博士論文

論文題目

Studies on Fano manifolds and vector bundles

(Fano 多様体とベクトル束の研究)

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Contents

Preface

This thesis concerns classification problems of Fano manifolds in certain restricted classes, which are formulated in terms of vector bundles. We will particularly discuss two problems, which can be regarded as generalizations of Mori's solution to the Hartshorne conjecture, which solution asserts that projective manifolds with ample tangent bundles are projective spaces [**Mor79**].

The first one is related to positivity of tangent bundles. Nefness is an algebro-geometric notion of semi-positivity for vector bundles. In 1991, Campana and Peternell [**CP91**] started the study of projective manifolds with nef tangent bundle, and they conjectured that

(1) up to an étale cover, each projective manifold X with nef tangent bundle admits a smooth Fano fibration $f\colon X \to A$ over an Abelian variety A and
(2) the fibers of f or Fano manifolds with nef tangent bundle are rational homogeneous manifolds.

In the paper they also verified their conjectures in dimension up to 3, and later Demailly-Peternell-Schneider proved the first part (1) in arbitrary dimension [**DPS94**]. Nowadays the remaining part (2) of their conjecture is called the *Campana-Peternell conjecture*, which is the subject of Part 1 of this thesis.

The second generalization is stated in terms of adjunction theory. The following condition for pairs $(X, \mathscr{E})$ is firstly introduced by Mukai in his problem list [**Muk88**]: X is a Fano manifold and $\mathscr{E}$ is an ample vector bundle on X whose adjoint bundle $K_X + c_1(\mathscr{E})$ is trivial. A typical example of such pairs is $(\mathbb{P}^n, T_{\mathbb{P}^n})$, hence the classification of such pairs can be regarded as a generalization of Mori's result. Besides, such pairs are related to various objects: Fano manifolds with large index, Fano manifolds with projective bundle structure, generalized polarized pairs.

The rank of Mukai pair is an analogous notion of the index of Fano manifold. Based on the classification of Fano manifolds with large index, Mukai made conjectures on the strucutures of such pairs with $\operatorname{rank} \mathscr{E} \geq \dim X$, and later his conjecture was solved independently by Fujita, Peternell and Ye-Zhang [**Fuj92**, **Pet90**, **Pet91**, **YZ90**]. Peternell-Szurek-Wiśniewski also classified the case $\operatorname{rank} \mathscr{E} = \dim X - 1$ [**PSW92b**], [**Wiś89b**, for $n = 3$] (cf. [**Occ05**]). As for the case $\operatorname{rank} \mathscr{E} = \dim X - 2$, Novelli and Occhetta treated the classification problem when $\dim X = 4$ [**NO07**]. Part 2 of this thesis deals the classification problem of such pairs with $\operatorname{rank} \mathscr{E} = \dim X - 2$ and $\dim X \geq 5$.

This thesis consists of two parts, preceded by a preliminary chapter. In the preliminary chapter, we recall definitions and basic properties of certain vector bundles, which will be used throughout this thesis.

Part 1, whose subject is the Campana-Peternell conjecture for Fano manifolds with large Picard numbers, consists of three chapters with an introduction to this part. We include in Chapter 1 preliminaries on this subject. In preceding works [**CP91**, **CP93**, **Hwa06**, **Mok02**, **Wat14a**, **Kan15**], the Campana-Peternell conjecture is checked up to dimension 5. Generalizing these results, we show in Chapter 2 that the Campana-Peternell conjecture for n-folds with $\rho_X > n - 5$ is true. In Chapter 3, we study the relation between nefness of tangent bundles and extremal rays.

Part 2 consists of one chapter, which deals Mukai's problem on the classification of pairs $(X, \mathscr{E})$ with $\operatorname{rank} \mathscr{E} = \dim X - 2$.

Chapters 2, 3 and 4 are based on papers [**Kan16b**], [**Kan16a**] and [**Kan17**] respectively.

Acknowledgements

First, the author wishes to express his deepest gratitude to Professor Hiromichi Takagi, his supervisor, and Professor Yoichi Miyaoka, his supervisor during the master's program, for their encouragement, comments and suggestions.

Also the author would like to thank Professors Marco Andreatta, Kento Fujita, János Kollár, Gianluca Occhetta, Keiji Oguiso, Thomas Peternell, Eiichi Sato, Luis E. Solá Conde and Kiwamu Watanabe, and Doctors Sho Ejiri, Takeru Fukuoka and Fumiaki Suzuki for various discussions, helpful comments or suggestions.

For Chapter 2, the author is especially grateful to Professor Kiwamu Watanabe for his helpful comments and suggestions, and for sending us his preprint of [**Wat15**].

For Chapter 4, the author would especially like to thank Professor Gianluca Occhetta for his invaluable comments and discussions, particularly about how to use minimal rational curves and minimal sections. Also the author is deeply indebted to Professor Yoichi Miyaoka for helping the author to improve the exposition of the introduction and suggesting the terminology "Mukai pairs".

Main part of Chapter 4 was done during the author's stay at the University of Trento with financial support from the FMSP program at the Graduate School of Mathematical Sciences, the University of Tokyo. The author is also grateful to the institution for the hospitality.

The author is a JSPS Research Fellow and he is supported by the Grant-in-Aid for JSPS fellows (JSPS KAKENHI Grant Number 15J07608). This work was supported by the Program for Leading Graduate Schools, MEXT, Japan.

Preliminaries

The purpose of this preliminary chapter is to fix the notations and to give the definitions of certain vector bundles, which will be used throughout this thesis.

In Sect. 2–6, we recall the definitions and properties of the spinor bundles, the Ottaviani bundles, the null correlation bundle on $\mathbb{P}^3$, the Cayley bundle on $\mathbb{Q}^5$ and the universal bundles on Grassmannians, based on [**OSS80**, **Ott88**, **Ott90**, **EH16**, **Ful98**].

Notations and Conventions.

CONVENTION 0.0.1. We work over the field of complex numbers. Given a vector bundle $\mathscr{E}$ on a manifold Y, we will denote by $\mathbb{P}_Y(\mathscr{E}) = \mathbb{P}(\mathscr{E})$ the Grothendieck projectivization of the vector bundle. For a projectivized vector bundle $\mathbb{P}(\mathscr{E})$, we will denote by $\xi_{\mathscr{E}}$ the relative tautological divisor. A morphism is called a $\mathbb{P}^r$*-bundle* if it is isomorphic to the projection of some projectivized vector bundle. On the other hand, a projective morphism between varieties is called a *smooth* $\mathbb{P}^r$*-fibration*, or simply a $\mathbb{P}^r$*-fibration*, if it is a smooth morphism whose fibers are isomorphic to $\mathbb{P}^r$.

NOTATION 0.0.2. We will use the following notations:

$\mathbb{Q}^n$	smooth hyperquadric of dimension n.
$\mathrm{Gr}(2,5)$	Grassmannian of 2-dimensional subspaces in a 5-dimensional vector space.
V_5	general hyperplane section of $\mathrm{Gr}(2,5)$ embedded into $\mathbb{P}^9$ via the Plücker embedding, which is unique up to isomorphism (see [**Fuj81**]).
$\mathrm{LG}(3,6)$	Lagrangian Grassmannian of three dimensional subspaces in a six dimensional vector space.
$S_k \simeq \mathrm{OG}(k, 2k+1)$	spinor variety, which is isomorphic to the orthogonal Grassmannian of k-dimensional isotropic subspaces in a $(2k+1)$-dimensional vector space.
$\mathrm{Fl}(i_1, \dots, i_m; k)$	variety of flags $(W_1 \subset \cdots \subset W_m)$ in a k-dimensional vector space with $\dim W_m = i_m$.
$\mathscr{S}_{\mathbb{Q}}$	spinor bundle on $\mathbb{Q}^{2k+1}$.
$\mathscr{S}_{\mathbb{Q}} = \mathscr{S}_{\mathbb{Q},i}$ $(i = 1, 2)$	spinor bundles on $\mathbb{Q}^{2k}$.
$\mathscr{G}_{\mathbb{Q}}$	Ottaviani bundle on $\mathbb{Q}^5$ or $\mathbb{Q}^6$.
$\mathscr{C} = \mathscr{C}_{\mathbb{Q}}$	Cayley bundle on $\mathbb{Q}^5$.
$\mathscr{N}$	null-correlation bundle on $\mathbb{P}^3$.
$\mathscr{S}_{\mathrm{Gr}}$ (resp. $\mathscr{Q}_{\mathrm{Gr}}$)	universal subbundle (resp. quotient bundle) on the Grassmannian $\mathrm{Gr}(2,5)$.

$\mathscr{S}_{V_5}$ (resp. $\mathscr{Q}_{V_5}$) restriction of the universal subbundle (resp. quotient bundle) to V_5.

For the definitions and properties of these bundles, we refer the reader to [**OSS80**, **Ott88**, **Ott90**, **EH16**, **Ful98**] and Sections 2–6 below.

1. Preliminaries on hyperquadrics

Let $\mathbb{Q}^n$ be the hyperquadric of dimension $n \geq 3$. The following facts about linear subspaces and the Chow ring of $\mathbb{Q}^n$ can be found in e.g. [**GH78**, Chapter 6], [**LVdV84**, pp. 16–17], [**Ott88**, Sect. 1], [**Fri83**].

1.0.1. *Linear subspaces on hyperquadrics.*

PROPOSITION 0.1.1. *Let $\mathbb{Q}^n$ be the hyperquadric of dimension $n \geq 3$.*

(1) *Assume that $n = 2k$ is even. Then the following hold:*
  (a) *There exists a linear subspace $\mathbb{P}^k$ and each linear subspace $\mathbb{P}^k$ is a maximal linear subspace.*
  (b) *The parameter space of linear subspaces $\mathbb{P}^k$ has two connected components $S_{k,1}$ and $S_{k,2}$.*
  (c) *$S_{k,i}$ is an irreducible component of the orthogonal Grassmannian* $\mathrm{OG}(k+1, 2k+2)$*, and isomorphic to the* spinor variety $S_k \simeq \mathrm{OG}(k, 2k+1)$.
  (d) *S_k is a Fano manifold of dimension $\dfrac{k(k+1)}{2}$ with Picard number one.*

  Thus, for each component of the parameter space, there is the following diagram:

(0.1.1.1)
$$\begin{tikzcd} & \mathscr{U}_i \arrow[dl, "p_i"'] \arrow[dr, "q_i"] & \\ \mathbb{Q}^{2k} & & S_{k,i} \simeq S_k, \end{tikzcd}$$

where $q_i \colon \mathscr{U}_i \to S_{k,i}$ is the universal $\mathbb{P}^k$-bundle and p_i is the evaluation morphism.

(2) *Assume that $n = 2k+1$ is odd. Then the following hold:*
  (a) *There exists a linear subspace $\mathbb{P}^k$ and each linear subspace $\mathbb{P}^k$ is a maximal linear subspace.*
  (b) *The parameter space of linear subspaces $\mathbb{P}^k$ is isomorphic to the spinor variety S_{k+1}.*

  Thus there is the following diagram:

(0.1.1.2)
$$\begin{tikzcd} & \mathscr{U} \arrow[dl, "p"'] \arrow[dr, "q"] & \\ \mathbb{Q}^{2k+1} & & S_{k+1}, \end{tikzcd}$$

where $q \colon \mathscr{U} \to S_{k+1}$ is the universal $\mathbb{P}^k$-bundle and p is the evaluation morphism.

REMARK 0.1.2. It is known that $S_1 \simeq \mathbb{P}^1$, $S_2 \simeq \mathbb{P}^3$ and $S_3 \simeq \mathbb{Q}^6$ (see e.g. [**FH91**, Section 23.3] [**LVdV84**, pp. 16]).

1.0.2. *Chow ring of hyperquadrics.* Let $A(\mathbb{Q}^n) = \bigoplus A^i(\mathbb{Q}^n)$ be the Chow ring of the hyperquadric $\mathbb{Q}^n$ and $H_{\mathbb{Q}} \in A^1(\mathbb{Q}^n)$ the class of the hyperplane section. Then, since $n \geq 3$, $A^1(\mathbb{Q}^n) \simeq \operatorname{Pic}(\mathbb{Q}^n)$ is generated by $H_{\mathbb{Q}}$.

DEFINITION 0.1.3. Let $\mathbb{Q}^n$ be the hyperquadric of dimension $n \geq 3$.

(1) If $n = 2k$ is even, then we will denote by $P_{\mathbb{Q},i} \in A^k(\mathbb{Q}^{2k})$ the class of a k-plane in the family $S_{k,i}$.
(2) If $n = 2k+1$ is odd, then we will denote by $P_{\mathbb{Q}} \in A^{k+1}(\mathbb{Q}^{2k+1})$ the class of a k-plane in the family S_{k+1}.

PROPOSITION 0.1.4. *Let $\mathbb{Q}^n$ be the hyperquadric of dimension $n \geq 3$.*

(1) *Assume that $n = 2k$ is even. Then the Chow ring $A(\mathbb{Q}^{2k})$ is generated by the classes $H_{\mathbb{Q}}$, $P_{\mathbb{Q},1}$ and $P_{\mathbb{Q},2}$ with the following relations:*
   (a) $H_{\mathbb{Q}}^k = P_{\mathbb{Q},1} + P_{\mathbb{Q},2}$,
   (b) $H_{\mathbb{Q}} \cdot P_{\mathbb{Q},1} = H_{\mathbb{Q}} \cdot P_{\mathbb{Q},2}$,
   (c) $P_{\mathbb{Q},1} \cdot P_{\mathbb{Q},2} = \begin{cases} 1 & k \equiv 1 \bmod 2, \\ 0 & k \equiv 0 \bmod 2, \end{cases}$
   (d) $P_{\mathbb{Q},1}^2 = \begin{cases} 0 & k \equiv 1 \bmod 2, \\ 1 & k \equiv 0 \bmod 2. \end{cases}$
(2) *Assume that $n = 2k+1$ is odd. Then the Chow ring $A(\mathbb{Q}^{2k+1})$ is generated by the classes $H_{\mathbb{Q}}$ and $P_{\mathbb{Q}}$ with the following relations:*
   (a) $H_{\mathbb{Q}}^{k+1} = 2P_{\mathbb{Q}}$,
   (b) $H_{\mathbb{Q}}^k \cdot P_{\mathbb{Q}} = 1$.

If $2i \neq n$, then we will identify an element in $A^i(\mathbb{Q}^n) \simeq \mathbb{Z}$ with an integer. An element $aP_{\mathbb{Q},1} + bP_{\mathbb{Q},2} \in A^k(\mathbb{Q}^{2k})$ is identified with the pair of integers (a, b).

2. Spinor bundles on hyperquadrics

We recall the definition and facts about the Spinor bundles on hyperquadrics [**Ott88**].

Since the spinor variety S_k is a Fano manifold of Picard number one, we have $\operatorname{Pic} S_k \simeq \mathbb{Z}$. Let $\mathcal{O}_{S_k}(1)$ be the ample generator of the Picard group.

DEFINITION 0.2.1. Let $\mathbb{Q}^n$ be the hyperquadric of dimension $n \geq 3$.

(1) Assume that $n = 2k$ is even. Then, for $i = 1$ or 2, the *Spinor bundle* $\mathscr{S}_{\mathbb{Q},i}$ is defined as the dual of the vector bundle $p_{i*}q_i^*\mathcal{O}_{S_k}(1)$:
$$\mathscr{S}_{\mathbb{Q},i} := (p_{i*}q_i^*\mathcal{O}_{S_k}(1))^*.$$
We will write it simply $\mathscr{S}_{\mathbb{Q}}$ if no confusion arises.
(2) Assume that $n = 2k+1$ is odd. Then the *Spinor bundle* $\mathscr{S}_{\mathbb{Q}}$ is defined as the dual of the vector bundle $p_*q^*\mathcal{O}_{S_{k+1}}(1)$:
$$\mathscr{S}_{\mathbb{Q}} := (p_*q^*\mathcal{O}_{S_{k+1}}(1))^*.$$

PROPOSITION 0.2.2 ([**Ott88**, Theorem 2.8]). *The dual of the spinor bundle is generated by the global sections.*

Proposition 0.2.3 ([**Ott88**, Remark 2.9]). *Let $\mathbb{Q}^n$ be the hyperquadric of dimension $n = 5$ or 6.*

(1) *If $n = 6$, then:*

$$(c_1(\mathscr{S}_{\mathbb{Q}}), c_2(\mathscr{S}_{\mathbb{Q}}), c_3(\mathscr{S}_{\mathbb{Q}}), c_4(\mathscr{S}_{\mathbb{Q}})) = (-2, 2, (-2, 0), 0) \text{ or } (-2, 2, (0, -2), 0).$$

(2) *If $n = 5$, then:*

$$(c_1(\mathscr{S}_{\mathbb{Q}}), c_2(\mathscr{S}_{\mathbb{Q}}), c_3(\mathscr{S}_{\mathbb{Q}}), c_4(\mathscr{S}_{\mathbb{Q}})) = (-2, 2, -2, 0).$$

3. Ottaviani bundles on $\mathbb{Q}^5$ and $\mathbb{Q}^6$

Based on [**Ott88**, Sect. 3], we define the Ottaviani bundles as follows:

Definition 0.3.1. Let $\mathbb{Q}^n$ be the hyperquadric of dimension $n = 5$ or 6.

(1) The *Ottaviani bundle* $\mathscr{G}_{\mathbb{Q}}$ on $\mathbb{Q}^6$ is defined as a stable vector bundle of rank three with Chern classes

$$(c_1(\mathscr{G}_{\mathbb{Q}}), c_2(\mathscr{G}_{\mathbb{Q}}), c_2(\mathscr{G}_{\mathbb{Q}})) = (2, 2, (2, 0)) \text{ or } (2, 2, (0, 2)).$$

(2) The *Ottaviani bundle* $\mathscr{G}_{\mathbb{Q}}$ on $\mathbb{Q}^5$ is defined as a stable vector bundle of rank three with Chern classes

$$(c_1(\mathscr{G}_{\mathbb{Q}}), c_2(\mathscr{G}_{\mathbb{Q}}), c_3(\mathscr{G}_{\mathbb{Q}})) = (2, 2, 2).$$

Note that there is the following diagram as in (0.1.1.2):

(0.3.1.1)
$$\begin{tikzcd} & \mathscr{U} \arrow[dl, "p"'] \arrow[dr, "q"] & \\ \mathbb{Q}^5 & & S_3 \simeq \mathbb{Q}^6. \end{tikzcd}$$

The pairs $(\mathbb{Q}^6, \mathscr{G}_{\mathbb{Q}})$ and $(\mathbb{Q}^5, \mathscr{G}_{\mathbb{Q}})$ are unique up to isomorphism. In fact we have the following:

Proposition 0.3.2 ([**Ott88**, Sect. 3]). *Let $\mathbb{Q}^n$ be the hyperquadric of dimension $n = 5$ or 6, and let $\mathscr{F}$ be a vector bundle of rank three on $\mathbb{Q}^n$.*

(1) *Assume that $n = 6$. Then the following are equivalent, up to isomorphism of pairs $(\mathbb{Q}^6, \mathscr{F})$.*
   (a) *$\mathscr{F}$ is the Ottaviani bundle.*
   (b) *There is the following exact sequence:*
   $$0 \to \mathcal{O}_{\mathbb{Q}} \to \mathscr{S}^*_{\mathbb{Q}} \to \mathscr{F} \to 0.$$
   (c) *$\mathscr{F} \simeq q_* p^* \mathcal{O}_{\mathbb{Q}^5}(1)$, where the notations are as in* (0.3.1.1)*.*
   (d) *$\mathscr{F}|_{\mathbb{Q}^5}$ is the Ottaviani bundle on $\mathbb{Q}^5$.*

(2) *Assume that $n = 5$. Then the following are equivalent, up to isomorphism of pairs $(\mathbb{Q}^5, \mathscr{F})$.*
   (a) *$\mathscr{F}$ is the Ottaviani bundle.*
   (b) *There is the following exact sequence:*
   $$0 \to \mathcal{O}_{\mathbb{Q}} \to \mathscr{S}^*_{\mathbb{Q}} \to \mathscr{F} \to 0.$$
   (c) *There is an embedding $\mathbb{Q}^5 \subset \mathbb{Q}^6$ as a hyperplane section and $\mathscr{F} = \mathscr{G}_{\mathbb{Q}^6}|_{\mathbb{Q}^5}$.*

Remark 0.3.3. By the above proposition, $\mathscr{G}_{\mathbb{Q}}$ is generated by global sections, the other contraction of $\mathbb{P}(\mathscr{G}_{\mathbb{Q}})$ is defined by the tautological divisor $\xi_{\mathscr{G}_{\mathbb{Q}}}$ and the contraction is of fiber type.

4. Null correlation bundle on projective three space

We use the following as the definition of the null correlation bundle on $\mathbb{P}^3$:

DEFINITION 0.4.1. The *null correlation bundle* $\mathscr{N} = \mathscr{N}_{\mathbb{P}^3}$ on $\mathbb{P}^3$ is defined as a stable vector bundle of rank two with Chern classes.

$$(c_1(\mathscr{N}), c_1(\mathscr{N})) = (0, 1).$$

REMARK 0.4.2. In fact, null correlation bundles are defined on each projective space $\mathbb{P}^{2k+1}$ with odd dimension (see [**OSS80**]). Since we do not need the definition in full generality, we omit the details.

Also the above definition is different from the definition in [**OSS80**]. However those definitions are the same by [**OSS80**, Lemma 4.3.2].

There is the following diagram as in (0.1.1.2):

(0.4.2.1)
$$\begin{tikzcd} & \mathscr{U} \arrow[dl, "p"'] \arrow[dr, "q"] & \\ \mathbb{Q}^3 & & S_2 \simeq \mathbb{P}^3 \end{tikzcd}$$

PROPOSITION 0.4.3 ([**SW90a**, Propositions 2.6 and Proposition 3.4]). *Let $\mathscr{F}$ be a vector bundle of rank two on $\mathbb{P}^3$. Then the following are equivalent, up to isomorphism of pairs $(\mathbb{P}^3, \mathscr{F})$:*

(1) *$\mathscr{F}$ is the null correlation bundle.*
(2) *$\mathscr{F}(1) = q_*p^*\mathcal{O}_{\mathbb{Q}^3}(1)$, where p and q are as in (0.4.2.1).*

5. Cayley bundle on hyperquadric of dimension five

We recall the definition of the Cayley bundle on $\mathbb{Q}^5$ [**Ott90**].

DEFINITION 0.5.1. The *Cayley bundle* $\mathscr{C} = \mathscr{C}_{\mathbb{Q}^5}$ on $\mathbb{Q}^5$ is defined as a stable vector bundle of rank two with Chern classes

$$(c_1(\mathscr{C}), c_1(\mathscr{C})) = (-1, 1).$$

Let $K(G_2)$ be the 5-dimensional contact homogeneous manifold of type G_2. Then there is the following diagram with two $\mathbb{P}^1$-bundles p and q:

(0.5.1.1)
$$\begin{tikzcd} & W \arrow[dl, "p"'] \arrow[dr, "q"] & \\ K(G_2) & & \mathbb{Q}^5, \end{tikzcd}$$

where W is the complete flag variety of type G_2.

PROPOSITION 0.5.2 ([**Ott90**]). *Let $\mathscr{F}$ be a vector bundle of rank two on $\mathbb{Q}^5$. Then the following are equivalent, up to isomorphism of pairs $(\mathbb{Q}^5, \mathscr{F})$:*

(1) *$\mathscr{F}$ is the Cayley bundle.*
(2) *$\mathscr{F}(2) = q_*p^*\mathcal{O}_{K(G_2)}(1)$.*
(3) *There exists the following exact sequence:*

$$0 \to \mathscr{F}(1) \to \mathscr{G}_{\mathbb{Q}} \to \mathcal{O}_{\mathbb{Q}^5}(1) \to 0.$$

Thus W is isomorphic to $\mathbb{P}_{\mathbb{Q}^5}(\mathscr{C})$.

PROPOSITION 0.5.3 ([**Ott90**, Theorem 3.5]). *Let $\mathscr{C}$ be the Cayley bundle on $\mathbb{Q}^5$. Then $\mathscr{C}(1)$ is not nef.*

6. Universal bundles on Grassmannians

Let k and n be integers with $0 < k < n$.

DEFINITION 0.6.1. We will denote by $\mathrm{Gr}(k, n)$ the Grassmannian variety of k-dimensional subspaces in an n-dimensional vector space.

The universal subbundle (resp. quotient bundle) on $\mathrm{Gr}(k, n)$ is denoted by $\mathscr{S}_{\mathrm{Gr}} = \mathscr{S}_{\mathrm{Gr}(k,n)}$ (resp. $\mathscr{Q}_{\mathrm{Gr}} = \mathscr{Q}_{\mathrm{Gr}(k,n)}$).

By definition, there exists the following exact sequence:

$$0 \to \mathscr{S}_{\mathrm{Gr}} \to \mathcal{O}^{\oplus n} \to \mathscr{Q}_{\mathrm{Gr}} \to 0.$$

It is known that $\mathrm{Pic}(\mathrm{Gr}(k, n)) \simeq \mathbb{Z}$. We will denote by $\mathcal{O}_{\mathrm{Gr}}(1)$ the ample generator of $\mathrm{Pic}(\mathrm{Gr}(k, n))$. See e.g. [**EH16**] or [**Ful98**], for the following facts:

PROPOSITION 0.6.2. *The following hold:*

(1) $\mathscr{S}^*_{\mathrm{Gr}}$ *and* $\mathscr{Q}_{\mathrm{Gr}}$ *are globally generated.*
(2) $\det \mathscr{S}^*_{\mathrm{Gr}} = \det \mathscr{Q}_{\mathrm{Gr}} = \mathcal{O}_{\mathrm{Gr}}(1)$.
(3) $T_{\mathrm{Gr}(k,n)} \simeq \mathscr{S}^*_{\mathrm{Gr}} \otimes \mathscr{Q}_{\mathrm{Gr}}$.
(4) $\det T_{\mathrm{Gr}(k,n)} = \mathcal{O}_{\mathrm{Gr}}(n)$.

Part 1

On Fano manifolds with nef tangent bundle

CHAPTER 1

Introduction and Preliminaries on Part 1

Introduction to Part 1

Main theme of this part is the following conjecture, particularly for the case $\rho_X > 1$:

Conjecture 1.0.1 (Campana-Peternell conjecture [**CP91**]). Fano manifolds X with nef tangent bundle are rational homogeneous manifolds.

This conjecture is known to be true in dimension ≤ 5. [**CP91**, **CP93**, **Hwa06**, **Mok02**, **Wat14a**, **Kan15**]. For other results or relevant materials about Conjecture 1.0.1, we refer the reader to the survey article [**MOSC$^+$15**] and references therein.

Let us call a Fano manifold with nef tangent bundle a *CP manifold* for brevity. In [**CP91**, **CP93**, **Wat14a**], Conjecture 1.0.1 for manifolds with $\rho_X > 1$ are discussed inductively as follows: If the Picard of a CP manifold is greater than one, then there is an elementary contraction $f\colon X \to Y$. Demailly-Peternell-Schneider contraction theorem [**DPS94**, Theorem 5.2] asserts that the contraction is smooth and hence the fibers F and image Y are again CP manifolds (see Proposition 1.2.3). Moreover $\rho_F = 1$ and $\dim F < \dim X$. Thus, by results in smaller dimension with Picard number one, we may assume by induction that F is a rational homogeneous manifold. By repeating this procedure, we can reduce the study of X to the study of the composites of elementary contractions ($X \to X_1 \to \cdots \to X_m$) with rational homogeneous fibers.

Thus in the above inductive approach the following condition and questions for Fano manifolds arise:

Condition ($*$). For every sequence of elementary contractions

$$X \xrightarrow{f_1} X_1 \xrightarrow{f_2} \cdots \xrightarrow{f_{m-1}} X_{m-1} \xrightarrow{f_m} X_m,$$

each f_i is a rational homogeneous fibration. Here a contraction is called a *rational homogeneous fibration* if it is smooth and every fiber is a rational homogeneous manifold.

Question 1.0.2 (cf. [**CP92**, Problem 6.4]).

(1) *Does Condition ($*$) imply the nefness of the tangent bundle of X?*
(2) *Does Condition ($*$) imply the homogeneity of X?*

The relation between these conditions are as follows:

$$\begin{tikzcd}
& \text{homogeneous} \arrow[dl, Rightarrow] \arrow[dr, Rightarrow] & \\
\text{nef tangent bundle} \arrow[ur, bend left, "\text{CP conj.}"] \arrow[rr, bend right, "\text{CP conj. for } \rho = 1"'] & & \text{Condition } (*) \arrow[ll, dashed, "\text{Question 1.0.2 (1)}"] \arrow[ul, dashed, bend right, "\text{Question 1.0.2 (2)}"']
\end{tikzcd}$$

The main result of Chapter 2 is the following

THEOREM 1.0.3. *Conjecture 1.0.1 is true if* $\rho_X > \dim X - 5$.

REMARK 1.0.4. Moreover, by the same proof, we see that Fano manifolds with Condition $(*)$ and $\rho_X > \dim X - 5$ are rational homogeneous manifolds (Theorem 3.2.2). Thus the above three conditions are equivalent if $\rho_X > \dim X - 5$. For example these three conditions are equivalent in dimension at most five and Question 1.0.2 is affirmative in dimension at most six (Note that if $\rho_X = 1$, then Question 1.0.2 is trivial).

On the other hand the objective of Chapter 3 is to give a negative answer to Question 1.0.2 and to classify Fano manifolds with Condition $(*)$ when $\rho_X = \dim X - 5$.

Let us observe briefly that Question 1.0.2 (2) is not true in general. In [**Ott88**, **Ott90**], Ottaviani constructed a Fano manifold X_0 with the following condition by using the action of the exceptional group of type G_2 on $\mathbb{Q}^5$:

- X_0 is a Fano 7-fold with $\rho_{X_0} = 2$ and its elementary contractions π and p are $\mathbb{P}^2$-bundles over $\mathbb{Q}^5$:

$$\begin{tikzcd}
& X_0 \arrow[dl, "\pi"'] \arrow[dr, "p"] & \\
\mathbb{Q}^5 & & \mathbb{Q}^5.
\end{tikzcd}$$

  In particular, X_0 satisfies Condition $(*)$.

We can see that X_0 as above is not homogeneous by using the classification of rational homogeneous manifolds, hence Question 1.0.2 (2) is not true in general.

Moreover, in Chapter 3, we will show that Question 1.0.2 (1) is also not true in general:

PROPOSITION 1.0.5. *The tangent bundle* T_{X_0} *is not nef.*

The main result of Chapter 3 shows that the negative answer to Question 1.0.2 when $\rho_X > n - 6$ is essentially given by the above X_0:

THEOREM 1.0.6. *Let* X *be a Fano* n*-fold with Condition* $(*)$ *and* $\rho_X > n - 6$. *Then* X *is either*

(1) *a rational homogeneous manifold or*
(2) $\left(\mathbb{P}^1\right)^{n-7} \times X_0$.

0.1. The outline of this part is as follows: In the rest of this chapter, we will provide preliminaries concerning this part. First, we present some generalities on intersection numbers with the relative anticanonical divisor on a projectivized vector bundle. To study such intersection numbers, we introduce two invariants $d_i(\mathscr{E})$ and $\Delta_i(\mathscr{E})$ of a vector bundle $\mathscr{E}$ as a variant of the definition of the Segre classes and Chern classes. These results are used later to study Fano bundles of rank bigger than three (Chapter 2 Subsection 1.2 and Chapter 3 Subsection 1.2). We also includes in this chapter facts about contractions of Fano manifolds with nef tangent bundle in [**DPS94**,**SCW04**,**MOSCW15**], and we present generalizations of these results to those for Fano manifolds with Condition $(*)$.

In Chapter 2, we will prove Theorem 1.0.3. In Chapter 3, we will prove Proposition 1.0.5 and Theorem 1.0.6.

1. Preliminaries: Classes $d_i(\mathscr{E})$ and $\Delta_i(\mathscr{E})$ for a vector bundle $\mathscr{E}$

In this section we introduce two invariants $d_i(\mathscr{E})$ and $\Delta_i(\mathscr{E})$ of a vector bundle $\mathscr{E}$ as a variant of the definition of Segre classes and Chern classes. Before the definition, we review the definition of Segre classes and Chern classes. For more details we refer the reader to [**Ful98**]. Note that our $\mathbb{P}(\mathscr{E})$ is $P(\mathscr{E}^*)$ in [**Ful98**] and that our odd Segre classes differ in sign from those in [**Ful98**]. Let $\mathscr{E}$ be a vector bundle of rank r on a projective manifold Y. Then the i-th Segre class $s_i(\mathscr{E})$ is defined by the equation

$$s_i(\mathscr{E}) = \pi_*\left(\xi_{\mathscr{E}}^{r-1+i}\right),$$

where $\pi\colon \mathbb{P}(\mathscr{E}) \to Y$ is the natural projection and $\xi_{\mathscr{E}}$ is the tautological divisor. Then the i-th Chern class $c_i(\mathscr{E})$ is defined to be the i-th coefficient of $\left(\sum_{i=0}^{\infty}(-1)^i s_i(\mathscr{E})t^i\right)^{-1}$. It is well known that Chern classes vanish for $i > r$. Hence, by the equations $c_i(\mathscr{E}) = 0$ for $i > r$, we can describe $s_i(\mathscr{E})$ for $i > r$ explicitly by $s_1(\mathscr{E}), \ldots, s_r(\mathscr{E})$. Therefore on the projectivized vector bundle $\mathbb{P}(\mathscr{E})$ of dimension n, the intersection numbers $\xi_{\mathscr{E}}^{n-i}\cdot\pi^*D_1\cdots\pi^*D_i$ are expressed in terms of intersections between $s_1(\mathscr{E}), \ldots, s_r(\mathscr{E})$ and $D_1, \ldots, D_i$ for $0 \leq i \leq n$.

In this manner Segre classes and Chern classes are suitable to describe intersection numbers with the tautological divisor. In some cases, however, the Segre and Chern classes are not adequate to study $\mathbb{P}(\mathscr{E})$ because they change when twisting $\mathscr{E}$ by line bundles. To avoid this, we use the relative anticanonical divisor $-K_\pi$ instead of the tautological divisor $\xi_{\mathscr{E}}$. The relative anticanonical divisor $-K_\pi$ or the "*normalized hyperplane class* $-K_\pi/r$" of a projectivized bundle is used effectively for the first time in Miyaoka's work [**Miy87**] (cf. [**Yas11**]). Nakayama in [**Nak04**, §6.b] defines basically the same invariants as below.

1.1. Definition of $d_i(\mathscr{E})$ and $\Delta_i(\mathscr{E})$. Let Y be a smooth projective variety of positive dimension and $\mathscr{E}$ a vector bundle of rank r on Y. Set $X \coloneqq \mathbb{P}(\mathscr{E})$ and let $\pi\colon X \to Y$ be the natural projection. We will denote by $\xi_{\mathscr{E}}$ the class of the tautological divisor on $\mathbb{P}(\mathscr{E})$. Then we have $-K_\pi = r\,\xi_{\mathscr{E}} - \pi^*c_1(\mathscr{E})$, where $-K_\pi$ is the relative anticanonical divisor for π. Let n be the dimension of X.

By the definition of Segre classes, we have

$$\pi_*\left((-K_\pi)^{r-1+i}\right) = r^{r-1}\sum_{j=0}^{i}(-1)^{i-j}\binom{r-1+i}{r-1+j}r^j\, s_j(\mathscr{E})s_1(\mathscr{E})^{i-j}. \tag{1.1.0.1}$$

Motivated by this, we define the classes $d_i(\mathscr{E})$ and $\Delta_i(\mathscr{E})$ as follows:

Definition 1.1.1. Let the notation be as above.

(1) Set

$$d_i(\mathscr{E}) \coloneqq \frac{\pi_*\left((-K_\pi)^{r-1+i}\right)}{r^{r-1}}.$$

(2) Set

$$d_t(\mathscr{E}) \coloneqq \sum_{i=0}^{\infty} d_i(\mathscr{E})t^i$$

and

$$\Delta_t(\mathscr{E}) \coloneqq d_{-t}(\mathscr{E})^{-1}.$$

Then $\Delta_i(\mathscr{E})$ is defined to be the i-th coefficient of $\Delta_t(\mathscr{E})$.

Remark 1.1.2.

(1) We have

$$d_i(\mathscr{E}) = \sum_{j=0}^{i}(-1)^{i-j}\binom{r-1+i}{r-1+j}r^j\, s_j(\mathscr{E})s_1(\mathscr{E})^{i-j}$$

by (1.1.0.1).

(2) By definition, we have the following for $D_1, D_2 \in N^1(Y)$:

$$(-K_\pi + \pi^*D_1)^i \cdot \pi^*D_2^{n-i} = r^{r-1}\sum_{k=0}^{i}\binom{i}{i-k}d_{k+1-r}(\mathscr{E})\cdot D_1^{i-k}D_2^{n-i}.$$

(3) For later usage, we write down $d_i(\mathscr{E})$ for small i explicitly:
  (a) $d_0(\mathscr{E}) = 1$,
  (b) $d_1(\mathscr{E}) = 0$,
  (c) $d_2(\mathscr{E}) = \dfrac{r(r-1)}{2}c_1(\mathscr{E})^2 - r^2\, c_2(\mathscr{E}) = \dfrac{r}{2}\mathit{\Delta}$, where $\mathit{\Delta}$ is the discriminant of the vector bundle $\mathscr{E}$,
  (d) $d_3(\mathscr{E}) = \dfrac{r(r-1)(r-2)}{3}c_1(\mathscr{E})^3 - r^2(r-2)\, c_1(\mathscr{E})c_2(\mathscr{E}) + r^3\, c_3(\mathscr{E})$.

