SOME PROBLEMS OF STOCHASTIC CALCULUS RELATED TO MATHEMATICAL FINANCE

(和訳:数理ファイナンスに関係する確率論の諸問題)

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論文の内容の要旨

論文題目　SOME PROBLEMS OF STOCHASTIC CALCULUS RELATED TO MATHEMATICAL FINANCE
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本論文においては　(i) Ritchken and Sankarasubramanian [17] [18] 型の金利期間構造のモデル　と、(ii) α-パーセンタイル·オプションの価格　の数学的側面が議論される。(i) に関して、より一般的なモデルを与え、その数学的構造を明らかにし、(Proposition 1.4) さらに、関係する確率積分方程式の解の爆発条件があたえられる (Theorem 3.1, 4.2)。これはモデルの妥当性に深く関係する (Remark 1.5,1.6)。(ii) に関しては、ドリフトのついたブラウン運動にかんする逆正弦法則をもとめている (Theorem 5.1)。それを用いて、α-パーセンタイル·オプションの価格が決定される (Theorem 6.1)。

以下では $(\Omega, \mathcal{F}, P, \{\mathcal{F}_t\}_{t\in[0,\infty)})$ を適当な確率空間の４つ組みとし、同値マルチンゲール測度とよばれる P^* の存在を仮定し、（ある種の経済学的均衡の概念と同値であることが知られている c.f. Harrison and Pliska [8] , Duffie [6]) W を P^*−ブラウン運動とする。

$r_t, t \in [0,\infty)$ （連続正値 $\{\mathcal{F}_t\}_{t\in[0,\infty)}$− 適合過程）を瞬間的短期金利としたとき、満期が Tである債権の時刻 t での価格 $p(t,T)$ は、

Propsition 1.1.

$$p(t,T) = E^*\left(e^{-\int_t^T r_s\, ds}\,|\mathcal{F}_t\right). \tag{0.1}$$

ここで、$E^*(\cdot)$　は P^*に対する期待値。

で与えられるが、このとき、右辺の条件つき期待値が r_t （と幾つかの状態変数）の簡単な関数でかかれていることが望ましい。これは r_t をマルコフ過程とすることが1つの解決法であるが、本論文では単に連続マルチンゲールを基礎に次のようなモデルを考えた。

$$\hat{X}_t \overset{def}{=} \int_0^t \xi_s [M]_s ds + M_t, \tag{0.2}$$

$$\bar{X}_t \overset{def}{=} \xi_t \hat{X}_t, \tag{0.3}$$

$$r_t \equiv \xi_t(\eta_t + \hat{X}_t) = \xi_t \eta_t + \bar{X}_t. \tag{0.4}$$

すると

Proposition 1.4.

$$p(t,T) = \exp\left\{ -\left(\int_s^u \xi_u du\right) \hat{X}_t - \frac{1}{2}\left(\int_s^u \xi_u du\right)^2 [M]_t \right\} \cdot e^{-\int_t^T \xi_s \eta_s ds}. \tag{0.5}$$

を得る。
（ただしここでξ, ν は deterministic な過程である。）

(0.2) の式を方程式で与えようとすれば、例えば

$$X_t = \eta(t) + \int_0^t \xi(s) \left(\int_0^s \sigma(u, X_u)^2 du\right) ds + \int_0^t \sigma(s, X_s) dW_s \tag{0.6}$$

となるが、$\sigma(x)$ の x 無限大でのオーダーが、$\sqrt{x}$ より大きいと、(0.6) のドリフト項が linear growth condition を破ることになる。しかし、数理ファイナンスにおいてはそういうものがよく扱われるので、詳しく調べる必要がでてくる。結果は、

Theorem 3.1. $\sigma(x) \equiv \sigma x^\gamma$, $\xi_. \equiv \xi, \eta_t \equiv x + \eta t$ とする。$\gamma > \dfrac{1}{2}$ のとき、*(0.6)* の解は確率1で爆発する。

Theorem 4.2. $\sigma(x) \equiv (xL(x))^{\frac{1}{2}}$ とする。ここで $L(x)$ は *slowly varying function* である。

(i) $\displaystyle\int_1^{+\infty} \frac{dx}{x\sqrt{L(x)}} = +\infty$ 　ならば、 $P(\mathfrak{e} < \infty) = 0$。

(ii) $\displaystyle\int_1^{+\infty} \frac{dx}{x\sqrt{L(x)}} < +\infty$ ならば、$P(\mathfrak{e} < \infty) = 1$.

ただし、ここで $\mathfrak{e}$ は、Xの無限大への到達時刻、すなわち爆発時刻である。

Theorem 3.1 の証明は、

$$S(x,y) = \frac{1}{1-\gamma}x^{1-\gamma} + y^{\frac{1}{\delta}} \quad x > 0,\ y > 0. \tag{0.7}$$

というスケール変換を用いて1次元マルコフ型の評価に持ち込むことによってなされる。

Theorem 4.2 の証明は、
$\sigma(x)^2 \equiv a(x)$ とし、$C_t \equiv \int_0^t a(X_s)\,ds$ の逆関数を A_t、すなわち、

$$A_t = \inf\{s, C_s > t\}. \tag{0.8}$$

として、A_t による時間変更を Xに対して施した確率過程 Y_tを考えることによってなされる。

すなわち、$Y_t \overset{def}{=} X_{A_t}$ である。このとき、

$$Y_t = x + \int_0^{A_t} C_s ds + \int_0^{A_t} (a(X_s))^{\frac{1}{2}}\, dW_s$$

$$= x + \int_0^t s dA_s + \int_0^{A_t} (a(X_s))^{\frac{1}{2}}\, dW_s. \tag{0.9}$$

であり、(0.9) の右辺第2項は$\mathcal{F}_{A_t}$ に適合したブラウン運動であることにより、結局、Y は、

$$Y_t = x + \int_0^t \frac{s}{a(Y_s)}\, ds + B_t, \quad x > 0 \tag{0.10}$$

の解となる。(ここで Bは新しいブラウン運動である。)

この方程式を変形して、

$$Z_t = x + \int_0^t \frac{s}{a(Z_s + B_s)}\, ds \tag{0.11}$$

とし、Bをノイズとみなして、常微分方程式だと思うことにする。以下、そもそも slowly varying function は、無限大の近傍でのみ決まるのであるから、law of iterated logarithm とあわせて、比較定理を繰り返し用いることにより、Theorem を得る。

本論文の後半はα−パーセンタイル·オプションの公正な価格についてである。そのなかで最も基本的な定理は、

Theorem 5.1.

$\mu > 0$ に対して、

$$P_0^*\left(\int_0^t 1_{\{W_s+\mu t>0\}} ds < y\right) = \frac{1}{2}\int_0^{ty} \left(\sqrt{\frac{2}{\pi s}}\exp\left(-\frac{\mu^2}{2}s\right) - 2\mu\Phi(\mu\sqrt{s})\right)$$
$$\times \left\{\left(2\mu + \sqrt{\frac{2}{\pi(t-s)}}\exp\left(-\frac{\mu^2}{2}(t-s)\right)\right) - 2\mu\Phi(\mu\sqrt{t-s})\right\} ds,$$

ここで、Φ は正規分布の末尾の分布関数、すなわち

$$\Phi(x) = \int_x^\infty \frac{1}{\sqrt{2\pi}}\exp\left(-\frac{y^2}{2}\right) dy.$$

である。

証明は Feynman-Kac の公式を用いてなされる。また、これを直接α−パーセンタイルの分布に書き換えると、

Theorem 8.1 (Dassios [4]).

$$M(\alpha,t) \stackrel{d}{=} \pm L_\pm(\alpha t) \mp L_\mp((1-\alpha)t).$$

という表現を得る。これについて若干の考察が本論文セクション7と8で与えられている。

SOME PROBREMS OF STOCHASTIC CLACULUS RELATED TO MATHEMATICAL FINANCE

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0. INTRODUCTION

In this paper we will study some probrems related to mathematical finance. Section 1-4 are taken from [1]. In section 1, a new interest rate models given. This is in fact an extention of the one given by Ritchken and Sankarasubramanian [17] [18]. In section 2,3,4, we will deal with a case of a stochastic integral equation. In section 2 pathwise uniqueness of the solution will be discussed. We should remark that in [17] and [18], the global existence of the solution is not clear. In section 3 and 4, some explosion tests will be given.

