博士論文

Representation theoretical approach to Schramm-Loewner evolution

（シュラム・レヴナー発展に対する表現論的アプローチ）

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Abstract

Growth processes have been proven to give frameworks that describe various equilibrium and non-equilibrium phenomena exhibited in nature. Examples of such growth processes we consider in this thesis are variants of Schramm-Loewner evolution (SLE), which was introduced by Schramm as subsequent scaling limit of loop erased random walks and uniform spanning trees. It has been shown to describe an interface of clusters in several critical systems in two dimensions including critical percolation and Ising models at criticality.

We have another framework to investigate two dimensional critical systems. It is two dimensional conformal field theory, which has been one of the most powerful tools in wide variety of fields from condensed matter physics to string theory, and in mathematics. A milestone of CFT prediction on a critical system is Cardy's formula, which gives the crossing probability for a critical percolation in two dimensions from computation of correlation functions in CFT. Cardy's formula was proved by Smirnov to be a theorem, while the derivation by Cardy has not been verified.

Since SLE and CFT are different frameworks that describe the same phenomena, they are expected to be bridged to each other. Connection between SLE and CFT has been studied under the name of SLE/CFT correspondence from various points of view. A significant development is the *group theoretical* formulation of SLE by Bauer and Bernard, which suggests an elegant way of constructing SLE local martingales from a representation of the Virasoro algebra.

The importance of SLE/CFT correspondence is bidirectional. In one direction, one can compute an SLE local martingale from computation in CFT, while in the other direction, SLE gives data of CFT such as partition functions and correlation functions, which in some sense may be interpreted as construction of a nontrivial quantum field theory as a result of path integral. Thus precise understanding SLE/CFT correspondence helps us develop theories of both SLE and CFT.

This thesis is aimed at generalizing the group theoretical formulation of SLE to representation theories of other algebras that describe internal symmetry of CFT as well as space-time symmetry, and correspondingly generalizing the notion of SLE. Such a direction of generalization will deepen our understanding of connection between SLE and CFT, and will help us settle the SLE/CFT correspondence in the sense of Bauer and Bernard in a more fundamental and general theory.

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Chapter 1

Background and Overview

We see in this chapter an underlying motivation of this thesis to understand properties of physical systems at criticality. One tool to investigate such a system is conformal field theory (CFT), which in particular in two dimensions has been proved to be powerful in vast fields of physics from condensed matter physics to string theory and in mathematics as well. Two dimensional CFT [BPZ84, DFMS97] is distinguished from other quantum field theories in that its symmetry described by an infinite dimensional Lie algebra allows exact computation of several correlation functions. Other approaches came from probability theory that concern the existence of scaling limit of lattice models, its conformal invariance and Markovian property [Smi06]. A milestone was introduction of Scramm-Loewner evolution (SLE) that characterizes an interface of clusters in a critical system as a solution of a stochastic differential equation [Sch00]. This thesis concerns a part of SLE/CFT correspondence that connects the notions of CFT and SLE.

1.1 Scaling Limit of critical percolation

We begin with a consideration on the (site) percolation problem in two dimensions. The description in this section partly follows [Wer03, Wer08, Smi06]. Let L_ϵ be a regular periodic lattice of lattice spacing ϵ that is embedded in the two dimensional infinite plane, and E_ϵ be the set of bonds in this lattice, namely it consists of pairs (x,y) of sites in the lattice such that the standard distance between x and y are ϵ under identification $(x,y)=(y,x)$. We attach a random variable σ_x on a site $x\in L_\epsilon$ that takes value $+1$ (open) with probability p and 0 (closed) with probability $1-p$, and suppose that these random variables are mutually independent on different sites Then the configuration space of the percolation problem on this lattice is given by $\{1,0\}^{L_\epsilon}$, and a probability measure $\mathbb{P}$ is defined as the direct product of ones attached to each edges. A configuration is denoted by σ so that σ_x can be regarded as its value at the site x. A sequence $(x_1,\cdots,x_n)$ of sites in L_ϵ is said to be *a path* if $(x_i,x_{i+1})\in E_\epsilon$ for every $i=1,\cdots,n-1$, and a path $(x_1,\cdots,x_n)$ is said to be *an open path* if $\sigma_{x_i}=1$ for every $i=1,\cdots,n$. A subset $V_\epsilon\subset L_\epsilon$ is *connected* if for any two points $x,y\in V_\epsilon$ one can take a path $(x,x_2,\cdots,x_{n-1},y)$ so that $x_2,\cdots,x_{n-1}\in V_\epsilon$ and it is *open connected* if such a path is taken as an open path. For a site x in L_ϵ and a configuration σ, we denote by $C_x(\omega)$ the maximal open connected subset that contains x, and call it *the open cluster* containing x. The percolation probability for the probability space $(\{1,0\}^{L_\epsilon},\mathbb{P})$ is defined by

$$P_{\mathrm{perc}}(p):=\mathbb{P}(\{\omega||C_x(\omega)|=\infty\}), \tag{1.1.1}$$

where $|C_x(\omega)|$ denotes the cardinality of the set $C_x(\omega)$. Here $\mathbb{P}(B)$ is the probability that the event B occurs. Notice that the percolation probability is independent of choice of a site x to define the open cluster. Since the percolation probability depends on p, we define the critical probability p_{c} by

$$p_{\mathrm{c}} := \inf\{p \in [0,1] | P_{\mathrm{perc}}(p) > 0\}. \tag{1.1.2}$$

For several lattices, it can be proved that the critical probability takes nontrivial value in $(0,1)$. For instance, the percolation problem on the equilateral triangular lattice exhibits $p_{\mathrm{c}} = \frac{1}{2}$ [Gri99, Kes82].

We focus on the scaling limit $\epsilon \to 0$ of the critical percolation, which is the percolation problem at the critical probability $p = p_{\mathrm{c}}$. To construct a well-defined problem, we take a simply connected domain Ω and four distinct points a, b, c, d on the boundary of Ω that are aligned counterclockwise so that the arcs (ab) and (cd) are two non-intersecting boundary segments. We approximate the domain Ω by a sublattice Ω_ϵ of L_ϵ and boundary points a, b, c, d by sites $a_\epsilon, b_\epsilon, c_\epsilon, d_\epsilon$ in Ω_ϵ. We shall consider the continuous limit $\epsilon \to 0$ while the domain Ω and its boundary points a, b, c, d are fixed. Since the number of sites in Ω_ϵ diverges as we take this limit, observables like, say the number of open clusters in Ω_ϵ, does not converge. Thus we have to choose observables that converge in the continuous limit. A candidate for such one is the crossing probability $P(\Omega_\epsilon, (a_\epsilon b_\epsilon), (c_\epsilon d_\epsilon))$ that one finds an open cluster in Ω_ϵ that touches the both segments $(a_\epsilon b_\epsilon)$ and $(c_\epsilon d_\epsilon)$.

Cardy's formula [Car92, Car01] concerns the scaling limit $P(\Omega, (ab), (cd))$ of the crossing probability $P(\Omega_\epsilon, (a_\epsilon b_\epsilon), (c_\epsilon d_\epsilon))$ as $\epsilon \to 0$. The idea of deriving Cardy's formula is divided into two steps. In the first step, we identify the percolation problem as the $Q \to 1$ limit of the Q-state Potts model, which is mapped to a Fortuin-Kasteleyn random cluster model [FK72]. Then the probability of a certain configuration occurs is proportional to

$$p^{\#\text{ of open sites}}(1-p)^{\#\text{ of closed sites}}Q^{\#\text{ of clusters}} \tag{1.1.3}$$

where $p \in [0,1]$ is a given parameter of the model. Thus we can see that the Fortuin-Kasteleyn random cluster model reduces to the percolation problem at $Q = 1$. This identification allows us to express the crossing probability by means of partition functions under boundary conditions. Indeed, if we denote the partition function under boundary conditions "α" on the arc (ab) and "β" on (cd) by $Z_{\alpha\beta}(Q)$, crossing events contribute only to partition functions of type $Z_{\alpha\alpha}(Q)$, while non-crossing events contribute to both partition functions $Z_{\alpha\alpha}$ and $Z_{\alpha\beta}(Q)$. Thus the crossing probability $P(\Omega, (ab), (cd))$ is proportional to

$$\lim_{Q\to 1}(Z_{\alpha\alpha}(Q) - Z_{\alpha\beta}(Q)). \tag{1.1.4}$$

The second step, which is more essential in the derivation, consists of the assertion that a partition function under a certain boundary condition is computed as the correlation function of boundary condition changing operators in the framework of boundary conformal field theory (BCFT) [Car84, Car86]. Namely, a partition function under boundary condition $Z_{\alpha\beta}(Q)$ has the following presentation

$$Z_{\alpha\beta}(Q) = Z(Q) \left\langle \phi_{(f|\alpha)}(a)\phi_{(\alpha|f)}(b)\phi_{(f|\beta)}(c)\phi_{(\beta|f)}(d) \right\rangle. \tag{1.1.5}$$

Here the field $\phi_{(\mathrm{bc}_2|\mathrm{bc}_1)}(z)$ changes boundary condition from bc_1 to bc_2 at the point z where it is inserted, and the symbol f denotes the "free" boundary condition. The partition function without boundary conditions is denoted simply by $Z(Q)$. Since boundary condition changing operators are assumed to behave as primary fields with respect to coordinate transformations, their four-point function is computed with help of "null vectors" and found to be proportional to a hypergeometric function.

From the above derivation, the crossing probability should behave in a conformally covariant way. With this observation, Carleson restated Cardy's formula in the following way.

Conjecture 1.1.1 (Cardy's formula, Carleson's version)**.** *Let Ω be a simply connected domain that is not $\mathbb{C}$ itself, and a, b, c, d be distinct boundary points aligned in the counterclockwise order. If we conformally map this domain to the equilateral triangle ABC so that the boundary points a, b, c are mapped to A, B, C, respectively, then the point d is mapped to a point D in the edge CA. Then we have*

$$P(\Omega, (ab), (cd)) = \frac{|CD|}{|CA|}, \tag{1.1.6}$$

where $|CD|$ and $|CA|$ denote the length of edges.

Theorem 1.1.2 (Smirnov [Smi01])**.** *Cardy's formula holds for the crossing probability obtained as a scaling limit of the critical percolation on the equilateral triangular lattice.*

1.2 Schramm-Loewner Evolution

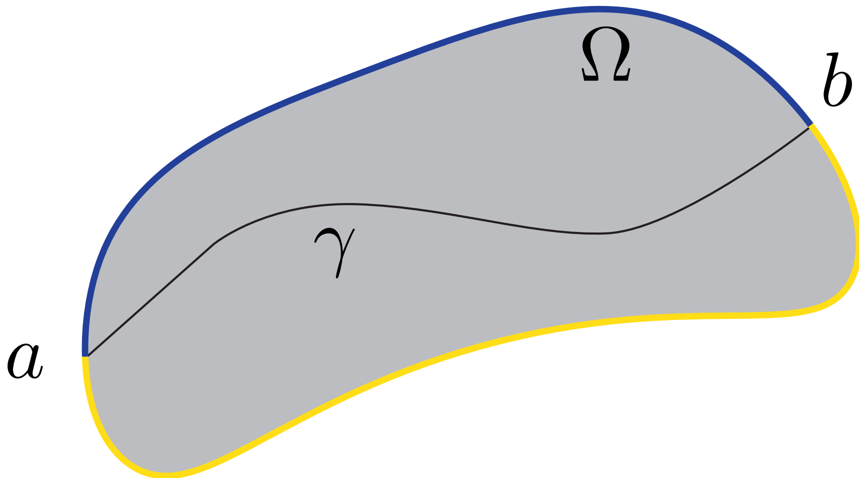

Figure 1.1: Boundary of clusters

There is another approach to the critical percolation that connect it to the notion of Schramm-Loewner evolution. Again we take a simply connected domain Ω in the two dimensional plane and two distinct boundary points a, b (Fig: 1.1). For a given lattice spacing ϵ, we approximate Ω by a sublattice Ω_ϵ of L_ϵ and boundary points a, b by sites a_ϵ, b_ϵ of L_ϵ. We also name the arc going counterclockwise from a to b $\mathcal{O}$ and the opposite arc $\mathcal{C}$, then the arcs are approximated by boundaries of L_ϵ denoted by $\mathcal{O}_\epsilon$ and $\mathcal{C}_\epsilon$, respectively. Under the boundary condition that the random value σ takes $+1$ on $\mathcal{O}_\epsilon$ and 0 on $\mathcal{C}_\epsilon$, one can find an interface growing from a_ϵ to b_ϵ that forms a boundary of the open cluster connected to $\mathcal{O}_\epsilon$. Then one obtains a probability measure $\mu_\epsilon(\Omega_\epsilon, a_\epsilon, b_\epsilon)$ that is supported on paths connecting a_ϵ to b_ϵ. We expect that the one-parameter family of probability measures $\mu_\epsilon(\Omega_\epsilon, a_\epsilon, b_\epsilon)$ labeled by the lattice spacing ϵ has a well-defined limit as $\epsilon \to 0$. Indeed, it converges to the SLE(6)-measure described below.

We first introduce (non-stochastic) Loewner evolution. Let $\mathbb{H} = \{z \in \mathbb{Z} | \mathrm{Im} z > 0\}$ be the upper half plane, and $\gamma : (0, \infty) \to \mathbb{H}$ be a parametrized curve going from 0 to ∞. At each time t, there is a uniformizaton map $g_t : \mathbb{H} \backslash \gamma(0, t] \to \mathbb{H}$, where $\gamma(0, t] = \{\gamma(s) | s \in (0, t]\}$ is a slit. Such a uniformization map is uniquely determined by *hydrodynamically* normalizing it as $g_t(z) = z + \alpha(t)/z + O(z^{-2})$ at the infinite point, where α is a strictly increasing map. After rescaling the time variable, we can normalize g_t so that

$$g_t(z) = z + \frac{2t}{z} + O(z^{-2}). \tag{1.2.1}$$

Loewner's theorem states that such a uniformization map is characterized as a solution of a differential equation.

Theorem 1.2.1 (Loewner [Löw23]). *For an appropriate parametrized curve γ in $\mathbb{H}$, there is a real valued continuous function w such that the corresponding uniformization map g_t satisfies the differential equation*

$$\frac{d}{dt} g_t(z) = \frac{2}{g_t(z) - w(t)} \tag{1.2.2}$$

under the initial condition $g_0(z) = z$.

Replacing the *driving force $w(t)$* by a Brownian motion X_t of covariance κ, we obtain the Schramm-Loewner evolution SLE(κ):

$$\frac{d}{dt} g_t(z) = \frac{2}{g_t(z) - X_t}, \quad g_0(z) = z, \tag{1.2.3}$$

which was originally introduced by Schramm in [Sch00]. Note that a sample path of a Brownian motion is almost surely continuous, thus the above equation can be a well-defined one. Inverting discussion that reaches to the non-stochastic Loewner evolution, a solution of SLE(κ) becomes a uniformization map $g_t : \mathbb{H} \backslash K_t \to \mathbb{H}$ for a certain hull $K_t \subset \mathbb{H}$ (Fig. 1.2), and the evolution of the hull K_t is organized into a parametrized curve γ in $\mathbb{H}$ (Fig. 1.3). Thus the SLE produces a measure on the space of curves in $\mathbb{H}$ from 0 to ∞, which we call *the SLE measure*. After its introduction, several properties of SLE were unveiled in probability theoretical context [RS05, LSW01a, LSW01b, LSW02b, LSW02a, Law04].

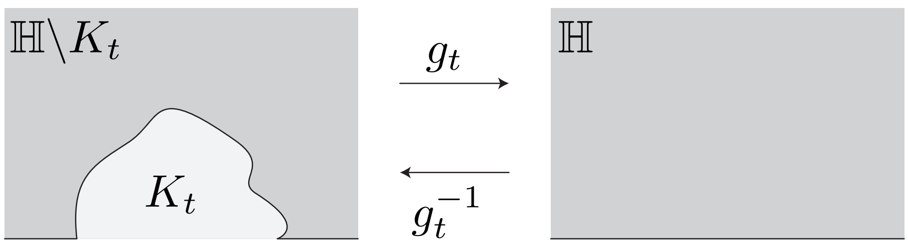

Figure 1.2: Uniformization map

In fact Schramm also introduced in [Sch00] another version of SLE, *radial* SLE, while SLE in Eq.(1.2.3) is called *chordal* SLE. For the definition of radial SLE, the upper half

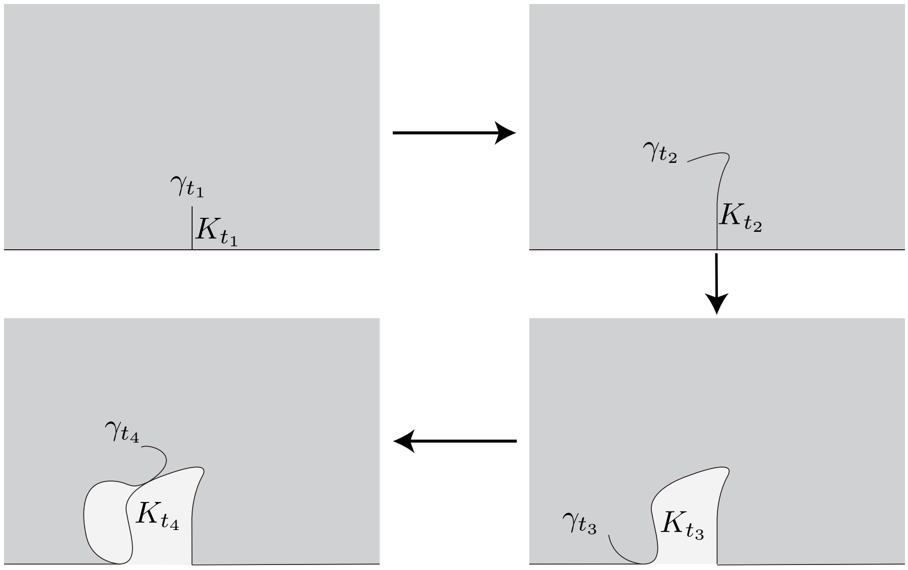

Figure 1.3: Evolution of hulls

plane $\mathbb{H}$ is replaced by the unit disc $\mathbb{D} = \{z \in \mathbb{C} | |z| < 1\}$. Radial SLE is an equation on a uniformization map $g_t : \mathbb{D} \backslash K_t \to \mathbb{D}$ for time evolving hull $K_t \subset \mathbb{D}$ such that

$$\frac{d}{dt} g_t(z) = -g_t(z) \frac{g_t(z) + W_t}{g_t(z) - W_t}, \quad g_0(z) = z, \tag{1.2.4}$$

where W_t is a Brownian motion on the boundary $\partial \mathbb{D}$ of the unit disc. In the following, we do not consider radial SLE and call chordal SLE simply as SLE.

To explain significant feature of the SLE measure, we introduce the notions of the conformal invariance and the domain Markov property of measures on curves. Let Ω be a simply connected domain in $\mathbb{C}$ that is not $\mathbb{C}$ itself, and a and b be two distinct points on the boundary of Ω. We consider random curves in Ω from a to b described by a probability measure $\mu(\Omega, a, b)$. For a conformal map $\phi : \Omega \to \phi(\Omega)$, a curve in Ω from a to b is mapped to a curve in $\phi(\Omega)$ from $\phi(a)$ to $\phi(b)$. Thus we can pull-back the probability measure $\mu(\Omega, a, b)$ by ϕ, namely the probability that a curve γ from $\phi(a)$ to $\phi(b)$ is given by

$$(\phi^* \mu(\Omega, a, b))(\gamma) = \mu(\Omega, a, b)(\phi^{-1}(\gamma)). \tag{1.2.5}$$

Definition 1.2.2. A family of probability measures $\mu(\Omega, a, b)$ labeled by a simply connected domain Ω and points a and b on the boundary of Ω on curves in Ω from a to b is conformal invariant if for any conformal map ϕ, we have

$$\phi^* \mu(\Omega, a, b) = \mu(\phi(\Omega), \phi(a), \phi(b)). \tag{1.2.6}$$

Let Ω be a simply connected domain and a and b be boundary points of Ω. We fix an internal point a' of Ω and a continuous path γ' from a to a'. Then we can obtain the conditioned measure from $\mu(\Omega, a, b)$ on curves from a to b that coincide with γ' until they arrive at a'. We denote such the measure obtained from $\mu(\Omega, a, b)$ conditioned on a curve γ' from a to a' by $\mu(\Omega, a, b)|_{\gamma'}$.

Definition 1.2.3. A family of probability measures $\mu(\Omega, a, b)$ labeled by a simply connected domain Ω and points a and b on the boundary of Ω on curves in Ω from a to b

possesses the domain Markov property if we have

$$\mu(\Omega, a, b)|_{\gamma'} = \mu(\Omega\backslash\gamma', a', b) \tag{1.2.7}$$

for any γ' starting from a and arriving at an internal point a' of Ω.

If a family of probability measures $\mu(\Omega, a, b)$ is conformal invariant, its properties are governed by a single probability measure $\mu(\mathbb{H}, 0, \infty)$ on continuous curves in the upper half plane $\mathbb{H}$ starting from 0 and arriving at ∞. If we take an internal point a and a curve γ from 0 to a, the complement of γ in $\mathbb{H}$ is conformally equivalent to $\mathbb{H}$ via a uniformization map $G_\gamma : \mathbb{H}\backslash\gamma \to \mathbb{H}$ that sends a to 0. Thus if a family of probability measure $\mu(\Omega, a, b)$ is conformal invariant, the domain Markov property on them is rephrased so that for any curve γ from 0 to an internal point a in $\mathbb{H}$, we have

$$(G_\gamma^{-1})^* \mu(\mathbb{H}, 0, \infty) = \mu(\mathbb{H}, 0, \infty)|_\gamma. \tag{1.2.8}$$

To a family of probability measures $\mu(\Omega, a, b)$ that possesses the conformal invariance and the domain Markov property, we associate a Loewner evolution g_t with a random driving force $w(t)$. By an argument by Schramm [Sch00] or a refined version by Smirnov [Smi06], it is shown that $w(t)$ have to be a Brownian motion of a certain covariance κ, or equivalently, g_t have to be the SLE(κ).

In the beginning of this section, we defined a one-parameter family of probability measures $\mu_\epsilon(\Omega_\epsilon, a_\epsilon, b_\epsilon)$ on interfaces in the approximated domain Ω_ϵ going from a_ϵ to b_ϵ from the critical percolation.

Theorem 1.2.4 (Convergence to SLE(6) [Smi01])**.** *Let Ω_ϵ be a sublattice in a triangular lattice of lattice spacing ϵ that approximates a connected domain Ω, and a_ϵ and b_ϵ be sites in Ω_ϵ that approximate boundary points a and b of Ω. The family of probability measures $\mu_\epsilon(\Omega_\epsilon, a_\epsilon, b_\epsilon)$ that is defined via the critical percolation on Ω_ϵ converges to the probability measure $\mu(\Omega, a, b)$ of SLE(6).*

1.3 On other models

There are several other models on lattices of which laws of interfaces have been shown or conjectured to converge to some SLE(κ) measures as the lattice spacing approaches to zero. In this section, we see some examples of such models.

1.3.1 Loop gas model

We only consider a hexagonal lattice Ω_ϵ that approximates a domain Ω, and take sites a_ϵ and b_ϵ that approximate boundary points a and b of Ω. A loop gas model on Ω_ϵ is specified by two parametes $n \geq 0$ and $x > 0$, which are called loop-weight and edge-weight, respectively. It gives a probability for a configuration consisting of a curve running from a_ϵ to b_ϵ and non-intersecting simple loops to be proportional to

$$n^{\#\text{ of loops}} x^{\text{length of loops}}. \tag{1.3.1}$$

If we take $n = 1$ and $x = 1$, we obtain the site percolation on a triangular lattice, noticing that a hexagonal lattice is the dual lattice of a triangular lattice.

Conjecture 1.3.1 (Nienhuis [Nie82, Nie84]). *Let $n \in [0, 2]$. The loop model has a critical value $x_c(n)$ and exhibits critical behavior at $x = x_c(n)$ and $x \in (x_c(n), \infty)$. Moreover the critical value $x_c(n)$ is given by*

$$x_c(n) = \frac{1}{\sqrt{2 + \sqrt{2 + n}}}. \tag{1.3.2}$$

We comment that this conjecture was derived by mapping the model to a Coulomb gas model, in which charged particles interact via logarithmic potential, thus its critical behavior is supported by the mechanism by Kosterlitz and Thouless [KT73, Kos74].

The criticality of a loop gas model has been established for $n = 1$, in which case the model is equivalent to the Ising model on a triangular lattice by taking $x = e^{-2\beta}$ with β being the inverse temperature. Indeed, the critical value is $\beta_c = \frac{1}{4}\log 3$, which leads to $x_c(1) = \frac{1}{\sqrt{3}}$ as was conjectured. We also comment that the case of $n = 0$ is equivalent to a self-avoiding random walk promising $0^{\#\text{ of loops}} = 0$ if (# of loops) ≥ 1 and 1 if there is no loop.

Conjecture 1.3.2 (Kager, Nienhuis [KN04]). *At $x = x_c(n)$, the law of the intergace running from a_ϵ to b_ϵ converges to SLE(κ)-measure for $\kappa = 4\pi/(2\pi - \mathrm{Arccos}(-n/2))$ as the lattice spacing ϵ goes to zero. At $x \in (x_c(n), \infty)$, it converges to SLE(κ)-measure for $\kappa = 4\pi/\mathrm{Arccos}(-n/2)$. Here the inverse function* Arccos *of the cosine takes value in* $[0, \pi]$.

1.3.2 Fortuin-Kasteleyn random cluster model

We have already seen the Fortuin-Kasteleyn random cluster model as an expression of the Q-state Potts model. It gives the probability for a certain configuration as in Eq.(1.1.3). We again consider this model on a lattice Ω_ϵ of lattice spacing ϵ that approximates a domain Ω, and take two sites a_ϵ and b_ϵ that approximate boundary points a and b of Ω. We also impose the boundary conditions that sites on the boundary segment going counterclockwise from a_ϵ to b_ϵ are "open" and ones on the opposite boundary segments are "closed". Then we can find in any configuration an interface running from a_ϵ to b_ϵ.

It was discussed in [SD87] that for $Q \in [0, 4]$ a Fortuin-Kasteleyn random cluster model at criticality should be equivalent to the "low temperature regime" $x \in (x_c(n), \infty)$ of the loop gas model for $n = \sqrt{Q}$. Thus Conjecture 1.3.2 leads to the following conjecture that was stated explicitly in [RS05].

Conjecture 1.3.3 (Rohde, Schramm [RS05]). *For $Q \in [0, 4]$, the law of the interface at criticality converges to the SLE(κ)-measure for $\kappa = 4\pi/\mathrm{Arccos}(-\sqrt{Q}/2)$.*

This conjecture has been proved for $Q = 0$ in [LSW04] and for $Q = 2$, which is the FK-Ising model, in [CDCH+14]. In the same paper [CDCH+14], the law of interfaces in the spin Ising model has been also proved to converge to SLE(3).

1.4 Connection between SLE and CFT

We have looked at two approaches to the critical percolation problem. The first one was Cardy's formula that stems from the boundary CFT picture, and the second one was the notion of SLE. Since CFT and SLE are two different frameworks to treat the same critical systems, they should be bridged in some sense (Fig. 1.4).

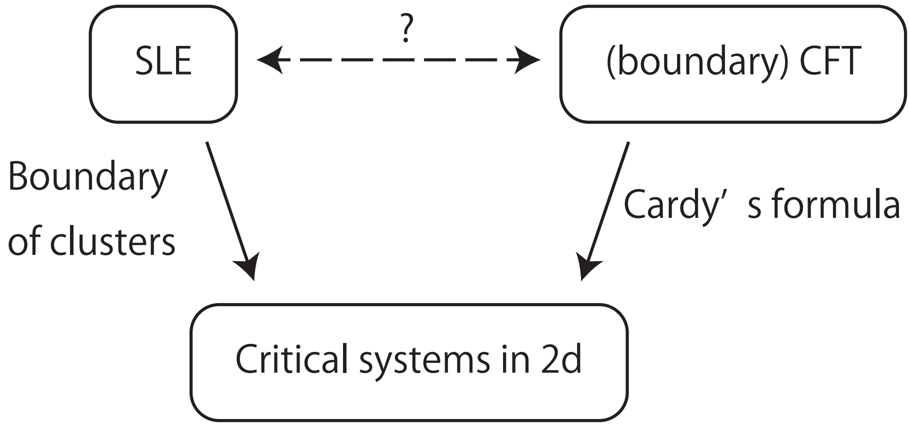

Figure 1.4: Connection between SLE and CFT

Several approaches trying to relate these two notions have been carried under the name of SLE/CFT correspondence. The first significant result was the group theoretical formulation of SLE/CFT correspondence originated by Bauer and Bernard in [BB02, BB03a, BB03b, BB04a], which related SLE and CFT via a random process on an infinite dimensional Lie group. Though we are planning to review the group theoretical formulation of SLE/CFT correspondence for the Virasoro algebra in Chapter 2, we shall make a rough sketch of it here. They focused the fact that a solution $g_t(z)$ of the SLE equation in Eq.(1.2.3) can be regarded as a random transformation of formal coordinates at infinity, thus can be associated with a random process $\mathscr{G}_t$ on the *lower Borel subgroup* of Virasoro group, of which the Lie algebra is the Virasoro algebra, under the initial condition that $\mathscr{G}_0$ is the unit element. We denote this target group by $\mathrm{Aut}_+\mathcal{O}$. Moreover, $\mathscr{G}_t$ acts on a highest weight representation of the Virasoro algebra. In particular, $\mathscr{G}_t\,|h,c\rangle$ for a certain highest weight vector $|h,c\rangle$ of highest weight h and central charge c is a random process on the corresponding highest weight irreducible representation $L(h,c)$, or precisely its formal completion. Bauer and Bernard found that the random process $\mathscr{G}_t\,|h,c\rangle$ in the representation space can be a local martingale, of which the stochastic differential equation does not have a term proportional to dt, for a certain choice of h and c due to presence of a null vector in the the Verma module that covers the irreducible representation $L(h,c)$. Thus if we evaluate the random process $\mathscr{G}_t\,|c,h\rangle$ by a dual vector, we obtain a probably nontrivial local martingale associated with a solution of SLE. We can summarize at least a part of the SLE/CFT correspondence in the sense of Bauer and Bernard by stating that *a computation on a state space of CFT leads to a local martingale associated with SLE.*

Here we have introduced the notion of local martingales as stochastic processes of which increments do not contain terms proportional to dt. A significant property of a local martingale is that its expectation value does not depend on time. In general, it is a difficult task to compute the expectation value of a stochastic process at some time, but for a local martingale, the expectation value at any time can be computed as the expectation value at the initial time. In a certain case, it may be possible to discuss that the defining function of an event is a local martingale, and in such a case, the probability of the event is easily computed as a remarkable application of the property of local martingales.

Although we obtain local martingales that are associated with the solution of SLE from SLE/CFT correspondence in the sense of Bauer and Bernard, it does not mean that we can *solve* the SLE equation. Indeed we do not know the explicit form the probability distribution function for SLE. Nevertheless, we can compute some kinds of probability associated with SLE due to SLE/CFT correspondence, and that is just the importance of SLE/CFT correspondence.

Notice that the group theoretical formulation of SLE/CFT correspondence sketched above consults only a single representation $L(c, h)$ and more CFT-like objects such as conformal blocks involving multiple representations does not appear. This can be also interpreted as that we are treating *a local theory on a point*. A significance of the group theoretical formulation is that in spite of its simplicity from the viewpoint of CFT, it allows us to obtain several local martingales associated with SLE. Since the existence of a null vector is the essence for the mechanism of generating local martingales, one shall consider other null vectors, other representations, and even other algebras in natural generalizations of the group theoretical formulation. Indeed stochastic differential equations have been constructed associated with more general null vectors in highest weight representations of the Virasoro algebra [LR04], ones for the $\mathcal{N} = 1$ superconformal algebra [Ras04b, NR05], and logarithmic representations of the Virasoro algebra [Ras04a, MARR04]. These works are objected to generalize the notion of SLE along the framework of SLE/CFT correspondence by generalizing the CFT side.

1.5 Other approaches to SLE/CFT correspondence

One of the significant properties of an SLE measure is the conformal restriction property [LSW03]. In [FW03], it has been found that on a space of functions of which values give some probability concerning a random process, an action of the Witt algebra is defined only assuming the conformal restriction property of the random process with highest weight h being specified by the parameter characterizing the conformal restriction property. The random process has appeared to associate to the SLE(κ) for some κ if the representation of the Witt algebra is degenerate at the weight $h + 2$.

Friedrich and Kalkkinen observed in [FK04, Fri04] that a correlation function, which is a section of a certain line bundle over the moduli space of Riemann surfaces, acts in the same way as the probability measure with the conformal restriction property under *conditioning*, and expected that a correlation function of a CFT gave some probability concerning SLE. A similar picture, in which an SLE measure is identified as a section of a determinant bundle over the moduli space of Riemann surfaces, has been suggested by Kontsevich in [Kon03].

Recently, Dubédat [Dub15b,Dub15a] also constructed an SLE measure by means of the localization technique, and identified its partition function with a highest weight vector in a highest weight representation of the Virasoro algebra.

1.6 Issues in this thesis

In Sect. 1.4, we have explained outline of SLE/CFT correspondence in the sense of Bauer and Bernard [BB02,BB03a,BB03b,BB04a], which connects SLE to representation theory of the Virasoro algebra. Their formulation of SLE/CFT correspondence allows to rederive Cardy's formula, and appears to be useful in description of cluster of boundaries in statistical mechanics models to which corresponding CFTs are associated with representations of the Virasoro algebra. Such models include loop gas models and Fortuin-Kasteleyn random cluster models described in Sect. 1.3. There are, however, other models that

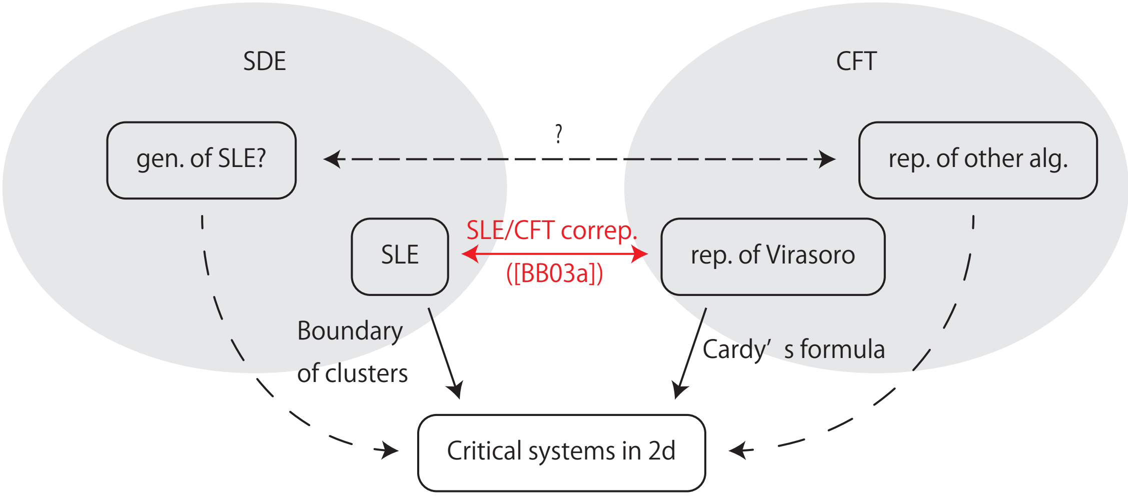

Figure 1.5: Generalization of SLE/CFT correspondence

are described by CFTs associated with other algebras. For example, Affleck and Haldane [AH87] proposed that quantum spin chains are believed to be described by Wess-Zumino-Witten (WZW) theories, which are associated with representations of affine Lie algebras. In statistical mechanics, it is known that several kinds of vertex models [Bax08] and loop modles [dGP04] are equivalent to spin chains, thus these models are expected to be described by WZW theories. Our aim in this thesis to generalize SLE/CFT correspondence to connection between stochastic differential equations and representations of other algebras than the Virasoro algebra (Fig. 1.5).

In the last of Sect. 1.4, we saw several directions of generalization of the notion of SLE. In this thesis, we propose a systematic manner to generalize the notion of SLE along this line. The fundamental strategy is to replace the target group of the random process $\mathscr{G}_t$ by a larger group that describes internal symmetry as well as the space-time symmetry. We can find such a group of internal symmetry as a group consisting of exponentiated elements of a (completed) current Lie algebra associated with a vertex algebra, which is sketched below. Given a vertex algebra V, we have a state-field correspondence map that sends a vector $A \in V$ to an $\mathrm{End}(V)$-valued formal power series $Y(A,z) = \sum_{n\in\mathbb{Z}} A_{(n)} z^{-n-1}$. Then operators of the form $A_{(n)}$ that arise as coefficients of fields satisfy commutation relation

$$[A_{(m)}, B_{(n)}] = \sum_{k=0}^{\infty} \binom{m}{k} (A_{(k)}B)_{(m+n-k)} \tag{1.6.1}$$

for $A, B \in V$ and $m, n \in \mathbb{Z}$. The current Lie algebra $U(V)$ of the vertex algebra V is roughly defined as a Lie algebra consisting of coefficients of fields that correspond to vectors in V with relations being defined by Eq.(1.6.1). If the vertex algebra V is $\mathbb{Z}$-graded, the current Lie algebra $U(V)$ is also $\mathbb{Z}$-graded. Note that even if the current Lie algebra is $\mathbb{Z}$-graded, each homogeneous space is not finite-dimensional.

If the vertex algebra V is conformal, its current Lie algebra $U(V)$ has the Virasoro algebra as a Lie subalgeba, since the Virasoro generators appear in coefficients of the Virasoro field. As we will see in Chapter 2, a certain subalgebra that is "half" of the Virasoro algebra denoted by $\mathrm{Der}_+\mathcal{O}$ plays a crucial role in the theory of SLE. Indeed, an infinite dimensional Lie group $\mathrm{Aut}_+\mathcal{O}$ is the Lie group corresponding to $\mathrm{Der}_+\mathcal{O}$. This Lie algebra consists of infinitesimal transformations of a formal coordinate on the formal disc, thus it concerns space-time symmetry, but symmetry that is encoded in a single vertex algebra is described by a greater Lie algebra $U(V)$.

We call a Lie subalgebra of $U(V)$ that is normalized by $\mathrm{Der}_+\mathcal{O}$ an internal symmetry following the terminology in [FBZ04]. Assuming exponentiation of an element in an internal symmetry makes sense to generate a Lie group $\mathcal{G}$, we have a semi-direct product group $\mathrm{Aut}_+\mathcal{O}\ltimes\mathcal{G}$, which consists of compatible transformations of space-time and internal degrees of freedom. This group becomes the target group of a random process $\mathscr{G}_t$ under our consideration.

