博士論文

On effects of curvatures of curves, surfaces and graphs

(曲線、曲面およびグラフの曲率の効果について)

三浦 達哉
Tatsuya Miura

The University of Tokyo,
Graduate School of Mathematical Sciences

Abstract

In this dissertation we study several problems on curves, surfaces, and graphs. Broadly speaking, our problems are included in the fields of analysis and geometry. More precisely, our studies are related to nonlinear analysis, ordinary and partial differential equations, calculus of variations, differential geometry, and geometric topology. The treated problems are motivated by both pure mathematics and physics as materials science. Mathematically, a point in common between the problems is that curvatures play crucial roles. We consistently study the properties, as shapes or singularities, of solutions to the problems.

In Chapter 1 we address quite classical boundary value problems on elastic curves. Namely, we consider the minimizing problems for the following two energies; the total squared curvature energy defined for planar curves of clamped endpoints and fixed length, and the modified total squared curvature energy defined for planar curves of clamped endpoints. Our main result indicates a new theoretical connection between elastic curve problems and phase transition problems. By using this observation, we reveal the precise shapes of global minimizers in a "straightening" limit, and moreover obtain a uniqueness result in a certain class of constraint parameters.

In Chapter 2 and Chapter 3 we deal with a free boundary problem on elastic curves. The problem is formulated as the minimizing problem for the modified total squared curvature energy with a contact potential term. The energy is defined for planar periodic curves constrained above a graph substrate. Mathematically, this is a kind of obstacle problem, and accordingly free boundary problem. This problem is motivated by materials science, more precisely, adhesion problems on thin objects as filaments or membranes. Our main concern is the graph representations of global minimizers.

In Chapter 2 we exhibit some sufficient conditions of parameters which ensure that any global minimizer is represented by a graph, and examples of parameters which guarantee that any global minimizer is overhanging, i.e., not represented by a graph. The proofs rely on energy estimates and geometric classifications of the possible states of admissible curves.

In Chapter 3 we study the case that the bending rigidity is sufficiently small, which is not included in the study in Chapter 2. We prove that this case also ensures the graph representations of minimizers. The main tool is a singular limit method as Γ-convergence. In this part we essentially use the results obtained in Chapter 1.

In Chapter 4 we study a completely different problem. Our main result characterizes the cut locus of a given open set with C^1-boundary in the Euclidean space. For a given open set, the cut locus is defined as the closure of the set of points where the distance function from the boundary of the open set is not differentiable. This

singular set appears in various contexts as differential geometry or partial differential equations. We prove that the cut locus is characterized in terms of a generalized radius of curvature of the boundary and its lower semicontinuous envelope. This result is a generalization of a classical characterization for an open set with C^2-boundary. Our proof is based on more geometric arguments than classical proofs; namely, we use comparisons of functions and the homotopy theory as mapping degree.

In Chapter 5, another different problem is treated. We study the smoothing effect of the mean curvature flow equation, which is an evolution equation defined for hypersurfaces. Our main result provides an example of a mean-convex mean curvature flow in the sense of level-set flow which is not smoothed out instantly, i.e., developing infinitely many singular epochs near the initial time. The constructed initial surface is an axisymmetric topological sphere, smooth except a single singular point, and mean-convex in the sense of White. Our example is useful to contrast some previous known results on the smoothing effect of the mean curvature flow.

All the chapters are independent of each other in principle; in particular, they include their own introduction and reference sections. As a general rule, the notations, definitions, and citations are valid only in each of the chapters. In particular, the reference numbers make sense only within each chapter since some papers are cited in plural chapters with different reference numbers. Some results are cross-referenced, but in that case we explicitly describe the chapter number.

Finally, we mention that this dissertation is basically a collection of the author's previous publications made adjustments. Chapter 1 and Chapter 2 correspond to [1] and [2], respectively. Chapter 3 is based on [3]; however, compared with [3], in Chapter 3 we give major extensions and more advanced conclusions. Chapter 4 and Chapter 5 correspond to [4] and [5], respectively.

[1] T. Miura, *Elastic curves and phase transitions*, to be submitted.

[2] T. Miura, *Overhanging of membranes and filaments adhering to periodic graph substrates*, to appear in Physica D: Nonlinear Phenomena.

[3] T. Miura, *Singular perturbation by bending for an adhesive obstacle problem*, Calc. Var. Partial Differential Equations (2016) 55:19, in press.

[4] T. Miura, *A characterization of cut locus for C^1 hypersurfaces*, NoDEA Nonlinear Differential Equations Appl. (2016) 23:60, in press.

[5] T. Miura, *An example of a mean-convex mean curvature flow developing infinitely many singular epochs*, J. Geom. Anal., 26 (2016), no. 4, 3019–3026.

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Contents

Chapter 1

Boundary value problems for elastic curves

In this chapter we indicate a new theoretical connection between two classical theories; elastic curves and phase transitions. Using this observation, we reveal the asymptotic shape of planar curves of clamped endpoints minimizing the modified total squared curvature as tension tends to infinity. We prove that any sequence of minimizing curves converges to the borderline elastica near the endpoints in a rescaled sense, and moreover becomes almost straight elsewhere. The borderline elastica is well-known as a typical solution to the equation of elastica, but our analogy indicates that it can also be regarded as a transition layer. Applying this result, we also prove that a similar convergence holds in a straightening process for a classical elastic rod problem, which is posed by D. Bernoulli and L. Euler in the 18th century. As a byproduct, the uniqueness of global minimizers is proved for a certain region of constraint parameters.

Keywords: Euler's elastica; Bending energy; Boundary value problem; Phase transition; Singular perturbation; Asymptotic expansion.

1.1 Introduction

This chapter addresses two problems on elastic curves. The first problem is the minimizing problem of the total squared curvature, so-called *bending energy*,

$$\mathcal{B}[\gamma] = \int_\gamma \kappa^2 ds, \tag{1.1.1}$$

where γ is a planar curve of fixed length and clamped endpoints, i.e., the positions and the tangential directions at the endpoints are fixed as in Figure 1.1. Here s denotes the arc length parameter and κ denotes the (signed) curvature. The second problem is the minimizing problem for the modified total squared curvature,

$$\mathcal{E}_\varepsilon[\gamma] = \varepsilon^2 \int_\gamma \kappa^2 ds + \int_\gamma ds, \tag{1.1.2}$$

where γ is a planar curve of clamped endpoints (and variable length). In this chapter, we call the first problem *inextensible problem* and the second one *extensible problem*. These problems are one-dimensional but higher order and strongly nonlinear, and hence there are a number of unclear points even today. We mainly study the profiles of global minimizers in both the problems.

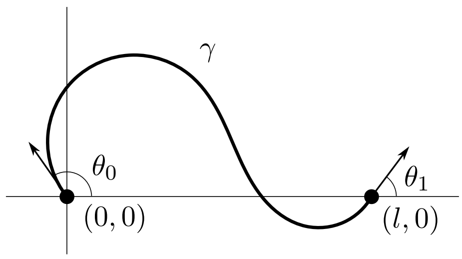

FIGURE 1.1: Clamped curve.

1.1.1 Euler's elastica: the origin

The inextensible problem is motivated to determine the shapes of inextensible and flexible elastic rods of clamped endpoints. This problem has a quite long history; it is originated at least 270 years ago. Historically, Daniel Bernoulli is the first to provide the completely same formulation as our problem (although a basic concept has been posed by Jacob Bernoulli in earlier times). The formulation appears in his letter to Euler in 1742. In response to it, in 1744, Euler provided the first study on the inextensible problem [27]. He derived ordinary differential equations for solution curves (i.e., critical points) and moreover classified the types of solutions qualitatively. The solution curves are called *Euler's elasticae* nowadays. For more details of the history, see e.g. [43, 49, 55, 66, 70, 74].

Although Euler derived the equations in terms of Cartesian coordinates at that time, it would be more convenient to adopt a modern and simpler intrinsic form expressed in terms of the curvature. By the classical Lagrange multiplier method, for any critical point γ in the inextensible problem, there is a multiplier $\lambda \in \mathbb{R}$ such that the curve γ is also a critical point of the energy

$$\int_\gamma \kappa^2 ds + \lambda \int_\gamma ds$$

among curves satisfying the same clamped boundary condition. Calculating the first variation, we find that the signed curvature κ of γ satisfies

$$2\partial_s^2\kappa + \kappa^3 - \lambda\kappa = 0.$$

In this paper we call it *elastica equation*. It is known that the elastica equation is uniquely solved for any given multiplier λ and initial values $\kappa(0)$ and $\kappa'(0)$. Moreover, all solutions are expressed in terms of the Jacobi elliptic functions. Figure 1.2 exhibits a classification of basic patterns of elasticae. The cases (i)–(iii) (resp. (iii)–(v)) correspond to that the curvatures are expressed in terms of dn functions (resp. cn functions). The cases (i) and (v) are a line and a circle. The case (ii) is called *non-inflectional* or *orbitlike* elastica. The case (iv) is called *inflectional* or *wavelike* elastica; this class includes a typical closed curve, so-called figure-of-eight elastica. The case (iii) is called *critical* or *borderline* elastica, and the only case having no periodicity. The borderline elastica plays a crucial role in our results. For more details on basic facts of elasticae, see e.g. [4, 11, 49, 72].

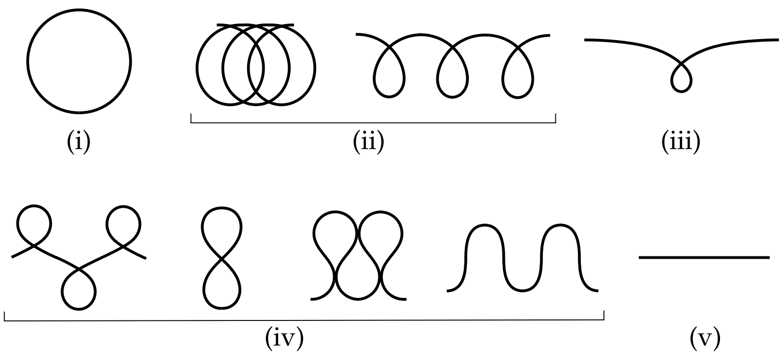

FIGURE 1.2: Basic patterns of elasticae.

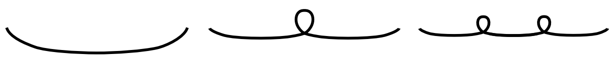

FIGURE 1.3: Loops.

1.1.2 Shape of clamped elastica: problems

The elastica problem is already "solved" as above, at least, at the level of equation. Notwithstanding, it is still difficult to perceive the exact shapes of solution curves for arbitrary given constraints. One reason is that our clamped boundary condition does not fix any of the parameters λ, $\kappa(0)$, and $\kappa'(0)$. The fact is that our problem admits infinitely many local minimizers (stable critical points) as e.g. in Figure 1.3; this can be easily proved by a winding number argument (see Appendix 1.A). Although there is a general formula describing the relations between our constraints and solution curves (cf. [46, 47]), the formula is given as involved simultaneous trancendental equations (including elliptic functions and elliptic integrals), and not necessarily direct evidence for a clear understanding of the shapes of solution curves in general.

For example, it is nontrivial which choice of constraints admits an embedded solution, i.e., a solution curve without self-intersections. The nontriviality is clear since the presence or absence of self-intersections is not determined by constraints. In other words, for fixed constraints, there may be local minimizers with and without self-intersections as in Figure 1.3. Hence, it is necessary to take the energy minimality into account in the self-intersection problem.

Another interesting question is to ask the number of infection points, i.e., points where the sign of the curvature changes. As a pioneering work on inflection points, in 1906, Born proved that any solution curve without inflection point is stable [9]. Recently, a series of papers [66, 67, 69] revisits the elastica problem in view of optimal control. In particular, Sachkov [67] states that any stable solution has at most two inflection points. The upper bound two is optimal since a well-known buckling example as in Figure 1.4 may be a global minimizer in a certain case. The figure-of-eight is also an example of local minimizer with two inflection points. As will be discussed below, it is quite nontrivial to exactly know the number of inflection points even if we restrict ourselves to considering only global minimizers.

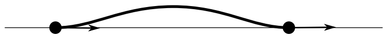

FIGURE 1.4: Buckling.

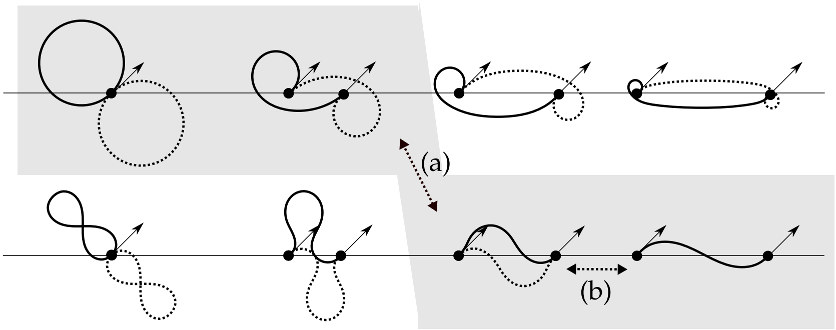

FIGURE 1.5: Formal observation for a straightening process.

In addition, it is worth mentioning that the uniqueness of global minimizers is not expected in general. It is a long-standing open problem to determine the region of constraint parameters which ensures the uniqueness (see e.g. [69]).

In the rest of this subsection, to clarify the above problems, we formally observe an example case of straightening by seeing Figure 1.5. The case of closed curves, which corresponds to the left end of Figure 1.5, is well-understood. The papers [5, 41, 68] show the following facts; any closed critical point is an n-wrapped circle or an n-wrapped figure-of-eight, any local minimizer is an n-wrapped circle or the 1-wrapped figure-of-eight, and the only global minimizer is the 1-wrapped circle. However, even just changing the distance of the endpoints from the closed case, we would observe complicated deformation of minimizers. Each row in Figure 1.5 describes "continuous" deformation from a closed elastica. Since the two boundary angles are given to be same in this figure, the dotted curves have the same energies as the corresponding bold curves, respectively. The gray region in Figure 1.5 indicates expected global minimizers. The point (a) indicates a topological change. The point (b) indicates a symmetry breaking, and accordingly a change of the number of global minimizers. The number of inflection points would also change at the typical points (a) and (b). We emphasize again that Figure 1.5 is just a formal observation and incomplete. For example, rotated figure-of-eights and n-wrapped circles are also local minimizers in the left end. To the author's knowledge, there is no general result to determine typical points as (a) or (b) rigorously (cf. [3, 29, 69]).

1.1.3 Straightened elastica: main results

The purpose of this chapter is to obtain precise conclusions on the shapes of global minimizers for generic boundary conditions. As discussed above, a general conclusion is not expected for fully general constraints. In this chapter we focus on a straightening problem, i.e., the limit that the distance of the endpoints tends to the length of a curve while the tangential angles at the endpoints are fixed.

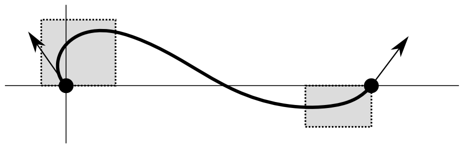

FIGURE 1.6: Straightened elastica.

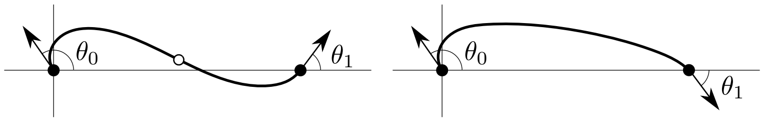

FIGURE 1.7: Straightened elasticae with and without inflection point.

Even if we focus on the straightening problem, the inextensible problem is however not easy to tackle directly. The main reason is the number of constraints. To circumvent this difficulty, we first consider the singular limit $\varepsilon \to 0$ for the extensible problem. Considering this limit is physically natural. In fact, the constant ε^2 is interpreted as bending rigidity divided by tension, and we expect that straightened elastic curves have very high tension. The extensible problem is relatively tractable in the sense that the multiplier ε^2 is a priori fixed. By using our results about the extensible problem, we also obtain similar results for the inextensible problem in the straightening limit.

Our main result (Theorem 1.2.3) states that, in the extensible problem, any sequence of global minimizers is straightened as $\varepsilon \to 0$ as in Figure 1.6 for an arbitrary given boundary condition. More precisely, for small ε, any minimizer bends at the scale of ε near the endpoints and is almost straight elsewhere, i.e., the tangent vectors are almost rightward. In addition, if we rescale a sequence of minimizers at an endpoint, then the rescaled curves locally converge to a part of the borderline elastica. The proof of these results is of most importance in this chapter; we use a theoretical analogy to the phase transition theory, as explained precisely in the next subsection. Our result also implies other more qualitative properties (Theorem 1.2.10). For instance, as a direct corollary, we find that any minimizer has no self-intersection for any small ε. In addition, combining our result with expressions of the curvatures by elliptic functions, we determine the exact number of inflection points for generic boundary angles providing that ε is small. The number is zero or one, and depends only on the signs of boundary angles as in Figure 1.7. Furthermore, in the case of no inflection point, we prove the uniqueness of global minimizers (Theorem 1.2.11). Our proof uses a change of variables which rephrases the minimizing problem in terms of the radius of curvatures parametrized by the tangential angles. In other words, we use a coordinate induced by the Gauss map, which is often used for the analysis of convex curves. The change of variables yields a "convexification" of the minimizing problem, which directly implies the uniqueness. Such a convexification has been already used in Born's stability analysis [9]. Our main contribution is an a priori guaranty of the convexity of global minimizers and determining the total variations of the tangential angles.

We then prove that similar results are also valid in the straightening limit for the inextensible problem. Generally speaking, by the Lagrange multiplier method, it is clear that there is some kind of relation between the extensible and inextensible problems at the level of critical points. In this paper, we investigate the precise relation of them at the level of global minimizers. We prove that the inextensible problem in the straightening limit is reduced to the extensible problem in the limit $\varepsilon \to 0$. At this time, our result is proved only in a subsequential sense in the general case (Theorem 1.2.12). However, we succeed to prove a full convergence result in a "convex" case (Theorem 1.2.14). For the fully general case there would remain an essential difficulty, which crucially relates to the uniqueness problem.

It would be noteworthy that our results deal with generic boundary angles, and do not impose any restrictive assumptions for curves as symmetry or the graph representation. Another remarkable novelty is to conclude the uniqueness at a certain level of generality.

1.1.4 Phase transition: a new perspective

As mentioned, the main feature of our result is to indicate a somewhat direct theoretical connection between the (extensible) elastic problem and the phase transition theory.

We briefly recall the studies on phase transition energies. The minimizing problem of a potential energy perturbed by a gradient term, as

$$E_\epsilon[u] = \epsilon^2 \int_\Omega |\nabla u|^2 + \int_\Omega W(u),$$

has been widely studied, in particular, in view of the van der Waals-Cahn-Hilliard theory of phase transitions [15, 75]. Here $\Omega \subset \mathbb{R}^n$ is a certain open set. The potential function W is often taken as the double-well potential $W(u) = (1-u^2)^2$, and the volume constraint $\int_\Omega u = M$ is often imposed.

In the phase transition problem, for small ϵ, the values of a minimizer should be almost separated into the phases 1 and -1 to minimize the potential energy. Moreover, if a minimizer needs to have a transition between the two phases due to the volume constraint, then the area of "interface" is expected to be minimized due to the effect of perturbation. These expectations are proved by Carr-Gurtin-Slemrod [14] in a one-dimensional case, and by Modica [58] and Sternberg [73] in higher-dimensional cases. The higher-dimensional results [58, 73] are described in terms of Γ-convergence, which is introduced by De Giorgi in 1970's (see e.g. [10, 20]). The Γ-convergence result particularly implies a first order expansion of the minimum value of E_ϵ as $\epsilon \to 0$. Moreover, it also implies that, up to a subsequence, any sequence of minimizers u_ϵ converges in L^1 to a characteristic function $u_0 \in BV(\Omega; \{-1, 1\})$ of which total variation is minimized among functions $u \in BV(\Omega; \{-1, 1\})$ with $\int_\Omega u = M$. Some stronger convergence results are also known, even for local minimizers [13] or critical points [33] with certain boundedness; roughly speaking, a locally uniform convergence holds except interfaces. Furthermore, at least formally, one expects that the transition part of a minimizer is close to a rescaled "transition layer" solution. In fact, in the particular case that $\Omega = (-1, 1)$ and $M = 0$, it is easy to prove that the rescaled minimizer $\hat{u}_\epsilon(x) = u_\epsilon(\epsilon x)$ is nothing but a transition layer, i.e., a solution to $|u'|^2 = W(u)$, as in Figure 1.8.

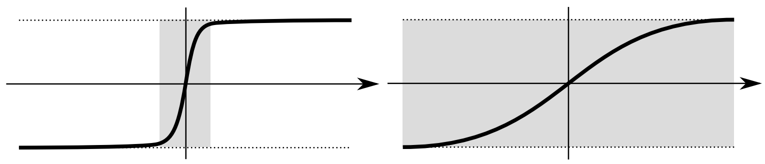

FIGURE 1.8: A minimizer u_ε and a transition layer.

Finally, it should be mentioned that a basic strategy for the above higher dimensional results [58, 73] has been provided in the earlier paper by Modica and Mortola [59]. The paper deals with an unconstrained problem for the periodic potential $W(u) = \sin^2(\pi u)$, which is more directly relative to our problem.

We now go back to our elastic curve problem. For a curve γ as in Figure 1.1, we denote its length by L and represent the modified total squared curvature in terms of its tangential angle function $\vartheta : [0, L] \to \mathbb{R}$ (i.e., $\partial_s\gamma = (\cos\vartheta, \sin\vartheta)$) as

$$\mathcal{E}_\varepsilon[\gamma] = \varepsilon^2 \int_0^L |\partial_s\vartheta|^2 ds + \int_0^L ds = \varepsilon^2 \int_0^L |\partial_s\vartheta|^2 ds + \int_0^L (1 - \cos\vartheta) ds + l,$$

where l is the fixed distance of the endpoints. The last equality follows since $\int_0^L \cos\vartheta ds$ is nothing but the difference of the x-coordinates at the endpoints. The above expression indicates that $\mathcal{E}_\varepsilon$ can be regarded as a one-dimensional phase transition energy with the periodic potential $W(\theta) = 1 - \cos\theta$ $(= 2\sin^2(\theta/2))$. All the stable phases $\theta \in 2\pi\mathbb{Z}$ correspond to the rightward tangent vector. Of course, there are some differences from the original phase transition problems; the integration interval $[0, L]$ is not fixed, and the number of constraints (due to the clamped boundary condition) are greater than the above volume constraint. Nevertheless, for our extensible problem, we obtain a first order expansion of the energy minimum, which is essentially similar to the phase transition problem (Lemma 1.3.1). In this part we rely on the assumption that the lengths of curves are unconstrained. By using the expansion, we reveal the precise convergence of minimizers as $\varepsilon \to 0$. In particular, near the endpoints, the rescaled tangential angles converge to a part of transition layer, i.e., a solution to $|\partial_s\vartheta|^2 = 1 - \cos\vartheta$ (Figure 1.9). The curve corresponding to a transition layer has one loop, and is nothing but the borderline elastica (Figure 1.2 (iii)). Thus we give a new interpretation of this typical elastica in view of the phase transition theory.

We note that our proof also essentially relies on the concept of Γ-convergence although this chapter includes no explicit statement. One may obtain a more general Γ-convergence result such that the function space of the limits of the tangential angles contains general $2\pi\mathbb{Z}$-valued BV-functions, but we do not state it in this chapter to avoid digressing from our main subject.

1.1.5 Related problems and remarks

We finally mention some related problems and several remarks.

Elastic curve problems are classical but still ongoing. The minimization of total squared curvature is studied not only in the plane but also in other manifolds or

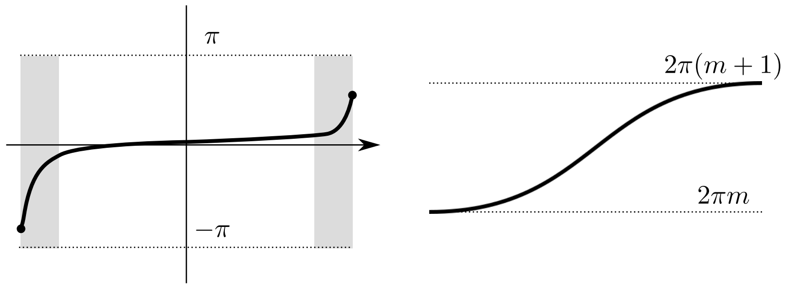

FIGURE 1.9: Tangential angle of a minimizer and a transition layer.

higher-dimensional spaces (e.g. [37, 39, 40, 41, 72]). In particular, there remain many open problems on elastic knots (see e.g. [31, 32]).

Boundary value problems are well-studied for "free" elasticae (e.g. [22, 23, 44, 48, 51]). Free elasticae are defined as critical points of the total squared curvature without length constraint unlike our problem. In this case we encounter another difficulty that there is no global minimizer in general. On the other hand, the corresponding equation is reduced, has no multiplier, and possesses a nice scale invariance; for a solution curve, any dilated curve also satisfies the same equation.

Free elasticae are also referred as one-dimensional Willmore surfaces. Willmore surfaces are critical points of the Willmore energy, i.e., the total squared mean curvature. For recent developments on this higher dimensional problems, we pick up some recent papers [7, 8, 35, 38, 54] (see also references therein). Boundary value problems are also studied for Willmore surfaces (e.g. [16, 63, 71]). In particular, Willmore surfaces of revolution are studied more precisely (e.g. [6, 18, 19, 26, 52]). This case is more related to our problems since the corresponding equation in terms of the hyperbolic curvature is reduced to our elastica equation.

There are various other points of view even in the plane. For closed curves, Gage's classical result of isoperimetric inequality type [30] is recently generalized in [12] and [28] independently. For open curves, a well-studied topic is a bifurcation problem concerning buckling (see e.g. [1, 2, 50]). The stability of post-buckling elasticae is even now a central issue (e.g. [29, 34, 53, 67, 70]). Obstacle type problems are also studied in various settings; confined closed curves [25, 21], graph curves above obstacles [17], and adhesion problems [36, 56, 57]. In particular, the author studied a singular limit for an adhesion problem in the paper [56], from which some ideas in the present paper come. However, at that time, the author just derived a result of Γ-convergence for graph curves, and in fact was not aware of the direct connection to phase transitions.

Concerning the terminology "phase transition", one may suppose that our viewpoint is not new in elastic problems since the "phase-field method" is often used even for elastic problems (see e.g. [24, 25, 65]). The phase-field method is also crucially based on the concept of phase transitions, but it is completely different from our concept. Basically, the phase-field method approximates an objective n-dimensional surface by an "interface" of a smoothed characteristic function defined in $(n+1)$-dimension.

It is worth noting that our philosophy is similar to Ni and Takagi's celebrated study [61] (see also [60, 62]). They prove that, for a singularly perturbed elliptic equation with small perturbation, any solution of least energy has one peak at a boundary point. In addition, the treated equation is essentially same as our elastica equation in one-dimension. Although the imposed constraints, considered energies, and obtained results are different, the concepts considering a limit of least energy solutions and "localizing" the effect of energy are in common.

Last but not least, we do not claim that this paper is the first to point out that the borderline elastica appears near the endpoints in the straightening limit. In fact, this has been indicated in Audoly and Pomeau's book in physics [4, Section 4.4.1] from a viewpoint of boundary layer analysis. However, our result would be the first to provide a mathematical proof on the rescaled convergence, and moreover to determine the precise rate of magnification in the rescaling.

1.1.6 Organization

All the main results of this chapter are collected in Section 1.2. The results are sequentially proved in subsequent Sections 1.3, 1.4, 1.5, and 1.6. We indicate the positions of the proofs in Section 1.2 more precisely.

1.2 Preliminaries and main results

1.2.1 Extensible problem

Let $I = (0,1)$ be the open unit interval and $\bar{I} = [0,1]$ be its closure. For a smooth regular curve $\gamma : \bar{I} \to \mathbb{R}^2$ we denote the length by $\mathcal{L}[\gamma]$, and the total squared curvature by $\mathcal{B}[\gamma]$ as (1.1.1). Then, for $\varepsilon > 0$, the modified total squared curvature (1.1.2) is represented as

$$\mathcal{E}_\varepsilon[\gamma] := \varepsilon^2 \mathcal{B}[\gamma] + \mathcal{L}[\gamma].$$

Hereafter, we use both the original parameter $t \in \bar{I}$ and the arc length parameterization $s \in [0, \mathcal{L}[\gamma]]$ as the situation demands. For a regular curve $\gamma \in C^\infty(\bar{I};\mathbb{R}^2)$, we often denote its arc length reparameterization by $\tilde{\gamma} : [0, \mathcal{L}[\gamma]] \to \mathbb{R}^2$.

Let $l > 0$ and $\theta_0, \theta_1 \in [-\pi, \pi]$. We say that a curve $\gamma \in C^\infty(\bar{I};\mathbb{R}^2)$ is *admissible* if γ is regular and constant speed, i.e., $|\dot{\gamma}| \equiv \mathcal{L}[\gamma] > 0$, and moreover satisfies the clamped boundary condition:

$$\begin{aligned} \gamma(0) &= (0,0), \quad \dot{\gamma}(0) = \mathcal{L}[\gamma](\cos\theta_0, \sin\theta_0), \\ \gamma(1) &= (l,0), \quad \dot{\gamma}(1) = \mathcal{L}[\gamma](\cos\theta_1, \sin\theta_1). \end{aligned} \tag{1.2.1}$$

We denote the set of all admissible curves by $\mathcal{A}_{\theta_0,\theta_1,l} \subset C^\infty(\bar{I};\mathbb{R}^2)$.

For $\varepsilon > 0$, we consider the following minimizing problem

$$\min_{\gamma \in \mathcal{A}_{\theta_0,\theta_1,l}} \mathcal{E}_\varepsilon[\gamma]. \tag{1.2.2}$$

The existence of minimizers follows by a direct method in the calculus of variations and a bootstrap argument (Appendix 1.A). Our purpose is to know the shape of global minimizers, i.e., a curve γ_ε such that $\mathcal{E}_\varepsilon[\gamma_\varepsilon] = \min_{\gamma \in \mathcal{A}_{\theta_0,\theta_1,l}} \mathcal{E}_\varepsilon[\gamma]$.

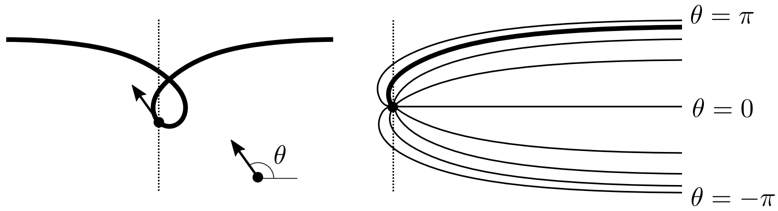

FIGURE 1.10: Borderline elastica with initial angle.

Our main theorem states that any sequence of global minimizers γ_ε converges as $\varepsilon \to 0$ to a part of the borderline elastica near each endpoint in a rescaled sense, and becomes almost straight elsewhere, as in Figure 1.6. To state the main theorem, we define borderline elasticae with initial angles as in Figure 1.10.

Definition 1.2.1 (Tangential angle)**.** For a smooth regular curve γ defined on an interval $\bar{J} = [0, T]$ (or $\bar{J} = [0, \infty)$) we denote by ϑ_γ a continuous representation of the *tangential angle*. More precisely, ϑ_γ is a smooth function on $\bar{J}$ such that the vectors $\dot{\gamma}(t)$ and $(\cos\vartheta_\gamma(t), \sin\vartheta_\gamma(t))$ are in a same direction for any $t \in \bar{J}$.

For a given curve, the tangential angle is unique if we fix the initial value $\vartheta_\gamma(0)$. In other words, it is unique up to the addition of constants in $2\pi\mathbb{Z}$.

Definition 1.2.2 (Borderline elastica with initial angle)**.** For $\theta \in [-\pi, \pi]$, we say that a smooth curve $\gamma_B^\theta : [0, \infty) \to \mathbb{R}^2$ parameterized by the arc length s is the *borderline elastica with initial angle θ* if

$$\gamma_B^\theta(0) = (0, 0), \quad \vartheta_{\gamma_B^\theta}(0) = \theta, \quad \lim_{s\to\infty} \vartheta_{\gamma_B^\theta}(s) = 0,$$

and moreover $|\partial_s \vartheta_{\gamma_B^\theta}|^2 = 1 - \cos\vartheta_{\gamma_B^\theta}$ holds in $(0, \infty)$. Such a curve is uniquely determined for any given $\theta \in [-\pi, \pi]$. (See also Definition 1.3.12.)

We are now in a position to state our main theorem.

Theorem 1.2.3 (Straightening result for extensible problem)**.** *Fix any convergent sequences $l_\varepsilon \to l$ in $(0, \infty)$ and $\theta_0^\varepsilon \to \theta_0$, $\theta_1^\varepsilon \to \theta_1$ in $[-\pi, \pi]$. Let γ_ε be a minimizer of $\mathcal{E}_\varepsilon$ in $\mathcal{A}_{\theta_0^\varepsilon, \theta_1^\varepsilon, l_\varepsilon}$ for $\varepsilon > 0$. Let $\tilde{\gamma}_\varepsilon$ be the arc length parameterization of γ_ε. Then the following statements hold.*

(1) Let $\hat{\gamma}_\varepsilon(\hat{s}) := \varepsilon^{-1}\tilde{\gamma}_\varepsilon(\varepsilon\hat{s})$. If $|\theta_0| < \pi$, then $\hat{\gamma}_\varepsilon$ converges to $\gamma_B^{\theta_0}$ in C^∞_{loc} as $\varepsilon \to 0$. If $|\theta_0| = \pi$, then for any subsequence of $\{\hat{\gamma}_\varepsilon\}_\varepsilon$ there is a subsequence $\{\hat{\gamma}_{\varepsilon'}\}_{\varepsilon'}$ such that $\hat{\gamma}_{\varepsilon'}$ converges to γ_B^π or $\gamma_B^{-\pi}$ in C^∞_{loc} as $\varepsilon' \to 0$.

(2) Denote the length of γ_ε by L_ε. Let $K_{c\varepsilon} = [c\varepsilon, L_\varepsilon - c\varepsilon]$. Then

$$\limsup_{\varepsilon\to 0} \max_{s\in K_{c\varepsilon}} |\partial_s \tilde{\gamma}_\varepsilon(s) - (1, 0)| \le 4e^{-\frac{c}{\sqrt{2}}}.$$

Theorem 1.2.3 is proved in Section 1.4. To prove this theorem, we first prove a key step in Section 1.3, namely, a first order expansion of the energy minimum. By using the expansion, in Section 1.4, we first prove the rescaled convergence (1) in a weak sense, and then complete the proof of the almost straightness (2). Finally, we improve the regularity of the rescaled convergence by using explicit expressions of the curvatures by elliptic functions.

We give some remarks on the main theorem.

Remark 1.2.4. To be more precise, the above C^∞_{loc}-convergence means that for any $c > 0$ the restricted rescaled curve $\hat{\gamma}_\varepsilon|_{[0,c]}$ converges to $\gamma_B^{\theta_0}|_{[0,c]}$ in $C^\infty([0,c];\mathbb{R}^2)$ as $\varepsilon \to 0$. The rescaled curve $\hat{\gamma}_\varepsilon(\hat{s})$ is defined for $\hat{s} \in [0, L_\varepsilon/\varepsilon]$, and hence at least in $[0, l_\varepsilon/\varepsilon]$. Since $l_\varepsilon/\varepsilon \to \infty$, for any fixed $c > 0$ there is $\varepsilon_c > 0$ such that for any $\varepsilon \in (0, \varepsilon_c)$ the curve $\hat{\gamma}_\varepsilon$ is defined at least in $[0, c]$. Thus, the convergence of $\hat{\gamma}_\varepsilon|_{[0,c]}$ is well-defined for any $c > 0$.

Remark 1.2.5. Theorem 1.2.3 states a rescaled convergence only at the origin. However, by symmetry, we immediately find that a similar rescaled convergence is valid for the other endpoint $(l_\varepsilon, 0)$ in the following sense. Let $\tilde{\gamma}^*_\varepsilon$ be the backward reparameterization of half-rotated $\tilde{\gamma}_\varepsilon$ about the point $(l_\varepsilon/2, 0) \in \mathbb{R}^2$. Let $\hat{\gamma}^*_\varepsilon(\hat{s}) := \varepsilon^{-1}\tilde{\gamma}^*_\varepsilon(\varepsilon\hat{s})$. Then $\hat{\gamma}^*_\varepsilon$ converges to the borderline elastica with initial angle θ_1 in the same sense as (1) in Theorem 1.2.3.

Remark 1.2.6. By Theorem 1.2.3, for any fixed $c > 0$, a minimizer $\tilde{\gamma}_\varepsilon$ is controlled in $[0, c\varepsilon]$ by (1) and $[c\varepsilon, L_\varepsilon - c\varepsilon]$ by (2) for any small $\varepsilon > 0$. Moreover, by symmetry, $\tilde{\gamma}_\varepsilon$ is also controlled in $[L_\varepsilon - c\varepsilon, L_\varepsilon]$. Hence, the whole shape of $\tilde{\gamma}_\varepsilon$ is controlled as $\varepsilon \to 0$.

Remark 1.2.7. In the case that $|\theta_0| = \pi$, the rescaled convergent limits are not unique. This is natural because, for example, if we additionally assume that $|\theta_1| = \pi$ or $\theta_1 = 0$, then there are two different minimizers $\gamma_\varepsilon = (x_\varepsilon, y_\varepsilon)$ and $\gamma'_\varepsilon = (x_\varepsilon, -y_\varepsilon)$. If $|\theta_1| \in (0, \pi)$, then there remains a possibility to obtain the uniqueness, but we then need a higher order expansion of the energy than our first order expansion.

Remark 1.2.8. Theorem 1.2.3 is valid only for global minimizers since there are local minimizers with loops (as in Figure 1.3) as shown in Appendix 1.A.

Remark 1.2.9. In Theorem 1.2.3, the boundary condition is perturbed as $l_\varepsilon \to l$, $\theta_0^\varepsilon \to \theta_0$, and $\theta_1^\varepsilon \to \theta_1$. However, the effects do not appear in the conclusion. This means that our result is "stable" for the perturbation. This stability would be useful for free boundary problems as in [56, 57]; in fact, we use this stability in Chapter 3.

From our viewpoint, the cases that $|\theta_0| = \pi$ and $|\theta_1| = \pi$ are critical and rather complicated than other generic cases. Moreover, the case that $\theta_0 = \theta_1 = 0$ is trivial but also critical in a sense. In this chapter, we often assume the following generic angle condition:

$$\theta_0, \theta_1 \in (-\pi, \pi), \quad |\theta_0| + |\theta_1| > 0. \tag{1.2.3}$$

This condition excludes the above critical cases.

By using Theorem 1.2.3, we also obtain more qualitative properties of global minimizers for small ε. We define an *inflection point* of a solution curve as a point (except the endpoints) where the sign of the curvature changes. This is well-defined since the curvature of any non-straight solution curve is represented by a nonzero elliptic function (see Proposition 1.4.5).

Theorem 1.2.10 (Qualitative properties)**.** *Fix any convergent sequences $l_\varepsilon \to l$ in $(0,\infty)$ and $\theta_0^\varepsilon \to \theta_0$, $\theta_1^\varepsilon \to \theta_1$ in $[-\pi,\pi]$. Then there is $\bar\varepsilon > 0$ such that for any $\varepsilon \in (0,\bar\varepsilon)$ any minimizer γ_ε of $\mathcal{E}_\varepsilon$ in $\mathcal{A}_{\theta_0^\varepsilon,\theta_1^\varepsilon,l_\varepsilon}$ has no self-intersection. In addition, if we suppose (1.2.3), then the following statements hold.*

(1) *If $\theta_0\theta_1 < 0$, then there is $\bar\varepsilon > 0$ such that for any $\varepsilon \in (0,\bar\varepsilon)$ any minimizer γ_ε has no inflection point, and moreover the total variation of $\vartheta_{\gamma_\varepsilon}$ is $|\theta_0^\varepsilon| + |\theta_1^\varepsilon|$.*

(2) *If $\theta_0\theta_1 \geq 0$, then there is $\bar\varepsilon > 0$ such that for any $\varepsilon \in (0,\bar\varepsilon)$ any minimizer γ_ε has exact one inflection point. Moreover, the total variation of $\vartheta_{\gamma_\varepsilon}$ converges to $|\theta_0|+|\theta_1|$ as $\varepsilon \to 0$.*

Theorem 1.2.10 is proved in Section 1.5. This theorem roughly states that for any small ε any minimizer is a straightened C-shaped or S-shaped curve as in Figure 1.7. In particular, for angles such that $|\theta_0|, |\theta_1| < \pi/2$, any minimizer is represented by the graph of a function for small ε.

We finally state that, if $\theta_0\theta_1 < 0$ holds in the generic angle condition, then the energy $\mathcal{E}_\varepsilon$ admits a unique global minimizer for any small ε. This theorem is also proved in Section 1.5.

Theorem 1.2.11 (Uniqueness)**.** *Fix any convergent sequences $l_\varepsilon \to l$ in $(0,\infty)$ and $\theta_0^\varepsilon \to \theta_0$, $\theta_1^\varepsilon \to \theta_1$ in $[-\pi,\pi]$ with (1.2.3) and $\theta_0\theta_1 < 0$. Then there is $\bar\varepsilon > 0$ such that for any $\varepsilon \in (0,\bar\varepsilon)$ the energy $\mathcal{E}_\varepsilon$ admits a unique minimizer in $\mathcal{A}_{\theta_0^\varepsilon,\theta_1^\varepsilon,l_\varepsilon}$.*

1.2.2 Inextensible problem

By using the above results, we also obtain a straightening result for the classical problem of inextensible elastic rods.

Let $0 < l < L$ and $\theta_0, \theta_1 \in [-\pi,\pi]$. Let $\mathcal{A}^L_{\theta_0,\theta_1,l} \subset \mathcal{A}_{\theta_0,\theta_1,l}$ be the set of admissible curves $\gamma \in \mathcal{A}_{\theta_0,\theta_1,l}$ of fixed length $\mathcal{L}[\gamma] = L$. Recall that the inextensible problem is formulated as

$$\min_{\gamma\in\mathcal{A}^L_{\theta_0,\theta_1,l}} \mathcal{B}[\gamma]. \tag{1.2.4}$$

We are concerned with the shapes of straightened elastic rods, i.e., the asymptotic shape of minimizers as the distance of the endpoints is enlarged as $l \uparrow L$ while the length L and the angles θ_0, θ_1 are fixed. We prove that in the limit $l \uparrow L$ we can rephrase (1.2.4) in terms of (1.2.2) at least in a subsequential sense.

Theorem 1.2.12 (Straightening result for inextensible problem: general case)**.** *Let $L > 0$ and $\theta_0, \theta_1 \in [-\pi,\pi]$ with $|\theta_0| + |\theta_1| > 0$. Then there are sequences $l_n \uparrow L$ and $\varepsilon_n \downarrow 0$ as $n \to \infty$ such that, for any minimizer γ_n of $\mathcal{B}$ in $\mathcal{A}^L_{\theta_0,\theta_1,l_n}$, the dilated curve $\frac{L}{l_n}\gamma_n$ is a minimizer of $\mathcal{E}_{\varepsilon_n}$ in $\mathcal{A}_{\theta_0,\theta_1,L}$. Moreover,*

$$\lim_{n\to\infty} \frac{L-l_n}{\varepsilon_n} = 4\sqrt{2}\left(\sin^2\frac{\theta_0}{4} + \sin^2\frac{\theta_1}{4}\right).$$

We remark that the distance of the endpoints of $\frac{L}{l_n}\gamma_n$ is fixed as L. The dilation is just for the normalization to fix the endpoints of curves. It is not effective since the magnification rate L/l_n converges to 1.

Theorem 1.2.12 is proved in Section 1.6. This theorem implies that similar straightening results to Theorem 1.2.3 and Theorem 1.2.10 are also valid for the classical inextensible problem, at least in a subsequential straightening process. In particular, minimizers bend at the scale ε_n in a straightening process $l_n \uparrow L$. The last equation in Theorem 1.2.12 means that the leading order term of ε_n is completely determined by $L - l_n$ and the angles θ_0 and θ_1.

Remark 1.2.13. The case $\theta_0 = \theta_1 = 0$ is quite different from others, both physically and mathematically. This case corresponds to buckling (Figure 1.4) but not straightening. In addition, if $\theta_0 = \theta_1 = 0$, then the extensible problem admits only the trivial segment minimizer, but such a segment is not admissible in the inextensible problem (except $l = L$). Hence, the problem (1.2.4) can not be read as (1.2.2).

Theorem 1.2.12 requires to take a subsequence. It is expected to be a technical assumption, but at this time we have no proof of a full convergence for the general case. As mentioned, the difficulty is crucially due to the lack of general theory for the uniqueness of minimizers in the extensible problem. In fact, if a given boundary condition guarantees the uniqueness as $\varepsilon \to 0$, then Theorem 1.2.12 is valid in a full convergence sense. This issue is discussed in Section 1.6 more precisely.

We finally state that, thanks to Theorem 1.2.11, if the generic angle condition is satisfied and $\theta_0\theta_1 < 0$ holds, then the uniqueness is also valid for the straightened inextensible rods, and moreover Theorem 1.2.12 holds in a full convergence sense as follows.

Theorem 1.2.14 (Straightening result for inextensible problem: convex case)**.** *Let $L > 0$ and $\theta_0, \theta_1 \in [-\pi, \pi]$ with (1.2.3) and $\theta_0\theta_1 < 0$. Then there is $\bar{l} \in (0, L)$ such that for any $l \in (\bar{l}, L)$ the energy $\mathcal{B}$ admits a unique minimizer in $\mathcal{A}^L_{\theta_0,\theta_1,l}$. Moreover, there is a strictly decreasing function $\tilde{\varepsilon} : (\bar{l}, L) \to (0, \infty)$ such that, for any $l \in (\bar{l}, L)$ and a unique minimizer γ_l of $\mathcal{B}$ in $\mathcal{A}^L_{\theta_0,\theta_1,l}$, the dilated curve $\frac{L}{l}\gamma_l$ is a minimizer of $\mathcal{E}_{\tilde{\varepsilon}(l)}$ in $\mathcal{A}_{\theta_0,\theta_1,L}$, and*

$$\lim_{l \uparrow L} \frac{L - l}{\tilde{\varepsilon}(l)} = 4\sqrt{2}\left(\sin^2\frac{\theta_0}{4} + \sin^2\frac{\theta_1}{4}\right).$$

1.3 Asymptotic expansion of the energies of minimizers

In this section, we prove a key step for our rescaled convergence: an asymptotic expansion of the energies of minimizers as $\varepsilon \to 0$. Throughout this section, we fix convergent sequences $l_\varepsilon \to l$ in $(0, \infty)$ and $\theta_0^\varepsilon \to \theta_0$, $\theta_1^\varepsilon \to \theta_1$ in $[-\pi, \pi]$.

Lemma 1.3.1. *Let $\gamma_\varepsilon \in \mathcal{A}_{\theta_0^\varepsilon,\theta_1^\varepsilon,l_\varepsilon}$ be a minimizer of $\mathcal{E}_\varepsilon$ in $\mathcal{A}_{\theta_0^\varepsilon,\theta_1^\varepsilon,l_\varepsilon}$ for $\varepsilon > 0$. Then*

$$\mathcal{E}_\varepsilon[\gamma_\varepsilon] - l_\varepsilon - 8\sqrt{2}\left(\sin^2\frac{\theta_0^\varepsilon}{4} + \sin^2\frac{\theta_1^\varepsilon}{4}\right)\varepsilon = o(\varepsilon) \quad \textit{as} \quad \varepsilon \to 0.$$

In the rest of this section we prove the above lemma. Note that it suffices to prove that, for any sequence of minimizers,

$$\limsup_{\varepsilon \to 0} \frac{\mathcal{E}_\varepsilon[\gamma_\varepsilon] - l_\varepsilon}{\varepsilon} \leq 8\sqrt{2}\left(\sin^2\frac{\theta_0}{4} + \sin^2\frac{\theta_1}{4}\right) \tag{1.3.1}$$

and

$$\liminf_{\varepsilon\to 0} \frac{\mathcal{E}_\varepsilon[\gamma_\varepsilon]-l_\varepsilon}{\varepsilon} \geq 8\sqrt{2}\left(\sin^2\frac{\theta_0}{4}+\sin^2\frac{\theta_1}{4}\right). \tag{1.3.2}$$

We define an energy functional $\mathcal{F}_\varepsilon$ for any smooth regular curve γ by

$$\mathcal{F}_\varepsilon[\gamma] = \int_0^{\mathcal{L}[\gamma]} \left(\varepsilon|\partial_s\vartheta_{\tilde{\gamma}}|^2 + \frac{1}{\varepsilon}(1-\cos\vartheta_{\tilde{\gamma}})\right) ds, \tag{1.3.3}$$

where $\theta_{\tilde{\gamma}}$ is the tangential angle of the arc length parameterization $\tilde{\gamma}$ of γ. Note that $\mathcal{F}$ is well-defined since this energy is invariant by the addition of constants of $2\pi\mathbb{Z}$ to $\vartheta_{\tilde{\gamma}}$. Moreover, we notice that for any $\gamma \in \mathcal{A}_{\theta_0^\varepsilon,\theta_1^\varepsilon,l_\varepsilon}$ the relation

$$\mathcal{F}_\varepsilon[\gamma] = \frac{\mathcal{E}_\varepsilon[\gamma]-l_\varepsilon}{\varepsilon}$$

holds since

$$\int_0^{\mathcal{L}[\gamma]} |\partial_s\vartheta_{\tilde{\gamma}}|^2 ds = \int_\gamma \kappa^2 ds, \quad \int_0^{\mathcal{L}[\gamma]} ds = \int_\gamma ds, \quad \int_0^{\mathcal{L}[\gamma]} \cos\vartheta_{\tilde{\gamma}} ds = l_\varepsilon.$$

Note that $\mathcal{F}_\varepsilon$ may be regarded as a phase transition energy since the latter term can be interpreted as a periodic potential of infinitely many wells. The representation $\mathcal{F}_\varepsilon$ is essentially used in this chapter.

The following lemma is obvious by definition, but frequently used in this chapter.

Lemma 1.3.2. *Let N be a positive integer and $t_0 < \cdots < t_N$. Let $\bar{J} = [t_0, t_N]$ and $\bar{J}_i = [t_i, t_{i+1}]$ for $i = 0, \ldots, N-1$. For any $\varepsilon > 0$ and any smooth constant speed curve $\gamma : \bar{J} \to \mathbb{R}^2$,*

$$\mathcal{F}_\varepsilon[\gamma] = \sum_{i=0}^{N-1} \mathcal{F}_\varepsilon[\gamma|_{\bar{J}_i}]$$

and each term of the right-hand sum is nonnegative. In particular, $\mathcal{F}_\varepsilon[\gamma] \geq \mathcal{F}_\varepsilon[\gamma|_{\bar{J}_i}]$ for each i.

1.3.1 Weighted total variation

The following weighted variation function is also frequently used in this chapter.

Definition 1.3.3 (Weighted variation of tangential angle)**.** Define a strictly increasing function $V \in C^1(\mathbb{R})$ by

$$V(\theta) := \int_0^\theta 2\sqrt{1-\cos\phi}\, d\phi.$$

Remark 1.3.4 (Calculation of weighted variation). By the half-angle formula, for any $\theta \in [-\pi, \pi]$ we calculate

$$V(\theta) = \mathrm{sign}(\theta) \cdot 8\sqrt{2}\sin^2\frac{\theta}{4}.$$

By the periodicity, for any $m \in \mathbb{Z}$ and $\theta \in [(2m-1)\pi, (2m+1)\pi)$ we have

$$V(\theta) = \mathrm{sign}([\![\theta]\!]) \cdot 8\sqrt{2}\sin^2\frac{[\![\theta]\!]}{4} + 8\sqrt{2}m,$$

where $[\![\theta]\!]$ denotes a unique angle in $[-\pi, \pi)$ so that $\theta - [\![\theta]\!] \in 2\pi\mathbb{Z}$. Hereafter, we frequently use the notation $[\![\cdot]\!]$ in this sense.

The weighted variation is essential for our arguments since the following lower estimate holds.

Lemma 1.3.5. *For any $\varepsilon > 0$ and smooth regular curve γ parameterized by the arc length s, we have*

$$\mathcal{F}_\varepsilon[\gamma] \geq \int_0^{\mathcal{L}[\gamma]} |\partial_s(V \circ \vartheta_\gamma)| ds \geq |V(\vartheta_\gamma(\mathcal{L}[\gamma])) - V(\vartheta_\gamma(0))| .$$

Proof. The first inequality follows by the definition of $\mathcal{F}$ and the inequality $\varepsilon X^2 + \varepsilon^{-1}Y^2 \geq 2|X||Y|$. The last inequality follows by the triangle inequality. $\square$

To compute the above lower bound, the following lemma is useful.

Lemma 1.3.6. *Let $\theta, \theta' \in \mathbb{R}$. Then the following inequality holds:*

$$|V(\theta) - V(\theta')| \geq 8\sqrt{2} \left| \sin^2 \frac{[\![\theta]\!]}{4} - \sin^2 \frac{[\![\theta']\!]}{4} \right| .$$

The equality is attained if and only if $\theta, \theta' \in [m\pi, (m+1)\pi]$ for some $m \in \mathbb{Z}$.

