On multi-products of Fourier integral operators

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ON MULTI-PRODUCTS OF FOURIER INTEGRAL OPERATORS

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0. Introduction

We denote the Fourier integral operators on $\boldsymbol{R}^n$ with phase function ϕ_j and symbol $p_j \in \boldsymbol{S}_\rho^{m_j}$ by $I(\phi_j, p_j)$, $j = 1, 2, \ldots, L+1$. If all the canonical maps ω_j associated with phase functions ϕ_j are sufficiently close to the identity, the composite canonical map $\omega_{L+1}\omega_L \cdots \omega_1$ is also near the identity. Moreover, we have

$$I(\phi, q) = I(\phi_{L+1}, p_{L+1}) I(\phi_L, p_L) \cdots I(\phi_1, p_1)\,, \tag{0.1}$$

for some phase function ϕ and some symbol $q \in \boldsymbol{S}^{\sum_{j=1}^{L+1} m_j}$. (cf. L. Hörmander [6]) Here the correspondence of the symbols $(p_{L+1}, p_L, \ldots, p_1) \to q$ is multi-linear. In [9],[10],[12], H. Kumano-go-Taniguchi theorem gives the following estimate for the symbol q ; that is, for any non-negative integers l, l', there exist a positive constant $C_{l,l'}$ and positive integers l_1, l_1' such that

$$|q|_{l,l'}^{(\sum_{j=1}^{L+1} m_j)} \leq (C_{l,l'})^L \prod_{j=1}^{L+1} |p_j|_{l_1,l_1'}^{(m_j)}\,, \tag{0.2}$$

where $|\cdot|_{l,l'}^{(m)}$ denotes the semi-norm of $\boldsymbol{S}_\rho^m$.

This estimate is useful in the calculus of Fourier integral operators. In [9],[10],[12], this estimate was applied to construct a fundamental solution for hyperbolic systems. Slight modification of this estimate was applied to construct a fundamental solution for Schrödinger equations. (cf. D. Fujiwara [1] $\sim$ [4], H. Kitada and H. Kumano-go [8], N.

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Kumano-go [11]) However, in their proofs, they used the inverse of the Fourier integral operators whose symbols are equal to 1. Therefore, their condition on phase functions was complicated, and the canonical maps associated with those phase functions must be very close to the identity. Recently, in [5], D. Fujiwara, N. Kumano-go and K. Taniguchi have given a more direct proof of this estimate and simplified the condition on phase functions in the case for Schrödinger equations. However they are not successful in the original case for hyperbolic systems. The aim of this paper is to give a more direct proof and to simplify the condition on phase functions in the original case for hyperbolic systems.

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1. Statement of results

In order to state our main theorems, we recall some definitions for Fourier integral operator in H. Kumano-go and K. Taniguchi [9],[10],[12].

Definition 1.1 (Classes of symbols S_ρ^m).

Let $m \in \boldsymbol{R}$ and $1/2 \leq \rho \leq 1$. We say that a C^∞-function $p(x,\xi)$ on $\boldsymbol{R}_x^n \times \boldsymbol{R}_\xi^n$ belongs to the class of symbols S_ρ^m, if, for any α, β, there exists a constant $C_{\alpha,\beta}$ such that

$$|\partial_x^\beta \partial_\xi^\alpha p(x,\xi)| \leq C_{\alpha,\beta} \langle \xi \rangle^{m+(1-\rho)|\beta|-\rho|\alpha|}, \tag{1.1}$$

where $\langle \xi \rangle = \sqrt{1+|\xi|^2}$.

Remark. For $p \in S_\rho^m$, we define semi-norms $|p|_{l,l'}^{(m)}$, $l, l' = 0, 1, 2, \ldots$ by

$$|p|_{l,l'}^{(m)} = \max_{|\alpha| \leq l,\, |\beta| \leq l'} \sup_{(x,\xi)} \frac{|\partial_x^\beta \partial_\xi^\alpha p(x,\xi)|}{\langle \xi \rangle^{m+(1-\rho)|\beta|-\rho|\alpha|}}. \tag{1.2}$$

Then S_ρ^m is a Fréchet space with these semi-norms.

Definition 1.2 (Classes of phase functions $P_\rho(t, \{\kappa_l\}_{l=0}^\infty)$).

Let $\{\kappa_l\}_{l=0}^\infty$ be an increasing sequence of positive constants and $t > 0$. We say that a real-valued C^∞-function $\phi(x, \xi)$ on $R_x^n \times R_\xi^n$ belongs to the class of phase functions $P_\rho(t, \{\kappa_l\}_{l=0}^\infty)$, if $\phi(x, \xi)$ satisfies the following :

$$|\partial_x^\beta \partial_\xi^\alpha \phi(x, \xi)| \le \kappa_{|\alpha+\beta|} t \langle \xi \rangle^{1-|\alpha|} \quad (\,|\alpha + \beta| \le 1\,), \tag{1.3}$$

$$|\partial_x^\beta \partial_\xi^\alpha \phi(x, \xi)| \le \kappa_{|\alpha+\beta|} t \langle \xi \rangle^{2\rho-1+(1-\rho)|\beta|-\rho|\alpha|} \quad (\,|\alpha + \beta| \ge 2\,). \tag{1.4}$$

Definition 1.3 (Fourier integral operators).

Let $p \in S_\rho^m$ and $\phi \in P_\rho(t, \{\kappa_l\}_{l=0}^\infty)$. We define the Fourier integral operator $I(\phi, p)$ with symbol p and phase function ϕ by

$$I(\phi, p)u(x) = \int_{R^{2n}} e^{i(x-y)\xi + i\phi(x,\xi)} p(x, \xi) u(y) dy đ\xi \quad (\, đ\xi = (2\pi)^{-n} d\xi \,), \tag{1.5}$$

for $u \in \mathcal{S}$, where $\mathcal{S}$ denotes the Schwartz class of rapidly decreasing C^∞-functions on R^n. The integrals of the right-hand side do not necessarily converge absolutely. We understand integrals of this type as oscillatory integrals. For the details about oscillatory integrals, see Chapter 1 §6 *in H. Kumano-go* [9].

Let $I(\phi_j, p_j)$, $j = 1, 2, \ldots, L+1$ be Fourier integral operators. Then, the composite of these Fourier integral operators is given by

$$\begin{aligned} &I(\phi_{L+1}, p_{L+1}) I(\phi_L, p_L) \cdots I(\phi_1, p_1) u(x_{L+1}) \\ &= \int_{R^{2n}} e^{i(x_{L+1} - x_0)\xi_0} K(x_{L+1}, \xi_0) u(x_0) dx_0 đ\xi_0 , \end{aligned} \tag{1.6}$$

where

$$K(x_{L+1}, \xi_0) = \int_{R^{2nL}} e^{i\Phi} \prod_{j=1}^{L+1} p_j(x_j, \xi_{j-1}) \prod_{j=1}^{L} dx_j đ\xi_j , \tag{1.7}$$

and

$$\Phi = \sum_{j=1}^{L} (x_{j+1} - x_j)(\xi_j - \xi_0) + \sum_{j=1}^{L+1} \phi_j(x_j, \xi_{j-1}). \tag{1.8}$$

For simplicity, we will consider a slightly more general oscillatory integrals of the following form:

$$\begin{aligned} &\mathbb{I}(\Phi, p)(x_{L+1}, \xi_0) \\ &= \int_{R^{2nL}} e^{i\Phi} p(x_{L+1}, \xi_L, x_L, \xi_{L-1}, \ldots, x_1, \xi_0) \prod_{j=1}^{L} dx_j đ\xi_j , \end{aligned} \tag{1.9}$$

where $p = p(x_{L+1}, \xi_L, x_L, \xi_{L-1}, \ldots, x_1, \xi_0)$ belongs to the class of multiple symbols $S_\rho^{\bar{m}_{L+1}}$ defined as follows.

Definition 1.4 (Classes of multiple symbols $S_\rho^{\widetilde{m}_{L+1}}$).

Let $\widetilde{m}_{L+1} = (m_{L+1}, m_L, \ldots, m_1) \in \boldsymbol{R}^{L+1}$ and $1/2 \le \rho \le 1$. We say that a C^∞-function $p = p(x_{L+1}, \xi_L, x_L, \xi_{L-1}, \ldots, x_1, \xi_0)$ on $\boldsymbol{R}^{2n(L+1)}$ belongs to the class of multiple symbols $S_\rho^{\widetilde{m}_{L+1}}$, if for any $\widetilde{\alpha} = (\alpha_L, \alpha_{L-1}, \ldots, \alpha_0)$ and $\widetilde{\beta} = (\beta_{L+1}, \beta_L, \ldots, \beta_1)$, there exists a constant $C_{\widetilde{\alpha},\widetilde{\beta}}$ such that

$$\begin{aligned} &\Big|\prod_{j=1}^{L+1} \partial_{x_j}^{\beta_j} \partial_{\xi_{j-1}}^{\alpha_{j-1}} p(x_{L+1}, \xi_L, \ldots, x_1, \xi_0)\Big| \\ &\le C_{\widetilde{\alpha},\widetilde{\beta}} \prod_{j=1}^{L+1} \langle \xi_{j-1} \rangle^{m_j + (1-\rho)|\beta_j| - \rho|\alpha_{j-1}|} . \end{aligned} \tag{1.10}$$

Remark.

