A generalization of local class field theory by using K-groups III

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To the memory of Takuro Shintani

In this paper, we shall study the class field theory of a complete discrete valuation field K whose residue field F is an algebraic function field in one variable over a finite field. We shall define in §2 a topological abelian group $C_2(K)$ called the K_2-idele class group of K, which plays the role of the usual idele class group in the class field theory of F. Our aim is to prove the following theorem which was stated in the introduction of [3].

THEOREM 1. *There is a canonical homomorphism* $C_2(K)\to \mathrm{Gal}(K^{\mathrm{ab}}/K)$ *which induces an isomorphism* $C_2(K)/N_{L/K}C_2(L)\cong \mathrm{Gal}(L/K)$ *for each finite abelian extension* L *of* K, *where* $N_{L/K}: C_2(L)\to C_2(K)$ *is the canonical norm homomorphism. The correspondence* $L\mapsto N_{L/K}C_2(L)$ *induces a bijection from the set of all finite abelian extensions of* K *to the set of all open subgroups of* $C_2(K)$ *of finite indices.*

We shall also define the K_1-idele class group $C_1(K)$ which can be used to describe the Brauer group of K.

THEOREM 2. *The Brauer group* $\mathrm{Br}(K)$ *of* K *is canonically isomorphic to the group of all continuous homomorphisms from* $C_1(K)$ *to the discrete group* $\boldsymbol{Q}/\boldsymbol{Z}$.

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§1. A preliminary for liftings of rings.

In this section, we prove the following lemma.

LEMMA 1. *Let* p *be a prime number,* A *a ring,* I *a nilpotent ideal of* A *such that* $p\cdot 1_A\in I$. *Let* $R=A/I$, *and let* $h: R\to S$ *be a ring homomorphism satisfying the following condition* (C).

(C) *The* S*-homomorphism* $R^{(p)}\otimes_R S\to S^{(p)}$; $x\otimes y\mapsto h(x)y^p$ *is bijective, where*

$R^{(p)}$ denotes the ring over R with the underlying ring R and with the structural map $R \to R^{(p)}$; $x \mapsto x^p$, and $S^{(p)}$ denotes the ring over S defined similarly.

Then, there exists a formally etale ring B over A such that B/IB is isomorphic to S over R. If S is flat over R, this ring B over A is characterized by the properties that it is flat over A and B/IB is isomorphic to S over R.

Here, as in Grothendieck [1] Ch. 0 §19, we call a ring B over a ring A formally etale if and only if for any ring C over A and for any nilpotent ideal J of C, the induced map $\mathrm{Hom}_A(B, C) \to \mathrm{Hom}_A(B, C/J)$ is bijective. Hence, if B is as in Lemma 1 and $u: B/IB \cong S$ is an R-isomorphism, and if (B', u') is another pair satisfying the same condition as (B, u), there exists a unique A-isomorphism $v: B \cong B'$ such that $u = u' \circ (v \otimes A/I)$.

DEFINITION 1. Let A, I, R and S be as in Lemma 1. We call the above ring B endowed with a fixed R-isomorphism $B/IB \cong S$, *the canonical lifting of S over* A (with respect to the ideal I). When the assumption that I is nilpotent is replaced by the more general assumption $A = \varprojlim A/I^n$, we call $\varprojlim B_n$ the canonical lifting of S over A, where each B_n denotes the canonical lifting of S over A/I^n.

PROOF OF LEMMA 1. Assume $I^n = 0$. Let r be an integer such that $r \geqq n-1$, N a sufficiently large number, and let $\phi_r: W_N(R) \to A$ be the homomorphism $(x_0, x_1, \cdots) \mapsto \sum_{i=0}^{r} p^i \tilde{x}_i^{p^{r-i}}$, where $W_N(R)$ denotes the ring of p-Witt vectors over R of length N and $\tilde{x}_i$ denotes a representative in A of x_i for each i. (Note that $\tilde{x}_i^{p^{r-i}}$ is independent of the choice of the representative.) Define the ring B over A by $B = A \otimes_{W_N(R)} W_N(S)$, where we regard A as a ring over $W_N(R)$ via ϕ_r. To see that B has the properties mentioned in Lemma 1, it suffices to prove the following

LEMMA 2. *Let p be a prime number, R a ring of characteristic p (i.e. $p \cdot 1_R = 0$), and $h: R \to S$ a ring homomorphism satisfying the condition (C) in Lemma 1. Then, the induced homomorphism $W_N(R) \to W_N(S)$ is formally etale for any $N \geqq 0$. If S is assumed furthermore to be flat over R, $W_N(S)$ is flat over $W_N(R)$.*

PROOF. For any ring A of characteristic p, let $\mathfrak{F}: W_N(A) \to W_N(A)$ be the homomorphism $(x_0, \cdots, x_{N-1}) \mapsto (x_0^p, \cdots, x_{N-1}^p)$, and for $r \geqq 0$, let $W_N(A)^{(\mathfrak{F}^r)}$ be the ring over $W_N(A)$ with the underlying ring $W_N(A)$ and with the structural map $\mathfrak{F}^r = \mathfrak{F} \circ \cdots \circ \mathfrak{F}$ (r times). It follows easily from the condition (C) that the $W_N(S)$-homomorphism