Proof. Fix $\theta, \theta' \in \mathbb{R}$. Then there exists unique $\theta^* \in \mathbb{R}$ so that $|[\![\theta^*]\!]| = |[\![\theta']\!]|$ and $\theta^*, \theta \in [m\pi, (m+1)\pi]$ for some $m \in \mathbb{Z}$. By periodicity, we have $|\theta - \theta'| \geq |\theta - \theta^*|$, and hence

$$|V(\theta) - V(\theta')| \geq |V(\theta) - V(\theta^*)|.$$

By Remark 1.3.4, the right-hand term is calculated as

$$|V(\theta) - V(\theta^*)| = 8\sqrt{2} \left| \sin^2 \frac{[\![\theta]\!]}{4} - \sin^2 \frac{[\![\theta^*]\!]}{4} \right| .$$

Since $\sin^2([\![\theta^*]\!]/4) = \sin^2([\![\theta']\!]/4)$, the desired inequality holds. In view of the first inequality, the equality is attained if and only if $\theta' = \theta^*$, i.e., $\theta, \theta' \in [m\pi, (m+1)\pi]$ for some $m \in \mathbb{Z}$. The proof is complete. $\square$

1.3.2 Lower bound for the modified squared curvature

In this subsection we prove the liminf inequality (1.3.2), that is, the following proposition.

Proposition 1.3.7. *Let $\gamma_\varepsilon \in \mathcal{A}_{\theta_0^\varepsilon, \theta_1^\varepsilon, l_\varepsilon}$ be a minimizer of $\mathcal{E}_\varepsilon$ in $\mathcal{A}_{\theta_0^\varepsilon, \theta_1^\varepsilon, l_\varepsilon}$ for $\varepsilon > 0$. Then*

$$\liminf_{\varepsilon \to 0} \mathcal{F}_\varepsilon[\gamma_\varepsilon] \geq 8\sqrt{2} \left(\sin^2 \frac{\theta_0}{4} + \sin^2 \frac{\theta_1}{4} \right) .$$

We first confirm basic convergences on a sequence of minimizers.

Proposition 1.3.8. *Let $\gamma_\varepsilon \in \mathcal{A}_{\theta_0^\varepsilon, \theta_1^\varepsilon, l_\varepsilon}$ be a minimizer of $\mathcal{E}_\varepsilon$ in $\mathcal{A}_{\theta_0^\varepsilon, \theta_1^\varepsilon, l_\varepsilon}$ for $\varepsilon > 0$. Then the length L_ε of γ_ε converges to l, and the curve γ_ε uniformly converges to the segment $\bar{\gamma}(t) = (lt, 0)$, $t \in \bar{I}$, as $\varepsilon \to 0$.*

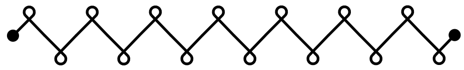

FIGURE 1.11: An example of a curve of which tangent vector is not rightward.

Proof. Let $L_\varepsilon = \mathcal{L}[\gamma_\varepsilon]$ be the length (speed) of γ_ε. It is easy to confirm that $\mathcal{E}_\varepsilon[\gamma_\varepsilon] \to l$ as $\varepsilon \to 0$ since we can easily construct a sequence of curves $\gamma'_\varepsilon \in \mathcal{A}_{\theta_0^\varepsilon,\theta_1^\varepsilon,l_\varepsilon}$ such that $\mathcal{E}_\varepsilon[\gamma'_\varepsilon] \to l$ by using circular arcs of radius ε and a segment.. Since $l_\varepsilon \leq L_\varepsilon \leq \mathcal{E}_\varepsilon[\gamma_\varepsilon]$ and $l_\varepsilon \to l$, the length (speed) L_ε also converges to l. In addition, since the speeds L_ε are bounded as $\varepsilon \to 0$, the curves γ_ε are equicontinuous as $\varepsilon \to 0$. Moreover, since the endpoint $\gamma_\varepsilon(0) = (0,0)$ is fixed and the lengths are bounded, we also find that the curves γ_ε are uniformly bounded as $\varepsilon \to 0$. Thus, by the Arzelà-Ascoli theorem, up to a subsequence of any subsequence, γ_ε uniformly converges to a continuous curve joining $(0,0)$ to $(l,0)$. Since $L_\varepsilon \to l$ and γ_ε is constant speed, the convergent limit must be the segment $\bar{\gamma}$. Hence, γ_ε fully converges to the segment $\bar{\gamma}$. The proof is complete. □

For such a convergent sequence, the following elementary lemma holds.

Lemma 1.3.9. *Let $l_\varepsilon \to l$ be a convergent sequence in $(0,\infty)$ and $\gamma_\varepsilon : \bar{I} \to \mathbb{R}^2$ be a smooth (regular) constant speed curves joining $(0,0)$ to $(l_\varepsilon,0)$ for $\varepsilon > 0$. Suppose that γ_ε uniformly converges to the segment $\bar{\gamma}(t) = (lt,0)$, and moreover the length L_ε of γ_ε converges to l as $\varepsilon \to 0$. Then for any open subinterval $J \subset I$ there is a sequence of times $\{t_\varepsilon\}_\varepsilon \subset J$ such that $[\![\vartheta_{\gamma_\varepsilon}(t_\varepsilon)]\!] \to 0$ as $\varepsilon \to 0$.*

Proof. We suppose the contradiction that there are $\delta > 0$, a subinterval $J = (t_0,t_1) \subset I$, and a sequence $\varepsilon_j \to 0$ such that $|[\![\vartheta_{\gamma_{\varepsilon_j}}(t)]\!]| \geq \delta$ for any j and $t \in J$. By this assumption, the x-component of γ_{ε_j} satisfies

$$\limsup_{j\to\infty}(x_{\varepsilon_j}(t_1) - x_{\varepsilon_j}(t_0)) = \limsup_{j\to\infty} L_{\varepsilon_j} \int_J \cos\vartheta_{\gamma_{\varepsilon_j}}\,dt \leq l(t_1-t_0)(\cos\delta) < l(t_1-t_0),$$

where the convergence $L_{\varepsilon_j} \to l$ is used. On the other hand, since γ_{ε_j} converges to the segment $\bar{\gamma}(t) = (lt,0)$, we immediately have

$$\lim_{j\to\infty}(x_{\varepsilon_j}(t_1) - x_{\varepsilon_j}(t_0)) = l(t_1-t_0).$$

This is a contradiction. □

Remark 1.3.10. The above lemma is elementary but should be slightly noted, since there is an example of a sequence of curves such that the sequence uniformly converges to a segment but the tangent vectors maintain a certain distance from the rightward vector anywhere. Such an example is constructed as in Figure 1.11, namely, as "sawtooth" curves of which edges are modified by loops, so that the number of the tooths diverges and the loops rapidly degenerate to points in the limit. Hence, the length convergence is an essential assumption.

We are now in a position to prove Proposition 1.3.7

Proof of Proposition 1.3.7. By Proposition 1.3.8 and Lemma 1.3.9, for $\varepsilon > 0$ there is $t_\varepsilon \in I$ such that $[\![\vartheta_{\gamma_\varepsilon}(t_{\varepsilon'})]\!] \to 0$ as $\varepsilon \to 0$. Then, by Lemma 1.3.5 and Lemma 1.3.6, we find that

$$\begin{aligned}\mathcal{F}_\varepsilon[\gamma_\varepsilon] &= \mathcal{F}_\varepsilon[\gamma_\varepsilon|_{[0,t_\varepsilon]}] + \mathcal{F}_\varepsilon[\gamma_\varepsilon|_{[t_\varepsilon,1]}] \\ &\geq 8\sqrt{2}\left|\sin^2\frac{[\![\vartheta_{\gamma_\varepsilon}(0)]\!]}{4} - \sin^2\frac{[\![\vartheta_{\gamma_\varepsilon}(t_\varepsilon)]\!]}{4}\right| \\ &\quad +8\sqrt{2}\left|\sin^2\frac{[\![\vartheta_{\gamma_\varepsilon}(t_\varepsilon)]\!]}{4} - \sin^2\frac{[\![\vartheta_{\gamma_\varepsilon}(1)]\!]}{4}\right|.\end{aligned}$$

Since

$$\sin^2\frac{[\![\vartheta_{\gamma_\varepsilon}(0)]\!]}{4} = \sin^2\frac{\theta_0^\varepsilon}{4}, \quad \sin^2\frac{[\![\vartheta_{\gamma_\varepsilon}(1)]\!]}{4} = \sin^2\frac{\theta_1^\varepsilon}{4},$$

and the convergences $\theta_0^\varepsilon \to \theta_0$, $\theta_1^\varepsilon \to \theta_1$, $[\![\vartheta_{\gamma_\varepsilon}(t_\varepsilon)]\!] \to 0$ hold as $\varepsilon \to 0$, we obtain

$$\varliminf_{\varepsilon\to 0} \mathcal{F}_\varepsilon[\gamma_\varepsilon] \geq 8\sqrt{2}\left(\sin^2\frac{\theta_0}{4} + \sin^2\frac{\theta_1}{4}\right).$$

The proof is complete. □

1.3.3 Construction of curves with energy convergence

In this subsection we prove that the limsup inequality (1.3.1) holds for any sequence of minimizers. To this end, it suffices to construct a suitable sequence of test curves so that the energies converge to the right-hand term of (1.3.1).

Proposition 1.3.11. *There is a sequence of curves $\gamma'_\varepsilon \in \mathcal{A}_{\theta_0^\varepsilon,\theta_1^\varepsilon,l_\varepsilon}$ such that*

$$\lim_{\varepsilon\to 0} \mathcal{F}_\varepsilon[\gamma'_\varepsilon] = 8\sqrt{2}\left(\sin^2\frac{\theta_0}{4} + \sin^2\frac{\theta_1}{4}\right). \tag{1.3.4}$$

This immediately implies (1.3.1) for any sequence of minimizers $\{\gamma_\varepsilon\}_\varepsilon$ since $\mathcal{F}_\varepsilon[\gamma_\varepsilon]$ is bounded above by $F_\varepsilon[\gamma'_\varepsilon]$ for a curve γ'_ε in Proposition 1.3.11.

For the proof, we construct suitable curves which are "optimally bending" as $\varepsilon \to 0$ near the endpoints. Some ideas are similar to the author's previous paper [56].

In view of phase transitions, near the endpoints, the rescaled tangential angles are expected to be close to transition layers for the phase transition energy $\mathcal{F}_\varepsilon$. Hence, we consider the following ordinary differential equations:

$$\partial_s\varphi_+(s) = \sqrt{1-\cos\varphi_+(s)}, \quad \partial_s\varphi_-(s) = -\sqrt{1-\cos\varphi_-(s)}. \tag{1.3.5}$$

For any initial values $\varphi_\pm(0) \in \mathbb{R}$, these equations are solved uniquely and globally in $s \in \mathbb{R}$. When $\varphi_\pm(0) \in 2\pi\mathbb{Z}$, the solutions are constant functions. In the case that $\varphi_\pm(0) = \pm\pi$, the solutions are represented as

$$\bar{\varphi}_\pm(s) := \pm 4\arctan\left(e^{\frac{s}{\sqrt{2}}}\right). \tag{1.3.6}$$

The function $\bar{\varphi}_+$ is strictly increasing with $\lim_{s\to\pm\infty}\bar{\varphi}_+(s) = \pi \pm \pi$ and its graph possesses point symmetry at $(0, \bar{\varphi}_+(0)) = (0, \pi)$. Any other solution to (1.3.5) is of the form $\bar{\varphi}_\pm(s+s_0) + 2\pi m$, where $s_0 \in \mathbb{R}$ and $m \in \mathbb{Z}$.

An important property of the above solutions is that for any $s_0 < s_1$, by (1.3.5), the following energy identity holds:

$$
\begin{aligned}
\text{(1.3.7)} \qquad \int_{s_0}^{s_1} \left(|\partial_s\bar{\varphi}_\pm|^2 + (1-\cos\bar{\varphi}_\pm)\right) ds &= \pm\int_{s_0}^{s_1} 2\partial_s\bar{\varphi}_\pm\sqrt{1-\cos\bar{\varphi}_\pm}ds \\
&= \pm\int_{s_0}^{s_1} \partial_s(V\circ\bar{\varphi}_\pm)ds \\
&= \pm(V\circ\bar{\varphi}_\pm(s_1) - V\circ\bar{\varphi}_\pm(s_0)), \\
&= |V(\bar{\varphi}_\pm(s_1)) - V(\bar{\varphi}_\pm(s_0))|,
\end{aligned}
$$

where V is the weighted variation function. The last equality follows since V is nondecreasing and $\bar{\varphi}_+$ (resp. $\bar{\varphi}_-$) is increasing (resp. decreasing).

A non-straight unit speed curve of which tangential angle satisfies (1.3.5) is nothing but the borderline elastica; in fact, concerning (1.3.6) for example, we compute the curvature as

$$\bar{\kappa}_\pm(s) = \partial_s\bar{\varphi}_\pm(s) = \pm\sqrt{2}\,\mathrm{sech}\frac{s}{\sqrt{2}}.$$

(See e.g. [72] to confirm that the above expression corresponds to the borderline elastica.) By (1.3.5) and (1.3.6), the borderline elasticae $\bar{\gamma}_\pm = (\bar{x}_\pm, \bar{y}_\pm)$ such that $\bar{\gamma}_\pm(0) = (0,0)$ and $\partial_s\bar{\gamma}_\pm(0) = (-1,0)$ are explicitly parameterized as

$$
\begin{aligned}
\bar{x}_\pm(s) &= \int_0^s \cos\bar{\varphi}_\pm = s - \int_0^s (1-\cos\bar{\varphi}_\pm) = s \mp \int_0^s \partial_s\bar{\varphi}_\pm\sqrt{1-\cos\bar{\varphi}_\pm} \\
&= s - \sqrt{2}\int_0^s \partial_s\bar{\varphi}_\pm \sin\frac{\bar{\varphi}_\pm}{2} = s + 2\sqrt{2}\cos\frac{\bar{\varphi}_\pm(s)}{2} = s - 2\sqrt{2}\tanh\frac{s}{\sqrt{2}}, \\
\bar{y}_\pm(s) &= \int_0^s \sin\bar{\varphi}_\pm = \mp\int_0^{|s|}\sqrt{1-\cos^2\bar{\varphi}_\pm} = -\int_0^{|s|}\partial_s\bar{\varphi}_\pm\sqrt{1+\cos\bar{\varphi}_\pm} \\
&= \sqrt{2}\int_0^{|s|}\partial_s\bar{\varphi}_\pm\cos\frac{\bar{\varphi}_\pm}{2} = 2\sqrt{2}\left(\sin\frac{\bar{\varphi}_\pm(s)}{2} \mp 1\right) = \pm 2\sqrt{2}\left(\mathrm{sech}\frac{s}{\sqrt{2}} - 1\right).
\end{aligned}
$$

Using the borderline elasticae, we can construct a sequence of curves satisfying (1.3.4). For the sake of convenience, we prepare a precise definition of borderline elasticae, which is equivalent to Definition 1.2.2.

Definition 1.3.12 (Borderline elastica with initial angle)**.** Let $\theta \in [-\pi,\pi]$. A function $\vartheta_B^\theta : [0,\infty) \to \mathbb{R}$ is called *borderline angle function with initial angle* θ if ϑ_B^θ is a solution to either of the equations (1.3.5) such that $\vartheta_B^\theta(0) = \theta$ and $\vartheta_B^\theta(s) \to 0$ as $s \to \infty$. Such a function is uniquely determined for any $\theta \in [-\pi,\pi]$.

Similarly, a smooth curve $\gamma_B^\theta : [0,\infty) \to \mathbb{R}^2$ parameterized by the arc length is called *borderline elastica with initial angle* θ if $\gamma_B^\theta(0) = (0,0)$ and its tangential angle $\vartheta_{\gamma_B^\theta}$ is the borderline angle function with initial angle θ in the above sense.

Lemma 1.3.13. *Let $\alpha \in (0,1)$ and $\theta_\varepsilon \to \theta$ be a convergent sequence in $[-\pi,\pi]$. Then there is a sequence of smooth regular curves $\gamma_\varepsilon = (x_\varepsilon, y_\varepsilon)$ parameterized by the arc length $s \in [0,\varepsilon^\alpha]$ such that the following conditions hold:*

(1) $\gamma_\varepsilon(0) = (0,0)$, $-2\sqrt{2}\varepsilon \le x_\varepsilon(s) \le \varepsilon^\alpha$ and $|y_\varepsilon(s)| \le 2\sqrt{2}\varepsilon$ for $s \in [0,\varepsilon^\alpha]$.

(2) $\vartheta_{\gamma_\varepsilon}(0) = \theta_\varepsilon$ and $\lim_{\varepsilon\to 0}\vartheta_{\gamma_\varepsilon}(\varepsilon^\alpha) = 0$.

(3) $\lim_{\varepsilon\to 0}\mathcal{F}_\varepsilon[\gamma_\varepsilon] = 8\sqrt{2}\sin^2(\theta/4)$.

Proof. We prove this lemma by using a part of the rescaled borderline elastica: we define the curve γ_ε so that $\mathcal{L}[\gamma_\varepsilon] = \varepsilon^\alpha$ and $\gamma_\varepsilon(s) = \varepsilon\gamma_B^{\theta_\varepsilon}(s/\varepsilon)$ for $s \in (0, \varepsilon^\alpha)$, where $\gamma_B^{\theta_\varepsilon}$ is the borderline elastica with initial angle θ_ε in Definition 1.3.12. Note that $\vartheta_{\gamma_\varepsilon}(s) = \vartheta_B^{\theta_\varepsilon}(s/\varepsilon)$. By the aforementioned properties of the borderline elastica, it is straightforward to confirm the conditions (1) and (2). It should be noted that $\varepsilon^{\alpha-1}\to\infty$ as $\varepsilon\to 0$ by $\alpha<1$, and hence $\vartheta_{\gamma_\varepsilon}(\varepsilon^\alpha) = \vartheta_B^{\theta_\varepsilon}(\varepsilon^{\alpha-1})$ converges to zero as $\varepsilon\to 0$. The last condition (3) follows by the energy identity (1.3.7):

$$
\begin{aligned}
\mathcal{F}_\varepsilon[\gamma_\varepsilon] &= \int_0^{\varepsilon^\alpha}\left(\varepsilon|\partial_s\vartheta_{\gamma_\varepsilon}|^2 + \frac{1}{\varepsilon}(1-\cos\vartheta_{\gamma_\varepsilon})\right)ds \\
&= \int_0^{\varepsilon^{\alpha-1}}\left(|\partial_{s'}\vartheta_B^{\theta_\varepsilon}|^2 + (1-\cos\vartheta_B^{\theta_\varepsilon})\right)ds' \quad (s'=s/\varepsilon) \\
&= |V(\vartheta_B^{\theta_\varepsilon}(0)) - V(\vartheta_B^{\theta_\varepsilon}(\varepsilon^{\alpha-1}))| \xrightarrow{\varepsilon\to 0} |V(\theta)-V(0)| = 8\sqrt{2}\sin^2\frac{\theta}{4}.
\end{aligned}
$$

The last equation follows by Lemma 1.3.6. □

Lemma 1.3.14. *Let $A_\varepsilon = (a_\varepsilon^x, a_\varepsilon^y), B_\varepsilon = (b_\varepsilon^x, b_\varepsilon^y) \in \mathbb{R}^2$ be points such that $A_\varepsilon \to (0,0)$ and $B_\varepsilon \to (l,0)$ as $\varepsilon\to 0$ for some $l>0$. Let $\theta_\varepsilon^A, \theta_\varepsilon^B$ be angles converging to zero as $\varepsilon\to 0$. Suppose that $|a_\varepsilon^y| + |b_\varepsilon^y| = o(\varepsilon^{1/2})$ as $\varepsilon\to 0$. Then there is a sequence of smooth curves γ_ε of length L_ε, parameterized by the arc length $s\in[0,L_\varepsilon]$, such that the boundary conditions*

$$\gamma_\varepsilon(0) = A_\varepsilon,\ \gamma(L_\varepsilon) = B_\varepsilon,\ \partial_s\gamma_\varepsilon(0) = (\cos\theta_\varepsilon^A, \sin\theta_\varepsilon^A),\ \partial_s\gamma_\varepsilon(L_\varepsilon) = (\cos\theta_\varepsilon^B, \sin\theta_\varepsilon^B)$$

hold, the length L_ε converges to l, and moreover

$$\lim_{\varepsilon\to 0}\mathcal{F}_\varepsilon[\gamma_\varepsilon] = 0.$$

Proof. We first note that it suffices to construct a sequence of curves of class C^1 and piecewise C^2 by a standard mollifying argument. We construct γ_ε as in Figure 1.12; namely, we use circular arcs of radius ε near the endpoints, and connect them by a segment.

By using circular arcs of radius ε and central angles ϕ_ε^A, ϕ_ε^B such that $\phi_\varepsilon^A, \phi_\varepsilon^B \to 0$ near the endpoints (and noting Lemma 1.3.2), we can modify the boundary conditions as $A'_\varepsilon, B'_\varepsilon, \theta_\varepsilon^{A'}, \theta_\varepsilon^{B'}$ such that $A'_\varepsilon, B'_\varepsilon$ satisfy the same assumptions as $A_\varepsilon, B_\varepsilon$, and $\theta_\varepsilon^{A'} = \theta_\varepsilon^{B'} = 0$ for any small $\varepsilon>0$. Note that the energy $\mathcal{F}_\varepsilon$ of the circular arc parts γ_ε^c tends to be zero as $\varepsilon\to 0$ since

$$\varepsilon\int_{\gamma_\varepsilon^c}\kappa^2 ds = \varepsilon\cdot\frac{1}{\varepsilon^2}\cdot\varepsilon(\phi_\varepsilon^A+\phi_\varepsilon^B)\to 0,\quad \frac{1}{\varepsilon}\int_{\gamma_\varepsilon^c}ds = \frac{1}{\varepsilon}\cdot\varepsilon(\phi_\varepsilon^A+\phi_\varepsilon^B)\to 0,$$

$$\frac{1}{\varepsilon}\left|\int_{\gamma_\varepsilon^c}\cos\vartheta_{\gamma_\varepsilon^c}ds\right| \le \frac{1}{\varepsilon}\int_{\gamma_\varepsilon^c}ds \to 0.$$

Then, by using again circular arcs of radius ε such that the central angles converge to zero, we may assume that the boundary conditions $A''_\varepsilon, B''_\varepsilon, \theta_\varepsilon^{A''}, \theta_\varepsilon^{B''}$ allow a segment which is compatible with the conditions.

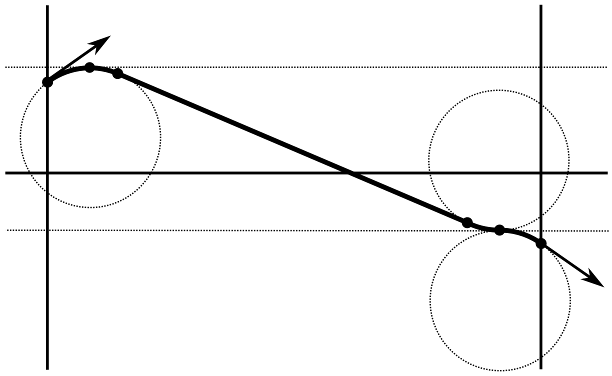

FIGURE 1.12: Construction of a curve for Lemma 1.3.14.

The energy $\mathcal{F}_\varepsilon$ of the segment γ_ε^s joining $A''_\varepsilon = (a_\varepsilon^{x''}, a_\varepsilon^{y''})$ to $B''_\varepsilon = (b_\varepsilon^{x''}, b_\varepsilon^{y''})$ also satisfies $\mathcal{F}_\varepsilon[\gamma_\varepsilon^s] \to 0$. In fact, the curvature of the segment is zero, and

$$\begin{aligned} \frac{1}{\varepsilon}\int_{\gamma_\varepsilon^s}(1-\cos\vartheta_{\gamma_\varepsilon^s})ds &= \frac{1}{\varepsilon}\left(\sqrt{|a_\varepsilon^{x''}-b_\varepsilon^{x''}|^2+|a_\varepsilon^{y''}-b_\varepsilon^{y''}|^2}-|a_\varepsilon^{x''}-b_\varepsilon^{x''}|\right) \\ &= \varepsilon^{-1}o(|a_\varepsilon^{y''}-b_\varepsilon^{y''}|^2)=o(1)\to 0 \end{aligned}$$

since $|a_\varepsilon^{x''}-b_\varepsilon^{x''}| \to l > 0$ and $|a_\varepsilon^{y''}|+|b_\varepsilon^{y''}| = o(\varepsilon^{1/2})$. The proof is now complete. □

By using the above lemmas, we complete the proof of Proposition 1.3.11.

Proof of Proposition 1.3.11. As mentioned in the proof of Lemma 1.3.14, it suffices to construct a sequence of curves of class C^1 and piecewise C^2 by a standard mollifying argument. We construct a sequence $\{\gamma'_\varepsilon\}_\varepsilon$ as in Figure 1.13.

Fix any $\alpha \in (0,1)$. Let ε be small as $\varepsilon^\alpha < l$. To construct γ'_ε, we use the curves in Lemma 1.3.13 near the endpoints and connect them suitably by Lemma 1.3.14. Namely, denoting the curves of Lemma 1.3.13 with $\theta = \theta_i^\varepsilon$ $(i=0,1)$ by γ_ε^i, we define γ'_ε so that

$$\gamma'_\varepsilon(s) = \begin{cases} \gamma_\varepsilon^0(s), & s \in [0,\varepsilon^\alpha], \\ \gamma''_\varepsilon(s-\varepsilon^\alpha), & s \in [\varepsilon^\alpha, L'_\varepsilon - \varepsilon^\alpha], \\ (l_\varepsilon, 0) - \gamma_\varepsilon^1(L'_\varepsilon - s), & s \in [L'_\varepsilon - \varepsilon^\alpha, L'_\varepsilon], \end{cases}$$

where the connecting part γ''_ε is taken as in Lemma 1.3.14 of which boundary conditions are suitably set so that γ'_ε is of class C^1 (the length L'_ε is a posteriori defined). Note that the points and tangential angles at $s = \varepsilon^\alpha$ and $s = L'_\varepsilon - \varepsilon^\alpha$ satisfy the assumptions in Lemma 1.3.14 by Lemma 1.3.13. Then, since Lemma 1.3.2 implies that

$$\mathcal{F}_\varepsilon[\gamma'_\varepsilon] = \mathcal{F}_\varepsilon[\gamma'_\varepsilon|_{[0,\varepsilon^\alpha]}] + \mathcal{F}_\varepsilon[\gamma'_\varepsilon|_{[\varepsilon^\alpha, L'_\varepsilon-\varepsilon^\alpha]}] + \mathcal{F}_\varepsilon[\gamma'_\varepsilon|_{[L'_\varepsilon-\varepsilon^\alpha, L'_\varepsilon]}],$$

Lemma 1.3.13 and Lemma 1.3.14 imply that the constructed curve γ'_ε satisfies the energy convergence (1.3.4). In particular, we note that

$$\mathcal{F}[\gamma'_\varepsilon|_{[L'_\varepsilon-\varepsilon^\alpha, L'_\varepsilon]}] = \mathcal{F}_\varepsilon[\gamma_\varepsilon^1|_{[0,\varepsilon^\alpha]}]$$

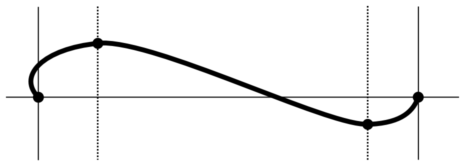

FIGURE 1.13: Construction of a curve for Proposition 1.3.11.

since the combination of the backward reparameterization and the half-rotation for a curve maintains the value of $\mathcal{F}_\varepsilon$ (the translation also maintains $\mathcal{F}_\varepsilon$ obviously). The proof is now complete. $\square$

1.4 Convergence of minimizers

In this section, we prove Theorem 1.2.3 by using results in the previous section. The rescaled convergence part is first proved in a weak sense, more precisely, the H^2-weak sense of curves. The almost straightness part is then fully proved. For these parts we mainly use properties of the energy. After that, we improve the regularity of our rescaled convergence; in this regularity part we strongly use properties of the equation of elastica.

1.4.1 Rescaled weak convergence to borderline elasticae near the endpoints

We first prove (1) of Theorem 1.2.3 in a weak sense.

The following fact is an essential step of our proof.

Lemma 1.4.1. *Let $c>0$ and $\vartheta \in H^1(0,c)$. Suppose that $\vartheta(0)\in[-\pi,\pi]$ and*

$$8\sqrt{2}\left(\sin^2\frac{\vartheta(0)}{4}-\sin^2\frac{[\![\vartheta(c)]\!]}{4}\right)\geq\int_0^c\left(|\vartheta'|^2+(1-\cos\vartheta)\right),$$

where $[\![\cdot]\!]$ is defined in Remark 1.3.4. Then, in the above inequality, the equality is attained. Moreover, if $|\vartheta(0)|<\pi$, the function ϑ is the borderline angle function with initial angle $\vartheta(0)$ (in the sense of Definition 1.3.12). In the case that $|\vartheta(0)|=\pi$, up to the addition of a constant in $\{0,\pm 2\pi\}$, the function ϑ is either the borderline angle function with initial angle π or $-\pi$.

Proof. By the inequality $X^2+Y^2\geq 2|X||Y|$,

$$\int_0^c\left(|\vartheta'|^2+(1-\cos\vartheta)\right)\geq\int_0^c 2|\vartheta'|\sqrt{1-\cos\vartheta}=\int_0^c|(V\circ\vartheta)'|. \tag{1.4.1}$$

By the triangle inequality,

$$\int_0^c|(V\circ\vartheta)'|\geq|V(\vartheta(0))-V(\vartheta(c))|. \tag{1.4.2}$$

Moreover, by Lemma 1.3.6,

$$|V(\vartheta(0)) - V(\vartheta(c))| \geq 8\sqrt{2}\left|\sin^2 \frac{[\![\vartheta(0)]\!]}{4} - \sin^2 \frac{[\![\vartheta(c)]\!]}{4}\right| \tag{1.4.3}$$

$$\geq 8\sqrt{2}\left(\sin^2 \frac{\vartheta(0)}{4} - \sin^2 \frac{[\![\vartheta(c)]\!]}{4}\right). \tag{1.4.4}$$

The last inequality follows by the definition of absolute value and the assumption that $\vartheta(0) \in [-\pi, \pi]$, i.e., $|[\![\vartheta(0)]\!]| = |\vartheta(0)|$.

Then, by the assumption, it turns out that in all the above inequalities (1.4.1)–(1.4.4) the equalities are attained. The equality in (1.4.1) implies $|\vartheta'|^2 = 1 - \cos\vartheta$ for a.e. in $[0, c]$. The equality in (1.4.2) implies that $(V \circ \vartheta)'$ does not change the sign, i.e., ϑ is monotone. Thus, ϑ satisfies either of the equations (1.3.5) in the classical sense.

By the above fact, the proof is complete when $\vartheta(0) = 0$ since the solution of (1.3.5) is unique in this case. Moreover, if $|\vartheta(0)| = \pi$, we also obtain the conclusion by noting the symmetry of the solutions. In the case that $0 < |\vartheta(0)| < \pi$, there are still two possibilities on ϑ since there are two solutions to (1.3.5). One solution is the desired borderline angle function; in this case the function $|\vartheta|$ is strictly decreasing. The other one corresponds to the case that $|\vartheta|$ is strictly increasing. However, since $\vartheta(0) \in (-\pi, \pi)$, Lemma 1.3.6 and the equality in (1.4.3) imply that $\vartheta(c) \in [-\pi, \pi]$. In addition, by the equality in (1.4.4) and $[\![\vartheta(0)]\!] = \vartheta(0) \in (-\pi, \pi)$, we find that $|[\![\vartheta(c)]\!]| \leq |\vartheta(0)|$. In particular, $|[\![\vartheta(c)]\!]| < \pi$, and hence $[\![\vartheta(c)]\!] = \vartheta(c)$. Consequently, $|\vartheta(c)| \leq |\vartheta(0)|$. Thus the function $|\vartheta|$ is decreasing, and hence ϑ is nothing but the borderline angle function with initial angle $\vartheta(0)$. The proof is now complete. $\square$

We are now in a position to prove the (weak) rescaled convergence. We prove it in terms of the tangential angle.

Proposition 1.4.2. *Let $\{\gamma_\varepsilon\}_\varepsilon$ be a sequence as in Theorem 1.2.3. Let $\tilde{\gamma}_\varepsilon$ be the arc length parameterization of γ_ε. Let $\vartheta_{\tilde{\gamma}_\varepsilon}$ be the unique tangential angle such that $\vartheta_{\tilde{\gamma}_\varepsilon}(0) = \theta_0^\varepsilon$. Fix any $c > 0$. Define the rescaled tangential angle $\hat{\vartheta}_\varepsilon \in C^\infty([0, c])$ as $\hat{\vartheta}_\varepsilon(\hat{s}) := \vartheta_{\tilde{\gamma}_\varepsilon}(\varepsilon\hat{s})$ for any small ε (so that $\varepsilon c < l_\varepsilon$). Then, for any subsequence of $\{\hat{\vartheta}_\varepsilon\}_\varepsilon$ there is a subsequence $\{\hat{\vartheta}_{\varepsilon'}\}_{\varepsilon'}$ such that $\hat{\vartheta}_{\varepsilon'}$ converges to some $\vartheta_* \in H^1(0, c)$ weakly in $H^1(0, c)$.*

Moreover, if $|\theta_0| < \pi$, the function ϑ_* is the (unique) borderline angle function with initial angle θ_0 (in the sense of Definition 1.3.12), and hence the convergence is valid in the full convergence sense. If $|\theta_0| = \pi$, up to the addition of a constant in $\{0, \pm 2\pi\}$, the function ϑ_* is either the borderline angle function with initial angle π or $-\pi$.

Proof. We decompose the curve $\tilde{\gamma}_\varepsilon(s)$ into the part $s \in [0, c\varepsilon]$ and $s \in [c\varepsilon, L_\varepsilon]$ (where $L_\varepsilon = \mathcal{L}[\gamma_\varepsilon]$). By Lemma 1.3.2, the energy $\mathcal{F}_\varepsilon[\gamma_\varepsilon]$ is also decomposed as

$$\mathcal{F}_\varepsilon[\gamma_\varepsilon] = \mathcal{F}_\varepsilon[\tilde{\gamma}_\varepsilon|_{[0,c\varepsilon]}] + \mathcal{F}_\varepsilon[\tilde{\gamma}_\varepsilon|_{[c\varepsilon,L_\varepsilon]}]. \tag{1.4.5}$$

By Lemma 1.3.1, the energy convergence (1.3.4) holds. Moreover, since $\hat{\vartheta}_\varepsilon(0) = \theta_0^\varepsilon \to \theta_0$ and

$$\mathcal{F}_\varepsilon[\gamma_\varepsilon] \geq \mathcal{F}_\varepsilon[\tilde{\gamma}_\varepsilon|_{[0,c\varepsilon]}] = \int_0^c \left(|\partial_{\hat{s}}\hat{\vartheta}_\varepsilon|^2 + (1 - \cos\hat{\vartheta}_\varepsilon)\right) d\hat{s} \geq \int_0^c |\partial_{\hat{s}}\hat{\vartheta}_\varepsilon|^2 d\hat{s},$$

the sequence $\{\hat{\vartheta}_\varepsilon\}_\varepsilon$ is bounded in $H^1(0,c)$ as $\varepsilon \to 0$. Thus, for any subsequence there is a subsequence (without relabeling) such that $\hat{\vartheta}_\varepsilon$ weakly converges to some function $\vartheta_* \in H^1(0,c)$ as $\varepsilon \to 0$, and hence $\hat{\vartheta}_\varepsilon$ uniformly converges to ϑ_* in $[0,c]$ by the Sobolev embedding.

We next prove

$$\liminf_{\varepsilon\to 0} \mathcal{F}_\varepsilon[\tilde{\gamma}_\varepsilon|_{[c\varepsilon,L_\varepsilon]}] \geq 8\sqrt{2}\left(\sin^2\frac{[\![\vartheta_*(c)]\!]}{4} + \sin^2\frac{\theta_1}{4}\right). \tag{1.4.6}$$

Notice that $\vartheta_{\tilde{\gamma}_\varepsilon}(c\varepsilon)$ $(= \hat{\vartheta}_\varepsilon(c))$ converges to $\vartheta_*(c)$ as $\varepsilon \to 0$ since $\hat{\vartheta}_\varepsilon$ uniformly converges to ϑ_* in $[0,c]$. Moreover, by Proposition 1.3.8 and Lemma 1.3.9, there exists a sequence of $s_\varepsilon \in [c\varepsilon, L_\varepsilon]$ such that $[\![\vartheta_{\gamma_\varepsilon}(s_\varepsilon)]\!] \to 0$ as $\varepsilon \to 0$. Then, by Lemma 1.3.2, Lemma 1.3.5, and Lemma 1.3.6, we find that

$$\begin{aligned} &\mathcal{F}_\varepsilon[\tilde{\gamma}_\varepsilon|_{[c\varepsilon,L_\varepsilon]}] = \mathcal{F}_\varepsilon[\tilde{\gamma}_\varepsilon|_{[c\varepsilon,s_\varepsilon]}] + \mathcal{F}_\varepsilon[\tilde{\gamma}_\varepsilon|_{[s_\varepsilon,L_\varepsilon]}] \\ \geq\ & 8\sqrt{2}\left|\sin^2\frac{[\![\vartheta_{\tilde{\gamma}_\varepsilon}(c\varepsilon)]\!]}{4} - \sin^2\frac{[\![\vartheta_{\tilde{\gamma}_\varepsilon}(s_\varepsilon)]\!]}{4}\right| + 8\sqrt{2}\left|\sin^2\frac{[\![\vartheta_{\tilde{\gamma}_\varepsilon}(s_\varepsilon)]\!]}{4} - \sin^2\frac{[\![\vartheta_{\tilde{\gamma}_\varepsilon}(L_\varepsilon)]\!]}{4}\right|. \end{aligned}$$

Since $|[\![\vartheta_{\tilde{\gamma}_\varepsilon}(L_\varepsilon)]\!]| \to \theta_1$, taking the limit $\varepsilon \to 0$, we obtain (1.4.6).

Combining the energy limit (1.3.4) with (1.4.5) and (1.4.6), we have

$$\begin{aligned} 8\sqrt{2}\left(\sin^2\frac{\theta_0}{4} - \sin^2\frac{[\![\vartheta_*(c)]\!]}{4}\right) &\geq \limsup_{\varepsilon\to 0} \mathcal{F}_\varepsilon[\gamma_\varepsilon|_{[0,c\varepsilon]}] \\ &= \limsup_{\varepsilon\to 0} \int_0^c \left(|\partial_{\hat{s}}\hat{\vartheta}_\varepsilon|^2 + (1-\cos\hat{\vartheta}_\varepsilon)\right) d\hat{s}. \end{aligned}$$

Moreover, since $\hat{\vartheta}_\varepsilon$ converges to ϑ_* weakly in $H^1(0,c)$, we also have

$$\liminf_{\varepsilon\to 0} \int_0^c \left(|\partial_{\hat{s}}\hat{\vartheta}_\varepsilon|^2 + (1-\cos\hat{\vartheta}_\varepsilon)\right) d\hat{s} \geq \int_0^c \left(|\vartheta_*'|^2 + (1-\cos\vartheta_*)\right).$$

Therefore, the function ϑ_* satisfies the assumption of Lemma 1.4.1, which implies the conclusion. The proof is complete. □

Since the endpoint $\gamma_\varepsilon(0) = (0,0)$ is fixed, we find that any sequence of minimizers converges to the borderline elastica in a weak sense.

1.4.2 Almost straightness except the endpoints

In this subsection, we prove (2) of Theorem 1.2.3 by using the above weak convergence. We improve the regularity of the weak convergence from the next subsection.

Since $|(\cos\theta, \sin\theta) - (1,0)| \leq |\theta|$ for $\theta \in [-\pi,\pi]$, we find that

$$|\partial_s\tilde{\gamma}_\varepsilon(s) - (1,0)| \leq |[\![\vartheta_{\tilde{\gamma}_\varepsilon}(s)]\!]|.$$

Hence, it suffices to prove the following proposition.

Proposition 1.4.3. *Let $\{\gamma_\varepsilon\}_\varepsilon$ be a sequence in Theorem 1.2.3. Let $\tilde{\gamma}_\varepsilon$ be the arc length parameterization of γ_ε. Fix any $c > 0$. Let $K_{c\varepsilon} = [c\varepsilon, L_\varepsilon - c\varepsilon]$ for any small ε (so that*

$\varepsilon < l_\varepsilon/c$), where $L_\varepsilon = \mathcal{L}[\gamma_\varepsilon]$. Then

$$\limsup_{\varepsilon\to 0} \max_{s\in K_{c\varepsilon}} |[\![\vartheta_{\tilde{\gamma}_\varepsilon}(s)]\!]| \leq 4e^{-\frac{c}{\sqrt{2}}}.$$

Proof. By Proposition 1.4.2 and symmetry, the angles $[\![\vartheta_{\tilde{\gamma}_\varepsilon}(c\varepsilon)]\!]$ and $[\![\vartheta_{\tilde{\gamma}_\varepsilon}(L_\varepsilon - c\varepsilon)]\!]$ converge as $\varepsilon \to 0$, and moreover

$$\lim_{\varepsilon\to 0} |[\![\vartheta_{\tilde{\gamma}_\varepsilon}(c\varepsilon)]\!]| = |[\![\vartheta_B^{\theta_0}(c)]\!]| \leq |\theta_0|, \quad \lim_{\varepsilon\to 0} |[\![\vartheta_{\tilde{\gamma}_\varepsilon}(L_\varepsilon - c\varepsilon)]\!]| = |[\![\vartheta_B^{\theta_1}(c)]\!]| \leq |\theta_1|,$$

where $\vartheta_B^{\theta_i}$ is the borderline angle function with initial angle θ_i for $i = 0, 1$. Notice that

$$|[\![\vartheta_B^{\theta_i}(c)]\!]| \leq |[\![\vartheta_B^{\pi}(c)]\!]| = |\bar{\varphi}_\pm(c) \mp 2\pi| = |\bar{\varphi}_\pm(-c)| = 4\arctan\left(e^{-\frac{c}{\sqrt{2}}}\right).$$

by the representation (1.3.6). Since $\arctan X \leq X$ for $X \geq 0$, we see that, for $i = 0, 1$,

$$|[\![\vartheta_B^{\theta_i}(c)]\!]| \leq 4e^{-\frac{c}{\sqrt{2}}}.$$

Thus it suffices to prove that

$$\limsup_{\varepsilon\to 0} \max_{s\in K_{c\varepsilon}} |[\![\vartheta_{\tilde{\gamma}_\varepsilon}(s)]\!]| = \max\{|[\![\vartheta_B^{\theta_0}(c)]\!]|, |[\![\vartheta_B^{\theta_1}(c)]\!]|\} =: \theta_c^*.$$

Note that $\theta_c^* \in (0, \pi)$. We prove it by contradiction, so we assume that there would exist $\delta \in (0, \pi - \theta_c^*)$, a sequence $\varepsilon_j \to 0$, and $s_j \in \mathring{K}_{c\varepsilon_j} := (c\varepsilon_j, L_{\varepsilon_j} - c\varepsilon_j)$ such that

$$\lim_{j\to\infty} |[\![\vartheta_{\tilde{\gamma}_{\varepsilon_j}}(s_j)]\!]| = \theta_c^* + \delta \in (\theta_c^*, \pi).$$

By taking a subsequence if necessary, we may assume that s_j converges. Then, by Proposition 1.3.8 and Lemma 1.3.9, there is a sequence of $s_j^* \in \mathring{K}_{c\varepsilon_j}$ such that $s_j^* \neq s_j$ and $[\![\vartheta_{\tilde{\gamma}_{\varepsilon_j}}(s_j^*)]\!] \to 0$ as $j \to \infty$. We then cut the arc length interval $[0, L_{\varepsilon_j}]$ at the points $c\varepsilon_j$, s_j, s_j^* and $L_{\varepsilon_j} - c\varepsilon_j$ and decompose the curve γ_{ε_j} into the corresponding five parts. (Note that the order of s_j and s_j^* may change as $j \to \infty$.) By using Lemma 1.3.5 and Lemma 1.3.6 for each of the parts and applying Lemma 1.3.2, we find that

$$\liminf_{j\to\infty} \mathcal{F}_{\varepsilon_j}[\gamma_{\varepsilon_j}] \geq 8\sqrt{2}\left(\sin^2\frac{\theta_0}{4} + \sin^2\frac{\theta_1}{4} - 2\sin^2\frac{\theta_c^*}{4} + 2\sin^2\frac{\theta_c^* + \delta}{4}\right).$$

However, this contradicts the energy convergence (1.3.4). The proof is complete. □

1.4.3 Jacobi elliptic functions and elastica equation

In the rest of this section we improve the regularity of the weak convergence in Proposition 1.4.2. To this end we use some properties of elliptic functions. In this subsection we briefly recall some properties of elliptic functions, and expressions of solutions to the elastica equation by elliptic functions.

We first recall that any minimizer satisfies the following elastica equation.

Proposition 1.4.4 (e.g. [11, 72]). *Let γ_ε be any minimizer of $\mathcal{E}_\varepsilon$ in $\mathcal{A}$ (with any boundary condition) and $\tilde{\gamma}$ be the arc length parameterization. Then its signed curvature $\kappa = \partial_s\vartheta_{\tilde{\gamma}}$*

satisfies

$$\varepsilon^2(2\partial_s^2\kappa + \kappa^3) - \kappa = 0. \tag{1.4.7}$$

It is well-known that any solution of the above equation is solved in terms of the Jacobi elliptic functions. We briefly recall the definitions and some properties of elliptic functions (see e.g. [42] for details).

Let $F(\xi, k)$ be the incomplete elliptic integral of the first kind of modulus $k \in (0,1)$:

$$F(\xi; k) := \int_0^\xi \frac{dt}{\sqrt{1-t^2}\sqrt{1-k^2t^2}}.$$

Let $K(k)$ be the complete elliptic integral of the first kind, i.e., $K(k) := F(1; k)$.

The function $\mathrm{sn}(x, k)$ is defined so that $x = F(\mathrm{sn}(x,k); k)$ for $|x| \le K(k)$, and $\mathrm{sn}(x,k) = -\mathrm{sn}(x + 2K(k), k)$ for $x \in \mathbb{R}$. Note that $\mathrm{sn}(\cdot, k)$ is an odd $2K(k)$-antiperiodic function and, in $[-K(k), K(k)]$, strictly increasing from -1 to 1.

The function $\mathrm{cn}(x, k)$ is defined as a unique smooth function such that $\mathrm{cn}(0, k) = 1$ and $\mathrm{cn}^2(x,k) + \mathrm{sn}^2(x,k) = 1$ for $x \in \mathbb{R}$. Note that $\mathrm{cn}(\cdot, k)$ is an even $2K(k)$-antiperiodic function and, in $[0, 2K(k)]$, strictly decreasing from 1 to -1.

The function $\mathrm{dn}(x, k)$ is defined as a unique smooth function such that $\mathrm{dn}(0, k) = 1$ and $\mathrm{dn}^2(x,k) + k^2\,\mathrm{sn}^2(x,k) = 1$. Note that $\mathrm{dn}(\cdot, k)$ is a positive even $2K(k)$-periodic function and, in $[0, K(k)]$, strictly decreasing from 1 to $\sqrt{1-k^2}$.

For $k = 0$, the functions sn, cn, dn are interpreted as sin, cos, 1, respectively. For $k = 1$, they are interpreted as tanh, sech, sech, respectively.

The following derivative formulae hold: for $k \in [0,1]$,

$$\mathrm{sn}' = \mathrm{cn}\,\mathrm{dn}, \quad \mathrm{cn}' = -\mathrm{sn}\,\mathrm{dn}, \quad \mathrm{dn}' = -k^2\,\mathrm{sn}\,\mathrm{cn}\,. \tag{1.4.8}$$

We finally recall that any solution to the equation (1.4.7) is expressed by an elliptic function.

Proposition 1.4.5 (e.g. [45]). *For any given $\varepsilon > 0$ and initial values $\kappa(0) = a_0$ and $\partial_s\kappa(0) = b_0$, the equation (1.4.7) is uniquely solved in $\mathbb{R}$. Moreover, the solution is given by either*

(1) $\kappa(s) = A\,\mathrm{cn}(\alpha s + \beta, k)$, where $k \in [0,1]$ is modulus, $A\,\mathrm{cn}(\beta, k) = a_0$, $-A\alpha\,\mathrm{sn}(\beta,k)\,\mathrm{dn}(\beta,k) = b_0$, $A^2 = 4k^2\alpha^2$, and $\varepsilon^2(A^2 - 2\alpha^2) = 1$, or

(2) $\kappa(s) = A\,\mathrm{dn}(\alpha s + \beta, k)$, where $k \in [0,1]$ is modulus, $A\,\mathrm{dn}(\beta,k) = a_0$, $-A\alpha p^2\,\mathrm{sn}(\beta,k)\,\mathrm{cn}(\beta,k) = b_0$, $A^2 = 4\alpha^2$, and $\varepsilon^2(A^2 - 2\alpha^2k^2) = 1$.

If $(a_0^2 - 2\varepsilon^{-2})a_0^2 + 4b_0^2 \ge 0$ then the solution is (1), and otherwise (2).

Since $\|\mathrm{cn}\|_\infty = \|\mathrm{dn}\|_\infty = 1$, the above solution κ satisfies $\|\kappa\|_\infty \le |A|$. We call the number $|A|$ *virtual maximum* of κ, since the maximum $|A|$ may not be attained in a finite interval.

Remark 1.4.6. Since ε^2 is positive, in the case of cn, the modulus has a lower bound as $k \in (1/\sqrt{2}, 1]$.

1.4.4 Boundedness of higher derivatives

For improving the regularity of the weak convergence in Proposition 1.4.2, it suffices to prove that any higher order derivative of the rescaled tangential angle is (locally) bounded as $\varepsilon \to 0$. We prove the boundedness by using the expression by elliptic functions.

Proposition 1.4.7. *Let $c > 0$ and $\hat{\vartheta}_\varepsilon \in C^\infty([0,c])$ be the rescaled tangential angle function in Proposition 1.4.2 for $\varepsilon > 0$ with $c\varepsilon < l_\varepsilon$. Then for any positive integer k the sequence of $\hat{\vartheta}_\varepsilon$ is bounded in $C^k([0,c])$ as $\varepsilon \to 0$. Accordingly, the H^1-weak convergence in Proposition 1.4.2 is improved to the C^∞-convergence.*

Proof. Let $\kappa_\varepsilon(s) = \partial_s \vartheta_{\tilde{\gamma}_\varepsilon}(s)$ be the signed curvature of the original minimizer γ_ε. Recall that κ_ε satisfies (1.4.7). Hence, the rescaled curvature $\hat{\kappa}_\varepsilon$ defined by

$$\hat{\kappa}_\varepsilon(\hat{s}) := \partial_{\hat{s}} \hat{\vartheta}_\varepsilon(\hat{s}) = \varepsilon \kappa_\varepsilon(\varepsilon \hat{s})$$

satisfies the normalized elastica equation:

$$2\partial_{\hat{s}}^2 \hat{\kappa}_\varepsilon + \hat{\kappa}_\varepsilon^3 - \hat{\kappa}_\varepsilon = 0.$$

By Proposition 1.4.5, the rescaled curvature $\hat{\kappa}_\varepsilon$ is of the form either (1) or (2) with $\varepsilon = 1$. Thus, it suffices to prove that the virtual maximum $|\hat{A}_\varepsilon|$ of $\hat{\kappa}_\varepsilon$ and the coefficient $\hat{\alpha}_\varepsilon$ of the variable is bounded as $\varepsilon \to 0$; in fact, by the derivative formulae (1.4.8) and the fact that all the elliptic functions and modulus $\hat{k}_\varepsilon$ are bounded above by 1, any derivative of $\hat{\kappa}$ is bounded by a polynomial of $|\hat{A}_\varepsilon|$ and $|\hat{\alpha}_\varepsilon|$. Moreover, by the relations in Proposition 1.4.5 (with $\varepsilon = 1$), the boundedness of $|\hat{A}_\varepsilon|$ and of $|\hat{\alpha}_\varepsilon|$ are equivalent. Hence, it suffices to prove that $|\hat{A}_\varepsilon|$ is bounded as $\varepsilon \to 0$.

We now assume the contradiction that a subsequence (not relabeled) of the virtual maximum $|\hat{A}_\varepsilon|$ of $\hat{\kappa}_\varepsilon$ diverges to infinity as $\varepsilon \to 0$. We prove that this assumption contradicts the fact that the sequence of $\hat{\kappa}_\varepsilon$ is bounded in $L^2(0,c)$ (by Proposition 1.4.2). By the relations of constants in Proposition 1.4.5 for $\hat{\kappa}_\varepsilon$, the assumption that $|\hat{A}_\varepsilon| \to \infty$ implies that only the case (1) occurs for any small ε. Hence, the following relations hold:

$$\hat{\kappa}_\varepsilon(\hat{s}) = \hat{A}_\varepsilon \operatorname{cn}(\hat{\alpha}_\varepsilon \hat{s} + \hat{\beta}_\varepsilon, \hat{k}_\varepsilon), \quad \hat{k}_\varepsilon^2 = \frac{\hat{A}_\varepsilon^2}{2(\hat{A}_\varepsilon^2 - 1)}, \quad \hat{\alpha}_\varepsilon^2 = \frac{\hat{A}_\varepsilon^2 - 1}{2}.$$

Then we calculate

$$\|\hat{\kappa}_\varepsilon\|_{L^2(0,c)}^2 = \frac{\hat{A}_\varepsilon^2}{|\hat{\alpha}_\varepsilon|} \int_{\hat{\beta}_\varepsilon}^{\hat{\alpha}_\varepsilon c + \hat{\beta}_\varepsilon} |\operatorname{cn}(x, \hat{k}_\varepsilon)|^2 dx.$$

Since $\hat{\alpha}_\varepsilon \to \infty$ and $\hat{k}_\varepsilon \to 1/\sqrt{2}$, for any small ε the interval $[\hat{\beta}_\varepsilon, \hat{\alpha}_\varepsilon c + \hat{\beta}_\varepsilon]$ includes one period $4K(\hat{k}_\varepsilon)$ of $\operatorname{cn}(x, \hat{k}_\varepsilon)$:

$$\int_{\hat{\beta}_\varepsilon}^{\hat{\alpha}_\varepsilon c + \hat{\beta}_\varepsilon} |\operatorname{cn}(x, \hat{k}_\varepsilon)|^2 dx \geq \int_0^{4K(\hat{k}_\varepsilon)} |\operatorname{cn}(x, \hat{k}_\varepsilon)|^2 dx.$$

By the dominated convergence theorem and $K(\hat{k}_\varepsilon) \to K(1/\sqrt{2})$, the right-hand term converges to a positive value, namely,

$$\int_0^{4K(1/\sqrt{2})} |\operatorname{cn}(x, 1/\sqrt{2})|^2 dx.$$

Since $\hat{A}_\varepsilon^2/|\hat{\alpha}_\varepsilon| \to \infty$, the L^2-norm $\|\hat{\kappa}_\varepsilon\|_{L^2(0,c)}$ diverges to infinity. This is a contradiction.