(1) For $p \in S_\rho^{\widetilde{m}_{L+1}}$, we define semi-norms $|p|_{l,l'}^{(\widetilde{m}_{L+1})}$, $l, l' = 0, 1, 2, \ldots$ by

$$|p|_{l,l'}^{(\widetilde{m}_{L+1})} = \max_{|\alpha_{j-1}| \le l,\, |\beta_j| \le l'} \sup_{\boldsymbol{R}^{2n(L+1)}} \frac{\Big|\prod_{j=1}^{L+1} \partial_{x_j}^{\beta_j} \partial_{\xi_{j-1}}^{\alpha_{j-1}} p\Big|}{\prod_{j=1}^{L+1} \langle \xi_{j-1} \rangle^{m_j + (1-\rho)|\beta_j| - \rho|\alpha_{j-1}|}} . \tag{1.11}$$

Then $S_\rho^{\widetilde{m}_{L+1}}$ is a Fréchet space with these semi-norms.

(2) For $p_j \in S_\rho^{m_j}$, $j = 1, 2, \ldots, L+1$, if we set

$$p = \prod_{j=1}^{L+1} p_j(x_j, \xi_{j-1}) , \tag{1.12}$$

then we have $p \in S_\rho^{\widetilde{m}_{L+1}}$. Furthermore we have

$$|p|_{l,l'}^{(\widetilde{m}_{L+1})} \le \prod_{j=1}^{L+1} |p_j|_{l,l'}^{(m_j)} . \tag{1.13}$$

Our first main theorem is the following:

Theorem 1.5 (Main theorem 1).

Let $\{\kappa_l\}_{l=0}^\infty$ be an increasing sequence of positive constants and $M \ge 0$.
Set $T = \min\{1/(7\sqrt{n}\kappa_1), 1/(4n\kappa_2)\}$. Then there exists a constant C such that

$$|\mathbb{I}(\Phi, p)| \le C^L |p|_{l_0,l_0'}^{(\widetilde{m}_{L+1})} \langle \xi_0 \rangle^{\sum_{j=1}^{L+1} m_j} , \tag{1.14}$$

for $\sum_{j=1}^{L+1} t_j \le T$, $\sum_{j=1}^{L+1} |m_j| \le M$, $p \in S_\rho^{\widetilde{m}_{L+1}}$ and $\phi_j \in \boldsymbol{P}_\rho(t_j, \{\kappa_l\}_{l=0}^\infty)$, where $l_0 = n+1$, $l_0' = \Big[2M\Big] + 2n + 1$ and the constant C depends only on M and $\{\kappa_l\}_{l=0}^\infty$, but not on L.

In order to state our second main theorem, we state the following proposition.

Proposition 1.6.

Let $\{\kappa_l\}_{l=0}^{\infty}$ *be an increasing sequence of positive constants. Let* $\phi_j \in \boldsymbol{P}_\rho(t_j, \{\kappa_l\}_{l=0}^{\infty})$ *for* $j = 1, 2, \ldots, L+1$. *Assume that* $\sum_{j=1}^{L+1} t_j \le 1/(4n\kappa_2)$.

(1) *For* $(x, \xi) \in \boldsymbol{R}^{2n}$, *the equation*

$$\begin{cases} 0 = -(x_j - x_{j+1}) + \partial_{\xi_j}\phi_{j+1}(x_{j+1}, \xi_j)\,, \\ 0 = -(\xi_j - \xi_{j-1}) + \partial_{x_j}\phi_j(x_j, \xi_{j-1})\,, \\ j = 1, 2, \ldots, L, \quad x_{L+1} = x, \quad \xi_0 = \xi\,, \end{cases} \tag{1.15}$$

has a unique solution $\{x_j, \xi_j\}_{j=1}^{L} = \{x_j^*, \xi_j^*\}_{j=1}^{L}(x, \xi)$.

(2) *There exists an increasing sequence of positive constants* $\{\kappa_l'\}_{l=0}^{\infty}$ *such that*

$$\Phi^* \in \boldsymbol{P}_\rho(\sum_{j=1}^{L+1} t_j, \{\kappa_l'\}_{l=0}^{\infty})\,, \tag{1.16}$$

where $\Phi^*(x, \xi)$ *is the function defined by*

$$\Phi^*(x, \xi) = \sum_{j=1}^{L}(x_{j+1}^* - x_j^*)(\xi_j^* - \xi_0) + \sum_{j=1}^{L+1}\phi_j(x_j^*, \xi_{j-1}^*)\,, \tag{1.17}$$

with $x_{L+1}^* = x$ *and* $\xi_0^* = \xi$.

Our second main theorem is the following:

Theorem 1.7 (Main theorem 2).

Let $\{\kappa_l\}_{l=0}^{\infty}$ *be an increasing sequence of positive constants and* $M \ge 0$.
Set $T = \min\{1/(7\sqrt{n}\kappa_1), 1/(4n\kappa_2)\}$.

(1) *For* $\sum_{j=1}^{L+1} t_j \le T$, $p \in \boldsymbol{S}_\rho^{\bar{m}_{L+1}}$ *and* $\phi_j \in \boldsymbol{P}_\rho(t_j, \{\kappa_l\}_{l=0}^{\infty})$, *set*

$$q(x_{L+1}, \xi_0) = e^{-i\Phi^*(x_{L+1}, \xi_0)}\mathbb{I}(\Phi, p)(x_{L+1}, \xi_0)\,. \tag{1.18}$$

Then $q \in \boldsymbol{S}_\rho^{\bar{m}_{L+1}}$.

(2) *Furthermore, for any* l, l', *there exists a constant* $C_{l,l'}$ *such that*

$$|q|_{l,l'}^{(\sum_{j=1}^{L+1} m_j)} \le (C_{l,l'})^L |p|_{l_1, l_1'}^{(\bar{m}_{L+1})}\,, \tag{1.19}$$

for $\sum_{j=1}^{L+1} t_j \le T$, $\sum_{j=1}^{L+1}|m_j| \le M$, $p \in \boldsymbol{S}_\rho^{\bar{m}_{L+1}}$ *and* $\phi_j \in \boldsymbol{P}_\rho(t_j, \{\kappa_l\}_{l=0}^{\infty})$, *where* $l_1 = n + 1 + l + l'$, $l_1' = \left[2M + 2\rho l + 2(1+\rho)l'\right] + 2n + 1 + l'$ *and the constant* $C_{l,l'}$ *depends only on* M *and* $\{\kappa_l\}_{l=0}^{\infty}$, *but not on* L.

Then H. Kumano-go-Taniguchi theorem follows from the theorem above.

Theorem 1.8 (H. Kumano-go-Taniguchi theorem).

Let $\{\kappa_l\}_{l=0}^{\infty}$ be an increasing sequence of positive constants and $M \geq 0$.
Set $T = \min\{1/(7\sqrt{n}\kappa_1), 1/(4n\kappa_2)\}$.

(1) *For $\sum_{j=1}^{L+1} t_j \leq T$, $p_j \in \boldsymbol{S}_\rho^{m_j}$ and $\phi_j \in \boldsymbol{P}_\rho(t_j, \{\kappa_l\}_{l=0}^{\infty})$, there exists a symbol $q \in \boldsymbol{S}_\rho^{\sum_{j=1}^{L+1} m_j}$ such that*

$$I(\Phi^*, q) = I(\phi_{L+1}, p_{L+1}) I(\phi_L, p_L) \dots I(\phi_1, p_1)\,, \tag{1.20}$$

where Φ^* is the function defined in Proposition 1.6.

(2) *Furthermore, for any l, l', there exists a constant $C_{l,l'}$ such that*

$$|q|_{l,l'}^{(\sum_{j=1}^{L+1} m_j)} \leq (C_{l,l'})^L \prod_{j=1}^{L+1} |p_j|_{l_1,l_1'}^{(m_j)}\,, \tag{1.21}$$

for $\sum_{j=1}^{L+1} t_j \leq T$, $\sum_{j=1}^{L+1} |m_j| \leq M$, $p_j \in \boldsymbol{S}_\rho^{m_j}$ and $\phi_j \in \boldsymbol{P}_\rho(t_j, \{\kappa_l\}_{l=0}^{\infty})$, where $l_1 = n + 1 + l + l'$, $l_1' = \left[2M + 2\rho l + 2(1+\rho)l'\right] + 2n + 1 + l'$ and the constant $C_{l,l'}$ depends only on M and $\{\kappa_l\}_{l=0}^{\infty}$, but not on L.

Remark. Note the condition $\sum_{j=1}^{L+1} t_j \leq T$ on phase functions. In our proof, the right-hand side T of this inequality depends only on κ_1, κ_2 and n. However, in the original proof, T depends on $\kappa_1, \dots, \kappa_l$ and n, with some large integer $l > 2$ depending on n. Therefore, in the original proof, T must be chosen very small.

2. Some Lemmas

In this section, we state two important lemmas needed later. First lemma is found in H. Kumano-go and K. Taniguchi [9],[10].

Lemma 2.1.

Let $A = (a_{jk})$ be an $L \times L$ real matrix. If there exists a constant $0 \leq c < 1$ such that

$$\sum_{k=1}^{L} |a_{jk}| \leq c\,, \tag{2.1}$$

for any $j = 1, 2, \ldots, L$, then we have

$$(1-c)^L \le \det(I - A) \le (1+c)^L , \tag{2.2}$$

where I denotes the unit matrix.

Proof. By induction. See Proposition 5.3 in Chapter 10 §5 of H. Kumano-go [9]. □

Second lemma is slight modification of Proposition 3.3 in D. Fujiwara, N. Kumano-go and K. Taniguchi [5].