Section 5 and 6 are taken from [2]. In section 5, tha arc-sine law for a Brownian motion with drift will be given. With this, in section 6, we will calculate the fair price for a new type of a path-dependent option. The idea of this option is originated from Miura [15].

Section 7 and 8 are taken from [3]. Dassios [5] derived an interesting relationship between law of Brownian quantile and its running mixima. This was done through the Feynman-Kac method. There are atempts to avoid Feynman-Kac. Embrechet, Rogers and Yor [19] is among these. In section 7 we will give a modification of Williams' formula, and with this, we will have Dassios' representation in section 8.

1. A NEW INTEREST RATE MODEL

Let us consider the financial bond market model. Let $(\Omega, \mathcal{F}, P, \{\mathcal{F}_t\}_{t\in[0,\infty)})$ be an appripriate filtered probability space.

Set the model as follows:

(i) $p(\cdot, T)$ (prices of zero coupon bond with maturity T) is an $\{\mathcal{F}_t\}$- adapted continuous semimartingale up to time $T \in (0, \infty)$ and assume some differentiability in Tso that the folloing would be valid.

(ii) $f(\cdot, T)$ (instantanious forward rates) is given by

$$f(t,T) \stackrel{def}{=} -\frac{\partial}{\partial T} \log p(t,T), \ \ t \leq T.$$

(iii) (the short rate process) $r_t \stackrel{def}{=} f(t,t)$.

In the context of mathematical finance, once we are given 'the equinalent martingale measure' P^* and the short rate process, we can compute the bond price at time t by the following formula. (See e.g. Duffie [6])

Propsition 1.1.

$$p(t,T) = E^* \left(e^{-\int_t^T r_s \, ds} | \mathcal{F}_t \right). \tag{1.1}$$

where $E^*(\cdot)$ denotes an expectation with respect to P^*.

Here we give a rather general model for the bond market.

Let M. be a continuous local martingale and $\xi. \in L^1_{loc}[0, \infty)$.

Set

$$\hat{X}_t \stackrel{def}{=} \int_0^t \xi_s [M]_s ds + M_t.* \tag{1.2}$$

and

$$\bar{X}_t \stackrel{def}{=} \xi_t \hat{X}_t. \tag{1.3}$$

Lemma 1.2. *Then we have*

$$-\int_0^u \bar{X}_t dt = N^u_u - \frac{1}{2}[\ N^u]_u \tag{1.4}$$

where

$$N^u_t \stackrel{def}{=} -\int_0^t \left(\int_s^u \xi_u du \right) dM_s. \tag{1.5}$$

(Proof).

*For a martingale $N.$, $[N].$ denotes its quadraic variation process.

$$\begin{aligned}\int_0^u \bar{X}_t dt &= \int_0^u \xi_t M_t dt + \int_0^u \xi_t \left(\int_0^t \xi_s [M]_s ds\right) dt \\ &= \int \xi_t \left(\int_0^t dM_s\right) dt + \int_0^u \xi_t \left(\int_0^t \xi_s \left(\int_0^s d[M]_v\right) ds\right) dt \\ &= \int_0^u \left(\int_t^u \xi_s ds\right) dM_t + \int_0^u \xi_t dt \int_0^t \left(\int_v^u \xi_s ds\right) d[M]_v \\ &= \int_0^u \left(\int_t^u \xi_s ds\right) dM_t + \int_0^u d[M]_t \int_t^u \xi_t \left(\int_v^t \xi_s ds\right) dt \\ &= \int_0^u \left(\int_t^u \xi_s ds\right) dM_t + \frac{1}{2}\int_0^u \left(\int_t^u \xi_s ds\right)^2 d[M]_t. \end{aligned} \tag{1.6}$$

□

We set the model by giving the equivalent martingale measure implicitly and the fluctuation of the short rate process r_t as follows:

$$r_t \equiv \xi_t(\eta_t + \hat{X}_t) = \xi_t \eta_t + \bar{X}_t. \tag{1.7}$$

Assumption 1.3. *For all $u > 0$*

$$E^* \left[\exp \frac{1}{2}[N^u]_t\right] < \infty \tag{1.8}$$

holds for all $0 \le t \le u$.

Proposition 1.4. *Under the the assumption 3.3, the bond prices are given by the following.*

$$p(t,T) = \exp\left\{-\left(\int_s^u \xi_u du\right)\hat{X}_t - \frac{1}{2}\left(\int_s^u \xi_u du\right)^2 [M]_t\right\} \cdot e^{-\int_t^T \xi_s \eta_s ds}. \tag{1.9}$$

(Proof). By propsiton 1.1, it suffices to calculate

$$E^*\left(e^{-\int_t^T r_s\, ds} | \mathcal{F}_t\right) = e^{\int_0^t r_s ds} E^*\left(e^{-\int_0^T r_s\, ds} | \mathcal{F}_t\right). \tag{1.10}$$

By (1.4) and (1.5),

$$e^{-\int_0^u r_s ds} = \mathcal{E}(N^u)_u e^{-\int_0^u \xi_s \eta_s ds}. \tag{1.11}$$

where $\mathcal{E}(\cdot)$ denotes its exponential semimartingale. By (1.10) and (1.11),

$$p(t,T) = \frac{\mathcal{E}(N^T)_t}{\mathcal{E}(N^t)_t} e^{-\int_t^T \xi_s \eta_s ds}. \tag{1.12}$$

Easy calculation leads to (1.9). □

REMARK.1.5. Ritchken and Sankarasubramanian [17] [18] 's interest rate model has also the form of (1.9). But theirs are ristricted the case when the short rate is a solution of a 2 dimentional (Markovian) stochastic differential equation which is in fact a 1-dimentional stochastic integal equation. As we shall see the sections to come, this S.I.E. might explode in many cases including the ones given in [17], [18].

REMARK.1.6. This model has some superiority to others. That is, (i) the bond prices are the function of M_t and $[M]_t$. (ii) the short rate process never go negative. (iii) it can be 'n-state Markov' if we wish it to. (iv) and it has the memory of the past. Takahashi [22] reported that it would be better to use S.I.E. in stead of simple 1-dim S.D.E. if we wish to model the actual data. (In [22] , the price processes are modelled by those slightly different from ours. See Kannan-Bharucha-Reid [14].)

2. A Stochastic Integral Equation

In this section we will study rather special case of (1.7) (or (1.2)).

Let $(\Omega, \mathcal{F})$ be an appropriate measurable space and let us consider the following stochastic integral equation.

$$X_t = \eta(t) + \int_0^t \xi(s) \left(\int_0^s \mu(u, X_u) du \right) ds + \int_0^t \sigma(s, X_s) dW_s \tag{2.1}$$

where W. denotes a one dimentional Brownian motion on a filtered probability space $(\Omega, \mathcal{F}, P, \{\mathcal{F}_t\}_{t\in[0,\infty)})$, P is a probability measure and $\{\mathcal{F}_t\}_{t\in[0,\infty)}$ is an augmented filteration,

$$\eta(\cdot), \xi(\cdot) : [0,\infty] \to \mathbb{R}^+, continuous \tag{2.2}$$

and

$\sigma, \mu : [0, \infty) \times \mathbb{R}^+ \to \mathbb{R}$, *jointly measurable and satisfies the following conditions.*

(a) For each $t > 0$,

$$|\sigma(s,x) - \sigma(s,y)|^2 \leq \rho(|x-y|) \tag{2.3}$$

holds for every $s \leq t$. Here ρ stands for a Borel function from $]0,\infty[$ to $]0,\infty[$ such that

$$\int_{0+} \frac{da}{\rho(a)} = +\infty.$$

(b) (Local Lipschitz Condition) For each $t > 0$ and integer n, there exists a constant $L_t^n > 0$ such that

$$|(\mu(s,x) - \mu(s,y)| \leq L_t^n |x-y| \tag{2.4}$$

holds for every $s \leq t$ and $|x| \leq n, |y| \leq n$.

REMARK 2.1. It should be noted that (2.1) includes the short rate of the R-S model. Of course we can say that it is 2-state Markovian. But it is driven by a Brownian motion, so it would be better to think it as an integral equation.

We define solutions for (2.1) as usual in the weak sense but up to an explosion time. The exisistence of weak solutions up to an explosion time follows from Skorohod [21]'s result by slight modifications. (See Ikeda and Watanabe [9].) By Yamada-Watanabe [23]'s theory we also able obtain the unique strong solution from weak existence and pathwise uniqueness.