(4) By definition, we have

$$\Delta_i(\mathscr{E}) = \sum_{\substack{j_1+\cdots+j_k=i,\\ j_l>0}}(-1)^{i-k}d_{j_1}(\mathscr{E})\cdots d_{j_k}(\mathscr{E}). \tag{1.1.2.1}$$

Hence, for small i, classes $\Delta_i(\mathscr{E})$ are written down explicitly as follows:
  (a) $\Delta_0(\mathscr{E}) = 1$,
  (b) $\Delta_1(\mathscr{E}) = 0$,
  (c) $\Delta_2(\mathscr{E}) = -d_2(\mathscr{E})$,
  (d) $\Delta_3(\mathscr{E}) = d_3(\mathscr{E})$,
  (e) $\Delta_4(\mathscr{E}) = -d_4(\mathscr{E}) + d_2(\mathscr{E})^2$,
  (f) $\Delta_5(\mathscr{E}) = d_5(\mathscr{E}) - 2\, d_2(\mathscr{E})d_3(\mathscr{E})$.

We establish a vanishing of $\Delta_i(\mathscr{E})$ and "Grothendieck's relation" for $-K_\pi$ in the next proposition.

PROPOSITION 1.1.3. *$\Delta_i(\mathscr{E}) = 0$ for $i > r$ and*

$$\sum_{i=0}^{r}(-1)^i(-K_\pi)^{r-i}\pi^*\Delta_i(\mathscr{E}) = 0.$$

PROOF. Set

$$\widetilde{\Delta}_i(\mathscr{E}) := \begin{cases} \displaystyle\sum_{k=0}^{i}(-1)^{i-k}\binom{r-k}{i-k}r^k c_k(\mathscr{E})c_1(\mathscr{E})^{i-k} & \text{if } i \le r, \\ 0 & \text{if } i > r. \end{cases}$$

Note that

$$a_{k,j} := \sum_{k\le i\le j}(-1)^{i+j-k}\binom{r-i}{r-j}\binom{r-k}{i-k} = \begin{cases} (-1)^j & \text{if } k = j, \\ 0 & \text{otherwise.} \end{cases}$$

By a direct calculation, we have

$$\begin{aligned} &\sum_{i=0}^{r}(-1)^i(-K_\pi)^{r-i}\pi^*\widetilde{\Delta}_i(\mathscr{E}) \\ &= \sum_{0\le k\le j\le r} a_{k,j}r^{r-j+k}\xi_{\mathscr{E}}^{r-j}\pi^*\left(c_1(\mathscr{E})^{j-k}c_k(\mathscr{E})\right) \\ &= r^r\sum_{i=0}^{r}(-1)^i\xi_{\mathscr{E}}^{r-i}\pi^*c_i(\mathscr{E}). \end{aligned}$$

Hence it is zero by the usual Grothendieck relation.

Therefore, for every nonnegative integer m, we have

$$\sum_{i=0}^{r} d_{m+1-i}(\mathscr{E})\cdot(-1)^i\widetilde{\Delta}_i(\mathscr{E}) = 0.$$

This implies that $\Delta_i(\mathscr{E}) = \widetilde{\Delta}_i(\mathscr{E})$. This completes the proof. □

REMARK 1.1.4. Note that $d_1(\mathscr{E}) = 0$. By the above proposition and Remark 1.1.2 (4), $d_i(\mathscr{E})$ for $i > r$ is written in terms of $d_2(\mathscr{E})\dots, d_r(\mathscr{E})$. For example, if $r = 3$, we have

$$d_4(\mathscr{E}) = d_2(\mathscr{E})^2 \text{ and } d_5(\mathscr{E}) = 2\,d_2(\mathscr{E})\cdot d_3(\mathscr{E}).$$

1.2. Slopes of Fano bundles. In this subsection, we assume that $\rho_Y = 1$ and $\mathscr{E}$ is a Fano bundle, i.e. $\mathbb{P}(\mathscr{E})$ is a Fano manifold. Then Y is also a Fano manifold by [**SW90a**, Theorem 1.6] or [**KMM92a**, Corollary 2.9]. We write $\mathrm{Pic}(Y) = \mathbb{Z}\,H_Y$ with the ample generator H_Y.

DEFINITION 1.1.5 ([**MOSC14**, Definition 2.1]). The *slope* τ for the pair $(Y, \mathscr{E})$ is the real number τ such that $-K_\pi + \tau\,\pi^*H_Y$ is nef but not ample.

Then, by [**MOSC14**, Proposition 2.4], [**KMM92a**, Corollary 2.8], the Kawamata rationality theorem and the Kawamata-Shokurov base point free theorem [**KMM87**, **KM98**], we have the following:

PROPOSITION 1.1.6 ([**MOSC14**, Proposition 2.4 and Remark 2.9]).

(1) *$\tau = 0$ if and only if $X \simeq \mathbb{P}^{r-1} \times Y$.*
(2) *$0 \leq \tau < r_Y$, where r_Y is the Fano index of Y.*
(3) *$\tau \in \mathbb{Q}$.*
(4) *$-K_\pi + \tau\,\pi^* H_Y$ is a semiample divisor and defines another contraction $p\colon X \to Z$.*

Then we have $\kappa(-K_\pi + \tau\,\pi^* H_Y) = \dim Z$, where $\kappa(-K_\pi + \tau\,\pi^* H_Y)$ is the Kodaira dimension of $-K_\pi + \tau\,\pi^* H_Y$. In particular $(-K_\pi + \tau\,\pi^* H_Y)^i \cdot \pi^* H_Y^{n-i} = 0$ for $i > \kappa(-K_\pi + \tau\,\pi^* H_Y)$. Hence we have the following by Remark 1.1.2 (2):

PROPOSITION 1.1.7. *For $i > \kappa(-K_\pi + \tau\,\pi^* H_Y)$, we have*

$$\sum_{k=0}^{i} \binom{i}{i-k} d_{k+1-r}(\mathscr{E}) \cdot H_Y^{n-k}\,\tau^{i-k} = 0.$$

2. Preliminaries: Contractions of CP manifolds

In this section, we collect some results which we use later.

DEFINITION 1.2.1 ([**MOSC$^+$15**, Definition 1.4]). A Fano manifold X is said to be a *CP manifold* if the tangent bundle of X is nef.

CP manifolds with dimension at most five are classified by the works [**CP91**, **CP93**, **CMSB02**, **Hwa06**, **Kan15**, **Miy04**, **Mok02**, **Wat14a**]:

THEOREM 1.2.2. *Let X be a CP manifold of dimension at most five. Then X is a rational homogeneous manifold.*

In this case, the explicit form of X as in the table of Remark 2.3.5 is also known.

Contractions of CP manifolds are similar to those of rational homogeneous manifolds (cf. [**CP91**, Proposition 2.11], [**DPS94**, Theorem 5.2] or [**SCW04**, Theorem 4.4]):

PROPOSITION 1.2.3 ([**MOSCW15**, Proposition 4] or [**MOSC$^+$15**, Section 3]). *Let X be a CP manifold and $\pi\colon X \to Y$ a contraction. Then the following properties hold:*

(1) *The morphism π and Y are smooth. In particular, the fibers and Y are CP manifolds.*
(2) *$\rho_X \leq \dim X$.*
(3) *The Picard number of a π-fiber F is $\rho_X - \rho_Y$ and*

$$j_*\big(\mathrm{NE}(F)\big) = \mathrm{NE}(X) \cap j_*\big(N_1(F)\big),$$

where $j\colon F \to X$ is the inclusion.
(4) *$\mathrm{NE}(X)$ is simplicial.*

Recently, in [**OSCWW17**], a characterization of complete flag manifolds was obtained by G. Occhetta, L.E. Solá Conde, K. Watanabe and J.A. Wiśniewski. We briefly recall the results of [**OSCWW17**]. For details, we refer the reader to [**MOSCW15**], [**OSCWW17**].

Definition 1.2.4 ([**MOSCW15**, Definition 1]). A Fano manifold M is said to be an *FT manifold* if every elementary contraction of M is a smooth $\mathbb{P}^1$-fibration.

Remark 1.2.5. In [**MOSCW15**, Definition 1], an FT manifold M is required to be a CP manifold. Thus the above definition slightly differs from the original definition. However the definition of FT manifolds here coincides with that in [**MOSCW15**] by the following theorem.

Theorem 1.2.6 ([**OSCWW17**, Theorem 1.2]). *An FT manifold M is a complete flag variety, that is, $M \simeq G/B$ where G is a semisimple group and B is a Borel subgroup.*

Furthermore, in [**MOSCW15**], the following property of FT manifolds is proved:

Proposition 1.2.7 ([**MOSCW15**, Proposition 5]). *Let X be a CP manifold. If there exists a contraction $\pi\colon X \to M$ onto an FT manifold M, then $X \simeq F \times M$ and π is the second projection, where F is a fiber of π.*

As a corollary, we have:

Corollary 1.2.8. *Let $n > 0$ and $k \geq 0$ be integers. Assume that Conjecture 1.0.1 is true for any CP manifold Y with $\dim Y < n$ and $\dim Y - \rho_Y \leq k$.*

Then Conjecture 1.0.1 is true for CP n-fold X with $n - \rho_X \leq k$ which admits a contraction $f\colon X \to M$ onto an FT manifold M.

3. Preliminaries: Contractions of Fano manifolds with Condition $(*)$

In this subsection, we generalize some results in Sect. 2 to those for Fano manifolds with Condition $(*)$.

Proposition 1.3.1 (cf. Proposition 1.2.3). *Let X be a Fano manifold with Condition $(*)$ and $\pi\colon X \to Y$ a contraction. Then the following hold:*

(1) *The morphism π is smooth and Y is a Fano manifold with Condition $(*)$.*
(2) $\rho_X \leq \dim X$
(3) $\rho_F = \rho_X - \rho_Y$ *and* $j_*\big(\mathrm{NE}(F)\big) = \mathrm{NE}(X) \cap j_*\big(N_1(F)\big)$ *for a π-fiber F, where $j\colon F \to X$ is the inclusion.*
(4) $\mathrm{NE}(X)$ *is simplicial.*
(5) *The fibers of π are Fano manifolds with Condition $(*)$.*

Proof. (1) We may reduce to the case $\rho(X/Y) = 1$ by induction on $\rho(X/Y)$. The first assertion follows from the definition of $(*)$. Hence Y is a Fano manifold by [**KMM92a**, Corollary 2.9]. Then, since X satisfies Condition $(*)$, Y also satisfies Condition $(*)$.

(2) The assertion follows from the induction on ρ_X.

(3), (4) Note that T_X is g-nef for every elementary contraction g. These follow from the same argument as in [**MOSCW15**, Proposition 4].

(5) By adjunction, F is a Fano manifold. By (3), every elementary contraction of F is induced by the elementary contraction of X. Hence the assertion follows by induction on $\rho(X/Y)$. $\square$

PROPOSITION 1.3.2 (cf. Proposition 1.2.7 for CP manifolds). *Let X be a Fano manifold with Condition $(*)$. Assume that there exists a contraction $\pi\colon X \to M$ onto an FT manifold M. Then $X \simeq F \times M$ and π is the second projection, where F is a fiber of π.*

Note that in the above proposition M is a complete flag manifold by Theorem 1.2.6.

PROOF OF PROPOSITION 1.3.2. The same argument as in the proof of [**MOSCW15**, Proposition 5] does work in this case. $\square$

CHAPTER 2

Fano n-folds with nef tangent bundle and Picard number greater than $n-5$

Introduction

In this chapter we will prove the following:

THEOREM 2.0.1. *Conjecture 1.0.1 is true in dimension n with Picard number $\rho_X > n-5$.*

REMARK 2.0.2. More generally Fano n-folds with Condition $(*)$ and Picard number $\rho_X > n-5$ are rational homogeneous manifolds by the same proof given below (see Theorem 3.2.2).

See Remark 2.3.5 for the explicit form of manifolds in the above theorem.

A similar result in the case where $\rho_X > n-4$ is independently obtained by K. Watanabe [**Wat15**]. The idea of Watanabe's proof and ours in the case where the manifold in question has large dimension are essentially the same; the idea is to use results of R. Muñoz, G. Occhetta, L.E. Solá Conde, K. Watanabe and J.A. Wiśniewski [**MOSCW15**, **OSCWW17**].

We explain the idea in more details: Given a CP n-fold X ($n \geq 6$) with Picard number $\rho_X > n-5$, we have a non-trivial contraction $f\colon X \to Y$. The fibers and the target Y are CP manifolds by Proposition 1.2.3. Furthermore, if the dimension of X is large enough, we can show that they have a contraction onto a CP manifold M whose elementary contractions are smooth $\mathbb{P}^1$-fibrations. Then, by results of R. Muñoz, G. Occhetta, L.E. Solá Conde, K. Watanabe and J.A. Wiśniewski, M is a complete flag variety and $X \simeq F \times M$, where F is a fiber of the contraction. Hence we can prove Theorem 2.0.1 by an inductive approach.

On the other hand, in lower dimensional cases, a CP manifold does not admit a contraction onto an FT manifold in general. Hence we need to treat them by case-by-case argument. The main difficult part is to prove that Conjecture 1.0.1 is true for CP 6-folds X with $\mathbb{P}^r$-bundle structure. This is essentially done in Section 1.

In Section 1, following the notion of Fano bundle, we introduce the notion of *CP bundle*. A vector bundle $\mathscr{E}$ is said to be a *CP bundle* if $\mathbb{P}(\mathscr{E})$ is a CP manifold. In [**MOSC14**], severe restrictions on the pair $(Y, \mathscr{E})$ of Fano bundles are obtained by the numerical conditions on τ (see, for instance, [**MOSC14**, the proof of Proposition 4.4]). Also in the present chapter, the numerical conditions on slopes play an important role.

As a byproduct of the proof of Theorem 2.0.1, we obtained the following:

THEOREM 2.0.3 (=Theorem 2.3.1 and Proposition 3.2.1). *Let X be a CP n-fold (resp. a Fano n-fold with Condition $(*)$). Assume $2\rho_X + 1 \geq n$. Then one of the following holds:*

(1) *$X \simeq Y \times M$, where Y is a CP manifold (resp. a Fano manifold with Condition $(*)$) and M is a complete flag variety.*
(2) *$X \simeq \left(\mathbb{P}^2\right)^{\rho_X}$, $\left(\mathbb{P}^2\right)^{\rho_X-1} \times \mathbb{P}^3$, $\left(\mathbb{P}^2\right)^{\rho_X-1} \times \mathbb{Q}^3$, $\left(\mathbb{P}^2\right)^{\rho_X-2} \times \mathbb{P}_{\mathbb{Q}^4}(\mathscr{S}_{\mathbb{Q},i})$ or $\left(\mathbb{P}^2\right)^{\rho_X-2} \times \mathbb{P}(T_{\mathbb{P}^3})$.*

Note that in the second case X is a rational homogeneous manifold.

1. CP Bundles

A vector bundle $\mathscr{E}$ on a manifold Y is called a *Fano bundle* if the projectivization $\mathbb{P}(\mathscr{E})$ is a Fano manifold, and they are classified in several cases. For more details and results, we refer the reader to [**MOSC14**] and references therein.

DEFINITION 2.1.1. A vector bundle $\mathscr{E}$ on a manifold Y is said to be a *CP bundle* if the projectivization $\mathbb{P}(\mathscr{E})$ is a CP manifold.

REMARK 2.1.2.

(0) If $\mathscr{E}$ is a CP bundle over Y, then Y is a CP manifold by Proposition 1.2.3.
(1) If Y is a rational manifold or a curve, then the Brauer group of Y is trivial and hence all smooth $\mathbb{P}^{r-1}$-fibration over Y is a $\mathbb{P}^{r-1}$-bundle (see, e.g. [**Wat14a**, Proposition 2.5]).

1.1. Triviality of CP bundles.

We prove several characterizations of triviality of CP bundles.

PROPOSITION 2.1.3. *Let $\mathscr{E}$ be a CP bundle of rank r over a manifold Y and $\pi\colon \mathbb{P}(\mathscr{E}) \to Y$ the natural projection. Then the following are equivalent:*

(1) *$\mathbb{P}(\mathscr{E})$ is trivial.*
(2) *The relative anticanonical divisor $-K_\pi$ is nef.*
(3) *For every rational curve $f\colon \mathbb{P}^1 \to Y$, the base change of π by f is trivial.*
(4) *For every rational curve $f\colon \mathbb{P}^1 \to Y$ whose image generates an extremal ray, the base change of π by f is trivial.*
(5) *For every elementary contraction $f\colon Y \to Z$ and every fiber F of f, the base change of π over F is trivial.*
(6) *$\mathscr{E}$ splits into a direct sum of line bundles.*
(7) *$\mathscr{E} \simeq \mathscr{L}^{\oplus r}$ for a line bundle $\mathscr{L}$.*

PROOF. Set $X := \mathbb{P}(\mathscr{E})$.

The implications (1) $\Rightarrow$ (2) $\Rightarrow$ (3) $\Rightarrow$ (4) and (1) $\Rightarrow$ (7) $\Rightarrow$ (6) are obvious.

(4) $\Rightarrow$ (5) and (3) $\Rightarrow$ (1). By Proposition 1.2.3, the fiber F and Y are Fano manifolds. Hence they are rationally connected [**KMM92a**, Theorem 0.1]. The assertion follows from [**MOSC14**, Proposition 2.4].

(5) $\Rightarrow$ (2). It is enough to see that $-K_\pi$ is nef on every extremal ray of $\mathrm{NE}(X)$. Let R be the ray corresponding to π. Obviously $-K_\pi$ is nef on R.

On the other hand, by Proposition 1.2.3, there is a one-to-one correspondence between the set of rays in $\mathrm{NE}(X)$ which are not R and the set of rays of $\mathrm{NE}(Y)$. Hence the assertion follows.

We already see that the first five of the conditions are equivalent. Hence we may assume that Y has Picard number one.

(6) $\Rightarrow$ (1). Since $\mathrm{Pic}(Y) \simeq \mathbb{Z}$, we can write $\mathscr{E} = \bigoplus \mathcal{O}(a_i)$ for some integers $a_1 \leq \cdots \leq a_r$. By twisting with a line bundle, we may assume that $a_1 = \cdots = a_s = 0$ and $a_{s+1} \neq 0$ for some integer $s \geq 1$. If $s < r$, then the relative tautological divisor ξ is nef and big but not ample, which contradicts the fact that every contraction of $\mathbb{P}(\mathscr{E})$ is of fiber type. Hence we have $r = s$, which completes the proof. □

1.2. CP 6-folds which admit projective space bundle structures. We restrict our attention to pairs $(Y, \mathscr{E})$ with $\dim \mathbb{P}(\mathscr{E}) = 6$ and the Picard number of Y is one.

Using the classification of Fano bundles of rank two, we have the following:

PROPOSITION 2.1.4. *Let $\mathscr{E}$ be a CP bundle of rank two over $Y \simeq \mathbb{P}^5$, $\mathbb{Q}^5$ or $K(G_2)$. Then $\mathbb{P}(\mathscr{E})$ is a rational homogeneous manifold. In particular, $X \simeq \mathbb{P}^1 \times \mathbb{P}^5$, $\mathbb{P}^1 \times \mathbb{Q}^5$, $\mathbb{P}^1 \times K(G_2)$ or $\mathbb{P}(\mathscr{C})$, where $\mathscr{C}$ is the Cayley bundle on $\mathbb{Q}^5$.*

PROOF. By Proposition 2.1.3 (6) $\Rightarrow$ (1), we may assume that $\mathscr{E}$ is indecomposable. If $Y \simeq \mathbb{P}^5$ or $\mathbb{Q}^5$, then the assertion follows from [**APW94**, Main Theorem 2.4]. If $Y \simeq K(G_2)$, then another elementary contraction of X is a $\mathbb{P}^1$-bundle by Proposition 1.2.3 and [**MOSC14**, Lemma 6.1]. The assertion follows from [**MOSC14**, Theorem 6.5] or [**Wat14b**, Theorem 1.1]. □

The main result of this section is the following:

THEOREM 2.1.5. *Let $\mathscr{E}$ be a CP bundle of rank 3 on $Y \simeq \mathbb{P}^4$ or $\mathbb{Q}^4$. Then $\mathbb{P}(\mathscr{E}) \simeq \mathbb{P}^2 \times \mathbb{P}^4$ or $\mathbb{P}^2 \times \mathbb{Q}^4$.*

Since X is a CP manifold with Picard number $\rho_X = 2$, we have another elementary contraction $p \colon X \to Z$. Note that Z is a CP manifold and p is a smooth morphism by Proposition 1.2.3. Furthermore, we have $\dim Z \leq 4$. Otherwise, X is a $\mathbb{P}^1$-bundle over a rational homogeneous 5-fold Z. This contradicts Proposition 2.1.4.

PROOF OF THEOREM 2.1.5 FOR THE CASE $Y \simeq \mathbb{P}^4$.

If $Y \simeq \mathbb{P}^4$, then the hyperplane section H_Y generates the Chow ring of Y. We identify each class in $A^i(Y)$ with an integer.

By Proposition 1.1.7 and Remark 1.1.4, we have

$$10\tau^3 + 5d_2(\mathscr{E})\tau + d_3(\mathscr{E}) = 0, \tag{2.1.5.1}$$

$$15\tau^4 + 15d_2(\mathscr{E})\tau^2 + 6d_3(\mathscr{E})\tau + d_2(\mathscr{E})^2 = 0. \tag{2.1.5.2}$$

By using (2.1.5.1) and (2.1.5.2), we have

$$d_2(\mathscr{E}) = \frac{15\tau^2 \pm 9\tau^2\sqrt{5}}{2}.$$

Note that $d_2(\mathscr{E}) \in \mathbb{Q}$. Hence $\tau = 0$, i.e. $-K_\pi$ is nef. Therefore X is isomorphic to $\mathbb{P}^2 \times \mathbb{P}^4$ by Proposition 1.1.6. $\square$

Proof of Theorem 2.1.5 for the case $Y \simeq \mathbb{Q}^4$.

If $Y \simeq \mathbb{Q}^4$, then the hyperplane section H_Y and two planes $P_{1,Y}, P_{2,Y}$ generate the Chow ring of Y. We will denote by L_Y the class of a line on $\mathbb{Q}^4$. The intersection products of them are as follows: $H_Y^2 = P_{1,Y} + P_{2,Y}$, $H_Y \cdot P_{i,Y} = L_Y$, $H_Y \cdot L_Y = 1$, $P_{i,Y}^2 = 1$ and $P_{1,Y} \cdot P_{2,Y} = 0$. We identify the classes $d_i(\mathscr{E})$ (resp. $c_i(\mathscr{E})$) with an integer d_i (resp. c_i), except for $d_2(\mathscr{E})$ (resp. $c_2(\mathscr{E})$), which we identify with a pair of integer (a, b) (resp. (a', b')). Set

$$\delta := d_2(\mathscr{E}).H_Y^2 = a + b,$$
$$\beta := d_2(\mathscr{E})^2 = a^2 + b^2 = \frac{\delta^2 - (a-b)^2}{2}.$$

By twisting $\mathscr{E}$ with a line bundle, we may assume that $c_1 = 1$, 2 or 3.

By Remark 1.1.2 (2) and Remark 1.1.4, we obtain

$$(-K_\pi + \tau\pi^* H_Y)^4 \cdot \pi^* {H_Y}^2 = 9\left(12\tau^2 + \delta\right), \tag{2.1.5.3}$$
$$(-K_\pi + \tau\pi^* H_Y)^5 \cdot \pi^* H_Y = 9\left(20\tau^3 + 5\delta\tau + d_3\right), \tag{2.1.5.4}$$
$$(-K_\pi + \tau\pi^* H_Y)^6 = 9\left(30\tau^4 + 15\delta\tau^2 + 6d_3\tau + \beta\right). \tag{2.1.5.5}$$

By Proposition 1.1.7 and $\dim Z \leq 4$, we have two equalities:

$$20\tau^3 + 5\delta\tau + d_3 = 0, \tag{2.1.5.6}$$
$$30\tau^4 + 15\delta\tau^2 + 6d_3\tau + \beta = 0. \tag{2.1.5.7}$$

By (2.1.5.6) and (2.1.5.7) and the definition of β, we have

$$\delta = 15\,\tau^2 \pm \sqrt{405\tau^4 - (a-b)^2}. \tag{2.1.5.8}$$

By Proposition 1.1.6, it suffices to show that $\tau = 0$. Assume by contradiction that $\tau \neq 0$ in the sequel.

Lemma 2.1.6. *The following hold:*

(1) $\tau = 1$, 2 *or* 3,
(2) $|a - b| = 9\tau^2$ *or* $18\tau^2$,
(3) $\delta = 15\tau^2 \pm \dfrac{162\tau^4}{|a-b|}$.

Proof. (1) By Proposition 1.1.6, it is enough to see that $\tau \in \mathbb{Z}$. Note that τ is a solution of the equations (2.1.5.6) and (2.1.5.7). Since each coefficient in (2.1.5.6) or (2.1.5.7) is integer and $\tau \in \mathbb{Q}$ by Proposition 1.1.6, we can write $\tau = m/10$ with $m \in \mathbb{Z}$. Then, by (2.1.5.7),

$$3m^4 = 10\left(15\delta m^2 + 60d_3 m + 100\beta\right).$$

So we have $m \equiv 0 \mod 10$.

(2) By (2.1.5.8), there exists an integer k such that $405\tau^4 - (a-b)^2 = k^2$. Therefore, for each $\tau = 1$, 2 or 3, we have (2).

Now, the assertion (3) follows from (2.1.5.8). $\square$

In any case, we have $(-K_\pi + \tau\pi^* H_Y)^4 \cdot \pi^* {H_Y}^2 = 9\left(12\tau^2 + \delta\right) > 0$. Hence $\dim Z = 4$. Therefore, $Z \simeq \mathbb{P}^4$ or $\mathbb{Q}^4$ and p is a $\mathbb{P}^2$-bundle by classification of CP surfaces and 4-folds. If $Z \simeq \mathbb{P}^4$, then we may apply Theorem 2.1.5 for $p\colon X \to Z$, which we have already shown. Then we have $X \simeq \mathbb{P}^2 \times \mathbb{P}^4$, a contradiction. Hence $Z \simeq \mathbb{Q}^4$:

$$\begin{tikzcd} & X \arrow[dl, "\pi"'] \arrow[dr, "p"] & \\ Y \simeq \mathbb{Q}^4 & & Z \simeq \mathbb{Q}^4. \end{tikzcd}$$

There exists a rank 3 vector bundle $\mathscr{F}$ over $Z \simeq \mathbb{Q}^4$ such that $X \simeq \mathbb{P}(\mathscr{F})$. We may assume that $c_1(\mathscr{F}) = 1$, 2 or 3. We will denote by $\xi_{\mathscr{E}}$ (resp. $\xi_{\mathscr{F}}$) the class of tautological bundle on $\mathbb{P}(\mathscr{E})$ (resp. $\mathbb{P}(\mathscr{F})$).

LEMMA 2.1.7. *The following hold:*

(1) $\tau = c_1$. *In particular, $\mathscr{E}$ is nef but not ample.*
(2) $c_2(\mathscr{E}) = \left({c_1}^2, 0\right)$ *or* $\left(0, {c_1}^2\right)$, *and* $c_3 = 0$.
(3) $9\left(12\tau^2 + \delta\right) = 3^4 {c_1}^2$.

PROOF. (1) Note that $\mathscr{E}$ is nef but not ample if and only if $\tau = c_1$.

Assume to the contrary that $\tau \neq c_1$. Then, since $|\tau - c_1| = 1$ or 2, $-K_\pi + \tau\pi^* H_Y$ is not a multiple of another divisor. Hence we have

$$-K_\pi + \tau\pi^* H_Y = p^* H_Z, \tag{2.1.7.1}$$

where H_Z is the ample generator of $\mathrm{Pic}(Z)$.

Since $-K_Z = 4H_Z$, we have $p^*\left(-K_Z\right) = 12\xi_{\mathscr{E}} + (4\tau - 4c_1)\,\pi^* H_Y$. Hence, we have

$$-K_p = -K_X - p^*\left(-K_Z\right) = -3p^* H_Z + (4 - \tau)\,\pi^* H_Y.$$

Therefore the slope τ_Z for the pair $(Z, \mathscr{F})$ is 3.

Then, by Lemma 2.1.6 (2) and (3) for $(Z, \mathscr{F})$, we have

$$\delta_Z := d_2(\mathscr{F}) \cdot {H_Z}^2 = (3c_1(\mathscr{F})^2 - 9c_2(\mathscr{F})).H_Z^2 \equiv 0 \mod 9.$$

Hence $c_1(\mathscr{F}) = 3$.

Since $\tau_Z = c_1(\mathscr{F})$, the divisor $\xi_{\mathscr{F}}$ is nef but not ample. Hence $\xi_{\mathscr{F}} = \pi^* H_Y$. By (2.1.7.1),

$$3\xi_{\mathscr{E}} = (c_1 - \tau)\,\xi_{\mathscr{F}} + p^* H_Z.$$

This contradicts the fact that $(\xi_{\mathscr{F}}, p^* H_Z)$ is a $\mathbb{Z}$-basis of $\mathrm{Pic}(X)$.

(2) Note that $c_2(\mathscr{E}) = a' P_{1,Y} + b' P_{2,Y}$. By (1), we have $a' \geq 0$, $b' \geq 0$ and $\tau = c_1$. Also we have

$$|a - b| = 9\left|a' - b'\right|$$

and

$$\delta = 6c_1^2 - 9(a' + b').$$

Then, by Lemma 2.1.6 and the definition of δ, we have the first assertion. The second assertion follows from the equation (2.1.5.6).

(3) The assertion follows since $\delta = -\tau^2$. $\square$

Then, by Lemma 2.1.7 (1) and the symmetry of $\pi\colon X \to Y$ and $p\colon X \to Z$, we have $\xi_{\mathscr{E}} = p^*H_Z$, $\xi_{\mathscr{F}} = \pi^*H_Y$ and $-K_\pi + \tau\pi^*H_Y = 3\xi_{\mathscr{E}}$. Hence

$$\begin{aligned}
(p^*H_Z)^4 \cdot {\xi_{\mathscr{F}}}^2 &= {\xi_{\mathscr{E}}}^4 \cdot {\pi^*H_Y}^2 \\
&= \frac{(-K_\pi + \tau\pi^*H_Y)^4 \cdot {\pi^*H_Y}^2}{3^4} \\
&= \frac{27\left(4\tau^2 + \delta\right)}{3^4} \text{ by (2.1.5.3)} \\
&= c_1^2.
\end{aligned}$$

The last equation follows from Lemma 2.1.7 (3).

By Lemma 2.1.6 (1), we have $c_1^2 = 1$, 4 or 9. On the other hand, since p is a $\mathbb{P}^2$-bundle over $\mathbb{Q}^4$, we have $(p^*H_Z)^4 . {\xi_{\mathscr{F}}}^2 = 2$. This gives a contradiction. Hence $\tau = 0$, completing the proof. □

2. Products of CP manifolds

In this section, we prove Proposition 2.2.1 below, which we use to prove that a certain CP manifold is a product of CP manifolds.

PROPOSITION 2.2.1. *Let X be a smooth projective variety and suppose that, for some integer $r \geq 2$, there exist a smooth contraction $f\colon X \to Y$ of relative dimension $r-1$ and another contraction $g\colon X \to Z$ onto an $r-1$-dimensional manifold Z.*

Assume that g does not contract any curve contained in an f-fiber, and assume moreover that one of the following holds:

(1) *f is a smooth $\mathbb{P}^{r-1}$-fibration, or*
(2) *every f-fiber is a smooth hyperquadric of dimension $r-1 \geq 3$.*

Then $-K_f$ is nef.

REMARK 2.2.2. If f is a $\mathbb{P}^{r-1}$-*bundle* and X is a Fano manifold, then the proposition follows from [**NO07**, Lemma 4.1].

PROOF. In any case, f is a smooth Fano contraction with fibers $\mathbb{P}^n$ or $\mathbb{Q}^n$. Hence f is an elementary Fano contraction, for instance by [**Wiś91b**, Proposition 1.3]. Thus $\rho_X = \rho_Y + 1$ by [**KMM87**, Lemma 3.5]. Note that, for any f-fiber F, $g|_F$ is a finite surjective morphism since g does not contract any F-fiber and $\rho_F = 1$.

Take a curve C in X and its normalization $\widetilde{C} \to C$. Then by base changing the morphism f over $\widetilde{C}$, we have the following diagram with a section s of $f_{\widetilde{C}}$:

$$\begin{tikzcd}
X_{\widetilde{C}} \arrow[r, "h"] \arrow[d, "f_{\widetilde{C}}"'] & X \arrow[r, "g"] \arrow[d, "f"] & Z \\
\widetilde{C} \arrow[r] & Y &
\end{tikzcd}$$

Let $X_{\widetilde{C}} \xrightarrow{g_{\widetilde{C}}} Z_{\widetilde{C}} \to Z$ be the Stein factorization of $g \circ h$. Then $\dim Z_{\widetilde{C}} = r-1$ since $g|_F$ is a finite surjective morphism for any f-fiber F. Because $-K_{f_{\widetilde{C}}}.s(\widetilde{C}) = -K_f.C$, we may assume that Y is a smooth projective curve.

We introduce an invariant τ which satisfies $-K_f + \tau F$ is nef but not ample, where F is the numerical equivalence class of an f-fiber (cf. Definition 1.1.5). Because the Picard number of X is two, the nef cone of X is spanned by $-K_f + \tau F$ and F. Since $\dim X > \dim Z$, the contraction g is defined by some nef but not ample divisor. Hence, by our assumption, $-K_f + \tau F \equiv_{\mathrm{num}} g^*D$ for some ample divisor $D \in N^1(Z)$.

First we treat the case where f is a smooth $\mathbb{P}^{r-1}$-fibration. In this case f is a smooth $\mathbb{P}^{r-1}$-bundle (see Remark 2.1.2 (1)). Hence, we have $X \simeq \mathbb{P}(\mathscr{E})$ for some vector bundle $\mathscr{E}$ over Y. We will denote by $F \in N^1(X)$ the class of a fiber and by $\xi \in N^1(X)$ the class of the tautological divisor.

Since $\dim Z = r - 1$, we have $(-K_f + \tau F)^r = 0$. Note that $\xi^r = \deg(\det \mathscr{E})$ and $-K_f \equiv_{\mathrm{num}} r\xi - \deg(\det \mathscr{E})F$. Hence we have $\tau = 0$, namely, $-K_f$ is nef.

Next, we treat the case where every f-fiber is a smooth quadric. We will denote by $F \in N^1(X)$ the class of an f-fiber.

By [**Ara09**, Proposition 21], there exists a triple $(\mathscr{E}, \mathscr{L}, s)$ which satisfies the following:

(1) $\mathscr{E}$ (resp. $\mathscr{L}$) is a vector bundle of rank $r+1$ (resp. a line bundle) over Y.
(2) $q \in H^0(S^2\mathscr{E} \otimes \mathscr{L})$.
(3) X is a zero scheme of q in $\mathbb{P}(\mathscr{E})$

Set $d := \deg(\det \mathscr{E})$ and $\ell := \deg \mathscr{L}$. Since f is smooth, we have

$$-2d = (r+1)\,\ell. \tag{2.2.2.1}$$

By adjunction, we have $-K_f \equiv_{\mathrm{num}} (r-1)\,\xi_X - (d+\ell)\,F$, where ξ_X is the restriction of the tautological divisor ξ on $\mathbb{P}(\mathscr{E})$.

Since $\dim Z = r - 1$, we have $(-K_f + \tau F)^r = 0$, that is,

$$\big((r-1)\,\xi_X\big)^r + r\,(\tau - d - \ell)\,\big((r-1)\,\xi_X\big)^{r-1} \cdot F = 0. \tag{2.2.2.2}$$

Note that $X \equiv_{\mathrm{num}} 2\xi + \ell F'$ in $N^1(\mathbb{P}(\mathscr{E}))$ and $\xi^{r+1} = d$, where F' is the class of a fiber of $\mathbb{P}(\mathscr{E}) \to Y$. Hence we have ${\xi_X}^r = 2d + \ell$ and ${\xi_X}^{r-1} \cdot F = 2$. Therefore, by (2.2.2.1) and (2.2.2.2), we have $\tau = 0$. □

3. CP Manifolds with large Picard number

In this section, we prove Theorems 2.3.3 and 2.3.4, which will complete our proof of Theorem 2.0.1. Theorem 2.3.3 was obtained independently by K. Watanabe. See also [**BCDD03**, Proposition 2.4], [**NO07**, Proposition 5.1] or [**Wat14a**, Proposition 2.3] for the case $n - \rho_X = 0$ or 1. We include our proof of them for completeness of our treatment.

First we prove the following:

Theorem 2.3.1. *Let X be a CP n-fold which does not admit a contraction onto an FT manifold. Then $n \geq 2\rho_X$. Furthermore, the following hold:*

(1) *If the equality holds, then $X \simeq \left(\mathbb{P}^2\right)^{\rho_X}$.*
(2) *If $n = 2\rho_X + 1$, then $X \simeq \left(\mathbb{P}^2\right)^{\rho_X - 1} \times \mathbb{P}^3$, $\left(\mathbb{P}^2\right)^{\rho_X - 1} \times \mathbb{Q}^3$, $\left(\mathbb{P}^2\right)^{\rho_X - 2} \times \mathbb{P}_{\mathbb{Q}^4}(\mathscr{S}_{\mathbb{Q},i})$ or $\left(\mathbb{P}^2\right)^{\rho_X - 2} \times \mathbb{P}(T_{\mathbb{P}^3})$.*

Remark 2.3.2. In fact, Theorem 2.3.1 is true if we replace X a Fano manifold with Condition $(*)$ (see Proposition 3.2.1).