Let N and L be positive integers, and $x \in \boldsymbol{R}^N$. For $j = 1, 2, \ldots, L$, let

$$P_j = \sum_{\beta_j \le \gamma_j,\, |\beta_j| \le 1} a_{j,\beta_j}(x) \partial_x^{\beta_j} , \tag{2.3}$$

where $\gamma_j \in \{0,1\}^N \subset \boldsymbol{N}_0^N$ and $a_{j,\beta_j}(x) \in C^\infty(\boldsymbol{R}^N)$. Furthermore we assume the following properties:

1° There exists a positive integer Γ independent of N and of L such that

$$|\gamma_j| \le \Gamma , \tag{2.4}$$

for $j = 1, 2, \ldots, L$.

2° There exists a positive integer K independent of N and of L such that

$$\sharp\Big\{ j = 1, 2, \ldots, L;\ \ \partial_x^{\beta_{L+1}} a_{j,\beta_j}(x) \not\equiv 0 \Big\} \le K , \tag{2.5}$$

for $\beta_j \le \gamma_j$, $|\beta_j| \le 1$, $j = 1, 2, \ldots, L$ and $0 \ne \beta_{L+1} \le \gamma_{L+1}$.

Then we get the following lemma:

Lemma 2.2.

(1) *The product of the operators $P_L P_{L-1} \cdots P_1$ is of the form*

$$\begin{aligned} &P_L P_{L-1} \cdots P_1 \\ &= \sum_{\{\beta_j\}_{j=1}^L}^{\prime} \sum_{\{\alpha_j\}_{j=0}^L}^{\prime\prime} C(\{\beta_j\}_{j=1}^L, \{\alpha_j\}_{j=0}^L) \Big(\prod_{j=1}^L \partial_x^{\alpha_j} a_{j,\beta_j}(x) \Big) \partial_x^{\alpha_0} . \end{aligned} \tag{2.6}$$

Here $\sum'_{\{\beta_j\}_{j=1}^L}$ *is the summation with respect to* $\{\beta_j\}_{j=1}^L$ *such that* $\beta_j \le \gamma_j$ *and* $|\beta_j| \le 1$ *for* $j = 1, 2, \dots, L$. $\sum''_{\{\alpha_j\}_{j=0}^L}$ *is the summation with respect to* $\{\alpha_j\}_{j=0}^L$ *such that* $\sum_{j=0}^L \alpha_j = \sum_{j=1}^L \beta_j$ *and* $\alpha_L = 0$.

(2) *Furthermore, there exists a constant* C *independent of* N *and of* L *such that*

$$\sum'_{\{\beta_j\}_{j=1}^L} \sum''_{\{\alpha_j\}_{j=0}^L} C(\{\beta_j\}_{j=1}^L, \{\alpha_j\}_{j=0}^L) \le C^L . \tag{2.7}$$

We can choose $C \le (1 + \Gamma(K+1))$.

Proof. By induction. See Proposition 3.3 in D. Fujiwara, N. Kumano-go and K. Taniguchi [5]. □

3. Proof of Theorem 1.5

Proof of Theorem 1.5.

1°. From (1.8), for $j = 1, 2, \dots, L$, we have

$$\begin{aligned} \partial_{\xi_j}\Phi &= -(x_j - x_{j+1}) + \partial_{\xi_j}\phi_{j+1}(x_{j+1}, \xi_j) , \\ \partial_{x_j}\Phi &= -(\xi_j - \xi_{j-1}) + \partial_{x_j}\phi_j(x_j, \xi_{j-1}) . \end{aligned} \tag{3.1}$$

Let

$$\begin{aligned} M_j &= \frac{1 - i\langle\xi_j\rangle^{1/2}(\partial_{\xi_j}\Phi)\langle\xi_j\rangle^{1/2}\partial_{\xi_j}}{1 + |\langle\xi_j\rangle^{1/2}(\partial_{\xi_j}\Phi)|^2} , \\ N_j &= \frac{1 - i\langle\xi_{j-1}\rangle^{-1/2}(\partial_{x_j}\Phi)\langle\xi_{j-1}\rangle^{-1/2}\partial_{x_j}}{1 + |\langle\xi_{j-1}\rangle^{-1/2}(\partial_{x_j}\Phi)|^2} . \end{aligned} \tag{3.2}$$

We denote the adjoint operators of M_j and of N_j respectively by M_j^* and by N_j^*. Then we can write

$$\begin{aligned} M_j^* &= a_j^1(x_{j+1}, \xi_j, x_j)\partial_{\xi_j} + a_j^0(x_{j+1}, \xi_j, x_j) , \\ N_j^* &= b_j^1(\xi_j, x_j, \xi_{j-1})\partial_{x_j} + b_j^0(\xi_j, x_j, \xi_{j-1}) , \end{aligned} \tag{3.3}$$

where

$$
\begin{aligned}
a_j^1(x_{j+1},\xi_j,x_j) &= \frac{i\langle\xi_j\rangle^{1/2}(\partial_{\xi_j}\Phi)}{1+|\langle\xi_j\rangle^{1/2}(\partial_{\xi_j}\Phi)|^2}\langle\xi_j\rangle^{1/2}\,,\\
a_j^0(x_{j+1},\xi_j,x_j) &= \frac{1}{1+|\langle\xi_j\rangle^{1/2}(\partial_{\xi_j}\Phi)|^2}\\
&\quad+\partial_{\xi_j}\Big(\frac{i\langle\xi_j\rangle^{1/2}(\partial_{\xi_j}\Phi)}{1+|\langle\xi_j\rangle^{1/2}(\partial_{\xi_j}\Phi)|^2}\langle\xi_j\rangle^{1/2}\Big)\,,
\end{aligned}
\tag{3.4}
$$

and

$$
\begin{aligned}
b_j^1(\xi_j,x_j,\xi_{j-1}) &= \frac{i\langle\xi_{j-1}\rangle^{-1/2}(\partial_{x_j}\Phi)}{1+|\langle\xi_{j-1}\rangle^{-1/2}(\partial_{x_j}\Phi)|^2}\langle\xi_{j-1}\rangle^{-1/2}\,,\\
b_j^0(\xi_j,x_j,\xi_{j-1}) &= \frac{1}{1+|\langle\xi_{j-1}\rangle^{-1/2}(\partial_{x_j}\Phi)|^2}\\
&\quad+\partial_{x_j}\Big(\frac{i\langle\xi_{j-1}\rangle^{-1/2}(\partial_{x_j}\Phi)}{1+|\langle\xi_{j-1}\rangle^{-1/2}(\partial_{x_j}\Phi)|^2}\langle\xi_{j-1}\rangle^{-1/2}\Big)\,.
\end{aligned}
\tag{3.5}
$$

2°. We note the formula $\langle\xi+\eta\rangle\le|\eta|+\langle\xi\rangle$. Then, when $|\xi_j-\xi_{j-1}|\le\frac{1}{2}\langle\xi_{j-1}\rangle$, we have

$$
2^{-1}\langle\xi_{j-1}\rangle\le\langle\xi_j\rangle\le2\langle\xi_{j-1}\rangle\,. \tag{3.6}
$$

And when $|\xi_j-\xi_{j-1}|>\frac{1}{2}\langle\xi_{j-1}\rangle$, we have

$$
\begin{aligned}
|\partial_{x_j}\Phi| &\ge |\xi_j-\xi_{j-1}|-\sqrt{n}\kappa_1 t_j\langle\xi_{j-1}\rangle\\
&\ge(1-2\sqrt{n}\kappa_1 t_j)|\xi_j-\xi_{j-1}|\\
&\ge(1-2\sqrt{n}\kappa_1 T)3^{-1}\langle\xi_j\rangle\,.
\end{aligned}
\tag{3.7}
$$

Using (3.6) and (3.7), we get the following estimates for derivatives of b_j^1 and b_j^0 : For any α_j, β_j, α_{j-1}, there exists a constant $C_{\alpha_j,\beta_j,\alpha_{j-1}}$ independent of j such that

$$
\begin{aligned}
|\partial_{\xi_j}^{\alpha_j}\partial_{x_j}^{\beta_j}\partial_{\xi_{j-1}}^{\alpha_{j-1}}b_j^1(\xi_j,x_j,\xi_{j-1})| &\le C_{\alpha_j,\beta_j,\alpha_{j-1}}\frac{1}{(1+|\langle\xi_{j-1}\rangle^{-1/2}(\partial_{x_j}\Phi)|^2)^{1/2}}\\
&\qquad\cdot\langle\xi_j\rangle^{-|\alpha_j|/2}\langle\xi_{j-1}\rangle^{-1/2+|\beta_j|/2-|\alpha_{j-1}|/2}\,,\\
|\partial_{\xi_j}^{\alpha_j}\partial_{x_j}^{\beta_j}\partial_{\xi_{j-1}}^{\alpha_{j-1}}b_j^0(\xi_j,x_j,\xi_{j-1})| &\le C_{\alpha_j,\beta_j,\alpha_{j-1}}\frac{1}{(1+|\langle\xi_{j-1}\rangle^{-1/2}(\partial_{x_j}\Phi)|^2)^{1/2}}\\
&\qquad\cdot\langle\xi_j\rangle^{-|\alpha_j|/2}\langle\xi_{j-1}\rangle^{|\beta_j|/2-|\alpha_{j-1}|/2}\,.
\end{aligned}
\tag{3.8}
$$