(1) $$W_N(R)^{(\mathfrak{F}^r)} \otimes_{W_N(R)} W_N(S) \longrightarrow W_N(S)^{(\mathfrak{F}^r)}; \quad x \otimes y \longrightarrow W_N(h)(x) \cdot \mathfrak{F}^r(y)$$

is bijective for any $r \geqq 0$. Let C be a ring over $W_N(R)$ and J a nilpotent ideal of C containing $p \cdot 1_C$. For the proof of the first assertion in Lemma 2, it suffices to prove that any $W_N(R)$-homomorphism φ: $W_N(S) \to C/J$ can be uniquely lifted to a $W_N(R)$-homomorphism $\tilde{\varphi}$: $W_N(S) \to C$. Take a sufficiently large number r. By virtue of the bijectivity of the above homomorphism (1), $\tilde{\varphi}$ can be defined to be the $W_N(R)$-homomorphism such that

$$\tilde{\varphi}(\mathfrak{F}^r(s_0, \cdots, s_{N-1})) = \sum_{i=0}^{N-1} p^i \widetilde{\varphi(s_i)}^{p^{r-i}} \quad \text{for any } (s_0, \cdots, s_{N-1}) \in W_N(S),$$

where $\widetilde{\varphi(s_i)}$ denotes any representative of $\varphi(s_i)$ in C.

Next assume furthermore that S is flat over R. For the second assertion of Lemma 2, it suffices to prove that $\mathrm{Tor}_1^{W_N(R)}(M, W_N(S)) = 0$ for all $W_N(R)$-modules M. For each $i = 0, \cdots, N$, let $I_i = \mathrm{Ker}(W_N(R) \to W_i(R))$. Since I_1 is nilpotent, we may assume $I_1 M = 0$. Since $S = W_N(S)/I_1 W_N(S)$ is flat over $R = W_N(R)/I_1$, we are reduced to proving $\mathrm{Tor}_1^{W_N(R)}(R, W_N(S)) = 0$. For this, by a downward induction starting with $i = N$, we can prove $\mathrm{Tor}_1^{W_N(R)}(W_N(R)/I_i, W_N(S)) = 0$ for all i, as a consequence of the bijectivity of the homomorphisms

$$I_i/I_{i+1} \otimes_{W_N(R)} W_N(S) \longrightarrow I_i W_N(S)/I_{i+1} W_N(S).$$

§ 2. Adeles and ideles.

Let F be an algebraic function field in one variable over a finite field, and let K be a complete discrete valuation field with residue field F. If we denote by $A(F)$ the adele ring of F, the canonical homomorphism $F \to A(F)$ satisfies the condition (C) in Lemma 1. Hence we can define the canonical lifting of $A(F)$ over the valuation ring O_K of K, which we denote by $A(O_K)$. We define the adele ring $A(K)$ of K to be $A(O_K) \otimes_{O_K} K$. For $q \geqq 0$, we call $K_q(A(K))$ the K_q-idele group of K (here K_* denotes Quillen's K-functor), and call the cokernel $C_q(K)$ of $K_q(K) \to K_q(A(K))$ the K_q-idele class group of K. We shall be concerned only with the cases $q = 1, 2$.

If L is a finite extension of K, then $A(L) = A(K) \otimes_K L$. So, we have norm homomorphisms $K_q(L) \to K_q(K)$ and $K_q(A(L)) \to K_q(A(K))$, which induce a norm homomorphism $N_{L/K}$: $C_q(L) \to C_q(K)$.

It is possible to write the adele ring $A(K)$ in the form of a restricted product with respect to the places of F. For each place v of F, let F_v be the v-adic completion of F. Let O_{K_v} be the canonical lifting of F_v over O_K and let $K_v = O_{K_v} \otimes_{O_K} K$. Then, O_{K_v} is a complete discrete valuation ring with residue field F_v, and K_v is the field of fractions of O_{K_v}. The canonical projection $A(F)\to F_v$ induces a K-homomorphism $A(K)\to K_v$ by the formally etale property, and thus we obtain a homomorphism $A(K)\to \prod_{\text{all } v} K_v$. We determine the image of this homomorphism in Lemma 4. We need the following lemma.

LEMMA 3. *Let k be a perfect field of characteristic $p>0$, F a finitely generated field over k, and K a complete discrete valuation field with residue field F. Then, there exists a subring I of O_K and a prime element π of K contained in I, having the following properties; $I=\varprojlim I/\pi^n I$, the canonical homomorphism $I/\pi I\to F$ is injective and its image is a finitely generated smooth ring over k having F as its field of fractions, and O_K is the canonical lifting of F over I with respect to the ideal πI of I.*

The proof will be postponed for a while. Now let K and F be as before, let k be the finite constant field of F, and let $\mathfrak{X}_F$ be the smooth proper curve over k whose rational function field coincides with F. Then we can find a subring I and a prime element π of K in I satisfying the conditions in Lemma 3. The image of $I/\pi I$ in F is the affine ring $O(U)$ of some non-empty affine open subscheme U of $\mathfrak{X}_F$. For each closed point v of U, which we regard as a place of F in the usual way, let I_v be the canonical lifting of O_{F_v} over I. Then, I_v is a two dimensional complete regular local ring with finite residue field. Let $J_v=I_v[\pi^{-1}]$, and let $J=I[\pi^{-1}]$. Thus, corresponding to $O(U), F, O_{F_v}, F_v, A(F)$, we have $I, O_K, I_v, O_{K_v}, A(O_K)$ as their liftings, and $J, K, J_v, K_v, A(K)$ by adding π^{-1}, respectively (note that I_v and J_v are defined only for $v\in U$).