The explosion time $\mathfrak{e}$ is the first hittig time of the solution X to the trap. In this paper we say that pathwise uniqueness up to an explotion time holds for (2.1) if for any two weak solution (up to an explosion time) of common initial value and common Brownian motion (relative to possiobly different firlerations) (X, W) and $(\tilde{X}, W)$,

$$P[X_t = \tilde{X}_t \,;\, 0 \leq {}^{\forall}t < \mathfrak{e}] = 1.$$

Theorem 2.2. *Under the conditions (2.2)-(2.4) pathwise uniqueness up to an explosion time holds for (2.1)* $\{\infty\}$ *as a trap.*

Let X^1 and X^2 be two solutions (with respect to the same Brownian moton and $X_0^1 = X_0^1 a.s.$) of (2.1) under the condition (2.3). Let

$$\tau_n^i \stackrel{def}{=} inf\{t \,|\, |X_t^i| \geq n\}, \;\; i = 1, 2, \; n \in \mathbb{N},$$

and

$$\tau_n \stackrel{def}{=} \tau_n^1 \wedge \tau_n^2.$$

To prove the theorem we use the following lemma from Revuz and Yor [16].

Lemma 2.3. *Fix an integer n. Then*

$$L_t^0(X^1 - X^2) = 0, \quad 0 \le {}^\forall t \le \tau_n.$$

Here we denote by L^0 the Local time at 0.

(Proof of the theorem 2.2).

We will show that

$$P[X_t^1 = X_t^2 \,;\, 0 \le {}^\forall t \le \tau_n] = 1., \quad {}^\forall n \in \mathbb{N}.$$

Thanks to Tanaka's formula and lemma 2.3, for $t > 0$,

$$|X_{t\wedge\tau_n}^1 - X_{t\wedge\tau_n}^2| = \int_0^{t\wedge\tau_n} sgn(X_s^1 - X_s^2) d(X_s^1 - X_s^2) \tag{2.5}$$

$$= \int_0^{t\wedge\tau_n} sgn(X_s^1 - X_s^2) \left(\sigma(s, X_s^1) - \sigma(s, X_s^2)\right) dW_s$$
$$+ \int_0^{t\wedge\tau_n} sgn(X_s^1 - X_s^2) \xi_s \left(\int_0^s (\mu(u, X_u^1) - \mu(u, X_u^2)) du \right) ds. \tag{2.6}$$

Since the stochastic integral term of (2.6) is bounded,

$$E \,|\, X_t^1 - X_t^2 \,| \le E \int_0^{t\wedge\tau_n} \xi_s \left(\int_0^s |\mu(u, X_u^1) - \mu(u, X_u^2)| du \right) ds \tag{2.7}$$

(by integration by parts)

$$= E \left(\int_0^{t\wedge\tau_n} \xi_s ds \right) \left(\int_0^{t\wedge\tau_n} |\mu(s, X_s^1) - \mu(s, X_s^2)| ds \right)$$
$$- E \int_0^{t\wedge\tau_n} \left(\int_0^s \xi_u du \right) |\, \mu(s, X_s^1) - \mu(s, X_s^2) \,| \, ds \tag{2.8}$$

(since ξ is positive and continuous, there exisit a positive constant C_t^n)

$$\le C_t^n \, E \int_0^{t\wedge\tau_n} |\, \mu(s, X_s^1) - \mu(s, X_s^2) \,| \, ds \tag{2.9}$$

$$\le C_t^n E \int_0^{t\wedge\tau_n} L_t^n \,|\, X_s^1 - X_s^1 \,| \, ds \tag{2.10}$$

$$\le C_t^n L_t^n E \int_0^t |\, X_{s\wedge\tau_n}^1 - X_{s\wedge\tau_n}^2 \,| \, ds. \tag{2.11}$$

By Gronwall's lemma,

$$X^1_{t\wedge\tau_n} = X^2_{t\wedge\tau_n}, \quad a.s.$$

Letting $n \uparrow \infty$, we get the desired result. □

To prove lemma 2.3, we need another lemma. (See also Revuz-Yor [16])

Lemma 2.4. *If $X.$ is a continuous semimartingale such that for some $\varepsilon > 0$ and every t,*

$$A_t \stackrel{def}{=} \int_0^t 1_{\{0<X_s\le\varepsilon\}}\rho(X_s)^{-1}d[X]_s < \infty, \quad a.s. \tag{2.12}$$

Then $L^0 \equiv 0$.

(Proof).

Fix $t > 0$. By the ocupation time formula,

$$A_t = \int_0^\varepsilon \rho(a)^{-1}\, L^a_t(X)da. \tag{2.13}$$

If $L^0_t(X)$ did not vanish a.s., as $L^a_t(X)$ converges to $L^0_t(X)$ when a decreases to zero, we would get $A_t = \infty$ with positive probability, which is a contradiction. □

(Proof of lemma 2.3).

Fix an integer n. We have

$$\begin{aligned} X^1_{t\wedge\tau_n} - X^2_{t\wedge\tau_n} &= \int_0^{t\wedge\tau_n} \left(\sigma(s,X^1_s) - \sigma(s,X^2_s)\right) dW_s \\ &+ \int_0^{t\wedge\tau_n} \xi_s \left(\int_0^s \left(\mu(u,X^1_u) - \mu(u,X^2_u)\right) du\right) ds. \end{aligned} \tag{2.14}$$

Therefore

$$\begin{aligned} &\int_0^{t\wedge\tau_n} \rho(X^1_s - X^2_s)^{-1} 1_{\{X^1_s > X^2_s\}} d[X^1 - X^2]_s \\ &= \int_0^{t\wedge\tau_n} \rho(X^1_s - X^2_s)^{-1} \left(\sigma(s,X^1_s) - \sigma(s,X^2_s)\right)^2 1_{\{X^1_s > X^2_s\}} ds. \end{aligned} \tag{2.15}$$

By (1.3)

$$\int_0^{t\wedge\tau_n} \rho(X^1_s - X^2_s)^{-1} 1_{\{X^1_s > X^2_s\}} d[X^1 - X^2]_s \le t. \tag{2.16}$$

Lemma 2.4 and (2.13) asserets lemma 2.4. □

3. Tests for explosion I

In this section we will deal with the very special cases of (2.1), namely

$$\sigma(x) \equiv \sigma x^\gamma,\ \mu(x) \equiv \sigma^2 x^{2\gamma} \tag{3.1}$$

where σ is a positive constant and $\dfrac{1}{2} \le \gamma \le 1$.

The qustion is whether the solutions for (3.1) would explode in finite time or not. A part of the answer is:

Theorem 3.1. *Assume(3.1), and* $\xi_. \equiv \xi, \eta_t \equiv x + \eta t$ *where* ξ *and* η *are positive constants. Then the solution for (2.1) explodes in finite time if* $\gamma \neq \dfrac{1}{2}$.

Let X_t be the unique solution of (2.1) and

$$Y_t \overset{def}{=} \sigma^2 \xi \int_0^t X_s^{2\gamma} ds. \tag{3.2}$$

Then

$$dX_t = (Y_t + \eta)\, dt + \sigma^2 X_t^2 dW_t. \tag{3.3}$$

We set a 'scale function' as follows;

$$S(x,y) = \frac{1}{1-\gamma} x^{1-\gamma} + y^{\frac{1}{\delta}} \quad x > 0,\ y > 0. \tag{3.4}$$

Then by Itô's formula,

$$S(X_t, Y_t) \equiv \sigma W_t + \int_0^t \mathcal{L}S(X_s, Y_s) ds \tag{3.5}$$

where

$$\mathcal{L} \equiv \frac{1}{2}\sigma^2 x^2 \frac{\partial^2}{\partial x^2} + (y+\eta)\frac{\partial}{\partial x} + \xi \sigma^2 x^{2\gamma} \frac{\partial}{\partial y}.$$

This time

$$\mathcal{L}S(x,y) = -\frac{1}{x^{1-\gamma}}\left(\frac{\sigma^2 \gamma}{2}\right) + \frac{y+\eta}{x^\gamma} + \xi\sigma^2 x^{2\gamma}\left(\frac{1}{\delta} y^{\frac{1}{\delta}-1}\right). \tag{3.6}$$

Set

$$G(R) \overset{def}{=} \inf_{S(x,y)=R} \mathcal{L}S(x,y), \qquad R > 0. \tag{3.7}$$

We will show that

Lemma 3.2. *there exisit positive constants C_1, C_2, C_3 so that for all $R > 0$,*

$$G(R) > C_1 R^{\frac{\delta+1}{3}} - C_2 R^{\delta - \frac{\gamma}{1-\gamma}} - C_3. \tag{3.8}$$

To estimate (3.7) we use the following elementary lemmas.

