学位論文

Aspects of anomalies in 6d superconformal field theories
(六次元超共形場の理論の量子異常の諸相)

平成２９年１２月　博士（理学）申請

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Abstract

In this thesis, the author analyze various aspects of anomalies of global symmetries of 6d superconformal field theories (SCFTs). Although the string theory predicts the existence of many 6d SCFTs, it is in general very hard to compute the physical quantities of these theories. However, the quantum anomalies, which capture the quantum-mechanical violation of the global symmetries of the theory, are exactly computable for all known 6d SCFTs. The purpose of this thesis is to extract non-trivial consequences about 6d SCFTs from the computed anomalies.

First of all, the author derived the conditions that a given 6d SCFT can be Higgsed to a collection of free hypermultiplets or to the $\mathcal{N}=(2,0)$ theory, respectively. These conditions are stated in term of the endpoint of a 6d SCFT, a string of positive integers which can be assigned to any 6d SCFT and specifies a particular sub-branch of the tensor branch. The derivation is based on the matching of the gravitational anomaly on the generic point of the Higgs branch and the endpoint of the tensor branch. The resulting conditions can be simply interpreted in terms of M5-branes probing the singularity in M-theory.

As another application of such an anomaly matching on the moduli space, the author also obtained the list of possible simple Lie groups whose one-instanton moduli space can become a Higgs branch of some 6d SCFTs. The answer fits well to the intuition based on the string theory construction of 6d SCFTs. As a byproduct, the author found a new way to compute the anomaly polynomial of the rank-1 E-string theory.

Another method to compute the anomalies of 6d SCFTs is the anomaly inflow, which does not require the knowledge about the moduli space of 6d SCFTs. By using the anomaly inflow, the author found the general anomaly formula for the chiral anomalies supported on the string in 6d $\mathcal{N}=(1,0)$ theories. The formula can be used to any strings in any 6d SCFTs and helps to study further about the properties of the worldsheet theories. Moreover, the author obtained a new interpretation of some properties of 6d SCFTs in terms of the strings.

As another application of the inflow technique, the author determined the Chern-Simons terms localized on the frozen singularity in M-theory, which is one of the still mysterious objects in M-theory. This is done by rewiring the anomaly polynomial computed from a matching on the tensor branch in a more M-theoretic form. The expression involves a new "Euler number" whose M-

theoretic meaning is worth of studying.

In summary, the author showed the usefulness of the anomalies in studying the various properties of 6d SCFTs, by concretely presenting several new interesting results.

Contents

Chapter 1

Introduction and summary

1.1 6d SCFTs

Quantum field theory (QFT) is one of the most important concepts in theoretical physics. For example, the standard model of the particle physics is very successful in describing the elementary particles of this world. Although our spacetime is (seemingly) four-dimensional, QFTs in lower spacetime dimension are also worth of studying. In fact, these lower dimensional QFTs not only serve as toy models of those in 4d, but also exhibit interesting dynamics on their own. Moreover, in condensed matter physics, many QFTs in $d < 4$ actually arise as long distance limits of various lattice models.

Then, is there any motivation to study about QFTs in $d > 4$? A common answer to this question is that QFTs in $d > 4$ are necessarily free and uninteresting. For example, the Yang-Mills coupling g in higher dimensions has the positive mass dimension and hence is non-reormalizable. However, progress in string theory in the 90s revealed that this naive answer is not necessarily true. In fact, the string theory predicts that there are many interesting field theories in $d = 5, 6$. These theories arise as a ultraviolet fixed point of an apparently non-renormalizable supersymmetric gauge theory in $d = 5, 6$. As a fixed point of a supersymmetric RG flow, these theories possess the superconformal symmetry in $d = 5, 6$ which unifies both the conformal symmetry and the supersymmetry. We will call these theories as 5d/6d superconformal field theories (SCFTs).

In string theory, a 5d SCFTs is engineered by a brane web in type IIB [1], or a compactification of M-theory on a Calabi-Yau threefold [2]. On the other hand, a 6d SCFTs is engineered by M5-branes probing an M-theory singularity [3], or a compactification of F-theory on an elliptic Calabi-Yau threefold [4, 5]. Recent studies try to classify such SCFTs arising from string theory. The classification of 6d SCFTs from the F-theory viewpoint is almost finished in [6, 7, 8], while the classification of 5d SCFTs was initiated in [9, 10].

In this thesis, we focus on 6d SCFTs, not on 5d SCFTs for the following reasons. First, six is believed to be the maximal spacetime dimension in which we can have an interacting SCFT, and hence we expect that some special features might exist at such a "boundary" of the dimension. Second, many (conjecturally all) 5d SCFTs arise as a S^1 compacfication of some 6d SCFT, and hence it is more economical to move to 5d SCFTs after studying enough about 6d SCFTs. Third, since the classification of 6d SCFTs is almost finished, we can systematically list 6d SCFTs which have the properties we focus on.

From their first discovery, 6d SCFTs are considered to be one of the most interesting topics in string theory. Some of the reasons why are given as follows. First, their light degrees of freedom contain strings, while those of ordinary field theories are particles. For example, the instantons of 6d gauge theories are stringy objects, whose tensions decrease to zero toward a UV fixed point. To find a better framework incorporating such light (and strongly interacting) strings into an ordinary field theory may help to find out the formulation of a strongly coupled string theory. Second, many 6d SCFTs describe the dynamics of M5-branes, which are still mysterious extended objects in the M-theory. The field theoretical study of 6d SCFTs shed light on our understanding of the M-theory. Third, 6d SCFTs can be used as a way to organize and understand various dynamical properties of lower dimensional field theories. For example, many S-dualities of 4d $\mathcal{N}=2$ field theories originate from the fact that they are obtained from the compacfication of some 6d SCFT on a Riemann surface [11].

With these goals in mind, this thesis investigates various properties of 6d SCFTs.

1.2 Anomalies and 6d SCFTs.

The tool of this thesis to extract useful and interesting information about 6d SCFTs is the anomaly of global symmetries [12] . We start by recalling the anomaly of global symmetries in QFTs. It is the phase variation of the partition function in the presence of non-trivial background fields (See textbook [13], or articles [14, 15]). We consider the D-dimensional quantum field theory with the global symmetry $\mathcal{G}$ and couple it to the background metric $g_{\mu\nu}$ and the gauge field A_μ for $\mathcal{G}$. Then the partition function $Z[g_{\mu\nu}, A_\mu]$ of the theory may change by the phase under the background gauge transformation;

$$Z[g_{\mu\nu}^{\xi_\mu}, A_\mu^g] = \exp\left(2\pi i \int_{X_D} I_D^{(1)}[g_{\mu\nu}, A_\mu; \xi_\mu, g]\right) Z[g_{\mu\nu}, A_\mu], \tag{1.2.1}$$

where ξ_μ and g is the parameter of the gauge transformation of $g_{\mu\nu}$ and A_μ, respectively. The I_D is the anomaly associated with the diffeomorphism and the global symmetry $\mathcal{G}$. The subscript (1) represents the fact that $I_D^{(1)}$ is first order in the gauge parameters ξ_μ, g.

The BRST formalism of the anomaly implies that it is convenient to work with the anomaly polynomial I_{D+2} rather than the anomaly itself. It is related to the anomaly via the descent equations;

$$I_{D+2} = dI^{(0)}_{D+1}, \tag{1.2.2}$$

$$\delta I^{(0)}_{D+1} = dI^{(1)}_{D}, \tag{1.2.3}$$

where d is the exterior derivative and δ is the gauge variation. The $(D+1)$ form $I^{(0)}_{D+1}$ is called the Chern-Simons form associated with the anomaly. The superscript (0) indicates that this quantity is independent of the gauge parameters. The anomaly polynomial I_{D+2} is a gauge invariant functional of the background gauge fields and the background metric.

Why anomalies are useful and important in the study of 6d SCFTs? The string theory predicts the existence of many 6d SCFTs and their qualitative features, such as their global symmetries and the gauge theory description. However, in order to obtain better understandings of 6d SCFTs, we need quantitative aspects of these theories. Anomaly is a convenient physical quantity to compute and to examine its physical consequences. Let us explain the importance of anomalies in the study of 6d SCFTs in more detail.

Anomalies are exactly computable. First, they are in fact exactly computable. The 6d SCFTs are necessarily strongly coupled and it is in general very hard to compute the values of physical quantities, such as correlation functions of local operators. However, the anomalies are associated with the topology of the gauge fields and receive little modifications during the RG flow [12], making the exact computation possible [1].

A general method to compute the anomaly polynomial of all the known 6d SCFTs has already been found [28, 29], as will be reviewed in Chapter 3 of this thesis. In this method, one uses the description of 6d SCFTs on the moduli space of vacua called as tensor branch. On the tensor branch, the theory is described by a system of almost free tensor multiplets, non-abelian gauge fields and hypermultiplets. The precise matter content on the tensor branch can be easily read off from the F-theory realization of the 6d theory. Combining the ordinary one-loop contributions from the massless multiplets and those from the Green-Schwarz coupling of tensor fields, we can uniquely determine the full anomaly polynomial of a given 6d SCFT. The fact that they are systematically computable for all known 6d SCFTs make the anomaly a valuable tool to study the 6d SCFTs.

Constraints from anomalies. The anomalies can be used to obtain constraints for 6d SCFTs. As already noticed in the 90s, the gauge anomaly cancelation puts a severe constraint for a given 6d

[1] Other physical quantities of 6d SCFTs considered in the literature include; conformal central charges [16, 17], some supersymmetric partition functions [18, 19, 20, 21, 22, 23], superconformal multiplets and deformations [24, 25]. The numerical bootstrap computation was performed for 6d SCFTs in [26, 27].

gauge theory to define a well-defined field theory. For example, as will be explained in Chapter 2, the 6d pure gauge theory is only consistent for gauge groups $G = \mathrm{SU}(2), \mathrm{SU}(3), \mathrm{SO}(8), F_4, E_{6,7,8}$. The cancelation of gauge anomalies is also important for the classification of 6d SCFTs from F-theory in [6, 7, 8].

As a generalization of such an anomaly constraint, we will point out in this thesis that the anomalies can also be used to find out all the 6d SCFTs which have a specific Higgs branch structure [30, 31]. Here Higgs branch is the moduli space of vacua where the $\mathrm{SU}(2)_R$ and the flavor symmetry of the 6d theory is spontaneously broken due to the Higgsing. We compare the anomalies on the origin and on the generic point of the putative Higgs branch. If the matching equation does not have a solution, we can conclude that 6d SCFTs with the conjectural Higgs branch do not exist.

In Chapter 4, we consider 6d SCFTs which can be Higgsed to a collection of hypermultiplets and 6d SCFTs which can be Higgsed to the $\mathcal{N} = (2,0)$ theory [31]. We will call such 6d SCFTs as very Higgsable or Higgsable to $\mathcal{N} = (2,0)$ theory, respectively. As an example, let us consider the Higgs branch of the E-string theory and the $\mathcal{N} = (2,0)$ theory. The E-string theory, which is a worldvolume theory of an M5 probing the E_8 wall, has the Higgs branch where the M5s are dissolved into the finite-sized E_8 instanton on the wall. This branch is described by the free hypermultiplets only. On the contrary, the $\mathcal{N} = (2,0)$ theory has no such a Higgs branch; there is no way to eliminate the tensor modes supported on the worldvolume on the M5s. Of course, the $\mathcal{N} = (2,0)$ theory is Higgsable to $\mathcal{N} = (2,0)$ theory.

There are several reasons to be interested in such two classes of 6d SCFTs. First of all, we will find that many interesting 6d SCFTs which can be engineered in M-theory fall into these classes. Secondly, we can concretely obtain the full list of 6d SCFTs which are very Higgsable or Higgable to $\mathcal{N} = (2,0)$ theory, by comparing the gravitational anomaly on the origin and the generic point of the Higgs branch. The full list and the Higgsing process of these theories turn out to have an interesting interpretation in terms of the branes. As a byproduct, we can derive the simple Higgs branch dimension formulas for these 6d SCFTs. Moreover, these 6d theories have nice behaviors under the T^2 compactification [32, 33]. For example, 4d $\mathcal{N} = 2$ theory obtained from the T^2 compactification of 6d theories Higgsable to $\mathcal{N} = (2,0)$ theory of type G contains the vector multiplet G which has an exactly marginal coupling. The gauge coupling of G exhibits the S-duality, identified with the $\mathrm{SL}(2,\mathbb{Z})$ acting on T^2.

In Chapter 5, we will determine the full list of 6d SCFTs which have the one-instanton moduli space of G, denoted as M_G, as its Higgs branch [30]. For example, when $G = E_8$, the 6d theory with this Higgs branch is already known; E-string theory. However, for $G = E_{6,7}$, we do not know any string theory constructions of such 6d theories. This should be compared with the case of 4d $\mathcal{N} = 2$ theories, where we know the existence of 4d $\mathcal{N} = 2$ SCFTs with Higgs branch M_{E_n}. In Chapter 5, we will find that most Gs are excluded since there is no solution to the anomaly matching equation. In particular, $G = E_{6,7}$ examples cannot exist since they contradict with the anomaly matching. Moreover, for $G = E_8$, we reproduce the anomaly of the rank-1 E-string theory, which gives another purely field theoretical way of computing its anomaly.

Strings and anomalies. The anomalies can be used to extract the useful information about the string in 6d SCFTs. As already mentioned, the self-dual strings are important ingredients of the 6d theories. To obtain a better understanding of strings, as in the analysis of ordinary perturbative string theory, we need the worldsheet theories on the strings. When the 6d theory is engineered by a chain of perturbative D-branes, it is straightforward to obtain a 2d $\mathcal{N}=(0,4)$ quiver gauge theory describing the strings. However, if a 6d theory is engineered by using (non-perturbative) F-theory sevenbranes, it is in general difficult to find such a gauge theory description.

We will present a general way to obtain a partial information about the worldsheet theories of strings in 6d $\mathcal{N}=(1,0)$ theories; the central charges of the worldsheet theory. In fact, combining the anomaly polynomial computed by the method reviewed in Section 3.1 with the anomaly inflow technique, also reviewed in Section 3.3, we can determine the anomaly 4-form associated with any types of strings in any 6d theories. When a gauge theory description of the worldsheet theory is available, we can check that our formula correctly reproduces the anomalies of the 2d gauge theory.

There are several applications of thus obtained anomaly formula of the strings. From the central charges of the string, one can check whether a conjectural gauge theory description of the worldsheet theory is right or not [34, 35]. Moreover, as pointed out in [35, 36], the elliptic genus of the string can be computed from the anomaly 4-form of the string, which in turn determines the BPS invariants of the 6d theory [18, 19, 20, 21, 22, 23]. However, in this thesis, we will present more simple applications of the string anomalies in Section 6.3. These include a new understanding of the ADE classification of 6d $\mathcal{N}=(2,0)$ theory, and the E_8 flavor symmetry of the E-string theory [37]. We also find a physical explanation of the curious numerology of exceptional groups.

M-theory and anomalies. When a 6d theory is a worldvolume theory on M5s, probably on top of other singularities in M-theory, we can compute the anomalies of the theory in a more geometric way; anomaly inflow [38]. In this method, the anomalies are computed by integrating the 11d Chern-Simons term in M-theory appropriately around the worlvolume of the M5s.

It has an advantage that the detailed knowledge about the 6d SCFT such as the matter contents on the tensor branch, is not necessary. However, it can only be applied to limited examples [39, 40, 41, 28] such as the $\mathcal{N}=(2,0)$ theory, the E-string theory and the (unfrozen) conformal matter theories. In Section 3.3 and Section 7.2, we will check that the anomaly inflow correctly reproduces the anomalies computed by the more field theoretical method in Section 3.1.

In order to apply the anomaly inflow to the branes, we need the Chern-Simons term in string theory. Conversely, if we already know the anomalies of the theory on the branes by another method, we can determine the precise Chern-Simons term in string theory. We will apply this idea to the frozen singularities in M-theory in Chapter 7.

The ALE singularity in M-theory has several frozen variants, whose properties are not fully understood yet. However, the tensor branch structure of the worldvolume theory on M5s probing such a frozen singularity can easily be obtained from the results in [3]. By applying the general method in Section 3.1, it is also straightforward to compute the anomalies of such 6d theories.

By rewriting the thus obtained anomaly polynomial into a form similar to the one obtained from the anomaly inflow, we can determine the 7d Chern-Simons term supported on the frozen singularity in M-theory, as explained in Section 7.3. In particular, we will find the quite exotic Euler invariant associated with such singularities. Therefore, we have obtained a new knowledge about the frozen singularities in M-theory, from the analysis of anomalies of 6d theories.

Compactification and anomalies. Although we do not treat the compactification of 6d theories in this thesis, we briefly add a comment here. Anomalies of 6d theories can be used to extract the central charges of a 4d $\mathcal{N}=1,2$ theory obtained by the compactification on a Riemann surface [42]. Let us suppose that we already have an explicit description of the 4d theory by using, for example, a chain of dualities or a guess from the symmetries. Then, we can check the validity of the proposed theory by comparing the central charges computed from 6d and those computed from the explicit description. This is a typical application of anomalies to the compactification of 6d SCFTs [43, 44, 45].

1.3 Summary and future directions

We briefly summarize the purposes and conclusions of this thesis. Moreover, we show several future directions of the thesis.

Summary. We used the anomaly of global symmetries to extract physical information about 6d SCFTs. We will find that purely field theoretical considerations of the anomaly lead to several new results about 6d SCFTs, which cannot be obtained just from the string theory consideration.

After introducing the important examples of 6d SCFTs in Chapter 2, we review how to compute the anomaly in Chapter 3. The computation is based on the purely field theoretical idea; 't Hooft anomaly matching. Due to this anomaly matching, we can compute the anomaly of 6d SCFTs from the effective gauge theory description on the tensor branch. In some examples, we also compare the results with the more string theoretical computation, anomaly inflow, and find the perfect agreement.

The new results are contained in the latter half of the thesis; Chapter 4, 5, 6, 7. In Chapter 4, based on [31], we consider the Higgsability of 6d SCFTs. We find the condition that a given 6d SCFT can be Higgsed to free hypermultiplets or to $\mathcal{N}=(2,0)$ theory. The conditions thus obtained admit an elegant M-theory interpretation. In Chapter 5, based on [30], we determine possible G such that a 6d SCFT whose Higgs branch is M_G, the one-instanton moduli space of gauge group G, can exist. The result is consistent with the string theory prediction of 6d SCFTs. In Chapter 6, based on [37], we obtain a partial but useful information about the strings in 6d $\mathcal{N}=(1,0)$ theories; central charges of the worldsheet theory. Our method is based on the anomaly inflow and is applicable to any known strings in 6d theories. We find that the central charges computed accordingly explain some mysterious features of 6d theories. In Chapter 7, based on [31], we compute a new physical quantity in M-theory; the Chern-Simons terms associated with the frozen singularity. This confirms that the studies of anomalies can shed light on the study of M-theory itself.

Future directions. There are several directions worth studying further. We find many examples of very Higgsable or Higgsable to $\mathcal{N}=(2,0)$ 6d SCFTs in Chapter 4. These notions were originally introduced in [32, 33] since the T^2 compactification of such theories are simpler than other 6d theories. Therefore, it is interesting to study the T^2 compactification of these 6d SCFTs.

In Chapter 5, we find putative anomaly polynomial of 6d SCFT whose Higgs branch is $M_{\mathrm{SU}(3)}$. It is interesting to study further whether this theory indeed exists or not. This is because if such a theory exist, the T^2 compactification leads to the Argyres-Douglas theory of type H_2. Although some Argyres-Douglas theories are known to arise as the T^2 compactification of 6d $\mathcal{N}=(1,0)$ theories, it is not clear whether the H_2 does or not.

In Chapter 6, we now have the general formula for the anomalies of the strings in 6d $\mathcal{N}=(1,0)$ theories. It is without a doubt interesting to use this formula to explore the worldsheet theories of the string further. In particular, it is very nice if we can find a way to compute the scattering amplitudes of such strings, mimicking the ordinary perturbative string theory.

In Chapter 7, we find the "Euler number" associated with a frozen singularity in M-theory. However, the M-theoretical meaning of this number is still unclear. The frozen singularity of M-theory should be investigated well enough so that we can interpret this number naturally. Moreover, in a similar way, we might be able to study the properties of the intersection of the ALE singularity and the E_8 wall, which is also a mysterious object in M-theory.

1.4 Organization of thesis

We explain the contents of each chapters in more detail.

Chapter 2. Chapter 2 is the review of the basic facts about 6d SCFTs which will be used in the thesis. In Section 2.1, we recall the massless multiplets in 6d supersymmetry. We also list the anomalies of such multiplets. In Section 2.2, we review the effective gauge theory description of 6d SCFTs on the tensor branch. We also briefly explain the Green-Schwarz mechanism in 6d, which plays the key role in the anomaly computation.

In Section 2.3, we show some simple examples of interacting 6d SCFTs. We first explain how the cancelation of the gauge anomaly constrain the gauge group of 6d $\mathcal{N}=(1,0)$ pure gauge theories and the number of fundamental hypers of 6d $\mathcal{N}=(1,0)$ $\mathrm{SU}(2)$ gauge theories. This is the typical application of anomalies to 6d SCFTs, which will appear several times in this thesis; exclude putative 6d theories satisfying some conditions. In this section, we also recall the $\mathcal{N}=(2,0)$ theory and the E-string theory.

In Section 2.4, we review the most important example of 6d SCFTs used in this thesis; the M5s probing the ALE singularity of M-theory, called as conformal matter theories. The tensor branch of these theories are quite non-trivial and interesting. We also explain how various 6d theories can be obtained by the deformation of these theories.

In Section 2.5, we review how F-theory can be used to construct many 6d SCFTs. In particular, we explain a rough classification of 6d SCFTs based on the endpoint. Endpoint is a name for the configuration of curves, which can be assigned to any 6d SCFTs and specifies a structure of the particular sub-branch of the tensor branch. We also compute the endpoints of conformal matter and related theories reviewed in Section 2.4.

Chapter 3. Chapter 3 is the review of the methods to compute the anomaly polynomial of 6d SCFTs. In Section 3.1, we introduce the general field theoretical method to compute the anomalies of 6d SCFTs. As examples, we will compute the anomalies of $\mathcal{N}=(2,0)$ theory, E-string theory, and pure 6d $\mathcal{N}=(1,0)$ theory, introduced in Section 2.3.

In Section 3.2, we derive the explicit form of the Green-Schwarz couplings of each tensor multiplets, for general 6d SCFTs engineered by F-theory. By using the thus obtained Green-Schwarz coupling, we can compute the full anomaly polynomial of all known 6d SCFTs.

In Section 3.3, we explain another method to compute the anomalies of some 6d SCFTs; anomaly inflow. Although this method is only applicable to 6d theories which arise as the world-volume theory on M5s, with or without other singularities, the various techniques used during the computation are very interesting. This method will be also used in Chapter 6 to determine the anomalies of the stringy defect in 6d theories. We also compare the result of the inflow for

$\mathcal{N}=(2,0)$ theory and E-string theory, with the anomaly polynomial obtained in a field theoretical way in Section 3.1.