The current Lie algebra itself is of course an example of such an internal symmetry, but it is too large to handle. A convenient internal symmetry comes from the loop algebra of a finite dimensional Lie algebra. For instance, the current Lie algebra of the Heisenberg vertex algebra contains the Heisenberg algebra as a Lie subalgebra. In Chapter 3, we take the half of the Heisenberg algebra as an internal symmetry, which is exponentiated to generate the positive loop space of a torus. Another example is for an affine vertex algebra. A current Lie algebra of an affine vertex algebra contains the corresponding affine Lie algebra as a Lie subalgebra. We take the half of the affine Lie algebra as an internal symmetry in Chapter 4. In this case its exponentiation gives the positive loop group of a simple Lie group.

We shall comment that SLEs corresponding to WZW theories have been already considered by authors of [BGLW05, ABI11] following the correlation function formulation of SLE, which is different from the group theoretical formulation. Their conclusion is that if primary fields of a WZW theory evolve according to given stochastic differential equations, their correlation function of a certain form is a local martingale. Their formulation, however, is not satisfactory to us in three points: (1) their stochastic differential equations along internal degrees of freedom seem to be *ad hoc*, (2) random processes along internal degrees of freedom are not constructed in a concrete way and (3) local martingales that are associated with the solution is hard to write down. As we will see in Chap. 4, we have overcome these points in our formulation. Namely in our formulation, (1) stochastic differential equations naturally appear from an annihilator of a highest weight vector, (2) random processes along internal degrees of freedom are concretely constructed in the case of $\widehat{\mathfrak{sl}_2}$ and (3) local martingales that are associated with the solution can be written down in an explicit way.

Organization

Chapter 2 is devoted to a review of what is known for SLE corresponding to the Virasoro algebra. It contains a precise description of the group theoretical formulation of SLE that was sketched in Sect. 1.4. and further developments of the theory of SLE/CFT correspondence, including formulation of multiple SLEs and SLE(κ,ρ), and Virasoro module structure of a space of SLE local martingales.

In Chapter 3, we formulate the notion of SLE that associates to Heisenberg vertex algebras. We take the positive loop algebra of a finite dimensional commutative Lie algebra as an internal symmetry, and construct a random process on an infinite dimensional Lie group. We also compute some local martingales that associate to the solution of the SLE process.

In Chapter 4, we extend the formulation in Chapter 3 to the case of affine vertex algebras. We only consider an affine Lie algebra of a finite dimensional simple Lie algebra. In this case we consider the positive loop algebra of the finite dimensional Lie algebra as an internal symmetry, and construct a random process on an infinite dimensional Lie group. We present it in the most concrete form in the case that the finite dimensional Lie algebra is $\mathfrak{sl}_2$, and write down several local martingales associating to the SLE process. We also investigate the affine Lie algebra symmetry of the space of local martingales.

In Chapter 5, we make discussion on our results and see perspectives that lie beyond our works. In particular, we discuss possibility of constructing examples associated with representation theory of other algebras, including a lattice vertex algebra that does not associates to a root lattice and an affine Lie superalgebra.

In Appendix A, we recall the notion of an Ito process on a Lie group at an intuitive level. In particular, we see the standard form of a stochastic differential equation for such a random process.

Appendix B is devoted to an exposition of the theory of vertex (operator) algebras that is used throughout this thesis. It also contains the definitions of some significant vertex algebras and a current Lie algebra associating to a given vertex algebra.

Appendix C contains some computational details referred in Chap. 2. We can find there derivation of differential equations on an operator that encodes a coordinate transformation on a vertex algebra, which is essentially used in unveiling a Virasoro module structure on a space of SLE local martingales.

In Appendix D, we present computations referred in Chap. 4 including detailed construction of SLE associating to representation theory of $\widehat{\mathfrak{sl}}_2$, derivation of differential equations on an operator that transforms a local coordinate and internal degrees of freedom, and derivation of operators that define an $\widehat{\mathfrak{sl}}_2$-module structure on a space of SLE local martingales.

Appendix E shows some computations for other examples of generalization of SLE. Ww try to construct SLEs that corresponds to a lattice vertex algebra associated with the lattice $L = \mathbb{Z}\alpha$ for $(\alpha|\alpha) = 4$, and an affine Lie superalgebra.

Chapter 2

Virasoro SLE

This chapter is a review part of what is known for connection between the notion of SLE and representation theory of the Virasoro algebra. We make a detailed description of the group theoretical formulation of SLE, and try to explain how computations in the framework of representation theory allows us to obtain local martingales associated with SLE. The group theoretical formulation of SLE then is found to open a natural way to generalize the notion of SLE. We will see an example of a generalization of this line carried by Lesage and Rasmussen in [LR04]. We also review further developments of the theory of SLE/CFT correspondence following the group theoretical formulation, which include the formulation of multiple SLE and SLE(κ, ρ).

2.1 Virasoro algebra and its representations

Let $\mathrm{Diff}_+(S^1)$ be the group of orientation preserving diffeomorphisms on the unit circle S^1. This group has structure of an infinite dimensional Lie group, and its Lie algebra consists of smooth vector fields on S^1 and is denoted by $\mathrm{Vect}(S^1)$. The complexification $\mathbb{C} \otimes \mathrm{Vect}(S^1)$ contains a significant Lie subalgebra $\mathrm{Witt} = \mathbb{C}[z, z^{-1}]\partial_z$, which is called the Witt algebra. Here we have realized the unit circle so that $S^1 = \{e^{i\theta} \in \mathbb{C} | 0 \leq \theta < 2\pi\}$ and introduced the variable z as $z = e^{i\theta}$. The Witt algebra has a basis $\{\ell_n\}_{n \in \mathbb{Z}}$ with $\ell_n = -z^{n+1}\partial_z$, and we find that the Lie algebra structure of the Witt algebra is described in this basis as $[\ell_m, \ell_n] = (m-n)\ell_{m+n}$. The second cohomology group $H^2(\mathrm{Witt}, \mathbb{C})$ of the Witt algebra is one dimensional spanned by the Gel'fand-Fuchs 2-cocycle defined by $(f(z)\partial_z, g(z)\partial_z) \mapsto \frac{1}{12}\mathrm{Res}_z f'''(z)g(z)$. Thus the Witt algebra admits the universal central extention by a one dimensional center, $0 \to \mathbb{C}C \to \mathrm{Vir} \to \mathrm{Witt} \to 0$, which is called the Virasoro algebra. We denote the preimage of ℓ_n in the Witt algebra by L_n, then the commutation relation among them is described as

$$[L_m, L_n] = (m-n)L_{m+n} + \frac{m^3 - m}{12}\delta_{m+n,0}C. \tag{2.1.1}$$

We only consider highest weight representations of the Virasoro algebra that are constructed in the following manner. Let us decompose the Virasoro algebra into subalgebras $\mathrm{Vir} = \mathrm{Vir}_{>0} \oplus \mathrm{Vir}_0 \oplus \mathrm{Vir}_{<0}$, where $\mathrm{Vir}_0 = \mathbb{C}L_0 \oplus \mathbb{C}C$ and $\mathrm{Vir}_{\gtrless 0} = \bigoplus_{\pm n > 0} \mathbb{C}L_n$. We also set $\mathrm{Vir}_{\geq 0} = \mathrm{Vir}_0 \oplus \mathrm{Vir}_{>0}$. For a pair $(c, h) \in \mathbb{C}^2$, let $\mathbb{C}_{(c,h)} = \mathbb{C}\mathbf{1}_{(c,h)}$ be a one dimensional representation of $\mathrm{Vir}_{\geq 0}$ on which C and L_0 act as multiplication by c and h, respectively. The highest weight Verma module $M(c, h)$ of highest weight (c, h) is defined by induction $M(c, h) = U(\mathrm{Vir}) \otimes_{U(\mathrm{Vir}_{\geq 0})} \mathbb{C}_{(c,h)}$, which is isomorphic to $U(\mathrm{Vir}_{<0}) \otimes \mathbb{C}_{(c,h)}$ as a vector space or a $\mathrm{Vir}_{<0}$-module. The numbers c and h in the highest weight are called the central charge and the conformal weight of the highest weight Verma module $M(c, h)$, respectively. Since we will only treat highest weight representations, we call a highest weight Verma module simply a Verma module. The highest weight vector $1 \otimes \mathbf{1}_{(c,h)}$ is denoted by $|c, h\rangle$. It is clear by construction that a Verma module $M(c, h)$ decomposes into direct sum of eigenspaces of L_0 so that $M(c, h) = \bigoplus_{n \in \mathbb{Z}_{\geq 0}} M(c, h)_{h+n}$, where we have defined $M(c, h)_\lambda = \{v \in M(c, h) | L_0 v = \lambda v\}$ for $\lambda \in \mathbb{C}$.

Let $\omega : \mathrm{Vir} \to \mathrm{Vir}$ be the linear anti-involution defined by $\omega(L_n) = L_{-n}$ and $\omega(C) = C$. Then there is a unique symmetric form $\langle \cdot | \cdot \rangle : M(c, h) \times M(c, h) \to \mathbb{C}$ satisfying $\langle Xu | v \rangle = \langle u | \omega(X) v \rangle$ for $u,\ v \in M(c, h)$ and $X \in \mathrm{Vir}$, and normalized as $\langle c, h | c, h \rangle = 1$. Here we have simply written $\langle |c, h\rangle \,|\, |c, h\rangle \rangle$ as $\langle c, h | c, h \rangle$. A similar use of notation will appear repeatedly in this thesis. The radical $J(c, h) := \mathrm{rad}\, \langle \cdot | \cdot \rangle$ of this symmetric form is shown to be the maximal proper submodule of the Verma module $M(c, h)$, thus the quotient $L(c, h) = M(c, h)/J(c, h)$ is the irreducible representation of the Virasoro algebra of highest weight (c, h). The symmetric bilinear form induces a nondegenerate bilinear form on the irreducible representation $L(c, h)$ that is denoted by the same bracket $\langle \cdot | \cdot \rangle$. For a generic highest weight, $J(c, h) = \{0\}$ and the Verma module $M(c, h)$ is itself irreducible. Highest weights (c, h) for which $J(c, h)$ becomes nontrivial are encoded in Kac's determinant formula [Kac78, KRR13]. We call an element in $J(c, h)$ a null vector.

Among other irreducible modules, that of highest weight $(c, 0)$ denoted by $L(c, 0)$ above has special feature that it carries a structure of a vertex operator algebra (VOA). We simply denote this VOA by L_c and call it the Virasoro VOA of central charge c. Detailed exposition of vertex operator algebra structure on L_c is presented in Appendix B, and we shall sketch the argument here. The vacuum vector is the highest weight vector $|0\rangle = |c, 0\rangle$, and it is generated by a conformal vector $L_{-2} |0\rangle$ that is transferred to the Virasoro field $L(z) = \sum_{n \in \mathbb{Z}} L_n z^{-n-2}$ under the state-field correspondence map. The nondenegrate bilinear form $\langle \cdot | \cdot \rangle$ on L_c is an invariant bilinear form on the VOA, namely, it satisfies

$$\langle Y(a, z) u | v \rangle = \langle u | Y(e^{zL_1} (-z^{-2})^{L_0} a, z^{-1}) v \rangle \tag{2.1.2}$$

for $a, u, v \in L_c$.

It has been shown that the Virasoro VOA L_c is rational if and only if the central charge has the form of $c = c_{p,q} = 1 - 6\frac{(p-q)^2}{pq}$ with (p, q) is a pair of coprime integers larger than 2. The minimal choice of (p, q) seems to be $(2, 3)$, but in this case the central charge becomes zero and it is easily shown that $L_{c=0} = \mathbb{C} |0\rangle$ is one-dimensional. Thus we often assume that (p, q) is not chosen as $(2, 3)$. The irreducible modules of $L_{c_{p,q}}$ are exhausted by $L(c_{p,q}, h_{p,q;r,s})$ with $0 < r < p$ and $0 < s < q$, where $h_{p,q;r,s}$ is defined by $h_{p,q;r,s} = \frac{(sp-rq)^2 - (p-q)^2}{4pq}$. In case that L_c is not rational, irreducible representations $L(c, h)$ of the Virasoro algebra are all of irreducible modules of L_c. Note that in this case, it is not sufficient to list all irreducible modules for understanding representation theory of the Virasoro VOA.

2.2 Conformal transformation

Let $\mathcal{O} = \mathbb{C}[[w]] = \varprojlim \mathbb{C}[w]/(w^n)$ be a complete topological $\mathbb{C}$-algebra and $D = \mathrm{Spec}\mathcal{O}$ be the formal disc. A continuous automorphism ρ of $\mathcal{O}$ is identified with the image of the topological generator w of $\mathcal{O}$ by the same automorphism ρ. Under this identification, the group $\mathrm{Aut}\mathcal{O}$ of continuous automorpshims of $\mathcal{O}$ is realized as

$$\mathrm{Aut}\mathcal{O} \simeq \{a_1 w + a_2 w^2 + \cdots | a_1 \in \mathbb{C}^\times,\ a_i \in \mathbb{C},\ i \geq 2\}. \tag{2.2.1}$$

Indeed, a nonzero constant term is prohibited to preserve the algebra $\mathcal{O}$, and $a_1 \neq 0$ is required for existence of inverse. The group law is defined by $(\rho * \mu)(w) = \mu(\rho(w))$ for $\rho,\ \mu \in \mathrm{Aut}\mathcal{O}$. The purpose of this section is to define representations of this group on a vertex operator algebra and its modules that are significant in application to the theory of SLE.

It is shown that the Lie algebra of $\mathrm{Aut}\mathcal{O}$ is that of vector fields $\mathrm{Der}_0\mathcal{O} = w\mathbb{C}[[w]]\partial_w$. The same Lie algebra is also constructed as a completion of a Lie subalgebra $\bigoplus_{n=0}^{\infty} \mathbb{C}L_n$ of the Virasoro algebra, which is isomorphic to the subalgebra $\bigoplus_{n=0}^{\infty} \mathbb{C}\ell_n = w\mathbb{C}[w]\partial_w$ of the Witt algebra. Since a subalgebra $w^n\mathbb{C}[w]\partial_w$ in $w\mathbb{C}[w]\partial_w$ is an ideal, the quotient $(w\mathbb{C}[w]\partial_w)/(w^n\mathbb{C}[w]\partial_w) \simeq w\mathbb{C}[w]/(w^n)\partial_w$ carries a Lie algebra structure, and moreover, we have a family of projections $w\mathbb{C}[w]/(w^m)\partial_w \to w\mathbb{C}[w]/(w^n)\partial_w$ for $m > n$. The projective limit $\varprojlim w\mathbb{C}[w]/(w^n)\partial_w$ of this projective system of Lie algebras is nothing but the desired Lie algebra $\mathrm{Der}_0\mathcal{O}$. Since for an arbitrary vector v in a vertex operator algebra V or its module M, we have $L_n v = 0$ for $n \gg 0$, we have a well-defined action of $\mathrm{Der}_0\mathcal{O}$ on V and M.

There is a significant subgroup $\mathrm{Aut}_+\mathcal{O}$ of $\mathrm{Aut}\mathcal{O}$ that is described as $\mathrm{Aut}_+\mathcal{O} \simeq \{w + a_2 w^2 + \cdots | a_i \in \mathbb{C},\ i \geq 2\}$. It is shown that the Lie algebra of this subgroup is $\mathrm{Der}_+\mathcal{O} = w^2\mathbb{C}[[w]]\partial_w$ that is a Lie subalgebra of $\mathrm{Der}_0\mathcal{O}$.

We shall exponentiate the action of the Lie algebra $\mathrm{Der}_0\mathcal{O}$ to the action of the Lie group $\mathrm{Aut}\mathcal{O}$. This is possible if L_n for $n > 1$ act locally nilpotently and L_0 is diagonalizable with integer eigenvalues, former of which is automatically holds for a highest weight representation, and latter of which is true if the conformal weight in the highest weight is integer. On such a highest weight representation of the Virasoro algebra, we construct the linear operator $R(\rho)$ for $\rho \in \mathrm{Aut}\mathcal{O}$ that defines a representation of $\mathrm{Aut}\mathcal{O}$. For an automorphism $\rho \in \mathrm{Aut}\mathcal{O}$, we uniquely find v_i, $i \geq 0$, such that

$$\rho(w) = \exp\left(\sum_{i>0} v_i w^{i+1}\partial_w\right) v_0^{w\partial_w} \cdot w. \tag{2.2.2}$$

Here the exponentiation of the Euler vector field is just defined by $v_0^{w\partial_w} \cdot w = v_0$. The above expression of ρ is nothing but specification of its action on $\mathcal{K} = \mathbb{C}((w))$ defined by $(\rho.F)(w) = f(\rho(w))$ for $F(w) \in \mathcal{K}$, where the group law of invertible operators on $\mathcal{K}$ is defined by composition. The first few of v_i for a given ρ are computed by comparing coefficients of each powers of w so that

$$v_0 = \rho'(0), \qquad v_1 = \frac{1}{2}\frac{\rho''(0)}{\rho'(0)}, \qquad v_2 = \frac{1}{6}\frac{\rho'''(0)}{\rho'(0)} - \frac{1}{4}\left(\frac{\rho''(0)}{\rho'(0)}\right)^2, \qquad \cdots.$$

Let V a vertex operator algebra. Then for an automorphism $\rho \in \mathrm{Aut}\mathcal{O}$, the following operator is well-defined in $\mathrm{End}(V)$

$$R(\rho) = \exp\left(-\sum_{i>0} v_i L_i\right) v_0^{-L_0}, \tag{2.2.3}$$

and satisfies $R(\rho)R(\mu) = R(\rho * \mu)$.

We investigate the behavior of a field $Y(A, z)$ on a vertex operator algebra V under the adjoint action by $R(\rho)$. Let $L(z) = \sum_{n\in\mathbb{Z}} L_n z^{-n-2}$ be the Virasoro field, then we have

$$[L(z), Y(A, w)] = \sum_{m\geq -1} Y(L_m A, w)\partial_w^{(m+1)}\delta(z - w), \tag{2.2.4}$$

which implies

$$[L_n, Y(A,w)] = \sum_{m\geq -1} \binom{n+1}{m+1} Y(L_m A, w) w^{n-m}. \tag{2.2.5}$$

For $\mathbf{v} = -\sum_{n\in\mathbb{Z}} v_n L_n$ such that $v_n = 0$ for $n \ll 0$, we have

$$[\mathbf{v}, Y(A,w)] = -\sum_{m\geq -1} \left(\partial_w^{(m+1)} v(w)\right) Y(L_m A, w), \tag{2.2.6}$$

where $v(w) = \sum_{n\in\mathbb{Z}} v_n w^{n+1}$.

Proposition 2.2.1. *For $A \in V$ and $\rho \in \mathrm{Aut}\mathcal{O}$, we have*

$$Y(A,w) = R(\rho) Y(R(\rho_w)^{-1} A, \rho(w)) R(\rho)^{-1}. \tag{2.2.7}$$

Here $\rho_w(t) = \rho(w+t) - \rho(w)$.

Proof. We denote by $\mathrm{Fie}(V)$ the space of fields on V. The state field correspondence map $Y(-,w)$ is regarded as an element in $\mathrm{Hom}(V, \mathrm{Fie}(V))$. For an automorphism $\rho \in \mathrm{Aut}\mathcal{O}$, we define an endomorphism T_ρ on $\mathrm{Hom}(V, \mathrm{Fie}(V))$ by

$$(T_\rho \cdot X)(A,w) := R(\rho) X(R(\rho_w)^{-1} A, \rho(w)) R(\rho)^{-1} \tag{2.2.8}$$

for $X \in \mathrm{Hom}(V, \mathrm{Fie}(V))$ and $A \in V$. Then this assignment $\rho \mapsto T_\rho$ is a group homomorphism. Indeed, we have

$$\begin{aligned}
&(T_\rho \cdot (T_\mu \cdot X))(A,w) \\
&= R(\rho)(T_\mu \cdot X)(R(\rho_w)^{-1} A, \rho(w)) R(\rho)^{-1} \\
&= R(\rho) R(\mu) X(R(\mu_{\rho(w)})^{-1} R(\rho_w)^{-1} A, \mu(\rho(w))) R(\mu)^{-1} R(\rho)^{-1}.
\end{aligned}$$

Notice that

$$\begin{aligned}
(\rho_w * \mu_{\rho(w)})(t) = \mu_{\rho(w)}(\rho_w(t)) &= \mu(\rho(w) + \rho_w(t)) - \mu(\rho(w)) \\
&= \mu(\rho(w) + \rho(w+t) - \rho(w)) - \mu(\rho(w)) \\
&= (\rho * \mu)_w(t)
\end{aligned}$$

to obtain

$$(T_\rho \cdot (T_\mu \cdot X))(A,w) = (T_{\rho*\mu} \cdot X)(A,w). \tag{2.2.9}$$

Since the exponential map $\mathrm{Der}_0\mathcal{O} \to \mathrm{Aut}\mathcal{O}$ is surjective, we can assume ρ to be infinitesimal. For an infinitesimal transformation $\rho(w) = w + \epsilon v(w) + o(\epsilon)$ with $v(w) = \sum_{n\geq 0} v_n w^{n+1}$, we have

$$R(\rho) = \mathrm{Id} + \epsilon \mathbf{v} + o(\epsilon), \tag{2.2.10}$$

where $\mathbf{v} = -\sum_{n\geq 0} v_n L_n$. The associated transformation $\rho_w(t)$ is approximated upto linear order of ϵ by

$$\begin{aligned}
\rho_w(t) = \rho(w+t) - \rho(w) &= w + t + \epsilon v(w+t) - w - \epsilon v(w) + o(\epsilon) \\
&= t + \epsilon \sum_{m\geq 0} \partial^{(m+1)} v(w) t^{m+1} + o(\epsilon).
\end{aligned}$$

Thus $R(\rho_w)^{-1}$ becomes

$$R(\rho_w)^{-1} = \mathrm{Id} + \epsilon \sum_{n\geq 0} \partial^{(n+1)} v(w) L_n + o(\epsilon). \tag{2.2.11}$$

We now show that the state-field correspondence map $Y(-,w)$ is fixed under the action of T_ρ up to linear order of ϵ.

$$
\begin{aligned}
&(T_\rho \cdot Y)(A,w)\\
&= (\mathrm{Id} + \epsilon \mathbf{v}) Y \left(\left(\mathrm{Id} + \epsilon \sum_{n\geq 0} \partial^{(n+1)} v(w) L_n \right) A, w + \epsilon + v(w) \right) (\mathrm{Id} - \epsilon \mathbf{v})\\
&= Y(A,w) + \epsilon \left([\mathbf{v}, Y(A,w)] + v(w)\partial Y(A,w) + \sum_{n\geq 0} \partial^{(n+1)} v(w) Y(L_n A, w) \right)\\
&= Y(A,w).
\end{aligned}
$$

□

Corollary 2.2.2. *Let $A \in V$ be a primary vector of conformal weight h, i.e., it satisfies $L_n A = 0$ for $n > 0$ and $L_0 A = hA$ For an automorphism $\rho \in \mathrm{Aut}\mathcal{O}$, we have*

$$
Y(A,w) = R(\rho) Y(A, \rho(w)) R(\rho)^{-1} (\rho'(w))^h. \tag{2.2.12}
$$

Proof. For a primary vector A of conformal weight h, the one dimensional space $\mathbb{C}A$ is preserved by the operator $R(\rho_w)$, where the presentation of $R(\rho_w)$ is given by

$$
R(\rho_w) = \exp\left(-\sum_{j>0} v_j(w) L_j \right) v_0(w)^{-L_0} \tag{2.2.13}
$$

with $v_j(w)$ being chosen so that

$$
\rho_w(t) = \exp\left(\sum_{j>0} v_j(w) t^{j+1} \partial_t \right) v_0(w)^{t\partial_t} \cdot t. \tag{2.2.14}
$$

Since A is primary, the nontrivial effect comes from the action by L_0, thus we have $R(\rho_w)A = v_0(w)^{-h}A$, where $v_0(w)$ is computed as $v_0(w) = \partial_t \rho_w(t=0) = \rho'(w)$, which implies that $R(\rho_w)^{-1}A = (\rho'(w))^h A$. □

One of important fields that are not primary is the Virasoro field $L(w) = Y(L_{-2}\left|0\right\rangle, w)$, which transforms as follows.

Proposition 2.2.3. *Let $L(w)$ be the Virasoro field. We have*

$$
L(w) = R(\rho) L(\rho(w)) R(\rho)^{-1} (\rho'(w))^2 + \frac{c}{12}(S\rho)(w). \tag{2.2.15}
$$

Here $c \in \mathbb{C}$ is the central charge and $(S\rho)(w)$ is the Schwarzian derivative defined by

$$
(S\rho)(w) = \frac{\rho'''(w)}{\rho'(w)} - \frac{3}{2}\left(\frac{\rho''(w)}{\rho'(w)} \right)^2. \tag{2.2.16}
$$

Proof. It is clear that the space $\mathbb{C}L_{-2}\left|0\right\rangle \oplus \mathbb{C}\left|0\right\rangle$ is preserved by the operator $R(\rho_w)$, thus we first compute the inverse $R(\rho_w)^{-1}$ on this space. Let $v_j(w) \in \mathbb{C}[[w]]$ be chosen so that

$$
\rho_w(t) = \exp\left(\sum_{j>0} v_j(w) t^{j+1} \partial_t \right) v_0(w)^{t\partial_t} \cdot t, \tag{2.2.17}
$$

then $R(\rho_w)$ is expressed as

$$R(\rho_w) = \exp\left(-\sum_{j>0} v_j(w) L_j\right) v_0(w)^{-L_0}. \tag{2.2.18}$$

The matrix form of this operator on $\mathbb{C}L_{-2}|0\rangle \oplus \mathbb{C}|0\rangle$ is expressed in this basis

$$R(\rho_z) = \begin{pmatrix} v_0(w)^{-2} & 0 \\ -\frac{c}{2} v_0(w)^{-2} v_2(w) & 1 \end{pmatrix}, \tag{2.2.19}$$

and its inverse is

$$R(\rho_w)^{-1} = \begin{pmatrix} v_0(w)^2 & 0 \\ \frac{c}{2} v_2(w) & 1 \end{pmatrix} = \begin{pmatrix} (\rho'(w))^2 & 0 \\ \frac{c}{12}(S\rho)(w) & 1 \end{pmatrix}, \tag{2.2.20}$$

which implies the desired result. □

In application to the theory of SLE, we regard the formal disc introduced here as the formal neighborhood at the infinity, and have to reformulate whole ingredients so to be associated with the coordinate $z = \frac{1}{w}$ at 0. While an automorphism ρ sends w to $\rho(w) = a_1 w + a_2 w^2 + \cdots$, the same automorphism sends z to $1/\rho(1/z)$. If we expand the image in $z\mathbb{C}[[z^{-1}]]$, we can also identify the group Aut$\mathcal{O}$ with

$$\mathrm{Aut}\mathcal{O} \simeq \{b_1 z + b_0 + b_{-1} z^{-1} + \cdots | b_1 \in \mathbb{C}^\times,\ b_i \in \mathbb{C},\ i \leq 0\} \tag{2.2.21}$$

The infinite series in $z\mathbb{C}[[z^{-1}]]$ that is identified with an automorphism ρ will be denoted by $\rho(z)$. In the following, we regard formal variables z and w as formal coordinate at 0 and the infinity, respectively, and $\rho(z)$ and $\rho(w)$ as infinite series identified with an automorphism ρ via identification Eq.(2.2.1) and Eq.(2.2.21), respectively.

Under realization Eq.(2.2.21) of the group Aut$\mathcal{O}$, its subgroup $\mathrm{Aut}_+\mathcal{O}$ consists of formal series $z + b_0 + b_{-1}z^{-1} + \cdots$ with $b_i \in \mathbb{C}$ for $i \leq 0$, and Lie algebras are realized as $\mathrm{Der}_+\mathcal{O} = \mathbb{C}[[z^{-1}]]\partial_z$ and $\mathrm{Der}_0\mathcal{O} = z\mathbb{C}[[z^{-1}]]\partial_z$.

Although a vertex operator algebra V is a representation of the Lie algebra $\mathrm{Der}_0\mathcal{O}$ by assigning $-z^{n+1}\partial_z \to L_n$ for $n \leq 0$. this representation cannot be exponentiated to give a representation of Aut$\mathcal{O}$ in an algebraically closed way. Nevertheless, we can define well-defied operators that represent the group Aut$\mathcal{O}$ on the completion of the vector space. Let $V = \bigoplus_{n\in\mathbb{Z}} V_n$ be the $\mathbb{Z}$-gradation of the vertex operator algebra V. Then we define its formal completion by $\overline{V} = \prod_{n\in\mathbb{Z}} V_n$. Recall that $V_n = 0$ for sufficiently small n.

For a given $\rho \in \mathrm{Aut}\mathcal{O}$, we can uniquely find numbers v_i ($i \leq 0$) that satisfy

$$\exp\left(\sum_{j<0} v_j z^{j+1}\partial_z\right) v_0^{z\partial_z} \cdot z = \rho(z). \tag{2.2.22}$$

Then the operator $G(\rho)$ defined by

$$G(\rho) = \exp\left(-\sum_{j<0} v_j L_j\right) v_0^{-L_0} \tag{2.2.23}$$

is a well-defined one on $\overline{V}$ and define a representation of Aut$\mathcal{O}$. Indeed, the part $v_0^{-L_0}$ behaves as multiplication by v_0^{-n} when restricted on V_n , and L_j with $j < 0$ strictly raises the degree, while the $\mathbb{Z}$-gradation on V is bounded from below.

We investigate the covariance property of a field $Y(A,z)$ under the adjoint action by $G(\rho)$. For $v(z)=\sum_{n\in\mathbb{Z}} v_n z^{n+1} \in \mathbb{C}((z^{-1}))$, we have

$$[\mathbf{v}, Y(A,z)] = \sum_{m\geq -1} \partial^{(m+1)} v(z) Y(L_m A, z), \tag{2.2.24}$$

with $\mathbf{v} = -\sum_{n\in\mathbb{Z}} v_n L_n$, but here the both sides belong to $\mathrm{End}(\overline{V})[z,z^{-1}]$.

Proposition 2.2.4. *For $A\in V$ and $\rho \in \mathrm{Aut}\mathcal{O}$, we have*

$$Y(A,z) = G(\rho) Y(R(\rho_z)^{-1}A, \rho(z)) G(\rho)^{-1}. \tag{2.2.25}$$

On a V-module on which eigenvalues of L_0 are not integers, the whole group $\mathrm{Aut}\mathcal{O}$ cannot act, while its subgroup $\mathrm{Aut}_+\mathcal{O}$ can act. In application to SLE, this subgroup is sufficient since a solution of the SLE equation is always normalized so that its expansion around the infinity begins from z.

For a certain operator T on a VOA V, we tentatively define its adjoint operator T^* by the character that $\langle Tu|v\rangle = \langle u|T^* v\rangle$ for $u,v\in V$. In this terminology, the operator $G(\rho)$ defined above is nothing but the inverse of the adjoint operator of $R(\rho)$, although $G(\rho)$ is not an operator on a VOA but on its formal completion.

2.3 Appearance of SLE equation

One of ways to find relationship between SLE and representation theory of the Virasoro algebra is the *group theoretical* formulation of SLE/CFT correspondence, which we recall in this section. A fundamental object is a random process ρ_t on the infinite dimensional Lie group $\mathrm{Aut}_+\mathcal{O}$. A random process on a Lie group induces one on the space of operators on a representation space. Let us take $(\gamma, \mathcal{K}=\mathbb{C}((z^{-1})))$ as a representation of $\mathrm{Aut}_+(\mathcal{O})$ defined by $(\gamma(\rho)F)(z) = F(\rho(z))$. Following description of a random process on a Lie group presented in Appendix A, we can assume that the induced random process on $\mathrm{Aut}\mathcal{K}$ satisfies the stochastic differential equation

$$\gamma(\rho_t)^{-1} d\gamma(\rho_t) = \left(2z^{-1}\partial_z + \frac{\kappa}{2}\partial_z^2\right) dt - \partial_z dB_t \tag{2.3.1}$$

under the initial condition $\gamma(\rho_0)=\mathrm{Id}$. Here B_t is the Brownian motion of covariance κ. Then we observe that $\gamma(\rho_t)z = \rho_t(z)$ satisfies the stochastic differential equation

$$d\rho_t(z) = \frac{2}{\rho_t(z)} dt - dB_t \tag{2.3.2}$$

under the initial condition $\rho_0(z)=z$. If we introduce $g_t(z)=\rho_t(z)+B_t$, we find that $g_t(z)$ satisfies the stochastic differential equation

$$\frac{d}{dt} g_t(z) = \frac{2}{g_t(z)-B_t}. \tag{2.3.3}$$

Moreover, since $B_0=0$, we have $g_0(z)=z$. Thus $g_t(z)$ is nothing but a solution of the original SLE.

We have just derived the SLE equation from a random process on the Lie group $\mathrm{Aut}_+\mathcal{O}$. This manner of formulation enables to obtain several local martingales associated with the solution of the SLE equation. Let us consider the object $G(\rho_t)\left|c,h\right\rangle$, which is regarded as a random process on $\overline{L(c,h)}$, of which increment is

$$d(G(\rho_t)\left|c,h\right\rangle) = G(\rho_t)\left(\left(-2L_{-2}+\frac{\kappa}{2}L_{-1}^2\right)\left|c,h\right\rangle dt + L_{-1}\left|c,h\right\rangle dB_t\right). \tag{2.3.4}$$

Thus if the vector $\chi = \left(-2L_{-2}+\frac{\kappa}{2}L_{-1}^2\right)\left|c,h\right\rangle$ is a null vector in the Verma module $M(c,h)$, the random process $G(\rho_t)\left|c,h\right\rangle$ is a local martingale. Notice that χ is a null vector if and only it is a singular vector, conditions for which is that we have $c = 1-\frac{3(\kappa-4)^2}{2\kappa}$ and $h = \frac{6-\kappa}{2\kappa}$. Thus for such choice of (c,h), the random process $G(\rho)\left|c,h\right\rangle$ in $\overline{L(c,h)}$ is a local martingale, and produces several local martingales associated with the solution $\rho_t(z)$ of the SLE equation. An example is given by $\langle c,h|L(z)G(\rho_t)|c,h\rangle$, where $L(z)$ is the Virasoro field on $L(c,h)$. From Prop. 2.2.4, we see that the Virasoro field behaves as

$$L(z) = G(\rho)L(\rho(z))G(\rho)^{-1}(\rho'(z))^2 + \frac{c}{12}(S\rho)(z) \tag{2.3.5}$$

under transformation by an automorphism $\rho \in \mathrm{Aut}_+\mathcal{O}$. Since the dual of the highest weight vector $\langle c,h|$ is invariant under the right action by $G(\rho)$, we find that

$$\langle c,h|L(z)G(\rho)|c,h\rangle = h\left(\frac{\rho_t'(z)}{\rho_t(z)}\right)^2 + \frac{c}{12}(S\rho_t)(z) \tag{2.3.6}$$

is a local martingale. We can show that such a quantity is indeed a local martingale by a standard Ito calculus, but the group theoretical formulation of SLE/CFT correspondence in this section further clarifies its representation theoretical origin.

Since the solution g_t of the original SLE is also described as $g_t(z) = (\rho_t * (z+B_t))(z)$, the operator $G(g_t)$ corresponding to g_t is written as $G(g_t) = G(f_t)e^{-B_tL_{-1}}$. Let $\mathcal{Y}(-,z)$ be an intertwining operator of type $\binom{L(c,h)}{L(c,h)\ \ L_c}$, then $\mathcal{Y}(\left|c,h\right\rangle, z)$ is a primary field, which is applied to the vacuum vector $\left|0\right\rangle$ to yields $\mathcal{Y}(\left|c,h\right\rangle, z)\left|0\right\rangle = e^{zL_{-1}}\left|c,h\right\rangle$. If we are allowed to substitute the Brownian motion B_t in the formal variable z, and further apply the operator, we have

$$G(g_t)\mathcal{Y}(\left|c,h\right\rangle, B_t)\left|0\right\rangle = G(\rho_t)\left|c,h\right\rangle, \tag{2.3.7}$$

which is a local martingale for a certain choice of (c,h). The left hand side will turn out to be a convenient form of the same local martingale in revealing a Virasoro module structure on a space of SLE local martingales.

We also comment that the group theoretical formulation of a radial SLE is also established in [BB04a].

2.4 Further developments

2.4.1 Correlation function formulation and multiple SLEs

The authors of [BBK05] suggested another formulation of SLE/CFT correspondence, which we call the correlation function formulation, than the group theoretical formulation, which leaded to a generalization of SLE to a "multiple" one. The work in this formulation is to investigate when the object

$$\mathcal{M}_t = \frac{\langle \prod_\alpha \varphi_{\delta_\alpha}(g_t(y_\alpha)) \prod_{i=1}^N \psi_i(X_t^{(i)})\rangle}{\langle \prod_{i=1}^N \psi(X_t^{(i)})\rangle} \prod_\alpha (g_t'(y_\alpha))^{\delta_\alpha} \tag{2.4.1}$$

becomes a local martingale. We shall explain ingredients of this object. The bracket $\langle \cdots \rangle$ is the expectation value in boundary CFT on the upper half plane $\mathbb{H}$. The field ψ_i are boundary condition changing operators and φ_δ are boundary primary fields of conformal weight δ. The points y_α are distinct tips of SLE slits in $\mathbb{H}$, and conformal map g_t satisfies the stochastic differential equation

$$dg_t(z) = \sum_{i=1}^{N} \frac{2dq_i}{g_t(z) - X_t^{(i)}}, \tag{2.4.2}$$

where $X_t^{(i)}$ satisfies $dX_t^{(i)} = dM_t^{(i)} + F_t^{(i)}$ with a local martingale $M_t^{(i)}$ with quadratic variation κq_i and a deterministic drift term $F_t^{(i)}$. The stochastic differential equation in Eq.(2.4.2) is regarded as a generalization of SLE, called multiple SLE.

Let $Z(x_1, \cdots, x_N) = \langle \prod_{i=1}^{N} \psi_i(x_i) \rangle$ be the partition function. Then the correlation function $\mathcal{M}_t$ is a local martingale if and only if the drift terms $F_t^{(i)}$ are given by

$$F_t^{(i)} = \kappa dq_i (\partial_{x_i} \log Z)(X_t^{(1)}, \cdots, X_t^{(N)}) + \sum_{j \neq i} \frac{2dq_j}{X_t^{(i)} - X_t^{(j)}}. \tag{2.4.3}$$

The correlation function formulation reduces to the group theoretical formulation in the case of $N = 1$. Indeed, if $N = 1$, the partition function becomes a constant, and a boundary condition changing operator ψ is realized by an intertwining operator $\mathcal{Y}(-, z)$ of type $\binom{L(c,h)}{L(c,h) \;\; L_c}$. Thus the object $\mathcal{M}_t$ is proportional to

$$\langle u | \prod_{\alpha} \varphi_\alpha(y_\alpha) G(g_t) \mathcal{Y}(|c, h\rangle, X_t) |0\rangle \tag{2.4.4}$$

for a certain vector u, which is shown to be a local martingale for a proper choice of (c, h) in the group theoretical formulation. In comparison to a multiple SLE for N being larger than 2, the case of $N = 1$ is often called the *single* SLE.