Lemma 3.3. *(i) Let $k_1, k_2 > 0,\ \delta_1 < \delta_2 < 0$ and*

$$f(x) = k_1 x^{\delta_1} - k_2 x^{\delta_2}. \quad x > 0.$$

Then for $x > 0$,

$$f(x) \geq k_1^{\frac{\delta_2}{\delta_2-\delta_1}} k_2^{-\frac{\delta_1}{\delta_2-\delta_1}} \left(\frac{\delta_1}{\delta_2}\right)^{\frac{\delta_1}{\delta_2-\delta_1}} \left(1 - \frac{\delta_1}{\delta_2}\right). \tag{3.9}$$

(ii) Let $k_1, k_2 > 0,\ \delta_1 < 0 < \delta_2$ and

$$f(x) = k_1 x^{\delta_1} + k_2 x^{\delta_2}. \quad x > 0.$$

Then for $x > 0$,

$$f(x) \geq k_1^{\frac{\delta_2}{\delta_2-\delta_1}} k_2^{-\frac{\delta_1}{\delta_2-\delta_1}} \left(\frac{-\delta_1}{\delta_2}\right)^{\frac{\delta_1}{\delta_2-\delta_1}} \left(1 + \frac{\delta_1}{\delta_2}\right). \tag{3.10}$$

Equality holds when

$$x = \left(\frac{k_1 |\delta_1|}{k_2 \delta_2}\right)^{\frac{1}{\delta_2-\delta_1}}. \tag{3.11}$$

(Proof of lemma 3.2).

We first remark that under the constraint that

$$R = \frac{1}{1-\gamma} x^{1-\gamma} + y^{\frac{1}{\delta}}, \quad x > 0,\ y > 0,\ R > 0,$$

there are bounds for both x and y, i.e;

$$0 < x < ((1-\gamma)R)^{\frac{1}{1-\gamma}}, \quad 0 < y < R^{\delta}. \tag{3.12}$$

Then we have

$$\frac{y}{x^\gamma} = \left(R - \frac{1}{1-\gamma}x^{1-\gamma}\right)^\delta \tag{3.13}$$

$$\geq \frac{R^\delta}{x^\gamma} - \frac{\delta}{1-\gamma}\frac{R^{\delta-1}}{x^{2\gamma-1}}. \tag{3.14}$$

And by (3.13)

$$\frac{1}{\delta}y^{\frac{1}{\delta}-1} > \frac{1}{\delta}R^{1-\delta}. \tag{3.15}$$

Let c_1, c_2 be positive constant such that $c_1 + c_2 = 1$. By lemma 2.3 (ii), and since $1 + \left(\frac{-\gamma}{2\gamma}\right) > 0$,

$$c_1\frac{R^\delta}{x^\gamma} + \xi\sigma^2 x^{2\gamma}\frac{1}{\delta}R^{1-\delta} \geq {}^\exists C_1 R^{\frac{\delta+1}{3}}. \tag{3.16}$$

Similarly by lemma 3.3 (i),

$$c_2\frac{R^\delta}{x^\gamma} - \frac{\delta}{1-\gamma}\frac{R^{\delta-1}}{x^{2\gamma-1}} \geq -{}^\exists C_2 R^{\delta-\frac{\gamma}{1-\gamma}}. \tag{3.17}$$

and

$$\frac{\eta}{x^\gamma} - \frac{1}{x^{1-\gamma}}\left(\frac{\sigma^2\gamma}{2}\right) \geq -{}^\exists C_3. \tag{3.18}$$

By (3.14)-(3.18) we get (3.8).

(Proof of theorem 3.1).

Let $\hat{R}_.$ be the solution of the following stochastic differential equation.

$$\begin{cases} d\hat{R}_t = \sigma dW_t + (C_1\hat{R}_t^\lambda - C_2\hat{R}_t - C_3)\,dt, \\ \hat{R}_0 = \dfrac{1}{1-\gamma}x^{1-\gamma} \quad ,\lambda \geq 1. \end{cases} \tag{3.19}$$

Denote its explosion time by $\mathfrak{t}$. By Feller's explosion test, (See e.g. Ikeda- Watanabe [9])

$$P[\mathfrak{t} < \infty] = \begin{cases} 1, & \text{if } \lambda > 1, \\ 0, & \text{if } \lambda = 1. \end{cases} \tag{3.20}$$

And by the comparison theorem (See Yamada [23], and Ikeda Watanabe [9]) and lemma 3.2,

$$S(X_t, Y_t) > \hat{R}_t, \ a.s. \tag{3.21}$$

Since $S(X_t, Y_t) \uparrow \infty$ implies $X_t \uparrow \infty$ we have

$$P(\mathfrak{e} < \mathfrak{t}) = 1.$$

We can take $\frac{\delta + 1}{3} > 1$ and $\delta - \frac{\gamma}{1-\gamma} = 1$ if and only if $\gamma > \frac{1}{2}$. This implies

$$P(\mathfrak{e} < \infty) = 1, \quad \text{if } \gamma > \frac{1}{2}. \quad \square$$

4. Tests for explosion II

In this section we will study the order more closely.

Let $L(x)$ be a slowly varying function on $[0, \infty)$; i.e. it is real valued, positive, measurable and $L(\lambda x) \sim L(x)^*$ for each $\lambda > 0$;

such that

(a) $\lim_{x \to \infty} L(x) = +\infty$,

(b) $L(x(L(x))^{-1}) \sim L(x)$,

(c) $L(x) \geq 1 \ \ \forall x \geq 0$,

and

(d) $(L(\sqrt{x}))^{-1}$satisfies Lipschitz condition.

Remark 4.1. (c) implies that for arbitrary $\tilde{L} \sim L$ and $0 \leq \alpha \leq 1$, $L\left(x(\tilde{L}(x))^{-\alpha}\right) \sim L(x)$. (See e.g. E.Seneta [20].)

Let $a(x) \stackrel{def}{=} xL(x)$ and $X.$ be the unique solution for

$$X_t = x + \int_0^t \int_0^s a(X_u) du ds + \int_0^t (a(X_s))^{\frac{1}{2}} dW_s, \quad x > 0. \tag{4.1}$$

*By $f(x) \sim g(x)$ we mean $\lim_{x \to \infty} \frac{f(x)}{g(x)} = 1$.

Theorem 4.2.

(i) If $\int_1^{+\infty} \frac{dx}{x\sqrt{L(x)}} = +\infty$*, then* $P(\mathfrak{e} < \infty) = 0$.

(ii) If $\int_1^{+\infty} \frac{dx}{x\sqrt{L(x)}} < +\infty$*, then* $P(\mathfrak{e} < \infty) = 1$.

Proof.

Let $C_t \overset{def}{=} \int_0^t a(X_s)ds$ and let A_t be the right continuous inverse of C_t ; i.e.

$$A_t = \inf\{s, C_s > t\}. \tag{4.2}$$

We will consider the time changed process $Y_t \overset{def}{=} X_{A_t}$ instead of X_t itself.
We have

$$\begin{aligned} Y_t &= x + \int_0^{A_t} C_s ds + \int_0^{A_t} (a(X_s))^{\frac{1}{2}} \, dW_s \\ &= x + \int_0^t s dA_s + \int_0^{A_t} (a(X_s))^{\frac{1}{2}} \, dW_s. \end{aligned} \tag{4.3}$$

By (4.2)

$$A_t = \int_0^t \frac{ds}{a(Y_s)}, \tag{4.4}$$

and the last term of (4.3) is an $\mathcal{F}_{A_t}$- Brownian motion.
Consequently, $Y.$ is a solution for

$$Y_t = x + \int_0^t \frac{s}{a(Y_s)} \, ds + B_t, \quad x > 0 \tag{4.5}$$

where B denotes a new Brownian motion.
To see its uniqueness, let us consider $V_t \overset{def}{=} Y_t^2$.
We have by Itô's formula,

$$V_t = x^2 + 2\int_0^t V_s^{\frac{1}{2}} \, dB_s + \int_0^t \left(\frac{2s}{L(\sqrt{V_s})} + 1 \right) ds. \tag{4.6}$$

By the assumption (d), the above has the unique solution (in the strong sense). Moreover, we see that the point 0 is at worst instantaneously reflecting boundary.