Chapter 4. From this chapter, we will use anomalies to extract physical consequences about 6d SCFTs. In this chapter, we list all the 6d SCFTs which have the specified Higgs properties by using the anomaly matching on the Higgs branch, following the original paper [31].

In Section 4.1, we introduce the class of 6d SCFTs which we will investigate in this chapter; very Higgsable and Higgsable to $\mathcal{N}=(2,0)$. These two classes were originally introduced in the context of T^2 compactification of 6d SCFTs in [32, 33]. However, we slightly extended the notion based compared to those references. We show some examples of these 6d theories.

In Section 4.2 and Section 4.3, we will give a complete list of 6d SCFTs which are very Higgsable or Higgsable to $\mathcal{N}=(2,0)$, respectively. The argument is based on matching of the gravitational anomaly. We compare the gravitational anomaly computed at a generic point on the Higgs branch and at the endpoint of the tensor branch. It should be noted that the constraints for the Higgsability are written in terms of the endpoint configuration. As a byproduct, we obtain a Higgs branch dimension formula for these classes of theories.

In Section 4.4, we will interpret the list and the Higgs branch dimension formula in the previous sections in terms of the M-theory. In fact, they are geometrically explained for conformal matter and related theories; the Higgs branch correspond to taking the M5s away from the singularity.

Chapter 5. This chapter consider SCFTs which has the one-instanton moduli space of G as its Higgs branch, following the original article [30]. In Section 5.1, we point out that for such SCFTs, we can determine the full anomaly polynomial, including the R and G symmetry part. This is because we have enough knowledge about the symmetry breaking and the hypermultiplet spectrum on M_G.

In Section 5.2, we apply the method outlined in Section 5.1 to 6d $\mathcal{N}=(1,0)$ theories. The only known examples of 6d SCFTs with Higgs branch M_G are the E-string theory of rank-1 for $G=E_8$ and the free hypermultiplets gauge by $\mathbb{Z}_2$ for $G=\mathrm{Sp}(n)$. In fact, we can exclude all other cases of G (except for $G=\mathrm{SU}(3)$) by the fact that there is no solution to the anomaly matching equation. For $G=E_8,\mathrm{Sp}(N)$, we can reproduce the anomalies of the E-string theory and the hypermultiplets.

In Section 5.3, we apply the same method to 2d $\mathcal{N}=(0,4)$ theories. This problem is interesting since we know many examples of 2d theories with Higgs branch M_G as the worldsheet theories of an instanton-string in 6d $\mathcal{N}=(1,0)$ gauge theories. As expected, we can reproduce the anomalies of such worldsheet theories by the anomaly matching on Higgs branch.

Chapter 6. In this chapter, we consider the strings in 6d $\mathcal{N}=(1,0)$ theories and compute their chiral anomalies, following the reference [37]. In Section 6.1, we will derive the formula for the anomaly 4-form of the string by using the inflow from 6d. The key ingredient in the computation is the Green-Schwarz 4-form I_i of the 6d theory. The formula thus obtained can be used to any bound state of strings in any 6d $\mathcal{N}=(1,0)$ theory.

In Section 6.2, we check the validity of the formula found in Section 6.1. When the 6d theory can be engineered by the intersecting D-branes in string theory, we can often determine the matter contents on the worldsheet theory on the strings. For such cases, we compute the anomalies from the matter contents and find the perfect agreement with our formula.

In Section 6.3, we show several applications of our anomaly formula. First of all, we reproduce the ADE classification of $\mathcal{N}=(2,0)$ theories, slightly extending the known argument [77]. Secondly, we give a partial field-theoretical explanation of the emergent E_8 symmetry in E-string theory. Finally, we give a physical "derivation" of the curious formula for the exceptional groups, mentioned in Section 2.3.1.

Chapter 7. In this chapter, we obtain a new information about M-theory from the computation of anomalies of frozen conformal matter theories, following [31]. In Section 7.1, we review the anomaly polynomial of unfrozen conformal matters, computed by the method in Section 3.1.

In Section 7.2, we review how the anomaly polynomial in the previous section can be reproduced from the inflow computation. The important ingredient in the computation is the Chern-Simons terms supported on the ALE singularity.

In Section 7.3, we compute the anomaly polynomial of frozen conformal matter theories, i.e. the worldvolume theories of M5s probing a frozen singularity in M-theory. We use the field theoretical method in Section 3.1, and find that the result can be rewritten in a form presented in Section 7.2. This in turn allows us to determine the Chern-Simons term supported on the frozen singularity. We also introduce a new geometric quantity associated with such a singularity.

Appendices. In Appendices, we will collect some mathematical facts used in the thesis.

Acknowledgement

The author would like to express thanks to Noppadol Mekareeya, Kantaro Ohmori, Yuji Tachikawa, Alessandro Tomasiello, Kazuya Yonekura and Gabi Zafrir for many valuable discussions and collaborations. Especially, the author would like to acknowledge his supervisor Yuji Tachikawa for many useful advices during the PhD course. The author also thanks the high energy physics group at University of Milano-Biccoca for hospitality during part of the work. The author also thanks to the judges of the thesis, namely, Yutaka Matsuo, Kentaro Hori, Mitsuhiro Kato, Mio Murao and Hiroshi Sakamoto for their useful comments. The author also would like to express thanks to his friends and families. The author is partially supported by the Programs for Leading Graduate Schools, MEXT, Japan (via the Leading Graduate Course for Frontiers of Mathematical Sciences and Physics), and by JSPS Research Fellowship for Young Scientists.

Chapter 2

Construction of 6d SCFTs

2.1 Free 6d supersymmetric multiplets

In this section, we briefly summarize the free multiplets in 6d supersymmetry and their anomaly polynomials for later use. See [46] for the computation of anomalies.

6d $\mathcal{N}=(1,0)$ multiplets. $\mathcal{N}=(1,0)$ is the minimal amount of 6d supersymmetry. It is generated by two symplectic Majorana-Weyl spinors and has eight supercharges. The R-symmetry group is $\mathrm{SU}(2)_R$, which rotates the two spinors.

Massless representations of $\mathcal{N}=(1,0)$ supersymmetry are labeled by their quantum numbers under the $\mathrm{SO}(4) \sim \mathrm{SU}(2) \times \mathrm{SU}(2)$ little group. Let us list some of them with low spins.

- tensor multiplet; $(1,0) + 2(\frac{1}{2},0) + (0,0)$; a self-dual 2-form, fermions and a real scalar.
- vector multiplet; $(\frac{1}{2},\frac{1}{2}) + 2(0,\frac{1}{2})$; a vector field and fermions.
- hypermultiplet; $2(\frac{1}{2},0) + 4(0,0)$; four real scalars and fermions.

The factor of two in the fermion multiplicities comes from the fact that a Weyl spinor in 6d is always complex. The 2-form field in the tensor multiplet has self dual field strength $H_3 = *H_3$.

There are two typical components of the moduli space of vacua; tensor branch and Higgs branch. The tensor branch is parametrized by the vev of the scalars in the tensor multiplets while the Higgs branch is parametrized by those in the hypermultiplets. A generic theory on the tensor branch is a non-abelian gauge theory coupled to tensor multiplets. A generic theory on the Higgs branch is free hypermultiplets (plus some tensors). We note that there can be mixed branches while there are no Coulomb branches.

6d $\mathcal{N}=(2,0)$ multiplets. $\mathcal{N}=(2,0)$ is the maximal amount of 6d supersymmetry and is generated by four symplectic Majorana-Weyl spinors. The R-symmetry group is $\mathrm{USp}(4)_R \sim \mathrm{SO}(5)_R$. The only massless representation of $\mathcal{N}=(2,0)$ supersymemtry with spin less than two is the $\mathcal{N}=(2,0)$ tensor multiplet. It consists of an $\mathcal{N}=(1,0)$ tensor multiplet and an $\mathcal{N}=(1,0)$ hypermultiplet. The five scalars transform as $\mathbf{5_v}$ of $\mathrm{SO}(5)_R$ while the four Weyl fermions are in the spinor representation of $\mathrm{SO}(5)_R$.

Anomaly polynomial of free multiplets. As a preliminary to the next chapter, we list here the anomaly polynomials of free multiplets of 6d minimal supersymmetry. The relevant multiplets are tensor, vector and hypermultplet. First, we recall that a spin-$\frac{1}{2}$ (complex) chiral fermion ψ in the representation ρ in the global symmetry $\mathcal{G}$ has the anomaly polynomial

$$I_8^{\text{fermion}} = \hat{A}(T)\,\mathrm{tr}_\rho\, e^{iF}|_8, \tag{2.1.1}$$

where T is the tangent bundle of the spacetime and F is the curvature two-form of $\mathcal{G}$. [1] We note that $1/2$ factor is needed in front of (2.1.1) when the fermion is Majorana.

The 2-form potential B_2 in the tensor multiplet also contributes to the gravitational anomaly. In fact, it transforms chirally under the Lorentz group due to the self-duality. The anomaly polynomial is

$$I_8^{\text{2-form}} = \frac{16p_1^2 - 112p_2}{5760}. \tag{2.1.3}$$

Using (2.1.1) and (2.1.3), we can find the anomaly polynimials for 6d multiplets.

- Hypermultiplet with representation ρ

$$\frac{\mathrm{tr}_\rho F^4}{24} + \frac{\mathrm{tr}_\rho F^2 p_1(T)}{48} + d_\rho \frac{7p_1^2(T) - 4p_2(T)}{5760} \tag{2.1.4}$$

- Vector multiplet with group G

$$-\frac{\mathrm{tr}_{\text{adj}} F^4 + 6c_2(R)\mathrm{tr}_{\text{adj}} F^2 + d_G c_2(R)^2}{24} - \frac{(\mathrm{tr}_{\text{adj}} F^2 + d_G c_2(R))p_1(T)}{48} - d_G \frac{7p_1^2(T) - 4p_2(T)}{5760} \tag{2.1.5}$$

- Tensor multiplet

$$\frac{c_2(R)^2}{24} + \frac{c_2(R)p_1(T)}{48} + \frac{23p_1(T)^2 - 116p_2(T)}{5760} \tag{2.1.6}$$

[1]The $\hat{A}(T)$ is the A-roof genus whose first few values are given as

$$\hat{A}_0 = 1, \quad \hat{A}_1 = -\frac{1}{24}p_1, \quad \hat{A}_2 = \frac{1}{5760}(-4p_2 + 7p_1^2). \tag{2.1.2}$$

- $\mathcal{N}=(2,0)$ tensor multiplet

$$\frac{p_1(N)^2+4p_2(N)}{192}-\frac{p_1(N)p_1(T)}{96}+\frac{p_1(T)^2-4p_2(T)}{192} \tag{2.1.7}$$

Here d_ρ and d_G are the dimensions of representation ρ and group G, respectively. Moreover, $p_{1,2}(N)$ in the anomaly polynomial of an $\mathcal{N}=(2,0)$ tensor multiplet are the Pontrjagin classes of the $\mathrm{SO}(5)_R$ R-symmetry bundle.

2.2 Effective action on tensor branch and the Green-Schwarz mechanism

The effective theory on the tensor branch of 6d SCFT is given by a non-abelian gauge theory coupled to hypermultiplets and tensor multiplets. In this thesis, the most important terms in the effective Lagrangian is given as

$$L_{\text{eff}}=2\pi\int\eta^{ij}\Big(-\frac{1}{2}d\phi_i\wedge\star d\phi_j-\frac{1}{2}dB_i\wedge\star dB_j \\ +\phi_i(\frac{1}{4}\operatorname{Tr}F_j\wedge\star F_j+\cdots)+B_i(\frac{1}{4}\operatorname{Tr}F_j\wedge F_j+\cdots)\Big). \tag{2.2.1}$$

which describes the non-minimal coupling of the tensors (B_i,ϕ_i) to the vector multiplets (A_i). Here the action of B_i is somewhat formal because its field strength is self-dual. The terms $(\frac{1}{4}\operatorname{Tr}F_j\wedge\star F_j+\cdots)$ is constructed out of the metric and the gauge fields. In the following we simply denote them as I_j. We note that the part containing ϕ_i is related to the B_i part by supersymmetry. The physical meaning of η^{ij} will be explained later.

At this point, we should note that the self-duality of the 2-form fields require the modification of the Bianchi identity given as

$$dH_i=I_i=\frac{1}{4}\operatorname{Tr}F_i\wedge F_i+\cdots \tag{2.2.2}$$

where H_i is the 3-form field strength for the potential B_i.

Charge paring matrix η^{ij}. The matrix in the effective action (2.2.1) is the charge paring matrix of strings in 6d. We will explain its basic properties in the following.

Let us start by reviewing the charge paring in 4d. We choose the normalization of the 4d Dirac-Zwanziger pairing of particles with dyonic charges $q=(e,m)$ and $q'=(e',m')$ to be

$$\langle q,q'\rangle_{4d}=em'-e'm\in\mathbb{Z}. \tag{2.2.3}$$

With this choice, the quantity $\langle q, q' \rangle \hbar/2$ is the angular momentum carried by the electromagnetic fields. Note also that the pairing is anti-symmetric in 4d.

Let us consider the n_T self-dual 2-form fields in 6d. Then, the charges of the self-dual strings take values in an n_T dimensional lattice Λ. In 6d, the pairing is symmetric: for $q, q' \in \Lambda$, $\langle q, q' \rangle_{6d} = \langle q', q \rangle_{6d}$. We choose the normalization of $\langle q, q' \rangle_{6d}$ by using the compactification to 4d by T^2. Namely, the self-dual string of charge q wrapping the cycle $mA + nB$ of T^2 becomes a dyonic particle in 4d with the charge $q(mA + nB)$. Then, the normalization of the paring of 6d strings is determined by the ralation

$$\langle qA, q'B \rangle_{4d} = \langle q, q' \rangle_{6d} \langle A, B \rangle_{T^2}, \tag{2.2.4}$$

where $\langle A, B \rangle_{T^2}$ is the intersection number of A and B on T^2. The most important fact here is that the lattice Λ must be integral and unimodular under this normalization of pairng [47].

Let us explicitly introduce the coordinates $q = (q_i)_{i=1,\ldots,n_T} \in \Lambda$ in the charge lattice. Then, the charge paring can be specify by a symmetric matrix η^{ij} whose definition is given as

$$\langle q, q' \rangle_{6d} = \eta^{ij} q_i q'_j. \tag{2.2.5}$$

Moreover, Bianchi identity for the self-dual 3-form field strengths H_i of the 2-form fields is given as

$$dH_i = q_i \prod_{a=2,3,4,5} \delta(x_a) dx_a, \tag{2.2.6}$$

when a self-dual string of charge q sits at $x_{a=2,3,4,5} = 0$. The 4-form I_i in (2.2.2) is interpreted as the smooth version of the delta functions and the string in the zero-sized instanton.

Green-Schwarz contribution to the anomaly. The anomalies listed in the previous section come from the one-loop box diagrams with internal massless chiral fields and with four external gauge bosons or gravitons. These should be called as the “one-loop” part of the anomaly.

In addition to the one-loop part of the anomalies, there can be another source of the anomaly for 6d $\mathcal{N}= (1, 0)$ theories [48, 49, 50]. This comes from the non-minimal coupling (2.2.1) and the modified Bianchi identity in (2.2.2). In fact, under the condition (2.2.2), the gauge transformation of the B_i is written as $\delta B_i = I_i^{(1)}$. By substituting this gauge variation in (2.2.1), we obtain the additional piece of anomalies

$$I^{\text{GS}} = \frac{1}{2} \eta^{ij} I_i I_j. \tag{2.2.7}$$

Diagramatically, the contribution (2.2.7) comes from the tree diagram with the internal 2-form field. This is the Green-Schwarz mechanism for 6d $\mathcal{N}= (1, 0)$ theories [48, 49, 50] and we call I^{GS} as the “Green-Schwarz” part of the anomalies. We should note that the precise expression of I_i is undetermined up to now.

2.3 Simple examples of 6d SCFTs

In this section, we present some simple examples of 6d SCFTs.

2.3.1 "Field theoretical" examples

We consider a 6d $\mathcal{N}=(1,0)$ gauge theory with a G vector mutlplet and a hypermultiplet in the representation $\mathbf{R}$ coupled to one tensor multplet. We can argue that most of the choices of G and $\mathbf{R}$ are inconsistent just by the analysis on gauge anomalies [51, 52].

$\mathrm{SU}(2)$ **with fundamental hypers.** As a simple example, let us consider $G = \mathrm{SU}(2)$ and N_f fundamental hypermultiplets [51]. Then, the total gauge anomaly of the vector and the hyper is given as

$$I^{\text{one-loop}} = \frac{1}{2}\frac{N_f - 16}{6}\left(\frac{1}{4}\operatorname{Tr} F^2_{\mathrm{SU}(2)}\right)^2. \tag{2.3.1}$$

The Green-Schwarz contribution from the tensor multiplet

$$I^{\mathrm{GS}} = \frac{n}{4}\left(\frac{1}{4}\operatorname{Tr} F^2_{\mathrm{SU}(2)}\right)^2, \tag{2.3.2}$$

should cancel the gauge anomaly (2.3.1) where n is some positive integer. Comparing (2.3.1) and (2.3.2), this is only possible when $N_f = 4, 10$. Therefore, we found that the most $\mathrm{SU}(2)$ gauge theories in 6d are inconsistent. The cases of $N_f = 4$ or $N_f = 10$ have a stringy realization; take an F-theory compacification on a complex 2-dimensional base with a $\mathbb{P}^1$ of self-intersection -2 or -1 with a gauge algebra $\mathfrak{su}_2$ on it [53].

Minimal 6d $\mathcal{N}=(1,0)$ theories. As a next example, we consider pure gauge theories in 6d. We call them as "minimal 6d $\mathcal{N}=(1,0)$ theories". We denote the gauge group by G. In order to cancel the gauge anomaly of the vector multiplet by the Green-Schwarz contribution like (2.3.2), we need to have the identity relating $\mathrm{tr}_{\text{adj}} F_G^4$ to $(\operatorname{Tr} F_G^2)^2$. This is only possible for $G = \mathrm{SU}(2), \mathrm{SU}(3), \mathrm{SO}(8)$ and all exceptional groups.

In these cases, we have

$$-\frac{1}{24}\mathrm{tr}_{\text{adf}} F_G^4 = -\frac{w_G}{2}\left(\frac{1}{4}\operatorname{Tr} F_G^2\right)^2, \tag{2.3.3}$$

where w_G is listed in the appendix. This anomaly indeed can be cancelled by the term (2.3.2) with $n = w_G$. Since n must be an integer, $G = \mathrm{SU}(2), G_2$ are excluded. The full list is shown in Table 2.1. These theories are obtained by the F-theory on a base complex surface with an isolated

n	3	4	5	6	8	12
G	SU(3)	SO(8)	F_4	E_6	E_7	E_8
$h^\vee_G$	3	6	9	12	18	30

Table 2.1: Allowed gauge group G for pure 6d gauge theories. We also show n in the Green-Schwarz coupling and the dual Coxeter number $h^\vee_G$ for G. We see a relation $h^\vee = 3(n-2)$.

$\mathbf{P}^1$ of self-intersection $-n$, with the choice of the minimal gauge algebra on it [4, 5, 53]. From the Table 2.1, we can find the curious numerology

$$h^\vee_G = 3(n-2), \tag{2.3.4}$$

for the appearing gauge groups.

Generalizations. The classification of the solution for the gauge anomaly cancelation was given in [52] in the case of a simple gauge group and one tensor multiplet plus hypermultiplets. All the anomaly free models found there can be geometrically engineered using F-theory [53].

2.3.2 $\mathcal{N}=(2,0)$ theory

The $\mathcal{N}=(2,0)$ theory is the maximally supersymmetric field theory in 6d. This theory is constructed from the string theory as follows. Consider type IIB superstring compactified on $\mathbb{R}^{1,5} \times \mathbb{C}^2/\Gamma_G$ where Γ_G is a discrete subgroup of $\mathrm{SU}(2)$ acting on $\mathbb{C}^2$ by the multiplication of 2×2 matrix. Here the subscript G denotes the ADE Lie group associated with each discrete subgroup of $\mathrm{SU}(2)$ by the Macky correspondence. By taking the gravity-decoupling limit, we obtain the superconformal 6d $\mathcal{N}=(2,0)$ theory of type G [54].

Blowing up the singularity of $\mathbb{C}^2/\Gamma_G$, we obtain a chain of S^2's on $\mathbb{C}^2$ intersecting each other according to the Dynkin diagram of G. The local geometry around each S^2 is given by $\mathcal{O}(-2) \to \mathbb{P}^1$. The blownup geometry represents the tensor branch of the 6d theory where the only massless fields are $\mathcal{N}=(2,0)$ tensor multiplets. The BPS self-dual strings on the tensor branch come from the D3 branes wrapped around these S^2's. The tension is proportional to the volume of S^2's. When we blow down all the S^2's simultaneously, the strings become tensionless and the 6d theory is superconformal.

M-theory construction. If $G = A_{n-1}$, the singularity is T-dual to n NS5 branes in type IIA superstring, or n M5 branes in M-theory. Therefore, the $\mathcal{N}=(2,0)$ theory of type A_{n-1} is the worldvolume theory on coincident n M5 branes [55]. In this picture, the self-dual strings are the membranes suspended between two M5 branes. If we separate the M5 branes, the strings acquire the tension.

The $\mathcal{N}=(2,0)$ theory of type D_n also admits the M-theoretic realization as the orbifold [56]. Namely, put coincident n M5 branes and divide the $\mathbb{R}^5$ transverse to the M5 branes by the $\mathbb{Z}_2$ action $x^i \to -x^i, i = 6 \cdots 10$. Then the worldvolume theory on the M5 branes is the $\mathcal{N}=(2,0)$ theory of type D_n. The M-theory construction of the $\mathcal{N}=(2,0)$ theory of type E_n is given in [57] as the non-geometric orbifold.

2.3.3 E-string theory

Let us first consider M-theory on the orbifold $\mathbb{R}^{1,9} \times S^1/\mathbb{Z}_2$ [58, 59] where $\mathbb{Z}_2$ acts on S^1 as $x_{10} \to -x_{10}$ and on the M-theory 3-form potential as $C \to -C$. The action on the potential is necessary for the invariance of the M-theory Chern-Simons terms. This orbifolding preserves the 10d $\mathcal{N}=(1,0)$ supersymmetry among the original 11d supersymmetry.

The fixed points of this orbifold consist of two distinct 10d hyperplanes in the 11d spacetime. Hořava and Witten proposed in [58] that on the single 10d hyperplane there lives a 10d super Yang-Mills theory with gauge group E_8. In the following, we will call them as end-of-the-world E_8 brane or M9 brane. In fact, M-theory on $S^1/\mathbb{Z}_2$ is dual to the $E_8 \times E_8$ heterotic string theory. The length of the 11th direction is proportional to the heterotic string coupling constant. The $E_8 \times E_8$ vector multiplet in the heterotic theory becomes two E_8 vector multiplets that propagate on each of the two hyperplanes.

