K be as in the hypothesis of Th.* 1, *p a prime number, χ an element of X_K of order p, and L the cyclic extension of K of degree p corresponding to χ. Then,* $\{\chi, ?\}$ *induces an isomorphism from $\mathfrak{K}_N(K)/\mathfrak{N}_{L/K}\mathfrak{K}_N(L)$ onto $H^{N+1}(K)_p$.*

Now, we prove this assertion by induction on N. The case $N=0$ is easy and so we assume $N\geqq 1$. Let F and E be the residue fields of K and L, respectively. We consider again the cases (A)$\sim$(D) taking N as q.

In the case (A), we may regard χ as an element of the subgroup X_F of X_K. By the above Corollary, we have $\mathfrak{K}_N(F)=\mathfrak{N}_{E/F}\mathfrak{K}_N(E)$. Thus we have the following commutative diagram, which proves the assertion (H) in this case by induction on N.

$$\begin{array}{ccc} \mathfrak{K}_{N-1}(F)/\mathfrak{N}_{E/F}\mathfrak{K}_{N-1}(E) & \xrightarrow{\{\chi,\ ?\}} & H^N(F)_p \\ \wr\Vert\downarrow & & \wr\Vert\downarrow h_K^F \\ \mathfrak{K}_N(K)/\mathfrak{N}_{L/K}\mathfrak{K}_N(L) & \xrightarrow{\{\chi,\ ?\}} & H^{N+1}(K)_p\,. \end{array}$$

In the case (B), $\chi=\{\theta, \pi\}$ for some non-zero element θ of $H^0(K)_p=H^0(F)_p$ and for some prime element π of K (cf. §3.2 Lemma 2). We have a commutative diagram

$$\begin{array}{ccc} \mathfrak{K}_N(F)/p\cdot\mathfrak{K}_N(F) & \xrightarrow{h_{p,F}^N} & H^N(F,\ \mu_p^{\otimes N})\cong H^N(F)_p \\ \wr\Vert\downarrow & & \wr\Vert\downarrow (-1)^N\cdot h_K^F \\ \mathfrak{K}_N(K)/\mathfrak{N}_{L/K}\mathfrak{K}_N(L) & \xrightarrow{\{\chi,\ ?\}} & H^{N+1}(K)_p\,, \end{array}$$

where $H^N(F, \mu_p^{\otimes N}) \cong H^N(F)_p$ is the isomorphism induced by θ. Hence, in this case, the assertion (H) follows from Corollary to Lemma 14 below.

LEMMA 14. *Let K be a complete discrete valuation field with residue field F. Let $q \geqq 1$.*

(1) *If m is an integer invertible in F, $h^q_{m,K}$ is bijective if and only if $h^q_{m,F}$ and $h^{q-1}_{m,F}$ are bijective.*

(2) *Assume* $\mathrm{ch}(F)=p>0$, $\mathrm{ch}(K)=0$, $[F: F^p]=p^{q-2}$, *and that K contains a primitive p-th root of 1. Then, $h^q_{p,K}$ is bijective if $h^F_{K,p}: P_1^{q-2}(F) \to H^q(K, \mu_p^{\otimes(q-1)})$ is bijective.*

PROOF. The proof of (2) is contained at the end of § 1.3.

COROLLARY. *Let N and K be as in the hypothesis of Th. 1. Then, $h^{N+1}_{p,K}$ is bijective for any prime number p which is invertible in K.*

PROOF. This follows from the above Lemma 14, Cor. to Prop. 2, and § 1.1 Th. 2 by induction on N.

In the cases (C) and (D), the assertion (H) follows from the following commutative diagram by induction on N.

$$\begin{array}{ccc} \Omega_F^{N-1}/(1-\gamma)\Omega_F^{N-1} & \cong & H^N(F)_p \\ f\downarrow & & \wr\!\!\parallel\downarrow \pm h_K^F \\ \mathfrak{K}_N(K)/\mathfrak{N}_{L/K}\mathfrak{K}_N(L) & \xrightarrow{\{\chi,\ ?\}} & H^{N+1}(K)_p \end{array}$$

The commutativity (up to sign) of this diagram is a consequence of

LEMMA 15. *Let K be a complete discrete valuation field with residue field F. Let p be a prime number, χ an element of X_K of order p, L/K the cyclic extension corresponding to χ, σ the element of* $\mathrm{Gal}(L/K)$ *such that $\chi(\sigma)=1/p$. Suppose* $\mathrm{ch}(F)=p$ *and that L/K is not unramified. Let $h \in L^*$, $a=\sigma(h)h^{-1}-1$, and $b=N_{L/K}(a)$. Then, for any $u \in O_K$,*

$$\{\chi,\ 1+ub\} = -\{i(\bar{u}),\ N_{L/K}(h)\} \quad in\ H^2(K),$$

where $i(\bar{u})$ denotes the image of $\bar{u}$ under the canonical homomorphism

$$F \longrightarrow F/\{x^p-x \mid x \in F\} \cong (X_F)_p \xrightarrow{\subset} (X_K)_p\,.$$