Proof. First, we prove by induction on n that every CP n-fold with $2\rho_X > n$ admits a contraction onto an FT manifold. The case $n = 1$ is trivial.

Assume $n > 1$. By Proposition 1.2.3, we have a sequence of smooth elementary contractions

$$X = X_0 \to X_1 \to \cdots \to X_{2\rho_X - n} \cdots \to X_{\rho_X - 1} \to X_{\rho_X} = \text{point}, \tag{2.3.2.1}$$

where each X_i is a CP manifold of dimension $\leq n - i$ with Picard number $\rho_X - i$.

If $\dim X_{2\rho_X - n} = 2n - 2\rho_X$ for every sequence (2.3.2.1), then $X_{2\rho_X - n-1}$ is an FT manifold.

Otherwise $\dim X_{2\rho_X - n} < 2n - 2\rho_X$ for some sequence (2.3.2.1). Then we have

$$\dim X_{2\rho_X - n} < 2n - 2\rho_X = 2\rho_{X_{2\rho_X - n}}.$$

Thus by inductive hypothesis $X_{2\rho_X - n}$ admits a contraction onto an FT manifold, and then so does X.

Next, we prove (1) and (2).

(1) we proceed by induction on n. If $n = 2$, then $X \simeq \mathbb{P}^2$ and the assertion holds. Hence we assume $n > 2$. Then, by Proposition 1.2.3, there exists a smooth elementary contraction $f\colon X \to Y$.

If $\dim Y < n-2$, then $2\rho_Y > \dim Y$. Hence Y admits a contraction onto an FT manifold. This contradicts our hypothesis. Hence $\dim Y \geq n - 2$ for every elementary contraction $f\colon X \to Y$. Furthermore, since X is not an FT manifold, there exists an elementary contraction $f\colon X \to Y$ with $\dim Y = n - 2$. Then, by inductive hypothesis, $Y \simeq \left(\mathbb{P}^2\right)^{\rho_Y}$. Hence f is a $\mathbb{P}^2$-bundle. Furthermore f is trivial on each factor $\mathbb{P}^2$ of Y by Theorem 1.2.2. Hence $X \simeq \left(\mathbb{P}^2\right)^{\rho_X}$ by Proposition 2.1.3 (5) $\Rightarrow$ (1).

(2) We proceed by induction on n. If $n = 3$ or 5, the assertion follows from Theorem 1.2.2. Hence we assume that $n > 5$. By our hypothesis, there exists an elementary contraction $f\colon X \to Y$ with $n - 3 \leq \dim Y \leq n - 2$.

If $\dim Y = n - 3$, then $Y \simeq \left(\mathbb{P}^2\right)^{\rho_X - 1}$ by (1). Let $g\colon X \to Z$ be the elementary contraction such that the following diagram is commutative:

$$\begin{tikzcd}
X \arrow[r, "g"] \arrow[d, "f"'] & Z \arrow[d] \\
Y \simeq \mathbb{P}^2 \times \left(\mathbb{P}^2\right)^{\rho_X - 2} \arrow[r, "\mathrm{pr}_2"] \arrow[d, "\mathrm{pr}_1"'] & \left(\mathbb{P}^2\right)^{\rho_X - 2} \arrow[d] \\
\mathbb{P}^2 \arrow[r] & \text{point}
\end{tikzcd}$$

Then, by the classification of CP 5-fold with Picard number two, every fiber of $\mathrm{pr}_2 \circ f$ is isomorphic to $\mathbb{P}^2 \times \mathbb{P}^3$ or $\mathbb{P}^2 \times \mathbb{Q}^3$. Thus g is a $\mathbb{P}^2$-fibration.

Hence we may find an elementary contraction $f\colon X \to W$ with $\dim W = n - 2$. Then $W \simeq \left(\mathbb{P}^2\right)^{\rho_X - 2} \times \mathbb{P}^3$, $\left(\mathbb{P}^2\right)^{\rho_X - 2} \times \mathbb{Q}^3$, $\left(\mathbb{P}^2\right)^{\rho_X - 3} \times \mathbb{P}_{\mathbb{Q}^4}(\mathscr{S}_{\mathbb{Q},i})$,

$\left(\mathbb{P}^2\right)^{\rho_X-3} \times \mathbb{P}(T_{\mathbb{P}^3})$ by inductive hypothesis. In any case f is a $\mathbb{P}^2$-bundle. Furthermore, if the last three cases occur, then f is trivial on any fiber of the elementary contractions of W by the classification of CP m-folds with $m = 3$, 4 or 5. Hence $X \simeq \mathbb{P}^2 \times W$ by Proposition 2.1.3 (5) $\Rightarrow$ (1).

Hence we may assume $W \simeq \left(\mathbb{P}^2\right)^{\rho_X-2} \times \mathbb{P}^3$. Let $g\colon X \to V$ be the elementary contraction such that the following diagram is commutative:

$$\begin{tikzcd}
X \arrow[r, "g"] \arrow[d, "f"'] & V \arrow[d] \\
W \simeq \mathbb{P}^2 \times \left(\left(\mathbb{P}^2\right)^{\rho_X-3} \times \mathbb{P}^3\right) \arrow[r, "\mathrm{pr}_2"] \arrow[d, "\mathrm{pr}_1"'] & \left(\left(\mathbb{P}^2\right)^{\rho_X-3} \times \mathbb{P}^3\right) \arrow[d] \\
\mathbb{P}^2 \arrow[r] & \text{point}
\end{tikzcd}$$

Then by the classification of CP 4-fold, every fiber of $\mathrm{pr}_2 \circ f$ is isomorphic to $\mathbb{P}^2 \times \mathbb{P}^2$. Thus g is a $\mathbb{P}^2$ -fibration. By inductive hypothesis, we have $V \simeq \left(\mathbb{P}^2\right)^{\rho_X-2} \times \mathbb{P}^3$, $\left(\mathbb{P}^2\right)^{\rho_X-2} \times \mathbb{Q}^3$, $\left(\mathbb{P}^2\right)^{\rho_X-3} \times \mathbb{P}_{\mathbb{Q}^4}(\mathscr{S}_{\mathbb{Q},i})$, $\left(\mathbb{P}^2\right)^{\rho_X-3} \times \mathbb{P}(T_{\mathbb{P}^3})$. Hence g is a $\mathbb{P}^2$-bundle. Since every g-fiber is not contracted by $\mathrm{pr}_1 \circ f$, it follows from Proposition 2.2.1 that $-K_g$ is nef. Hence the assertion follows from Proposition 2.1.3 (2) $\Rightarrow$ (1). $\square$

THEOREM 2.3.3. *Let X be a CP n-fold with $n - \rho_X \leq 3$. Then X is a rational homogeneous manifold.*

PROOF. We may assume that $2\rho_X + 2 \leq n$ by Corollary 1.2.8, Theorem 2.3.1 and induction on n. Then, by the inequality $2\rho_X + 2 \leq n \leq \rho_X + 3$, we have $n \leq 4$ and the assertion follows from Theorems 1.2.2. $\square$

Finally, we prove the following:

THEOREM 2.3.4. *Let X be a CP n-fold with $n - \rho_X = 4$. Then X is a rational homogeneous manifold.*

PROOF. By Corollary 1.2.8, Theorems 2.3.1 and 2.3.3 and induction on n, we may assume that $2\rho_X + 2 \leq n$, and hence $n \leq 6$. The case $n = 5$ follows from Theorem 1.2.2.

Assume that $n = 6$. Then $\rho_X = 2$, and there are two different smooth elementary contractions $f\colon X \to Y$ and $g\colon X \to Z$. Without loss of generality, we may assume that $\dim Y \geq \dim Z$. Furthermore, $\dim Y \geq 3$ by the inequality $\dim X \leq \dim Y + \dim Z$.

If $\dim Y = 5$, then Y is isomorphic to $\mathbb{P}^5$, $\mathbb{Q}^5$ or $K(G_2)$ by Theorem 1.2.2. Since Y is rational, f is a $\mathbb{P}^1$-bundle. Hence X is a rational homogeneous manifold by Proposition 2.1.4.

If $\dim Y = 4$, then Y is isomorphic to $\mathbb{P}^4$ or $\mathbb{Q}^4$ by Theorem 1.2.2. Since Y is rational, f is a $\mathbb{P}^2$-bundle. Hence X is isomorphic to $\mathbb{P}^2 \times \mathbb{P}^4$ or $\mathbb{P}^2 \times \mathbb{Q}^4$ by Theorem 2.1.5.

In the remaining case, we have $\dim Y = \dim Z = 3$. Hence $Y \simeq \mathbb{P}^3$ or $\mathbb{Q}^3$, and $Z \simeq \mathbb{P}^3$ or $\mathbb{Q}^3$ by Theorem 1.2.2. Then, by Proposition 2.2.1, $-K_f$ and $-K_g$ are nef. Because $\rho_X = 2$, the nef cone of X is spanned by $\{-K_f, -f^*K_X\}$ or $\{-K_g, -g^*K_Y\}$. Note that $-K_X = -K_f - f^*K_X =$

$-K_g - g^*K_Y$ and $-f^*K_X \neq -g^*K_Y$. Hence we have $-K_f = -g^*K_Y$. Therefore we have $-K_X = -f^*K_Y - g^*K_Z$. Then, by purity of branch locus, $(f,g) : X \to Y \times Z$ is étale. Since $Y \times Z$ is simply connected, we have $X \simeq Y \times Z$. This completes the proof. $\square$

REMARK 2.3.5. Here we include the explicit form of manifolds as in Theorem 2.0.1:

ρ_X	X
$n-4$	$\left(\mathbb{P}^1\right)^{n-5} \times \left[\mathbb{P}^5,\ \mathbb{Q}^5 \text{ or } K(G_2)\right]$,
	$\left(\mathbb{P}^1\right)^{n-6} \times \left[\mathbb{P}(\mathscr{C}),\ \mathbb{P}^2 \times \mathbb{P}^4,\ \mathbb{P}^2 \times \mathbb{Q}^4,\ \left(\mathbb{P}^3\right)^2,\ \mathbb{P}^3 \times \mathbb{Q}^3 \text{ or } \left(\mathbb{Q}^3\right)^2\right]$,
	$\left(\mathbb{P}^1\right)^{n-7} \times \left[\left(\mathbb{P}^2\right)^2 \times \mathbb{P}^3,\ \left(\mathbb{P}^2\right)^2 \times \mathbb{Q}^3,\ \mathbb{P}^2 \times \mathbb{P}_{\mathbb{Q}^4}(\mathscr{S}_{\mathbb{Q},i}),\ \mathbb{P}^2 \times \mathbb{P}(T_{\mathbb{P}^3}),\right.$
	$\left.\mathbb{P}(\mathscr{N}) \times \mathbb{P}^3 \text{ or } \mathbb{P}(\mathscr{N}) \times \mathbb{Q}^3\right]$,
	$\left(\mathbb{P}^1\right)^{n-7} \times \mathbb{P}(T_{\mathbb{P}^2}) \times \left[\mathbb{P}^4 \text{ or } \mathbb{Q}^4\right]$,
	$\left(\mathbb{P}^1\right)^{n-8} \times \left[\left(\mathbb{P}(\mathscr{N})\right)^2,\ \mathbb{P}(\mathscr{N}) \times \left(\mathbb{P}^2\right)^2,\ \left(\mathbb{P}^2\right)^4 \text{ or } \mathbb{P}^2 \times \mathrm{Fl}(1,2,3;4)\right]$,
	$\left(\mathbb{P}^1\right)^{n-8} \times \mathbb{P}(T_{\mathbb{P}^2}) \times \left[\mathbb{P}_{\mathbb{Q}^4}(\mathscr{S}_{\mathbb{Q},i}),\ \mathbb{P}(T_{\mathbb{P}^3}),\ \mathbb{P}^2 \times \mathbb{P}^3 \text{ or } \mathbb{P}^2 \times \mathbb{Q}^3\right]$,
	$\left(\mathbb{P}^1\right)^{n-9} \times \mathbb{P}(T_{\mathbb{P}^2}) \times \left[\left(\mathbb{P}^2\right)^3,\ \mathbb{P}^2 \times \mathbb{P}(\mathscr{N}) \text{ or } \mathrm{Fl}(1,2,3;4)\right]$,
	$\left(\mathbb{P}^1\right)^{n-9} \times \left(\mathbb{P}(T_{\mathbb{P}^2})\right)^2 \times \left[\mathbb{P}^3 \text{ or } \mathbb{Q}^3\right]$,
	$\left(\mathbb{P}^1\right)^{n-10} \times \left(\mathbb{P}(T_{\mathbb{P}^2})\right)^2 \times \left[\mathbb{P}(\mathscr{N}) \text{ or } \left(\mathbb{P}^2\right)^2\right]$,
	$\left(\mathbb{P}^1\right)^{n-11} \times \left(\mathbb{P}(T_{\mathbb{P}^2})\right)^3 \times \mathbb{P}^2$,
	$\left(\mathbb{P}^1\right)^{n-12} \times \left(\mathbb{P}(T_{\mathbb{P}^2})\right)^4$
$n-3$	$\left(\mathbb{P}^1\right)^{n-4} \times \left[\mathbb{P}^4 \text{ or } \mathbb{Q}^4\right]$,
	$\left(\mathbb{P}^1\right)^{n-5} \times \left[\mathbb{P}_{\mathbb{Q}^4}(\mathscr{S}_{\mathbb{Q},i}),\ \mathbb{P}(T_{\mathbb{P}^3}),\ \mathbb{P}^2 \times \mathbb{P}^3 \text{ or } \mathbb{P}^2 \times \mathbb{Q}^3\right]$,
	$\left(\mathbb{P}^1\right)^{n-6} \times \left[\left(\mathbb{P}^2\right)^3,\ \mathbb{P}^2 \times \mathbb{P}(\mathscr{N}) \text{ or } \mathrm{Fl}(1,2,3;4)\right]$,
	$\left(\mathbb{P}^1\right)^{n-6} \times \mathbb{P}(T_{\mathbb{P}^2}) \times \left[\mathbb{P}^3 \text{ or } \mathbb{Q}^3\right]$,
	$\left(\mathbb{P}^1\right)^{n-7} \times \mathbb{P}(T_{\mathbb{P}^2}) \times \left[\mathbb{P}(\mathscr{N}) \text{ or } \left(\mathbb{P}^2\right)^2\right]$,
	$\left(\mathbb{P}^1\right)^{n-8} \times \left(\mathbb{P}(T_{\mathbb{P}^2})\right)^2 \times \mathbb{P}^2$,
	$\left(\mathbb{P}^1\right)^{n-9} \times \left(\mathbb{P}(T_{\mathbb{P}^2})\right)^3$
$n-2$	$\left(\mathbb{P}^1\right)^{n-3} \times \left[\mathbb{P}^3 \text{ or } \mathbb{Q}^3\right]$,
	$\left(\mathbb{P}^1\right)^{n-4} \times \left[\mathbb{P}(\mathscr{N}) \text{ or } \left(\mathbb{P}^2\right)^2\right]$,
	$\left(\mathbb{P}^1\right)^{n-5} \times \mathbb{P}(T_{\mathbb{P}^2}) \times \mathbb{P}^2$,
	$\left(\mathbb{P}^1\right)^{n-6} \times \left(\mathbb{P}(T_{\mathbb{P}^2})\right)^2$
$n-1$	$\left(\mathbb{P}^1\right)^{n-2} \times \mathbb{P}^2$,
	$\left(\mathbb{P}^1\right)^{n-3} \times \mathbb{P}(T_{\mathbb{P}^2})$
n	$\left(\mathbb{P}^1\right)^{n}$

CHAPTER 3

Extremal rays and nefness of tangent bundles

Introduction

In Introduction to Part 1 we have introduced the following condition for Fano manifolds in relation to the inductive approach to Conjecture 1.0.1:

Condition $(*)$. For every sequence of elementary contractions

$$X \xrightarrow{f_1} X_1 \xrightarrow{f_2} \cdots \xrightarrow{f_{m-1}} X_{m-1} \xrightarrow{f_m} X_m,$$

each f_i is smooth and every fiber of f_i is a rational homogeneous manifold.

Later we will briefly see that, by the same argument given in Chapter 2, Fano n-folds with Condition $(*)$ and Picard number $\rho_X > n-5$ are rational homogeneous manifolds (Theorem 3.2.2). For this fact, one might hope that every Fano manifold with Condition $(*)$ would be a rational homogeneous manifold. However, this speculation is not true in general. In fact, we clarify that the Fano 7-fold X_0 constructed by Ottaviani [**Ott88**, **Ott90**] satisfies the following properties (see Theorem 3.1.2 for details):

(1) X_0 is a Fano 7-fold with Picard number two which admits two different smooth $\mathbb{P}^2$-fibrations π and p over the five dimensional quadrics:

(3.0.0.1)
$$\begin{tikzcd} & X_0 \arrow[dl, "\pi"'] \arrow[dr, "p"] & \\ \mathbb{Q}^5 & & \mathbb{Q}^5. \end{tikzcd}$$

In particular, X_0 satisfies Condition $(*)$.

(2) The tangent bundle T_{X_0} is not nef. In particular, X_0 is not homogeneous.

Moreover, we show that a Fano 7-fold with property (1) is unique up to isomorphism (for a more precise statement, see Theorem 3.1.6). In particular, the existence of X_0 shows that smooth Fano n-folds with Condition $(*)$ and $\rho_X > n-6$ are not necessarily rational homogeneous.

The purpose of this chapter is to classify Fano n-folds with Condition $(*)$ and Picard number $\rho_X > n-6$:

Theorem 3.0.1. *Let X be a Fano n-fold with Condition $(*)$ and $\rho_X > n-6$. Then X is either*

(1) *a rational homogeneous manifold or*
(2) $\left(\mathbb{P}^1\right)^{n-7} \times (X_0$ *as in* (3.0.0.1)$)$.

By a theorem of Demailly, Peternell and Schneider [**DPS94**, Theorem 5.2] (see also [**SCW04**, Theorem 4.4], [**MOSCW15**, Proposition 4]), a Fano manifold X with nef tangent bundle satisfies Condition $(*)$ if one assumes Conjecture 1.0.1 for k-folds with Picard number one and $k \leq \dim X - \rho_X + 1$. Note that Conjecture 1.0.1 is true in dimension at most five. Hence, as a corollary of Theorem 3.0.1, we obtain a result with respect to Conjecture 1.0.1:

COROLLARY 3.0.2. *If Conjecture 1.0.1 for* 6*-folds with* $\rho_X = 1$ *is true, then Conjecture 1.0.1 is true for* n*-folds with* $\rho_X > n - 6$.

Note that the above example X_0 as in (3.0.0.1) also gives a negative answer to the following problem for $q = 1$ (cf. [**Yas12**, **Yas14**]):

PROBLEM 3.0.3 ([**CP92**, Problem 6.4]). *Let* X *be a Fano manifold. If* $\bigwedge^q T_X$ *is nef on every extremal rational curve, then is* $\bigwedge^q T_X$ *nef?*

A significant progress concerning Conjecture 1.0.1 and Problem 3.0.3 for $q = 1$ is obtained by Muñoz, Occhetta, Solá-Conde, Watanabe and Wiśniewski [**MOSCW15**, **OSCWW17**]. They show that FT manifolds, i.e. Fano manifolds whose elementary contractions are smooth $\mathbb{P}^1$-fibrations, are complete flag manifolds. In particular, Problem 3.0.3 for $q = 1$ and Conjecture 1.0.1 are affirmative for such Fano manifolds.

We explain an outline of this chapter: In Sect. 1, we give two descriptions of the manifold X_0 as in (3.0.0.1) and then a characterization of X_0 is established. Here, as in [**MOSC14**, **Wat14b**] and Chapter 2 , slopes for Fano bundles (see Definition 1.1.5) and numerical conditions on slopes play important roles. In Sect. 3, we complete the proof of Theorem 3.0.1 for $\rho_X = n - 5$ and Corollary 3.0.2.

1. A characterization of Ottaviani bundle

1.1. Ottaviani bundle and the family of special planes on the five dimensional quadric. We identify the Chern class $c_i(\mathscr{E})$ of a vector bundle $\mathscr{E}$ on $\mathbb{Q}^5$ with an integer c_i.

DEFINITION 3.1.1. Let $\mathscr{G}_{\mathbb{Q}}$ be the Ottaviani bundle on $\mathbb{Q}^5$, i.e. a stable vector bundle of rank three with Chern classes $(c_1, c_2, c_3) = (2, 2, 2)$. In this chapter, we denote by X_0 the projectivized Ottaviani bundle $\mathbb{P}(\mathscr{G}_{\mathbb{Q}})$.

By Proposition 0.3.2 there is the following exact sequence on $\mathbb{Q}^5$:

$$0 \to \mathscr{O}_{\mathbb{Q}^5} \to \mathscr{S}^*_{\mathbb{Q}} \to \mathscr{G}_{\mathbb{Q}} \to 0.$$

Set $Y := \mathbb{Q}^5$. The surjection $\mathscr{S}^*_{\mathbb{Q}} \to \mathscr{G}_{\mathbb{Q}}$ gives an immersion of projectivized vector bundles $i\colon X_0 := \mathbb{P}_Y(\mathscr{G}_{\mathbb{Q}}) \to \mathbb{P}_Y(\mathscr{S}^*_{\mathbb{Q}})$. By the definition of the spinor bundle, there exists a smooth $\mathbb{P}^2$-bundle $p'\colon \mathbb{P}_Y(\mathscr{S}^*_{\mathbb{Q}}) \to S_3 \simeq \mathbb{Q}^6$, where S_3 is the spinor variety. Let $\mathbb{P}(\mathscr{G}_{\mathbb{Q}}) \xrightarrow{p} Z \xrightarrow{h} S_3$ be the Stein factorization of $p' \circ i$:

$$\begin{tikzcd}
 & X_0 = \mathbb{P}_Y(\mathscr{G}_{\mathbb{Q}}) \arrow[r, hook, "i"] \arrow[dl, "\pi"'] \arrow[drr, "p"] & \mathbb{P}_Y(\mathscr{S}^*_{\mathbb{Q}}) \arrow[dll, "\pi'"'] \arrow[drr, "p'"] & & \\
Y = \mathbb{Q}^5 & & & Z \arrow[r, "h"] & S_3 \simeq \mathbb{Q}^6
\end{tikzcd}$$

We use the same notations as in Chapter 1 Sect. 1 (e.g. H_Y is the ample generator of $\mathrm{Pic}(Y)$). Note that the Chern classes (c_1, c_2, c_3, c_4) of $\mathscr{S}^*_{\mathbb{Q}}$ are $(2, 2, 2, 0)$ and that p is not an isomorphism since $\dim X_0 > \dim Z$.

THEOREM 3.1.2. *The following hold:*

(1) *p is a $\mathbb{P}^2$-bundle over the 5-dimensional quadric $Z \simeq \mathbb{Q}^5$. In particular, X_0 satisfies Condition $(*)$.*
(2) *The tangent bundle of X_0 is not nef.*

PROOF. By the definition of the spinor bundle, we have $p'^*\mathcal{O}_{\mathbb{Q}^6}(1) \simeq \mathcal{O}_{\mathbb{P}(\mathscr{S}^*_{\mathbb{Q}})}(1)$ and hence the vector bundle $\mathscr{S}^*_{\mathbb{Q}}$ is nef but not ample. Thus the slope for the pair $(Y, \mathscr{S}^*_{\mathbb{Q}})$ is two and p' is defined by the semiample divisor $-K_{\pi'} + 2\pi'^*H_Y$. Therefore the morphism p is defined by the semiample divisor $(-K_{\pi'} + 2\pi'^*H_Y)|_{X_0}$, which is equivalent to $4\,\xi_{\mathscr{G}_{\mathbb{Q}}}$.

Because $\dim Z < \dim X_0$, the divisor $-K_\pi + 2\pi^*H_Y = 3\xi_{\mathscr{G}_{\mathbb{Q}}}$ is nef but not ample. This implies that $\mathscr{G}_{\mathbb{Q}}$ is a Fano bundle whose slope τ is two.

By a direct calculation using Remark 1.1.2 (2), Remark 1.1.2 (3) and Remark 1.1.4, we have $(-K_\pi + 2\pi^*H_Y)^6 \cdot \pi^*H_Y = 0$ and $(-K_\pi + 2\pi^*H_Y)^5 \cdot \pi^*H_Y^2 \neq 0$. Hence we have $\dim Z = 5$, h is an immersion and p is the base change of p' over Z. Since p' is a $\mathbb{P}^2$-bundle, p is also a $\mathbb{P}^2$-bundle. Furthermore Z is a linear section $\mathbb{Q}^5$ of $\mathbb{Q}^6$ since the normal bundle of X_0 in $\mathbb{P}(\mathscr{S}^*_{\mathbb{Q}})$ is $\mathcal{O}_{\mathbb{P}(\mathscr{S}^*_{\mathbb{Q}})}(1)|_{X_0} \simeq p'^*\mathcal{O}_{\mathbb{Q}^6}(1)|_{X_0}$. Hence X_0 is a Fano 7-fold with Picard number two which satisfies Condition $(*)$:

$$\begin{tikzcd} & X_0 \arrow[dl, "\pi"'] \arrow[dr, "p"] & \\ Y \simeq \mathbb{Q}^5 & & Z \simeq \mathbb{Q}^5. \end{tikzcd}$$

By Proposition 0.5.2, there exists the following exact sequence on $Y = \mathbb{Q}^5$:

$$0 \to \mathscr{C}(1) \to \mathscr{G}_{\mathbb{Q}} \to \mathcal{O}_{\mathbb{Q}^5}(1) \to 0,$$

where $\mathscr{C}$ is the Cayley bundle on $\mathbb{Q}^5$. Then the surjection $\mathscr{G}_{\mathbb{Q}} \to \mathcal{O}_{\mathbb{Q}^5}(1)$ gives a section $S \simeq \mathbb{Q}^5 \subset X_0$ of π with normal bundle $N_{S/X_0} \simeq \mathscr{C}^* \simeq \mathscr{C}(1)$, which is *not* nef by Proposition 0.5.3. Hence the tangent bundle of X_0 is *not* nef since the normal bundle N_{S/X_0} is a quotient of the tangent bundle. $\square$

REMARK 3.1.3. In [**Pan13**], a smooth projective variety is called convex if

$$H^1(\mu^*T_{X_0}) = 0$$

for every morphism $\mu\colon \mathbb{P}^1 \to X_0$, and Pandharipande proved that a convex, rationally connected smooth complete intersection is a homogeneous manifold. Note that X_0 is not convex in the sense of [**Pan13**]. Indeed the restriction of $\mathscr{C}(1)$ on a special line in $\mathbb{Q}^5 \simeq S$ is $\mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}(2)$ [**Ott90**, Theorem 3.5]. Hence if we take a double cover of the special line, we have a morphism $\mu\colon \mathbb{P}^1 \to X_0$ with $H^1(\mu^*T_{X_0}) \neq 0$

DEFINITION 3.1.4 ([**Ott90**, Section 1]). Let $\mathbb{O}$ be the complexified Cayley octonions, which is an algebra generated by 1, e_1, e_2, $\ldots e_7$ with the following relations: (1) $e_i^2 = -1$, (2) $e_i \cdot e_j = -e_j \cdot e_i$ for $i \neq j$, (3) $e_1 \cdot e_2 = e_3$,

(4) $e_1 \cdot e_4 = e_5$, (5) $e_1 \cdot e_7 = e_6$, (6) $e_2 \cdot e_5 = e_7$, (7) $e_2 \cdot e_4 = e_6$, (8) $e_3 \cdot e_4 = e_7$, (9) $e_3 \cdot e_6 = e_5$.

It is known that the automorphism group of $\mathbb{O}$ is a semisimple group of type G_2, and that the group acts on the variety of projectivized elements with null-square. The equations for the variety of projectivized elements with null-square is

$$x_0 = \sum_{i=1}^{7} x_i^2 = 0.$$

Hence it is naturally isomorphic to the five dimensional quadric $\mathbb{Q}^5$.

The *special plane on* $\mathbb{Q}^5$ through a point $y \in \mathbb{Q}^5$ is defined to be

$$\Pi_y := \{\, x \in \mathbb{Q}^5 \mid x \cdot y = 0 \,\}.$$

Set

$$X := \{\, (x, y) \in \mathbb{Q}^5 \times \mathbb{Q}^5 \mid x \cdot y = 0 \,\}$$

and let $p_1 \colon X \to Y := \mathbb{Q}^5$ be the first projection and $p_2 \colon X \to Z := \mathbb{Q}^5$ the second projection. We call this X the *family of special planes on* $\mathbb{Q}^5$.

Then every p_2-fiber over $y \in Z \simeq \mathbb{Q}^5$ defines the special plane $\Pi_y \subset Y \simeq \mathbb{Q}^5$. Hence $p_2 \colon X \to Z$ is a $\mathbb{P}^2$-bundle. By the symmetry, $p_1 \colon X \to Y$ is also a $\mathbb{P}^2$-bundle. Hence we have the following:

PROPOSITION 3.1.5. *The family of special planes X satisfies Condition $(*)$ and admits two different $\mathbb{P}^2$-bundles over 5-dimensional quadrics $\mathbb{Q}^5$.*

In the next subsection, we will show that the above two manifolds in Theorem 3.1.2 and Proposition 3.1.5 are isomorphic.

1.2. A characterization of projectivized Ottaviani bundle.

THEOREM 3.1.6. *Let X be a Fano 7-fold with Picard number two which satisfies Condition $(*)$. Assume that X has a smooth $\mathbb{P}^2$-fibration $\pi \colon X \to Y$. Then X is isomorphic to $\mathbb{P}^2 \times Y$ or X_0 as in Definition 3.1.1.*

In particular the manifolds of Theorem 3.1.2 and Proposition 3.1.5 are isomorphic.

The rest of this section is occupied with our proof of Theorem 3.1.6. Let X be a manifold as in Theorem 3.1.6. Then Y is a 5-dimensional rational homogeneous manifold with Picard number one by Condition $(*)$. By the classification of rational homogeneous 5-folds, we have $Y \simeq \mathbb{P}^5$, $\mathbb{Q}^5$ or $K(G_2)$, where $K(G_2)$ is the 5-dimensional Fano contact homogeneous manifold of type G_2. Since the Brauer group of Y is trivial there exists a vector bundle $\mathscr{E}$ over Y such that $X \simeq \mathbb{P}(\mathscr{E})$. By Condition $(*)$, there is another smooth elementary rational homogeneous fibration $p \colon X \to Z$ over a rational homogenous manifold Z:

$$\begin{tikzcd} & X \arrow[dl, "\pi"'] \arrow[dr, "p"] & \\ Y & & Z. \end{tikzcd}$$

If the slope τ for the pair $(Y, \mathscr{E})$ is zero, then $X \simeq \mathbb{P}^2 \times Y$ by Proposition 1.1.6.

Hence, in the rest of this section we assume that the slope τ for the pair $(Y, \mathscr{E})$ is nonzero and we will show that X is isomorphic to X_0 as in Definition 3.1.1.

NOTATION 3.1.7. In any case, we have $A^i(Y)_{\mathbb{Z}} \simeq \mathbb{Z}$ for each i. We fix effective generators of $A^i(Y)_{\mathbb{Z}}$ as follows: $A^0(Y)_{\mathbb{Z}} = \mathbb{Z}\,[Y]$, $A^1(Y)_{\mathbb{Z}} = \mathbb{Z}\,H_Y$, $A^2(Y)_{\mathbb{Z}} = \mathbb{Z}\,\Sigma_Y$, $A^3(Y)_{\mathbb{Z}} = \mathbb{Z}\,P_Y$, $A^4(Y)_{\mathbb{Z}} = \mathbb{Z}\,\ell_Y$, $A^5(Y)_{\mathbb{Z}} = \mathbb{Z}\,\{\text{point}\}$. Therefore there exists a triple of integers (n_Y, m_Y, d_Y) which satisfies $H_Y^2 = n_Y\,\Sigma_Y$, $H_Y \cdot \Sigma_Y = m_Y\,P_Y$ and $H_Y^5 = d_Y$. We identify each class in $A^i(Y)_{\mathbb{Z}}$ with an integer.

We will write $d_2(\mathscr{E}) = a \cdot H_Y^2 = a\,n_Y \cdot \Sigma_Y$ and $d_3(\mathscr{E}) = b \cdot H_Y^3 = b\,n_Y m_Y P_Y$ with rational numbers a and b. Note that $d_2(\mathscr{E}) = a\,n_Y \in \mathbb{Z}$ and $d_3(\mathscr{E}) = b\,n_Y m_Y \in \mathbb{Z}$.

REMARK 3.1.8. We have the following:

(1) $(n_Y, m_Y, d_Y) = (1, 1, 1)$ if $Y \simeq \mathbb{P}^5$,
(2) $(n_Y, m_Y, d_Y) = (1, 2, 2)$ if $Y \simeq \mathbb{Q}^5$,
(3) $(n_Y, m_Y, d_Y) = (3, 2, 18)$ if $Y \simeq K(G_2)$.

Note that (n_Y, m_Y, d_Y) for $Y \simeq K(G_2)$ can be determined by the following two $\mathbb{P}^1$-bundle structures as in (0.5.1.1) [**Ott90**, 1.3]:

$$\begin{tikzcd} & \mathbb{P}_{\mathbb{Q}^5}(\mathscr{C}) \arrow[dl] \arrow[dr] & \\ \mathbb{Q}^5 & & K(G_2). \end{tikzcd}$$

LEMMA 3.1.9. $\dim Z \leq 5$.

PROOF. Assume to the contrary $\dim Z = 6$. Then the other contraction p is a $\mathbb{P}^1$-fibration over Z (in fact it is a $\mathbb{P}^1$-bundle since Z is rational). Then, since $b_i(X) = b_{i-4}(Y) + b_{i-2}(Y) + b_i(Y)$ and $b_i(X) = b_{i-2}(Z) + b_i(Z)$, we have $b_4(Z) = 2$ and $b_6(Z) = 1$. This contradicts the hard Lefschetz theorem. □

Hence, by Remark 1.1.4 and Proposition 1.1.7, we have the following:

$$f(\tau) = 21\,\tau^5 + 35a\,\tau^3 + 21b\,\tau^2 + 7a^2\,\tau + 2ab = 0, \tag{3.1.9.1}$$

$$g(\tau) = 15\,\tau^4 + 15a\,\tau^2 + 6b\,\tau + a^2 = 0. \tag{3.1.9.2}$$

Then we have $R(f, g) = 0$, where $R(f, g)$ is the resultant. This is equivalent to

$$0 = 9a(216\,b^2 + 49\,a^3)(250047\,b^4 - 222804\,a^3 b^2 + 132496\,a^6). \tag{3.1.9.3}$$

LEMMA 3.1.10. *The following hold:*

(1) $a = -6\,k^2$ *and* $b = 7\,k^3$ *for some* $0 \neq k \in \mathbb{Z}$.
(2) $\tau = 2k$.
(3) *Up to twisting* $\mathscr{E}$ *with a line bundle, we have* $\tau = c_1(\mathscr{E})$ *and the following possibilities for* $(Y; c_1(\mathscr{E}), c_2(\mathscr{E}), c_3(\mathscr{E}))$:
  (a) $(\mathbb{P}^5; 2, 2, 1)$,
  (b) $(\mathbb{P}^5; 4, 8, 8)$,
  (c) $(\mathbb{Q}^5; 2, 2, 2)$,
  (d) $(\mathbb{Q}^5; 4, 8, 16)$,

(e) $(K(G^2); 2, 6, 6)$.

(4) $\dim Z = 5$.

REMARK 3.1.11. In the proof of Lemma 3.1.10, we use only the conditions:

(1) $\tau \neq 0$
(2) $Y \simeq \mathbb{P}^5$, $\mathbb{Q}^5$ or $K(G_2)$,
(3) $\dim Z \leq 5$,
(4) $\mathbb{P}(\mathscr{E})$ is a Fano manifold.

Note that, by the definition of τ, $\mathscr{E}$ is semiample if $\tau = c_1(\mathscr{E})$.

PROOF OF LEMMA 3.1.10. (1) First observe that $a \neq 0$. Indeed if $a = 0$ then the equation $f(\tau) = g(\tau) = 0$ gives $\tau = 0$, which contradicts our assumption $\tau \neq 0$. Hence equation (3.1.9.3) gives

$$216\, b^2 + 49\, a^3 = 0$$

or

$$250047\, b^4 - 222804\, a^3 b^2 + 132496\, a^6 = 0.$$

If the latter equation holds, then we have $b \notin \mathbb{Q}$, which gives a contradiction. Hence the former case occurs, or equivalently we have

$$216\, n_Y {d_3}^2 + 49\, m_Y^2 {d_2}^3 = 0.$$

Note that d_2, $d_3 \in \mathbb{Z}$ and (n_Y, m_Y) are described as in Remark 3.1.8. For each case we can solve the equation and the first assertion follows.

(2) By (1) and the equations (3.1.9.1) and (3.1.9.2), we have

$$(\tau - 2k)^2(\tau + k)(\tau^2 + 3\, k\tau - k^2) = 0,$$
$$(\tau - 2k)(5\,\tau^3 + 10\, k\tau^2 - 10\, k^2\tau - 6\, k^3) = 0.$$

This gives the second assertion.

(3) By Proposition 1.1.6 and our assumption $\tau \neq 0$, we have $0 < \tau = 2k < r_Y$. Hence we have

$$\begin{cases} k = 1 & \text{if } Y \simeq K(G_2), \\ k = 1, 2 & \text{otherwise.} \end{cases}$$

Since $\operatorname{rank} \mathscr{E} = 3$, we may assume that $1 \leq c_1(\mathscr{E}) \leq 3$ if $k = 1$, and that $4 \leq c_1(\mathscr{E}) \leq 6$ if $k = 2$.