We can add an M5 brane in the M-theory on $S^1/\mathbb{Z}_2$ [60]. If we put coincident Q M5 branes on top of the end-of-the-world E_8 brane, the worldvolume theory is a 6d $\mathcal{N}=(1,0)$ SCFT, called as the E-string theory of rank-Q. The tensor branch of the E-string theory corresponds to the taking M5-branes off the end-of-the-world E_8 brane. On the generic point of the tensor branch, the massless spectrum consists only of Q tensor multiplets. The BPS string on the tensor branch is given by the membrane suspended between the end-of-the-world E_8 brane and the M5 brane. This BPS string carries a E_8 current algebra on its worldvolume.

The Higgs branch is the Q-instanton moduli space of E_8. We note that the dimension of the Higgs branch is given as

$$\dim_{\mathbb{H}}^{\text{CFT}} \text{ Higgs} = Qh^{\vee}_{E_8} - 1 = 30Q - 1. \tag{2.3.5}$$

There exists a transition that a single M5-brane on the end-of-the-world E_8 brane becomes a finite sized instanton of E_8, called as "small instanton transition". This trades one tensor multiplet with 29 hypermultiplets as can be seen from (2.3.5).

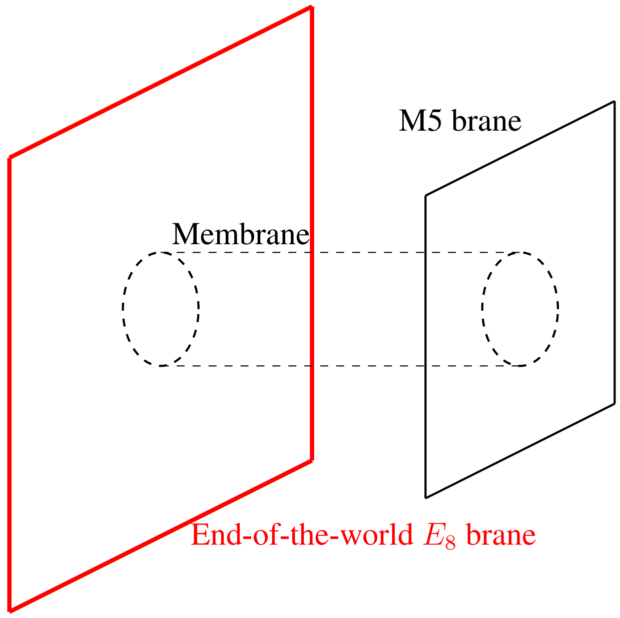

Figure 2.1: Rank-1 E-string theory on tensor branch and its BPS string

2.4 Conformal matter theories

2.4.1 M5 fractionation on the singularity

Let us consider N M5-branes at probing the $\mathbb{R}\times\mathbb{C}^2/\Gamma_G$ singularity where G is a simply-laced group. The worldvolume theory is a 6d $\mathcal{N}=(1,0)$ theory which we will denote as $\mathcal{T}_G(N-1)$. As a domain wall theory in 7d G SYM, the theory $\mathcal{T}_G(N-1)$ has the $G_L\times G_R$ flavor symmetry where $G_{L,R}$ is isomorphic to G.

IIA analysis. Let us analyze the tensor branch structure of these theories. When $G=\mathrm{SU}(k)$, the singularity $\mathbb{C}^2/\mathbb{Z})k$ has a circle direction along which we can reduce M-theory to type IIA. Then, the system becomes the N NS5s on top of the k D6s. Then, the tensor branch description is the linear quiver $[\mathrm{SU}(k)]$—$(\mathrm{SU}(k))\cdots(\mathrm{SU}(k))$—$[\mathrm{SU}(k)]$ where — represents the bifundamental hypermultiplets and $(\mathrm{SU}(k))$ represents a gauge group and a tensor multiplet controlling the gauge coupling or positions of M5s along the $\mathbb{R}$ direction.

When $G=\mathrm{SO}(2k)$, we can similarly reduce to IIA, obtaining the N NS5s probing k D6s with $\mathrm{O6}^-$. However, it is known that an NS5 can fractionate on the $\mathrm{O6}^-$ plane to two half NS5s[61]. When crossing the first half NS5, the $\mathrm{O6}^-$ becomes $\mathrm{O6}^+$, and then the $\mathrm{O6}^+$ again becomes $\mathrm{O6}^-$ after crossing the second half NS5. Since the D-brane charge of $\mathrm{O6}^-$ and $\mathrm{O6}^+$ differs by 4, the gauge group in between the two half NS5s is now $\mathrm{Sp}(n-4)$. Then, the tensor branch description is the linear quiver $[\mathrm{SO}(2k)$—$(\mathrm{USp}(k-4))$—$(\mathrm{SO}(2k))\cdots(\mathrm{USp}(k-4))$—$[\mathrm{SO}(2k)]$ where —

represents a bifumdamental half-hypermultiplet. There are $2N-1$ gauge groups and associated tensor multiplets. Note also if the group $\mathrm{USp}(k-4)$ was $\mathrm{SO}(2k)$ as one naively thought, there was no possible hypermultplets allowed by gauge anomaly cancellation.

F-theory dual. What happens when $G = E_n$? Since the duality to IIA does not help, we must use more sophisticated technique. The first option is to use the duality to F-theory [3]. Let us briefly recall this duality in the case of single M5 for simplicity and find that it is dual to two F-theory seven-branes of type G intersecting at a point.

First, we realize the $\mathbb{C}^2/\Gamma_G$ singularity in the elliptic fibration over $\mathbb{C}$. Namely, the torus fiber receives certain $\mathrm{SL}(2,\mathbb{Z})$ monodromy as we move around the origin of $\mathbb{C}$. Then, the M-theory spacetime now becomes an elliptic fibration form $\mathbb{R}^{1,4} \times S^1 \times \mathbb{R} \times (\mathbb{C} \times T^2)_g$ where g represents the monodromy. Here we additionally compactified the $\mathbb{R}^{1,5}$ to $\mathbb{R}^{1,4} \times S^1$, which we make large at the final step. Note also that there is an M5.

Then, we reduce to IIA along S^1 to obtain $\mathbb{R}^{1,4} \times \mathbb{R} \times (\mathbb{C} \times T^2)_g$ with a D4 at the origin of $\mathbb{R} \times (\mathbb{C} \times T^2)_g$. We further take a double T-dual along the elliptic fiber T^2 to obtain $\mathbb{R}^{1,4} \times \mathbb{R} \times (\mathbb{C} \times T^2)_g$ with a D6 at the origin. Then, we lift the configuration again to M-theory. The $\mathbb{R} \times (\mathbb{C} \times T^2)_g$ part now becomes the $\mathbb{C}^2 \times T^2$. The D6 becomes a smooth geometry $\mathbb{C}^2/\mathbb{Z}_1 \sim \mathbb{C}^2$ whose coordinates we denote as (u, v). On $\mathbb{C}^2$, there are now two cigars $\Sigma_{1,2}$ intersecting at $u = v = 0$ and the elliptic fibration degenerates on each cigar. We can safely set $\Sigma_1 = \{u = 0\}$ and $\Sigma_2 = \{v = 0\}$. Therefore, the total geometry of M-theory is $\mathbb{R}^{1,4} \times (\mathbb{C}^2 \times T^2)$ with $\Sigma_{1,2}$ in $\mathbb{C}^2$. After we make the original S^1 large again, this is nothing but the F-theory configuration with two seven-branes of type G wrapped on $\Sigma_{1,2}$ and intersecting at the origin $u = v = 0$.

Here we briefly recall what the F-theory compactification to six dimension is. This is IIB on $\mathbb{R}^{1,5} \times B_2$ with seven-branes wrapped around $\mathbb{R}^{1,5} \times \Sigma_a$. Here B_2 is a complex surface and Σ_a is a compact or non-compact divisor of B_2. Let consider the IIB axio-dilaton field τ as a function on B_2. Due to the $\mathrm{SL}(2,\mathbb{Z})$ duality of IIB, the B_2 and τ defines a threefold Y_3 which is an elliptic fibration over B_2. In order to preserve the minimal supersymmetry on $\mathbb{R}^{1,5}$, the Y_3 must be Calabi-Yau. The axio-dilaton field τ receives non-trivial monodromy when we go around a loop transverse to each Σ_a in B_2. In terms of the geometry of Y_3, this means that the elliptic fibration becomes singular along the divisor Σ_a on B_2 whose degeneration type is specified by the type of sevenbrane. In the last paragraph, we have used the important fact that F-theory on $\mathbb{R}^{1,4} \times S^1 \times B_2$ is dual to M-theory on $\mathbb{R}^{1,4} \times Y_3$.

Let us describe the geometry of Y_3 more concretely. In general, the elliptic fibration Y_3 on B_2 can be described in the Weierstrass form;

$$y^2 = x^3 + f(u,v)x + g(u,v), \tag{2.4.1}$$

where u, v is the local coordinates of B_2 and (2.4.1) represents the fibration of the torus over B_2.

On various divisors Σ_a of $\mathbb{C}^2$, the elliptic fiber degenerates, i.e. the discriminant $\Delta = 4f^3 + 27g^2$ vanishes. The type of degeneration is classified in terms of the order of vanishing of (f, g, Δ) along the divisor Σ_a;

Type	G	ord(f)	ord(g)	ord(Δ)
I_0	none	≥ 0	≥ 0	0
I_k	$\mathfrak{su}(k)$ or $\mathfrak{sp}(\lfloor k/2 \rfloor)$	0	0	$k \geq 2$
II	none	≥ 1	1	2
III	$\mathfrak{su}(2)$	1	≥ 2	3
IV	$\mathfrak{su}(3)$ or $\mathfrak{su}(2)$	≥ 2	2	4
I_0^*	$\mathfrak{so}(8)$ or $\mathfrak{so}(7)$ or $\mathfrak{g}_2$	≥ 2	≥ 3	6
I_k^*	$\mathfrak{so}(2k+8)$ or $\mathfrak{so}(2k+7)$	2	3	$k+6 \geq 7$
IV^*	$\mathfrak{e}_6$ or $\mathfrak{f}_4$	≥ 3	4	8
III^*	$\mathfrak{e}_7$	3	≥ 5	9
II^*	$\mathfrak{e}_8$	≥ 4	5	10

(2.4.2)

As we already mentioned, the degeneration of the fiber along Σ_a means that we have a 7-brane wrapped along $\mathbb{R}^{1,5} \times \Sigma_a$. The G in Table 2.4.2 means that the gauge algebra on the brane is as written. There are two or three possible choices of the gauge algebra for I_k, IV, I_k^* and IV^*. This happens when the degenerate fiber undergoes the monodromy corresponding to the outer-automorphism of a simply-laced algebra when we go around a loop along Σ_a. To determine whether or not this reduction of the gauge algebra happens, we need to apply the Tate's algorithm to the Weierstrass form (2.4.1), whose detail we do not explain in this thesis.

To conclude this subsection, we explicitly write the Weierstrass form of the F-theory geometry for $\mathcal{T}_G(0)$ when $G = E_{6,7,8}$. As we already argued, the relevant configuration involves the base $\mathbb{C}^2$ with two non-compact 7-branes of type G on $\Sigma_1 = \{u = 0\}$ and $\Sigma_2 = \{v = 0\}$. From the Table 2.4.2, they are simply written as follows;

$$
\begin{aligned}
(E_6, E_6) &: \ y^2 = x^3 + u^4 v^4, \\
(E_7, E_7) &: \ y^2 = x^3 + u^3 v^3 x, \\
(E_8, E_8) &: \ y^2 = x^3 + u^5 v^5.
\end{aligned}
\tag{2.4.3}
$$

Tensor branch structure. The 6d theory $\mathcal{T}_G(0)$ is now trapped at the intersection $\{u = v = 0\}$. Then, we come back to the original question; what is tensor branch of the theory $\mathcal{T}_G(0)$ for $G = E_n$? The key fact is that a point of contact between two seven-branes of type $G = E_n$ (and D_n) has a singularity not present in Kodaira's classification. For example, when $G = E_6$, the order of vanishing of (f, g, Δ) over $u = 0$ or $v = 0$ is $(3, 4, 8)$ and at $u = v = 0$, the order of vanishing is enhanced to $(6, 8, 16)$, not present in the Table 2.4.2. This can be partially cured by blowing up the contact point. This is done by using the change of coordinates by $t = u/v$ where the point $u = v = 0$ is replaced by the whole $\mathbb{CP}^1$ parametrized by t. The new $\mathbb{CP}^1$ has self-intersection -1 in $\mathbb{C}^2$ and the order of vanishing of (f, g, Δ) is reduced by $(4, 6, 12)$.

For example, when $G = E_6$, the polynomials vanish with the orders $(6, 8, 16)$ at the origin, as can be seen from Table 2.4.2. After blowing up the intersection point, we have

$$[E_6][E_6] \to [E_6]\overset{IV}{1}[E_6] \to [E_6]\overset{I_0}{1}\overset{IV}{3}\overset{I_0}{1}[E_6], \tag{2.4.4}$$

where $[E_6]$ represents the noncompact $[E_6]$ sevenbrane while $1, 3$ represents the new $\mathbb{CP}^1$ with self-intersection -1 or -3, respectively. After the first blowup, the order of vanishing on the (-1) curve is $(2, 2, 4)$, indicating that the elliptic fiber is IV. However, the intersection of two curves still has a singularity not in the Kodaira classification; order $(5, 6, 12)$. Then, we need to blow up again two intersection points, obtaining the final expression in (2.4.4). The elliptic fiber along (-1) curves is now non-singular I_0.

The self-intersection number of the curve with the IV fiber changes from (-1) to (-3) after the second step in (2.4.4). In fact, this is due to a result of classic algebraic geometry; when we blowup a point on B_2, the self-intersection of the neighboring curves changes by 1. Explicitly, $\ldots nm \ldots \to \ldots (n+1)1(m+1)\ldots$ where $n, m, 1$ represents a curve with self-intersection $-n, -m, -1$, respectively. This can be derived as follows. Let us consider a $-n$ curve Σ and blow up a point on Σ. When we denote the exceptional divisor as E, the new curve is $\Sigma_{\text{new}} = \Sigma - E$. Then the self-intersection becomes

$$\Sigma_{\text{new}} \cdot \Sigma_{\text{new}} = \Sigma \cdot \Sigma - 2\Sigma \cdot E + E \cdot E = -(n+1), \tag{2.4.5}$$

because $\Sigma \cdot \Sigma = -n$ and $E \cdot E = -1$.

We can repeat the same repeated blowups for general $\mathcal{T}_G(N-1)$. The result is[2]

$$\begin{aligned}
G = \mathrm{SU}(k):&\quad [\mathrm{SU}(k)]\overset{\mathfrak{su}_k}{2}\ \ldots\ \overset{\mathfrak{su}_k}{2}[\mathrm{SU}(k)]\\
G = \mathrm{SO}(2k):&\quad [\mathrm{SO}(2k)]\ \overset{\mathfrak{usp}_{2k-8}}{1}\ \overset{\mathfrak{so}_{2k}}{4}\ \ldots\ [\mathrm{SO}(2k)]\\
G = E_6:&\quad [E_6]\ 1\ \overset{\mathfrak{su}_3}{3}\ 1\ \overset{\mathfrak{e}_6}{6}\ \ldots\ [E_6]\\
G = E_7:&\quad [E_7]\ 1\ \overset{\mathfrak{su}_2}{2}\ \overset{\mathfrak{so}_7}{3}\ \overset{\mathfrak{su}_2}{2}\ 1\ \overset{\mathfrak{e}_7}{8}\ \ldots\ [E_7]\\
G = E_8:&\quad [E_8]\ 1\ 2\ \overset{\mathfrak{su}_2}{2}\ \overset{\mathfrak{g}_2}{3}\ 1\ \overset{\mathfrak{f}_4}{5}\ 1\ \overset{\mathfrak{g}_2}{3}\ \overset{\mathfrak{su}_2}{2}\ 2\ 1\ \overset{\mathfrak{e}_8}{12}\ldots\ [E_8]\,,
\end{aligned} \tag{2.4.7}$$

[2] A brief notation that we will use sometimes is

$$[G]\text{——}(G)\text{——}\ldots\text{——}(G)\text{——}[G]\,. \tag{2.4.6}$$

For example, the $N = 2$, $G = E_8$ case is (2.5.1), which in the notation of (2.4.6) written as $[E_8]$——(E_8)——$[E_8]$.

where each sequence of curves is repeated $N-1$ times (and in particular the total number of G gauge factors is $N-1$). The two outermost copies of $[G]$ represent a $G \times G$ flavor symmetry. We note after the repeated blowdown of (-1) curves in (2.4.7), the configuration becomes a sequence of $N-1$ (-2)-curves. This is as expected since an M5 is dual to a (-2) curve in F-theory, as we already remarked when introducing the $\mathcal{N}=(2,0)$ theory.

In (2.4.7), we explicitly show the gauge algebra over each curve, not the Kodaira type of the fiber as in (2.4.4). It should be noted that we can only determine the number of blowups needed, the self-intersection number of new curves, and the order of vanishing of (f, g, Δ) by the procedure outlined above. To determine the gauge algebras over the new curves, for example $\mathfrak{su}(3)$ or $\mathfrak{su}(2)$ in Table 2.4.2, we have to apply the Tate's algorithm starting from the explicit Weierstrass form as in (2.4.3), which is beyond what is needed in this thesis.

We further add comments on (2.4.7). First, the -1 curve without a gauge algebra on it represents the rank-1 E-string theory. The E_8 flavor symmetry of the E-string theory is gauged according to the product subalgebras of $\mathfrak{e}_8$ such as $\mathfrak{e}_6 \times \mathfrak{su}_3$, $\mathfrak{e}_7 \times \mathfrak{su}_2$, and $\mathfrak{f}_4 \times \mathfrak{g}_2$ in (2.4.7). The two consecutive curves 12 without gauge algebras in the $G = E_8$ case represents the rank-2 E-string theory, whose flavor symmetry $\mathfrak{e}_8 \times \mathfrak{su}_2$ is fully gauged. When two curves with gauge algebras $\mathfrak{g}_{1,2}$ are adjacent in (2.4.7), there is a bifudamental multiplet of $\mathfrak{g}_1 \times \mathfrak{g}_2$, trapped at the intersection. The spectrum of hypermultiplets in (2.4.7) are summarized as follows;

- When $G = \mathrm{SU}(k)$, there are N bifundamental hypermultplets of $\mathfrak{su}_k \times \mathfrak{su}_k$.
- When $G = \mathrm{SO}(2k)$, there are $2N$ bifundamental half-hypermultplets of $\mathfrak{so}_{2k} \times \mathfrak{usp}_{2k-8}$.
- When $G = E_6$, there are no hypermultiplets.
- When $G = E_7$, there are $2N$ half-hypermultplets transforming $(\mathbf{2}, \mathbf{7}+\mathbf{1})$ under $\mathfrak{su}_2 \times \mathfrak{so}_7$.
- When $G = E_8$, there are $2N$ half-hypermultplets transforming $(\mathbf{2}, \mathbf{7}+\mathbf{1})$ under $\mathfrak{su}_2 \times \mathfrak{g}_2$.

Here we omit the indices which distinguish the isomorphic but distinct gauge algebras appearing in (2.4.7), expecting that this causes no confusion. We note that with these choices of hypermulepltets, the gauge anomalies in the quiver (2.4.7) are precisely cancelled.

M-theory interpretaion. The theories in (2.4.7) was called in [3] as "conformal matter theory".[3] We find from (2.4.7) that for $\mathfrak{g} \neq \mathfrak{su}_k$, two $\mathfrak{g}$ gauge algebras are connected by a non-trivial 6d SCFT with its own tensor branch. This is due to the repeated blowup as in (2.4.4).

[3] To be more precise, these theories are sometimes called (G, G) conformal matter, to highlight that they are used to connect two G gauge groups. There also exist (G, G') chains, some of which we will encounter in later. To simplify the notation, we denote the theories in (2.4.7) as $\mathcal{T}_G(N-1)$.

From the M-theory point of view, the scalar in a tensor multiplet represents the motion of an M5 on the $\mathbb{R}$. Since there are now several tensor multiplets even in a single M5, it is natural to conjecture that the M5 has now split in several fractions on the $G = D_n, E_n$ singularity. As we already mentioned, for the $G = SO(2k)$ case, this is known from IIA. For the $G = E_k$ cases, we see from (2.4.7) that the number of fractions $\mathfrak{f}$ is 4 for E_6, 6 for E_7, 12 for E_8. Summarizing,

$$\mathfrak{f}(SU(k)) = 1\,, \qquad \mathfrak{f}(SO(2k)) = 2\,, \qquad \mathfrak{f}(E_6) = 4\,, \qquad \mathfrak{f}(E_7) = 6\,, \qquad \mathfrak{f}(E_8) = 12\,. \quad (2.4.8)$$

Another aspect of (2.4.7) is that, when we cross an M5 fraction, the gauge algebra $\mathfrak{g}$ is broken to a small subalgebra $\mathfrak{g}_{\rm fr}$ (possibly even trivial). Such a variant of the ALE singularity whose gauge algebra is smaller than the simply-laced $\mathfrak{g}$ is called as a "frozen" version of the singularity. This "freezing" of the singularity is characterized by a discrete flux of the M-theory 3-form field C [62, 63]:

$$\int_{S^3/\Gamma_G} C = \frac{n}{d} \qquad \text{mod } 1, \tag{2.4.9}$$

where S^3/Γ_G denotes the orbifolded unit sphere around the singularity in $\mathbb{C}^2/\Gamma_G$, and n, d are coprime. The denominator $d = d_{G\to G_{\rm fr}}$ is shown in the following table (see [62, Table 14]) and [63, Eq. (1.1), Table 1]:

G	$G_{\rm fr}$	$d_{G\to G_{\rm fr}}$
$SO(2k+8)$	$USp(2k)$	2
E_6	$SU(3)$	2
	$\varnothing$	3
E_7	$SO(7)$	2
	$SU(2)$	3
	$\varnothing$	4
E_8	F_4	2
	G_2	3
	$SU(2)$	4
	$\varnothing$	5
	$\varnothing$	6

(2.4.10)

The interpretation of the discrete flux (2.4.9) in terms of the Chern-Simons invariant of flat G bundle on T^3, called as "gauge triples" in [62] will be presented in Appendix B. There, we will show the argument presented in [32] where the system is compactified on T^3 and then dualized so that the M5s become fractional instantons of G on $\mathbb{R}\times T^3$.