PROOF. Let $t=v_K(b)$. Then, $1 \leqq t < \infty$. By Chapter I § 1 (17), we have $U_K^{(t+1)} \subset N_{L/K}(L^*)$ and

$$(*) \qquad N_{L/K}(1+xa) \equiv 1+(x^p-x)b \mod U_K^{(t+1)}$$

for any $x \in O_K$. Hence, if $\bar{u} \in \{x^p - x \mid x \in F\}$, $\{\chi, 1+ub\}=0$ by § 3.2 Cor. to Lemma 1. Assume $\bar{u} \notin \{x^p - x \mid x \in F\}$. Let T/K the unramified cyclic extension of degree p corresponding to $i(\bar{u})$, v an element of O_T such that $\bar{v}^p - \bar{v} = \bar{u}$, M the composite field $L \underset{K}{\otimes} T$, τ the unique element of $\mathrm{Gal}(M/K)$ such that $\tau|_L = \sigma$ and $\overline{\tau(v)} = \bar{v}+1$, L' the fixed field of τ in M. Since $\chi(\tau) = i(\bar{u})(\tau) = 1/p$, it follows that L'/K is the cyclic extension corresponding to $\chi - i(\bar{u})$. Since M/L' is unramified, the sequence $U_M^{(n)} \xrightarrow{\tau-1} U_M^{(n)} \xrightarrow{N_{M/L'}} U_{L'}^{(n)}$ is exact for any $n \geqq 1$. From this, we can deduce that there is an element w of O_M such that $\bar{w} = \bar{v}$ and $\tau(1+wa) \cdot (1+wa)^{-1} = 1+a$. We have $(1+wa)h^{-1} \in L'$ and

$$N_{L'/K}((1+wa)h^{-1}) = N_{M/T}(1+wa) \cdot N_{L/K}(h)^{-1}$$
$$\equiv (1+ub) \cdot N_{L/K}(h)^{-1} \mod U_K^{(i+1)}$$

by the above (*). Hence (note that $U_K^{(1)} \subset N_{T/K}(T^*)$), by Cor. to Lemma 1, we have

$$0 = \{\chi - i(\bar{u}), (1+ub) \cdot N_{L/K}(h)^{-1}\} = \{\chi, 1+ub\} + \{i(\bar{u}), N_{L/K}(h)\},$$

which proves Lemma 15.

Thus, we have proved the assertion (H), and hence, Ψ_K induces an isomorphism $\mathfrak{K}_N(K)/\mathfrak{N}_{L/K}\mathfrak{K}_N(L) \cong \mathrm{Gal}(L/K)$ for any finite abelian extension L of K, and Conjecture 1 holds in the case $k=K$ (K is as in Th. 1) and $q=N$.

Lastly, we prove the uniqueness of Ψ_K. Let Ψ' be a homomorphism $\mathfrak{K}_N(K) \to \mathrm{Gal}(K^{ab}/K)$ having the following properties (1) and (2).

(1) For any finite abelian extension L of K, the homomorphism $\mathfrak{K}_N(K) \to \mathrm{Gal}(L/K)$ induced by Ψ' annihilates $\mathfrak{N}_{L/K}\mathfrak{K}_N(L)$.

(2) For each $i=1, \cdots, N$, let π_i be a lifting to K of a prime element of k_i. Then, the image of $\Psi'(\{\pi_1, \cdots, \pi_N\})$ under the canonical homomorphism $\mathrm{Gal}(K^{ab}/K) \to \mathrm{Gal}(k^{ab}/k)$ coincides with the Frobenius automorphism over k.

Since $\mathfrak{K}_N(K)$ is generated by elements of the form $\{\pi_1, \cdots, \pi_N\}$ as above, it is sufficient to prove $\Psi'(a) = \Psi_K(a)$ for any $a = \{\pi_1, \cdots, \pi_N\}$. Let S be the fixed field of $\Psi_K(a)$ in K^{ab}, and T the unramified extension of K corresponding to k_s/k. Then, for any finite extension L/K contained in S, $\Psi_K(a)|_S = 1$ shows $a \in \mathfrak{N}_{L/K}\mathfrak{K}_N(L)$. Hence by (1), we have $\Psi'(a)|_S = 1$. Since $\Psi'(a)|_T = \Psi_K(a)|_T$ by (2) and since $K^{ab} = S \cdot T$, $\Psi'(a) = \Psi_K(a)$ on K^{ab}.

Now we completed the proof of Th. 1.

Concerning Th. 2, it remains to prove $\Upsilon_K \circ \iota_K = \Psi_K$ in the case $\mathrm{ch}(K) = p > 0$. Let $\Psi' = \Upsilon_K \circ \iota_K$. Then, by § 3.2 Lemma 6 (1) (i) and by § 2.4 Cor. 2 to Prop. 5, Ψ' has the above property (1). Furthermore, by § 3.2 Lemma 6 (2) and by § 2.4 Cor. 1 to Prop. 5, Ψ' has the above property (2). Hence $\Psi' = \Psi_K$ as was seen above.

§ 3.4. The Brauer groups.

In this § 3.4, we prove the following Prop. 3 and Prop. 4.

For any field k, let $X_k \otimes k^* \to \mathrm{Br}(k)$; $\chi \otimes a \mapsto (\chi, a)$ be the pairing in Serre [20] Ch. XIV § 1.

PROPOSITION 3. *Let $N \geqq 1$, and let K be as in § 3.1 Th. 1. Then, there is a canonical injective homomorphism*

$$\Phi_K : \mathrm{Br}(K) \longrightarrow \mathrm{Hom}(\mathfrak{K}_{N-1}(K), \boldsymbol{Q}/\boldsymbol{Z})$$

such that $\Phi_K((\chi, a))(b) = \chi(\Psi_K(\{a, b\}))$ for any $\chi \in X_K$, $a \in K^*$ and $b \in \mathfrak{K}_{N-1}(K)$.