By (1) and Remark 1.1.2 (3), the following hold:

$$-6\, k^2 = 3\, c_1(\mathscr{E})^2 - \frac{9}{n_Y}\, c_2(\mathscr{E}),$$
$$7\, k^2 = 2\, c_1(\mathscr{E})^3 - \frac{9}{n_Y}\, c_1(\mathscr{E})c_2(\mathscr{E}) + \frac{27}{n_Y m_Y}\, c_3(\mathscr{E}).$$

We have the assertion by solving these equations for each case.

(4) Since $\tau = 2\, k$, we have $k \neq 0$. By a direct calculation with Remark 1.1.2 and Remark 1.1.4, we obtain

$$(-K_\pi + \tau\, \pi^* H_Y)^5 \pi^* H_Y^2 = 54k^2 \neq 0.$$

Hence $\dim Z = 5$. □

By Lemma 3.1.10 (4) and Condition $(*)$, Z is also a rational homogeneous 5-fold with $\rho_Z = 1$ and p is an elementary rational homogeneous fibration of relative dimension two. Hence $Z \simeq \mathbb{P}^5$, $\mathbb{Q}^5$ or $K(G_2)$, and p is a smooth $\mathbb{P}^2$-fibration by the classification of rational homogeneous manifolds. Since the Brauer group of Z is trivial, p is a $\mathbb{P}^2$-bundle. Hence there exists a rank three vector bundle $\mathscr{F}$ on Z such that $X \simeq \mathbb{P}_Z(\mathscr{F})$. Therefore we have the following diagram:

$$\begin{tikzcd} & \mathbb{P}_Y(\mathscr{E}) = X = \mathbb{P}_Z(\mathscr{F}) \arrow[dl, "\pi"'] \arrow[dr, "p"] & \\ (Y, \mathscr{E}) & & (Z, \mathscr{F}), \end{tikzcd}$$

where p is the natural projection. Twisting with a line bundle, we may assume that $(Z, \mathscr{F})$ also satisfies the conditions in Lemma 3.1.10.

We use a similar notation for $A^i(Z)$ as in Notation 3.1.7.

Lemma 3.1.12. *Only Lemma 3.1.10 (3) (c) occurs.*

Proof. In any case of Lemma 3.1.10 (3), $\mathscr{E}$ and also $\mathscr{F}$ are nef but not ample since the slope τ for each bundle is equal to its first Chern class. Hence $\xi_{\mathscr{E}} = p^*H_Z$ and $\xi_{\mathscr{F}} = \pi^*H_Y$.

By the Grothendieck relation,

$$n_Z m_Z\, p^*P_Z = p^*H_Z^3 = \xi_{\mathscr{E}}^3 = c_1(\mathscr{E})\,\xi_{\mathscr{E}}^2 \cdot \pi^*H_Y - c_2(\mathscr{E})\,\xi_{\mathscr{E}} \cdot \pi^*\Sigma_Y + c_3(\mathscr{E})\pi^*P_Y.$$

We also have the following:

$$\xi_{\mathscr{F}} \cdot p^*\Sigma_Z = n_Z\, \xi_{\mathscr{E}}^2\, \pi^*H_Y,$$
$$\xi_{\mathscr{F}}^2 \cdot p^*H_Z = n_Y\, \xi_{\mathscr{E}}\, \pi^*\Sigma_Y.$$

Because the triples $(p^*P_Z,\, \xi_{\mathscr{F}} \cdot p^*\Sigma_Z,\, \xi_{\mathscr{F}}^2 \cdot p^*H_Z)$ and $(\pi^*P_Y,\, \xi_{\mathscr{E}} \cdot \pi^*\Sigma_Y,\, \xi_{\mathscr{E}}^2 \cdot \pi^*H_Y)$ are $\mathbb{Z}$-bases of $A^3(X)$, the following matrix is unimodular:

$$\begin{pmatrix} 0 & n_Y & 0 \\ n_Z & 0 & 0 \\ \dfrac{c_1(\mathscr{E})}{n_Z m_Z} & -\dfrac{c_2(\mathscr{E})}{n_Z m_Z} & \dfrac{c_3(\mathscr{E})}{n_Z m_Z} \end{pmatrix}.$$

From this it follows that

(1) Lemma 3.1.10 (3) (a) occurs and $Z \simeq \mathbb{P}^5$ or
(2) Lemma 3.1.10 (3) (c) occurs and $Z \simeq \mathbb{Q}^5$.

However the first one cannot happen by [**Sat85**]. □

The following completes our proof of Theorem 3.1.6.

Lemma 3.1.13. *Let $(Y, \mathscr{E})$ be as in Lemma 3.1.10 (3) (c), then $\mathscr{E}$ is stable.*

Proof. It is enough to show that $H^0(\mathscr{E}(-1)) = H^0(\mathscr{E}^*) = 0$.

Because $\xi_{\mathscr{E}}$ defines the other contraction of fiber type $p\colon X \to Z$, we have $\xi_{\mathscr{E}} \in \mathrm{Psef}(X) \setminus \mathrm{Big}(X)$, where $\mathrm{Psef}(X)$ is the pseudoeffective cone of X and $\mathrm{Big}(X)$ the big cone of X. Hence we have $0 = H^0\left(\mathcal{O}(\xi_{\mathscr{E}} - \pi^*H_Y)\right) = H^0(\mathscr{E}(-1))$.

Note that $\mathscr{E}$ is a 2-ample vector bundle because p is a $\mathbb{P}^2$-bundle. Hence $H^0(\mathscr{E}^*) = H^5(\omega_Y \otimes \mathscr{E}) = 0$ by the Sommese vanishing theorem [**Som78**, Proposition 1.14]. □

Hence $\mathscr{E}$ is the Ottaviani bundle. This completes the proof of Theorem 3.1.6.

2. Fano manifolds with Condition $(*)$ and $n \le 2\rho_X + 1$

Here we generalize some results in Chapter 2 to those for Fano manifolds with Condition $(*)$. Note that, by replacing Proposition 1.2.3 with Proposition 1.3.1, we see that Propositions 2.1.3 is true for a vector bundle $\mathscr{E}$ whose projectivization satisfies Condition $(*)$.

Here we include the following proposition, which is a variant of Theorem 2.3.1:

PROPOSITION 3.2.1 (cf. Theorem 2.3.1 for CP manifolds). *Let X be a Fano n-fold with Condition $(*)$. If $n \le 2\rho_X + 1$, then one of the following holds:*

(1) *$X \simeq Y \times M$, where Y is a Fano manifold with Condition $(*)$ and M is a complete flag manifold.*
(2) *$X \simeq (\mathbb{P}^2)^{\rho_X}$, $(\mathbb{P}^2)^{\rho_X - 1} \times \mathbb{P}^3$, $(\mathbb{P}^2)^{\rho_X - 1} \times \mathbb{Q}^3$, $(\mathbb{P}^2)^{\rho_X - 2} \times \mathbb{P}_{\mathbb{Q}^4}(\mathscr{S}_{\mathbb{Q},i})$ or $(\mathbb{P}^2)^{\rho_X - 2} \times \mathbb{P}(T_{\mathbb{P}^3})$. In particular X is homogeneous in this case.*

PROOF. The proof of this proposition proceeds by induction on n and, once the assertion in the case of $n = 4$ or 5 with Picard number two is proved, then the same argument as in the proof of Theorem 2.3.1 works if we replace Proposition 1.2.3 with Proposition 1.3.1, (note that the assertion in the case of $\rho_X = 1$ is trivial by our definition of Condition $(*)$). On the other hand, the same argument to classify CP n-folds ($n = 4$, 5) with Picard number two in [**CP93**, **Wat14a**] does work to deduce the assertion in the case of $n = 4$ or 5 with Picard number two. Here we sketch the argument shortened by using the result of [**OSCWW17**] (= Theorem 1.2.6).

Let X be a Fano n-fold with Picard number two which satisfies Condition $(*)$. Then there exist two elementary contractions $\pi\colon X \to Y$ and $p\colon X \to Z$:

$$\begin{tikzcd} & X \arrow[dl, "\pi"'] \arrow[dr, "p"] & \\ Y & & Z. \end{tikzcd}$$

By Condition $(*)$ and Proposition 1.3.1, all fibers of π and p, Y and Z are rational homogeneous manifolds with Picard number one. We have $\dim Y + \dim Z \ge \dim X$, and we may assume that $\dim Y \ge \dim Z$.

CASE. $n = 4$

Then we have (a) $\dim Y = 3$ or (b) $\dim Y = \dim Z = 2$.

Assume $\dim Y = 3$. If $\dim Z = 3$, then, by Theorem 1.2.6, X is a complete flag manifold, hence it is homogeneous. If $\dim Z \le 2$, then $X \simeq \mathbb{P}^1 \times Y$ by the classification of Fano bundles of rank two on $\mathbb{P}^3$ and $\mathbb{Q}^3$ [**SW90a**, **SSW91**].

Assume $\dim Y = \dim Z = 2$. Then by [**NO07**, Lemma 4.1] $X \simeq \mathbb{P}^2 \times \mathbb{P}^2$.

CASE. $n = 5$

Then we have (a) $\dim Y = 4$ or (b) $\dim Y = 3 \geq \dim Z \geq 2$.

Assume that $\dim Y = 4$. Note that the projectivization of the stable vector bundle of rank two on $\mathbb{Q}^4$ with Chern classes $c_1 = -1$ and $c_2 = (1,1)$ does not satisfy Condition $(*)$ (see for instance the proof of [**Wat14a**, Lemma 3.8]). Hence by [**APW94**] we have $X \simeq \mathbb{P}^1 \times Y$ or $\mathbb{P}_{\mathbb{Q}^4}(\mathscr{S}_{\mathbb{Q},i})$, where $\mathscr{S}_{\mathbb{Q},i}$ is one of the spinor bundles on $\mathbb{Q}^4$.

Assume that $\dim Y = 3$. Then, by [**OW02**, Theorem 2] and [**NO07**, Lemma 4.1], $X \simeq \mathbb{P}^2 \times Y$ or $\mathbb{P}(T_{\mathbb{P}^3})$. □

2.1. The case $\rho_X > n-5$. We briefly check Theorem 3.0.1 in the case $\rho_X > n-5$.

THEOREM 3.2.2. *Let X be a Fano n-fold with Condition $(*)$ and $\rho_X > n-5$. Then X is a rational homogeneous manifold.*

PROOF. Note that the assertion for $\rho_X = 1$ is trivial by the definition of Condition $(*)$. The assertion follows from the same argument to classify CP manifolds with $\rho_X > 1$ and $\rho_X > n-5$ in Chapter 2. Here we include only a sketch of the proof.

By Proposition 3.2.1 and induction on n, we may assume that $n > 2\rho_X + 1$. Since $\rho_X > n-5$, we have $n = 6$ and $\rho_X = 2$.

This case is treated in the proof of Theorem 2.3.4. To apply the same argument, we need to check that Propositions 2.1.4 and 2.1.5 are valid only assuming that $\mathscr{E}$ is a vector bundle whose projectivization satisfies Condition $(*)$. Note that the argument given in Chapter 2 Section 1 only uses Proposition 1.2.3 and the classification of CP manifolds of dimension ≤ 5 with Picard number one. Thus, by replacing Proposition 1.2.3 with Proposition 1.3.1, we see that Propositions 2.1.4 and 2.1.5 are valid for $\mathscr{E}$ whose projectivization satisfies Condition $(*)$.

□

3. Case $\rho_X = n-5$

In this section, we shall complete our proof of Theorem 3.0.1 in the case $\rho_X = n-5$. Before proving it, we include here the classification of Fano manifold of dimension ≤ 6 with Condition $(*)$ for convenience of the readers. This is equivalent to the classification of rational homogeneous manifolds of dimension ≤ 6 by Theorem 3.2.2.

PROPOSITION 3.3.1. *Fano manifold of dimension ≤ 6 with Condition $(*)$ is one of the following:*

$\dim X$	ρ_X	X
6	1	$\mathbb{P}^6$, $\mathbb{Q}^6$, $\mathrm{Gr}(2,5)$ *or* $\mathrm{LG}(3,6)$
	2	$\mathbb{P}^1 \times \mathbb{P}^5$, $\mathbb{P}^1 \times \mathbb{Q}^5$, $\mathbb{P}^1 \times K(G_2)$, $\mathbb{P}(\mathscr{C})$, $\mathbb{P}^2 \times \mathbb{P}^4$, $\mathbb{P}^2 \times \mathbb{Q}^4$, $(\mathbb{P}^3)^2$, $\mathbb{P}^3 \times \mathbb{Q}^3$ *or* $(\mathbb{Q}^3)^2$

	3	$(\mathbb{P}^1)^2 \times \mathbb{P}^4$, $(\mathbb{P}^1)^2 \times \mathbb{Q}^4$, $\mathbb{P}^1 \times \mathbb{P}_{\mathbb{Q}^4}(\mathscr{S}_{\mathbb{Q},i})$, $\mathbb{P}^1 \times \mathbb{P}(T_{\mathbb{P}^3})$, $\mathbb{P}^1 \times \mathbb{P}^2 \times \mathbb{P}^3$, $\mathbb{P}^1 \times \mathbb{P}^2 \times \mathbb{Q}^3$, $(\mathbb{P}^2)^3$, $\mathbb{P}^2 \times \mathbb{P}(\mathscr{N})$, $\mathrm{Fl}(1,2,3;4)$, $\mathbb{P}(T_{\mathbb{P}^2}) \times \mathbb{P}^3$ *or* $\mathbb{P}(T_{\mathbb{P}^2}) \times \mathbb{Q}^3$
	4	$(\mathbb{P}^1)^3 \times \mathbb{P}^3$, $(\mathbb{P}^1)^3 \times \mathbb{Q}^3$, $(\mathbb{P}^1)^2 \times \mathbb{P}(\mathscr{N})$, $(\mathbb{P}^1)^2 \times (\mathbb{P}^2)^2$, $\mathbb{P}^1 \times \mathbb{P}(T_{\mathbb{P}^2}) \times \mathbb{P}^2$ *or* $(\mathbb{P}(T_{\mathbb{P}^2}))^2$
	5	$(\mathbb{P}^1)^4 \times \mathbb{P}^2$ *or* $(\mathbb{P}^1)^3 \times \mathbb{P}(T_{\mathbb{P}^2})$
	6	$(\mathbb{P}^1)^6$
5	1	$\mathbb{P}^5$, $\mathbb{Q}^5$ *or* $K(G_2)$
	2	$\mathbb{P}^1 \times \mathbb{P}^4$, $\mathbb{P}^1 \times \mathbb{Q}^4$, $\mathbb{P}_{\mathbb{Q}^4}(\mathscr{S}_{\mathbb{Q},i})$, $\mathbb{P}(T_{\mathbb{P}^3})$, $\mathbb{P}^2 \times \mathbb{P}^3$ *or* $\mathbb{P}^2 \times \mathbb{Q}^3$
	3	$(\mathbb{P}^1)^2 \times \mathbb{P}^3$, $(\mathbb{P}^1)^2 \times \mathbb{Q}^3$, $\mathbb{P}^1 \times \mathbb{P}(\mathscr{N})$ $\mathbb{P}^1 \times (\mathbb{P}^2)^2$ *or* $\mathbb{P}(T_{\mathbb{P}^2}) \times \mathbb{P}^2$
	4	$(\mathbb{P}^1)^3 \times \mathbb{P}^2$ *or* $(\mathbb{P}^1)^2 \times \mathbb{P}(T_{\mathbb{P}^2})$
	5	$(\mathbb{P}^1)^5$
4	1	$\mathbb{P}^4$ *or* $\mathbb{Q}^4$
	2	$\mathbb{P}^1 \times \mathbb{P}^3$, $\mathbb{P}^1 \times \mathbb{Q}^3$, $\mathbb{P}(\mathscr{N})$ *or* $(\mathbb{P}^2)^2$
	3	$(\mathbb{P}^1)^2 \times \mathbb{P}^2$ *or* $\mathbb{P}^1 \times \mathbb{P}(T_{\mathbb{P}^2})$
	4	$(\mathbb{P}^1)^4$
3	1	$\mathbb{P}^3$ *or* $\mathbb{Q}^3$
	2	$\mathbb{P}^1 \times \mathbb{P}^2$ *or* $\mathbb{P}(T_{\mathbb{P}^2})$
	3	$(\mathbb{P}^1)^3$
2	1	$\mathbb{P}^2$
	2	$(\mathbb{P}^1)^2$
1	1	$\mathbb{P}^1$

First we show the assertion for 7-folds with Picard number two (Proposition 3.3.3).

NOTATION 3.3.2. Let X be a Fano 7-fold with Picard number two which satisfies Condition $(*)$. Then there are two different elementary contractions $\pi\colon X \to Y$ and $p\colon X \to Z$:

$$\begin{tikzcd} & X \arrow[dl, "\pi"'] \arrow[dr, "p"] & \\ Y & & Z. \end{tikzcd}$$

We have $\dim Y + \dim Z \geq \dim X$ and may assume that $\dim Y \geq \dim Z$.

PROPOSITION 3.3.3. *Let the notation be as above. Then X is one of the following:*

(1) $\mathbb{P}^r \times Y$ *or* $\mathbb{Q}^3 \times Y$, *where Y is a rational homogeneous manifold.*
(2) *Flag varieties* $F(1,2;5)$ *or* $F(1,4;5)$.
(3) X_0 *as in Definition 3.1.1.*

In particular, Theorem 3.0.1 holds in this case.

PROOF. By the equations $\dim Y + \dim Z \geq \dim X$ and $\dim Y \geq \dim Z$, we have $\dim Y \geq 4$.

First assume $\dim Y = 6$. Then Y is isomorphic to $\mathbb{P}^6$, $\mathbb{Q}^6$, Grassmannian $\mathrm{Gr}(2,5)$ or Lagrangian Grassmannian $\mathrm{LG}(3,6)$. Moreover every π-fiber is isomorphic to $\mathbb{P}^1$. Since the Brauer group of Y is trivial, $\pi\colon X \to Y$ is a projective bundle $\mathbb{P}(\mathscr{E}) \to Y$, where $\mathscr{E}$ is a Fano bundle of rank two on Y. Note that the fourth Betti number $b_4(\mathrm{LG}(3,6)) = 1$. Then, by the classification of Fano bundles of rank two [**APW94**], [**MOSC12a**] and [**MOSC14**, Lemma 6.1 and Theorem 6.5], $\mathscr{E}$ is a direct sum of line bundles or the universal subbundle on $\mathrm{Gr}(2,5)$ up to twist with a line bundle.

If $\mathscr{E}$ splits, then $\mathscr{E} \simeq \mathcal{O}^{\oplus 2}$ since the other contraction p is of fiber type by Condition $(*)$. Hence $X \simeq \mathbb{P}^1 \times Y$.

If $\mathscr{E}$ is the universal subbundle on $\mathrm{Gr}(2,5)$, then $X \simeq \mathrm{Fl}(1,2;5)$.

Second assume $\dim Y = 5$. Then by Condition $(*)$, p is a $\mathbb{P}^2$-bundle and hence the assertion follows from Theorem 3.1.6.

Finally we assume that $\dim Y = 4$. Then π is a $\mathbb{P}^3$-bundle or a smooth $\mathbb{Q}^3$-fibration on Y, and $\dim Z = 3$ or 4.

If $\dim Z = 3$, then $-K_\pi$ is nef by Proposition 2.2.1, and hence $-K_\pi$ defines the contraction p. Then, for a p-fiber F', we have

$$-K_{F'} = -K_X|_{F'} = (-\pi^* K_Y - K_\pi)|_{F'} = -\pi^* K_Y|_{F'}.$$

Therefore the morphism $F' \to Y$ is étale, and hence isomorphism. This implies that $X \simeq Y \times Z$, and the assertion follows since Y and Z are homogeneous by Condition $(*)$.

Hence we may assume that $\dim Z = 4$. In this case it is enough to show that π and p are smooth $\mathbb{P}^3$-fibrations by [**OW02**, Theorem 2]. Assume to the contrary that one of the morphisms is a smooth $\mathbb{Q}^3$-fibration. We may assume that π is a smooth $\mathbb{Q}^3$-fibration. Then, by the Serre spectral sequence, $H^2(X,\mathbb{Z}) \to H^2(F,\mathbb{Z})$ is surjective, where F is a π-fiber. Hence there exists a vector bundle $\mathscr{E}$ of rank five on Y such that X is a (relative) quadric bundle in $\mathbb{P}(\mathscr{E})$, more precisely;

(1) $X \subset \mathbb{P}(\mathscr{E})$ and $X \in |2\,\xi + m\,\varphi^* H_Y|$, where ξ is the tautological divisor on $\mathbb{P}(\mathscr{E})$ and H_Y is the ample generator of $\mathrm{Pic}(Y)$.
(2) $\mathscr{E}^* \simeq \mathscr{E}(mH_Y)$ by the section $s \in H^0(S^2\mathscr{E}(mH_Y))$ corresponding to $X \in |2\,\xi + m\,\varphi^* H_Y|$.

$$\begin{tikzcd} X \arrow[r, hook, "i"] \arrow[d, "\pi"'] & \mathbb{P}(\mathscr{E}) \arrow[dl, "\varphi"] \\ Y. & \end{tikzcd}$$

Since the rank of $\mathscr{E}$ is odd, m is an even number by (2). Hence we may assume that $m = 0$ by twisting $\mathscr{E}$ with a line bundle.

Note that, since $\mathscr{E} \simeq \mathscr{E}^*$, odd Chern classes of $\mathscr{E}$ vanish. Thus the Grothendieck relation of $\mathbb{P}(\mathscr{E})$ reads

$$\xi^5 + \varphi^* c_2(\mathscr{E}) \cdot \xi^3 + \varphi^* c_4(\mathscr{E}) \cdot \xi = 0. \tag{3.3.3.1}$$

Also note that $-K_\pi = 3\xi|_X$ by the adjunction.

Let τ be the slope for $\pi\colon X \to Y$, that is the number τ such that $-K_\pi + \tau\pi^* H_Y$ is nef but not ample (cf. Subsection 1.2 for projectivized vector bundles). Then g is defined by the divisor $-K_\pi + \tau\pi^* H_Y$, and hence

$$(-K_\pi + \tau\pi^* H_Y)^6 \cdot \pi^* H_Y = (-K_\pi + \tau\pi^* H_Y)^5 \cdot \pi^* H_Y^2 = 0.$$

Since $X \in |2\xi|$, we can rewrite these as follows:

$$(3\xi + \tau\varphi^* H_Y)^6 \cdot \varphi^* H_Y \cdot \xi = (3\xi + \tau\varphi^* H_Y)^5 \cdot \varphi^* H_Y^2 \cdot \xi = 0 \text{ on } \mathbb{P}(\mathscr{E}).$$

Combining with the Grothendieck relation (3.3.3.1), we have

$$10H_Y^4\,\tau^2 - 3^2 c_2(\mathscr{E}) H_Y^2 = 0,$$
$$10H_Y^4\,\tau^3 - 3^3 c_2(\mathscr{E}) H_Y^2\,\tau = 0.$$

This implies that $\tau = 0$ and $c_2(\mathscr{E})H_Y^2 = 0$. Hence $-K_\pi$ is nef but not ample, and some multiple of it defines the other contraction p. It follows from the Grothendieck relation (3.3.3.1) and $c_2(\mathscr{E})H_Y^2 = 0$ that $(-K_\pi)^4 \cdot \pi^* H_Y^3 = 0$. Hence we have $\dim Z = 3$, which contradicts our assumption $\dim Z = 4$. $\square$

Second we show the assertion in the case of 8-folds with Picard number three:

Proposition 3.3.4. *Let X be a Fano 8-fold with Picard number three which satisfies Condition $(*)$. Then X is a complete flag manifold or a product of two Fano manifolds with Condition $(*)$. In particular, Theorem 3.0.1 holds in this case.*

Proof. By the assumption on X, there exists the following commutative diagram whose arrows are pairwise distinct elementary rational homogeneous fibrations by Proposition 1.3.1:

(3.3.4.1)
$$\begin{tikzcd}
X \arrow[rr] \arrow[dr, "f_1"] \arrow[dd, "f_2"'] & & X_3 \arrow[dr] \arrow[dd] & \\
& X_1 \arrow[rr] \arrow[dd, "g_1"] & & X_{3,1} \arrow[dd] \\
X_2 \arrow[rr] \arrow[dr, "g_2"'] & & X_{2,3} \arrow[dr] & \\
& X_{1,2} \arrow[rr] & & \mathrm{pt}.
\end{tikzcd}$$

Set $h := g_1 \circ f_1 = g_2 \circ f_2$.

If every elementary contraction of X is a smooth $\mathbb{P}^1$-fibrations, then X is a complete flag manifold by Theorem 1.2.6, and the assertion follows. If X admits a contraction π onto a complete flag manifold M, then X is isomorphic to the product of a π-fiber and M by Proposition 1.3.2, and the assertion follows also in this case.

Therefore, we may assume that X has an elementary contraction which is not a smooth $\mathbb{P}^1$-fibration and that X does not admit a contraction onto a complete flag manifold. By renumbering if necessary, we may assume that $\dim X_1 \leq 6$. Then, by our assumption and Propositions 1.3.1 and 3.3.1, X_1 is one of the following homogeneous manifolds:

(1) $\mathbb{P}^2 \times \mathbb{P}^4$, $\mathbb{P}^2 \times \mathbb{Q}^4$, $\mathbb{P}^2 \times \mathbb{P}^3$, $\mathbb{P}^2 \times \mathbb{Q}^3$, $(\mathbb{P}^2)^2$,
(2) $\mathbb{P}_{\mathbb{Q}^4}(\mathscr{S}_{\mathbb{Q},i})$, $\mathbb{P}(T_{\mathbb{P}^3})$,
(3) $\mathbb{P}^3 \times \mathbb{Q}^3$, $(\mathbb{P}^3)^2$, $(\mathbb{Q}^3)^2$.

If case (1) or (2) occurs, then we may assume that g_1 is a $\mathbb{P}^2$-bundle. Every h-fiber is a Fano manifold with Condition $(*)$ of dimension 4, 5 or 6 by Proposition 1.3.1 and it admits a contraction onto $\mathbb{P}^2$. Then every h-fiber is isomorphic to $\mathbb{P}^2 \times$ (an f_1-fiber) by Proposition 3.3.1. Hence the square on the left of (3.3.4.1) is a Cartesian product and f_2 is a $\mathbb{P}^2$-bundle.

Therefore, if Case (1) occurs, then we have $X \simeq X_{3,1} \times X_2$ since the squares on the left and the front are Cartesian products.

On the other hand, if Case (2) occurs, then we have $X_{1,2} \simeq \mathbb{P}^3$. Then (an f_1-fiber) $\simeq$ (a g_2-fiber) $\simeq \mathbb{P}^3$ or $\mathbb{Q}^3$. Hence $X_2 \simeq X_{1,2} \times$ (a g_2-fiber) and $X_{2,3} \simeq$ (a g_2-fiber) by Propositions 1.3.1 and 3.3.1. Then the square on the bottom of (3.3.4.1) is a Cartesian product and $X \simeq X_1 \times \mathbb{P}^3$ or $X_1 \times \mathbb{Q}^3$.

Assume that Case (3) occurs. If the square on the left of (3.3.4.1) is a Cartesian product, then $X \simeq X_2 \times X_{3,1}$ and the assertion follows. Hence we may assume that the square on the left is not a Cartesian product. Then h-fibers are isomorphic to $\mathbb{P}_{\mathbb{Q}^4}(\mathscr{S}_{\mathbb{Q},i})$ or $\mathbb{P}(T_{\mathbb{P}^3})$ by Propositions 1.3.1 and 3.3.1. In particular we have that g_1 is a $\mathbb{P}^3$-bundle. In this case, $f_2\colon X \to X_2$ is the family of linear subspaces in g_1-fibers. By the universality of Hilbert schemes, we have $X \simeq X_{1,2} \times \mathbb{P}_{\mathbb{Q}^4}(\mathscr{S}_{\mathbb{Q},i})$ or $X_{1,2} \times \mathbb{P}(T_{\mathbb{P}^3})$. $\square$

We prove the following, which completes the proof of our main theorem:

THEOREM 3.3.5. *Let X be a Fano n-fold with Condition $(*)$ and Picard number $\rho_X = n-5$. Then X is a rational homogenous manifold or $(\mathbb{P}^1)^{n-7} \times$ (X_0 as in Definition 3.1.1).*

PROOF. We proceed by induction on n. The assertion in the case $n = 6$ follows from the definition and the assertion for the cases $n = 7$ or $n = 8$ follows from Propositions 3.3.3 and 3.3.4. If $n > 8$, then $n \leq 2\rho_X + 1$ holds. Therefore, by Proposition 3.2.1, X is homogeneous or a product $Y \times M$, where M is a complete flag manifold. In the latter case, since

$$\dim Y - \rho_Y \leq \dim X - \rho_X = 5,$$

Y is a rational homogenous manifold or $(\mathbb{P}^1)^{\dim Y - 7} \times X_0$ by our inductive hypothesis.

If Y is a rational homogenous manifold, then the assertion follows. If Y is isomorphic to $(\mathbb{P}^1)^{\dim Y - 7} \times X_0$, then $\dim M = \rho_M$. Hence $M \simeq (\mathbb{P}^1)^{\dim X - \dim Y}$ by [**BCDD03**, Proposition 2.4], [**NO07**, Proposition 5.1] or [**Wat14a**, Proposition 2.3]. This completes the proof. $\square$

Finally, we prove Corollary 3.0.2.

PROOF OF COROLLARY 3.0.2. By Theorem 3.0.1, it is enough to show that every CP n-fold with $\rho_X > n-6$ satisfies Condition $(*)$. The proof proceeds by induction on ρ_X. Note that every CP manifold of dimension ≤ 5 with Picard number one is a rational homogeneous manifold by Theorem 1.2.2. Hence, by our assumption, every CP manifold of dimension ≤ 6 with Picard number one is a rational homogeneous manifold, and hence satisfies Condition $(*)$.

Let X be a CP n-fold with $\rho_X > n-6$ and $\rho_X > 1$. Suppose that

$$X \xrightarrow{f_1} X_1 \xrightarrow{f_2} \cdots \xrightarrow{f_{m-1}} X_{m-1} \xrightarrow{f_m} X_m$$

is a sequence of elementary contractions. Then X_1 is again a CP manifold by Proposition 1.2.3. Hence by our inductive hypothesis X_1 satisfies Condition $(*)$, and hence f_i for $i \geq 2$ are rational homogeneous fibrations. Also, by Proposition 1.2.3, every f_1-fiber F is a CP manifold with Picard number one and $\rho_{X_1} \leq \dim X_1$. Then we have

$$\dim F = \dim X - \dim X_1 \leq \dim X - \rho_{X_1} = \dim X - \rho_X + 1 < 7.$$

Hence F is a rational homogeneous fibration by our assumption. This completes the proof. $\square$

Part 2

On Fano manifolds and ample vector bundles

CHAPTER 4

Classification of Mukai pairs with corank 3

Introduction

A *Mukai pair* of dimension n and rank r is, by definition, a pair $(X, \mathscr{E})$ of a smooth Fano n-fold X and an ample vector bundle $\mathscr{E}$ of rank r on X with $c_1(X) = c_1(\mathscr{E})$. Study of such pairs was proposed by Mukai [**Muk88**] in relation to Fano manifolds with large index or based on Mori's solution to the Hartshorne conjecture.

The rank r of Mukai pairs is related to the *indices* of Fano manifolds. The *Fano index*, or simply the *index*, of a Fano manifold X is the greatest integer which divides $c_1(X)$ in $\operatorname{Pic}(X)$. If the index of a Fano n-fold X is r, then $(X, \bigoplus \mathcal{O}(d_i H_X))$ gives a Mukai pair of dimension n and rank $\leq r$, where $H_X \coloneqq -K_X/r$, $d_i > 0$ and $r = \sum d_i$. Thus the study of Fano n-folds of index r is essentially the same as the study of Mukai pairs $(X, \mathscr{E})$ of dimension n and rank $\leq r$ such that $\mathscr{E}$ splits into a direct sum of line bundles (Mukai pairs of split type). Conversely, by associating $\mathbb{P}(\mathscr{E})$ with $(X, \mathscr{E})$, we obtain a one-to-one correspondence between Mukai pairs $(X, \mathscr{E})$ of dimension n and rank r, and Fano $(n + r - 1)$-folds of index r with $\mathbb{P}^{r-1}$-bundle structures (see e.g. [**NO07**, Proposition 3.3] for a proof).

It is known that the index r_X of a Fano n-fold X satisfies $r_X \leq n + 1$, and the nonnegaitve integer $n - r_X + 1$ is called the *coindex* of X. As is well known, the structure of X is simpler if the coindex is small, hence we can conduct detailed analysis of X provided its coindex is small enough. For example, a classical theorem of Kobayashi-Ochiai shows that Fano manifolds with coindex 0 or 1 is isomorphic to projective space $\mathbb{P}^n$ or hyperquadric $\mathbb{Q}^n$, respectively [**KO73**]. Fujita gave a complete list of Fano manifolds with coindex 2 (*del Pezzo manifolds*) [**Fuj82a**, **Fuj82b**], while Mukai classified Fano manifolds with coindex 3 (*Mukai manifolds*) [**Muk89**] (cf. [**Mel99**, **Amb99**]).

In keeping with the above observation, the *corank* of a Mukai pair $(X, \mathscr{E})$ of dimension n and rank r is analogously defined to be the integer $c = n - r + 1$, and one can expect that the classification of Mukai pairs of corank c is possible if c is small enough. Since there exists a rational curve C on X such that $n + 1 \geq -K_X.C = c_1(\mathscr{E}).C \geq r$ [**Mor79**], the corank of a given Mukai pair $(X, \mathscr{E})$ is nonnegative. For $(X, \mathscr{E})$ with the smallest or the second smallest corank $c = 0$ or 1, Mukai made explicit conjectures on their structure, which were confirmed independently by Fujita, Peternell and Ye-Zhang:

Theorem 4.0.1 ([**Fuj92**, **Pet90**, **Pet91**, **YZ90**]).

(1) *A Mukai pair $(X, \mathscr{E})$ of dimension n and rank $n + 1$ is isomorphic to* $\left(\mathbb{P}^n, \mathcal{O}(1)^{\oplus n+1}\right)$.

(2) *A Mukai pair* $(X, \mathscr{E})$ *of dimension* n *and rank* n *is isomorphic to either*

$$(\mathbb{P}^n, T_{\mathbb{P}^n}),\ \big(\mathbb{P}^n, \mathcal{O}(2) \oplus \mathcal{O}(1)^{\oplus n-1}\big) \text{ or } \big(\mathbb{Q}^n, \mathcal{O}(1)^{\oplus n}\big).$$

Thus $(\mathbb{P}^n, T_{\mathbb{P}^n})$ is the unique Mukai pair of non-split type with corank $c \leq 1$. The case corank $c = 2$ was treated by Peternell-Szurek-Wiśniewski:

Theorem 4.0.2 ([**Wiś89b**] for the case $n = 3$; [**PSW92b**] for higher dimension (cf. [**Occ05**])). *Let* $(X, \mathscr{E})$ *be a Mukai pair of dimension* n *and rank* $n - 1$*. Then:*

(1) X *is isomorphic to either* $\mathbb{P}^n$, $\mathbb{Q}^n$*, a del Pezzo manifold or* $\mathbb{P}^1 \times \mathbb{P}^2$ $(n = 3)$.
(2) $(X, \mathscr{E})$ *of non-split type (i.e.,* $\mathscr{E}$ *is not a direct sum of line bundles) is isomorphic to one of the following four pairs:*
   (a) $\big(\mathbb{P}^3, \mathscr{N}(2)\big)$*, where* $\mathscr{N}$ *is the null-correlation bundle.*
   (b) $\Big(\mathbb{Q}^4, \mathscr{S}^*_{\mathbb{Q}}(1) \oplus \mathcal{O}(1)\Big)$*, where* $\mathscr{S}^*_{\mathbb{Q}}$ *is the dual of spinor bundle.*
   (c) $\Big(\mathbb{Q}^3, \mathscr{S}^*_{\mathbb{Q}}(1)\Big)$.
   (d) $\big(\mathbb{P}^1 \times \mathbb{P}^2, p_1^*\mathcal{O}(1) \otimes p_2^* T_{\mathbb{P}^2}\big)$.

It is noteworthy that in the above list appear the null-correlation bundle and spinor bundles, which are closely related to representation theory. This fact implies that we may find out further interesting vector bundles and their interplay with geometry of homogeneous spaces in the course of classification of Mukai pairs of larger corank.

Such an anticipation in mind, we extend in this chapter the preceding classification results to the next case corank $c = 3$:

Theorem 4.0.3. *Let* $(X, \mathscr{E})$ *be a Mukai pair of dimension* $n \geq 5$ *and rank* $n - 2$*. Then:*

(1) X *is isomorphic to either* $\mathbb{P}^n$, $\mathbb{Q}^n$*, a del Pezzo manifold, a Mukai manifold or* $\mathbb{P}^2 \times \mathbb{P}^3$ $(n = 5)$.
(2) $(X, \mathscr{E})$ *of non-split type is isomorphic to one of the following eight pairs:*
   (a) $\Big(\mathbb{Q}^6, \mathscr{S}^*_{\mathbb{Q}}(1)\Big)$.
   (b) $\big(\mathbb{Q}^6, \mathscr{G}_{\mathbb{Q}}(1) \oplus \mathcal{O}(1)\big)$.
   (c) $\big(\mathbb{Q}^5, \mathscr{G}_{\mathbb{Q}}(1)\big)$.
   (d) $(\mathrm{Gr}(2,5), \mathscr{S}^*_{\mathrm{Gr}}(1) \oplus \mathcal{O}(1) \oplus \mathcal{O}(1))$.
   (e) $(\mathrm{Gr}(2,5), \mathscr{Q}_{\mathrm{Gr}}(1) \oplus \mathcal{O}(1))$.
   (f) $\big(V_5, \mathscr{S}^*_{V_5}(1) \oplus \mathcal{O}(1)\big)$.
   (g) $(V_5, \mathscr{Q}_{V_5}(1))$.
   (h) $\big(\mathbb{P}^2 \times \mathbb{P}^3, p_1^*\mathcal{O}(1) \otimes p_2^* T_{\mathbb{P}^3}\big)$.