Furthermore, we get the following estimates for derivatives of a_j^1 and a_j^0 :
For any α_j, there exists a constant C_{α_j} independent of j such that

$$
\begin{aligned}
|\partial_{\xi_j}^{\alpha_j} a_j^1(x_{j+1},\xi_j,x_j)| &\le C_{\alpha_j} \frac{1}{(1+|\langle\xi_j\rangle^{1/2}(\partial_{\xi_j}\Phi)|^2)^{1/2}} \langle\xi_j\rangle^{1/2-|\alpha_j|/2}, \\
|\partial_{\xi_j}^{\alpha_j} a_j^0(x_{j+1},\xi_j,x_j)| &\le C_{\alpha_j} \frac{1}{(1+|\langle\xi_j\rangle^{1/2}(\partial_{\xi_j}\Phi)|^2)^{1/2}} \langle\xi_j\rangle^{-|\alpha_j|/2}.
\end{aligned}
\tag{3.9}
$$

3°. We take $\chi \in C_0^\infty(\boldsymbol{R}^n)$ such that

$$
0 \le \chi \le 1 \quad \text{and} \quad \chi(x) = \begin{cases} 1 & (\,|x| \le 1/3\,) \\ 0 & (\,|x| \ge 1/2\,) \end{cases}.
\tag{3.10}
$$

For $R = 0,1,2,\dots,L$ and $0 = j_0 < j_1 < \cdots < j_R < j_{R+1} = L+1$, let

$$
\begin{aligned}
\chi_{j_0,j_1,\dots,j_R} = &\prod_{r=1}^{R+1} \prod_{j=j_{r-1}+1}^{j_r-1} \chi\Big((\xi_j - \xi_{j_{r-1}})/\langle\xi_{j_{r-1}}\rangle\Big) \\
&\cdot \prod_{r=1}^{R} \Big(1 - \chi\Big((\xi_{j_r} - \xi_{j_{r-1}})/\langle\xi_{j_{r-1}}\rangle\Big)\Big).
\end{aligned}
\tag{3.11}
$$

Then we have

$$
\mathbb{I}(\Phi,p) = \sum_{R=0}^{L} \sum_{0=j_0<j_1<\cdots<j_R<j_{R+1}=L+1} \mathbb{I}(\Phi, \chi_{j_0,j_1,\dots,j_R}p).
\tag{3.12}
$$

4°. We consider $\mathbb{I}(\Phi,\chi_{j_0,j_1,\dots,j_R}p)$. Set $J = \Big[2M\Big] + 2n + 1$. Integrating by parts, we have

$$
\mathbb{I}(\Phi,\chi_{j_0,j_1,\dots,j_R}p) = \mathbb{I}(\Phi, p^\circ_{j_0,j_1,\dots,j_R}),
\tag{3.13}
$$

where

$$
\begin{aligned}
p^\circ_{j_0,j_1,\dots,j_R} = &(M_L^*)^{n+1}(M_{L-1}^*)^{n+1}\cdots(M_1^*)^{n+1} \\
&\cdot (N_L^*)^J (N_{L-1}^*)^J \cdots (N_1^*)^J \chi_{j_0,j_1,\dots,j_R} p.
\end{aligned}
\tag{3.14}
$$

Therefore, by Lemma 2.2, there exists a constant C_1 such that

$$
\begin{aligned}
&|p^\circ_{j_0,j_1,\dots,j_R}| \le (C_1)^L |p|^{(\widetilde{m}_{L+1})}_{n+1,J} \langle\xi_0\rangle^{m_1} \\
&\cdot \prod_{r=1}^{R+1} \prod_{j=j_{r-1}+1}^{j_r-1} \Big\{ \frac{1}{(1+\langle\xi_j\rangle|\partial_{\xi_j}\Phi|^2)^{(n+1)/2}} \cdot \frac{1}{(1+\langle\xi_{j-1}\rangle^{-1}|\partial_{x_j}\Phi|^2)^{J/2}} \langle\xi_j\rangle^{m_{j+1}} \Big\} \\
&\cdot \prod_{r=1}^{R} \Big\{ \frac{1}{(1+\langle\xi_{j_r}\rangle|\partial_{\xi_{j_r}}\Phi|^2)^{(n+1)/2}} \cdot \frac{1}{(1+\langle\xi_{j_r-1}\rangle^{-1}|\partial_{x_{j_r}}\Phi|^2)^{J/2}} \langle\xi_{j_r}\rangle^{m_{j_r+1}} \Big\}.
\end{aligned}
\tag{3.15}
$$

5°. For $r = 1, 2, \dots, R+1$ and $j = j_{r-1}+1, j_{r-1}+2, \dots, j_r - 1$, we note that

$$|\xi_j - \xi_{j_{r-1}}| \le \frac{1}{2}\langle \xi_{j_{r-1}} \rangle, \tag{3.16}$$

on the support of $p^\circ_{j_0,j_1,\dots,j_R}$. Using the formula $\langle \xi + \eta \rangle \le |\eta| + \langle \xi \rangle$, we get

$$2^{-1}\langle \xi_{j_{r-1}} \rangle \le \langle \xi_j \rangle \le 2\langle \xi_{j_{r-1}} \rangle, \tag{3.17}$$

for $r = 1, 2, \dots, R+1$ and $j = j_{r-1}+1, j_{r-1}+2, \dots, j_r - 1$ on the support of $p^\circ_{j_0,j_1,\dots,j_R}$. Therefore, there exists a constant C_2 such that

$$\begin{aligned}
&\left|p^\circ_{j_0,j_1,\dots,j_R}\right| \le (C_2)^L |p|^{(\tilde{m}_{L+1})}_{n+1,J} \langle \xi_0 \rangle^{\sum_{j=1}^{L+1} m_j} \\
&\cdot \prod_{j=j_R+1}^{L} \left\{ \frac{1}{(1+\langle \xi_{j_R} \rangle |\partial_{\xi_j}\Phi|^2)^{(n+1)/2}} \cdot \frac{1}{(1+\langle \xi_{j_R} \rangle^{-1} |\partial_{x_j}\Phi|^2)^{J/2}} \right\} \\
&\cdot \prod_{r=1}^{R} \prod_{j=j_{r-1}+1}^{j_r-1} \left\{ \frac{1}{(1+\langle \xi_{j_{r-1}} \rangle |\partial_{\xi_j}\Phi|^2)^{(n+1)/2}} \cdot \frac{1}{(1+\langle \xi_{j_{r-1}} \rangle^{-1} |\partial_{x_j}\Phi|^2)^{(J-2M)/4}} \right\} \\
&\cdot \prod_{r=1}^{R} \frac{1}{(1+\langle \xi_{j_r} \rangle |\partial_{\xi_{j_r}}\Phi|^2)^{(n+1)/2}} \cdot \prod_{r=1}^{R} \langle \xi_{j_r} \rangle^{-(J-2M)/4} \\
&\cdot \prod_{r=1}^{R} \frac{\langle \xi_{j_r} \rangle^{\sum_{j=j_r+1}^{L+1} m_j} \langle \xi_{j_{r-1}} \rangle^{-\sum_{j=j_r+1}^{L+1} m_j}}{\prod_{j=j_{r-1}+1}^{j_r} (1+\langle \xi_{j_{r-1}} \rangle^{-1} |\partial_{x_j}\Phi|^2)^M} \\
&\cdot \prod_{r=1}^{R} \frac{\langle \xi_{j_r} \rangle^{(J-2M)/4}}{\prod_{j=j_{r-1}+1}^{j_r} (1+\langle \xi_{j_{r-1}} \rangle^{-1} |\partial_{x_j}\Phi|^2)^{(J-2M)/4}} .
\end{aligned} \tag{3.18}$$

6°. We note that

$$|\xi_{j_r} - \xi_{j_{r-1}}| \ge \frac{1}{3}\langle \xi_{j_{r-1}} \rangle, \tag{3.19}$$

for $r = 1, 2, \dots, R$ on the support of $p^\circ_{j_0,j_1,\dots,j_R}$. Using the formula $\langle \xi + \eta \rangle \le |\eta| + \langle \xi \rangle$, we have

$$|\xi_{j_r} - \xi_{j_{r-1}}| \ge \frac{1}{4}\langle \xi_{j_r} \rangle, \tag{3.20}$$

for $r = 1, 2, \ldots, R$ on the support of $p^\circ_{j_0,j_1,\ldots,j_R}$. Furthermore, noting (3.17) and (3.19), we have

$$
\begin{aligned}
&\prod_{j=j_{r-1}+1}^{j_r} (1 + \langle \xi_{j_{r-1}} \rangle^{-1} |\partial_{x_j} \Phi|^2)^{1/2} \\
&\geq 2^{-(j_r - j_{r-1})/2} \prod_{j=j_{r-1}+1}^{j_r} (1 + \langle \xi_{j_{r-1}} \rangle^{-1/2} |\partial_{x_j} \Phi|) \\
&\geq 2^{-(j_r - j_{r-1})/2} \langle \xi_{j_{r-1}} \rangle^{-1/2} \sum_{j=j_{r-1}+1}^{j_r} |\partial_{x_j} \Phi| \\
&\geq 2^{-(j_r - j_{r-1})/2} \langle \xi_{j_{r-1}} \rangle^{-1/2} \sum_{j=j_{r-1}+1}^{j_r} \Big(|\xi_j - \xi_{j-1}| - \sqrt{n}\kappa_1 t_j \langle \xi_{j-1} \rangle \Big) \\
&\geq 2^{-(j_r - j_{r-1})/2} \langle \xi_{j_{r-1}} \rangle^{-1/2} \sum_{j=j_{r-1}+1}^{j_r} \Big(|\xi_j - \xi_{j-1}| - 2\sqrt{n}\kappa_1 t_j \langle \xi_{j_{r-1}} \rangle \Big) \\
&\geq 2^{-(j_r - j_{r-1})/2} 3^{-1/2} (1 - 6\sqrt{n}\kappa_1 T) |\xi_{j_r} - \xi_{j_{r-1}}|^{1/2},
\end{aligned}
\tag{3.21}
$$

for $r = 1, 2, \ldots, R$ on the support of $p^\circ_{j_0,j_1,\ldots,j_R}$.