PROPOSITION 4. *Let N and K be as in § 3.1 Th. 2. Then for each $1 \leqq q \leqq N+1$, there is a unique homomorphism*

$$\Theta_K^q : H^q(K) \longrightarrow \mathrm{Hom}(K_{N+1-q}(K), \boldsymbol{Q}/\boldsymbol{Z})$$

such that $\Theta_K^q(\{\chi, a_1, \cdots, a_{q-1}\})(b) = \chi \circ \Upsilon_K(\{a_1, \cdots, a_{q-1}, b\})$ for all $\chi \in X_K = H^1(K)$, $a_1, \cdots, a_{q-1} \in K^*$, and $b \in K_{N+1-q}(K)$. Here Υ_K is the homomorphism $K_N(K) \to \mathrm{Gal}(K^{\mathrm{ab}}/K)$ defined in § 3.2.

REMARK 2. If $\mathrm{ch}(K) = p > 0$, the homomorphism Φ_K in Prop. 3 is characterized by the mentioned relation with Ψ_K, for $\mathrm{Br}(K)$ is generated by elements of the form (χ, a) $(\chi \in X_K,\ a \in K^*)$ in this case. It is probable that for any field k, $H^q(k)$ $(q \geqq 1)$ is generated by elements of the form $\{\chi, a_1, \cdots, a_{q-1}\}$ $(\chi \in X_k = H^1(k),\ a_1, \cdots, a_{q-1} \in k^*)$.

We begin with the following Lemma.

LEMMA 16. *Let k be a field of characteristic $p > 0$, and let $n \geqq 0$. Then, there is a canonical isomorphism*

$$P_n^1(k) \xrightarrow{\cong} \mathrm{Br}(k)_{p^n};\ \{\chi, a\} \mapsto (\chi, a) \quad (\chi \in P_n^0(k),\ a \in k^*),$$

where we identify $P_n^0(k)$ with $(X_k)_{p^n}$ via Witt theory [25].

COROLLARY. *For any field k, $H^2(k)$ is canonically isomorphic to* $\mathrm{Br}(k)$.

PROOF OF LEMMA 16. The pairing $(X_k)_{p^n} \otimes k^* \to \mathrm{Br}(k)_{p^n}$; $\chi \otimes a \mapsto (\chi, a)$ induces via Witt theory a pairing $P_n^0(k) \otimes k^* \to \mathrm{Br}(k)_{p^n}$. By Teichmüller [24] Satz 1, this pairing induces a homomorphism $P_n^1(k) \to \mathrm{Br}(k)_{p^n}$; $\{\chi, a\} \mapsto (\chi, a)$. It remains to prove the bijectivity of this homomorphism. By induction on n, we may assume $n=1$. But in this case, $P_1^1(k) \to \mathrm{Br}(k)_p$ is the homomorphism $\Omega_k/(1-\gamma)\Omega_{k,d=0} \to \mathrm{Br}(k)_p$ induced from the exact sequence

$$0 \longrightarrow k_s^* \xrightarrow{p} k_s^* \xrightarrow{x \mapsto \frac{dx}{x}} \Omega_{k_s, d=0} \xrightarrow{1-\gamma} \Omega_{k_s} \longrightarrow 0$$

by taking the Galois cohomology, and hence is bijective.

PROOF OF PROPOSITION 3. The homomorphism Φ_K is defined by the isomorphism $\mathrm{Br}(K) \cong H^2(K)$ (the above Corollary) and by the pairing

$$H^2(K) \otimes \mathfrak{K}_{N-1}(K) \xrightarrow{w \otimes a \mapsto \{w, a\}} H^{N+1}(K) \overset{h_K}{\cong} \boldsymbol{Q}/\boldsymbol{Z}.$$

The relation with Ψ_K is clear. The injectivity of Φ_K is non-trivial, and is proved as follows. It is sufficient to prove for each prime number p, that the homomorphism $\mathrm{Br}(K)_p \to \mathrm{Hom}(\mathfrak{K}_{N-1}(K), \boldsymbol{Q}/\boldsymbol{Z})$ induced by Φ_K is injective. We prove this by induction on N. Assume $w \in \mathrm{Br}(K)_p$ and $\Phi_K(w)=0$, and let L/K be a finite Galois extension of K such that $w_L=0$. (We hope to prove $w=0$.) Let K' be the fixed field of a Sylow-p-subgroup of $\mathrm{Gal}(L/K)$. Then, $\mathrm{Br}(K)_p \to \mathrm{Br}(K')_p$ is injective, and there is a sequence $K'=E_0 \subset \cdots \subset E_n=L$ such that E_i is a cyclic extension of E_{i-1} of degree p for each $i=1, \cdots, n$. Since $\Phi_{E_i}(w_{E_i})=\Phi_K(w)\circ \mathfrak{N}_{E_i/K}=0$, it follows that we may assume the extension L/K to be cyclic and of degree p. Let χ be an element of X_K to which L corresponds. Since the sequence

$$L^* \xrightarrow{N_{L/K}} K^* \xrightarrow{(\chi, ?)} \mathrm{Br}(K) \longrightarrow \mathrm{Br}(L)$$

is exact, all things are reduced to the following assertion (I).

(I) *Let p be a prime number, $\chi \in (X_K)_p$, $\chi \neq 0$ and L/K the cyclic extension of degree p corresponding to χ. Then the homomorphism*

$$\varphi_{\chi,K}: K^*/N_{L/K}(L^*) \longrightarrow \mathrm{Hom}(\mathfrak{K}_{N-1}(K), \boldsymbol{Q}/\boldsymbol{Z});$$
$$a \longmapsto (b \mapsto \Phi_K(\{\chi, a\})(b)=\chi\circ \Psi_K(\{a, b\}))$$

is injective.