(see Notation 0.0.2 for the symbols used here)

Remark 4.0.4. In Theorem 4.0.3, the missing cases $n = 3$ and 4 were (almost) settled by preceding works. If $n = 3$ and $r = 1$, then $\mathscr{E} = \mathcal{O}(-K_X)$ and the classification of such Mukai pairs is simply the classification of Fano 3-folds, which was completed by milestone articles by Fano, Iskovskih, Shokurov, Fujita, Mori and Mukai (see [**IP99**] and references therein). 4-dimensional Mukai pairs $(X, \mathscr{E})$ of rank 2 corresponds to Fano 5-folds of index 2 with $\mathbb{P}^1$-bundle structures. Novelli and Occhetta gave a list of all possible candidates of such 5-folds in [**NO07**]. One of the candidates therein, unfortunately, is not yet known to actually exist.

0.1. Study of *generalized polarized pairs* gives another motivation to investigate Mukai pairs. A pair $(X, \mathscr{E})$ is called a *generalized polarized pair* of dimension n and rank r if X is a smooth projective n-fold and $\mathscr{E}$ is an ample vector bundle of rank r. The adjoint divisor $K_X + c_1(\mathscr{E})$ is attached to a given generalized polarized pair $(X, \mathscr{E})$, and a fundamental problem in this field is to determine when the *adjoint divisor* $K_X + c_1(\mathscr{E})$ satisfy positivity (e.g., ampleness or nefness) or to distinguish generalized polarized pairs whose adjoint divisors lack positivity from general ones. Such a problem is carried out in a number of papers, including [**Wiś89a**, **YZ90**, **Zha91**, **Fuj92**, **ABW92**, **Zha96**, **AM97**, **Ohn06**, **Tir13**]. In [**AM97**], Andreatta and Mella studied the case $r = n - 2$ and they clarified when the adjoint divisor is not nef. Also, assuming that $K_X + c_1(\mathscr{E})$ is nef but not ample, they (roughly) described the structure of the contraction defined by the adjoint divisor. Understandably the contraction can be trivial, which implies that $(X, \mathscr{E})$ is a Mukai pair [**AM97**, Theorem 5.1 (2) (i)]. Our result gives a detailed classification in such a case.

Also, given a generalized polarized pair $(X, \mathscr{E})$ of dimension n and rank r, the geometry of the zero locus S of a section $s \in H^0(\mathscr{E})$ is studied in several context, provided that S has the expected dimension $n - r$. For example, in [**Lan96**, Corollary 1.3], it is proved that if S as above is a minimal surface of Kodaira dimension $= 0$, then S is a K3 surface and $(X, \mathscr{E})$ is a Mukai pair of corank 3. Thus:

Corollary 4.0.5. *Let $(X, \mathscr{E})$ be a generalized polarized pair of dimension $n \geq 5$ and rank $n - 2$. Suppose that there is a K3 surface $S \subset X$ which is a zero locus of a section $s \in H^0(\mathscr{E})$. Then $(X, \mathscr{E})$ is one of the pairs as in Theorem 4.0.3.*

0.2. We sketch an outline of this chapter. Let $(X, \mathscr{E})$ be a pair as in Theorem 4.0.3. Then the *length* l_X is defined as the minimum anticanonical degree of *free* rational curves on X (see Definition 4.1.1). The length l_X is at most $n + 1$ by Mori's theorem. In addition, the existence of the bundle $\mathscr{E}$ implies that l_X is at least $n - 2$;

$$l_X \in \{\, n - 2, \dots, n + 1 \,\}.$$

The proof is roughly divided into four cases depending on the value l_X.

In Section 1, we treat some easy cases with preliminaries on family of rational curves. Firstly the case $\rho_X \geq 2$ is settled (Proposition 4.1.4), which allow us to assume $\rho_X = 1$ in the sequel. Then $\mathbb{P}(\mathscr{E})$ is a Fano manifold with Picard number two and index $n - 2$. Secondly we treat the case $l_X = n - 2$ (Proposition 4.1.10). Thirdly we deal the case $\ell(R_\varphi) > n - 2$ (Proposition 4.1.14), where R_φ is the extremal ray which is not contracted by the projection $\pi\colon \mathbb{P}(\mathscr{E}) \to X$ and $\ell(R_\varphi)$ is the *length* of the extremal ray. Note that $\ell(R_\varphi) \geq n - 2$ since the index $r_{\mathbb{P}(\mathscr{E})} = n - 2$.

From the above, we can assume three conditions $\rho_X = 1$, $l_X \geq n - 1$ and $\ell(R_\varphi) = n - 2$ in the remaining sections. We also include in Section 1 a construction of sections of the projection $\pi\colon \mathbb{P}(\mathscr{E}) \to X$.

Section 2 provides two characterizations of Ottaviani bundle on $\mathbb{Q}^5$, based on [**Ott88**] and Chapter 3 Subsection 1.2.

In Section 3, we will see which rational curves are contracted by φ. More precisely, we will prove that *minimal lifts* of minimal rational curves to the projective bundle $\mathbb{P}(\mathscr{E})$ are contracted by φ, or equivalently the $\mathbb{Q}$-bundle $\mathscr{E}(K_X/l_X)$ is semiample (Theorem 4.3.2, cf. [**PSW92b**, Sect. 3]).

In Section 4, we will treat the case $l_X \geq n$. In this case, by numerical characterizations of projective space and hyperquadric [**CMSB02**, **Miy04**] (cf. [**Keb02**, **DH17**]), X is isomorphic to $\mathbb{P}^n$ or $\mathbb{Q}^n$. The result in Sect. 3 implies that $\mathscr{E}(-1)$ is nef. First we will show that $\mathscr{E}(-1)$ is globally generated. Then we immediately see that $\mathscr{E}$ splits by [**SU14**, **AM13**, **Tir13**] unless $X \simeq \mathbb{Q}^6$ or $\mathbb{Q}^5$. Finally we will deal the case $X \simeq \mathbb{Q}^6$ or $\mathbb{Q}^5$. Here the characterization of Ottaviani bundles plays an important role.

In Sections 5 and 6, the case $l_X = n-1$ is discussed, and the proof of Theorem 4.0.3 will be completed. The crucial case is where φ is of fiber type, which will be treated in Section 6. The key step is to prove $\dim X \leq 6$ (Proposition 4.6.2), and the main ingredients of the proof are

(1) Chain connectedness of X by the images of φ-fibers and
(2) Miyaoka's criterion on semistability of vector bundles on curves [**Miy87**].

Notation 4.0.6. We work over the field of complex numbers and use the following notations:

(1) $\mathbb{P}(\mathscr{E})$ is the Grothendieck projectivisation of the bundle $\mathscr{E}$.
(2) $\pi\colon \mathbb{P}(\mathscr{E}) \to X$ is the natural projection.
(3) $\xi_{\mathscr{E}} = \xi$ is the relative tautological divisor of $\mathbb{P}(\mathscr{E})$.
(4) If $\rho_X = 1$, then H_X is the ample generator of the Picard group of X.
(5) If $\rho_X = 1$, then R_φ is the extremal ray of $\mathrm{NE}(\mathbb{P}(\mathscr{E}))$ which is different from the ray associated to π, and φ is the contraction of R_φ.
(6) $\mathrm{Exc}(\varphi)$ is the exceptional locus of φ.
(7) Given a projective manifold V with an ample (not necessarily very ample) line bundle $\mathcal{O}_V(1)$, we will denote by $\mathcal{O}_V(a_1^{b_1}, \dots, a_m^{b_m}) = \mathcal{O}(a_1^{b_1}, \dots, a_m^{b_m})$ the vector bundle $\mathcal{O}_V(a_1)^{\oplus b_1} \oplus \cdots \oplus \mathcal{O}_V(a_m)^{\oplus b_m}$.
(8) For a closed subvariety $W \subset V$, we will denote by $\mathrm{NE}(W, V)$ the subcone generated by the classes of the effective curves on W.
(9) For a morphism $f\colon V \to W$ between varieties and a coherent sheaf $\mathscr{M}$ on W, we will denote by $\mathscr{M}|_V$ the pullback $f^*\mathscr{M}$.

1. Preliminaries

The purpose of this section is to present some preliminaries and prove Theorem 4.0.3 in the following cases (Propositions 4.1.4, 4.1.10 and 4.1.14):

(1) $\rho_X > 1$,
(2) $\rho_X = 1$ and $l_X = n-2$ (see Definition 4.1.1),
(3) $\rho_X = 1$ and $\ell(R_\varphi) \neq n-2$ (see Definition 4.1.11).

1.1. Anticanonical degrees of rational curves. In this chapter, the image C of the projective line $\mathbb{P}^1$, or the normalization map $f\colon \mathbb{P}^1 \to C \subset X$ is called a *rational curve*.

DEFINITION 4.1.1. Let X be a Fano manifold.

(1) A rational curve $f\colon \mathbb{P}^1 \to X$ is called *free* if f^*T_X is nef.
(2) (a) The *index* r_X of X is defined as:

$$r_X \coloneqq \max\{\, k \in \mathbb{Z} \mid -K_X = kH \text{ for some ample divisor } H\}.$$

(b) The *pseudoindex* i_X is the minimum anticanonical degree of rational curves:

$$i_X \coloneqq \min\{\, -K_X.C \mid C \text{ is a rational curve on } X\,\}.$$

(c) The *(global) length* l_X is the minimum anticanonical degree of *free* rational curves:

$$l_X \coloneqq \min\{\, -K_X.C \mid C \text{ is a free rational curve on } X\,\}.$$

By these definitions and [**Mor79**] or [**Kol96**, Theorem 5.14], it holds:

$$n + 1 \geq l_X \geq i_X \geq r_X \geq 1.$$

Fano manifolds with large index $r_X \geq n-2$ are classified in [**KO73**, **Fuj82a**, **Fuj82b**, **Muk89**].

In [**CMSB02**, **Miy04**] (cf. [**Keb02**, **DH17**]), numerical characterizations of projective spaces and hyperquadrics are established:

THEOREM 4.1.2. *Let X be a Fano manifold with $l_X \geq n$ and $\rho_X = 1$. Then $X \simeq \mathbb{P}^n$ or $\mathbb{Q}^n$.*

LEMMA 4.1.3. *Let $(X, \mathscr{E})$ be a pair as in Theorem 4.0.3 and $f\colon \mathbb{P}^1 \to X$ a rational curve of anticanonical degree $d \leq n+1$. Then $d \geq n-2$ and the following hold:*

(1) *If $d = n+1$, then $f^*\mathscr{E} \simeq \mathcal{O}(4, 1^{n-3})$, $\mathcal{O}(3, 2, 1^{n-4})$ or $\mathcal{O}(2^3, 1^{n-5})$.*
(2) *If $d = n$, then $f^*\mathscr{E} \simeq \mathcal{O}(3, 1^{n-3})$ or $\mathcal{O}(2^2, 1^{n-4})$.*
(3) *If $d = n-1$, then $f^*\mathscr{E} \simeq \mathcal{O}(2, 1^{n-3})$.*
(4) *If $d = n-2$, then $f^*\mathscr{E} \simeq \mathcal{O}(1^{n-2})$.*

In particular, we have $i_X \geq n-2$.

PROOF. By the Grothendieck theorem every vector bundle on $\mathbb{P}^1$ splits, i.e., it is a direct sum of line bundles, whence $f^*\mathscr{E} \simeq \mathcal{O}(a_1, \ldots, a_{n-2})$ for $a_i \in \mathbb{Z}$. Since $\mathscr{E}$ is ample with $c_1(\mathscr{E}) = c_1(X)$, each a_i is positive and $\sum a_i = d$. Now the assertion is clear. □

1.1.1. *Case $\rho_X > 1$.* Here we settle Theorem 4.0.3 for $\rho_X > 1$:

PROPOSITION 4.1.4. *Let $(X, \mathscr{E})$ be a pair as in Theorem 4.0.3. Assume $\rho_X > 1$. Then:*

(1) *$X \simeq \mathbb{P}^3 \times \mathbb{P}^3$, $\mathbb{P}^2 \times \mathbb{P}^3$, $\mathbb{P}^2 \times \mathbb{Q}^3$, $\mathbb{P}_{\mathbb{P}^3}(\mathcal{O}(1, 0^2))$ or $\mathbb{P}(T_{\mathbb{P}^3})$,*
(2) *$\mathscr{E}$ splits unless $(X, \mathscr{E}) \simeq (\mathbb{P}^2 \times \mathbb{P}^3, p_1^*\mathcal{O}(1) \otimes p_2^*T_{\mathbb{P}^3})$.*

PROOF. From [**Wiś90**, Theorem A] and the assumption $\rho_X > 1$, it follows

$$i_X \leq \frac{1}{2}n + 1.$$

Since $i_X \geq n-2$ by Lemma 4.1.3, we have $n \leq 7$. Moreover, if $n=6$, then the assertion follows from [**AM97**, Lemma 5.3].

If $n=5$, then by [**Fuj16**] X is isomorphic to one of the following:

(1) $\mathbb{P}_{\mathbb{P}^3}(\mathcal{O}(0^3)) \simeq \mathbb{P}^2 \times \mathbb{P}^3$,
(2) $\mathbb{P}_{\mathbb{Q}^3}(\mathcal{O}(0^3)) \simeq \mathbb{P}^2 \times \mathbb{Q}^3$,
(3) $\mathbb{P}_{\mathbb{P}^3}(\mathcal{O}(1,0^2))$,
(4) $\mathbb{P}(T_{\mathbb{P}^3})$.

Note that in each case X admits a $\mathbb{P}^2$-bundle structure $q\colon X \to Y$ with the relative tautological line bundle $\mathcal{O}_q(1)$.

By adjunction, $c_1(\mathscr{E}|_{\mathbb{P}^2}) = c_1(\mathbb{P}^2)$ for each q-fiber $\mathbb{P}^2$. Thus, by Theorem 4.0.1, $\mathscr{E}|_{\mathbb{P}^2} \simeq \mathcal{O}(1^3)$ for each q-fiber $\mathbb{P}^2$. Hence $\mathscr{E}_Y := q_*(\mathscr{E} \otimes \mathcal{O}_q(-1))$ is a vector bundle of rank three with $q^*\mathscr{E}_Y \simeq \mathscr{E} \otimes \mathcal{O}_q(-1)$. Since $\mathscr{E}$ is a Fano bundle, the bundle $\mathscr{E}_Y$ is also a Fano bundle by [**SW90b**, Theorem 1.6] or [**KMM92a**, Corollary 2.9].

If $X \simeq \mathbb{P}(T_{\mathbb{P}^3})$, then there is another $\mathbb{P}^2$-bundle structure $q'\colon X \to Y' \simeq \mathbb{P}^3$ which parametrizes planes on $Y \simeq \mathbb{P}^3$, and $\mathscr{E} \otimes \mathcal{O}_q(-1)$ is q'-relatively trivial by the same reason as above. This implies that $\mathscr{E}_Y$ is trivial on any hyperplane $\mathbb{P}^2$ on Y. Hence $\mathscr{E}_Y$ is trivial by Horrocks' criterion [**Hor64**], [**OSS80**, Theorem 2.3.2].

In the remaining cases there is a section $\widetilde{Y}$ of q with $\mathcal{O}_q(1)|_{\widetilde{Y}} \simeq \mathcal{O}_{\widetilde{Y}}$. Thus we have

$$\mathscr{E}_Y \simeq q^*\mathscr{E}_Y|_{\widetilde{Y}} \simeq (q^*\mathscr{E}_Y \otimes \mathcal{O}_q(1))|_{\widetilde{Y}} \simeq \mathscr{E}|_{\widetilde{Y}}.$$

Therefore $\mathscr{E}_Y$ is an ample vector bundle with

- $c_1(\mathscr{E}_Y) = c_1(Y)$ if $X \simeq \mathbb{P}_{\mathbb{P}^3}(\mathcal{O}(0^3))$ or $\mathbb{P}_{\mathbb{Q}^3}(\mathcal{O}(0^3))$.
- $c_1(\mathscr{E}_Y) = c_1(Y) - 1$ if $X \simeq \mathbb{P}_{\mathbb{P}^3}(\mathcal{O}(1,0^2))$.

Theorem 4.0.1 implies $\mathscr{E}_Y$ splits unless $X \simeq \mathbb{P}_{\mathbb{P}^3}(\mathcal{O}(0^3))$ and $\mathscr{E}_Y \simeq T_{\mathbb{P}^3}$, and the assertion follows. □

1.2. Families of rational curves. For accounts of families of rational curves, our basic references are [**Kol96**, **Deb01**].

Definition 4.1.5. Let X be a Fano manifold and $\mathrm{Ratcurves}^n(X)$ the normalization of the scheme parametrizing rational curves on X.

(1) An irreducible component of $\mathrm{Ratcurves}^n(X)$ is called a *family of rational curves.*

If M is a family of rational curves on X, then there is the following diagram:

$$\begin{array}{ccc} U & \xrightarrow{e} & X \\ {\scriptstyle p}\downarrow & & \\ M, & & \end{array}$$

where $p\colon U \to M$ is the *universal family* and $e\colon U \to X$ is the *evaluation morphism.*

Let M be a family of rational curves on X as above.

(2) The family M is called *unsplit* if it is proper.
(3) The family M is called *dominating* (resp. *covering*) if the morphism e is dominating (resp. surjective).

(4) X is said to be *chain connected* by rational curves in the family M if any two points in X can be connected by a chain of rational curves in this family M.

PROPOSITION 4.1.6 ([**Mor79**], [**Kol96**, Chapter II, Theorems 1.2 and 2.15]). *Let X be a Fano manifold of dimension n, M a family of rational curves on X and C a rational curve belonging to the family M. Then* $\dim M \geq (-K_X).C + n - 3$.

PROPOSITION 4.1.7. *Let $(X, \mathscr{E})$ be a pair as in Theorem 4.0.3 and $\rho_X = 1$. Then there exists an unsplit covering family of rational curves with $(-K_X)$-degree l_X on X. Moreover X is chain connected by rational curves in this family.*

PROOF. By the definition of l_X, there exists a dominating family of rational curves of anticanonical degree l_X on X. If $l_X \geq n$, then $X \simeq \mathbb{P}^n$ or $\mathbb{Q}^n$ by Proposition 4.1.2. Then the family parametrizes lines on X and the assertion follows. Therefore we may assume that $l_X < n$.

Assume that this family is not unsplit. Then there exists a rational curve of $(-K_X)$-degree $\leq l_X/2 < n/2$. By Lemma 4.1.3, we have $n - 2 \leq i_X < \frac{n}{2}$, which implies $n < 4$. This contradicts our assumption $n \geq 5$.

Note that $\rho_X = 1$. The chain connectedness by rational curves in this family follows from [**Deb01**, Proof of Proposition 5.8] or [**KMM92b**, Proof of Lemma 3]. □

DEFINITION 4.1.8. Let $(X, \mathscr{E})$ be a pair as in Theorem 4.0.3 with $\rho_X = 1$.

(1) By taking all the families M_j of rational curves of anticanonical degree l_X, we have the following diagram:

$$\begin{tikzcd} U := \coprod U_j \arrow[r, "e := \coprod e_j"] \arrow[d, "p := \coprod p_j"'] & X \\ M := \coprod M_j, & \end{tikzcd}$$

where $p_j : U_j \to M_j$ is the universal family over M_j and $e_j : U_j \to X$ is the evaluation morphism for each j.

(2) We call a rational curve in one of this family a *minimal rational curve* on X.

(3) The vector bundle $\mathscr{E}$ is said to be *uniform* (resp. *uniform at a point* $x \in X$) if the isomorphism classes of bundles $\mathscr{E}|_{\mathbb{P}^1}$ do not depend on minimal rational curves $f : \mathbb{P}^1 \to X$ (resp. minimal rational curves $f : \mathbb{P}^1 \to X$ such that $x \in f(\mathbb{P}^1)$).

REMARK 4.1.9.

(1) By Proposition 4.1.7 there exists at least one unsplit covering family of rational curves of $(-K_X)$-degree l_X on X. Hence the evaluation morphism e is surjective.

(2) If $l_X \geq n$, then $X \simeq \mathbb{P}^n$ or $\mathbb{Q}^n$ by Proposition 4.1.2. Thus M is the family of lines and hence irreducible.

(3) If $l_X \leq n - 1$ then we do not know a priori whether the family M is irreducible or not. Also each family M_j may not be covering.

(4) If $l_X \leq n-1$ then each family M_j is unsplit by the proof of Proposition 4.1.7. Also $\mathscr{E}$ is uniform by Lemma 4.1.3.

1.2.1. *Case $\rho_X = 1$ and $l_X = n-2$.* Now Theorem 4.0.3 follows in the case of $\rho_X = 1$ and $l_X = n-2$:

PROPOSITION 4.1.10. *Let $(X, \mathscr{E})$ be a pair as in Theorem 4.0.3. If $\rho_X = 1$ and $l_X = n-2$, then X is a Mukai manifold and $\mathscr{E} \simeq \mathcal{O}(1^{n-2})$.*

PROOF. By Proposition 4.1.7, there is an unsplit covering family of rational curves of $(-K_X)$-degree $n-2$ and X is chain connected by rational curves in this family. Also $\mathscr{E}$ is uniform by Lemma 4.1.3. Thus the assertion follows from [**AW01**, Proposition 1.2]. □

1.3. Length of the other contraction of $\mathbb{P}(\mathscr{E})$. Let $(X, \mathscr{E})$ be a pair as in Theorem 4.0.3 with $\rho_X = 1$. Then $\mathbb{P}(\mathscr{E})$ is a Fano manifold with $\rho_X = 2$ and hence there exists another elementary contraction $\varphi\colon \mathbb{P}(\mathscr{E}) \to Y$ by the Kawamata-Shokurov base point free theorem [**KMM87**, **KM98**]. We will denote by R_φ the ray contracted by φ and H_X the ample generator of the Picard group of X.

Note that $-K_X = (n-2)\xi_{\mathscr{E}}$ and hence the index $r_{\mathbb{P}(\mathscr{E})}$ is $n-2$.

DEFINITION 4.1.11. The length $\ell(R_\varphi)$ is defined as the minimum anticanonical degree of rational curves contracted by φ:

$$\ell(R_\varphi) := \min\left\{ -K_{\mathbb{P}(\mathscr{E})}.C \mid C \text{ is a rational curve on } \mathbb{P}(\mathscr{E}) \text{ with } [C] \in R_\varphi \right\}.$$

Since the index $r_{\mathbb{P}(\mathscr{E})}$ is $n-2$, we have $\ell(R_\varphi) \geq n-2$.

We will denote by $\mathrm{Exc}(\varphi)$ the exceptional locus of φ. Then the inequality of Ionescu and Wiśniewski [**Ion86**, Theorem 0.4], [**Wiś91a**, Theorem 1.1] implies:

LEMMA 4.1.12. *Let F be a fiber of φ and E an irreducible component of $\mathrm{Exc}(\varphi)$ such that $F \subset E$. Then $\dim F \leq n$ and*

$$\dim E + \dim F \geq 2n - 4 + \ell(R_\varphi) \geq 3n - 6.$$

PROOF. Since the morphism $F \to X$ is finite, it holds $\dim F \leq n$. The last assertion follows from [**Ion86**, Theorem 0.4], [**Wiś91a**, Theorem 1.1] and the fact $\ell(R_\varphi) \geq n-2$. □

PROPOSITION 4.1.13. *Let $(X, \mathscr{E})$ be a pair as in Theorem 4.0.3 and $\rho_X = 1$. Assume that $\ell(R_\varphi) = n-2$. Then there exists an ample line bundle $\mathscr{L}$ on $\mathbb{P}(\mathscr{E})$ such that $K_{\mathbb{P}(\mathscr{E})} + (n-2)\mathscr{L}$ defines the contraction φ.*

PROOF. If $\ell(R_\varphi) = n-2$, then there is a rational curve C_φ on $\mathbb{P}(\mathscr{E})$ with $[C_\varphi] \in R_\varphi$ and $\xi.C_\varphi = 1$. Then $\mathscr{L} := (\pi^* H_X.C_\varphi + 1)\xi - \pi^* H_X$ satisfies the desired properties. □

On the other hand, the following proposition deal the case $\ell(R_\varphi) \neq n-2$:

PROPOSITION 4.1.14. *Let $(X, \mathscr{E})$ be a pair as in Theorem 4.0.3 and $\rho_X = 1$. Then the following are equivalent:*

(1) $\ell(R_\varphi) \neq n-2$.
(2) $\mathscr{E}|_{\mathbb{P}^1} \simeq \mathcal{O}(2^3)$ *for every minimal rational curve* $f\colon \mathbb{P}^1 \to X$.
(3) $(X, \mathscr{E}) \simeq (\mathbb{P}^5, \mathcal{O}(2^3))$.

Proof. The implications (3) $\Rightarrow$ (1) and (3) $\Rightarrow$ (2) are obvious. The implication (2) $\Rightarrow$ (3) follows from the same argument as in the proof of Proposition 4.1.10.

(1) $\Rightarrow$ (3). Assume that $\ell(R_\varphi) \neq n-2$. Then $\ell(R_\varphi) \geq 2(n-2)$ since $r_{\mathbb{P}(\mathscr{E})} = n-2$. Lemma 4.1.12 implies

$$\dim E \geq 2n-4-\dim F + \ell(R_\varphi) \geq n-4+\ell(R_\varphi) \geq 3n-8.$$

Since $\dim E \leq \dim \mathbb{P}(\mathscr{E}) = 2n-3$, this is possible only if

$$n=5,\ \dim E = \dim \mathbb{P}(\mathscr{E}),\ \dim F = 5 \text{ and } \ell(R_\varphi) = 6.$$

In this case, the morphism φ is of fiber type and, since $\dim F = 5$ for any φ-fiber, it holds $\dim Y = 2$. Then $\mathscr{E} \simeq \mathcal{O}(a^3)$ for some positive integer a by [**NO07**, Lemma 4.1].

In this case $\mathbb{P}(\mathscr{E}) \simeq \mathbb{P}^2 \times X$ and the contraction φ is the first projection. Thus $i_X = \ell(R_\varphi) = 6$. Hence $X \simeq \mathbb{P}^5$ by Theorem 4.1.2. Since $\mathscr{E} \simeq \mathcal{O}(a^3)$ and $c_1(\mathscr{E}) = c_1(X)$, we have $\mathscr{E} \simeq \mathcal{O}(2^3)$. □

1.4. Sections of the projective bundle $\mathbb{P}(\mathscr{E})$. In this subsection, *minimal lifts* of a minimal rational curves, which can be regarded as a notion of local sections of φ, are defined and family of such curves are constructed. Also we will see how global sections of π are constructed by using minimal lifts.

The following ensures the existence of a minimal lift, which will be defined soon later.

Proposition 4.1.15. *Let $(X, \mathscr{E})$ be a pair as in Theorem 4.0.3 with $\rho_X = 1$ and $\ell(R_\varphi) = n-2$. There exists a rational curve $\widetilde{C}$ on $\mathbb{P}(\mathscr{E})$ with $\xi_{\mathscr{E}}.\widetilde{C} = 1$ and $\pi(\widetilde{C})$ is a minimal rational curve.*

Proof. Let $f\colon \mathbb{P}^1 \to C \subset X$ be a minimal rational curve. By taking the base change of π by f, we obtain the following commutative diagram:

$$\begin{tikzcd} \mathbb{P}(\mathscr{E}|_{\mathbb{P}^1}) \arrow[r] \arrow[d, "\pi_{\mathbb{P}^1}"'] \arrow[rr, bend left, "\varphi_{\mathbb{P}^1}"] & \mathbb{P}(\mathscr{E}) \arrow[r, "\varphi"'] \arrow[d, "\pi"] & Y \\ \mathbb{P}^1 \arrow[r, "f"] & X. & \end{tikzcd} \tag{4.1.15.1}$$

There exists at least one minimal rational curve such that $\mathscr{E}|_{\mathbb{P}^1}$ has a direct summand $\mathcal{O}(1)$. Otherwise, $n=5$ and $\mathscr{E}|_{\mathbb{P}^1} \simeq \mathcal{O}(2^3)$ for every minimal rational curve by Lemma 4.1.3 and the assumption $n \geq 5$. Then $\ell(R_\varphi) = 6$ by Proposition 4.1.14, which contradicts our assumption $\ell(R_\varphi) = n-2$.

Then the section of $\pi_{\mathbb{P}^1}$ corresponding to the direct summand $\mathcal{O}(1)$ gives a rational curve $\widetilde{C}$ with the desired properties. □

Let $(X, \mathscr{E})$ be a pair as in Theorem 4.0.3 with $\rho_X = 1$ and $\ell(R_\varphi) = n-2$, and $\widetilde{f} : \mathbb{P}^1 \to \widetilde{C} \subset \mathbb{P}(\mathscr{E})$ a rational curve on $\mathbb{P}(\mathscr{E})$. Set $f := \pi \circ \widetilde{f}$ and $C := \pi(\widetilde{C}) \subset X$. Assume that $f\colon \mathbb{P}^1 \to C \subset X$ is a minimal rational curve, or equivalently $\pi^*(-K_X).\widetilde{C} = l_X$.

Definition 4.1.16. Let the notation be as above.

(1) The rational curve $\widetilde{f} : \mathbb{P}^1 \to \widetilde{C} \subset \mathbb{P}(\mathscr{E})$ or $\widetilde{C}$ itself is called a *minimal lift* of a minimal rational curve $f : \mathbb{P}^1 \to C$ if $\xi_{\mathscr{E}}.\widetilde{C} = 1$.
(2) We denote by $\widetilde{M} = \coprod \widetilde{M}_i$ the union of all the families $\widetilde{M}_i$ of minimal lifts $\widetilde{C}$ of minimal rational curves:

$$\begin{tikzcd} \widetilde{M} \coloneqq \coprod \widetilde{M}_i & \widetilde{U} \coloneqq \coprod \widetilde{U}_i \arrow[l, "\widetilde{p} \coloneqq \coprod \widetilde{p}_i"'] \arrow[r, "\widetilde{e} \coloneqq \coprod \widetilde{e}_i"] & \mathbb{P}(\mathscr{E}) \arrow[r, "\varphi"'] \arrow[d, "\pi"] & Y \\ M = \coprod M_j & U = \coprod U_j \arrow[l, "p"'] \arrow[r, "e"] & X, & \end{tikzcd}$$

where $\widetilde{p}_i : \widetilde{U}_i \to \widetilde{M}_i$ is the universal family and $\widetilde{e}_i$ is the evaluation morphism.

Remark 4.1.17.

(1) By the definition, a rational curve $\widetilde{f} : \mathbb{P}^1 \to \widetilde{C} \subset \mathbb{P}(\mathscr{E})$ on $\mathbb{P}(\mathscr{E})$ is a minimal lift of a minimal rational curve if $\pi^*(-K_X).\widetilde{C} = l_X$ and $\xi_{\mathscr{E}}.\widetilde{C} = 1$. Therefore, since $\rho_{\mathbb{P}(\mathscr{E})} = 2$, the class $[\widetilde{C}] \in N_1(\mathbb{P}(\mathscr{E}))$ does not depend the choice of $\widetilde{C}$ or C.
(2) In some literature, $\widetilde{C}$ as above is called a *minimal section* of the rational curve C. However we do not know whether $\widetilde{C}$ is isomorphic to C or not. Thus we will use the above terminology, though it is not common in the literature.

We will frequently use the following generalization of [**PSW92b**, Claim 4.1.1] to construct a section of π:

Lemma 4.1.18. *Let $(X, \mathscr{E})$ be a pair as in Theorem 4.0.3 with $\rho_X = 1$ and $\ell(R_\varphi) = n - 2$. Let $\widetilde{C}$ be a minimal lift of minimal rational curve as in Definition 4.1.16.*

Suupose that $V \subset \mathbb{P}(\mathscr{E})$ is a closed subvariety of dimension n such that

$$\mathrm{NE}(V, \mathbb{P}(\mathscr{E})) \subset \langle \mathbb{R}_{\geq 0}[\widetilde{C}], R_\varphi \rangle.$$

Then $l_X = r_X$, $\mathrm{NE}(V, \mathbb{P}(\mathscr{E})) = \mathbb{R}_{\geq 0}[\widetilde{C}]$ and V is a section of π corresponding to the following exact sequence:

$$0 \to \mathscr{E}_1 \to \mathscr{E} \to \mathcal{O}_X(1) \to 0.$$

Proof. The following argument is based on [**PSW92b**, Proof of Claim 4.1.1]. Note that $\pi_V : V \to X$ is finite by our assumption on the Kleiman-Mori cone. Let $\bar{V}$ be the normalization of V and $\pi_{\bar{V}}$ the composite $\bar{V} \to V \to X$. Set $S \coloneqq \pi_{\bar{V}}(\mathrm{Sing}(\bar{V}))$ and $\bar{S} \coloneqq \pi_{\bar{V}}^{-1}(S)$.

Then the function $x \mapsto \#(\pi_{\bar{V}}^{-1}(x))$ is lower semicontinuous on $X \setminus S$ and $\pi_{\bar{V}}$ is étale over $x \in X \setminus S$ if $\#(\pi_{\bar{V}}^{-1}(x)) = \deg \pi_{\bar{V}}$.

Let C be a general minimal rational curve and $\bigcup_{i=1}^m \widetilde{C}_i$ the union of all 1-dimensional irreducible components of $\pi_V^{-1}(C)$, where m is the number of 1-dimensional components. Note that

$$\mathrm{NE}(\pi^{-1}(C), \mathbb{P}(\mathscr{E})) = \langle R_\pi, \mathbb{R}_{\geq 0}[\widetilde{C}] \rangle.$$

Then, by our assumption on the Kleiman-Mori cone, we have $[\widetilde{C}_i] \in \mathbb{R}_{\geq 0}[\widetilde{C}]$. Hence, if we take the normalization $\mathbb{P}^1 \to C$, the curves $\widetilde{C}_i$ are images of some minimal sections of $\mathbb{P}(\mathscr{E}|_{\mathbb{P}^1}) \to \mathbb{P}^1$. Hence $\#(\pi_{\bar{V}}^{-1}(x)) \geq m$ for $x \in C$ and the equality holds for general $x \in C$.

Assume that $\pi_{\bar{V}}$ is not étale. Then the branch locus of $\pi_{\bar{V}}$ is a divisor $B \subset X$ by purity of branch locus. Since C is general and $\rho_X = 1$, we have $C \not\subset B$ and $C \cap B \neq \emptyset$. Since S has codimension at least two, a general minimal rational curve C does not meet S by [**KMM92b**, Lemma 2], [**Kol96**, II. Proposition 3.7]. This contradicts the semicontinuities. Hence $\pi_{\bar{V}}$ is étale and hence isomorphism since X is simply connected. Therefore $V = \bar{V}$ is a section of π, which restricts to a minimal section on the normalization $f\colon \mathbb{P}^1 \to X$ of each minimal rational curve. Thus $\mathrm{NE}(V, \mathbb{P}(\mathscr{E})) = \mathbb{R}_{\geq 0}[\widetilde{C}]$.

Corresponding to the section V, there is an exact sequence:

$$0 \to \mathscr{E}_1 \to \mathscr{E} \to \mathscr{L} \to 0,$$

where $\mathscr{L}$ is ample line bundle such that $\mathscr{L}|_{\mathbb{P}^1} = \mathcal{O}(1)$ for every minimal rational curve $f\colon \mathbb{P}^1 \to X$. Thus $\mathscr{L} \simeq \mathcal{O}(H_X)$ and hence $l_X = r_X$, which completes the proof. □

2. Ottaviani bundles and Fano manifolds with two $\mathbb{P}^2$-bundles

Here we provide characterizations of the Ottaviani bundle on $\mathbb{Q}^5$, based on [**Ott88, Ott90**] and Chapter 3 Section 2.

Let us consider the pair $(\mathbb{Q}^5, \mathscr{G}_{\mathbb{Q}}(1))$. By Theorem 3.1.2, the other contraction φ of $\mathbb{P}(\mathscr{G}_{\mathbb{Q}})$ is a $\mathbb{P}^2$-bundle. This phenomenon arising with $(\mathbb{Q}^5, \mathscr{G}_{\mathbb{Q}}(1))$ is intractable in our argument. Our general strategy is to find or to look at φ-fibers F whose dimensions are larger than expected. Since the index $r_{\mathbb{P}(\mathscr{E})}$ is $n-2$, we have $\dim F \geq n-3$ by Lemma 4.1.12 and in the above case the dimension of fibers are smallest as possible.

In the Peternell-Szurek-Wiśniewski classification with $r = n-1$, there is a similar possibility with two $\mathbb{P}^2$-bundle structures [**PSW92b**, Proposition 7.4 (iii)], and the possibility is excluded later in [**Wiś94, Occ05**]. On the other hand, in our case, W as above actually has two $\mathbb{P}^2$-bundle structures and compensates the case.

To overcome the difficulties arising when we deal with this situation, we establish two characterizations of the Ottaviani bundle. Theorem 4.2.1 is crucial in the proof of Theorem 4.0.3 for the case $X \simeq \mathbb{Q}^5$ or $\mathbb{Q}^6$ (Section 4). Also Proposition 4.2.2 will be applied to the most difficult situation in the proof of Theorem 4.3.2.