A fractional M5 brane is then a domain wall dividing two frozen (or unfrozen) singularities with different values of the discrete flux (2.4.9). Let the singular locus be at $x^{8,9,10,11} = 0$ and consider a fractional M5 which sits at $x^{7,8,9,10,11} = 0$. The value of discrete flux on the domain

$x^7 < 0$ will be denoted as r_1 and on the domain $x^7 > 0$ denoted as $r_2(\geq r_1)$. Then, the 3-form charge of the fractional M5 is given as

$$\left|\int_S \mathrm{d}C\right| = \int_{\{\varepsilon\}\times S^3/\Gamma_G} C - \int_{\{-\varepsilon\}\times S^3/\Gamma_G} C = r_2 - r_1, \tag{2.4.11}$$

where $\varepsilon > 0$ and $S = [-\varepsilon, \varepsilon] \times S^3/\Gamma_G \cup \{\varepsilon\} \times D^3/\Gamma_G \cup \{-\varepsilon\} \times D^3/\Gamma_G$ is the 5-sphere surrounding the fractional M5.

2.4.2 Variants of conformal matter theories

In the previous section, we introduced the conformal matter theories $\mathcal{T}_G(N-1)$, describing the N M5s on the singularity $\mathbb{C}^2/\Gamma_G$ where G is a simply-laced algebra. In this section, we will introduce the two variants of them; T-brane theories and frozen conformal matter theories.

T-brane theories. Conformal matter theories can be decorated by adding on the two outermost curves a feature that is not seemingly described by the geometry of F-theory; a T-brane [64]. This is defined as follows. The transverse fluctuation of an F-theory seven-brane Σ_a on B_2 are parameterized by a Higgs field and we can make a nilpotent vev to it. Since it does not change the eigenvalues and thus the position of the brane, at first sight such a deformation has no effect. However, it does nevertheless have physical content when we introduce a nilpotent pole for the Higgs field describing the non-compact flavor 7-brane. Such a nilpotent deformation which has physical consequences is called as a T-brane (deformation).

This possibility was originally suggested by duality with IIA configurations involving D8-branes [3]. The 6d theory $T_{SU(k)}(N-1)$ has a realization in IIA as an NS5-D6 intersection. But in IIA one can also introduce D8s [65, 66], and the possible combinatorics to introduce them so that we still get a 6d SCFT are summarized by two Young diagrams $Y_{\mathrm{L,R}}$; we then call these theories as $\mathcal{T}_{SU(k)}(\{Y_{\mathrm{L}}, Y_{\mathrm{R}}\}, N-1)$.

This has a natural interpretation in terms of the Nahm equation living on the D6. Indeed, introducing D8s is equivalent to introducing poles for those equations with a nilpotent residue on the left/rightmost of D6s. The poles are parameterized by two Young diagrams $Y_{L,R}$. Going to IIB by a T-duality, this becomes a pole for the Higgs field on the flavor seven-branes $[\mathrm{SU}(K)]_{L,R}$.

Since things work well for $G = \mathrm{SU}(k)$. it is natural to conjecture that it is also possible to decorate any conformal matter theory (even for $G \neq \mathrm{SU}(k)$) by two nilpotent elements $Y_{\mathrm{L,R}} \in G$, obtaining a more general set of theories

$$\mathcal{T}_G(\{Y_{\mathrm{L}}, Y_{\mathrm{R}}\}, N-1)\ . \tag{2.4.12}$$

A tensor branch description of all the theories (2.4.12) was obtained in [67] for any simply-laced Lie group G. The Higgs branch RG flows starting from $\mathcal{T}_G(N)$ were analyzed in detail there and it was found that there is a bijective correspondence between the web of RG flows from $\mathcal{T}_G(N)$ and the Hasse diagram of nilpotent elements of G. Moreover, It was shown in [68] that the difference in the Higgs moduli space dimension between the theort $\mathcal{T}_G(N-1)$ and its T-brane deformation $\mathcal{T}_G(\{Y_{\rm L}, Y_{\rm R}\}, N-1)$ is exactly equal to $d_{Y_{\rm L}} + d_{Y_{\rm R}}$, the sum of the dimensions of the nilpotent orbits associated to $Y_{\rm L,R}$.

Frozen conformal matters. In this thesis, we are also interested in "incomplete" versions of the theories in (2.4.7) — namely, to the 6d quivers that are obtained from taking some of the outermost fractions (i.e. tensor multiplets) to infinity. For example, for $G = E_7$ we can take to infinity the two outermost fractions on the left and the three outermost on the right, and we obtain the configuration

$$[SU(2)] \overset{\mathfrak{so}_7}{3} \overset{\mathfrak{su}_2}{2} \; 1 \; \overset{\mathfrak{e}_7}{8} \ldots \overset{\mathfrak{e}_7}{8} \; 1 \; \overset{\mathfrak{su}_2}{2} \; [SO(7)] \, . \tag{2.4.13}$$

The quiver now ends on the left and on the right with a partially frozen singularity. For this reason, we will sometimes call this general class of theories as *frozen conformal matter*.

2.5 F-theoretic construction

2.5.1 Generic point on tensor branch

In the previous section, we have encountered F-theory quivers such as

$$[E_8] \; 1 \; 2 \; \overset{\mathfrak{su}_2}{2} \; \overset{\mathfrak{g}_2}{3} \; 1 \; \overset{\mathfrak{f}_4}{5} \; 1 \; \overset{\mathfrak{g}_2}{3} \; \overset{\mathfrak{su}_2}{2} \; 2 \; 1 \; \overset{\mathfrak{e}_8}{12} \; 1 \; 2 \; \overset{\mathfrak{su}_2}{2} \; \overset{\mathfrak{g}_2}{3} \; 1 \; \overset{\mathfrak{f}_4}{5} \; 1 \; \overset{\mathfrak{g}_2}{3} \; \overset{\mathfrak{su}_2}{2} \; 2 \; 1 \; [E_8] \tag{2.5.1}$$

which actually describes 2 M5s on a Γ_{E_8} singularity, each of which has broken down in 12 fractions. Here [] represent the flavor symmetry. Each numbers n are $\mathbb{CP}^1$ with self-intersection $-n$ on the base (in this case $\mathbb{C}^2$) of the F-theory. Adjacent curves meet each other at a point on $\mathbb{C}^2$. The gauge algebra on the number n specifies the gauge algebra on the 7-brane wrapping that curve, which corresponds to the type of degeneration of elliptic fiber in Table 2.4.2.

It is then natural to use the F-theory to systematically construct six-dimensional theories [6, 7]. We consider the F-theory on $\mathbb{R}^{1,5} \times Y_3$ where Y_3 is the elliptic Calabi-Yau threefold whose base is $\mathbb{C}^2$. On the base $\mathbb{C}^2$, we have a tree (in fact, most often a chain) of $\mathbb{CP}^1$s: $\{\Sigma_i\}$. The F-theory seven brane with gauge algebra $\mathfrak{g}_i$ wraps on $\mathbb{R}^{1,5} \times \Sigma_i$. [4] This configuration specifies a 6d effective field theory consisting of tensor/vector/hyper multiplets.

The data defining the 6d theory is the geometry of curves Σ_i on $\mathbb{C}^2$ and the types of sevenbranes $\mathfrak{g}_i$ wrapping each curves. There are several constraints for these datas for a 6d SCFT. We will describe these constraints as follows. For more details, see [6, 7].

[4] When Σ_i is non-compact, the seven-brane is a flavor brane.

- The configuration of curves on $\mathbb{C}^2$ is characterized by the adjacency matrix η^{ij}. This is an $n_T \times n_T$ matrix, where n_T is the number of curves in the chain, defined by

$$\eta^{ij} = \begin{cases} n_i & i = j \, , \\ -1 & |i-j| = 1 \, , \end{cases} \tag{2.5.2}$$

where n_i is minus the self-intersection number for the i-th curve in the chain.

In order for a configuration to define a 6d SCFT, it is required that we can contract all the curves to a point. Otherwise, we cannot reach the conformal phase of the theory. This condition is equivalent to the fact that the matrix η^{ij} must be positive-definite. For example, the adjacency matrix of the configuration 313 is positive-definite and can be used as a base for a 6d SCFT. However, the seemingly similar configuration 13131 does not have positive-definite η^{ij} and cannot be used in constructing a 6d SCFT.

It was further argued in [6] that for a 6d SCFT, such a base consists of a configuration of curves built from

1. ADE diagram of -2 curves,
2. "non-Higgsable clusters" given by $3, 4, 5, 6, 7, 8, (12), 23, 232$ or 223,

possibly joined by -1 curves. In fact, we can easily check that the configurations in (2.4.7) satisfy this constraint.

- For each curve C, we have a choice of gauge algebra $\mathfrak{g}$ on it. However, we cannot freely choose the algebra. This follows from the algebra geometric results of the surface. The first is the adjunction formula

$$(K + C) \cdot C = 2g - 2, \tag{2.5.3}$$

where K is the canonical divisor of the base and g is the genus of the curve C (in this thesis $g = 0$). The second is the Zariski decomposition of a (effective) divisor A of the surface B_2;

$$C \cdot C < 0, \quad A \cdot C < 0 \quad \to \quad A = C + X \tag{2.5.4}$$

where X is some residual component of A. This decomposition gives us a sufficient condition for the C to be a component of A.

As example, let us consider a single curve C with self-intersection -8. Then, it follows from (2.5.3) that $K \cdot C = 6$. Applying (2.5.4) to divisors $-4K, -6K$ and $-12K$, we find that (f, g, Δ) vanishes at least $(3, 5, 9)$ on C. Then if follows from the Table 2.4.2 that the gauge algebra on C is at least $\mathfrak{e}_7$. We note that we can choose a more degenerate elliptic fiber. In this case, the only enhancement is to $\mathfrak{e}_8$.

The only curves without gauge algebra on them is -1 or -2 curve. A -2 curve without gauge algebra represents the $\mathcal{N}=(2,0)$ theory of type A_1. A -1 curve without gauge algebra represents the E-string theory of rank-1. On the contrary, a $-n$ curve with $n>2$ must be accompanied by a gauge algebra on it.

More details about the most generic choice of the elliptic fiber for each configuration of curves and possible enhancements of it can be found in [6, 7].

Taking account these constraints, the (quite lengthy) list of configuration of curves which defines a 6d SCFT was obtained in [7].

2.5.2 Endpoint configuration

So far we have described a 6d theory on a generic point of its tensor branch. It is sometimes useful to consider a non-generic locus that one can obtain by "blowing down" the -1 curves, that is by shrinking them to zero size. As we already mentioned, after the blowdown the self-intersection of the neighboring curves changes by 1: namely, $\ldots n1m\ldots\to\ldots(n-1)(m-1)\ldots$. This procedure might create new (-1)-curves, that can be shrunk as well. The locus on the tensor branch of the theory where there are no longer any (-1)-curve is called as *endpoint*.

For example for (2.5.1) it can be checked that the endpoint consists of a single curve:

$$[E_8]\ \overset{\mathfrak{e}_8}{2}\ [E_8]\,. \tag{2.5.5}$$

It represents the locus where the 24 M5 fractions have coalesced in two full M5. We note that the adjacency matrix can also be difined to the endpoint; we will call η^{end} the resulting $n_T^{\text{end}}\times n_T^{\text{end}}$-matrix.

In [6], a classification of all possible endpoints for a 6d SCFT was obtained. In fact, the configuration of curves of the endpoint is a minimal resolution of the orbifold singularity $\mathbb{C}^2/\Gamma_{\text{discrete}}$. Here the discrete subgroups Γ_{discrete} is contained in $U(2)$. Not all the possible discrete subgroups of $U(2)$ arise in endpoint configurations. The allowed discrete subgroups are all the ADE subgroups of $\mathrm{SU}(2)$ and certain generalizations of the A-type and D-type subgroup in $U(2)$ which we label as $A(x_1,\cdots x_r)$ and $D(y|x_1,\cdots,x_l)$.

The subgroup $A(x_1,\cdots x_r)$ is cyclic of order p with generator acting on $(z_1,z_2)\in\mathbb{C}^2$ by

$$(z_1,z_2)\to(\omega z_1,\omega^q z_2),\ \text{where}\ \omega=\exp(2\pi i/p) \tag{2.5.6}$$

Here p and q are given by the expansion into the continued fraction:

$$\frac{p}{q}=x_1-\frac{1}{x_2-\cdots\frac{1}{x_r}}. \tag{2.5.7}$$

The subgroup $D(y|x_1, \cdots, x_l)$ is generated by the cyclic group

$$A(x_l, \cdots, x_1, 2y-2, x_1, \cdots, x_l) \tag{2.5.8}$$

and an element Λ which sends $(z_1, z_2) \to (z_2, -z_1)$. The geometry of the minimal resolution of the orbifold $\mathbb{C}^2/\Gamma_{\text{discrete}}$ for $A(x_1, \cdots, x_r)$ and $D(y|x_1, \cdots x_r)$ is given by

$$\mathcal{C}_{\text{end}} = x_1 \cdots x_r \tag{2.5.9}$$

for $A(x_1, \cdots x_r)$ and

$$\mathcal{C}_{\text{end}} = \begin{array}{|c|c|c|} \hline & 2 & \\ \hline 2 & y & x_1 \cdots x_r \\ \hline \end{array} \tag{2.5.10}$$

for $D(y|x_1, \cdots x_r)$.

Let us list all the endpoint configurations. It consists of four classes; ordinary ADE type, generalized A-type, generalized D-type and outliers.

Ordinary ADE-type. The discrete subgroup is the ADE subgroup of $\mathrm{SU}(2)$. The endpoint configuration consists of curves with self-intersection -2, intersecting each other according to an ADE graph.

Generalized A-type. The discrete subgroup is $A(x_1, \cdots x_r)$, but not all the r-tuple $x_1, \cdots x_r$ is realized. The possible configuration is the following

$$\mathcal{C}_{\text{end}} = x_1 x_2 x_3 x_4 x_5 \ A_N \ y_5 y_4 y_3 y_2 y_1 \tag{2.5.11}$$

where $A_N,\ N \geq 0$ is a Dynkin diagram with 2 at each node, and $2 \leq x_i \leq x_i^{\text{max}}$, $2 \leq y_i \leq y_i^{\text{max}}$. x_i^{max} and y_i^{max} is given by

$$\begin{aligned} x_1^{\text{max}} x_2^{\text{max}} x_3^{\text{max}} x_4^{\text{max}} x_5^{\text{max}} &\in \{7, 24, 223, 2223, 22223\} \\ y_1^{\text{max}} y_2^{\text{max}} y_3^{\text{max}} y_4^{\text{max}} y_5^{\text{max}} &\in \{7, 42, 322, 3222, 32222\} \end{aligned} \tag{2.5.12}$$

Generalized D-type. The discrete subgroup is $D(y|x_1, \cdots, x_l)$, but again not all the l-tuple is realized. The possible configuration is following:

$$D_N \, z_2 z_1 \tag{2.5.13}$$

where $D_N,\ N \geq 2$ is a Dynkin diagram with 2 at each node. z_1 and z_2 are given by

$$z_2 z_1 \in \{32, 24, 23\} \tag{2.5.14}$$

Outliers. There exists some outliers of the infinite families presented above. They occur in the A-type series for nine or fewer nodes. For the list, see [6, Eq. (5.12)].

2.5.3 Endpoints of conformal matter theories

In this subsection, we examine the endpoints of conformal matter and related theories. We will find that from the viewpoint of the endpoint classification, the conformal matte theories are general enough to represent most of the endpoints.

Conformal matter theories. From the list in (2.4.7), we can easily find that the endpoints of 6d theory $\mathcal{T}_G(N-1)$ is a linear chain

$$[G] \overset{\mathfrak{g}}{2} \cdots \overset{\mathfrak{g}}{2} \; [G] \tag{2.5.15}$$

where 2 is repeated $N-1$ times. This is natural from the Higgs branch RG flow. When we Higgs the $G, \mathfrak{g}$ symmetries in (2.5.15), we have a linear chain of -2 curves, which defined an $\mathcal{N}=(2,0)$ theory of type $\mathrm{SU}(N)$. In the M-theory picture, this corresponds to moving stacks of M5s away from the singularity, giving us the $\mathcal{N}=(2,0)$ theory of type $\mathrm{SU}(N)$ as expected.

T-brane theories. One can check from [67] that all the T-brane theories have $2\ldots2$ as its endpoint. This fact can be explained by their (conjectured) origin as a decoration of the original sequence of curves by two poles with nilpotent residues. In fact, nilpotent Higgs fields is not expected to change the geometry. Moreover, it was argued in [67] that all the theories with $2\ldots2$ endpoint are T-brane theories, possibly up to short outliers.

Frozen conformal matters. This class is more general than the ordinary "unfrozen" conformal matter whose endpoint is given by (2.5.15). The endpoint is no longer a sequence of -2 curves. For example, for (2.4.13) it is $232^{n-2}3$, where n is the number of $\mathfrak{e}_7$ gauge algebras and $2^n \equiv \underbrace{2\ldots2}_{n}$.

If we check the endpoints for all frozen conformal matter, we can find all the (generalized) A-type endpoints, as classified in [6]. The rule to obtain the endpoint can be summarized as follows;

$$a_1 \underbrace{(G)\text{—}(G)\text{—}\ldots(G)}_{\#\text{ of }(G)=n} a_2^t \qquad \mapsto \qquad \mathcal{C}_{\text{end}} = e(a_1)2^{n-2}e(a_2)^t \;, \tag{2.5.16}$$

where ^{t} denotes inversion of order, and e is described by Table 2.2. When $n=1$, (2.5.16) is changed to the following rule:

$$\cdots x2^{-1}y\cdots \equiv \cdots(x+y-2)\cdots \;. \tag{2.5.17}$$

The general rules (2.5.16) and (2.5.17) are enough to cover all the possible frozen conformal matter theories, except for those that do not include any copies of the unfrozen gauge algebra $\mathfrak{g}$ at all. These cases can also be treated sepately.

$SO(2k)$:

a	1	$\varnothing$
$e(a)$	2	3

E_6 :

a	131	31	1	$\varnothing$
$e(a)$	2	23	3	4

E_7 :

a	12321	2321	321	21	1	$\varnothing$
$e(a)$	2	223	23	3	4	5

E_8 :

a	12231513221	2231513221	231513221	31513221	1513221	513221
$e(a)$	2	22223	2223	223	23	33
a	13221	3221	221	21	1	$\varnothing$
$e(a)$	3	24	4	5	6	7

Table 2.2: The map e used in the general endpoint result (2.5.16).

For example, the endpoint of (2.4.13) can be obtained using (2.5.16) as follows. In this case, $a_1 = 321$, and $a_2^t = 12$. Table 2.2 gives $e(a_1) = 23$, $e(a_2) = 3$, and then $e(a_2)^t = 3$. So now the endpoint is $232^{n-2}3$ as was remarked. For $n = 1$, we have to use (2.5.17), which gives $232^{-1}3 = 24$.

Looking at Table 2.2, we see that the $e(a)$ cover all the possible $x_1x_2x_3x_4x_5$ and $y_1y_2y_3y_4y_5$ in the endpoint classification in (2.5.11) and (2.5.12). Moreover, with the rule (2.5.17) we can also find all the "rigid outliers" of [6, Eq. (5.12)].

Remarks. We conclude this subsection with several remarks. Although we can obtain all the generalized A-type endpoints from the conformal matter theories, there are generalized D-type such as (2.5.13) and (2.5.14) in the full list of endpoints. It is interesting to ask whether these bifurcated endpoints can be obtained by decorating the conformal matter theories by orientifold-type objects.

Since many different theories can share the same endpoint, the frozen conformal matter theories are far from being the most general 6d SCFTs. However, as we mentioned in in Section 2.4.2, the T-branes theories (2.4.12 seem to exhaust all the possible 6d SCFTs with endpoint 2^{N-1}. This suggests that one could similarly obtain all the 6d SCFTs with generalized A-type endpoint by introducing the T-branes to frozen conformal matter theories. The cases associated with nilpotent hierarchies for G_2 and F_4 were already examined in [67].

Chapter 3

Computations of anomalies of 6d SCFTs

3.1 Anomaly matching on tensor branch

In this section, we will develop a systematic field theoretical way to compute the anomaly polynomial of 6d $\mathcal{N}=(1,0)$ theories. The basic ingredient is the 't Hooft anomaly matching [12], which states that 't Hooft anomalies are matched between the UV and IR theory of the RG flow.

Let us consider the RG flow initiated by moving onto the generic point on the tensor branch. The UV theory is the 6d SCFT whose anomaly polynomial we would like to compute. The IR theory is a collection of almost free massless tensor/vector/hyper multiplets of the 6d supersymmetry. The essential point here is that this RG flow does not break any symmetry other than the conformal symmetry. Therefore, the whole chiral anomalies of the SCFT can be found on the generic tensor branch due to the anomaly matching [28, 29].[1]

As we saw in Section 2.2 the anomaly on the tensor branch has two sources: the one-loop contribution and the Green-Schwarz contribution. The one-loop anomaly is the sum of (2.1.4), (2.1.5) and (2.1.6). The Green-Schwarz contribution is (2.2.7), and the total anomaly polynomial of the SCFT is simply given by $I^{\text{SCFT}} = I^{\text{GS}} + I^{\text{one-loop}}$.

Since the 4-form I_i in (2.2.7) is unknown by far, the problem of computing the anomaly polynomial of a 6d SCFT reduces to the determination of I_i's. We adopt the following two complementary methods to determine them:

1. If there is no gauge group whose coupling is controlled by the tensor multiplet scalar, we compactify the system on S^1, and determine the induced Chern-Simons term in 5d. We can determine uniquely the 6d Green-Schwarz term from the 5d Chern-Simons terms.

2. If there is a gauge group whose coupling is controlled by the tensor multiplet scalar, the re-

[1] Since the conformal symmetry is spontaneously broken along the flow, we need to include the contribution of the Wess-Zumino-Witten terms of the dilaton in order to match the conformal anomaly between UV and IR.

quirement of the cancellation of the gauge, mixed gauge-R and gauge-gravitational anomalies uniquely fixes the Green-Schwarz term.

3.1.1 $\mathcal{N}=(2,0)$ theory and E-string theory

In this subsection, we will consider the 6d theories whose tensor branch does not contain any gauge groups. There are two possibilities; the $\mathcal{N}=(2,0)$ theories of arbitrary type and the E-string theories of general rank. It is surprising that the S^1 compactification of these theories are known explicitly, hence we can determine the Green-Schwarz couplings from the 5d Chern-Simons terms.

First of all, we explain the precise relationship between the 6d Green-Schwarz terms and the 5d Chern-Simons terms. When we reduce n self-dual fields H_i along S^1, we obtain n Abelian gauge fields A_i. The kinetic term for the H_is in (2.2.1) is reduced to the Yang-Mills coupling;

$$\frac{1}{2R}\eta^{ij}F_i \wedge \star F_j, \tag{3.1.1}$$

where R is the radius of S^1. We have used the relation $F_{\mu\nu} = 2\pi R \cdot H_{\mu\nu 5}$ which just follows from the quantization condition of each field strengths.