Fortunately, by Serre [20] Ch. V §3, and by Ch. I §1 of this paper, the group $K^*/N_{L/K}(L^*)$ can be known exactly. Let F and E be the residue fields of K and L respectively. We may assume $N \geqq 2$, for the case $N=1$ is well known in the usual local class field theory. There are four cases (A)~(D).

(A) Assume that L/K is unramified. Then, we can regard χ as an element of $X_F \subset X_K$. We have a commutative diagram

$$\begin{array}{ccc} K^*/N_{L/K}(L^*) & \xrightarrow{\varphi_{\chi,K}} & \mathrm{Hom}(\mathfrak{K}_{N-1}(K)/\mathfrak{U}_{N-1}^{(1)}(K), \boldsymbol{Q}/\boldsymbol{Z}) \\ \wr\Vert & & \wr\Vert \\ F^*/N_{E/F}(E^*) \oplus \boldsymbol{Z}/p\boldsymbol{Z} & \xrightarrow{\varphi_{\chi,F} \oplus \text{“}\chi\circ\Psi_F\text{''}} & \mathrm{Hom}(\mathfrak{K}_{N-2}(F), \boldsymbol{Q}/\boldsymbol{Z}) \oplus \mathrm{Hom}(\mathfrak{K}_{N-1}(F), \boldsymbol{Q}/\boldsymbol{Z}), \end{array}$$

where "$\chi\circ\Psi_F$" denotes the homomorphism $1\mapsto\chi\circ\Psi_F$. Hence $\varphi_{\chi,K}$ is injective in this case by induction on N.

(B) Assume that L/K is totally ramified and $\mathrm{ch}(F)\neq p$. Then F contains a primitive p-th root of 1. We have a commutative diagram

$$\begin{array}{ccc} K^*/N_{L/K}(L^*) & \xrightarrow{\varphi_{\chi,K}} & \mathrm{Hom}(\mathfrak{K}_{N-1}(K)/p\cdot\mathfrak{K}_{N-1}(K),\ \boldsymbol{Q}/\boldsymbol{Z}) \\ \wr\Vert & & \downarrow \\ F^*/(F^*)^p \cong (X_F)_p & \xrightarrow{\text{by } \Psi_F} & \mathrm{Hom}(\mathfrak{K}_{N-1}(F),\ \boldsymbol{Q}/\boldsymbol{Z}) \end{array}$$

where $F^*/(F^*)^p\cong(X_F)_p$ is the isomorphism in Kummer theory. Since the homomorphism $X_F\to\mathrm{Hom}(\mathfrak{K}_{N-1}(F),\ \boldsymbol{Q}/\boldsymbol{Z})$ induced by Ψ_F is injective by § 3.1 Th. 1, $\varphi_{\chi,K}$ is injective in this case.

(C) Assume that L/K is totally ramified and $\mathrm{ch}(F)=p$. Let t be as in § 3.3 Proofs of Prop. 1 and Th. 1 case (C). If $i+j>t$,

$$\chi\circ\Psi_K(\{U_K^{(i)},\ \mathfrak{U}_{N-1}^{(j)}(K)\})\subset\chi\circ\Psi_K(\mathfrak{U}_N^{(t+1)}(K))=0\,,$$

for $\mathfrak{U}_N^{(t+1)}(K)\subset\mathfrak{N}_{L/K}\mathfrak{K}_N(L)$. Hence $\varphi_{\chi,K}$ induces the upper horizontal arrow in the following each diagram below. Let $h:\ \Omega_F^{N-1}\to\boldsymbol{Q}/\boldsymbol{Z}$ be the composite

$$\Omega_F^{N-1}\longrightarrow\Omega_F^{N-1}/(1-\gamma)\Omega_F^{N-1}\cong P_1^{N-1}(F)\longrightarrow H^N(F)\cong\boldsymbol{Q}/\boldsymbol{Z}\,,$$

and note that h is not the zero map. By § 3.3 Lemma 15 and by the computation of $N_{L/K}:\ L^*\to K^*$ in [20] Ch. V § 3, the following diagrams are commutative for some suitable choices of φ_*^1 and φ_*^{N-1} (cf. § 1.3 Lemma 6 (2)).

$$\begin{array}{ccc} K^*/(U_K^{(1)}\cdot N_{L/K}(L^*)) & \xrightarrow{\varphi_{\chi,K}} & \mathrm{Hom}\,(\mathfrak{U}_{N-1}^{(t)}(K)/\mathfrak{U}_{N-1}^{(t+1)}(K),\ \boldsymbol{Q}/\boldsymbol{Z}) \\ \wr\Vert & & \Big\downarrow \text{by } \varphi_t^{N-1}(*,\ 0) \\ F^*/(F^*)^p & \xrightarrow{x\mapsto h\left(\frac{dx}{x}\wedge ?\right)} & \mathrm{Hom}\,(\Omega_F^{N-2},\ \boldsymbol{Q}/\boldsymbol{Z})\,. \end{array}$$

$$\begin{array}{ccc} (U_K^{(n)}\cdot N_{L/K}(L^*))/(U_K^{(n+1)}\cdot N_{L/K}(L^*)) & \xrightarrow{\varphi_{\chi,K}} & \mathrm{Hom}\,(\mathfrak{U}_{N-1}^{(t-n)}(K)/\mathfrak{U}_{N-1}^{(t-n+1)}(K),\ \boldsymbol{Q}/\boldsymbol{Z}) \\ \wr\Vert & & \Big\downarrow \text{by } \varphi_{t-n}^{N-1}(*,\ 0) \\ F/F^p & \xrightarrow{x\mapsto h\left(\frac{dx}{x}\wedge ?\right)} & \mathrm{Hom}\,(\Omega_F^{N-2},\ \boldsymbol{Q}/\boldsymbol{Z}) \end{array}$$