2.1. Ottaviani bundle. We need the following characterization of the Ottaviani bundle on $\mathbb{Q}^5$.

THEOREM 4.2.1. *Let $\mathscr{F}$ be a vector bundle of rank three on $X \simeq \mathbb{P}^5$ or $\mathbb{Q}^5$. Then the following are equivalent:*

(1) *$X \simeq \mathbb{Q}^5$ and $\mathscr{F}$ is the Ottaviani bundle.*
(2) *$(X, \mathscr{E} := \mathscr{F}(1))$ is a pair as in Theorem 4.0.3 and the other contraction $\varphi\colon \mathbb{P}(\mathscr{E}) \to Y$ is of fiber type with $\ell(R_\varphi) = 3$.*

PROOF. (1) ⇒ (2). This is a consequence of Remark 0.3.3 and Proposition 4.1.14.

(2) $\Rightarrow$ (1). Assume that $\mathscr{F}$ satisfies (2). Then we have $\dim Y \leq 5$ by Lemma 4.1.12. By Lemma 3.1.10 and Lemma 3.1.11, up to twist with a line bundle, $\mathscr{F}$ is semiample and one of the following holds:

(1) $X \simeq \mathbb{P}^5$ and $(c_1(\mathscr{F}), c_2(\mathscr{F}), c_3(\mathscr{F})) = (2,2,1)$ or $(4,8,8)$,
(2) $X \simeq \mathbb{Q}^5$ and $(c_1(\mathscr{F}), c_2(\mathscr{F}), c_3(\mathscr{F})) = (2,2,2)$ or $(4,8,16)$.

The condition $c_1(\mathscr{F}(1)) = c_1(X)$ implies $X \simeq \mathbb{Q}^5$ and

$$(c_1(\mathscr{F}), c_2(\mathscr{F}), c_3(\mathscr{F})) = (2,2,2).$$

Thus it remains to prove that $\mathscr{F}$ is stable. The stability of $\mathscr{F}$ is equivalent to the conditions $H^0(\mathscr{F}(-1)) = 0$ and $H^0(\mathscr{F}^*) = 0$. Since the other contraction of $\mathbb{P}(\mathscr{F})$, which is defined by the semiample divisor $\xi_{\mathscr{F}}$, is of fiber type, we have $H^0(\mathscr{F}(-1)) = H^0(\xi_{\mathscr{F}} - \pi^* H_X) = 0$. On the other hand, if $H^0(\mathscr{F}^*) \neq 0$, then the section defines a subbundle $\mathcal{O} \subset \mathscr{F}^*$ by [**CP91**, Proposition 1.2 (12)]. This contradicts the fact that $c_3(\mathscr{F}) \neq 0$. Therefore we also have $H^0(\mathscr{F}^*) = 0$. □

2.2. Fano manifolds with two $\mathbb{P}^2$-bundles. Let $\mathscr{G}_{\mathbb{Q}}$ be the Ottaviani bundle on $X \simeq \mathbb{Q}^5$. By Theorem 3.1.2 and Theorem 3.1.6, $\mathbb{P}_X(\mathscr{G}_{\mathbb{Q}})$ is a Fano 7-fold with Picard number two, which has a symmetric structure; the other elementary contraction φ of $\mathbb{P}_X(\mathscr{G}_{\mathbb{Q}})$ is a $\mathbb{P}^2$-bundle over $Y \simeq \mathbb{Q}^5$ and it is again the projectivization of the Ottaviani bundle:

$$\begin{array}{ccc} & \mathbb{P}_X(\mathscr{G}_{\mathbb{Q}}) \simeq \mathbb{P}_Y(\mathscr{G}_{\mathbb{Q}}) & \\ \swarrow^{\pi} & & \searrow^{\varphi} \\ X \simeq \mathbb{Q}^5 & & Y \simeq \mathbb{Q}^5. \end{array}$$

There is a closed subvariety $V \subset \mathbb{P}_X(\mathscr{G}_{\mathbb{Q}})$ such that V is a section of both projection π and φ. Indeed, by Proposition 0.5.2, there is the following exact sequence on X:

$$0 \to \mathscr{C}(1) \to \mathscr{G}_{\mathbb{Q}} \to \mathcal{O}_X(1) \to 0,$$

where $\mathscr{C}$ is the Cayley bundle on $X \simeq \mathbb{Q}^5$. Thus there is a section $V \subset \mathbb{P}_X(\mathscr{G}_{\mathbb{Q}})$ of π corresponding to the exact sequence. Note that the other contraction φ is defined by the relative tautological divisor $\xi_{\mathscr{G}_{\mathbb{Q}}}$. Thus V is also a section of φ.

The following characterizes Fano manifolds with the above properties.

Proposition 4.2.2. *Let W be a Fano manifold with Picard number two. Assume that two elementary contractions p_i $(i = 1, 2)$ are $\mathbb{P}^2$-bundles and there exists a closed subvariety $V \subset W$ which is a section for both projections p_i. Then W is one of the following:*

(1) $\mathbb{P}^2 \times \mathbb{P}^2$,
(2) $\mathbb{P}(T_{\mathbb{P}^3})$,
(3) $\mathbb{P}(\mathscr{G}_{\mathbb{Q}})$ *over* $\mathbb{Q}^5$.

Proof. Let $p_1 \colon W \to X$ and $p_2 \colon W \to Y$ be the two $\mathbb{P}^2$-bundle. Let $\psi : \widetilde{W} \to W$ be the blow up of W along V, E the exceptional divisor and R_ψ the extremal ray of ψ. Then each $(p_i \circ \psi)$-fiber is the Hirzebruch surface $\mathbb{F}_1$. Hence $p_i \circ \psi$ contracts $K_{\widetilde{W}}$-negative face of dimension 2, which is spanned by R_ψ and the other ray R_i. By contracting extremal rays R_i, we have two contractions $\widetilde{p}_1 \colon \widetilde{W} \to \widetilde{X}$ and $\widetilde{p}_2 \colon \widetilde{W} \to \widetilde{Y}$ as in the following diagram:

$$\begin{tikzcd}
 & E \arrow[d, hook, "j"] \arrow[drrr] & & & & \\
 & \widetilde{W} \arrow[dl, "\widetilde{p}_1"'] \arrow[dr, "\widetilde{p}_2"] \arrow[drrr, "\psi"] & & & V \arrow[d, hook, "i"] & \\
\widetilde{X} \arrow[dr, "g_1"'] \arrow[drrr, "f_1"] & & \widetilde{Y} \arrow[dl, "g_2"] \arrow[drrr, "f_2"] & & W \arrow[dl, "p_1"'] \arrow[dr, "p_2"] & \\
 & Z & & X & & Y
\end{tikzcd}$$

As each $(p_i \circ \psi)$-fiber is $\mathbb{F}_1$, $\widetilde{p}_i$ and f_i are smooth $\mathbb{P}^1$-fibrations and $\widetilde{p}_i \circ j$ are isomorphisms. By [**Kan15**, Theorem 2.2 and Remark 2.3] there exist two smooth elementary contractions g_i such that $g_1 \circ \widetilde{p}_1 = g_2 \circ \widetilde{p}_2$ and each fiber of $g_i \circ \widetilde{p}_i$ is isomorphic to a complete flag manifold of Picard number two.

Note that $E \simeq \widetilde{X} \simeq \widetilde{Y}$. Let F be a $g_i \circ \widetilde{p}_i$-fiber. Then both $\widetilde{p}_1|_F$ and $\widetilde{p}_2|_F$ are $\mathbb{P}^1$-bundles, and $E \cap F$ is a section for both $\mathbb{P}^1$-bundles. Hence each $g_i \circ \widetilde{p}_i$-fiber is isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$ and g_i are smooth $\mathbb{P}^1$-fibrations. This implies that $\widetilde{X}$ and $\widetilde{Y}$ are isomorphic to a complete flag manifold of Picard number two by Theorem 1.2.6, and hence X and Y are isomorphic to a rational homogeneous manifold of dimension at most five. Then the assertion follows from the classification given in Propositions 3.3.1 and 3.3.3 $\square$

3. Comparison theorem

In the rest of this chapter, we assume the following by virtue of Propositions 4.1.4, 4.1.10, 4.1.14:

SETTING 4.3.1. $(X, \mathscr{E})$ is a pair as in Theorem 4.0.3 with $\rho_X = 1$, $l_X \geq n-1$ and $\ell(R_\varphi) = n-2$.

We use the notations as in Definitions 4.1.8 and 4.1.16. In this section we will prove that every minimal lift $\widetilde{C}$ of a minimal rational curve C is contracted by φ:

THEOREM 4.3.2. *Let $(X, \mathscr{E})$ be a pair as in Setting 4.3.1. Then $\mathbb{R}_{\geq 0}[\widetilde{C}] = R_\varphi$ and hence $l_X\xi + \pi^*K_X = l_X\xi - \pi^*c_1(\mathscr{E})$ is a supporting divisor of the contraction φ.*

In [**PSW92b**, (3.1)], the corresponding statement is called the comparison lemma. An outline of the proof is similar to that in [**PSW92b**, Sect. 3]; In Subsection 3.3, we show that $\mathrm{Exc}(\varphi) \cap \widetilde{e}(\widetilde{U}) \neq \emptyset$ (Proposition 4.3.9) and then, assuming $\mathbb{R}_{\geq 0}[\widetilde{C}] \neq R_\varphi$, obtain a contradiction by studying the relation between $\widetilde{e}(\widetilde{U})$ and $\mathrm{Exc}(\varphi)$ in Subsection 3.4.

In our case, since the index of $\mathbb{P}(\mathscr{E})$ becomes smaller, there are more possibilities of the contraction φ and hence we need to treat them in more details, particularly when φ is a small contraction in Subsection 3.3 or φ is of fiber type with small dimensional fibers in Subsection 3.4. We deal these cases by using an application of Mori's bend and break argument (Lemma 4.3.4), several splitting criteria (which will be proved in Subsection 3.2) and the characterization of the Ottaviani bundle (Proposition 4.2.2). Also Professor Gianluca Occhetta kindly suggested the author to apply results from the studies on the Mukai conjecture [**ACO04**, **Occ06**, **CO06**] to prove Theorem 4.3.2 in Subsection 3.4.

Before the proof of Theorem 4.3.2, we prove a corollary, which is a consequence of Theorem 4.3.2:

COROLLARY 4.3.3. *Let $(X, \mathscr{E})$ be a pair as in Setting 4.3.1, $i \colon F \to \mathbb{P}(\mathscr{E})$ a morphism from a projective variety F and D_F the divisor $\xi|_F$. Assume that $(\varphi \circ i)(F)$ is a point. Then the following hold:*

(1) *$\Omega_\pi|_F$ and $\mathscr{E}|_F(-D_F)$ are nef vector bundles with first Chern classes $(l_X - n + 2)D_F$. Moreover $\mathscr{E}|_F(-D_F)$ is semiample.*
(2) *There is the following exact sequence:*

$$0 \to \Omega_\pi|_F \to \mathscr{E}|_F(-D_F) \to \mathcal{O}_F \to 0.$$

PROOF. By restricting the relative Euler sequence, we have the exact sequence in (2). Thus $c_1(\Omega_\pi|_F) = c_1(\mathscr{E}|_F(-D_F))$. If $\mathscr{E}|_F(-D_F)$ is semiample, then it is nef and hence $\Omega_\pi|_F$ is also nef by [**CP91**, Proposition 1.2 (8)]. Therefore it is enough to show that $\mathscr{E}|_F(-D_F)$ is a semiample vector bundle with first Chern class $(l_X - n + 2)D_F$.

By Theorem 4.3.2, $l_X\xi - \pi^*c_1(\mathscr{E})$ defines the contraction φ. Since F is contracted to a point by φ, the divisor $(l_X\xi - \pi^*c_1(\mathscr{E}))|_F = l_X D_F - c_1(\mathscr{E}|_F)$ is trivial. Thus $l_X D_F = c_1(\mathscr{E}|_F)$. Therefore $c_1(\mathscr{E}|_F(-D_F)) = (l_X - n + 2)D_F$.

Also, on $\mathbb{P}(\mathscr{E}|_F)$, we have

$$\begin{aligned} l_X \xi_{\mathscr{E}|_F(-D_F)} &= l_X \xi_{\mathscr{E}|_F} - \pi^*(l_X D_F) \\ &= l_X \xi_{\mathscr{E}|_F} - \pi^* c_1(\mathscr{E}|_F) \\ &= (l_X \xi - \pi^* c_1(\mathscr{E}))|_{\mathbb{P}(\mathscr{E}|_F)}. \end{aligned}$$

The last divisor is semiample by Theorem 4.3.2. Hence $\mathscr{E}|_F(-D_F)$ is semiample. □

In the rest of this section, we will prove Theorem 4.3.2.

3.1. Inequalities. Let E be an irreducible component of $\mathrm{Exc}(\varphi)$ and set $E_x \coloneqq E \cap \pi^{-1}(x)$ for $x \in \pi(E)$ and $\widetilde{e}(\widetilde{U})_x \coloneqq \widetilde{e}(\widetilde{U}) \cap \pi^{-1}(x)$ for $x \in X$.

Later we will prove $E \cap \widetilde{e}(\widetilde{U}) \neq \emptyset$, or equivalently $E_x \cap \widetilde{e}(\widetilde{U})_x \neq \emptyset$ for some point $x \in X$. Since $\pi^{-1}(x) \simeq \mathbb{P}^{n-3}$, the assertion follows if $\dim E_x + \dim \widetilde{e}(\widetilde{U})_x \geq n - 3$.

For $x \in \pi(E)$, we have

$$\dim E_x \geq \dim E - n, \tag{4.3.3.1}$$

Note that $e(U) = X$ by Proposition 4.1.7. Thus, for every point $x \in X$, there is a minimal rational curve $C \ni x$. For $x \in X$, we define M_x to be the set of all minimal rational curve through x:

$$M_x \coloneqq \left\{ f \colon \mathbb{P}^1 \to X \mid f(\mathbb{P}^1) \text{ is a minimal rational curve through } x \right\},$$

and set

$$m_x \coloneqq \max \{ m \mid \mathcal{O}(1^m) \text{ is a direct summand of } f^*\mathscr{E} \text{ for some } [f] \in M_x \}.$$

Then, for each point $x \in X$,

$$\dim \widetilde{e}(\widetilde{U})_x \geq m_x - 1. \tag{4.3.3.2}$$

Also the following follows from Lemma 4.1.3:

$$m_x - 1 \geq 2n - 5 - l_X. \tag{4.3.3.3}$$

In particular,

$$\dim \widetilde{e}(\widetilde{U})_x \geq 2n - 5 - l_X. \tag{4.3.3.4}$$

The following enables us to obtain a better lower bound of $\dim E_x$ in a subtle case.

LEMMA 4.3.4. *Assume that φ is a small contraction and $n = 5$. If $\mathbb{R}_{\geq 0}[\widetilde{C}] \neq R_\varphi$, then there exists a closed subvariety $N \subset \mathrm{Exc}(\varphi)$ of dimension ≥ 4 with $\dim \pi(N) = \dim N - 1$. In particular, inequality (4.3.3.1) is strict for $x \in \pi(N)$.*

PROOF. By Lemma 4.3.8, the morphism $E \to \varphi(E)$ is equidimensional of relative dimension four and $\dim \varphi(E) = 1$. Take two general points $y_1, y_2 \in \varphi(E)$ and set $F_i := (\varphi|_E)^{-1}(y_i)$.

The family of the lines contained in the π-fibers is given by the following diagram:

$$\begin{tikzcd}
\mathbb{P}(T_{\pi'}) \simeq \mathbb{P}(T_\pi) \arrow[r, "f"] \arrow[d, "g"'] & \mathbb{P}(\mathscr{E}) \arrow[r, "\varphi"] \arrow[d, "\pi"'] & Y \\
\mathbb{P}(\mathscr{E}^*) \arrow[r, "\pi'"] & X, &
\end{tikzcd}$$

where g is the universal family and f is the evaluation morphism.

Since $\pi(F_1)$ and $\pi(F_2)$ are effective divisors and $\rho_X = 1$, we have $\pi(F_1) \cap \pi(F_2) \neq \emptyset$. Hence there exists at least a line ℓ contained in a π-fiber which intersects with both F_1 and F_2. Thus $g(f^{-1}(F_1)) \cap g(f^{-1}(F_2)) \neq \emptyset$, which has dimension ≥ 3 by the Serre inequality. Let W be a 3-dimensional component of $g(f^{-1}(F_1)) \cap g(f^{-1}(F_2))$. Set $N := f(g^{-1}(W))$.

Since two distinct points in a π-fiber defines a unique line in the π-fiber, the morphism $\pi'|_W$ is finite. Hence $\dim N = \dim W + 1 \geq 4$ and $\dim \pi(N) = \dim N - 1$.

On the other hand the $(\varphi \circ f)$-image of each g-fiber over W passes through y_1 and y_2. Hence $\dim \varphi(N) = 1$ by Mori's bend and break argument [**Kol96**, Chapter II, Theorem 5.4]. This implies $N \subset \mathrm{Exc}(\varphi)$. □

3.2. Splitting criteria. In this subsection, we provide three splitting criteria. As we mentioned, if $\dim \widetilde{e}(\widetilde{U})_x$ is enough large, then it will intersect with $\mathrm{Exc}(\varphi)$. The following criteria enables us to deal the case where $\dim \widetilde{e}(\widetilde{U})_x$ is rather small.

PROPOSITION 4.3.5. *Let $(X, \mathscr{E})$ be a pair as in Setting 4.3.1 with $X \simeq \mathbb{P}^n$ or $X \simeq \mathbb{Q}^n$. Assume that $\dim \widetilde{e}(\widetilde{U}) = 3n - 5 - l_X$ and $\mathscr{E}$ is uniform of type*

$$\mathcal{O}(2^{-n+2+l_X}, 1^{2n-4-l_X}).$$

Then $\mathscr{E}$ splits.

PROOF. The proof proceeds similarly to that of [**MOSC12b**, Proof of Theorem 3.1]. Details are as follows:

Since $\mathscr{E}$ is uniform of type $\mathcal{O}(2^{-n+2+l_X}, 1^{2n-4-l_X})$, we have the following exact sequence of vector bundles on U:

$$0 \to \mathscr{F} \to e^*(\mathscr{E}(-1)) \to Q' := (p^* p_* e^*(\mathscr{E}^*(1)))^* \to 0, \tag{4.3.5.1}$$

which restricts on each p-fiber to

$$0 \to \mathcal{O}(1^{-n+2+l_X}) \to \mathcal{O}(1^{-n+2+l_X}, 0^{2n-4-l_X}) \to \mathcal{O}(0^{2n-4-l_X}) \to 0.$$

This gives a morphism $g\colon U \to \mathrm{Gr}(-n+2+l_X, \mathscr{E})$, where $\mathrm{Gr}(-n+2+l_X, \mathscr{E})$ is the Grassmannian of subbundles in $\mathscr{E}$.

Now $\widetilde{U}$ is naturally isomorphic to $\mathbb{P}(Q')$ and the evaluation morphism $\widetilde{e}$ is the morphism corresponding to the surjection $e^*\mathscr{E} \to Q' \to 0$. Since every fiber of the morphism $\widetilde{e}(\widetilde{U}) \to X$ is of dimension $2n-5-l_X$, the morphism $g(U) \to X$ is generically finite. Note that the evaluation morphism e is a contraction of an extremal ray since M is the family of lines on $\mathbb{P}^n$ or $\mathbb{Q}^n$ ($n \geq 5$). Thus the morphism g factors through the evaluation morphism e. This implies that there exists the following exact sequence on X:

$$0 \to S \to \mathscr{E}(-1) \to Q \to 0,$$

which restricts on U to (4.3.5.1). Hence S and Q are direct sums of line bundles by [**Sat76**, **KS02**] or [**AW01**, Proposition 1.2]. Therefore $\mathscr{E} \simeq \mathcal{O}(2^{-n+2+l_X}, 1^{2n-4-l_X})$. □

PROPOSITION 4.3.6. *Let $(X, \mathscr{E})$ be a pair as in Setting 4.3.1. Assume that $X \simeq \mathbb{P}^n$ and there exists a point $x \in X$ such that equality holds in* (4.3.3.3)*. Then $\mathscr{E}$ splits.*

PROOF. Since equality holds in (4.3.3.3), $\mathscr{E}$ is uniform at the point $x \in X$. Thus the assertion follows from [**Sat76**, Main Theorem and Remark 2.1]. □

PROPOSITION 4.3.7. *Let $(X, \mathscr{E})$ be a pair as in Setting 4.3.1. Then $\mathscr{E}$ splits if one of the following holds:*

(1) *$X \simeq \mathbb{P}^6$ and every fiber of the morphism $\widetilde{e}(\widetilde{U}) \to X$ has dimension ≤ 1.*
(2) *$X \simeq \mathbb{P}^5$, $\dim \widetilde{e}(\widetilde{U}) \leq 5$ and there is no line C such that $\mathscr{E}|_C \simeq \mathcal{O}(4, 1^{n-3})$.*

PROOF. The proof proceeds in several steps.

STEP 1. If $\mathscr{E}$ is uniform at a point $x \in X$, then $\mathscr{E}$ splits by [**Sat76**, Main Theorem and Remark 2.1]. Thus we may assume that $\mathscr{E}$ is not uniform at every point $x \in X$, and hence for each point $x \in X$ there exists a line $C \ni x$ such that $\mathscr{E}|_C \not\simeq \mathcal{O}(2^3, 1^{n-5})$ by Lemma 4.1.3. Thus inequality (4.3.3.3) is strict and so is inequality (4.3.3.4).

STEP 2. We will prove that there is no line C such that $\mathscr{E}|_C \simeq \mathcal{O}(4, 1^{n-3})$. If (2) holds, then the assertion is already assumed. If (1) holds, then every fiber of the morphism $\widetilde{e}(\widetilde{U}) \to X$ has dimension ≤ 1. Hence by (4.3.3.2) the assertion follows.

STEP 3. Hence we have $\mathscr{E}|_C \simeq \mathcal{O}(3,2,1^{n-4})$ for special lines C, and $\mathscr{E}|_C \simeq \mathcal{O}(2^3,1^{n-5})$ for general lines C by Lemma 4.1.3. Set

$$M_{\text{jump}} := \{\, [C] \in M \mid \mathscr{E}|_C \simeq \mathcal{O}(3,2,1^{n-4})\,\},$$

which is a closed subset of M (see e.g. [**OSS80**, Lemma 3.2.2]), and $U_{\text{jump}} := p^{-1}(M_{\text{jump}})$.

The morphism $e|_{U_{\text{jump}}}$ is surjective, since $\mathscr{E}$ is not uniform at any point. Hence there exists an irreducible component M^0_{jump} of M_{jump} such that $e|_{U^0_{\text{jump}}}$ is surjective, where $U^0_{\text{jump}} := p^{-1}(M^0_{\text{jump}})$. Therefore we have the following diagram with a surjection $e_0 := e|_{U^0_{\text{jump}}}$:

$$\begin{tikzcd}
M^0_{\text{jump}} \arrow[d, hook] & U^0_{\text{jump}} \arrow[l, "p_0"'] \arrow[r, "e_0"] \arrow[d, hook] & X \\
M & U. \arrow[l, "p"'] \arrow[ur, "e"'] &
\end{tikzcd}$$

STEP 4. There exists the following exact sequence of vector bundles on U^0_{jump}:

$$0 \to \mathscr{F} \to e_0^*\left(\mathscr{E}(-1)\right) \to \mathscr{G} := (p_0^* p_{0*} e_0^*\left(\mathscr{E}^*(1)\right))^* \to 0, \tag{4.3.7.1}$$

which restricts on each p_0-fiber to

$$0 \to \mathcal{O}(2,1) \to \mathcal{O}(2,1,0^{n-4}) \to \mathcal{O}(0^{n-4}) \to 0.$$

Then the exact sequence gives the following commutative diagram,

$$\begin{tikzcd}
\mathbb{P}(\mathscr{G}) \arrow[r, "\widetilde{e}_0"] \arrow[d] & \mathbb{P}(\mathscr{E}) \arrow[r, "\varphi"'] \arrow[d, "\pi"'] & Y \\
U^0_{\text{jump}} \arrow[r, "e_0"] & X. &
\end{tikzcd}$$

The image $\widetilde{e}_0(\mathbb{P}(\mathscr{G}))$ is the union of all minimal lifts over the minimal rational curves belonging to M^0_{jump}. Also a morphism $U^0_{\text{jump}} \to \mathrm{Gr}(2,\mathscr{E})$ is induced by sequence (4.3.7.1) (Note that if (2) holds then $\mathbb{P}(\mathscr{G}) \simeq U^0_{\text{jump}}$ and $\mathbb{P}(\mathscr{E}) \simeq \mathrm{Gr}(2,\mathscr{E})$).

STEP 5. If (1) holds, then every fiber of the morphism $\widetilde{e}(\widetilde{U}) \to X$ has dimension ≤ 1, so does for every fiber of the morphism $\widetilde{e}_0(\mathbb{P}(\mathscr{G})) \to X$. This implies that the morphism $\widetilde{e}_0(\mathbb{P}(\mathscr{G})) \to X$ is equidimensional of relative dimension 1. Thus the morphism $U^0_{\text{jump}} \to \mathrm{Gr}(2,\mathscr{E})$ is finite over X.

If (2) holds, then since $\dim \widetilde{e}(\widetilde{U}) \leq 5$, the image of the corresponding morphism $U^0_{\text{jump}} \to \mathrm{Gr}(2,\mathscr{E})$ is *generically* finite over X.

STEP 6. Here we will prove that every fiber of e_0 is connected. Moreover if $n = 5$ then e^0 is equidimensional.

Now $X \simeq \mathbb{P}^n$ and thus e is a projective bundle of relative dimension $n-1 = 4$ or 5. Thus the assertion follows if $\dim U^0_{\text{jump}} \geq n+3$. Note that if $n = 5$ then $(e^0)^{-1}(x)$ is equidimensional. Otherwise $e^{-1}(x) = (e^0)^{-1}(x)$, which implies that $\mathscr{E}$ is uniform at the point $x \in X$, which contradicts our assumption in Step 1.

Thus it is enough to show:

Claim 4.3.7.1. $\dim U^0_{\mathrm{jump}} \geq n+3$.

Proof of Claim. Consider the dual projective bundle $\pi' \colon \mathbb{P}(\mathscr{E}^*) \to X$. There is a one-to-one correspondence between the rational curves $C \subset X$ such that $[C] \in M_{\mathrm{jump}}$ and the rational curves $\widetilde{C} \subset \mathbb{P}(\mathscr{E}^*)$ satisfies $\xi_{\mathscr{E}^*}.\widetilde{C} = -3$ and $(\pi'^* H_X).\widetilde{C} = 1$. Indeed if C is a jumping line on X, then the lift $\widetilde{C} \subset \mathbb{P}(\mathscr{E}^*)$ corresponding to the direct summand $\mathcal{O}(-3) \subset \mathscr{E}^*|_C$ satisfies $\xi_{\mathscr{E}^*}.\widetilde{C} = -3$ and $(\pi'^* H_X).\widetilde{C} = 1$. Conversely, if a rational curve $\widetilde{C}$ in $\mathbb{P}(\mathscr{E}^*)$ satisfies $\xi_{\mathscr{E}^*}.\widetilde{C} = -3$ and $(\pi'^* H_X).\widetilde{C} = 1$, then the image $C = \pi'(\widetilde{C})$ is a line on X and $\widetilde{C}$ is a section corresponding to a surjection $\mathscr{E}^*|_C \to \mathcal{O}(-3)$. Hence C is a jumping line for $\mathscr{E}$. Also the correspondence is one-to-one.

Thus the family of rational curves on $\mathbb{P}(\mathscr{E}^*)$ with $\xi_{\mathscr{E}^*}.\widetilde{C} = -3$ and $(\pi'^* H_X).\widetilde{C} = 1$ is isomorphic to the normalization of M_{jump}. By counting the dimension of the family of rational curves on $\mathbb{P}(\mathscr{E}^*)$ by Proposition 4.1.6, we have $\dim U^0_{\mathrm{jump}} \geq n+3$. □

Step 7. By applying the rigidity lemmas [**Kol96**, Chapter II. Proposition 5.3] and [**KM98**, Lemma 1.6] to the case (1) and (2) respectively, we see that the morphism $U^0_{\mathrm{jump}} \to \mathrm{Gr}(2, \mathscr{E})$ factors through e_0. This implies that there exists the following exact sequence on X:

$$0 \to S \to \mathscr{E}(-1) \to Q \to 0,$$

such that the pull back of the sequence by e_0 coincides (4.3.7.1). Since $\mathscr{E}$ is ample, so is $Q(1)$. By restricting each p_0-fiber, we see that $c_1(Q(1)) = n-4$. Since $\operatorname{rank} Q = n-4$, the bundle Q is uniform. Note that there is no line C such that $\mathscr{E}|_C \simeq \mathcal{O}(4, 1^{n-3})$. Thus $\mathscr{E}$ is a uniform vector bundle, which contradicts our assumption that $\mathscr{E}$ is not uniform. This completes the proof.

□

3.3. Exceptional locus of φ and locus of minimal lifts. The following is a consequence of Lemma 4.1.12:

Lemma 4.3.8. *Let $(X, \mathscr{E})$ be a pair as in Setting 4.3.1, E an irreducible component of* $\mathrm{Exc}(\varphi)$ *and F an irreducible component of a φ-fiber contained in E. Assume that $\mathbb{R}_{\geq 0}[\widetilde{C}] \neq R_\varphi$.*

Then $\dim F \leq n-1$ *and one of the following holds:*

(1) *φ is of fiber type and* $\dim F \geq n-3$*,*
(2) *φ is a divisorial contraction and* $\dim F \geq n-2$*,*
(3) *φ is a small contraction,* $\dim E = 2n-5$ *and* $\dim F = n-1$.

Proof. If there is a φ-fiber F of dimension n, then the morphism $\varphi_{\mathbb{P}^1}$ in diagram (4.1.15.1) contracts at least one curve, which is one of the minimal sections of $\pi_{\mathbb{P}^1}$. This contradicts our assumption $\mathbb{R}_{\geq 0}[\widetilde{C}] \neq R_\varphi$. Hence $\dim F \leq n-1$. The remaining assertion follows from Lemma 4.1.12. □

Proposition 4.3.9. *Let $(X, \mathscr{E})$ be a pair as in Setting 4.3.1. Then* $\mathrm{Exc}(\varphi) \cap \widetilde{e}(\widetilde{U}) \neq \emptyset$.

Proof. Assume to the contrary $\mathrm{Exc}(\varphi) \cap \widetilde{e}(\widetilde{U}) = \emptyset$. Then obviously $\mathbb{R}_{\geq 0}[\widetilde{C}] \neq R_\varphi$ and hence the assumption of Lemma 4.3.8 holds. Also φ is not of fiber type. Hence $\dim E = 2n-4$ or $2n-5$. Moreover $\mathscr{E}$ does not split since $\mathbb{R}_{\geq 0}[\widetilde{C}] \neq R_\varphi$.

Since $\pi^{-1}(x) = \mathbb{P}^{n-3}$, we have $E_x \cap \widetilde{e}(\widetilde{U})_x \neq \emptyset$ for $x \in \pi(E)$ if

$$\dim E_x + \dim \widetilde{e}(\widetilde{U})_x \geq n-3.$$

Therefore, by our assumption $\operatorname{Exc}(\varphi) \cap \widetilde{e}(\widetilde{U}) = \emptyset$, we have

$$n-4 \geq \dim E_x + \dim \widetilde{e}(\widetilde{U})_x \tag{4.3.9.1}$$

for $x \in \pi(E)$.

By the above inequality and inequalities (4.3.3.1)–(4.3.3.3) the following holds for $x \in \pi(E)$:

$$\begin{aligned} n-4 &\geq \dim E_x + \dim \widetilde{e}(\widetilde{U})_x \\ &\geq (\dim E - n) + (m_x - 1) \\ &\geq (\dim E - n) + (2n - 5 - l_X). \end{aligned} \tag{4.3.9.2}$$

On the other hand, we have $(\dim E - n) + (2n-5-l_X) \geq n-6$ by Lemma 4.3.8. Thus

$$n-4 \geq (\dim E - n) + (2n-5-l_X) \geq n-6.$$

We will divide the proof into four cases depending on the value $(\dim E - n) + (2n-5-l_X)$. Note that there are only finite possibilities for triplets $(n, l_X, \dim E)$, since $n \geq 5$, $l_X \in \{\, n-1, \dots, n+1 \,\}$ and $\dim E = 2n-4$ or $2n-5$.

Case. $(\dim E - n) + (2n-5-l_X) = n-4$.

This case occurs if and only if $(n, l_X, \dim E) = (5,4,5)$, $(5,5,6)$, $(6,6,7)$, $(6,7,8)$ or $(7,8,9)$.

Since $(\dim E - n) + (2n-5-l_X) = n-4$, inequality (4.3.9.2) gives

$$\dim E_x + \dim \widetilde{e}(\widetilde{U})_x = (\dim E - n) + (m_x - 1) = n-4$$

Thus inequalities (4.3.3.1)–(4.3.3.4) become equalities. Hence $E \to X$ is surjective and every fiber is equidimensional of dimension $\dim E - \dim X$. Also the equality in (4.3.3.2) implies that $\mathscr{E}$ is a uniform vector bundle of type $\mathcal{O}(2^{-n+2+l_X}, 1^{2n-4-l_X})$.

If $(n, l_X, \dim E) = (5,4,5)$, then Lemma 4.3.4 gives a contradiction to the fact that the morphism $E \to X$ is equidimensional.

In the other cases, we have $X \simeq \mathbb{P}^n$ or $\mathbb{Q}^n$ by Lemma 4.1.2. Also $\mathscr{E}$ is uniform of type

$$\mathcal{O}(2^{-n+2+l_X}, 1^{2n-4-l_X})$$

and the equality holds in (4.3.3.4). Thus Proposition 4.3.5 gives a contradiction to the fact that $\mathscr{E}$ does not split.

Case. $(\dim E - n) + (2n-5-l_X) = n-5$.

This case occurs if and only if $(n, l_X, \dim E) = (5,5,5)$, $(5,6,6)$ or $(6,7,7)$.

Claim 4.3.9.1. Inequalities (4.3.3.1) and (4.3.3.4) can not be strict at the same time.

Proof of Claim. Otherwise the following inequality gives a contradiction:

$$n-4 \geq \dim E_x + \widetilde{e}(\widetilde{U})_x \geq (\dim E - n + 1) + (2n - 5 - l_X + 1) \geq n-3.$$

□

Subcase. $(n, l_X, \dim E) = (5, 5, 5)$.

In this case $X \simeq \mathbb{Q}^5$ by Proposition 4.1.2. If there is a point $x \in X$ such that $\mathscr{E}$ is uniform at the point x, then $\mathscr{E}$ splits by [**KS02**, Theorem 4.1]. This contradicts the fact that $\mathscr{E}$ does not split. Thus, for every point $x \in X$, $\mathscr{E}$ is not uniform at x and hence there exists a line C such that $x \in C$ and $\mathscr{E}|_C \simeq \mathcal{O}(3, 1^2)$ by Lemma 4.1.3. Thus inequality (4.3.3.3) is strict for each point $x \in X$ and hence inequality (4.3.3.4) is also strict.

By Lemma 4.3.4, there exists a subvariety $N \subset E$ such that $\pi(N)$ has dimension ≥ 3 and $\pi|_N$ is of fiber type. Thus inequality (4.3.3.1) is also strict for $x \in \pi(N)$. This contradicts Claim 4.3.9.1.

Subcase. $(n, l_X, \dim E) = (5, 6, 6)$ or $(6, 7, 7)$.

In this case $X \simeq \mathbb{P}^n$ by Proposition 4.1.2. We will prove that one of the assumption in Proposition 4.3.7 holds. By Proposition 4.3.6, we may assume that inequality (4.3.3.3) is strict for every $x \in X$ and so is inequality (4.3.3.4).

By Claim 4.3.9.1, the equality holds in (4.3.3.1) for every $x \in \pi(E)$. Therefore the morphism $E \to X$ is surjective and equidimensional of relative dimension one. Since $E \cap \widetilde{e}(\widetilde{U}) = \emptyset$, every fiber of the morphism $\widetilde{e}(\widetilde{U}) \to X$ has dimension $\leq n-5$. Thus there is no line C such that $\mathscr{E}|_C \simeq \mathcal{O}(4, 1^{n-3})$ by (4.3.3.2).

Case. $(\dim E - n) + (2n - 5 - l_X) = n - 6$.

This case occurs if and only if $(n, l_X, \dim E) = (5, 6, 5)$. In this case $X \simeq \mathbb{P}^n$ by Theorem 4.1.2. We will prove that the assumption (2) in Proposition 4.3.7 holds.

It holds $\dim \widetilde{e}(\widetilde{U}) \leq 5$. Otherwise $\dim \widetilde{e}(\widetilde{U}) > 5$. Thus $\widetilde{e}(\widetilde{U})$ contains at least a divisor D. Since $\mathrm{Exc}(\varphi) \cap \widetilde{e}(\widetilde{U}) = \emptyset$, we have $D = \varphi^*\varphi_* D$. Since $\rho_Y = 1$, $\varphi_* D$ is an ample Cartier divisor on Y. However by Lemma 4.3.8 we have $\dim \varphi(\mathrm{Exc}(\varphi)) \geq n-4 \geq 1$ and hence $\varphi_* D \cap \varphi(\mathrm{Exc}(\varphi)) \neq \emptyset$. This contradicts the assumption $\mathrm{Exc}(\varphi) \cap \widetilde{e}(\widetilde{U}) = \emptyset$.