Therefore, there exists a constant C_3 such that

$$
\begin{aligned}
&\left| p^\circ_{j_0,j_1,\ldots,j_R} \right| \leq (C_3)^L |p|^{(\widetilde{m}_{L+1})}_{n+1,J} \langle \xi_0 \rangle^{\sum_{j=1}^{L+1} m_j} \\
&\cdot \prod_{j=j_R+1}^{L} \left\{ \frac{1}{(1 + \langle \xi_{j_R} \rangle |\partial_{\xi_j} \Phi|^2)^{(n+1)/2}} \cdot \frac{1}{(1 + \langle \xi_{j_R} \rangle^{-1} |\partial_{x_j} \Phi|^2)^{J/2}} \right\} \\
&\cdot \prod_{r=1}^{R} \prod_{j=j_{r-1}+1}^{j_r - 1} \left\{ \frac{1}{(1 + \langle \xi_{j_{r-1}} \rangle |\partial_{\xi_j} \Phi|^2)^{(n+1)/2}} \cdot \frac{1}{(1 + \langle \xi_{j_{r-1}} \rangle^{-1} |\partial_{x_j} \Phi|^2)^{(J-2M)/4}} \right\} \\
&\cdot \prod_{r=1}^{R} \frac{1}{(1 + \langle \xi_{j_r} \rangle |\partial_{\xi_{j_r}} \Phi|^2)^{(n+1)/2}} \cdot \prod_{r=1}^{R} \langle \xi_{j_r} \rangle^{-(J-2M)/4}.
\end{aligned}
\tag{3.22}
$$

7°. For $r = 1, 2, \ldots, R+1$ and $j = j_{r-1}+1, j_{r-1}+2, \ldots, j_r - 1$, let

$$
\begin{aligned}
z_j &= \partial_{\xi_j} \Phi = -(x_j - x_{j+1}) + \partial_{\xi_j} \phi_{j+1}(x_{j+1}, \xi_j), \\
\zeta_j &= \partial_{x_j} \Phi = -(\xi_j - \xi_{j-1}) + \partial_{x_j} \phi_j(x_j, \xi_{j-1}).
\end{aligned}
\tag{3.23}
$$

For simplicity, we set $k = j_{r-1} + 1$, $k' = j_r - 1$ and

$$
\begin{aligned}
&\widetilde{x}_{k,k'} = (x_k, x_{k+1}, \ldots, x_{k'}), \quad \widetilde{\xi}_{k,k'} = (\xi_k, \xi_{k+1}, \ldots, \xi_{k'}), \\
&\widetilde{z}_{k,k'} = (z_k, z_{k+1}, \ldots, z_{k'}), \quad \widetilde{\zeta}_{k,k'} = (\zeta_k, \zeta_{k+1}, \ldots, \zeta_{k'}).
\end{aligned}
\tag{3.24}
$$

Then we have

$$
\begin{aligned}
&\frac{\partial(\widetilde{z}_{k,k'},\widetilde{\zeta}_{k,k'})}{\partial(\widetilde{x}_{k,k'},\widetilde{\xi}_{k,k'})}\\
&=-\begin{pmatrix}\Delta_{k'-k+1} & 0\\ 0 & {}^t\Delta_{k'-k+1}\end{pmatrix}+\begin{pmatrix}\Lambda^1_{k,k'} & \Lambda^2_{k,k'}\\ \Lambda^3_{k,k'} & \Lambda^4_{k,k'}\end{pmatrix},
\end{aligned}
\tag{3.25}
$$

where $\Delta_{k'-k+1}$, $\Lambda^1_{k,k'}$, $\Lambda^2_{k,k'}$, $\Lambda^3_{k,k'}$ and $\Lambda^4_{k,k'}$ are $(n(k'-k+1))\times(n(k'-k+1))$ matrices defined by

$$
\Delta_{k'-k+1}=\begin{pmatrix}1 & -1 & 0 & \dots & 0\\ 0 & 1 & -1 & \ddots & \vdots\\ 0 & 0 & 1 & \ddots & 0\\ \vdots & \ddots & \ddots & \ddots & -1\\ 0 & \dots & 0 & 0 & 1\end{pmatrix},
\tag{3.26}
$$

$$
\Lambda^1_{k,k'}=\begin{pmatrix}0 & \partial_{x_{k+1}}\partial_{\xi_k}\phi_{k+1} & 0 & \dots & 0\\ 0 & 0 & \partial_{x_{k+2}}\partial_{\xi_{k+1}}\phi_{k+2} & \ddots & \vdots\\ 0 & 0 & 0 & \ddots & 0\\ \vdots & \ddots & \ddots & \ddots & \partial_{x_{k'}}\partial_{\xi_{k'-1}}\phi_{k'}\\ 0 & \dots & 0 & 0 & 0\end{pmatrix},
\tag{3.27}
$$

$$
\Lambda^2_{k,k'}=\begin{pmatrix}\partial^2_{\xi_k}\phi_{k+1} & 0 & 0 & \dots & 0\\ 0 & \partial^2_{\xi_{k+1}}\phi_{k+2} & 0 & \ddots & \vdots\\ 0 & 0 & \partial^2_{\xi_{k+2}}\phi_{k+3} & \ddots & 0\\ \vdots & \ddots & \ddots & \ddots & 0\\ 0 & \dots & 0 & 0 & \partial^2_{\xi_{k'}}\phi_{k'+1}\end{pmatrix},
\tag{3.28}
$$

$$
\Lambda^3_{k,k'}=\begin{pmatrix}\partial^2_{x_k}\phi_k & 0 & 0 & \dots & 0\\ 0 & \partial^2_{x_{k+1}}\phi_{k+1} & 0 & \ddots & \vdots\\ 0 & 0 & \partial^2_{x_{k+2}}\phi_{k+2} & \ddots & 0\\ \vdots & \ddots & \ddots & \ddots & 0\\ 0 & \dots & 0 & 0 & \partial^2_{x_{k'}}\phi_{k'}\end{pmatrix},
\tag{3.29}
$$

and

$$
\Lambda^4_{k,k'}=\begin{pmatrix}0 & 0 & 0 & \dots & 0\\ \partial_{\xi_k}\partial_{x_{k+1}}\phi_{k+1} & 0 & 0 & \ddots & \vdots\\ 0 & \partial_{\xi_{k+1}}\partial_{x_{k+2}}\phi_{k+2} & 0 & \ddots & 0\\ \vdots & \ddots & \ddots & \ddots & 0\\ 0 & \dots & 0 & \partial_{\xi_{k'-1}}\partial_{x_{k'}}\phi_{k'} & 0\end{pmatrix}.
\tag{3.30}
$$

Furthermore we can write

$$\begin{aligned}
&\det \frac{\partial(\widetilde{z}_{j_{r-1}+1,j_r-1}, \widetilde{\zeta}_{j_{r-1}+1,j_r-1})}{\partial(\widetilde{x}_{j_{r-1}+1,j_r-1}, \widetilde{\xi}_{j_{r-1}+1,j_r-1})} \\
&= (-1)^{2n(j_r-j_{r-1}-1)} \det \begin{pmatrix} \Delta_{j_r-j_{r-1}-1} & 0 \\ 0 & {}^t\Delta_{j_r-j_{r-1}-1} \end{pmatrix} \\
&\cdot \det \left\{ I - \begin{pmatrix} \Lambda^5_{j_{r-1}+1,j_r-1} & \Lambda^6_{j_{r-1}+1,j_r-1} \\ \Lambda^7_{j_{r-1}+1,j_r-1} & \Lambda^8_{j_{r-1}+1,j_r-1} \end{pmatrix} \right\},
\end{aligned} \tag{3.31}$$

where

$$\begin{aligned}
\Lambda^5_{j_{r-1}+1,j_r-1} &= (\Delta_{j_r-j_{r-1}-1})^{-1} \Lambda^1_{j_{r-1}+1,j_r-1}\,, \\
\Lambda^6_{j_{r-1}+1,j_r-1} &= \langle \xi_{j_{r-1}} \rangle \cdot (\Delta_{j_r-j_{r-1}-1})^{-1} \Lambda^2_{j_{r-1}+1,j_r-1}\,, \\
\Lambda^7_{j_{r-1}+1,j_r-1} &= \langle \xi_{j_{r-1}} \rangle^{-1} \cdot ({}^t\Delta_{j_r-j_{r-1}-1})^{-1} \Lambda^3_{j_{r-1}+1,j_r-1}\,, \\
\Lambda^8_{j_{r-1}+1,j_r-1} &= ({}^t\Delta_{j_r-j_{r-1}-1})^{-1} \Lambda^4_{j_{r-1}+1,j_r-1}\,.
\end{aligned} \tag{3.32}$$

Hence, by Lemma 2.1 and (3.17), we have

$$\begin{aligned}
&(1-3n\kappa_2 T)^{2n(j_r-j_{r-1}-1)} \\
&\le \det \frac{\partial(\widetilde{z}_{j_{r-1}+1,j_r-1}, \widetilde{\zeta}_{j_{r-1}+1,j_r-1})}{\partial(\widetilde{x}_{j_{r-1}+1,j_r-1}, \widetilde{\xi}_{j_{r-1}+1,j_r-1})} \\
&\le (1+3n\kappa_2 T)^{2n(j_r-j_{r-1}-1)}\,,
\end{aligned} \tag{3.33}$$

for $r = 1, 2, \ldots, R+1$ on the support of $p^\circ_{j_0,j_1,\ldots,j_R}$.