$(1\leqq n<t)$.

$$\begin{array}{ccc}
(U_K^{(t)}\cdot N_{L/K}(L^*))/N_{L/K}(L^*) & \xrightarrow{\varphi_{\chi,K}} & \mathrm{Hom}\,(\mathfrak{K}_{N-1}(K)/\mathfrak{U}^{(1)}_{N-1}(K),\ \boldsymbol{Q}/\boldsymbol{Z}) \\
\wr\| & & \downarrow \\
F/\{x^p-x \mid x\in F\}\cong(X_F)_p & \xrightarrow{\text{by } \Psi_F} & \mathrm{Hom}\,(\mathfrak{K}_{N-1}(F),\ \boldsymbol{Q}/\boldsymbol{Z})\,.
\end{array}$$

Hence the upper horizontal arrow in each diagram is injective, and this proves the injectivity of $\varphi_{\chi,K}$.

(D) Assume that E/F is a purely inseparable extension of degree p. Let h, g, σ, a, b and t be as in § 3.3 Proofs of Prop. 1 and Th. 1 case (D). Then, Chapter I § 1 (cf. Remark 1 at the end of that section) shows $U_K^{(t+1)}\subset N_{L/K}(L^*)$, and shows that there are diagrams which are commutative for some choices of φ_*^1 and φ_*^{N-1} by § 3.3 Lemma 15:

$$\begin{array}{ccc}
K^*/(U_K^{(1)}\cdot N_{L/K}(L^*)) & \xrightarrow{\varphi_{\chi,K}} & \mathrm{Hom}(\mathfrak{U}^{(t)}_{N-1}(K)/\mathfrak{U}^{(t+1)}_{N-1}(K),\ \boldsymbol{Q}/\boldsymbol{Z}) \\
\wr\| & & \big\downarrow \text{by } \varphi_t^{N-1} \\
\boldsymbol{Z}/p\boldsymbol{Z}\oplus F^*/(E^*)^p & \longrightarrow & \mathrm{Hom}(\Omega_F^{N-2},\ \boldsymbol{Q}/\boldsymbol{Z})\oplus\mathrm{Hom}(\Omega_F^{N-3},\ \boldsymbol{Q}/\boldsymbol{Z}) \\
m\oplus u & \longrightarrow & m\cdot h\left(\frac{d\bar{g}}{\bar{g}}\wedge ?\right)\oplus h\left(\frac{du}{u}\wedge\frac{d\bar{g}}{\bar{g}}\wedge ?\right),
\end{array}$$

$$\begin{array}{ccc}
(U_K^{(n)}\cdot N_{L/K}(L^*))/(U_K^{(n+1)}\cdot N_{L/K}(L^*)) & \xrightarrow{\varphi_{\chi,K}} & \mathrm{Hom}\,(\mathfrak{U}^{(t-n)}_{N-1}(K)/\mathfrak{U}^{(t-n+1)}_{N-1}(K),\ \boldsymbol{Q}/\boldsymbol{Z}) \\
\wr\| & & \big\downarrow \text{by } \varphi_{t-N}^{N-1}(*,\ 0) \\
F & \xrightarrow{x\mapsto h\left(x\frac{d\bar{g}}{\bar{g}}\wedge ?\right)} & \mathrm{Hom}\,(\Omega_F^{N-2},\ \boldsymbol{Q}/\boldsymbol{Z}) \\
 & & (1\leqq n<t,\ p\nmid n)\,,
\end{array}$$

$$\begin{array}{ccc}
(U_K^{(n)}\cdot N_{L/K}(L^*))/(U_K^{(n+1)}\cdot N_{L/K}(L^*)) & \xrightarrow{\varphi_{\chi,K}} & \mathrm{Hom}\,(\mathfrak{U}^{(t-n)}_{N-1}(K)/\mathfrak{U}^{(t-n+1)}_{N-1}(K),\ \boldsymbol{Q}/\boldsymbol{Z}) \\
\wr\| & & \big\downarrow \text{by } \varphi_{t-n}^{N-1}(0,\ *) \\
F/E^p & \xrightarrow{x\mapsto h\left(dx\wedge\frac{d\bar{g}}{\bar{g}}\wedge ?\right)} & \mathrm{Hom}\,(\Omega_F^{N-3},\ \boldsymbol{Q}/\boldsymbol{Z}) \\
 & & (1\leqq n<t,\ p\mid n)\,.
\end{array}$$

Hence the upper horizontal arrow in each diagram is injective. It remains to prove the injectivity of

$$(U_K^{(t)}\cdot N_{L/K}(L^*)/N_{L/K}(L^*) \xrightarrow{\varphi_{\chi,K}} \mathrm{Hom}(\mathfrak{K}_{N-1}(K)/\mathfrak{U}_{N-1}^{(t)}(K), \boldsymbol{Q}/\boldsymbol{Z})$$