The modified Bianchi identity (2.2.2) is reduced to

$$d(\frac{1}{2\pi R} \star F_i) = I_i. \tag{3.1.2}$$

Therefore, the 5d Chern-Simons term is written using I_i as

$$\frac{1}{2\pi}S^{\text{CS}} = \eta^{ij}A_i I_j = A_i I^i. \tag{3.1.3}$$

Induced Chern-Simons terms. We consider a 5d fermion ψ with a mass term $m\psi\psi$ which has charge q under a $\mathrm{U}(1)$ field and couples to a non-abelian background gauge field F_G in a representation ρ and to the metric. Integrating out this fermion ψ induces the 5d Chern-Simons terms

$$\frac{1}{2}(\operatorname{sign} m)qA \wedge \left(\frac{1}{2}\operatorname{tr}_\rho F_G^2 + \frac{1}{24}d_\rho p_1(T)\right). \tag{3.1.4}$$

Although this follows from a careful computation of the triangle diagrams [69], here we show another derivation based on the counting of fermion zero-modes. Namely, we introduce nontrivial F_G and/or nontrivial metric on the spatial slice of 5d spacetime. In this instanton background, the fermion ψ has $\nu = -\operatorname{tr}_\rho F^2/2 - d_\rho p_1(T)/24$ zero modes. It is known that under the existence of fermion zero-modes, the electric charge of the fermion changes. In fact, each zero mode shifts the electric charge by $(\operatorname{sign} m)q/2$. Then the worldline Lagrangian for ψ has an additional coupling $(\operatorname{sign} m)(q\nu/2)A$, which indicates the 5d Chern-Simons term (3.1.4).

$\mathcal{N}=(2,0)$ **theory.** We will compute the anomaly polynomial of 6d $\mathcal{N}=(2,0)$ theory of arbitrary type $G = A_r$, D_r and E_r. Its S^1 compactification is the 5d $\mathcal{N}=2$ SYM with gauge group G. The charge lattice of string η^{ij} is given by the Cartan matrix of G.

We go to the tensor branch where the R-symmetry is broken to $\mathrm{SU}(2)_L \times \mathrm{SU}(2)_R \simeq \mathrm{SO}(4)_R \subset \mathrm{SO}(5)_R$. In 5d SYM, this corresponds to giving a vev to only $\phi^{I=5}$ out of five scalars ϕ^I in 5d SYM, where I is acted by $\mathrm{SO}(5)_R$. We consider a generic vev $v \in \mathfrak{h}$, where $\mathfrak{h}$ is the Cartan subalgebra of $\mathfrak{g}$, such that the gauge symmetry is broken to $\mathrm{U}(1)^r$.

We note that the fermion mass term is written as

$$\psi\Gamma^I\phi^I\psi \tag{3.1.5}$$

where ψ is in the spinor representation of $\mathrm{SO}(5)_R$. Then, the massive spectrum of the 5d theory is as follows;

- For each root $\alpha \in \mathfrak{h}$, we have a massive $\mathcal{N}=1$ vector multiplet, charged only under $\mathrm{SU}(2)_R$ and with the real mass $-v\cdot\alpha$.
- For each root $\alpha \in \mathfrak{h}$, we have a massive $\mathcal{N}=1$ hypermultiplet, charged only under $\mathrm{SU}(2)_L$ and with the real mass $+v\cdot\alpha$.

If we choose to use v to separate all the roots α into the positive roots and the negative roots, then the induced Chern-Simons terms for the $\mathrm{U}(1)^r$ field A valued in $\mathfrak{h}$ is given as

$$\frac{1}{2}\sum_{\alpha>0}(\alpha\cdot A)\left[(c_2(L)+\frac{2}{24}p_1(T)) - (c_2(R)+\frac{2}{24}p_1(T))\right] = \rho\cdot A(c_2(L)-c_2(R)) \tag{3.1.6}$$

where $\rho \mod \frac{1}{2}\sum_{\alpha>0}\alpha$ is the Weyl vector. From this Chern-Simons term, we find that the Green-Schwarz contribution to the anomaly is given as

$$\frac{1}{2}\langle\rho,\rho\rangle(c_2(L)-c_2(R))^2 = \frac{h^\vee_G d_G}{24}(c_2(L)-c_2(R))^2 \tag{3.1.7}$$

where we used the strange formula of Freudenthal and de Vries.

Therefore, adding the contribution from r_G free $\mathcal{N}=(2,0)$ tensor multiplets, the anomaly polynomial of the 6d $\mathcal{N}=(2,0)$ theory of type G is given by

$$I_G^{\mathcal{N}=(2,0)} = \frac{h^\vee_G d_G}{24}p_2(N) + r_G I^{\mathcal{N}=(2,0)\text{ tensor}}, \tag{3.1.8}$$

Here we used the formula $\chi_4(N) = c_2(L) - c_2(R)$, $p_2(N) = \chi_4(N)^2$ to lift the $\mathrm{SU}(2)_L \times \mathrm{SU}(2)_R$ bundle to $\mathrm{SO}(5)_R$.

The anomaly of Q coincident M5-branes is obtained by adding the contribution of a free $\mathcal{N}=(2,0)$ tensor to (3.1.8) with $G = \mathrm{SU}(Q)$;

$$I^{Q\text{ M5s}} = \frac{Q^3}{24}p_2(N) - QI_8 \tag{3.1.9}$$

where

$$I_8 = \frac{1}{48}(p_2(N) + p_2(T) - \frac{1}{4}(p_1(N) - p_1(T))^2). \quad (3.1.10)$$

We note that the Q^3 scaling law of the degrees of freedom on M5 branes, expected from holography [70] is reproduced from purely field theoretical analysis.

E-string theory. We will compute the anomalies of the E-string theory of rank Q. During the computation, we keep the hypermultiplet corresponding to the center-of-mass motion of M5s parallel to the E_8 wall. We move to the point of the tensor branch where all the M5s are coincident but apart from the E_8 wall. There is one tensor multiplet controlling the distance of the stack of M5s from the E_8 wall. Note that the matrix η in (2.2.5) for this tensor is just Q.

We compactify this setup on S^1 with a holonomy to break the E_8 symmetry to $\mathrm{SO}(16)$. Since $\mathrm{SO}(16)$ is a maximal rank subgroup of E_8, we can reconstruct the E_8 anomalies from those of $\mathrm{SO}(16)$. The 5d theory is now given by $\mathcal{N}=1$ $\mathrm{USp}(2Q)$ theory with an antisymmetric hypermultiplet[2] and 8 hypermultiplets in the fundamental representation.

We are now on the Coulomb branch where the $\mathrm{USp}(2Q)$ is broken to $\mathrm{U}(Q)$ and the tensor explained above now becomes a gauge field A which is the $\mathrm{U}(1) \subset \mathrm{U}(Q)$. Then, we need to determine its Chern-Simons term. The multiplet spectrum is summarized as follows;

- A massless $\mathcal{N}=4$ $\mathrm{U}(Q)$ vector multipltets which consists of Q^2 vector multiplets and Q^2 hypermultiplets.
- Q massive hypermultiplets in the vector representation of $\mathrm{SO}(16)$ with $\mathrm{U}(1)$ charge 1. These come from 8 hypermultiplets in the fundamental representation of $\mathrm{USp}(2Q)$.
- $Q^2 + Q$ massive vector multiplets and $Q^2 - Q$ hypermultiplets, both with $\mathrm{U}(1)$ charge 2. These come from the vector multiplet and the antisymmetric hypermultiplet of $\mathrm{USp}(2Q)$, respectively.

Therefore, the induced Chern-Simons term is given as

$$\begin{aligned}\frac{1}{2}A \wedge \Big[Q(\frac{\mathrm{Tr}\,F^2}{2} + \frac{16p_1(T)}{24}) + 2(Q^2 - Q)\frac{1}{2}(c_2(L) + \frac{2p_1(T)}{24})\\ - 2(Q^2 + Q)\frac{1}{2}(c_2(R) + \frac{2p_1(T)}{24})\Big] = \eta A \wedge (\frac{Q}{2}\chi_4(N) + I_4) \quad (3.1.11)\end{aligned}$$

where $\eta = Q$ and

$$I_4 = \frac{1}{4}(\mathrm{Tr}\,F^2 + p_1(T) + p_1(N)). \quad (3.1.12)$$

We also used the fact that the vector is charged under the $\mathrm{SU}(2)_R$ and the hypers in the antisymmetric representation is charged under the $\mathrm{SU}(2)_L$,

[2] This multiplet contains $2Q^2 - Q$ hypermultiplets in total.

Using this Chern-Simons term, we can find the anomaly polynomial of E-string theory;

$$\begin{aligned} I^{\text{E-string, rank } Q\text{+free hyper}} &= I^{Q \text{ M5s}} + \frac{Q}{2}(\frac{Q}{2}\chi_4(N) + I_4)^2 \\ &= \frac{Q^3}{6}\chi_4(N)^2 + \frac{Q^2}{2}\chi_4(N)I_4 + Q(\frac{1}{2}I_4^2 - I_8). \end{aligned} \tag{3.1.13}$$

Here I_8 is as given above. Note that since (3.1.13) contains the free center-of-mass hypermultiplet, we always need to subtract that contribution from (3.1.13) when the E-string theory is used as a matter content,.

3.1.2 Minimal 6d $\mathcal{N}=(1,0)$ theories

In this subsection, we will consider the 6d theories whose tensor branch contain any gauge groups. As an example, we compute the anomalies of minimal 6d $\mathcal{N}=(1,0)$ theories, introduce in Section 2.3.1.

Let us start from the generality. We consider a point on the tensor branch where the effective theory consists of t vector multiplets in gauge group G_A, $A = 1, \cdots, t$, and t tensor multiplets controlling the coupling constants of G_A, together with a number of charged "bifundamental matter contents". These "bifundamental matter" can either be Lagrangian hypermultiplets or another 6d SCFT whose flavor symmetries are gauged by G_A. For 6d theories in [6], the strongly coupled "bifundamental matter" is always the E-string theories of rank-1 or 2. We further assume that anomalies of the "bifundamental matter" are already known.

Now, the "one-loop" anomaly (i.e. the anomaly without Green-Schwarz contribution) on the tensor branch is given by

$$I^{\text{one-loop}} = \sum_A I^{\text{vec}}_{F_A} + \sum_{A,B} I^{\text{matter}}_{F_A,F_B} + tI^{\text{tensor}}. \tag{3.1.14}$$

It contains pure and mixed gauge anomalies,

$$I^{\text{one-loop}} \supset -\frac{1}{32}c^{AB}\,\text{Tr}\,F_A^2\,\text{Tr}\,F_B^2 - \frac{1}{4}X^A\,\text{Tr}\,F_A^2, \tag{3.1.15}$$

where X^A consists of only background flavor and gravity fields. By assumption, we already know the coefficients c^{AB} and X^A.

One needs to cancel these anomalies by the Green-Schwarz contribution,

$$\frac{1}{2}\eta^{ij}I_iI_j. \tag{3.1.16}$$

Here η^{ij} is the symmetric matrix introduced in (2.2.5). The anomaly cancellation requires,

$$I_i = \frac{1}{4}d_i^A\,\text{Tr}\,F_A^2 + (\eta^{-1})_{ij}(d^{-1})^j_A X^A, \qquad d_i^A d_j^B \eta^{ij} = c^{AB}. \tag{3.1.17}$$

where we have assumed that the matrix c^{AB} has the maximal rank t so that the matrix d_i^A is invertible. We also used the assumption that the number of free tensor multiplets and the gauge groups G_A is same.

While the matrix d_i^A is not completely determined, we can uniquely determine the Green-Schwarz contribution in terms of c^{AB} and X^A;

$$\frac{1}{2}\eta^{ij}I_iI_j = \frac{1}{32}c^{AB}\operatorname{Tr}F_A^2\operatorname{Tr}F_B^2 + \frac{1}{4}X^A\operatorname{Tr}F_A^2 + \frac{1}{2}(c^{-1})_{AB}X^AX^B. \tag{3.1.18}$$

Combining (3.1.15) and (3.1.18), the Green-Schwarz contribution to the anomaly polynomial of the SCFT is given by

$$I^{GS} = \frac{1}{2}(c^{-1})_{AB}X^AX^B. \tag{3.1.19}$$

Minimal 6d $\mathcal{N}=(1,0)$ theories. Let us compute the R-symmetry anomalies of minimal 6d $\mathcal{N}=(1,0)$ theories, namely the theories with one-dimesinonal tensor branch with a vector multiplet of gauge group $G = \mathrm{SU}(3)$, $\mathrm{SO}(8)$, F_4, $E_{6,7,8}$, reviewed in Section 2.3.1.

The one-loop anomaly polynomial on the tensor branch is written as

$$I^{\text{one-loop}} = I^{\text{vec}} + I^{\text{tensor}} = -\frac{1}{24}\left(\frac{3n}{4}(\operatorname{Tr}F^2)^2 + 6h_G^\vee\operatorname{Tr}F^2c_2(R) + d_Gc_2(R)^2\right) + \frac{1}{24}c_2(R)^2. \tag{3.1.20}$$

where n is defined in Section 2.3.1.

In order to cancel the pure and mixed gauge anomalies in (3.1.20), the unique choice of the Green-Schwarz contribution is

$$I^{\text{GS}} = \frac{n}{2}\left(\frac{1}{4}\operatorname{Tr}F^2 + \frac{h_G^\vee}{n}c_2(R)\right)^2. \tag{3.1.21}$$

Therefore the total R-symmetry anomaly is

$$I^{\text{tot}} = I^{\text{one-loop}} + I^{\text{GS}} = \left(\frac{(h_G^\vee)^2}{2n} - \frac{d_G-1}{24}\right)c_2(R)^2. \tag{3.1.22}$$

3.2 Green-Schwarz coupling for general 6d SCFTs

In this section, we will determine the Green-Schwarz coupling for general 6d $\mathcal{N}=(1,0)$ theories constructed by F-theory as in Section 2.5. Namely, we give a method to extract the Green-Schwarz term I_i associated with each compact curves Σ_i on B_2.

Non-R symmetry terms in I_i. Let us recall how to determine the pieces proportional to $p_1(T)$ and $\operatorname{Tr}F^2$ in I_i [50]. We will denote the configuration of curves on B_2 by $\mathcal{C}$, which consists of compact or non-compact curves Σ_a wrapped by sevenbranes.[3]

[3] We will use the index $a, b, \cdots$ to label all components of $\mathcal{C}$, while $i, j, \cdots$ label only compact ones.

The 6d self-dual 3-form field strengths H_i is the reduction of the spacetime 5-form field strength F_5 in type IIB string on Σ_i;

$$F_5 = H_i \wedge \omega^i \tag{3.2.1}$$

where ω_i is the Poincaré dual of Σ_i. The Bianchi identity for the 5-form field strength F_5 is

$$\mathrm{d}F_5 = Z, \tag{3.2.2}$$

for some 6-form Z consisting of background field strength is

$$\mathrm{d}H_i = I_i, \quad \eta^{ij} I_j = -\int_B Z \wedge \omega^i, \tag{3.2.3}$$

where we used the definition of $\eta^{ij} = -\int_B \omega^i \wedge \omega^j = -\Sigma_i \cdot \Sigma_j$. We extend this intersection form to include η^{ia}, intersections between compact and non-compact curves. Then, the contributions for the anomaly polynomial from these Green-Schwarz terms are[4]

$$I^{\mathrm{GS}} = -\frac{1}{2}\int_B Z^2 = \frac{1}{2}\eta^{ij} I_i I_j. \tag{3.2.4}$$

The anomalous coupling due to the sevenbranes are given by

$$S_{\text{anom}} = \int_{\mathbb{R}^{1,5}\times B_2} B_4 \wedge \sum_a \left(\frac{1}{4}\operatorname{Tr} F_a^2 - \frac{N_a}{48} p_1(T)\right) \wedge \omega^a \tag{3.2.5}$$

where F_a is the gauge field on Σ_a, $N_a = \mathrm{ord}(\Delta)|_{\Sigma_a}$ is the number of sevenbranes wrapping Σ_a and B_4 is the potential for the 5-form field strength F_5. Here the Calabi-Yau condition for the total space is [4]

$$c_1(B) = -\frac{1}{12}\sum_a N_a \omega^a, \tag{3.2.6}$$

which means that the localized curvature on the sevenbranes cancels the positve curvature of the base B_2 and the total space becomes Calabi-Yau. Then, substituting (3.2.6) to (3.2.5), we obtain

$$S_{\text{anom}} = \int_{\mathbb{R}^{1,5}\times B_2} B_4 \wedge \left(\frac{1}{4} c_1(B) \wedge p_1(T) + \frac{1}{4}\sum_a \omega^a \operatorname{Tr} F_a^2\right). \tag{3.2.7}$$

Then, the 10d Green-Schwarz term Z is

$$Z = \frac{1}{4} c_1(B) \wedge p_1(T) + \frac{1}{4}\sum_a \omega^a \operatorname{Tr} F_a^2 \tag{3.2.8}$$

where F_a is the field strength on the sevenbranes wrapping Σ_a. So we obtain

$$\eta^{ij} I_j = \frac{1}{4}(\eta^{ia} \operatorname{Tr} F_a^2 - K^i p_1(T)), \quad K^i := \int_B c_1(B) \wedge \omega^i = 2 - \eta^{ii}, \tag{3.2.9}$$

up to the $c_2(R)$-dependent terms.

[4] We use a convention that F_5 is anti-self-dual so that the 6d fields H_i become self-dual, because ω^i are anti-self-dual.The minus sign in front of $\frac{1}{2}\int Z^2$ comes from this anti-self-dual (instead of self-dual) property of F_5.

R-symmetry term in I_i. Next, we consider the terms associated to the $\mathrm{SU}(2)_R$ R-symmetry, which are not directly visible by the geometry of F-theory construction. Each Green-Schwarz term I_i should contain contributions proportional to $c_2(R)$ to cancel mixed gauge-$\mathrm{SU}(2)_R$ anomalies, so we write

$$\eta^{ij} I_j = \frac{1}{4}(\eta^{ia} \operatorname{Tr} F_a^2 - K^i p_1(T)) + y^i c_2(R). \tag{3.2.10}$$

Our next task is to determine the coefficients y^i. The contribution to the mixed anomalies between gauge and R symmetries from the Green-Schwarz terms are

$$I^{\mathrm{GS}} \supset \frac{1}{4} y^i \operatorname{Tr} F_i^2 c_2(R). \tag{3.2.11}$$

Let us consider a a curve Σ_i with a nontrivial gauge group G_i. Then, we have a vector multiplet which contributes the anomaly of R and G_i by

$$I^{\mathrm{vec}} = -\frac{1}{24}(\frac{3}{4} w_{G_i} (\operatorname{Tr} F^2)^2 + 6 h^\vee_{G_i} \operatorname{Tr} F^2 c_2(R) + d_{G_i} c_2(R)^2), \tag{3.2.12}$$

where $3w_{G_i}/4$ is the coefficient converting $\operatorname{tr}_{\mathrm{adj}} F^4$ to $(\operatorname{Tr} F^2)^2$.

To cancel the pure and mixed anomalies in (3.2.12), the unique choice of the Green-Schwarz term is to take

$$I^{\mathrm{GS}} = \frac{w_G}{2}(\frac{1}{4} \operatorname{Tr} F^2 + \frac{h^\vee_G}{w_G} c_2(R))^2. \tag{3.2.13}$$

Therefore, we can conclude that $y^i = h^\vee_{G_i}$ for the curve Σ_i.

For -1 and -2 curves without gauge groups, we cannot determine y^i with this method. One can avoid this problem by using the anomalies of E-string theory of rank-1 or 2, computed in the previous section.

In fact, let us blow down a -1 curve Σ_i without gauge algebra in the configuration of curves. Before the curve was shrunk, the massless matter associated is just a tensor multiplet. After the blowdown, the massless spectrum includes the E-string theory of rank-1. Then, the difference of the "one-loop" part of the anomaly before and after of the blowdown is given by

$$\Delta I^{\text{one-loop}} = I^{\mathrm{rank1}}_{\mathrm{E-string}} - I^{\mathrm{tensor}} = \frac{1}{2}\left(c_2(R) - \frac{1}{4} p_1(T) - \frac{1}{4} \operatorname{Tr} F^2_{E_8}\right)^2. \tag{3.2.14}$$

In order to obtain a consistent anomaly before and after, the mismatch (3.2.14) should be accounted by the Green-Schwarz coupling associated with the -1 curve;

$$\frac{1}{2}(I^i)^2 = \frac{1}{2}\left(y^i c_2(R) - \frac{1}{4} p_1(T) - \frac{1}{4} \operatorname{Tr} F^2_{i-1} - \frac{1}{4} \operatorname{Tr} F^2_{i+1}\right)^2, \tag{3.2.15}$$

where we have assumed that the -1 curve Σ_i touches two curves Σ_{i-1} and Σ_{i+1} with gauge symmetries G_{i-1} and G_{i+1}, respectively. Here $G_{i-1} \times G_{i+1} \subset E_8$. Comparing (3.2.14) and (3.2.15), we find that we must have $y^i = 1$ for the -1 curve without gauge algebra. From a similar computation of rank-2 E-string theory, we find that we also have $y^i = 1$ for the -2 curve without gauge algebra.

3.3 Comparison with the inflow

In this section, we compare the anomaly polynomial of $\mathcal{N}=(2,0)$ theory and E-string theory obtained in the previous section with the results of anomaly inflow in M-theory.

Since this technique is used throughout the thesis, let us recall what the anomaly inflow is. We consider a defect W in the spacetime M whose worldvolume supports chiral zero modes with anomalies $I^{\text{zero-modes}}$. When the defect serves as a singular source for the field strength $G = dC$ and there is a Chern-Simons coupling for G in M, there are additional contribution I_{inflow} to the anomalies of the worldvlume theory on W in addition to $I^{\text{zero-modes}}$. This phenomenon is called as anomaly inflow [38].

We would like to review how this works in the context of the eleven-dimensional supergravity, i.e. low-energy effective description of the M-theory. The massless bulk fields include a three-form potential C_3 and we denote whose field strength as $G_4 = dC_3$. Let us put an M5-brane X_6 in the spacetime X_{11}. Since the M5 is a magnetic source for G_4, the Bianchi identity is modified as

$$dG_4 = \delta^{(5)}(X_6 \hookrightarrow X_{11}), \tag{3.3.1}$$

where $\delta^{(5)}$ is the 5-form representing the position of the M5 in the spacetime. On the other hand, it is well-known [5] that there is a non-minimal Chern-Simons term

$$S_{CI_8} = -2\pi \int_{X_{11}} C \wedge I_8, \qquad I_8 = \frac{1}{48}\left[p_2(TX_{11}) - \frac{1}{4}p_1^2(TX_{11})\right], \tag{3.3.2}$$

that is linear in C_3 in the low-energy limit of M-theory. The descent formula implies that the gauge variation of the Chern-Simons term (3.3.2) is given as

$$\delta S_{CI_8} = -2\pi \int_{X_{11}} dG_4 \wedge I_6^{(1)}, \tag{3.3.3}$$

and by using (3.3.1), we have

$$\delta S_{CI_8} = -2\pi \int_{X_6} I_6^{(1)}|_{X_6}. \tag{3.3.4}$$

Therefore, we obtain an additional piece of anomaly localized on the M5-brane, $I_{CI_8}^{\text{inflow}} = -I_8|_{X_6}$.