Therefore, there exists a constant C_4 such that

$$\begin{aligned}
&\left| p^\circ_{j_0,j_1,\ldots,j_R} \prod_{r=1}^{R+1} \det \frac{\partial(\widetilde{x}_{j_{r-1}+1,j_r-1}, \widetilde{\xi}_{j_{r-1}+1,j_r-1})}{\partial(\widetilde{z}_{j_{r-1}+1,j_r-1}, \widetilde{\zeta}_{j_{r-1}+1,j_r-1})} \right| \le (C_4)^L |p|^{(\widetilde{m}_{L+1})}_{n+1,J} \langle \xi_0 \rangle^{\sum_{j=1}^{L+1} m_j} \\
&\cdot \prod_{j=j_R+1}^{L} \left\{ \frac{\langle \xi_{j_R} \rangle^{n/2}}{(1+\langle \xi_{j_R} \rangle |z_j|^2)^{(n+1)/2}} \cdot \frac{\langle \xi_{j_R} \rangle^{-n/2}}{(1+\langle \xi_{j_R} \rangle^{-1} |\zeta_j|^2)^{J/2}} \right\} \\
&\cdot \prod_{r=1}^{R} \prod_{j=j_{r-1}+1}^{j_r-1} \left\{ \frac{\langle \xi_{j_{r-1}} \rangle^{n/2}}{(1+\langle \xi_{j_{r-1}} \rangle |z_j|^2)^{(n+1)/2}} \cdot \frac{\langle \xi_{j_{r-1}} \rangle^{-n/2}}{(1+\langle \xi_{j_{r-1}} \rangle^{-1} |\zeta_j|^2)^{(J-2M)/4}} \right\} \\
&\cdot \prod_{r=1}^{R} \frac{\langle \xi_{j_r} \rangle^{n/2}}{(1+\langle \xi_{j_r} \rangle |x_{j_r} - x_{j_r+1} - \partial_{\xi_{j_r}} \phi_{j_r+1}(x_{j_r+1}, \xi_{j_r})|^2)^{(n+1)/2}} \\
&\cdot \prod_{r=1}^{R} \langle \xi_{j_r} \rangle^{-n/2-(J-2M)/4}\,.
\end{aligned} \tag{3.34}$$

8°. We change the variables :

$$(\widetilde{x}_{j_{r-1}+1,j_r-1},\widetilde{\xi}_{j_{r-1}+1,j_r-1}) \Longrightarrow (\widetilde{z}_{j_{r-1}+1,j_r-1},\widetilde{\zeta}_{j_{r-1}+1,j_r-1}),$$

for $r=1,2,\dots,R+1$. Here we note that x_{j_r+1} is a function depending only on $x_{j_{r+1}}$, $\widetilde{z}_{j_r+1,j_{r+1}-1}$, $\widetilde{\zeta}_{j_r+1,j_{r+1}-1}$ and ξ_{j_r}, for $r=1,2,\dots,R$.

Hence we integrate in the following order. First we integrate by x_{j_1}, x_{j_2}, $\dots$, x_{j_R}. Secondly we integrate by $\widetilde{z}_{j_{r-1}+1,j_r-1}$, $\widetilde{\zeta}_{j_{r-1}+1,j_r-1}$, $r=1,2,\dots,R+1$. Thirdly we integrate by ξ_{j_R}, $\xi_{j_{R-1}}$,$\dots$,ξ_{j_1}. Then there exists a constant C_5 such that

$$|\mathbb{I}(\Phi,p^\circ_{j_0,j_1,\dots,j_R})| \le (C_5)^L |p|^{(\widetilde{m}_{L+1})}_{n+1,J}\langle\xi_0\rangle^{\sum_{j=1}^{L+1} m_j}. \tag{3.35}$$

Therefore we have

$$\begin{aligned}|\mathbb{I}(\Phi,p)| &\le \sum_{R=0}^{L}\sum_{0=j_0<j_1<\dots<j_R<j_{R+1}=L+1} |\mathbb{I}(\Phi,p^\circ_{j_0,j_1,\dots,j_R})| \\ &\le (2C_5)^L |p|^{(\widetilde{m}_{L+1})}_{n+1,J}\langle\xi_0\rangle^{\sum_{j=1}^{L+1} m_j}.\end{aligned} \tag{3.36}$$

□

Furthermore, looking over the proof above again, we can easily get the following corollary.

Corollary 3.1.

Let $\{\kappa_l\}_{l=0}^\infty$ be an increasing sequence of positive constants and $M\ge 0$.
Set $T=\min\{1/(7\sqrt{n}\kappa_1),1/(4n\kappa_2)\}$.

(1) *There exists a constant C such that*

$$|\mathbb{I}(\Phi,\chi_0 p)| \le C^L |p|^{(\widetilde{m}_{L+1})}_{l_0,l_0}\langle\xi_0\rangle^{\sum_{j=1}^{L+1} m_j}, \tag{3.37}$$

for $\sum_{j=1}^{L+1} t_j \le T$, $\sum_{j=1}^{L+1}|m_j|\le M$, $p\in \boldsymbol{S}_\rho^{\widetilde{m}_{L+1}}$ and $\phi_j\in \boldsymbol{P}_\rho(t_j,\{\kappa_l\}_{l=0}^\infty)$, where $l_0=n+1$ and the constant C depends only on M and $\{\kappa_l\}_\infty$, but not on L.

(2) *There exists a constant C such that*

$$|\mathbb{I}(\Phi,\chi_{j_0,j_1,\dots,j_R} p)| \le C^L |p|^{(\widetilde{m}_{L+1})}_{l_0,l_0}\langle\xi_0\rangle^{\sum_{j=1}^{L+1} m_j-(M-\sum_{j=1}^{L+1}|m_j|)}, \tag{3.38}$$

for $\sum_{j=1}^{L+1} t_j \le T$, $\sum_{j=1}^{L+1}|m_j|\le M$, $p\in \boldsymbol{S}_\rho^{\widetilde{m}_{L+1}}$, $\phi_j\in \boldsymbol{P}_\rho(t_j,\{\kappa_l\}_{l=0}^\infty)$, $R=1,2,\dots,L$ and $0=j_0<j_1<\dots<j_R<j_{R+1}=L+1$, where $l_0=n+1$, $l_0'=\left[2M\right]+2n+1$ and the constant C depends only on M and $\{\kappa_l\}_{l=0}^\infty$, but not on L.

4. Proof of Proposition 1.6

Proof of Proposition 1.6.

1°. First we assume that the solution $\{x_j^*, \xi_j^*\}_{j=1}^L$ of (1.15) exists. Then we have

$$\begin{aligned} |\xi_j^* - \xi_{j-1}^*| &\le \sqrt{n}\kappa_1 t_j \langle \xi_{j-1}^* \rangle \\ &\le \sqrt{n}\kappa_1 t_j \Big\{ \sum_{k=1}^{L} |\xi_k^* - \xi_{k-1}^*| + \langle \xi_0 \rangle \Big\}, \end{aligned} \tag{4.1}$$

for $j = 1, 2, \ldots, L$. Hence we get

$$\sum_{j=1}^{L} |\xi_j^* - \xi_{j-1}^*| \le \frac{\sqrt{n}\kappa_1 \sum_{j=1}^{L} t_j}{1 - \sqrt{n}\kappa_1 \sum_{j=1}^{L} t_j} \langle \xi_0 \rangle \le \frac{1}{2} \langle \xi_0 \rangle . \tag{4.2}$$

Therefore, the solution $\{x_j^*, \xi_j^*\}_{j=1}^L$ of (1.15) satisfies

$$|\xi_j^* - \xi_0| \le \frac{1}{2} \langle \xi_0 \rangle , \tag{4.3}$$

for $j = 1, 2, \ldots, L$.

2°. For $(\widetilde{x}_{1,L}, \widetilde{\xi}_{1,L}) \in \boldsymbol{R}^{2nL}$, we introduce the norms $||^t(\widetilde{x}_{1,L}, \widetilde{\xi}_{1,L})||_\infty^{\xi_0}$, $||^t(\widetilde{x}_{1,L}, \widetilde{\xi}_{1,L})||_1^{\xi_0}$ given by

$$\begin{aligned} ||^t(\widetilde{x}_{1,L}, \widetilde{\xi}_{1,L})||_\infty^{\xi_0} &= \max_{j=1,2,\ldots,L} |x_j| + \langle \xi_0 \rangle^{-1} \max_{j=1,2,\ldots,L} |\xi_j| , \\ ||^t(\widetilde{x}_{1,L}, \widetilde{\xi}_{1,L})||_1^{\xi_0} &= \sum_{j=1}^{L} \{ |x_j| + \langle \xi_0 \rangle^{-1} |\xi_j| \} . \end{aligned} \tag{4.4}$$

Let $\Omega_\infty^{\xi_0}$ be the norm space $(\boldsymbol{R}^{2nL}, || \cdot ||_\infty^{\xi_0})$, and $\Omega_1^{\xi_0}$ the norm space $(\boldsymbol{R}^{2nL}, || \cdot ||_1^{\xi_0})$. Let $\Theta_\infty^{\xi_0}$ be the closed set of $\Omega_\infty^{\xi_0}$ given by

$$\Theta_\infty^{\xi_0} = \Big\{ (\widetilde{x}_{1,L}, \widetilde{\xi}_{1,L}) \in \Omega_\infty^{\xi_0} ; \ \ |\xi_j - \xi_0| \le \frac{1}{2} \langle \xi_0 \rangle , \ \ j = 1, 2, \ldots, L \Big\} . \tag{4.5}$$