2.4.2 Conformal field theory of SLE(κ, ρ)

The correlation function formulation also enables to find relation between conformal field theory and the notion of SLE(κ, ρ), which was introduced in [LSW03, Dub05] as a generalization of SLE. An SLE(κ, ρ) is specified by data $\kappa \geq 0$, $\rho = (\rho_1, \cdots, \rho_n) \in \mathbb{R}^n$ and boundary points $x_1, \cdots, x_n \in \mathbb{R} = \partial\mathbb{H}$. The defining equation is

$$\partial g_t(z) = \frac{2}{g_t(z) - X_t}, \quad g_0(z) = z, \tag{2.4.5}$$

where X_t (is not just a Brownian motion but) satisfies

$$dX_t = dB_t + \sum_{j=1}^{n} \frac{\rho_j}{X_t - g_t(x_j)} dt \tag{2.4.6}$$

with B_t being a Brownian motion of covariance κ.

Associating to SLE(κ, ρ), Kytölä [Kyt06] have found that the following object in CFT becomes a local martingale:

$$\frac{\left\langle \prod_{i=1}^{m} \varphi_{h_i}(g_t(y_i)) \varphi_{\delta_\infty}(\infty) \prod_{j=1}^{n} \varphi_{\delta_j}(g_t(x_j)) \varphi_{\delta_0}(X_t) \right\rangle}{\left\langle \varphi_{\delta_\infty}(\infty) \prod_{j=1}^{n} \varphi_{\delta_j}(g_t(x_j)) \varphi_{\delta_0}(X_t) \right\rangle} \prod_{i=1}^{m} (g_t'(y_i))^{h_i} \tag{2.4.7}$$

for a certain choice of primary fields φ, where $y_i \in \mathbb{R} = \partial\mathbb{H}$.

We also comment that Schramm and Wilson [SW05] have shown that radial SLE(κ) and chordal SLE(κ,ρ) for $\rho = \kappa - 6$ are mapped to each other under a Möbius coordinate transformation.

2.4.3 Virasoro symmetry of SLE local martingales

As we have already seen, the vector valued random process $\mathscr{G}_t\,|c,h\rangle$ in $\overline{L(c,h)}$ is a local martingale, and thus we can obtain a local martingale of the form $\langle u|\mathscr{G}_t|c,h\rangle$ by taking the inner product of it with a vector $u \in L(c,h)$. Since a vector u is arbitrarily taken from a representation space of the Virasoro algebra, on the space of local martingales obtained as *values* of $\mathscr{G}_t\,|c,h\rangle$ the Virasoro algebra acts. Such a representation of the Virasoro algebra has been reported in [BB03b, BB04b] and made clear by Kytölä in [Kyt07].

Although Kytölä has clarified a Virasoro module structure of the space of local martingales for wide variants of SLE including multiple SLE and SLE(κ,ρ), we review his result in the simplest example of the single SLE. When we expand the solution $g_t(z)$ of the SLE equation in Eq.(2.3.3) as $g_t(z) = z + \sum_{n\le 0} g_n(t) z^n$, we have infinitely many $\mathbb{C}$-valued random processes $g_n(t)$, $n \le 0$, associated to $g_t(z)$. Let $\mathcal{F} = C^\infty(\mathbb{R},\mathbb{C})[f_0, f_{-1}, \cdots]$ be the space of polynomials in infinitely many variables with coefficients in differentiable functions. We shall seek an element $Q(x, f_0, f_{-1}, \cdots)$ in $\mathcal{F}$ such that its evaluation $Q(B_t, g_0(t), g_{-1}(t), \cdots)$ at $x = B_t$, $f_0 = g_0(t)$, $f_{-1} = g_{-1}(t)$, $\cdots$ is a local martingale. The main idea is that by regarding an automorphism $\rho \in \mathrm{Aut}_+\mathcal{O}$ as a variable of $G(\rho)$, we obtain for each vector $u \in L(c,h)$ an object $Q_u(x, f_0, f_{-1}, \cdots) = \langle u|G(g)\mathcal{Y}(|c,h\rangle, x)|0\rangle \in \mathbb{C}[[x]][f_0, f_{-1}, \cdots]$, namely we have a linear map $L(c,h) \to \mathbb{C}[[x]][f_0, f_{-1}, \cdots]$. Although analyticity of $Q_u(x, f_0, f_{-1}, \cdots)$ in x is a priori nontrivial and thus it is not clear whether we can substitute the Brownian motion B_t in the variable x, if it is possible, then as we have seen in the previous section, the object Q_u evaluated at the Brownian motion and the solutoin of SLE equation $Q_u(B_t, g_0(t), g_{-1}(t), \cdots) = \langle u|G(g_t)\mathcal{Y}(|c,h\rangle, B_t)|0\rangle$ is a local martingale. Here since $u \in L(c,h)$ is arbitrary, an object $Q_{L_k u}$ for some $L_k \in \mathrm{Vir}$ also provides a local martingale if it is evaluated at the Brownian motion and the solution of the SLE. If we can write $Q_{L_k u} = \mathscr{L}_k Q_u$ for some operator $\mathscr{L}_k$ acting on Q_u, we may say that the Virasoro algebra acts on a space of local martingales associated to the SLE.

To find an operator $\mathscr{L}_k$, one first notice that the Virasoro field $L(z)$ transforms under action of an automorphism $\rho \in \mathrm{Aut}_+\mathcal{O}$ as

$$L(z) = G(f)L(f(z))G(f)^{-1}f'(z)^2 + \frac{c}{12}(Sf)(z), \tag{2.4.8}$$

which helps transferring the Virasoro generator L_k from left to right of $G(f)$ so that

$$L_{-k}G(f) = \sum_{m\in\mathbb{Z}} \left(\mathrm{Res}_z z^{-k+1} f(z)^{-m-2} f'(z)^2\right) G(f)L_m + \frac{c}{12}\mathrm{Res}_z z^{-k+1}(Sf)(z). \tag{2.4.9}$$

As will be shown in Appendix C, the operator $G(f)$ satisfies the following system of differential equations

$$\frac{\partial G(f)}{\partial f_n} = -\sum_{j<0} \left(\mathrm{Res}_z z^n f(z)^{-j-2} f'(z)\right) G(f)L_j \tag{2.4.10}$$

for $n \le 0$ which is inverted to express $G(f)L_m$ for $m \le -1$ in terms of $G(f)$ in f_n. As a consequence we obtain

$$G(f)L_m = -\sum_{n\le 0} \left(\mathrm{Res}_z z^{-n-1} f(z)^{m+1}\right) \frac{\partial G(f)}{\partial f_n} \tag{2.4.11}$$

for $m \leq -1$. By using the intertwining relation $[L_m, \mathcal{Y}(|c,h\rangle, x)] = x^m(h(m+1) + x\partial_x)\mathcal{Y}(|c,h\rangle, x)$ and the fact that the vacuum vector is invariant under the action of L_m for $m \geq -1$, we can write down the operator $\mathscr{L}_k$ explicitly as

$$\begin{aligned}\mathscr{L}_k = &-\sum_{n\leq 0}\left(\mathrm{Res}_z\mathrm{Res}_u\frac{z^{-k+1}u^{-n-1}f'(z)^2}{f(u)-f(z)}\right)\frac{\partial}{\partial f_n}\\ &+\mathrm{Res}_z z^{-k+1}f'(z)^2\left(\frac{h}{(f(z)-x)^2}+\frac{1}{f(z)-x}\frac{\partial}{\partial x}\right)\\ &+\frac{c}{12}\mathrm{Res}_z z^{-k+1}(Sf)(z). \end{aligned}\tag{2.4.12}$$

Here we used the convention that a rational function like $\frac{1}{z-u}$ is expanded in the region $|z| > |u|$. It is not trivial that such obtained operators $\mathscr{L}_k$ define a representation of the Virasoro algebra on the space $\mathcal{F}$, but they do in fact.

Proposition 2.4.1. *The assignment $L_k \mapsto \mathscr{L}_k$ from* Vir *to* $\mathrm{End}(\mathcal{F})$ *defines a representation of the Virasoro algebra of central charge c.*

For a given $Q(x, f_0, f_{-1}, \cdots) \in \mathcal{F}$, its value at $x = B_t$ and $f_n = g_n(t)$ satisfies the stochastic differential equation

$$\begin{aligned}&dQ(B_t, g_0(t), g_{-1}(t), \cdots)\\ &= (\mathcal{A}Q)(B_t, g_0(t), g_{-1}(t), \cdots)dt + \frac{\partial Q}{\partial x}(B_t, g_0(t), g_{-1}(t), \cdots)dB_t,\end{aligned}\tag{2.4.13}$$

where $\mathcal{A}$ is an operator on $\mathcal{F}$ defined by

$$\mathcal{A} = \frac{\kappa}{2}\frac{\partial^2}{\partial x^2} + 2\sum_{n\leq 0}\left(\mathrm{Res}_z\frac{z^{-n-1}}{f(z)-x}\right)\frac{\partial}{\partial f_n}.\tag{2.4.14}$$

It is clear that in this discription, an element in $\ker\mathcal{A} \subset \mathcal{F}$ produces a local martingale if the Brownian motions and $g_n(t)$ are substituted, thus the space $\ker\mathcal{A}$ is one interesting for our purpose.

An astonishing fact is that the space $\ker\mathcal{A}$ is preserved under the action of the Virasoro algebra. Indeed, we have the following proposition.

Proposition 2.4.2. *Let $\mathscr{L}(\zeta) = \sum_{k\in\mathbb{Z}}\mathscr{L}_k\zeta^{-n-2}$ be the generating function of the Virasoro algebra acting on $\mathcal{F}$. Then we have*

$$[\mathscr{L}(\zeta), \mathcal{A}] = -2\zeta^{-4}\frac{f'(\zeta^{-1})^2}{(f(\zeta^{-1})-x)^2}\mathcal{A}.\tag{2.4.15}$$

Combining the facts that an element in $\ker\mathcal{A}$ produces an local martingale associated with the solution of the SLE equation, and that the space $\ker\mathcal{A}$ is preserved under the action of the Virasoro algebra, we can claim that a space of local martingales associated with the SLE equation, called SLE local martingales in short, has a symmetry described by the Virasoro algebra. A subtle problem is that $\ker\mathcal{A}$ may be smaller than the space

$$\mathcal{M} = \{Q(x, f_0, f_{-1}, \cdots) \in \mathcal{F} | Q(B_t, g_0(t), g_{-1}(t), \cdots) \text{ is a local martingale}\},\tag{2.4.16}$$

since there may be an element Q that does not annihilated by $\mathcal{A}$, but produces a local martingale when evaluated at the Brownian motion and the solution of the SLE equation. The difference between $\ker\mathcal{A}$ and $\mathcal{M}$ is not clear but both spaces are representations of the Virasoro algebra.

2.4.4 Connection to Logarithmic Conformal Field Theories

As we have already commented, Kytölä in [Kyt07] established a Virasoro module structure of a space of local martingales associated with several variants of SLE that cover the notion of multiple SLE and SLE(κ, ρ). Structure of such obtained Virasoro module was investigated in [Kyt09] for some examples and staggered Virasoro modules [KR09] were found as Virasoro submodules. A staggered module is an example of logarithmic modules, on which the zero mode L_0 of the Virasoro algebra is not diagonalizable. Such a module of course does not appear as a quotient of a Verma module, but can be constructed by extension of modules. Considering this kind of modules gives a direction of generalizing the notion of CFT to that of logarithmic CFT (LCFT), which stems from the discovery of logarithmic operators in CFT [Gur93]. History and development of LCFT can be found in the special issue in Journal of Physics A beginning from [GRR13]. Following the work by Kytölä in [Kyt09], the connection between SLE and LCFT has been investigated in various ways [MR08, SAPR09, SC09].

2.5 Generalization of SLE-type process

In the group theoretical formulation of SLE, we consider a random process $\mathscr{G}_t$ on $\mathrm{Aut}_+\mathcal{O}$ that produces a local martingale $G(\rho_t)\,|c,h\rangle$ in $\overline{L(c,h)}$ due to existence of a null vector of the form $(-2L_{-2} + \frac{\kappa}{2}L_{-1}^2)\,|c,h\rangle$ in the Verma module $M(c,h)$. Since the only essence in this construction is the existence of a null vector, it is natural to generalize the notion of SLE to one corresponding to more general null vectors.

Lesage and Rasmussen [LR04] have constructed stochastic differential equations that correspond to null vectors of the form

$$\left(-2L_{-2n} + \frac{\kappa}{2}L_{-n}^2\right)|c,h\rangle \tag{2.5.1}$$

in the Verma module $M(c,h)$ for a positive integer n. Although they have captured stochastic differential equations via transformation rule of primary fields, we can now find them in a more direct way. Let ρ_t be a random process on $\mathrm{Aut}_+\mathcal{O}$ that induces a random process $G(\rho_t)$ satisfying

$$G(\rho_t)^{-1}dG(\rho_t) = \left(-2L_{-2n} + \frac{\kappa}{2}L_{-n}^2\right)dt + L_{-n}dB_t, \tag{2.5.2}$$

where the covariance of the Brownian motion B_t is κ. Then the $\mathrm{Aut}\mathcal{K}$-valued random process $\gamma(\rho_t)$ satisfies

$$\gamma(\rho_t)^{-1}d\gamma(\rho_t) = \left(2z^{-2n+1}\partial_z + \frac{\kappa}{2}(-z^{-n+1}\partial_z)^2\right)dt - z^{-n+1}\partial_z. \tag{2.5.3}$$

If we apply the both sides to $z \in \mathcal{K}$, we obtain a stochastic differential equation

$$d\rho_t(z) = \left(2 + \frac{\kappa}{2}(-n+1)\right)\rho_t(z)^{-2n+1}dt - \rho_t(z)^{-n+1}dB_t, \tag{2.5.4}$$

which is also derived in [LR04].

A left problem is on the existence of a null vector of the form in Eq.(2.5.1). Lesage and Rasmussen [LR04] also find an example of such a null vector

$$\left(-2L_{-4} + \frac{10}{3}L_{-2}^2\right)|0\rangle \tag{2.5.5}$$

in the vacuum Verma module of the Virasoro algebra at the Yang-Lee singularity, which is a BPZ minimal model specified by $(p,q) = (2,5)$ that leads to the central charge $c = -\frac{22}{5}$.

Chapter 3

Heisenberg SLE

In this chapter, we construct SLEs associating to representation theory of Heisenberg algebras. We take as internal symmetry group a positive loop group of a torus, then a random process on the semidirect product group of space-time symmetry and internal symmetry naturally leads to a generalization of SLE, and allows us to compute several local martingales. Cardy [Car06] pointed out that local martingales associating to SLE(κ, ρ) is tightly connected to representation theory of the Heisenberg algebra, but our construction is much different from that of SLE(κ, ρ).

3.1 Heisenberg algebras and their representations

Let $\mathfrak{h}$ be a finite dimensional vector space, and $(\cdot|\cdot)$ be a symmetric bilinear form on $\mathfrak{h}$. The Heisenberg algebra associated with this vector space is an infinite dimensional Lie algebra $\widehat{\mathfrak{h}} = \mathfrak{h} \otimes \mathbb{C}[\zeta, \zeta^{-1}] \oplus \mathbb{C}K$ with Lie bracket being given by

$$[H_1(m), H_2(n)] = m(H_1|H_2)\delta_{m+n,0}K, \quad [K, \widehat{\mathfrak{h}}] = 0. \tag{3.1.1}$$

Here we denote $H \otimes \zeta^n$ for $H \in \mathfrak{h}$ and $n \in \mathbb{Z}$ by $H(n)$. To construct a Fock representation, we take subalgebra $\widehat{\mathfrak{h}}_{\geq 0} = \mathfrak{h} \otimes \mathbb{C}[\zeta] \oplus \mathbb{C}K$. Then for $\lambda \in \mathfrak{h}^*$, a one dimensional space $\mathbb{C}v_\lambda$ becomes a representation of $\widehat{\mathfrak{h}}_{\geq 0}$, on which $H(0)$ acts as multiplication by $\langle \lambda, H \rangle$ for $H \in \mathfrak{h}$, K acts as identity, and $\mathfrak{h} \otimes \mathbb{C}[\zeta]\zeta$ acts trivially. We construct a Fock representation of the Heisenberg algebra by induction

$$M_{\mathfrak{h}}(1, \lambda) = \mathrm{Ind}_{\widehat{\mathfrak{h}}_{\geq 0}}^{\widehat{\mathfrak{h}}} \mathbb{C}v_\lambda = U(\widehat{\mathfrak{h}}) \otimes_{U(\widehat{\mathfrak{h}}_{\geq 0})} \mathbb{C}v_\lambda. \tag{3.1.2}$$

We denote the highest weight vector $1 \otimes v_\lambda$ by the same symbol v_λ and do not distinguish an element of the Heisenberg algebra and its action on $M_{\mathfrak{h}}(1, \lambda)$. Then this space is spanned by elements of the form $H_1(-j_1) \cdots H_k(-j_k)v_\lambda$, where $H_1, \cdots, H_k \in \mathfrak{h}$ and $j_1, \cdots, j_k > 0$. In the sequel, we identify $\mathfrak{h}^*$ with $\mathfrak{h}$ via the nondegenerate form $(\cdot|\cdot)$.

For $H \in \mathfrak{h}$, the corresponding *current* field is a $\mathrm{End}(M_{\mathfrak{h}}(1, \lambda))$-valued formal power series in z, defined by

$$H(z) = \sum_{n \in \mathbb{Z}} H(n) z^{-n-1}. \tag{3.1.3}$$

Then the commutation relation of two current field $H_1(z)$ and $H_2(w)$ is computed as

$$[H_1(z), H_2(w)] = (H_1|H_2)\partial_w \delta(z - w) \tag{3.1.4}$$

in $\mathrm{End}(M_{\mathfrak{h}}(1,\lambda))[[z,z^{-1},w,w^{-1}]]$. Here $\delta(z-w)=\sum_{n\in\mathbb{Z}} z^{-n-1}w^n$ is the formal delta disribution. The nonnegative power part $H(z)_+$ and negative power part $H(z)_-$ of a current field $H(z)$ are defined by

$$H(z)_+=\sum_{n\le -1} H(n)z^{-n-1},\quad H(z)_-=\sum_{n\ge 0} H(n)z^{-n-1}. \tag{3.1.5}$$

For two current field $H_1(z)$ and $H_2(z)$, their naïve product "$H_1(z)H_2(z)$" is not well defined in $\mathrm{End}(M_{\mathfrak{h}}(1,\lambda))[[z,z^{-1}]]$ due to the singularity in their operator product expansion. Instead their *normal ordered product* is defined by

$$:H_1(z)H_2(z): = H_1(z)_+H_2(z)+H_2(z)H_1(z)_- \tag{3.1.6}$$

as an element of $\mathrm{End}(M_{\mathfrak{h}}(1,\lambda))$.

Let $\{H_i\}_{i=1}^{\ell}$ be an orthonormal basis of $\mathfrak{h}$ with respect to the bilinear form $(\cdot|\cdot)$. We take a field

$$L(z)=\frac{1}{2}\sum_{i=1}^{\ell} :H_i(z)^2: \ +\alpha\partial b(z) \tag{3.1.7}$$

for some $b\in\mathfrak{h}$ such that $(b|b)=1$ and $\alpha\in\mathbb{C}$, then the coefficients L_n in the expansion $L(z)=\sum_{n\in\mathbb{Z}} L_n z^{-n-2}$ define a representation of the Virasoro algebra of central charge $c_{\ell,\alpha}=\ell-12\alpha^2$, namely they satisfy

$$[L_m,L_n]=(m-n)L_{m+n}+\frac{m^3-m}{12}\delta_{m+n,0}c_{\ell,\alpha}. \tag{3.1.8}$$

We find that

$$[L_m,H(n)]=-nH(m+n)+\alpha m(m+1)(b|H)\delta_{m+n,0} \tag{3.1.9}$$

for $H\in\mathfrak{h}$ and $m,n\in\mathbb{Z}$, and in particular $[L_0,H(n)]=-nH(n)$, which implies that the operator $H(n)$ raises eigenvalue of L_0 by $-n$. We also find that the highest weight vector v_λ of $M_{\mathfrak{h}}(1,\lambda)$ is an eigenvector of L_0 with eigenvalue $h_\lambda=\frac{1}{2}(\lambda|\lambda)^2-\alpha(\lambda|b)$. Thus $M_{\mathfrak{h}}(1,\lambda)$ is decomposed into direct sum of eigenspaces of L_0 so that eigenvalues take form of $h_\lambda+n$ for $n\ge 0$.

3.2 Annihilator of the highest weight vector

We shall seek an annihilating operator of the highest weight vector v_λ of degree 2. We first compute vectors $L_{-2}v_\lambda$ and $L_{-1}^2v_\lambda$. By the concrete expression of L_n, they are computed as

$$L_{-2}v_\lambda=\left(\frac{1}{2}\sum_{i=1}^{\ell} H_i(-1)^2+\lambda(-2)+\alpha b(-2)\right)v_\lambda, \tag{3.2.1}$$

$$L_{-1}^2 v=\big(\lambda(-1)^2+\lambda(-2)\big)v_\lambda. \tag{3.2.2}$$

We assume that λ is proportional H_1 with coefficient being again written as λ, and that $b=H_1$. Then we can realize any conformal weight h_λ by varying λ and α. Notice that we can also obtain any central charge c by choosing the dimension of $\mathfrak{h}$ and tuning α. Under these assumptions, we have

$$\left(-2L_{-2}+\frac{\kappa}{2}L_{-1}^2\right)v_\lambda=\left(-(1-2\lambda(\lambda+\alpha))H_1(-1)^2-\sum_{i=2}^{\ell} H_i(-1)^2\right)v_\lambda, \tag{3.2.3}$$

for $\kappa=\frac{4(\lambda+\alpha)}{\lambda}$. Thus we have found an operator that annihilates v_λ of a suitable form.

Proposition 3.2.1. *The following operator annihilates the highest weight vector v_λ*

$$-2L_{-2} + \frac{\kappa}{2}L_{-1}^2 + \frac{1}{2}\sum_{i=1}^{\ell} \tau_i H_i(-1)^2, \tag{3.2.4}$$

where $\kappa = \frac{4(\lambda+\alpha)}{\lambda}$, $\tau_1 = 2 - 4\lambda(\lambda+\alpha)$ and $\tau_i = 2$ for $i \geq 2$.

It can be seen that both κ and τ_1 are positive if $\alpha > 0$ and $-\alpha < \lambda < 0$.

3.3 Construction of random process

Let $T^\ell = (\mathbb{C}^\times)^\ell$ be an ℓ-dimensional torus, which is a commutative Lie group, and $T^\ell(\mathcal{O}) = T^\ell[[\zeta^{-1}]]$ be the positive (in variable ζ^{-1}) loop group of T^ℓ. We also take a subgroup $T^\ell_+\mathcal{O}$ consisting elements that coincide with the unit element modulo $T^\ell[[\zeta^{-1}]]\zeta^{-1}$. The Lie algebras of $T^\ell\mathcal{O}$ and $T^\ell_+\mathcal{O}$ are $\mathfrak{h}(\mathcal{O}) = \mathfrak{h}[[\zeta^{-1}]]$ and $\mathfrak{h}_+(\mathcal{O}) = \mathfrak{h}[[\zeta^{-1}]]\zeta^{-1}$, respectively, where $\mathfrak{h}$ is identified with a commutative Lie algebra that is used in definition of the Heisenberg algebra. The Lie algebra $\mathrm{Der}_0\mathcal{O}$ acts on $\mathfrak{h}[[\zeta^{-1}]]$ by $[L_n, H\otimes f(\zeta)] = -H\otimes \zeta^{n+1}\partial_\zeta f(\zeta)$, and the group of automorphism $\mathrm{Aut}\mathcal{O}$ also acts on $T^\ell(\mathcal{O})$ by transformation of the variable. Consequently, we obtain a semi-direct product of groups $\mathrm{Aut}\mathcal{O} \ltimes T^\ell(\mathcal{O})$ and its Lie algebra $\mathrm{Der}_0\mathcal{O} \ltimes \mathfrak{h}(\mathcal{O})$. It is noted that the subgroup $T^\ell_+(\mathcal{O})$ is normaized by $\mathrm{Aut}_+\mathcal{O}$, thus their semi-direct product is also defined.

It is obvious that the Lie algebra $\mathfrak{h}(\mathcal{O})$ does not act on a Fock space $M_\mathfrak{h}(1,\lambda)$. Let $M_\mathfrak{h}(1,\lambda) = \bigoplus_{n\in\mathbb{Z}_{\geq 0}} M_\mathfrak{h}(1,\lambda)_{h_\lambda+n}$ be the eigenspace decomposition with respect to the operator L_0. Then the formal completion of $M_\mathfrak{h}(1,\lambda)$ is defined by $\overline{M_\mathfrak{h}(1,\lambda)} = \prod_{n\in\mathbb{Z}_{\geq 0}} M_\mathfrak{h}(1,\lambda)_{h_\lambda+n}$. The Lie algebra $\mathfrak{h}(\mathcal{O})$ acts on this completion. We next exponentiate this action to define an action of $T^\ell(\mathcal{O})$. Since an element in $\mathfrak{h}[[\zeta^{-1}]]\zeta^{-1}$ strictly raises the eigenvalue of L_0, the action of $\mathfrak{h}[[\zeta^{-1}]]\zeta^{-1}$ can be exponentiated. The zero mode $H\otimes\zeta^0$ can be exponentiated only if $\lambda\in\mathbb{Z}$, but for our purpose it is sufficient to define an action of $T^\ell_+(\mathcal{O})$. Moreover, this action is compatible with the action of $\mathrm{Aut}_+\mathcal{O}$ in the sense that we have obtained an action of $\mathrm{Aut}_+\mathcal{O} \ltimes T^\ell_+(\mathcal{O})$.

We shall construct a random process $\mathscr{G}_t$ on $\mathrm{Aut}_+\mathcal{O} \ltimes T^\ell_+(\mathcal{O})$ that produces a local martingale on $\overline{M_\mathfrak{h}(1,\lambda)}$ when applied to the highest weight vector v_λ. Such a random process can be found in the following manner. Let $B^{(i)}_t$ for $i = 0, 1, \cdots, \ell$ be mutually independent Brownian motions with covariance κ for $B^{(0)}_t$ and τ_i for $B^{(i)}_t$ with $i = 1, \cdots, \ell$. Then we seek a random process $\mathscr{G}_t$ satisfying the stochastic differential equation

$$\mathscr{G}_t^{-1}d\mathscr{G}_t = \left(-2L_{-2} + \frac{\kappa}{2}L_{-1}^2 + \frac{1}{2}\sum_{i=1}^{\ell}\tau_i H_i(-1)^2\right)dt + L_{-1}dB^{(0)}_t + \sum_{i=1}^{\ell} H_i(-1)dB^{(i)}_t. \tag{3.3.1}$$

Proposition 3.3.1. *Let κ and τ_i be positive real number as in Proposition 3.2.1, and $\mathscr{G}_t$ be the random process on $\mathrm{Aut}_+\mathcal{O} \ltimes T^\ell_+(\mathcal{O})$ satisfying Eq.(3.3.1). Then $\overline{M_\mathfrak{h}(1,\lambda)}$-valued random process $\mathscr{G}_t v_\lambda$ is a local martingale.*

We give the random process $\mathscr{G}_t$ in a more explicit form. Let ρ_t be a random process on $\mathrm{Aut}_+\mathcal{O}$ that generate the SLE equation, then we already know that the random process of corresponding transformation operators $G(\rho_t)$ on satisfies the stochastic differential equation

$$G(\rho_t)dG(\rho_t) = \left(-2L_{-2} + \frac{\kappa}{2}L_{-1}^2\right)dt + L_{-1}dB^{(0)}_t. \tag{3.3.2}$$

We put an ansatz that $\mathscr{G}_t$ is expressed in the form of $\mathscr{G}_t = e^{\mathbf{h}_t^1} \cdots e^{\mathbf{h}_t^\ell} G(\rho_t)$, where we have set $\mathbf{h}_t^i = H_i \otimes h_t^i(\zeta)$ with $h_t^i(\zeta)$ being a random process on $\mathbb{C}[[\zeta^{-1}]]\zeta^{-1}$. We claim that $\rho_t(z)$ and $h_t^i(\zeta)$ are degrees of freedom of the random process under construction, and derive stochastic differential equations they satisfy, while that for ρ_t is nothing but the SLE equation. To derive stochastic differential equations on $h_t^i(\zeta)$, we first note that a current field $H(z)$ is transformed under adjoint action by $G(\rho)$ as

$$H(z) = G(\rho)H(\rho(z))G(\rho)^{-1}\rho'(z) - \alpha(H_1|H)\frac{\rho''(z)}{\rho'(z)}, \tag{3.3.3}$$

which implies that

$$G(\rho)H(-1)G(\rho)^{-1} = H \otimes \rho(\zeta)^{-1}. \tag{3.3.4}$$

Here we used the fact that $\frac{\rho^{-1\prime\prime}(z)}{\rho^{-1\prime}(z)}$ does not have a constant term in z, thus the anomaly term does not affect. From this transformation rule, we conclude that the part $Q_t = e^{\mathbf{h}_t^1} \cdots e^{\mathbf{h}_t^\ell}$ satisfies the stochastic differential equations

$$Q_t^{-1}dQ_t = \frac{1}{2}\sum_{i=1}^{\ell} \tau_i (H_i \otimes \rho_t(\zeta)^{-1})^2 dt + \sum_{i=1}^{\ell} H_i \otimes \rho_t(\zeta)^{-1} dB_t^{(i)}, \tag{3.3.5}$$

which is possible if we impose the following stochastic differential equations on $h_t^i(\zeta)$.

Proposition 3.3.2. *Under ansatz $\mathscr{G}_t = e^{\mathbf{h}_t^1} \cdots e^{\mathbf{h}_t^\ell} G(\rho_t)$, the random processes $h_t^i(\zeta)$ satisfy*

$$dh_t^i(\zeta) = \frac{1}{\rho_t(\zeta)} dB_t^{(i)} \tag{3.3.6}$$

for $i = 1, \cdots, \ell$.

Thus the random processes $h_t^i(\zeta)$ are completely determined by the solution of SLE so that $h_t^i(\zeta) = \int_0^t \frac{dB_s^{(i)}}{\rho_s(\zeta)}$.

3.4 Local martingales

The local martingale $\mathscr{G}_t v_\lambda$ on $\overline{M_{\mathfrak{h}}(1,\lambda)}$ generates local martingales when taken inner product with any vectors in $M_{\mathfrak{h}}(1,\lambda)$. To describe them explicitly, we first investigate how a current field $H(z)$ and the Virasoro field $L(z)$ are transformed under adjoint action by $\mathscr{G}_t$. First a current field $H(z)$ transforms under adjoint action by $e^{-\mathbf{h}_t^1}$ as $e^{-\mathbf{h}_t^1} H(z) e^{\mathbf{h}_t^1} = H(z) - (H_1|H)\partial h_t^1(z)$, which implies

$$Q_t^{-1} H(z) Q_t = H(z) - \sum_{i=1}^{\ell} (H_i|H)\partial h_t^i(z). \tag{3.4.1}$$

Since the transformation rule of $H(z)$ under adjoint action by $G(\rho_t)^{-1}$ has been already obtained, we have

$$\mathscr{G}_t^{-1} H(z) \mathscr{G}_t = H(\rho_t(z))\rho_t'(z) - \alpha(H_1|H)\frac{\rho_t''(z)}{\rho_t'(z)} - \sum_{i=1}^{\ell}(H_i|H)\partial h_t^i(z). \tag{3.4.2}$$

This can be used to write down a local martingale $\langle v_\lambda, H(z)\mathscr{G}_t v_\lambda \rangle$.

Theorem 3.4.1. *Let ρ_t be the solution of SLE(κ) and h_t^i be the solutions of Eq.(3.3.6). Then the following quantity is a local martingale.*

$$\langle v_\lambda, H(z)\mathscr{G}_t v_\lambda\rangle = \lambda(H_1|H)\frac{\rho_t'(z)}{\rho_t(z)} - \alpha(H_1|H)\frac{\rho_t''(z)}{\rho_t'(z)} - \sum_{i=1}^{\ell}(H_i|H)\partial h_t^i(z). \qquad (3.4.3)$$

We move on to derive the transformation rule for the Virasoro field $L(z)$. The commutation relation between a current field $H(z)$ and the Virasoro filed $L(z)$ implies

$$[H\otimes h(\zeta), L(z)] = \partial h(z)H(z) + \alpha(H_1|H)\partial^2 h(z), \qquad (3.4.4)$$

for any $h(\zeta)\in\mathbb{C}[[\zeta^{-1}]]\zeta^{-1}$. If we take the adjoint by $H\otimes h(\zeta)$ once more, we obtain

$$[H\otimes h(\zeta), [H\otimes h(\zeta), L(z)]] = (H|H)\partial h(z)^2, \qquad (3.4.5)$$

which leads to

$$e^{-H\otimes h(\zeta)}L(z)e^{H\otimes h(\zeta)} = L(z) - \partial h(z)H(z) - \alpha(H_1|H)\partial^2 h(z) + \frac{1}{2}(H|H)\partial h(z)^2. \quad (3.4.6)$$

Note that $\{H_i\}_{i=1}^{\ell}$ is an orthonormal basis, thus the corresponding currents $H_i(z)$ are mutually commutative. This enables us to compute the quantity $Q_t^{-1}L(z)Q_t$ so that

$$Q_t^{-1}L(z)Q_t = L(z) - \sum_{i=1}^{\ell}\partial h_t^i(z)H_i(z) - \alpha\partial^2 h_t^1(z) + \frac{1}{2}\sum_{i=1}^{\ell}\partial h_t^i(z)^2. \qquad (3.4.7)$$

When we further take adjoint by $G(\rho_t)^{-1}$ on it, we obtain

$$\begin{aligned}\mathscr{G}_t^{-1}L(z)\mathscr{G}_t =& L(\rho_t(z))\partial\rho_t(z)^2 - \sum_{i=1}^{\ell}\partial h_t^i(z)\partial\rho_t(z)H_i(\rho_t(z)) \\ &+ \frac{c}{12}(S\rho_t)(z) + \alpha\partial h_t^1(z)\frac{\partial^2\rho_t(z)}{\partial\rho_t(z)} - \alpha\partial^2 h_t^1(z) + \frac{1}{2}\sum_{i=1}^{\ell}\partial h_t^i(z)^2. \qquad (3.4.8)\end{aligned}$$

This relation again helps us write down a local martingale $\langle v_\lambda, L(z)G_t v_\lambda\rangle$ associated with the solution $\rho_t(z)$ and $h_t^i(z)$ of the SLE equation.

Theorem 3.4.2. *Let ρ_t be the solution of SLE(κ) and h_t^i be the solutions of Eq.(3.3.6). Then the following quantity is a local martingale.*

$$\begin{aligned}\langle v_\lambda, L(z)\mathscr{G}_t v_\lambda\rangle =& h_\lambda\left(\frac{\partial\rho_t(z)}{\rho_t(z)}\right)^2 + \partial h_t^1(z)\left(\alpha\frac{\partial^2\rho_t(z)}{\partial\rho_t(z)} - \lambda\frac{\partial\rho_t(z)}{\rho_t(z)}\right) \\ &+ \frac{c}{12}(S\rho_t)(z) - \alpha\partial^2 h_t^1(z) + \frac{1}{2}\sum_{i=1}^{\ell}\partial h_t^i(z)^2. \qquad (3.4.9)\end{aligned}$$

Since on our representation space $M_{\mathfrak{h}}(1,\lambda)$ the Virasoro field is realized by using current fields, the local martingale $\langle v_\lambda, L(z)\mathscr{G}_t v_\lambda\rangle$ has another description. From the transformation rule of a current field $H(z)$, its positive and negative power parts are transformed as

$$\begin{aligned}\mathscr{G}_t^{-1}H(z)_+\mathscr{G}_t =& \sum_{m\in\mathbb{Z}}\mathrm{Res}_w\frac{\partial\rho_t(w)\rho_t(w)^{-m-1}}{w-z}H(m)\\ &-\mathrm{Res}_w\frac{1}{w-z}\left(\alpha(H_1|H)\frac{\partial^2\rho_t(w)}{\partial\rho_t(w)}+\sum_{i=1}^{\ell}(H_i|H)\partial h_t^i(w)\right), \qquad (3.4.10)\\ \mathscr{G}_t^{-1}H(z)_-\mathscr{G}_t =& \sum_{m\in\mathbb{Z}}\mathrm{Res}_w\mathrm{Res}_w\frac{\partial\rho_t(w)\rho_t(w)^{-m-1}}{z-w}H(m)\\ &-\mathrm{Res}_w\frac{1}{z-w}\left(\alpha(H_1|H)\frac{\partial^2\rho_t(w)}{\partial\rho_t(w)}+\sum_{i=1}^{\ell}(H_i|H)\partial h_t^i(w)\right). \qquad (3.4.11)\end{aligned}$$

Thus the local martingale associated with the normal ordered product $:H(z)^2:$ is computed as

$$\begin{aligned}\langle v_\lambda, :H(z)^2:\ \mathscr{G}_t v_\lambda\rangle =&(H|H)\mathrm{Res}_w\left[\frac{\partial\rho_t(w)}{w-z}\partial_z\left(\frac{1}{\rho_t(w)-\rho_t(z)}\right)-\frac{\partial\rho_t(w)}{z-w}\partial_z\left(\frac{\rho_t(z)\rho_t(w)^{-1}}{\rho_t(z)-\rho_t(w)}\right)\right]\\ &+(\lambda(H_1|H))^2\left(\frac{\partial\rho_t(z)}{\rho_t(z)}\right)^2\\ &-2\lambda(H_1|H)\left(\alpha(H_1|H)\frac{\partial^2\rho_t(z)}{\partial\rho_t(z)}+\sum_{i=1}^{\ell}(H_i|H)\partial h_t^i(z)\right)\frac{\partial\rho_t(z)}{\rho_t(z)}\end{aligned} \qquad (3.4.12)$$

This enables us to derive another form of the local martingale $\langle v_\lambda, L(z)\mathscr{G}v_\lambda\rangle$ so that

$$\begin{aligned}\langle v_\lambda, L(z)\mathscr{G}_t v_\lambda\rangle =&\frac{\ell}{2}\mathrm{Res}_w\left[\frac{\partial\rho_t(w)}{w-z}\partial_z\left(\frac{1}{\rho_t(w)-\rho_t(z)}\right)-\frac{\partial\rho_t(w)}{z-w}\partial_z\left(\frac{\rho_t(z)\rho_t(w)^{-1}}{\rho_t(z)-\rho_t(w)}\right)\right]\\ &+h_\lambda\left(\frac{\partial\rho_t(z)}{\rho_t(z)}\right)^2-\lambda\frac{\partial\rho_t(z)}{\rho_t(z)}\partial h_t^1(z)-\alpha\partial^2h_t^1(z)\\ &-\alpha^2\left(\frac{\partial^3\rho_t(z)}{\partial\rho_t(z)}-\left(\frac{\partial^2\rho_t(z)}{\partial\rho_t(z)}\right)^2\right). \qquad (3.4.13)\end{aligned}$$

Comparing this with the same quantity, which is seemingly different, derived previously, we obtain a nontrivial equality among random processes

$$\begin{aligned}&\frac{\ell}{2}\mathrm{Res}_w\left[\frac{\partial\rho_t(w)}{w-z}\partial_z\left(\frac{1}{\rho_t(w)-\rho_t(z)}\right)-\frac{\partial\rho_t(w)}{z-w}\partial_z\left(\frac{\rho_t(z)\rho_t(w)^{-1}}{\rho_t(z)-\rho_t(w)}\right)\right]\\ &-\alpha^2\left(\frac{\partial^3\rho_t(z)}{\partial\rho_t(z)}-\left(\frac{\partial^2\rho_t(z)}{\partial\rho_t(z)}\right)^2\right)\\ &=\alpha\partial h_t^1(z)\frac{\partial^2\rho_t(z)}{\partial\rho_t(z)}+\frac{c}{12}(S\rho_t)(z)+\frac{1}{2}\sum_{i=1}^{\ell}(\partial h_t^i(z))^2. \qquad (3.4.14)\end{aligned}$$

3.5 Summary

In this chapter, we take a positive loop group of a torus as an internal symmetry group, on which the group of space-time symmetry acts as transformation of the loop variable.