By Hilbert's Satz 90, the sequence $L^* \xrightarrow{\sigma-1} L^* \xrightarrow{N_{L/K}} K^*$ is exact. From this and Chapter I §1, we can deduce that the sequence

$$0 \longrightarrow F^* \xrightarrow{\subset} E^* \xrightarrow{\bar{x}\mapsto\sigma(x)x^{-1}} U_L^{(tp^{-1})}/U_L^{(tp^{-1}+1)} \xrightarrow{N_{L/K}} U_K^{(t)}/U_K^{(t+1)}$$

is exact and the homomorphism $N_{L/K}: U_L^{(tp^{-1})}/U_L^{(tp^{-1}+1)} \to U_K^{(t)}/U_K^{(t+1)}$ is $1+\sum_{i=0}^{p-1} x_i h^i a \mapsto 1+(x_0^p - x_0 + \sum_{i=1}^{p-1} x_i^p g^i)b$ (each $x_i \in O_K$). Since this sequence is still exact when we replace F, E, K and L by F_s, E_s, K_{nr} and L_{nr} respectively, it is continued as

$$U_L^{(tp^{-1})}/U_L^{(tp^{-1}+1)} \xrightarrow{N_{L/K}} U_K^{(t)}/U_K^{(t+1)} \xrightarrow{1+xb\mapsto\left[x\frac{d\bar{g}}{\bar{g}}\right]} \mathrm{Br}(F)_p$$

by taking the Galois cohomology. Here, for each $\omega\in\Omega_F$, $[\omega]$ denotes the image of ω under the canonical isomorphism $\Omega_F/(1-\gamma)\Omega_{F,d=0}\cong\mathrm{Br}(F)_p$. Now, the following commutative diagram completes the proof of the injectivity of $\varphi_{\chi,K}$ by induction on N.

$$\begin{array}{ccc} (U_K^{(t)}\cdot N_{L/K}(L^*))/N_{L/K}(L^*) & \xrightarrow{\varphi_{\chi,K}} & \mathrm{Hom}(\mathfrak{K}_{N-1}(K)/\mathfrak{U}_{N-1}^{(t)}(K), \boldsymbol{Q}/\boldsymbol{Z}) \\ & \searrow \qquad \swarrow & \\ & \mathrm{Hom}(\mathfrak{K}_{N-2}(F), \boldsymbol{Q}/\boldsymbol{Z}). & \end{array}$$

$$1+xb \mapsto \Phi_F\left(\left[\bar{x}\frac{d\bar{g}}{\bar{g}}\right]\right)$$

Thus we have proved the assertion (I) and hence Prop. 3.

PROOF OF PROPOSITION 4. Let K be as in §3.1 Th. 2. First we define the p-primary part of Θ_K^q. Let $\mathrm{Res}_K: \hat{C}K_{N+1}(K)\to\hat{C}K_1(\boldsymbol{F}_p)$ the homomorphism defined in §3.2. Then, the induced homomorphism

$$\hat{C}K_q(K) \longrightarrow \mathrm{Hom}(K_{N+1-q}(K), \hat{C}K_1(\boldsymbol{F}_p))\,;\; w \longmapsto (a\mapsto\mathrm{Res}_K(\{w, a\}))$$

is compatible with the actions of $W(\boldsymbol{F}_p)$, V_n ($n\geqq 1$) and F_n ($n\geqq 1$) (on $\hat{C}K_q(K)$ and $\hat{C}K_1(\boldsymbol{F}_p)$). Hence we obtain a homomorphism

$$\theta_K^q: T\hat{C}K_q(K) \longrightarrow \mathrm{Hom}(K_{N+1-q}(K), W^{(p)}(\boldsymbol{F}_p)) = \mathrm{Hom}(K_{N+1-q}(K), \boldsymbol{Z}_p),$$

which satisfies

(1) $\theta_K^q \circ V_p = p \cdot \theta_K^q$ (2) $\theta_K^q \circ F_p = \theta_K^q$.
Furthermore, we can prove
(3) $\theta_K^q(\{T\hat{C}K_{q-1}(K), T\})=0$
as follows. By easy computation, for any $i \geqq 0$, we have

$$V_{p^i}(\{T\hat{C}K_{q-1}(K), T\}) \subset \mathrm{filt}^{2i} T\hat{C}K_q(K).$$

Since the image of $\mathrm{filt}^{2i}T\hat{C}K_q(K)$ in $K_q(K[T]/(T^{p^{2i}-1}))$ is zero, the image of $\theta_K^q \circ V_{p^i}(\{T\hat{C}K_{q-1}(K), T\})$ in $\mathrm{Hom}(K_{N+1-q}(K), K_1(\boldsymbol{F}_p[T]/(T^{p^{2i}-1}))$ is zero by the definition of the residue homomorphisms. By the above (1), this implies

$$p^i \cdot \theta_K^q(\{T\hat{C}K_{q-1}(K), T\}) \subset \mathrm{Hom}(K_{N+1-q}(K), p^{2i}\boldsymbol{Z}_p) \quad \text{for all } i \geqq 0,$$

and hence we have $\theta_K^q(\{T\hat{C}K_{q-1}(K), T\})=0$. By § 2.2 Cor. 4 to Prop. 2, these (2) and (3) show that θ_K^q induces a homomorphism

$$P_n^{q-1}(K) \longrightarrow \mathrm{Hom}(K_{N+1-q}(K), \boldsymbol{Z}/p^n\boldsymbol{Z});$$
$$\{w, a_1, \cdots, a_{q-1}\} \longmapsto (b \mapsto \mathrm{Res}_K(\{w, a_1, \cdots, a_{q-1}, b\})$$

for each $n \geqq 0$, and in the limit, a homomorphism

$$H^q(K)(p) \longrightarrow \mathrm{Hom}(K_{N+1-q}(K), \boldsymbol{Q}/\boldsymbol{Z}).$$

We adopt this homomorphism as the p-primary part of Θ_K^q.