Instead of (4.5) we will consider the following (random) ordinary differential equation for each $\omega \in \Omega$.

$$Z_t = x + \int_0^t \frac{s}{a(Z_s + B_s)}\, ds \tag{4.7}$$

which in turn means $Z_t = Y_t - B_t$.
We shall have

$$Z_t \sim t\,(L(t))^{-\frac{1}{2}}, \quad a.s.. \tag{4.8}$$

Assume that (4.8) holds. Then $\dfrac{Z_t}{t} + \dfrac{B_t}{t} \sim (L(t))^{-\frac{1}{2}}, \ \ a.s.,$
and

$$\begin{aligned} a(Y_t) &= (Z_t + B_t)L(Z_t + B_t) \\ &= t\left(\frac{Z_t}{t} + \frac{B_t}{t}\right) L\left(t\left(\frac{Z_t}{t} + \frac{B_t}{t}\right)\right) \\ &\sim t\,(L(t))^{\frac{1}{2}}, \quad a.s.. \end{aligned} \tag{4.9}$$

On the other hand, $A_\infty < \infty$ is equivalent to $\mathfrak{e} < \infty$ because $\lim_{x\to\infty} a(x) = +\infty$.
Combining these and (4.4), we have the assertion of Theorem.
Now we turn to prove (4.8). First we shall see that $L(Z_t + B_t)$ is slowly varying. This is done by the follwing

Lemma 4.3.

Let $\bar{\Omega} = \{\omega \in \Omega, \lim_{t\to+\infty} \dfrac{B_t(\omega)}{t} = 0\}$ *and fix* $\omega \in \bar{\Omega}$.

(i) There exists a positive constant $M(\omega)$ *such that*

$$Z_t(\omega) < x + M(\omega)\, t. \tag{4.10}$$

(ii) There exists a slowly varying function $L^* \sim L$ *such that*

$$Z_t(\omega) > x + t\,(L^*(t))^{-1}. \tag{4.11}$$

Proof of Lemma 4.3.
Since $\lim_{t\to+\infty} \dfrac{B_t(\omega)}{t} = 0$, there exists a positive constant K such that $\left|\dfrac{B_t(\omega)}{t}\right| < K$.

Let Z_t^0 and be the solution for

$$Z_t^0 = x + \int_0^t \frac{s}{Z_s^0 + B_s}\, ds, \tag{4.12}$$

and Z_t^1 be the solution for

$$Z_t^1 = x + \int_0^t \frac{s}{Z_s^1 - Ks}\, ds. \tag{4.13}$$

Since $L(x) \geq 1$, we have $Z_t^0 \geq Z_t$ for all $t \in [0, \infty)$ and since $\frac{B_t}{t} > Z_t^0$ for all $t \in [0, \infty)$.

Let $M \equiv K + 1$ and $Z_t^2 \overset{def}{=} x + Mt$. Then we have

$$\begin{aligned}
(Z_t^2)' &= M \\
&= M \frac{t}{Z_t^2 - Kt} \frac{Z_t^2 - Kt}{t} \\
&= \frac{t}{Z_t^2 - Kt} \left(\frac{x}{t} + 1 \right) (K + 1) \qquad (4.14) \\
&> \frac{t}{Z_t^2 - Kt}. \qquad (4.15)
\end{aligned}$$

By the comparison theorem we have $Z_t^2 > Z_t^1$ which proves the first part of Lemma 4.3.

To prove the latter part, we first observe that

$$\begin{aligned}
Z_t + B_t &= t \left(\frac{Z_t}{t} + \frac{B_t}{t} \right) \\
&\leq t \left(\frac{x}{t} + M + K \right) \\
&= x + t(M + K). \qquad (4.16)
\end{aligned}$$

Let us introduce a new slowly varying function $\bar{L}$ as follows.

$$\bar{L}(x) \overset{def}{=} \sup_{0 \leq y \leq x} L(y).$$

Then $\bar{L}$ is monotone, $\bar{L} \geq L$ and $\bar{L} \sim L$. (See E. Seneta.)
We have

$$\frac{t}{a(Z_t + B_t)} = \frac{t}{Z_t + B_t} \frac{1}{L(Z_t + B_t)}$$

since $\bar{L} \geq L$

$$\geq \frac{t}{Z_t + B_t} \frac{1}{\bar{L}(Z_t + B_t)} \tag{4.17}$$

by (4.16) and monotonicity of $\bar{L}$

$$> \frac{t}{x + t(M+K)} \frac{1}{\bar{L}(x + t(M+K))} \tag{4.18}$$

$$\sim (L(t))^{-1}. \tag{4.19}$$

Let

$$Z_t^3 \stackrel{def}{=} x + \int_0^t \frac{s}{x + s(M+K)} \frac{1}{\bar{L}(x + s(L+K))} \, ds. \tag{4.20}$$

Then by the comparison theorem we see that $Z_t^3 < Z_t$. On the other hand, it is well known that $\int^x N(y)\, dy \sim xN(x)$ holds for arbitrary slowly varying function N. (See E. Seneta [20].)

Hence we have $Z_t^3 \sim t(L(t))^{-1}$. This proves (ii) of Lemma4.2. □

REMARK 4.4. It shold be noted that $P(\bar{\Omega}) = 1$. So we have the assertions with probability one.

By the above lemma we have

$$\underline{L}\left(t\left((L^*(t))^{-1} + \frac{B_t + x}{t}\right)\right) \leq L(Z_t + B_t) \leq \bar{L}\left(t\left(M + \frac{B_t + x}{t}\right)\right). \tag{4.21}$$

where $\underline{L}(x) \stackrel{def}{=} \inf_{x \leq y < +\infty} L(y)$. It is also known that $\underline{L}$ is monotone, $\underline{L} \leq L, and \underline{L} \sim L$. (See E. Seneta [20].)

Since $\underline{L} \sim \bar{L}$, we see that

$$L(Z_t + B_t) \sim L(t) \text{ with probability one.} \tag{4.22}$$

Finally, let $U_t \stackrel{def}{=} (Z_t)^2$. Then we have

$$\begin{aligned}(U_t)' &= 2Z_t(Z_t)' \\ &= 2\frac{Z_t}{Z_t+B_t}\frac{t}{L(Z_t+B_t)} \\ &\sim \frac{2t}{L(t)}. \quad a.s.. \end{aligned} \tag{4.23}$$

We need the following

Lemma 4.5 (Karamata [12]). *If N is slowly varying on $[c,\infty)$, then for each $k>-1$*

$$\lim_{x\to+\infty}\frac{x^{k+1}N(x)}{\int_c^x y^k N(y)\,dy} = k+1. \tag{4.24}$$

By (4.23) and (4.24) we have

$$U_t \sim t^2(L(t))^{-1}. \tag{25}$$

This proves (4.8), □

5. A GENERALIZED ARC-SIN LAW

The following two sections are taken from [3]. In this section, W_t denotes a atandard Brownian motion starting at 0, $\mathcal{F}_t$ denotes its canonical filteration and P_0 denotes its probability measure. Let

$$A(t,x\,;\mu) = \frac{1}{t}\int_0^t 1_{\{W_s+\mu s<x\}}\,ds, \quad \mu>0, t>0, X\in\mathbb{R}^1. \tag{5.1}$$

Then we have the following theorem.

Theorem 5.1.

