論文の内容の要旨

Representation theoretical approach to Schramm-Loewner evolution

（シュラム・レヴナー発展に対する表現論的アプローチ）

越田 真史

Background

Growth processes have been proven to give frameworks that describe various equilibrium and non-equilibrium phenomena exhibited in nature. Examples of such growth processes we consider in this thesis are variants of Schramm-Loewner evolution (SLE), which was introduced by Schramm in [Sch00] as subsequent scaling limit of loop erased random walks and uniform spanning trees. It is a stochastic differential equation on a uniformization map $g_t : \mathbb{H}\backslash K_t \to \mathbb{H}$ for evolving hulls $K_t \subset \mathbb{H}$ in the upper half plane $\mathbb{H}$, defined by

$$\frac{d}{dt} g_t(z) = \frac{2}{g_t(z) - X_t}, \quad g_0(z) = z. \tag{1}$$

Here X_t is a Brownian motion on the real axis, which is the boundary of the upper half plane $\mathbb{H}$. It has been shown to describe an interface of clusters in several critical systems in two dimensions including critical percolation [Smi01] and Ising models at criticality [CDCH$^+$14].

We have another framework to investigate two dimensional critical systems. It is two dimensional conformal field theory (CFT) [BPZ84], which has been one of the most powerful tools in wide variety of fields from condensed matter physics to string theory, and in mathematics as well. The notion of CFT has also been extended to the case that a system has nonempty boundaries and boundary conditions are imposed as boundary conformal field theory (BCFT) [Car86]. A milestone of (B)CFT prediction on a critical system is Cardy's formula [Car92], which gives the crossing probability for a critical percolation in two dimensions from computation of correlation functions in BCFT. Cardy's formula has been proven by Smirnov [Smi01] to be a theorem, while the derivation by Cardy has not been verified.

Since SLE and CFT are different frameworks that describe the same phenomena, they are expected to be bridged to each other in some sense. Connection between SLE and CFT has been studied under the name of SLE/CFT correspondence from various points of view. A significant development is the *group theoretical* formulation of SLE by Bauer and Bernard [BB02, BB03], which proposes an elegant way of constructing SLE local martingales from a representation of the Virasoro algebra. Their strategy was to encode the coordinate transformations described by SLE on an irreducible representation space of the Virasoro algebra of a certain highest weight by an operator valued random process $\mathscr{G}_t$ acting on the representation space. Discovery by Bauer and Bernard was that if the Verma module of the same highest weight of the Virasoro algebra has a null vector in the space of conformal weight higher than the highest weight by 2, then the random process $\mathscr{G}_t$ produces a representation-space-valued local martingale when applied to the highest weight vector. In other words, such a random process consists of infinitely many local martingales that is associated with the solution of SLE. A local martingale is by definition a random process of which increment does not have a drift term. A significant property of a local martingale is that its expectation value is independent of time, which is why to find local martingale is a crucial task in study of a random process. We can summarize at least a part of SLE/CFT correspondence in the sense of Bauer and Bernard by stating that *a computation on a state space of CFT leads to local martingales associated with SLE*.

The importance of SLE/CFT correspondence is bidirectional. In one direction, one can compute an SLE local martingale from computation in CFT, while in the other direction, SLE gives data of CFT such as partition functions and correlation functions, which in some sense may be interpreted as construction of a nontrivial quantum field theory as a result of path integral. Thus precise understanding SLE/CFT correspondence helps us develop theories of both SLE and CFT.

Issue

The group theoretical formulation of SLE has established the connection between SLE and CFT and enabled to obtain local martingales from a state space of CFT. Although some CFT has internal degrees of freedom as well as space-time ones, the analogous group theoretical formulation of SLE that corresponds to such a CFT is absent. This thesis is aimed at generalizing the group theoretical formulation of SLE to representation theories of other algebras than the Virasoro algebra that describe internal symmetry of CFT as well as space-time symmetry, and correspondingly generalizing the notion of SLE. Such a direction of generalization will deepen our understanding of connection between SLE and CFT, and will help us settle the SLE/CFT correspondence in the sense of Bauer and Bernard in a more fundamental and general theory.

Idea

The notion of vertex operator algebra (VOA) is a mathematical counterpart of CFT. For a given VOA, we can associate with it a Lie algebra that is generated by coefficients of field operators, which is called the current Lie algebra of the VOA. Since any VOA has by definition a Virasoro field (stress-energy tensor), a current Lie algebra contains the Virasoro algebra of some fixed central charge as a Lie subalgebra. Thus we can say that the Virasoro algebra acts on a current Lie algebra. We call any Lie subalgebra of a current Lie algebra that is normalized by the action of the Virasoro algebra a Lie algebra of *internal symmetry* following the terminology of [FBZ04]. For a Lie algebra of internal symmetry, we can take a semi-direct product of it with the Virasoro algebra. and correspondingly we obtain a semi-direct product of Lie groups of space-time symmetry and internal symmetry. We expect that a random process on this Lie group yields a generalization of SLE that involves internal symmetry as the ordinary SLE is generated by a random process on the group of space-time symmetry.