The corresponding representation theory is one of a Heisenberg algebra. We construct SLE associated with a Fock representation of a Heisenberg algebra of any rank from a random process on the semi-direct product group of space-time and internal symmetries, and compute some local martingales that originate from an annihilator of the highest weight vector. Cardy [Car06] once pointed out that the notion of SLE(κ, ρ) is related to representation theory of a Heisenberg algebra, but our construction is very different from that of SLE(κ, ρ) in that ours is a generalization of the group theoretical formulation while a group theoretical formulation of SLE(κ, ρ) is not known, and indeed, SLE we construct is different from SLE(κ, ρ).

Chapter 4

Affine SLE

In this chapter, we extend the formulation in Chapter 3 to the case of affine vertex algebras. We only consider an affine Lie algebra of a finite dimensional simple Lie algebra. In this case we consider the positive loop algebra of the finite dimensional Lie algebra as an internal symmetry, and construct a random process on an infinite dimensional Lie group. We present it in the most concrete form in the case that the finite dimensional Lie algebra is $\mathfrak{sl}_2$, and write down several local martingales associating to the SLE process. We also investigate the affine Lie algebra symmetry of the space of local martingales. A part of the content of this chapter has been also reported in an original paper [Kos17].

4.1 SLE for Wess-Zumino-Witten theories

CFTs that are associated with representation theory of affine Lie algebras are known as Wess-Zumino-Witten (WZW) theories [WZ71, Wit84, KZ84]. SLEs corresponding to WZW theories have been considered in [BGLW05, ABI11]. Let us shortly review their approach. Let $\mathfrak{g}$ be a finite dimensional simple Lie algebra and $k \in \mathbb{C}$ be a level. They start from an object

$$\mathcal{M}_t = \frac{\langle \phi_\Lambda(z_t)\phi_{\lambda_1}(z_1)\cdots\phi_{\lambda_N}(z_N)\phi_{\lambda_1^*}(\bar{z}_1)\cdots\phi_{\lambda_N^*}(\bar{z}_N)\phi_{\Lambda^*}(\infty)\rangle^{\mathfrak{g}}}{\langle \phi_\Lambda(z_t)\phi_{\Lambda^*}(\infty)\rangle^{\mathfrak{g}}} \tag{4.1.1}$$

Here ϕ_λ is the primary field corresponding to weight λ, with convention that λ^* denotes the dual representation of λ. The points $z_1, \cdots, z_N$ are put on the upper half plane and z_t is the tip of the SLE slit defined by $z_t = \rho_t^{-1}(0)$, where $\rho_t(z)$ satisfies $d\rho_t(z) = \frac{2dt}{\rho_t(z)} - B_t$ with B_t being the Brownian motion of covariance κ. The numerator takes values in the $\mathfrak{g}$-invariant subspace of $L(\Lambda) \otimes L(\lambda_1) \otimes \cdots \otimes L(\Lambda)^*$, where $L(\lambda)$ is the irreducible representation of $\mathfrak{g}$ of highest weight λ. The denominator takes values in $\mathfrak{g}$-invariant subspace of $L(\Lambda) \otimes L(\Lambda)^*$, which is one-dimensional due to Schur's Lemma.

Since a primary field of a WZW theory has internal degrees of freedom, random evolution of a primary field involves ones along the internal degrees of freedom. The authors of [BGLW05, ABI11] suggest the following stochastic differential equation:

$$d\phi_{\lambda_i}(w_i) = \mathcal{G}_i \phi_{\lambda_i}(w_i), \tag{4.1.2}$$

where $w_i = \rho_t(z_i)$ and

$$\mathcal{G}_i = dt\left(\frac{2}{w_i}\partial_{w_i} - \frac{\tau C_i}{2w_i^2}\right) - dB_t\partial_{w_i} + \left(\frac{1}{w_i}\sum_a d\theta^a t_i^a + \frac{\tau}{2w_i^2}\sum_a t_i^a t_i^a dt\right). \tag{4.1.3}$$

Here $\{t^a\}$ is a basis of $\mathfrak{g}$ and $\{t_i^a\}$ are their representation matrices on $L(\lambda_i)$. Random processes θ^a are independent Brownian motions of covariance τ. The number C_i is the value of the Casimir on the representation $L(\lambda_i)$.

The claim of [BGLW05, ABI11] is that the random process $\mathcal{M}_t$ is a local martingale for a certain choice of κ and τ, and Eq. (4.1.2) is a generalization of SLE so to correspond to a WZW theory. We shall comment that their formulation has been extended to multiple SLEs in [Sak13] and to coset WZW theories in [Naz12, Fuk].

As we have already mentioned in Chap. 1, their formulation is not satisfactory to us in three points: (1) their stochastic differential equations along internal degrees of freedom seem to be *ad hoc*, (2) random processes along internal degrees of freedom are not constructed in a concrete way and (3) local martingales that are associated with the solution is hard to write down. We overcome these difficulties in our formulation in this chapter.

4.2 Affine Lie algebras and their representations

In this section, we recall basics of an affine Lie algebra and their representation theory. Let $\mathfrak{g}$ be a finite dimensional simple complex Lie algebra and $(\cdot|\cdot) : \mathfrak{g} \times \mathfrak{g} \to \mathbb{C}$ be a nondegenerate symmetric invariant bilinear form on $\mathfrak{g}$. The affinization $\widehat{\mathfrak{g}}$ of $\mathfrak{g}$ is defined by $\widehat{\mathfrak{g}} = \mathfrak{g} \otimes \mathbb{C}[\zeta, \zeta^{-1}] \oplus \mathbb{C}K$ with Lie bracket being defined by

$$[X(m), Y(n)] = [X, Y](m+n) + m(X|Y)\delta_{m+n,0}K, \quad [K, \widehat{\mathfrak{g}}] = \{0\}, \tag{4.2.1}$$

where we denote $X \otimes \zeta^n$ by $X(n)$ for $X \in \mathfrak{g}$ and $n \in \mathbb{Z}$. Let M be a finite dimensional representation of the finite dimensional Lie algebra $\mathfrak{g}$. Then we lift the action of $\mathfrak{g}$ to an action of a Lie subalgebra $\mathfrak{g} \otimes \mathbb{C}[\zeta] \oplus \mathbb{C}K$ of the affine Lie algebra so that $\mathfrak{g} \otimes \zeta^0$ acts naturally, $\mathfrak{g} \otimes \zeta\mathbb{C}[\zeta]$ acts trivially, and K acts as multiplication by a complex number k. Then we obtain a representation $\widehat{M}_k$ of the affine Lie algebra $\widehat{\mathfrak{g}}$ by

$$\widehat{M}_k = \mathrm{Ind}_{\mathfrak{g}\otimes\mathbb{C}[\zeta]\oplus\mathbb{C}K}^{\widehat{\mathfrak{g}}} M = U(\widehat{\mathfrak{g}}) \otimes_{U(\mathfrak{g}\otimes\mathbb{C}[\zeta]\oplus\mathbb{C}K)} M. \tag{4.2.2}$$

Here introduced complex number k is called the level of the representation. By the Poincaré-Birkhoff-Witt theorem, $\widehat{M}_k$ is isomorphic to $U(\mathfrak{g} \otimes \zeta^{-1}\mathbb{C}[\zeta^{-1}]) \otimes_{\mathbb{C}} M$ as a vector space or a $U(\mathfrak{g} \otimes \zeta^{-1}\mathbb{C}[\zeta^{-1}])$-module.

We fix a Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$, and let $\Pi^\vee = \{\alpha_i^\vee, \cdots, \alpha_\ell^\vee\} \subset \mathfrak{h}$ be the set of simple coroots of $\mathfrak{g}$. Then the fundamental weights $\varpi_i \in \mathfrak{h}^*$ for $i = 1, \cdots, \ell$ are defined by $\langle \varpi_i, \alpha_j^\vee \rangle = \delta_{ij}$, and span the weight lattice $P = \bigoplus_{i=1}^{\ell} \mathbb{Z}\varpi_i$. A weight $\Lambda \in P$ is called dominant if $\langle \Lambda, \alpha_i^\vee \rangle \geq \mathbb{Z}_{\geq 0}$ for all $i = 1, \cdots, \ell$. We denote the set of dominant weight by P_+. Finite dimensional irreducible representations of $\mathfrak{g}$ are labeled by P_+, namely, for a dominant weight $\Lambda \in P_+$, there is a finite dimensional irreducible representation $L(\Lambda)$ of $\mathfrak{g}$ with highest weight Λ, and conversely, the highest weight of a finite dimensional irreducible representation of $\mathfrak{g}$ is dominant. For an irreducible representation $L(\Lambda)$ of $\mathfrak{g}$, we can construct a representation $\widehat{L(\Lambda)}_k$ of $\widehat{\mathfrak{g}}$ in the manner described in the previous paragraph. Note that although $L(\Lambda)$ is irreducible as a representation of $\mathfrak{g}$, $\widehat{L(\Lambda)}_k$ is not necessarily an irreducible representation of $\widehat{\mathfrak{g}}$, then we denote by $L_{\mathfrak{g},k}(\Lambda)$ the irreducible quotient of $\widehat{L(\Lambda)}_k$ as a representation of $\widehat{\mathfrak{g}}$.

On a representation space $L_{\mathfrak{g},k}(\Lambda)$ of the affine Lie algebra $\widehat{\mathfrak{g}}$, we can define an action of the Virasoro algebra through the Segal-Sugawara construction. We normalize the bilinear form so that $(\theta|\theta) = 2$, where θ is the highest root of $\mathfrak{g}$, and assume $k \neq -h^\vee$, where $h^\vee$ is the dual Coxeter number of $\mathfrak{g}$. Let $\{X_a\}_{a=1}^{\dim \mathfrak{g}}$ be an orthonormal basis of $\mathfrak{g}$ with respect to $(\cdot|\cdot)$. Then the operators $L_n \in \mathrm{End}(L_{\mathfrak{g},k}(\Lambda))$ for $n \in \mathbb{Z}$ acting on $L_{\mathfrak{g},k}(\Lambda)$ that are defined by

$$L_n = \frac{1}{2(k+h^\vee)} \sum_{a=1}^{\dim \mathfrak{g}} \sum_{k\in\mathbb{Z}} :X_a(n-k)X_a(k): \tag{4.2.3}$$

give an action of the Virasoro algebra of central charge $c_k = \frac{k \dim \mathfrak{g}}{k+h^\vee}$. Here the normal ordered product $:A(p)B(q):$ is defined by $A(p)B(q)$ for $p < q$ and $B(q)A(p)$ for $p \geq q$. Moreover a vector $v_\Lambda \in L(\Lambda) \hookrightarrow L_{\mathfrak{g},k}(\Lambda)$ is an eigenvector of L_0 corresponding to an eigenvalue $h_\Lambda = \frac{(\Lambda|\Lambda+2\rho)}{2(k+h^\vee)}$, with $\rho = \sum_{i=1}^{\ell} \Lambda_i$, and L_0 is diagonalizable on $L_{\mathfrak{g},k}(\Lambda)$ so that $L_{\mathfrak{g},k}(\Lambda) = \bigoplus_{n\in\mathbb{Z}_{\geq 0}} L_{\mathfrak{g},k}(\Lambda)_{h_\Lambda+n}$ with each $L_{\mathfrak{g},k}(\Lambda)_h$ being the eigenspace of L_0 corresponding to an eigenvalue h. We shall remark that this action of the Virasoro algebra is compatible with the action of $\widehat{\mathfrak{g}}$ in the sense that $[L_n, A\otimes f(\zeta)] = -A\otimes \zeta^{n+1}\frac{df(\zeta)}{d\zeta}$.

Among representations $L_{\mathfrak{g},k}(\Lambda)$, we can equip on $L_{\mathfrak{g},k}(0)$ a VOA structure. The vacuum vector is $|0\rangle = 1 \otimes \mathbf{1}$, where $\mathbf{1}$ spans the one-dimensional representation $L(0)$ of $\mathfrak{g}$. Let $\{X_a\}_{a=1}^{\dim \mathfrak{g}}$ be a basis of $\mathfrak{g}$, then this VOA is strongly generated by vectors $X_a(-1)|0\rangle$. In the following, we call this VOA the affine VOA of $\mathfrak{g}$ with level k and denote it by $L_{\mathfrak{g},k}$.

4.3 Internal symmetry

Let G be a finite dimensional simple complex Lie group of which Lie algebra is $\mathfrak{g}$. To construct an SLE equation associated with a representation of an affine Lie algebra, we consider the positive loop group $G(\mathcal{O}) = G[[\zeta^{-1}]]$ of G as a group of internal symmetry. A significant subgroup $G_+(\mathcal{O})$ consists of elements that are the unit element modulo $G[[\zeta^{-1}]]\zeta^{-1}$. The Lie algebras of $G(\mathcal{O})$ and $G_+(\mathcal{O})$ are $\mathfrak{g}[[\zeta^{-1}]]$ and $\mathfrak{g}[[\zeta^{-1}]]\zeta^{-1}$, respectively. The group of automorphisms $\mathrm{Aut}\mathcal{O}$ acts on $G(\mathcal{O})$ to define a semi-direct product $\mathrm{Aut}\mathcal{O} \ltimes G(\mathcal{O})$. Moreover, the subgroup $\mathrm{Aut}_+\mathcal{O}$ normalizes $G_+(\mathcal{O})$, thus their semi-direct product $\mathrm{Aut}_+\mathcal{O} \ltimes G_+(\mathcal{O})$ is also defined.

On a representation $L_{\mathfrak{g},k}(\Lambda)$ of the affine Lie algebra $\widehat{\mathfrak{g}}$, the Lie algebra $\mathfrak{g}\otimes\mathbb{C}[[\zeta^{-1}]]$ cannot act, but its formal completion $\overline{L_{\mathfrak{g},k}(\Lambda)} = \prod_{n\in\mathbb{Z}_{\geq 0}} L_{\mathfrak{g},k}(\Lambda)_{h_\Lambda+n}$ can be a representation of $\mathfrak{g}\otimes\mathbb{C}[[\zeta^{-1}]]$. It is also obvious that the action of $\mathfrak{g}\otimes\mathbb{C}[[\zeta^{-1}]]$ is exponentiated to define an action of $G(\mathcal{O})$. Indeed, an element in $\mathfrak{g}\otimes\zeta^{-1}\mathbb{C}[[\zeta^{-1}]]$ strictly raises degree, and a zero-mode element $X\otimes\zeta^0$ is exponentiated to be an action of $e^X \in G$ while each homogeneous space is a representation of the finite dimensional Lie group G. Moreover, this action of $G(\mathcal{O})$ is compatible to the action of $\mathrm{Aut}\mathcal{O}$. Thus $\mathrm{Aut}_+\mathcal{O} \ltimes G_+(\mathcal{O})$ acts on $\overline{L_{\mathfrak{g},k}(\Lambda)}$.

We investigate how each field is transformed under the adjoint action of $e^{\mathbf{a}}$ where $\mathbf{a} = A\otimes a(\zeta) \in \mathfrak{g}\otimes\mathbb{C}[[\zeta^{-1}]]$. We compute the commutator $[\mathbf{a}, Y(B,w)]$ for $B \in L_{\mathfrak{g},k}(\Lambda)$. From the OPE formula

$$[Y(A(-1)|0\rangle, z), Y(B,w)] = \sum_{k\geq 0} Y(A(k)B,w)\partial_w^{(k)}\delta(z-w), \tag{4.3.1}$$

we obtain

$$[A(n), Y(B,w)] = \sum_{k\geq 0} \binom{n}{k} w^{n-k} Y(A(k)B,w). \tag{4.3.2}$$

Thus the desired commutator is computed as

$$[\mathbf{a}, Y(B,w)] = Y(\mathbf{a}_w B, w), \tag{4.3.3}$$

where $\mathbf{a}_w = \sum_{k\geq 0} \partial^{(k)} a(w) A(k)$. This enables us to obtain the following transformation formula.

$$Y(B,w) = e^{\mathbf{a}} Y(e^{-\mathbf{a}_w} B, w) e^{-\mathbf{a}} \tag{4.3.4}$$

Now we assume $a(z) \in \mathbb{C}[[z^{-1}]]$ and $A \in \mathfrak{g}$, and compute $e^{-\mathbf{a}_w} X(-1)\,|0\rangle$ for some $X \in \mathfrak{g}$ to investigate the transformation rule of $Y(X(-1)\,|0\rangle\,, z)$ under the adjoint action by $e^{\mathbf{a}}$. The action of $\mathbf{a}_w$ on $X(-1)\,|0\rangle$ gives

$$\begin{aligned}
\mathbf{a}_w X(-1)\,|0\rangle &= \sum_{k=0}^{\infty} \partial^{(k)} a(w) A(k) X(-1)\,|0\rangle \\
&= a(w) A(0) X(-1)\,|0\rangle + \partial a(w) A(1) X(-1)\,|0\rangle \\
&= a(w)(\mathrm{ad}A)(X)(-1)\,|0\rangle + k(A|X)\partial a(w)\,|0\rangle\,.
\end{aligned} \tag{4.3.5}$$

Applying $\mathbf{a}_w$ once more, we have

$$\begin{aligned}
\mathbf{a}_w^2 X(-1)\,|0\rangle &= a(w)^2 (\mathrm{ad}A)^2(X)(-1)\,|0\rangle + (\partial a(w)) a(w) k(A|[A,X])\,|0\rangle \\
&= a(w)^2 (\mathrm{ad}A)^2(X)(-1)\,|0\rangle\,,
\end{aligned} \tag{4.3.6}$$

where we have used the invariance of the bilinear form $(A|[A,X]) = ([A,A]|X) = 0$, and inductively, we have

$$\mathbf{a}_w^n X(-1)\,|0\rangle = a(w)^n (\mathrm{ad}A)^n (X)(-1)\,|0\rangle \tag{4.3.7}$$

for $n \geq 2$. Thus we can see that

$$e^{-\mathbf{a}_w} X(-1)\,|0\rangle = (e^{-a(w)\mathrm{ad}A} X)(-1)\,|0\rangle - k(A|X)\partial a(w)\,|0\rangle\,, \tag{4.3.8}$$

which implies that

$$Y(X(-1)\,|0\rangle\,, w) = e^{\mathbf{a}} Y\left((e^{-a(w)\mathrm{ad}A} X)(-1)\,|0\rangle\,, w \right) e^{-\mathbf{a}} - k(A|X)\partial a(w). \tag{4.3.9}$$

It is also convenient to write down the formula for the object like $e^{-\mathbf{a}} X \otimes x(\zeta) e^{\mathbf{a}}$, where $x(\zeta) \in \mathbb{C}((\zeta^{-1}))$ and $\mathbf{a}$ and $X \in \mathfrak{g}$ are taken as above. It becomes

$$\begin{aligned}
e^{-\mathbf{a}} X \otimes x(\zeta) e^{\mathbf{a}} &= \mathrm{Res}_w \sum_{n\in\mathbb{Z}} (e^{-a(w)\mathrm{ad}A} X) \otimes \zeta^n w^{-n-1} x(w) - k(X|Y)\mathrm{Res}_w \partial a(w) x(w). \\
&= \sum_{m=0}^{\infty} \frac{(-1)^m}{m!} (\mathrm{ad}A)^m (X) \otimes a(\zeta)^m x(\zeta) - k(X|Y)\mathrm{Res}_w \partial a(w) x(w).
\end{aligned} \tag{4.3.10}$$

We next investigate the transformation rule of the Virasoro field $L(z)$ under the action of $G(\mathcal{O})$. To this end we compute $e^{-\mathbf{a}_z} L_{-2}\,|0\rangle$ where $\mathbf{a} = A \otimes a(\zeta) \in \mathfrak{g}(\mathcal{O})$ and correspondingly $\mathbf{a}_z = \sum_{k\geq 0} \partial^{(k)} a(z) A(k)$. Notice that the OPE

$$[L(z), Y(A(-1)\,|0\rangle\,, w)] = Y(A(-1)\,|0\rangle\,, w)\partial_w \delta(z-w) + \partial Y(A(-1)\,|0\rangle\,, w)\delta(z-w) \tag{4.3.11}$$

is equivalent to

$$[Y(A(-1)\,|0\rangle\,, z), L(w)] = Y(A(-1)\,|0\rangle\,, w)\partial_w \delta(z-w), \tag{4.3.12}$$

which implies

$$A(n) L_{-2}\,|0\rangle = \begin{cases} A(-1)\,|0\rangle\,, & n = 1, \\ 0, & n \in \mathbb{Z}_{\geq 0}\backslash\{1\}. \end{cases} \tag{4.3.13}$$

Thus we have

$$-\mathbf{a}_z L_{-2}\left|0\right\rangle = -\partial a(z) A(-1)\left|0\right\rangle. \tag{4.3.14}$$

If we apply $-\mathbf{a}_z$ once more time, it becomes

$$(-\mathbf{a}_z)^2 L_{-2}\left|0\right\rangle = k(\partial a(z))^2 (A|A)\left|0\right\rangle. \tag{4.3.15}$$

Then we obtain the following transformation formula:

$$L(z) = e^{\mathbf{a}} L(z) e^{-\mathbf{a}} - \partial a(z) e^{\mathbf{a}} Y(A(-1)\left|0\right\rangle, z) e^{-\mathbf{a}} + \frac{k(A|A)(\partial a(z))^2}{2}. \tag{4.3.16}$$

Formulae in $\mathfrak{g} = \mathfrak{sl}_2$ case

We now specialize our attention to the case of $\mathfrak{g} = \mathfrak{sl}_2$ and explicitly write down formulae Eq.(4.3.9) and Eq.(4.3.10). We take as a standard basis of $\mathfrak{sl}_2$

$$E = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \qquad H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad F = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \tag{4.3.17}$$

and denote $E \otimes e(\zeta)$, $H \otimes h(\zeta)$ and $F \otimes f(\zeta)$ for $e(\zeta), h(\zeta), f(\zeta) \in \mathbb{C}[[\zeta^{-1}]]$ simply by $\mathbf{e}$, $\mathbf{h}$ and $\mathbf{f}$, respectively. We also write a current field $Y(X(-1)\left|0\right\rangle, z)$ by $X(z)$ for $X \in \mathfrak{g}$.

(1) $X = A = H$.

$$H(z) = e^{\mathbf{h}} H(z) e^{-\mathbf{h}} - 2k\partial h(z),$$
$$e^{-\mathbf{h}} H \otimes x(\zeta) e^{\mathbf{h}} = H \otimes x(\zeta) - 2k\mathrm{Res}_w \partial h(w) x(w).$$

(2) $X = H$, $A = E$.

$$H(z) = e^{\mathbf{e}} H(z) e^{-\mathbf{e}} + 2e(z)E(z),$$
$$e^{-\mathbf{e}} H \otimes x(\zeta) e^{\mathbf{e}} = H \otimes x(\zeta) + 2E \otimes e(\zeta) x(\zeta).$$

(3) $X = H$, $A = F$.

$$H(z) = e^{\mathbf{f}} H(z) e^{-\mathbf{f}} - 2f(z)F(z),$$
$$e^{-\mathbf{f}} H \otimes x(\zeta) e^{\mathbf{f}} = H \otimes x(\zeta) - 2F \otimes f(\zeta) x(\zeta).$$

(4) $X = E$, $A = H$.

$$E(z) = e^{-2h(z)} e^{\mathbf{h}} E(z) e^{-\mathbf{h}},$$
$$e^{-\mathbf{h}} E \otimes x(\zeta) e^{\mathbf{h}} = E \otimes e^{-2h(\zeta)} x(\zeta).$$

(5) $X = A = E$.

$$E(z) = e^{\mathbf{e}} E(z) e^{-\mathbf{e}},$$
$$e^{-\mathbf{e}} E \otimes x(\zeta) e^{\mathbf{e}} = E \otimes x(\zeta).$$

(6) $X = E$, $A = F$.

$$E(z) = e^{\mathbf{f}} E(z) e^{-\mathbf{f}} + f(z) e^{\mathbf{f}} H(z) e^{-\mathbf{f}} - f(z)^2 e^{\mathbf{f}} F(z) e^{-\mathbf{f}} - k\partial f(z),$$
$$e^{-\mathbf{f}} E \otimes x(\zeta) e^{\mathbf{f}} = E \otimes x(\zeta) + H \otimes f(\zeta) x(\zeta) - F \otimes f(\zeta)^2 x(\zeta) - k\mathrm{Res}_w \partial f(w) x(w).$$

(7) $X = F$, $A = H$.

$$F(z) = e^{2h(z)} e^{\mathbf{h}} F(z) e^{-\mathbf{h}},$$
$$e^{-\mathbf{h}} F \otimes x(\zeta) e^{\mathbf{h}} = F \otimes e^{2h(\zeta)} x(\zeta).$$

(8) $X = F$, $A = E$.

$$F(z) = e^{\mathbf{e}} F(z) e^{-\mathbf{e}} - e(z) e^{\mathbf{e}} H(z) e^{-\mathbf{e}} - e(z)^2 E(z) - k\partial e(z),$$
$$e^{-\mathbf{e}} F \otimes x(\zeta) e^{\mathbf{e}} = F \otimes x(\zeta) - H \otimes e(\zeta) x(\zeta) - E \otimes e(\zeta)^2 x(\zeta) - k\mathrm{Res}_w \partial e(w) x(w).$$

(9) $X = A = F$.

$$F(z) = e^{\mathbf{f}} F(z) e^{-\mathbf{f}},$$
$$e^{-\mathbf{f}} F \otimes x(\zeta) e^{\mathbf{f}} = F \otimes x(\zeta).$$

4.4 Construction of random process

4.4.1 General Lie algebras $\mathfrak{g}$

We shall construct a random process that is a generalization of SLE with internal symmetry described by $G_+(\mathcal{O})$. Such a random process is expected to be induced from a random process on an infinite dimensional Lie group $\mathrm{Aut}_+\mathcal{O} \ltimes G_+(\mathcal{O})$. To decide a direction of designing a random process on this group, we first make an observation on an annihilator of the vacuum vector in the vacuum representation $L_{\mathfrak{g},k}(0)$. Since we have defined a representation of the Virasoro algebra by the Segal-Sugawara construction, we have $L_{-2}|0\rangle = \frac{1}{2(k+h^\vee)} \sum_{r=1}^{\dim \mathfrak{g}} X_r(-1)^2 |0\rangle$. Combining the fact that the vacuum vector is translation invariant, we see that the operator

$$-2L_{-2} + \frac{\kappa}{2} L_{-1}^2 + \frac{1}{k+h^\vee} \sum_{r=1}^{\dim \mathfrak{g}} X_r(-1)^2 \tag{4.4.1}$$

annihilates the vacuum vector for arbitrary κ. We now assume without support that we can find a vector $v_\Lambda \in L(\Lambda)$ in the top space of a higher spin representation $L_{\mathfrak{g},k}(\Lambda)$ that is annihilated by an operator

$$-2L_{-2} + \frac{\kappa}{2} L_{-1}^2 + \frac{\tau}{2} \sum_{r=1}^{\dim \mathfrak{g}} X_r(-1)^2 \tag{4.4.2}$$

with parameters κ and τ being finely tuned positive numbers. The existence of such a vector and its annihilator of the above form will be discussed later.

A random process $\mathscr{G}_t$ on $\mathrm{Aut}_+\mathcal{O} \ltimes G_+(\mathcal{O})$ we should consider is now obvious. It satisfies the stochastic differential equation

$$\mathscr{G}_t^{-1} d\mathscr{G}_t = \left(-2L_{-2} + \frac{\kappa}{2} L_{-1}^2 + \frac{\tau}{2} \sum_{r=1}^{\dim \mathfrak{g}} X_r(-1)^2 \right) dt + L_{-1} dB_t^{(0)} + \sum_{r=1}^{\dim \mathfrak{g}} X_r(-1) dB_t^{(r)}, \tag{4.4.3}$$

where $B_t^{(i)}$ for $i = 0, 1, \cdots, \dim \mathfrak{g}$ are mutually independent Brownian motions with covariance κ for $B_t^{(0)}$ and τ for $B_t^{(r)}$ with $r = 1, \cdots, \dim \mathfrak{g}$. We comment that an idea of considering a random process on such an infinite dimensional Lie group as $\mathrm{Aut}_+\mathcal{O} \ltimes G_+(\mathcal{O})$ has already appeared in [Ras07], but it lacks a principle of writing down a stochastic differential equation based on an annihilating operator, and it does not include the classical SLE in the coordinate transformation part.

Proposition 4.4.1. *Assume that the highest weight vector v_Λ of $L_{\mathfrak{g},k}(\Lambda)$ is annihilated by the operator in Eq.(4.4.2). Then for a random process $\mathscr{G}_t$ on $\mathrm{Aut}_+\mathcal{O} \ltimes G_+(\mathcal{O})$ satisfying Eq.(4.4.3), the random process $\mathscr{G}_t v_\Lambda$ in $\overline{L_{\mathfrak{g},k}(\Lambda)}$ is a local martingale.*

We can write the random process $\mathscr{G}_t$ as $\mathscr{G}_t = \Theta_t G(\rho_t)$ where the random process ρ_t on $\mathrm{Aut}_+\mathcal{O}$ induces the SLE(κ) and Θ_t is a random process on $G_+(\mathcal{O})$.

Proposition 4.4.2. *Under the ansatz $\mathscr{G}_t = \Theta_t G(\rho_t)$ described above, the random process Θ_t on $G_+(\mathcal{O})$ satisfies the stochastic differential equation*

$$\Theta_t^{-1} d\Theta_t = \frac{\tau}{2} \sum_{r=1}^{\dim \mathfrak{g}} (X_r \otimes \rho_t(\zeta)^{-1})^2 dt + \sum_{r=1}^{\dim \mathfrak{g}} X_r \otimes \rho_t(\zeta)^{-1} dB_t^{(r)}. \tag{4.4.4}$$

Proof. The action of the Virasoro algebra on an affine Lie algebra, which is described by the relation $[L_n, X(m)] = -mX(n+m)$, implies the transformation formula

$$G(\rho) X \otimes f(\zeta) G(\rho)^{-1} = X \otimes f(\rho(\zeta)) \tag{4.4.5}$$

for $f(\zeta) \in \mathbb{C}((\zeta^{-1}))$ and $\rho \in \mathrm{Aut}_+\mathcal{O}$. If we apply this formula in the case that $f(\zeta) = \zeta^{-1}$, we obtain the desired result. □

This equation has already appeared in an equivalent form in the correlation function formulation of SLEs corresponding to WZW models [BGLW05, ABI11]. Let $\mathcal{Y}(-, z)$ be an intertwining operator of type $\binom{L_{\mathfrak{g},k}(\Lambda_3)}{L_{\mathfrak{g},k}(\Lambda_1)\ L_{\mathfrak{g},k}(\Lambda_2)}$, and $v \in L(\Lambda_1)$ be a primary vector in the top space of $L_{\mathfrak{g},k}(\Lambda_1)$. If we take adjoint of the primary field $\mathcal{Y}(v, z)$ by $\mathscr{G}_t^{-1}$, we obtain

$$\mathscr{G}_t^{-1} \mathcal{Y}(v, z) \mathscr{G}_t = \mathcal{Y}(\Theta_t^{-1}(z) v, \rho_t(z)) (\partial \rho_t(z))^{h_{\Lambda_1}}. \tag{4.4.6}$$

Here the object $\Theta_t^{-1}(z)$ is a random process on the group of $z^{-1}\mathbb{C}[[z^{-1}]]$-points in G that is obtained by substituting $\zeta = z$ in Θ_t^{-1}. From the identity $\Theta_t^{-1}\Theta_t = \mathrm{Id}$, the stochastic differential equation on $\Theta_t^{-1}(z)$ becomes

$$d\Theta_t^{-1}(z)\Theta_t(z) = \frac{\tau}{2} \sum_{r=1}^{\dim \mathfrak{g}} (\rho_t(z)^{-1} X_r)^2 - \sum_{r=1}^{\dim \mathfrak{g}} \rho_t(z)^{-1} X_r dB_t^{(r)}. \tag{4.4.7}$$

Apart from the Jacobian part, the right hand side of Eq.(4.4.6) is the random transformation of a primary field in Eq.(4.1.2) considered in the correlation function formulation of SLEs [BGLW05, ABI11], which seemed to be *ad hoc*, while it naturally appears in the group theoretical formulation presented here.

4.4.2 Specialization to $\mathfrak{sl}_2$

It is not sufficient to derive the stochastic differential equation on Θ_t for writing down local martingales associated to SLE. Now we specialize our attention to the case of $\mathfrak{g} = \mathfrak{sl}_2$. To construct the random process Θ_t in a sufficiently explicit way, we make an ansatz that it is written as $\Theta_t = e^{\mathbf{e}_t} e^{\mathbf{h}_t} e^{\mathbf{f}_t}$, where $\mathbf{e}_t = E \otimes e_t(\zeta)$, $\mathbf{h}_t = H \otimes h_t(\zeta)$, $\mathbf{f}_t = F \otimes f_t(\zeta)$ are random processes on $\mathfrak{g}_+(\mathcal{O})$ associated with $\mathbb{C}[[\zeta^{-1}]]\zeta^{-1}$-valued random processes $e_t(\zeta)$, $h_t(\zeta)$ and $f_t(\zeta)$. Then we shall derive stochastic differential equations on $e_t(\zeta)$, $h_t(\zeta)$ and $f_t(\zeta)$. To this end, we assume the stochastic differential equations on them as

$$dx_t(\zeta) = \overline{x}_t(\zeta) dt + \sum_{r=1}^{3} x_t^r(\zeta) dB_t^{(r)}, \quad x = e, h, f. \tag{4.4.8}$$

Since $X(n)$ with $n<0$ are mutually commutative for a fixed $X\in\mathfrak{sl}_2$, the increment of the random process Θ_t is computed by the standard Ito calculus, and we can determine data $\overline{x}_t(\zeta)$ and $x^{(r)}_t(\zeta)$ so the increment of Θ_t to be the desired form in Eq.(4.4.4). After computation that is presented in Appendix D, we obtain the following:

Proposition 4.4.3. *Under the ansatz $\Theta_t=e^{\mathbf{e}_t}e^{\mathbf{h}_t}e^{\mathbf{f}_t}$ described above, the stochastic differential equation in Eq.(4.4.4) implies the following set of stochastic differential equations:*

$$\begin{aligned} de_t(\zeta)=&-\frac{e^{2h_t(\zeta)}}{\sqrt{2}\rho_t(\zeta)}dB^{(2)}_t-\frac{ie^{2h_t(\zeta)}}{\sqrt{2}\rho_t(\zeta)}dB^{(3)}_t, && (4.4.9)\\ dh_t(\zeta)=&-\frac{\tau}{2\rho_t(\zeta)^2}dt\\ &-\frac{1}{\sqrt{2}\rho_t(\zeta)}dB^{(1)}_t+\frac{f_t(\zeta)}{\sqrt{2}\rho_t(\zeta)}dB^{(2)}_t+\frac{if_t(\zeta)}{\sqrt{2}\rho_t(\zeta)}dB^{(3)}_t. && (4.4.10)\\ df_t(\zeta)=&-\frac{\sqrt{2}f_t(\zeta)}{\rho_t(\zeta)}dB^{(1)}_t-\frac{1-f_t(\zeta)^2}{\sqrt{2}\rho_t(\zeta)}dB^{(2)}_t+\frac{i(1+f_t(\zeta)^2)}{\sqrt{2}\rho_t(\zeta)}dB^{(3)}_t. && (4.4.11)\end{aligned}$$

4.5 Search for a vector v_Λ

We have assumed that there exists a vector v_Λ in the top space of $L_{\mathfrak{g}.k}(\Lambda)$ that is annihilated by the operator in Eq.(4.4.2) with finely tuned parameters κ and τ. In this section we see examples of such vectors. As we have already seen, the vacuum vector $|0\rangle$ is annihilated by the operator in Eq.(4.4.2) for $\tau=\frac{2}{k+h^\vee}$ and arbitrary κ. Thus we shall search for an example in a spin representation.

Now we again focus on the case of $\mathfrak{g}=\mathfrak{sl}_2$ and moreover we assume that the level is $k=1$. In this case, the vacuum representation $L_{\mathfrak{sl}_2,1}$ is isomorphic as a VOA to the lattice vertex operator algebra V_Q associated with the root lattice $Q=\mathbb{Z}\alpha$, $(\alpha|\alpha)=2$ of $\mathfrak{sl}_2$. The isomorphism is described by

$$E(z)\mapsto\Gamma_\alpha(z),\qquad H(z)\mapsto\alpha(z),\qquad F(z)\mapsto\Gamma_{-\alpha}(z). \tag{4.5.1}$$

Here $\alpha(z)$ is the free Bose field and $\Gamma_{\pm\alpha}(z)$ are the vertex operators associated with $\pm\alpha\in Q$. This isomorphism of VOAs is called the Frenkel-Kac construction of an affine VOA [FK80], of which an exposition is also contained in Appendix B. The dominant weights of level $k=1$ are exhausted by 0 and the fundamental weight ϖ such that $(\varpi|\alpha)=1$. The spin-$\frac{1}{2}$ representation $L_{\mathfrak{g},1}(\varpi)$ corresponding to ϖ is also realized as a module of the lattice VOA V_Q by $V_{Q+\varpi}$ that is defined by

$$V_{Q+\varpi}=\bigoplus_{\beta\in Q}M_{\mathbb{C}\otimes_{\mathbb{Z}}Q}(1,0)\otimes e^{\beta+\varpi}. \tag{4.5.2}$$

Here $M_{\mathbb{C}\otimes_{\mathbb{Z}}Q}(1,0)$ is the Heisenberg Fock space introduced in Chapter 3. Let the top space of $L_{\mathfrak{sl}_2,1}(\varpi)$ be realized as $L(\varpi)=\mathbb{C}v_0\oplus\mathbb{C}v_1$ so that $Ev_0=0$. Then the isomorphism $L_{\mathfrak{sl}_2,1}(\varpi)\simeq V_{Q+\varpi}$ is determined by

$$v_0\mapsto e^{\varpi},\quad v_1\mapsto e^{-\varpi}. \tag{4.5.3}$$

We show that both v_0 and v_1 is annihilated by an operator of the form in Eq.(4.4.2). Let $\mathcal{Y}(-,z)$ be the intertwining operator of type $\binom{L_{\mathfrak{sl}_2,1}(\varpi)}{L_{\mathfrak{sl}_2,1}(\varpi)\ \ L_{\mathfrak{sl}_2,1}}$. Then we have $\mathcal{Y}(e^{\pm\varpi},z)=$

$\Gamma_{\pm\varpi}(z)$, where $\Gamma_{\pm\varpi}(z)$ are generalized vertex operators associated with $\pm\varpi$. Such a realization of an intertwining operator allows us to obtain

$$L_{-2}e^{\pm\varpi} = L_{-1}^2 e^{\pm\varpi} = \left(\frac{1}{4}\alpha(-1)^2 \pm \frac{1}{2}\alpha(-2)\right) e^{\pm\varpi} \tag{4.5.4}$$

by computation of operator product expansions. In the case of $\mathfrak{g} = \mathfrak{sl}_2$, we have

$$\sum_{r=1}^{3} X_r(-1)^2 = \frac{1}{2}H(-1)^2 + E(-1)F(-1) + F(-1)E(-1). \tag{4.5.5}$$

It is obvious that $E(-1)e^{\varpi} = 0$ from $\Gamma_\alpha(z)\Gamma_\varpi(w) = (z-w)\Gamma_{\alpha,\varpi}(z,w)$. On the other hand, $F(-1)$ nontrivially acts on e^{ϖ} and further applying $E(-1)$, we have $E(-1)F(-1)e^{\varpi} = \alpha(-2)e^{\varpi}$. Combining them we can see that

$$\left(-2L_{-2} + \frac{\kappa}{2}L_{-1}^2 + \frac{\tau}{2}\sum_{r=1}^{3} X_r(-1)^2\right) e^{\varpi} = 0 \tag{4.5.6}$$

if the relation $\kappa + 2\tau - 4 = 0$. Computation for $e^{-\varpi}$ is carried in the analogous way. We have $F(-1)e^{-\varpi} = 0$, while $F(-1)E(-1)e^{-\varpi} = -\alpha(-2)e^{-\varpi}$, which leads us to

$$\left(-2L_{-2} + \frac{\kappa}{2}L_{-1}^2 + \frac{\tau}{2}\sum_{r=1}^{3} X_r(-1)^2\right) e^{-\varpi} = 0 \tag{4.5.7}$$

if the parameters κ and τ satisfies the same relation $\kappa + 2\tau - 4 = 0$ as in the case of e^{ϖ}.