However, this is not the whole story of the anomaly inflow in M-theory. Indeed, the zero-mode on a single M5-brane is a free $\mathcal{N}=(2,0)$ tensor multiplet whose anomalies we already cited in

[5]The interaction S_{CI_8} cannot be seen in the naive supergravity action up to two derivatives. The existence of such a term can be inferred from the following facts. First, a one-loop effect in type IIA superstring theory [71] is known to induce the $B \wedge I_8$ coupling, which is the type IIA descendant of the S_{CI_8}. Second, this interaction is necessary to cancel the gravitational anomaly of an M5 [72]. Third, on a general 11d spin manifold, without the S_{CI_8} term, another Chern-Simons coupling S_{CGG} not well defined due to the shifted quantization condition of M-theory 3-form C [73].

(2.1.7). The sum of these two contributions are

$$I^{\text{zero-modes}}_{(2,0)\text{-tensor}} + I^{\text{inflow}}_{CI_8} = \frac{1}{24} p_2(N) \neq 0, \tag{3.3.5}$$

where N is the normal bundle of X_6 in X_{11}. As a consistent quantum theory, we expect that there is a further anomaly localized on W_6 which cancels the normal bundle contribution (3.3.5). Indeed, there is another Chern-Simons term cubic in C_3,

$$S_{CGG} = \frac{2\pi}{6} \int_{X_{11}} C \wedge G \wedge G, \tag{3.3.6}$$

in the 11d supergravity, which is related to the kinetic term for C_3 via the supersymmetry. It is not straightforward to compute the contribution of (3.3.6) to the anomaly on X_6, since it is nonlinear in C_3 and the argument around (3.3.4) cannot be applied.

Nevertheless, a prescription to compute the anomaly inflow from (3.3.6) which correctly cancel the normal bundle anomaly (3.3.5) was found in [39]. It involves two improvements,

- The singular behavior of G_4 around X_6 is modeled by using the global angular form so as to reproduce the modified Bianchi identity (3.3.1).
- The definition of the Chern-Simons term (3.3.6) is slightly modified so that the integration avoids the X_6 on which the G_4 is singular.

In the following sections, we will adopt this prescription and compute the anomaly of M5-branes with various types of singularities in M-theory.

Remark. In order to simplify the computation in the following sections, we prefer to represent the coupling (3.3.6) as

$$S_{CGG} = \frac{2\pi}{6} \int_{Y_{12}} G \wedge G \wedge G. \tag{3.3.7}$$

where Y_{12} is a 12d manifold whose boundary ∂Y_{12} is equal to X_{11}. We also rewrite the term (3.3.2) using Y_{12}:

$$S_{CI_8} = -2\pi \int_{Y_{12}} G \wedge I_8. \tag{3.3.8}$$

It is straightforward to reproduce the results in the following sections by using the original expressions (3.3.2) and (3.3.6).

3.3.1 $\mathcal{N}=(2,0)$ theory

In this section, we review the procedure of [39, 40] to obtain the anomaly polynomial of Q coincident M5-branes. Let us denote the local coordinates transverse to the M5s as $y_i, i = 1, \cdots, 5$ such that the worldvolume of the M5s is given by $X_6 = \{y_i = 0\}$. In this coordinate system, the modified Bianchi identity for G_4 is written as

$$dG = Q \prod_{i=1}^{5} \delta(y_i) dy_i. \tag{3.3.9}$$

In the presence of M5s, the 11d Chern-Simons terms (3.3.7) and (3.3.8) is singular around the worldvolume X_6. Such a contribution gives rise to an anomaly inflow toward X_6, which should be cancelled by the anomalies of chiral zero-modes supported on the M5s. When $Q > 1$, this assumption determines the anomalies of the mysterious $\mathcal{N}=(2,0)$ theory on M5s.

In order to extract the singular behavior of (3.3.7) and (3.3.8), we analyze how the field strength G behaves around the M5s. The solution of (3.3.9) near $y \sim 0$ is

$$G = \frac{Q}{2} e_4 + (\text{regular}), \tag{3.3.10}$$

where (regular) represents non-singular terms at $y = 0$. Here, e_4 is the global angular form of the normal bundle of X_6 with the normalization $\int_{S^4} e_4 = 2$. It can be explicitly written as

$$\begin{aligned} e_4(y) =& \frac{1}{32\pi^2} \epsilon_{a_1 \cdots a_5} \Big[(D\hat{y})^{a_1} (D\hat{y})^{a_2} (D\hat{y})^{a_3} (D\hat{y})^{a_4} \hat{y}^{a_5} \\ & - 2F^{a_1 a_2} (D\hat{y})^{a_3} (D\hat{y})^{a_4} \hat{y}^{a_5} + F^{a_1 a_2} F^{a_3 a_4} \hat{y}^{a_5} \Big]. \end{aligned} \tag{3.3.11}$$

using the normalized coordintes $\hat{y}^a = y^a/|y|$, a covariant exterior derivative D of $\mathrm{SO}(5)_R$ rotational symmetry around $y = 0$, and the field strength $F^{a_1 a_2}$ of $\mathrm{SO}(5)_R$. Several properties of the global angular form are listed in Appendix C.

Now we can determine the inflow from the Chern-Simons terms. Since G is singular on the worldvolume of M5s, we remove a small tubular neighborhood of X_6 in the integral of Chern-Simons terms. In the following, we will denote a tubular neighborhood of a submanifold M with radius ϵ as $D_\epsilon(M)$. The precise definition of the Chern-Simons terms which take into account of the singular behavior of G is given as

$$\int_{Y_{12}} \to \lim_{\epsilon \to 0} \int_{Y_{12} \setminus D_\epsilon(Y_7)}. \tag{3.3.12}$$

Here, Y_7 is a submanifold of Y_{12} whose boundary is X_6 and the orientation is such that $\partial(Y_{12} \setminus D_\epsilon(Y_7)) = -\partial D_\epsilon(Y_7)$.

Because e_4 is closed, it can be locally written as $e_4 = de_3^{(0)}$ Now, we extract the most singular part from the Chern-Simons term S_{CGG}^{bulk};

$$\begin{aligned} S_{CGG}^{\text{bulk}} =& \frac{2\pi}{6} \lim_{\epsilon\to 0} \int_{Y_{12}\setminus D_\epsilon(Y_7)} G^{\text{bulk}} \wedge G^{\text{bulk}} \wedge G^{\text{bulk}} \\ \sim& 2\pi \cdot \frac{Q^3}{48} \lim_{\epsilon\to 0} \int_{Y_{12}\setminus D_\epsilon(Y_7)} e_4^3 = -2\pi \cdot \frac{Q^3}{48} \lim_{\epsilon\to 0} \int_{\partial D_\epsilon(Y_7)} e_3^{(0)} e_4^2 \\ =& -2\pi \cdot \frac{Q^3}{24} \int_{Y_7} p_2(N)^{(0)}, \end{aligned} \tag{3.3.13}$$

where $dp_2(N)^{(0)} = p_2(N)$. We have used the Stoke theorem in the second line and second equation of (C.5) in the last line. Thus the inflow toward the M5s is $-(Q^3/24)p_2(N)$. Similarly, the singular part of S_{CI_8} can be extracted as follows;

$$S_{CI_8}|_{\text{sing}} \sim 2\pi Q \lim_{\epsilon\to 0} \int_{\partial D_\epsilon(Y_7)} \left(\frac{e_3^{(0)}}{2} \wedge I_8 \right) = 2\pi Q \int_{Y_7} I_7^{(0)}. \tag{3.3.14}$$

Therefore, the inflow from the eleven-dimensional Chern-Simons terms is given by

$$-\frac{Q^3}{24} p_2(N) + QI_8, \tag{3.3.15}$$

and then the anomaly polynomial $A_{\text{M5}}(Q)$ of the 6d theory on Q coincident M5s is given by

$$A_{\text{M5}}(Q) = \frac{Q^3}{24} p_2(N) - QI_8. \tag{3.3.16}$$

This is precisely the anomaly polynomial of M5s obtained by the tensor branch anomaly matching in (3.1.9).

Remarks. We make a brief comment on the inflow computation of the anomaly of $\mathcal{N}=(2,0)$ theories of other types. The D_n (2,0) theories are simply realized by putting n M5-branes on top of the M-theoretic orientifold. The inflow calculation was performed in [74] and agrees with (3.1.8). The E_n (2,0) theories are realized by putting M5s on top of a non-geometric background as recently found in [57]. The inflow computation in this non-geometric setup has not been performed yet. It would be also interesting to reproduce (3.1.8) by the anomaly inflow in type IIB supergravity on $X_6 \times \mathbb{C}^2/\Gamma_G$.

3.3.2 E-string theory

Anomalies on the E_8 wall. Let us determine the anomalies $A_{E_8\text{-brane}}$ supported on the E_8 wall in M-theory [59]. We take the E_8 wall located at $x_{10} = 0$. Then the Bianchi identity is modified around the wall as:

$$dG = \delta(x_{10}) dx_{10} I_4. \tag{3.3.17}$$

Then, near the wall, the 4-form field strength G behaves as

$$G \sim e_0 I_4 + \text{(regular)} \tag{3.3.18}$$

where (regular) represents non-singular terms at $x_{10} = 0$ and e_0 is just a step function which is 1 for $x_{10} > 0$ and -1 for $x_{10} < 0$. Hořava and Witten argued that the concrete form of the 4-form I_4 is given as

$$I_4 = \frac{1}{4}(\operatorname{Tr} F^2 + p_1(TX_{11})) \tag{3.3.19}$$

where F is the field strength of the E_8 gauge field on the E_8 wall. We note that while $\frac{1}{4}\operatorname{Tr} F^2$ is the correctly normalized instanton number, $\frac{1}{4}p_1(TX_{11})$ takes values in half-integers.[6] This gravitational factor comes from the gravitino boundary condition at the wall.

Repeating the same calculation as in the previous section, we find the singular part of the Chern-Simons terms near the wall as,

$$S_{CGG}|_{\text{sing}} + S_{CI_8}|_{\text{sing}} = -\int_{Y_{11}} I_3^{(0)}(\frac{1}{6}I_4^2 - I_8). \tag{3.3.20}$$

Therefore the anomaly polynomial $A_{E_8\text{-brane}}$ supported on the wall is

$$A_{E_8\text{-brane}} = I_4\left(\frac{1}{6}I_4^2 - I_8\right). \tag{3.3.21}$$

We can check that this coincides with the anomaly of an E_8 vector multiplet plus one half the anomaly of the supergravity multiplet. Then, adding the contribution from the other E_8 wall, we reproduce the anomalies of massless field contents of $E_8 \times E_8$ heterotic superstring. This is a piece of the evidence that the strong coupling limit of $E_8 \times E_8$ heterotic superstring becomes M-theory orbifold, as discussed in [59].

Inflow computation. Let us compute the anomaly polynomial of the E-string theory of rank Q. Near the M5-branes put on the wall, the singular behavior for G is

$$G \sim e_0 I_4 + 2Q\frac{e_4}{2} + \text{(regular)}, \tag{3.3.22}$$

where the M5 charge is $2Q$, including the contributions from the mirror images.

Now, the field strength G has two types of singularities: the first term in (3.3.22) is singular along Y_{11}, and the second term in (3.3.22) is singular along Y_7. Then, the properly modified Chern-Simons terms are:

$$S_{CS} = \lim_{\epsilon_{1,2}\to 0} 2\pi \int_{Y_{12}\backslash(D_{\epsilon_1}(Y_{11})\cup D_{\epsilon_2}(Y_7))/\mathbb{Z}_2} \left(\frac{1}{6}G\wedge G\wedge G - G\wedge I_8\right). \tag{3.3.23}$$

[6]For any 11d spin manifold X_{11}, the first Pontgyagin number $p_1(TX_{11})$ is always even.

The singular part which does not contain the second term of (3.3.22) are cancelled by the anomalies on the wall (3.3.21). The remaining singular part of (3.3.23) localizes to the boundary of $D_{\epsilon_2}(Y_7)$, and should be canceled by the anomalies of the E-string theory of rank Q. This remainder is given by

$$
\begin{aligned}
&S_{CS}|_{\text{sing}} + \int_{Y_{11}} A^{(0)}_{E_8\text{-brane}} \\
&= 2\pi \int_{\partial D(Y_7)/\mathbb{Z}_2} \left(-\frac{Q^3}{6} e_3^{(0)} e_4^2 - \frac{Q^2}{2} e_3^{(0)} e_4 I_4 e_0 - \frac{Q}{2} e_3^{(0)} I_4^2 + Q e_3^{(0)} I_8 \right) \\
&= 2\pi \int_{Y_7} \left(-\frac{Q^3}{6} p_2^{(0)} - \frac{Q^2}{2} \chi_4 I_3^{(0)} - \frac{Q}{2} I_4 I_3^{(0)} + Q I_7^{(0)} \right), \qquad (3.3.24)
\end{aligned}
$$

where we used the formulas (C.5), (C.6) in the last line, noting that the fiber is $S^4/\mathbb{Z}_2$, instead of S^4.

Then, the anomaly polynomial $A_{E_8+\text{free}}(Q)$ of the E-string theory of rank Q (plus free center-of-mass hyper multiplet) is

$$
A_{E_8+\text{free}}(Q) = \frac{Q^3}{6} p_2(N) + \frac{Q^2}{2} \chi_4(N) I_4 + Q \left(\frac{1}{2} I_4^2 - I_8 \right), \qquad (3.3.25)
$$

which coincides with (3.1.13).

Chapter 4

Anomaly constraints for Higgsability of 6d SCFTs

4.1 Classes of 6d theories

In Chapter 3, we have used the anomaly matching on the tensor branch to determine the full anomaly polynomials of 6d SCFTs. In this chapter, we focus on the anomaly matching on the Higgs branch. Since the R and flavor symmetries are broken on the Higgs branch, it is difficult to compute those part of the anomaly polynomial. However, we can safely compute the gravitational anomaly of the SCFT on the Higgs branch. In this chapter, we will use the anomaly matching on the Higgs branch to extract the interesting information about the Higgsability of 6d SCFTs.

We will first introduce some terminology following [31] (see also [32, 33]).

Very Higgsable (vH) theories. These are the SCFTs whose Higgs branch is such that at its generic point the theory flows to a collection of free hypermultiplets, without any tensors:

$$\text{vH SCFT} \quad \to \quad \text{free hypermultiplets .} \tag{4.1.1}$$

In this thesis, we call this process, where tensors are lost in favor of hypermultiplets, as a "small instanton transition" . We divide very Higgsable 6d theories in two subclasses: Obviously very Higgsable and Hiddenly very Higgsable.

1. **Obviously very Higgsable (OvH) theories.** This class was first introduced in [32]. These theories are such that one can directly identify as very Higgsable by looking at the F-theory realization: all of the compact cycles producing the tensor multiplets can be removed by repeated blow-downs of (-1)-curves. Examples of such theories include

   - The theory of free hypermultiplets.

- The E-string theory which describes a single M5 on an E_8 wall in M-theory. The F-theory construction involves only a single (-1)-curve. It has a Higgs branch of dimension 29 as can be seen from the M-theory description. Indeed, the tensor branch corresponds to pulling away the M5 from the wall, while the Higgs branch corresponds to turning the M5 into a finite-sized E_8 instanton. Since such an instanton cannot be pulled off the wall, we see that the tensor multiplet has been lost on the Higgs branch.
- The E-string theory of rank-N which describes N M5s on the E_8 wall. The F-theory realization consists of a single (-1)-curve followed by $N-1$ (-2)-curves.
- The worldvolume theory of a single or multiple M5-branes on $\mathbb{C}^2/\Gamma_G$ on an E_8 wall.
- The worldvolume theory $\mathcal{T}_G(0)$ of a single M5-brane on $\mathbb{C}^2/\Gamma_G$, also called as the minimal conformal matter theory of type (G,G), where G is a simply-laced group.
- Certain theories that describe fractional M5-branes on $\mathbb{C}^2/\Gamma_G$, including
  - $(E_7, \mathrm{SO}(7))$ minimal conformal matter describing $1/2$ M5-branes on $\mathbb{C}^2/\Gamma_{E_7}$;
  - (E_8, F_4) minimal conformal matter describing $1/2$ M5-branes on $\mathbb{C}^2/\Gamma_{E_8}$; and
  - (E_8, G_2) minimal conformal matter describing $1/3$ M5-branes on $\mathbb{C}^2/\Gamma_{E_8}$.

2. **Hiddenly very Higgsable (HvH) theories.** These are very Higgsable theories which are not obviously very Higgsable. We will obtain an F-theory characterization in the next section. Examples of such theories include

   - The worldvolume theory of a single M5-brane probing the completely or partially frozen $\mathrm{SO}(2k)$ or $E_{6,7,8}$ singularity. The gauge group on a partially frozen singularity is a non-simply-laced group $G_{\mathrm{fr}} = G_2, F_4, \mathrm{USp}(2k)$. We will examine these theories in more detail in section 4.4.

Higgsable to $\mathcal{N}=(2,0)$ (HN) theories. We can slightly generalized the notion of vH 6d theories. A 6d theory is HN if its Higgs branch is such that at its generic point the theory flows to an $\mathcal{N}=(2,0)$ theory plus free hypermultiplets:

$$\text{HN SCFT} \;\to\; \mathcal{N}=(2,0) \text{ of type } \mathfrak{g} \;+\; \hat{d} \text{ hypers.} \tag{4.1.2}$$

We again divide the class in two subclasses: obvious or hidden.

1. **Obviously Higgsable to $\mathcal{N}=(2,0)$ (OHN) theories.** This class of theories was originally defined in terms of F-theory [33]; these theories have the endpoint which consists of only (-2)-curves. It is quite obvious that such theories indeed have the Higgs branch RG flow as (4.1.2). We denote the number of (-2)-curves at the endpoint by $\mathfrak{n}$ for this class of theories.

Then, the number $\mathfrak{n}$ is precisely the rank of the $\mathcal{N}=(2,0)$ theory in (4.1.2). Examples include

- The worldvolume theory of multiple M5-branes on $\mathbb{C}^2/\Gamma_G$, also known as the non-minimal conformal matter of type (G,G), with G a simply-laced group.
- The T-brane theories. These theories are obtained by the nilpotent Higgsing associated with a pair of the nilpotent orbits (μ_L,μ_R) of a sufficiently long chain of the conformal matter of type (G,G), where G is simply-laced.

2. **Hiddenly Higgsable to $\mathcal{N}=(2,0)$ (HHN) theories.** While the theory admits a RG flow (4.1.2), the endpoint does not consist of only (-2)-curves . The possible endpoints will be classified in the next section. Examples include

   - The theories describing multiple M5-branes probing a completely or partially frozen $SO(2k)$ or $E_{6,7,8}$ singularity (or their T-brane descendants).

The main purpose of this chapter is to give an F-theoretic characterization of hiddenly vH or HN theories by using the anomaly matching on the Higgs branch. As a byproduct, we will find the formula for the Higgs branch dimension for these theories.

4.2 Constraints for the very Higgsable

In this section, we obtain a necessary condition for a 6d SCFT to be an obviously or hiddenly very Higgsable. Since the diffeomorphism group remains unbroken during the flow (4.1.1), the gravitational anomalies can be matched in both sides.

We compute the gravitational anomaly of the 6d SCFT by moving onto its endpoint. The tensor branch flow from the vH SCFT to the endpoint is as follows:

$$\text{vH SCFT} \quad \to \quad \oplus_i \text{OvH}_i \ + \ n_V^{\text{end}} \text{ vectors} \ + \ n_H^{\text{end}} \text{ hypers} \tag{4.2.1}$$

where at the endpoint there are a collection of OvH theories (denoted as OvH_i), n_V^{end} vector multiplets and n_H^{end} tensor multiplets. The configuration of the tensor multiplets at the endpoint is encoded in an integral, symmetric and positive definite matrix η^{ij}_{end} with $i,j=1,\ldots,n_T^{\text{end}}$.

Then, the gravitational anomaly at the endpoint is

$$I^{\text{end}} = I^{\text{GS}} + n_T^{\text{end}} I^{\text{tensor}} + n_V^{\text{end}} I^{\text{vector}} + \sum_i I^{\text{OvH}_i}, \tag{4.2.2}$$

where each term is as follows:

- The Green-Schwarz contribution $I^{\rm GS}$ to the gravitational anomaly at the endpoint is

$$\frac{n_{\rm GS}^{\rm end}}{32}p_1(T)^2, \tag{4.2.3}$$

where

$$n_{\rm GS}^{\rm end} \equiv \sum_{i,j=1}^{n_T^{\rm end}} (\eta_{\rm end}^{-1})_{ij}(2-\eta_{\rm end}^{ii})(2-\eta_{\rm end}^{jj}) \ . \tag{4.2.4}$$

- The contribution of tensor/vector multiplet is given as

$$I^{\rm tensor} = \frac{23p_1(T)^2 - 116p_2(T)}{5760}, \qquad I^{\rm vector} = -\frac{7p_1(T)^2 - 4p_2(T)}{5760}. \tag{4.2.5}$$

- The contribution $I^{{\rm OvH}_i}$ is the gravitational anomalies of each ${\rm OvH}_i$ theory. As was proved in [32] by using the inductive method, it can be written as

$$I^{{\rm OvH}_i} = d_H^{{\rm OvH}_i}\frac{7p_1(T)^2 - 4p_2(T)}{5760} \tag{4.2.6}$$

by using some positive integer $d_H^{{\rm OvH}_i}$. More explicitly, the integer $d_H^{\rm end}$ is written as

$$d_H^{{\rm OvH}_i} = 29n_T^{{\rm OvH}_i} + n_H^{{\rm OvH}_i} - n_V^{{\rm OvH}_i} \tag{4.2.7}$$

where $n_{T,H,V}^{{\rm OvH}_i}$ is the number of tensor/hyper/vector multiplets of the ${\rm OvH}_i$ theory.

Substituting these contributions to (4.2.2), we obtain the full gravitational anomaly

$$\begin{aligned} &\frac{1}{5760}\left(180n_{\rm GS}^{\rm end} + 23n_T^{\rm end} - 7n_V^{\rm end} + 7\sum_i d_H^{{\rm OvH}_i}\right)p_1(T)^2 \\ &-\frac{1}{5760}\left(116n_T^{\rm end} - 4n_V^{\rm end} + 4\sum_i d_H^{{\rm OvH}_i}\right)p_2(T) \ . \end{aligned} \tag{4.2.8}$$

From the assumption that the theory is very Higgsable, the gravitational anomaly can also be written as

$$I^{\rm end} = d_H\frac{7p_1(T)^2 - 4p_2(T)}{5760} \ . \tag{4.2.9}$$

Here we denote the number of hypermultiplets in the right hand side of (4.1.1) as d_H. Then, d_H is nothing but the dimension of the Higgs branch of the vH theory in question at the origin of the tensor branch.