LEMMA 4. *The homomorphism $A(K)\to \prod_v K_v$ is injective, and its image consists of all elements $(a_v)_v$ such that $\mathrm{ord}_{K_v}(a_v)$ is bounded below when v varies, and such that for any $n\geqq 1$, $a_v \bmod m_{K_v}^n$ belongs to $(J_v+m_{K_v}^n)/m_{K_v}^n$ for almost all $v\in U$. (m_{K_v} is the maximal ideal of O_{K_v}.)*

This follows from the characterization of the canonical lifting in the flat case in Lemma 1.

Now we consider the idele groups. Recall that for any commutative ring R, the group $R^\times$ of all invertible elements in R is regarded canonically as a direct summand of $K_1(R)$, and there is a canonical homomorphism $R^\times\otimes R^\times\to K_2(R)$; $x\otimes y\mapsto \{x, y\}$.

LEMMA 5. (1) $K_1(A(K))=A(K)^\times$.

(2) *$K_2(A(K))$ is generated by symbols $\{x, y\}$ ($x, y\in A(K)^\times$) and all relations*

between symbols are generated by

$$\{f, 1-f\}=0 \text{ for all } f\in A(K) \text{ such that } f, 1-f\in A(K)^{\times}.$$

COROLLARY. *The homomorphism* $K_1(A(K))=A(K)^{\times}\to\prod_v K_v^{\times}$ *is injective, and the image consists of all elements* $(a_v)_v$ *such that the image of the map* $v\mapsto \mathrm{ord}_{K_v}(a_v)\in \boldsymbol{Z}$ *is finite, and such that for any* $n\geq 1$, $a_v \bmod (1+m_{K_v}^n)$ *belongs to the image of* $J_v^{\times}$ *in* $K_v^{\times}/(1+m_{K_v}^n)$ *for almost all* $v\in U$.

PROOF OF LEMMA 5. According to [2], for $k\geqq 0$, we say that a commutative ring R is k-fold stable if R satisfies the following property. For any k pairs (a_i, b_i) $(1\leqq i\leqq k)$ of elements of R such that $a_iR+b_iR=R$ for $1\leqq i\leqq k$, there is an element r of R such that $a_ir+b_i\in R^{\times}$ for all i. By using the definition $K_1(R)=\varinjlim_n GL_n(R)/E_n(R)$ where $E_n(R)$ denotes the subgroup of $GL_n(R)$ generated by elementary matrices, we see without difficulty that $K_1(R)=R^{\times}$ if R is 1-fold stable. On the other hand, if R is 5-fold stable, by Kallen [2], $K_2(R)$ is generated by symbols and all the relations of symbols are generated by the Matsumoto relation $\{f, 1-f\}=0$. As is easily seen, $A(K)$ is k-fold stable for any k.

Lastly we define a canonical topology on $K_q(A(K))$ for $q=1, 2$. Take a prime element π of K. Then, we have a direct decomposition

$$\prod_v{}' \boldsymbol{Z}\times A(O_K)^{\times}\longrightarrow A(K)^{\times};\quad ((n_v)_v,\ (a_v)_v)\longmapsto(\pi^{n_v}a_v)_v$$

where we regard $A(K)$ as a subring of $\prod_{\text{all } v} K_v$ and $\prod_v{}'\boldsymbol{Z}$ denotes the subgroup of the direct product $\prod_{\text{all } v}\boldsymbol{Z}$ consisting of all elements $(n_v)_v$ such that the image of the map $v\mapsto n_v\in\boldsymbol{Z}$ is finite. We endow $\prod_v{}'\boldsymbol{Z}$ with the topology induced by the product topology on $\prod_{\text{all } v}\boldsymbol{Z}$ (each $\boldsymbol{Z}$ is regarded to be discrete). On the other hand, we endow $A(O_K)^{\times}$ with the following topology.

For each $n\geqq 1$, let $A(O_K/m_K^n)$ be the canonical lifting of $A(F)$ over O_K/m_K^n. Then, $A(O_K)^{\times}=\varprojlim A(O_K/m_K^n)^{\times}$. We first define a locally compact topology of $A(O_K/m_K^n)^{\times}$ for each n, and then take the inverse limit of these topologies as the topology of $A(O_K)^{\times}$. Take $I\subset O_K$ and $U\subset\mathfrak{X}_F$ as before. Then, for any place v of F, $(O_{K_v}/m_{K_v}^n)^{\times}$ is regarded as a locally compact group by [3] §7. The group $A(O_K/m_K^n)^{\times}$ is isomorphic to the restricted product $\prod_{\text{all } v}{}'(O_{K_v}/m_{K_v}^n)^{\times}$ with respect to the open compact subgroups $(I_v/m_K^nI_v)^{\times}$ $(v\in U)$ of $(O_{K_v}/m_{K_v}^n)^{\times}$, and hence it is endowed naturally with the topology of the restricted product. It is not difficult to see that this topology is independent of the choices of I and U.