We summarize the above computation as follows:

Proposition 4.5.1. *Let Λ be the fundamental weight of $\mathfrak{sl}_2$, and the fundamental representation of $\mathfrak{sl}_2$ be described by $L(\Lambda) = \mathbb{C}v_\Lambda \oplus \mathbb{C}Fv_\Lambda$. Here v_Λ is the highest weight vector of highest weight Λ. We also denote the vector Fv_Λ by $v_{-\Lambda}$. Then we have in $L_{\mathfrak{sl}_2,1}(\Lambda)$*

$$\left(-2L_{-2} + \frac{\kappa}{2}L_{-1}^2 + \frac{\tau}{2}\sum_{r=1}^{3} X_r(-1)^2\right) v_{\pm\Lambda} = 0 \tag{4.5.8}$$

if the relation $\kappa + 2\tau - 4 = 0$ holds.

4.6 Local martingales

We write down in this section several local martingales associated with SLE with affine symmetry that are generated by a local martingale $\mathscr{G}_t v_\Lambda$ in $\overline{L_{\mathfrak{sl}_2,k}(\Lambda)}$. We treat the case that Λ is the fundamental weight of $\mathfrak{sl}_2$ and $k = 1$ and use the description $L(\Lambda) = \mathbb{C}v_\Lambda \oplus \mathbb{C}v_{-\Lambda}$ of the fundamental weight as in Prop. 4.5.1

Firstly we write down transformation formulae for current fields $X(z)$ for $X = E, H, F$ under adjoint action by $\mathscr{G}_t^{-1}$.

Lemma 4.6.1.

$$
\begin{aligned}
\mathscr{G}_t^{-1}E(z)\mathscr{G}_t =& e^{-2h_t(z)}\partial\rho_t(z)E(\rho_t(z)) + e^{-2h_t(z)}f_t(z)\partial\rho_t(z)H(\rho_t(z)) \\
&- e^{-2h_t(z)}f_t(z)^2\partial\rho_t(z)F(\rho_t(z)) - k\partial f_t(z), \qquad (4.6.1)\\
\mathscr{G}_t^{-1}H(z)\mathscr{G}_t =& 2e^{-2h_t(z)}e_t(z)\partial\rho_t(z)E(\rho_t(z)) + (1+2e^{-2h_t(z)}e_t(z)f_t(z))\partial\rho_t(z)H(\rho_t(z)) \\
&- (2f_t(z)+2e^{-2h_t(z)}e_t(z)f_t(z)^2)\partial\rho_t(z)F(\rho_t(z)) \\
&- k(2\partial h_t(z) + 2e^{-2h_t(z)}e_t(z)\partial f_t(z)), \qquad (4.6.2)\\
\mathscr{G}_t^{-1}F(z)\mathscr{G}_t =& - e^{-2h_t(z)}e_t(z)^2\partial\rho_t(z)E(\rho_t(z)) \\
&- (e_t(z) + e^{-2h_t(z)}e_t(z)^2f_t(z))\partial\rho_t(z)H(\rho_t(z)) \\
&+ (2e_t(z)f_t(z) + e^{-2h_t(z)}e_t(z)^2f_t(z)^2)\partial\rho_t(z)F(\rho_t(z)) \\
&+ k(2e_t(z)\partial f_t(z) + e^{-2h_t(z)}e_t(z)^2\partial f_t(z) - \partial e_t(z)). \qquad (4.6.3)
\end{aligned}
$$

This will allow us to compute local martingales of the form $\langle v_{\pm\Lambda}|X(z)\mathscr{G}_t v_{\pm\Lambda}\rangle$ for $X = E, H, F$.

Theorem 4.6.2. *Let Λ be the fundamental weight of $\mathfrak{sl}_2$, and the fundamental representation of $\mathfrak{sl}_2$ be described by $L(\Lambda) = \mathbb{C}v_\Lambda \oplus \mathbb{C}v_{-\Lambda}$ as in Prop. 4.5.1. We assume that κ and τ be positive real numbers satisfying the relation $\kappa + 2\tau - 4 = 0$. For the solution $\rho_t(z)$ of SLE(κ) and random processes $e_t(z)$, $h_t(z)$ and $f_t(z)$ satisfying the stochastic differential equations in Prop. 4.4.3, the following quantities are local martingales.*

(1) $X = E$.

$$\langle v_\Lambda|E(z)\mathscr{G}_t v_\Lambda\rangle = e^{-2h_t(z)}f_t(z)\frac{\partial\rho_t(z)}{\rho_t(z)} - \partial f_t(z), \qquad (4.6.4)$$

$$\langle v_{-\Lambda}|E(z)\mathscr{G}_t v_\Lambda\rangle = -e^{-2h_t(z)}f_t(z)^2\frac{\partial\rho_t(z)}{\rho_t(z)}, \qquad (4.6.5)$$

$$\langle v_\Lambda|E(z)\mathscr{G}_t v_{-\Lambda}\rangle = e^{-2h_t(z)}\frac{\partial\rho_t(z)}{\rho_t(z)}, \qquad (4.6.6)$$

$$\langle v_{-\Lambda}|E(z)\mathscr{G}_t v_{-\Lambda}\rangle = -e^{-2h_t(z)}f_t(z)\frac{\partial\rho_t(z)}{\rho_t(z)} - \partial f_t(z). \qquad (4.6.7)$$

(2) $X = H$.

$$
\begin{aligned}
\langle v_\Lambda|H(z)\mathscr{G}_t v_\Lambda\rangle =& (1+2e^{-2h_t(z)}e_t(z)f_t(z))\frac{\partial\rho_t(z)}{\rho_t(z)} \\
&- (2\partial h_t(z) + 2e^{-2h_t(z)}e_t(z)\partial f_t(z)), \qquad (4.6.8)\\
\langle v_{-\Lambda}|H(z)\mathscr{G}_t v_\Lambda\rangle =& - (2f_t(z) + 2e^{-2h_t(z)}e_t(z)f_t(z)^2)\frac{\partial\rho_t(z)}{\rho_t(z)}, \qquad (4.6.9)\\
\langle v_\Lambda|H(z)\mathscr{G}_t v_{-\Lambda}\rangle =& 2e^{-2h_t(z)}e_t(z)\frac{\partial\rho_t(z)}{\rho_t(z)}, \qquad (4.6.10)\\
\langle v_{-\Lambda}|H(z)\mathscr{G}_t v_{-\Lambda}\rangle =& - (1+2e^{-2h_t(z)}e_t(z)f_t(z))\frac{\partial\rho_t(z)}{\rho_t(z)} \\
&- (2\partial h_t(z) + 2e^{-2h_t(z)}e_t(z)\partial f_t(z)). \qquad (4.6.11)
\end{aligned}
$$

(3) $X = F$.

$$\langle v_\Lambda | F(z) \mathscr{G}_t v_\Lambda \rangle = -\left(e_t(z) + e^{-2h_t(z)} e_t(z)^2 f_t(z)\right) \frac{\partial \rho_t(z)}{\rho_t(z)} + \left(2 e_t(z) \partial f_t(z) + e^{-2h_t(z)} e_t(z)^2 \partial f_t(z) - \partial e_t(z)\right), \tag{4.6.12}$$

$$\langle v_{-\Lambda} | F(z) \mathscr{G}_t v_\Lambda \rangle = \left(2 e_t(z) f_t(z) + e^{-2h_t(z)} e_t(z)^2 f_t(z)^2\right) \frac{\partial \rho_t(z)}{\rho_t(z)}, \tag{4.6.13}$$

$$\langle v_\Lambda | F(z) \mathscr{G}_t v_{-\Lambda} \rangle = - e^{-2h_t(z)} e_t(z)^2 \frac{\partial \rho_t(z)}{\rho_t(z)}, \tag{4.6.14}$$

$$\langle v_{-\Lambda} | F(z) \mathscr{G}_t v_{-\Lambda} \rangle = \left(e_t(z) + e^{-2h_t(z)} e_t(z)^2 f_t(z)\right) \frac{\partial \rho_t(z)}{\rho_t(z)} + \left(2 e_t(z) \partial f_t(z) + e^{-2h_t(z)} e_t(z)^2 \partial f_t(z) - \partial e_t(z)\right). \tag{4.6.15}$$

Proof. By assumption, we have that $\mathscr{G}_t v_{\pm\Lambda}$ are local martingale in $\overline{L_{\mathfrak{sl}_2,1}(\Lambda)}$ from Proposition 4.4.1 and Proposition 4.5.1. Thus the quantities $\langle u | \mathscr{G}_t | v_{\pm\Lambda} \rangle$ are local martingales. Noticing that $\langle v_{\pm\Lambda} | \mathscr{G}_t = \langle v_{\pm\Lambda} |$ and using the formula in Lemma 4.6.1, we obtain the desired result. The detailed computation is presented in Appendix D. □

We also compute the local martingales $\langle v_{\pm\Lambda} | L(z) \mathscr{G}_t v_{\pm\Lambda} \rangle$ for the Virasoro field $L(z)$. The Virasoro field is found to be transformed under adjoint action by $\mathscr{G}_t^{-1}$ as follows.

Lemma 4.6.3.

$$\begin{aligned} \mathscr{G}_t^{-1} L(z) \mathscr{G}_t = & (\partial \rho_t(z))^2 L(\rho_t(z)) \\ & - e^{-2h_t(z)} \partial e_t(z) \partial \rho_t(z) E(\rho_t(z)) \\ & - \left(\partial h_t(z) + e^{-2h_t(z)} f_t(z) \partial e_t(z)\right) \partial \rho_t(z) H(\rho_t(z)) \\ & - \left(\partial f_t(z) - 2 f_t(z) \partial h_t(z) - e^{-2h_t(z)} f_t(z)^2 \partial e_t(z)\right) \partial \rho_t(z) F(\rho_t(z)) \\ & + k\left((\partial h_t(z))^2 + e^{-2h_t(z)} \partial e_t(z) \partial f_t(z)\right) + \frac{c}{12} (S\rho_t)(z). \end{aligned} \tag{4.6.16}$$

Theorem 4.6.4. *Let Λ be the fundamental weight of $\mathfrak{sl}_2$, and the fundamental representation of $\mathfrak{sl}_2$ be described by $L(\Lambda) = \mathbb{C} v_\Lambda \oplus \mathbb{C} v_{-\Lambda}$ as in Prop. 4.5.1. We assume that κ and τ be positive real numbers satisfying the relation $\kappa + 2\tau - 4 = 0$. For the solution $\rho_t(z)$ of SLE(κ) and random processes $e_t(z)$, $h_t(z)$ and $f_t(z)$ satisfying the stochastic differential equations in Prop. 4.4.3, the following quantities are local martingales.*

$$\langle v_\Lambda | L(z) \mathscr{G}_t v_\Lambda \rangle = \frac{1}{4} \left(\frac{\partial \rho_t(z)}{\rho_t(z)}\right)^2 - \left(\partial h_t(z) + e^{-2h_t(z)} f_t(z) \partial e_t(z)\right) \frac{\partial \rho_t(z)}{\rho_t(z)} + \left((\partial h_t(z))^2 + e^{-2h_t(z)} \partial e_t(z) \partial f_t(z)\right) + \frac{1}{12} (S\rho_t)(z), \tag{4.6.17}$$

$$\langle v_{-\Lambda} | L(z) \mathscr{G}_t v_\Lambda \rangle = -\left(\partial f_t(z) - 2 f_t(z) \partial h_t(z) - e^{-2h_t(z)} f_t(z)^2 \partial e_t(z)\right) \frac{\partial \rho_t(z)}{\rho_t(z)}, \tag{4.6.18}$$

$$\langle v_\Lambda | L(z) \mathscr{G}_t v_{-\Lambda} \rangle = - e^{-2h_t(z)} \partial e_t(z) \frac{\partial \rho_t(z)}{\rho_t(z)}, \tag{4.6.19}$$

$$\langle v_{-\Lambda} | L(z) \mathscr{G}_t v_{-\Lambda} \rangle = \frac{1}{4} \left(\frac{\partial \rho_t(z)}{\rho_t(z)}\right)^2 + \left(\partial h_t(z) + e^{-2h_t(z)} f_t(z) \partial e_t(z)\right) \frac{\partial \rho_t(z)}{\rho_t(z)} + \left((\partial h_t(z))^2 + e^{-2h_t(z)} \partial e_t(z) \partial f_t(z)\right) + \frac{1}{12} (S\rho_t)(z). \tag{4.6.20}$$

Proof. The proof is analogous to on of Theorem 4.6.2. We note that on $L_{\mathfrak{sl}_2,1}(\Lambda)$, the central charge is $c = 1$ and the conformal weight of the highest weight vector v_Λ is $\frac{1}{4}$. □

4.7 Symmetry of the space of local martingales

In the previous section, we saw that a local martingale $\mathscr{G}_t v_\Lambda$ that takes its value in $\overline{L_{\mathfrak{sl}_2,k}(\Lambda)}$ generates several local martingales. We shall describe this situation from a different point of view.

Let $\mathcal{Y}(-,z)$ be an intertwining operator of type $\binom{L_{\mathfrak{sl}_2,k}(\Lambda)}{L_{\mathfrak{sl}_2,k}(\Lambda)\ \ L_{\mathfrak{sl}_2,k}}$. Then for a vector $v \in L_{\mathfrak{g},k}(\Lambda)$, we have $\mathcal{Y}(v,z)\,|0\rangle = e^{zL_{-1}}v$. This implies that for a vector $v_\Lambda \in L(\Lambda)$ in the top space of $L_{\mathfrak{g},k}(\Lambda)$ that is annihilated by an operator of the form of Eq.(4.4.2),

$$\mathscr{G}_t v_\Lambda = \Theta_t G(g_t)\mathcal{Y}(v_\Lambda, B_t)\,|0\rangle \tag{4.7.1}$$

is a local martingale.

In the Virasoro case in Chapter 2, we saw that the space of SLE local martingales is equipped with a Virasoro modules structure, which is constructed as a subspace of $C^\infty(\mathbb{R},\mathbb{C})[g_0, g_{-1}, \cdots]$. In affine case, we analogously find an affine Lie algebra module structure on the space of local SLE martingales. The idea is similar to the case of the Virasoro algebra. For a generic element in $\mathrm{Aut}_+\mathcal{O} \ltimes G_+(\mathcal{O})$ and an intertwining operator $\mathcal{Y}(-,z)$ of type $\binom{L_{\mathfrak{g},k}(\Lambda)}{L_{\mathfrak{g},k}(\Lambda)\ \ L_{\mathfrak{g},k}}$, the quantity

$$Q_u = \langle u|\mathscr{G}\mathcal{Y}(-,x)|0\rangle \in L(\Lambda)^*[g_{n+1}, e_n, h_n, f_n | n<0][[x]] =: \mathcal{F}_{\mathrm{aff}}(\Lambda) \tag{4.7.2}$$

for any vector $u \in L_{\mathfrak{g},k}(\Lambda)$ gives a local martingale when we evaluate g_n, e_n, h_n, f_n at the solution of SLE, and x at the Brownian motion of covariance κ. Thus we may find the space of local martingales as a subspace of $\mathcal{F}_{\mathrm{aff}}(\Lambda)$. Since u is arbitrarily taken, the quantity $Q_{X(\ell)u}$ for $X \in \mathfrak{sl}_2$ has the same property. Thus if we find a operator $\mathscr{X}_\ell$ such that $Q_{X(\ell)u} = \mathscr{X}_\ell Q_u$, it can describe affine Lie algebra symmetry of a space of local martingales in $\mathcal{F}_{\mathrm{aff}}(\Lambda)$. The derivation of the operators $\mathscr{X}_\ell$ is presented in Appendix D, and we only write down the results.

$$\begin{aligned}
\mathscr{E}_\ell = &-\sum_{n\le -1}\mathrm{Res}_z\mathrm{Res}_w \frac{w^{-n-1}e^{2h(w)}e^{-2h(z)}z^{-\ell}g'(z)}{g(w)-g(z)}\frac{\partial}{\partial e_n} \\
&-\sum_{n\le -1}\mathrm{Res}_z\mathrm{Res}_w \frac{w^{-n-1}e^{-2h(z)}(f(z)-f(w))z^{-\ell}g'(z)}{g(w)-g(z)}\frac{\partial}{\partial h_n} \\
&+\sum_{n\le -1}\mathrm{Res}_z\mathrm{Res}_w \frac{w^{-n-1}e^{-2h(z)}(f(z)-f(w))^2z^{-\ell}g'(z)}{g(w)-g(z)}\frac{\partial}{\partial f_n} \\
&+\mathrm{Res}_z\frac{e^{-2h(z)}z^{-\ell}g'(z)}{g(z)-x}\pi(E) \\
&+\mathrm{Res}_z\frac{e^{-2h(z)}f(z)z^{-\ell}g'(z)}{g(z)-x}\pi(H) \\
&-\mathrm{Res}_z\frac{e^{-2h(z)}f(z)^2z^{-\ell}g'(z)}{g(z)-x}\pi(F) \\
&+k\mathrm{Res}_z\partial f(z)e^{-2h(z)}z^{-\ell}.
\end{aligned} \tag{4.7.3}$$

$$
\begin{aligned}
\mathscr{H}_\ell = & -2\sum_{n\le -1}\mathrm{Res}_z\mathrm{Res}_w\frac{w^{-n-1}e^{2h(w)}e^{-2h(z)}e(z)z^{-\ell}g'(z)}{g(w)-g(z)}\frac{\partial}{\partial e_n}\\
& -\sum_{n\le -1}\mathrm{Res}_z\mathrm{Res}_w\frac{w^{-n-1}(1+2e^{-2h(z)}(f(z)-f(w)))z^{-\ell}g'(z)}{g(w)-g(z)}\frac{\partial}{\partial h_n}\\
& -2\sum_{n\le -1}\mathrm{Res}_z\mathrm{Res}_w\frac{w^{-n-1}(f(w)-f(z)-e^{-2h(z)}e(z)(f(w)-f(z))^2)z^{-\ell}g'(z)}{g(w)-g(z)}\frac{\partial}{\partial f_n}\\
& +2\mathrm{Res}_z\frac{e^{-2h(z)}e(z)z^{-\ell}g'(z)}{g(z)-x}\pi(E)\\
& +\mathrm{Res}_z\frac{(1+2e^{-2h(z)}e(z)f(z))z^{-\ell}g'(z)}{g(z)-x}\pi(H)\\
& -2\mathrm{Res}_z\frac{(1+e^{-2h(z)}e(z)f(z))f(z)z^{-\ell}g'(z)}{g(z)-x}\pi(F)\\
& +2k\mathrm{Res}_z(\partial h(z)-\partial f(z)e^{-2h(z)}e(z))z^{-\ell}.
\end{aligned}
\tag{4.7.4}
$$

$$
\begin{aligned}
\mathscr{F}_\ell = & \sum_{n\le -1}\mathrm{Res}_z\mathrm{Res}_w\frac{w^{-n-1}e^{2h(w)}e^{-2h(z)}e(z)^2z^{-\ell}g'(z)}{g(w)-g(z)}\frac{\partial}{\partial e_n}\\
& -\sum_{n\le -1}\mathrm{Res}_z\mathrm{Res}_w\frac{w^{-n-1}(1+e^{-2h(z)}e(z)(f(w)-f(z)))e(z)z^{-\ell}g'(z)}{g(w)-g(z)}\frac{\partial}{\partial h_n}\\
& -\sum_{n\le -1}\mathrm{Res}_z\mathrm{Res}_w w^{-n-1}\Bigg[\frac{e^{2h(z)}+2e(z)(f(z)-f(w))}{g(w)-g(z)}\\
& \qquad\qquad +\frac{e^{-2h(z)}e(z)^2(f(z)-f(w))^2}{g(w)-g(z)}\Bigg]z^{-\ell}g'(z)\frac{\partial}{\partial f_n}\\
& -\mathrm{Res}_z\frac{e^{-2h(z)}e(z)^2z^{-\ell}g'(z)}{g(z)-x}\pi(E)\\
& -\mathrm{Res}_z\frac{(1+e^{-2h(z)}e(z)f(z))e(z)z^{-\ell}g'(z)}{g(z)-x}\pi(H)\\
& +\mathrm{Res}_z\frac{(e^{2h(z)}+2e(z)f(z)+e^{-2h(z)}e(z)^2f(z)^2)z^{-\ell}g'(z)}{g(z)-x}\pi(F)\\
& -\mathrm{Res}_z(2\partial h(z)e(z)-\partial e(z)+\partial f(z)e^{-2h(z)}e(z)^2)z^{-\ell}.
\end{aligned}
\tag{4.7.5}
$$

Here the representation $\pi : \mathfrak{sl}_2 \to \mathrm{End}(L(\Lambda)^*)$ is defined by $(\pi(X)\phi)(v) = -\phi(Xv)$ for $X \in \mathfrak{sl}_2$, $\phi \in L(\Lambda)^*$ and $v \in L(\Lambda)$.

We can also derive operators $\mathscr{L}_\ell$ that associate with the action of the Virasoro algebra such that $\langle L_\ell u|\mathscr{G}\mathcal{Y}(-,x)|0\rangle = \mathscr{L}_\ell \langle u|\mathscr{G}\mathcal{Y}(-,x)|0\rangle$. While the detailed derivation is postponed to the appendix, it yields

$$
\begin{aligned}
\mathscr{L}_\ell = &- \sum_{n\leq 0} \mathrm{Res}_z \mathrm{Res}_w \frac{z^{-\ell+1} w^{-n-1} g'(z)^2}{g(w)-g(z)} \frac{\partial}{\partial g_n} \\
&- \sum_{n\leq -1} \mathrm{Res}_z \mathrm{Res}_w \frac{z^{-\ell+1} w^{-n-1} e^{2h(w)} e^{-2h(z)} \partial e(z) g'(z)}{g(w)-g(z)} \frac{\partial}{\partial e_n} \\
&- \sum_{n\leq -1} \mathrm{Res}_z \mathrm{Res}_w \frac{z^{-\ell+1} w^{-n-1} (\partial h(z) + e^{-2h(z)} \partial e(z)(f(z)-f(w))) g'(z)}{g(w)-g(z)} \frac{\partial}{\partial h_n} \\
&- \sum_{n\leq -1} \mathrm{Res}_z \mathrm{Res}_w z^{-\ell+1} w^{-n-1} \left[\frac{\partial f(z) - 2\partial h(z)(f(z)-f(w))}{g(w)-g(z)} \right. \\
&\qquad\qquad \left. - \frac{e^{-2h(z)} \partial e(z) (f(z)-f(w))^2}{g(w)-g(z)} \right] g'(z) \frac{\partial}{\partial f_n} \\
&+ \mathrm{Res}_z z^{-\ell+1} g'(z)^2 \left(\frac{h}{(g(z)-x)^2} + \frac{1}{g(z)-x} \frac{\partial}{\partial x} \right) \\
&+ \mathrm{Res}_z \frac{z^{-\ell+1} e^{-2h(z)} \partial e(z) g'(z)}{g(z)-x} \pi(E) \\
&+ \mathrm{Res}_z \frac{z^{-\ell+1} (\partial h(z) + e^{-2h(z)} f(z) \partial e(z)) g'(z)}{g(z)-x} \pi(H) \\
&+ \mathrm{Res}_z \frac{z^{-\ell+1} (\partial f(z) - 2f(z)\partial h(z) - e^{-2h(z)} f(z)^2 \partial e(z)) g'(z)}{g(z)-x} \pi(F) \\
&+ \mathrm{Res}_z z^{-\ell+1} \left(\frac{c}{12} (Sg)(z) + k(\partial h(z)^2 + e^{-2h(z)} \partial f(z) \partial e(z)) \right). \qquad (4.7.6)
\end{aligned}
$$

For a vector $v \in L(\Lambda)$ in the top space of $L_{\mathfrak{sl}_2,k}(\Lambda)$, the corresponding local martingale Q_v is a constant function in x that takes value $\langle v|-\rangle \in L(\Lambda)^*$. Applying the operators $\mathscr{X}_\ell$ on elements Q_v for $v \in L(\Lambda)$, we obtain all local martingales that are generated by a random process $\mathscr{G}_t$ on $\mathrm{Aut}_+\mathcal{O} \ltimes G_+(\mathcal{O})$.

Theorem 4.7.1. *Assume that we have an operator of the form in Eq.(4.4.2) that annihilates the highest weight vector of $L_{\mathfrak{sl}_2,k}(\Lambda)$. Let $\mathcal{U}$ be the subspace of $\mathcal{F}_{\mathrm{aff}}(\Lambda)$ that is obtained by applying operators $\mathscr{X}_\ell$ for $\mathscr{X} = \mathscr{E}, \mathscr{H}, \mathscr{F}$ and $\ell \in \mathbb{Z}$ to elements of the form $\langle u|-\rangle \in L(\Lambda)^*$ for $u \in L(\Lambda)$. Then an element of $\mathcal{U}$ gives a local martingale when the solution of SLE(κ) and the stochastic differential equations in Proposition 4.4.3 being substituted. Namely, for an element $f(g_n, e_n, h_n, f_n) \in \mathcal{U}$,*

$$f(g_n(t), e_n(t), h_n(t), f_n(t))(u) \qquad (4.7.7)$$

is a local martingale for an arbitrary $u \in L(\Lambda)$. Here

$$g_t(z) = \sum_{n\leq 0} g_n(t) z^n \qquad (4.7.8)$$

satisfies SLE(κ) and

$$e_t(z) = \sum_{n<0} e_n(t) z^n, \tag{4.7.9}$$

$$h_t(z) = \sum_{n<0} h_n(t) z^n, \tag{4.7.10}$$

$$f_t(z) = \sum_{n<0} f_n(t) z^n \tag{4.7.11}$$

satisfy the stochastic differential equations in Proposition 4.4.3.

4.8 Generalization

We have constructed SLE-type growth processes associated with representation of affine Lie algebras. The most relevant foundation of our construction is the existence of a vector v_Λ that is annihilated by an operator of the form of Eq.(4.4.2). We can generalize our construction to obtain stochastic processes associated with more general annihilator, say of the form

$$-2L_{-2n} + \frac{\kappa}{2} L_{-n}^2 + \frac{\tau}{2} \sum_{r=1}^{\dim \mathfrak{g}} X_r(-n)^2 \tag{4.8.1}$$

for some positive integer n.

4.8.1 Construction of random processes

If there is a vector v_Λ annihilated by the operator in Eq.(4.8.1), it motivates us to consider a random process $\mathscr{G}_t$ on $\mathrm{Aut}_+\mathcal{O} \ltimes G_+(\mathcal{O})$ that satisfies the stochastic differential equation

$$\mathscr{G}_t^{-1} d\mathscr{G}_t = \left(-2L_{-2n} + \frac{\kappa}{2} L_{-n}^2 + \frac{\tau}{2} \sum_{r=1}^{\dim \mathfrak{g}} X_r(-n)^2 \right) dt + L_{-n} dB_t^{(0)} + \sum_{r=1}^{\dim \mathfrak{g}} X_r(-n) dB_t^{(r)}. \tag{4.8.2}$$

Here $B_t^{(i)}$ with $i = 0, 1, \cdots, \dim \mathfrak{g}$ are mutually independent Brownian motions with covariance κ for $B_t^{(0)}$ and τ for $B_t^{(r)}$ with $r = 1, \cdots, \dim \mathfrak{g}$. If we write $\mathscr{G}_t$ as $\mathscr{G}_t = \Theta_t G(\rho_t)$ where Θ_t and ρ_t are random processes on $G_+(\mathcal{O})$ and $\mathrm{Aut}_+\mathcal{O}$, respectively, we obtain stochastic differential equations on them as

$$d\rho_t(z) = \frac{(2 + \frac{\kappa}{2}(-n+1))dt}{\rho_t(z)^{2n-1}} - \frac{dB_t^{(0)}}{\rho_t(z)^{n-1}}, \tag{4.8.3}$$

$$\Theta_t^{-1} d\Theta_t = \frac{\tau}{2} \sum_{r=1}^{\dim \mathfrak{g}} (X_r \otimes (\rho_t(\zeta))^{-n})^2 dt + \sum_{r=1}^{\dim \mathfrak{g}} X_r \otimes (\rho_t(\zeta))^{-n} dB_t^{(r)}. \tag{4.8.4}$$

4.8.2 Examples of annihilators

We give examples of annihilator of the form in Eq.(4.8.1) for $n = 2$ acting on the basic representations of $\widehat{\mathfrak{sl}}_2$ and $\widehat{\mathfrak{sl}}_3$.

Basic representation of $\widehat{\mathfrak{sl}}_2$

As we have already seen, the basic representation $L_{\mathfrak{sl}_2,1}$ is isomorphic to the lattice vertex operator algebra V_Q associated with the root lattice Q of $\mathfrak{sl}_2$. Our problem is whether there is an annihilator of the vacuum vector of the form

$$-2L_{-4} + \frac{\kappa}{2}L_{-2}^2 + \frac{\tau}{2}\sum_{r=1}^{3}(X_r(-2))^2 \tag{4.8.5}$$

with κ and τ being certainly chosen.

Let us compute each ingredient of a candidate for an annihilator. The Virasoro field $L(z)$, which is given by the Segal-Sugawara construction, coincides with the standard one in the lattice VOA.

$$Y(L_{-2}\left|0\right\rangle, z) = L(z) = \frac{1}{4} :\alpha(z)^2: \ . \tag{4.8.6}$$

By the translation covariance of fields we have

$$Y(L_{-4}\left|0\right\rangle, z) = \frac{1}{2}\partial^2 L(z) = \frac{1}{4}\left(:\partial^2\alpha(z)\alpha(z): \ + :\partial\alpha(z)^2: \right). \tag{4.8.7}$$

From the strong reconstruction theorem, we have $Y(L_{-2}^2\left|0\right\rangle, z) =:L(z)^2:$, which is computed by using Wick's theorem so that

$$\mathcal{Y}(L_{-2}^2\left|0\right\rangle, z) = \frac{1}{4} :\partial^2\alpha(z)\alpha(z): \ + \frac{1}{16} :\alpha(z)^4: \ . \tag{4.8.8}$$

Again the strong reconstruction theorem determines the field corresponding to the vector of the form $X(-2)Y(-2)\left|0\right\rangle$ with $X, Y \in \mathfrak{sl}_2$ to be $:\partial X(z)\partial Y(z):$. The case of $X = Y = H$ is obviously gives $Y(H(-2)^2\left|0\right\rangle, z) =:\partial\alpha(z)^2:$. The cases of $X = E$, $Y = F$ and $X = F$, $Y = E$ are also handled by means of Wick computation involving vertex operators so that

$$Y(E(-2)F(-2)\left|0\right\rangle, z) = \frac{1}{12}\left(\partial^3\alpha(z) + 2 :\partial^2\alpha(z)\alpha(z): \ +3 :\partial\alpha(z)^2: \ - :\alpha(z)^4: \right) \tag{4.8.9}$$

$$Y(F(-2)E(-2)\left|0\right\rangle, z) = \frac{1}{12}\left(-\partial^3\alpha(z) + 2 :\partial^2\alpha(z)\alpha(z): \ +3 :\partial\alpha(z)^2: \ - :\alpha(z)^4: \right) \tag{4.8.10}$$

Combining these results and noticing that the state-field correspondence $Y(-, z)$ is injective, we can see that

$$\left(-2L_{-4} + \frac{4}{3}L_{-2}^2 + \frac{1}{2}\sum_{r=1}^{3}(X_r(-2))^2\right)\left|0\right\rangle = 0 \tag{4.8.11}$$

in $L_{\mathfrak{sl}_2,1}$. Thus $\mathscr{G}_t\left|0\right\rangle$ corresponding to $n = 2$, $\kappa = \frac{8}{3}$, and $\tau = 1$ is a local martingale.

Basic representation of $\mathfrak{sl}_3$

Now we move on to searching an annihilator in the basic representation of $\widehat{\mathfrak{sl}}_3$ of the form

$$-2L_{-4} + \frac{\kappa}{2}L_{-2}^2 + \frac{\tau}{2}\sum_{r=1}^{8}(X_r(-2))^2 \tag{4.8.12}$$

with certain κ and τ. We take the set of Chevalley generators of $\mathfrak{sl}_3$ as

$$E_1 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad H_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad F_1 = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \tag{4.8.13}$$

$$E_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \qquad H_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \qquad F_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \tag{4.8.14}$$

where H_1 and H_2 span a Cartan subalgebra. With this Cartan subalgebra being fixed, the basic representation $L_{\mathfrak{sl}_3,1}$ of $\widehat{\mathfrak{sl}}_3$ is isomorphic to the lattice VOA $V_{Q_{A_2}}$ associated with the root lattice $Q_{A_2} = \mathbb{Z}\alpha_1 + \mathbb{Z}\alpha_2$ of $\mathfrak{sl}_3$ by the assignment

$$E_i(z) \mapsto \Gamma_{\alpha_i}(z), \qquad H_i(z) \mapsto \alpha_i(z), \qquad F_i(z) \mapsto \Gamma_{-\alpha_i}(z) \tag{4.8.15}$$

for $i = 1, 2$. By similar but more complicated computation than in the case of $\widehat{\mathfrak{sl}}_2$, we find that

$$\left[-2L_{-4} + \frac{6}{5}L_{-2}^2 + \frac{2}{5}\sum_{r=1}^{8}(X_r(-2))^2 \right] |0\rangle = 0 \tag{4.8.16}$$

in $L_{\mathfrak{sl}_3,1}$. Thus the random process $\mathscr{G}_t\,|0\rangle$ in $L_{\mathfrak{sl}_3,1}$ is a martingale if $\kappa = \frac{12}{5}$ and $\tau = \frac{4}{5}$.

From the above observations for $\widehat{\mathfrak{sl}}_2$ and $\widehat{\mathfrak{sl}}_3$, we make a conjecture that there is an operator of the form of

$$-2L_{-4} + \frac{\kappa}{2}L_{-2}^2 + \frac{\tau}{2}\sum_{r=1}^{n^2-1}(X_r(-2))^2 \tag{4.8.17}$$

that annihilates the vacuum vector in the basic representation $L_{\mathfrak{sl}_n,1}$ of $\widehat{\mathfrak{sl}}_n$. Here $\{X_r\}_{r=1}^{n^2-1}$ is an orhonormal basis of $\mathfrak{sl}_n$ with respect to the trace form $(X|Y) = \mathrm{Tr}(XY)$ for $X, Y \in \mathfrak{sl}_n$.

One may ask whether we have an annihilator acting on spin representations. It can be found that the situation is more subtle for a spin representation. It may be impossible to construct a desired annihilator that acts on $L_{\mathfrak{g},k}(\Lambda)$ other thatn $\Lambda = 0$. Indeed, we have conformed that there does not exists an operator of the form of Eq.(4.8.5) that annihilate a vector in the top space of $L_{\mathfrak{sl}_2,1}(\varpi)$.

4.9 Summary

In this chapter, we construct SLE associated with representation theory of an affine Lie algebra of a finite dimensional simple Lie algebra. The corresponding group of internal symmetry is the positive loop group of the corresponding finite dimensional simple Lie group, on which the group of space-time symmetry again acts as transformation of the loop variable. We consider a random process on the semi-direct product group of space-time and internal symmetries and derive the corresponding SLE.

Compared to another formulation of SLE corresponding to WZW theory in [BGLW05, ABI11], our formulation reveals how the stochastic differential equations naturally arise from consideration on an annihilating operator of a highest weight vector. We have also constructed random processes along internal degrees of freedom in the case of $\widehat{\mathfrak{sl}}_2$ and written down local martingales that are associated with the solution in an explicit way, which was not carried in [BGLW05, ABI11].

We propose a way of generalization of SLE associating to representation theory of affine Lie algebras. As examples we construct stochastic differential equations associating to operators of degree 4 that annihilate the vacuum vectors in the basic representations of $\widehat{\mathfrak{sl}}_2$ and $\widehat{\mathfrak{sl}}_3$.

Chapter 5

Discussions and Perspectives

In this thesis, we have proposed generalization of SLE that corresponds to representations of Heisenberg algebras and affine Lie algebras. As is illustrated in Chap. 1 and Chap. 2, SLE/CFT correspondence in the sense of Bauer and Bernard [BB02, BB03a, BB03b, BB04a] allows us to compute local martingales associated with SLE from a representation of the Virasoro agebra. Our achievement is to generalize this notion of SLE/CFT correspondence to connection between stochastic differential equations and representations of other algebras than the Virasoro algebra (Fig. 5.1).