The improvement of the regularity of convergence is obvious since, by the boundedness of higher order derivatives, the Arzelà-Ascoli theorem implies the desired C^∞-convergence. The proof is now complete. □

We shall complete the proof of Theorem 1.2.3.

Proof of Theorem 1.2.3. Let $\{\gamma_\varepsilon\}_\varepsilon$ be any sequence of minimizers as in the assumption. For the part (1), since the position of $\gamma_\varepsilon(0)$ is fixed at the origin, it suffices to prove (1) in terms of the tangential angles. This follows by Proposition 1.4.2 and Proposition 1.4.7. The almost straightness part (2) is proved in Proposition 1.4.3, which is also in terms of the tangential angles. The proof is now complete. □

1.5 Qualitative properties

In this section we prove Theorem 1.2.10 and Theorem 1.2.11 by using Theorem 1.2.3. In this part we also use expressions of the curvatures of solutions by elliptic functions.

1.5.1 Self-intersection and inflection point

We first confirm that any minimizer has no self-intersection in the limit $\varepsilon \to 0$.

Proposition 1.5.1. *Let $\{\gamma_\varepsilon\}_\varepsilon$ be any sequence of minimizers as in Theorem 1.2.3. Then there is $\bar{\varepsilon} > 0$ such that for any $\varepsilon \in (0, \bar{\varepsilon})$ the curve γ_ε has no self-intersection.*

Proof. Fix sufficiently large $c > 0$ so that $4e^{-\frac{c}{\sqrt{2}}} < 1$ and the x-component of $\gamma_B^{\theta_i}(c)$ is positive for $i = 0, 1$, where $\gamma_B^{\theta_i}$ is the borderline elastica with initial angle θ_i. Decompose the domain of the arc length parameterized curve $\tilde{\gamma}_\varepsilon$ into $[0, c\varepsilon]$, $[c_\varepsilon, L_\varepsilon - c\varepsilon]$, and $[L_\varepsilon - c\varepsilon, L_\varepsilon]$. Then, for any small ε, the curve $\tilde{\gamma}_\varepsilon$ has no self-intersection in each of the parts by Theorem 1.2.3. Moreover, for any small ε, the parts $\tilde{\gamma}_\varepsilon|_{(0,c\varepsilon)}$, $\tilde{\gamma}_\varepsilon|_{(c_\varepsilon, L_\varepsilon - c\varepsilon)}$, $\tilde{\gamma}_\varepsilon|_{(L_\varepsilon - c\varepsilon, L_\varepsilon)}$ are respectively included in the sets

$$\{x < \tilde{x}_\varepsilon(c\varepsilon)\}, \quad \{\tilde{x}_\varepsilon(c\varepsilon) < x < \tilde{x}_\varepsilon(L_\varepsilon - c\varepsilon)\}, \quad \{\tilde{x}_\varepsilon(L_\varepsilon - c\varepsilon) < x\},$$

where $\tilde{x}_\varepsilon$ denotes the x-component of $\tilde{\gamma}_\varepsilon$. This implies that there is no self-intersection in the whole of $\tilde{\gamma}_\varepsilon$ for small ε. □

We next determine the number of the inflection points, i.e., the sign changes of the curvature. Recall that the curvatures of all nontrivial (non-straight) solution curves are represented by non-zero elliptic functions, and hence their sign changes are well-defined if $|\theta_0| + |\theta_1| > 0$. In particular, all the zeroes of the curvature (except the endpoints) are nothing but the sign changes.

The key step is to prove that the number of the inflection points are bounded above by one for any small ε. This upper bound is valid even for the critical cases $|\theta_0| = \pi$ and $|\theta_1| = \pi$.

Proposition 1.5.2. *Let $\{\gamma_\varepsilon\}_\varepsilon$ be any sequence of minimizers as in Theorem 1.2.3. Suppose that $|\theta_0| + |\theta_1| > 0$. Then there is $\bar{\varepsilon} > 0$ such that for any $\varepsilon \in (0, \bar{\varepsilon})$ the curve γ_ε has at most one inflection point.*

Proof. By symmetry, we may assume that $\theta_0 > 0$ without loss of generality. We suppose the contradiction that there is a sequence $\varepsilon_j \to 0$ such that γ_{ε_j} has at least two inflection points. Recall that the signed curvature κ_ε of $\tilde{\gamma}_\varepsilon$ is represented by an elliptic function as in Proposition 1.4.5. Since κ_{ε_j} has a zero (and $\kappa_\varepsilon \not\equiv 0$ by $\theta_0 \neq 0$), it is of the form

$$\kappa_{\varepsilon_j}(s) = A_j \,\mathrm{cn}(\alpha_j s + \beta_j, k_j),$$

where $k_j \in (0,1)$, $A_j \neq 0$, and $\alpha_j \neq 0$. We take the smallest two zeroes $s_0^j, s_2^j \in (0, L_{\varepsilon_j})$ of κ_{ε_j} with $s_0^j < s_2^j$. By the $2K$-antiperiodicity of cn, we find that

$$s_2^j = s_0^j + 2K(k_j)/|\alpha_j|.$$

We now extend the curvature function $\kappa_{\varepsilon_j}(s)$ as a $2K(k_j)$-antiperiodic function defined for any $s \in \mathbb{R}$ by using the elliptic function cn; we use the same notation κ_{ε_j} for the extended curvature. Let

$$s_{\pm 1}^j := s_0^j \pm K(k_j)/|\alpha_j|.$$

By the periodicity of cn, the curvature κ_{ε_j} takes its maximum or minimum at $s_{\pm 1}^j$. Take arbitrary large $c > 0$. Since $\theta_0 > 0$, Theorem 1.2.3 implies that the rescaled curvature $\hat{\kappa}_\varepsilon(\hat{s}) := \varepsilon \kappa_\varepsilon(\varepsilon \hat{s})$, defined for $\hat{s} \in [0, c]$, smoothly converges to $\partial_s \vartheta_B^{\theta_0}$, where

$$\partial_s \vartheta_B^{\theta_0}(\hat{s}) = -\sqrt{2}\,\mathrm{sech}\left(\frac{\hat{s} + s_{\theta_0}}{\sqrt{2}}\right),$$

and $s_{\theta_0} > 0$ is a unique constant. Thus, for any small ε_j, the curvature κ_{ε_j} is negative and increasing in $[0, c\varepsilon_j]$. Hence, for any small ε_j the interval $[0, c\varepsilon_j]$ is included in $[s_{-1}^j, s_0^j]$. In particular, $s_0^j > c\varepsilon_j$. Moreover, we have $s_0^j - s_{-1}^j \geq c\varepsilon_j$, and hence $s_2^j - s_1^j \geq c\varepsilon_j$. Since $s_2^j < L_{\varepsilon_j}$, we also find that $s_1^j < L_{\varepsilon_j} - c\varepsilon_j$. Combining with $s_0^j > c\varepsilon_j$, we see that $[s_0^j, s_1^j] \subset [c\varepsilon_j, L_{\varepsilon_j} - c\varepsilon_j]$. Noting the periodicity of cn, we have

$$\begin{aligned}
2\left(\limsup_{j\to\infty} \max_{s \in [c\varepsilon_j, L_{\varepsilon_j} - c\varepsilon_j]} |\vartheta_{\tilde{\gamma}_{\varepsilon_j}}(s)|\right) &\geq \limsup_{j\to\infty} |\vartheta_{\tilde{\gamma}_{\varepsilon_j}}(s_1^j) - \vartheta_{\tilde{\gamma}_{\varepsilon_j}}(s_0^j)| \\
&= \limsup_{j\to\infty} |\vartheta_{\tilde{\gamma}_{\varepsilon_j}}(s_0^j) - \vartheta_{\tilde{\gamma}_{\varepsilon_j}}(s_{-1}^j)| \\
&\geq \lim_{j\to\infty} |\vartheta_{\tilde{\gamma}_{\varepsilon_j}}(c\varepsilon_j) - \vartheta_{\tilde{\gamma}_{\varepsilon_j}}(0)| \\
&= |\vartheta_B^{\theta_0}(c) - \vartheta_B^{\theta_0}(0)| = \theta_0 - \vartheta_B^{\theta_0}(c).
\end{aligned}$$

The last term tends to $\theta_0 > 0$ as $c \to \infty$. This contradicts (2) in Theorem 1.2.3. □

By using the above upper bound, we determine the exact number of the inflection points providing the generic angle condition.

Proposition 1.5.3. *Let $\{\gamma_\varepsilon\}_\varepsilon$ be any sequence of minimizers as in Theorem 1.2.3. Suppose the generic angle condition (1.2.3). If $\theta_0\theta_1 \geq 0$ (resp. $\theta_0\theta_1 < 0$), then there is $\bar{\varepsilon} > 0$ such that for any $\varepsilon \in (0, \bar{\varepsilon})$ the curve γ_ε has exact one inflection point (resp. no inflection point).*

Proof. By symmetry, we may assume that $\theta_0 \in (0, \pi)$ without loss of generality. Let κ_ε denote the curvature of a minimizer γ_ε

In the case that $\theta_0\theta_1 < 0$, we easily find that $\kappa_\varepsilon(0)\kappa_\varepsilon(1) < 0$ for any small ε by (1) in Theorem 1.2.3. Hence, κ_ε has at least one sign change for any small ε. By Proposition 1.5.2, κ_ε has exact one sign change.

In the case that $\theta_0\theta_1 > 0$, we similarly find that $\kappa_\varepsilon(0)\kappa_\varepsilon(1) > 0$ for any small ε. Hence, κ_ε has either no sign change or at least two sign changes. By Proposition 1.5.2, κ_ε has no sign change.

We finally prove that, if $\theta_0\theta_1 = 0$, i.e., $\theta_1 = 0$, then the curvature has (at least) one sign change for any small ε. We notice that, by (1) in Theorem 1.2.3 and symmetry, the straightness (2) in Theorem 1.2.3 extends to the endpoint $(l_\varepsilon, 0)$, i.e., for any $c > 0$,

$$\limsup_{\varepsilon \to 0} \max_{s \in [c\varepsilon, L_\varepsilon]} |\partial_s \tilde{\gamma}_\varepsilon(s) - (1,0)| \leq 4e^{-c/\sqrt{2}}.$$

Let $c > 0$ be sufficiently large so that for any small ε the x-component of $\partial_s\tilde{\gamma}_\varepsilon$ is positive in $[c\varepsilon, L_\varepsilon]$. By (1) in Theorem 1.2.3, the assumption that $\theta_0 > 0$ implies that the y-components of $\tilde{\gamma}_\varepsilon(c\varepsilon)$ and $\partial_s\tilde{\gamma}_\varepsilon(c\varepsilon)$ are positive for any small ε. Then the curve $\tilde{\gamma}_\varepsilon|_{[c\varepsilon, L_\varepsilon]}$ is represented as the graph of a function u_ε defined on an interval $[a_\varepsilon, b_\varepsilon]$ such that

$$u_\varepsilon(a_\varepsilon) > 0, \quad u_\varepsilon'(a_\varepsilon) > 0, \quad u_\varepsilon(b_\varepsilon) = 0, \quad u_\varepsilon'(b_\varepsilon) = 0.$$

By these boundary conditions, the second derivative u_ε'' must have a zero in $(a_\varepsilon, b_\varepsilon)$. Since a zero of u_ε'' corresponds to a sign change of κ_ε, we find that κ_ε has a sign change for any small ε. By Proposition 1.5.2, κ_ε has exact one sign change. The proof is now complete. □

Remark 1.5.4. In the last part of the above proof, the graph representation is essential. In particular, for any nonzero vectors $v_0, v_1 \in \mathbb{R}^2$, there is a non-graph (looping) smooth regular curve $\gamma : \bar{I} \to \mathbb{R}^2$ without inflection point such that $\gamma(0) = (0,0)$, $\gamma(1) = (1,0)$, $\dot{\gamma}(0) = v_0$ and $\dot{\gamma}(1) = v_1$.

Remark 1.5.5. The critical cases $|\theta_0| = \pi$ and $|\theta_1| = \pi$ are omitted since our results are not sufficient to determine the curvature signs near the endpoints. However, for the special case that $|\theta_0| = \pi$ and $\theta_1 = 0$ (or left and right reversed), the same argument implies that any minimizer has exact one inflection point for small ε.

We shall complete the proof of Theorem 1.2.10.

Proof of Theorem 1.2.10. Proposition 1.5.1 and Proposition 1.5.3 directly imply the self-intersection part and the inflection point part in Theorem 1.2.10, respectively. For example, if there would be a sequence $\varepsilon_j \to 0$ and a sequence of minimizers $\{\gamma_{\varepsilon_j}\}_j$ having self-intersections, then Proposition 1.5.1 would immediately imply a contradiction. Moreover, combining Proposition 1.5.3 with Theorem 1.2.3, we immediately obtain the part on the total variation of the tangential angle in Theorem 1.2.10. The proof is now complete. □

1.5.2 Uniqueness

We finally prove the uniqueness result as in Theorem 1.2.11.

For $l > 0$ and $\theta_0, \theta_1 \in \mathbb{R}$ with $\theta_0 \neq \theta_1$, we denote by $\tilde{\mathcal{A}}_{\theta_0,\theta_1,l}$ the set of all smooth constant speed curves joining $(0,0)$ to $(l,0)$ such that the tangential angles are strictly monotone functions from θ_0 to θ_1. Notice that $\tilde{\mathcal{A}}_{\theta_0,\theta_1,l} \subset \mathcal{A}_{\theta_0,\theta_1,l}$ if $\theta_0, \theta_1 \in [-\pi, \pi]$. We remark that the constraint of $\tilde{\mathcal{A}}_{\theta_0,\theta_1,l}$ completely fixes the variation of the tangential angle of a curve unlike our original clamped boundary condition.

The following statement is a key step for the proof.

Proposition 1.5.6. *Let $l > 0$ and $\theta_0, \theta_1 \in \mathbb{R}$ with $\theta_0 \neq \theta_1$. Then, for any $\varepsilon > 0$ the energy $\mathcal{E}_\varepsilon : \tilde{\mathcal{A}}_{\theta_0,\theta_1,l} \to (0,\infty)$ admits at most one minimizer in $\tilde{\mathcal{A}}_{\theta_0,\theta_1,l}$.*

To prove Proposition 1.5.6, we convexify our minimizing problem by using the radius of curvatures parameterized by the (monotone) tangential angles. As mentioned in Introduction, this idea is classical (see e.g. Born's stability analysis [9]).

Proof of Proposition 1.5.6. We may assume that $\theta_0 < \theta_1$ without loss of generality. For any $\gamma \in \tilde{\mathcal{A}}_{\theta_0,\theta_1,l}$, we can define the radius of curvature function $\rho : [\theta_0, \theta_1] \to (0,\infty)$ parameterized by the tangential angle as $\rho(\phi) := 1/\kappa(\vartheta_{\tilde{\gamma}}^{-1}(\phi))$, where $\tilde{\gamma}$ is the arc length parameterization of γ and $\kappa(s) = \partial_s \vartheta_{\tilde{\gamma}}(s)$. For any $\varepsilon > 0$ and $\gamma \in \tilde{\mathcal{A}}_{\theta_0,\theta_1,l}$, the energy $\mathcal{E}_\varepsilon$ is represented as

$$\mathcal{E}_\varepsilon[\gamma] = \int_0^{\mathcal{L}[\gamma]} \left(\varepsilon^2\kappa^2 + 1\right) ds = \int_{\theta_0}^{\theta_1} \left(\frac{\varepsilon^2}{\rho} + \rho\right) d\phi =: \tilde{\mathcal{E}}_\varepsilon[\rho].$$

In particular, for any fixed ε, the energy $\tilde{\mathcal{E}}_\varepsilon$ is strictly convex with respect to ρ since $\rho > 0$ and the integrand $f(\rho) = \varepsilon^2/\rho + \rho$ is strictly convex in $(0,\infty)$. Moreover, the constraints on the positions of γ at the endpoints

$$\int_0^{\mathcal{L}[\gamma]} \cos\vartheta_{\tilde{\gamma}} ds = l, \quad \int_0^{\mathcal{L}[\gamma]} \sin\vartheta_{\tilde{\gamma}} ds = 0,$$

are also expressed in terms of ρ as

$$\int_{\theta_0}^{\theta_1} \rho\cos\phi d\phi = l, \quad \int_{\theta_0}^{\theta_1} \rho\sin\phi d\phi = 0. \tag{1.5.1}$$

We now denote by $\tilde{\mathcal{R}}_{\theta_0,\theta_1,l}$ the set of all functions $\rho \in C^\infty([\theta_0,\theta_1];(0,\infty))$ satisfying (1.5.1). Clearly, the set $\tilde{\mathcal{R}}_{\theta_0,\theta_1,l}$ is convex. Moreover, by the above arguments, we find that the minimizing problem of $\mathcal{E}_\varepsilon : \tilde{\mathcal{A}}_{\theta_0,\theta_1,l} \to (0,\infty)$ is equivalent to the minimizing problem of $\tilde{\mathcal{E}}_\varepsilon : \tilde{\mathcal{R}}_{\theta_0,\theta_1,l} \to (0,\infty)$. More explicitly, there is a bijection Φ from $\tilde{\mathcal{R}}_{\theta_0,\theta_1,l}$ to $\tilde{\mathcal{A}}_{\theta_0,\theta_1,l}$ such that for any $\varepsilon > 0$ and $\rho \in \tilde{\mathcal{R}}_{\theta_0,\theta_1,l}$ the equality $\mathcal{E}_\varepsilon[\Phi(\rho)] = \tilde{\mathcal{E}}_\varepsilon[\rho]$ holds. In addition, we easily find that the energy $\tilde{\mathcal{E}}_\varepsilon : \tilde{\mathcal{R}}_{\theta_0,\theta_1,l} \to (0,\infty)$ admits at most one minimizer since $\tilde{\mathcal{E}}_\varepsilon$ is a strictly convex functional defined on a convex set. Therefore, we also find that the energy $\mathcal{E}_\varepsilon : \tilde{\mathcal{A}}_{\theta_0,\theta_1,l} \to (0,\infty)$ admits at most one minimizer. The proof is now complete. □

We shall complete the proof of Theorem 1.2.11.

Proof of Theorem 1.2.11. By Theorem 1.2.10, there is $\bar{\varepsilon} > 0$ such that, for any $\varepsilon \in (0, \bar{\varepsilon})$ and any minimizer of $\mathcal{E}_\varepsilon$ in $\mathcal{A}_{\theta_0^\varepsilon,\theta_1^\varepsilon,l_\varepsilon}$, the tangential angle is strictly monotone from θ_0^ε to θ_1^ε, that is, the curve γ_ε belongs to $\tilde{\mathcal{A}}_{\theta_0^\varepsilon,\theta_1^\varepsilon,l_\varepsilon}$. Since $\tilde{\mathcal{A}}_{\theta_0^\varepsilon,\theta_1^\varepsilon,l_\varepsilon}$ is included in $\mathcal{A}_{\theta_0^\varepsilon,\theta_1^\varepsilon,l_\varepsilon}$, any minimizer of $\mathcal{E}_\varepsilon$ in $\mathcal{A}_{\theta_0^\varepsilon,\theta_1^\varepsilon,l_\varepsilon}$ also minimizes $\mathcal{E}_\varepsilon$ in $\tilde{\mathcal{A}}_{\theta_0^\varepsilon,\theta_1^\varepsilon,l_\varepsilon}$. Therefore, Proposition 1.5.6 implies the desired uniqueness. □

Remark 1.5.7. As explained precisely in Appendix 1.A, for any fixed $l > 0$ and $\theta_0, \theta_1 \in [-\pi, \pi]$, the set of admissible curves $\mathcal{A}_{\theta_0,\theta_1,l}$ is decomposed into the sets $\mathcal{A}_{\theta_0,\theta_1,l,m}$ by winding number $m \in \mathbb{Z}$. For each m, the set $\mathcal{A}_{\theta_0,\theta_1,l,m}$ is defined to fix the variation of the tangential angle as

$$\vartheta_\gamma(1) - \vartheta_\gamma(0) = \theta_1 - \theta_0 + 2\pi m.$$

It is known that, for any inflectional elastica (i.e., cn-solution) of finite length, the range of its tangential angle is included in an interval of which width is less than 2π (see e.g. [11]). Hence, if $|m| > 1$, then $|\vartheta_\gamma(1) - \vartheta_\gamma(0)| \geq 2\pi$, and hence any critical point in $\mathcal{A}_{\theta_0,\theta_1,l,m}$ must be a non-inflectional elastica (i.e., dn-solution). Therefore, for $|m| > 1$, by the same convexification as above, we find that $\mathcal{E}_\varepsilon$ admits a unique minimizer in $\mathcal{A}_{\theta_0,\theta_1,l,m}$. For $|m| \leq 1$, there may be multiple candidates of minimizers.

1.6 Connection of inextensible and extensible problems

In this section we prove Theorem 1.2.12 and Theorem 1.2.14. The relation between the problems (1.2.2) and (1.2.4) is not so trivial at the level of global minimizers. As already mentioned, the case that $\theta_0 = \theta_1 = 0$ is omitted since it is not possible to express the inextensible problem (1.2.4) by the extensible problem (1.2.2).

1.6.1 Length of minimizers of the modified total squared curvature

We shall confirm some properties of the minimum values of energy and the lengths of minimizers in the extensible problem. Throughout this subsection, we fix $l > 0$ and $\theta_0, \theta_1 \in [-\pi, \pi]$ with $|\theta_0| + |\theta_1| > 0$, and denote $\mathcal{A}_{\theta_0,\theta_1,l}$ by $\mathcal{A}$ simply.

We first confirm basic properties of the minimum function

$$m(\varepsilon) = \min_{\gamma \in \mathcal{A}} \mathcal{E}_\varepsilon[\gamma].$$

We extend the function m to the origin as $m(0) = l$.

Proposition 1.6.1. *The minimum function m is strictly increasing and continuous in $[0, \infty)$. Moreover, m is locally semi-convex in $(0, \infty)$.*

Proof. First we note that $m(\varepsilon) > l$ for $\varepsilon > 0$ and $m(\varepsilon) \to l$ as $\varepsilon \to 0$ by Lemma 1.3.1 and the assumption that $|\theta_0| + |\theta_1| > 0$. Let $0 < \varepsilon_0 < \varepsilon_1$. By taking a minimizer $\gamma_1 \in \mathcal{A}$ of $\mathcal{E}_{\varepsilon_1}$, we find the strict monotonicity

$$m(\varepsilon_0) \leq \mathcal{E}_{\varepsilon_0}[\gamma_1] < \mathcal{E}_{\varepsilon_1}[\gamma_1] = m(\varepsilon_1).$$

Moreover, for any $\varepsilon > 0$ and $\delta \in \mathbb{R}$ with small $|\delta|$, taking any minimizer $\gamma_\varepsilon \in \mathcal{A}$ of $\mathcal{E}_\varepsilon$, we have

$$m(\varepsilon + \delta) \leq \mathcal{E}_{\varepsilon+\delta}[\gamma_\varepsilon] = \mathcal{B}[\gamma_\varepsilon]\delta^2 + 2\varepsilon\mathcal{B}[\gamma_\varepsilon]\delta + m(\varepsilon).$$

This relation and the monotonicity imply the remaining conclusions. □

We define a set-valued function $\tilde{L}$ as

$$\tilde{L}(\varepsilon) := \{\mathcal{L}[\gamma] \mid \gamma \in \mathcal{A} \text{ is a minimizer of } \mathcal{E}_\varepsilon\} \tag{1.6.1}$$

for $\varepsilon \in (0,\infty)$, and extend $\tilde{L}$ to the origin by $\tilde{L}(0) = \{l\}$. (Note that the definition depends on the constraints l, θ_0, θ_1.) By the existence of minimizers (Appendix 1.A), the set $\tilde{L}(\varepsilon)$ is nonempty for any $\varepsilon > 0$. Moreover, we notice that $\tilde{L}(\varepsilon) \subset (l,\infty)$ for $\varepsilon > 0$.

Proposition 1.6.2. *The set-valued function $\tilde{L}$ is nondecreasing in the sense that, for any $0 \le \varepsilon_0 < \varepsilon_1$, any $L_0 \in \tilde{L}(\varepsilon_0)$ and $L_1 \in \tilde{L}(\varepsilon_1)$ satisfy $L_0 \le L_1$.*

Proof. Fix such ε_0, ε_1, L_0 and L_1. The case $\varepsilon_0 = 0$ is obvious since $m(\varepsilon_1) > l$ so we assume that $\varepsilon_0 > 0$. By the definition of $\tilde{L}$, for $i = 0, 1$, there is a minimizer $\gamma_i \in \mathcal{A}$ of $\mathcal{E}_{\varepsilon_i}$ with length L_i. Then, noting the minimality of γ_0 and γ_1, we have

$$\mathcal{E}_{\varepsilon_0}[\gamma_0] \le \mathcal{E}_{\varepsilon_0}[\gamma_1], \quad \mathcal{E}_{\varepsilon_1}[\gamma_1] \le \mathcal{E}_{\varepsilon_1}[\gamma_0],$$

that is,

$$\varepsilon_0^2\mathcal{B}[\gamma_0] + L_0 \le \varepsilon_0^2\mathcal{B}[\gamma_1] + L_1, \quad \varepsilon_1^2\mathcal{B}[\gamma_1] + L_1 \le \varepsilon_1^2\mathcal{B}[\gamma_0] + L_0.$$

Combining these inequalities, we obtain $(\varepsilon_1^2 - \varepsilon_0^2)(L_1 - L_0) \ge 0$, which implies $L_0 \le L_1$. □

Recall that $\tilde{L}(\varepsilon)$ is nonempty for any ε. Moreover, as in [64], it is known that $\tilde{L}(\varepsilon)$ is a finite set. Hence, the following upper and lower envelopes of $\tilde{L}$, which are single-valued functions, are well-defined:

$$L^*(\varepsilon) := \max\{L \mid L \in \tilde{L}(\varepsilon)\}, \quad L_*(\varepsilon) := \min\{L \mid L \in \tilde{L}(\varepsilon)\}.$$

Proposition 1.6.3. *The function L^* (resp. L_*) is nondecreasing and upper (resp. lower) semicontinuous.*

Proof. Notice that the monotonicity in $[0,\infty)$ follows by Proposition 1.6.2. Moreover, the continuity at the origin follows by the length convergence in Proposition 1.3.8. Hence, it suffices to prove the semicontinuity for any fixed $\varepsilon > 0$. We prove only the upper semicontinuity.

For any $\delta \in \mathbb{R}$ with small $|\delta|$, we take a minimizer $\gamma_{\varepsilon+\delta} \in \mathcal{A}$ of $\mathcal{E}_{\varepsilon+\delta}$ so that $L^*(\varepsilon + \delta) = \mathcal{L}[\gamma_{\varepsilon+\delta}]$. Then, since the sequence $\{\gamma_{\varepsilon+\delta}\}_\delta$ is H^2-bounded by their minimality, for any subsequence there is a subsequence $\{\gamma_{\varepsilon+\delta'}\}_{\delta'}$ converging to a regular H^2-curve γ' weakly in H^2 and strongly in C^1; in particular, $\mathcal{L}[\gamma_{\varepsilon+\delta'}] \to \mathcal{L}[\gamma']$. Noting the H^2-weak lower semicontinuity of $\mathcal{E}_\varepsilon$ and Proposition 1.6.1, we have

$$\mathcal{E}_\varepsilon[\gamma'] \le \liminf_{\delta'\to 0} \mathcal{E}_\varepsilon[\gamma_{\varepsilon+\delta'}] = \liminf_{\delta'\to 0} \mathcal{E}_{\varepsilon+\delta'}[\gamma_{\varepsilon+\delta'}] = \liminf_{\delta'\to 0} m(\varepsilon+\delta') = m(\varepsilon),$$

which implies that γ' is a minimizer of $\mathcal{E}_\varepsilon$ (and hence $\gamma' \in \mathcal{A}$ by Appendix 1.A). Then we find that

$$\lim_{\delta'\to 0} L^*(\varepsilon+\delta') = \lim_{\delta'\to 0} \mathcal{L}[\gamma_{\varepsilon+\delta'}] = \mathcal{L}[\gamma'] \le L^*(\varepsilon),$$

and hence we obtain the upper semicontinuity

$$\limsup_{\delta\to 0} L^*(\varepsilon+\delta) \le L^*(\varepsilon)$$

in the full limit sense. The proof is now complete. □

Combining Proposition 1.6.2 and Proposition 1.6.3, we see that the set of jump points

$$J = \{\varepsilon \in [0,\infty) \mid L^*(\varepsilon) > L_*(\varepsilon)\} = \{\varepsilon \in [0,\infty) \mid \tilde{L}(\varepsilon) \text{ is not a singleton}\}$$

consists of at most countably many elements, and moreover for any open set $U \subset [0,\infty)\setminus J$ the function L_* $(= L_*)$ is a strictly increasing continuous function on U.

We finally confirm a first order expansion of the lengths of minimizers with respect to ε.

Proposition 1.6.4. *Any sequence of $L_\varepsilon \in \tilde{L}(\varepsilon)$ satisfies, as $\varepsilon \to 0$,*

$$L_\varepsilon = l + 4\sqrt{2}\left(\sin^2\frac{\theta_0}{4} + \sin^2\frac{\theta_1}{4}\right)\varepsilon + o(\varepsilon).$$

Proof. Let $X_\varepsilon := \sqrt{\varepsilon\mathcal{B}[\gamma_\varepsilon]}$ and $Y_\varepsilon := \sqrt{(L_\varepsilon - l)/\varepsilon}$. By Lemma 1.3.1,

$$X_\varepsilon^2 + Y_\varepsilon^2 = \frac{\mathcal{E}_\varepsilon[\gamma_\varepsilon] - l}{\varepsilon} = 8\sqrt{2}\left(\sin^2\frac{\theta_0}{4} + \sin^2\frac{\theta_1}{4}\right) + o(1)$$

as $\varepsilon \to 0$. Moreover, by the Cauchy-Schwarz inequality,

$$\begin{aligned} 2X_\varepsilon Y_\varepsilon &= 2\left(\int_0^{L_\varepsilon} |\partial_s\vartheta_{\tilde{\gamma}_\varepsilon}|^2 ds\right)^{1/2}\left(\int_0^{L_\varepsilon}(1-\cos\vartheta_{\tilde{\gamma}_\varepsilon})ds\right)^{1/2} \\ &\ge \int_0^{L_\varepsilon} |\partial_s\vartheta_{\tilde{\gamma}_\varepsilon}| 2\sqrt{1-\cos\vartheta_{\tilde{\gamma}_\varepsilon}}ds = \int_0^{L_\varepsilon} |\partial_s(V\circ\vartheta_{\tilde{\gamma}_\varepsilon})|ds. \end{aligned}$$

By Lemma 1.3.9, there is a sequence of $s_\varepsilon \in [0, L_\varepsilon]$ such that $[\![\vartheta_{\tilde{\gamma}_\varepsilon}(s_\varepsilon)]\!] \to 0$. Hence, by the triangle inequality and Lemma 1.3.6, we find that

$$\begin{aligned} 2X_\varepsilon Y_\varepsilon &\ge \int_0^{s_\varepsilon} |\partial_s(V\circ\vartheta_{\tilde{\gamma}_\varepsilon})|ds + \int_{s_\varepsilon}^{L_\varepsilon} |\partial_s(V\circ\vartheta_{\tilde{\gamma}_\varepsilon})|ds \\ &\ge 8\sqrt{2}\left(\sin^2\frac{\theta_0}{4} + \sin^2\frac{\theta_1}{4}\right) - o(1) \end{aligned}$$

as $\varepsilon \to 0$. Therefore, $0 \le (X_\varepsilon - Y_\varepsilon)^2 \le o(1)$ as $\varepsilon \to 0$. Noting that X_ε and Y_ε are bounded as $\varepsilon \to 0$, we find that X_ε and Y_ε converges to a same value; hence, we find that

$$\frac{L_\varepsilon - l}{\varepsilon} = Y_\varepsilon^2 \to 4\sqrt{2}\left(\sin^2\frac{\theta_0}{4} + \sin^2\frac{\theta_1}{4}\right)$$

as $\varepsilon \to 0$. The proof is complete. □

1.6.2 Connection of inextensible and extensible problems: fixed endpoints

We prove a prototype of Theorem 1.2.12, which connects the inextensible problem to the extensible problem under a fixed clamped boundary condition. This prototype deals with "shortening" ($L \downarrow l$) but not straightening ($l \uparrow L$); in the next subsection, we give a statement in terms of straightening.

Proposition 1.6.5. *Let $L > l$ and $\theta_0, \theta_1 \in [-\pi, \pi]$ with $|\theta_0| + |\theta_1| > 0$. Let $\tilde{L}$ be the length function (1.6.1) for l, θ_0, θ_1. Then, for any $\varepsilon > 0$ such that $L \in \tilde{L}(\varepsilon)$, any minimizer of $\mathcal{B}$ in $\mathcal{A}^L_{\theta_0,\theta_1,l}$ is a minimizer of $\mathcal{E}_\varepsilon$ in $\mathcal{A}_{\theta_0,\theta_1,l}$.*

Proof. Let γ be a minimizer of $\mathcal{B}$ in $\mathcal{A}^L_{\theta_0,\theta_1,l}$ and $\varepsilon > 0$ satisfy $L \in \tilde{L}(\varepsilon)$. Then, by $L \in \tilde{L}(\varepsilon)$, there exists a minimizer γ' of $\mathcal{E}_\varepsilon$ in $\mathcal{A}_{\theta_0,\theta_1,l}$ with $\mathcal{L}[\gamma'] = L$ ($= \mathcal{L}[\gamma]$). Since γ minimizes $\mathcal{B}$ in $\mathcal{A}^L_{\theta_0,\theta_1,l}$, we have $\mathcal{B}[\gamma] \leq \mathcal{B}[\gamma']$ and hence $\mathcal{E}_\varepsilon[\gamma] \leq \mathcal{E}_\varepsilon[\gamma']$. Since γ' minimizes $\mathcal{E}_\varepsilon$ in $\mathcal{A}_{\theta_0,\theta_1,l}$, so does γ. □

We are now in a position to state the following Theorem 1.6.6, which ensures that the inextensible problem in the shortening limit is read as the extensible problem.

Theorem 1.6.6. *Let $l > 0$ and $\theta_0, \theta_1 \in [-\pi, \pi]$ with $|\theta_0| + |\theta_1| > 0$. Let $\tilde{L}$ be the length function (1.6.1) for l, θ_0, θ_1. Let $L_\varepsilon \downarrow l$ be a sequence such that there is $\bar{\varepsilon} > 0$ such that $L_\varepsilon \in \tilde{L}(\varepsilon)$ for any $\varepsilon \in (0, \bar{\varepsilon})$. Then any minimizer γ_ε of $\mathcal{B}$ in $\mathcal{A}^{L_\varepsilon}_{\theta_0,\theta_1,l}$ is a minimizer of $\mathcal{E}_\varepsilon$ in $\mathcal{A}_{\theta_0,\theta_1,l}$. Moreover, as $\varepsilon \to 0$,*

$$\lim_{\varepsilon \to 0} \frac{L_\varepsilon - l}{\varepsilon} = 4\sqrt{2}\left(\sin^2 \frac{\theta_0}{4} + \sin^2 \frac{\theta_1}{4}\right).$$

Proof. An immediate corollary of Proposition 1.6.4 and Proposition 1.6.5. □

1.6.3 Dilation

We finally prove Theorem 1.2.12 and Theorem 1.2.14 via Theorem 1.6.6 and simple dilation arguments. We use the following elementary facts, the proofs of which are omitted.

Lemma 1.6.7. *Let $\theta_0, \theta_1 \in [-\pi, \pi]$ and $0 < \lambda < \Lambda$. Then a curve γ is a minimizer of $\mathcal{B}$ in $\mathcal{A}^\Lambda_{\theta_0,\theta_1,\lambda}$ if and only if the curve $\Lambda\gamma/\lambda$ is a minimizer of $\mathcal{B}$ in $\mathcal{A}^{\Lambda^2/\lambda}_{\theta_0,\theta_1,\Lambda}$.*

Lemma 1.6.8. *Let $\epsilon > 0$, $\theta_0, \theta_1 \in [-\pi, \pi]$ and $0 < \lambda < \Lambda$. Then a curve γ is a minimizer of $\mathcal{E}_\epsilon$ in $\mathcal{A}_{\theta_0,\theta_1,\Lambda}$ if and only if the curve $\lambda\gamma/\Lambda$ is a minimizer of $\mathcal{E}_{\lambda\epsilon/\Lambda}$ in $\mathcal{A}_{\theta_0,\theta_1,\lambda}$.*

Proof of Theorem 1.2.12. Recall that the constants L, θ_0, θ_1 are given in the assumption. Let $\tilde{L}$ be the length function defined as (1.6.1) for L, θ_0, θ_1. Notice that $L'_\varepsilon \to L$ holds as $\varepsilon \downarrow 0$ for any sequence of $L'_\varepsilon \in \tilde{L}(\varepsilon)$ by Proposition 1.6.4; in particular, there are sequences $L'_n \downarrow L$ and $\varepsilon_n \downarrow 0$ as $n \to \infty$ such that $L'_n \in \tilde{L}(\varepsilon_n)$ for any n. Then, by Theorem 1.6.6 with $l = L$, any minimizer of $\mathcal{B}$ in $\mathcal{A}^{L'_n}_{\theta_0,\theta_1,L}$ is a minimizer of $\mathcal{E}_{\varepsilon_n}$ in $\mathcal{A}_{\theta_0,\theta_1,L}$, and moreover

$$\lim_{n \to \infty} \frac{L'_n - L}{\varepsilon_n} = 4\sqrt{2}\left(\sin^2 \frac{\theta_0}{4} + \sin^2 \frac{\theta_1}{4}\right).$$

We now define l_n as $l_n := L^2/L'_n$. We confirm that the sequences $l_n \uparrow L$ and $\varepsilon_n \downarrow 0$ satisfy the desired properties. Let γ_n be any minimizer of $\mathcal{B}$ in $\mathcal{A}^L_{\theta_0,\theta_1,l_n}$. By Lemma

1.6.7 with $\lambda = l_n$ and $\Lambda = L$, the curve $\frac{L}{l_n}\gamma_n$ is a minimizer of $\mathcal{B}$ in $\mathcal{A}^{L'_n}_{\theta_0,\theta_1,L}$. Hence, by Theorem 1.6.6, the curve $\frac{L}{l_n}\gamma_n$ is a minimizer of $\mathcal{E}_{\varepsilon_n}$ in $\mathcal{A}_{\theta_0,\theta_1,L}$. Thus the first assertion is confirmed. Moreover, since $L'_n = L^2/l_n$, we have

$$\lim_{n\to\infty}\frac{L-l_n}{\varepsilon_n} = \lim_{n\to\infty}\frac{L'_n-L}{\varepsilon_n}\cdot\frac{l_n}{L} = \lim_{n\to\infty}\frac{L'_n-L}{\varepsilon_n} = 4\sqrt{2}\left(\sin^2\frac{\theta_0}{4}+\sin^2\frac{\theta_1}{4}\right),$$

which is nothing but the last assertion. The proof is now complete. □

Remark 1.6.9. In the above proof we need to take a subsequence since the "continuity" of $\tilde{L}$ is not guaranteed in general even in a neighborhood of the origin. Once the continuity is ensured, then there is no need to take a subsequence as shown in the following proof.

Proof of Theorem 1.2.14. Recall that the constants L, θ_0, θ_1 with (1.2.3) and $\theta_0\theta_1 < 0$ are given in the assumption. By Theorem 1.2.11, there is $\bar{\varepsilon} > 0$ such that for any $\varepsilon \in (0, \bar{\varepsilon})$ the energy $\mathcal{E}_\varepsilon$ admits a unique minimizer in $\mathcal{A}_{\theta_0,\theta_1,L}$.

Let $\tilde{L}$ be the length function defined as (1.6.1) for fixed L, θ_0, θ_1. Then, by the above uniqueness, $\tilde{L}$ is a single-valued function in $[0, \bar{\varepsilon})$, and hence the lower semicontinuous envelope L_* is a continuous nondecreasing function in $[0, \bar{\varepsilon}]$. Then, in particular, the function $L_* : [0, \bar{\varepsilon}] \to [L, L_*[\bar{\varepsilon}]]$ is surjective, and hence we can define a function $\tilde{\varepsilon}' : [L, L_*(\bar{\varepsilon})] \to [0, \bar{\varepsilon}]$ so that $L_* \circ \tilde{\varepsilon}'$ is the identity map on $[L, L_*(\bar{\varepsilon})]$. Note that $\tilde{\varepsilon}'$ is a strictly increasing function since L_* is nondecreasing. In addition, by Theorem 1.6.6 with $l = L$, for any $L' \in (L, L_*(\bar{\varepsilon}))$, any minimizer of $\mathcal{B}$ in $\mathcal{A}^{L'}_{\theta_0,\theta_1,L}$ is a minimizer of $\mathcal{E}_{\tilde{\varepsilon}'(L')}$ in $\mathcal{A}_{\theta_0,\theta_1,L}$, and moreover

$$\lim_{L'\downarrow L}\frac{L'-L}{\tilde{\varepsilon}'(L')} = 4\sqrt{2}\left(\sin^2\frac{\theta_0}{4}+\sin^2\frac{\theta_1}{4}\right).$$

In particular, for any $L' \in (L, L_*(\bar{\varepsilon}))$ the energy $\mathcal{B}$ admits a unique minimizer in $\mathcal{A}^{L'}_{\theta_0,\theta_1,L}$ (since $\mathcal{E}_{\tilde{\varepsilon}'(L')}$ admits a unique minimizer in $\mathcal{A}_{\theta_0,\theta_1,L}$).

Now we set $\bar{l} := L^2/L_*(\bar{\varepsilon})$. Define a function $\tilde{\varepsilon} : [\bar{l}, L] \to [0, \bar{\varepsilon}]$ by $\tilde{\varepsilon}(l) := \tilde{\varepsilon}'(L^2/l)$. Notice that $\tilde{\varepsilon}$ is strictly decreasing. Then, by Lemma 1.6.7, for any $l \in (\bar{l}, L)$ and any minimizer γ_l of $\mathcal{B}$ in $\mathcal{A}^L_{\theta_0,\theta_1,l}$, the dilated curve $\frac{L}{l}\gamma_l$ minimizes $\mathcal{B}$ in $\mathcal{A}^{L^2/l}_{\theta_0,\theta_1,L}$. Since $L < L^2/l < L_*(\bar{\varepsilon})$, the desired uniqueness holds by the above arguments. In addition, we find that the curve $\frac{L}{l}\gamma_l$ also minimizes $\mathcal{E}_{\tilde{\varepsilon}(l)}$ in $\mathcal{A}_{\theta_0,\theta_1,L}$. Moreover, we also find that

$$\lim_{l\uparrow L}\frac{L-l}{\tilde{\varepsilon}(l)} = \lim_{l\uparrow L}\frac{L^2/l-L}{\tilde{\varepsilon}'(L^2/l)}\cdot\frac{l}{L} = \lim_{L'\downarrow L}\frac{L'-L}{\tilde{\varepsilon}'(L')} = 4\sqrt{2}\left(\sin^2\frac{\theta_0}{4}+\sin^2\frac{\theta_1}{4}\right).$$

The proof is now complete. □

Remark 1.6.10. It is not claimed that the above function $\tilde{\varepsilon}$ (or $\tilde{\varepsilon}'$) is continuous. The continuity is ensured if the length function $\tilde{L}$ (or equivalently L_*) is strictly increasing.

Appendices

1.A Existence of minimizers

Fix $l > 0$ and $\theta_0, \theta_1 \in [-\pi, \pi]$. We say that $\gamma \in H^2(I; \mathbb{R}^2) \subset C^1(\bar{I}; \mathbb{R}^2)$ is *H^2-admissible* if γ is of constant speed and satisfying the boundary condition (1.2.1). We denote the set of H^2-admissible curves by $\mathcal{X}$. Note that the H^2-weak topology is stronger than C^1-topology; hence, in particular, the set $\mathcal{X}$ is H^2-weakly closed in $H^2(I; \mathbb{R}^2)$.

Theorem 1.A.1. *Let $\mathcal{X}' \subset \mathcal{X}$ be an H^2-weakly closed subset. Then the functional $\mathcal{E}_\varepsilon = \varepsilon^2 \mathcal{B} + \mathcal{L}$ defined on $\mathcal{X}'$ attains its minimum in $\mathcal{X}'$.*

Proof. The proof is straightforward. Since any $\gamma \in \mathcal{X}'$ is of constant speed, we have the following representations:

$$\mathcal{L}[\gamma] \equiv |\dot{\gamma}| \geq l, \quad \mathcal{B}[\gamma] = \frac{1}{\mathcal{L}[\gamma]^3} \int_I |\ddot{\gamma}(t)|^2 dt.$$

By the above relations and the boundary condition, we find that a minimizing sequence is H^2-bounded. Since $\mathcal{E}_\varepsilon$ is lower semicontinuous with respect to the H^2-weak topology, a standard direct method argument implies the existence of a minimizer. $\square$

Moreover, if $\mathcal{X}'$ admits any local perturbation, then we find that any minimizer $\gamma \in \mathcal{X}'$ is of class C^∞ by a standard bootstrap argument.

By the above results, the problems (1.2.2) and (1.2.4) admit minimizers. (In the case of fixed length, we use the Lagrange multiplier method to modify the length constraint.)

In addition, it is also proved that there are infinitely many local minimizers with different winding numbers in a sense. Here $\gamma \in \mathcal{X}$ is a local minimizer of the energy $\mathcal{E}_\varepsilon$ if there is $\delta > 0$ such that $\mathcal{E}_\varepsilon[\gamma] \leq \mathcal{E}_\varepsilon[\gamma']$ for any $\gamma' \in \mathcal{X}$ with $\|\gamma - \gamma'\|_{H^2} \leq \delta$. To state the above fact, we use a kind of winding number; for $\gamma \in \mathcal{X}$ we define $\mathcal{N}[\gamma] \in \mathbb{Z}$ as

$$\mathcal{N}[\gamma] = \frac{1}{2\pi} \left(\int_\gamma \kappa ds + \theta_0 - \theta_1 \right),$$

where κ is the counterclockwise signed curvature ($\kappa = \partial_s \vartheta_{\tilde{\gamma}}$). We notice that the functional $\mathcal{N}$ is $\mathbb{Z}$-valued and continuous with respect to the H^2-weak and -strong topologies. Thus for any $m \in \mathbb{Z}$ the set $\mathcal{X}_m = \{\gamma \in \mathcal{X} \mid \mathcal{N}[\gamma] = m\}$ is open and closed in $\mathcal{X}$ both weakly and strongly. Since $\mathcal{X}_m$ is weakly closed, by Theorem 1.A.1, the energies $\mathcal{E}_\varepsilon$ defined on $\mathcal{X}_m$ and $\mathcal{B}$ defined on $\mathcal{X}_m \cap \mathcal{X}^L$ attain their minimizers, where $L > l$ and $\mathcal{X}^L = \{\gamma \in \mathcal{X} \mid \mathcal{L}[\gamma] = L\}$. Moreover, the set $\mathcal{X}_m$ is strongly open, and hence such minimizers are local minimizers on $\mathcal{X}$ or $\mathcal{X}^L$, respectively.

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Chapter 2

A free boundary problem for elastic curves I: Formulation and graph representation

In this chapter we mathematically study membranes and filaments adhering to periodic patterned substrates in a one-dimensional model. The problem is formulated by the minimizing problem of an elastic energy with a contact potential on graph substrates. Global minimizers (ground states) are mainly considered in view of their graph representations. Our main results exhibit sufficient conditions for the graph representation and examples of situations where any global minimizer must overhang.

Keywords: Adhesion; Euler's elastica; Obstacle problem; Contact potential; Free boundary problem; Graph representation.

2.1 Introduction

The figuration of elastic bodies is complicated to comprehend, in particular, if external factors and constraints are taken into consideration. Our study is devoted to a theoretical study of slender elastic bodies adhering to solid substrates.

The contact and adhesion problems between soft objects and solid substrates appear in various contexts. For example, complex adhesion patterns are observed when soft nano-objects, as graphene [24, 38] or carbon nanotubes [9], are sheeted on rough patterned substrates. The adhesion property is also known for vesicles (cf. [36]). More broadly, in contact mechanics [16], it is a central question to ask how elastic bodies contact rough substrates [30, 41]. This question is relevant for many motivating problems as rubber friction [28] or adhesion in biological systems as geckos [14, 29, 33]. Recently, there are remarkable progresses in "elasto-capillary" problems [34]. The elasto-capillary problems essentially relate to our problem in the sense that they are focused on the competition between elasticity and adhesiveness.

2.1.1 Our model

In this chapter we mathematically study the adhesion problems of filaments and membranes in a one-dimensional setting, as in [31]. To be more precise, we consider the minimizing problem of the energy

$$\mathcal{E}[\gamma] = \int_\gamma ds \left[\frac{C}{2}\kappa^2(s) + \sigma\left(\gamma(s)\right)\right] \tag{2.1.1}$$

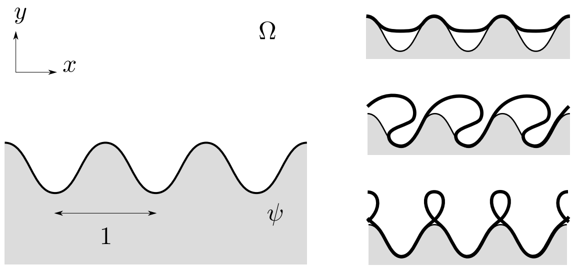

FIGURE 2.1: Periodic substrate function ψ and periodic admissible curves. Admissible curves may overhang or self-intersect.

defined for planar curves γ. Here κ and s denote the curvature and arc length parameter, respectively. Admissible curves γ (corresponding to elastic bodies) are constrained in the upper side of a given λ-periodic substrate function ψ_λ as in Figure 2.1. The constant $C > 0$ corresponds to the bending rigidity. The contact potential σ is defined as $\sigma = \sigma_F$ in the free part and $\sigma = \sigma_B$ in the bounded part, where $0 < \sigma_B < \sigma_F$ are constants. The constants σ_B and σ_F correspond to tension or surface energies. (See Section 2.2 for details.)

Our energy is a simple generalization of the modified total squared curvature, so-called Euler's elastica energy (see [2, 3, 4, 19, 20] and also [21, 35, 37]), so that an adhesion effect (contact potential) is included. Its minimization invokes a free boundary problem of the elastica equation, i.e., the free part of any minimizing curve satisfies the curvature equation $C(\kappa_{ss} + \kappa^3/2) - \sigma_F \kappa = 0$. The free boundary conditions are concerned with curvature jumps (see [31] and also [18, 27, 34, 36]). Our model can be regarded as an elastic version of wetting problems (cf. [10, 22]).

Our model concerns only the bending modes of filaments or membranes and neglects the stretching modes. As mentioned in [31], the underlying physical assumptions are that elastic bodies are sufficiently thin, vary only in one direction, and move along substrates freely (no friction). The stretching modes should be taken into account in fully two-dimensional models, even for thin films without friction (see e.g. [15, 38]).

2.1.2 Our goal

The local laws (as the elastica equation or boundary conditions) are well-known in our model since similar models have been widely studied (e.g. in [18, 27, 34, 36]). Our fundamental goal is to know the whole shapes of minimizers in our model. However, it is not realistic to determine the exact whole shapes of minimizers for arbitrary parameters and a substrate. This chapter focuses on whether minimizers are represented by the graphs of functions or not.

Whether minimizers are graphs or have overhangs is an important assertion on the shapes. In fact, the absence of overhangs guarantees that the shape of a solution is not so "complex", in particular, there is no self-intersection. Conversely, the presence of overhangs implies the possibility of self-intersections. Once membranes or filaments self-intersect, then other mechanisms (not taken into account in our model) may yield more complex shapes as rackets [6, 7, 42] (see also [34]).

An a priori guarantee of the graph representation is also important for the theoretical study. Such a guarantee rigorously justifies the graph setting, i.e., the assumption to consider only graph curves as admissible curves. The graph setting yields strong topological and morphological constraints, and hence makes the analysis considerably simpler. In fact, there are theoretical studies [17, 23, 31] concerning the whole shapes of minimizers in our model, but all of them rely on the graph setting. The paper [31] particularly depends on the graph setting since its analysis crucially uses the small slope approximation.

2.1.3 Main results

The present paper gives a rigorous study on the graph representations of global minimizers (ground states). A theoretical reason to consider only global minimizers is that the shapes of local minimizers (metastable states) may be more complicated even for parameters ensuring the graph representations of global minimizers (see Section 2.5 for details). The assumption of global minimality would be however appropriate for some experimental situations, for example, thin films on substrates with wetting fluids at the interfaces (almost no friction) as in [15]. In addition, as a mathematical assumption, we assume that curves γ and a substrate ψ_λ have a same period λ.