Matching the gravitational anomalies (4.2.8) and (4.2.9), we obtain two equations;

$$\begin{aligned} 180n_{\rm GS}^{\rm end} + 23n_T^{\rm end} - 7n_V^{\rm end} + 7\sum_i d_H^{{\rm OvH}_i} &= 7d_H, \\ 116n_T^{\rm end} - 4n_V^{\rm end} + 4\sum_i d_H^{{\rm OvH}_i} &= 4d_H. \end{aligned} \tag{4.2.10}$$

From the second equation, we have the dimension formula of the Higgs branch;

$$d_H = 29n_T^{\rm end} + \left(\sum_i d_H^{{\rm OvH}_i}\right) - n_V^{\rm end}\,. \tag{4.2.11}$$

This fourmula implies that the Higgs branch dimension d_H of the vH theory at the origin of the tensor branch can be computed using the endpoint data, including $n_T^{\rm end}$ and $n_V^{\rm end}$. The quantity $\sum_i d_H^{{\rm OvH}_i}$ should be viewed at the "effective" number of hypermultiplets at the endpoint.

By plugging (4.2.11) into (4.2.10), we obtain a nontrivial constraint for the possible tensor branch structure of the endpoint;

$$n_T^{\rm end} = n_{\rm GS}^{\rm end} = \sum_{i,j} (\eta_{\rm end}^{-1})_{ij}(2-\eta_{\rm end}^{ii})(2-\eta_{\rm end}^{jj})\,, \tag{4.2.12}$$

where we have used the definition of $n_{\rm GS}^{\rm end}$ in (4.2.4). Note that for an OvH theory, the constraint (4.2.12) is trivial since both sides are zero.

Remark. We can also obtain an equation similar to (4.2.11), but relating the Higgs branch dimension d_H at the origin to data of the field theory at a generic point on the tensor branch. We denote the number of the tensor multiplets, hypermultiplets and the vector multiplets at a generic point on the tensor branch as $n_{T,H,V}$.

Since the Green-Schwarz term does not contribute to the coefficient of $p_2(T)$, it can be matched between the d_H free hypermultiplets and the collection of free massless multiplets at a generic point on the tensor branch. Then, we obtain another Higgs branch dimension formula;

$$d_H = 29n_T + n_H - n_V. \tag{4.2.13}$$

4.2.1 Characterization of HvH theories in terms of F-theory

We examine the solutions to (4.2.12) in detail. We first present some examples and then give the complete list of solutions. The M-theory interpretation of the list will be presented in Section 4.4.

Theories with an endpoint of -2 curves. We first note that the constraint cannot be satisfied when the endpoint consists only of -2 curves, since the right hand side of (4.2.12) is zero, while the left hand side is positive. This prevents a number of 6d $\mathcal{N}=(1,0)$ theories from being very Higgsable. For example, we find that the worldvolume theories on multiple M5-branes on $\mathbb{C}^2/\Gamma_G$ cannot have the RG flow (4.1.1).

Theories with a single curve. Let us consider the theory engineered in F-theory by a single curve with self-intersection $-n$. Then, such a theory is very Higgsable only if n satisfies the constraint

$$\frac{1}{n}(n-2)^2 = 1, \tag{4.2.14}$$

whose solution is $n = 1, 4$.

The solution $n = 1$, with a generic elliptic fibration over the curve, is nothing but the rank-1 E-string theory, which is indeed known to be OvH. The transition (4.1.1) results in 29 free hypermultiplets.

The case of $n = 4$ is more interesting. If we assume that the elliptic fibration over the curve is the most generic one, the theory on the tensor branch consists only of a vector multiplet of $\mathrm{SO}(8)$. However, at the origin of the tensor branch, this theory has a one-dimensional Higgs branch! Indeed, the formula (4.1.1) says that $d_H = 1$ as $n_V^{\text{end}} = 28$, $n_T^{\text{end}} = 1$ and $d_H^{\text{OvH}_i} = 0$. We will present an M-theoretic interpretation of the Higgs branch later in Section 4.4.

Theories with two curves. Next we consider a theory of two curves with the self-intersection $-n$ and $-m$. The intersection matrix and its inverse are given by

$$\eta = \begin{pmatrix} m & -1 \\ -1 & n \end{pmatrix}, \quad \eta^{-1} = \frac{1}{mn-1}\begin{pmatrix} n & 1 \\ 1 & m \end{pmatrix}. \tag{4.2.15}$$

Then, the constraint (4.2.12) becomes

$$\frac{mn(m+n-6)+8}{mn-1} = 2, \tag{4.2.16}$$

whose solutions are $(m,n) = (1,2)$, $(1,5)$, $(2,5)$. The first two cases reduce to a single (-1) or (-4) curve after blowing down the (-1)-curve. The solution $(2,5)$ is more interesting: in Section 4.4, we will find vH theories with this endpoint.

General analysis. The constraint (4.2.12) quickly becomes complicated in the case of three or more curves. However, as explained in Section 4.1, the full list of possible endpoints of 6d SCFTs was already obtained in [6]. So we can simply check whether (4.2.12) is satisfied for each endpoints in that paper.

The full list of solutions is given as

$$4\,, \qquad 52\,, \qquad 352\,, \qquad 622\,, \qquad 7222\,, \qquad 82222\,. \tag{4.2.17}$$

The first two cases already appear in the examples above. We will give a M-theory interpretation to the list (4.2.17) in Section 4.4.

4.3 Constraints for the Higgsable to $\mathcal{N}=(2,0)$ theories

In this section, we obtain a necessary condition for a 6d SCFT to be an obviously or hiddenly Higgsable to $\mathcal{N}=(2,0)$ theories. In the following, for simplicity, we will use the notation

$$\mathfrak{n} = \text{rank } \mathfrak{g} \ . \tag{4.3.1}$$

Combining the Higgs branch RG flow in (4.1.2) with the flow of the $\mathcal{N}=(2,0)$ theory;

$$\text{N=(2,0) of type } \mathfrak{g} \ \rightarrow \ \mathfrak{n} \text{ copies of } (\text{N=(1,0) hyper} + \text{N=(1,0) tensor}), \tag{4.3.2}$$

we obtain the following RG flow for the HN theory in question,

$$\text{HN SCFT} \ \rightarrow \ d_H \text{ hypers} + \mathfrak{n} \text{ tensors} \ , \tag{4.3.3}$$

with

$$d_H = \hat{d} + \mathfrak{n} \ . \tag{4.3.4}$$

We will refer to d_H as the CFT Higgs branch dimension, since it can be interpreted as the dimension of the Higgs branch at the origin of the tensor branch.

Constraint for HN. By matching the coefficient $p_1(T)^2$ of the gravitational anomaly of the right hand side of (4.3.3) with the effective theory at the endpoint, we find that

$$n_T^{\text{end}} = n_{\text{GS}}^{\text{end}} + \mathfrak{n} = \sum_{i,j} (\eta_{\text{end}}^{-1})_{ij} (2 - \eta_{\text{end}}^{ii})(2 - \eta_{\text{end}}^{jj}) + \mathfrak{n} \, . \tag{4.3.5}$$

Then, a necessary condition for an SCFT to be HN is that $n_T^{\text{end}} - n_{\text{GS}}^{\text{end}}$ should be an integer. We also note that $n_{\text{GS}} - n_T$ is invariant under blowdown. So we have

$$n_{\text{GS}}^{\text{end}} - n_T^{\text{end}} = n_{\text{GS}} - n_T \ , \tag{4.3.6}$$

where now $n_{\text{GS}} = \sum_{i,j=1}^{n_T} \eta_{ij}^{-1}(2-\eta^{ii})(2-\eta^{jj})$. So the constraint (4.3.5) can be imposed at a generic point on the tensor branch, by requiring that $n_{\text{GS}} - n_T$ be an integer.

Higgs branch dimension formula. Just like in (4.2.13), we can match the coefficient of $p_2(T)$ in the gravitational anomaly of the right hand side of (4.3.3) with the effective theory on the generic point of the tensor branch:

$$d_H + 29\mathfrak{n} = 29 n_T + n_H - n_V \ , \tag{4.3.7}$$

where $n_{T,H,V}$ are the number of the tensor multiplets, hypermultiplets and the vector multiplets at a generic point on the tensor branch. This expression can be rewritten by using (4.3.5) and (4.3.6) as

$$\boxed{\dim_{\mathbb{H}}^{\text{CFT}}\ \text{Higgs} = n_H - n_V + 29 n_{\text{GS}}.} \tag{4.3.8}$$

Here and in the following, we will use $\dim_{\mathbb{H}}^{\text{CFT}}$ Higgs$(\mathcal{T}^{6d})$ to denote the Higgs branch dimension at the origin of the tensor branch of the 6d theory $\mathcal{T}^{6d}$.

Remark. There is another expression for the Higgs branch dimension. In fact, for an HN theory, there should be a tensor branch flow

$$\text{HN SCFT} \ \to \ \oplus_i \, \text{vH}_i + \mathfrak{n}\ \text{tensors} + \tilde{n}_V\ \text{vectors}, \tag{4.3.9}$$

where vH_i are collections of (obviously or hiddenly) vH theories. The matching of the coefficient $p_2(T)$ yields

$$d_H = \sum_i d_H^{\text{vH}_i} - \tilde{n}_V\ . \tag{4.3.10}$$

4.3.1 Characterization of HHN theories in terms of F-theory

We examine the solutions to (4.3.5) in detail. We first present some examples and then give the complete list of solutions. The M-theory interpretation of the list will be presented in Section 4.4.

OHN theory. For an OHN theory, the endpoint consists only of (-2)-curves and the number of these curves are equal to $\text{rank}(G) = \mathfrak{n}$. Therefore, $n_T^{\text{end}} = \mathfrak{n}$ and $\sum_{i,j} (\eta_{\text{end}}^{-1})_{ij} (2 - \eta_{\text{end}}^{ii})(2 - \eta_{\text{end}}^{jj}) = 0$ for such a theory. The constraint (4.3.5) is satisfied, as expected.

Non-minimal frozen conformal matters. In Section 4.4, we will see that the worldvolume theories on multiple M5-branes on a completely or partially frozen singularity are hiddenlly HN. Here we show an example; two M5s on a frozen D_4 singularity. A generic point of the tensor branch is represented as

$$[1] \ \overset{\mathfrak{so}(8)}{4} \ 1 \ \overset{\mathfrak{so}(8)}{4} \ [1]. \tag{4.3.11}$$

For this configuration, we can easily check that (4.3.5) is satisfied with $\mathfrak{n} = 1$. In fact, this theory has a Higgs branch flow to $\mathcal{N} = (2,0)$ theory of type $\mathfrak{su}(2)$.

General analysis. We can find the most general solution to (4.3.5) by going through the list of endpoints provided by [6]. This is just like we did for (4.2.12) at the end of Section 4.2.1.

The full list is provided by

$$32^{\mathfrak{n}-1}3\,,\qquad 42^{\mathfrak{n}-1}32\,,\qquad 332^{\mathfrak{n}-1}42\,,\qquad 52^{\mathfrak{n}-1}322\,,\qquad 62^{\mathfrak{n}-1}3222\,,\qquad 72^{\mathfrak{n}-1}32222\,, \tag{4.3.12}$$

where $2^{\mathfrak{n}-1} \equiv \underbrace{2\ldots2}_{\mathfrak{n}-1}$, and $\mathfrak{n}$ is nothing but the same number appearing in the constraint (4.3.5).

We should note that all $e(a)$ in Table 2.2 appears on one side of an element of the list (4.3.12), with a companion on the other side (which is itself for $e(a)$=3). It should be also noted that (4.2.17) can be obtained by setting $\mathfrak{n} = 0$ in (4.3.12) and applying the rule (2.5.17). We will give an M-theory interpretation to (4.3.12) in the next section.

4.4 Examples: conformal matters and related theories

4.4.1 Conformal matter theories

In this section, we will apply the results of the previous sections to the worldvolume theories of M5-branes on singularities. We start from quite simple cases, i.e. the 6d theory $\mathcal{T}_G(N-1)$ for simply laced G and their T-brane deformations.

Conformal matter theories. As we already remarked, these theories are OHN. In fact, after blowing down all (-1) curves, we have a linear sequence of (-2) curves; $n_{\rm GS} = 0$, and $\mathfrak{n} = N-1$. From (4.3.8), the Higgs branch dimension when all the M5s are coincident is

$$\dim_{\mathbb{H}}^{\rm CFT} \text{Higgs}(\mathcal{T}_G(\mathfrak{n})) = \mathfrak{n} + \dim(G) + 1\ . \tag{4.4.1}$$

This formula has an M-theoretic interpretation. Since each full M5 can be pulled away from the singularity, they contribute $N = \mathfrak{n} + 1$ to the formula (4.4.1). Note that fractional M5s cannot move away from the singularity. The extra piece $\dim(G)$ comes from the coupling to the 7d gauge field. In fact, if we pull away all the M5s from $\mathbb{C}^2/\Gamma_G$, the flavor symmetry is broken from $G_L \times G_R$ to $G_{\rm diag}$ and this should eat $\dim(G)$ hypermultiplets.

T-brane theories. All the T-brane theories obtained in [67] from conformal matter theories are also actually HHN. This fact can be seen from their realization in F-theory: they are obtained by taking a conformal matter theory $\mathcal{T}_G()N-1$ and adding a pole for the Hitchin field at the two outermost (-2) curves. Moreover, the reference [68] find a compact formula for the Higgs branch dimension by using (4.4.1), namely

$$\dim_{\mathbb{H}}^{\rm CFT} \text{Higgs}(\mathcal{T}_G(\{Y_{\rm L}, Y_{\rm R}\}, \mathfrak{n})) = \mathfrak{n} + \dim(G) + 1 - d_{Y_{\rm L}} - d_{Y_{\rm R}}\ . \tag{4.4.2}$$

4.4.2 (Partially) frozen conformal matter theories

Let us remember our definition of frozen conformal matter in Section 2.4.2; they are obtained by taking to infinity some of the outermost tensor multiplets in the tensor branch description of a conformal matter theory. Then, we will determine which frozen conformal matter theories are very Higgsable or Higgsable to $\mathcal{N}=(2,0)$.

vH frozen conformal matter theories. The possible hiddenly vH endpoints are listed in (4.2.17). From (2.5.16), (2.5.17), we find that all the allowed possibilities have an exactly $\mathfrak{f}(G)-1$ curves before going to the endpoint. Here $\mathfrak{f}$ is the number of fractions defined in (2.4.8).

For example, for $G = E_6$, these are

$$[1] \overset{\mathfrak{su}(3)}{3} \; 1 \; \overset{\mathfrak{e}_6}{6} \; [1] \,, \qquad [SU(3)] \; 1 \; \overset{\mathfrak{e}_6}{6} \; 1 \; [SU(3)] \,. \tag{4.4.3}$$

These are both obtained from the middle part of Figure 4.1 by keeping four neighboring M5 fractions in the middle and sending to infinity all the others fractions. The same pattern is found for any G. Note that the left and right flavor symmetries of the theory is same. Hence, we will denote these vH theories as $\mathcal{T}^{\text{fr}}_{G\to G_{\text{fr}}}(0)$, where G_{fr} is the flavor symmetry on either side. For example, the configurations in (4.4.3) are $\mathcal{T}^{\text{fr}}_{E_6\to\varnothing}(0)$ and $\mathcal{T}^{\text{fr}}_{E_6\to SU(3)}(0)$.

The M-theoretic interpretation of the fact that these theories are very Higgsable is that a transition such as the one in the lower part of Figure 4.1 can occur. Here, starting from the unfrozen G singularity, we make a full M5 from $\mathfrak{f}$ fractions which are not taken in the original order. One can take the full M5 thus formed off the singularity, leaving behind a partially or totally frozen singularity. In this sense, the theories $\mathcal{T}^{\text{fr}}_{G\to G_{\text{fr}}}(0)$ represent an M5 probing a partially frozen singularity.

To conclude, et us list ilustrative examples of theories $\mathcal{T}^{\text{fr}}_{G\to G_{\text{fr}}}(0)$;

- Non-simply-laced G_{fr}; looking at (2.4.7), we find that the non-simply-laced possibilities for G_{fr} in $\mathcal{T}^{\text{fr}}_{G\to G_{\text{fr}}}(0)$ are SO(7), USp($2k$), G_2, F_4. The F-theory quivers for these theories are

$$\begin{aligned}
[G_2]\text{——}[G_2]: &\quad [G_2] \overset{\mathfrak{su}_2}{2} \; 2 \; 1 \; \overset{\mathfrak{e}_8}{12} \; 1 \; 2 \; \overset{\mathfrak{su}_2}{2} \; \overset{\mathfrak{g}_2}{3} \; 1 \; \overset{\mathfrak{f}_4}{5} \; 1 \; [G_2] \,, \\
[F_4]\text{——}[F_4]: &\quad [F_4] \; 1 \; \overset{\mathfrak{g}_2}{3} \; \overset{\mathfrak{su}_2}{2} \; 2 \; 1 \; \overset{\mathfrak{e}_8}{12} \; 1 \; 2 \; \overset{\mathfrak{su}_2}{2} \; \overset{\mathfrak{g}_2}{3} \; 1 \; [F_4] \,, \\
[\text{Sp}(k)]\text{——}[\text{Sp}(k)]: &\quad [\text{Sp}(k)] \; \overset{\mathfrak{so}_{2k+8}}{4} \; [\text{Sp}(k)] \,.
\end{aligned} \tag{4.4.4}$$

  If we go to the endpoints of the theories in (4.4.4) by blowing down all (-1) curves, we can easily find that the result does not consist only of (-2) curves: we get 52 for the $G_{\text{fr}} = G_2$ case, and 4 for $G_{\text{fr}} = F_4$, $\text{Sp}(k)$. However, these configurations satisfy the constraint (4.2.12) for a theory to be hiddenly very-Higgsable.

- Complete freezing: $G_{\text{fr}} = \varnothing$. It is also possible to have $G_{\text{fr}} = \varnothing$. The F-theory quivers of all the possible cases are

$$\begin{aligned}
G = \text{SO}(8): &\qquad [1]\ \overset{\mathfrak{so}(8)}{4}\ [1]\,, \\
G = E_6: &\qquad [1]\ \overset{\mathfrak{su}(3)}{3}\ 1\ \overset{\mathfrak{e}_6}{6}\ [1]\,, \\
G = E_7: &\qquad [1]\ \overset{\mathfrak{su}(2)}{2}\ \overset{\mathfrak{so}(7)}{3}\ \overset{\mathfrak{su}(2)}{2}\ 1\ \overset{\mathfrak{e}_7}{8}\ [1]\,, \\
G = E_8: &\qquad [1]\ 2\ \overset{\mathfrak{su}(2)}{2}\ \overset{\mathfrak{g}_2}{3}\ 1\ \overset{\mathfrak{f}_4}{5}\ 1\ \overset{\mathfrak{g}_2}{3}\ \overset{\mathfrak{su}(2)}{2}\ 2\ 1\ \overset{\mathfrak{e}_8}{12}\ [1]\,,
\end{aligned} \tag{4.4.5}$$

and two more possibilities for the $G = E_8$ case to have $G_{\text{fr}} = \varnothing$:

$$\begin{aligned}
&[2]\ \overset{\mathfrak{su}(2)}{2}\ \overset{\mathfrak{g}_2}{3}\ 1\ \overset{\mathfrak{f}_4}{5}\ 1\ \overset{\mathfrak{g}_2}{3}\ \overset{\mathfrak{su}(2)}{2}\ 2\ 1\ \overset{\mathfrak{e}_8}{12}\ 1\ [2]\,, \\
&[1]\ \overset{\mathfrak{f}_4}{5}\ 1\ \overset{\mathfrak{g}_2}{3}\ \overset{\mathfrak{su}(2)}{2}\ 2\ 1\ \overset{\mathfrak{e}_8}{12}\ 1\ 2\ \overset{\mathfrak{su}(2)}{2}\ \overset{\mathfrak{g}_2}{3}\ [1]\,.
\end{aligned} \tag{4.4.6}$$

We will call the two in (4.4.6) as "exotically frozen"[1]. They correspond to the case $d_{E_8\to\varnothing} = 5$ in Table 2.4.10, while the case $G = E_8$ in (4.4.5) corresponds to $d_{E_8\to\varnothing} = 6$.

HN frozen conformal matter theories. Let us find all the HN frozen conformal matter theories. The possible endpoints are listed in (4.3.12). We find that all the HN frozen conformal matter theories are in fact chains of vH theories such as (4.4.3). In particular, the quiver at the generic point of the tensor branch consists of $N\mathfrak{f} - 1$ curves and represent $N\mathfrak{f}$ fractions. We will denote these theories as $\mathcal{T}^{\text{fr}}_{G\to G_{\text{fr}}}(N-1)$. They represent $N = \mathfrak{n} + 1$ M5-branes probing a partially frozen singularity.

By applying the Higgs branch dimension formula in each cases, we can see that the dimension is compactly expressed as

$$\dim^{\text{CFT}}_{\mathbb{H}} \text{Higgs}(\mathcal{T}_{G\to G_{\text{fr}}}(\mathfrak{n})) = \mathfrak{n} + \dim(G_{\text{fr}}) + 1\,. \tag{4.4.7}$$

Notice that this expression is just like (4.4.1), but with G replaced by G_{fr}. This is as expected from the M-theoretic construction; the $\mathfrak{n} + 1 = N$ summand in (4.4.1) represents the moduli of the full M5-branes, while the $\dim(G)$ summand comes from the coupling to the 7d G_{fr} gauge field.

As a rather simple example, we consider the chain which represents two full M5s on the partially frozen E_8 singularity;

$$[F_4]\ 1\ \overset{\mathfrak{g}_2}{3}\ \overset{\mathfrak{su}_2}{2}\ 2\ 1\ \overset{\mathfrak{e}_8}{12}\ 1\ 2\ \overset{\mathfrak{su}_2}{2}\ \overset{\mathfrak{g}_2}{3}\ 1\ \overset{\mathfrak{f}_4}{5}\ 1\ \overset{\mathfrak{g}_2}{3}\ \overset{\mathfrak{su}_2}{2}\ 2\ 1\ \overset{\mathfrak{e}_8}{12}\ 1\ 2\ \overset{\mathfrak{su}_2}{2}\ \overset{\mathfrak{g}_2}{3}\ 1\ [F_4]\,. \tag{4.4.8}$$

The endpoint is 33. This indeed appears in the list of possibilities (4.3.12) for $\mathfrak{n} = 1$. One can check this directly as well: $\eta^{\text{end}} = \begin{pmatrix} 3 & -1 \\ -1 & 3 \end{pmatrix}$, $\eta^{-1}_{\text{end}} = \frac{1}{8}\begin{pmatrix} 3 & 1 \\ 1 & 3 \end{pmatrix}$, $n_T^{\text{end}} = 2$; thus $\mathfrak{n} = 2 - 1 = 1$.