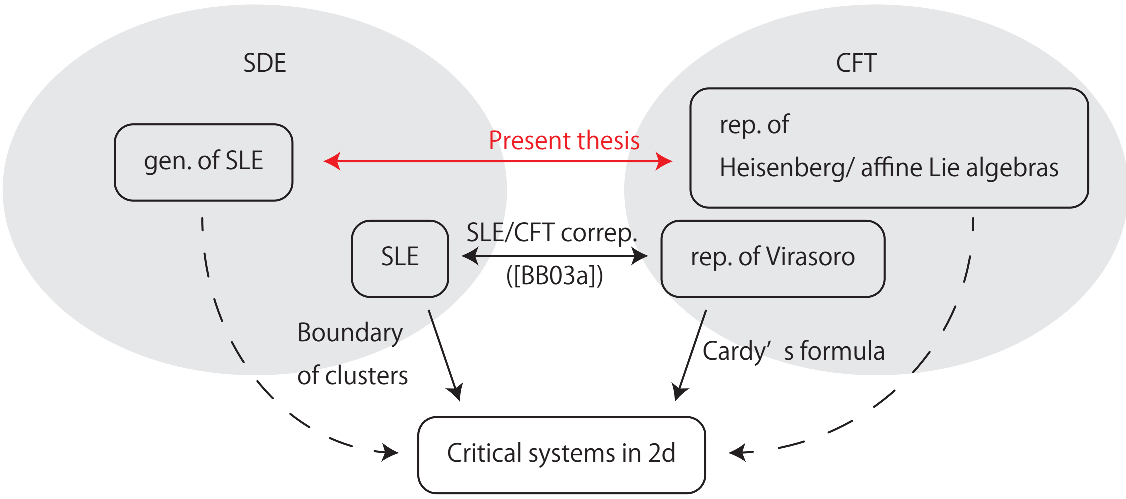

Figure 5.1: Generalization of SLE/CFT correspondence

Our strategy is to extend a random process on an infinite dimensional Lie group $\mathrm{Aut}_{+}\mathcal{O}$ that is naturally connected to SLE associated with the Virasoro algebra to a random process on a larger group $\mathrm{Aut}_{+}\mathcal{O} \ltimes \mathcal{G}$, where $\mathcal{G}$ governs an internal symmetry.

We have applied our formulation in two cases in which we have taken a positive loop group of a torus T^{ℓ} in Chapter 3 and a simple Lie group G in Chapter 4 as internal symmetries, respectively. In Chapter 3, we have constructed a random process on $\mathrm{Aut}_{+}\mathcal{O}\ltimes T_{+}^{\ell}(\mathcal{O})$ that has led us to a generalized SLE. We have carried all computation concretely, including derivation of several local martingales associated with the solution of a generalized SLE. In Chapter 4, we have also considered a random process on $\mathrm{Aut}_{+}\mathcal{O} \ltimes G_{+}(\mathcal{O})$, and written down a stochastic differential equation of the random process Θ_t on $G_{+}(\mathcal{O})$ that emerges when we write the whole random process $\mathscr{G}_t$ in the form $\Theta_t G(\rho_t)$ with $G(\rho_t)$ being the random process on $\mathrm{Aut}_+\mathcal{O}$ producing the ordinary SLE. An equivalent form of the stochastic differential equation on Θ_t has been already written down in [BGLW05, ABI11] in a different approach. We pointed out that a stochastic differential equation in this form is far from sufficient for more detailed analysis. We specialize our attention to the case of $G = SL(2)$ and write down the stochastic differential equation on $\mathscr{G}_t$ in the most concrete form, which allows us to compute local martingales associated with its solution. We also found that the space of such obtained local martingales admits an action of the affine Lie algebra $\widehat{\mathfrak{sl}}_2$. This symmetry of the space of local martingales allows us to systematically obtain local martingales associated with SLE, which is hard in probability theoretical manner.

As is mentioned above, SLEs corresponding to WZW theories have been already considered by authors of [BGLW05, ABI11] following the correlation function formulation of SLE, which is different from the group theoretical formulation. They stated that if primary fields of a WZW theory evolve according to given stochastic differential equations, their correlation function of a certain form is a local martingale. We have suggested the following issues against these works. Firstly their stochastic differential equations along internal degrees of freedom seem to be *ad hoc* and the leading principle to them is not clear to us. Secondly, even though the stochastic differential equations along internal degrees of freedom are written down in [BGLW05, ABI11], the target space of this random process is infinite dimensional, which is difficult to handle out. This gives rise to the third problem: local martingales that are associated with the solution is hard to write down. As we saw in Chap. 4, we have overcome these points in our formulation. In the group theoretical formulation developed in this thesis, we have found that the stochastic differential equations along internal degrees of freedom naturally appear from consideration on an annihilator of a highest weight vector. We also introduce suitable "coordinates" on the infinite dimensional group and constructed random processes along internal degrees of freedom in the case of $\widehat{\mathfrak{sl}_2}$, which has makes it possible to write down local martingales that are associated with the solution in an explicit way.

In Chap. 4, we have considered a random process $\mathscr{G}_t$ on an infinite dimensional Lie group $\mathrm{Aut}_+\mathcal{O} \ltimes G_+(\mathcal{O})$, of which the dt term in its increment is an annihilating operator

$$-2L_{-2} + \frac{\kappa}{2}L_{-1}^2 + \frac{\tau}{2}\sum_{i=1}^{\dim\mathfrak{g}} X_i(-1)^2 \tag{5.0.1}$$

of the highest weight vector. This annihilator is chosen by the following principle. Firstly, our construction should derive the ordinary SLE in the coordinate transformation part, which forces an annihilator to have the part $-2L_{-2} + \frac{\kappa}{2}L_{-1}^2$. Secondly, the operator of the above form indeed annihilates the vacuum vector due to the Sugawara construction of the Virasoro generators. The third term of the annihilator has room for generalization, which we shall discuss. We can allow the covariance τ to depend on i, namely an annihilator of the form

$$-2L_{-2} + \frac{\kappa}{2}L_{-1}^2 + \frac{1}{2}\sum_{i=1}^{\dim\mathfrak{g}} \tau_i X_i(-1)^2 \tag{5.0.2}$$

can be considered. Indeed, for an Heisenberg algebra treated in Chap.3, such an annihilator naturally appeared in Eq.(3.2.4). We can also deform the annihilator by adding a term like $X(-2)$ for $X \in \mathfrak{g}$. Such a deformation will be inevitable if we twist the Virasoro generators by a derivative of a current field. The problem whether annihilators generalized in these ways indeed annihilate the highest weight vector, of course, requires case-by-case investigation.

A possible application of our construction of SLEs corresponding to Heisenberg algebras and affine Lie algebras is to derive generalization of Cardy's formula. In case of the Virasoro algebra, SLE/CFT correspondence rederives Cardy's formula [BB03b]. We shall discuss possibility toward generalization of Cardy's formula. This work will be two-folded. One is to find an appropriate scaling limit of a model of statistical physics in which a kind of cluster interface is described SLE derived in our formulation. Candidates for such models of statistical physics may include quantum spin chains [AH87], kinds of vertex models [Bax08] and loop models [dGP04], since these models are expected to be described by WZW theories in their scaling limit. Interestingly, a Peano curve in a 6-vertex model has been shown to be described by SLE(κ) in a recent work [KMSW17]. An important point to be considered is that our SLE trace has internal degrees of freedom (Fig.5.2) which forces us to find a scaling limit that captures such internal degrees of freedom as well as a cluster interface itself. Also since boundary conditions for a WZW theory are imposed on its current fields [Gaw02], correspondingly, appropriate boundary conditions should be imposed in a model of statistical physics. The other is to make discussion to identify an object like

$$\langle u|\mathcal{Y}(A_1,z_1)\cdots\mathcal{Y}(A_n,z_n)\mathscr{G}_t|v_\Lambda\rangle \tag{5.0.3}$$

with the defining function of an event associated with the solution of SLE derived in this thesis. Here $\mathcal{Y}(-,z)$ is an intertwining operator and $\mathscr{G}_t\,|\Lambda\rangle$ is a representation-space-valued local martingale constructed in this thesis. If such a discussion is possible, the probability of the event is computed as the expectation value of the above quantity, which is time independent and thus reduces to a purely algebraic quantity

$$\langle u|\mathcal{Y}(A_1,z_1)\cdots\mathcal{Y}(A_n,z_n)|v_\Lambda\rangle \tag{5.0.4}$$

and may be computed.

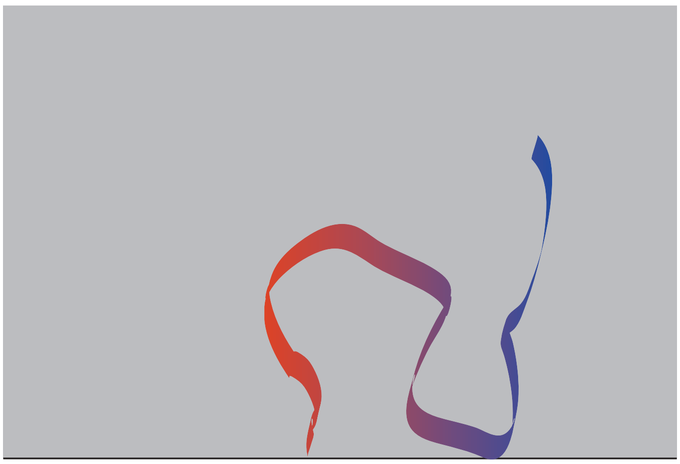

Figure 5.2: SLE trace with internal degrees of freedom

It is natural to seek other examples of generalization of SLE involving more general internal symmetry $\mathcal{G}(V)$ that is encoded on a vertex algebra V. Both examples $T_+^\ell(\mathcal{O})$ and $G_+(\mathcal{O})$ we treated in this thesis are of infinite dimensional, but are loop spaces of finite dimensional spaces, thus easier to treat than general internal symmetry group $\mathcal{G}(V)$. For another internal symmetry group $\mathcal{G}(V)$ than loop spaces, we can write down a stochastic differential equation on a random process Θ_t on $\mathcal{G}(V)$ for a given annihilator of a vector in the top space in a proper form, but it appears to be difficult in general to construct the random process in a sufficiently concrete form to compute local martingales. An example of a stochastic differential equation on Θ_t is presented in Appendix E for the lattice vertex algebra V_L associated with $L = \mathbb{Z}\alpha$ such that $(\alpha|\alpha) = 4$. The other example of internal symmetry presented in Appendix E is a loop group of a Lie supergroup, where the associated representation theory is one of an affine Lie superalgebra. The construction of SLE corresponding to a representation of an affine Lie superalgebra will involve a technical difficulty in designing a vector-valued local martingale. This difficulty stems from the fact that an odd element in an Lie superalgebra or its action cannot be exponentiated by itself to give an element in the corresponding Lie supergroup, and instead, one has to exponentiate an odd element with a Grassmann number. Such a realization of Lie supergroup is suggested in [BK70,Kac77a,Kac77b] and computed for $OSP(1|2)$ in [BT81]. Geometrically, this realization is equivalent to viewing a set of Grassmann-algebra-valued points of a Lie supergroup. In Appendix E, we will see that we can obtain a vector-valued local martingale after integration over Grassmann numbers, and this technique may open a way of constructing SLE corresponding to a representation of an affine Lie superalgeba. Another interesting examples of internal symmetry also includes a twisted affine Lie algebra. Since the Sugawara construction also works for a twisted affine Lie algebra, a parallel construction to ours in this thesis will be possible for a twisted affine Lie algebra.

It will be an important direction to better understand examples in Appendix E and to find an example of internal symmetry of which structure has been made clear to handle out for further development of theory of SLE from representation theory.

It has been made clear that the group theoretical formulation of SLE is well explained by representation theory of infinite dimensional Lie algebras and the theory of vertex algebras. A next task for us is to construct a *group theoretical formulation of multiple SLE*, while what is meant by the words has not fixed yet. In the group theoretical formulation of SLE, the random process of the form $\mathscr{G}_t v$ in a (completion of) representation space of an infinite dimensional Lie algebra is a local martingale due to existence of an annihilator of the vector v. In the present formulation of multiple SLE, the fundamental object is a random process $\mathcal{M}_t$ defined as a correlation function, which becomes a local martingale if it is associated with a solution of multiple SLE in a proper way. While $\mathcal{M}_t$ is checked to be a local martingale by direct computation, why *such an object should be a local martingale* is less clear than the case of a single SLE. At a glance, multiple SLE involves multiple modules and thus is related to a multiple conformal block. If we understand the martingale property of $\mathcal{M}_t$ in a more natural way, it will deepen our understanding of connection of SLE and CFT, in particular CFT in the sense of [MS89, TUY92, BKJ01]. Moreover, such an approach will enable to *multiplize* SLE that is constructed only for a single case.

Appendix A

Ito processes on Lie groups

This appendix devoted to a short description of Ito processes on Lie groups. A detailed exposition on this matter can be found in [Chi12, App14].

Let G be a finite dimensional complex Lie algebra and $\mathfrak{g}$ be its Lie algebra. A strategy to construct an Ito process on the Lie group G may be exponentiating an Ito process on the Lie group $\mathfrak{g}$. We take for convenience of description a basis $\{X_i\}_{i=1}^{\dim\mathfrak{g}}$ of $\mathfrak{g}$. Then an Ito process X_t on $\mathfrak{g}$ expanded in this basis so that $X_t = \sum_{i=1}^{\dim\mathfrak{g}} x_t^i X_i$, where x_t^i are Ito processes that are characterized by stochastic differential equations they satisfy of the form of

$$dx_t^i = \overline{x}_t^i dt + \sum_{j\in I_{\mathrm{B}}} x_{(j)t}^i dB_t^{(j)}. \tag{A.0.1}$$

Here $\overline{x}_t^i$ and $x_{(j)t}^i$ are random process with proper finiteness properties, and $B_t^{(j)}$ are mutually independent Brownian motions labeled by a set I_{B}. We set the covariance of $B_t^{(j)}$ as κ_j. Then we can obtain a random process g_t on G by exponentiating X_t as $g_t = \exp(X_t)$, but it is not easy to write down the stochastic differential equation on g_t due to noncommutativity in the Lie algebra $\mathfrak{g}$.

Instead, we construct a random process on G via the McKean-Gangolli injection [McK05]. In this approach, we identify the value X_t at each time t as a left invariant vector field on G, and a random process g_t on G evolves along this random vector field. Then the infinitesimal time evolution of g_t is described by

$$g_{t+dt} = g_t \exp\left(\sum_{i=1}^{\dim\mathfrak{g}} dx_t^i X_i\right). \tag{A.0.2}$$

To write down the stochastic differential equation on such constructed g_t, we take finite dimensional faithful representation V of $\mathfrak{g}$. Then on the vector space V is defined an action of G by exponentiating the action of $\mathfrak{g}$. In the following, we do not distinguish an element of $\mathfrak{g}$ from its action on V. When we expand the exponential function in Eq. (A.0.2) and notice that quadratic terms in dx_t^i may give contribution proportional to dt, we obtain a stochastic differential equation

$$g_t^{-1}dg_t = \left(\sum_{i=1}^{\dim\mathfrak{g}} \overline{x}_t^i X_i + \frac{1}{2}\sum_{j\in I_{\mathrm{B}}} \kappa_j \left(\sum_{i=1}^{\dim\mathfrak{g}} x_{(j)t}^i X_i\right)^2\right) dt + \sum_{i=1}^{\dim\mathfrak{g}} \sum_{j\in I_{\mathrm{B}}} x_{(j)t}^i dB_t^{(j)}. \tag{A.0.3}$$

We regard this equation as the standard form of stochastic differential equations on Ito processes on Lie groups.

We have to handle a random process on an infinite dimensional Lie group in application to SLE. The construction above can be naturally extended to infinite dimensional setting. Let $\mathfrak{g}$ be an infinite dimensional Lie algebra and G be the corresponding Lie group. Examples of such infinite dimensional Lie group include the group of coordinate transformations $\mathrm{Aut}\mathcal{O}$ on a formal disc, loop groups of finite dimensional Lie groups and their semi-direct products. In typical cases, a faithful representation V of $\mathfrak{g}$ is infinite dimensional, thus it is in general nontrivial whether the action of $\mathfrak{g}$ on V is exponentiated to an action of G, but we assume that it is. The validity of this assumption can be verified for each example. We also assume that an infinite sum that appears in Eq.(A.0.2) in the case of $\dim \mathfrak{g} = \infty$ makes sense. Then the McKean-Gangolli injection works to construct a random process on the Lie group G from an Ito process on $\mathfrak{g}$, and stochastic differential equation of the form Eq.(A.0.3) characterizes the random process.

Appendix B

Notes on vertex algebras

In this chapter, we recall the notion of vertex algebras which is useful, though not necessary if we carry case-by-case computation. Detailed expositions of theory of vertex algebras can be found in [Kac98, FBZ04]. The appendix of [IK11] is also useful.

B.1 Definiton of vertex algebras

Let V be a vector space. A field on V is a series $a(z) = \sum_{n\in\mathbb{Z}} a_{(n)} z^{-n-1}$ in a formal variable z with coefficients $a_{(n)}$ being in $\mathrm{End}(V)$ such that for any $v \in V$ we have $a_{(n)}v = 0$ for $n \gg 0$. Equivalently, a field is a linear map from V to $V((z)) = V[[z]][z^{-1}]$. We let the space of fields be denoted by $\mathrm{Fie}(V) := \mathrm{Hom}_{\mathbb{C}}(V, V((z)))$.

Definition B.1.1. A vertex algebra is a quadruple $(V, |0\rangle, T, Y)$ of a vector space V, a distinguished vector $|0\rangle \in V$, an operator $T \in \mathrm{End}(V)$, and a linear operator $Y \in \mathrm{Hom}(V, \mathrm{Fie}(V))$, on which the following axioms are imposed:

(VA1) (translation covariance)

$$[T, Y(a, z)] = \partial Y(a, z) \tag{B.1.1}$$

(VA2) (vacuum axioms)

$$T|0\rangle = 0, \qquad Y(|0\rangle, z) = \mathrm{Id}_V, \qquad Y(a, z)|0\rangle\big|_{z=0} = a. \tag{B.1.2}$$

(VA3) (locality)

$$(z - w)^N [Y(a, z), Y(b, w)] = 0, \quad N \gg 0. \tag{B.1.3}$$

Here we have denoted the image of $a \in V$ via Y by $Y(a, z)$.

Remark B.1.2. (1) In the right hand side of the second equation in Eq.(B.1.2), Id_V is a formal series in z with coefficient Id_V for z^0 and zero for others. Thus it is equivalent to $|0\rangle_{(n)} = \delta_{n,-1}\mathrm{Id}_V$.

(2) The statement of the third equation in Eq.(B.1.2) contains the requirement that we can really substitute $z = 0$, not being just a requirement on the constant term. Thus it is equivalent to $a_{(n)}|0\rangle = 0$ $(n > -1)$ and $a_{(-1)}|0\rangle = a$.

(3) The operation of T is determined by applying both sides of Eq.(B.1.1) to the vacuum $|0\rangle$ and setting $z = 0$ and using the first equation in Eq.(B.1.2) so that $Ta = a_{(-2)}|0\rangle$.

(4) It is clear that the state-field correspondence map $Y : V \to \mathrm{Fie}(V)$ is injective.

We often denote a vertex algebra $(V, |0\rangle, T, Y)$ simply by V.

Definition B.1.3. A vertex algebra V is said to be $\mathbb{Z}$-graded if it admits a $\mathbb{Z}$-gradation $V = \bigoplus_{n\in\mathbb{Z}} V_n$ such that $|0\rangle \in V_0$, $TV_n \subset V_{n+1}$, and $(V_h)_{(n)}(V_{h'}) \subset V_{h+h'-n-1}$ for any $h, h', n \in \mathbb{Z}$. We say that a vector in V_h has conformal weight h.

Definition B.1.4. A vector $\omega \in V$ is a conformal vector of central charge c if the coefficients of $Y(\omega, z) = \sum_{n\in\mathbb{Z}} L_n z^{-n-2}$ define a representation of the Virasoro algebra of central charge c, or explicitly satisfy the commutation relation

$$[L_m, L_n] = (m-n)L_{m+n} + \frac{c}{12}(m^3 - m)\delta_{m+n,0}, \tag{B.1.4}$$

we have $L_{-1} = T$, and L_0 is diagonalizable on V. A vertex algebra endowed with a conformal vector ω is called a conformal vertex algebra of rank c. The field $Y(\omega, z)$ is called a Virasoro field of the conformal vertex algebra V.

Definition B.1.5 (Vertex operator algebra)**.** A $\mathbb{Z}$-graded conformal vertex algebra $(V = \bigoplus_{n\in\mathbb{Z}} V_n, \omega)$ is called a vertex operator algebra if we have

- $L_0|_{V_n} = n\mathrm{id}_{V_n}$ for all $n \in \mathbb{Z}$.
- $\dim V_n < \infty$ for all $n \in \mathbb{Z}$.
- There exists $N \in \mathbb{Z}$ such that $V_n = \{0\}$ for $n < N$.

B.1.1 Strong Reconstruction Theorem

Let $a(z)$ and $b(z)$ be two fields on a vector space V. Their naïve product $a(z)b(z)$ may not be a well-defined field, since its expansion in powers of z may involve infinite sum in each coefficient. Instead, we can define the *normal ordered product* of them. A field $a(z) = \sum_{n\in\mathbb{Z}} a_{(n)} z^{-n-1}$ is decomposed into *positive (creating)* and *netagive (annihilating)* parts defined by

$$a(z)_+ = \sum_{n\le -1} a_{(n)} z^{-n-1}, \quad a(z)_- = \sum_{n>-1} a_{(n)} z^{-n-1}. \tag{B.1.5}$$

Then the normal order product of fields $a(z)$ and $b(s)$ is defined by

$$:a(z)b(s): := a(z)_+ b(z) + b(z)a(z)_-, \tag{B.1.6}$$

is a field due to the field conditions on $a(z)$ and $b(z)$. The normal order of several fields is also defined inductively by

$$:a_1(z_1)\cdots a_{n-1}(z_{n-1})a_n(z_n): :=:a_1(z_1)(:\cdots(:a_{n-1}(z_{n-1})a_n(z_n):)\cdots:): \;. \tag{B.1.7}$$

We shall ask for a minimal data to determine a structure of a single vertex algebra. We prepare the following data:

- A vector space V,
- A vector $|0\rangle \in V$,
- A collection of vectors labeled by a countable and ordered set S, $\{a^\alpha\}_{\alpha\in S}$ and a collection of fields $\{a^\alpha(z) = \sum_{n\in\mathbb{Z}} a^\alpha_{(n)} z^{-n-1}\}_{\alpha\in S}$,

on which the following conditions are imposed:

(1) $a^\alpha(z)\left|0\right\rangle|_{z=0} = a^\alpha$ for all $\alpha \in S$.

(2) $T\left|0\right\rangle = 0$, $[T, a^\alpha(z)] = \partial a^\alpha(z)$ for all $\alpha \in S$.

(3) All fields $a^\alpha(z)$ are mutually local.

(4) $V = \mathrm{Span}_{\mathbb{C}}\left\{ a^{\alpha_1}_{(j_1)} \cdots a^{\alpha_k}_{(j_k)}\left|0\right\rangle \,\middle|\, j_i < 0, \alpha_1 \leq \cdots \leq \alpha_k, k \in \mathbb{Z}_{\geq 0} \right\}$.

The strong reconstruction theorem says that the above data is sufficient to construct a vertex algebra.

Theorem B.1.6 (Strong Reconstruction Theorem). *(1) By setting*

$$Y(a^{\alpha_1}_{(j_1)} \cdots a^{\alpha_k}_{(j_k)}\left|0\right\rangle, z) := :\partial^{-j_1-1}a^{\alpha_1}(z) \cdots \partial^{-j_k-1}a^{\alpha_k}(z):\,, \tag{B.1.8}$$

with $\partial^{(n)} = \frac{1}{n!}\partial^n$ *and* $Y(\left|0\right\rangle, z) = \mathrm{id}_V$*, one defines a vertex algebra structure on* V*.*

(2) This is a unique vertex algebra structure on V *which satisfies conditions above (1)-(4) and*

$$Y(a^\alpha, z) = a^\alpha(z). \tag{B.1.9}$$

(3) Moreover, if V *is* $\mathbb{Z}$*-graded,* $\left|0\right\rangle \in V_0$*,* $\deg T = 1$*,* a^α *are homogeneous and* $a^\alpha(z)$ *have conformal dimension* $\deg a^\alpha$*, then one defines a* $\mathbb{Z}$*-graded vertex algebra structure on* V*.*

B.2 Modules and intertwining operators

B.2.1 Definitions

Definition B.2.1. Let $(V, \left|0\right\rangle, T, Y, \omega)$ be a vertex operator algebra. A weak V-module is a pair (M, Y^M) of a vector space M and a linear map $Y^M : V \to \mathrm{End}(M)[[z, z^{-1}]]$ satisfying the following conditions:

- $Y^M(\left|0\right\rangle, z) = \mathrm{id}_M$.
- For arbitrary $A \in V$ and $v \in M$,

$$Y^M(A, z)v \in M((z)).$$

- For arbitrary $A, B \in V$ and $m, n \in \mathbb{Z}$,

$$\begin{aligned}
&\mathrm{Res}_{z-w} Y^M(Y(A, z-w)B, w) i_{w,z-w} z^m (z-w)^n \\
&= \mathrm{Res}_z Y^M(A, z) Y^M(B, w) i_{z,w} z^m (z-w)^n \\
&\quad - \mathrm{Res}_z Y^M(B, w) Y^M(A, z) i_{w,z} z^m (z-w)^n.
\end{aligned}$$

For a weak V-module (M, Y^M), the image of $A \in V$ by Y^M is expressed as

$$Y^M(A, z) = \sum_{n \in \mathbb{Z}} A^M_{(n)} z^{-n-1} \tag{B.2.1}$$

with $A^M_{(n)} \in \mathrm{End}(M)$.

Definition B.2.2. Let M be a weak V-module. A subspace $N \subset M$ is a weak V-submodule if $Y^M(A,z)N \subset N((z))$ for any $A \in V$. For a weak V-submodule $N \subset M$, the quotient space M/N is naturally a weak V-module via

$$Y^{M/N}(A,z)[m] = [Y^M(A,z)m].$$

Proposition B.2.3. *Let M be a weak V-module. Then*

(1) All fields $Y^M(A,z)$ are mutually local.

(2) $Y^M(TA,z) = \partial Y^M(A,z)$.

Proof. We apply u^{-m-1} to the both sides of the third axiom in the definition of a weak V-module and take sum over $m \in \mathbb{Z}$. Then the left hand side gives

$$\sum_{j=0}^{\infty} Y^M(A_{(n+j)}B,w)\partial_w^{(j)}\delta(u-w),$$

while the right hand side gives

$$Y^M(A,u)Y^M(B,w)i_{u,w}(u-w)^n - Y^M(B,w)Y^M(A,u)i_{w,u}(u-w)^n.$$

In particular, for $n=0$, we have

$$[Y^M(A,z),Y^M(B,w)] = \sum_{j=0}^{\infty} Y^M(A_{(j)}B,w)\partial_w^{(j)}\delta(z-w), \tag{B.2.2}$$

which implies the fields $Y^M(A,z)$ and $Y^M(B,w)$ are mutually local proving (1).

We define the n-th product among fields by

$$\begin{aligned}&Y^M(A,w)_{(n)}Y^M(B,w)\\&= \mathrm{Res}_z\left(Y^M(A,z)Y^M(B,w)i_{z,w}(z-w)^n - Y^M(B,w)Y^M(A,z)i_{w,z}(z-w)^n\right)\end{aligned}$$

for $n \in \mathbb{Z}$. Then it is clear that

$$Y^M(A,w)_{(n)}Y^M(B,w) = Y^M(A_{(n)}B,w)$$

for $n \in \mathbb{Z}$. In particular, for $B = |0\rangle$ and $n=-2$, we have

$$Y^M(A,w)_{(-2)}\mathrm{id} = Y^M(TA,w).$$

On the other hand, by definition of the n-th product among fields,

$$\begin{aligned}&Y^M(A,w)_{(-2)}\mathrm{id}\\&= \sum_{j=0}^{\infty}\binom{-2}{j}A^M_{(-2-j)}(-w)^j - \sum_{j=0}^{\infty}\binom{-2}{j}A^M_{(j)}(-w)^{-j-2}\\&= \sum_{j=-2}^{-\infty}(-j-1)A^M_{(j)}w^{-j-2} + \sum_{j=0}^{\infty}(-j-1)A^M_{(j)}w^{-j-2}\\&= \partial Y^M(A,w)\end{aligned}$$

providing (2). $\square$

Remark B.2.4. Eq.(B.2.2) says that any OPE in a weak V-module M is computed from the vertex algebra structure in V. In particular, the field $L^M(z) = Y^M(\omega, z)$ behaves as a Virasoro field, namely it satisfies

$$L^M(z)L^M(w) \sim \frac{c/2}{(z-w)^4} + \frac{2L^M(w)}{(z-w)^2} + \frac{\partial L^M(w)}{z-w}. \tag{B.2.3}$$

Some of other consequences following Eq. (B.2.2) are listed below for convenience.

$$[L^M_{-1}, Y^M(A,z)] = \partial Y^M(A,z),$$
$$[L^M_0, Y^M(A,z)] = z\partial_z Y^M(A,z) + Y^M(L_0A,z).$$

Definition B.2.5. Let V be a vertex operator algebra. An admissible V-module is a weak V-module (M, Y^M) such that

- M is $\mathbb{Z}_{\geq 0}$-graded
$$M = \bigoplus_{n\in\mathbb{Z}_{\geq 0}} M_n.$$
- If $A \in V_m$, then $Y^M(A,z)$ has the conformal dimension m, *i.e.*, $\deg A^M_{(n)} = -n + m - 1$, or equivalently,
$$A^M_{(n)} M_k \subset M_{k-n+m-1}.$$

If $Y^M(A,z)$ has the conformal dimension Δ, it is convenient to expand $Y^M(A,z)$ as

$$Y^M(A,z) = \sum_{n\in\mathbb{Z}} A^M_n z^{-n-\Delta} \tag{B.2.4}$$

so that $\deg A^M_n = -n$.

Definition B.2.6. Let V be a vertex operator algebra and $\omega \in V$ be the conformal vector of V. A ordinary V-module is a weak V-module M such that

- L^M_0 in the expansion
$$Y^M(\omega, z) = \sum_{n\in\mathbb{Z}} L^M_n z^{-n-2}$$
is diagonalizable on M.
- In the L^M_0-eigenspace decomposition
$$M = \bigoplus_{\lambda\in\mathbb{C}} M_\lambda,$$
$\dim M_\lambda < \infty$ for all $\lambda \in \mathbb{C}$. Moreover, for arbitrary $\lambda \in \mathbb{C}$, $M_{\lambda-n} = 0$ for $n \gg 0$.

Definition B.2.7. An admissible V-module M is simple if 0 and M are the only $\mathbb{Z}_{\geq 0}$-graded submodules.

Definition B.2.8. A vertex operator algebra V is rational if every admissible V-module is a direct sum of simple admissible V-modules.

Proposition B.2.9. *Let V be a rational vertex operator algebra. Then*

(1) V has finitely many isomorphism classes of simple admissible V-modules.

(2) An admissible V-module is an ordinary V-module.

B.2.2 Intertwining operators

Definition B.2.10. Let M^1, M^2 and M^3 be $\mathbb{Z}_{\geq 0}$-graded weak V-modules. An intertwining operator of type $\begin{pmatrix} & M_3 & \\ M_1 & & M_2 \end{pmatrix}$ is a linear operator

$$\mathcal{Y}(-,z) : M^1 \to \operatorname{Hom}(M^2, M^3)z^K := \left\{ \sum_{a \in K} v_a z^a \middle| v_\alpha \in \operatorname{Hom}(M^2, M^3) \right\}, \tag{B.2.5}$$

where $K = \bigcup_i (\alpha_i + \mathbb{Z})$ with finitely many $\alpha_i \in \mathbb{C}$ being chosen associated with M^1, M^2 and M^3 that satisfies the following properties:

- For any $A \in V$, $v \in M^1$ and $m, n \in \mathbb{Z}$ we have

$$\begin{aligned} &\operatorname{Res}_{z-w} \mathcal{Y}(Y^{M^1}(A, z-w)v, w) i_{w,z-w} z^m (z-w)^n \\ &= \operatorname{Res}_z Y^{M^3}(A, z) \mathcal{Y}(v, w) i_{z,w} z^m (z-w)^n \\ &\quad - \operatorname{Res}_z \mathcal{Y}(v, w) Y^{M^2}(A, z) i_{w,z} z^m (z-w)^n. \end{aligned}$$

- For any $v \in M^1$, we have

$$\mathcal{Y}(L_{-1}v, z) = \frac{d}{dz} \mathcal{Y}(v, z). \tag{B.2.6}$$

B.3 Examples

B.3.1 Virasoro vertex algebra

Let Vir be the Virasoro algebra defined by $\mathrm{Vir} = \bigoplus_{n \in \mathbb{Z}} \mathbb{C}L_n \oplus \mathbb{C}C$ with Lie bracket

$$[L_m, L_n] = (m-n)L_{m+n} + \frac{m^3 - m}{12} \delta_{m+n,0} C, \quad [C, \mathrm{Vir}] = \{0\}. \tag{B.3.1}$$

To construct a representation of the Virasoro algebra, we decompose it into $\mathrm{Vir} = \mathrm{Vir}_{<0} \oplus \mathrm{Vir}_0 \oplus \mathrm{Vir}_{>0}$, where $\mathrm{Vir}_{\lesseqgtr 0} = \bigoplus_{n \lesseqgtr 0} \mathbb{C}L_n$ and $\mathrm{Vir}_0 = \mathbb{C}L_0 \oplus \mathbb{C}C$. We also take a subalgebra $\mathrm{Vir}_{\geq 0} = \mathrm{Vir}_{>0} \oplus \mathrm{Vir}_0$. For $(c, h) \in \mathbb{C}^2$, let $\mathbb{C}_{c,h} = \mathbb{C}\mathbf{1}_{c,h}$ be a one-dimensional representation of $\mathrm{Vir}_{\geq 0}$ on which C and L_0 act as multiplication by c and h, respectively, and $\mathrm{Vir}_{>0}$ acts trivially. Then the Verma module $M(c, h)$ of highest weight (c, h) is defined by

$$M(c, h) = \operatorname{Ind}_{\mathrm{Vir}_{\geq 0}}^{\mathrm{Vir}} \mathbb{C}_{c,h} = U(\mathrm{Vir}) \otimes_{U(\mathrm{Vir}_{\geq 0})} \mathbb{C}_{c,h}. \tag{B.3.2}$$

The irreducible quotient of a Verma module $M(c, h)$ is denoted by $L(c, h)$.

The Verma module $M(c, 0)$ of highest weight $(c, 0)$ has a submodule generated by $L_{-1}\mathbf{1}_{c,0}$. Then the vacuum representation V_c of Virasoro algebra is defined by

$$V_c := M(c, 0)/U(\mathrm{Vir}^-)L_{-1}\mathbf{1}_{c,0}. \tag{B.3.3}$$

Now we prepare the ingredient of a vertex algebra structure on V_c.

- $|0\rangle = \mathbf{1}_{c,0}$,
- $T = L_{-1}$,
- $S = \{*\}$, $a^* = \omega = L_{-2}\,|0\rangle$ and $a^*(z) = \sum_{n \in \mathbb{Z}} L_n z^{-n-2}$.

From this data, we construct a vertex algebra structure on V_c by

$$Y(L_{j_1}\cdots L_{j_k}\,|0\rangle\,, z) = :\partial^{(-j_1-2)}T(z)\cdots\partial^{(-j_k-2)}T(z): , \tag{B.3.4}$$

with $T(z) = Y(\omega, z)$. Moreover, V is $\mathbb{Z}$-graded by

$$\deg(L_{j_1}\cdots L_{j_k}\,|0\rangle) = -\sum_{i=1}^{k} j_i. \tag{B.3.5}$$

Then $\omega \in V_2$ and $\deg L_n = -n$, implying V is equipped with a $\mathbb{Z}$-graded vertex algebra. We also see ω is a conformal vector, and V is a vertex operator algebra. It is obvious that the maximal proper submodule of V_c as a Vir-module is a vertex subalgebra. Thus the irreducible representation $L(c, 0)$ of the Virasoro algebra also carries a vertex algebra structure and we denote this vertex algebra by L_c.

The Zhu algebra $A(V_c)$ of V_c is the space of polynomial $\mathbb{C}[x]$ of a variable x, where x is the image of ω via the quotient map $V_c \to A(V_c) = V_c/O(V_c)$. An irreducible representation of $A(V_c)$ is an evaluation module $\mathrm{ev}_h : \mathbb{C}[x] \ni f(z) \mapsto f(h) \in \mathbb{C} \simeq \mathrm{End}(\mathbb{C})$ with some $h \in \mathbb{C}$. The irreducible $\mathbb{Z}_{\geq 0}$-graded weak V_c-module corresponding to ev_h is $L(c, h)$. It is obvious that V_c is not rational, but for a certain central charge, V_c has a nontrivial maximal proper submodule generated by a single vector and the irreducible quotient L_c becomes rational.

Theorem B.3.1 (Wang [Wan93])**.** *If the central charge c takes value*

$$c = c_{p,q} = 1 - \frac{6(p-q)^2}{pq} \tag{B.3.6}$$

for relatively prime integers p and q greater or equal to 2, the corresponding simple vertex algebra L_c is rational.

B.3.2 Heisenberg vertex algebra

Let $\mathfrak{h}$ be a finite dimensional vector space and $(\cdot|\cdot)$ be a nondegenerate symmetric bilinear form on $\mathfrak{h}$. The Heisenberg algebra $\widehat{\mathfrak{h}}$ associated with the vector space $\mathfrak{h}$ is defined by $\widehat{\mathfrak{h}} = \mathfrak{h} \otimes \mathbb{C}[\zeta, \zeta^{-1}] \oplus \mathbb{C}K$ with Lie bracket

$$[H_1(m), H_2(n)] = m(H_1|H_2)\delta_{m+n,0}K, \quad [K, \widehat{\mathfrak{h}}] = \{0\}. \tag{B.3.7}$$

Here we denote $H \otimes \zeta^m$ by $H(m)$ for $H \in \mathfrak{g}$ and $m \in \mathbb{Z}$. We decompose the Heisenberg algebra as $\widehat{\mathfrak{h}} = \widehat{\mathfrak{h}}_{<0} \oplus \widehat{\mathfrak{h}}_0 \oplus \widehat{\mathfrak{h}}_{>0}$ with $\widehat{\mathfrak{h}}_{\lessgtr 0} = \bigoplus_{n \lessgtr 0} \mathfrak{h} \otimes \zeta^n$ and $\widehat{\mathfrak{h}}_0 = \mathfrak{h} \otimes \zeta^0 \oplus \mathbb{C}K$. We also take a subalgebra $\widehat{\mathfrak{h}}_{\geq 0} = \widehat{\mathfrak{h}}_{>0} \oplus \widehat{\mathfrak{h}}_0$. For $\lambda \in \mathfrak{h}^*$, let $\mathbb{C}_\lambda = \mathbb{C}\mathbf{1}_\lambda$ be a one-dimensional representation of $\widehat{\mathfrak{h}}_{\geq 0}$ on which $H(0)$ acts by multiplication by $\langle \lambda, H\rangle$ for $H \in \mathfrak{h}$, K acts as identity and $\widehat{\mathfrak{h}}_{>0}$ acts trivially. Then the Heisenberg Fock space $M(1, \lambda)$ is defined by

$$M(1, \lambda) = \mathrm{Ind}_{\widehat{\mathfrak{h}}_{\geq 0}}^{\widehat{\mathfrak{h}}} \mathbb{C}_\lambda = U(\widehat{\mathfrak{h}}) \otimes_{U(\widehat{\mathfrak{h}}_{\geq 0})} \mathbb{C}_\lambda. \tag{B.3.8}$$

Note that a Heisenberg Fock space is irreducible representation of the Heisenberg algerbra.