To describe our results, it is convenient to recall the typical length scale $\ell = \sqrt{C/\sigma_F}$, which compares bending rigidity and surface tension. The scale ℓ is called the elasto-capillary length e.g. in [15, 34]. As mentioned in [15, 34], the scale ℓ appears as a typical bending scale of an elastic body. We also use the length scale $r = \|\psi_\lambda''\|_\infty^{-1}$ which is the reciprocal of the maximum of the second derivative. The scale r roughly corresponds to the minimal bending scale of ψ_λ. Moreover, the dimensionless ratio $\alpha = \sigma_B/\sigma_F$ is also important since it corresponds to adhesiveness.

Global minimizers are flat in many limiting cases; dominant bending effect ($C = \infty$), no adhesion ($\sigma_B = \sigma_F$) or flat substrate ($\psi = 0$). Hence, the graph representation is expected at least nearly the above cases. Indeed, Theorem 2.3.3 and Theorem 2.3.4 give explicit conditions ensuring that global minimizers must be graphs. The first condition is described as $\alpha^{-1} - 1 \ll (\ell/\lambda)^2$. In particular, this condition is satisfied as the limits $C \to \infty$ and $\sigma_B \to \sigma_F$. The second condition is described as $(r/\lambda)^2 \gg \alpha^{-1} + (\ell/\lambda)^{-2}$. In particular, this condition is satisfied as the limit $r \to \infty$, which means a second order flatness of ψ_λ. Our proof uses only energy arguments; we compare the energies of all non-graph curves and special graph competitors.

On the other hand, even if ψ is smooth of class C^∞, it turns out that there are situations such that global minimizers are overhanging, i.e., not represented by graphs. The mechanism of overhangs is involved, so we deal with only special substrates like "fakir carpets" (see the figures in Section 2.4). Our result indicates that the wave height length scale H and dimensionless "deviation" $\Delta := \min\{\lambda, H\}/(\lambda + 2H)$ of a fakir carpet appear as characteristic quantities. More precisely, as a main result (Theorem 2.4.4), we rigorously prove that global minimizers must overhang if ψ_λ is smooth but shaped like a fakir carpet and moreover the relations $r \ll \ell \ll \min\{\lambda, H\}$ and $\alpha \ll \Delta$ are satisfied. Our proof is based on a geometric viewpoint to classify possible global states of non-overhanging curves, and an energy estimate for each of the cases. A special overhanging competitor is then constructed in view of the optimal bending scale ℓ. We notice that the condition $r \ll \ell$ requires that ℓ is not arbitrary small for overhangs. However, we also prove that if such substrates are Lipschitz (i.e., folding singularly $r = 0$), then ℓ can be arbitrary small for a fixed substrate (Theorem 2.4.7). To this end, we need further discussion for local bending structure, but we still use only energy arguments.

2.1.4 Related mathematical results

In the rest of this section, reviewing related mathematical literature, we see that in our one-dimensional problem both the contact potential and the total squared curvature play crucial roles for overhangs.

There is much mathematical literature of first order energies with contact potentials on flat substrates (see e.g. [1, 5, 39, 40] for graphs, [5, 22] for the boundary of sets, and references therein). The problems in the cited papers roughly correspond to our problem with $C = 0$ and $\psi_\lambda \equiv 0$ (but in higher dimensions). In first order cases, solutions may have edge singularities at the free boundary and the contact angle θ satisfies Young's equation $\cos\theta = \sigma_B/\sigma_F$. In higher dimensional cases, the contact potential may imply the loss of graph representation even in first order cases (cf. [39]). However, although our substrates are not flat, our problem is one-dimensional and periodic, so the graph setting would be still suitable while $C = 0$.

To our knowledge, there is little mathematical literature of higher order problems with contact potentials except the aforementioned papers [17, 23]. In [23], the author obtains an energy expansion as $C \to 0$. The paper [17] studies a discretization of our model and proposes numerical results. As already mentioned, the papers [17, 23] assume that admissible curves are graphs.

The total squared curvature is higher order and a main reason of the loss of graph representation. In fact, it is well-known that there are non-graph solutions to the elastica equation, which our minimizers obey in the free part (see e.g. the figures in [4, 21]). Thus, if we impose suitable fixed boundary conditions, it is not difficult to prove that a global minimizer of the modified total squared curvature overhangs. Our problem is a free boundary problem, and hence the graph representation problem is more involved.

We finally mention that, in dynamical problems of curves (without substrates), the graph representations of solutions have also been concerned. Although the L^2-gradient flow of the length energy (curve shortening flow) preserves the graph property [12], one of the modified total squared curvature (curve shortening-straightening flow [20, 25, 26, 32]) may lose in the middle even in the periodic setting [13]. However, in such a periodic case (without external factors), stationary global minimizers are only straight lines. Our problem takes an adhesion effect into account and thus even global minimizers may not be graphs.

2.1.5 Organization

This chapter is organized as follows. Basic notation and definitions are prepared in Section 2.2. Section 2.3 provides some sufficient conditions for the graph representations of global minimizers. In Section 2.4, we prove the existence of situations where global minimizers must overhang. In Section 2.5, we give further discussion on overhangs and also mention self-intersections and local minimizers.

2.2 Preliminaries: curves, energy and quantities

In this section we prepare notation of admissible curves and the total energy and then formulate our problem. For simplicity, throughout this chapter, we impose normalizations with respect to the wavelength and tension. In the last subsection we mention the relation between our normalized problem and original physical quantities.

2.2.1 Definition of admissible curves

Let $\psi : \mathbb{R} \to \mathbb{R}$ be a continuous function with 1-periodicity, that is, $\psi(x) = \psi(x+1)$ for any $x \in \mathbb{R}$. Let $\Omega \subset \mathbb{R}^2$ be the strict epigraph of ψ:

$$\Omega := \left\{(x,y) \in \mathbb{R}^2 \mid y > \psi(x)\right\}.$$

Denote its closure by $\overline{\Omega} = \{y \geq \psi(x)\}$ and the boundary by $\partial\Omega = \{y = \psi(x)\}$. The set $\overline{\Omega}$ corresponds to the upper side of the substrate ψ.

Let I be the open interval $(0,1)$. We denote by H^2_Ω the set of all curves $\gamma \in H^2(I;\mathbb{R}^2)$ such that γ is regular and confined in $\overline{\Omega}$, that is, $|\dot{\gamma}(t)| > 0$ and $\gamma(t) \in \overline{\Omega}$ for any $t \in \bar{I}$. The H^2-Sobolev setting corresponds to the square integrability of curvature. Recall that any regular H^2 curve is a regular C^1 curve by Sobolev embedding and hence γ and $\dot{\gamma}$ are defined pointwise in $\bar{I}$ (including the endpoints).

Moreover we say that a curve $\gamma = (x,y) \in H^2_\Omega$ is admissible if it satisfies the following periodic boundary condition:

$$x(0) = 0,\ x(1) = 1,\ y(0) = y(1),\ \dot{\gamma}(0) = \dot{\gamma}(1).$$

We denote the set of admissible curves by $\mathcal{A} \subset H^2_\Omega$. We remark that the set $\mathcal{A}$ consists of the restrictions to I of regular curves $\gamma = (x,y) \in H^2_{loc}(\mathbb{R};\overline{\Omega})$ such that $x(0) = 0$ and $\gamma(t+1) = \gamma(t) + (1,0) \in \mathbb{R}^2$ for any $t \in \mathbb{R}$.

In this setting, admissible curves may have any self-intersections, and thus it would not be compatible with membrane problems. However, we easily confirm that all the results in this chapter are valid in a membrane setting (see Section 2.5 for details).

2.2.2 Definition of the total energy

For any admissible curve $\gamma \in \mathcal{A}$ we define the length of one period by

$$L_\gamma := \int_I |\dot{\gamma}(t)|dt \geq 1. \tag{2.2.1}$$

The lower bound follows by the periodicity, in particular, by $x(0) = 0$ and $x(1) = 1$.

Using the arc length parameterization $0 \leq s \leq L_\gamma$, for given constants $\varepsilon > 0$ and $0 < \alpha < 1$, we define the total energy of one period:

$$E[\gamma] := \int_0^{L_\gamma} \left[\varepsilon^2|\gamma_{ss}(s)|^2 + \Theta(\gamma(s))\right] ds, \tag{2.2.2}$$

where $\Theta : \overline{\Omega} \to \mathbb{R}$ is defined as $\Theta \equiv 1$ in Ω and $\Theta \equiv \alpha$ on $\partial\Omega$. Note that $|\gamma_{ss}|^2$ is equal to the squared curvature κ^2.

2.2.3 Minimizing problem

Our problem is formulated as

$$\min_{\gamma \in \mathcal{A}} E[\gamma]. \tag{2.2.3}$$

Our purpose is to know the shapes of global minimizers, i.e., curves $\gamma \in \mathcal{A}$ satisfying $E[\gamma] = \min_{\mathcal{A}} E$. Hereafter a global minimizer is often called a minimizer simply.

The problem (2.2.3) is determined by the quantities $\varepsilon > 0$ and $0 < \alpha < 1$ and the substrate ψ. The quantity ε corresponds to the normalized elasto-capillary length scale (or bending scale) of minimizing curves. The coefficient α corresponds to adhesiveness. The smaller α is, the easier the curves become to adhere. From Section 2.3, changing the parameters ε, α and ψ, we consider whether minimizers are represented as graphs or not.

We shall state the existence of solutions to the problem (2.2.3).

Theorem 2.2.1. *The problem (2.2.3) admits a minimizer.*

This is proved by a basic direct method in the calculus of variations. However, we need some careful arguments to prove it rigorously and thus postpone the precise proof until 2.A. In general, the uniqueness is not expected in this problem.

We have simple bounds for the minimum of E:

$$\alpha \leq \min_{\mathcal{A}} E \leq 1. \tag{2.2.4}$$

The upper bound follows since the trivial straight line $\bar{\gamma}(t) = (t, c)$, where c is a constant larger than the maximum of ψ, belongs to $\mathcal{A}$ and satisfies $E[\bar{\gamma}] = 1$. The lower bound follows since for any $\gamma \in \mathcal{A}$ we have $L_\gamma \geq 1$ and also

$$E[\gamma] \geq \int_\gamma \alpha ds = \alpha L_\gamma. \tag{2.2.5}$$

2.2.4 On normalization of the problem

As mentioned, our problem is normalized with respect to the wavelength and tension. To clarify the normalizations, we consider the relation between our normalized problem and the original one. In the original problem, we only assume that admissible curves and a substrate have a same periodicity $\lambda > 0$.

If the substrate ψ_λ and admissible curves γ_λ have a general period $\lambda > 0$ and tension in the energy is also a general positive number, then as in Introduction our original problem is the minimization of

$$\mathcal{E}[\gamma_\lambda] := \int_0^{L_{\gamma_\lambda}} \left[\frac{C}{2} |(\gamma_\lambda)_{ss}(s)|^2 + \sigma(\gamma_\lambda(s)) \right] ds, \tag{2.2.6}$$

where $C > 0$ and σ is defined as $\sigma \equiv \sigma_F$ in $\{y > \psi_\lambda(x)\}$ and $\sigma \equiv \sigma_B$ on $\{y = \psi_\lambda(x)\}$ with $0 < \sigma_B < \sigma_F$.

Normalizing ψ_λ and γ_λ by rescaling as $\psi(x) = \lambda^{-1}\psi_\lambda(\lambda x)$ and $\gamma(s) = \lambda^{-1}\gamma_\lambda(\lambda s)$, we have

$$\mathcal{E}[\gamma_\lambda] = \lambda\sigma_F E[\gamma],$$

where the dimensionless quantities $\varepsilon > 0$ and $0 < \alpha < 1$ in E are defined as

$$\varepsilon := \frac{1}{\lambda}\sqrt{\frac{C}{2\sigma_F}}, \quad \alpha := \frac{\sigma_B}{\sigma_F}. \tag{2.2.7}$$

Since we only used a similarity transformation, the shapes of curves and a substrate are maintained. Thus, the minimizing problem of (2.2.6) is equivalent to (2.2.3) up to rescaling. We finally recall that ε has the same scale as the normalized elasto-capillary length scale, i.e., $\varepsilon \sim \ell/\lambda$.

2.3 Graph solutions

In this section we prove that, under suitable conditions for ε, α, and ψ, the problem (2.2.3) admits only graph minimizers.

We shall give the definition of graph curves.

Definition 2.3.1 (Graph curves)**.** We say that $\gamma = (x, y) \in \mathcal{A}$ is a graph curve if $x'(t) > 0$ for any $t \in \bar{I}$.

Remark 2.3.2. By the periodicity, the condition $x'(t) > 0$ is equivalent to $x'(t) \neq 0$. Any graph curve γ is represented by an H^2 function in the y-direction; more precisely, there exists a 1-periodic function $u \in H^2_{loc}(\mathbb{R})$ such that its graph curve $(\cdot, u(\cdot)) \in \mathcal{A}$ is a reparameterization of γ.

2.3.1 Statements and discussion

We first observe the following limiting cases; $\varepsilon = \infty$, $\alpha = 1$ and $\psi \equiv 0$. We easily notice that in all the cases minimizers are only straight lines. Indeed, in the case that $\varepsilon = \infty$ or $\alpha = 1$, our energy is regarded as the (modified) total squared curvature, which admits only straight line minimizers under the periodicity. Moreover, in the case that $\psi \equiv 0$, it is trivial that a unique minimizer is the completely adhering straight line.

By the above observation, when $\varepsilon \gg 1$, $\alpha \approx 1$ or $\psi \approx 0$, we expect that any minimizer is nearly flat and hence a graph curve. In fact, the following two statements hold.

Theorem 2.3.3. *Suppose that $(\pi^2\varepsilon^2 + 1)\alpha \geq 1$. Then, independently of ψ, any minimizer of (2.2.3) is a graph curve.*

Theorem 2.3.4. *Suppose that ψ is of class C^2 and $\|\psi''\|_\infty^2 \leq \frac{8\pi^2}{8/\alpha+1/\varepsilon^2}$. Then any minimizer of (2.2.3) is a graph curve.*

Remark 2.3.5. Theorem 2.3.3 immediately implies that, if we fix ε and take $\alpha \approx 1$, or fix α and take $\varepsilon \gg 1$, then any minimizer is a graph curve.

In view of the original physical quantities (2.2.7), the condition that $(\pi^2\varepsilon^2+1)\alpha \geq 1$ in Theorem 2.3.3 is read as

$$\frac{\pi^2}{2}\frac{C}{\lambda^2\sigma_F} = \frac{\pi^2}{2}\left(\frac{\ell}{\lambda}\right)^2 \geq \frac{\sigma_F}{\sigma_B} - 1.$$

This is enough to indicate the following two qualitative features: if the effect of adhesion is weak ($\sigma_B \to \sigma_F$), or the effect of bending is strong ($C \to \infty$), then any minimizer must be a graph curve.

Remark 2.3.6. Theorem 2.3.4 states that, for any ε and α which may be small, if the substrate ψ is sufficiently flat in a second order sense $\psi'' \approx 0$, then our problem still admits only graph curve minimizers.

Recall that the sup norm $\|\psi''\|_\infty = \max_{x\in\mathbb{R}} |\psi''(x)|$ is also a dimensionless quantity since $\|\psi''\|_\infty = \lambda/r$, where $r = \|\psi_\lambda''\|_\infty^{-1}$ corresponds to the minimal bending scale of the original substrate ψ_λ. By (2.2.7), the condition in Theorem 2.3.4 can be also expressed by the original quantities.

2.3.2 Proof of graph representation

In this section we prove Theorem 2.3.3 and Theorem 2.3.4. We first obtain a lower bound for the energies of non-graph curves. This is a key step to prove our theorems.

Proposition 2.3.7. *Any non-graph curve $\gamma \in \mathcal{A}$ satisfies*

$$E[\gamma] > \min\left\{1, (\pi^2\varepsilon^2+1)\alpha\right\}.$$

Proof. By (2.2.5), any curve $\gamma \in \mathcal{A}$ with $L_\gamma > 1/\alpha$ satisfies $E[\gamma] \geq \alpha L_\gamma > 1$. Thus it suffices to prove that any non-graph curve $\gamma \in \mathcal{A}$ with $1 \leq L_\gamma \leq 1/\alpha$ satisfies $E[\gamma] > (\pi^2\varepsilon^2+1)\alpha$. By the Cauchy-Schwarz inequality, we have

$$E[\gamma] \geq \varepsilon^2 \int_\gamma \kappa^2 ds + \alpha L_\gamma \geq \frac{\varepsilon^2}{L_\gamma}\left(\int_\gamma |\kappa| ds\right)^2 + \alpha L_\gamma.$$

Moreover, since γ is non-graph and periodic, its total absolute curvature (i.e., the total variation of its tangential angle) is larger than π. Noting that $1 \leq L_\gamma \leq 1/\alpha$, we have

$$E[\gamma] > \frac{\varepsilon}{1/\alpha}\pi^2 + \alpha = (\pi^2\varepsilon^2+1)\alpha.$$

The proof is complete. □

We are in a position to prove the theorems.

Proof of Theorem 2.3.3. Proposition 2.3.7 and the assumption $(\pi^2\varepsilon^2+1)\alpha \geq 1$ imply that any non-graph curve $\gamma \in \mathcal{A}$ satisfies $E[\gamma] > 1$. By the upper bound in (2.2.4), such a curve is not a minimizer. □

Proof of Theorem 2.3.4. By Theorem 2.3.3, we may assume $(\pi^2\varepsilon^2 + 1)\alpha < 1$. Thus, by Proposition 2.3.7, we only need to prove that any $\gamma \in \mathcal{A}$ such that $E[\gamma] > (\pi^2\varepsilon^2 + 1)\alpha$ is not a minimizer.

We compare such γ with the completely adhering competitor $\tilde{\gamma} := (\cdot, \psi(\cdot)) \in \mathcal{A}$, which is a graph curve since ψ is of class C^2. Noting that $2\|\psi'\|_\infty \le \|\psi''\|_\infty$ by the 1-periodicity, we find that the curve $\tilde{\gamma}$ satisfies

$$\begin{aligned} E[\tilde{\gamma}] &= \varepsilon^2 \int_I \frac{|\psi''|^2}{(1+|\psi'|^2)^{5/2}} + \alpha \int_I \sqrt{1+|\psi'|^2} \\ &\le \varepsilon^2 \|\psi''\|_\infty^2 + \alpha \left(1 + \frac{1}{2}\left(\frac{\|\psi''\|_\infty}{2}\right)^2\right). \\ &= \left(\frac{8\varepsilon^2+\alpha}{8\alpha}\|\psi''\|_\infty^2 + 1\right)\alpha. \end{aligned}$$

The assumption on $\|\psi''\|_\infty$ immediately implies that $\frac{8\varepsilon^2+\alpha}{8\alpha}\|\psi''\|_\infty^2 \le \pi^2\varepsilon^2$ and hence $E[\gamma] > E[\tilde{\gamma}]$. Therefore, the curve γ does not minimize E. $\square$

Remark 2.3.8. In the proof of Proposition 2.3.7, using the inequality of arithmetic and geometric means, we also have another type of lower bound as

$$E[\gamma] > \frac{\varepsilon^2}{L_\gamma}\pi^2 + \alpha L_\gamma \ge 2\pi\varepsilon\sqrt{\alpha}.$$

Thus the condition in Theorem 2.3.3 can be replaced by $4\varepsilon^2\alpha \ge 1$. Although this condition is meaningful quantitatively, it is not sharp enough to obtain the qualitative property that any minimizer is a graph curve for any fixed ε and $\alpha \approx 1$.

2.4 Overhanging solutions

In this section we show that there is a combination of ε, α, and ψ such that any minimizer must overhang.

Definition 2.4.1 (Overhanging)**.** We say that a curve $\gamma = (x, y) \in \mathcal{A}$ is overhanging if there exists $t \in \bar{I}$ such that $x'(t) < 0$.

Remark 2.4.2. By the periodicity of $\gamma \in \mathcal{A}$, there is $t \in \bar{I}$ such that $x'(t) > 0$ in general, and thus any overhanging curve must have "turns" in the x-direction.

Heuristically, overhanging solutions should appear in order to circumvent sharp mountain folds of substrates as in Figure 2.2 since minimizing curves should bend as the scale ε in principle.

However, this is a kind of local necessary condition and in general the global shape formation of curves is very complicated. In order to find overhanging minimizers rigorously, we deal with a special substrate (fakir carpet), which is simple enough to analyze.

In what follows, we first give a formal discussion for a very singular substrate, and then rigorously prove the existence of overhanging minimizers for smooth or Lipschitz substrates.

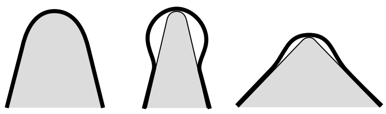

FIGURE 2.2: Curves near mountain folds. Minimizing curves should bend at the scale ε and hence overhang to circumvent more sharp folds (center). Curves would not overhang for folds with large bending scale (left) or small slope (right).

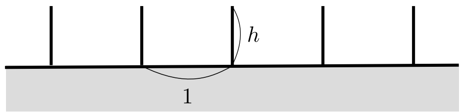

FIGURE 2.3: Fakir carpet of height h and the 1-periodicity.

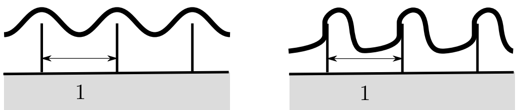

FIGURE 2.4: Non overhanging curves not touching the base part.

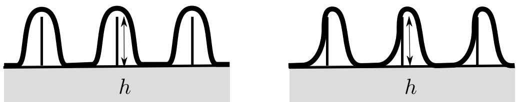

FIGURE 2.5: Non overhanging curves touching the base part.

2.4.1 For fakir carpets: strategy

In this subsection we give an intuitive explanation by formally taking a singular substrate as in Figure 2.3; ψ is the fakir carpet of height h and period 1, which is the most simple substrate with a singularly sharp mountain fold (but no longer a continuous function). For a fakir carpet substrate, we obtain a general lower bound for the energy of all non-overhanging curves and show that, under suitable assumptions on the smallness of ε and α, there is a special overhanging competitor so that its energy is lower than any non-overhanging curve.

We first obtain a lower bound for non-overhanging curves. In the present setting, it turns out that, for any ε and α, any non-overhanging curve $\gamma \in \mathcal{A}$ satisfies

$$E[\gamma] \geq \min\{1, h\}. \tag{2.4.1}$$

In fact, any non-overhanging curve γ is either, not touching the base part of the fakir carpet as in Figure 2.4, or touching as in Figure 2.5. Note that in both cases γ touches at most one side of the needle as in the figures since γ is not overhanging. (To touch both sides, the curve must have a singularity.) In the former case (Figure 2.4), the curve γ has the free part of length at least 1, i.e., $E[\gamma] \geq 1$. In the latter case (Figure 2.5), the curve γ has the free part of length at least h, i.e., $E[\gamma] \geq h$. Consequently, any non-overhanging curve satisfies (2.4.1).

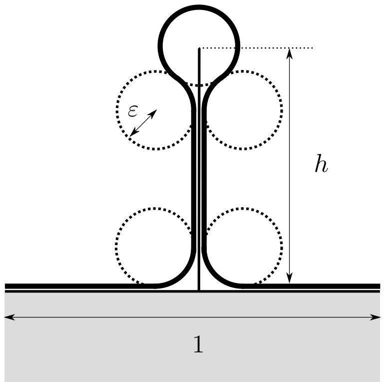

FIGURE 2.6: Overhanging competitor above the fakir carpet. The curve consists of the adhering straight parts and the non-adhering circular arc parts of radius ε.

On the other hand, providing that ε is sufficiently small as $\varepsilon < \min\{1,h\}/5$, we can define an overhanging competitor $\hat{\gamma} \in \mathcal{A}$ as in Figure 2.6, which is almost adhering to the fakir carpet and bending in the free (non-adhering) part as circular arcs of radius ε. Then $\hat{\gamma}$ satisfies

$$E[\hat{\gamma}] < (1+2h)\alpha + 20\pi\varepsilon. \tag{2.4.2}$$

In fact, the total length of the bounded (adhering) part $\hat{\gamma}_B$ is less than $1+2h$, that is, $E[\hat{\gamma}_B] < (1+2h)\alpha$, and in the free part $\hat{\gamma}_F$ the energy $E[\hat{\gamma}_F]$ is bounded as

$$\int_{\hat{\gamma}_F} \left[\varepsilon^2\kappa^2 + 1\right] ds < 10\pi\varepsilon \left[\varepsilon^2 \frac{1}{\varepsilon^2} + 1\right] = 20\pi\varepsilon,$$

where $10\pi = 5 \times 2\pi$ is a rough upper bound for the total angle of the circular arcs.

Combining (2.4.1) and (2.4.2), we see that the conditions

$$\alpha < \Delta := \frac{\min\{1,h\}}{1+2h}, \quad \varepsilon < \frac{(1+2h)(\Delta-\alpha)}{20\pi}$$

imply $E[\gamma] > E[\hat{\gamma}]$, which means that the energy of any non-overhanging curve γ is strictly higher than the overhanging competitor $\hat{\gamma}$. In conclusion, for any fakir carpet ψ, if α and ε are sufficiently small as $\alpha \ll \Delta$ and $\varepsilon \ll \min\{1,h\}$, then any minimizer must overhang.

Finally, we remark that for any $h>0$ the inequality

$$\Delta = \frac{\min\{1,h\}}{1+2h} \leq \frac{1}{3}$$

holds, and the equality is attained if and only if $h=1$. This means that, at least in our method, the case of height 1 allows the optimal (highest) upper bound for α or ε to observe overhangs. The dimensionless quantity Δ may be read as a "deviation" of the hall of a fakir carpet. Indeed, the hall is the square when Δ takes the maximum $1/3$ ($h=1$), and the halls become thin rectangles as $\Delta \downarrow 0$ ($h \to 0$ or $h \to \infty$). Thus, the

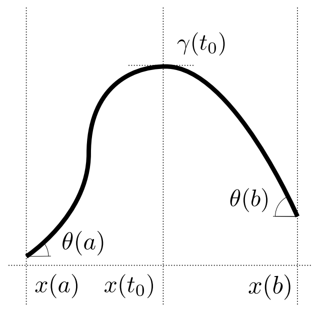

FIGURE 2.7: Curve $\gamma = (x, y)$ under the assumption of Lemma 2.4.3.

more a hall deviates from the square, the smaller the ε and α are necessary to be for the presence of overhangs.

2.4.2 For smooth substrates

A similar consideration is valid for a smooth but fakir carpet like substrate ψ as in Figure 2.8. The main difference from the singular case is that, in the smooth case, curves may touch both the walls of substrates. Thus we need to state that if a non-overhanging curve touches both the wall parts of a "thin" needle then the total energy is sufficiently high. To this end, we prepare a general lemma concerning a lower bound for the bending energies of non-overhanging curves as in Figure 2.7. The lower bound only depends on the width of curves in the x-direction and the tangential angles at the endpoints.

We define the following nonnegative even function:

$$f(\theta) := \left| \int_0^\theta \sqrt{\cos\varphi}\, d\varphi \right| = \int_0^{|\theta|} \sqrt{\cos\varphi}\, d\varphi.$$

Moreover, for a regular curve γ, we define the tangential angle $\theta(t) \in (-\pi, \pi]$ at $t \in \bar{I}$ so that $\dot{\gamma}(t) = |\dot{\gamma}(t)|(\cos\theta(t), \sin\theta(t))$. Then we have the following

Lemma 2.4.3. *Let $J = (a, b)$ be a bounded interval and $\gamma = (x, y) \in H^2(J; \mathbb{R}^2)$ be a regular curve such that $x'(t) \geq 0$ for any $t \in J$ and $y'(t_0) = 0$ for some $t_0 \in J$. Then the following inequality holds:*

$$\int_\gamma \kappa^2 ds \geq \frac{[f(\theta(a)) + f(\theta(b))]^2}{x(b) - x(a)}.$$

Proof. We first assume that a curve $\gamma = (x, y)$ satisfies $x'(t) > 0$ for any $t \in \overline{J}$. Then the curve is represented by some function $u \in H^2(x(a), x(b))$ such that $u'(x(a)) = \tan\theta(a)$, $u'(x(b)) = \tan\theta(b)$, $u'(x(t_0)) = 0$ and

$$\int_\gamma \kappa^2 ds = \int_{x(a)}^{x(b)} \frac{|u''(z)|^2}{(1 + u'(z)^2)^{5/2}} dz.$$

The Cauchy-Schwarz inequality yields that

$$\begin{aligned}
&\int_{x(a)}^{x(b)} \frac{|u''(z)|^2}{(1+u'(z)^2)^{5/2}} dz \\
\geq\ & \frac{1}{x(b)-x(a)} \left(\int_{x(a)}^{x(b)} \frac{|u''(z)|}{(1+u'(z)^2)^{5/4}} dz \right)^2 .
\end{aligned}$$

By change of variables, we have

$$\begin{aligned}
&\int_{x(a)}^{x(b)} \frac{|u''(z)|}{(1+u'(z)^2)^{5/4}} dz \\
\geq\ & \left| \int_{\tan\theta(a)}^{0} \frac{dw}{(1+w^2)^{5/4}} \right| + \left| \int_{0}^{\tan\theta(b)} \frac{dw}{(1+w^2)^{5/4}} \right| \\
=\ & f(\theta(a)) + f(\theta(b)),
\end{aligned}$$

and thus we obtain the desired lower bound.

For general $\gamma = (x, y)$ with $x' \geq 0$, we obtain the same conclusion by considering modified curves as $\gamma_\delta(t) = (x(t) + \delta t, y(t))$ for small $\delta > 0$ and taking the limit $\delta \to 0$. Note that $x(b) > x(a)$ holds even in this case by the assumption of γ. $\square$

We now state and prove the main theorem. Let $h > 0$ and $0 < 2\delta < \min\{1, h\}$. A 1-periodic function ϕ is called δ-smooth fakir carpet of height h if ϕ is as in Figure 2.8, namely, of class C^∞ and satisfies

(1) $\phi(x) = \phi(1-x)$ for any $x \in [0, 1/2]$,

(2) $\phi \equiv 0$ in $[0, 1/2-\delta]$ and $\phi(1/2) = h$,

(3) $\phi' \geq 0$ in $[0, 1/2]$,

(4) $\phi''(x) = 0$ while $\delta \leq \phi(x) \leq h - \delta$.

Moreover, we define its base and wall parts as in Figure 2.9; the base part is the part with $y = \phi(x) = 0$ and the left (resp. right) wall part is the part with $y = \phi(x)$, $\delta \leq y \leq h - \delta$ and $\phi'(x) > 0$ (resp. $\phi'(x) < 0$). All the parts are straight. Note that $\delta \gtrsim \|\phi''\|_\infty^{-1}$.

Theorem 2.4.4. *Let $h > 0$ and $\Delta := \frac{\min\{1,h\}}{1+2h}$. Then for any $\alpha < \Delta$ and $\varepsilon < \frac{(1+2h)(\Delta-\alpha)}{20\pi}$ there exists $0 < \bar{\delta} < \varepsilon$ such that, for any δ-smooth fakir carpet substrate ψ_δ of height h with $0 < \delta < \bar{\delta}$, any minimizer of (2.2.3) is overhanging.*

Proof. Fix any $\alpha < \Delta$ and $\varepsilon < \frac{(1+2h)(\Delta-\alpha)}{20\pi}$. For any small $0 < \delta \ll \varepsilon$, we take a substrate ψ_δ of δ-smooth fakir carpet of height h. Take the overhanging competitor $\hat{\gamma}_\delta \in \mathcal{A}$ as in Figure 2.10. Then, by the similar way to obtain (2.4.2), we see that

$$E[\hat{\gamma}_\delta] \leq (1+2h)\alpha + 20\pi\varepsilon.$$

By the assumptions on α and ε, we have

$$E[\hat{\gamma}_\delta] \leq \min\{1, h\} - c,$$

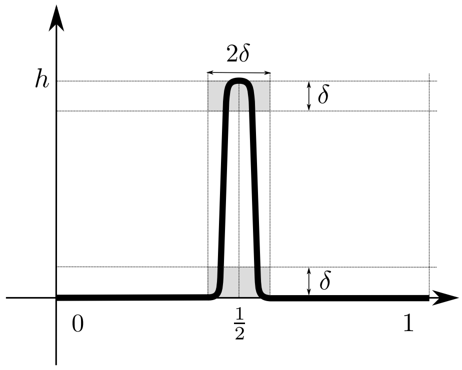

FIGURE 2.8: A 1-period part of the smooth but fakir carpet like substrate. The function is bending only in the gray regions and otherwise straight.

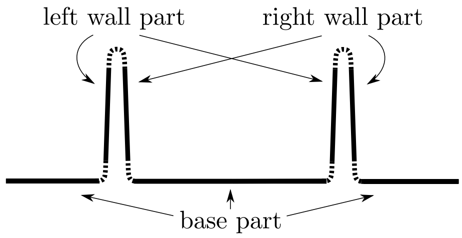

FIGURE 2.9: Base part and left and right wall parts of δ-smooth fakir carpets.

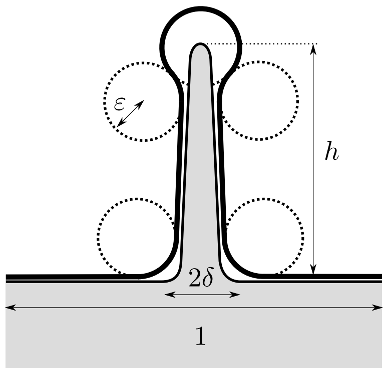

FIGURE 2.10: Overhanging competitor $\hat{\gamma}_\delta$ above the smooth substrate ψ_δ defined for any small $\delta \ll \varepsilon$. The curve $\hat{\gamma}_\delta$ is adhering to the substrate only in the base and wall parts and otherwise bending as circular arc of radius ε. For any small δ the curve $\hat{\gamma}_\delta$ is overhanging.

where $c > 0$ is some constant independent of δ. Therefore it suffices to prove that

$$\liminf_{\delta \downarrow 0} \inf_{\gamma} E[\gamma] \geq \min\{1, h\}, \tag{2.4.3}$$

where the infimum is taken over all non-overhanging curves in the upper side of ψ_δ. Indeed, if this is proved then there exists $0 < \bar{\delta} < \varepsilon$ such that for any $0 < \delta < \bar{\delta}$ any non-overhanging curve γ satisfies $E[\hat{\gamma}_\delta] < E[\gamma]$.

Notice that any (non-overhanging) curve $\gamma \in \mathcal{A}$ belongs to at least one of the following three cases:

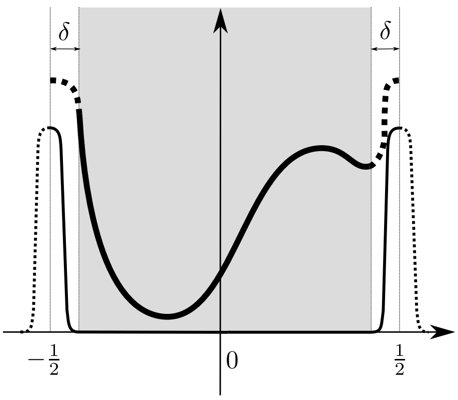

FIGURE 2.11: A curve not touching the base part in $(-\frac{1}{2}, \frac{1}{2})$. To avoid the base part, any periodic curve $\gamma \in \mathcal{A}$ must cross the gray region without touching the graph of ψ_δ.

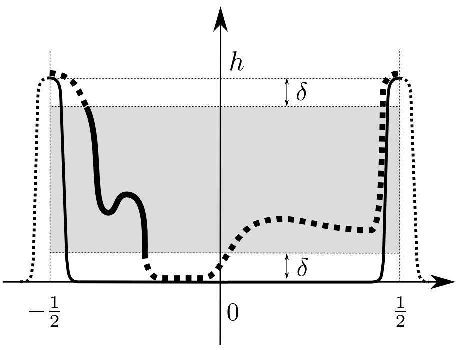

FIGURE 2.12: A curve touching the base part but avoiding the right wall part in $(-\frac{1}{2}, \frac{1}{2})$. To touch the base part and avoid the right (or left) wall part, any periodic non-overhanging curve $\gamma \in \mathcal{A}$ must cross the gray region without touching the graph of ψ_δ at least one time.

1. γ does not touch the base part (Figure 2.11),
2. γ touches the base part but not the left nor right wall part (Figure 2.12),
3. γ touches both the left and right wall parts.

We prove (2.4.3) for all the cases 1, 2, and 3.

Case 1. By the periodicity, as in Figure 2.11, any curve $\gamma = (x, y) \in \mathcal{A}$ may be regarded as satisfying $x(0) = -1/2$ and $x(1) = 1/2$. Then the condition of Case 1 implies that γ passes through the region $\{|x| < 1/2 - \delta\}$ freely. Hence we have $E[\gamma] \geq 1 - 2\delta$, which implies (2.4.3).

Case 2. Similarly, as in Figure 2.12, we may regard any curve $\gamma = (x, y) \in \mathcal{A}$ as satisfying $x(0) = -1/2$ and $x(1) = 1/2$, and hence $y(0) = y(1) \geq h$. Then the condition of Case 2 and the fact that γ is non-overhanging imply that γ passes through the region $\{\delta < y < h - \delta\}$ freely at least one time. Hence we have $E[\gamma] \geq h - 2\delta$, which implies (2.4.3).

Case 3. For any non-overhanging $\gamma \in \mathcal{A}$ touching both the wall parts (tangentially), there are $t_1, t_2 \in I$ such that the part of γ from t_1 to t_2 satisfies the assumption of Lemma

2.4.3 with $x(t_2) - x(t_1) \leq 2\delta$ and $|\theta(t_1)| = |\theta(t_2)| = \theta_\delta$, where $\theta_\delta > 0$ is the slope angle of the left wall part. Then Lemma 2.4.3 implies that

$$E[\gamma] \geq \varepsilon^2 \int_\gamma \kappa^2 ds \geq \varepsilon^2 \frac{4f(\theta_\delta)^2}{x(t_2) - x(t_1)} \geq \frac{2\varepsilon^2 f(\theta_\delta)^2}{\delta}$$

and especially (2.4.3). The proof is now complete. □

Theorem 2.4.4 indicates that the smallness of the height of ψ does not imply the graph representations of minimizers. In this view, we can simplify the statement as

Corollary 2.4.5. *For any $h > 0$, there exist ε, α, and smooth ψ of height h such that any minimizer of (2.2.3) must overhang.*

In addition, as mentioned in the previous subsection, $h = 1$ gives the optimal upper bound $1/3$ for α in our method. In this view, Theorem 2.4.4 is simplified as

Corollary 2.4.6. *For any $0 < \alpha < 1/3$, there exist ε and smooth ψ such that any minimizer of (2.2.3) must overhang.*

2.4.3 For Lipschitz substrates

Finally, for small α, we give an example of a Lipschitz (singularly folding) substrate with large slope such that any minimizer must be overhanging for "any" small ε. This kind of uniformity is mathematically important. An intuitive meaning of this result has been given in Introduction.

We shall state it as a proposition. Let $h > 0$ and $0 < 2\delta < \min\{1, h\}$. A 1-periodic function ϕ is called δ-Lipschitz fakir carpet of height h if

$$\phi(x) := \max\left\{0, h - \left|\frac{h}{\delta}x - \frac{1}{2}\right|\right\}$$

for $x \in [0, 1]$. We also define the base and wall parts as well as the smooth case; namely, the base part is the part with $y = \phi(x) = 0$ and the left (resp. right) part is the part with $y = \phi(x)$, $\delta < y < h - \delta$ and $\phi(x)' > 0$ (resp. $\phi'(x) < 0$).

Theorem 2.4.7. *Let $h > 0$ and $\alpha < \Delta := \frac{\min\{1,h\}}{1+2h}$. Then there exist $\bar{\varepsilon} > 0$ and $\bar{\delta} > 0$ such that, for any $0 < \varepsilon < \bar{\varepsilon}$ and the δ-Lipschitz fakir carpet substrate ψ_δ of height h with any $0 < \delta < \bar{\delta}$, any minimizer of (2.2.3) is overhanging.*

Proof. Noting the condition of α, in the same way as Case 1 and Case 2 in Theorem 2.4.4, we see that there are $\delta_0 > 0$ and $\varepsilon_0 > 0$ such that, for any $0 < \delta < \delta_0$ and $0 < \varepsilon < \varepsilon_0$, any non-overhanging curve is necessary to touch the left and right wall parts in order to minimize E. Note that for arbitrary small $\varepsilon > 0$ an overhanging competitor as in Figure 2.10 is well-defined since the substrate ψ_δ is folding singularly.

To complete the proof, we shall prove that, for any small δ, ε, and any non-overhanging γ touching both the wall parts, there is an overhanging competitor $\hat{\gamma}$ such that $E[\gamma] > E[\hat{\gamma}]$.

Fix arbitrary $0 < \delta < \delta_0$ and $0 < \varepsilon < \varepsilon_0$ and take any non-overhanging $\gamma = (x, y) \in \mathcal{A}$ touching both the wall parts. Then there are times $t_1 < t_2$ such that γ touches the

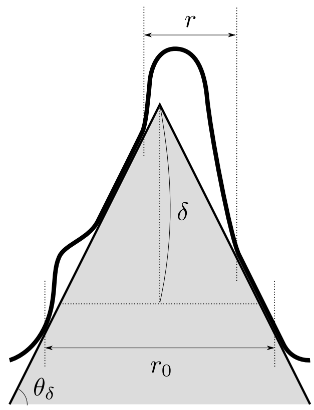

FIGURE 2.13: Non overhanging curve γ touching the left and right wall parts. There are two touching points $\gamma(t_1)$ and $\gamma(t_2)$ of height less than $h - \delta$. There are also two points such that γ is tangent to the left and right slope there but does not touch ψ_δ between them.

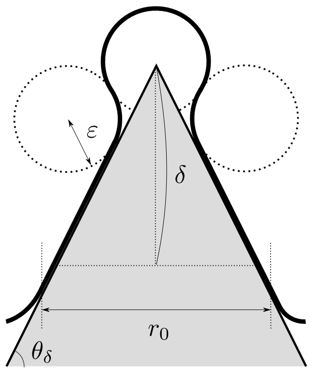

FIGURE 2.14: The curve of Figure 2.13 modified from $\gamma(t_1)$ to $\gamma(t_2)$. The modified part consists of straightly adhering parts and a freely bending part with radius ε. This curve is well-defined and not self-intersecting whenever $\varepsilon \ll \delta$ and moreover overhanging whenever $\theta_\delta > \pi/3$.

left (resp. right) wall part at t_1 (resp. t_2). Define $t_3 \in [t_1, t_2]$ (resp. $t_4 \in [t_1, t_2]$) as the supremum (resp. infimum) of time $t \in [t_1, t_2]$ such that $\gamma(t)$ touches ψ_δ and $x(t), 1/2$ (resp. $x(t) > 1/2$). Note that $\theta(t_3) = -\theta(t_4) = \theta_\delta$, where $\theta_\delta > 0$ denotes the slope angle of ψ_δ. Moreover, in (t_3, t_4) the curve γ does not touch ψ_δ except at the vertex $(1/2, \psi_\delta(1/2))$. Denote $r_0 = x(t_2) - x(t_1)$ and $r = x(t_4) - x(t_3)$ as in Figure 2.13.

Then, by Lemma 2.4.3 and the fact that γ circumvents the vertex of ψ_δ freely (except the vertex), the energy E of the part of γ from t_3 to t_4 is bounded below as

$$E[\gamma|_{[t_3,t_4]}] \geq \varepsilon^2 \frac{4f(\theta_\delta)^2}{r} + \frac{r}{\cos\theta_\delta}.$$

In addition, the part from t_1 to t_3 and from t_4 to t_2 is totally bounded below as

$$E[\gamma|_{[t_1,t_3]}] + E[\gamma|_{[t_4,t_2]}] \geq \alpha \cdot \frac{r_0 - r}{\cos\theta_\delta}$$

since the energies of $\gamma|_{[t_1,t_3]}$ and $\gamma|_{[t_4,t_2]}$ are more than or equal to the energies of the completely adhering straight lines joining the endpoints of $\gamma|_{[t_1,t_3]}$ and $\gamma|_{[t_4,t_2]}$, respectively. Therefore, the part of γ from t_1 to t_2 is bounded below as

$$\begin{aligned} E[\gamma|_{[t_1,t_2]}] &\geq \frac{\varepsilon^2 4f(\theta_\delta)^2}{r} + \frac{r}{\cos\theta_\delta} + \frac{\alpha(r_0 - r)}{\cos\theta_\delta} \\ &= \varepsilon^2 4f(\theta_\delta)^2 \frac{1}{r} + \frac{1-\alpha}{\cos\theta_\delta} r + \frac{\alpha r_0}{\cos\theta_\delta} \\ &\geq \left(2\sqrt{4f(\theta_\delta)^2 \frac{1-\alpha}{\cos\theta_\delta}}\right)\varepsilon + \frac{\alpha r_0}{\cos\theta_\delta}, \end{aligned}$$

which does not depend on r.

On the other hand, providing that δ and ε are sufficiently small as $\theta_\delta > \pi/3$ and $\varepsilon < \delta/3$, the competitor $\hat{\gamma}$ constructed by modifying γ in (t_1, t_2) as in Figure 2.14 is well-defined and overhanging. The energy of $\hat{\gamma}$ from t_1 to t_2 is bounded above as

$$\begin{aligned} E[\hat{\gamma}|_{[t_1,t_2]}] &\leq 6\pi\varepsilon\left[\varepsilon^2 \frac{1}{\varepsilon^2} + 1\right] + \alpha \cdot \frac{r_0}{\cos\theta_\delta} \\ &= 12\pi\varepsilon + \frac{\alpha r_0}{\cos\theta_\delta}. \end{aligned}$$

In the outside of (t_1, t_2), the curves $\hat{\gamma}$ and γ coincide.

Consequently, noting that for any small $\delta > 0$

$$2\sqrt{4f(\theta_\delta)^2 \frac{1-\alpha}{\cos\theta_\delta}} > 12\pi,$$

we have $E[\gamma] > E[\hat{\gamma}]$ for any small δ and ε. The proof is now complete. □

2.5 Discussion

In this last section, we give some further remarks and discussions.

2.5.1 Small bending scale

We first discuss the graph representations of minimizers for small bending scale ε. Recall that Theorem 2.3.3 states that large bending scale $\varepsilon \gg 1$ implies the graph representation independently of ψ. This theorem is relatively easy to prove since, if $\varepsilon \gg 1$, the periodic boundary condition is effective and hence non-graph curves must have large energies. On the other hand, the case that $\varepsilon \ll 1$ is not easy to obtain the graph representation rigorously since there is no large difference in the energies of graph and non-graph curves. In fact, Theorem 2.4.4 and Theorem 2.4.7 state that overhanging minimizers exist when ε is small and the minimal bending scale $r = \|\psi''\|_\infty^{-1}$ of ψ is much smaller. However, we may expect the graph representation when $\varepsilon \ll r$ by the following formal observation. When ψ is smooth and $\varepsilon = 0$, minimizers would be Lipschitz functions with straight free parts as in Figure 2.15. They have edge singularities at the contact points as valley folds and the minimal distance d among the singularities is bounded below (by a constant depending on α and ψ). Thus, for small $\varepsilon \ll \min\{d, r\}$, any minimizer would be obtained by modifying such valley folds smoothly as in Figure 2.15. Moreover, in contrast to mountain folds (Figure 2.2),

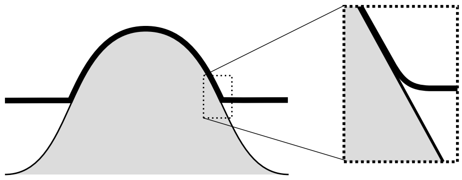

FIGURE 2.15: Minimizer on a smooth substrate for small ε. If $\varepsilon = 0$ then the minimizer has valley fold singularities as the left. When ε is small the singularities would be modified as smooth curves with scale ε as the right.

the modification of valley folds would not require to increase the slopes. Hence any minimizer is expected to be a graph curve. This observation is rigorously justified in the next chapter (Chapter 3).

2.5.2 Flat substrates

Theorem 2.3.4 states that a second order flatness of ψ implies the graph representations of minimizers; for any $\varepsilon > 0$ and $0 < \alpha < 1$ there is $k > 0$ such that if ψ satisfies $\|\psi''\|_\infty \leq k$ then any minimizer is a graph curve. The problem would become more difficult if we replace ψ'' with ψ' or ψ.

Another interesting problem is the following uniform and strong version: is there $k > 0$ such that for any ε, α, and ψ with $\|\psi''\|_\infty \leq k$ or $\|\psi'\|_\infty \leq k$ any minimizer is a graph curve? Notice that Theorem 2.4.7 states that any smallness of $\|\psi\|_\infty$ does not imply the above conclusion.

2.5.3 On self-intersections

Self-intersections are more difficult to occur than overhangs in the sense that any self-intersecting curve in $\mathcal{A}$ must be overhanging. Here we say that $\gamma \in \mathcal{A}$ has a self-intersection if there are $0 \leq t_1 < t_2 < 1$ and $m \in \mathbb{Z}$ such that $\gamma(t_1) = \gamma(t_2) + (m, 0) \in \mathbb{R}^2$. This definition is suitable for our periodic setting; if we take $\tilde{\gamma} \in H^2_{loc}(\mathbb{R}; \mathbb{R}^2)$ such that $\tilde{\gamma}(t+1) = \tilde{\gamma}(t) + (1, 0)$ for any $t \in \mathbb{R}$, then $\tilde{\gamma}$ is not injective if and only if the restriction $\gamma = \tilde{\gamma}|_I \in \mathcal{A}$ has a self-intersection in the above sense. In this chapter we proved the existence of overhanging minimizers, but it is not clear whether there exist self-intersecting minimizers in our setting.

Our admissible curves $\gamma \in \mathcal{A}$ may have any self-intersection (self-contact and self-crossing) as mentioned, and thus our problem would be a priori suitable only for filaments but not membranes. To make our problem compatible with membranes, we especially need to exclude self-crossing curves. To this end, let $\mathcal{A}_* \subset \mathcal{A}$ be the H^2-weak closure of the set of curves without self-intersection. This set is compatible with membranes since $\mathcal{A}_*$ consists of non self-intersecting curves and limits of such curves at least in C^1; especially, any curve of $\mathcal{A}_*$ is not self-crossing but only self-contacting. Then all the results in this chapter are valid even if we replace $\mathcal{A}$ with $\mathcal{A}_*$ in the problem (2.2.3) since all the competitors used in our proof have no self-intersection. The existence of minimizers is proved in the same way as 2.A since $\mathcal{A}_*$ is H^2-weakly closed.

We mention that this kind of self-contact setting has been considered in e.g. [8, 11]. In particular, the paper [8] proves that, for confined closed free elasticae, (i) any convex confinement admits only convex minimizers, which especially have no self-intersection, and (ii) there is a confinement with two halls which admits a self-contacting minimizer. These results indicate that whether minimizers have self-contacts crucially relates to the simplicity of confinements. Our graph confinements are simple but the effect of adhesion make curves easier to form complicated shapes, and hence the self-intersection problem becomes more nontrivial.

2.5.4 Local minimizers

We next give a theoretical discussion on local minimizers. A curve $\gamma_0 \in \mathcal{A}$ is called a local minimizer if there is $\delta > 0$ such that $E[\gamma] \geq E[\gamma_0]$ for any $\gamma \in \mathcal{A}$ with $\|\gamma - \gamma_0\|_{H^2} \leq \delta$.

A straight line not touching ψ is obviously a graph local minimizer in any case. However, the straight line touching ψ is not necessarily a local minimizer. The existence of graph local minimizers touching ψ is not trivial.

Moreover, it is shown that there are infinitely many self-intersecting local minimizers in $\mathcal{A}$ by using a kind of winding number (as in Section 1.A of Chapter 1). For $\gamma \in \mathcal{A}$, the winding number $N_\gamma \in \mathbb{Z}$ is defined so that $2\pi N_\gamma$ is equal to the total curvature, or equivalently

$$2\pi N_\gamma = \theta(1) - \theta(0),$$

where $\theta : \bar{I} \to \mathbb{R}$ is a continuous representation of tangential angle (unique up to addition by a constant of $2\pi\mathbb{Z}$). The winding number is obviously continuous with respect to the C^1-topology, and hence continuous with respect to the weak and strong H^2-topologies. We denote by $\mathcal{A}_n \subset \mathcal{A}$ the set of all curves with $N_\gamma = n$. Then, since N_γ is discrete-valued and (weakly and strongly) continuous on $\mathcal{A}$, for any $n \in \mathbb{Z}$ the set $\mathcal{A}_n$ is open and closed in $\mathcal{A}$ with respect to both the weak and strong H^2-topologies. Since $\mathcal{A}_n$ is weakly closed, in the same way as 2.A, we can prove that for any $n \in \mathbb{Z}$ there is a minimizer of E among $\mathcal{A}_n$. Then, since $\mathcal{A}_n$ is strongly open, it turns out that such a minimizing curve is nothing but a local minimizer in the whole space $\mathcal{A}$. Any curve with $N_\gamma \neq 0$ has a self-intersection, and thus there are infinitely many self-intersecting (and overhanging) local minimizers in $\mathcal{A}$. In the membrane setting $\mathcal{A}_*$, the above argument does not work since the winding number of any curve is zero, and thus the existence of overhanging local minimizers is nontrivial.