(i) We have

$$P_0(A(t,0;\mu) < y) = \frac{1}{2}\int_0^{ty} \left(\sqrt{\frac{2}{\pi s}}\exp\left(-\frac{\mu^2}{2}s\right) - 2\mu\Phi(\mu\sqrt{s})\right) \times \left\{\left(2\mu + \sqrt{\frac{2}{\pi(t-s)}}\exp\left(-\frac{\mu^2}{2}(t-s)\right)\right) - 2\mu\Phi(\mu\sqrt{t-s})\right\} ds,$$

where Φ denotes the tail of the distribution function of the normal distribution; that is,

$$\Phi(x) = \int_x^\infty \frac{1}{\sqrt{2\pi}}\exp\left(-\frac{y^2}{2}\right) dy.$$

(ii) More generally, we have

$$P_0(A(t,x;\mu) < y) = \int_0^{t\alpha} h(s,x;\mu)\phi(t-s, y-\frac{s}{t};\mu)\, ds, \quad x \neq 0,$$

where $h(s,x;\mu)$ denotes the density of the first hitting time of $B_t + \mu t$ to x; that is,

$$h(s,x;\mu) = \frac{|x|}{\sqrt{2\pi s}}\exp\left(\frac{(|x|-\mu s)^2}{2s}\right)$$

and

$$\phi(t,x;\mu) = P_0(A(t,0;\mu) < x).$$

Proof. First we prove assertion (i).Let

$$u(x) = E_x\int_0^\infty \exp(-\zeta t)\exp(-\lambda A(t,0;\mu))dt, \quad x \in \mathbb{R}^1. \tag{5.2}$$

Then the Feynman-Kac formula (cf. Kac [11] and Itô and Mckean [10]) claims that $u(x)$ is the unique bounded solution of the equation

$$-\frac{1}{2}u'' - \mu u' + \zeta u + \lambda 1_{\{x>0\}}u = 1, \quad \zeta > 0, \lambda > 0. \tag{5.3}$$

By solving (3),

$$u(0) = \frac{\mu}{2}\frac{1}{\zeta}\frac{\sqrt{\mu^2+2(\zeta+\lambda)}}{\zeta+\lambda} - \frac{\mu}{2}\frac{1}{\zeta+\lambda}\frac{\sqrt{\mu^2+2\zeta}}{\zeta} + \frac{1}{2}\frac{\sqrt{\mu^2+2(\zeta+\lambda)}}{\zeta+\lambda}\frac{\sqrt{\mu^2+2\zeta}}{\zeta} - \frac{\mu^2}{2}\frac{1}{\zeta(\zeta+\lambda)}. \tag{5.4}$$

Since we have, by inverting the Laplace transform (see, e.g., Widder[27]),

$$\begin{aligned}\frac{\sqrt{\mu^2+2(\zeta)}}{\zeta} &= \frac{1}{\zeta}\int_0^{\zeta}\frac{d\lambda}{\sqrt{\mu^2+2\lambda}}+\frac{1}{\zeta}\mu \\ &= \int_0^{\infty}\exp(-\zeta t)\left(\int_t^{\infty}\frac{\exp(-(\frac{\mu^2}{2})s)}{\sqrt{2\pi s^3}}ds+\mu\right)dt, \end{aligned} \tag{5.5}$$

we get

$$\begin{aligned}u(0) = \int_0^{\infty}\exp(-\zeta t)\int_0^t \frac{exp(-\lambda s)}{2} \\ \times\left(2\mu+\int_{t-s}^{\infty}\frac{\exp(-(\frac{\mu^2}{2})\tau)}{\sqrt{2\pi\tau^3}}d\tau\right) \\ \times\left(\int_s^{\infty}\frac{\exp(-(\frac{\mu^2}{2})\tau)}{\sqrt{2\pi\tau^3}}d\tau\right)dsdt, \end{aligned} \tag{5.6}$$

Comparing (5.6) and (5.2), we get

$$\begin{aligned}P_0(A(t,0;\mu)<y) = \int_0^{ty}\frac{1}{2}\left(2\mu+\int_{t-s}^{\infty}\frac{\exp(-(\frac{\mu^2}{2})\tau)}{\sqrt{2\pi\tau^3}}d\tau\right) \\ \times\left(\int_s^{\infty}\frac{\exp(-(\frac{\mu^2}{2})\tau)}{\sqrt{2\pi\tau^3}}d\tau\right)ds. \end{aligned} \tag{5.7}$$

Integrating by parts, we get

$$\begin{aligned}\int_s^{\infty}\frac{\exp(-(\frac{\mu^2}{2})\tau)}{\sqrt{2\pi\tau^3}}d\tau = \sqrt{\frac{2}{\pi s}}\exp\left(-\frac{\mu^2}{2}s\right) \\ -2\mu\int_{\mu\sqrt{s}}^{\infty}\frac{1}{\sqrt{2\pi}}\exp\left(-\frac{\tau^2}{2}\right)d\tau. \end{aligned} \tag{5.8}$$

So we have the assretion (i).

Assertion (ii) follows directly from the strong Markov property of $B_t+\mu t$. □

REMARK 5.2. Since $A(t,x;-\mu)\overset{law}{=}1-A(t,-x;\mu)$, we now obtain Theorem5.1 for all $\mu\in\mathbb{R}^1$.

Let us consider the Black-Scholes model (c.f. Black and Sholes[4]): The stock price X_t is a geometric Brownian motion and the bond price b_t is nonstochastic; that is,

$$X_t = X_0 \exp(\sigma W_t + (\mu - \frac{1}{2}\sigma^2)t), \quad X_0 > 0, \sigma > 0, \mu \in \mathbb{R}^1, \tag{6.1}$$

$$b_t = b_0 \exp(rt), \quad r \geq 0, b_0 > 0. \tag{6.2}$$

We define the α-percentile option $(m(T;\alpha) - c)^+, c > 0$, and present the procing formulae for this.

Here we think of "pricing" as the stochastic integral representation of the option with respect to the discounted stock price under martingale measure (c.f. Harrison and Pliska [8])

We define a discounted price process Z_t by setting

$$Z_t = b_t^{-1} X_t. \tag{6.3}$$

let us introduce aprobability measure P_0^* under which Z_t is a martingale and let E_0^* denote its expectation. Let π be the price of the option, ζ_t be the amount of stock and ν_t be the amount of bond.

Then we have a stochastic representation of the option as follows:

$$b_T^{-1}(m(T;\alpha) - c)^+ = \pi + \int_0^T \zeta_t dZ_t \tag{6.4}$$

where

$$\pi = E_0^* \left((m(T;\alpha) - c)^+ b_T^{-1} \right), \tag{6.5}$$

and we have

$$\nu_t = E_0^* \left((m(T;\alpha) - c)^+ b_T^{-1} | \mathcal{F}_t \right) - \zeta_t Z_t. \tag{6.6}$$

We can give the following formulae for π, ζ_t, ν_t by virtue of Theorem 5.1

Theorem 6.1. *We have*

$$\pi = b_T^{-1} \int_c^{\infty} G\left(T, \sigma^{-1} \log \frac{y}{X_0}; \alpha, \frac{r}{\sigma} - \frac{1}{2}\sigma\right) dy$$
$$+ c b_T^{-1} G\left(T, \sigma^{-1} \log \frac{c}{X_0}; \alpha, \frac{r}{\sigma} - \frac{1}{2}\sigma\right), \tag{6.7}$$

$$\zeta_t = -\frac{b_T^{-1}}{\sigma Z_t} \int_0^{\infty} \frac{\partial G}{\partial x}\left(T-t, \sigma^{-1} \log \frac{b_0 y}{Z_t}; \frac{T\alpha - C_t}{T-t},, \frac{r}{\sigma} - \frac{1}{2}\sigma\right) dy \tag{6.8}$$

$$\nu_t = \int_c^{\infty} G\left(T-t, \sigma^{-1} \log \frac{b_0 y}{Z_t}; \frac{T\alpha - C_t}{T-t}, \frac{r}{\sigma} - \frac{1}{2}\sigma\right) dy$$
$$- \zeta_t Z_t + c b_t^{-1} G\left(T-t, \sigma^{-1} \log \frac{b_0 c}{Z_t}; \frac{T\alpha - C_t}{T-t},, \frac{r}{\sigma} - \frac{1}{2}\sigma\right), \tag{6.9}$$

where

$$G(t, x; \alpha, \mu) \equiv \int_0^{t\alpha} h(s.x; \mu) \phi\left(t-s, \alpha - \frac{s}{t}; \mu\right) ds, \tag{6.10}$$

$\frac{\partial G}{\partial x}$ *denotes the derivative with respect to the second variable and*

$$C_t = A\left(t, \sigma \log x; \frac{r}{\sigma} - \frac{1}{2}\sigma\right). \tag{6.11}$$

REMARK 6.2. To calculate $\frac{\partial G}{\partial x}$ we observe the following facts:

(i) $s \cdot h(s, x, \mu)$ is uniformly integrable.

(ii)$\frac{1}{s}\left(\varphi\left(t-s, \alpha - \frac{s}{t}, \mu\right) - \varphi(t, \alpha, \mu)\right)$ is uniformly integrable since φ is differentiable at $s = 0$.