We endow $A(K)^{\times}$ with the product topology of the above topologies of $\prod_v{}'\boldsymbol{Z}$ and $A(O_K)^{\times}$. This topology is independent of the choice of the prime element π

of K.

We endow $K_2(A(K))$ with the finest topology which is compatible with the group structure and for which the map $A(K)^\times \times A(K)^\times \to K_2(A(K))$; $(x, y) \mapsto \{x, y\}$ is continuous.

Lastly we endow $C_1(K)$ and $C_2(K)$ with the topologies as quotients of $K_1(A(K))$ and $K_2(A(K))$ respectively.

Now we give the proof of Lemma 3.

If $\mathrm{ch}(K)=p$, we can suppose $K=F((T))$. Let R be any finitely generated smooth domain over k having F as its field of fractions. It suffices to define $I=R[[T]]$ and $\pi=T$. Now, suppose $\mathrm{ch}(K)=0$. First, consider the case that p is a prime element in K. Since k is perfect, there is a smooth integral domain R over k having F as its field of fractions, and an etale k-homomorphism $k[X_1, \cdots, X_m] \to R$ for some $m \geqq 0$. Let I be the canonical lifting of R over $A=\varprojlim (W_n(k)[X_1, \cdots, X_m])$ with respect to the ideal pA, and let $\pi=p$. Since O_K is the canonical lifting of F over A, I can be naturally regarded as a subring of O_K. Next, in general, there is a complete discrete valuation field K' having p as a prime element such that K is a totally ramified extension of K' of finite degree (cf. Nagata [5] 31.12). Hence, it suffices to apply Lemma 7 below.

Notation 1. In Definition 1, in the case $S=R[b^{-1}]$ for an element b of R, we denote by $A_{\langle b \rangle}$ the canonical lifting of S over A.

LEMMA 6. *Suppose that π is a non-zero-divisor of a ring I such that $I=\varprojlim I/\pi^n I$ and $R=I/\pi I$ is an integral domain of characteristic $p>0$. Let $D^\circ=\varinjlim_{\substack{b \in R \\ b \neq 0}} I_{\langle b \rangle}$, F the field of fractions of R, and D the canonical lifting of F over I with respect to the ideal πI. Then, D° is a Henselian discrete valuation ring and is dense in the complete discrete valuation ring D. Consequently, if K° and K denote the fields of fractions of D° and D, and K_s° and K_s denote their separable closures, respectively, the restriction $\mathrm{Gal}(K_s/K) \to \mathrm{Gal}(K_s^\circ/K^\circ)$ is bijective.*

The proof of the fact that D° is Henselian goes just as the well known proof of the fact that a complete local ring is Henselian.

LEMMA 7. *Let I, π, R and K be as in Lemma 6, and let L be a finite separable extension of K. Then, there are b, B and Π satisfying the following conditions.*

(1) *b is a non-zero element of R.*

(2) *B is a subring of O_L containing $I_{\langle b \rangle}$, and is free as an $I_{\langle b \rangle}$-module.*

(3) *The homomorphism* $B\otimes_{I_{\langle b\rangle}}O_K\to O_L$ *is bijective.*

(4) Π *is an element of* B *such that* $\Pi^{e_{L/K}}\in B^{\times}\pi$, *where* $e_{L/K}$ *denotes the index of the ramification of* L/K.

If L/K *is totally ramified, there are* b, B *and* Π *which satisfy the conditions* (1)$\sim$(4) *and the following condition; the induced homomorphism* $I_{\langle b\rangle}/\pi I_{\langle b\rangle}\to B/\Pi B$ *is bijective.*

PROOF. Let K° be as in Lemma 6. By Lemma 6, there are a finite separable extension L° of K° and a K-isomorphism $L^{\circ}\otimes_{K^{\circ}}K\cong L$. If one reforms the conditions in Lemma 7 by replacing K by K° and replacing L by L°, it is easy to define b, B and Π which satisfy the reformed conditions. But such b, B and Π satisfy the original conditions at the same time.

§3. The reciprocity maps.

Let K and F be as at the beginning of §2. In this section, we define canonical homomorphisms $C_2(K)\to \mathrm{Gal}(K^{\mathrm{ab}}/K)$ and $\mathrm{Br}(K)\to \mathrm{Hom}(C_1(K), \boldsymbol{Q}/\boldsymbol{Z})$.

For a field k and $q\geqq 0$, let $H^q(k)$ be the group defined in [4] §3.2. In particular, $H^1(k)$ is the group of all continuous characters of the compact abelian group $\mathrm{Gal}(k^{\mathrm{ab}}/k)$, and $H^2(k)=\mathrm{Br}(k)$.

For each place v of F, the field K_v defined in §2 is a 2 dimensional local field, that is, a complete discrete valuation field whose residue field is also a complete discrete valuation field with finite residue field. Hence by [4] §3, we have a canonical isomorphism $h_{K_v}\colon H^3(K_v)\cong \boldsymbol{Q}/\boldsymbol{Z}$, and a canonical pairing

$$H^q(K_v)\otimes K_{3-q}(K_v)\longrightarrow H^3(K_v)\cong \boldsymbol{Q}/\boldsymbol{Z};\quad x\otimes a\longmapsto h_{K_v}(\{x, a\})$$

for $q=1, 2$. In what follows, let $q=1$ or 2. We shall prove the following lemma.