2.5.5 Periodic boundary condition

We finally give a brief remark on periodicity. In this chapter we assumed that admissible curves and minimizers have a same period, but the paper [17] proposes a numerical example of a global minimizer of a period several times a substrate period. Hence, physically, a more natural assumption is that, if an original substrate has a period λ, then a minimizer γ has the period $n\lambda$ for a positive integer n. It is not easy to determine n for a general case. However, in terms of scale, our assumption would be formally justified. In fact, the elasto-capillary length scale $\ell = \sqrt{C/\sigma_F}$ may be interpreted as the optimal bending scale of a minimizer (as in [15, 34]), and thus we would formally expect that a minimizing curve crosses over several periods of a substrate if and only if the scales ℓ and λ balance each other out ($\ell \sim \lambda$), where λ is the original substrate period (wavelength). In our normalized setting, this balance is described as $\varepsilon \sim 1$. The main concerns in this chapter are the cases that $\varepsilon \gg 1$ and $\varepsilon \ll 1$ (even though our results give more precise conditions), and hence our periodic assumption would not be restrictive from this viewpoint. The case that $\varepsilon \sim 1$ is of course more interesting and challenging, but our study is a first step and does not address the precise analysis.

Appendices

2.A Existence of minimizers

We confirm the existence theorem (Theorem 2.2.1) by a direct method in the calculus of variations. In this part we deal with a more general σ; we assume that the values of σ may depend on the positions in $\partial\Omega$. However, we still assume that $\sigma \geq \sigma_B$ holds for a positive constant $\sigma_B > 0$, and $\sigma \equiv \sigma_F$ in Ω for a constant $\sigma_F > \sigma_B$.

Proof of Theorem 2.2.1. We first note that $\inf_{\mathcal{A}} E \leq \sigma_F$ by (2.2.4) and the case $\inf_{\mathcal{A}} E = \sigma_F$ is trivial since a trivial straight line competitor is nothing but a minimizer. Thus we may assume that $\inf_{\mathcal{A}} E < \sigma_F$.

Take a minimizing sequence $\{\gamma_n\}_n \subset \mathcal{A}$ such that

$$\sigma_F > E[\gamma_n] \to \inf_{\mathcal{A}} E \ (\geq \sigma_B).$$

Without loss of generality, we may assume that all the curves are of constant speed. In this case, the total energy of γ_n is represented as

$$E[\gamma_n] = \frac{C}{2L_{\gamma_n}^3} \int_I |\ddot{\gamma}_n(t)|^2 dt + L_{\gamma_n} \int_I \sigma(\gamma_n(t)) dt.$$

Now we obtain the boundedness of $\{\gamma_n\}_n$ in $H^2(I;\mathbb{R}^2)$. Since $L_{\gamma_n}\sigma_B \leq E[\gamma_n]$, the sequence $\{L_{\gamma_n}\}_n$ is bounded. Thus, since γ_n is of constant speed, the sequence of $\|\dot{\gamma}_n\|_2 = L_{\gamma_n}$ is also bounded. Moreover, since $\frac{C}{2L_{\gamma_n}^3}\|\ddot{\gamma}_n\|_2^2 \leq E[\gamma_n]$, the sequence of $\|\ddot{\gamma}_n\|_2$ is also bounded. Finally, since $E[\gamma_n] < \sigma_F$, we see that all the curves γ_n must touch $\partial\Omega$. Combining this fact with the uniformly boundedness of length and the periodic boundary condition, we find that the sequence of $\|\gamma_n\|_\infty$ is bounded, and thus the sequence of $\|\gamma_n\|_2$ is also bounded. Therefore, the sequence $\{\gamma_n\}_n$ is bounded in $H^2(I;\mathbb{R}^2)$.

Noting that $H^2(I;\mathbb{R}^2)$ is compactly embedded in $C^1(\bar{I};\mathbb{R}^2)$, there exists $\gamma \in H^2(I;\mathbb{R}^2)$ such that, up to a subsequence (not relabeled), γ_n converges to γ in C^1 and weakly in H^2. Notice that $\gamma \in \mathcal{A}$, the curve γ is of constant speed, and $L_{\gamma_n} \to L_\gamma \geq 1$. It only remains to prove $\liminf_{n\to\infty} E[\gamma_n] \geq E[\gamma]$. The lower semicontinuity of σ and Fatou's lemma imply

$$\liminf_{n\to\infty} \int_I \sigma(\gamma_n(t)) dt \geq \int_I \sigma(\gamma(t)) dt.$$

Moreover, $\liminf_{n\to\infty} \|\ddot{\gamma}_n\|_2 \geq \|\ddot{\gamma}\|_2$ holds since $\ddot{\gamma}_n \to \ddot{\gamma}$ weakly in L^2. Noting the convergence of length, we obtain the lower semicontinuity of E. Consequently, the curve γ is a minimizer. $\square$

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Chapter 3

A free boundary problem for elastic curves II: Singular limit

This chapter is a direct continuation of Chapter 2. In this chapter we deal with the almost same adhesion problem on elastic curves as Chapter 2 for smooth substrates. The only difference from the previous chapter is that the adhesion coefficient may be inhomogeneous, that is, its values may depend on the positions. In this chapter we consider the case of small bending rigidity. For a general periodic graph substrate of class C^2, we first study the case of no bending rigidity precisely, and then obtain a singular limit result in terms of Γ-convergence in the small bending rigidity limit. The singular limit result is applied to obtaining the precise convergences of sequences of global minimizers. As a corollary, we prove that any global minimizer is represented by a graph providing that the bending rigidity is small.

Keywords: Free boundary problem; Euler's elastica; Obstacle problem; Γ-convergence; Singular perturbation; Graph representation.

3.1 Introduction

3.1.1 Problem

Let us briefly recall our adhesion problem. Let $\Omega = \{y > \psi(x)\} \subset \mathbb{R}^2$, where $\psi \in C^2(\mathbb{R})$ is a given function (substrate) with the periodicity $\psi(x+1) = \psi(x)$. We consider only C^2-substrates in this chapter. For a planar curve γ constrained in the closure $\overline{\Omega}$, the total energy is defined as

$$E_\varepsilon[\gamma] = \varepsilon^2 \int_\gamma \kappa^2 ds + \int_\gamma \Theta(\gamma) ds,$$

where κ denotes the curvature, and s denotes the arc length parameter. Here $\varepsilon > 0$ is a given constant, and the function $\Theta : \overline{\Omega} \to \mathbb{R}$ is a contact potential function defined as $\Theta \equiv 1$ in Ω and $\Theta(x,y) = \alpha(x,y)$ on $\partial\Omega$, where $\alpha : \partial\Omega \to (0,1)$ is a given uniformly continuous function with the periodicity $\alpha(x+1,y) = \alpha(x,y)$. Unlike Chapter 2, the adhesion coefficient α may be inhomogeneous (i.e., depending on the position). Then our problem is formulated as

$$\min_{\gamma \in X^{2,2}} E_\varepsilon[\gamma],$$

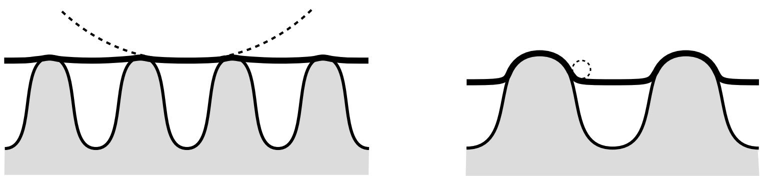

FIGURE 3.1: Graph minimizers for $\varepsilon \to 1$ (left) and $\varepsilon \ll r$ (right), where 1 is the period of the substrate ψ and $r = 1/\|\psi''\|_\infty$.

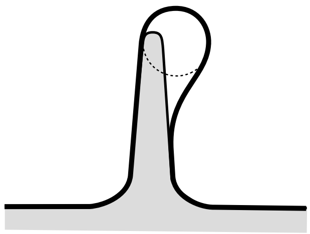

FIGURE 3.2: An overhanging minimizer for $r \ll \varepsilon \ll 1$.

where $X^{2,2}$ is a certain space of periodic $W^{2,2}$-curves (see Section 3.4). See Chapters 1, 2 or the paper [9] for more details of the backgrounds and references on our problem.

We also recall that, at least formally, the constant ε corresponds to the "minimal bending scale", i.e., for any minimizing curve, the minimal radius of curvature in the free part would have the scale ε. As shown in the previous chapter, comparing the constant ε and typical quantities of ψ plays an important role for the shape analysis on minimizers. In the previous chapter, we especially proved the following facts; if $\varepsilon \to \infty$, then any minimizer is a graph curve (Figure 3.1 left); and if $r \ll \varepsilon \ll 1$ (where $r := 1/\|\psi''\|_\infty$ is the minimal bending scale of ψ), then there is a chance that a minimizer overhangs (Figure 3.2).

3.1.2 Main results

The purpose of this chapter is to study the remaining case $\varepsilon \ll r$. In this case, in view of the minimal bending scale, we expect that any minimizer is represented by a graph as in the right of Figure 3.1. In this chapter we prove that this expectation is true in the sense that, for any fixed α and ψ of class C^2, there is small $\bar{\varepsilon} > 0$ such that for any $\varepsilon \in (0, \bar{\varepsilon})$ any minimizer of E_ε is represented by a graph (Corollary 3.6.1).

The above intuitive expectation is however not easy to justify rigorously. For the justification, we take the following steps; we first prove that in the case that $\varepsilon = 0$ any minimizer is represented by a Lipschitz graph (Figure 3.3 right); we then prove that for any subsequence of minimizers of E_ε there is a subsequence converging to a minimizer of E_0 as $\varepsilon \to 0$ in a certain sense (Figure 3.3 left). Our convergence result is enough sharp to conclude the graph representations of minimizers for small ε. In what follows, we explain the above steps more precisely. Throughout this paper, we deal with constant speed curves in principle.

If $\varepsilon = 0$, then our problem degenerates in the sense that the higher order term of the energy vanishes. In this case, the setting of $W^{2,2}$-curves is not suitable, and it is appropriate to take admissible curves as Lipschitz curves. In fact, due to the adhesion

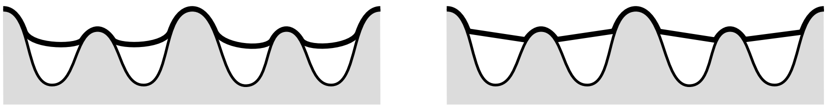

FIGURE 3.3: Minimizers for small ε (left), and $\varepsilon = 0$ (right).

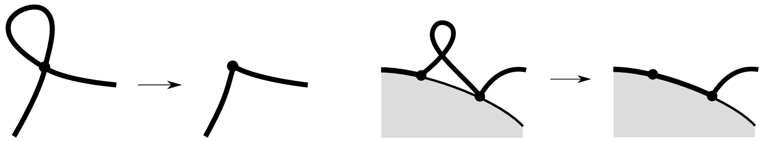

FIGURE 3.4: Cutting a loop (left), and replacing a free part with an adhering curve (right): the procedures are always allowed if $\varepsilon = 0$.

effect, any minimizer is expected to have "edge" singularities at the contact points as in the right of Figure 3.3. Our first theorem (Theorem 3.3.7) states that any minimizer is represented as the graph of a Lipschitz function as in the right of Figure 3.3. To be more precise, we first prove that any minimizer has a finite number of segment parts (free parts). This property is naturally expected since the effect of adhesion implies that a "small free part" should be replaced by an adhering curve in order to minimize E_0 as in the right of Figure 3.4. We then confirm that any minimizer has no self-intersection. This is also natural since any "loop" should be cut to minimize E_0 as in the left of Figure 3.4. By using the fact of no self-intersection, we prove that any minimizer is represented by the graph of a Lipschitz function. In this part we employ the Jordan curve theorem to control the global behavior of minimizers. Moreover, we also prove that the angles θ of edges are determined by the adhesion coefficient α at the contact points $p \in \partial\Omega$; any contact angle θ satisfies Young's equation $\cos\theta = \alpha(p)$. We mention that similar facts have been proved in the author's previous paper [9], but the paper [9] deals with only graph curves. In the present chapter, the arguments are extended to general non-graph curves. In particular, obtaining the graph representation of a minimizer is a totally new part.

We then study the convergence of minimizers of E_ε as $\varepsilon \to 0$. Since our general curve setting implies an L^∞-compactness, it is easy to prove that any sequence of minimizers of E_ε has a subsequence converging to a minimizer of E_0 in L^∞. However, this convergence does not imply the graph representations of minimizers. Our goal is to prove an "adhesion convergence", i.e., up to a subsequence, any sequence of minimizers $\{\gamma_\varepsilon\}_\varepsilon$ converges to a minimizer γ of E_0 in the sense that γ_ε has the same number of contact points as γ for small ε, and moreover the positions of the contact points of γ_ε converge to those of γ as $\varepsilon \to 0$ (see Definition 3.5.1 for details). Combining the adhesion convergence with the straightening result in Chapter 1, we obtain the graph representation theorem as in Corollary 3.6.1.

The adhesion convergence is proved by a first order expansion of E_ε. For minimizers γ_ε, it is not difficult to prove the zeroth order energy expansion, i.e., as $\varepsilon \to 0$,

$$E_\varepsilon[\gamma_\varepsilon] = \min E_0 + o(1). \tag{3.1.1}$$

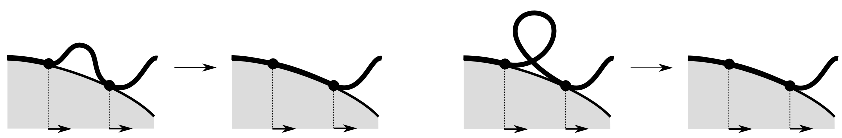

FIGURE 3.5: Replacing free parts with adhering curves: these are allowed even if $\varepsilon > 0$ while the endpoints are "ordered".

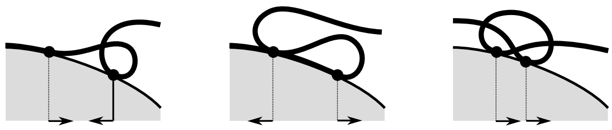

FIGURE 3.6: Examples of free parts that may not be replaced by adhering curves when $\varepsilon > 0$.

This expansion implies the following facts; up to a subsequence, the sequence of γ_ε converges to a minimizer γ of E_0 in L^∞, and moreover γ_ε "almost" adheres to the substrate near the part where γ adheres. If admissible curves are supposed to be graph curves, then the above facts imply the adhesion convergence with comparative ease. In fact, in the graph setting, we may always replace a small free part with an adhering curve as in the left of Figure 3.5, and this procedure makes the energy E_ε strictly smaller while ε is sufficiently small; hence, we find that for small ε a minimizer is not "almost" but "completely" adhering to the substrate near the part where γ adheres (as in Figure 3.3). However, our admissible curves are not necessarily to be graph curves, and hence the problem is more involved. In particular, the above replacing procedure is not necessarily allowed in general; Figure 3.5 and Figure 3.6 indicate some allowable and non-allowable cases, respectively. Indeed, there are sequences of admissible curves which satisfy (3.1.1) but the keep having small free parts as in Figure 3.6. In other words, the zeroth order expansion (3.1.1) is not enough sharp to exclude such planar tangles.

In order to confirm that minimizers γ_ε of E_ε have no planar tangles, we obtain the following first order energy expansion (Corollary 3.4.7):

$$E_\varepsilon[\gamma_\varepsilon] = \min E_0 + m\varepsilon + o(\varepsilon) \tag{3.1.2}$$

as $\varepsilon \to 0$, where $m \geq 0$ is a constant depending on ψ and α. As shown in Chapter 1, the first order expansion of an elastic energy is expected to control the precise shapes of minimizers. We prove that in fact the expansion (3.1.2) is enough sharp to exclude planar tangles. Accordingly, we obtain our main result (Theorem 3.5.2) which states that any L^∞-convergent sequence of minimizers satisfies the adhesion convergence.

The expansion (3.1.2) is a direct corollary of Theorem 3.4.6, which is one of our main results. Theorem 3.4.6 provides the first order asymptotic expansion of the energy E_ε as

$$E_\varepsilon = E_0 + \varepsilon F + o(\varepsilon)$$

in the sense of Γ-convergence (cf. [1, 5]) with respect to the L^∞-topology. The singular limit energy F is defined for any minimizer γ of E_0 as

$$F[\gamma] = \int_{C_\gamma} 4(\sqrt{2} - \sqrt{1+\alpha})d\mathcal{H}^0,$$

where $C_\gamma \subset \partial\Omega$ denotes the set of the contact points of γ, and $\mathcal{H}^0$ denotes the zero-dimensional Hausdorff measure. The energy F counts the number of contact points weighted by the adhesion coefficient α. The constant m in (3.1.2) is nothing but the minimum of F. Theorem 3.4.6 is a generalization of the main theorem in the aforementioned paper [9], which provides the same asymptotic expansion in a graph setting (with a different boundary condition and a slightly stronger topology). We emphasize that our generalization is not a mere graph-to-curve extension, but quite meaningful. In fact, thanks to our general curve setting, the Γ-expansion result obtained in the present paper is compatible with the equi-coerciveness of E_ε. Such a compatibility is not valid in the previous $W^{1,1}$-graph setting in [9] due to the lack of compactness. The compatibility is important since it is not until the equi-coerciveness (compactness) holds that any sequence of minimizers of E_ε has a subsequence converging to a minimizer of F; the Γ-expansion itself implies only that any "convergent" sequence of minimizers of E_ε converges to a minimizer of F. Minimizing F yields a selection principle among minimizers of E_0 as stated in [9], and our generalization is the first result to completely justify the selection principle for E_ε as $\varepsilon \to 0$.

3.1.3 Organization

We prepare definitions and state basic properties on E_0 in Section 3.2. We then prove precise properties of minimizers of E_0 in Section 3.3. We prove the Γ-expansion of E_ε in Section 3.4. By using the expansion we prove the main theorem of adhesion convergence in Section 3.5. Finally, we prove the graph representation result in Section 3.6.

3.2 Preliminaries on the energy E_0

We first prepare some terminologies for general curves. Let J be a bounded open interval, and $\bar{J}$ be its closure. Let $W^{1,\infty}(J;\mathbb{R}^2)$ be the set of vector-valued $W^{1,\infty}$-Sobolev curves. We denote the length of $\gamma \in W^{1,\infty}(J;\mathbb{R}^2)$ by

$$\mathcal{L}[\gamma] = \int_J |\dot{\gamma}(t)|dt.$$

A curve $\gamma \in W^{1,\infty}(J;\mathbb{R}^2)$ is called *regular* if there is a positive constant $c > 0$ such that $|\dot{\gamma}| \geq c$ a.e. in J. Any regular curve has a positive length. We remark that a curve $\gamma \in W^{1,\infty}(J;\mathbb{R}^2)$ may not be regular in general. A curve $\gamma \in W^{1,\infty}(J;\mathbb{R}^2)$ is called *constant speed* if there is $L \geq 0$ such that $|\dot{\gamma}| = L$ a.e. in J. In this case, the constant L is nothing but the length, but often called the speed of γ.

A well-known fact identifies the set $W^{1,\infty}(J;\mathbb{R}^2)$ with the set $\mathrm{Lip}(\bar{J};\mathbb{R}^2)$, namely, the set of vector-valued Lipschitz functions defined on the closure of J. Hence, we may regard that any curve $\gamma \in W^{1,\infty}(J;\mathbb{R}^2)$ is defined pointwise in $\bar{J}$ (up to the endpoints). Moreover, any Lipschitz curve $\gamma \in \mathrm{Lip}(\bar{J};\mathbb{R}^2)$ is differentiable almost everywhere in $\bar{J}$ by Rademacher's theorem, and this classical derivative coincides with the weak derivative almost everywhere. Theses facts are frequently used in this paper without notice.

We also recall the following compactness, which is a direct consequence of the Arzelà-Ascoli theorem (see Appendix 3.A).

Lemma 3.2.1. *Let $\{\gamma_k\}_k \subset W^{1,\infty}(J;\mathbb{R}^2)$ be a bounded sequence in $W^{1,\infty}$. Then there are $\gamma \in W^{1,\infty}(J;\mathbb{R}^2)$ and a subsequence $\{\gamma_{k'}\}_{k'}$ such that $\gamma_{k'} \to \gamma$ in L^∞. Moreover,*

$$\liminf_{k'\to\infty} \|\dot{\gamma}_{k'}\|_\infty \geq \|\dot{\gamma}\|_\infty.$$

Now we shall define our problem. Throughout this paper, we assume that a domain $\Omega \subset \mathbb{R}^2$ is admissible in the following sense.

Definition 3.2.2 (Admissible domain)**.** A domain $\Omega \subset \mathbb{R}^2$ with C^2-boundary $\partial\Omega$ is called *admissible* if $\Omega = \{y > \psi(x)\}$, where $\psi \in C^2(\mathbb{R})$ is a function such that $\psi(x+1) = \psi(x)$ for any $x \in \mathbb{R}$ and ψ attains its maximum at the origin.

For any given admissible $\Omega \subset \mathbb{R}^2$, there is a (sufficiently small) positive constant $R_\Omega > 0$ such that the curvature of the boundary $\kappa_{\partial\Omega}$ satisfies

$$\max_{p\in\partial\Omega} |\kappa_{\partial\Omega}(p)| \leq 1/R_\Omega,$$

and for any $p \in \partial\Omega$ the set $\partial\Omega \cap B_{R_\Omega}(p)$ is path-connected and represented by the graph of a C^2-function in the normal direction of $\partial\Omega$. Here $B_{R_\Omega}(p) \subset \mathbb{R}^2$ denotes the open ball of radius R_Ω centered at p. We call R_Ω *admissible constant* of Ω.

Remark 3.2.3 (Geodesic in boundary). Some of our arguments in this paper may work for more general open sets. However, in this paper, we restrict ourselves to only admissible domains in order to avoid difficulties due to the topological generality of boundaries. For an admissible domain Ω, it is obvious that for any two different points in the boundary $p_0, p_1 \in \partial\Omega$, there exists a constant speed C^2-curve $\gamma : [t_0, t_1] \to \partial\Omega$ such that $\gamma(t_0) = p_0$ and $\gamma(t_1) = p_1$. Such a curve is unique up to the choice of domain $[t_0, t_1]$. We call such γ *geodesic* in $\partial\Omega$ joining p_0 to p_1.

Remark 3.2.4. The assumption that ψ attains its maximum at the origin is just a normalization due to the periodicity, and not restrictive.

Let I be the open unit interval $(0,1)$. We say that a curve $\gamma \in W^{1,\infty}(I;\mathbb{R}^2)$ is *confined* in $\overline{\Omega}$ if $\gamma \in W^{1,\infty}(I;\overline{\Omega})$, or in other words, $\gamma(t) \in \overline{\Omega}$ for any $t \in \bar{I}$. A curve $\gamma = (x,y) \in W^{1,\infty}(I;\mathbb{R}^2)$ is called *admissible* if γ is constant speed and confined in $\overline{\Omega}$, and moreover satisfies the following periodic boundary condition:

$$x(0) = 0, \; x(1) = 1, \; y(0) = y(1).$$

Denote by $X^{1,\infty} \subset W^{1,\infty}(I;\mathbb{R}^2)$ the set of all admissible curves.

For a curve $\gamma \in X^{1,\infty}$, we define the total energy by

$$E_0[\gamma] := \int_\gamma \Theta(\gamma) ds := \int_I \Theta(\gamma(t))|\dot{\gamma}(t)|dt.$$

Here $\Theta : \overline{\Omega} \to \mathbb{R}$ is the following lower semicontinuous contact potential:

$$\Theta(x,y) = \begin{cases} 1 & \text{in } \Omega, \\ \alpha(x,y) & \text{on } \partial\Omega, \end{cases}$$

where $\alpha : \partial\Omega \to (0,1)$ is a uniformly continuous function satisfying the 1-periodicity with respect to $x \in \mathbb{R}$, i.e., $\alpha(x,y) = \alpha(x+1,y)$ for any $(x,y) \in \mathbb{R}^2$. Notice that $0 < \underline{\alpha} \leq \bar{\alpha} < 1$, where

$$\underline{\alpha} := \min_{(x,y)\in\partial\Omega} \alpha(x,y), \quad \bar{\alpha} := \max_{(x,y)\in\partial\Omega} \alpha(x,y).$$

For a curve γ defined on an interval J, and for a subset $J' \subset J$ (at least measurable), the notation $\gamma|_{J'}$ denotes the restriction of γ to J. The energy E_0 is also defined for a restricted curve.

We exhibit some basic properties on the energy E_0, the proofs of which are given in Appendix 3.A. First, we often use the constant speed reparameterization hereafter based on the following lemma.

Lemma 3.2.5. *For any $\gamma \in W^{1,\infty}(J;\mathbb{R}^2)$, which may not be regular, there is a constant speed reparameterization of γ, i.e., there are a nondecreasing continuous surjective function $\tau : \bar{J} \to \bar{J}$ and a constant speed curve $\hat{\gamma} \in W^{1,\infty}(J;\mathbb{R}^2)$ such that $\gamma = \hat{\gamma} \circ \tau$. Moreover,*

$$E_0[\gamma] = E_0[\hat{\gamma}].$$

Since Θ is lower semicontinuous, the energy E_0 satisfies the following L^1-lower semicontinuity.

Lemma 3.2.6. *Let $\{\gamma_k\}_k \subset W^{1,\infty}(J;\overline{\Omega})$ and $\gamma \in L^\infty(J;\overline{\Omega})$. Suppose that $\{\gamma_k\}_k$ is bounded in $W^{1,\infty}$ and $\gamma_k \to \gamma$ in L^1. Then $\gamma_k \to \gamma$ in L^∞ and $\gamma \in W^{1,\infty}(J;\overline{\Omega})$ with*

$$\liminf_{k\to\infty} \|\dot{\gamma}_k\|_\infty \geq \|\dot{\gamma}\|_\infty.$$

Moreover,

$$\liminf_{k\to\infty} E_0[\gamma_k] \geq E_0[\gamma].$$

In addition, if a convergent sequence of constant speed curves satisfies the energy convergence for E_0, then some other quantities are also controlled.

Lemma 3.2.7. *Let $\{\gamma_k\}_k \subset W^{1,\infty}(I;\overline{\Omega})$ be a sequence of constant speed curves and $\gamma \in W^{1,\infty}(I;\overline{\Omega})$. Suppose that $\gamma_k \to \gamma$ in L^∞ and*

$$\lim_{k\to\infty} E_0[\gamma_k] = E_0[\gamma].$$

Then

$$\lim_{k\to\infty} E_0[\gamma_k|_J] = E_0[\gamma|_J]$$

holds for any open subinterval $J \subset I$. Moreover, γ is also constant speed, and the speed of γ_k converges to the speed of γ as $k \to \infty$.

Finally, we state the existence theorem of minimizers of E_0. The proof is given in Appendix 3.B.

Theorem 3.2.8. *The energy $E_0 : X^{1,\infty} \to (0, \infty)$ attains its minimum in $X^{1,\infty}$.*

3.3 Properties of minimizers of E_0

In this subsection we prove some properties of minimizers of E_0. In particular, any minimizer is represented by the graph of a Lipschitz function.

3.3.1 Statement on properties of minimizers

To describe our statement, we prepare some definitions.

Definition 3.3.1 (Coincidence set and free boundary)**.** Let J be an open interval and $\gamma \in W^{1,\infty}(J; \overline{\Omega})$. Recall that γ has a unique representation of Lipschitz continuous curve defined pointwise in $\bar{J}$. We define the *non-coincidence set* and *coincidence set* by

$$\{\gamma \in \Omega\} := \{t \in \bar{J} \mid \gamma(t) \in \Omega\}, \quad \{\gamma \in \partial\Omega\} := \{t \in \bar{J} \mid \gamma(t) \in \partial\Omega\},$$

respectively. Moreover, the topological boundary of the non-coincidence set $\partial\{\gamma \in \Omega\}$ in $\bar{J}$ is called *free boundary*.

Since γ is continuous, the sets $\{\gamma \in \Omega\}$ and $\{\gamma \in \partial\Omega\}$ are open and closed in $\bar{J}$, respectively. Note that the disjoint union of the coincidence and non-coincidence sets is nothing but $\bar{J}$.

Definition 3.3.2 (Partition of time interval)**.** Let $N \geq 0$ be a nonnegative integer. We call closed intervals $K_0, \dots, K_N \subset \bar{I}$ *partition* if there are numbers $0 = t_0 < t_1 < \cdots < t_{2N+1} = 1$ such that $K_j = [t_{2j}, t_{2j+1}]$ for $0 \leq j \leq N$. In the case that $N > 0$, we often use the notation $U_j = (t_{2j-1}, t_{2j})$ for $1 \leq j \leq N$ as the *complement* of the partition. We call N *partition number*.

Definition 3.3.3 (Partitional regular)**.** A curve $\gamma \in X^{1,\infty}$ is called *partitional regular* if there is a partition $K_0, \dots, K_N$ such that the coincidence set $\{\gamma \in \partial\Omega\}$ is equal to the (disjoint) union of $K_0, \dots, K_N$, and the curve γ is of class C^1 in the non-coincidence set $\{\gamma \in \Omega\}$ if $N > 0$.

Remark 3.3.4. Note that for any partitional regular curve its partition is unique. If the partition number N is positive, then the (disjoint) union of the complement $U_j = (t_{2j-1}, t_{2j})$, $j = 1, \dots, N$, is nothing but the non-coincidence set. Moreover, the set of the points $t_1, \dots, t_{2N}$ is nothing but the free boundary set. If $N = 0$, then the non-coincidence set and the free boundary are empty, and hence the above interpretations are also valid in this case.

Definition 3.3.5 (Contact angle)**.** Let $\gamma \in X^{1,\infty}$ be a partitional regular curve with positive partition number $N > 0$. Recall that the free boundary set consists of a finite number of times $t_1, \dots, t_{2N} \in I$ which are the points of partition except $t_0 = 0$ and $t_{2N+1} = 1$. We define *contact points* $p_1, \dots, p_{2N}$ by $p_i := \gamma(t_i)$. For any contact point p_i, there is a unique angle $\theta_i \in [-\pi, \pi)$ such that

$$R_{\theta_i}\dot{\gamma}(t_i-) = \dot{\gamma}(t_i+),$$

where

$$\dot{\gamma}(t_i\pm) := \lim_{t \to t_i \pm 0} \dot{\gamma}(t)$$

and R_θ denotes the counterclockwise rotation matrix through angle θ. We call θ_i *contact angle* at p_i.

Definition 3.3.6 (Ordered partitional regular)**.** A partitional regular curve $\gamma \in X^{1,\infty}$ is *ordered partitonal regular* if either, the partition number N is zero, or N is positive and the contact points $p_1, \dots, p_{2N}$ satisfies $0 < x_{p_1} < \cdots < x_{p_{2N}} < 1$, where x_p denotes the x-component of p.

We are now in a position to state our main theorem in this section.

Theorem 3.3.7 (Minimizers for admissible domains)**.** *Any minimizer $\gamma \in X^{1,\infty}$ of E_0 is ordered partitional regular. In addition, the restriction $\gamma|_{K_j}$ is a geodesic in $\partial\Omega$ for any $0 \leq j \leq N$, where $K_0, \dots, K_N$ are the partition. If $N > 0$, then the restriction $\gamma|_{U_j}$ is a segment for any $1 \leq j \leq N$, where $U_1, \dots, U_N$ are the complement. Furthermore, for any $1 \leq i \leq 2N$, the contact angle θ_i at the contact point p_i satisfies $\cos\theta_i = \alpha(p_i)$ and $\theta_i \in (0, \pi/2)$.*

We remark that for any given $p \in \partial\Omega$ there is a unique candidate of the contact angle at p by the above contact angle condition. The above theorem immediately implies that any minimizer is represented by the graph of a function, and especially ensures that there is a minimizer in a graph setting.

Corollary 3.3.8 (Lipschitz graph representation)**.** *For any minimizer $\gamma \in X^{1,\infty}$ of E_0, there is a Lipschitz function $u : \bar{I} \to \mathbb{R}$ with $u(0) = u(1)$ such that γ is the constant speed reparameterization of the graph curve of u.*

3.3.2 Finitely adhesion property

In this subsection we prove that any minimizer adheres to a substrate at most finitely many times.

Lemma 3.3.9. *Let J be an bounded interval. Let $p_0, p_1 \in \partial\Omega$, and $X^{1,\infty}_{p_0,p_1}$ be the set of all constant speed curves in $W^{1,\infty}(J; \partial\Omega)$ joining p_0 to p_1. Then the energy $E_0 : X^{1,\infty}_{p_0,p_1} \to (0,\infty)$ attains its minimum in $X^{1,\infty}_{p_0,p_1}$. Moreover, a minimizer is the unique geodesic in $\partial\Omega$ joining p_0 to p_1.*

Proof. If a curve $\gamma \in X^{1,\infty}_{p_0,p_1}$ is not injective, then we can make a new curve γ' by cutting the "loop" (and reparameterization) such that $\gamma' \in X^{1,\infty}_{p_0,p_1}$ and $E_0[\gamma'] < E_0[\gamma]$. Hence, the only minimizer is the unique injective curve in $X^{1,\infty}_{p_0,p_1}$, which is nothing but the constant speed curve $\gamma \in C^2(\bar{J}; \partial\Omega)$ joining p_0 to p_1. □

Lemma 3.3.10. *Let $p_0, p_1 \in \partial\Omega$ with $r := |p_1 - p_0| < R_\Omega$. Let J be a bounded open interval and $\gamma_\Omega \in C^2(\bar{J}; \partial\Omega)$ be the unique geodesic joining p_0 to p_1 in $\partial\Omega$. Then $\gamma_\Omega(\bar{J}) \subset B_{R_\Omega}(p_0)$, and the length of γ_Ω satisfies*

$$\mathcal{L}[\gamma_\Omega] \leq \frac{r}{1 - r/R_\Omega}.$$

Proof. By the definition of admissible constant R_Ω, the boundary $\partial\Omega$ is connected and represented by a graph in the open ball $B_{R_\Omega}(p_0)$; there is a Euclidean transformation $\Phi : \mathbb{R}^2 \to \mathbb{R}^2$ with $\Phi(p_0) = O := (0,0)$ such that the set $\Phi(\partial\Omega \cap B_{R_\Omega}(p_0))$ coincides with $\{y = \phi(x)\} \cap B_{R_\Omega}(O)$ for some function $\phi \in C^2(\mathbb{R})$ with $\phi(0) = \phi'(0) = 0$. Since $r = |p_1 - p_0| < R_\Omega$, the point p_1 belongs to the open ball $B_{R_\Omega}(p_0)$. Thus, there is $x_1 \in U$ such that $\Phi(p_1) = (x_1, \phi(x_1))$. We may assume that $x_1 \geq 0$ without loss of generality. Moreover, we may assume that $x_1 > 0$ since the case that $x_1 = 0$, i.e., $p_0 = p_1$, is trivial.

By the above graph representation, we see that the unique geodesic γ_Ω in $\partial\Omega$ joining p_0 to p_1 is a reparameterization of the graph curve of $\phi : [0, x_1] \to \mathbb{R}$ up to the transformation Φ. Hence, in particular, $\gamma_\Omega(\bar{J}) \subset B_{R_\Omega}(p_0)$. Moreover, the length of γ_Ω is represented as

$$\mathcal{L}[\gamma_\Omega] = \int_0^{x_1} \sqrt{1 + \phi'^2}\, dx.$$

Recall that the upward curvature κ_ϕ of the graph curve of ϕ satisfies

$$\left(-\frac{1}{\sqrt{1 + \phi'^2}} \right)' = \frac{\phi''}{(1 + \phi'^2)^{3/2}} = \kappa_\phi.$$

Integrating it from 0 to $x \in [0, x_1]$ and noting that $\phi'(0) = 0$, $|x_1| \leq r$, and $\kappa_\phi \leq 1/R_\Omega$, we have

$$1 - \frac{1}{\sqrt{1 + \phi'(x)^2}} = \int_0^x \kappa_\phi\, dx \leq |x_1| \left(\max_{[0,x_1]} \kappa_\phi \right) \leq \frac{r}{R_\Omega},$$

and hence

$$\max_{x \in [0,x_1]} \sqrt{1 + \phi'^2(x)} \leq \frac{1}{1 - r/R_\Omega}.$$

This implies that

$$\mathcal{L}[\gamma_\Omega] \leq |x_1| \left(\max_{x \in [0,x_1]} \sqrt{1 + \phi'^2(x)} \right) \leq \frac{r}{1 - r/R_\Omega}.$$

The proof is complete. □

The following lemma states that a "small free part" must adhere to a substrate in order to minimize E_0. For a Lebesgue measurable set $A \subset \mathbb{R}$ we denote the measure by $|A|$.

Lemma 3.3.11. *Let $J = (t_0, t_1)$ be a bounded open interval. Let $\gamma \in W^{1,\infty}(J; \overline{\Omega})$ be any curve of constant speed $L > 0$ such that $\gamma(t_0), \gamma(t_1) \in \partial\Omega$, $\{\gamma \in \Omega\} = J$, and*

$$L|J| \ (= \mathcal{L}[\gamma]) \leq R_\Omega(1 - \bar{\alpha}(1 + \delta))$$

for some $\delta \in (0, 1/\bar{\alpha} - 1)$*. Then the unique geodesic* $\gamma_\Omega \in C^2(\bar{J}; \partial\Omega)$ *joining* $\gamma(t_0)$ *to* $\gamma(t_1)$ *satisfies*

$$E_0[\gamma_\Omega] \leq \frac{1}{1+\delta} E_0[\gamma].$$

Moreover, the speed L_Ω *of* γ_Ω *satisfies* $0 \leq L_\Omega < L/\bar{\alpha}$*.*

Proof. Since $J \subset \{\gamma \in \Omega\}$, the energy $E_0[\gamma]$ is nothing but the length:

$$E_0[\gamma] = L|J| > 0.$$

Let $r := |\gamma(t_1) - \gamma(t_0)|$. The case $r = 0$ is obvious (since $E_0[\gamma_\Omega] = 0$ and $L_\Omega = 0$) so we may assume that $r > 0$. Since γ_Ω lies in $\partial\Omega$, we easily find that $E_0[\gamma_\Omega] \leq \bar{\alpha}\mathcal{L}[\gamma_\Omega]$. Since $L|J| < R_\Omega$ by the assumption, Lemma 3.3.10 implies the upper bound

$$\mathcal{L}[\gamma_\Omega] = L_\Omega|J| \leq \frac{r}{1 - r/R_\Omega},$$

and hence

$$E_0[\gamma_\Omega] \leq \bar{\alpha}L_\Omega|J| \leq \frac{\bar{\alpha}r}{1 - r/R_\Omega}.$$

Since $r \leq L|J|$ and $r < R_\Omega(1 - \bar{\alpha}(1+\delta))$, the above right-hand term is estimated as

$$\frac{\bar{\alpha}r}{1 - r/R_\Omega} \leq \frac{L|J|}{1+\delta} = \frac{1}{1+\delta} E_0[\gamma],$$

and hence $E_0[\gamma_\Omega] \leq \frac{1}{1+\delta} E_0[\gamma]$. This is the first conclusion. Moreover, the intermediate estimates (and $|J| > 0$) imply that $L_\Omega \leq (L/\bar{\alpha})/(1+\delta) < L/\bar{\alpha}$. □

The above lemma implies the finite adhesion property of minimizers.

Proposition 3.3.12. *Let* $\gamma \in X^{1,\infty}$ *be a minimizer of* E_0*. Then the number of the connected components of* $\{\gamma \in \Omega\}$ *is finite.*

Proof. If the number of the connected components of $\{\gamma \in \Omega\}$ is infinite, then for any small $\delta > 0$ there is a connected component J of $\{\gamma \in \Omega\}$, which is an open interval, such that $\gamma(\partial J) \subset \partial\Omega$ and $|J| < \delta$. Let δ be taken so that $\delta \leq R_\Omega(1 - \bar{\alpha})/L$ ab initio, where L is the speed of γ. Then the curve $\gamma|_J$ satisfies the assumption of Lemma 3.3.11, and hence the geodesic γ_Ω in $\partial\Omega$ joining the endpoints of $\gamma|_J$ satisfies $E_0[\gamma|_J] > E_0[\gamma_\Omega]$. Then the curve γ replaced by γ_Ω in J has a strictly smaller energy than γ; hence, so does its constant speed reparameterization. This contradicts the minimality of γ. □

In the rest of this subsection, we state that for an almost adhering curve, which may have an infinitely many free parts, the corresponding geodesic has a smaller energy. This result is not used in this section, but used later (Lemma 3.5.7).

Lemma 3.3.13. *Let* $J = (t_0, t_1)$ *be a bounded open interval and* $\gamma \in W^{1,\infty}(J; \overline{\Omega})$ *be any curve of constant speed* $L > 0$ *such that* $\gamma(t_0), \gamma(t_1) \in \partial\Omega$*. Suppose that any open subinterval* $J' \subset \{\gamma \in \Omega\}$ *satisfies*

$$L|J'| < R_\Omega(1 - \bar{\alpha}).$$

Then the unique geodesic $\gamma_\Omega \in C^2(\bar{J};\partial\Omega)$ in $\partial\Omega$ joining $\gamma(t_0)$ to $\gamma(t_1)$ satisfies

$$E_0[\gamma_\Omega] \leq E_0[\gamma].$$

Proof. The case that $\{\gamma \in \Omega\} = \emptyset$ is trivial by Lemma 3.3.9; hence, we may assume that $\{\gamma \in \Omega\} \neq \emptyset$. Since $\{\gamma \in \Omega\}$ is an open set in $\mathbb{R}$, it consists of at most countably many disjoint open intervals $J_1, J_2, \cdots \subset J$. Moreover, since $\sum_{i=1}^{\infty} |J_i| \leq |J| < \infty$, we have $\lim_{i\to\infty} \sum_{k=i}^{\infty} |J_k| = 0$. For each interval $J_i = (t_0^i, t_1^i)$, the curve γ touches $\partial\Omega$ at the endpoints, i.e., $\gamma(t_0^i), \gamma(t_1^i) \in \partial\Omega$. Thus, the assumption that $L|J_i| < R_\Omega(1-\bar{\alpha})$ and Lemma 3.3.11 imply that the unique geodesic $\gamma_\Omega^i \in C^2(\bar{J}_i;\partial\Omega)$ in $\partial\Omega$ joining $\gamma(t_0^i)$ to $\gamma(t_1^i)$ satisfies $E_0[\gamma_\Omega^i] < E_0[\gamma|_{J_i}]$ and $0 \leq L^i < L/\bar{\alpha}$, where L^i denotes the speed of γ_Ω^i.

We then define a sequence of Lipschitz curves $\gamma_i \in W^{1,\infty}(J;\overline{\Omega})$ by $\gamma_0 := \gamma$ and

$$\gamma_i := \begin{cases} \gamma_\Omega^i & \text{in } J_i \\ \gamma_{i-1} & \text{otherwise.} \end{cases}$$

Since $E_0[\gamma_i] - E_0[\gamma_{i-1}] = E_0[\gamma_\Omega^i] - E_0[\gamma|_{J_i}] < 0$, the energy $E_0[\gamma_i]$ strictly decreases as i increases.

If the number N of J_i is finite, then γ_N is a Lipschitz curve joining $\gamma(t_0)$ to $\gamma(t_1)$ along $\partial\Omega$. Notice that $E_0[\gamma] = E_0[\gamma_0] > E_0[\gamma_N]$. We denote by $\hat{\gamma}_N$ the constant speed reparameterization of γ_N. By Lemma 3.2.5, $E_0[\gamma_N] = E_0[\hat{\gamma}_N]$. Moreover, since $\hat{\gamma}_N(\bar{J}) \subset \partial\Omega$, Lemma 3.3.9 immediately implies that $E_0[\hat{\gamma}_N] \geq E_0[\gamma_\Omega]$. Therefore, we find that $E_0[\gamma] > E_0[\gamma_\Omega]$, which is the desired conclusion.

The remaining case is that the number of J_i is infinite. Even in this case, the sequence $\{\gamma_i\}_i \subset W^{1,\infty}(J;\mathbb{R}^2)$ is bounded in $W^{1,\infty}$; in fact,

$$\|\dot{\gamma}_i\|_\infty \leq \max\{L_i, L\} \leq L/\bar{\alpha}$$

and the endpoints $\gamma_i(\partial J)$ are fixed. Moreover, $\{\gamma_i\}_i$ is a Cauchy sequence in $L^1(J;\mathbb{R}^2)$; in fact, Lemma 3.3.10 implies that $\gamma_\Omega^i(J_i) \subset B_{R_\Omega}(\gamma(t_0^i))$ and hence

$$\|\gamma_i - \gamma_{i+1}\|_{L^1} = \int_{J_i} |\gamma - \gamma_\Omega^i| \leq \int_{J_i} \left(|\gamma| + |\gamma(t_0^i)| + |\gamma_\Omega^i - \gamma(t_0^i)|\right) \leq |J_i|(2\|\gamma\|_\infty + R_\Omega).$$

Since $\lim_{i\to\infty} \sum_{k=i}^{\infty} |J_k| = 0$, the sequence $\{\gamma_i\}_i$ is Cauchy. Thus there is a limit $\gamma_\infty \in L^1(J;\mathbb{R}^2)$ such that $\gamma_i \to \gamma_\infty$ in L^1. Then Lemma 3.2.6 implies that γ_∞ belongs to $W^{1,\infty}(J;\mathbb{R}^2)$, the L^∞-convergence $\gamma_i \to \gamma_\infty$ holds, and

$$\liminf_{i\to\infty} E_0[\gamma_i] \geq E_0[\gamma_\infty].$$

In particular, $E_0[\gamma] > E_0[\gamma_\infty]$. Notice that γ_∞ has the same endpoints as γ, i.e, the curve γ_∞ joins p_0 to p_1. We also notice that $\gamma_\infty(\bar{J}) \subset \partial\Omega$ by definition of γ_i. Then, by the same argument as in the finite case, we conclude that $E_0[\gamma] > E_0[\gamma_\Omega]$. The proof is now complete. $\square$

3.3.3 Partitional regularity and contact angle condition

In the previous subsection, it is proved that for any minimizer $\gamma \in X^{1,\infty}$ the number of the connected components of $\{\gamma \in \Omega\}$ is at most finite. Hence, $\{\gamma \in \partial\Omega\}$ consists of the disjoint union of closed intervals of positive width or isolated single points. To obtain the partitional regularity, we need to prove that a minimizer γ is smooth in $\{\gamma \in \Omega\}$ (in fact straight), the curve γ touches $\partial\Omega$ at least near ∂I, and any connected component of $\{\gamma \in \partial\Omega\}$ has a positive width (not a point).

To uniformly deal with periodic curves at the endpoints and interior points of I, it is convenient to use the following periodic extension.

Definition 3.3.14 (Periodic extension of curves)**.** For a curve $\gamma \in X^{1,\infty}$ we define its (unique) *periodic extension* $\tilde{\gamma} : \mathbb{R} \to \mathbb{R}^2$ so that $\gamma(t) = \tilde{\gamma}(t)$ for $t \in I$ and $\tilde{\gamma}(t) = \tilde{\gamma}(t) + (1,0)$ for $t \in \mathbb{R}$.

For a periodic extension curve, the coincidence set $\{\tilde{\gamma} \in \partial\Omega\}$ and non-coincidence set $\{\tilde{\gamma} \in \Omega\}$ are similarly defined. In particular, each connected component of $\{\tilde{\gamma} \in \Omega\}$ is an open interval in $\mathbb{R}$. We notice that if γ is a minimizer of E_0 then its periodic extension $\tilde{\gamma}$ also minimizes E_0 locally, i.e., $E_0[\tilde{\gamma}|_J] \leq E_0[\zeta]$ for any open interval J with $|J| \leq 1$ and any $\zeta \in W^{1,\infty}(J;\overline{\Omega})$ with $\tilde{\gamma} = \zeta$ on ∂J.

The following proposition states that a minimizer γ is straight in $\{\gamma \in \Omega\}$, which especially implies the smoothness of γ in $\{\gamma \in \Omega\}$.

Proposition 3.3.15. *Let $\gamma \in X^{1,\infty}$ be a minimizer of E_0 and $\tilde{\gamma}$ be its periodic extension. Then, for each (open) connected component U of $\{\tilde{\gamma} \in \Omega\}$, the curve $\tilde{\gamma}|_U$ is an open segment.*

Proof. Since $U \subset \{\tilde{\gamma} \in \Omega\}$, the curve $\tilde{\gamma}|_U$ minimizes the length functional locally, thus the assertion follows by the standard minimal surface theory. (Since our problem is one-dimensional, it is also proved by an elementary argument.) □

We next prove that the adhesion effect implies that the segment parts do not contact $\partial\Omega$ tangentially.

Lemma 3.3.16. *Let $\gamma \in X^{1,\infty}$ be a minimizer of E_0. Let $J \subset \{\gamma \in \Omega\}$ be an open interval. Suppose that $\gamma(t_0) \in \partial\Omega$ for an endpoint $t_0 \in \partial J$. Then the segment $\gamma|_J$ is not tangent to $\partial\Omega$ at $\gamma(t_0)$.*

Proof. We prove by contradiction. Suppose that $\gamma|_J$ would be tangent to $\partial\Omega$ at $p_0 := \gamma(t_0)$. To obtain a contradiction, we suitably perturb γ in $\overline{B_{\bar{r}}(p_0)}$, where $\bar{r}$ is sufficiently small to be less than R_Ω and the length of the segment $\gamma|_J$. Since we only consider a local perturbation, we may suppose the following assumptions; the point p_0 is the origin, $\gamma|_J$ is the segment ℓ_γ joining the origin to $(\bar{r}, 0) \in \mathbb{R}^2$ in $B_{\bar{r}}(p_0)$, the domain Ω is the epigraph of ψ in $B_{\bar{r}}(p)$, and $\psi'(0) = \psi(0) = 0$. Any other case is reduced to the above case by a Euclidean transformation.

Then there is a sequence $\delta_j \downarrow 0$ such that for any j the (open) segment joining $(\delta_j, \psi(\delta_j))$ to $(\bar{r}, 0)$ is included in Ω. Denote the segment by ℓ_j. Moreover, we denote the geodesic in $\partial\Omega$ joining the origin to $(\delta_j, \psi(\delta_j))$ by γ_Ω^j.

We now perturb γ; we replace the segment part ℓ_γ by γ_Ω^j and ℓ_j. Note that the perturbed curves remain admissible up to the reparameterization. We denote such a curve by γ_j. Then we calculate the first variation of E_0 as

$$\begin{aligned} E_0[\gamma_j] - E_0[\gamma] &= E_0[\gamma_\Omega^j] + E_0[\ell_j] - E_0[\ell_\gamma] \\ &= \int_0^{\delta_j} \tilde{\alpha}(x)\sqrt{1+\psi'^2(x)}dx + \sqrt{(\bar{r}-\delta_j)^2 + \psi'(\delta_j)^2} - \bar{r} \\ &= (\tilde{\alpha}(0)-1)\delta_j + o(\delta_j), \end{aligned}$$

as $\delta_j \to 0$, where $\tilde{\alpha}(x) := \alpha(x, \psi(x))$. Since $\tilde{\alpha}(0) \leq \bar{\alpha} < 1$, the leading order term is negative, and hence $E_0[\gamma_j] < E_0[\gamma]$ for any large j. This contradicts the minimality of γ. □

In particular, the above contact condition implies the following

Lemma 3.3.17. *Let $\gamma \in X^{1,\infty}$ be a minimizer of E_0. Then there is $t \in \bar{I}$ such that $\gamma(t) \in \partial\Omega$.*

Proof. If there would not exist such $t \in \bar{I}$, then γ would be a segment parallel with the x-axis by Proposition 3.3.15 and the periodic boundary condition. We shift γ downward to touch $\partial\Omega$. This procedure does not increase the energy. Then we would have a new minimizer that touches $\partial\Omega$ tangentially. This contradicts Lemma 3.3.16. □

Then we have the following statement as a maximum principle.

Lemma 3.3.18. *Any minimizer $\gamma = (x, y) \in X^{1,\infty}$ of E_0 satisfies*

$$\max_{t\in\bar{I}} y(t) = y(0) = \max_{x\in\bar{I}} \psi(x) \ (= \psi(0)).$$

In particular, $\partial I \subset \{\gamma \in \partial\Omega\}$, i.e., γ touches $\partial\Omega$ at the endpoints.

Proof. We easily notice that $\max_{t\in\bar{I}} y(t) \geq y(0) \geq \max_{x\in\bar{I}} \psi(x)$. Thus it suffices to prove that $\max_{t\in\bar{I}} y(t) \leq \max\psi$. We consider the periodic extension $\tilde{\gamma} = (\tilde{x}, \tilde{y})$ of γ and prove that $\max_{t\in\mathbb{R}} \tilde{y}(t) \leq \max\psi$. Obviously, it suffices to confirm the estimate only for the non-coincidence set, i.e.,

$$\sup_{t\in\{\tilde{\gamma}\in\Omega\}} \tilde{y}(t) \leq \max\psi.$$

By Lemma 3.3.17, the set $\{\tilde{\gamma} \in \partial\Omega\}$ is nonempty. Hence, by periodicity, any connected component J of $\{\tilde{\gamma} \in \Omega\}$ is a bounded open interval. Moreover, by Proposition 3.3.15, the curve $\tilde{\gamma}|_J$ is a segment joining points in $\partial\Omega$. This implies that $\sup_{t\in J} \tilde{y}(t) \leq \max\psi$. Since J is an arbitrary connected component of $\{\tilde{\gamma} \in \Omega\}$, the proof is complete. □

Proposition 3.3.12, Lemma 3.3.16 and Lemma 3.3.18 immediately imply the following

Lemma 3.3.19. *Let $\gamma = (x, y) \in X^{1,\infty}$ be a minimizer. Then there are closed intervals $K, K' \subset \{\gamma \in \partial\Omega\}$ of positive width such that $0 \in K$ and $1 \in K'$.*

The above fact especially implies that any connected component of $\{\gamma \in \Omega\}$ is an open interval in $\mathbb{R}$, i.e., $\{\gamma \in \Omega\}$ does not include the endpoints of $\bar{I}$.