On the Heisenberg Fock space $M(1, 0)$ corresponding to $\lambda = 0$, we equip a structure of a vertex algebra. For a basis $\{H_i\}_{i=1}^{\ell}$ of $\mathfrak{h}$, this vertex algebra is generated by $\{H_i(-1)\,|0\rangle\}_{i=1}^{\ell}$, where we write $|0\rangle = \mathbf{1}_0$. The field corresponding to $H_i(-1)\,|0\rangle$ is defined by

$$H_i(z) = \sum_{n \in \mathbb{Z}} H_i(n) z^{-n-1}. \tag{B.3.9}$$

Then the strong reconstruction theorem determines the vertex algebra structure on $M(1,0)$ so that

$$Y(H_{i_1}(-j_1)\cdots H_{i_k}(-j_k)\left|0\right\rangle, z) =:\partial^{(j_1-1)}H_{i_1}(z)\cdots\partial^{(j_k-1)}H_{i_k}(z): \ . \tag{B.3.10}$$

Let $\{H_i\}_{i=1}^{\ell}$ be an orthonormal basis of $\mathfrak{h}$. Then the vector

$$\omega = \frac{1}{2}\sum_{i=1}^{\ell} H_i(-1)^2\left|0\right\rangle \tag{B.3.11}$$

is a conformal vector for $M(1,0)$ and equips the vertex algebra $M(1,0)$ with a structure of vertex operator algebra.

The Zhu algebra $A(M(1,0))$ for $M(1,0)$ is a space of polynomials $\mathbb{C}[x_1,\cdots,x_\ell]$ where each variable x_i is the image of $H_i(-1)\left|0\right\rangle$. An irreducible $A(M(1,0))$-module is an evaluation module by $x_i=\lambda_i$ and corresponding irreducible $M(1,0)$-module is the Heisenberg Fock space $M(1,\lambda)$ with $\langle\lambda, H_i\rangle = \lambda_i$. It is obvious that the Heisenberg vertex algebra is not rational.

B.3.3 Affine vertex algebra

Let $\mathfrak{g}$ be a finite dimensional simple Lie algebra and $(\cdot|\cdot)$ be a nondegenerate symmetric invariant bilinear form on $\mathfrak{g}$. Note that if we allow $\mathfrak{g}$ to be reductive, the affine vertex algebras we are now defining contains the Heisenberg vertex algebras as special cases. The affine Lie algebra $\widehat{\mathfrak{g}}$ is defined by $\widehat{\mathfrak{g}} = \mathfrak{g}\otimes\mathbb{C}[\zeta,\zeta^{-1}]\oplus\mathbb{C}K$ with Lie bracket

$$[X(m),Y(n)] = [X,Y](m+n) + m(X|Y)\delta_{m+n,0}K, \quad [K,\widehat{\mathfrak{g}}] = \{0\}, \tag{B.3.12}$$

where we denote $X\otimes\zeta^m$ by $X(m)$ for $X\in\mathfrak{g}$ and $m\in\mathbb{Z}$.

Let us fix a Cartan subalgebra $\mathfrak{h}\subset\mathfrak{g}$. Correspondingly, we denote the set of simple coroots $\Pi = \{\alpha_i^\vee,\cdots,\alpha_\ell^\vee\}$. The fundamental weights $\varpi_i\in\mathfrak{h}^*$ for $i=1,\cdots,\ell$ are defined by $\langle\varpi_i,\alpha_j^\vee\rangle = \delta_{ij}$ and span the weight lattice $P=\bigoplus_{i=1}^{\ell}\mathbb{Z}\varpi_i$. Finite dimensional irreducible representations of $\widehat{\mathfrak{g}}$ are classified by the set of dominant weights $P_+ = \{\Lambda\in P|\,\langle\lambda,\alpha_i^\vee\rangle\geq 0\}$, of which an element Λ corresponds to the irreducible representation $L(\Lambda)$ of highest weight Λ. On a finite dimensional irreducible representation $L(\Lambda)$ with $\Lambda\in P_+$ of $\mathfrak{g}$, we define an action of a subalgebra $\mathfrak{g}\otimes\mathbb{C}[\zeta]\oplus\mathbb{C}K$ of the affine Lie algebra so that $\mathfrak{g}\otimes\mathbb{C}[\zeta]\zeta$ acts trivially, $\mathfrak{g}\otimes\zeta^0$ acts naturally and K acts as multiplication by a complex number k. Then the Weyl module associating $L(\Lambda)$ of level k is defined by

$$V_{\mathfrak{g},k}(\Lambda) = \mathrm{Ind}^{\widehat{\mathfrak{g}}}_{\mathfrak{g}\otimes\mathbb{C}[\zeta]\oplus\mathbb{C}K}L(\Lambda). \tag{B.3.13}$$

Even though the representation $L(\Lambda)$ of $\mathfrak{g}$ is irreducible, its induced module $V_{\mathfrak{g},k}(\Lambda)$ is not necessarily irreducible as a representation of $\widehat{\mathfrak{g}}$. Then the irreducible quotient is denoted by $L_{\mathfrak{g},k}(\Lambda)$.

We equip a vertex algebra structure on $V_{\mathfrak{g},k}(\Lambda=0)$. The irreducible representation $L(0)$ of $\mathfrak{g}$ is one-dimensional, spanned by, say, a vector $\left|0\right\rangle$. Let $\{X_i\}_{i=1}^{\dim\mathfrak{g}}$ be a basis of $\mathfrak{g}$ and take a set of vectors $\{X_i(-1)\left|0\right\rangle\}_{i=1}^{\dim\mathfrak{g}}$ in $V_{\mathfrak{g},k}(0)$. We assert that the field corresponding to $X_i(-1)\left|0\right\rangle$ is given by

$$X_i(z) = \sum_{n\in\mathbb{Z}} X_i(n)z^{-n-1}. \tag{B.3.14}$$

Then from the strong reconstruction theorem, a vertex algebra structure on $V_{\mathfrak{g},k}(0)$ is uniquely determined so that

$$Y(X_{i_1}(-j_1)\cdots X_{i_k}(-j_k)\,|0\rangle\,,z) =:\partial^{(j_1-1)}X_{i_1}(z)\cdots\partial^{(j_k-1)}X_{i_k}(z): \ . \tag{B.3.15}$$

The vertex algebra $V_{\mathfrak{g},k}(0)$ is simply denoted by $V_{\mathfrak{g},k}$. Since the maximal proper submodule of $V_{\mathfrak{g},k}(0)$ is an ideal of the vertex algebra $V_{\mathfrak{g},k}$, the irreducible quotient $L_{\mathfrak{g},k}(0)$ also carries a vertex algebra structure, and is denoted by $L_{\mathfrak{g},k}$. Both vertex algebras $V_{\mathfrak{g},k}$ and $L_{\mathfrak{g},k}$ are called the affine vertex algebra associated with $\mathfrak{g}$ with level k, but it is convenient to call $V_{\mathfrak{g},k}$ a *universal* affine vertex algebra and $L_{\mathfrak{g},k}$ a *irreducible* affine vertex algebra.

Let $\{X_i\}_{i=1}^{\ell}$ be an orthonormal basis of $\mathfrak{g}$ with respect to $(\cdot|\cdot)$. We normalize the bilinear form so that $(\theta|\theta)=2$ with θ being the highest root and assume that $k\neq -h^\vee$, where $h^\vee$ is the dual Coxeter number of $\mathfrak{g}$. Then the vector

$$\omega=\frac{1}{2(k+h^\vee)}\sum_{i=1}^{\ell}X_i(-1)^2\,|0\rangle \tag{B.3.16}$$

in $V_{\mathfrak{g},k}$ or $L_{\mathfrak{g},k}$ is a conformal vector of central charge $c_k=\frac{k\dim\mathfrak{g}}{k+h^\vee}$. This construction of a conformal vector is called the *Segal-Sugawara* construction. Then both $V_{\mathfrak{g},k}$ and $L_{\mathfrak{g},k}$ become vertex operator algebras.

The Zhu algebra $A(V_{\mathfrak{g},k})$ of the affine vertex algebra associated with $\mathfrak{g}$ of level k is isomorphic to the universal enveloping algebra $U(\mathfrak{g})$, where the image of $X_{i_1}(-j_1)\cdots X_{i_k}(-j_k)\,|0\rangle$ is $(-1)^{j_1+\cdots+j_k}X_{i_j}\cdots X_{i_1}\in U(\mathfrak{g})$. Thus a $\mathbb{Z}_{\geq 0}$-graded weak $V_{\mathfrak{g},k}$-module is obtained by induction from a representation of $\mathfrak{g}$. In particular, simple $V_{\mathfrak{g},k}$-modules are exhausted by $L_{\mathfrak{g},k}(\Lambda)$ with $\Lambda\in P_+$. If the kernel of $V_{\mathfrak{g},k}\twoheadrightarrow L_{\mathfrak{g},k}$ is nontrivial, all of $V_{\mathfrak{g},k}$-modules are not $L_{\mathfrak{g},k}$-modules. For a positive integer level $k\in\mathbb{Z}_{>0}$, the list of simple $L_{\mathfrak{g},k}$-modules are obtained.

Theorem B.3.2 (Frenkel-Zhu [FZ92])**.** *Let $k\in\mathbb{Z}_{>0}$. The simple $L_{\mathfrak{g},k}$-modules are exhausted by $L_{\mathfrak{g},k}(\Lambda)$ with $\Lambda\in P_+^k$, where P_+^k is the set of dominant weights of level k defined by*

$$P_+^k=\{\Lambda\in P_+|(\theta|\Lambda)\leq k\}. \tag{B.3.17}$$

B.3.4 Lattice vertex algebra

Let L be a non-degenerate even lattice of rank ℓ, namely, it is a free $\mathbb{Z}$-module of rank ℓ endowed with a non-degenerate $\mathbb{Z}$-bilinear form $(\cdot|\cdot): L\times L\to\mathbb{Z}$, such that $(\alpha|\alpha)=2$ for $\alpha\in L$. There uniquely exists a cohomology class $[\epsilon]\in H^2(L,\mathbb{C}^\times)$ satisfying

$$\epsilon(\alpha,0)=\epsilon(0,\alpha)=1, \tag{B.3.18}$$

$$\epsilon(\alpha,\beta)=(-1)^{(\alpha|\beta)+|\alpha|^2|\beta|^2}\epsilon(\beta,\alpha) \tag{B.3.19}$$

for $\alpha,\beta\in L$. Here we denote $|\alpha|^2=(\alpha|\alpha)$. Notice that conditions Eq. (B.3.18) and Eq. (B.3.19) are independent of the choice of a representative ϵ of $[\epsilon]$. In can be shown that we can choose a 2-cocycle $\epsilon\in[\epsilon]$ so that it takes values in $\{\pm 1\}$ [Kac98, Remark 5.5a]. We let ϵ be such a 2-cocycle in the following. Let $\mathbb{C}_\epsilon[L]$ be the ϵ-twisted group algebra of L, which is

$$\mathbb{C}_\epsilon[L]=\bigoplus_{\alpha\in L}\mathbb{C}e^\alpha \tag{B.3.20}$$

as a vector space with multiplication defined by

$$e^\alpha e^\beta=\epsilon(\alpha,\beta)e^{\alpha+\beta} \tag{B.3.21}$$

for $\alpha, \beta \in L$.

We set $\mathfrak{h} = \mathbb{C} \otimes_{\mathbb{Z}} L$ and extend the symmetric $\mathbb{Z}$-bilinear form $(\cdot|\cdot)$ on L to a symmetric $\mathbb{C}$-bilinear form on $\mathfrak{h}$. Then we obtain the corresponding Heisenberg algebra $\widehat{\mathfrak{h}}$ and its vacuum representation $M_{\mathfrak{h}}(1,0)$ of level 1. The lattice vertex algebra V_L associated to L is

$$V_L = M_{\mathfrak{h}}(1,0) \otimes \mathbb{C}_\epsilon[L] \tag{B.3.22}$$

as a vector space. We define the action of $\widehat{\mathfrak{h}}$ on V_L by

$$H(m).(s \otimes e^\alpha) := (H(m) + \delta_{m,0}(H|\alpha))s \otimes e^\alpha \tag{B.3.23}$$

for $H \in \mathfrak{h}$, $m \in \mathbb{Z}$, $s \in M_{\mathfrak{h}}(1,0)$, and $\alpha \in L$. We also define the action of $\mathbb{C}_\epsilon[L]$ on V_L by

$$e^\beta.(s \otimes e^\alpha) := \epsilon(\beta,\alpha)s \otimes e^{\alpha+\beta} \tag{B.3.24}$$

for $\alpha, \beta \in L$ and $s \in M_{\mathfrak{h}}(1,0)$. The lattice vertex algebra is generated by vectors $H(-1)|0\rangle \otimes e^0$ with $H \in \mathfrak{h}$ and $|0\rangle \otimes e^\alpha$ with $\alpha \in L$, of which the corresponding fields are given by

$$H(z) = \sum_{n \in \mathbb{Z}} H(n) z^{-n-1}, \tag{B.3.25}$$

$$\Gamma_\alpha(z) = e^\alpha z^{\alpha(0)} e^{-\sum_{j<0} \frac{z^{-j}}{j}\alpha(j)} e^{-\sum_{j>0} \frac{z^{-j}}{j}\alpha(j)}, \tag{B.3.26}$$

respectively. Then by the strong reconstruction theorem, V_L admits a unique structure of a vertex algebra.

Let $\{H_i\}_{i=1}^{\ell}$ be an orthonormal basis of $\mathfrak{h}$ with respect to $(\cdot|\cdot)$. Then the vector

$$\omega = \frac{1}{2}\sum_{i=1}^{\ell} H_i(-1)|0\rangle \otimes e^0 \tag{B.3.27}$$

is a conformal vector of central charge ℓ.

The Zhu algebra $A(V_L)$ of the lattice vertex algebra V_L is computed in [DLM97] so that $A(V_L)$ is generated by E_α ($\alpha \in L$) and $\mathfrak{h}$ with relations

$$E_0 = 1 \quad \text{(identity)}, \tag{B.3.28}$$

$$HH' - H'H = 0, \qquad H, H' \in \mathfrak{h}, \tag{B.3.29}$$

$$HE_\alpha - E_\alpha H = (H|\alpha)E_\alpha, \qquad H \in \mathfrak{h},\ \alpha \in L. \tag{B.3.30}$$

The irreducible V_L-modules are classified by elements of L^*/L [Don93]. Here L^* is the dual lattice of L in $\mathfrak{h}$, then L is naturally a sublattice of L^*. For $\varpi \in L^*$, we can construct a V_L-module in the following way. Let $\mathbb{C}[L+\varpi]$ be a vector space spanned by elements of $L+\varpi$ so that $\mathbb{C}[L+\varpi] = \bigoplus_{\beta \in L} e^{\beta+\varpi}$, on which the Heisenberg algebra $\widehat{\mathfrak{h}}_{\geq 0}$ as $H(m)e^{\beta+\varpi} = 0$ for $m > 0$ and $H(0)e^{\beta+\varpi} = (H|\beta+\varpi)e^{\beta+\varpi}$ with $H \in \mathfrak{h}$ and $\beta \in L$. Then the V_L-module $V_{L+\varpi}$ is constructed as

$$V_{L+\varpi} = \mathrm{Ind}_{\widehat{\mathfrak{h}}_{\geq 0}}^{\widehat{\mathfrak{h}}} \mathbb{C}[L+\varpi], \tag{B.3.31}$$

on which the action of V_L is defined in a obvious way. It is also clear that $V_{L+\varpi}$ depends only on the equivalence class $[\varpi]$ of ϖ in L^*/L.

B.3.5 Frenkel-Kac construction

One of the most significant examples of lattice vertex algebras are one associated with root lattices of ADE type, which are isomorphic to irreducible affine vertex algebras associated with the corresponding Lie algebras. We shall explain these examples.

Let $\mathfrak{g}$ be a finite dimensional simple Lie algebra of ADE type and fix its Cartan subalgebra $\mathfrak{h}$. Correspondingly we denote the set of roots by Δ, and the root lattice by $Q = \mathbb{Z}\Delta$. Let $(\cdot|\cdot)$ be the nondegenerate symmetric invariant bilinear form on $\mathfrak{g}$ normalized so that $(\theta|\theta) = 2$ for the highest root θ. Let $\Pi = \{\alpha_1, \cdots, \alpha_\ell\}$ be the set of simple roots, then they are basis for the root lattice. We also denote the root space decomposition of $\mathfrak{g}$ by $\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha\in\Delta} \mathfrak{g}_\alpha$, where $\mathfrak{g}_\alpha = \mathbb{C}E_\alpha$ is the root space of the root $\alpha \in \Delta$ spanned by normalized vector E_α so that $(E_\alpha|E_{-\alpha}) = 1$, and the set of simple coroots by $\Pi^\vee = \{\alpha_1^\vee, \cdots, \alpha_\ell^\vee\}$.

Theorem B.3.3 (Frenkel-Kac [FK80])**.** *There is an isomorphism $L_{\mathfrak{g},k} \to V_Q$ of vertex algebras such that*

$$\alpha_i^\vee(-1)\,|0\rangle \mapsto \alpha_i(-1)\,|0\rangle\,, \quad E_\alpha(-1)\,|0\rangle \mapsto e^\alpha, \quad \alpha \in \Delta. \tag{B.3.32}$$

B.4 Current Lie algebras

In this section, we introduce the notion of the current Lie algebra associated with a vertex operator algebra. As a preliminary, we see an example of a commutative vertex algebra.

Example B.4.1. Let $V = \mathbb{C}[t, t^{-1}]$ be the space of Laurent polynomials, and define $Y(-, z) : V \to \mathrm{End}(V)[[z^\pm]]$ by

$$Y(t^m, z)t^n = (e^{z\partial_t} t^m)t^n = \sum_{k=0}^{\infty} \binom{m}{k} t^{m+n-k} z^k. \tag{B.4.1}$$

Then $(V, 1, Y)$ gives an example of a (commutative) vertex algebra. We also notice that the translation operator coincides with the derivative in t. Indeed, we have

$$Tt^m = \mathrm{Res}_z z^{-2} Y(t^m, z)1 = mt^{m-1}. \tag{B.4.2}$$

To each vertex algebra V, we assign a Lie algebra $\mathscr{L}(V)$ of coefficients of fields on V. To define $\mathscr{L}(V)$, we first see that the following proposition.

Proposition B.4.2. *Let V be a vertex algebra. Then $V/\mathrm{Im}T$ is a Lie algebra with Lie bracket being defined by*

$$[\overline{A}, \overline{B}] := \overline{A_{(0)}B}. \tag{B.4.3}$$

Here $\overline{A}$ is the image of $A \in V$ by the projection $V \twoheadrightarrow V/\mathrm{Im}T$.

Proof. The well-defined ness of the bracket is checked by noticing that $(TA)_{(0)} = [T, A_{(0)}] = 0$. The bilinearity of the bracket is obvious. The skew-symmetry property in a vertex algebra reads

$$Y(A, z)B = e^{zT} Y(B, -z)A. \tag{B.4.4}$$

Taking the residue in z of the both sides, we obtain

$$A_{(0)}B = -B_{(0)}A - \sum_{k=1}^{\infty} \frac{(-1)^k}{k!} T^k B_{(k)} A, \tag{B.4.5}$$

which implies $\overline{A_{(0)}B} = -\overline{B_{(0)}A}$ in $V/\mathrm{Im}T$. The commutation relation in a vertex algebra is

$$[A_{(m)}, B_{(n)}] = \sum_{k=0}^{\infty} \binom{m}{k} (A_{(k)}B)_{(m+n-k)}. \tag{B.4.6}$$

In particular case of $m = n = 0$, we have

$$[A_{(0)}, B_{(0)}] = (A_{(0)}B)_{(0)}, \tag{B.4.7}$$

which leads to the Jacobi identity of the bracket:

$$\begin{aligned}
[[\overline{A}, \overline{B}], \overline{C}] &= \overline{(A_{(0)}B)_{(0)}C} = \overline{[A_{(0)}, B_{(0)}]C} \\
&= \overline{A_{(0)}(B_{(0)}C)} - \overline{B_{(0)}(A_{(0)}C)} \\
&= [\overline{A}, [\overline{B}, \overline{C}]] - [\overline{B}, [\overline{A}, \overline{C}]].
\end{aligned}$$

Thus we conclude that $V/\mathrm{Im}T$ is a Lie algebra. □

We have obtained a Lie algebra $V/\mathrm{Im}T$, but this is not the desired Lie algebra $\mathscr{L}(V)$, which is constructed below.

Definition B.4.3. Let $(V, \mathbf{1}_V, Y^V)$ and$(W, \mathbf{1}_W, Y^W)$ be vertex algebras. Then $(V \otimes W, \mathbf{1}_V \otimes \mathbf{1}_W, Y^{V\otimes W})$ defined by

$$Y^{V\otimes W}(A \otimes B, z) := Y^V(A, z) \otimes Y^W(B, z), \quad A \in V, \ B \in W \tag{B.4.8}$$

is a vertex algebra, called the tensor product of V and W.

The translation on the tensor product $V \otimes W$ of vertex algebras is given by

$$T_{V\otimes W} = T_V \otimes \mathrm{id} + \mathrm{id} \otimes T_W. \tag{B.4.9}$$

Indeed, we have

$$\begin{aligned}
T_{V\otimes W}(A \otimes B) &= \mathrm{Res}_z z^{-2} Y^V(A, z)\mathbf{1}_V \otimes Y^W(B, z)\mathbf{1}_W \\
&= \sum_{n\in\mathbb{Z}} A_{(n)}\mathbf{1}_V \otimes B_{(-n-3)}\mathbf{1}_W \\
&= A_{(-1)}\mathbf{1}_V \otimes B_{(-2)}\mathbf{1}_W + A_{(-2)}\mathbf{V} \otimes B_{(-1)}\mathbf{1}_W \\
&= A \otimes TB + TA \otimes B.
\end{aligned}$$

Definition B.4.4. Let V be a vertex algebra. Its current Lie algebra $\mathscr{L}(V)$ is defined by

$$U(V) := V \otimes \mathbb{C}[t, t^{-1}]/\mathrm{Im}(T \otimes \mathrm{id} + \mathrm{id} \otimes \partial_t). \tag{B.4.10}$$

Let us describe the structure of the current Lie algebra $U(V)$. We denote by $A_{[m]}$ the image of $A \otimes t^m$ to $U(V)$. Since $TA \otimes t^m + mA \otimes t^{m-1}$ is projected to the zero vector in $U(V)$, we have

$$(TA)_{[m]} = -mA_{[m-1]}. \tag{B.4.11}$$

The Lie bracket is defined by means of the 0-th product in the vertex algebra. Here we observe that

$$\begin{aligned}
(A \otimes t^m)_{(0)}(B \otimes t^n) &= \mathrm{Res}_z \left(\sum_{\ell\in\mathbb{Z}} A_{(\ell)}Bz^{-\ell-1} \right) \otimes (e^{z\partial_t} t^m) t^n \\
&= \sum_{k=0}^{\infty} \binom{m}{k} A_{(k)}B \otimes t^{m+n-k},
\end{aligned}$$

which determines the Lie algebra structure on $U(V)$ so that

$$[A_{[m]}, B_{[n]}] = \sum_{k=0}^{\infty} \binom{m}{k} (A_{(k)}B)_{[m+n-k]}. \tag{B.4.12}$$

From the above observation, it is obvious that

$$U(V) \ni A_{[m]} \mapsto A_{(m)} \in \mathrm{End}(V) \tag{B.4.13}$$

is a homomorphims of Lie algebras, or in other words, the vertex algebra V is a representation of its current Lie algebra $U(V)$. More generally, any module M over the vertex algebra is a representation of the current Lie algebra.

If the vertex algebra V is $\mathbb{Z}$-graded, the current Lie algebra $U(V)$ is also $\mathbb{Z}$-graded by defining $\deg A_{[m]} := -m + \deg A - 1$. Note that the Lie bracket on $U(V)$ is compatible with this grading. An important point for the $\mathbb{Z}$-graded structure of the current Lie algebra is that each homogeneous subspace is infinite dimensional in general.

Appendix C

Computations in Virasoso SLE

In this appendix, we present some computational details referred in Chapter 2.

C.1 Differential equations on $G(\rho)$

We have obtained a representation of a group of automorphisms $\mathrm{Aut}_+\mathcal{O}$ on the formal completion $\overline{V}$ of a vertex operator algebra V denoted by $G : \mathrm{Aut}_+\mathcal{O} \to \mathrm{Aut}(\overline{V})$. Since each automorphism $\rho \in \mathrm{Aut}_+\mathcal{O}$ is identified with a formal power series $\rho(z) = z + a_0 + a_{-1}z^{-1} + \cdots$, infinite number of coefficients a_i, can be regarded as coordinates on $\mathrm{Aut}_+\mathcal{O}$. Then we shall ask for differential equations in these variables that $G(\rho)$ satisfies. The key feature for derivation of such differential equations is that G is a homomorphism of groups, *i.e.* it satisfies $G(\rho * \mu) = G(\rho)G(\mu)$. We take an infinitesimal automorphism as μ so that $\mu(z) = z + \epsilon v(z) + o(\epsilon)$. Here $v(z) = \sum_{j=-1}^{-\infty} v_j z^{j+1}$ and $o(\epsilon)$ is an infinitesimally small quantity such that we have $\lim_{\epsilon \to 0} \frac{o(\epsilon)}{\epsilon} = 0$. Then the corresponding operator $G(\mu)$ is expressed in linear order of ϵ as $G(\mu) = \mathrm{Id} + \epsilon \mathbf{v} + o(\epsilon)$, where we have defined $\mathbf{v} = -\sum_{j<0} v_j L_j$. On the other hand, the automorphism $\rho * \mu$ is an infinitesimally perturbed one around ρ that is expressed as $(\rho * \mu)(z) = \mu(\rho(z)) = \rho(z) + \epsilon v(\rho(z)) + o(\epsilon)$. Here we can choose $v(z)$ so that $v(\rho(z)) = z^n$ for $n \le 0$ by designing coefficients v_j as $v_j = \mathrm{Res}_z z^n \rho(z)^{-j-2}\rho'(z)$, and for such choice of $v(z)$, we can regard the perturbation around ρ is exactly along z^n. Thus the derivative of $G(\rho)$ in a_n is computed as

$$\frac{\partial G(\rho)}{\partial a_n} = \lim_{\epsilon \to 0} \frac{1}{\epsilon} \left(G(\rho * \mu) - G(\rho) \right) = -\sum_{j<0} \left(\mathrm{Res}_z z^n \rho(z)^{-j-2} \rho'(z) \right) G(\rho) L_j \qquad \text{(C.1.1)}$$

for $n \le 0$. Note that the coefficients v_j vanishes unless $j \ge n-1$, thus the summation actually runs in this range.

We next invert this relation, namely, we express $G(\rho)L_k$ for some $k \le -1$ as a linear combination in $\frac{\partial G(\rho)}{\partial a_n}$ for $n \le 0$. To this end, we shall look for numbers b_n with $n \le 0$ such that $\sum_{n=0}^{-\infty} b_n \frac{\partial R(\rho)}{\partial a_n} = R(\rho)L_k$, but such numbers are determined by requiring $b(z) = \sum_{n=0}^{-\infty} b_n z^n = -(\rho(z))^{k+1}$ so that $b_n = -\mathrm{Res}_z \rho(z)^{k+1} z^{-n-1}$. Indeed, for such choice, we have $\mathrm{Res}_z b(z)\rho(z)^{-j-2}\rho'(z) = -\delta_{j,k}$, leading to

$$G(\rho)L_k = -\sum_{n=0}^{-\infty} \left(\mathrm{Res}_z \rho(z)^{k+1} z^{-n-1} \right) \frac{\partial G(\rho)}{\partial a_n}, \qquad \text{(C.1.2)}$$

for $k \le -1$.

Appendix D

Computations in Affine SLE

This appendix contains computational details referred in Chapter 4

D.1 Differential equations on $\mathscr{G}$

We first derive differential equations satisfied by $\mathscr{G} = e^{\mathbf{e}}e^{\mathbf{h}}e^{\mathbf{f}}G(g)$. Here $\mathbf{e} = E \otimes e(\zeta)$, $\mathbf{h} = H \otimes h(\zeta)$ and $\mathbf{f} = F \otimes f(\zeta)$ are elements in $\mathfrak{g} \otimes \mathbb{C}[[\zeta^{-1}]]\zeta^{-1}$ with $e(\zeta) = \sum_{n<0} e_n \zeta^n$, $h(\zeta) = \sum_{n<0} h_n \zeta^n$, and $f(\zeta) = \sum_{n<0} f_n \zeta^n$. and $g \in \mathrm{Aut}_+\mathcal{O}$ is identified with a Laurant series $g(z) = z + \sum_{n\leq 0} g_n z^n$. By differentiating $\mathscr{G}$ by e_n we obtain

$$\frac{\partial \mathscr{G}}{\partial e_n} = e^{\mathbf{e}} E \otimes \zeta^n e^{\mathbf{h}} e^{\mathbf{f}} G(g). \tag{D.1.1}$$

After transferring $E \otimes \zeta^n$ to the rightest position, we have a differential equation

$$\begin{aligned}\mathscr{G}^{-1}\frac{\partial \mathscr{G}}{\partial e_n} =& E \otimes e^{-2h(g^{-1}(\zeta))} g^{-1}(\zeta)^n + H \otimes e^{-2h(g^{-1}(\zeta))} f(g^{-1}(\zeta)) g^{-1}(\zeta)^n \\ &- F \otimes e^{-2h(g^{-1}(\zeta))} f(g^{-1}(\zeta))^2 g^{-1}(\zeta)^n \end{aligned}\tag{D.1.2}$$

Similarly, we can compute derivatives of $\mathscr{G}$ in variables h_n and f_n as

$$\mathscr{G}^{-1}\frac{\partial \mathscr{G}}{\partial h_n} = H \otimes g^{-1}(\zeta)^n - 2F \otimes f(g^{-1}(\zeta)) g^{-1}(\zeta)^n, \tag{D.1.3}$$

$$\mathscr{G}^{-1}\frac{\partial \mathscr{G}}{\partial f_n} = F \otimes g^{-1}(\zeta)^n. \tag{D.1.4}$$

We invert these relations as we did in the case of the Virasoro algebra, namely, we express an object like $\mathscr{G} X \otimes \theta(\zeta)$ for a certain $\theta(\zeta) \in \mathbb{C}[[\zeta^{-1}]]\zeta^{-1}$ by linear combination of derivatives of $\mathscr{G}$. The case of $\mathscr{G} F \otimes \theta(\zeta)$ is straightforward by searching for an infinite series $a(z) = \sum_{n\leq -1} a_n z^n$ such that $a(g^{-1}(\zeta)) = \theta(\zeta)$. Such an infinite series is indeed obtained by setting $a_n = \mathrm{Res}_w w^{-n-1}\theta(g(w))$ which leads to

$$\mathscr{G} F \otimes \theta(\zeta) = \sum_{n\leq -1} \left(\mathrm{Res}_w w^{-n-1}\theta(g(w))\right) \frac{\partial \mathscr{G}}{\partial f_n}. \tag{D.1.5}$$

This relation helps us treat the case of $\mathscr{G} H \otimes \theta(\zeta)$ to give

$$\begin{aligned}\mathscr{G} H \otimes \theta(\zeta) =& \sum_{n\leq -1} \left(\mathrm{Res}_w w^{-n-1}\theta(g(w))\right) \frac{\partial \mathscr{G}}{\partial h_n} \\ &+ 2\sum_{n\leq -1} \left(\mathrm{Res}_w w^{-n-1} f(w)\theta(g(w))\right) \frac{\partial \mathscr{G}}{\partial f_n}. \end{aligned}\tag{D.1.6}$$

We can also express $\mathscr{G}E\otimes\theta(\zeta)$ as

$$\mathscr{G}E\otimes\theta(\zeta)=\sum_{n\leq -1}\left(\mathrm{Res}_w w^{-n-1}e^{2h(w)}\theta(g(w))\right)\frac{\partial\mathscr{G}}{\partial e_n}-\sum_{n\leq -1}\left(\mathrm{Res}_w w^{-n-1}f(w)\theta(g(w))\right)\frac{\partial\mathscr{G}}{\partial h_n}$$
$$-\sum_{n\leq -1}\left(\mathrm{Res}_w w^{-n-1}f(w)^2\theta(g(w))\right)\frac{\partial\mathscr{G}}{\partial f_n}. \tag{D.1.7}$$

D.2 Derivation of stochastic differential equations

We derive stochastic differential equations on $e(\zeta)$, $h(\zeta)$, and $f(\zeta)$ so that the random process $\mathscr{G}_t=e^{\mathbf{e}_t}e^{\mathbf{h}_t}e^{\mathbf{f}_t}G(\rho_t)$ satisfies

$$\mathscr{G}_t^{-1}d\mathscr{G}_t=\left(-2L_{-2}+\frac{\kappa}{2}L_{-1}^2+\frac{\tau}{2}\sum_{r=1}^{3}X_r(-1)^2\right)dt+L_{-1}dB_t^{(0)}-\sum_{r=1}^{3}X_r(-1)dB_t^{(r)}. \tag{D.2.1}$$

Here $\{X_r\}_{r=1}^3$ is an orthonormal basis of $\mathfrak{sl}_2$ defined by

$$X_1=\frac{1}{\sqrt{2}}H,\quad X_2=\frac{1}{\sqrt{2}}(E+F),\quad X_3=\frac{i}{\sqrt{2}}(E-F), \tag{D.2.2}$$

and $B_t^{(i)}$, $i=0,1,2,3$ are independent Brownian motions with covariance being given by

$$dB_t^{(0)}\cdot dB_t^{(0)}=\kappa dt,\quad dB_t^{(r)}\cdot dB_t^{(r)}=\tau dt,\quad r=1,2,3. \tag{D.2.3}$$

Since each element $X\otimes f(\zeta)$ in the affine Lie algebra transforms under adjoint action by $G(\rho_t)$ as $G(\rho_t)^{-1}X\otimes f(\zeta)G(\rho_t)=X\otimes f(\rho_t^{-1}(\zeta))$, it suffices to derive stochastic differential equations so that $\Theta_t=e^{\mathbf{e}_t}e^{\mathbf{h}_t}e^{\mathbf{f}_t}$ satisfies

$$\Theta_t^{-1}d\Theta_t=\frac{\tau}{2}\sum_{r=1}^{3}(X_r\otimes\rho_t(\zeta)^{-1})^2dt-\sum_{r=1}^{3}X_r\otimes\rho_t(\zeta)^{-1}dB_t^{(r)}. \tag{D.2.4}$$

We suppose that $e_t(\zeta)$, $h_t(\zeta)$, and $f_t(\zeta)$ satisfy

$$de_t(\zeta)=\overline{e}_t(\zeta)dt+\sum_{r=1}^{3}e_t^r(\zeta)dB_t^{(r)}, \tag{D.2.5}$$

$$dh_t(\zeta)=\overline{h}_t(\zeta)dt+\sum_{r=1}^{3}h_t^r(\zeta)dB_t^{(r)}, \tag{D.2.6}$$

$$df_t(\zeta)=\overline{f}_t(\zeta)dt+\sum_{r=1}^{3}f_t^r(\zeta)dB_t^{(r)}. \tag{D.2.7}$$

Then by an Ito calculus, we obtain

$$de^{\mathbf{e}_t}=e^{\mathbf{e}_t}\left(E\otimes\overline{e}_t(\zeta)+\frac{\tau}{2}(E\otimes e_t^r(\zeta))^2\right)dt+e^{\mathbf{e}_t}\sum_{r=1}^{3}E\otimes e_t^r(\zeta)dB_t^{(r)}, \tag{D.2.8}$$

$$de^{\mathbf{h}_t}=e^{\mathbf{h}_t}\left(H\otimes\overline{h}_t(\zeta)+\frac{\tau}{2}(H\otimes h_t^r(\zeta))^2\right)dt+e^{\mathbf{h}_t}\sum_{r=1}^{3}H\otimes h_t^r(\zeta)dB_t^{(r)}, \tag{D.2.9}$$

$$de^{\mathbf{f}_t}=e^{\mathbf{f}_t}\left(F\otimes\overline{f}_t(\zeta)+\frac{\tau}{2}(F\otimes f_t^r(\zeta))^2\right)dt+e^{\mathbf{f}_t}\sum_{r=1}^{3}F\otimes f_t^r(\zeta)dB_t^{(r)}. \tag{D.2.10}$$

The increment of Θ_t is also computed as

$$\begin{aligned}d\Theta_t =&(de^{\mathbf{e}_t})e^{\mathbf{h}_t}e^{\mathbf{f}_t} + e^{\mathbf{e}_t}(de^{\mathbf{h}_t})e^{\mathbf{f}_t} + e^{\mathbf{e}_t}e^{\mathbf{h}_t}(de^{\mathbf{f}_t})\\ &+ (de^{\mathbf{e}_t})(de^{\mathbf{h}_t})e^{\mathbf{f}_t} + (de^{\mathbf{e}_t})e^{\mathbf{h}_t}(de^{\mathbf{f}_t}) + e^{\mathbf{e}_t}(de^{\mathbf{h}_t})(de^{\mathbf{f}_t}).\end{aligned} \tag{D.2.11}$$

Terms in the increment $d\Theta_t$ proportional to increments of the Brownian motions are

$$\begin{aligned}\sum_{r=1}^{3}\Bigg(&E\otimes e^{-2h_t(\zeta)}e_t^r(\zeta)\\ &+H\otimes(e^{-2h_t(\zeta)}f_t(\zeta)e_t^r(\zeta)+h_t^r(\zeta))\\ &+F\otimes(f_t^r(\zeta)-e^{-2h_t(\zeta)}f_t(\zeta)^2e_t^r(\zeta)-2f_t(\zeta)h_t^r(\zeta))\Bigg)dB_t^{(r)}\end{aligned} \tag{D.2.12}$$

Comparing this to $\sum_{r=1}^{3} X_r \otimes \rho_t(\zeta)^{-1}dB_t^{(r)}$, we identify $e_t^r(\zeta)$, $h_t^r(\zeta)$ and $f_t^r(\zeta)$ as

$$e_t^1(\zeta)=0,\qquad h_t^1(\zeta)=-\frac{1}{\sqrt{2}\rho_t(\zeta)},\qquad f_t^1(\zeta)=-\frac{\sqrt{2}f_t(\zeta)}{\rho_t(\zeta)}, \tag{D.2.13}$$

$$e_t^2(\zeta)=-\frac{e^{2h_t(\zeta)}}{\sqrt{2}\rho_t(\zeta)},\qquad h_t^2(\zeta)=\frac{f_t(\zeta)}{\sqrt{2}\rho_t(\zeta)},\qquad f_t^2(\zeta)=-\frac{1-f_t(\zeta)^2}{\sqrt{2}\rho_t(\zeta)}, \tag{D.2.14}$$

$$e_t^3(\zeta)=-\frac{ie^{2h_t(\zeta)}}{\sqrt{2}\rho_t(\zeta)},\qquad h_t^3(\zeta)=\frac{if_t(\zeta)}{\sqrt{2}\rho_t(\zeta)},\qquad f_t^3(\zeta)=\frac{i(1+f_t(\zeta)^2)}{\sqrt{2}\rho_t(\zeta)}. \tag{D.2.15}$$

Then the term in the increment $d\Theta_t$ proportional to dt becomes

$$\begin{aligned}&E\otimes e^{-2h_t(\zeta)}\overline{e}_t(\zeta)\\ &+H\otimes\left(\overline{h}_t(\zeta)+e^{-2h_t(\zeta)}f_t(\zeta)\overline{e}_t(\zeta)+\frac{\tau}{2\rho_t(\zeta)^2}\right)\\ &+F\otimes\left(\overline{f}_t(\zeta)-e^{-2h_t(\zeta)}f_t(\zeta)^2\overline{e}_t(\zeta)-2f_t(\zeta)\overline{h}_t(\zeta)-\frac{\tau f_t(\zeta)}{\rho_t(\zeta)^2}\right)\\ &+\frac{\tau}{2}\sum_{r=1}^{3}(X_r\otimes\rho(\zeta)^{-1})^2.\end{aligned} \tag{D.2.16}$$

Comparing this to $\frac{\tau}{2}\sum_{r=1}^{3}(X_r\otimes\rho_t(\zeta)^{-1})^2$, we obtain

$$\overline{e}_t(\zeta)=0,\quad \overline{h}_t(\zeta)=-\frac{\tau}{2}\rho_t(\zeta)^{-2},\quad \overline{f}_t(\zeta)=0. \tag{D.2.17}$$

We can finally write down stochastic differential equations

$$de_t(\zeta)=-\frac{e^{2h_t(\zeta)}}{\sqrt{2}\rho_t(\zeta)}dB_t^{(2)}-\frac{ie^{2h_t(\zeta)}}{\sqrt{2}\rho_t(\zeta)}dB_t^{(3)}, \tag{D.2.18}$$

$$dh_t(\zeta)=-\frac{\tau}{2}\rho_t(\zeta)^{-2}dt-\frac{1}{\sqrt{2}\rho_t(\zeta)}dB_t^{(1)}+\frac{f_t(\zeta)}{\sqrt{2}\rho_t(\zeta)}dB_t^{(2)}+\frac{if_t(\zeta)}{\sqrt{2}\rho_t(\zeta)}dB_t^{(3)}, \tag{D.2.19}$$

$$df_t(\zeta)=-\frac{\sqrt{2}f_t(\zeta)}{\rho_t(\zeta)}dB_t^{(1)}-\frac{1-f_t(\zeta)^2}{\sqrt{2}\rho_t(\zeta)}dB_t^{(2)}+\frac{i(1+f_t(\zeta)^2)}{\sqrt{2}\rho_t(\zeta)}dB_t^{(3)}. \tag{D.2.20}$$

D.3 Derivation of operators $\mathscr{X}_\ell$

We first prepare formulae to compute $\mathscr{G}^{-1}X(-\ell)\mathscr{G}$ for $X \in \mathfrak{sl}_2$ and $\ell \in \mathbb{Z}$.