Next, the non-p part of Θ_K^q is defined by induction on N as follows. We may assume $N \geqq 1$ and $q \geqq 2$. Let F be the residue field of K, and π a fixed prime element of K. Then, by § 3.2 Lemma 2, we have an isomorphism

$$H^q(F)(\text{non-}p) \oplus H^{q-1}(F)(\text{non-}p) \xrightarrow{\cong} H^q(K)(\text{non-}p);$$
$$w_1 \oplus w_2 \longmapsto i_K^F(w_1) + \{i_K^F(w_2), \pi\}.$$

Hence, by induction on N, we can define a homomorphism

$$H^q(K)(\text{non-}p) \longrightarrow \mathrm{Hom}(K_{N+1-q}(K), \boldsymbol{Q}/\boldsymbol{Z})$$

by $i_K^F(w_1) + \{i_K^F(w_2), \pi\} \mapsto (b \mapsto \Theta_F^q(w_1)(\partial_F^K(b)) + \Theta_F^{q-1}(w_2)(\partial_F^K\{\pi, b\}))$.

Thus, we have defined the homomorphism Θ_K^q. The relation with Υ_K is clear on the p-primary part and can be easily checked by induction on N on the non-p part. The uniqueness of Θ_K^q follows from the fact that $H^q(K)$ $(q \geqq 1)$ is generated by elements of the form $\{\chi, a_1, \cdots, a_{q-1}\}$ $(\chi \in H^1(K), a_1, \cdots, a_{q-1} \in K^*)$, which can be proved easily by induction on N and by § 3.2 Lemma 2.

§ 3.5. The case $N=2$.

In this § 3.5, we prove Th. 3 stated in § 3.1. We have already proved this

Theorem in the mixed characteristic case in Chapter I. It is easily seen that the definitions of the homomorphisms Ψ_K and Φ_K of this Chapter II coincide with those of Chapter I §6 in this case. So, in the following, we consider only the case $\mathrm{ch}(K)=\mathrm{ch}(k_1)$.

Let K be as in the hypothesis of Th. 3 ($N=2$), $F=k_1$, and assume $\mathrm{ch}(K)=\mathrm{ch}(F)$. We have already seen in Th. 1 the injectivity of the homomorphism $X_K\to\mathrm{Hom}(K_2(K), \boldsymbol{Q}/\boldsymbol{Z})$ induced by Ψ_K, and that of the homomorphism $\Phi_K:\mathrm{Br}(K)\to\mathrm{Hom}(K^*, \boldsymbol{Q}/\boldsymbol{Z})$ in Prop. 3. It remains to prove the following assertions (i)~(v). In the following, the topologies of K^* and $K_2(K)$ mean the ones defined in Chapter I §7 in the case $\mathrm{ch}(K)=\mathrm{ch}(F)=p>0$, and the discrete topologies in the case $\mathrm{ch}(K)=\mathrm{ch}(F)=0$.

(i) For each $\chi\in X_K$, $\chi\circ\Psi_K: K_2(K)\to\boldsymbol{Q}/\boldsymbol{Z}$ is continuous.

(ii) For each continuous homomorphism $\varphi: K_2(K)\to\boldsymbol{Q}/\boldsymbol{Z}$ of finite order, there is an element χ of X_K such that $\varphi=\chi\circ\Psi_K$.

(iii) For each $w\in\mathrm{Br}(K)$, $\mathrm{Ker}(\Phi_K(w))=\mathrm{Nrd}(w/K)$.

(iv) For each $w\in\mathrm{Br}(K)$, $\Phi_K(w): K^*\to\boldsymbol{Q}/\boldsymbol{Z}$ is continuous.

(v) For each continuous homomorphism $\varphi: K^*\to\boldsymbol{Q}/\boldsymbol{Z}$ of finite order, there is an element w of $\mathrm{Br}(K)$ such that $\varphi=\Phi_K(w)$.

REMARK 3. Though we adopt above the discrete topologies in the case $\mathrm{ch}(K)=\mathrm{ch}(F)=0$, we can define more reasonable topologies, which take the usual topology of the residue field into account, if a ring homomorphism $f:\boldsymbol{Q}_p\to O_K$ is fixed satisfying the condition that the composite $\boldsymbol{Q}_p\xrightarrow{f}O_K\to O_K/m_K=F$ coincides with the canonical inclusion. Then, for each non-zero element t of m_K, O_K becomes a free module of finite rank over $\boldsymbol{Q}_p[[t]]$ via f. We can endow $\boldsymbol{Q}_p[[t]]$ with the product topology of the usual topology of $\boldsymbol{Q}_p$, and O_K with the topology as a finite product of copies of $\boldsymbol{Q}_p[[t]]$. This topology of O_K induces in the way of Chapter I §7, the topologies of K^* and $K_2(K)$. These topologies are independent of the choices of t and a basis of O_K over $\boldsymbol{Q}_p[[t]]$, but unfortunately, really depend on the choice of the homomorphism f, for the image under f of a transcendental basis $(b_i)_i$ of $\boldsymbol{Q}_p$ over $\boldsymbol{Q}$ can be arbitrarily chosen satisfying $f(b_i)\bmod m_K=b_i$ for all i.

REMARK 4. In the case $\mathrm{ch}(F)=p>0$, any continuous homomorphism $K^*\to\boldsymbol{Q}/\boldsymbol{Z}$ or $K_2(K)\to\boldsymbol{Q}/\boldsymbol{Z}$ is automatically of finite order. This can be proved just as in Chapter I §8.