We finally confirm that a minimizer does not touch $\partial\Omega$ as a point in I.

Proposition 3.3.20. *Let $\gamma \in X^{1,\infty}$ be a minimizer of E_0. Then for any connected component K of $\{\gamma \in \partial\Omega\}$, the set K is a closed interval of positive width, and the curve $\gamma|_K$ is a geodesic in $\partial\Omega$ of positive length.*

Proof. By Lemma 3.3.9, the curve $\gamma|_K$ is a geodesic in $\partial\Omega$ for any connected component K. In addition, Lemma 3.3.19 implies that K is not a point if K includes an endpoint of $\bar{I}$. Hence, the remaining part is to confirm that any $K \subset I$ (included in the interior of $\bar{I}$) is not a point.

If K would be a single point $\{t_0\}$, then it would be isolated in $\{\gamma \in \partial\Omega\}$ by Proposition 3.3.12, and hence there would be small $\bar{t} > 0$ such that

$$\gamma((t_0 - \bar{t}, t_0 + \bar{t})) \subset B_{R_\Omega}(\gamma(t_0)),$$

and the curve γ touches $\partial\Omega$ only at t_0 in $(t_0 - \bar{t}, t_0 + \bar{t})$. Then, both $\gamma|_{(t_0-\bar{t},t_0)}$ and $\gamma|_{(t_0,t_0+\bar{t})}$ are segments by Proposition 3.3.15. By Lemma 3.3.16, the segments are not tangent to $\partial\Omega$ at $\gamma(t_0)$. Hence, the curve γ would have an "edge" at $\gamma(t_0)$ such that, in the ball $B_{R_\Omega}(\gamma(t_0))$, the acute angle cone made by the edge is included in Ω. This edge obviously allows a local modification of γ in Ω to decrease the length, and hence decrease E_0. This contradicts the minimality of γ. □

At this time we know that any minimizer γ is partitional regular, and moreover γ is a segment in each connected component of $\{\gamma \in \Omega\}$. We now confirm the contact angle condition.

Proposition 3.3.21. *Let $\gamma \in X^{1,\infty}$ be a minimizer of E_0. For any contact point $p \in \partial\Omega$, the contact angle $\theta \in [-\pi, \pi)$ satisfies $\cos\theta = \alpha(p)$.*

Proof. For the contact angle condition, it suffices to see local properties for each contact angle. Thus we consider a sufficiently small interval $J = (t_0 - \bar{t}, t_0 + \bar{t}) \subset I$ so that $\gamma(t_0) = p$ and $\gamma(\bar{J}) \subset B_{R_\Omega}(p)$. Since γ is partitional regular, we may assume that the curve $\gamma|_{\bar{J}}$ is a geodesic in $\partial\Omega$ on a half of $\bar{J}$, and a segment in Ω on the other half part. Without loss of generality, we may assume that the geodesic part is the former part $[t_0 - \bar{t}, t_0]$. To obtain the contact angle condition, we suitably perturb γ.

By using a Euclidean transformation, we may assume that the contact point p is the origin, the domain Ω is the epigraph ψ in $B_{R_\Omega}(p)$, and $\psi(0) = \psi'(0) = 0$. Moreover, we may assume that the x-components of the geodesic part $\gamma|_{[t_0-\bar{t},t_0]}$ are nonpositive. Then the segment part $\gamma|_{[t_0,t_0+\bar{t}]}$ joins the origin to a point $(\bar{r}\cos\theta', \bar{r}\sin\theta') \in \mathbb{R}^2$ for some $\bar{r} \in (0, R_\Omega)$, where $\theta' = |\theta| \in [0, \pi]$ (not necessarily $\theta' = \theta$ due to a Euclidean transformation). Denote the above segment part by ℓ_γ. By Lemma 3.3.16, the segment ℓ_γ is not tangent to $\partial\Omega$, i.e., $\theta' \in (0, \pi)$; this property implies that for any $\delta \in \mathbb{R}$ with small $|\delta|$, the open segment ℓ_δ joining $(\delta, \psi(\delta))$ to $(\bar{r}\cos\theta, \bar{r}\sin\theta)$ is included in Ω. Moreover, for any small δ, we denote the geodesic in $\partial\Omega$ joining $\gamma(t_0 - \bar{t})$ to $(\delta, \psi(\delta))$ by γ_Ω^δ. Finally, we denote the x-component of the point $\gamma(t_0 - \bar{t})$ by $\bar{x} < 0$.

We now perturb γ; we define a sequence $\{\gamma_\delta\}_\delta$ by replacing γ by the geodesic γ_Ω^δ in $[t_0 - \bar{t}, t_0]$, and by the segment ℓ_δ in $[t_0, t_0 + \bar{t}]$ (and reparameterization). Then we calculate the first variation of E_0 as

$$\begin{aligned} E_0[\gamma_\delta] - E_0[\gamma] &= E_0[\gamma^\delta_\Omega] + E_0[\ell_\delta] - E_0[\ell_\gamma] \\ &= \int_{\bar{x}}^{\delta} \tilde{\alpha}(x)\sqrt{1+\psi'^2(x)}dx + \sqrt{(\bar{r}\cos\theta' - \delta)^2 + (\bar{r}\sin\theta' - \psi'(\delta))^2} - \bar{r} \\ &= (\tilde{\alpha}(0) - \cos\theta')\delta + o(\delta), \end{aligned}$$

as $\delta \to 0$, where $\tilde{\alpha}(x) := \alpha(x, \psi(x))$. By the minimality of γ, the leading order term vanishes, i.e., $\tilde{\alpha}(0) = \cos\theta'$. Since $\tilde{\alpha}(0) = \alpha(p)$ and $\cos\theta = \cos\theta'$, the proof is now complete. □

The above result and the positivity $\alpha > 0$ imply that $\theta \in (-\pi/2, \pi/2)$. However, the condition $\theta > 0$ is not proved yet. This condition immediately follows by the ordered partitional regularity, which is proved in the next subsection.

3.3.4 Self-intersection and ordered partitional regularity

We are now in a position to complete the proof of Theorem 3.3.7 by proving that any minimizer is ordered partitional regular. The order is a global property, so we need some topological arguments.

We first confirm that any minimizer has no self-intersection.

Lemma 3.3.22. *Any minimizer $\gamma \in X^{1,\infty}$ of E_0 has no self-intersection.*

Proof. We prove by contradiction. Suppose that there would be $t_0, t_1 \in \bar{I}$ such that $t_0 < t_1$ and $\gamma(t_0) = \gamma(t_1)$. By the periodic boundary condition of γ, we notice that $t_1 - t_0 < 1$, i.e., at least one of t_0 and t_1 is not in ∂I. Then, we can make a new curve γ' by cutting the "loop" (and reparameterization) so that $\gamma' \in X^{1,\infty}$ and $E_0[\gamma'] < E_0[\gamma]$. This contradicts the minimality of γ. □

This implies that any minimizer is included in the "one period" of a substrate.

Lemma 3.3.23. *The image of any minimizer $\gamma \in X^{1,\infty}$ of E_0 is included in the set*

$$[\Omega] := \{(x,y) \in \overline{\Omega} \mid 0 \le x \le 1,\ y \le \max\psi\}.$$

Moreover, any contact point $p \in \partial\Omega$ of a minimizer satisfies $0 < x_p < 1$, where x_p denotes the x-component of p.

Proof. We confirm that $\gamma(\bar{I}) \subset [\Omega]$. The y-component part is proved in Lemma 3.3.18. The x-component part follows by Lemma 3.3.18 and Lemma 3.3.22. In fact, if there is $t \in I$ such that the x-component of γ satisfies $x(t) < 0$ or $x(t) > 1$, then the intermediate value theorem implies that there is $t' \in I$ (not in ∂I) such that $x(t') = 0 = x(0)$ or $x(t') = 1 = x(1)$. By Lemma 3.3.18, $y(t') = y(0)$ or $y(t') = y(1)$. This contradicts Lemma 3.3.22.

Let p be a contact point. Since $\gamma(\bar{I}) \subset [\Omega]$, we find that $0 \le x_p \le 1$. If $x_p = 0$ or $x_p = 1$, then γ would touch ψ tangentially; this contradicts Lemma 3.3.16. Hence, $x_p \neq 0, 1$. The proof is now complete. □

The following lemma is an essential step for the ordered partitional regularity.

Lemma 3.3.24. *Let $\gamma \in X^{1,\infty}$ be a minimizer of E_0. Let $p, p' \in \partial\Omega$ be points with $p \neq p'$ such that, for a connected component U of $\{\gamma \in \Omega\}$, the curve $\tilde{\gamma}|_U$ is the segment joining p to p'. Let γ_Ω be the geodesic in $\partial\Omega$ joining p to p'. Then the curves γ and γ_Ω intersect only at p and p'.*

To prove the above lemma, we recall the Jordan curve theorem (see e.g. [8, Proposition 2B.1 (b)]).

Lemma 3.3.25 (Jordan curve theorem)**.** *For any simple closed curve $C : \bar{J} \to \mathbb{R}^2$ (where J is a bounded interval), the set $\mathbb{R}^2 \setminus C(\bar{J})$ consists of two disjoint connected components; one is bounded and the other is unbounded.*

We call the bounded part *inside* and the unbounded part *outside*.

Proof of Lemma 3.3.24. Without loss of generality, we may assume that $x_p < x_{p'}$. Lemma 3.3.23 implies that $0 < x_p < x_{p'} < 1$. We define a simple closed curve $C : [0, 4] \to \mathbb{R}^2$ so that

- $C|_{[0,1]}$ is the segment from p to p',
- $C|_{[1,2]}$ is the segment from p' to $p' + (0, -1)$,
- $C|_{[2,3]}$ follows the graph curve of $\psi - 1$ from $p' + (0, -1)$ to $p + (0, -1)$, and
- $C|_{[3,4]}$ is the segment from $p + (0, -1)$ to p.

We decompose the graph curve of ψ by C into the three parts;

$$\begin{aligned} \Gamma_1 &= \{y = \psi(x) \mid x < x_p\}, \\ \Gamma_2 &= \{y = \psi(x) \mid x_p < x < x_{p'}\}, \\ \Gamma_3 &= \{y = \psi(x) \mid x_{p'} < x\}. \end{aligned}$$

The set Γ_2 is nothing but the image of the geodesic γ_Ω except the endpoints. It is straightforward to confirm that Γ_2 is in the inside of C, and the other parts Γ_1 and Γ_3 are in the outside.

We now prove the assertion by contradiction. Suppose that there would be $q \in \Gamma_2$ and $t_q \in I$ such that $\gamma(t_q) = q$. Notice that $t_q \in I \setminus \overline{U}$ since $\gamma(U) \subset \Omega$ and $\gamma(\partial U) = \{p, p'\} \not\ni q$. Then there is a closed interval $\bar{J} \subset \bar{I} \setminus \overline{U}$ such that the endpoints of $\bar{J}$ consist of t_q and either 0 or 1. The curve $\gamma|_{\bar{J}}$ is a path which connects Γ_2 and either Γ_1 or Γ_3 (since $\gamma(0) \in \Gamma_1$ and $\gamma(1) \in \Gamma_3$). Then, by the Jordan curve theorem, $\gamma|_{\bar{J}}$ intersects C. Since $\gamma(\bar{J}) \subset \overline{\Omega}$ and moreover $C([0, 4]) \cap \overline{\Omega} = \gamma(\overline{U})$, the curves $\gamma|_{\bar{J}}$ and $\gamma|_{\overline{U}}$ intersect, i.e., there are $t_1 \in \bar{J} \subset \bar{I} \setminus \overline{U}$ and $t_2 \in \overline{U}$ such that $\gamma(t_1) = \gamma(t_2)$. This contradicts Lemma 3.3.22. The proof is now complete. □

We are now in a position to prove the ordered partitional regularity.

Proposition 3.3.26. *Let $\gamma \in X^{1,\infty}$ be a minimizer of E_0 of positive partition number $N > 0$. Then the x-components of the contact points $p_1, \dots, p_{2N} \in \partial\Omega$ satisfy $0 < x_{p_1} < \cdots < x_{p_{2N}} < 1$.*

Proof. For convenience, we define $p_0, p_{2N+1} \in \partial\Omega$ as $p_0 = \gamma(0)$ and $p_{2N+1} = \gamma(1)$. Notice that $x_{p_0} = 0$ and $x_{p_{2N+1}} = 1$. By Lemma 3.3.23, we know that the points $x_{p_1}, \dots, x_{2N}$ are included in the interval $(0,1)$. Recall that γ has no self-intersection. Hence, in particular, we find that $x_{p_i} \neq x_{p_{i'}}$ for any $i \neq i'$.

We first prove that, for $j = 0, \dots, N-1$, if $x_{p_{2j}} < x_{p_{2j+1}}$, then $x_{p_{2j+1}} < x_{p_{2j+2}}$. We prove it by contradiction, so suppose that $x_{p_{2j}}, x_{p_{2j+2}} < x_{p_{2j+1}}$. Then there are only the following two cases; $x_{p_{2j}} < x_{p_{2j+2}} < x_{p_{2j+1}}$ and $x_{p_{2j+2}} < x_{p_{2j}} < x_{p_{2j+1}}$. Recall that $\gamma|_{[t_{2j},t_{2j+1}]}$ is the geodesic joining p_{2j} to p_{2j+1} in $\partial\Omega$ and $\gamma|_{(t_{2j+1},t_{2j+2})}$ is a segment from p_{2j+1} to p_{2j+2} in Ω. Thus, the case that $x_{p_{2j+2}} < x_{p_{2j}} < x_{p_{2j+1}}$ contradicts Lemma 3.3.24. Moreover, if $x_{p_{2j}} < x_{p_{2j+2}} < x_{p_{2j+1}}$, then p_{2j+2} lies in the geodesic joining p_{2j} to p_{2j+1} along $\partial\Omega$, and hence γ has a self-intersection as $p_{2j+2} \in \gamma([t_{2j}, t_{2j+1}])$; this is also a contradiction.

Moreover, it is similarly proved that, for $j = 0, \dots, N-1$, if $x_{p_{2j+1}} < x_{p_{2j+2}}$, then $x_{p_{2j+2}} < x_{p_{2j+3}}$. Noting that $x_{p_0} = 0 < x_{p_1}$, the above order preservations imply that $x_{p_0} < \cdots < x_{p_{2N+1}}$. The proof is now complete. □

We shall complete the proof of Theorem 3.3.7.

Proof of Theorem 3.3.7. It follows by Propositions 3.3.12, 3.3.20, 3.3.21, and 3.3.26. □

3.4 Γ-convergence

Hereafter, we consider the minimizing problem for the original energy E_ε, where $\varepsilon > 0$. In this section we obtain a first order singular limit energy of E_ε as $\varepsilon \to 0$. To this end we utilize the notion of Γ-convergence.

3.4.1 Definition and basic properties of Γ-convergence

We first recall the definition of Γ-convergence. See [3, 4, 6] for more details.

Definition 3.4.1 (Γ-convergence and equi-coerciveness)**.** Let X be a metric space and $F_\varepsilon, F : X \to [0,\infty]$ be functionals. We say that F_ε Γ-*converges* to F on X as $\varepsilon \to 0$ if the following conditions hold.

(1) For any convergent sequence $u_\varepsilon \to u$ in X as $\varepsilon \to 0$,

$$\liminf_{\varepsilon \to 0} F_\varepsilon[u_\varepsilon] \geq F[u].$$

(2) For any $u \in X$, there exists a convergent sequence $u_\varepsilon \to u$ in X as $\varepsilon \to 0$ such that

$$\limsup_{\varepsilon \to 0} F_\varepsilon[u_\varepsilon] \leq F[u].$$

The functionals F_ε are *equi-coercive* if the following condition holds.

(3) If $\{u_\varepsilon\}_\varepsilon \subset X$ satisfies $\limsup_{\varepsilon\to 0} F_\varepsilon[u_\varepsilon] < \infty$, then there is a subsequence $\{u_{\varepsilon_j}\}_j$ converging to some $u \in X$ in X as $\varepsilon_j \to 0$.

It is well-known that the above conditions imply the following convergences.

Corollary 3.4.2 (e.g. [4, Theorem 2.1])**.** *Under the above conditions (1)–(3), we have*

$$\lim_{\varepsilon\to 0}\inf_X F_\varepsilon = \inf_X F.$$

Moreover, for any sequence of minimizers of F_ε, there is a subsequence converging to a minimizer of F in X as $\varepsilon \to 0$.

Remark 3.4.3. Our definition of equi-coerciveness is slightly different from the reference [4], but the difference does not make any change for the proof of Corollary 3.4.2. In addition, the reference [4] assumes the separability of X throughout, but this assumption is not used until [4, Theorem 2.1] at least. For Corollary 3.4.2, see e.g. also [3, Theorem 1.21] or [6, Theorem 7.4].

3.4.2 Γ-convergence for the energy E_ε

In this subsection we rigorously state our Γ-convergence result for the energy

$$E_\varepsilon[\gamma] = \varepsilon^2 \int_\gamma \kappa^2 ds + \int_\gamma \Theta(\gamma) ds,$$

where κ is the curvature, s is the arc length parameter, and Θ is the potential function defined in Section 3.2.

We first define the set of admissible curves for the energy E_ε.

Let J be a bounded open interval and $\bar{J}$ be its closure. Let $W^{2,2}(J;\mathbb{R}^2)$ be the set of all $W^{2,2}$-Sobolev curves, which may not be regular. Recall that $W^{2,2}(J;\mathbb{R}^2)$ is embedded in $C^1(\bar{J};\mathbb{R}^2)$. Hence, for any curve in $W^{2,2}(J;\mathbb{R}^2)$, the curve itself and its first derivative are defined pointwise in $\bar{J}$; in particular, its arc length parameterization is defined in the classical sense. For any regular $W^{2,2}$-curve, the curvature κ is defined in the L^2-sense.

We define the set of admissible curves for E_ε as the set of all curves $\gamma \in W^{2,2}(I;\mathbb{R}^2) \cap X^{1,\infty}$, where $I = (0,1)$, such that $\dot{\gamma}(0) = \dot{\gamma}(1)$. We denote the set of admissible curves by $X^{2,2}$. Notice that any $\gamma \in X^{2,2}$ is a constant speed C^1-regular curve confined in $\overline{\Omega}$, and moreover satisfies the periodic boundary condition in the first order sense.

We shall state the existence of minimizers for E_ε in this setting. The proof is given in Section 2.A of Chapter 2.

Theorem 3.4.4. *The energy $E_\varepsilon : X^{2,2} \to (0,\infty)$ attains its minimum in $X^{2,2}$.*

We now state our Γ-convergence result on E_ε. Let $m_0 := \min_{X^{1,\infty}} E_0$. Define $F_\varepsilon : L^\infty(I;\mathbb{R}^2) \to [0,\infty]$ as

$$F_\varepsilon[\gamma] := \begin{cases} \dfrac{E_\varepsilon[\gamma] - m_0}{\varepsilon} & \text{for } \gamma \in X^{2,2} \subset L^\infty(I;\mathbb{R}^2), \\ \infty & \text{otherwise.} \end{cases} \tag{3.4.1}$$

Let M_0 be the set of all minimizers of E_0, i.e.,

$$M_0 := \operatorname*{argmin}_{X^{1,\infty}} E_0 := \{\gamma \in X^{1,\infty} \mid E_0[\gamma] = m_0\}.$$

We define a limit energy functional $F : L^\infty(I;\mathbb{R}^2) \to [0,\infty]$ as

$$F[\gamma] := \begin{cases} \displaystyle\sum_{i=1}^{2N} 8\sqrt{2}\sin^2\frac{\theta_i}{4} & \text{for } \gamma \in M_0 \subset L^\infty(I;\mathbb{R}^2), \\ \infty & \text{otherwise}, \end{cases} \tag{3.4.2}$$

where N is the partition number of γ, and $\theta_1, \dots, \theta_{2N}$ denote the contact angles of γ. When $N = 0$, we interpret the sum as zero.

Remark 3.4.5. The half-angle formulae and the contact angle condition imply that

$$8\sqrt{2}\sin^2\frac{\theta_i}{4} = 4(\sqrt{2} - \sqrt{1+\cos\theta_i}) = 4(\sqrt{2} - \sqrt{1+\alpha(p_i)}),$$

where p_i denotes the i-th contact point. Hence, the energy F is also interpreted in terms of the adhesion coefficient α as in Introduction.

The main result of this section is stated as

Theorem 3.4.6 (Γ-convergence). *Let F_ε and F be as in (3.4.1) and (3.4.2). Then the functionals F_ε are equi-coercive, and F_ε Γ-converges to F on $L^\infty(I;\mathbb{R}^2)$ as $\varepsilon \to 0$.*

The functional F_ε admits a minimizer for any $\varepsilon > 0$, which is nothing but a minimizer of E_ε. Hence, in particular, Corollary 3.4.2 implies that the functional F admits a minimizer in $L^\infty(I;\mathbb{R}^2)$ (and hence, in M_0).

We summarize more important consequences of Theorem 3.4.6 and Corollary 3.4.2 as the following

Corollary 3.4.7 (Convergence of minimizers). *Let $\{\gamma_\varepsilon\}_\varepsilon \subset X^{2,2}$ be a sequence of minimizers of E_ε. Then there is a subsequence $\{\gamma_{\varepsilon_j}\}_{\varepsilon_j}$ and a minimizer γ of F (in particular, of E_0) such that, $\gamma_{\varepsilon_j} \to \gamma$ in L^∞, the speed of γ_{ε_j} also converges to the speed of γ, and moreover*

$$E_{\varepsilon_j}[\gamma_{\varepsilon_j}] = E_0[\gamma] + \varepsilon_j m_F + o(\varepsilon_j)$$

as $\varepsilon_j \to 0$, where $m_F := \min_{M_0} F$.

We confirm the above corollary by assuming Theorem 3.4.6.

Proof of Corollary 3.4.7. Under the assumption that Theorem 3.4.6 is valid, Corollary 3.4.2 directly implies the L^∞-convergence part and the minimum convergence part. Moreover, since $\lim_{\varepsilon_j\to 0} E_0[\gamma_{\varepsilon_j}] = E_0[\gamma]$ follows by the convergence of the minima, Lemma 3.2.7 implies the convergence of the speeds. $\square$

Remark 3.4.8. The above result is also valid even if the original sequence $\{\gamma_\varepsilon\}_\varepsilon$ is taken as a subsequence $\{\gamma_{\varepsilon'}\}_{\varepsilon'}$ of minimizers. This is because, if F_ε Γ-converges to F, then any subsequence $\{F_{\varepsilon'}\}_{\varepsilon'}$ also Γ-converges to F obviously.

In the rest of this section, we prove Theorem 3.4.6.

3.4.3 Equi-coerciveness

We first confirm the equi-coerciveness in Theorem 3.4.6.

Proof of the equi-coerciveness of Theorem 3.4.6. Let $\{\gamma_\varepsilon\}_\varepsilon \subset L^\infty(I;\mathbb{R}^2)$ such that

$$\limsup_{\varepsilon\to 0} F_\varepsilon[\gamma_\varepsilon] < \infty.$$

It suffices to prove that this sequence satisfies the assumption of Lemma 3.2.1 as $\varepsilon \to 0$.

Since $F_\varepsilon[\gamma_\varepsilon] < \infty$ for small ε, we find that $\gamma_\varepsilon \in X^{2,2}$ for any small ε. Moreover, since

$$F[\gamma_\varepsilon] = \varepsilon \int_\gamma \kappa^2 ds + \frac{1}{\varepsilon}(E_0[\gamma_\varepsilon] - m_0)$$

and both the two terms are nonnegative, the last term is bounded as $\varepsilon \to 0$ in particular, and hence

$$\lim_{\varepsilon\to 0} E_0[\gamma_\varepsilon] = m_0 < \infty.$$

Let L_ε $(= \|\dot{\gamma}_\varepsilon\|_\infty)$ be the speed of γ_ε. Since $\underline{\alpha} L_\varepsilon \leq E_0[\gamma_\varepsilon]$, the sequence of speed is bounded as $\varepsilon \to 0$; hence, $\|\dot{\gamma}_\varepsilon\|_\infty$ is bounded as $\varepsilon \to 0$.

Recall that, any curve in $X^{1,\infty}$ not touching $\partial\Omega$ satisfies $E_0[\gamma] \geq 1$, a non-touching segment $\bar{\gamma}$ attains $E_0[\bar{\gamma}] = 1$, and any minimizer touches $\partial\Omega$ by Theorem 3.3.7. Therefore, we find that $m_0 < 1$, and moreover for any small $\varepsilon > 0$ the curve γ_ε touches $\partial\Omega$. Then, as in the proof of Theorem 3.2.8, we find that $\|\gamma_\varepsilon\|_\infty$ is also bounded as $\varepsilon \to 0$. The proof is complete. □

3.4.4 Lower bound inequality

In this subsection we prove the lower bound inequality of Theorem 3.4.6, i.e., for any convergent sequence $\gamma_\varepsilon \to \gamma$ in $L^\infty(I;\mathbb{R}^2)$,

$$\liminf_{\varepsilon\to 0} F_\varepsilon[\gamma_\varepsilon] \geq F[\gamma].$$

Notice that it suffices to confirm the above property for a subsequence of any subsequence of $\{\gamma_\varepsilon\}_\varepsilon$. In this subsection we do not take care relabeling in such a procedure to simplify notation. (Careful readers may interpret the index ε as a subindex ε' in this subsection.)

We only consider a (sub)sequence satisfying the following assumption since any other case is trivial or reduced to this case.

Assumption 3.4.9. A sequence $\gamma_\varepsilon \to \gamma$ in $L^\infty(I;\mathbb{R}^2)$ is assumed so that $\{\gamma_\varepsilon\}_\varepsilon \subset X^{2,2}$, the curve γ is a minimizer of E_0 in $X^{1,\infty}$ with a positive partition number $N > 0$, and

$$\lim_{\varepsilon\to 0} E_0[\gamma_\varepsilon] = E_0[\gamma] \; (= m_0).$$

Lemma 3.4.10. *If the lower bound inequality of Theorem 3.4.6 is valid for any sequence satisfying Assumption 3.4.9, then it is valid for any sequence $\gamma_\varepsilon \to \gamma$ in $L^\infty(I;\mathbb{R}^2)$.*

Proof. Consider any convergent sequence $\gamma_\varepsilon \to \gamma$ in $L^\infty(I;\mathbb{R}^2)$. If γ is not a minimizer, then by the lower semicontinuity of Lemma 3.2.6 we have

$$\liminf_{\varepsilon\to 0} E_\varepsilon[\gamma_\varepsilon] \geq E_0[\gamma] > m_0,$$

and hence $\liminf_{\varepsilon\to0} F_\varepsilon[\gamma_\varepsilon] = \infty$; this case is trivial. By the same reason, the case that $\liminf_{\varepsilon\to0} E_\varepsilon[\gamma_\varepsilon] > E_0[\gamma]$ is also trivial. Moreover, if γ is a minimizer but the partition number is zero, then $F[\gamma] = 0$, so it is also trivial.

We finally consider the case that $\{\gamma_\varepsilon\}_\varepsilon \not\subset X^{2,2}$. The case that $\liminf_{\varepsilon\to0} F_\varepsilon[\gamma_\varepsilon] = \infty$ is trivial. If $\liminf_{\varepsilon\to0} F_\varepsilon[\gamma_\varepsilon] < \infty$, then there is a subsequence $\{\gamma_{\varepsilon_j}\}_j$ such that

$$\lim_{j\to\infty} F_{\varepsilon_j}[\gamma_{\varepsilon_j}] = \liminf_{\varepsilon\to0} F_\varepsilon[\gamma_\varepsilon] < \infty.$$

Then, since $F_{\varepsilon_j}[\gamma_{\varepsilon_j}] < \infty$ for any large j, we find that $\gamma_{\varepsilon_j} \in X^{2,2}$ for any large j. By relabeling, we can also reduce this case to Assumption 3.4.9. □

In the rest of this subsection, we prove the lower bound inequality for a sequence in Assumption 3.4.9.

Let $\mathcal{E}_\varepsilon$ be the modified total squared curvature energy, which is defined for any regular $W^{2,2}$-curve γ:

$$\mathcal{E}_\varepsilon[\gamma] := \varepsilon^2 \int_\gamma \kappa^2 ds + \int_\gamma ds.$$

Let l_γ denote the distance of the endpoints of γ. In addition, we define $\mathcal{F}_\varepsilon$ as

$$\mathcal{F}_\varepsilon[\gamma] := \frac{\mathcal{E}_\varepsilon[\gamma] - l_\gamma}{\varepsilon}.$$

Our temporary goal is to obtain the following lower bound.

Lemma 3.4.11. *Let $\gamma_\varepsilon \to \gamma$ be as in Assumption 3.4.9. Let $0 < t_1 < \cdots < t_{2N} < 1$ be the partition of γ. Then there is $\bar{\varepsilon} > 0$ such that for any $\varepsilon \in (0, \bar{\varepsilon})$ there are numbers $0 < t_1^\varepsilon < \cdots < t_{2N}^\varepsilon < 1$ such that the following properties hold; $J_j^\varepsilon = (t_{2j-1}^\varepsilon, t_{2j}^\varepsilon)$ is a connected component of $\{\gamma_\varepsilon \in \Omega\}$ for $j = 1, \ldots, N$, the convergences $t_i^\varepsilon \to t_i$ and $\gamma_\varepsilon(t_i^\varepsilon) \to \gamma(t_i)$ hold as $\varepsilon \to 0$ for $i = 1, \ldots, 2N$, and moreover for any $\varepsilon \in (0, \bar{\varepsilon})$*

$$F_\varepsilon[\gamma_\varepsilon] \geq \sum_{j=1}^N \mathcal{F}_\varepsilon[\gamma_\varepsilon|_{J_j^\varepsilon}].$$

To prove this lemma, we first confirm that the energy convergence on E_0 implies an "almost" adhesion convergence.

Lemma 3.4.12. *Let $\gamma_\varepsilon \to \gamma$ be an L^∞-convergent sequence in $X^{1,\infty}$ such that*

$$\lim_{\varepsilon\to0} E_0[\gamma_\varepsilon] = E_0[\gamma].$$

Then, for any open interval $J \subset \bar{I}$ with $J \subset \{\gamma \in \partial\Omega\}$, there is $\varepsilon_J > 0$ such that the curve γ_ε touches $\partial\Omega$ in J for any $\varepsilon \in (0, \varepsilon_J)$.

Proof. Fix J. We prove by contradiction; suppose that there would be $\varepsilon_j \to 0$ such that any γ_{ε_j} does not touch $\partial\Omega$ in J. Then $J \subset \{\gamma_{\varepsilon_j} \in \Omega\}$, and hence

$$E_0[\gamma_{\varepsilon_j}|_J] = L_{\varepsilon_j}|J| = L_{\varepsilon_j}\delta,$$

where L_{ε_j} is the speed of γ_{ε_j}. On the other hand, since $J \subset \{\gamma \in \partial\Omega\}$, we would have

$$E_0[\gamma|_J] \leq \bar{\alpha}L|J| = \bar{\alpha}L\delta,$$

where L is the speed of γ. Taking the limit $j \to \infty$ we obtain a contradiction to Lemma 3.2.7. □

We next confirm that any free segment connecting boundary points allows a small perturbation.

Lemma 3.4.13. *Let $\gamma \in X^{1,1}$ be a minimizer of E_0 of positive partition number $N > 0$ and $p_1, \dots, p_{2N}$ be the contact points. Then there is $r > 0$ such that, for any $1 \leq j \leq N$ and any points $p'_{2j-1} \in \partial\Omega \cap B_r(p_{2j-1})$ and $p'_{2j} \in \partial\Omega \cap B_r(p_{2j})$, the open segment from p'_{2j-1} to p'_{2j} is included in Ω.*

Proof. Note that any original segment from p_j to p_{j+1} is not tangent to $\partial\Omega$ at the endpoints, and moreover the segment does not touch $\partial\Omega$ except the endpoints. Hence, the segment can be perturbed within Ω. □

We are in a position to prove Lemma 3.4.11.

Proof of Lemma 3.4.11. We first prove that for any small $\varepsilon > 0$ there are $0 < t_1^\varepsilon < \cdots < t_{2N}^\varepsilon < 1$ such that $t_i^\varepsilon \to t_i$ as $\varepsilon \to 0$ for $i = 1, \dots, 2N$ and the interval $J_j^\varepsilon = (t_{2j-1}^\varepsilon, t_{2j}^\varepsilon)$ is a connected component of $\{\gamma_\varepsilon \in \Omega\}$ for $j = 1, \dots, N$. By Lemma 3.4.12, for any small $\delta > 0$ there is $\varepsilon_\delta > 0$ such that γ_ε with $\varepsilon \in (0, \varepsilon_\delta)$ touches $\partial\Omega$ in each of the intervals

$$(t_1 - \delta, t_1), (t_2, t_2 + \delta), \dots, (t_{2N-1} - \delta, t_{2N-1}), (t_{2N}, t_{2N} + \delta).$$

In addition, by the uniform convergence $\gamma_\varepsilon \to \gamma$, for given $\delta > 0$ there is $\varepsilon'_\delta > 0$ such that for any $\varepsilon \in (0, \varepsilon'_\delta)$ and any $j = 1, \dots, N$, we have $(t_{2j-1} + \delta, t_{2j} - \delta) \subset \{\gamma_\varepsilon \in \Omega\}$. The above facts imply that for any small $\varepsilon > 0$ there are open intervals $J_j^\varepsilon = (t_{2j-1}^\varepsilon, t_{2j}^\varepsilon)$, $j = 1, \dots, N$, such that they are connected components of $\{\gamma_\varepsilon \in \Omega\}$, and moreover $t_i^\varepsilon \to t_i$ for any $i = 1, \dots, 2N$. The convergence $\gamma_\varepsilon(t_i^\varepsilon) \to \gamma(t_i)$ immediately follows by $t_i^\varepsilon \to t_i$ and the uniform convergence $\gamma_\varepsilon \to \gamma$.

We finally prove the energy estimate. Define a new curve $\gamma'_\varepsilon \in W^{1,\infty}(I; \mathbb{R}^2)$ as, for $j = 1, \dots, N$, the curve γ'_ε is a segment from $\gamma_\varepsilon(t_{2j-1}^\varepsilon)$ to $\gamma_\varepsilon(t_{2j}^\varepsilon)$ in J_j^ε, and otherwise $\gamma'_\varepsilon = \gamma_\varepsilon$. Then, by Lemma 3.4.13, for any small $\varepsilon > 0$ the image of γ'_ε is included in $\overline{\Omega}$; in particular, the segment parts are included in Ω except the endpoints. Notice that $E_0[\hat{\gamma}'_\varepsilon] \geq E_0[\gamma]$, where $\hat{\gamma}'_\varepsilon$ is the constant speed parameterization of γ'_ε, since $\hat{\gamma}'_\varepsilon \in X^{1,\infty}$ and γ is a minimizer of E_0. Let J^ε be the union of disjoint intervals $J_1^\varepsilon, \dots, J_{2N}^\varepsilon$. Then we find that

$$\begin{aligned} E_0[\gamma_\varepsilon]-E_0[\gamma] &\geq E_0[\gamma_\varepsilon]-E_0[\hat{\gamma}'_\varepsilon] \\ &= E_0[\gamma_\varepsilon|_{J^\varepsilon}]-E_0[\gamma'_\varepsilon|_{J^\varepsilon}] \\ &= \int_{\gamma_\varepsilon|_{J^\varepsilon}} ds - \int_{\gamma'_\varepsilon|_{J^\varepsilon}} ds \\ &= \sum_{j=1}^N \left(\int_{\gamma_\varepsilon|_{J_j^\varepsilon}} ds - l_{\gamma_\varepsilon|_{J_j^\varepsilon}} \right). \end{aligned}$$

Therefore, we have

$$\begin{aligned} F_\varepsilon[\gamma_\varepsilon] &= \varepsilon \int_{\gamma_\varepsilon} \kappa^2 ds + \frac{1}{\varepsilon}(E_0[\gamma_\varepsilon]-E_0[\gamma]) \\ &\geq \varepsilon \int_{\gamma_\varepsilon} \kappa^2 ds + \frac{1}{\varepsilon}\sum_{j=1}^N \left(\int_{\gamma_\varepsilon|_{J_j^\varepsilon}} ds - l_{\gamma_\varepsilon|_{J_j^\varepsilon}} \right) \\ &\geq \sum_{j=1}^N \frac{1}{\varepsilon}\left(\mathcal{E}_\varepsilon[\gamma_\varepsilon|_{J_j^\varepsilon}] - l_{\gamma_\varepsilon|_{J_j^\varepsilon}} \right). \end{aligned}$$

The proof is now complete. □

To conclude the lower bound inequality, we confirm that the obtained lower bound in Lemma 3.4.11 is bounded below by the functional F as $\varepsilon \to 0$.

Lemma 3.4.14. *Let $\gamma_\varepsilon \to \gamma$ be as in Assumption 3.4.9 (and recall the notations in Lemma 3.4.11). Let $\theta_{2j-1}^\varepsilon, \theta_{2j}^\varepsilon \in [-\pi,\pi)$ be the angles between the curve $\gamma_\varepsilon|_{[t_{2j-1}^\varepsilon, t_{2j}^\varepsilon]}$ and the segment from $\gamma_\varepsilon(t_{2j-1}^\varepsilon)$ to $\gamma_\varepsilon(t_{2j}^\varepsilon)$ at the endpoints for $j=1,\dots,N$ (i.e., the three vectors*

$$\gamma_\varepsilon(t_{2j}^\varepsilon)-\gamma_\varepsilon(t_{2j-1}^\varepsilon),\ R_{\theta_{2j-1}^\varepsilon}\dot{\gamma}_\varepsilon(t_{2j-1}^\varepsilon),\ R_{(-\theta_{2j}^\varepsilon)}\dot{\gamma}_\varepsilon(t_{2j}^\varepsilon)$$

are in the same direction). Then the following statements hold.

(1) For any $i=1,\dots,2N$, any subsequence of $\{\theta_i^\varepsilon\}_\varepsilon$ has a subsequence converging to either $\theta_i \in (0,\pi/2)$ or $\theta_i - \pi \in (-\pi/2,-\pi)$, where θ_i is the i-th contact angle of γ.

(2) Let $\{\gamma_{\varepsilon'}\}_{\varepsilon'}$ be any subsequence of $\{\gamma_\varepsilon\}_\varepsilon$ such that $\theta_i^{\varepsilon'}$ converges to some θ_i^* for any $i = 1,\dots,2N$. (Here θ_i^* is either θ_i or $\theta_i-\pi$.) Then, for any $j=1,\dots,N$, each term of the lower bound in Lemma 3.4.11 is estimated as

$$\liminf_{\varepsilon'\to 0} \mathcal{F}_\varepsilon[\gamma_{\varepsilon'}|_{J_j^{\varepsilon'}}] \geq 8\sqrt{2}\left(\sin^2\frac{\theta_{2j-1}^*}{4}+\sin^2\frac{\theta_{2j}^*}{4}\right).$$

(3) In particular, the following estimate holds in the full limit $\varepsilon\to 0$:

$$\liminf_{\varepsilon\to 0}\sum_{j=1}^N \mathcal{F}_\varepsilon[\gamma_\varepsilon|_{J_j^\varepsilon}] \geq \sum_{i=1}^{2N} 8\sqrt{2}\sin^2\frac{\theta_i}{4} = F[\gamma].$$

Proof. We first prove (1). Note that γ_ε is of class $W^{2,2}$, and hence C^1. Thus, the curve touches $\partial\Omega$ at $\gamma_\varepsilon(t_{2j-1}^\varepsilon)$ and $\gamma_\varepsilon(t_{2j}^\varepsilon)$ tangentially. Recall that, by Lemma 3.4.11, the endpoints of the segment from $\gamma_\varepsilon(t^\varepsilon_{2j-1})$ to $\gamma_\varepsilon(t^\varepsilon_{2j})$ converge to the endpoints of the segment from $\gamma(t_{2j-1})$ to $\gamma(t_{2j})$. Therefore, for any $i = 1, \dots, 2N$, the angle θ^ε_i converges, up to a subsequence of any subsequence, to the contact angle $\theta_i \in (0, \pi/2)$ or the opposite angle $\theta_i - \pi \in (-\pi, -\pi/2)$.

The estimate in (2) follows by Proposition 1.3.7 in Chapter 1. More precisely, if we apply Proposition 1.3.7 to the (suitably translated and rotated) sequence of $\gamma_{\varepsilon'}|_{J^{\varepsilon'}_j}$, then we reach the conclusion. (Note that the curve $\gamma_{\varepsilon'}|_{J^{\varepsilon'}_j}$ may not be a minimizer in the sense of Proposition 1.3.7, but the energy $\mathcal{F}_\varepsilon[\gamma_{\varepsilon'}|_{J^{\varepsilon'}_j}]$ is not smaller than the energy of such a minimizer.)

We finally prove (3). For the full limit estimate, it suffices to confirm that for any subsequence of $\{\gamma_\varepsilon\}_\varepsilon$ there is a subsequence $\{\gamma_{\varepsilon'}\}_{\varepsilon'}$ such that the desired estimate holds. Notice that the result (1) implies that for any subsequence of $\{\gamma_\varepsilon\}_\varepsilon$ there is a subsequence $\{\gamma_{\varepsilon'}\}_{\varepsilon'}$ such that the assumption of (2) holds, and hence the estimate in (2) holds. Recall that the limit angle θ^*_i in (2) is either θ_i or $\theta_i - \pi$ for any $i = 1, \dots, 2N$. In addition, since $\theta_i/4 \in (0, \pi/8)$ and $(\theta_i - \pi)/4 \in (-\pi/4, -\pi/8)$, we immediately find that

$$\sin^2 \frac{\theta_i - \pi}{4} > \sin^2 \frac{\theta_i}{4}, \quad \text{and hence} \quad \sin^2 \frac{\theta^*_i}{4} \geq \sin^2 \frac{\theta_i}{4}.$$

This implies the desired estimate in (3). The proof is complete. □

We shall complete the proof of the lower bound inequality.

Proof of the lower bound inequality of Theorem 3.4.6. By Lemma 3.4.10, it suffices to consider a sequence as in Assumption 3.4.9. For such a sequence, Lemma 3.4.11 and Lemma 3.4.14 directly imply the desired lower bound inequality. □

3.4.5 Upper bound inequality

We finally complete the proof of Theorem 3.4.6 by proving the upper bound inequality.

Proof of the upper bound inequality of Theorem 3.4.6. Let $\gamma \in X^{1,\infty}$ be any minimizer of E_0. If the partition number N of γ is zero, i.e., γ is the graph curve of ψ, then the trivial sequence of $\gamma_\varepsilon := \gamma$ satisfies the conclusion. Suppose that $N > 0$. Let $0 = t_0 < \cdots < t_{2N+1} = 1$ be the partition. We construct a suitable sequence by modifying γ in the segment part $U_j = (t_{2j-1}, t_{2j})$ for any $j = 1, \dots, N$.

For each j, we take a minimizer γ^j_ε of $\mathcal{E}_\varepsilon$ among all smooth constant speed curves $\zeta : [t_{2j-1}, t_{2j}] \to \mathbb{R}^2$ satisfying $\zeta = \gamma$ and $\dot\zeta = \dot\gamma$ at t_{2j-1} and t_{2j}. Noting that $\partial\Omega$ is of class C^2 and the contact angles of γ are strictly positive, we find that Theorem 1.2.3 of Chapter 1 implies that for any j and small ε the minimizer γ^j_ε is included in Ω except the endpoints. Moreover, Lemma 1.3.1 of Chapter 1 implies that the sequence of γ^j_ε satisfies

$$\lim_{\varepsilon \to 0} \mathcal{F}_\varepsilon[\gamma^j_\varepsilon] = 8\sqrt{2} \left(\sin^2 \frac{\theta_{2j-1}}{4} + \sin^2 \frac{\theta_{2j}}{4} \right),$$

where θ_i denotes the i-th contact angle of γ. We then define γ_ε so that $\gamma_\varepsilon = \gamma^j_\varepsilon$ in $U_j = (t_{2j-1}, t_{2j})$ for any $j = 1, \dots, N$, and $\gamma_\varepsilon = \gamma$ elsewhere. It turns out that the sequence of the constant speed reparameterization $\hat\gamma_\varepsilon$ of γ_ε (defined for small ε) satisfies the desired upper bound inequality. In fact, since the difference between the energies E_0 of $\hat{\gamma}_\varepsilon$ and γ appears only in the free part, we find that

$$F_\varepsilon[\hat{\gamma}_\varepsilon] = \frac{E_\varepsilon[\hat{\gamma}_\varepsilon] - E_0[\gamma]}{\varepsilon} = \sum_{j=1}^{N} \mathcal{F}_\varepsilon[\gamma_\varepsilon^j].$$

Taking the limit $\varepsilon \to 0$, we reach the conclusion. □

3.5 Adhesion convergence

In this section, by using Theorem 3.4.6 (in particular, the minimum convergence in Corollary 3.4.7), we prove that any convergent sequence of minimizers of E_ε converges not only in L^∞ but also in the sense of adhesion convergence as follows.

Definition 3.5.1 (Adhesion convergence)**.** Let $\gamma \in X^{1,\infty}$ be partitional regular of partition $0 = t_0 < t_1 < \cdots < t_{2N} < t_{2N+1} = 1$. Let $\{\gamma_\varepsilon\}_\varepsilon \subset X^{2,2}$. We say that γ_ε *A-converges* to γ as $\varepsilon \to 0$ if there is $\bar{\varepsilon} > 0$ such that γ_ε is ordered partitional regular of partition number N for any $\varepsilon \in (0, \bar{\varepsilon})$, and moreover its partition $0 = t_0^\varepsilon < t_1^\varepsilon < \cdots < t_{2N}^\varepsilon < t_{2N+1}^\varepsilon = 1$ satisfies the convergence $t_i^\varepsilon \to t_i$ as $\varepsilon \to 0$ for any $1 \leq i \leq 2N$.

The main theorem of this section is the following

Theorem 3.5.2 (Adhesion convergence of minimizers)**.** *Let $\{\gamma_\varepsilon\}_\varepsilon \subset X^{2,2}$ be a sequence of minimizers of E_ε and $\gamma \in X^{1,\infty}$ be a minimizer of F such that $\gamma_\varepsilon \to \gamma$ in L^∞. Then γ_ε A-converges to γ as $\varepsilon \to 0$.*

The adhesion convergence is also defined for a subsequence $\{\gamma_{\varepsilon'}\}_{\varepsilon'}$ in the same way. In this subsection we also use the notation ε as the index even for a subsequence; in particular, Theorem 3.5.2 is valid for any sequence $\varepsilon' \to 0$ and any sequence $\{\gamma_{\varepsilon'}\}_{\varepsilon'} \subset X^{2,2}$ of minimizers of $E_{\varepsilon'}$ converging to a minimizer of F in L^∞.

3.5.1 Partitional regularization

In this subsection we confirm that a new notion "partitional regularization" is well-defined for any convergent sequence of minimizers of E_ε. This notion is important since, as shown in the next subsection, it turns out that a minimizer coincides with its partitional regularization for small ε.

We first prove that, for a sequence of minimizers, the angles in Lemma 3.4.14 fully converge to the corresponding contact angles.

Lemma 3.5.3. *Let $\{\gamma_\varepsilon\}_\varepsilon \subset X^{2,2}$ be a sequence of minimizers of E_ε and $\gamma \in X^{1,\infty}$ be a minimizer of F such that $\gamma_\varepsilon \to \gamma$ in L^∞. Suppose that the partition number N of γ is positive. (Note that γ also minimizes E_0 and*

$$\lim_{\varepsilon \to 0} E_0[\gamma_\varepsilon] = E_0[\gamma]$$

holds, and hence the assumption of Lemma 3.4.11 is satisfied.) Then, for any $i = 1, \ldots, 2N$, the angle θ_i^ε defined in Lemma 3.4.14 converges to the corresponding contact angle $\theta_i \in (0, \pi/2)$ of γ in the full limit $\varepsilon \to 0$.

Proof. Since γ_ε also minimizes F_ε, Corollary 3.4.7 implies that

$$\lim_{\varepsilon\to 0} F_\varepsilon[\gamma_\varepsilon] = F[\gamma] = \sum_{i=1}^{2N} 8\sqrt{2}\sin^2\frac{\theta_i}{4}.$$

On the other hand, by Lemma 3.4.11 and Lemma 3.4.14,

$$\lim_{\varepsilon\to 0} F_\varepsilon[\gamma_\varepsilon] \geq \liminf_{\varepsilon\to 0}\sum_{i=1}^{2N} 8\sqrt{2}\sin^2\frac{\theta_i^\varepsilon}{4} \geq \sum_{i=1}^{2N} 8\sqrt{2}\sin^2\frac{\theta_i}{4}.$$

Recalling that θ_i^ε converges to θ_i or $\theta_i - \pi$ up to a subsequence of any subsequence, and

$$\sin^2\frac{\theta_i - \pi}{4} > \sin^2\frac{\theta_i}{4},$$

we find that the lower bound estimate is also valid for each term, i.e., for any $i = 1,\dots,2N$,

$$\liminf_{\varepsilon\to 0}\sin^2\frac{\theta_i^\varepsilon}{4} \geq \sin^2\frac{\theta_i}{4}.$$

Since each term in the sum is positive, we find that

$$\lim_{\varepsilon\to 0}\sin^2\frac{\theta_i^\varepsilon}{4} = \sin^2\frac{\theta_i}{4}.$$

Therefore, we conclude that θ_i^ε fully converges to θ_i as $\varepsilon\to 0$. □

By using the above result, we can appropriately modify minimizers as partitional regular curves.

Corollary 3.5.4. *Suppose the same assumption as in Lemma 3.5.3. Using the same notations as in Lemma 3.4.11, we define a curve $\bar{\gamma}_\varepsilon : \bar{I} \to \overline{\Omega}$ for any $\varepsilon \in (0,\bar{\varepsilon})$ so that, $\bar{\gamma}_\varepsilon(0) = (0,\psi(0))$, $\bar{\gamma}_\varepsilon(1) = (1,\psi(1))$, $\bar{\gamma}_\varepsilon = \gamma_\varepsilon$ in $J_1^\varepsilon,\dots,J_N^\varepsilon$, and $\bar{\gamma}_\varepsilon$ is a constant speed geodesic in $\partial\Omega$ in each of the connected components $K_0^\varepsilon,\dots,K_N^\varepsilon$ of $\bar{I}\setminus J^\varepsilon$, where J^ε is the union of $J_1^\varepsilon,\dots,J_N^\varepsilon$. Then there exists $\bar{\varepsilon}' \in (0,\bar{\varepsilon})$ such that for any $\varepsilon \in (0,\bar{\varepsilon}')$ the constant speed reparameterization $\hat{\bar{\gamma}}_\varepsilon$ of $\bar{\gamma}_\varepsilon$ belongs to $X^{2,2}(I;\overline{\Omega})$. In particular,*

$$E_\varepsilon[\hat{\bar{\gamma}}_\varepsilon] \geq E_\varepsilon[\gamma_\varepsilon].$$

Proof. To confirm that the reparameterization of $\bar{\gamma}_\varepsilon$ belongs to $X^{2,2}$, it suffices to prove that, all the geodesic parts of $\bar{\gamma}_\varepsilon$ have positive lengths (speeds), and $\bar{\gamma}_\varepsilon$ is "smoothly" connected at all the connection points $\bar{\gamma}_\varepsilon(t_i^\varepsilon)$, $i = 1,\dots,2N$, i.e., the direction of the tangent vector $\bar{\gamma}_\varepsilon/|\bar{\gamma}_\varepsilon|$ is continuous at t_i^ε for any $i = 1,\dots,2N$.

Recall that the convergent limit γ in Lemma 3.5.3 is represented as a graph curve; in particular, the (x-components of the) points $\gamma(0),\gamma(t_1),\dots,\gamma(t_{2N}),\gamma(1)$ are ordered in the left-to-right direction. Hence, by the convergence $\gamma_\varepsilon(t_i^\varepsilon)\to\gamma(t_i)$ in Lemma 3.4.11, the points $\bar{\gamma}_\varepsilon(0),\bar{\gamma}_\varepsilon(t_1^\varepsilon),\dots,\bar{\gamma}_\varepsilon(t_{2N}^\varepsilon),\bar{\gamma}_\varepsilon(1)$ are also ordered in the left-to-right direction for any small $\varepsilon > 0$. Since the every other two adjacent points are the endpoints of the geodesic parts of $\bar{\gamma}_\varepsilon$, all the geodesic parts have positive lengths for any small $\varepsilon > 0$.

The remaining part is to confirm that the connection points are "smooth". We consider the half-limits of the tangent vector $\dot{\bar{\gamma}}_\varepsilon$ at t_i^ε for any $i = 1,\dots,2N$. By the above argument on the order, we also notice that for any small $\varepsilon > 0$ the geodesic parts of $\bar{\gamma}_\varepsilon$ follow $\partial\Omega$ in the left-to-right direction. In particular, noting that each of the connection points converges to the corresponding contact point of γ as $\varepsilon \to 0$, we find that for any small $\varepsilon > 0$ and $j = 1, \dots, N$ the half-limits (from the geodesic parts) $\dot{\bar{\gamma}}_\varepsilon(t^\varepsilon_{2j-1}-)$ and $\dot{\bar{\gamma}}_\varepsilon(t^\varepsilon_{2j-}+)$ make acute angles with the vector $\bar{\gamma}_\varepsilon(t^\varepsilon_{2j}) - \bar{\gamma}_\varepsilon(t^\varepsilon_{2j-1})$. On the other hand, the original curve γ_ε touches $\partial\Omega$ tangentially at the connection points; hence, for the direction of each of the half-limits $\dot{\bar{\gamma}}_\varepsilon(t^\varepsilon_{2j-1}+)$ $(= \dot{\gamma}_\varepsilon(t^\varepsilon_{2j-1}))$ and $\dot{\bar{\gamma}}_\varepsilon(t^\varepsilon_{2j}-)$ $(= \dot{\gamma}_\varepsilon(t^\varepsilon_{2j}))$, there are only two possibilities; the same direction in the corresponding half-limit from the geodesic part, or the opposite direction. By Lemma 3.5.3, we find that $\dot{\bar{\gamma}}_\varepsilon(t^\varepsilon_{2j-1}+)$ and $\dot{\bar{\gamma}}_\varepsilon(t^\varepsilon_{2j}-)$ also make acute angles with the vector $\bar{\gamma}_\varepsilon(t^\varepsilon_{2j}) - \bar{\gamma}_\varepsilon(t^\varepsilon_{2j-1})$; this means that the two half-limits are in the same direction at each connection point. Therefore, the constant speed reparameterization $\hat{\bar{\gamma}}_\varepsilon$ belongs to $X^{2,2}$.