LEMMA 8. (1) *For each* $x\in H^q(K)$ *and* $a\in K_{3-q}(A(K))$, $h_{K_v}(\{x_{K_v}, a_{K_v}\})=0$ *for almost all places* v *of* F. *(Here* x_{K_v} *(resp.* a_{K_v}*) denotes the image of* x *(resp.* a*) in* $H^q(K_v)$ *(resp.* $K_{3-q}(K_v)$*).)*

(2) *For* $x\in H^q(K)$ *and* $a\in K_{3-q}(K)$, $\sum_{\text{all } v} h_{K_v}(\{x_{K_v}, a_{K_v}\})=0$.

By this lemma, we obtain a canonical pairing

$$\langle\ ,\ \rangle_K\colon H^q(K)\otimes C_{3-q}(K)\longrightarrow \boldsymbol{Q}/\boldsymbol{Z};\quad (x, \bar{a})\longmapsto \sum_{\text{all } v} h_{K_v}(\{x_{K_v}, a_{K_v}\})$$

($a\in K_{3-q}(A(K))$ and $\bar{a}$ denotes its image in $C_{3-q}(K)$). This induces canonical homomorphisms $C_2(K)\to \mathrm{Gal}(K^{\mathrm{ab}}/K)$ and $\mathrm{Br}(K)\to \mathrm{Hom}(C_1(K), \boldsymbol{Q}/\boldsymbol{Z})$. The rest of this section is devoted to the proof of Lemma 8.

For any ring R, let $H^1(R)$ be the etale cohomology group $H^1(\mathrm{Spec}(R), \boldsymbol{Q}/\boldsymbol{Z})$ and let $H^2(R)$ be the Brauer group $\mathrm{Br}(R)$. Take $I\subset O_K$, $U\subset \mathfrak{X}_F$, π, $J=I[\pi^{-1}]$ as in §2, and let K° be the field given by Lemma 6. Then, $H^q(K^\circ)\cong H^q(K)$. Since H^q commutes with filtered direct limites, by replacing U by its non-empty open subset using the localization in the sense of §2 Def. 2, we see that each element of $H^q(K)$ comes from $H^q(J)$ for some choice of I, π and U. The first key point is the following

LEMMA 9. *If x comes from $H^q(J)$, $h_{K_v}(\{x_{K_v}, ?\})$: $K_{3-q}(K_v)\to \boldsymbol{Q}/\boldsymbol{Z}$ annihilates the image of $K_{3-q}(J_v)$ for any $v\in U$.*

This is proved by using the following reciprocity law of the 2 dimensional complete regular local ring I_v with finite residue field. For each prime ideal $\mathfrak{p}$ of height one of I_v, the $\mathfrak{p}$-adic completion $M_\mathfrak{p}$ of the field of fractions M of I_v is a 2 dimensional local field. Hence we have a canonical isomorphism $h_{M_\mathfrak{p}}$: $H^3(M_\mathfrak{p})\cong \boldsymbol{Q}/\boldsymbol{Z}$. The reciprocity law states that

$$\sum_{\mathfrak{p}} h_{M_\mathfrak{p}}(x_{M_\mathfrak{p}})=0 \quad \text{for any } x\in H^3(M)$$

where $\mathfrak{p}$ ranges over all prime ideals of height one of I_v. For the proof of this reciprocity law, cf. Saito [7] Ch. I. Let $\mathfrak{p}_0$ be the prime ideal πI_v. Then, $K_v=M_{\mathfrak{p}_0}$. For any $\mathfrak{p}\neq\mathfrak{p}_0$, J_v is contained in the valuation ring $O_{M_\mathfrak{p}}$ of $M_\mathfrak{p}$. By the class field theory of $M_\mathfrak{p}$, we see that the pairing $H^q(O_{M_\mathfrak{p}})\otimes K_{3-q}(O_{M_\mathfrak{p}})\to H^3(M_\mathfrak{p})$ is the zero map. Hence the reciprocity law shows that for any $x\in H^q(J_v)$ and $a\in K_{3-q}(J_v)$,

$$h_{K_v}(\{x_{K_v}, a_{K_v}\})=-\sum_{\mathfrak{p}\neq\mathfrak{p}_0} h_{M_\mathfrak{p}}(\{x_{M_\mathfrak{p}}, a_{M_\mathfrak{p}}\})=0.$$

The next key point is the following Lemma 10. For a discrete valuation field k and for $n\geqq 1$, let $U^nK_1(k)$ be the subgroup $1+m_k^n$ of $k^\times$ and let $U^nK_2(k)$ be the image of $U^nK_1(k)\otimes K_1(k)\to K_2(k)$.