Lemma D.3.1. *We set $\xi := g^{-1}(\zeta)$.*

$$\begin{aligned}
\mathscr{G}^{-1}E\otimes\zeta^{-\ell}\mathscr{G} =& E\otimes e^{-2h(\xi)}\xi^{-\ell} + H\otimes e^{-2h(\xi)}f(\xi)\xi^{-\ell}\\
&- F\otimes e^{-2h(\xi)}f(\xi)^2\xi^{-\ell} - k\mathrm{Res}_w \partial f(w)e^{-2h(w)}w^{-\ell}, \qquad \text{(D.3.1)}\\
\mathscr{G}^{-1}H\otimes\zeta^{-\ell}\mathscr{G} =& 2E\otimes e^{-2h(\xi)}e(\xi)\xi^{-\ell}\\
&+ H\otimes(1+2e^{-2h(\xi)}e(\xi)f(\xi))\xi^{-\ell}\\
&- 2F\otimes(f(\xi)+e^{-2h(\xi)}e(\xi)f(\xi)^2)\xi^{-\ell}\\
&- 2k\mathrm{Res}_w(\partial h(w)+\partial f(w)e^{-2h(w)}e(w))w^{-\ell}. \qquad \text{(D.3.2)}\\
\mathscr{G}^{-1}F\otimes\zeta^{-\ell}\mathscr{G} =& - E\otimes e^{-2h(\xi)}e(\xi)^2\xi^{-\ell}\\
&- H\otimes(e(\xi)+e^{-2h(\xi)}e(\xi)^2f(\xi))\xi^{-\ell}\\
&+ F\otimes(e^{2h(\xi)}+2e(\xi)f(\xi)+e^{-2h(\xi)}e(\xi)^2f(\xi)^2)\xi^{-\ell}\\
&+ k\mathrm{Res}_w(2\partial h(w)e(w)-\partial e(w)+\partial f(w)e^{-2h(w)}e(w)^2)w^{-\ell}. \qquad \text{(D.3.3)}
\end{aligned}$$

Next we express the objects like $\mathscr{G}X\otimes\theta(\zeta)\mathcal{Y}(v,x)|0\rangle$ for $X\in\mathfrak{sl}_2$ and $\theta(\zeta)\in\mathbb{C}((\zeta^{-1}))$ in a convenient form.

Lemma D.3.2.

$$\begin{aligned}
\mathscr{G}E\otimes\theta(\zeta)\mathcal{Y}(v,x)|0\rangle =& \Bigg(\sum_{n\le -1}\mathrm{Res}_z\mathrm{Res}_w\frac{w^{-n-1}e^{2h(w)}\theta(z)}{g(w)-z}\frac{\partial}{\partial e_n}\\
&\quad - \sum_{n\le -1}\mathrm{Res}_z\mathrm{Res}_w\frac{w^{-n-1}f(w)\theta(z)}{g(w)-z}\frac{\partial}{\partial h_n}\\
&\quad - \sum_{n\le -1}\mathrm{Res}_z\mathrm{Res}_w\frac{w^{-n-1}f(w)^2\theta(z)}{g(w)-z}\frac{\partial}{\partial f_n}\Bigg)\mathscr{G}\mathcal{Y}(v,x)|0\rangle\\
&+ \mathrm{Res}_z\frac{\theta(z)}{z-x}\mathscr{G}\mathcal{Y}(Ev,x)|0\rangle, \qquad \text{(D.3.4)}\\
\mathscr{G}H\otimes\theta(\zeta)\mathcal{Y}(v,x)|0\rangle =& \Bigg(\sum_{n\le -1}\mathrm{Res}_z\mathrm{Res}_w\frac{w^{-n-1}\theta(z)}{g(w)-z}\frac{\partial}{\partial h_n}\\
&\quad + 2\sum_{n\le -1}\mathrm{Res}_z\mathrm{Res}_w\frac{w^{-n-1}f(w)\theta(z)}{g(w)-z}\frac{\partial}{\partial f_n}\Bigg)\mathscr{G}\mathcal{Y}(v,x)|0\rangle\\
&+ \mathrm{Res}_z\frac{\theta(z)}{z-x}\mathscr{G}\mathcal{Y}(Hv,x)|0\rangle, \qquad \text{(D.3.5)}\\
\mathscr{G}F\otimes\theta(\zeta)\mathcal{Y}(v,x)|0\rangle =& \sum_{n\le -1}\mathrm{Res}_z\mathrm{Res}_w\frac{w^{-n-1}\theta(z)}{g(w)-z}\frac{\partial}{\partial f_n}\mathscr{G}\mathcal{Y}(v,x)|0\rangle\\
&+ \mathrm{Res}_z\frac{\theta(z)}{z-x}\mathscr{G}\mathcal{Y}(Fv,x)|0\rangle. \qquad \text{(D.3.6)}
\end{aligned}$$

For an intertwining operator $\mathcal{Y}(-,z)$ of type $\binom{L_{\mathfrak{sl}_2,k}(\Lambda)}{L_{\mathfrak{sl}_2,k}(\Lambda),\ L_{\mathfrak{sl}_2,k}}$, we regard $\langle u|\mathscr{G}\mathcal{Y}(-,x)|0\rangle$ as an element of $L(\Lambda)^*[g_{n+1},e_n,h_n,f_n|n<0][[x]]$. The dual space $L(\Lambda)^*$ is equipped with a representation π of $\mathfrak{sl}_2$ defined by $(\pi(X)\phi)(v) = -\phi(Xv)$ for $X \in \mathfrak{sl}_2$, $\phi \in L(\Lambda)^*$ and $v \in L(\Lambda)$.

We begin with the computation of $\langle E(\ell)u|\mathscr{G}\mathcal{Y}(v,x)|0\rangle$.

$$\langle E(\ell)u|\mathscr{G}\mathcal{Y}(v,x)|0\rangle = -\langle u|E(-\ell)\mathscr{G}\mathcal{Y}(v,x)|0\rangle = \mathscr{E}_\ell \langle u|\mathscr{G}\mathcal{Y}(v,x)|0\rangle\,, \tag{D.3.7}$$

where

$$\begin{aligned}
\mathscr{E}_\ell = &-\sum_{n\leq -1}\mathrm{Res}_z\mathrm{Res}_w \frac{w^{-n-1}e^{2h(w)}e^{-2h(z)}z^{-\ell}g'(z)}{g(w)-g(z)}\frac{\partial}{\partial e_n}\\
&-\sum_{n\leq -1}\mathrm{Res}_z\mathrm{Res}_w \frac{w^{-n-1}e^{-2h(z)}(f(z)-f(w))z^{-\ell}g'(z)}{g(w)-g(z)}\frac{\partial}{\partial h_n}\\
&+\sum_{n\leq -1}\mathrm{Res}_z\mathrm{Res}_w \frac{w^{-n-1}e^{-2h(z)}(f(z)-f(w))^2z^{-\ell}g'(z)}{g(w)-g(z)}\frac{\partial}{\partial f_n}\\
&+\mathrm{Res}_z\frac{e^{-2h(z)}z^{-\ell}g'(z)}{g(z)-x}\pi(E)\\
&+\mathrm{Res}_z\frac{e^{-2h(z)}f(z)z^{-\ell}g'(z)}{g(z)-x}\pi(H)\\
&-\mathrm{Res}_z\frac{e^{-2h(z)}f(z)^2z^{-\ell}g'(z)}{g(z)-x}\pi(F)\\
&+k\mathrm{Res}_z\partial f(z)e^{-2h(z)}z^{-\ell}.
\end{aligned} \tag{D.3.8}$$

We also obtain

$$\langle H(\ell)u|\mathscr{G}\mathcal{Y}(v,x)|0\rangle = -\langle u|H(-\ell)\mathscr{G}\mathcal{Y}(v,x)|0\rangle = \mathscr{H}_\ell \langle u|\mathscr{G}\mathcal{Y}(v,x)|0\rangle\,, \tag{D.3.9}$$

where

$$\begin{aligned}
\mathscr{H}_\ell = &-2\sum_{n\leq -1}\mathrm{Res}_z\mathrm{Res}_w \frac{w^{-n-1}e^{2h(w)}e^{-2h(z)}e(z)z^{-\ell}g'(z)}{g(w)-g(z)}\frac{\partial}{\partial e_n}\\
&-\sum_{n\leq -1}\mathrm{Res}_z\mathrm{Res}_w \frac{w^{-n-1}(1+2e^{-2h(z)}(f(z)-f(w)))z^{-\ell}g'(z)}{g(w)-g(z)}\frac{\partial}{\partial h_n}\\
&-2\sum_{n\leq -1}\mathrm{Res}_z\mathrm{Res}_w \frac{w^{-n-1}(f(w)-f(z)-e^{-2h(z)}e(z)(f(w)-f(z))^2)z^{-\ell}g'(z)}{g(w)-g(z)}\frac{\partial}{\partial f_n}\\
&+2\mathrm{Res}_z\frac{e^{-2h(z)}e(z)z^{-\ell}g'(z)}{g(z)-x}\pi(E)\\
&+\mathrm{Res}_z\frac{(1+2e^{-2h(z)}e(z)f(z))z^{-\ell}g'(z)}{g(z)-x}\pi(H)\\
&-2\mathrm{Res}_z\frac{(1+e^{-2h(z)}e(z)f(z))f(z)z^{-\ell}g'(z)}{g(z)-x}\pi(F)\\
&+2k\mathrm{Res}_z(\partial h(z)-\partial f(z)e^{-2h(z)}e(z))z^{-\ell}.
\end{aligned} \tag{D.3.10}$$

and

$$\langle F(\ell)u|\mathscr{G}\mathcal{Y}(v,x)|0\rangle = -\langle u|F(-\ell)\mathscr{G}\mathcal{Y}(v,x)|0\rangle = \mathscr{F}_\ell \langle u|\mathscr{G}\mathcal{Y}(v,x)|0\rangle\,, \tag{D.3.11}$$

where

$$\begin{aligned}
\mathscr{F}_\ell = &\sum_{n\le -1} \mathrm{Res}_z\mathrm{Res}_w \frac{w^{-n-1}e^{2h(w)}e^{-2h(z)}e(z)^2z^{-\ell}g'(z)}{g(w)-g(z)}\frac{\partial}{\partial e_n} \\
&- \sum_{n\le -1} \mathrm{Res}_z\mathrm{Res}_w \frac{w^{-n-1}(1+e^{-2h(z)}e(z)(f(w)-f(z)))e(z)z^{-\ell}g'(z)}{g(w)-g(z)}\frac{\partial}{\partial h_n} \\
&- \sum_{n\le -1} \mathrm{Res}_z\mathrm{Res}_w w^{-n-1}\left[\frac{e^{2h(z)}+2e(z)(f(z)-f(w))}{g(w)-g(z)}\right. \\
&\qquad\qquad \left. + \frac{e^{-2h(z)}e(z)^2(f(z)-f(w))^2}{g(w)-g(z)}\right] z^{-\ell}g'(z)\frac{\partial}{\partial f_n} \\
&- \mathrm{Res}_z \frac{e^{-2h(z)}e(z)^2z^{-\ell}g'(z)}{g(z)-x}\pi(E) \\
&- \mathrm{Res}_z \frac{(1+e^{-2h(z)}e(z)f(z))e(z)z^{-\ell}g'(z)}{g(z)-x}\pi(H) \\
&+ \mathrm{Res}_z \frac{(e^{2h(z)}+2e(z)f(z)+e^{-2h(z)}e(z)^2f(z)^2)z^{-\ell}g'(z)}{g(z)-x}\pi(F) \\
&- \mathrm{Res}_z(2\partial h(z)e(z)-\partial e(z)+\partial f(z)e^{-2h(z)}e(z)^2)z^{-\ell}.
\end{aligned} \tag{D.3.12}$$

D.4 Derivation of $\mathscr{L}_\ell$

We look for an operator $\mathscr{L}_\ell$ such that $\langle L_\ell u|\mathscr{G}\mathcal{Y}(v,x)|0\rangle = \mathscr{L}_\ell \langle u|\mathscr{G}\mathcal{Y}(v,x)|0\rangle$. We first prepare a lemma.

Lemma D.4.1. *We set $\xi = g^{-1}(\zeta)$.*

$$\begin{aligned}
\mathscr{G}^{-1}L_{-\ell}\mathscr{G} = &\sum_{m\in\mathbb{Z}} \left(\mathrm{Res}_z z^{-\ell+1}g(z)^{-n-2}g'(z)^2\right) L_m \\
&- E\otimes e^{-2h(\xi)}\partial e(\xi)\xi^{-\ell+1} \\
&- H\otimes (\partial h(\xi)+e^{-2h(\xi)}f(\xi)\partial e(\xi))\xi^{-\ell+1} \\
&- F\otimes (\partial f(\xi)-2f(\xi)\partial h(\xi)-e^{-2h(\xi)}f(\xi)^2\partial e(\xi))\xi^{-\ell+1} \\
&+ \mathrm{Res}_z z^{-\ell+1}\left(\frac{c}{12}(Sg)(z)+k(\partial h(z)^2+e^{-2h(z)}\partial f(z)\partial e(z))\right).
\end{aligned} \tag{D.4.1}$$

Notice that $\mathscr{G}$ satisfies the same differential equation as one in the case of the Virasoro algebra, thus we have

$$\mathscr{G}L_m = -\sum_{n\le 0}\left(\mathrm{Res}_z z^{-n-1}g(z)^{m+1}\right)\frac{\partial\mathscr{G}}{\partial g_n} \tag{D.4.2}$$

for $m\le -1$. Terms of type $\mathscr{G}X\otimes x(\zeta)$ for $X\in\mathfrak{sl}_2$ can be also expressed as derivatives of

$\mathscr{G}$ as is shown in Sect. D.1. Thus the desired operator $\mathscr{L}_\ell$ is specified as

$$
\begin{aligned}
\mathscr{L}_\ell = &-\sum_{n\le 0}\mathrm{Res}_z\mathrm{Res}_w\frac{z^{-\ell+1}w^{-n-1}g'(z)^2}{g(w)-g(z)}\frac{\partial}{\partial g_n}\\
&-\sum_{n\le -1}\mathrm{Res}_z\mathrm{Res}_w\frac{z^{-\ell+1}w^{-n-1}e^{2h(w)}e^{-2h(z)}\partial e(z)g'(z)}{g(w)-g(z)}\frac{\partial}{\partial e_n}\\
&-\sum_{n\le -1}\mathrm{Res}_z\mathrm{Res}_w\frac{z^{-\ell+1}w^{-n-1}(\partial h(z)+e^{-2h(z)}\partial e(z)(f(z)-f(w)))g'(z)}{g(w)-g(z)}\frac{\partial}{\partial h_n}\\
&-\sum_{n\le -1}\mathrm{Res}_z\mathrm{Res}_w z^{-\ell+1}w^{-n-1}\left[\frac{\partial f(z)-2\partial h(z)(f(z)-f(w))}{g(w)-g(z)}\right.\\
&\qquad\qquad\qquad\qquad\qquad\qquad\left.-\frac{e^{-2h(z)}\partial e(z)(f(z)-f(w))^2}{g(w)-g(z)}\right]g'(z)\frac{\partial}{\partial f_n}\\
&+\mathrm{Res}_z z^{-\ell+1}g'(z)^2\left(\frac{h}{(g(z)-x)^2}+\frac{1}{g(z)-x}\frac{\partial}{\partial x}\right)\\
&+\mathrm{Res}_z\frac{z^{-\ell+1}e^{-2h(z)}\partial e(z)g'(z)}{g(z)-x}\pi(E)\\
&+\mathrm{Res}_z\frac{z^{-\ell+1}(\partial h(z)+e^{-2h(z)}f(z)\partial e(z))g'(z)}{g(z)-x}\pi(H)\\
&+\mathrm{Res}_z\frac{z^{-\ell+1}(\partial f(z)-2f(z)\partial h(z)-e^{-2h(z)}f(z)^2\partial e(z))g'(z)}{g(z)-x}\pi(F)\\
&+\mathrm{Res}_z z^{-\ell+1}\left(\frac{c}{12}(Sg)(z)+k(\partial h(z)^2+e^{-2h(z)}\partial f(z)\partial e(z))\right). \qquad \text{(D.4.3)}
\end{aligned}
$$

Appendix E

Other examples

In this appendix, we show some computation in other constructions of SLE than ones presented in this thesis. We treat cases that internal symmetry is manifested by the current Lie algebra of a vertex algebra in Sect. E.1 and by an affine Lie superalgebra in Sect. E.2.

E.1 Lattice vertex algebra

that associates to the lattice vertex algebra V_L for a rank 1 lattice $L = \mathbb{Z}\alpha$ with $(\alpha|\alpha) = 4$. In the following, we denote the current field associating to $H \in \mathfrak{h} = \mathbb{C} \otimes_{\mathbb{Z}} L$ by $H(z)$, and the vertex operator associating to $\beta \in L$ by $\Gamma_\beta(z)$, which are written explicitly as

$$H(z) = \sum_{n\in\mathbb{Z}} H(n) z^{-n-1}, \tag{E.1.1}$$

$$\Gamma_\beta(z) = e^\beta z^{\beta(0)} \exp\left(-\sum_{j<0} \frac{z^{-j}}{j}\beta(j)\right) \exp\left(-\sum_{j>0} \frac{z^{-j}}{j}\beta(j)\right) = \sum_{n\in\mathbb{Z}} e^\beta_{(n)} z^{-n-1}. \tag{E.1.2}$$

We shall seek an annihilator of the vacuum vector with degree 4. The vector $\omega = \frac{1}{8}\alpha(-1)^2 |0\rangle$ is a conformal vector of central charge 1, which is equivalent to $L_{-2}|0\rangle = \frac{1}{8}\alpha(-1)^2|0\rangle$. By the translation covariance, we obtain

$$L_{-4}|0\rangle = \frac{1}{4}\alpha(-3)\alpha(-1)|0\rangle + \frac{1}{8}\alpha(-2)^2|0\rangle. \tag{E.1.3}$$

By a Wick computation, we also see that

$$\begin{aligned} L(z)L(w) =& \frac{1}{2(z-w)^4} + \frac{1}{4(z-w)^2} :\alpha(w)^2: + \frac{1}{4(z-w)} :\partial\alpha(w)\partial(w): \\ & \frac{1}{8} :\partial^2\alpha(w)\alpha(w): + \frac{1}{64} :\alpha(w)^4: + O(z-w), \end{aligned}$$

of which singular parts are equivalent to the Virasoro commutation relation and the constant term implies that

$$L_{-2}^2|0\rangle = \frac{1}{4}\alpha(-3)\alpha(-1)|0\rangle + \frac{1}{64}\alpha(-1)^4|0\rangle. \tag{E.1.4}$$

An OPE among vertex operators reads

$$\begin{aligned}\Gamma_\alpha(z)\Gamma_{-\alpha}(w) =& \frac{1}{(z-w)^4} + \frac{1}{(z-w)^3}\alpha(w) + \frac{1}{2(z-w)^2}(\partial\alpha(w) + :\alpha(w)^2:) \\ &+ \frac{1}{6(z-w)}(\partial^2\alpha(w) + 3:\partial\alpha(w)\alpha(w): + :\alpha(w)^3:) \\ &+ \frac{1}{24}(\partial^3\alpha(w) + 4:\partial^2\alpha(w)\alpha(w): +3:\partial\alpha(w)^2: +6:\partial\alpha(w)\alpha(w)^2: + :\alpha(w)^4:) \\ &+ O(z-w),\end{aligned}$$

from the constant term of which we obtain

$$e^{\alpha}_{(-1)}e^{-\alpha}_{(-1)}\left|0\right\rangle = \frac{1}{24}(6\alpha(-4) + 8\alpha(-3)\alpha(-1) + 3\alpha(-2)^2 + 6\alpha(-2)\alpha(-1)^2 + \alpha(-1)^4)\left|0\right\rangle. \tag{E.1.5}$$

From an analogous computation with α and $-\alpha$ being exchanged, we have

$$e^{\alpha}_{(-1)}e^{-\alpha}_{(-1)}\left|0\right\rangle = \frac{1}{24}(-6\alpha(-4) + 8\alpha(-3)\alpha(-1) + 3\alpha(-2)^2 - 6\alpha(-2)\alpha(-1)^2 + \alpha(-1)^4)\left|0\right\rangle. \tag{E.1.6}$$

Combining these results, we can see that the vacuum vector $\left|0\right\rangle$ of this lattice vertex algebra is annihilated by the following degree 4 operator:

$$-2L_{-4} + 4L_{-2}^2 + \frac{3}{4}\left(\frac{7}{12}\alpha(-2)^2 - e^{\alpha}_{(-1)}e^{-\alpha}_{(-1)} - e^{-\alpha}_{(-1)}e^{\alpha}_{(-1)}\right). \tag{E.1.7}$$

Let $U(V_L)_{<0}$ be the subalgebra of the current Lie algebra associated with the lattice vertex algebra V_L consisting elements with strictly negative degree. Then this subalgebra is normalized by the action of $\mathrm{Der}_+\mathcal{O}$ in the obvious way. An element in $U(V_L)_{<0}$ cannot be exponentiated to define an operator on V_L, but can be yielding an operator on the formal completion $\overline{V}_L$. The group of such obtained operators is denoted by $\mathcal{G}(V_L)_{<0}$, which the group $\mathrm{Aut}_+\mathcal{O}$ normalizes. We shall consider a random process on the semi-direct product group $\mathrm{Aut}_+\mathcal{O} \ltimes \mathcal{G}(V_L)_{<0}$ that leads to a generalization of SLE.

Let us first notice that by taking $X^1_{(n)} = \frac{i}{\sqrt{2}}(e^{\alpha}_{(n)} + e^{-\alpha}_{(n)})$ and $X^2_{(n)} = \frac{1}{\sqrt{2}}(e^{\alpha}_{(n)} - e^{-\alpha}_{(n)})$, we have

$$(X^1_{(n)})^2 + (X^2_{(n)})^2 = -e^{\alpha}_{(n)}e^{-\alpha}_{(n)} - e^{-\alpha}_{(n)}e^{\alpha}_{(n)}. \tag{E.1.8}$$

Thus the random process $\mathscr{G}_t$ on $\mathrm{Aut}_+\mathcal{O} \ltimes \mathcal{G}(V_L)_{<0}$ under consideration should satisfy the stochastic differential equation

$$\begin{aligned}\mathscr{G}_t^{-1}d\mathscr{G}_t =& \left[-2L_{-4} + 4L_{-2}^2 + \frac{1}{2}\left(\frac{3}{2}(X^1_{(-1)})^2 + \frac{3}{2}(X^2_{(-1)})^2 + \frac{7}{8}\alpha(-2)^2\right)^2\right]dt \\ &+ L_{-2}dB_t^{(0)} + X^1_{(-1)}dB_t^{(1)} + X^2_{(-1)}dB_t^{(2)} + \alpha(-2)dB_t^{(3)},\end{aligned} \tag{E.1.9}$$

where $B_t^{(i)}$ for $i = 0, 1, 2, 3$ are mutually independent Brownian motions such that

$$dB_t^{(0)} \cdot dB_t^{(0)} = 8dt, \quad dB_t^{(1)} \cdot dB_t^{(1)} = dB_t^{(2)} \cdot dB_t^{(2)} = \frac{3}{2}dt, \quad dB_t^{(3)} \cdot dB_t^{(3)} = \frac{7}{8}dt. \tag{E.1.10}$$

We assume the ansatz that $\mathscr{G}_t = \Theta_t G(\rho_t)$ where Θ_t is a random process on $\mathcal{G}(V_L)_{<0}$ and ρ_t is one on $\mathrm{Aut}_+\mathcal{O}$ that satisfies

$$G(\rho_t)^{-1}dG(\rho_t) = \left(-2L_{-4} + 4L_{-2}^2\right)dt + L_{-2}dB_t^{(0)}. \tag{E.1.11}$$

Then the image of z under the automorphism ρ_t satisfies

$$d\rho_t(z) = -\frac{2dt}{\rho_t(z)^3} - \frac{dB_t^{(0)}}{\rho_t(z)}. \tag{E.1.12}$$

Since the current $\alpha(z)$ and vertex operators $\Gamma_{\pm\alpha}(z)$ are Virasoro primary fields, they transform under the adjoint action by $G(\rho_t)^{-1}$ as

$$\alpha(z) = G(\rho_t)\alpha(\rho_t(z))G(\rho_t)^{-1}\partial\rho_t(z), \tag{E.1.13}$$

$$\Gamma_{\pm\alpha}(z) = G(\rho_t)\Gamma_{\pm\alpha}(\rho_t(z))G(\rho_t)^{-1}(\partial\rho_t(z))^2, \tag{E.1.14}$$

first of which implies that

$$G(\rho_t)\alpha(-2)G(\rho_t)^{-1} = \alpha \otimes (\rho_t(\zeta))^{-2}. \tag{E.1.15}$$

From Eq.(E.1.14), we also obtain

$$G(\rho_t)e^{\pm\alpha}_{(-1)}G(\rho_t)^{-1} = \mathrm{Res}_z \frac{\Gamma_{\pm\alpha}(z)}{\rho_t(z)\partial\rho_t(z)}. \tag{E.1.16}$$

Recall that the operators $e^{\pm\alpha}_{(n)}$ in the expansion of the vertex operators $\Gamma_{\pm\alpha}(z)$ represent the elements $\overline{e^{\pm\alpha}\otimes\zeta^n}$ in the current Lie algebra $U(V_L)$. Under this notion, we have

$$G(\rho_t)e^{\pm\alpha}_{(-1)}G(\rho_t)^{-1} = \overline{e^{\alpha}\otimes(\rho_t(\zeta)\partial\rho_t(\zeta))^{-1}}. \tag{E.1.17}$$

These observations lead us to the stochastic differential equation on Θ_t on $\mathcal{G}(V_L)_{<0}$ as follows.

$$\begin{aligned}\Theta_t^{-1}d\Theta_t =& \frac{1}{2}\left(\frac{3}{2}(\overline{X^1\otimes(\rho_t(\zeta)\partial\rho_t(\zeta))^{-1}})^2 + \frac{3}{2}(\overline{X^2\otimes(\rho_t(\zeta)\partial\rho_t(\zeta))^{-1}})^2 + \frac{8}{7}(\alpha\otimes\rho_t(\zeta)^{-2})^2\right)dt \\ &+ \overline{X^1\otimes(\rho_t(\zeta)\partial\rho_t(\zeta))^{-1}}dB_t^{(1)} + \overline{X^2\otimes(\rho_t(\zeta)\partial\rho_t(\zeta))^{-1}}dB_t^2 + \alpha\otimes\rho_t(\zeta)^{-2}dB_t^{(3)},\end{aligned} \tag{E.1.18}$$

where we have set $X^1 = \frac{i}{\sqrt{2}}(e^\alpha + e^{-\alpha})$ and $X^2 = \frac{1}{\sqrt{2}}(e^\alpha - e^{-\alpha})$.

E.2 Affine Lie superalgebra

A Lie super algebra is a $\mathbb{Z}_2$-graded vector space $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$ endowed with a Lie super-bracket $[\cdot,\cdot] : \mathfrak{g}^{\otimes 2} \to \mathfrak{g}$ that is compatible with the $\mathbb{Z}_2$-gradation, super-antisymmetric and satisfies the super Jacobi identity. An example of a Lie superalgebra is $\mathfrak{osp}(1|2)$ that is defined by

$$\mathfrak{osp}(1|2)_0 = \mathbb{C}H \oplus \mathbb{C}E \oplus \mathbb{C}F, \quad \mathfrak{osp}(1|2)_1 = \mathbb{C}e \oplus \mathbb{C}f, \tag{E.2.1}$$

with Lie bracket

$$\begin{aligned} &[H,E] = 2E^{\pm}, && [H,F] = -2F && [E,F] = H, \\ &[H,e] = e, && [H,f] = -f, && [e,f] = H, \\ &[E,f] = -e, && [F,e] = -f, && \\ &[e,e] = E, && [f,f] = -F. && \end{aligned}$$

Notice that the even part $\mathfrak{osp}(1|2)$ is isomorphic to $\mathfrak{sl}_2$.

For a finite dimensional simple Lie superalgebra $\mathfrak{g}$ with a supersymmetric even invariant bilinear form $(\cdot|\cdot)$ normalized so that the square length of a long root is 2, the corresponding (untwisted) affine Lie superalgebra is $\widehat{\mathfrak{g}} = \mathfrak{g} \otimes \mathbb{C}[\zeta, \zeta^{-1}] \oplus \mathbb{C}K$ with Lie bracket

$$[X(m), Y(n)] = [X, Y](m+n) + m(X|Y)\delta_{m+n,0}K, \quad [K, \widehat{\mathfrak{g}}] = \{0\}, \tag{E.2.2}$$

where we have denoted $X \otimes \zeta^m$ for $X \in \mathfrak{g}$ and $m \in \mathbb{Z}$ by $X(m)$. We can construct a representation of an affine Lie superalgebra by induction. Let M be a representation of a finite dimensional simple Lie superalgebra $\mathfrak{g}$. Then its Weyl module is defined by

$$\widehat{M_k} = \mathrm{Ind}^{\widehat{\mathfrak{g}}}_{\mathfrak{g}\otimes\mathbb{C}[\zeta]\oplus\mathbb{C}K} M, \tag{E.2.3}$$

on which the central element K acts as multiplication by $k \in \mathbb{C}$. On a representation of $\widehat{\mathfrak{g}}$ constructed in this way, we define an action of the Virasoro algebra via the Sugawara construction. Let $\{X_i\}_{i=1}^{\dim\mathfrak{g}}$ be an homogeneous basis of $\mathfrak{g}$ and $\{X^j\}_{j=1}^{\dim\mathfrak{g}}$ be the dual basis with respect to $(\cdot|\cdot)$. Note that since the bilinear form is even, the even part and the odd part is orthogonal with each other, and since it is supersymmetric, the square norm of an odd element is zero and thus we cannot take an orthonormal basis. Assuming $k \neq -h^\vee$, where $h^\vee$ is the dual Coxeter number of $\mathfrak{g}$, we define operators L_n for $n \in \mathbb{Z}$ by

$$L_n = \frac{1}{2(k+h^\vee)} \sum_{i=1}^{\dim\mathfrak{g}} \sum_{p\in\mathbb{Z}} (-1)^{|X_i|} :X_i(p)X^i(n-p): \; . \tag{E.2.4}$$

Then they define a representation of the Virasoro algebra of central charge $c_k = \frac{k\mathrm{sdim}\mathfrak{g}}{k+h^\vee}$.

We shall comment on the relation between Lie superalgebras and Lie supergroups. In the case of an ordinary Lie algebra, the corresponding Lie group is roughly generated by exponentiation of elements in the Lie algebra, namely $G = \langle e^X | X \in \mathfrak{g}\rangle$. This description is modified in the case of Lie superalgebra. For a Lie superalgebra $\mathfrak{g}$, we take its "Grassmann envelop " $\mathfrak{g}(\bigwedge)$, which is defined as follows. Let $\bigwedge = \bigwedge[\theta_1, \cdots, \theta_N]$ be the external (Grassmann) algebra with generators $\theta_1, \cdots, \theta_N$. Then it is $\mathbb{Z}_2$-graded as a vector space by assigning $\deg\theta_i = 1$. Then the Grassmann envelop of $\mathfrak{g}$ is the even part of the tensor product $\mathfrak{g} \otimes \bigwedge$, or more explicitly,

$$\mathfrak{g}(\bigwedge) = \mathfrak{g}_0 \otimes \bigwedge{}_0 \oplus \mathfrak{g}_1 \otimes \bigwedge{}_1. \tag{E.2.5}$$

The Grassmann envelop is an ordinary Lie algebra by defining

$$[X \otimes \eta_1, Y \otimes \eta_2] = [X, Y] \otimes \eta_1\eta_2 \tag{E.2.6}$$

Then the Lie supergroup corresponding to $\mathfrak{g}$ is identified as $G = \langle e^X | X \in \mathfrak{g}(\bigwedge)\rangle$.

Following these preliminaries, we shall construct a SLE with internal symmetry being manifested by an affine Lie superalgebra. In the Weyl module that is induced from the trivial representation spanned by the vacuum vector, we have the following annihilator of the vacuum vector:

$$-2L_{-2} + \frac{1}{k+h^\vee} \sum_{i=1}^{\dim\mathfrak{g}} (-1)^{|X_i|} X_i(-1)X^i(-1). \tag{E.2.7}$$

As we have done in Chap. 4, we assume that for another representation $L_{\mathfrak{g},k}(\Lambda)$ that is obtained as a quotient of a Weyl module $\widehat{M_k}$, we can find an annihilator of a vector v_Λ in the top space in the form of

$$-2L_{-2}+\frac{\kappa}{2}L_{-1}^2+\frac{\tau}{2}\sum_{i=1}^{\dim\mathfrak{g}}(-1)^{|X_i|}X_i(-1)X^i(-1) \tag{E.2.8}$$

with parameters κ and τ being finely tuned. We would like to design a random process $\mathscr{G}_t$ on $\mathrm{Aut}_+\ltimes G(\mathcal{O})$ that produces a local martingale when applied to the vector v_Λ. To see how this is possible, we concentrate on the case of $\mathfrak{g}=\mathfrak{osp}(1|2)$. In this case, the tentative annihilator has the form

$$-2L_{-2}+\frac{\kappa}{2}L_{-1}^2+\frac{\tau}{2}\left(\Omega_0+\Omega_1\right), \tag{E.2.9}$$

where

$$\Omega_0=\frac{1}{2}H(-1)^2+E(-1)F(-1)+F(-1)E(-1), \tag{E.2.10}$$

$$\Omega_1=-\frac{1}{2}(e(-1)f(-1)-f(-1)e(-1)). \tag{E.2.11}$$

In construction of SLE, one has to express Ω_0 and Ω_1 as sum of squared quantities. The "even" part Ω_0 raises no problem, it can be written as

$$\Omega_0=\left(\frac{1}{\sqrt{2}}H(-1)\right)^2+\left(\frac{1}{\sqrt{2}}(E(-1)+F(-1))\right)^2+\left(\frac{i}{\sqrt{2}}(E(-1)-F(-1))\right)^2. \tag{E.2.12}$$

It is easily seen that the "odd" term Ω_1 cannot have such expression as long as the coefficients are in $\mathbb{C}$. Instead we observe the following equation in the Grassmann envelop $\widehat{\mathfrak{osp}(1|2)}(\bigwedge)$ over $\bigwedge=\bigwedge[\theta_1,\theta_2]$:

$$\left(\frac{1}{\sqrt{2}}(\theta_1e(-1)+\theta_2f(-1))\right)^2=-\frac{1}{2}\theta_2\theta_1(e(-1)f(-1)-f(-1)e(-1)). \tag{E.2.13}$$

If we set $J_1=\frac{1}{\sqrt{2}}H(-1)$, $J_2=\frac{1}{\sqrt{2}}(E(-1)+F(-1))$, $J_3=\frac{i}{\sqrt{2}}(E(-1)-F(-1))$ and $J_4=\frac{1}{\sqrt{2}}(\theta_1e(-1)+\theta_2f(-1))$, the following equation is one on a random process on $\mathrm{Aut}_+\mathcal{O}\ltimes G_+(\mathcal{O})$:

$$\mathscr{G}_t^{-1}d\mathscr{G}_t=\left(-2L_{-2}+\frac{\kappa}{2}L_{-1}^2+\frac{\tau}{2}(\Omega_0+\theta_2\theta_1\Omega_1)\right)dt+L_{-1}dB_t^{(0)}+\sum_{i=1}^{4}J_1dB_t^{(i)}. \tag{E.2.14}$$

Note that this random process does not act on $\overline{L_{\mathfrak{g},k}(\Lambda)}$, but acts on $\overline{L_{\mathfrak{g},k}(\Lambda)}\otimes\bigwedge$. By assumption on an annihilator of the highest weight vector v_Λ, we observe that

$$\int d\theta_1d\theta_2\mathscr{G}_tv_\Lambda\otimes(1+\theta_2\theta_1) \tag{E.2.15}$$

is an $\overline{L_{\mathfrak{g},k}(\Lambda)}$-valued local martingale. This property will serve as an essential ingredient in construction of SLE corresponding to an affine Lie superalgebra.

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