By (i) and (ii), $h(s, x, \mu) \cdot \left(\varphi\left(t-s, \alpha - \frac{s}{t}, \mu\right) - \varphi(t, \alpha, \mu)\right)$ is uniformly integrable. Therefore, for $x > 0$,

$$\begin{aligned}
\frac{\partial G}{\partial x} &= \int_0^{t\alpha} \left(\frac{\partial}{\partial x} h(s, x, \mu)\right)\left(\varphi\left(t-s, \alpha - \frac{s}{t}, \mu\right) - \varphi(t, \alpha, \mu)\right) ds \\
&\quad + \frac{\partial}{\partial x} \int_0^{t\alpha} h(s, x, \mu) \varphi(t, \alpha, \mu) ds \\
&= \int_0^{t\alpha} \left(\frac{1}{x} - \frac{x}{s} + \mu\right) h(s, x, \mu) \left(\varphi\left(t-s, \alpha - \frac{s}{t}, \mu\right) - \varphi(t, \alpha, \mu)\right) ds \\
&\quad + \varphi(t, \alpha, \mu) \frac{\partial}{\partial x} \int_0^{t\alpha} \frac{x}{\sqrt{2\pi s}} e^{-\frac{x^2}{2s} + \mu x - \frac{s\mu^2}{2}} ds.
\end{aligned} \tag{6.12}$$

Again we observe:

1) $s \cdot \frac{x}{\sqrt{2\pi s}} e^{-\frac{x^2}{2s}+\mu x}$ is uniformly integrable.

2)$\frac{1}{s}\left(e^{-\frac{s\mu^2}{2}} - 1\right)$ is uniformly integrable. Then,

$$\text{(second term of (1))} = \varphi(t,\alpha,\mu) \int_0^{t\alpha} \frac{\partial}{\partial x}\left(\frac{x}{\sqrt{2\pi s}} e^{-\frac{x^2}{2s}+\mu x}\right)\left(e^{-\frac{s\mu^2}{2}} - 1\right) ds$$
$$+ \varphi(t,\alpha,\mu) \frac{\partial}{\partial x} \int_0^{t\alpha} \frac{x}{\sqrt{2\pi s}} e^{-\frac{x^2}{2s}+\mu x} ds. \tag{6.13}$$

$$\text{(first term of (2))} = \varphi(t,\alpha,\mu) \int_0^{t\alpha} \left(\frac{1}{x} - \frac{x}{s} + \mu\right)\left(h(s,x,\mu) - \frac{x}{\sqrt{2\pi s}} e^{-\frac{x^2}{2s}+\mu x}\right) ds. \tag{6.14}$$

$$\text{(second term of (2))}$$
$$= \varphi(t,\alpha,\mu)\left(\mu e^{\mu x} \int_0^{t\alpha} \frac{x}{\sqrt{2\pi s}} e^{-\frac{x^2}{2s}} ds + e^{\mu x} \frac{\partial}{\partial x} \int_0^{t\alpha} \frac{x}{\sqrt{2\pi s}} e^{-\frac{x^2}{2s}} ds\right). \tag{6.15}$$

Since

$$\text{(second term of (4))} = -2\varphi(t,\alpha,\mu) e^{\mu x} \sqrt{2\pi t\alpha} e^{-\frac{x^2}{2t\alpha}}. \tag{6.16}$$

By (6.12)-(6.16),we have

$$\frac{\partial G}{\partial x} = \int_0^{t\alpha} \left(\frac{1}{x} - \frac{x}{s} + \mu\right) h(s,x,\mu) \varphi\left(t-s, \alpha - \frac{s}{t}, \mu\right) ds$$
$$- e^{\mu x} \varphi(t,\alpha,\mu) \left(\int_0^{t\alpha} \left(\frac{1}{x} - \frac{x}{s}\right) \frac{x}{\sqrt{2\pi s}} e^{-\frac{x^2}{2s}} ds + 2 \frac{1}{\sqrt{2\pi t\alpha}} e^{-\frac{x^2}{2t\alpha}}\right). \tag{6.17}$$

Proof of Theorem 6.1.

Let $W_t^* \equiv W_t + \frac{(\mu - r)t}{\sigma}$. Then W_t^* is a Brownian motion under P_0^* and we have

$$Z_t = X_0 b_0^{-1} \exp(\sigma W_t^* - \frac{1}{2}\sigma^2 t)$$

and

$$X_t = X_0 \exp(\sigma W_t^* + (r - \frac{1}{2}\sigma^2) t).$$

Since

$$P_0^*(m(T;\alpha) > x) = P_0^*\left(A\left(T, \frac{1}{\sigma}\log\frac{x}{X_0}; \frac{r}{\sigma} - \frac{1}{2}\sigma\right) < \alpha\right), \tag{6.18}$$

we have from Theorem5.1 (ii),

$$P_0^*(m(T;\alpha) > x) = G\left(T, \frac{1}{\sigma}\log\frac{x}{X_0}; \alpha, \frac{r}{\sigma} - \frac{1}{2}\sigma\right). \tag{6.19}$$

Therefore, we get (6.7).

To obtain (6.8) and (6.9), we first observe

$$\begin{aligned} P_0^*(m(T;\alpha) > x|\mathcal{F}_t) &= P_0^*\left(A\left(T, \frac{1}{\sigma}\log\frac{x}{X_0}; \frac{r}{\sigma} - \frac{1}{2}\sigma\right) < \alpha|\mathcal{F}_t\right) \\ &= P_0^*\left(\int_t^T 1_{\{X_s \le x\}} ds < T\alpha - C_t|\mathcal{F}_t\right) \\ &= P_0^*\left(A\left(T-t, \frac{1}{\sigma}\log\frac{x}{X_0} - W_t^* + \frac{1}{2}\sigma t; \frac{r}{\sigma} - \frac{1}{2}\sigma\right) < \frac{T\alpha - C_t}{T-t}\right) \\ &= G\left(T-t, \frac{1}{\sigma}\log\frac{b_0 x}{Z_t}; \frac{T\alpha - C_t}{T-t}, \frac{r}{\sigma} - \frac{1}{2}\sigma\right). \end{aligned} \tag{6.20}$$

By integrating both sides of (6.20) with respect to x, we obtain $E^*((m(T;\alpha)-c)^+|\mathcal{F}_t)$.

Itô's formula claims that the integrand $\tilde{\zeta}_t$ should be the partial derivative of (6.20) with respect to Z_t, so we get (6.8) and then (6.9). $\square$

7. An Extention of Williams formula

The following two swctions are taken from [2].

Let $B_t, 0 \le t < \infty$ be a standard Brownian motion starting at 0 and let

$$A_+(t) = \int_0^t 1_{\{B_s \ge 0\}} ds$$

and

$$A_-(t) = \int_0^t 1_{\{B_s \le 0\}} ds, \quad 0 \le t < \infty.$$

Let $A_+^{-1}(t)$ and $A_-^{-1}(t)$ be the right-continuous inverse of $A_+(t)$ and $A_-(t)$, respectively, that is,

$$A_\pm^{-1}(t) \triangleq \inf\{s | A_\pm(s) > t\}.$$

Then $B_\pm(t) \triangleq \pm B(A_\pm^{-1}(t))$ are *mutually independent reflecting-barieer Brownian motions.* Let $L(t)$ be a local time at 0 of B_t, and let

$$L_\pm(t) = L(A_\pm^{-1}(t))$$

and

$$L_\pm^{-1}(t) = \inf\{s | L_\pm(s) > t\}, \qquad 0 \le t < \infty.$$

By Williams formulae we mean the following equivalence in law

$$A_+^{-1}(t) \stackrel{d}{=} t + L_-^{-1}(L_+(t)) \tag{7.1}$$

and

$$A_-^{-1}(t) \stackrel{d}{=} t + L_+^{-1}(L_-(t)). \tag{7.2}$$

Williams [28] 's proof of the Levy's arc-sine law relies upon the above formula. (see e.g. Karatzas-Shereve [13]) S.Watanabe [26] extended this approach to so-called generalized diffusion processes. But our results don't hold when B_t is not a Brownian motion (except for the constant drift case).