There is no line C with $\mathscr{E}|_C \simeq \mathcal{O}(4, 1^2)$. Otherwise, by the same argument of the proof of Claim 4.3.7.1, we have

$$\dim\{\, [C] \in M \mid \text{line } C \text{ with } \mathscr{E}|_C \simeq \mathcal{O}(4, 1^2) \,\} \geq 4.$$

By Lemma 4.3.4, there is a closed subvariety $N \subset E$ of dimension ≥ 4 with $\dim \pi(N) = \dim N - 1$. Hence there is a line C such that $C \cap \pi(N) \neq \emptyset$ and $\mathscr{E}|_C \simeq \mathcal{O}(4, 1^2)$. Take a point $x \in C \cap \pi(N)$. Then $\dim E_x \geq 1$. Also by (4.3.3.2) $\dim \widetilde{e}(\widetilde{U})_x \geq 1$. This contradicts (4.3.9.1).

Therefore the assumption (2) in Proposition 4.3.7 holds and hence $\mathscr{E}$ splits. This contradicts the fact $\mathscr{E}$ does not split. This completes the proof of Proposition 4.3.9. □

3.4. Proof of Theorem 4.3.2. By Proposition 4.3.9, there is a component $\widetilde{M}_0$ and a component F of a non-trivial π-fiber such that $\widetilde{e}(\widetilde{U}_0) \cap F \neq \emptyset$.

DEFINITION 4.3.10. Let X be a projective manifold, $Y \subset X$ a closed subvariety and $U \to M$ an unsplit family of rational curves on X. Then $\mathrm{Locus}(M)_Y$ (resp. $\mathrm{ChLocus}_k(M)_Y$) is defined to be the set of the points which can be connected to Y by a rational curve in M (resp. by a connected chain of rational curves in M with length k).

Then by [**ACO04**, Lemma 5.4], [**Occ06**, Lemma 3.2 and Remark 3.3] (cf. [**CO06**, Corollary 2.2 and Remark 2.4]) we have:

LEMMA 4.3.11. *Assume that* $\mathbb{R}_{\geq 0}[\widetilde{C}] \neq R_\varphi$. *Then the following hold:*

(1) $\dim \mathrm{Locus}(\widetilde{M}_0)_F \geq \dim(F \cap \mathrm{Locus}(\widetilde{M}_0)) + \dim \mathrm{Locus}(\widetilde{M}_0)_p$ *for a general point* $p \in F \cap \mathrm{Locus}(\widetilde{M}_0)$,
(2) $\dim \mathrm{Locus}(\widetilde{M}_0)_F \geq \dim F + n - 3$,
(3) $\mathrm{NE}(\mathrm{Locus}(\widetilde{M}_0)_F, \mathbb{P}(\mathscr{E})) \subset \langle \mathbb{R}_{\geq 0}[\widetilde{C}], R_\varphi \rangle$.

LEMMA 4.3.12. *Assume that* $\mathbb{R}_{\geq 0}[\widetilde{C}] \neq R_\varphi$. *Then*

$$n \geq \dim \mathrm{Locus}(\widetilde{M}_0)_F \geq \dim F + n - 3.$$

In particular $\dim F \leq 3$.

PROOF. Since $\mathbb{R}_{\geq 0}[\widetilde{C}] \neq R_\varphi$, the morphism $\mathrm{Locus}(\widetilde{M}_0)_F \to X$ is finite by Lemma 4.3.11 (3). Thus $n \geq \dim \mathrm{Locus}(\widetilde{M}_0)_F$. By Lemma 4.3.11 (2) we have

$$n \geq \dim \mathrm{Locus}(\widetilde{M}_0)_F \geq \dim F + n - 3,$$

and the assertion follows □

LEMMA 4.3.13. *Assume that* $\mathbb{R}_{\geq 0}[\widetilde{C}] \neq R_\varphi$. *Then one of the following hold:*

(1) $n = 6$, φ *is of fiber type and*
$$\dim \mathrm{Locus}(\widetilde{M}_0)_F = \dim F + 3 = 6.$$
(2) $n = 5$, φ *is a divisorial contraction and*
$$\dim \mathrm{Locus}(\widetilde{M}_0)_F = \dim F + 2 = 5.$$
(3) $n = 5$, φ *is of fiber type and*
$$5 \geq \dim \mathrm{Locus}(\widetilde{M}_0)_F \geq \dim F + 1 \geq 4.$$

PROOF. This follows from Lemmas 4.3.8 and 4.3.12 □

LEMMA 4.3.14. *Assume that* $\mathbb{R}_{\geq 0}[\widetilde{C}] \neq R_\varphi$ *and* $\dim \mathrm{Locus}(\widetilde{M}_0)_F = n$. *Let* V *be an* n*-dimensional component of* $\mathrm{Locus}(\widetilde{M}_0)_F$. *Then:*

(1) $X \simeq \mathbb{P}^n$,
(2) $\dim(V \cap F) = 0$,
(3) $\dim F \leq n - 3$,
(4) V *is a section of* π *corresponding to an exact sequence:*
$$0 \to \mathscr{E}_1 \to \mathscr{E} \to \mathcal{O}_X(1) \to 0.$$

PROOF. By Lemma 4.3.11 (3), $\mathrm{NE}(V, \mathbb{P}(\mathscr{E})) \subset \langle \mathbb{R}_{\geq 0}[\widetilde{C}], R_\varphi \rangle$. Therefore by Lemma 4.1.18 we have $l_X = r_X$ and V is a section of π corresponding to the following exact sequence:

$$0 \to \mathscr{E}_1 \to \mathscr{E} \to \mathcal{O}_X(1) \to 0.$$

Now $\mathrm{NE}(V, \mathbb{P}(\mathscr{E})) = \mathbb{R}_{\geq 0}[\widetilde{C}]$. Thus $\dim(V \cap F) = 0$.

Since $\dim(V \cap F) = 0$, there is a point $p \in F$ such that $V \subset \mathrm{Locus}(\widetilde{M_0})_p$. This implies that there is a point $x \in X$ such that $\mathrm{Locus}(M)_x = X$. Hence $X \simeq \mathbb{P}^n$ by [**KS99**, Corollary 4.2].

On the other hand the Serre inequality implies $\dim(V \cap F) \geq \dim V + \dim F - \dim \mathbb{P}(\mathscr{E}) = \dim F - n + 3$. Thus we have $0 \geq \dim F - n + 3$. □

LEMMA 4.3.15. *Neither Lemma 4.3.13 (1) nor (2) occurs.*

PROOF. If Lemma 4.3.13 (2) occurs, then $\dim F = 3$, which contradicts Lemma 4.3.14 (3).

Assume that Lemma 4.3.13 (1) occurs. We firstly prove that $\mathrm{Locus}(\widetilde{M_0})_F$ is equidimensional of dimension 6.

We have $\dim \widetilde{U}_0 \geq 11$ by Proposition 4.1.6. Hence each irreducible component of a fiber $(\varphi \circ \widetilde{e}_0)^{-1}(y)$ has dimension at least five. Hence each component of $\widetilde{p}_0((\varphi \circ \widetilde{e}_0)^{-1}(y))$ has dimension at least five.

On the other hand, by the proof of [**ACO04**, Lemma 5.4], the morphism $\widetilde{e}_0$ is finite on $\widetilde{p}_0^{-1}(\widetilde{p}_0((\varphi \circ \widetilde{e}_0)^{-1}(y))) \setminus (\varphi \circ \widetilde{e}_0)^{-1}(y)$. Thus each component of $\mathrm{Locus}(\widetilde{M_0})_F$ has dimension ≥ 6.

Hence, by Lemma 4.3.14, we have $\dim \mathrm{Locus}(\widetilde{M_0})_F \cap F = 0$. This is possible only if $\dim \mathrm{Locus}(\widetilde{M_0}) = 6$. Hence $\mathrm{Locus}(\widetilde{M_0})_F = \mathrm{Locus}(\widetilde{M_0})$.

Since φ is of fiber type, the same argument does work for any component $\widetilde{M_i}$. Thus $\widetilde{e}(\widetilde{U})$ is a finite union of sections of π and thus (4.3.3.3) becomes an equality. Then $\mathscr{E}$ splits by Proposition 4.3.6, which gives a contradiction to $\mathbb{R}_{\geq 0}[\widetilde{C}] \neq R_\varphi$. □

LEMMA 4.3.16. *Assume $\mathbb{R}_{\geq 0}[\widetilde{C}] \neq R_\varphi$. Then $n = 5$, φ is a $\mathbb{P}^2$-bundle and $l_X = 4$.*

PROOF. By Lemmas 4.3.13 and Lemma 4.3.15, $n = 5$, φ is of fiber type and

$$5 \geq \dim \mathrm{Locus}(\widetilde{M_0})_F \geq \dim F + 1 \geq 4.$$

Since $\mathbb{R}_{\geq 0}[\widetilde{C}] \neq R_\varphi$, $\widetilde{M}$ is not a covering family by [**CO06**, Lemma 2.4] (Note that $\widetilde{M}$ is an unsplit family). If $l_X \geq 5$, then $X \simeq \mathbb{P}^5$ or $\mathbb{Q}^5$ by Theorem 4.1.2, hence by Theorem 4.2.1 we have $(X, \mathscr{E}) \simeq (\mathbb{Q}^5, \mathscr{G}_{\mathbb{Q}})$. This contradicts the assumption $\mathbb{R}_{\geq 0}[\widetilde{C}] \neq R_\varphi$. Thus we have $l_X = 4$. Also by the assumption $\widetilde{e}(\widetilde{U}) \neq \mathbb{P}(\mathscr{E})$ and inequality (4.3.3.1), we may assume that $\dim \widetilde{e}(\widetilde{U}_0) = 6$.

The morphism $\varphi : \widetilde{e}(\widetilde{U}_0) \to Y$ is surjective. Otherwise there is a fiber F with $\dim \widetilde{e}(\widetilde{U}_0) \cap F \geq 2$. On the other hand $\dim \mathrm{Locus}(\widetilde{M_0})_p \geq 3$ for a general point $p \in \widetilde{e}(\widetilde{U}_0) \cap F$ since $\dim \widetilde{e}(\widetilde{U}_0) = 6$ and $\dim \widetilde{U}_0 \geq 8$ by Proposition 4.1.6. Hence $\dim \mathrm{Locus}(\widetilde{M_0})_F \geq 5$ by Lemma 4.3.11 (1). By Lemma 4.3.14, we have $X \simeq \mathbb{P}^5$. This contradicts $l_X = 4$.

Hence the divisor $D := \widetilde{e}(\widetilde{U})$ is ample and meets every fiber of φ. If there is a φ-fiber F with $\dim F \geq 3$, then we have $\dim \mathrm{Locus}(\widetilde{M_0})_F \geq 5$, which yields a contradiction again. Thus φ is a $\mathbb{P}^2$-bundle by Proposition 4.1.13 and [**Fuj87**, Lemma 2.12]. $\square$

By Lemma 4.3.16, φ is a $\mathbb{P}^2$-bundle, $n = 5$ and $l_X = 4$.

Set $\mathscr{E}_Y := \varphi_* \mathcal{O}_{\mathbb{P}(\mathscr{E})}(1)$. Then $(Y, \mathscr{E}_Y)$ is also a pair as in Theorem 4.0.3 and the following symmetric diagram is obtained:

$$\begin{tikzcd} & \mathbb{P}_X(\mathscr{E}) = \mathbb{P}_Y(\mathscr{E}_Y) \arrow[dl, "\pi"'] \arrow[dr, "\varphi"] & \\ X & & Y. \end{tikzcd}$$

We may assume that $(Y, \mathscr{E}_Y)$ is a pair as in Setting 4.3.1. In the rest of this proof we denote by C_X (resp. C_Y) a minimal rational curve on X (resp. Y) and by $\widetilde{C}_X$ (resp. $\widetilde{C}_Y$) a minimal lift over C_X (resp. C_Y). Set $R_X := \mathbb{R}_{\geq 0}[\widetilde{C}_X]$ and $R_Y := \mathbb{R}_{\geq 0}[\widetilde{C}_Y]$. If $R_Y = R_\pi$, namely Theorem 4.3.2 is true for $(Y, \mathscr{E}_Y)$, then Theorem 4.0.3 is true for the pair $(Y, \mathscr{E}_Y)$ by the argument given later in the subsequent sections. However there is no pair $(Y, \mathscr{E}_Y)$ as in this case. Hence we have $R_Y \neq R_\pi$ and hence $l_Y = 4$.

Proof of Theorem 4.3.2. To apply Proposition 4.2.2, we will construct a closed subvariety $V \subset \mathbb{P}_X(\mathscr{E})$ which is a section for both projection π and φ.

By [**Wat11**, Theorem 1.2], there is a point $x_1 \in X$ such that

$$\mathrm{ChLocus}_2(M)_{x_1} = X.$$

Hence for any point $x_2 \in X$, there are two minimal rational curves $C_{X,1}$ and $C_{X,2}$ with $x_1, x_2 \in C_{X,1} \cup C_{X,2}$ and $C_{X,1} \cap C_{X,2} \neq \emptyset$. Since minimal lifts over a fixed minimal rational curve sweep out a divisor in a π-fiber by Lemma 4.1.3, there are minimal lifts $\widetilde{C}_{X,1}$ and $\widetilde{C}_{X,2}$ with $\widetilde{C}_{X,1} \cap \widetilde{C}_{X,2} \neq \emptyset$. Hence we have $\dim \mathrm{ChLocus}_2(\widetilde{M})_{\pi^{-1}(x_1)} \geq 5$. Note that by [**CO06**, Corollary 2.2 and Remark 2.4] we have

$$\mathrm{NE}(\mathrm{ChLocus}_2(\widetilde{M})_{\pi^{-1}(x_1)}, \mathbb{P}(\mathscr{E})) \subset \langle R_\pi, R_X \rangle.$$

Thus there is a component V of $\mathrm{ChLocus}_2(\widetilde{M})_{\pi^{-1}(x_1)}$ such that the morphism $V \to Y$ is finite and hence surjective.

Claim 4.3.16.1. $R_X = R_Y$.

Proof of Claim. We will prove $[\widetilde{C}_X] = [\widetilde{C}_Y]$. Note that $\xi_{\mathscr{E}}.\widetilde{C}_X = \xi_{\mathscr{E}}.\widetilde{C}_Y$. Thus it is enough to see that $\pi^*(-K_X).\widetilde{C}_X = \pi^*(-K_X).\widetilde{C}_Y$.

Since $\dim \mathbb{P}(\mathscr{E}_Y|_{C_Y}) \cap V \geq 1$, we have

$$0 \neq \mathrm{NE}(V, \mathbb{P}(\mathscr{E})) \cap \mathrm{NE}(\mathbb{P}(\mathscr{E}_Y|_{C_Y}), \mathbb{P}(\mathscr{E})) = \langle R_\pi, R_X \rangle \cap \langle R_\varphi, R_Y \rangle.$$

Thus $\pi^*(-K_X).\widetilde{C}_X \geq \pi^*(-K_X).\widetilde{C}_Y$.

Note that by applying the same argument as above for the pair $(Y, \mathscr{E}_Y)$, we have

$$\pi(\mathrm{Locus}(\widetilde{M}_Y)) = X,$$

where $\widetilde{M_Y}$ is the union of the families of minimal lifts $\widetilde{C}_Y$. Hence the images of the minimal lifts $\widetilde{C}_Y$ define a covering family of rational curves on X. Hence we have $\pi^*(-K_X).\widetilde{C}_X \leq \pi^*(-K_X).\widetilde{C}_Y$ by the minimality of the anticanonical degree. Thus the assertion follows. $\square$

Then, by Lemma 4.1.18, V is a section of the morphism φ corresponding to the following sequence:

$$0 \to \mathscr{E}_{Y,1} \to \mathscr{E}_Y \to \mathcal{O}_Y(1) \to 0,$$

and $\mathrm{NE}(V, \mathbb{P}(\mathscr{E})) = R_X = R_Y$. Hence, again by Lemma 4.1.18, V is also a section of the morphism π corresponding to a sequence:

$$0 \to \mathscr{E}_1 \to \mathscr{E} \to \mathcal{O}_X(1) \to 0.$$

Thus V is a section for both projection π and φ. Then Proposition 4.2.2 and the fact $n \geq 5$ implies $X \simeq \mathbb{Q}^5$, which contradicts $l_X = 4$. $\square$

4. Case $l_X \geq n$

In this section, we will prove Theorem 4.0.3 for pairs $(X, \mathscr{E})$ with $l_X \geq n$. In this case, by Proposition 4.1.2, $X \simeq \mathbb{P}^n$ or $\mathbb{Q}^n$ and hence it is enough to prove the following:

THEOREM 4.4.1. *Let $(X, \mathscr{E})$ be a pair as in Theorem 4.0.3 with $X \simeq \mathbb{P}^n$ or $\mathbb{Q}^n$. Then $\mathscr{E}$ splits unless $(X, \mathscr{E})$ is isomorphic to a pair as in Theorem 4.0.3 (a)–(c).*

In this section, we will identify the i-th Chern class of a vector bundle with an integer if $A^i(X) \simeq \mathbb{Z}$.

By the following proposition, the proof of Theorem 4.4.1 is reduced to give a classification of nef vector bundles of rank $n-2$ on $\mathbb{P}^n$ (resp. $\mathbb{Q}^n$) with first Chern class three (resp. two):

PROPOSITION 4.4.2. *Let $(X, \mathscr{E})$ be a pair as in Theorem 4.0.3 with $X \simeq \mathbb{P}^n$ or $\mathbb{Q}^n$. Then $\mathscr{E}(-1)$ is a nef vector bundle of rank $n-2$ with $c_1(\mathscr{E}(-1)) = c_1(X) - n + 2$.*

PROOF. Since $c_1(\mathscr{E}) = c_1(X)$, we have $c_1(\mathscr{E}(-1)) = c_1(X) - n + 2$. Thus it is enough to show that $\mathscr{E}(-1)$ is nef.

If $\ell(R_\varphi) \neq n-2$, then by Proposition 4.1.14 we have $(X, \mathscr{E}) \simeq (\mathbb{P}^5, \mathcal{O}(2^3))$ and the assertion follows.

If $\ell(R_\varphi) = n-2$, then by Theorem 4.3.2 the divisor $l_X \xi_{\mathscr{E}} + \pi^* K_X$ is nef. Note that $l_X \xi_{\mathscr{E}} + \pi^* K_X = l_X \xi_{\mathscr{E}} - r_X \pi^* H_X$. Since $X \simeq \mathbb{P}^n$ or $\mathbb{Q}^n$, we have $l_X = r_X$. Hence $\xi_{\mathscr{E}} - \pi^* H_X$ is nef and the assertion follows. $\square$

For partial results or discussions on the classification of nef vector bundles on $\mathbb{P}^n$ or $\mathbb{Q}^n$ with $c_1(\mathscr{E}(-1)) = c_1(X) - n + 2$ without the condition on the rank, we refer the reader to [**OT14**, **Ohn14**, **Ohn16**, **Ohn17**].

4.1. Spannedness and adjunction. In this subsection, we slightly generalize the problem and consider the classification of nef vector bundles $\mathscr{F}$ on $\mathbb{P}^n$ or $\mathbb{Q}^n$ ($n \geq 3$) which satisfy

$$c_1(\mathscr{F}) + \operatorname{rank} \mathscr{F} \leq c_1(X). \tag{4.4.2.1}$$

PROPOSITION 4.4.3. *If a nef vector bundle $\mathscr{F}$ on $X \simeq \mathbb{P}^n$ or $\mathbb{Q}^n$ $(n \geq 3)$ satisfies* (4.4.2.1)*, then $\mathscr{F}$ is generated by global sections.*

PROOF. We will show the assertion by slightly modifying the argument in [**APW94**, Proof of Proposition 2.6]. First we will prove that

CLAIM 4.4.3.1. $H^i(\mathscr{F}(-i)) = 0$ for $0 < i < c_1(X)$.

PROOF OF CLAIM. If $c_1(X) > i \geq \operatorname{rank} \mathscr{F}$, then by the Le Potier vanishing theorem we have $H^i(X, \mathscr{F}(-i)) = 0$. Thus $H^i(X, \mathscr{F}(-i)) = 0$ for $c_1(X) > i \geq c_1(X) - c_1(\mathscr{F})$ by (4.4.2.1). On the other hand, if $c_1(X) - c_1(\mathscr{F}) > i > 0$, then we have

$$\begin{aligned}
&H^i(X, \mathscr{F}(-i)) \\
&= H^i(\mathbb{P}(\mathscr{F}), \xi_{\mathscr{F}} - i\pi^* H_X) \\
&= H^i(\mathbb{P}(\mathscr{F}), K_{\mathbb{P}(\mathscr{F})} + (r+1)\xi_{\mathscr{F}} + (c_1(X) - c_1(\mathscr{F}) - i)\pi^* H_X) \\
&= 0,
\end{aligned}$$

where the last vanishing follows from the Kodaira vanishing theorem on $\mathbb{P}(\mathscr{F})$. □

Hence the assertion follows if $X \simeq \mathbb{P}^n$ since $\mathscr{F}$ is 0-regular in the sense of Castelnuovo-Mumford.

Assume $X = \mathbb{Q}^n$. Then we already have $H^i(\mathscr{F}(-i)) = 0$ for $n > i > 0$. If $H^n(\mathscr{F}(-n)) = 0$, then the assertion follows as above.

Assume that $H^n(\mathscr{F}(-n)) \neq 0$, or $H^0(\mathscr{F}^*) \neq 0$ by the Serre duality. Then we have a section of $\mathscr{F}^*$ and hence a subbundle $\mathcal{O} \subset \mathscr{F}^*$ by [**CP91**, Proposition 1.2 (12)]. Then the bundle $\mathscr{F}' \coloneqq (\mathscr{F}^*/\mathcal{O})^*$ is nef by [**CP91**, Proposition 1.2 (8)], and $c_1(\mathscr{F}') = c_1(\mathscr{F})$. Hence $\mathscr{F}'$ satisfies the condition of this proposition. By a similar computation as above using the Kodaira vanishing theorem on $\mathbb{P}(\mathscr{F}')$, we have $H^1(X, \mathscr{F}') = 0$. Hence we have $\mathscr{F} = \mathcal{O} \oplus \mathscr{F}'$, and the assertion follows by induction on the rank. □

If $\operatorname{rank} \mathscr{F} \geq n$ in Proposition 4.4.3, then by using Theorem 4.0.1 we see that $(X, \mathscr{F})$ is isomorphic to

$$(\mathbb{P}^n, \mathcal{O}^{\oplus n+1}),\ (\mathbb{P}^n, \mathcal{O}(1, 0^{n-1})),\ (\mathbb{P}^n, \mathcal{O}^{\oplus n}),\ (\mathbb{P}^n, T_{\mathbb{P}^n}(-1)) \text{ or } (\mathbb{Q}^n, \mathcal{O}^{\oplus n}).$$

On the other hand, if $n > \operatorname{rank} \mathscr{F}$, then the following proposition enables us to reduce the study of $\mathscr{F}$ to a lower rank case $\operatorname{rank} \mathscr{F} = c_1(\mathscr{F}) - c_1(X) + n + 1$:

PROPOSITION 4.4.4. *Assume $n > \operatorname{rank} \mathscr{F} \geq c_1(\mathscr{F}) - c_1(X) + n + 1$ in Proposition 4.4.3. Then there exist the following exact sequences of vector bundles:*

$$\begin{gathered}
0 \to \mathcal{O} \to \mathscr{F}_0 \to \mathscr{F}_1 \to 0, \\
\vdots \\
0 \to \mathcal{O} \to \mathscr{F}_{k-1} \to \mathscr{F}_k \to 0,
\end{gathered}$$

where $\mathscr{F}_0 \coloneqq \mathscr{F}$ and $\operatorname{rank} \mathscr{F}_k = c_1(\mathscr{F}) - c_1(X) + n + 1$.

Proof. A similar proof is contained in [**Tir13**, Lemmas 2.4 and 2.7].

If $\operatorname{rank} \mathscr{F} = c_1(\mathscr{F}) - c_1(X) + n + 1$, then there is nothing to prove. Hence we assume $\operatorname{rank} \mathscr{F} > c_1(\mathscr{F}) - c_1(X) + n + 1$.

Since $\mathscr{F}$ is spanned by Proposition 4.4.3, the zero locus Z of a general section of $\mathscr{F}$ defines a smooth subscheme of dimension $n - \operatorname{rank} \mathscr{F} > 0$ if $Z \neq \emptyset$. Assume $Z \neq \emptyset$. Then by adjunction we have $-K_Z = (c_1(X) - c_1(\mathscr{F}))|_Z$ and, by our assumption, $-K_Z$ is ample. By [**KO73**] we have $\dim Z + 1 \geq r_Z$. Therefore $n - r + 1 \geq c_1(X) - c_1(\mathscr{F})$. This contradicts our assumption. Hence a general section of $\mathscr{F}$ defines a subbundle $\mathcal{O} \subset \mathscr{F}$, and the assertion follows by induction on the rank. □

4.2. Case $X \simeq \mathbb{P}^n$.

Proof of Theorem 4.4.1 for $X \simeq \mathbb{P}^n$. By Proposition 4.4.2, $\mathscr{F} := \mathscr{E}(-1)$ is a nef vector bundle with $c_1(\mathscr{F}) = 3$ and $\operatorname{rank} \mathscr{F} = n - 2$. Then $\mathscr{F}$ is globally generated by Proposition 4.4.3 and hence $\mathscr{F}$ is a direct sum of line bundles by [**SU14**, **AM13**] (cf. [**Tir13**, Corollary 2.5]). □

4.3. Case $X \simeq \mathbb{Q}^n$. In this subsection we assume that $X \simeq \mathbb{Q}^n$ ($n \geq 5$) and $\mathscr{F}$ is a nef vector bundle of rank $n - 2$ with $c_1(\mathscr{F}) = 2$. Then $\mathscr{F}$ is globally generated by Proposition 4.4.3. If $n \geq 7$, then $\mathscr{F}$ is a direct sum of line bundles by [**Tir13**, Corollary 2.8]. Therefore we further assume $n = 5$ or 6. Then $\mathscr{F}_k$ in Proposition 4.4.4 is a globally generated vector bundle of rank 3 with $c_1(\mathscr{F}_k) = 2$.

Proposition 4.4.5. *$\mathscr{F}_k$ splits or is isomorphic to the Ottaviani bundle.*

Proof. If $c_3(\mathscr{F}_k) = 0$, then a general section of $\mathscr{F}_k$ defines a subbundle $\mathcal{O} \subset \mathscr{F}_k$. Then the quotient $\mathscr{F}_{k+1}$ is a nef vector bundle of rank two with $c_1 = 2$. Thus it is a Fano bundle of rank two. Then, by [**APW94**], $\mathscr{F}_{k+1}$ and hence $\mathscr{F}_k$ splits.

Assume that $c_3(\mathscr{F}_k) \neq 0$. If $n = 6$ and the restriction of $\mathscr{F}_k$ to a general linear section $\mathbb{Q}^5$ is the Ottaviani bundle, then by Proposition 0.3.2 $\mathscr{F}_k$ is also the Ottaviani bundle on $\mathbb{Q}^6$. Note that $\mathscr{F} = \mathscr{F}_k$ if $n = 5$. Hence it is enough to show the following:

Claim 4.4.5.1. Assume that $n = 5$. If $c_3(\mathscr{F}) \neq 0$, then $\mathscr{F}$ is the Ottaviani bundle.

Proof of Claim. Set $\mathscr{E} := \mathscr{F}(1)$. Then the pair $(\mathbb{Q}^5, \mathscr{E})$ satisfies the condition of Setting 4.3.1 by Propsition 4.1.14. The semiample divisor $\xi_{\mathscr{E}} - \pi^* H_X = \xi_{\mathscr{F}}$ defines the contraction φ by Theorem 4.3.2. Let F be a component of a φ-fiber and $\bar{F}$ a resolution of F. By Corollary 4.3.3 (2), $c_3(\mathscr{F})|_{\bar{F}} = 0$ and hence $c_3(\mathscr{F}).\pi(F) = 0$. Since $c_3(\mathscr{F}) \neq 0$, we have $\dim F = \dim \pi(F) \leq 2$. By Lemma 4.1.12, we have $\dim E \geq \dim \mathbb{P}(\mathscr{E})$ and hence φ is of fiber type. The assertion follows from Theorem 4.2.1. □

This completes the proof of Proposition 4.4.5. □

Proof of Theorem 4.4.1 for $X \simeq \mathbb{Q}^n$. As we mentioned, $\mathscr{E}(-1)$ is a globally generated vector bundle of rank $n - 2$ on $\mathbb{Q}^n$ with $c_1(\mathscr{E}(-1)) = 2$, and we may assume $n = 5$ or 6. If $n = 5$, then the assertion follows from

Proposition 4.4.5. If $n = 6$, then there exists the following exact sequence by Proposition 4.4.4:

$$0 \to \mathcal{O} \to \mathscr{E}(-1) \to \mathscr{F}_1 \to 0.$$

By Proposition 4.4.5, $\mathscr{F}_1$ is a direct sum of line bundles or the Ottaviani bundle. In the former case the exact sequence splits and hence $\mathscr{E}$ is a direct sum of line bundles. In the latter case $\mathscr{E}(-1)$ is the dual of the Spinor bundle or $\mathscr{E}(-1) \simeq \mathcal{O} \oplus \mathscr{F}_1$ by [**Ott88**, Sect. 3]. Thus the assertion follows. □

5. Case $l_X = n-1$ and φ is birational

In this section, we will prove Theorem 4.0.3 under Setting 4.3.1 when $l_X = n-1$ and φ is a birational contraction:

THEOREM 4.5.1. *Let $(X, \mathscr{E})$ be a pair as in Setting 4.3.1. Assume that $l_X = n-1$ and φ is a birational contraction. Then $\mathscr{E}$ is a direct sum of line bundles.*

In this case $E \coloneqq \mathrm{Exc}(\varphi)$ is an irreducible divisor. Set $Z \coloneqq \varphi(E)$.

$$\begin{tikzcd} E \arrow[r] \arrow[d, hook] & Z \arrow[d, hook] \\ \mathbb{P}(\mathscr{E}) \arrow[r, "\varphi"] \arrow[d, "\pi"'] & Y \\ X. & \end{tikzcd}$$

LEMMA 4.5.2. *$E = \widetilde{e}(\widetilde{U})$ and $n-2 \geq \dim Z \geq n-4$.*

PROOF. By Theorem 4.3.2, minimal lifts over minimal rational curves are contracted by φ. Thus $\widetilde{e}(\widetilde{U}) \subset \mathrm{Exc}(\varphi)$. By Lemma 4.1.3 we have $\dim \widetilde{e}(\widetilde{U}) \geq 2n-4 = \dim \mathbb{P}(\mathscr{E}) - 1$. Hence $E = \widetilde{e}(\widetilde{U})$. By Lemma 4.1.12, we have $n \geq \dim F \geq n-2$ for a non-trivial φ-fiber F. Thus $n-2 \geq \dim Z \geq n-4$. □

LEMMA 4.5.3. *If $\dim Z = n-3$ or $n-2$, then $E \equiv \xi - a\pi^* H_X$ for some $a \in \mathbb{Z}$.*

PROOF. Let F be a component of a general non-trivial φ-fiber and set $D_F \coloneqq \xi|_F$. Then either

(1) $\dim Z = n-3$, F is normal and $\Delta(F, D_F) = 0$ or
(2) $\dim Z = n-2$ and $(F, D_F) \simeq (\mathbb{P}^{n-2}, \mathcal{O}(1))$

by [**And95**, Theorem 2.1] and Proposition 4.1.13. Also, by Theorem 4.3.2, $(n-1)D_F = -K_X|_F$.

Note that $\dim F = n-1 \geq 4$ in the former case, hence, by using the classification of varieties with small delta genus [**Fuj75**,**Fuj82b**], we see that there is a linear subspace $\mathbb{P}^2 \subset F$ through any point $p \in F$. Hence there is a morphism $j\colon \mathbb{P} \to X$ through a general point $x \in X$ with $j^*\mathcal{O}(-K_X) = \mathcal{O}_{\mathbb{P}}(n-1)$, where $\mathbb{P} \coloneqq \mathbb{P}^2 \subset F$ if $\dim Z = n-3$ or $\mathbb{P} \coloneqq F$ if $\dim Z = n-2$.

Let $f\colon \mathbb{P}(\mathscr{E}|_{\mathbb{P}}) \to \mathbb{P}(\mathscr{E})$ be the morphism obtained by taking the base change of j by π, and let $\mathbb{P}(\mathscr{E}|_{\mathbb{P}}) \xrightarrow{\varphi_{\mathbb{P}}} Y_{\mathbb{P}} \to Y$ be the Stein factorization of

$\varphi \circ f$. Set $E_{\mathbb{P}} \coloneqq \operatorname{Exc}(\varphi_{\mathbb{P}})$. Then there exists the following commutative diagram:

(4.5.3.1)
$$\begin{tikzcd}
E_{\mathbb{P}} \subset f^*E \arrow[r] \arrow[d, hook] & E \arrow[d, hook] & \\
\mathbb{P}(\mathscr{E}|_{\mathbb{P}}) \arrow[r, "f"'] \arrow[d, "\pi_{\mathbb{P}}"'] & \mathbb{P}(\mathscr{E}) \arrow[r, "\varphi"'] \arrow[d, "\pi"'] & Y \\
\mathbb{P} \arrow[r, "j"] & X. &
\end{tikzcd}$$

Since $j(\mathbb{P})$ passes through a general point of X, $\varphi_{\mathbb{P}}$ is not of fiber type. Since $\dim f^*E > \dim Z$, it holds that $f^*E \subset E_{\mathbb{P}}$. Thus we have $E_{\mathbb{P}} = \operatorname{Supp} f^*E$.

Now $\mathscr{E}|_{\mathbb{P}}(-1)$ is a nef vector bundle of rank $n-2$ with $c_1 = 1$ by Corollary 4.3.3 and $\varphi_{\mathbb{P}}$ is not of fiber type. Hence $\mathscr{E}|_{\mathbb{P}}(-1)$ is isomorphic to $\mathcal{O}(1, 0^{n-4})$ by [**SW90c**, **PSW92a**]. Thus $E_{\mathbb{P}}$ is a hyperplane in the $\pi_{\mathbb{P}}$-fiber over a general point. Hence the same holds for E and the assertion follows. □

Proposition 4.5.4. $\dim Z = n-4$.

Proof. Assume to the contrary that $\dim Z \geq n-3$. We use the same notation as in the proof of Lemma 4.5.3. Then $f^*E = E_{\mathbb{P}}$ and $\mathscr{E}|_{\mathbb{P}} \simeq \mathcal{O}(2, 1^{n-4})$ by the proof of Lemma 4.5.3. Hence we have $\mathcal{O}_{\mathbb{P}}(aj^*H_X) = \mathcal{O}_{\mathbb{P}}(2)$. This implies $\mathcal{O}_{\mathbb{P}^1}(aH_X|_{\mathbb{P}^1}) = \mathcal{O}_{\mathbb{P}^1}(2H_{\mathbb{P}^1})$ for a minimal rational curve $\mathbb{P}^1 \to X$.

Let $s\colon \mathcal{O} \to \mathscr{E}(-a)$ be a section corresponding to $E \in |\xi - a\pi^*H_X|$ and W the zero locus of the section s. Assume $W \neq \emptyset$. Then by Proposition 4.1.7 there is a minimal rational curve $f\colon \mathbb{P}^1 \to X$ such that $f(\mathbb{P}^1) \cap W \neq \emptyset$ and $f(\mathbb{P}^1) \not\subset W$. On the other hand, if $f\colon \mathbb{P}^1 \to X$ is a minimal rational curve, then the restriction of the section

$$f^*s\colon \mathcal{O}_{\mathbb{P}^1} \to f^*\mathscr{E}(-a) \simeq \mathcal{O}(0, (-1)^{n-3})$$

is non-vanishing or the zero morphism. This gives a contradiction. Hence s is a non-vanishing section.

Therefore the quotient $\mathscr{E}(-a)/\mathcal{O}$ is a uniform vector bundle of type $\mathcal{O}(-1^{n-3})$ and hence a direct sum of line bundles by [**AW01**, Proposition 1.2]. This implies that $\mathscr{E}$ is also a direct sum of line bundles and $\mathscr{E} \simeq \mathcal{O}_X(2, 1^{n-2})$. Then $\dim Z = n-4$, which contradicts our assumption that $\dim Z = n-2$ or $n-3$. □

Proof of Theorem 4.5.1. By Proposition 4.5.4, we have $\dim Z = n-4$ and any component of a non-trivial φ-fiber has dimension n by Lemma 4.1.12. Hence each n-dimensional component of a φ-fiber is a section of π by Lemma 4.1.18.

Let C be a minimal rational curve, $n\colon \mathbb{P}^1 \to C \subset X$ the normalization and $x \in \mathbb{P}^1$ a point. We fix a decomposition $\mathscr{E}|_{\mathbb{P}^1} \simeq \mathcal{O}_{\mathbb{P}^1}(2, 1^{n-3})$ as in Lemma 4.1.3. Then by taking a base change of the diagram, we obtain the following diagram:

$$\begin{tikzcd}
E_{\mathbb{P}^1} \simeq \mathbb{P}^1 \times \mathbb{P}^{n-4} \arrow[d] & & \\
\mathbb{P}(\mathscr{E}|_{\mathbb{P}^1}) \arrow[r, "m"] \arrow[d, "\pi_{\mathbb{P}^1}"'] & \mathbb{P}(\mathscr{E}) \arrow[r, "\varphi"] \arrow[d, "\pi"'] & Y \\
\mathbb{P}^1 \arrow[r, "n"] & X, &
\end{tikzcd}$$

where $E_{\mathbb{P}^1}$ is the subbundle of $\mathbb{P}(\mathscr{E}|_{\mathbb{P}^1})$ corresponding to the direct summand $\mathcal{O}_{\mathbb{P}^1}(1^{n-3}) \subset \mathscr{E}|_{\mathbb{P}^1}$.