Summary of Chapters

Chapter 3

In Chapter 3 of this thesis, we take the positive loop group of a torus as an internal symmetry group, on which the group of space-time symmetry acts as transformation of the loop variable. The corresponding representation theory is one of a Heisenberg algebra. We construct SLE associated with a Fock representation of a Heisenberg algebra of any rank from a random process on the semi-direct product group of space-time and internal symmetries, and compute some local martingales that originate from an annihilator of the highest weight vector. Cardy [Car06] once pointed out that the notion of SLE(κ, ρ) is related to representation theory of a Heisenberg algebra, but our construction is very different from that of SLE(κ, ρ) in that ours is a generalization of the group theoretical formulation while a group theoretical formulation of SLE(κ, ρ) is not known, and indeed, SLE we construct is different from SLE(κ, ρ).

Chapter 4

In Chapter 4, we construct SLE associated with representation theory of an affine Lie algebra of a finite dimensional simple Lie algebra. The corresponding group of internal symmetry is the positive loop group of the corresponding finite dimensional simple Lie group, on which the group of space-time symmetry again acts as transformation of the loop variable. We consider a random process on the semi-direct product group of space-time and internal symmetries and derive the corresponding SLE.

CFT that is associated with representation theory of an affine Lie algebra is known as Wess-Zumino-Witten (WZW) theory. The notion of SLE that corresponds to WZW theory has been considered in [BGLW05, ABI11], and the equations appearing in [BGLW05, ABI11] is reconstructed in our formulation. However, construction in [BGLW05, ABI11] is not sufficient in the following points: (1) their stochastic differential equations on internal degrees of freedom seem to be *ad hoc*, (2) the random process along internal degrees of freedom is not concretely constructed and (3) local martingales associated with the solution are difficult to write down.

We construct a random process in the internal symmetry group in the most concrete form in the case of $\mathfrak{sl}_2$, and compute local martingales that are obtained from the representation-space-valued local martingale. We also find out a module structure over the affine Lie algebra $\widehat{\mathfrak{sl}}_2$ on a space of local martingales. In comparison to the work in [BGLW05, ABI11], in our formulation, (1) the stochastic differential equations on internal degrees of freedom naturally arise, (2) the random process along internal degrees of freedom is concretely constructed in the case of $\widehat{\mathfrak{sl}}_2$ and (3) local martingales associated with the solution are written down explicitly in the case of $\widehat{\mathfrak{sl}}_2$.

We also propose a way of generalization of SLE associated with representation theory of affine Lie algebras. As examples, we construct stochastic differential equations associated with operators of degree 4 that annihilate the vacuum vectors in the basic representations of $\widehat{\mathfrak{sl}}_2$ and $\widehat{\mathfrak{sl}}_3$.

References

[ABI11] Anton Alekseev, Andrei Bytsko, and Konstantin Izyurov. On SLE martingales in boundary WZW models. *Lett. Math. Phys.*, 97:243–261, 2011.

[BB02] Michel Bauer and Denis Bernard. SLE_κ growth process and conformal field theories. *Phys. Lett. B*, 543:135–138, 2002.

[BB03] Michel Bauer and Denis Bernard. Conformal field theories of stochastic Loewner equations. *Commun. Math. Phys.*, 239:493–521, 2003.

[BGLW05] E. Bettelheim, I. A. Gruzberg, A. W. W. Ludwig, and P. Wiegmann. Stochastic Loewner evolution for conformal field theories with Lie group symmetries. *Phys. Rev. Lett.*, 95:251601, 2005.

[BPZ84] A. A. Belavin, A. M Polyakov, and A. B. Zamolodchikov. Infinite conformal symmetry in two-dimensional quantum field theory. *Nucl. Phys. B*, 241:333–380, 1984.

[Car86] J. L. Cardy. Effect of boundary conditions on the operator center of two-dimensional conformally invariant theories. *Nucl. Phys. B*, 275:200–218, 1986.

[Car92] John L. Cardy. Critical percolation in finite geometries. *J. Phys. A: Math. Gen.*, 25:L201–L206, 1992.

[Car06] John Cardy. $\mathrm{SLE}(\kappa, \rho)$ and conformal field theory, 2006. arXiv:math-ph/0412033.

[CDCH+14] Dmitry Chelkak, Hugo Duminil-Copin, Clément Hongler, Antti Kemppainen, and Stanislav Smirnov. Convergence of Ising interfaces to Schramm's SLE curves. *Comptes Rendus Mathematique*, 352:157–161, 2014. arXiv:1312.0533.

[FBZ04] Edward Frenkel and David Ben-Zvi. *Vertex Algebras and Algebraic Curves*, volume 88 of *Mathematical Surveys and Monographs*. American Methematical Society, 2nd edition, 2004.

[Sch00] Oded Schramm. Scaling limits of loop-erased random walks and uniform spanning trees. *Israel J. Math.*, 118:221–288, 2000.

[Smi01] Stanislav Smirnov. Critical percolation in the plane: conformal invariance, Cardy's formula, scaling limits. *C. R. Acad. Sci. Paris*, 333:239–244, 2001.