The last inequality in the statement is an immediate consequence of the minimality of γ_ε. The proof is now complete. □

We are in a position to define the partitional regularization of a minimizer.

Definition 3.5.5 (Partitional regularization)**.** Let $\{\gamma_\varepsilon\}_\varepsilon \subset X^{2,2}$ be a sequence of minimizers of E_ε, and $\gamma \in X^{1,\infty}$ be a minimizer of F such that $\gamma_\varepsilon \to \gamma$ in L^∞. Let N be the partition number of γ. If $N > 0$, then we define the curve $\bar{\gamma}_\varepsilon$ as a (unique) curve in Corollary 3.5.4 for $\varepsilon \in (0, \bar{\varepsilon}')$. If $N = 0$ (i.e., γ is the geodesic in $\partial\Omega$ joining $(0, \psi(0))$ to $(1, \psi(1))$), then for any $\varepsilon > 0$ we define $\bar{\gamma}_\varepsilon$ just as equal to γ. (In this case we interpret as $\varepsilon' = \infty$.) We call $\bar{\gamma}_\varepsilon$ *partitional regularization* of γ_ε.

3.5.2 Minimizers coincide with the partitional regularizations

As mentioned, in this subsection, we prove that the partitional regularization of a minimizer coincides with the minimizer itself for any small $\varepsilon > 0$. This immediately implies Theorem 3.5.2.

We first state that the difference between a minimizer and its partitional regularization tends to be small as $\varepsilon \to 0$ in the following sense.

Lemma 3.5.6. *Let $\{\gamma_\varepsilon\}_\varepsilon \in X^{2,2}$ be a sequence of minimizers of E_ε and $\gamma \in X^{1,\infty}$ be a minimizer of F such that $\gamma_\varepsilon \to \gamma$ in L^∞. Let $\bar{\gamma}_\varepsilon$ be the partitional regularization of γ_ε for $\varepsilon \in (0, \bar{\varepsilon}')$. Let M_ε be the supremum of the widths $|J|$ of all open intervals J in $\{\gamma_\varepsilon \in \Omega\} \cap \{\bar{\gamma}_\varepsilon \in \partial\Omega\}$, where $M_\varepsilon := 0$ if there is no such interval. Then $M_\varepsilon \to 0$ as $\varepsilon \to 0$.*

Proof. We prove by contradiction; suppose that there would be $r > 0$ and $\varepsilon_k \to 0$ such that $M_{\varepsilon_k} \geq 4r$ for any k. Then for any k there is an open interval $J_k = (t^k_0, t^k_1)$ in $\{\gamma_{\varepsilon_k} \in \Omega\} \cap \{\bar{\gamma}_{\varepsilon_k} \in \partial\Omega\}$ such that $|J_k| \geq 3r$. Since the intervals J_k are included in the compact set $\bar{I}$, the endpoints $t^k_0, t^k_1 \in \partial J_k \subset \{\bar{\gamma}_\varepsilon \in \partial\Omega\}$ converge to some limits $t_0, t_1 \in \bar{I}$ with $t_1 - t_0 \geq 3r$ up to a subsequence (without relabeling). Recall that, by Lemma 3.4.11, the closed set $\{\bar{\gamma}_\varepsilon \in \partial\Omega\}$ converges to a limit set included in $\{\gamma \in \partial\Omega\}$ as $\varepsilon \to 0$ (at least, the closure of the limit set coincides with $\{\gamma \in \partial\Omega\}$). Hence, in particular, $[t_0, t_1] \subset \{\gamma \in \partial\Omega\}$. Let $J := (t_0 + r, t_1 - r) \subset \{\gamma \in \partial\Omega\}$. Note that $|J| \geq r$. Then, by the convergences $t^k_0 \to t_0$ and $t^k_1 \to t_1$, for any large k (small ε_k) the interval J is included in J_k, and hence in $\{\gamma_{\varepsilon_k} \in \Omega\}$. Therefore, $J \subset \{\gamma_{\varepsilon_k} \in \Omega\} \cap \{\gamma \in \partial\Omega\}$ for any large k. This contradicts Lemma 3.4.12. □

We then prove that a part of minimizer does not have a smaller energy than a geodesic in $\partial\Omega$ of the same endpoints as follows.

Lemma 3.5.7. *Let $\{\gamma_\varepsilon\}_\varepsilon \subset X^{2,2}$ be a sequence of minimizers of E_ε and $\gamma \in X^{1,\infty}$ be a minimizer of F such that $\gamma_\varepsilon \to \gamma$ in L^∞. Let $\bar{\gamma}_\varepsilon$ be the partitional regularization of γ_ε for $\varepsilon \in (0, \bar{\varepsilon}')$. Let $\tilde{\gamma}_\varepsilon$ and $\tilde{\bar{\gamma}}_\varepsilon$ be the periodic extensions of γ_ε and $\bar{\gamma}_\varepsilon$, respectively. Then there is $\bar{\varepsilon}'' \in (0, \bar{\varepsilon}')$ such that, for any $\varepsilon \in (0, \bar{\varepsilon}'')$ and any open interval $J \subset \{\tilde{\bar{\gamma}}_\varepsilon \in \partial\Omega\}$ with $\tilde{\gamma}_\varepsilon(\partial J)$,*

$$E_0[\tilde{\gamma}_\varepsilon|_J] \geq E_0[\gamma_\Omega^{\varepsilon,J}],$$

where $\gamma_\Omega^{\varepsilon,J}$ denotes a geodesic in $\partial\Omega$ joining the points of $\tilde{\gamma}_\varepsilon(\partial J) \subset \partial\Omega$. If the partition number of γ is zero, then the above estimate also holds for $\varepsilon \in (0, \bar{\varepsilon}'')$ and any interval J in $\mathbb{R}$ of width 1.

Proof. By Lemma 3.5.6 and the fact that the speed L_ε of γ_ε converges (to the speed of γ), there is $\bar{\varepsilon}'' \in (0, \bar{\varepsilon}')$ such that $L_\varepsilon|J'| < R_\Omega(1-\bar{\alpha})$ holds for any $\varepsilon \in (0, \bar{\varepsilon}'')$ and any open interval $J' \subset \{\tilde{\gamma}_\varepsilon \in \Omega\} \cap \{\tilde{\bar{\gamma}}_\varepsilon \in \partial\Omega\}$. Fix any $\varepsilon \in (0, \bar{\varepsilon}'')$ and any open interval $J \subset \{\tilde{\bar{\gamma}}_\varepsilon \in \partial\Omega\}$ with $\tilde{\gamma}_\varepsilon(\partial J) \subset \partial\Omega$. Then Lemma 3.3.13 implies that $E_0[\tilde{\gamma}_\varepsilon] \geq E_0[\gamma_\Omega^{\varepsilon,J}]$, which is the desired estimate.

Suppose that the partition number of γ is zero. In this case $\{\tilde{\bar{\gamma}}_\varepsilon \in \partial\Omega\} = \mathbb{R}$. Then, by Lemma 3.5.6, there is $\bar{\varepsilon}'' \in (0, \bar{\varepsilon}')$ such that for any $\varepsilon \in (0, \bar{\varepsilon}'')$ any open interval $J' \subset \{\tilde{\gamma}_\varepsilon \in \Omega\}$ $(= \{\tilde{\gamma}_\varepsilon \in \Omega\} \cap \{\tilde{\bar{\gamma}}_\varepsilon \in \partial\Omega\})$ satisfies not only $L_\varepsilon|J'| < R_\Omega(1-\bar{\alpha})$ but also $|J'| < 1$. This implies that for any $\varepsilon \in (0, \bar{\varepsilon}'')$ the curve $\tilde{\gamma}_\varepsilon$ touches $\partial\Omega$ at some $t_\varepsilon \in \mathbb{R}$ (and hence at $t_\varepsilon + 1$ by periodicity); thus, Lemma 3.3.13 implies that $E_0[\tilde{\gamma}_\varepsilon] \geq E_0[\gamma_\Omega^{\varepsilon,J}]$ also holds for $J = (t_\varepsilon, t_\varepsilon + 1)$. By the periodicity of $\tilde{\gamma}_\varepsilon$, the interval J can be taken as any interval of width 1. The proof is complete. □

As the last preparation for the proof of Theorem 3.5.2, we state that, for small ε, original minimizers are represented by graphs at least on the coincidence sets of their partitional regularizations.

Lemma 3.5.8. *Let $\{\gamma_\varepsilon\}_\varepsilon \subset X^{2,2}$ be a sequence of minimizers of E_ε and $\gamma \in X^{1,\infty}$ be a minimizer of F such that $\gamma_\varepsilon \to \gamma$ in L^∞. Let $\bar{\gamma}_\varepsilon$ be the partitional regularization of γ_ε and $K_0^\varepsilon, \dots, K_N^\varepsilon$ be the partition of $\bar{\gamma}_\varepsilon$ for $\varepsilon \in (0, \bar{\varepsilon}')$. Then there is $\bar{\varepsilon}''' \in (0, \bar{\varepsilon}')$ such that for any $\varepsilon \in (0, \bar{\varepsilon}''')$ the curve γ_ε is represented by the graph of an H^2-function in each of $K_0^\varepsilon, \dots, K_N^\varepsilon$.*

Proof. We first suppose that $N > 0$. Let $\tilde{\gamma}_\varepsilon$ and $\tilde{\bar{\gamma}}_\varepsilon$ be the periodic extensions of γ_ε and $\bar{\gamma}_\varepsilon$. Let $\tilde{K}_0^\varepsilon$ be the union of K_0^ε and $K_N^\varepsilon - 1$, i.e.,

$$\tilde{K}_0^\varepsilon := \{t \in \mathbb{R} \mid t \in K_0^\varepsilon \ \text{ or } \ t+1 \in K_N^\varepsilon\},$$

and $\tilde{K}_j^\varepsilon := K_j^\varepsilon$ for any $j = 1, \dots, N-1$. Note that $\tilde{K}_0^\varepsilon$ is a closed interval including the origin, and $\tilde{\gamma}_\varepsilon$ touches $\partial\Omega$ at the endpoints of $\tilde{K}_0^\varepsilon, \dots, \tilde{K}_{N-1}^\varepsilon$. Since $E_\varepsilon[\gamma_\varepsilon] \leq E_\varepsilon[\hat{\gamma}_\varepsilon]$ by Corollary 3.5.4, where $\hat{\gamma}_\varepsilon$ is the constant reparameterization of $\bar{\gamma}_\varepsilon$, we find that, for any $\varepsilon \in (0, \bar{\varepsilon}')$,

$$\sum_{j=0}^{N-1} \left(E_\varepsilon[\tilde{\gamma}_\varepsilon|_{\tilde{K}_j^\varepsilon}] - E_\varepsilon[\tilde{\bar{\gamma}}_\varepsilon|_{\tilde{K}_j^\varepsilon}] \right) \leq 0. \tag{3.5.1}$$

Moreover, for any small $\varepsilon \in (0, \bar{\varepsilon}')$, each term is estimated from below as

$$
\begin{aligned}
E_\varepsilon[\tilde{\gamma}_\varepsilon|_{\tilde{K}_j^\varepsilon}] - E_\varepsilon[\tilde{\bar{\gamma}}_\varepsilon|_{\tilde{K}_j^\varepsilon}] &\geq E_0[\tilde{\gamma}_\varepsilon|_{\tilde{K}_j^\varepsilon}] - E_0[\tilde{\bar{\gamma}}_\varepsilon|_{\tilde{K}_j^\varepsilon}] - \varepsilon^2 \int_{\tilde{\bar{\gamma}}_\varepsilon|_{\tilde{K}_j^\varepsilon}} \kappa^2 ds \\
&\geq -\varepsilon^2 \int_I \frac{\psi''^2}{(1+\psi'^2)^{5/2}}
\end{aligned}
\tag{3.5.2}
$$

since $\tilde{\bar{\gamma}}_\varepsilon|_{\tilde{K}_j^\varepsilon}$ is a geodesic included in one period of ψ and $E_0[\tilde{\gamma}_\varepsilon|_{\tilde{K}_j^\varepsilon}] \geq E_0[\tilde{\bar{\gamma}}_\varepsilon|_{\tilde{K}_j^\varepsilon}]$ for any small ε by Lemma 3.5.7. Therefore, combining (3.5.1) and (3.5.2), we find that

$$
E_\varepsilon[\tilde{\gamma}_\varepsilon|_{\tilde{K}_j^\varepsilon}] - E_\varepsilon[\tilde{\bar{\gamma}}_\varepsilon|_{\tilde{K}_j^\varepsilon}] = o(\varepsilon^2)
\tag{3.5.3}
$$

as $\varepsilon \to 0$ for any $j = 0, \dots, N-1$.

We now suppose the contradiction that there would be a subsequence $\{\tilde{\gamma}_{\varepsilon_k}\}_k$ of $\{\tilde{\gamma}_\varepsilon\}_\varepsilon$ and a sequence of times $\{t_k\}_k$ in the union of $\tilde{K}_1^{\varepsilon_k}, \dots, \tilde{K}_N^{\varepsilon_k}$ such that for any k the x-component of $\dot{\tilde{\gamma}}_{\varepsilon_k}(t_k)$ is nonpositive. We extract a subsequence (without relabeling) so that there is j such that $t_k \in \tilde{K}_j^{\varepsilon_k}$ for any k. Then there are a connected component J_k' of $\{\tilde{\gamma}_{\varepsilon_k} \in \Omega\} \cap \tilde{K}_j^{\varepsilon_k}$ and a time $t_k' \in J_k'$ such that the x-component of $\dot{\tilde{\gamma}}_{\varepsilon_k}(t_k')$ is zero. Note that $\tilde{\gamma}_{\varepsilon_k}$ touches $\partial\Omega$ at the endpoints of J_k'. We decompose $\tilde{K}_j^{\varepsilon_k}$ into J_k' and $\tilde{K}_j^{\varepsilon_k} \setminus J_k'$. The part $\tilde{\gamma}_{\varepsilon_k}|_{\tilde{K}_j^{\varepsilon_k} \setminus J_k'}$ still satisfies

$$
E_{\varepsilon_k}[\tilde{\gamma}_{\varepsilon_k}|_{\tilde{K}_j^{\varepsilon_k} \setminus J_k'}] - E_{\varepsilon_k}[\tilde{\bar{\gamma}}_{\varepsilon_k}|_{\tilde{K}_j^{\varepsilon_k} \setminus J_k'}] \geq -\varepsilon_k^2 \int_I \frac{\psi''^2}{(1+\psi'^2)^{5/2}}
\tag{3.5.4}
$$

for any small ε_k by the same reason as (3.5.2). The part $\tilde{\gamma}_{\varepsilon_k}|_{J_k'}$ is estimated as

$$
\begin{aligned}
& E_{\varepsilon_k}[\tilde{\gamma}_{\varepsilon_k}|_{J_k'}] - E_{\varepsilon_k}[\tilde{\bar{\gamma}}_{\varepsilon_k}|_{J_k'}] \\
\geq\ & {\varepsilon_k}^2 \int_{\tilde{\gamma}_{\varepsilon_k}|_{J_k'}} \kappa^2 ds + \int_{\tilde{\gamma}_{\varepsilon_k}|_{J_k'}} ds - E_0[\tilde{\bar{\gamma}}_{\varepsilon_k}|_{J_k'}] - {\varepsilon_k}^2 \int_I \frac{\psi''^2}{(1+\psi'^2)^{5/2}}.
\end{aligned}
$$

By Lemma 3.3.11 and Lemma 3.5.6 (and the fact that the speed L_{ε_k} of $\tilde{\gamma}_\varepsilon$ converges), there are $\delta > 0$ and $\varepsilon_\delta > 0$ such that for any $\varepsilon_k \in (0, \varepsilon_\delta)$ we have

$$
E_0[\tilde{\bar{\gamma}}_{\varepsilon_k}|_{J_k'}] \leq \frac{1}{1+\delta} E_0[\tilde{\gamma}_{\varepsilon_k}|_{J_k'}] = \frac{1}{1+\delta} \int_{\tilde{\gamma}_{\varepsilon_k}|_{J_k'}} ds.
$$

Therefore, for any small ε_k,

$$
\begin{aligned}
& E_{\varepsilon_k}[\tilde{\gamma}_{\varepsilon_k}|_{J_k'}] - E_{\varepsilon_k}[\tilde{\bar{\gamma}}_{\varepsilon_k}|_{J_k'}] \\
\geq\ & {\varepsilon_k}^2 \int_{\tilde{\gamma}_{\varepsilon_k}|_{J_k'}} \kappa^2 ds + \frac{\delta}{1+\delta} \int_{\tilde{\gamma}_{\varepsilon_k}|_{J_k'}} ds - {\varepsilon_k}^2 \int_I \frac{\psi''^2}{(1+\psi'^2)^{5/2}} \\
\geq\ & 2\varepsilon_k \sqrt{\frac{\delta}{1+\delta}} \int_{\tilde{\gamma}_{\varepsilon_k}|_{J_k'}} |\kappa| ds - {\varepsilon_k}^2 \int_I \frac{\psi''^2}{(1+\psi'^2)^{5/2}}.
\end{aligned}
$$

Recall that the total absolute curvature $\int |\kappa| ds$ of a curve is nothing but the total variation of the tangential angle of the curve. Hence, by the assumption that the

x-component of $\dot{\tilde{\gamma}}_{\varepsilon_k}$ is zero at $t'_k \in J'_k$ and the fact that $\tilde{\gamma}_{\varepsilon_k}$ touches $\partial\Omega$ tangentially at the endpoints of J'_k, there is $r_\psi > 0$ such that for any ε_k

$$\int_{\tilde{\gamma}_{\varepsilon_k}|_{J'_k}} |\kappa| ds \geq r_\psi.$$

This implies that there is $C_{\delta,\psi} > 0$ such that

$$E_{\varepsilon_k}[\tilde{\gamma}_{\varepsilon_k}|_{J'_k}] - E_{\varepsilon_k}[\tilde{\bar{\gamma}}_{\varepsilon_k}|_{J'_k}] \geq C_{\delta,\psi}\varepsilon_k. \tag{3.5.5}$$

for any small ε_k. The sum of (3.5.4) and (3.5.5) contradicts (3.5.3). Therefore, the proof is complete in the case that $N > 0$.

We finally consider the case that $N = 0$. In this case, for any small ε the curve $\tilde{\gamma}_\varepsilon$ touches $\partial\Omega$ at some $t_\varepsilon \in \bar{I}$ (and hence at $t_\varepsilon + 1$) by Lemma 3.5.6. Then, letting $\tilde{K}_0^\varepsilon := (t_\varepsilon, t_\varepsilon + 1)$, we can proceed the same argument as in the case that $N = 1$, and then reach the conclusion. The proof is now complete. □

We finally complete the proof of Theorem 3.5.2 by showing that any minimizer γ_ε coincides with the partitional regularization $\bar{\gamma}_\varepsilon$ for any small ε.

Proof of Theorem 3.5.2. It suffices to prove that $\{\tilde{\gamma}_\varepsilon \in \partial\Omega\} = \{\tilde{\bar{\gamma}}_\varepsilon \in \partial\Omega\}$ for any small ε. By the definition of $\bar{\gamma}_\varepsilon$, we know that $\{\tilde{\gamma}_\varepsilon \in \partial\Omega\} \subset \{\tilde{\bar{\gamma}}_\varepsilon \in \partial\Omega\}$. Therefore, it suffices to prove that $\{\tilde{\gamma}_\varepsilon \in \Omega\} \cap \{\tilde{\bar{\gamma}}_\varepsilon \in \partial\Omega\}$ is empty for any small ε.

We suppose the contradiction that there would be a sequence $\varepsilon_k \to 0$ and (nonempty) connected component J_{ε_k} of $\{\tilde{\gamma}_{\varepsilon_k} \in \Omega\} \cap \{\tilde{\bar{\gamma}}_{\varepsilon_k} \in \partial\Omega\}$ for any small ε_k. By Lemma 3.5.6, $|J_{\varepsilon_k}| \to 0$ as $\varepsilon_k \to 0$. Recall that $\tilde{\gamma}_{\varepsilon_k}$ touches $\partial\Omega$ at the endpoints of J_{ε_k} and the speed L_{ε_k} of $\tilde{\gamma}_{\varepsilon_k}$ converges. Then, by Lemma 3.3.11, there is $\delta > 0$ such that, for any small ε_k,

$$E_0[\gamma_\Omega^{\varepsilon_k}] \leq \frac{1}{1+\delta} E_0[\tilde{\gamma}_{\varepsilon_k}|_{J_{\varepsilon_k}}],$$

where $\gamma_\Omega^{\varepsilon_k}$ is a geodesic in $\partial\Omega$ joining the points $\tilde{\gamma}_{\varepsilon_k}(\partial J_{\varepsilon_k})$. Since the speed $L_\Omega^{\varepsilon_k}$ of $\gamma_\Omega^{\varepsilon_k}$ satisfies $L_\Omega^{\varepsilon_k} \leq L_{\varepsilon_k}/\bar{\alpha}$, we also have

$$\varepsilon_k^2 \int_{\gamma_\Omega^{\varepsilon_k}} \kappa^2 ds \leq \frac{\varepsilon_k^2 L_\Omega^{\varepsilon_k} |J_{\varepsilon_k}|}{R_\Omega^2} \leq \frac{\varepsilon_k^2 L_{\varepsilon_k} |J_{\varepsilon_k}|}{R_\Omega^2 \bar{\alpha}} = \frac{\varepsilon_k^2}{R_\Omega^2 \bar{\alpha}} E_0[\tilde{\gamma}_{\varepsilon_k}|_{J_{\varepsilon_k}}],$$

and hence, for any small ε_k (as $\frac{\varepsilon_k^2}{R_\Omega^2 \bar{\alpha}} \leq \frac{\delta/2}{1+\delta}$),

$$E_{\varepsilon_k}[\gamma_\Omega^{\varepsilon_k}] \leq \frac{1+\delta/2}{1+\delta} E_0[\tilde{\gamma}_{\varepsilon_k}|_{J_{\varepsilon_k}}] < E_0[\tilde{\gamma}_{\varepsilon_k}|_{J_{\varepsilon_k}}] \leq E_{\varepsilon_k}[\tilde{\gamma}_{\varepsilon_k}|_{J_{\varepsilon_k}}].$$

Let $\tilde{\gamma}'_{\varepsilon_k}$ be the curve $\tilde{\gamma}_{\varepsilon_k}$ replaced by the geodesic $\gamma_\Omega^{\varepsilon_k}$ in J_{ε_k} and, in $J_{\varepsilon_k} + \mathbb{Z}$, replaced by the translated geodesics so that $\tilde{\gamma}'_{\varepsilon_k}$ has the 1-periodicity. By Lemma 3.5.8, we know that for any small ε_k the curve $\tilde{\gamma}_{\varepsilon_k}$ is represented as a graph in J_{ε_k}. Hence, the modification from $\tilde{\gamma}_{\varepsilon_k}$ to $\tilde{\gamma}'_{\varepsilon_k}$ is "smooth" in the sense that the constant speed reparameterization $\hat{\tilde{\gamma}}'_{\varepsilon_k}$ of $\tilde{\gamma}'_{\varepsilon_k}$ belongs to $X^{2,2}$. Moreover, by the above energy estimate, we find that

$$E_{\varepsilon_k}[\hat{\tilde{\gamma}}'_{\varepsilon_k}|_I] < E_{\varepsilon_k}[\tilde{\gamma}_{\varepsilon_k}|_I]$$

for any small ε_k. This contradicts the minimality of $\tilde{\gamma}_{\varepsilon_k}$. The proof is complete. $\square$

3.6 Graph representation result

In this last section, we confirm that any minimizer is represented by a graph for small ε by using the adhesion convergence result.

Corollary 3.6.1 (Graph representation). *There is $\bar{\varepsilon} = \bar{\varepsilon}(\alpha, \psi) > 0$ such that, for any $\varepsilon \in (0, \bar{\varepsilon})$, any minimizer of E_ε is represented by the graph of a function $u \in H^2(I)$ such that $\max u = \max \psi$, $\|u'\|_\infty \leq \|\psi'\|_\infty$, and moreover $u''(x) > 0$ for any $x \in \bar{I}$ with $u(x) > \psi(x)$.*

Proof. It suffices to prove that, for any sequence $\varepsilon_j \to 0$ and any sequence of minimizers $\{\gamma_{\varepsilon_j}\}_j$ of E_{ε_j}, there is a subsequence $\{\gamma_{\varepsilon_k}\}_k$ such that for any small ε_k the curve γ_{ε_k} is represented by the graph of some $u_k \in H^2(I)$ with the desired properties.

By Corollary 3.4.7, for any sequence of minimizers there are a subsequence $\{\gamma_{\varepsilon_k}\}_k$ and a minimizer γ of F such that $\gamma_{\varepsilon_k} \to \gamma$ in L^∞. Then, by Theorem 3.5.2, we also find that γ_{ε_k} A-converges to γ. By the adhesion convergence, in the case that $N = 0$, any curve γ_{ε_k} is the graph of ψ; this immediately implies the conclusion. Suppose that $N > 0$. Let $0 < t_1^{\varepsilon_k} < \cdots < t_{2N}^{\varepsilon_k} < 1$ be as in Definition 3.5.1 and $J_j^k := (t_{2j-1}^{\varepsilon_k}, t_{2j}^{\varepsilon_k})$ for $j = 1, \ldots, N$. For j and ε_k, we denote by $\gamma_{\varepsilon_k}^j$ a minimizer of $\mathcal{E}_{\varepsilon_k}$ among all smooth constant speed curves $\zeta : \overline{J_j^k} \to \mathbb{R}^2$ with the same positions and tangential angles as $\gamma_{\varepsilon_k}|_{J_j^k}$ at the endpoints. Noting that $\partial\Omega$ is of class C^2 and the contact angles of γ are strictly positive, we find that Theorem 1.2.3 in Chapter 1 implies that for any j and small ε_k the minimizer $\gamma_{\varepsilon_k}^j$ is included in Ω except the endpoints. Then, noting that γ_{ε_k} minimizes E_ε, we find that for any j and small ε_k the curve $\gamma_{\varepsilon_k}|_{J_j^k}$ is nothing but the minimizer $\gamma_{\varepsilon_k}^j$; in particular, the curve $\gamma_{\varepsilon_k}|_{J_j^k}$ is a convex curve (i.e., no sign change of the curvature) near the corresponding segment part of γ. This implies the conclusion, and the proof is now complete. $\square$

Appendices

3.A Lipschitz curves and line integrals

This section completes the proofs (or references) of the statements in §3.2. For any $\gamma \in W^{1,\infty}(J;\mathbb{R}^2)$ and Borel function $\rho : \mathbb{R}^2 \to \mathbb{R}$, we define the line integral of ρ along γ as

$$\int_\gamma \rho ds := \int_J \rho(\gamma(t))|\dot{\gamma}(t)|dt.$$

Lemma 3.A.1. *For any $\gamma \in W^{1,\infty}(J;\mathbb{R}^2)$ (not necessarily regular), there is a constant speed reparameterization of γ, i.e., there are a nondecreasing continuous surjective function $\tau : \bar{J} \to \bar{J}$ and a constant speed curve $\hat{\gamma} \in W^{1,\infty}(J;\mathbb{R}^2)$ such that $\gamma = \hat{\gamma} \circ \tau$. Moreover, for any Borel function $\rho : \mathbb{R}^2 \to \mathbb{R}$, the line integrals along γ and $\hat{\gamma}$ coincide:*

$$\int_\gamma \rho ds = \int_{\hat{\gamma}} \rho ds.$$

Proof. See [7, Section 3]. □

Lemma 3.A.2. *Let $\{\gamma_k\}_k \subset W^{1,\infty}(J;\mathbb{R}^2)$ be bounded in $W^{1,\infty}$. Then there are $\gamma \in W^{1,\infty}(J;\mathbb{R}^2)$ and a subsequence $\{\gamma_{k'}\}_{k'}$ such that $\gamma_{k'} \to \gamma$ in L^∞. Moreover,*

$$\|\dot{\gamma}\|_\infty \leq \liminf_{k'\to\infty} \|\dot{\gamma}_{k'}\|_\infty.$$

Proof. Since $\{\gamma_k\}_k \subset W^{1,\infty}(J;\mathbb{R}^2)$ is bounded in $W^{1,\infty}$, the Arzelà-Ascoli theorem implies that there is a subsequence $\{\gamma_{k'}\}_{k'}$ converging to some continuous curve γ in L^∞. Recall that, for any k' and $t_0, t_1 \in \bar{J}$ with $t_0 < t_1$,

$$|\gamma_{k'}(t_1) - \gamma_{k'}(t_0)| = \left|\int_{t_0}^{t_1} \dot{\gamma}_{k'}(t)dt\right| \leq \int_{t_0}^{t_1} |\dot{\gamma}_{k'}(t)|dt \leq \|\dot{\gamma}_{k'}\|_\infty |t_1 - t_0|.$$

By the uniform convergence (in particular, pointwise convergence),

$$|\gamma(t_1) - \gamma(t_0)| = \lim_{k'\to 0} |\gamma_{k'}(t_1) - \gamma_{k'}(t_0)| \leq \liminf_{k'\to\infty} \|\dot{\gamma}_{k'}\|_\infty |t_1 - t_0|.$$

Noting that $\liminf_{k'\to\infty} \|\dot{\gamma}_{k'}\|_\infty < \infty$, we reach the conclusions $\gamma \in W^{1,\infty}(J;\mathbb{R}^2)$ and $\|\dot{\gamma}\|_\infty \leq \liminf_{k'\to\infty} \|\dot{\gamma}_{k'}\|_\infty$. □

Lemma 3.A.3. *Let $\{\gamma_k\}_k \subset W^{1,\infty}(J;\mathbb{R}^2)$ and $\gamma \in L^1(J;\mathbb{R}^2)$. Suppose that $\{\gamma_k\}_k$ is bounded in $W^{1,\infty}$ and $\gamma_k \to \gamma$ in L^1. Then $\gamma_k \to \gamma$ in L^∞ and $\gamma \in W^{1,\infty}(J;\mathbb{R}^2)$ with*

$$\|\dot{\gamma}\|_\infty \leq \liminf_{k\to\infty} \|\dot{\gamma}_k\|_\infty.$$

Moreover, for any lower semicontinuous function $\rho : \mathbb{R}^2 \to [0,\infty]$,

$$\liminf_{k\to\infty} \int_{\gamma_k} \rho ds \geq \int_{\gamma} \rho ds.$$

Proof. Lemma 3.A.2 implies that any subsequence of $\{\gamma_k\}_k$ includes a subsequence converging to some curve $\zeta \in W^{1,\infty}(I;\mathbb{R}^2)$ in L^∞. By the assumption of L^1-convergence and the uniqueness of L^1-limit, the curve ζ is nothing but γ. Hence, $\gamma \in W^{1,\infty}(I;\mathbb{R}^2)$, and γ_k fully converges to γ in L^∞. The estimate in Lemma 3.A.2 is also valid in the full limit sense:

$$\|\dot{\gamma}\|_\infty \leq \liminf_{k\to\infty} \|\dot{\gamma}_k\|_\infty.$$

In the rest of the proof, we confirm the lower semicontinuity of the line integral for any fixed ρ satisfying the assumption.

We first assume that ρ is a bounded m-Lipschitz continuous function such that

$$\rho_* := \inf_{(x,y)\in\mathbb{R}^2} \rho(x,y) > 0.$$

Since $\{\gamma_k\}_k$ is bounded in $W^{1,\infty}$, the sequence $\{\dot{\gamma}_k\}_k$ is bounded in L^2 and hence relatively L^2-weakly compact. Since the L^2 space is separable, $\{\dot{\gamma}_k\}_k$ is relatively sequentially L^2-weakly compact. Therefore, $\dot{\gamma}_k \to \dot{\gamma}$ weakly in L^2 (in the full convergence sense, by the uniqueness of weak derivative); in particular, $\dot{\gamma}_k \to \dot{\gamma}$ weakly in L^1. Recall that any convex and L^1-strongly lower semicontinuous functional $\Phi : L^1(J;\mathbb{R}^2) \to \mathbb{R}$ is also L^1-weakly lower semicontinuous; in particular, Φ satisfies the sequential lower semicontinuity for $\dot{\gamma}_k \to \dot{\gamma}$: $\liminf_{k\to\infty} \Phi[\dot{\gamma}_k] \geq \Phi[\dot{\gamma}]$. Since ρ is bounded and $\rho_* > 0$, the functional $\Phi[\zeta] := \int_J \rho(\gamma(t))|\zeta(t)|dt$ is a norm equivalent to $\|\cdot\|_1$, and hence this functional is convex and L^1-strongly continuous. Thus

$$\liminf_{k\to\infty} \int_J \rho(\gamma(t))|\dot{\gamma}_k(t)|dt \geq \int_J \rho(\gamma(t))|\dot{\gamma}(t)|dt.$$

In addition, by the m-Lipschitz continuity of ρ,

$$\lim_{k\to\infty} \left| \int_J \rho(\gamma_k(t))|\dot{\gamma}_k(t)|dt - \int_J \rho(\gamma(t))|\dot{\gamma}_k(t)|dt \right| \leq \lim_{k\to\infty} m\|\gamma_k - \gamma\|_\infty \|\dot{\gamma}_k\|_\infty = 0.$$

Therefore,

$$\liminf_{k\to\infty} \int_{\gamma_k} \rho ds = \liminf_{k\to\infty} \int_J \rho(\gamma_k(t))|\dot{\gamma}_k(t)|dt \geq \int_J \rho(\gamma(t))|\dot{\gamma}(t)|dt = \int_\gamma \rho ds.$$

We next consider any lower semicontinuous function $\rho : \mathbb{R}^2 \to [0,\infty]$ with $\rho_* := \inf \rho > 0$. Then there is a sequence $\{\rho_m\}_m$ such that $\rho(p) = \sup_m \rho_m(p)$ for $p \in \mathbb{R}^2$ and any ρ_m is a bounded m-Lipschitz function with $\rho_m \geq \rho_*$; for example, we define ρ_m for $m \geq \rho_*$ as the cut-off inf-convolution of ρ, that is,

$$\rho_m(p) = \min\left\{ m, \inf_{q\in\mathbb{R}^2} (\rho(q) + m|p-q|) \right\}.$$

Let $\Phi[\zeta] := \int_\zeta \rho ds$ and $\Phi_m[\zeta] := \int_\zeta \rho_m ds$. Then, by the monotone convergence theorem, we find that $\Phi[\zeta] = \sup_m \Phi_m[\zeta]$ for any $\zeta \in W^{1,\infty}(L;\mathbb{R}^2)$. Recall that for any $m \geq \rho_*$ the functional Φ_m is L^1-lower semicontinuous on a bounded set of $W^{1,\infty}(L;\mathbb{R}^2)$ by the above argument. Hence, Φ is also L^1-lower semicontinuous on a bounded set of $W^{1,\infty}(L;\mathbb{R}^2)$. This implies the conclusion in this case.

We finally take any lower semicontinuous function $\rho : \mathbb{R}^2 \to [0,\infty]$. Let $\delta > 0$ and $\rho_\delta := \rho + \delta$. Then, by the above argument,

$$\liminf_{k\to\infty} \int_{\gamma_k} \rho_\delta ds \geq \int_\gamma \rho_\delta ds \geq \int_\gamma \rho ds.$$

Since there is $M > 0$ such that $\|\dot{\gamma}_k\|_\infty \leq M$,

$$\liminf_{k\to\infty} \int_{\gamma_k} \rho_\delta ds \leq \liminf_{k\to\infty} \left(\int_{\gamma_k} \rho ds + \delta \|\dot{\gamma}_k\|_\infty \right) \leq \liminf_{k\to\infty} \int_{\gamma_k} \rho ds + \delta M.$$

Taking the limit $\delta \to 0$, we obtain the conclusion. The proof is now complete. □

We remark that there is no need to suppose the $W^{1,\infty}$-boundedness for obtaining only the lower semicontinuity; see e.g. [2].

Lemma 3.A.4. *Let $\{\gamma_k\}_k \subset W^{1,\infty}(I;\mathbb{R}^2)$ be a sequence of constant speed curves and $\gamma \in W^{1,\infty}(I;\mathbb{R}^2)$ with $E_0[\gamma] < \infty$. Let $\rho : \mathbb{R}^2 \to [0,\infty]$ be a lower semicontinuous function such that*

$$\rho_* := \inf \rho > 0, \quad \rho^* := \sup \rho < \infty.$$

Suppose that $\gamma_k \to \gamma$ in L^∞ and

$$\lim_{k\to\infty} \int_{\gamma_k} \rho ds = \int_\gamma \rho ds.$$

holds. Then

$$\lim_{k\to\infty} \int_{\gamma_k|_J} \rho ds = \int_{\gamma|_J} \rho ds$$

holds for any subinterval $J \subset I$. Moreover, γ is also constant speed, and the speed of γ_k converges to the speed of γ as $k \to \infty$.

Proof. Notice that $\{\gamma_k\}_k$ is bounded in $W^{1,\infty}$ since $\int_{\gamma_k} \rho ds$ is bounded as $k \to \infty$ and

$$\int_{\gamma_k} \rho ds \geq L_k \rho_* = \rho_* \|\dot{\gamma}_k\|_\infty.$$

Fix $J \subset I$. Denote the interior set of the complement of J by J', which is the union of at most two open intervals. Then, by Lemma 3.A.3, the line integral is lower semicontinuous for $\gamma_k|_J \to \gamma|_J$ and $\gamma_k|_{J'} \to \gamma|_{J'}$. Moreover, the Lebesgue measure of the boundary of J' is zero. Then we have

$$\begin{aligned}\int_{\gamma|_J}\rho ds &\leq \liminf_{k\to\infty}\int_{\gamma_k|_J}\rho ds \leq \limsup_{k\to\infty}\int_{\gamma_k|_J}\rho ds\\ &= \limsup_{k\to\infty}\left(\int_{\gamma_k}\rho ds - \int_{\gamma_k|_{J'}}\rho ds\right) = \int_{\gamma}\rho ds - \liminf_{k\to\infty}\int_{\gamma_k|_{J'}}\rho ds\\ &\leq \int_{\gamma}\rho ds - \int_{\gamma|_{J'}}\rho ds = \int_{\gamma|_J}\rho ds,\end{aligned}$$

and thus the first assertion holds.

Let L_k be the speed of γ_k. Let $L_* := \liminf_{k\to\infty} L_k$ and $L^* := \limsup_{k\to\infty} L_k$. The case $L^* = (L_* =)\ 0$ is trivial (since γ_k degenerates to a single point and the energy converges to zero); thus, we may assume that $L^* > 0$. Then we notice that $L^* < \infty$ since $\rho_* > 0$ and

$$L^*\rho_* = \limsup_{k\to\infty} L_k\rho_* \leq \lim_{k\to\infty}\int_{\gamma_k}\rho ds = \int_{\gamma}\rho ds < \infty.$$

Let $J = (0,t)$ for any $t \in I$. Since $0 < L^* < \infty$, there is a subsequence $\{\gamma_{k'}\}_{k'} \subset \{\gamma_k\}_k$ such that $L^* = \lim_{k'\to\infty} L_{k'}$. Then we have

$$\begin{aligned}\int_{\gamma|_J}\rho ds &= \lim_{k\to 0}\int_{\gamma_k|_J}\rho ds = \lim_{k'\to\infty} L_{k'}\int_0^t \rho(\gamma_{k'}(r))dr\\ &\geq L^*\liminf_{k'\to\infty}\int_0^t \rho(\gamma_{k'}(r))dr.\end{aligned} \tag{3.A.1}$$

By Fatou's lemma and $L_* \leq L^*$,

$$L^*\liminf_{k'\to\infty}\int_0^t \rho(\gamma_{k'}(r))dr \geq L^*\int_0^t \rho(\gamma(r))dr \geq L_*\int_0^t \rho(\gamma(r))dr. \tag{3.A.2}$$

Then, by Lemma 3.A.2, we have $\|\dot\gamma\|_\infty \leq L_*$, and hence

$$L_*\int_0^t \rho(\gamma(r))dr \geq \int_0^t \rho(\gamma(r))|\dot\gamma(r)|dr = \int_{\gamma|_J}\rho ds. \tag{3.A.3}$$

Therefore, all the inequalities in (3.A.1), (3.A.2), and (3.A.3) hold as the equalities. In particular,

$$L^*\liminf_{k'\to\infty}\int_0^t \rho(\gamma_{k'}(r))dr = L_*\int_0^t \rho(\gamma(r))dr \geq \int_0^t \rho(\gamma(r))|\dot\gamma(r)|dr.$$

Then we find that $L^* = L_*$ since ρ is positive; in particular, the speed L_ε converges to L_*. Moreover, the Lebesgue differentiation theorem implies that $L_*\rho(\gamma(r)) = \rho(\gamma(r))|\dot\gamma(r)|$ holds for a.e. $r \in I$. Since ρ is positive, $L_* = |\dot\gamma(r)|$ holds for a.e. $r \in I$. This implies that γ is constant speed, and moreover the speed of γ coincides with the limit of the speed of γ_k. The proof is complete. □

3.B Existence of minimizers of E_0

We prove Theorem 3.2.8, i.e., the existence theorem of minimizers of E_0.

Proof of Theorem 3.2.8. We first note obvious facts; the set $\{E_0[\cdot] < \infty\} \subset X^{1,\infty}$ is nonempty, and $\inf_{X^{1,\infty}} E_0 > 0$. Hence we can take a minimizing sequence $\{\gamma_k\}_k \subset X^{1,\infty}$, i.e.,

$$\lim_{k\to\infty} E_0[\gamma_k] = m := \inf_{X^{1,\infty}} E_0 \in (0,\infty).$$

To apply Lemma 3.2.1, we confirm the $W^{1,\infty}$-boundedness of $\{\gamma_k\}_k$.

Let L_k be the speed of γ_k. Notice that

$$E_0[\gamma_k] \geq \underline{\alpha} L_k = \underline{\alpha}\|\dot{\gamma}_k\|_\infty.$$

Since $\underline{\alpha} > 0$ and $E_0[\gamma_k]$ is bounded, the sequence of $\|\dot{\gamma}_k\|_\infty$ is bounded.

We next confirm that the sequence of $\|\gamma_k\|_\infty$ is also bounded. Denote the x- and y-component of γ_k by x_k and y_k, respectively. Then, the boundary condition $x_k(0) = 0$ implies that

$$\|x_k\|_\infty = \max_{t\in\bar{I}} |x_k(t) - x_k(0)| \leq L_k \max_{t\in\bar{I}} |t| = L_k.$$

Since L_k is bounded, the x-components are bounded. The remaining part is to confirm the uniform boundedness for y_k. Without loss of generality, we may assume that any γ_k touches $\partial\Omega$. In fact, if not, then by shifting γ_k downward we obtain a new curve γ_k' touching $\partial\Omega$ such that $E_0[\gamma_k'] \leq E_0[\gamma_k]$; we notice that this procedure does not change the x-component and speed. Hence, there is $t_k \in \bar{I}$ such that $\gamma(t_k) \in \partial\Omega$. Thus we have

$$\|y_k\|_\infty \leq \max_{t\in\bar{I}} |y_k(t) - y_k(t_k)| + |y_k(t_k)| \leq L_k \max_{t\in\bar{I}} |t - t_k| + \sup_{\partial\Omega} |y| \leq L_k + \psi(0).$$

Since L_k is bounded, we find that the y-components are also bounded.

Therefore, by Lemma 3.2.1, there is a subsequence $\{\gamma_{k'}\}_{k'}$ converging to some $\gamma \in W^{1,\infty}(I;\mathbb{R}^2)$. We confirm that γ is nothing but a minimizer. We first notice that the L^∞-convergence $\gamma_{k'} \to \gamma$ immediately implies that the image of γ is included in $\overline{\Omega}$, and γ satisfies the same periodic boundary condition as γ_k. Moreover, Lemma 3.2.6 implies that

$$m = \lim_{k'\to\infty} E_0[\gamma_{k'}] \geq E_0[\gamma],$$

and hence $m = E_0[\gamma]$. Then, by Lemma 3.2.7, γ is also constant speed. The above facts imply that γ is admissible. Since $m = E_0[\gamma]$, the curve γ is nothing but a minimizer. $\square$

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Chapter 4

A characterization of cut locus for C^1 hypersurfaces

Let Ω be an open set in $\mathbb{R}^n$ with C^1-boundary and Σ be the skeleton of Ω, which consists of points where the distance function to $\partial\Omega$ is not differentiable. In this chapter we characterize the cut locus (ridge) $\overline{\Sigma}$, which is the closure of the skeleton, by introducing a generalized radius of curvature and its lower semicontinuous envelope. As an application we give a sufficient condition for vanishing of the Lebesgue measure of $\overline{\Sigma}$.

Keywords: Distance function; Eikonal equation; Singularity; Ridge; Cut locus; Radius of curvature.

4.1 Introduction

Let $n \geq 2$ and $\Omega \subset \mathbb{R}^n$ be an open set with nonempty boundary $\partial\Omega$. The *distance function* $d : \Omega \to (0, \infty)$ and the *metric projection* $\pi : \Omega \to \mathcal{P}(\partial\Omega)$ are defined by

$$d(x) := \inf_{\xi \in \partial\Omega} |\xi - x|, \quad \pi(x) := \{\xi \in \partial\Omega \mid d(x) = |\xi - x|\}.$$

The relation between the differentiability of the distance function and the number of elements of the metric projection is well-known:

Theorem 4.1.1 (e.g. [4, Corollary 3.4.5])**.** *The distance function d is differentiable at $x \in \Omega$ if and only if $\pi(x)$ is a singleton.*

Thus we define the singular set $\Sigma \subset \Omega$ called *skeleton* (or *medial axis*) by

$$\Sigma := \{x \in \Omega \mid \pi(x) \text{ is not a singleton}\}.$$

Besides the above characterization, several properties of the skeletons are known for general open sets Ω: the skeletons are C^2-rectifiable [2], in particular they have null Lebesgue measure, and they have the same homotopy type as Ω at least in the bounded case [13] (cf. [1]). Skeletons (medial axes) are also studied in view of image processing, see e.g. [3] and references therein. The generality of Ω is important to contain noisy cases.

In this chapter we study the set $\overline{\Sigma}$, which is the closure of the skeleton in Ω, called *cut locus* (or *ridge*). The complement of the cut locus concerns the differentiability of the distance function not only at points but also in neighborhoods of points.

Cut loci have also been studied in several (generalized) settings, see e.g. [7, 10, 12, 15]. The complement set $\Omega \setminus \overline{\Sigma}$ is the largest open set where the distance function is of class C^1. For general Ω the distance function is a unique nonnegative viscosity solution to the simplest Eikonal equation $|\nabla u| = 1$ with the zero Dirichlet condition (see e.g. [9, 14]). Since this is a first order equation, it is natural to know where the solution is of class C^1. We mention that cut loci also appear in the studies of other partial differential equations, see e.g. [5, 6, 11].

Unlike skeletons, some properties of cut loci crucially depend on the regularity of boundary. In particular, it is critical whether the boundary is of class C^2. If the boundary is at least C^2, then the cut locus behaves rather well [7, 10, 12, 15]; in particular, it still has null Lebesgue measure [7]. On the other hand, in [15, Section 3], Mantegazza-Mennucci give a pathological example of a planar convex $C^{1,1}$-domain such that the cut locus has positive Lebesgue measure.

Our purpose is to find a general theory for cut loci without the C^2 assumption, that is, with pathological cases. We emphasize that the theories in the above cited papers basically work only for regular sets at least C^2. In this paper, as a first step, we characterize the cut locus by a geometric quantity of the boundary for a general open set with C^1-boundary.

We first recall a characterization of cut loci by radius of curvature in the C^2 case:

Theorem 4.1.2 (e.g. [7]). *Let $\partial\Omega$ be of class C^2. Then $x \in \Omega \setminus \overline{\Sigma}$ if and only if $\pi(x)$ is a singleton and $d(x) < \rho(\xi)$, where $\pi(x) = \{\xi\}$.*

Here $\rho(\xi)$ is the classical inner radius of curvature of $\partial\Omega$ at ξ (see Remark 4.2.2 for the definition). Theorem 4.1.2 is well-known: the cited paper [7] proves it in terms of principal curvature κ instead of ρ (for the Minkowski distance). Theorem 4.1.2 means that the differentiability of d near a point depends on not only the global shape of $\partial\Omega$ but also the local shape as curvature. That curvature appears in the statement also tells the importance of the C^2 assumption.

The main result of this chapter is to characterize cut loci in the C^1 case by a kind of radius of curvature. To state it we should extend the definition of radius of curvature ρ. This can be easily achieved for general open sets by using locally inner touching spheres (see Definition 4.2.1). Unfortunately, by just this extension, the "if" part of the above characterization is not valid even in the $C^{1,1}$ case due to the loss of rigidity of C^2. Indeed, the example provided in [15, Section 3] is also a counterexample to this case. Nevertheless, we can characterize cut loci by taking a lower semicontinuous envelope of radius of curvature ρ_* as Definition 4.2.1. Our main result is the following:

Theorem 4.1.3. *Let $\partial\Omega$ be of class C^1. Then $x \in \Omega \setminus \overline{\Sigma}$ if and only if $\pi(x)$ is a singleton and $d(x) < \rho_*(\xi)$, where $\pi(x) = \{\xi\}$.*

Since ρ_* coincides with classical ρ in the C^2 case, Theorem 4.1.3 is a generalization of Theorem 4.1.2.

As an application of this characterization, in Section 4, we show that if $\partial\Omega$ is $C^{1,1}$ and almost C^2, then the Lebesgue measure of the cut locus vanishes.

We finally emphasize that our proof of Theorem 4.1.3 quite differs from that of C^2 case; it does not calculate second derivatives and is more geometric. The "if" part is proved by considering its contrapositive. The inequality $d(x) \geq \rho_*(\xi)$ is obtained for $x \in \overline{\Sigma} \setminus \Sigma$ by seeking a sequence of suitably "curving" points in $\partial\Omega$ converging to ξ. To state "curving" we use comparisons of functions. Our proof of this part depends on the local C^1-graph representation of $\partial\Omega$. The "only if" part is proved by showing that, for $x \in \Omega \setminus \overline{\Sigma}$, any point near ξ has a locally inner touching sphere with suitably large radius. To this end we use the homotopy theory as mapping degree. This part is proved for a wider class of Ω (see Definition 4.3.5). Another more conceptual and geometric proof is also given.

The organization of this chapter is as follows. Some notations and known results are prepared in Section 2. Theorem 4.1.3 is proved in Section 3. The proof is separated into the "if" part (Section 3.1) and the "only if" part (Section 3.2). As a corollary of Theorem 4.1.3, a sufficient condition for vanishing of the Lebesgue measures of cut loci is given in Section 4. Finally, we mention remarks for non-smooth cases in Section 5.

4.2 Envelope of inner radius of curvature

In this section, we prepare notation and review a known result.

We first prepare some notations. Let $B_r^m(x)$ denote an m-dimensional open ball of radius r centered at $x \in \mathbb{R}^m$ and $\overline{B}_r^m(x)$ denote the closure. Let $S_r^{m-1}(x)$ denote an $(m-1)$-dimensional sphere of radius r centered at $x \in \mathbb{R}^m$, that is, $S_r^{m-1}(x) = \partial B_r^m(x)$. Let $\mathcal{U}_x^m$ denote the set of open neighborhoods of x in $\mathbb{R}^m$.

Then we define a generalized inner radius of curvature as follows.

Definition 4.2.1. Let $\Omega \subset \mathbb{R}^n$ be an open set with nonempty boundary and $\xi \in \partial\Omega$. We say that an open ball $B_r^n(x) \subset \mathbb{R}^n$ is a *locally inner touching ball* at ξ if $\xi \in \partial B_r^n(x)$ and there exists a neighborhood $U \in \mathcal{U}_\xi^n$ such that $B_r^n(x) \cap U$ is contained in $\Omega \cap U$. We denote the set of locally inner touching balls at ξ by $\mathfrak{B}_\xi^n$. Then we define the *inner radius of curvature* at ξ by

$$\rho(\xi) := \begin{cases} \sup \left\{ r > 0 \,\middle|\, B_r^n(x) \in \mathfrak{B}_\xi^n \right\} & \text{if } \mathfrak{B}_\xi^n \neq \emptyset, \\ 0 & \text{if } \mathfrak{B}_\xi^n = \emptyset. \end{cases}$$

Note that $\rho : \partial\Omega \to [0, \infty]$. Moreover, we denote by ρ_* the lower semicontinuous envelope of ρ, that is, for $\xi \in \partial\Omega$

$$\rho_*(\xi) := \lim_{r \downarrow 0} \inf\{\rho(\eta) \mid \eta \in B_r^n(\xi) \cap \partial\Omega\}.$$

The function $\rho_* : \partial\Omega \to [0, \infty]$ is lower semicontinuous.