For $(\widetilde{x}_{1,L}, \widetilde{\xi}_{1,L}) \in \Theta_\infty^{\xi_0}$, we consider the mapping $\mathcal{F} : (\widetilde{x}_{1,L}, \widetilde{\xi}_{1,L}) \longmapsto (\widetilde{y}_{1,L}, \widetilde{\eta}_{1,L})$ defined by

$$^t(\widetilde{y}_{1,L}, \widetilde{\eta}_{1,L}) = \Delta^{-1} \Psi(x_{L+1}, \widetilde{x}_{1,L}, \widetilde{\xi}_{1,L}, \xi_0) , \tag{4.6}$$

where

$$\Delta = \begin{pmatrix} \Delta_L & 0 \\ 0 & {}^t\Delta_L \end{pmatrix}, \tag{4.7}$$

and

$$\begin{aligned}
&\Psi(x_{L+1}, \widetilde{x}_{1,L}, \widetilde{\xi}_{1,L}, \xi_0) \\
&= \begin{pmatrix} 0 \\ \vdots \\ 0 \\ x_{L+1} \\ \xi_0 \\ 0 \\ \vdots \\ 0 \end{pmatrix} + \begin{pmatrix} \partial_{\xi_1}\phi_2(x_2, \xi_1) \\ \partial_{\xi_2}\phi_3(x_3, \xi_2) \\ \vdots \\ \partial_{\xi_L}\phi_{L+1}(x_{L+1}, \xi_L) \\ \partial_{x_1}\phi_1(x_1, \xi_0) \\ \partial_{x_2}\phi_2(x_2, \xi_1) \\ \vdots \\ \partial_{x_L}\phi_L(x_L, \xi_{L-1}) \end{pmatrix}.
\end{aligned} \tag{4.8}$$

For $j = 1, 2, \ldots, L$, we note that

$$2^{-1}\langle \xi_0 \rangle \leq \langle \xi_j \rangle \leq 2\langle \xi_0 \rangle. \tag{4.9}$$

Hence, from (4.6), we have

$$\begin{aligned}
|\eta_j - \xi_0| &\leq \sum_{j=1}^{L} |\partial_{x_j}\phi_j(x_j, \xi_{j-1})| \leq \sum_{j=1}^{L} \sqrt{n}\kappa_1 t_j \langle \xi_{j-1} \rangle \\
&\leq 2\sqrt{n}\kappa_1 \sum_{j=1}^{L} t_j \langle \xi_0 \rangle \leq \frac{1}{2}\langle \xi_0 \rangle,
\end{aligned} \tag{4.10}$$

for $j = 1, 2, \ldots, L$. Therefore, the mapping $\mathcal{F} : \Theta_\infty^{\xi_0} \to \Theta_\infty^{\xi_0}$ is well-defined.

3°. Let

$$\Lambda(x_{L+1}, \widetilde{x}_{1,L}, \widetilde{\xi}_{1,L}, \xi_0) = \begin{pmatrix} \Lambda_{1,L}^1 & \Lambda_{1,L}^2 \\ \Lambda_{1,L}^3 & \Lambda_{1,L}^4 \end{pmatrix}. \tag{4.11}$$

For $(\widetilde{x}_{1,L}, \widetilde{\xi}_{1,L}), (\widetilde{x}'_{1,L}, \widetilde{\xi}'_{1,L}) \in \Theta_\infty^{\xi_0}$, let

$$\begin{aligned}
{}^t(\widetilde{y}_{1,L}, \widetilde{\eta}_{1,L}) &= \Delta^{-1}\Psi(x_{L+1}, \widetilde{x}_{1,L}, \widetilde{\xi}_{1,L}, \xi_0), \\
{}^t(\widetilde{y}'_{1,L}, \widetilde{\eta}'_{1,L}) &= \Delta^{-1}\Psi(x_{L+1}, \widetilde{x}'_{1,L}, \widetilde{\xi}'_{1,L}, \xi_0).
\end{aligned} \tag{4.12}$$

Then we have

$$
\begin{aligned}
&||{}^t(\widetilde{y}'_{1,L},\widetilde{\eta}'_{1,L}) - {}^t(\widetilde{y}_{1,L},\widetilde{\eta}_{1,L})||^{\xi_0}_\infty \\
&\le ||\Delta^{-1}||_{\Omega^{\xi_0}_1\to\Omega^{\xi_0}_\infty} \\
&\cdot \Big|\Big|\int_0^1 \Lambda(x_{L+1},\widetilde{x}_{1,L}+\theta(\widetilde{x}'_{1,L}-\widetilde{x}_{1,L}),\widetilde{\xi}_{1,L}+\theta(\widetilde{\xi}'_{1,L}-\widetilde{\xi}_{1,L}),\xi_0)d\theta\Big|\Big|_{\Omega^{\xi_0}_\infty\to\Omega^{\xi_0}_1} \\
&\cdot ||{}^t(\widetilde{x}'_{1,L},\widetilde{\xi}'_{1,L}) - {}^t(\widetilde{x}_{1,L},\widetilde{\xi}_{1,L})||^{\xi_0}_\infty\,.
\end{aligned}\tag{4.13}
$$

Clearly we have

$$
||\Delta^{-1}||_{\Omega^{\xi_0}_1\to\Omega^{\xi_0}_\infty} \le 1\,. \tag{4.14}
$$

Noting that

$$
\begin{aligned}
&\langle\xi_j+\theta(\xi'_j-\xi_j)\rangle \le (1-\theta)|\xi_j-\xi_0| + \theta|\xi'_j-\xi_0| + \langle\xi_0\rangle\,, \\
&\langle\xi_0\rangle \le (1-\theta)|\xi_j-\xi_0| + \theta|\xi'_j-\xi_0| + \langle\xi_j+\theta(\xi'_j-\xi_j)\rangle\,,
\end{aligned}\tag{4.15}
$$

we have

$$
2^{-1}\langle\xi_0\rangle \le \langle\xi_j+\theta(\xi'_j-\xi_j)\rangle \le 2\langle\xi_0\rangle\,, \tag{4.16}
$$

for $j=1,2,\ldots,L$. Hence we get

$$
\begin{aligned}
&\Big|\Big|\int_0^1 \Lambda(x_{L+1},\widetilde{x}_{1,L}+\theta(\widetilde{x}'_{1,L}-\widetilde{x}_{1,L}),\widetilde{\xi}_{1,L}+\theta(\widetilde{\xi}'_{1,L}-\widetilde{\xi}_{1,L}),\xi_0)d\theta\Big|\Big|_{\Omega^{\xi_0}_\infty\to\Omega^{\xi_0}_1} \\
&\le 3n\kappa_2\sum_{j=1}^{L+1} t_j < 1\,.
\end{aligned}\tag{4.17}
$$

By (4.13), (4.14) and (4.17), $\mathcal{F}$ is a contraction. Hence there exists a unique solution $\{x^*_j,\xi^*_j\}_{j=1}^L \in \Theta^{\xi_0}_\infty$ such that

$$
{}^t(\widetilde{x}^*_{1,L},\widetilde{\xi}^*_{1,L}) = \Delta^{-1}\Psi(x_{L+1},\widetilde{x}^*_{1,L},\widetilde{\xi}^*_{1,L},\xi_0)\,. \tag{4.18}
$$

Therefore, there exists a unique solution $\{x^*_j,\xi^*_j\}_{j=1}^L \in \Theta^{\xi_0}_\infty$ such that

$$
\begin{cases}
0 = -(x^*_j - x^*_{j+1}) + \partial_{\xi_j}\phi_{j+1}(x^*_{j+1},\xi^*_j)\,, \\
0 = -(\xi^*_j - \xi^*_{j-1}) + \partial_{x_j}\phi_j(x^*_j,\xi^*_{j-1})\,, \\
j=1,2,\ldots,L,\quad x^*_{L+1}=x_{L+1},\quad \xi^*_0=\xi_0\,.
\end{cases}\tag{4.19}
$$

4°. Clearly, from (4.19), we have

$$
\begin{aligned}
&|x_j^* - x_{j+1}^*| \le \sqrt{n}\kappa_1 t_{j+1}\,, \\
&|\xi_j^* - \xi_{j-1}^*| \le \sqrt{n}\kappa_1 t_j \langle \xi_{j-1}^* \rangle \le 2\sqrt{n}\kappa_1 t_j \langle \xi_0 \rangle\,,
\end{aligned}
\tag{4.20}
$$

for $j = 1, 2, \ldots, L$. Furthermore, for any α_0, β_{L+1} ($|\alpha_0 + \beta_{L+1}| \ge 1$), there exists a constant $C_{\alpha_0,\beta_{L+1}}$ such that

$$
\begin{aligned}
&|\partial_{x_{L+1}}^{\beta_{L+1}} \partial_{\xi_0}^{\alpha_0} (x_j^* - x_{j+1}^*)| \le C_{\alpha_0,\beta_{L+1}} t_{j+1} \langle \xi_0 \rangle^{-(1-\rho)+(1-\rho)|\beta_{L+1}|-\rho|\alpha_0|}\,, \\
&|\partial_{x_{L+1}}^{\beta_{L+1}} \partial_{\xi_0}^{\alpha_0} (\xi_j^* - \xi_{j-1}^*)| \le C_{\alpha_0,\beta_{L+1}} t_j \langle \xi_0 \rangle^{\rho+(1-\rho)|\beta_{L+1}|-\rho|\alpha_0|}\,,
\end{aligned}
\tag{4.21}
$$

for $j = 1, 2, \ldots, L$. Therefore we get (1.16). $\square$

5. Proof of Theorem 1.7

Proof of Theorem 1.7.