By Dassios formula we mean the equivalence in law between Brownian quantiles and indepedent sum of max and min. The distribution of the quantiles of Brownian motion (with drift) has been studied by [3] and M.Yor [25]. This is very closely related to the pricing of a sort of path-dependent options in mathematical finance. (see R.Miura [15])

Let

$$A_+(t,x) = \int_0^t 1_{\{B_s \ge x\}} ds, \quad 0 \le t < \infty, \ \ x \in \mathbf{R}$$

and

$$A_-(t,x) = \int_0^t 1_{\{B_s \le x\}} ds, \quad 0 \le t < \infty, \ \ x \in \mathbf{R}$$

and let $A_\pm^{-1}(t,x)$ be their right continuous inverse.

Theorem 7.1. *For each $t > 0$, we have the following equivalence in law:*

$$A_+^{-1}(t,x) \stackrel{d}{=} t + L_-^{-1}(L_+(t) + |x|) \tag{7.3}$$

and

$$A_-^{-1}(t,x) \stackrel{d}{=} t + L_+^{-1}(L_-(t) + |x|). \tag{7.4}$$

Proof. It suffices us to proof (2.1) when $x > 0$ by using the following formula.

$$E[e^{-\mu A_+^{-1}(t,x)}] = E[e^{-\mu\tau_x}]E[e^{-\mu A_+^{-1}(t)}] \tag{7.5}$$

where $\tau_x \triangleq \inf\{s|B_s > x\}$, the first hitting time at x.

Since τ_x, $L_\pm^{-1}$ are stable subordinators with exponent $\frac{1}{2}$, and by (7.1),

$$(\textit{Right hand side of } (7.5)) = e^{-\sqrt{2\mu}x}E[e^{-\mu(t+L_-^{-1}(L_+(t))}] \tag{7.6}$$

(by the independence of $\{L_-^{-1}, L_+\}$)

$$= e^{-\mu t}E[e^{-\sqrt{2\mu}(L_+(t)+x)}] \tag{7.7}$$

$$= E[e^{-\mu(t+L_-^{-1}(L_+(t)+x))}]. \tag{7.8}$$

By the uniqueness of Laplace transform, we get (7.3).

(proof of (7.5).)

$$\begin{aligned}
\int_0^\infty e^{-\mu t}E[e^{-\lambda A_+(t,x)}]dt &= \int_0^\infty e^{-\mu t}\left(\lambda E[\int_{A_+(t,x)}^\infty e^{-\lambda u}]\right)dt \\
&= \lambda E[\int_0^\infty e^{-\mu t}dt\int_0^\infty 1_{\{A_+(t,x)\le u\}}e^{-\lambda u}du] \\
&= \lambda E[\int_0^\infty\int_0^\infty 1_{\{A_+^{-1}(u,x)\ge t\}}e^{-\lambda u}e^{-\mu t}du] \\
&= \lambda E[\int_0^\infty e^{-\lambda u}du\int_0^{A_+^{-1}(u,x)} e^{-\mu t}dt] \\
&= \frac{1}{\mu} - \frac{\lambda}{\mu}E[\int_0^\infty e^{-\lambda u-\mu A_+^{-1}(u,x)}du].
\end{aligned} \tag{7.9}$$

On the other hand, by the Markov property of B_t

$$\begin{aligned}
\int_0^\infty e^{-\mu t}E_0[e^{-\lambda A_+(t,x)}]dt &= E_0[\int_0^{\tau_x} e^{-\mu t-\lambda A_+(t,x)}dt] + E_0[\int_{\tau_x}^\infty e^{-\mu t-A_+(t,x)}dt] \\
&= \frac{1}{\mu}(1 - E_0[e^{-\mu\tau_x}]) + E_0[e^{-\mu\tau_x}]E_x[\int_0^\infty e^{-\mu t-\lambda A_+(t,x)}dt]
\end{aligned}$$

(since $E_x[e^{-\lambda A_+(t,x)}] = E_0[e^{-\lambda A_+(t)}]$)

$$\begin{aligned}
&= E_0[e^{-\mu\tau_x}]\left(E_0[\int_0^\infty e^{-\mu t-\lambda A_+(t)}dt] - \frac{1}{\mu}\right) + \frac{1}{\mu} \\
&= -\frac{\lambda}{\mu}E_0[e^{-\mu\tau_x}]E_0[\int_0^\infty e^{-\lambda u-\mu A_+^{-1}(u)}du] + \frac{1}{\mu}
\end{aligned} \tag{7.10}$$

where we denote starting points of Brownian motions by subscript of E.

By (7.9) and (7.10) we get (7.5). □

8. Another proof of Dassios formula

Let us define *Brownan quantiles with parameter* α (in mathmatical finance α *percentiles)* as follows;

$$M(\alpha, t) \triangleq \inf\{x | A_-(t, x) > t\alpha\}, \quad 0 < \alpha < 1. \tag{8.1}$$

A.Dassios [2] proved the following equivalence in law for each t;

$$M(\alpha, t) \stackrel{d}{=} \max_{0 \le s \le \alpha t} B_s^1 + \min_{0 \le s \le (1-\alpha)t} B_s^2 \tag{8.2}$$

where B^1 and B^2 are two independent Brownian motions.

He proved above formula by Feynman-Kac method including the case when it has a constant drift. Now we take another way as we stated in section 7.

Theorem 8.1 (Dassios [4]). *We have the following equivalence in law for each* t.

$$M(\alpha, t) \stackrel{d}{=} \pm L_{\pm}(\alpha t) \mp L_{\mp}((1-\alpha)t). \tag{8.3}$$

Proof. For $s < 0$,

$$\begin{aligned} P(M(\alpha, t) < s) &= P(A_-(t, s) > \alpha t) \\ &= P(A_-^{-1}(\alpha t, s) < t) \end{aligned}$$

(by theorem 1)

$$\begin{aligned} &= P(\alpha t + L_+^{-1}(L_-(\alpha t) + |s|) < t) \\ &= P(L_+^{-1}(L_-(\alpha t) - s) < (1-\alpha)t) \\ &= P(L_-(\alpha t) - s < L_+((1-\alpha)t)) \\ &= P(L_-(\alpha t) - L_+((1-\alpha)t) < s). \end{aligned} \tag{8.4}$$

Since $A_-(t, x) \stackrel{d}{=} A_+(t, -x)$ and $L_+(t) \stackrel{d}{=} L_-(t)$, we get (3.3). □

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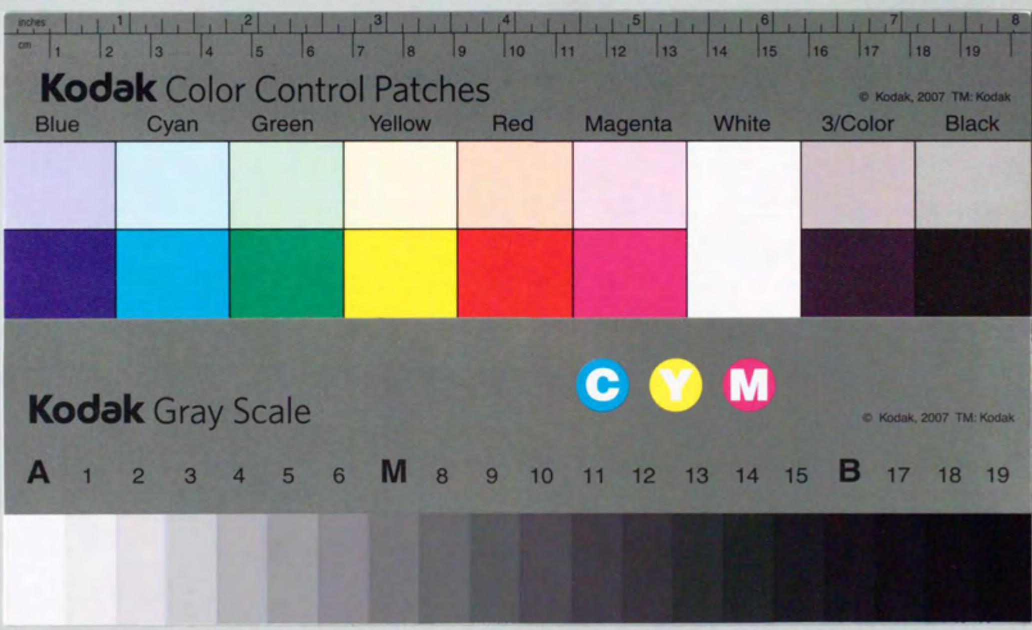

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