Corresponding to each direct summand $\mathcal{O}_{\mathbb{P}^1}(1)$, there are $n-3$ minimal sections $\widetilde{\mathbb{P}}^1_1, \dots \widetilde{\mathbb{P}}^1_{n-3}$ of $\pi_{\mathbb{P}^1}$.

Note that the morphism $\varphi \circ m \colon \mathbb{P}(\mathscr{E}|_{\mathbb{P}^1}) \to Y$ contracts $E_{\mathbb{P}^1}$. Hence there are sections $\widetilde{X}_i$ of π such that $m^{-1}(\widetilde{X}_i) = \widetilde{\mathbb{P}}^1_i$. Note that each section $\widetilde{X}_i$ defines a surjection $\mathscr{E} \to \mathcal{O}_X(1)$ and hence we have a morphism $a \colon \mathscr{E} \to \mathcal{O}_X(1^{n-3})$.

Claim 4.5.4.1. The morphism a is surjective.

Proof of Claim. The assertion is true on any point $x \in C$. Let C' be a minimal rational curve on X. Assume that the assertion is true at a point $x' \in C'$. Then the assertion is true for any point on C', since the bundles is isomorphic to $\mathcal{O}_{\mathbb{P}^1}(2, 1^{n-3})$ on the normalization. Hence the assertion follows from Proposition 4.1.7. □

By the above claim, we have the following exact sequence:

$$0 \to \mathcal{O}_X(2) \to \mathscr{E} \to \mathcal{O}_X(1^{n-3}) \to 0.$$

This sequence splits since $H^1(\mathcal{O}_X(1)) = 0$, and the assertion follows. □

6. Case $l_X = n-1$ and φ is of fiber type

This section deals with the remaining case where $l_X = n-1$ and φ is of fiber type:

Theorem 4.6.1. *Let $(X, \mathscr{E})$ be a pair as in Setting 4.3.1. Assume that $l_X = n-1$ and φ is of fiber type. Then the pair $(X, \mathscr{E})$ is isomorphic to one of the pairs (d)–(g) in Theorem 4.0.3.*

Let F be a general φ-fiber and set $D_F := \xi|_F$. By taking the base change of π by $\pi|_F$, we have the following diagram:

(4.6.1.1)
$$\begin{tikzcd}
\widetilde{F} \arrow[r, "\iota"] & \mathbb{P}(\mathscr{E}|_F) \arrow[r] \arrow[d, "\pi_F"'] \arrow[rr, bend left, "\theta_F"] & \mathbb{P}(\mathscr{E}) \arrow[r, "\varphi"'] \arrow[d, "\pi"'] & Y \\
& F \arrow[r, "\pi|_F"] & X, &
\end{tikzcd}$$

where $\widetilde{F}$ is the section of π_F corresponding to the original fiber F. Let

$$\mathbb{P}(\mathscr{E}|_F) \xrightarrow{\varphi_F} Y' \to Y$$

be the Stein factorization of θ_F. Then φ_F is defined by the semiample divisor $\xi_{\mathscr{E}|_F} - \pi_F^* D_F$ by the proof of Corollary 4.3.3.

6.1. Bounding the dimension of X. The first step of the proof is to show $n \leq 6$. In addition, $(\dim Y; F, \mathcal{O}(D_F), \mathscr{E}|_F)$ is also determined:

PROPOSITION 4.6.2. *Under the assumption of Theorem 4.6.1, we have $n \leq 6$ and the quadruple $(\dim Y; F, \mathcal{O}(D_F), \mathscr{E}|_F)$ is one of the following:*
(1) $(n; \mathbb{P}^{n-3}, \mathcal{O}_{\mathbb{P}^{n-3}}(1), \mathcal{O}(2, 1^{n-3}))$,
(2) $(n-1; \mathbb{Q}^{n-2}, \mathcal{O}_{\mathbb{Q}^{n-2}}(1), \mathscr{S}^*_{\mathbb{Q}}(1) \oplus \mathcal{O}(1^{n-4}))$,
where F is a general φ-fiber.

Note that, by Lemma 4.1.12, we have $n - 3 \leq \dim Y \leq n$ in this case.

LEMMA 4.6.3. $\dim Y \geq n - 1$.

PROOF. We have $\dim Y \neq n - 3$. Otherwise the projective bundle $\mathbb{P}(\mathscr{E})$ is trivial by [**NO07**, Lemma 4.1], which contradicts the fact that $\mathscr{E}|_{\mathbb{P}^1} \simeq \mathcal{O}(2, 1^{n-3})$ for a minimal rational curve $f\colon \mathbb{P}^1 \to X$.

Assume $\dim Y = n - 2$. Then a general φ-fiber F is a smooth projective manifold of dimension $n - 1$ with $-K_F = (n-2)\xi|_F$ by adjunction. Hence F is a del Pezzo manifold. Set $D_F := \xi|_F$ and $\mathcal{O}_F(1) := \mathcal{O}(D_F)$. Note that $(n-1)D_F = (\pi|_F)^*(-K_X)$ by Theorem 4.3.2.

By Corollary 4.3.3, $\mathscr{E}|_F(-1)$ is semiample and $c_1(\mathscr{E}|_F(-1)) = D_F$. Since $\dim Y = n - 2$, we have $(\xi_{\mathscr{E}|_F} - \pi_F^* D_F)^{n-1} = 0$.

The Kodaira vanishing theorem implies

$$H^i(F, \det(\mathscr{E}|_F(-1)) \otimes \mathcal{O}(K_F)) = 0$$

and

$$H^i(\mathbb{P}(\mathscr{E}|_F(-1)), t\xi_{\mathscr{E}|_F(-1)}) = 0$$

for $i > 0$ and $t > 0$. Also

$$H^0(F, \det(\mathscr{E}|_F(-1)) \otimes \mathcal{O}(K_F)) = H^0(F, \mathcal{O}_F(-n+3)) = 0.$$

Hence, by [**PSW92b**, Corollary 1.3], we have the following exact sequence:

$$0 \to \mathcal{O}_F(-1) \to \mathcal{O}_F^{\oplus n-1} \to \mathscr{E}|_F(-1) \to 0.$$

By dualizing this sequence, we see that the ample line bundle $\mathcal{O}_F(1)$ is generated by $n - 1$ sections. This contradicts $\dim F = n - 1$. $\square$

PROOF OF PROPOSITION 4.6.2. By Lemma 4.6.3 and Lemma 4.1.12 we have $\dim Y = n - 1$ or n. Note that $-K_F = (n-2)D_F$ by adjunction.

CASE. $\dim Y = n - 1$.

In this case $F \simeq \mathbb{Q}^{n-2}$ and $\mathcal{O}(D_F) \simeq \mathcal{O}(1)$ by the Kobayashi-Ochiai theorem. Note that φ_F is of fiber type since $\dim \mathbb{P}(\mathscr{E}) > \dim Y$.

By Corollary 4.3.3, $\Omega_\pi|_F$ is a nef vector bundle of rank $n - 3$ with $c_1(\mathscr{E}|_F(-1)) = 1$. Thus, by [**PSW92a**], the bundle $\Omega_\pi|_F$ is either

- a direct sum of line bundles,
- $\mathscr{S}^*_{\mathbb{Q}} \oplus \mathcal{O}$ with $n = 6$ or
- $\mathscr{S}^*_{\mathbb{Q}}$ with $n = 5$.

Hence, by Corollary 4.3.3 and [**Ott88**, Theorem 2.3], $\mathscr{E}|_F(-1)$ is either

- a direct sum of line bundles,
- $\mathscr{S}_{\mathbb{Q}}^* \oplus \mathcal{O}^{\oplus 2}$ with $n = 6$ or
- $\mathscr{S}_{\mathbb{Q}}^* \oplus \mathcal{O}$ with $n = 5$.

Since φ_F is a morphism of fiber type, the first case does not occur.

CASE. $\dim Y = n$.

In this case, $F \simeq \mathbb{P}^{n-3}$ and $\mathcal{O}(D_F) \simeq \mathcal{O}_{\mathbb{P}^{n-3}}(1)$ by Kobayashi-Ochiai theorem. Also φ is an adjunction theoretic scroll by Proposition 4.1.13. Thus the morphism φ is a smooth $\mathbb{P}^{n-3}$-bundle over a open subset Y^0 of Y. Set $\mathbb{P}(\mathscr{E})^0 := \varphi^{-1}(Y^0)$. We will denote by F_y a fiber $(\varphi^0)^{-1}(y) \simeq \mathbb{P}^{n-3}$ for $y \in Y^0$.

STEP 1. By Corollary 4.3.3, $\Omega_\pi|_F$ is a nef vector bundle with $c_1(\Omega_\pi|_F) = 1$. Hence $\Omega_\pi|_F \simeq T_{\mathbb{P}^{n-3}}(-1)$ or $\mathcal{O}(1, 0^{n-4})$ by Theorem 4.0.1. Therefore $\mathscr{E}|_F \simeq T_{\mathbb{P}^{n-3}} \oplus \mathcal{O}(1)$ or $\mathcal{O}(2, 1^{n-3})$ by Corollary 4.3.3. Thus one of the following holds:

- $\dim \operatorname{Im} \varphi_F = n-2$ and $\mathscr{E}|_F \simeq T_{\mathbb{P}^{n-3}} \oplus \mathcal{O}(1)$,
- $\dim \operatorname{Im} \varphi_F = 2n-6$ and $\mathscr{E}|_F \simeq \mathcal{O}(2, 1^{n-3})$.

Since $\dim \operatorname{Im} \varphi_{F_y}$ do not depend on $y \in Y^0$, the isomorphic classes of $\mathscr{E}|_{F_y}$ also do not depend on $y \in Y^0$. If the latter case occurs then $2n-6 \leq n$, or equivalently $n \leq 6$ and the assertion follows. Hence it is enough to show that $\mathscr{E}|_F \simeq \mathcal{O}(2, 1^{n-3})$. In the following we assume to the contrary that $\mathscr{E}|_F \simeq T_{\mathbb{P}^{n-3}} \oplus \mathcal{O}(1)$.

STEP 2. General two points in X can be connected by a chain of (π-images of) φ^0-fibers. In fact, since $\rho_X = 1$, general two points in X can be connected by a chain of lines contained in φ^0-fibers (see [**Deb01**, Proof of Proposition 5.8] or [**KMM92b**, Proof of Lemma 3]). Hence the assertion follows.

STEP 3. Let F_1 and F_2 be two φ^0-fibers. In this step, we show that $\dim(\pi(F_1) \cap \pi(F_2)) \geq 1$ if $\pi(F_1) \cap \pi(F_2) \neq \emptyset$.

Assume $\pi(F_1) \cap \pi(F_2) \neq \emptyset$ and take a point $x \in \pi(F_1) \cap \pi(F_2)$. Then there exists a point $p \in \pi^{-1}(x) \cap F_1$. Since φ_F is a morphism of fiber type, there exists a curve $C \subset \pi^{-1}(\pi(F_2))$ such that $p \in C$ and C is contracted by φ. Since F_1 is a fiber, we have $C \subset F_1$. Hence $\pi(C) \subset \pi(F_1) \cap \pi(F_2)$.

STEP 4. Set $V_y := \operatorname{Im} \theta_{F_y}$. Note that $\dim V_y = n-2$. Let C be the normalization of a curve contained in F_y. Then we have the following diagram:

$$\begin{tikzcd}
 & & & V_y \arrow[d, hook] \\
\mathbb{P}(\mathscr{E}|_C) \arrow[r] \arrow[d, "\pi_C"'] \arrow[rrru, bend left, "\theta_C"] & \mathbb{P}(\mathscr{E}|_{F_y}) \arrow[r] \arrow[d, "\pi_{F_y}"'] \arrow[rru, "\theta_{F_y}"] & \mathbb{P}(\mathscr{E}) \arrow[r, "\varphi"'] \arrow[d, "\pi"'] & Y \\
C \arrow[r] & F_y \arrow[r] & X. &
\end{tikzcd}$$

CLAIM 4.6.3.1. θ_C is surjective onto V_y.

PROOF. If θ_C is not surjective, then $\dim \theta_C(\mathbb{P}(\mathscr{E}|_C)) = n-3$. Hence $\mathscr{E}|_C$ is semistable by [**Miy87**, Theorem 3.1]. On the other hand $T_{\mathbb{P}^{n-3}} \subset \mathscr{E}|_{F_y}$ is a destabilizing subsheaf, which gives a contradiction. □

STEP 5. Fix general points $x_1, x_2 \in X^0$. Then there exists a point $y \in Y^0$ such that $x_1 \in \pi(F_y)$, and hence $\varphi(\pi^{-1}(x_1)) \subset V_y$.

By Step 2, x_1 and x_2 can be connected by a chain of φ^0-fibers. Then by Step 3 and 4 we have $\varphi(\pi^{-1}(x_2)) \subset V_y$. Hence $\varphi(\pi^{-1}(x)) \subset V_y$ for every general point $x \in X$, which contradicts the surjectivity of φ.

This completes the proof. □

6.2. Decomposition of $\mathscr{E}$. We now turn to prove that the bundle $\mathscr{E}$ admits a decomposition except for one case. Recall that each bundle $\mathscr{E}$ of pairs (d)–(f) in Theorem 4.0.3 is decomposable.

PROPOSITION 4.6.4. *The following hold:*

(1) *If Proposition 4.6.2 (1) occurs, then $r_X = n-1$ and $\mathscr{E} \simeq \mathscr{E}_1 \oplus \mathcal{O}(1^{n-4})$ with an ample vector bundle $\mathscr{E}_1$ of rank two.*
(2) *If Proposition 4.6.2 (2) occurs and $n = 6$, then $r_X = 5$ and $\mathscr{E} \simeq \mathscr{E}_1 \oplus \mathcal{O}(1)$ with an ample vector bundle $\mathscr{E}_1$ of rank three.*

PROOF. (1) Assume that Proposition 4.6.2 (1) occurs. Let F be a general φ-fiber and consider the following diagram:

$$\begin{tikzcd}
E \simeq \mathbb{P}(\mathcal{O}(1^{n-3})) \arrow[r] \arrow[d, hook] & E' \arrow[d, hook] & & \\
\mathbb{P}(\mathscr{E}|_F) \simeq \mathbb{P}(\mathcal{O}(2, 1^{n-3})) \arrow[r] \arrow[d, "\pi_F"'] \arrow[rrr, bend left, "\theta_F"] & \pi^{-1}(\pi(F)) \arrow[r, hook] \arrow[d] & \mathbb{P}(\mathscr{E}) \arrow[r, "\varphi"'] \arrow[d, "\pi"'] & Y \\
F \simeq \mathbb{P}^{n-3} \arrow[r, "\pi|_F"] & \pi(F) \arrow[r, hook] & X, &
\end{tikzcd}$$

where E is the subbundle corresponding to the direct summand $\mathcal{O}(1^{n-3}) \subset \mathscr{E}|_F$ and E' is the image of E in $\pi^{-1}(\pi(F))$. A *minimal section* of π_F is defined to be a section corresponding to a surjection $\mathscr{E}|_F \to \mathcal{O}(1)$. Since φ_F is defined by $\xi_{\mathscr{E}|_F} - \pi^* D_F$, the exceptional divisor of the contraction φ_F is E and hence each minimal section of π_F is contracted to a point by θ_F.

STEP 1. By Proposition 4.1.7 there exists a rational curve $[C] \in M$ such that $C \cap \pi(F) \neq \emptyset$ and $C \not\subset \pi(F)$. Let $x \in C \cap \pi(F)$ be a point. Then the deformations of minimal lifts $\widetilde{C}$ of C sweep out at least a divisor in $\pi^{-1}(x)$ by Lemma 4.1.3. Hence

$$\dim \bigcup_{\widetilde{C}} (\widetilde{C} \cap E' \cap \pi^{-1}(x)) \geq n-5. \tag{4.6.4.1}$$

Fix a minimal lift $\widetilde{C}$ with $\widetilde{C} \cap E' \cap \pi^{-1}(x) \neq \emptyset$ and let w be a point in $\widetilde{C} \cap E' \cap \pi^{-1}(x)$. If $\varphi^{-1}(\varphi(w))$ has dimension $n-3$, then φ is flat at $\varphi(w)$ by Proposition 4.1.13 and [**Fuj87**, Lemma 2.12]. The flatness at $\varphi(w)$ implies $\varphi^{-1}(\varphi(w)) \subset E'$ (In fact it is a projective bundle near φ and the above conclusion $\varphi^{-1}(\varphi(w)) \subset E'$ is trivial, but, here we use only flatness to apply a similar argument also for the case (2)). Thus $\widetilde{C} \subset \varphi^{-1}(\varphi(w)) \subset E'$. This contradicts the fact that $C \not\subset \pi(F)$. Hence φ is not equidimensional at w. By (4.6.4.1), the family of jumping fibers of φ has dimension at least $n-5$.

STEP 2. Let F' be a component of a jumping fiber of φ with $\dim F' \geq n-2$.

Assume that $\dim F' = n-2$. Then F' is isomorphic to $\mathbb{P}^{n-2}$ and $\mathcal{O}_{\mathbb{P}(\mathscr{E})}(1)|_{F'} \simeq \mathcal{O}_{\mathbb{P}^{n-2}}(1)$ by [**And95**, Theorem 2.1]. By Corollary 4.3.3, $\Omega_\pi|_{F'}$ is a nef vector bundle of rank $n-3$ with $c_1 = 1$ and hence isomorphic to $\mathcal{O}(1) \oplus \mathcal{O}^{\oplus n-4}$ by [**PSW92a**]. Thus $\mathscr{E}|_{F'}(-1) \simeq \mathcal{O}(1) \oplus \mathcal{O}^{\oplus n-3}$ by Corollary 4.3.3. Then by a similar argument to Step 1 we have a jumping fiber of dimension $\geq n-1$. Also note that if $n = 6$ then every jumping fiber has dimension $\geq n-1$, otherwise the inequality $\dim \operatorname{Im} \theta_{F'} = 2n-5 > n = \dim Y$ yields a contradiction.

STEP 3. Let F' be a component of a jumping fiber of φ with $\dim F' \geq n-1$ and F a general fiber. Then the image $\pi(F')$ contains a non-zero effective divisor on X. Since $\rho_X = 1$, we have $\pi(F) \cap \pi(F') \neq \emptyset$. Hence $\pi^{-1}(\pi(F)) \cap F' \neq \emptyset$ of dimension $\geq n-4$. Since θ_F contracts only minimal sections, there exists a minimal section $\widetilde{\mathbb{P}}^{n-3} \subset E$ of π_F such that the image P' in E' contains an $(n-4)$-dimensional component of $\pi^{-1}(\pi(F)) \cap F'$. Hence we have $P' \subset F'$. Since $\pi(P') = \pi(F)$ and F is a general fiber, a general point on X is contained in $\pi(F')$. Hence $\dim F' = n$.

STEP 4. Hence we have an $(n-5)$-dimensional family of jumping fibers of dimension n. Let V be an n-dimensional component of a fiber. Then $r_X = n-1$ and V is a section of π corresponding to the following exact sequence by Lemma 4.1.18:

$$0 \to \mathscr{E}_1' \to \mathscr{E} \to \mathcal{O}(1) \to 0.$$

Set $\mathscr{E}_1 := \mathscr{E}_1'$ if $n = 5$. If $n = 6$, then we can find in the same way another section V' with $V \cap V' = \emptyset$, and hence we have the following exact sequence:

$$0 \to \mathscr{E}_1 \to \mathscr{E} \to \mathcal{O}(1^2) \to 0.$$

Now $\mathscr{E}(-1)$ is a nef vector bundle by Theorem 4.3.2. Hence $\mathscr{E}_1(-1)$ is a nef vector bundle of rank two with $c_1(\mathscr{E}_1(-1)) = 1$ by [**CP91**, Proposition 1.2 (8)]. Then, by the Kodaira vanishing theorem on $\mathbb{P}(\mathscr{E}_1)$, we have $H^1(\mathscr{E}_1(-1)) = 0$. Therefore $\mathscr{E} \simeq \mathscr{E}_1 \oplus \mathcal{O}(1^{n-4})$. This completes the proof in the case where φ is an adjunction theoretic scroll.

(2) Assume that Proposition 4.6.2 (2) occurs and $n = 6$. Then consider the following diagram:

$$\begin{tikzcd}
E \simeq \mathbb{Q}^4 \times \mathbb{P}^1 \arrow[r] \arrow[d] & E' \arrow[d, hook] & & \\
\mathbb{P}(\mathscr{E}|_F) \arrow[r] \arrow[d, "\pi_F"'] \arrow[rrr, bend left, "\theta_F"] & \pi^{-1}(\pi(F)) \arrow[r, hook] \arrow[d] & \mathbb{P}(\mathscr{E}) \arrow[r, "\varphi"'] \arrow[d, "\pi"'] & Y \\
F \simeq \mathbb{Q}^4 \arrow[r, "\pi|_F"] & \pi(F) \arrow[r, hook] & X, &
\end{tikzcd}$$

where E is the subbundle corresponding to the direct summand $\mathcal{O}(1^2) \subset \mathscr{E}|_F$. Then the contraction φ_F is an adjunction theoretic scroll and each jumping fiber of the contraction is a section of π_F contained in E. By a similar argument to the above case the assertion follows also in this case. Note that φ is flat at a point $y \in Y$ if φ is equidimensional at y by [**ABW93**, Theorem B]. □

6.3. Index of X. By Proposition 4.6.4, we have already seen that the index of X is $n-1$ except for the case $n=5$ and $\dim Y = 4$. The same thing also holds in the remaining case:

PROPOSITION 4.6.5. *Assume that $n = 5$ and $\dim Y = 4$. Then r_X is four.*

PROOF. Set $a := 4/r_X \in \mathbb{Z}$. Since φ is defined by the semiample divisor $4\xi - r_X\pi^*H_X$ by Theorem 4.3.2, we have $\varphi^*H_Y = a\xi - \pi^*H_X$. Let $F \simeq \mathbb{Q}^3$ be a general fiber of φ.

Then, since $F \equiv_{\mathrm{num}} \varphi^*H_Y^4 = (a\xi - \pi^*H_X)^4$ and $r_X\pi^*H_X|_F = 4\xi|_F = 4D_F$, we have

$$(a\xi - \pi^*H_X)^4(r_X\pi^*H_X)^3 = 2^7.$$

This is equivalent to

$$\frac{4^3}{a^3}\left(a^4\left(c_1^2 - c_2\right)H_X^3 - 4a^3c_1H_X^4 + 6a^2H_X^5\right) = 2^7,$$

where $c_i = c_i(\mathscr{E})$.

On the other hand, since $\dim Y = 4$, we have $(a\xi - \pi^*H_X)^5 = 0$. This implies:

$$a^5(c_1^3 - 2c_1c_2 + c_3) - 5a^4(c_1^2 - c_2)H_X + 10a^3c_1H_X^2 - 10a^2H_X^3 = 0,$$
$$a^5(-c_1^2c_2 + c_1c_3 + c_2^2) - 5a^4(-c_1c_2 + c_3)H_X - 10a^3c_2H_X^2 + 5aH_X^4 = 0,$$
$$a^5(c_1^2c_3 - c_2c_3) - 5a^4c_1c_3H_X + 10a^3c_3H_X^2 - H_X^5 = 0.$$

Since $c_1 = r_XH_X = \dfrac{4}{a}H_X$, the above four equations are equivalent to the following:

$$\begin{aligned} 6H_X^5 - a^2c_2H_X^3 &= 2a, \\ 14a^2H_X^5 - 3a^4c_2H_X^3 + a^5c_3H_X^2 &= 0, \\ 5aH_X^5 - 6a^3c_2H_X^3 - a^4c_3H_X^2 + a^5c_2^2H_X &= 0, \\ H_X^5 - 6a^3c_3H_X^2 + a^5c_2c_3 &= 0. \end{aligned}$$

By solving these equations for H_X^5, $c_2H_X^3$, $c_3H_X^2$ and c_2c_3, we have:

$$\begin{aligned} H_X^5 &= \frac{18a + a^4c_2^2H_X}{35}, \\ c_2H_X^3 &= \frac{38 + 6a^3c_2^2H_X}{35a}, \\ c_3H_X^2 &= \frac{-138 + 4a^3c_2^2H_X}{35a^2}, \\ c_2c_3 &= \frac{-846 + 23a^3c_2^2H_X}{35a^4}. \end{aligned}$$

If $a = 4$, then $c_3H_X^2 = \dfrac{-69+16c_2^2H_X}{70}$, which cannot be an integer. This gives a contradiction. Also if $a = 2$, then the equation $c_3H_X^2 = \dfrac{-69+128c_2^2H_X}{280}$ gives a contradiction again. Hence we have $a = 1$ and the assertion follows. $\square$

6.4. Proof of Theorem 4.6.1. In any case, $(n-1)\xi_{\mathscr{E}} + \pi^*K_X = (n-1)(\xi_{\mathscr{E}} - \pi^*H_X)$. Therefore $\xi_{\mathscr{E}} - \pi^*H_X = \varphi^*H_Y$ for an ample Cartier divisor H_Y on Y.

Let us consider the following diagram unless $n = 5$ and $\dim Y = 4$:

$$\begin{tikzcd} \mathbb{P}(\mathscr{E}_1) \arrow[r, "\varphi_1"'] \arrow[d, "\pi_1"'] & Y_1 \\ X, & \end{tikzcd}$$

where φ_1 is obtained by taking the Stein factorization $\mathbb{P}(\mathscr{E}) \xrightarrow{\varphi_1} Y_1 \to Y$ of the composite $\mathbb{P}(\mathscr{E}_1) \to \mathbb{P}(\mathscr{E}) \xrightarrow{\varphi} Y$. Thus φ_1 is defined by the semiample divisor $(n-1)\xi_{\mathscr{E}_1} + \pi_1^*K_X$.

Since $r_X = n-1$, we have $(n-1)\xi_{\mathscr{E}_1} + \pi_1^*K_X = (n-1)(\xi_{\mathscr{E}_1} - \pi_1^*H_X)$. Thus $\xi_{\mathscr{E}_1} - \pi_1^*H_X = \varphi_1^*H_{Y_1}$ for an ample cartier divisor H_{Y_1} on Y_1.

LEMMA 4.6.6. *Let the notation be as above. Then φ_1 is defined by the semiample divisor $K_{\mathbb{P}(\mathscr{E}_1)} + (n-2)\xi_{\mathscr{E}_1}$, $\dim Y_1 = 4$ and*

(1) *If Proposition 4.6.2 (1) occurs, then general φ_1-fibers are isomorphic to $\mathbb{P}^{n-3}$.*
(2) *If Proposition 4.6.2 (2) occurs and $n = 6$, then and general φ_1-fibers are isomorphic to $\mathbb{Q}^4$.*

PROOF. The assertion on supporting divisor is only a computation. If $\dim Y_1$ is as stated, then the statement about fibers follows from adjunction and Kobayashi-Ochiai theorem.

Thus it is enough to see that $\dim Y_1 = 4$. By Proposition 4.6.4, $\mathscr{E}$ admits a decomposition $\mathscr{E} \simeq \mathscr{E}' \oplus \mathcal{O}(1)$. Then $\mathbb{P}(\mathscr{E}')$ is a divisor on $\mathbb{P}(\mathscr{E})$, which is linearly equivalent to $\xi_{\mathscr{E}} - \pi^*H_X = \varphi^*H_Y$. Thus $\dim \varphi(\mathbb{P}(\mathscr{E}')) = \dim Y - 1$. If Proposition 4.6.2 (1) occurs and $n = 6$, then, by repeating the procedure, we have the assertion on $\dim Y_1$. $\square$

Also we obtain the following diagram as in (4.6.1.1) for a general φ_1-fiber F:

$$(4.6.6.1)\qquad \begin{tikzcd} \widetilde{F} \arrow[r, "\iota"] & \mathbb{P}(\mathscr{E}_1|_F) \arrow[r] \arrow[d, "\pi_{1,F}"'] \arrow[rr, bend left, "\theta_{1,F}"] & \mathbb{P}(\mathscr{E}_1) \arrow[r, "\varphi_1"'] \arrow[d, "\pi_1"'] & Y_1 \\ & F \arrow[r, "\pi_1|_F"] & X. & \end{tikzcd}$$

Let $\mathbb{P}(\mathscr{E}_1|_F) \xrightarrow{\varphi_{1,F}} Y_1' \to Y_1$ be the Stein factorization of $\theta_{1,F}$.

Note that general φ_1-fiber F maps isomorphically on to φ-fiber. Thus:

(1) $\mathscr{E}_1|_F \simeq \mathcal{O}(2,1)$ if Proposition 4.6.2 (1) occurs.

(2) $\mathscr{E}_1|_F \simeq \mathscr{S}^*_{\mathbb{Q}}(1) \oplus \mathcal{O}(1)$ if Proposition 4.6.2 (2) occurs and $n=6$.

Hence Y'_1 is a projective space.

Proof of Theorem 4.6.1.

Case. $n=6$ and $\dim Y = 6$.

Then $\mathscr{E}$ is isomorphic to $\mathscr{E}_1 \oplus \mathcal{O}(1^2)$, $\dim Y_1 = 4$ and X is a del Pezzo manifold by Proposition 4.6.4 (1).

φ_1 is equidimensional. Otherwise there exists a jumping fiber of φ_1. Let F' be a component of the jumping fiber with $\dim F' \geq 4$. If $\dim F' = 4$, then F' is isomorphic to $\mathbb{P}^4$ and $\mathcal{O}_{\mathbb{P}(\mathscr{E}_1)}(1)|_{F'} \simeq \mathcal{O}_{\mathbb{P}^4}(1)$ by [**And95**, Theorem 2.1]. Then by a similar argument to the Step 2 of the proof of Proposition 4.6.4, we have $\mathscr{E}_1|_{F'} \simeq \mathcal{O}(2) \oplus \mathcal{O}(1)$, which yields a contradiction to $\dim Y_1 = 4$. Hence we have $\dim F' \geq 5$. Let F be a general fiber. Then $\pi_1(F) \cap \pi_1(F') \neq \emptyset$ since $\rho_X = 1$. Then $\pi_1^{-1}(\pi_1(F)) \cap F' \neq \emptyset$, hence $\dim \pi_1^{-1}(\pi_1(F)) \cap F' \geq 2$ by the Serre inequality. Thus we have $F \cap F' \neq \emptyset$ since $\varphi_{1,F}$ contracts only $\widetilde{F}$. This gives a contradiction and hence φ_1 is equidimensional.

By [**Fuj87**, Lemma 2.12], φ_1 is a projective bundle and Y_1 is smooth. Since $Y'_1 \simeq \mathbb{P}^4$, we have $Y_1 \simeq \mathbb{P}^4$ by [**Laz84**, Theorem 4.1].

Now $\varphi^* H_{Y_1} = \xi_{\mathscr{E}_1} - \pi_1^* H_X$, where H_{Y_1} is the ample generator of $\mathrm{Pic}(Y_1)$. Hence we have a surjection between vector bundles:

$$\mathcal{O}_X^5 \to \mathscr{E}_1(-1).$$

This gives a finite surjective morphism $j\colon X \to \mathrm{Gr}(2,5)$ with $j^*\mathscr{S}^*_{\mathrm{Gr}} = \mathscr{E}_1(-1)$ and hence $j^*\mathcal{O}(1) = \mathcal{O}_X(1)$. Thus j is an isomorphism.

Case. $n=5$ and $\dim Y = 5$.

Then $\mathscr{E}$ is isomorphic to $\mathscr{E}_1 \oplus \mathcal{O}(1)$, $\dim Y_1 = 4$ and X is a del Pezzo 5-fold of $\rho_X = 1$.

Let $F \simeq \mathbb{P}^2$ be a general φ_1-fiber. Then $\pi_1(F)$ does not meet $\mathrm{Bs}\,|H_X|$ since $\dim \mathrm{Bs}\,|H_X| \leq 0$ by [**Fuj82b**]. Note that $H_X|_F$ is linearly equivalent to the class of a line. Hence $\pi_1|_F$ is an isomorphism onto its image. Since F is a general fiber, $T_{\mathbb{P}(\mathscr{E}_1)}|_F$ is nef and hence $T_X|_F$ is also nef with the following diagram:

$$0 \to T_F \to T_X|_F \to N_{\pi_1(F)/X} \to 0.$$

This implies that the normal bundle $N_{\pi_1(F)/X}$ is a nef vector bundle of rank three with $c_1(N_{\pi_1(F)/X}) = 1$. Hence the normal bundle $N_{\pi_1(F)/X}$ is isomorphic to $\mathcal{O}(1,0^2)$ or $T_{\mathbb{P}^2}(-1) \oplus \mathcal{O}$ by [**SW90c**]. Then, by the above exact sequence, the Chern classes $(c_1(T_X|_F), c_2(T_X|_F))$ are $(4,6)$ or $(4,7)$. By using the classification of del Pezzo manifolds, we see that this is possible only if X is a linear section of $\mathrm{Gr}(2,5)$ (cf. [**NO11**]).

Set $\mathscr{F} := \mathscr{E}_1(-1)$. Then on $\mathbb{P}(\mathscr{F})$ we have $\xi^6_{\mathscr{F}} = \xi^5_{\mathscr{F}} = 0$. This is equivalent to $c_1(\mathscr{F})^4 - 3c_1(\mathscr{F})^2 c_2(\mathscr{F})^3 + c_2(\mathscr{F})^2 = 0$ and $c_1(\mathscr{F})^3 c_2(\mathscr{F}) - 2c_1(\mathscr{F})c_2(\mathscr{F})^2 = 0$. Set $c_2(\mathscr{F}) := a\sigma_{2,0} + b\sigma_{1,1}$, where $\sigma_{2,0}$ and $\sigma_{1,1}$ are restrictions of Schubert cycles on $\mathrm{Gr}(2,5)$. Then, since $c_1(\mathscr{F}) = H_X$, we have

$$5 - 9a - 6b + 2a^2 + 2ab + b^2 = 0, \tag{4.6.6.2}$$

$$3a + 2b - 4a^2 - 4ab - 2b^2 = 0. \tag{4.6.6.3}$$

By solving these equations we have $(a,b) = (0,1)$. In this case, the following holds:

$$(c_1(\mathscr{F}), c_2(\mathscr{F})) = (c_1(\mathscr{S}_X^*), c_2(\mathscr{S}_X^*)),$$

where $\mathscr{S}_X^*$ is the restriction of the universal subbundle $\mathscr{S}_{\mathrm{Gr}}^*$ on $\mathrm{Gr}(2,5)$. By the Kodaira vanishing theorem on $\mathbb{P}(\mathscr{F})$, we know that $\chi(\mathscr{F}) = h^0(\mathscr{F})$ and this is equal to $h^0(\mathscr{S}_X^*) = 5$ by the Riemann-Roch theorem. Now $h^0(H_{Y_1}) = h^0(\mathscr{F}) = 5$ and $H_{Y_1}^4 = \xi_{\mathscr{F}}^4.(\pi_1^* H_X)^2 = 1$. Hence the delta-genus $\Delta(Y_1, H_{Y_1})$ is zero and $\deg H_{Y_1} = 1$. This implies $Y_1 \simeq \mathbb{P}^4$ by [**KO73**, **Fuj75**, **Fuj82b**].

Therefore, similarly to the above case, we have a finite surjective morphism $j\colon X \to \mathrm{Gr}(2,5)$ with $j^*\mathscr{S}_{\mathrm{Gr}}^* = \mathscr{F}$ and hence $j^*\mathcal{O}(1) = \mathcal{O}_X(1)$. Thus j is an isomorphism onto its image.

Case. $n = 6$ and $\dim Y = 5$.

Then $\mathscr{E}$ is isomorphic to $\mathscr{E}_1 \oplus \mathcal{O}(1)$ and $\dim Y_1 = 4$.

In this case, φ_1 is equidimensional and hence a quadric fibration by [**ABW93**, Theorem B]. This can be seen as follows: Assume that there exists a jumping fiber of φ_1. Let F' be a component of the jumping fiber with $\dim F' \geq 5$ and F a general fiber. Then $\pi_1(F) \cap \pi_1(F') \neq \emptyset$. Hence $\pi_1^{-1}(\pi_1(F)) \cap F' \neq \emptyset$ of dimension ≥ 3 by the Serre inequality. Since the contraction defined by $\theta_{1,F}$ is a scroll with only one jumping fiber $\widetilde{F}$, we have $F \cap F' \neq \emptyset$, which gives a contradiction.

Thus φ_1 is equidimensional and hence Y_1 is smooth by [**ABW93**, Theorem B]. Since $Y_1' \simeq \mathbb{P}^4$, we have $Y_1 \simeq \mathbb{P}^4$ by [**Laz84**, Theorem 4.1].

Similarly to the above cases, this gives a finite surjective morphism $j\colon X \to \mathrm{Gr}(2,5)$ with $j^*\mathscr{Q}_{\mathrm{Gr}} = \mathscr{E}_1(-1)$ and $j^*\mathcal{O}(1) = \mathcal{O}_X(1)$, and hence j is an isomorphism.

Case. $n = 5$ and $\dim Y = 4$.

In this case, φ is equidimensional by a similar argument as above, and hence φ is a quadric fibration and Y is smooth by [**ABW93**, Theorem B].

Since the image of the contraction φ_F is $\mathbb{P}^4$, we have $Y \simeq \mathbb{P}^4$ by [**Laz84**].

Now $r_X = 4$ by Proposition 4.6.5 and hence $\xi - \pi^* H_X = \varphi^* H_Y$. Therefore we have a surjection $\mathcal{O}_X^5 \to \mathscr{E}(-1)$. This gives a finite surjective morphism $j\colon X \to \mathrm{Gr}(2,5)$ with $j^*\mathscr{Q}_{\mathrm{Gr}} = \mathscr{E}(-1)$ and $j^*\mathcal{O}(1) = \mathcal{O}_X(1)$. Therefore j is an isomorphism onto its image. This completes the proof. $\square$

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