論文の内容の要旨

On the difference between positivity and complete positivity of maps in quantum theory
(量子写像における正値性と完全正値性の差異)

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The composability of local systems in quantum theory places a non-trivial restriction on quantum state transformations. Transformations of quantum states have to be completely positive maps that transform quantum states to other valid quantum states even when the maps are applied on a part of entangled states. In contrast, positive maps are guaranteed to transform quantum states to other valid quantum states only when the input states are not entangled to other systems. Positive maps that are not completely positive play significant roles in physics, for example, to represent antiunitary symmetry transformations such as time-reversal and charge-conjugation, to describe non-Markovian quantum processes, and to characterise quantum entanglement. Chapter 2 provides a detailed introduction to positive maps that are not completely positive.

Positive and completely positive maps are also definable in classical probability theory where these two notions define the same class of maps. Therefore the gap between positive and completely positive maps characterises the fundamental difference between quantum and classical probability theories. In this thesis, we consider two kinds of theories that intermediate quantum and classical probability theories, and analyse the gap between positivity and completely positivity in these theories. This procedure broadens our understanding of differences between quantum theory and classical probability theory, and provide new insights into quantum theory itself regarding on positive but not completely positive maps. See Fig. 1 for a schematic representation of the two theories considered in this thesis.

We first consider a variant of quantum theory where the input states of maps are provided with their finite clones. Positive maps that are not completely positive such as state transposition $\rho \mapsto \rho^{\mathrm{T}}$ cannot be realised in quantum mechanics since they do not transform entangled states to valid quantum states. It is still impossible to construct machines to realise the action of positive but not completely positive maps even on the restricted set of quantum states that are not correlated to other systems. We investigate the gap between positivity and completely positivity by analysing the realisability of the action of positive maps on the states uncorrelated to other systems with respect to the number of clones. The gap closes when an infinite number of clones is provided, in the sense that it is possible to extract the classical description of the original state from the clones, and to produce the output state of any positive map. In other words, the action of positive maps on the uncorrelated states become realisable much like classical probability theory, if an infinite number of clones is provided.

In Chapter 3, we show that the gap does not completely close with the finite clones, by proving the necessity of infinitely many clones to realize a certain class of positive maps on the uncorrelated states. We provide two proofs of the necessity of infinite clones, one specifically for the state transposition, and the other for a class of positive maps that includes state transpositions on $d > 2$-dimensional spaces.

In Chapter 4, a special attention is paid on a mapping $\mathrm{T} \circ \Gamma \circ \mathrm{T}$ on quantum channels Γ which we call channel transposition, where T represents the transposition. Channel transposition can be regarded as a positive map on maps, and we show that it is impossible to construct a machine to realise channel transposition from a finite replicas of the unknown channel provided as the oracle. We find, however, realisability of channel transposition on a restricted class of channels, namely unitary transformations, from finite replicas. This result indicates that some of the gap between positivity and complete positivity closes with finite replicas, and also show a clear difference between the state transposition and the channel transposition.

The machine to realise channel transposition on unitary transformations finds its physical interpretation in fermionic systems. Under an identification of antisymmetric subspaces of tensor product Hilbert spaces and fermionic systems, the machine works as if exchanging particles and holes by a unitary operator. The realisability of channel transposition is related to the action of mode transformations on fermions and corresponding holes.

We further provide an application of our method of channel transposition to computation of quantum entanglement. Independently to the analysis on channel transposition, we find a link between the quantities defined by using conjugation operators such as concurrence, and the observables whose expectation values coincides with these quantities. These observables can be used to compute the corresponding quantities without the complete classical description on the state of interest. In particular, our method of channel transposition and its slight generalization can be used to express a known family of concurrence monotones that completely characterize the bipartite pure state entanglement by using conjugation operators, and we rediscover a known set of observables for directly measuring the family of concurrence monotones.

We secondly consider topos quantum theory to generalize positive maps in classical probability theory in Chapters 5 and 6. Topos quantum theory provides representations of quantum states as direct generalizations of the probability distribution, namely probability valuation. Since the analysis on transformations between valuations is currently missing, we employ category theory to find the canonical generalization of positive maps in classical probability theory to topos quantum theory, and investigate the properties of the resulting maps.

Before proceeding to the analysis on maps, we define composite systems in topos quantum theory by generalizing the composite systems in classical probability theory, and analyse the joint valuations therein in Chapter 5, since the defining difference between positivity and complete positivity arises in composite systems. It seems there is no unique generalization of the classical composite systems to topos quantum theory, and our definition of composite system leads to a bijective correspondence between joint valuations and positive over pure tensor states, rather than quantum states. Positive over pure tensor states have close relationship between positive quantum maps from which we deduce that positive quantum maps may all regarded as completely positive in topos quantum theory.

Instead of a direct analysis on the positive maps between valuations, we consider Markov chains in topos quantum theory in Chapter 6, motivated from the fact that classical Markov chains are recursively generated from positive maps applied on shorter Markov chains. Again category theory provides a straightforward generalization of classical Markov chains to topos quantum theory. We show several properties shared by Markov chains of classical probability theory and topos quantum theory. We find, however, an incompatibility between these shared properties and a certain monogamy property of quantum states. The positive over pure tensor states are also shown to have this monogamy property, and the incompatibility trivialize our Markov chains to product states. This consequence reveals that there only exist maps between different marginal systems that do not create any correlation. These maps are anticipated to be too trivial to ask their complete positivity.

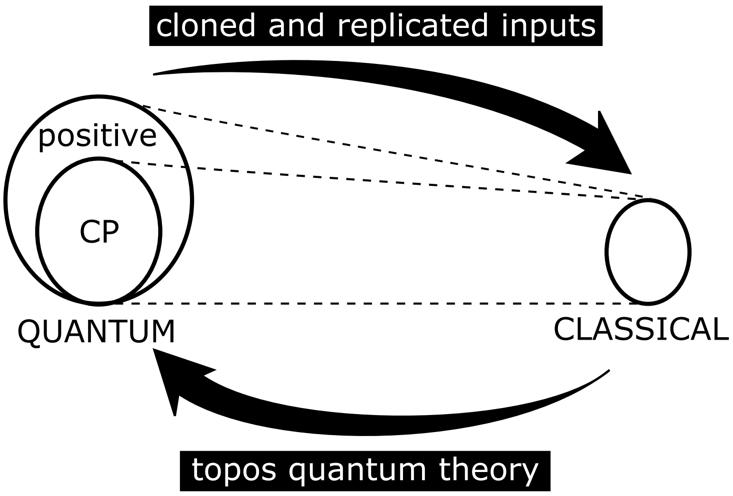

Figure 1: A schematic diagram representing two kinds of theories intermediating quantum and classical probability theories that considered in this thesis.