LEMMA 10. *For each $x\in H^q(K)$, there is an integer $n\geqq 1$, such that x_{K_v} annihilates $U^nK_{3-q}(K_v)$ for all places v of F.*

PROOF. There is a finite separable extension L of K such that the image x_L of x in $H^q(L)$ is zero. An easy observation of the norm homomorphism $N_{L/K}$: $L^\times\to K^\times$ shows that there is an integer $n\geqq 1$ such that $U^nK_1(K_v)\subset N_{L_v/K_v}(K_1(L_v))$ for all places v where $L_v=K_v\otimes_K L$, and hence $U^nK_2(K_v)\subset N_{L_v/K_v}(K_2(L_v))$ for all v. But for any $a\in K_{3-q}(L_v)$,

$$\{x, N_{L_v/K_v}(a)\}=\mathrm{Cor}_{L_v/K_v}(\{x_{L_v}, a\})=0 \quad \text{in} \quad H^3(K_v)$$

where Cor_{L_v/K_v} is the corestriction map $H^3(L_v)\to H^3(K_v)$.

Now Lemma 8 (1) follows from the above Lemma 9 and Lemma 10 combined with the description of $K_1(A(K))$ in Cor. to Lemma 5 and the surjectivity of $K_1(A(K))\otimes K_1(A(K))\to K_2(A(K))$.

PROOF OF LEMMA 8 (2). By [3] §5 and [4] §3.2 Lemma 3, we have a canonical isomorphism $H^3(K)\cong H^2(F)$. For each $x\in H^q(K)$ and $a\in K_{3-q}(K)$, $h_{K_v}(\{x_{K_v}, a_{K_v}\})$ is the image of $\{x, a\}\in H^3(K)$ under the homomorphism $H^3(K)\cong H^2(F)\to H^2(F_v)\cong \boldsymbol{Q}/\boldsymbol{Z}$. Hence Lemma 8 (2) follows from the usual Hasse's reciprocity law which states that the composite

$$H^2(F)\longrightarrow \bigoplus_{\text{all } v} H^2(F_v)\cong \bigoplus_{\text{all } v} \boldsymbol{Q}/\boldsymbol{Z}\xrightarrow{\text{sum}}\boldsymbol{Q}/\boldsymbol{Z}$$

is the zero map.

§4. Proofs of Th. 1 and Th. 2.

Let $q=1$ or 2. Our task is to prove the following (A) - (D).

(A) For any $x\in H^q(K)$, the homomorphism $\langle x, \rangle_K$: $C_{3-q}(K)\to\boldsymbol{Q}/\boldsymbol{Z}$ is continuous.

(B) The homomorphism $H^q(K)\to\mathrm{Hom}(C_{3-q}(K), \boldsymbol{Q}/\boldsymbol{Z})$ is injective.

(C) Any continuous homomorphism $C_{3-q}(K)\to\boldsymbol{Q}/\boldsymbol{Z}$ is of finite order, and comes from $H^q(K)$.

(D) For any finite abelian extension L of K, $C_2(K)\to\mathrm{Gal}(L/K)$ annihilates $N_{L/K}C_2(L)$ and induces $C_2(K)/N_{L/K}C_2(L)\cong\mathrm{Gal}(L/K)$.

Since the proofs of these facts are the simple modifications of the proofs of the corresponding facts in the class field theory of K_v in [3] §6, §8 and [4] §3.5, we shall often omit the details of the proofs.

First, (A) follows from the definition of the topology of $C_{3-q}(K)$ and the following facts; (1) For each place v of F, the homomorphism $h_{K_v}(\{x_{K_v}, ?\})$: $K_{3-q}(K_v)\to \boldsymbol{Q}/\boldsymbol{Z}$ is continuous ([3] §8, [4] §3.5), (2) For almost all $v\in U$, $h_{K_v}(\{x_{K_v}, ?\})$ annihilates the image of $K_{3-q}(J_v)\to K_{3-q}(K_v)$ by Lemma 9 (U and J_v are as in §2), (3) For some $n\geqq 1$, $h_{K_v}(\{x_{K_v}, ?\})$ annihilates $U^nK_{3-q}(K_v)$ for all v by Lemma 10.

To prove (B), it is sufficient to show that for any prime number l, the induced homomorphism $H^q(K)_l=\mathrm{Ker}(l\colon H^q(K)\to H^q(K))\to\mathrm{Hom}(C_{3-q}(K), \boldsymbol{Z}/l)$ is injective. The proof of this injectivity and that of the fact

(C′) Any continuous homomorphism $C_{3-q}(K)\to\boldsymbol{Z}/l$ comes from $H^q(K)_l$, are almost same as the proof of $H^q(K_v)_l\cong\mathrm{Hom}_{\mathrm{cont}}(K_{3-q}(K_v), \boldsymbol{Z}/l)$ given in [3] §6, §8 and [4] §3.5. Here instead of the results about the structure of $K_{3-q}(K_v)/lK_{3-q}(K_v)$

studied in [3] and [4], we use the similar results about the structure of $K_{3-q}(A(K))/lK_{3-q}(A(K))$. For each $n\geqq 1$, let $U^nK_1(A(K))$ be the subgroup $1+m_K^n A(O_K)$ of $K_1(A(K))=A(K)^\times$, and let $U^nK_2(A(K))$ be the image of $U^nK_1(A(K))\otimes K_1(A(K))\to K_2(A(K))$.