Now we prove the assertions (i)~(v). We shall omit the details of the proofs in the case where they are just the repetitions of those in Chapter I. First, we prove (iii). Let $w\in\mathrm{Br}(K)$. By §3.2 Lemma 1 (2) and Prop. 1 (2) (ii), we have the formula

$$\Phi_L(w_L)(a)=\Phi_K(w)(N_{L/K}(a)) \quad (w\in \mathrm{Br}(K),\ a\in L^*),$$

where L is any finite extension of K. By virtue of this formula, the proof of (iii) goes just as in Chapter I § 6. We do not repeat here the same argument. The proof of (iv) is the repetition of Chapter I § 8 Proof of Th. 1 Step 1, and (i) follows from (iv) just as in Chapter I § 8 Proof of Th. 2 Step 1. Lastly, we prove the existence theorems (ii) and (v). The case where the order of φ is invertible in F is easy, and so, we assume that $\mathrm{ch}(K)=p>0$ and that the order of φ is a power of p. Since X_k and $\mathrm{Br}(k)$ are p-divisible for any field k of characteristic $p>0$, we are reduced to the case $p\cdot\varphi=0$. Our task becomes to the surjectivity of the injective homomorphisms

$$\alpha:\ K/\{x^p-x\mid x\in K\} \longrightarrow \mathrm{Hom}_c\Big(K_2(K)/p\cdot K_2(K),\ \frac{1}{p}\boldsymbol{Z}/\boldsymbol{Z}\Big)$$

$$x \longmapsto \Big(\{y,\ z\}\mapsto h\Big(x\frac{dy}{y}\wedge\frac{dz}{z}\Big)\Big),$$

$$\beta:\ \Omega_K/(1-\gamma)\Omega_{K,d=0} \longrightarrow \mathrm{Hom}_c\Big(K^*/(K^*)^p,\ \frac{1}{p}\boldsymbol{Z}/\boldsymbol{Z}\Big)$$

$$\omega \longmapsto \Big(x\mapsto h\Big(\omega\wedge\frac{dx}{x}\Big)\Big).$$

where Hom_c denotes the group of all continuous homomorphisms, and h denotes the composite

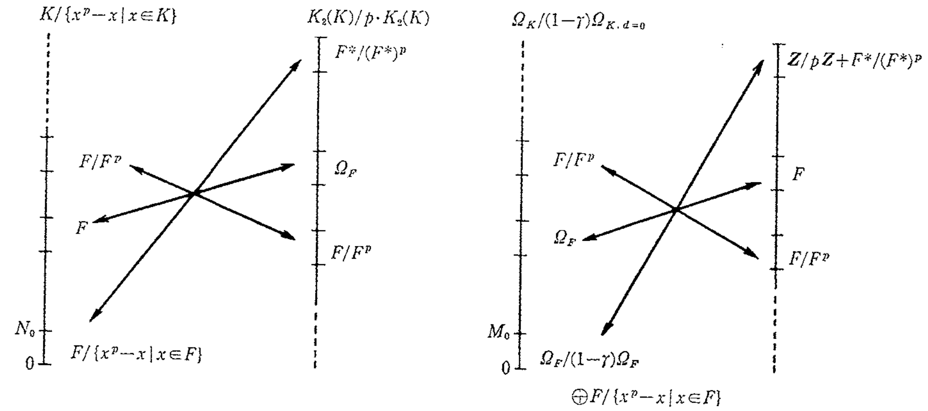
$$\Omega_K^2/(1-\gamma)\Omega_K^2\cong P_1^2(K) \longrightarrow H^3(K) \overset{h_K}{\cong} \boldsymbol{Q}/\boldsymbol{Z}.$$

For each $n\geqq -1$, let N_n be the image of $\{x\in K\mid v_K(x)\geqq -n\}$ in $K/\{x^p-x\mid x\in K\}$, and M_n the subgroup of $\Omega_K/(1-\gamma)\Omega_{K,d=0}$ defined in Chapter I § 4 Proof of Lemma 5. Recall that we have computed the groups M_n/M_{n-1} $(n\geqq 0)$ in that place. Similarly, we have $N_{-1}=0$, $N_0/N_{-1}\cong F/\{x^p-x\mid x\in F\}$ canonically, and

$$N_n/N_{n-1}\cong F/F^p;\ x\longmapsto \overline{xb^p} \bmod F^p \quad \text{if } n\geqq 1 \text{ and } p\mid n,$$

$$N_n/N_{n-1}\cong F;\ x\longmapsto \overline{xc} \quad \text{if } n\geqq 1 \text{ and } p\nmid n,$$

where b and c are fixed elements of K such that $v_K(b)=n/p$ and $v_K(c)=n$. On the other hand, the subquotients of $K_2(K)/p\cdot K_2(K)$ were computed in Chapter I § 2 Prop. 1 and those of $K^*/(K^*)^p$ were computed in Chapter I § 6 Figure 2. It is easy to prove that N_n annihilates $(\mathfrak{U}_2^{(n+1)}(K)+p\cdot K_2(K))/p\cdot K_2(K)$ via α and M_n annihilates $U_K^{(n+1)}(K^*)^p/(K^*)^p$ via β for any $n\geqq 0$. The analogues of Chapter I § 6 Figure 3 are obtained as follows.



Since a continuous homomorphism $K_2(K)\to\frac{1}{p}\boldsymbol{Z}/\boldsymbol{Z}$ $\left(\text{resp. } K^*\to\frac{1}{p}\boldsymbol{Z}/\boldsymbol{Z}\right)$ annihilates $\mathfrak{U}_2^{(n)}(K)$ (resp. $U_K^{(n)}$) for sufficiently large n as is easily seen, we can prove the surjectivity of α and β just as in Chapter I §8 Proof of Th. 2 Step 2.5 and Chapter I §8 Proof of Th. 1 Step 2.5, respectively.

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