Remark 4.2.2. If $\partial\Omega$ is of class C^2, then both ρ and ρ_* coincide with the classical inner radius of curvature $1/\kappa$, where

$$\kappa := \max\{0, \kappa_1, \ldots, \kappa_{n-1}\}.$$

Here $\kappa_1, \ldots, \kappa_{n-1}$ are the inner principal curvatures of $\partial\Omega$, and we interpret $1/\kappa = \infty$ when $\kappa = 0$.

Finally we review a well-known result about the continuity of the metric projection, which is frequently used in this chapter. The proof is elementary so is safely omitted.

Lemma 4.2.3. *For any open set $\Omega \subset \mathbb{R}^n$ with nonempty boundary, the map $\pi : \Omega \to \mathcal{P}(\partial\Omega)$ is set-valued upper semicontinuous. In particular, the induced map $\hat{\pi} : \Omega \setminus \Sigma \to \partial\Omega$ defined by $x \mapsto \xi \in \pi(x)$ is well-defined and continuous.*

4.3 Characterization of the cut locus by radius of curvature

In this section we prove our main theorem (Theorem 4.1.3). Let e_n denote the n-th unit vector of $\mathbb{R}^n$ and $\langle \cdot, \cdot \rangle$ denote the Euclidean inner product. Let $\widetilde{\mathrm{pr}} : \mathbb{R}^n \to \mathbb{R}^{n-1}$ be a map induced by the orthogonal projection to the hyperplane $\{\langle y, e_n \rangle = 0\}$, that is,

$$\widetilde{\mathrm{pr}}(y_1, \ldots, y_n) := (y_1, \ldots, y_{n-1}).$$

Throughout this section we fix these notations.

4.3.1 Upper bound for radius of curvature

In this subsection we prove the "if" part of Theorem 4.1.3 by proving its contrapositive. By the definition of Σ, it suffices to prove the following:

Proposition 4.3.1. *Let $x \in \overline{\Sigma} \setminus \Sigma$, $\pi(x) = \{\xi\}$ and $\partial\Omega$ be of class C^1 near ξ. Then the inequality $d(x) \geq \rho_*(\xi)$ holds.*

To prove this proposition we prepare the following two lemmas about locally touching spheres and circles for the subgraphs of functions.

Here we say that a continuous function f_1 on $A_1 \in \mathcal{U}_y^m$ touches a function f_2 on $A_2 \in \mathcal{U}_y^m$ from below (resp. above) at $y \in \mathbb{R}^m$ if the function $f_2 - f_1$ defined on $A_1 \cap A_2$ attains its local minimum (resp. maximum) at y, and $f_1(y) = f_2(y)$ holds. Moreover, we say that a function defined on an open set $A \subset \mathbb{R}^m$ is an upper semi-sphere function if there are $c_0 > 0$, $r_0 > 0$ and $x_0 \in \mathbb{R}^m$ such that

$$f(x) = \sqrt{r_0^2 - |x - x_0|^2} + c_0$$

holds for all $x \in A$. When $m = 1$, it is called an upper semi-circle function.

Lemma 4.3.2. *Let $h : [a, b] \to \mathbb{R}$ be a continuous function and $\tilde{h} : [a, b] \to \mathbb{R}$ be a part of an upper semi-circle function with radius $\tilde{r} > (b - a)/2$ such that $h \geq \tilde{h}$ in $[a, b]$ and $h = \tilde{h}$ on $\partial[a, b]$. Then there exists $c \in (a, b)$ such that any upper semi-circle function with radius larger than $\tilde{r}$ can not touch h from below at c.*

Proof. The case $h \equiv \tilde{h}$ is obvious; thus we assume $h \not\equiv \tilde{h}$. Since $h \geq \tilde{h}$, and $h - \tilde{h}$ is continuous on $[a, b]$, we can take a constant $\alpha := \max(h - \tilde{h}) > 0$. Then $\tilde{h} + \alpha \geq h$ holds in $[a, b]$ and, by the boundary condition, there exists $c \in (a, b)$ such that $\tilde{h}(c) + \alpha = h(c)$. In particular, the upper semi-circle function $\tilde{h} + \alpha$ with radius $\tilde{r}$ touches h from above at c (see Figure 4.1). This implies that any upper semi-circle function with radius larger than $\tilde{r}$ can not touch h from below at c. □

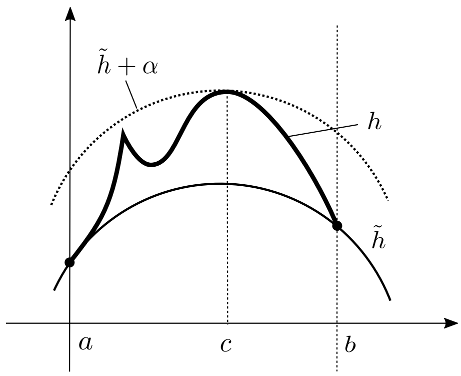

FIGURE 4.1: Continuous function on upper semi-circle.

Remark 4.3.3. In the above lemma, the fact is that the subgraph $\{y < h(x)\}$ does not admit a locally inner touching ball (disk) with radius larger than $\tilde{r}$ at $(c, h(c))$. We should be careful that, in general, the existence of a locally inner touching ball for the subgraph of a function f does not imply the existence of an upper semi-sphere function with the same radius touching f from below (the converse is generally valid). A counterexample is given by $f(x) = \mathrm{sign}(x)\sqrt{1-(|x|-1)^2}$. In this case any upper semi-circle function does not touch f from below at 0 since $\lim_{x\to\pm 0} f'(x) = \infty$ but the subgraph admits a locally inner touching ball with radius 1 at the origin. However, in particular, if f is differentiable at a point, then the two conditions are equivalent there. (The fact is that it is sufficient that f is touched by a cone function from above.)

Lemma 4.3.4. *Let $m \geq 1$, $x_0 \in \mathbb{R}^m$, $U \in \mathcal{U}^m_{x_0}$ and $f : U \to \mathbb{R}$ be a continuous function which is differentiable at x_0. Suppose that there exists an upper semi-sphere function with radius $\tilde{r}$ touching f from below at x_0. Then, for any open segment $I \subset U$ including x_0, there exists an upper semi-circle function with radius not smaller than $\frac{\tilde{r}}{\sqrt{1+|\nabla f(x_0)|^2}}$ touching $f|_I$ from below at x_0.*

Proof. Without loss of generality, we may assume that $|x_0| < \tilde{r}$, $I \subset B^m_{\tilde{r}}(0)$ and the upper semi-sphere in the assumption is represented by

$$\tilde{f}(x) = \sqrt{\tilde{r}^2 - |x|^2}.$$

Since $\tilde{f}$ touches f from below at x_0, we have $f \geq \tilde{f}$ near x_0 and

$$\nabla f(x_0) = \nabla \tilde{f}(x_0) = \frac{-x_0}{\sqrt{\tilde{r}^2 - |x_0|^2}}.$$

Now let $L \subset \mathbb{R}^n$ be the line through I, and $x_* \in L$ be a unique point so that $|x_*| = \min_{x\in L}|x|$. Since $x_0 \in I \subset L$, we notice that $|x_*| \leq |x_0|$. Moreover, for any $x \in I$, noting that $x - x_*$ is perpendicular to x_*, we have

$$\tilde{f}|_I(x) = \sqrt{\tilde{r}^2 - |x_*|^2 - |x-x_*|^2},$$

which implies that $\tilde{f}|_I$ is an upper semi-circle function with radius $\sqrt{\tilde{r}^2 - |x_*|^2}$. Therefore, noting that $\tilde{f}|_I$ touches $f|_I$ from below at $x_0 \in I$ and

$$\sqrt{\tilde{r}^2 - |x_*|^2} \geq \sqrt{\tilde{r}^2 - |x_0|^2} = \frac{\tilde{r}}{\sqrt{1 + |\nabla f(x_0)|^2}},$$

we obtain the conclusion. □

Now we prove Proposition 4.3.1. We may refer Figure 4.2 in the proof. Without loss of generality, we may assume $x = 0$ and $\xi = de_n$, where $d = d(x) > 0$. Then we can represent $\partial\Omega$ by a C^1-graph near ξ in the direction of e_n: there exist $0 < \varepsilon_0 < d$, a neighborhood $U_\xi \in \mathcal{U}_\xi^n$ and a C^1-function g on $B_{\varepsilon_0}^{n-1}(0)$ such that $g(0) = d$, $\nabla g(0) = 0$ and $\tilde{g}(B_{\varepsilon_0}^{n-1}(0)) = \partial\Omega \cap U_\xi$, where $\tilde{g}(\cdot) := (\cdot, g(\cdot))$.

Proof of Proposition 4.3.1. Since $x = 0 \in \overline{\Sigma}$, there exists a sequence $\{x_k\} \subset \Sigma$ such that $x_k \to 0$. Then for any k the set $\pi(x_k) \subset \partial\Omega$ has at least two elements. We denote them by ξ_k^1 and ξ_k^2. Since π is set-valued upper semicontinuous by Lemma 4.2.3, both ξ_k^1 and ξ_k^2 converge to ξ as $k \to \infty$. Thus we may assume that for any k

$$\xi_k^1, \xi_k^2 \in \tilde{g}(B_{\varepsilon_0}^{n-1}(0)) = \partial\Omega \cap U_\xi$$

and the n-th components of $\xi_k^1 - x_k$ and $\xi_k^2 - x_k$ are positive, that is, the points ξ_k^1 and ξ_k^2 lie in the upper semi-sphere part of $S_{d(x_k)}^{n-1}(x_k)$. Define $\xi_k^{\prime i} := \widetilde{\mathrm{pr}}(\xi_k^i) \in B_{\varepsilon_0}^{n-1}(0)$ for $i = 1, 2$. Noting that $\xi_k^{\prime 1} \neq \xi_k^{\prime 2}$, we can define I_k as the closed segment joining $\xi_k^{\prime 1}$ to $\xi_k^{\prime 2}$.

Since the radius of any circle obtained as a section of $S_{d(x_k)}^{n-1}(x_k)$ is at most $d(x_k)$, the function $g|_{I_k}$ satisfies the assumption of Lemma 4.3.2 for an upper semi-circle function with radius not larger than $d(x_k)$. Thus, by Lemma 4.3.2, there exists $\xi_k^{\prime 3} \in I_k \setminus \{\xi_k^{\prime 1}, \xi_k^{\prime 2}\}$ such that any upper semi-circle function with radius larger than $d(x_k)$ can not touch $g|_{I_k}$ from below at $\xi_k^{\prime 3}$.

Therefore, by the contrapositive of Lemma 4.3.4, any upper semi-sphere function with radius larger than $d(x_k)\sqrt{1 + |\nabla g(\xi_k^{\prime 3})|^2}$ can not touch g from below at $\xi_k^{\prime 3}$. Noting Remark 4.3.3 and that g is differentiable, we find that the above fact yields the nonexistence of locally inner touching balls with radius larger than $d(x_k)\sqrt{1 + |\nabla g(\xi_k^{\prime 3})|^2}$ at $\xi_k^3 := \tilde{g}(\xi_k^{\prime 3}) \in \partial\Omega$; thus we have

$$\rho(\xi_k^3) \leq d(x_k)\sqrt{1 + |\nabla g(\xi_k^{\prime 3})|^2}.$$

Since $d(x_k) \to d(x)$, $\xi_k^{\prime 3} \to 0$ and $\xi_k^3 \to \xi$ as $k \to \infty$ and g is of class C^1, we obtain

$$\rho_*(\xi) \leq \liminf_{k \to \infty} \rho(\xi_k^3) \leq d(x)\sqrt{1 + |\nabla g(0)|^2} = d(x),$$

which is the desired inequality. □

4.3.2 Lower bound for radius of curvature

In this subsection we prove the "only if" part of Theorem 4.1.3 for a class of $\Omega \subset \mathbb{R}^n$. The class contains all open sets with C^1-boundary. We first define this class.

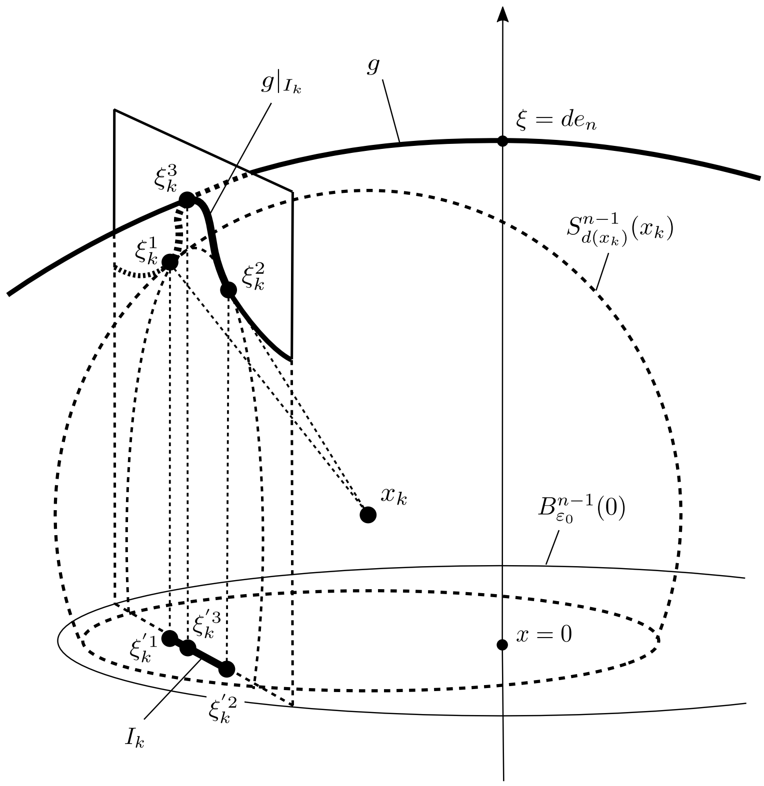

FIGURE 4.2: Panorama of the sequences: proof of Proposition 4.3.1.

Definition 4.3.5. Let $\Omega \subset \mathbb{R}^n$ be an open set with nonempty boundary. We say that a point $\eta \in \partial\Omega$ has *non-spreading inner perpendicular* if either, $\pi^{-1}(\{\eta\})$ is empty, or nonempty $\pi^{-1}(\{\eta\})$ is included in a (unique) line $L \subset \mathbb{R}^n$. In addition, if $\partial\Omega$ is locally represented by a graph near η in the direction of L, then we say that the point $\eta \in \partial\Omega$ has *non-spreading inner perpendicular with graph representation*.

Remark 4.3.6. Note that if $\eta \in \partial\Omega$ has non-spreading inner perpendicular and there exists $y \in \Omega$ such that $\eta \in \pi(y)$, then, even when $y \notin \pi^{-1}(\{\eta\})$, the pull-back $\pi^{-1}(\{\eta\})$ is not empty and the line L passes through y and η. That is because for any $t \in (0,1)$ the point $ty + (1-t)\eta$ belongs to $\pi^{-1}(\{\eta\})$.

Remark 4.3.7. For any open set with C^1-boundary, all points in its boundary have non-spreading inner perpendicular with graph representation (in this case L is in the normal direction of $\partial\Omega$). In addition, the same property is valid for some more non-smooth sets, for example general convex open sets.

The "only if" part follows by proving the following proposition.

Proposition 4.3.8. *Let $x \in \Omega \setminus \overline{\Sigma}$ and $\pi(x) = \{\xi\}$. Suppose that $\xi \in \partial\Omega$ has non-spreading inner perpendicular with graph representation. Then there exist $\delta > 0$ and $r_\delta > 0$ such that for any $\eta \in B^n_{r_\delta}(\xi) \cap \partial\Omega$ the inequality $d(x) + \delta \leq \rho(\eta)$ holds.*

To prove this, we prepare a key lemma (Lemma 4.3.10) about the image of the metric projection $\hat{\pi}$. To this end we check a basic property of continuous maps.

Lemma 4.3.9. *Let $m \geq 1$, $D \subset \mathbb{R}^m$ be a bounded region containing the origin, and $f : \overline{D} \to \mathbb{R}^m$ be a continuous map such that*

$$\langle f(y), y\rangle + |f(y)||y| > 0 \quad \textit{for any} \quad y \in \partial D. \tag{4.3.1}$$

Then there exists $\bar{r} > 0$ such that $B^m_{\bar{r}}(0) \subset f(D)$.

Proof. Define the homotopy between the identity map id and the function f by

$$H(y,t) := tf(y) + (1-t)y.$$

Then, by (4.3.1), it is easily confirmed that $|H(y,t)|^2 > 0$ for any (y,t) in $\partial D \times [0,1]$. Since $\partial D \times [0,1]$ is compact and H is continuous, there exists $\bar{r} > 0$ such that $|H| > \bar{r}$ on $\partial D \times [0,1]$. Then for any $z \in B^m_{\bar{r}}(0)$ we have $z \notin H(\partial D, [0,1])$ and hence

$$\deg(f, D, z) = \deg(\mathrm{id}, D, z) = 1$$

by the homotopy invariance of mapping degree (see e.g. [16, Section IV.2]). This implies that there exists $x \in D$ such that $f(x) = z$. Thus $B^m_{\bar{r}}(0) \subset f(D)$. ☐

Now let us go back in our situation (Proposition 4.3.8). We may assume again $x = 0$ and $\xi = de_n$, where $d = d(x) > 0$. Under the assumption of Proposition 4.3.8, $\partial\Omega$ has a local graph representation near $\xi = de_n$ in the direction of e_n: there exist $\varepsilon_0 > 0$, a neighborhood $U_\xi \in \mathcal{U}^n_\xi$ and a function g on $B^{n-1}_{\varepsilon_0}(0)$ such that $g(0) = d$ and $\tilde{g}(B^{n-1}_{\varepsilon_0}(0)) = \partial\Omega \cap U_\xi$, where $\tilde{g}(\cdot) := (\cdot, g(\cdot))$.

Since $\Omega \setminus \overline{\Sigma}$ is open, we may also assume $B^n_{\varepsilon_0}(0) \subset \Omega \setminus \overline{\Sigma}$. Moreover, we define a lower semi-sphere in $B^n_{\varepsilon_0}(0)$ by

$$S^{n-1}_{\varepsilon_1,-} := \{y \in B^n_{\varepsilon_0}(0) \mid \langle y, e_n\rangle \leq 0,\ |y| = \varepsilon_1\} \tag{4.3.2}$$

so that $\hat{\pi}(S^{n-1}_{\varepsilon_1,-}) \subset \partial\Omega \cap U_\xi$. Such $0 < \varepsilon_1 < \varepsilon_0$ can be taken since $\hat{\pi}$ is continuous by Lemma 4.2.3.

Then the following key lemma holds (see also Figure 4.3).

Lemma 4.3.10. *Suppose the assumption of Proposition 4.3.8 and let $S^{n-1}_{\varepsilon_1,-}$ be the semi-sphere defined as (4.3.2). Then there exists $\bar{r} > 0$ such that*

$$B^n_{\bar{r}}(\xi) \cap \partial\Omega \subset \hat{\pi}(S^{n-1}_{\varepsilon_1,-}).$$

Proof. Define a homeomorphism $p_1 : S^{n-1}_{\varepsilon_1,-} \to \overline{B}^{n-1}_{\varepsilon_1}(0)$ by the restriction of $\widetilde{\mathrm{pr}}$, namely $p_1 := \widetilde{\mathrm{pr}}|_{S^{n-1}_{\varepsilon_1,-}}$. If the map

$$\widetilde{\mathrm{pr}} \circ \hat{\pi} \circ p_1^{-1} : \overline{B}^{n-1}_{\varepsilon_1}(0) \to \mathbb{R}^{n-1}$$

satisfies the assumption of Lemma 4.3.9, then there exists small $0 < \bar{r} < \varepsilon_0$ such that $B^n_{\bar{r}}(\xi) \subset U_\xi$ and

$$B^{n-1}_{\bar{r}}(0) \subset \widetilde{\mathrm{pr}} \circ \hat{\pi} \circ p_1^{-1}(\overline{B}^{n-1}_{\varepsilon_1}(0)) = \widetilde{\mathrm{pr}}\left(\hat{\pi}(S^{n-1}_{\varepsilon_1,-})\right).$$

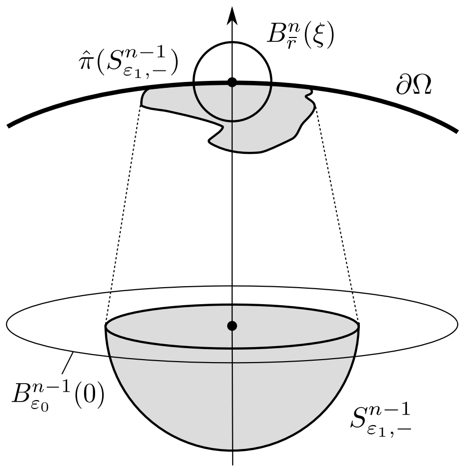

FIGURE 4.3: Image of the metric projection $\hat{\pi}(S^{n-1}_{\varepsilon_1,-})$.

Using these inclusions and noting that $\hat{\pi}(S^{n-1}_{\varepsilon_1,-}) \subset \partial\Omega \cap U_\xi$, we obtain the conclusion

$$B^n_{\bar{r}}(\xi) \cap \partial\Omega \subset \tilde{g}(B^{n-1}_{\bar{r}}(0)) \subset \tilde{g}\left(\widetilde{\mathrm{pr}}\left(\hat{\pi}(S^{n-1}_{\varepsilon_1,-})\right)\right) = \hat{\pi}(S^{n-1}_{\varepsilon_1,-}).$$

Therefore, it suffices to confirm that the map $\widetilde{\mathrm{pr}} \circ \hat{\pi} \circ p_1^{-1}$ satisfies the assumption of Lemma 4.3.9, where $\overline{D} = \overline{B}^{n-1}_{\varepsilon_1}(0)$. This map is obviously continuous by Lemma 4.2.3. The condition (4.3.1) is proved as follows. Fix any $y' \in \partial\overline{B}^{n-1}_{\varepsilon_1}(0)$ and denote

$$y := p_1^{-1}(y') = (y', 0) \in \mathbb{R}^n.$$

Clearly, $d(y) \le |\xi - y|$ holds. Moreover, we see that $d(y) < |\xi - y|$ since $\partial\Omega$ has non-spreading inner perpendicular. Indeed, the equality $d(y) = |\xi - y|$ implying $\hat{\pi}(y) = \xi$ can not be attained since, if so, then $\{0, y\} \subset \pi^{-1}(\{\xi\})$, but the three points 0, ξ, $y \in \mathbb{R}^n$ are not in alignment. Thus we find that $\hat{\pi}(y) \in B^n_{|\xi-y|}(y) \cap \partial\Omega$. In addition, we find that $\hat{\pi}(y) \notin \overline{B}^n_d(0)$ since $\overline{B}^n_d(0) \cap \partial\Omega = \{\xi\}$ and $\xi \notin B^n_{|\xi-y|}(y)$. Noting that any $z \in B^n_{|\xi-y|}(y) \setminus \overline{B}^n_d(0)$ satisfies $\langle \widetilde{\mathrm{pr}}(z), \widetilde{\mathrm{pr}}(y)\rangle > 0$ by the shape of $B^n_{|\xi-y|}(y) \setminus \overline{B}^n_d(0)$, we have

$$\langle \widetilde{\mathrm{pr}}\left(\hat{\pi}(y)\right), \widetilde{\mathrm{pr}}(y)\rangle = \langle \widetilde{\mathrm{pr}} \circ \hat{\pi} \circ p_1^{-1}(y'), y'\rangle > 0.$$

This immediately implies the condition (4.3.1). □

We are now in a position to prove the main proposition.

Proof of Proposition 4.3.8. By Lemma 4.3.10, there exists $\bar{r} > 0$ such that

$$B^n_{\bar{r}}(\xi) \cap \partial\Omega \subset \hat{\pi}(S^{n-1}_{\varepsilon_1,-}).$$

Therefore, for any $\eta \in B^n_{\bar{r}}(\xi) \cap \partial\Omega$ there exists $y_\eta \in S^{n-1}_{\varepsilon_1,-}$ such that $\hat{\pi}(y_\eta) = \eta$, which implies that $B^n_{|\eta-y_\eta|}(y_\eta)$ is an inner touching ball at $\eta \in \partial\Omega$. This yields that $\rho(\eta) \ge |\eta - y_\eta|$. In addition, by the definition of $S^{n-1}_{\varepsilon_1,-}$, there exist $\delta > 0$ and $0 < r_\delta < \bar{r}$ such that the distance between $B^n_{r_\delta}(\xi)$ and $S^{n-1}_{\varepsilon_1,-}$ is not smaller than $d(x) + \delta$. Then for any

$\eta \in B^n_{r_\delta}(\xi) \cap \partial\Omega$ we have
$$\rho(\eta) \geq |\eta - y_\eta| \geq d(x) + \delta.$$

The proof is complete. □

Remark 4.3.11. The above proof does not use the differentiability of the distance function. We have another proof of this part, which uses the implicit function theorem for the distance function (and requires slightly different assumptions) but is considerably shorter. The statement and proof are given in the rest of this section.

Proposition 4.3.12. *Let $x \in \Omega \setminus \overline{\Sigma}$ and $\pi(x) = \{\xi\}$. Suppose that there is a neighborhood U of ξ in $\partial\Omega$ such that any $\eta \in \partial\Omega \cap U$ has non-spreading inner perpendicular and $\partial\Omega \cap U$ is an $(n-1)$-dimensional topological manifold. Then there exist $\delta > 0$ and $r_\delta > 0$ such that for any $\eta \in B^n_{r_\delta}(\xi) \cap \partial\Omega$ the inequality $d(x) + \delta \leq \rho(\eta)$ holds.*

Proof. Since $x \in \Omega \setminus \overline{\Sigma}$, there is small $r > 0$ such that the function d is of class C^1 in $B^n_r(x) \subset \Omega \setminus \overline{\Sigma}$ and, by Lemma 4.2.3, $\hat{\pi}(B^n_r(x)) \subset \partial\Omega \cap U$. Define
$$\Gamma := \{y \in B^n_r(x) \mid d(y) = d(x)\}.$$

Noting that $|\nabla d| = 1$ if it is differentiable, we find that Γ is a C^1 hypersurface by the implicit function theorem. By the assumption of non-spreading inner perpendicular, the restriction map $\hat{\pi}|_\Gamma$ is injective. Hence, by the assumption of topological manifold and the invariance of domain theorem, the map $\hat{\pi}|_\Gamma$ is a homeomorphism to its image; in particular, the image of any neighborhood of x in Γ of $\hat{\pi}$ is a neighborhood of ξ in $\partial\Omega$. For any small open region neighborhood V of x in Γ, the boundary of $\hat{\pi}(V)$ in $\partial\Omega$ is strictly far from ξ; thus, even if the set V is slightly shifted in the direction away from $\partial\Omega$ (while $V \subset B^n_r(x)$), the image of $\hat{\pi}$ still contains a neighborhood of ξ. This is justified by the theory of mapping degree. The proof is now complete. □

4.4 A sufficient condition for vanishing of Lebesgue measure

In this section, for $\Omega \subset \mathbb{R}^n$ with $C^{1,1}$-boundary we give a sufficient condition that the Lebesgue measure of the cut locus $\overline{\Sigma}$ vanishes by utilizing Theorem 4.1.3. Let $\mathcal{L}^n$ and $\mathcal{H}^{n-1}$ denote the n-dimensional Lebesgue measure and the $(n-1)$-dimensional Hausdorff measure.

The following is the main theorem in this section.

Theorem 4.4.1. *Let $\Omega \subset \mathbb{R}^n$ be an open set with $C^{1,1}$-boundary. If there is an $\mathcal{H}^{n-1}$-negligible subset $\Gamma \subset \partial\Omega$ such that $\rho_* = \rho$ in $\partial\Omega \setminus \Gamma$, then $\mathcal{L}^n(\overline{\Sigma}) = 0$.*

By this theorem we immediately obtain:

Corollary 4.4.2. *Let $\Omega \subset \mathbb{R}^n$ be an open set with $C^{1,1}$-boundary. If there is an $\mathcal{H}^{n-1}$-negligible subset $\Gamma \subset \partial\Omega$ such that $\partial\Omega \setminus \Gamma$ is C^2, then $\mathcal{L}^n(\overline{\Sigma}) = 0$.*

Here that $\partial\Omega \setminus \Gamma$ is C^2 means that for any $\xi \in \partial\Omega \setminus \Gamma$ there is an open neighborhood $U \in \mathcal{U}^n_\xi$ such that $(\partial\Omega \setminus \Gamma) \cap U$ is a C^2-manifold. Of course, the assumptions in Theorem 4.4.1 and Corollary 4.4.2 exclude the example of Mantegazza-Mennucci [15, Section 3].

We now prove Theorem 4.4.1. Our proof is inspired by [7]. We first prepare the following inclusion relation.

Proposition 4.4.3. *Let $\partial\Omega$ be of class C^1. Define for $\xi \in \partial\Omega$ and $0 \le t < \infty$*

$$\Psi(\xi, t) := \xi + t\nu(\xi),$$

where $\nu(\xi)$ is the inner unit normal at ξ, and

$$\mathcal{S} := \{\Psi(\xi, t) \in \mathbb{R}^n \mid \xi \in \partial\Omega,\ \rho_*(\xi) < \infty,\ \rho_*(\xi) \le t \le \rho(\xi)\}.$$

Then $\overline{\Sigma} \setminus \Sigma \subset \mathcal{S}$.

Proof. Fix any $x \in \overline{\Sigma} \setminus \Sigma$ and let $\xi = \hat{\pi}(x) \in \partial\Omega$. By Theorem 4.1.3 we have

$$\rho_*(\xi) \le d(x) = |x - \xi| < \infty.$$

Noting that $B^n_{|x-\xi|}(x)$ is an inner touching ball at ξ of Ω, we also have

$$|x - \xi| \le \rho(\xi).$$

Since $\nu(\xi) = \frac{x-\xi}{|x-\xi|}$, we obtain $\Psi(\xi, |x - \xi|) = x$ and hence $x \in \mathcal{S}$. □

By the above proposition and the fact that $\mathcal{L}^n(\Sigma) = 0$ (see e.g. [2]), it suffices to confirm the following proposition in order to prove Theorem 4.4.1.

Proposition 4.4.4. *Let $\partial\Omega$ be of class $C^{1,1}$ and suppose that there is an $\mathcal{H}^{n-1}$-negligible subset $\Gamma \subset \partial\Omega$ such that $\rho_*(\xi) = \rho(\xi)$ holds for any $\xi \in \partial\Omega \setminus \Gamma$. Then $\mathcal{L}^n(\mathcal{S}) = 0$.*

Proof. Under the assumption, by Proposition 4.4.3 we notice that $\mathcal{S} \subset \mathcal{S}^1 \cup \mathcal{S}^2$, where

$$\mathcal{S}^1 := \{\Psi(\xi, t) \in \mathbb{R}^n \mid \xi \in \partial\Omega,\ t = \rho_*(\xi) < \infty\},$$

$$\mathcal{S}^2 := \{\Psi(\xi, t) \in \mathbb{R}^n \mid \xi \in \Gamma,\ t \ge 0\}.$$

Thus it suffices to confirm that $\mathcal{L}^n(\mathcal{S}^1) = \mathcal{L}^n(\mathcal{S}^2) = 0$.

Since $\partial\Omega$ is $C^{1,1}$, there exist countable local parametrizations: for $k \in \mathbb{N}$ there are open sets $U_k \subset \mathbb{R}^{n-1}$, $V_k \subset \mathbb{R}^n$, a compact set $C_k \subset U_k$ and a $C^{1,1}$-diffeomorphism $Y_k : U_k \to \partial\Omega \cap V_k$ such that $\partial\Omega = \bigcup_k Y_k(C_k)$. Then, noting that ν is Lipschitz since $\partial\Omega$ is $C^{1,1}$, we can define Lipschitz maps $\overline{\Psi}_k : \mathbb{R}^n \supset C_k \times [0, \infty) \to \mathbb{R}^n$ by

$$\overline{\Psi}_k(x', t) := \Psi(Y_k(x'), t) = Y_k(x') + t\nu(Y_k(x')).$$

Now we define a set $A^1_k \subset C_k \times [0, \infty)$ for any k by

$$A^1_k := \{(x', t) \in C_k \times [0, \infty) \mid t = \rho_*(Y_k(x'))\}.$$

Since A^1_k is a part of the graph of the lower semicontinuous function $\rho_* \circ Y'_k$, we have $\mathcal{L}^n(A^1_k) = 0$. Noting that $\mathcal{S}^1 = \bigcup_k \overline{\Psi}_k(A^1_k)$, we obtain

$$\mathcal{L}^n(\mathcal{S}^1) \le \sum_k \mathcal{L}^n(\overline{\Psi}_k(A^1_k)) \le \sum_k L^n_k \mathcal{L}^n(A^1_k) = 0,$$

where L_k is the Lipschitz constant of $\overline{\Psi}_k$ on C_k.

Next we define a set $A_k^2 \subset C_k \times [0,\infty)$ for any k by

$$A_k^2 := Y_k^{-1}(\Gamma \cap Y_k(C_k)) \times [0,\infty).$$

By the area formula (see e.g. [8]) and $\mathcal{H}^{n-1}(\Gamma) = 0$, we have

$$\mathcal{L}^{n-1}(Y_k^{-1}(\Gamma \cap Y_k(C_k))) \leq J_k^{-1}\mathcal{H}^{n-1}(\Gamma \cap Y_k(C_k)) = 0,$$

where $J_k := \min_{C_k} |JY_k| > 0$. Thus we have $\mathcal{L}^n(A_k^2) = 0$ and hence, similarly as above, $\mathcal{L}^n(\mathcal{S}^2) = 0$. $\square$

4.5 A counterexample with Lipschitz boundary

The statement of Theorem 4.1.3 does not hold in Lipschitz cases. A counterexample $\Omega \subset \mathbb{R}^2$ is simply given as the union of the unit disc $\{x_1^2 + x_2^2 < 1\}$ and the upper half plane $\{x_2 > 0\}$. Note that $\overline{\Sigma} = \{x_1 = 0\} \cap \{x_2 \geq 0\}$. Then, for example, the point $x = (1,2) \in \Omega \setminus \overline{\Sigma}$ satisfies $\hat{\pi}(x) = (1,0)$ but we find that $d(x) = 2$ and

$$\rho_*(\hat{\pi}(x)) = \lim_{\theta \uparrow 0} \rho((\cos\theta, \sin\theta)) = 1.$$

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Chapter 5

An example of a mean-convex flow developing infinitely many singular epochs

In this chapter, we give an example of a compact mean-convex hypersurface with a single singular point moved by mean curvature having a sequence of singular epochs (times) converging to zero.

Keywords: Mean curvature flow; Singularity; Smoothing effect; Mean-convex.

5.1 Introduction

The regularity and singularity of mean curvature flow, which is a one-parameter family of hypersurfaces in $\mathbb{R}^{n+1}$ moving by its mean curvature, have been studied by many authors. There is an excellent survey paper [8] on this issue from classical results to recent developments.

In particular it is well-studied for mean-convex flows, namely, mean curvature flows of hypersurfaces with positive mean curvature. A well-known conjecture about such flows is: any mean-convex flow from a smooth initial surface develops singularities only at finitely many epochs (for example see [20]).

The main result of this chapter, Theorem 5.2.1, shows that there is a chance that the set of singular epochs is not finite even if an initial surface has only one singular point. Such an example is rigorously given in Section 5.3 but rough shape of the initial surface is as drawn in Figure 5.1.

Our initial surface is constructed by dilation. Thus it is self-similar near the singularity. Using the self-similarity, we prove that the flow from the surface pinches at infinitely many epochs (times) $t_k \downarrow 0$ by comparing Angenent's doughnuts [2] and balls. One may be tempted to construct such a surface by using a rescaled periodic function. However, this simple idea does not work directly, although the idea of rescaling is important. Our construction looks slightly complicated because we have to connect a ball like shape in a suitable way. An advantage of our construction is that it is easy to confirm the desired properties like mean-convexity. The feature of our construction is explained in detail in Remark 5.3.7.

We describe the result in terms of the level set method introduced by Chen-Giga-Goto [5] and Evans-Spruck [10] (see a self-contained book by Giga [12] for details). This method can define a(n) *(generalized) interface evolution* of mean curvature flow for all times through singularities. The interface evolution is uniquely determined

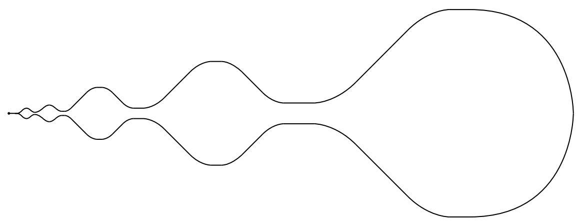

FIGURE 5.1: Example.

by a given initial surface, although in general it is not necessarily unique in the sense of "surface evolution". The reason is that interface evolutions can fatten, namely, have an interior point at some time. However, our example is now mean-convex in the sense of White [21], and hence it does not fatten.

In the rest of this section, we mention some related known results. Our example is useful to contrast them.

For any smooth compact initial surface, there is a unique classical solution of mean curvature flow at least locally in time (see e.g. [3, 17]). However it must develop singularities in finite time and it is complicated generally. The first non-simple singularity is given by Grayson [13] called "neck-pinching", which inspires the result of this chapter.

On the other hand, the mean curvature flow has a smoothing effect due to its parabolicity. A remarkable well-known result by Ecker and Huisken [9] is that any uniformly Lipschitz initial surface admits a classical solution of mean curvature flow locally in time. This result is proved by establishing local interior regularity estimates. Some other results are also known. For example, Evans and Spruck [11] proved a local interior regularity result for a level set flow provided that it is given locally as the graph of a continuous function. In addition, the recent works of Tonegawa and his co-authors [15, 18, 19] show the local existence of a classical solution for C^1 initial surfaces in terms of the Brakke flow (with a transport term). Our example suggests that these smoothing effects are crucially based on that an initial surface is locally represented by a graph.

As mentioned above, mean-convex flows are well-studied compared with general flows. There are many results about the size or nature of the singular set of mean-convex flows (see e.g. [6, 7, 21, 22, 23] or the subsection 2.3 in [8] for details). In particular, if $n = 2$, the mean-convex interface evolution is smooth for almost every time [21]. Our example shows that the set of singular times can be an infinite set (in a finite time interval).

The case that an initial surface is given by rotating a graph is studied even better in [1]. In particular, an axisymmetric compact smooth initial surface develops singularities only at finitely many epochs. Our example is axisymmetric, and thus also complements the above result.

Finally, we mention the case of curve shortening flow ($n = 1$). In this case, recently, Lauer proved that any finite length Jordan curve is smoothed out instantly [16]. This result is in marked contrast to our higher dimensional result ($n \geq 2$). In fact, our example is of finite area and the image of a continuous injection from the n-dimensional sphere, although the smoothing effect as in [16] is not valid.

5.2 An example of mean-convex hypersuface developing infinitly many singular epochs moved by mean curvature

In this section we state our main theorem rigorously. Throughout this chapter, for a given open set D_0 (resp. boundary $\Gamma_0 = \partial D_0$, closed set $E_0 = D_0 \cup \Gamma_0$) in $\mathbb{R}^{n+1}$, the set D (resp. Γ, E) in $\mathbb{R}^{n+1} \times [0, \infty)$ denotes the open (resp. interface, closed) evolution of mean curvature flow, and the set D_t (resp. Γ_t, E_t) in $\mathbb{R}^{n+1}$ denotes its cross-section at time $t > 0$. See [12] for details of the above definitions.

Here is our main theorem.

Theorem 5.2.1. *Let $n \geq 2$. There exists a compact connected axisymmetric initial hypersurface $\Gamma_0 \subset \mathbb{R}^{n+1}$, which is the boundary of some bounded open set D_0 of finite perimeter, satisfying the following conditions:*

(1) *all points except one point in Γ_0 are C^∞-regular and mean-convex points,*

(2) *the generated evolutions satisfy the monotonicity $E_{t+h} \subset D_t$ for any $t \geq 0$ and $h > 0$; in particular, Γ_t does not fatten for any $t \geq 0$,*

(3) *for any $\tau > 0$ there exists $0 < t < \tau$ such that Γ_t has a singularity.*

A point $\mathbf{x} \in \Gamma_0$ is called *C^∞-regular point* if there exists some open neighborhood U in $\mathbb{R}^{n+1}$ containing $\mathbf{x}$ such that $U \cap \Gamma_0$ is an embedded n-dimensional C^∞-hypersurface. A C^∞-regular point $\mathbf{x} \in \Gamma_0 = \partial D_0$ is called *mean-convex point* if the inward mean curvature at $\mathbf{x}$ is positive.

Remark 5.2.2. If we drop the connectivity, an example of initial surface developing infinitely many singular epochs is easily provided, for example, by the countable union of dwindling spheres converging to a point.

Remark 5.2.3. The monotonicity in (2) is the same as the mean-convexity of White [21]. This monotonicity directly implies that the interface evolution does not fatten so that the level set flow is nothing but a Brakke flow (see also [12, 14]).

5.3 Construction of an example

We construct an example concretely in order to prove Theorem 5.2.1. This construction is based on the comparison principle of mean curvature flow (Lemma 5.3.1) and two self-shrinking classical solutions (Examples 5.3.2 and 5.3.3). Using them, we can obtain a “neck-pinching” singularity as shown in [2].

Lemma 5.3.1 (Avoiding property)**.** *Let Γ, $\Gamma' \subset \mathbb{R}^{n+1} \times [0, \infty)$ be interface evolutions generated by compact initial surfaces Γ_0, $\Gamma'_0 \subset \mathbb{R}^{n+1}$, respectively. If Γ_0 and Γ'_0 are disjoint, then so are Γ and Γ'.*

Proof. See [12, Theorem 4.5.2, Lemma 4.5.13]. □

Example 5.3.2 (Spheres)**.** The n-sphere $S^n \subset \mathbb{R}^{n+1}$ shrinks to its center without changing the shape since the curvature is the same all around. The n-sphere with radius R disappears at time $R^2/2n$.

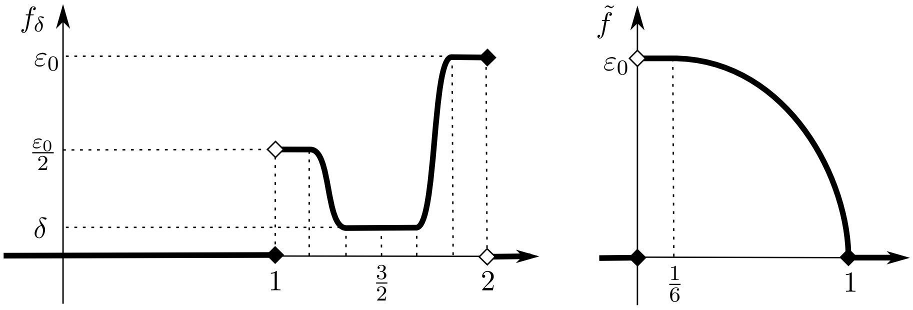

FIGURE 5.2: Graphs of f_δ and $\tilde{f}$.

Example 5.3.3 (Shrinking doughnuts)**.** For $n \geq 2$, Angenent [2] showed that there exists a self-shrinking doughnut $A^n \approx S^1 \times S^{n-1} \subset \mathbb{R}^{n+1}$. More precisely, A^n is created by rotating a suitable simple closed curve γ around the x_0-axis, where γ lies in the x_0x_1-plane with $x_1 > 0$, and is symmetric with respect to reflection in the x_1-axis. The doughnut A^n shrinks to its center without changing the shape and disappears in finite time. We define the radius of hole r and the thickness R of A^n by

$$r := \min\{x_1 \mid (x_0, x_1, 0, \ldots, 0) \in \gamma\}, \quad R := \max\{2x_0 \mid (x_0, x_1, 0, \ldots, 0) \in \gamma\}.$$

Now we construct our example. Let $\phi_0 : [0, \frac{1}{2}] \to [0, 1]$ be a nondecreasing function of class C^∞ such that $\phi_0 \equiv 0$ in $[0, \frac{1}{6}]$ and $\phi_0 \equiv 1$ in $[\frac{1}{3}, \frac{1}{2}]$. Fix a positive constant $\varepsilon_0 \in (0, 1)$ so that $(1 + \max|\phi_0''|)\varepsilon_0^2 < 1$. For $\delta \in (0, \frac{\varepsilon_0}{2})$ we define $f_\delta : \mathbb{R} \to [0, \varepsilon_0]$ by

$$f_\delta(x) := \begin{cases} (\varepsilon_0 - \delta)\phi_0(x - \frac{3}{2}) + \delta & (\frac{3}{2} < x \leq 2), \\ (\frac{\varepsilon_0}{2} - \delta)\phi_0(\frac{3}{2} - x) + \delta & (1 < x \leq \frac{3}{2}), \\ 0 & (\text{otherwise}). \end{cases}$$

Next, let Ω_0 be a planer convex domain in the xy-plane such that, Ω_0 is symmetric with respect to reflection in the x- and y-axis, and its boundary $\partial\Omega_0$ is of class C^∞, passes through the four points $(0, \pm\varepsilon_0), (\pm 1, 0) \in \mathbb{R}^2$, is straight in the region $\{|x| \leq \frac{1}{6}\}$, and has positive inner curvature at $(\pm 1, 0) \in \mathbb{R}^2$. Then we define $\tilde{f} : \mathbb{R} \to [0, \varepsilon_0]$ so that the graph curve $\{(x, \tilde{f}(x)) \mid x \in (0, 1]\}$ is contained in $\partial\Omega_0$ and $\tilde{f} \equiv 0$ elsewhere.

Finally, we define $F : \mathbb{R} \to [0, \varepsilon_0]$ by

$$F(x) := \tilde{f}(x - 2) + \sum_{k=0}^{\infty} 2^{-k} f_{\delta_0}(2^k x),$$

where $\delta_0 \in (0, \frac{\varepsilon_0}{2})$ is taken to be sufficiently small so that there exists a self-shrinking doughnut A_0^n with thickness $R_0 < \frac{1}{3}$ and radius of hole $r_0 > \delta_0$ such that A_0^n disappears earlier than the n-sphere with radius $\frac{\varepsilon_0}{12}$. Notice that F is self-similar in $[0, 2]$ in the sense that for $x \in [0, 1]$

$$F(2x) = 2F(x). \tag{5.3.1}$$

It turns out that the hypersurface $\widetilde{\Gamma}_0 \subset \mathbb{R}^{n+1}$ created by rotating the graph of F with respect to the x_0-axis (as Figure 5.1), namely $\widetilde{\Gamma}_0 := \partial \tilde{D}_0$ where

$$\widetilde{D}_0 := \left\{ (x_0, \ldots, x_n) \in \mathbb{R}^{n+1} \;\middle|\; F(x_0) > \sqrt{{x_1}^2 + \ldots + {x_n}^2} \right\},$$

satisfies all conditions of Theorem 5.2.1. The surface $\widetilde{\Gamma}_0$ lies in the region $\{0 \leq x_0 \leq 3\}$. The origin is only one singular point in $\widetilde{\Gamma}_0$.

We shall check that this compact connected axisymmetric surface $\widetilde{\Gamma}_0$ of finite area satisfies the conditions of Theorem 5.2.1. The following three propositions 5.3.4, 5.3.5 and 5.3.6 correspond to the three conditions (1), (2) and (3), respectively.

Proposition 5.3.4. *All points except the origin in $\widetilde{\Gamma}_0$ are C^∞-regular mean-convex points.*

Proof. It is easy to check that $\widetilde{\Gamma}_0$ is of class C^∞ except the origin. Thus we only confirm the mean-convexity. It suffices to confirm in the region $\{0 < x_0 \leq 2\}$ since $\widetilde{\Gamma}_0$ is convex in $\{2 < x_0 \leq 3\}$. Moreover, the mean-convexity is preserved by dilation, and hence we only need to confirm in $\{1 < x_0 \leq 2\}$.

The inward mean curvature of $\widetilde{\Gamma}_0$ in $\{0 < x_0 \leq 2\}$ is represented as

$$\frac{n-1}{F(x_0)\sqrt{1 + (F'(x_0))^2}} - \frac{F''(x_0)}{(\sqrt{1 + (F'(x_0))^2})^3}.$$

Therefore, to confirm its positivity it suffices to prove that for $1 < x \leq 2$ the inequality $F(x)F''(x) < 1$ holds. This inequality follows since in this case $F \equiv f_{\delta_0}$ holds and for any $\delta \in (0, \frac{\varepsilon_0}{2})$ we have $f_\delta \leq \varepsilon_0$ and

$$f_\delta'' \leq (\varepsilon_0 - \delta)|\phi_0''| + \delta \leq (1 + \max|\phi_0''|)\varepsilon_0 < \varepsilon_0^{-1}.$$

The last inequality follows from the definition of $\varepsilon_0 \in (0, 1)$. □

We denote the evolutions corresponding to $\widetilde{\Gamma}_0$ by $\widetilde{\Gamma}$, $\widetilde{D}$ and $\widetilde{E}$.

Proposition 5.3.5. *The monotonicity $\widetilde{E}_{t+h} \subset \widetilde{D}_t$ holds for any $t \geq 0$ and $h > 0$.*

Proof. By order preserving property [12, Theorem 4.5.2], it suffices to prove that $\widetilde{E}_t \subset \widetilde{D}_0$ for any $t > 0$. For any positive integer k we define $F_k : \mathbb{R} \to [0, \varepsilon_0]$ by

$$F_k(x) := \begin{cases} F(x) & (x > 2^{-k+1}), \\ \varepsilon_0 2^{-k} & (2^{-k} \leq x \leq 2^{-k+1}), \\ 2^{-k}\tilde{f}(1 - 2^k x) & (x < 2^{-k}), \end{cases}$$

and

$$\widetilde{D}_0^k := \left\{ (x_0, \ldots, x_n) \in \mathbb{R}^{n+1} \;\middle|\; F_k(x_0) > \sqrt{{x_1}^2 + \ldots + {x_n}^2} \right\}$$

(see Figure 5.3) and denote the corresponding closed evolution by $\widetilde{E}^k$. By definition, we find that all points in $\partial \widetilde{D}_0^k$ are C^∞-regular mean-convex points. Since the classical mean-convexity implies the monotonicity [12, Theorem 4.5.7], we have $\widetilde{E}_t^k \subset \widetilde{D}_0^k$ for all

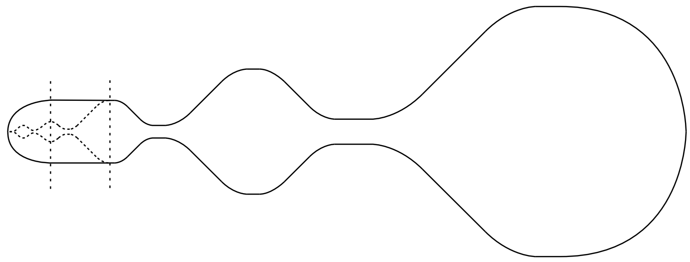

FIGURE 5.3: $\partial \widetilde{D}_0^k$.

k and $t > 0$. Hence, noting the convergence $\widetilde{D}_0^k \downarrow \widetilde{D}_0$, we find that for all $t > 0$

$$\bigcap_{k=0}^{\infty} \widetilde{E}_t^k \subset \bigcap_{k=0}^{\infty} \widetilde{D}_0^k = \widetilde{D}_0.$$

Using monotone convergence property [12, Theorem 4.5.4], we have $\widetilde{E}_t^k \downarrow \widetilde{E}_t$ for all $t > 0$, and thus we conclude that $\widetilde{E}_t \subset \widetilde{D}_0$ for any $t > 0$. $\square$

Proposition 5.3.6. *For any $\tau > 0$ there exists $0 < t < \tau$ such that $\widetilde{\Gamma}_t$ has a singularity.*

Proof. Denote $\mathbf{e}_0 := (1, 0, \ldots, 0) \in \mathbb{R}^{n+1}$. Seeing our construction of $\widetilde{\Gamma}_0$, we notice that $\widetilde{\Gamma}_0$ encloses the two n-spheres with radius $\frac{\varepsilon_0}{12}$ centered at $\mathbf{e}_0$ and $2\mathbf{e}_0$. Moreover, noting the definition of δ_0, we also notice that $\widetilde{\Gamma}_0$ is circled by a self-shrinking doughnut centered at $\frac{3}{2}\mathbf{e}_0$ disappearing earlier than the spheres. Then we find that the interface evolution $\widetilde{\Gamma}$ has a neck-pinching singularity at some time $t_0 \in (0, \frac{\varepsilon_0^2}{144n})$. By the self similarity (5.3.1), for any positive integer k we can take the two spheres with radius $\frac{\varepsilon_0}{12 \cdot 2^k}$ centered at $\frac{1}{2^k}\mathbf{e}_0$ and $\frac{1}{2^{k-1}}\mathbf{e}_0$ and the doughnut centered at $\frac{3}{2^{k+1}}\mathbf{e}_0$ disappearing earlier than the spheres as above. We thus obtain a sequence of singular times $\{t_k\}_k$ of $\tilde{\Gamma}$ such that $t_k \in (0, \frac{\varepsilon_0^2}{144n \cdot 4^k})$. Since $t_k \downarrow 0$, we reach the conclusion. $\square$

Remark 5.3.7. As mentioned in the introduction, we are tempted to construct the self-similar part of an initial surface by a simpler scaling argument, for example rotating some rescaled periodic function as

$$f(x) = x\left(\phi(\varepsilon \log x) + \delta\right),$$

where ϕ is a suitable nonnegative periodic function and ε, δ are sufficiently small positive numbers. This construction is simpler than ours and should provide a surface satisfying the main desired properties. Unfortunately, we then need to be careful to confirm the properties rigorously. For example in the proof of Proposition 5.3.5, we made a new surface by cutting and pasting smoothly. In addition, the obtained surface should enclose the original one and remain mean-convex. It is not trivial to confirm them for the simply constructed surface. However, the surface given in this chapter is partially just straight, and its overall shape is also clear, so that there is no need to be careful in such a process.

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