LEMMA 11. *Let p be the characteristic of F.*

(1) *$U^1K_q(A(K))$ is l-divisible for any prime number $l\neq p$.*

(2) $K_1(A(K))/U^1K_1(A(K))\cong \prod_v{}' \boldsymbol{Z}\oplus K_1(A(F))$,

$$K_2(A(K))/U^1K_2(A(K))\cong K_1(A(F))\oplus K_2(A(F)).$$

(3) $K_1(A(F))=A(F)^\times$. Let S be the subgroup of $K_2(A(F))$ generated by all elements $\{x, y\}$ such that $x, y\in \prod_{\text{all } v} O_{F_v}^\times$. Then, S is a divisible group and $K_2(A(F))/S\cong \bigoplus_{\text{all } v} K_1(\kappa(v))$ by the boundary map of the K-theory, where $\kappa(v)$ denotes the residue field of v. Moore's exact sequence gives an exact sequence $K_2(F)\to K_2(A(F))/S\to k^\times\to 0$ where k is the constant field of F.

(4) Let $q=1$ (resp. $q=2$). For $n\geqq 1$,

$$(U^nK_q(A(K))+pK_q(A(K)))/(U^{n+1}K_q(A(K))+pK_q(A(K)))$$

is isomorphic to $A(F)/\{x^p; x\in A(F)\}$ if $0<n<e_Kp/(p-1)$ and $p|n$, $A(F)$ (resp. $\Omega^1_{A(F)/Z}$) if $0<n<e_Kp/(p-1)$ and n is prime to p, $A(F)/\{x^p-ax; x\in A(F)\}$ (resp. $A(F)/\{x^p-ax; x\in A(F)\}\oplus\Omega^1_{A(F)/Z}/\{w-a\gamma(w); w\in\Omega^1_{A(F)/Z}\}$) if $n=e_Kp/(p-1)$ and this is an integer, where a is a certain element of F defined in [3] §2 Prop. 1 and γ denotes the Cartier operator, and 0 if $n>e_Kp/(p-1)$.

Note that $\Omega^1_{A(F)/Z}$ is isomorphic to the usual restricted product $\prod_v{}' \Omega^1_{F_v/Z}$ of the one dimensional F_v-spaces $\Omega^1_{F_v/Z}$.

The proof of the K_2 part of (2) uses the explicit presentations of $K_2(A(K))$ and $K_2(A(F))$ by symbols and Matsumoto relations (cf. the proof of Lemma 5; $A(F)$ is also k-fold stable for any k). The K_2 part of (4) is proved by using the method and the result in the computation of $(U^nK_2(K_v)+pK_2(K_v))/(U^{n+1}K_2(K_v)+pK_2(K_v))$ in [3] §2 Prop. 1.

By using this lemma and the theory of the autoduality of the adele ring $A(F)$ (which states for example, that the dual of $\Omega^1_{A(F)/Z}/\Omega^1_{F/Z}$ is F) instead of the theory of the autoduality of a locally compact field, we can prove the claims (B) and (C′) just as in [3] §6, §8 and [4] §3.5.

The finiteness of the order of a continuous character is proved by the same method as in the case of $K_{3-q}(K_v)$ by using the above lemma.

Now we show by induction on the order of a continuous character $\varphi\colon C_{3-q}(K)\to$

$\boldsymbol{Q}/\boldsymbol{Z}$, that φ comes from $H^q(K)$. Let l be a prime divisor of the order of φ. By induction, there is an element x of $H^q(K)$ such that $l\varphi=\langle x, \rangle_K$. If $l=\mathrm{ch}(K)$, $H^q(K)$ is l-divisible and hence $x=lx'$ for some $x'\in H^q(K)$. Then $\varphi-\langle x', \rangle_K$ is annihilated by l and we are reduced to the claim (C′). If $l\neq\mathrm{ch}(K)$, let $\delta\colon H^q(K)(l)\to H^{q+1}(K, \mu_l^{\otimes(q-1)})$ be the connecting homomorphism defined by the exact sequence

$$0\longrightarrow \mu_l^{\otimes(q-1)}\longrightarrow \varinjlim_n \mu_{l^n}^{\otimes(q-1)}\xrightarrow{\ l\ }\varinjlim_n \mu_{l^n}^{\otimes(q-1)}\longrightarrow 0.$$

($H^q(K)(l)$ denotes the l-primary part). By the class field theory of K_v, the restriction of φ to the component $K_2(K_v)$ of $K_2(A(K))$ is induced by an element y of $H^q(K_v)$. Hence $x_{K_v}=ly$ and this shows $\delta(x)_{K_v}=0$ in $H^{q+1}(K_v, \mu_l^{\otimes(q-1)})$ for all places v. Since $H^{q+1}(K, \mu_l^{\otimes(q-1)})\to \prod_{\text{all } v} H^{q+1}(K_v, \mu_l^{\otimes(q-1)})$ is injective (this fact is reduced to the case where K contains a primitive l-th root of 1, and then reduced to the injectivity of $H^{q+1}(K)\to\prod_{\text{all } v} H^{q+1}(K_v)$, which is contained in (B) in the case $q=1$ and follows from the injectivity of $H^2(F)\to\prod_{\text{all } v} H^2(F_v)$ in the case $q=2$), we have $\delta(x)=0$. Hence $x=lx'$ for some $x'\in H^q(K)$, and we can proceed as in the case $l=\mathrm{ch}(K)$ above.

Lastly, the first assertion of (D) is reduced to its local version in the local class field theory of K_v. For the rest, it is sufficient to show

(D′) If L is a cyclic extension of K of degree a prime l, the order of $C_2(K)/N_{L/K}C_2(L)$ is $\leqq l$.