1°. For $R = 0, 1, 2, \ldots, L$ and $0 = j_0 < j_1 < \cdots < j_R < j_{R+1} = L + 1$, let

$$
q_{j_0,j_1,\ldots,j_R} = e^{-i\Phi^*}\mathbb{I}(\Phi, \chi_{j_0,j_1,\ldots,j_R} p)\,. \tag{5.1}
$$

Then we have

$$
q = \sum_{R=0}^{L} \sum_{0=j_0<j_1<\cdots<j_R<j_{R+1}=L+1} q_{j_0,j_1,\ldots,j_R}\,. \tag{5.2}
$$

2°. We consider the case where $R = 0$. We can write

$$
\begin{aligned}
\partial_{\xi_0} q_0(x_{L+1}, \xi_0) &= e^{-i\Phi^*}\mathbb{I}(\Phi, \partial_{\xi_0}(\chi_0 p)) \\
&\quad + e^{-i\Phi^*}\mathbb{I}(\Phi, i(\partial_{\xi_0}\Phi - \partial_{\xi_0}\Phi^*)\chi_0 p)\,.
\end{aligned}
\tag{5.3}
$$

For $j = 1, 2, \ldots, L$, let

$$
\begin{aligned}
z_j &= \partial_{\xi_j}\Phi\,, \\
\zeta_j &= \partial_{x_j}\Phi\,.
\end{aligned}
\tag{5.4}
$$

Then we have

$$
\begin{aligned}
{}^t(\widetilde{z}_{1,L}, \widetilde{\zeta}_{1,L}) &= -\Delta\, {}^t(\widetilde{x}_{1,L}, \widetilde{\xi}_{1,L}) + \Psi(x_{L+1}, \widetilde{x}_{1,L}, \widetilde{\xi}_{1,L}, \xi_0)\,, \\
{}^t(0,0) &= -\Delta\, {}^t(\widetilde{x}^*_{1,L}, \widetilde{\xi}^*_{1,L}) + \Psi(x_{L+1}, \widetilde{x}^*_{1,L}, \widetilde{\xi}^*_{1,L}, \xi_0)\,.
\end{aligned}
\tag{5.5}
$$

Hence we can write

$$
\begin{aligned}
&{}^t(\widetilde{z}_{1,L}, \widetilde{\zeta}_{1,L}) \\
&= \Big(-\Delta + \int_0^1 \Lambda(x_{L+1}, \widetilde{x}^*_{1,L} + \theta(\widetilde{x}_{1,L} - \widetilde{x}^*_{1,L}), \widetilde{\xi}^*_{1,L} + \theta(\widetilde{\xi}_{1,L} - \widetilde{\xi}^*_{1,L}), \xi_0) d\theta \Big) \\
&\cdot {}^t(\widetilde{x}_{1,L} - \widetilde{x}^*_{1,L}, \widetilde{\xi}_{1,L} - \widetilde{\xi}^*_{1,L})\,.
\end{aligned}
\tag{5.6}
$$

Furthermore we have

$$
\begin{aligned}
&\partial_{\xi_0}\Phi - \partial_{\xi_0}\Phi^* \\
&= \Big(1 + \int_0^1 \partial_{x_1}\partial_{\xi_0}\phi_1(x_1^* + \theta(x_1 - x_1^*), \xi_0) d\theta\Big)(x_1 - x_1^*)\,.
\end{aligned}
\tag{5.7}
$$

Hence, by (5.6) and (5.7), there exist functions v_j and w_j, $j = 1, 2, \ldots, L$ such that

$$
\begin{aligned}
\partial_{\xi_0}\Phi - \partial_{\xi_0}\Phi^* &= \sum_{j=1}^{L} v_j z_j + \sum_{j=1}^{L} w_j \zeta_j \\
&= \sum_{j=1}^{L} v_j(\partial_{\xi_j}\Phi) + \sum_{j=1}^{L} w_j(\partial_{x_j}\Phi)\,.
\end{aligned}
\tag{5.8}
$$

Noting that

$$
2^{-1}\langle\xi_0\rangle \le \langle \xi_j^* + \theta(\xi_j - \xi_j^*)\rangle \le 2\langle\xi_0\rangle\,, \tag{5.9}
$$

for $j = 1, 2, \ldots, L$ on the support of χ_0, we get

$$
v_j\chi_0 p \in \boldsymbol{S}_\rho^{\widetilde{m}_{L+1}}\,, \quad w_j\chi_0 p \in \boldsymbol{S}_\rho^{\widetilde{m}_{L+1} - e_j}\,, \tag{5.10}
$$

where $e_j = (\overset{L+1}{0}, \ldots, 0, \overset{j}{1}, 0, \ldots, \overset{1}{0})$. Integrating by parts, we can write

$$
\begin{aligned}
\partial_{\xi_0} q_0(x_{L+1}, \xi_0) &= e^{-i\Phi^*}\mathbb{I}(\Phi, \partial_{\xi_0}(\chi_0 p)) \\
&- e^{-i\Phi^*}\sum_{j=1}^{L}\mathbb{I}(\Phi, \partial_{\xi_j}(v_j\chi_0 p)) - e^{-i\Phi^*}\sum_{j=1}^{L}\mathbb{I}(\Phi, \partial_{x_j}(w_j\chi_0 p))\,.
\end{aligned}
\tag{5.11}
$$

Hence, in general, it is enough to take $l_1 \geq n+1+l+l'$ and $l_1' \geq n+1+l+l'$.

3°. Next we consider the case where $R \neq 0$. We can write

$$\begin{aligned}
&\partial_{\xi_0} q_{j_0,j_1,\dots,j_R}(x_{L+1},\xi_0)\\
&= -i(\partial_{\xi_0}\Phi^*)e^{-i\Phi^*}\mathbb{I}(\Phi,\chi_{j_0,j_1,\dots,j_R}p)\\
&+ e^{-i\Phi^*}\mathbb{I}(\Phi, i(\partial_{\xi_0}\Phi)\chi_{j_0,j_1,\dots,j_R}p + \partial_{\xi_0}(\chi_{j_0,j_1,\dots,j_R}p)).
\end{aligned} \tag{5.12}$$

Integrating by parts, we can write

$$\begin{aligned}
&\partial_{\xi_0} q_{j_0,j_1,\dots,j_R}(x_{L+1},\xi_0)\\
&= -i(\partial_{\xi_0}\Phi^*)e^{-i\Phi^*}\mathbb{I}(\Phi,\chi_{j_0,j_1,\dots,j_R}p)\\
&+ e^{-i\Phi^*}\sum_{j=0}^{L}\mathbb{I}(\Phi, i(\partial_{\xi_j}\phi_{j+1})\chi_{j_0,j_1,\dots,j_R}p))\\
&+ e^{-i\Phi^*}\sum_{j=0}^{L}\mathbb{I}(\Phi, \partial_{\xi_j}(\chi_{j_0,j_1,\dots,j_R}p)).
\end{aligned} \tag{5.13}$$

Hence, in general, it is enough to take $l_1 \geq n+1+l$ and $l_1' \geq \left[2M+2\rho l+2(1+\rho)l'\right]+2n+1+l'$. □

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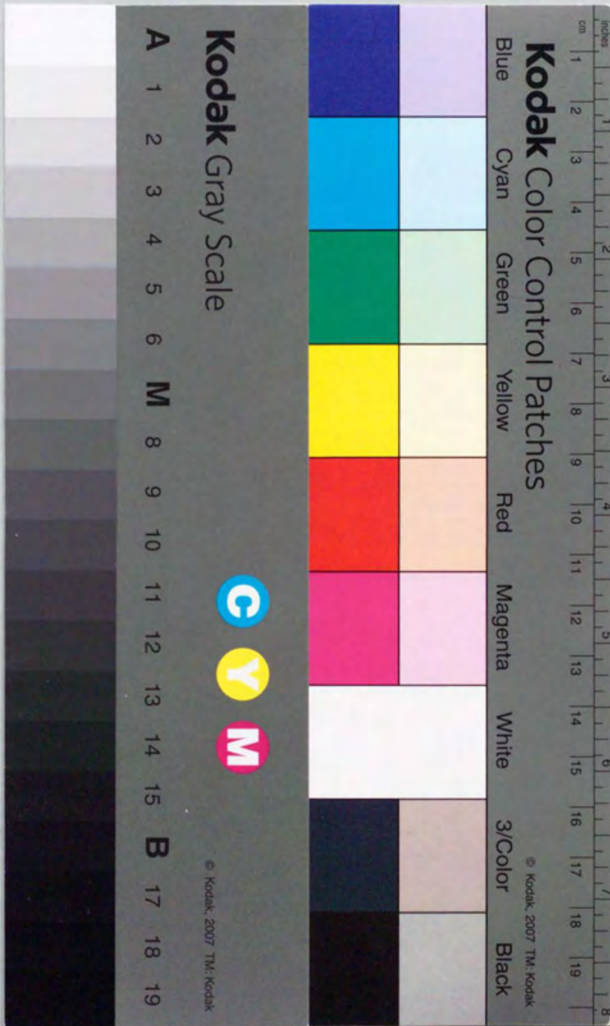
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