We present here only the proof of the case in which $l=p$ $(=\mathrm{ch}(F))$ and L/K is totally ramified. Let t be the ramification number of L/K. By using the computation of $N_{L/K}\colon L^\times\to K^\times$ in Serre [6] Ch. V §3 and by the above Lemma 11 (4), we have

(1) $K_2(A(K))\subset U^1K_2(A(K))+N_{A(L)/A(K)}K_2(A(L))$,

(2) $U^nK_2(A(K))\subset U^{n+1}K_2(A(K))+N_{A(L)/A(K)}K_2(A(L))$ if $1\leqq n<t$,

(3) $U^{t+1}K_2(A(K))\subset N_{A(L)/A(K)}K_2(A(L))$,

(4) If π_L is a prime element of L and g is an element of $O_K^\times$ whose residue class $\bar g$ is not in F^p, there is a commutative diagram

$$\begin{array}{ccc}
A(F) & \xrightarrow{(1-\gamma^{-1})(?\cdot d\bar g/\bar g)} & \Omega^1_{A(F)/\mathbf{Z}}/dA(F) \\
\wr\!\!\!\!\parallel\downarrow & & \downarrow \\
U^tK_1(A(L))/U^{t+1} & \xrightarrow{N_{A(L)/A(K)}(\{?, g\})} & U^tK_2(A(K))/(U^{t+1}K_2(A(K))+\{U^tK_1(A(K)), N_{L/K}(\pi_L)\})
\end{array}$$

in which the right vertical arrow is surjective. The similar commutative diagram exists when we replace $A(K)$, $A(L)$ and $A(F)$ by K, L and F respectively, and is compatible with the above diagram. Hence (D′) follows from the commutative diagram

$$\begin{tikzcd}
\Omega^1_{F/Z} \arrow[r, "1-\gamma^{-1}"] \arrow[d] & \Omega^1_{F/Z}/dF \arrow[r] \arrow[d] & \mathrm{Br}(F)_p \arrow[r] \arrow[d] & 0 \\
\Omega^1_{A(F)/Z} \arrow[r, "1-\gamma^{-1}"] & \Omega^1_{A(F)/Z}/dA(F) \arrow[r] & \bigoplus_v \mathrm{Br}(F_v)_p \arrow[r] \arrow[d] & 0 \\
 & & \frac{1}{p}\boldsymbol{Z}/\boldsymbol{Z} \arrow[d] & \\
 & & 0 &
\end{tikzcd}$$

in which all sequences are exact.

§5. The K_2-idele group of the restricted product type.

The following slight modification of the K_2-idele group also describes the class field theory of K. Take (I, π, U) as in §2. Let $\prod'_v K_2(K_v)$ be the subgroup of the product $\prod_{\text{all } v} K_2(K_v)$ consisting of all elements $(a_v)_v$ such that for any $n\geqq 1$, a_v is contained in the sum of $U^n K_2(K_v)$ and the image of $K_2(J_v)$ for almost all $v \in U$. Then, $\prod'_v K_2(K_v)$ is in fact independent of the choice of (I, π, U). There is a canonical homomorphism $K_2(A(K)) \to \prod'_v K_2(K_v)$. Let $C'_2(K)$ be the cokernel of $K_2(K) \to \prod'_v K_2(K_v)$. Then, the reciprocity homomorphism factors as $C_2(K) \to C'_2(K) \to \mathrm{Gal}(K^{\mathrm{ab}}/K)$, where the second homomorphism is defined by the same method in §3. By using Lemma 7, it is shown that the norm homomorphisms of K_2 induce a norm homomorphism $N_{L/K}\colon C'_2(L) \to C'_2(K)$ for a finite extension L of K. We define the topology of $C'_2(K)$ as follows. For $n \geqq 1$, let $V^n = \prod'_v K_2(K_v) \cap \prod_v U^n K_2(K_v) \subset \prod_v K_2(K_v)$. We define the topology of $(\prod'_v K_2(K_v))/V^n$ as follows. For each v, we regard $K_2(K_v)$ as a topological group by the method in [3] §7. For each v in U, let S_v be the image of $K_2(J_v)$ in $K_2(K_v)$. We endow S_v with the topology induced by the topology of $K_2(K_v)$. We endow $(\prod'_v K_2(K_v))/V^n$ with the finest topology which is compatible with the group structure and for which the canonical homomorphisms

$$\prod_{v\in U'} S_v \times \prod_{v\notin U'} K_2(K_v) \longrightarrow \left(\prod_v{}' K_2(K_v)\right)/V^n$$

are continuous for all non-empty open subsets U' of U. We endow $\prod'_v K_2(K_v)$ with the topology induced by the topologies of $(\prod'_v K_2(K_v))/V^n$ ($n \geqq 1$), and $C'_2(K)$ with the quotient topology.

The homomorphism $C_2(K) \to C_2'(K)$ is continuous and the image is dense.

The following result is proved in the same way as Th. 1, and in fact it is deduced from Th. 1 without difficulty.

THEOREM 1′. *All the assertions in Th.* 1 *are still valid when we replace* $C_2(K)$ *by* $C_2'(K)$.

This is the form of the class field theory of K stated in the summary [8].

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