[1]The term *exotic* is used here because, as we have discussed in Section 7.3, the theories (4.4.6) have more complicated anomaly formulae than those in (4.4.5).

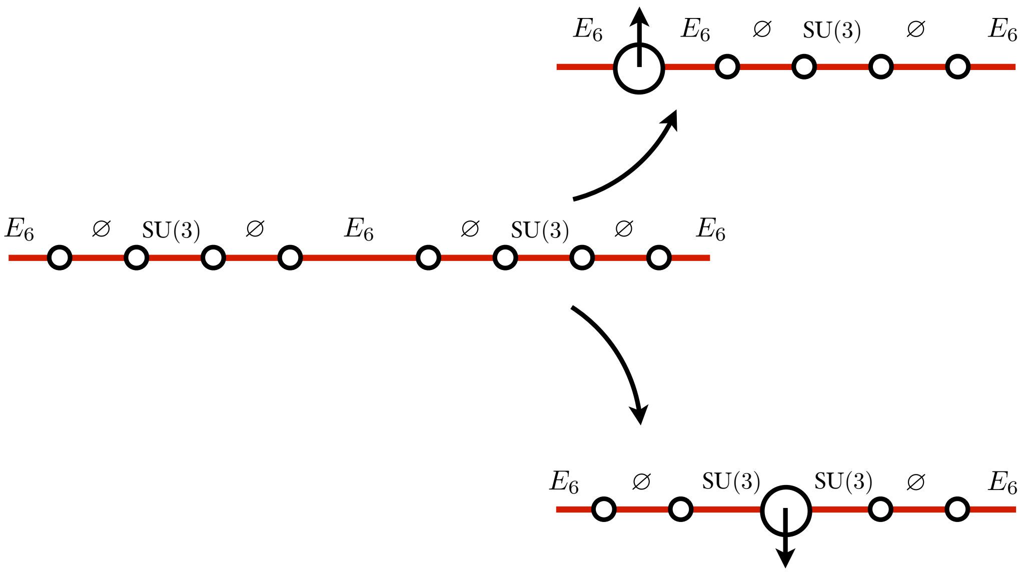

Figure 4.1: The central part of the picture represents fractional 2 M5-branes (dots) on a $\mathbb{R} \times \mathbb{R}^4/\Gamma_{E_6}$ singularity (red line). In this case each of the individual fractions is $1/4$ an ordinary M5. (To be precise, the M5 charges of the fractions are not the same. The fraction between E_6 and $\varnothing$ has charge $1/3$, while the one between $\varnothing$ and $SU(3)$ has charge $1/6$; see (2.4.11).) We also show the gauge groups (or lack thereof) on each segment between two fractional M5s. On the top part of the picture, we show a situation where the first four fractions have recombined into a full M5; the latter can now be pulled off the singularity. On the bottom part of the picture we see a different transition, where the fractions have come together in a different way.

Shorter OvH theories. We would like to comment on a few "shorter" OvH theories which we mentioned at the beginning of this chapter. They are incomplete conformal matter theories that do not consist of $\mathfrak{f}$ fractions. They are given as

$$\begin{aligned}
[E_6]\text{---}[\mathrm{SU}(3)] &: \quad [E_6]\ 1\ \ [\mathrm{SU}(3)]\ , \\
[E_7]\text{---}[\mathrm{SO}(7)] &: \quad [E_7]\ 1\ \overset{\mathfrak{su}(2)}{2}\ [\mathrm{SO}(7)]\ , \\
[E_8]\text{---}[G_2] &: \quad [E_8]\ 1\ 2\ \overset{\mathfrak{su}(2)}{2}\ [G_2]\ , \\
[E_8]\text{---}[F_4] &: \quad [E_8]\ 1\ 2\ \overset{\mathfrak{su}(2)}{2}\ \overset{\mathfrak{g}_2}{3}\ 1\ [F_4]\ .
\end{aligned} \tag{4.4.9}$$

These are all OvH theories, as one can easily check. The first describes $2 = \frac{1}{2}\mathfrak{f}(E_6)$ fractions, which is a "half M5" on an E_6 singularity. The second one describes $3 = \frac{1}{2}\mathfrak{f}(E_7)$ fractions, which is a "half M5" on an E_7 singularity. The third and fourth one describe one third and one half an M5 on top of the E_8 singularity.

The reason these theories are very Higgsable has nothing to do with the recombination of M5s shown in Figure 4.1. They seem to be related to a T-brane phenomenon: the Higgs branch flows activated by the nilpotent vevs of the Higgs field. In fact, for low numbers of M5s and for nilpotent elements with large enough orbit dimension of $G = \mathrm{SO}(2n), E_n$, this flow can sometime lead to the loss of all tensor multiplets.

T-brane theories. It is also possible to Higgs the theories $\mathcal{T}^{\mathrm{fr}}_{G\to G_{\mathrm{fr}}}(N-1)$ by two nilpotent elements $Y_{\mathrm{L,R}}$ in G_{fr}. We then obtain T-brane theories $\mathcal{T}^{\mathrm{fr}}_{G\to G_{\mathrm{fr}}}(\{Y_{\mathrm{L}}, Y_{\mathrm{R}}\}, N)$. These theories have not been fully worked out in general except for the cases $G_{\mathrm{fr}} = G_2$ and F_4 in [67]. It would be interesting to work out the properties of these theories.

4.4.3 Partial recombinations

In the previous sections, we only consider the Higgs moduli space at the origin of the tensor branch for several interesting theories. It would also be interesting to study about the Higgs moduli space on different non-generic loci of the tensor branch, where only some of the fractions are coinciding. In particular, will examine how the Higgs moduli space dimension formula (4.4.1) at the CFT point changes. In this subsection, we only consider the original unfrozen conformal matter theories $\mathcal{T}_G(N-1)$.

Full M5s not necessarily coincident. First of all, we can see that the dimension (4.4.1) is also valid at loci of the tensor branch where the M5 fractions coincide in groups of $\mathfrak{f}$, thus making N full M5s, but they need not be on top of each other.

In fact, if we have a sequence of CFT's connected by vectors and hypers, we expect d_H at that locus to be

$$\sum_i d_H(\mathrm{CFT}_i) + n_H - n_V \, . \tag{4.4.10}$$

Let us consider the non-generic locus of $\mathcal{T}_G(N-1)$ where N full M5s are formed but do not all coincide. The theory on this locus consists of N copies of $\mathcal{T}_G(0)$ connected by $(N-1)$ copies of a G gauge field. Thus (4.4.10) gives

$$(\dim(G)+1)N - \dim(G)(N-1) = \dim(G) + N \tag{4.4.11}$$

which coincides with (4.4.1) by using $N = \mathfrak{n}+1$. Thus the maximum Higgs moduli space dimension is already reached on this locus.

A simple interpretation of (4.3.7) inspired by the observation above is as follows. We rewrite it as in terms of the number of fractions $\mathfrak{f}$ (recall (2.4.8)) as $d_H = 29(\mathfrak{f}-1)\mathfrak{n} + n_H - n_V$. On a generic point of the tensor branch where all the fractions are separeted , the theory is weakly coupled and one expects simply $d_H = n_H - n_V$. Thus we can interpret that the Higgs moduli space increases every time one puts fractions on top of each other, up to a maximum when one makes $(\mathfrak{f}-1)\mathfrak{n}$ coincidences. This number of coincidences is already achieved when recombining the fractions in N full M5s, not necessarily making all the M5s coincident.

Partial coincidence of fractions. As a next step, we consider the Higgs moduli space at some loci of partial coincidence of fractions. For example, we take the locus where $2N$ half-M5s are put on the E_7 singularity;

$$[E_7]\text{---}(\mathrm{SO}(7))\text{---}(E_7)\cdots(E_7)\text{---}(\mathrm{SO}(7))\text{---}[E_7]\ . \tag{4.4.12}$$

As already mentioned, $[E_7]$—$[\mathrm{SO}(7)]$ is OvH and we use (4.3.8) to compute its Higgs branch dimension; $d_H = 8 - 3 + 2 \cdot 29 = 63$. Then, the formula (4.4.10) gives $(2N \times 63) - 21N - 133(N-1) = 133 - 28N = \dim(E_7) - 28N$. Comparing this with (4.4.1), we see that the Higgs moduli space dimension is smaller by $29N$. This is in accord with the interpretation in the last paragraph; the dimension increases by 29 whenever we coalesce two half M5s to a full M5.

Similarly, we can consider $[E_8]$—(F_4)—$(E_8)\cdots$—$[E_8]$, which is a sequence of half-M5s on the E_8 singularity. Here $[E_8]$—$[F_4]$ has $d_H = 136$, and (4.4.10) gives $(2N \times 136) - 52N - 248(N-1) = 248 - 28N = \dim(E_8) - 28N$. Again a dimension 29 is lost when splitting the M5s in half.

Chapter 5

One-instanton moduli space as Higgs branch of 6d SCFTs

5.1 Anomaly matching on Higgs branch

In the previous chapter, we focused on the gravitational part of the anomaly. For general theories, it is difficult to determine the R-symmetry and the flavor symmetry part of the anomaly from the anomaly matching on Higgs branch. In this chapter, however, we will point out that we can determine the full anomaly polynomial from the anomaly matching on Higgs branch for a special class of theories [30].

The theories we consider in this chapter are 6d $\mathcal{N}=(1,0)$ theories with a Higgs branch given by the one-instanton moduli space M_G of a group G. We will uniquely determine the anomaly polynomial of these theories and find that the most of the G is excluded by the consistency. We also consider 2d $\mathcal{N}=(0,4)$ theories whose Higgs branch is M_G, because these theories often appear on the string in 6d $\mathcal{N}=(1,0)$ theories [37].

In the rest of this section, we would like to explain more about the properties of M_G and how the anomaly matching on Higgs branch works in this class of theories.

Moduli space M_G. Let us review the geometric properties of M_G needed in the computation. The details can be found e.g. in [75]. The moduli space M_G, whose quaternionic dimension is $h^\vee_G - 1$, is smooth on a generic point, and the symmetry $\mathrm{SU}(2)_R \times G$ acting on M_G is broken to $\mathrm{SU}(2)_D \times G'$. Here, $\mathrm{SU}(2)_X$ is the $\mathrm{SU}(2)$ subgroup associated to the highest root of G and G' is its commutant within G. The $\mathrm{SU}(2)_D$ is the diagonal subgroup of $\mathrm{SU}(2)_R$ and $\mathrm{SU}(2)_X$.

In addition to the symmetry breaking, we can find the representation of the hypermultiplets, which correspond to the tangent space of M_G. Indeed, there is a neutral hypermultiplet and a charged half-hypermultiplet in a representation R under $\mathrm{SU}(2)_X \times G'$ at a generic point. The rule

G	$h^\vee$	G'	R'	short comment
$\mathrm{SU}(n)$	n	$\mathrm{U}(1)_F \times \mathrm{SU}(n-2)$	$(\mathbf{n-2})_{-n} \oplus (\overline{\mathbf{n-2}})_{-n}$	
$\mathrm{SO}(n)$	$n-2$	$\mathrm{SU}(2)_F \times \mathrm{SO}(n-4)$	$\mathbf{2}_F \otimes (\mathbf{n-4})$	
$\mathrm{Sp}(n)$	$n+1$	$\mathrm{Sp}(n-1)$	$\mathbf{2n-2}$	
E_6	12	$\mathrm{SU}(6)$	$\mathbf{20}$	3-index antisym.
E_7	18	$\mathrm{SO}(12)$	$\mathbf{32}$	chiral spinor.
E_8	30	E_7	$\mathbf{56}$	
F_4	9	$\mathrm{Sp}(3)$	$\mathbf{14'}$	3-index antisym. traceless.
G_2	4	$\mathrm{SU}(2)_F$	$\mathbf{4}$	3-index sym.

Table 5.1: The data. For $\mathrm{SU}(n)$, $\mathrm{U}(1)_F$ is normalized so that $\mathbf{n}$ splits as $(\mathbf{n-2})_{-2}$ and $\mathbf{2}_{n-2}$. For $\mathrm{SO}(n)$, n is assumed to be ≥ 5.

to determine R is

$$\mathfrak{g} = \mathfrak{g}' \oplus \mathfrak{su}(2) \oplus R. \tag{5.1.1}$$

Here R is always of the form of the doublet of $\mathrm{SU}(2)_X$ tensored with a representation R' of G'. The subgroup G' and the representation R' are listed in the Table 5.1.

Anomaly matching. With the help of the data explained above, we can full reconstruct the anomaly polynomial of 6d SCFT with Higgs branch M_G. We can focus on anomalies of the R-symmetry $\mathrm{SU}(2)_R$ and the flavor symmetry G that the supeconformal theory at the origin has.

When we move onto a generic point of the Higgs branch, we have free hypermultiplets whose unbroken symmetry is now $\mathrm{SU}(2)_D \times G'$. As mentioned earlier, $\mathrm{SU}(2)_D$ is the diagonal subgroup of $\mathrm{SU}(2)_R$ and $\mathrm{SU}(2)_X$. The number of hypermultiplets is given by $d_H = h^\vee - 1$ hypermultiplets whose details are as follows;

- One of them is neutral under the unbroken symmetry $\mathrm{SU}(2)_D \times G'$; indeed it is identified with changing the vev. By noting the fact that the scalars in a half-hyper are doublets of $SU(2)_R$, we can find that this hypermultiplet is in fact a half-hyper in the $\mathbf{2}$ of $SU(2)_X$.
- The remaining $d_H - 1$ hypers transform as a doublet of $SU(2)_X$ and in some representation R' of G' listed in Table 5.1. Since $SU(2)_X$ and $SU(2)_R$ are identified to be $\mathrm{SU}(2)_D$, this corresponds to have $d_H - 1$ free hypers in the representation R'.

Then, the anomaly matching on Higgs branch proceeds as follows;

1. The anomaly of G of the theory at the origin can be found from the anomaly of G' of the free hypermultiplets on the Higgs branch, as long as G' is nonempty.

2. This in turn determines the contribution of $\mathrm{SU}(2)_X$ to the anomaly of $\mathrm{SU}(2)_D$. Then, we can fix the anomaly of $\mathrm{SU}(2)_R$ of the original superconformal theory.

3. Even when G' is empty, we can still constrains the anomaly of $\mathrm{SU}(2)_R$ and G of the theory at the origin.

During the process above, we often find that the anomaly matching cannot be satisfied. If so, we can conclude that a superconformal theory with Higgs branch M_G does not exist.

5.2 Six-dimensional theories

In this section we use the method explained in Section 5.1 to which M_G can arise as a Higgs branch of 6d $\mathcal{N}=(1,0)$ SCFTs. We note that the argument in the following can only be applied under the assumption that the theories in question are superconformal.

We will find that the anomaly matching on Higgs branch can be done consistently only for

$$\mathrm{SU}(2), \quad \mathrm{SU}(3), \quad \mathrm{Sp}(n), \quad E_8, \quad \text{and} \quad G_2. \tag{5.2.1}$$

The features of each cases are summarized as follows;

- In the $\mathrm{SU}(2)$ case, we cannot completely determine the anomaly of the SCFT; we find a three-parameter family of solutions (5.2.10). When the parameters take special values, we can reproduce the anomaly of a free hypermultiplet gauged by $\mathbb{Z}_2$.
- In the $\mathrm{SU}(3)$ case, we can unambiguously determine the anomaly as in (5.2.13). But we do not know any example of 6d theories with these values of anomalies. Moreover, we either do not know the string theory construction of a 6d SCFT with Higgs branch $M_{\mathrm{SU}(3)}$.
- The $\mathrm{Sp}(n)$ case reproduces the anomaly polynomial of n free hypermultiplets gauged by $\mathbb{Z}_2$.
- The E_8 case reproduces the anomaly of the rank-1 E-string theory.
- The G_2 case is excluded by a more refined anomaly matching test using the global anomaly, as explained in Section 5.2.3

In the following subsections, we will show the detailed computations. The analysis is slightly different depending on whether G is classical or exceptional.

5.2.1 G is one of the exceptionals

We first consider the cases of exceptional groups. Since there is no independent quartic Casimir for exceptional groups, the anomaly polynomial at the origin can be written as

$$\begin{aligned} I_8^{\text{origin}} &= \alpha c_2(R)^2 + \beta c_2(R) p_1(T) + \gamma p_1(T)^2 + \delta p_2(T) \\ &\quad + \frac{1}{4} \operatorname{Tr} F_G^2 \left(\frac{\kappa}{4} \operatorname{Tr} F_G^2 + \lambda c_2(R) + \mu p_1(T) \right), \end{aligned} \tag{5.2.2}$$

where we have introduced the unknown coefficients $\alpha, \beta, \gamma, \delta, \kappa, \lambda, \mu$ to be determined below. On the generic point of the Higgs branch, using (D.2) we find that the anomaly polynomial (5.2.2) becomes

$$\begin{aligned} I_8^{\text{generic}} &= (\alpha + \kappa + \lambda) c_2(D)^2 + (\beta + \mu) c_2(D) p_1(T) + \gamma p_1(T)^2 + \delta p_2(T) \\ &\quad + \frac{m}{4} \operatorname{Tr} F_{G'}^2 \left(\frac{m\kappa}{4} \operatorname{Tr} F_{G'}^2 + (2\kappa + \lambda) c_2(D) + \mu p_1(T) \right). \end{aligned} \tag{5.2.3}$$

On the other hand, the anomaly of free hypers becomes

$$\begin{aligned} I_8^{\text{hypers}} &= \frac{1}{24} c_2(D)^2 + \frac{1}{48} c_2(D) p_1(T) + \frac{7(2 + d_{R'})}{11520} p_1(T)^2 - \frac{2 + d_{R'}}{2880} p_2(T) \\ &\quad + \frac{1}{48} \operatorname{tr}_{R'} F_{G'}^4 + \frac{T^{G'}(R')}{96} \operatorname{Tr} F_{G'}^2 p_1(T). \end{aligned} \tag{5.2.4}$$

when decomposing the characteristic classes of G into those of G'. In order to match (5.2.3) and (5.2.4), there should be no independent quartic Casimir invariant for G'.Therefore, we can excludes $G = E_6, E_7, F_4$ and we only consider the cases $G = G_2, E_8$.

When $G = E_8$. Since $\operatorname{tr}_{\mathbf{56}} F_{E_7}^4 = \frac{3}{2} (\operatorname{Tr} F_{E_7}^2)^2$ and $T^{E_7}(\mathbf{56}) = 6$, the anomaly (5.2.4) becomes

$$\begin{aligned} I_8^{\text{hyper}} &= \frac{1}{24} c_2(D)^2 + \frac{1}{48} c_2(D) p_1(T) + \frac{203}{5760} p_1(T)^2 - \frac{29}{1440} p_2(T) + \\ &\quad + \frac{1}{32} \left(\operatorname{Tr} F_{G'}^2 \right)^2 + \frac{1}{16} \operatorname{Tr} F_{G'}^2 p_1(T). \end{aligned} \tag{5.2.5}$$

Comparing (5.2.3) and (5.2.5), we can solve as

$$\alpha = \frac{13}{24},\ \beta = -\frac{11}{48},\ \gamma = \frac{203}{5760},\ \delta = -\frac{29}{1440},\ \kappa = \frac{1}{2},\ \lambda = -1,\ \mu = \frac{1}{4} \tag{5.2.6}$$

which coincides with the anomaly of rank-1 E-string theory determined in Section 3.3.2.

When $G = G_2$. Using $\mathrm{tr}_{\mathbf{4}} F^4_{\mathrm{SU}(2)} = \frac{41}{4}(\mathrm{Tr}\, F^2_{\mathrm{SU}(2)})^2$ and $T^{\mathrm{SU}(2)}(\mathbf{4}) = 5$, the anomaly (5.2.4) becomes

$$I_8^{\text{hypers}} = \frac{1}{24}c_2(D)^2 + \frac{1}{48}c_2(D)p_1(T) + \frac{7}{1920}p_1(T)^2 - \frac{1}{480}p_2(T) + \\ + \frac{41}{192}\left(\mathrm{Tr}\, F^2_{\mathrm{SU}(2)}\right)^2 + \frac{5}{96}\,\mathrm{Tr}\, F^2_{\mathrm{SU}(2)}p_1(T). \tag{5.2.7}$$

Comparing (5.2.3) and (5.2.7), we can solve the equations by

$$\alpha = \frac{91}{216},\ \beta = -\frac{7}{144},\ \gamma = \frac{7}{1920},\ \delta = -\frac{1}{480},\ \kappa = \frac{41}{108},\ \lambda = -\frac{41}{54},\ \mu = \frac{5}{72}. \tag{5.2.8}$$

When $G = \mathrm{SU}(2)$. Let us consider the $\mathrm{SU}(2)$ case, which is included in this subsection since the independent fourth order Casimir does not exist. In this case, the anomaly of the hypermultiplet is just

$$I_8^{\text{hypers}} = \frac{c_2(D)^2}{24} + \frac{c_2(D)p_1(T)}{48} + \frac{7p_1(T)^2 - 4p_2(T)}{5760}. \tag{5.2.9}$$

The anomaly polynomial of the SCFT can be written as (5.2.2) since there is no independent quartic Casimir. On a generic point of the Higgs branch $\mathrm{SU}(2)_G$ and $\mathrm{SU}(2)_R$ are identified as in (D.9). Matching the resulting anomaly polynomial with (5.2.9), we can solve the equations by

$$\begin{aligned} \alpha + \kappa + \lambda &= \frac{1}{24}, \\ \beta + \mu &= \frac{1}{48}, \\ \gamma &= \frac{7}{5760}, \delta = \frac{1}{1440}. \end{aligned} \tag{5.2.10}$$

In this case we cannot determine the anomaly polynomial completely. The only known 6d SCFT that have the Higgs branch $M_{\mathrm{SU}(2)}$ is the $\mathrm{O}(1)\times\mathrm{SU}(2)$ free hyper. The anomaly polynomial of this theory can be reproduced from (5.2.10) by setting $\alpha = \beta = \lambda = 0$.

When $G = \mathrm{SU}(3)$. The case of $\mathrm{SU}(3)$ is also included in this subsection since the fourth Casimir of $\mathrm{SU}(3)$ is zero and we can take the SCFT anomaly polynomial in the form of (5.2.2). Substituting the decomposition (D.10) to (5.2.2), we obtain

$$I_8^{\text{generic}} = (\alpha + \kappa + \lambda)c_2(D)^2 + (\beta + \mu)c_2(D)p_1(T) + \gamma p_1(T)^2 + \delta p_2(T) \\ - 3c_1(\mathrm{U}(1)_F)^2\Big((\lambda + 2\kappa)c_2(D) + \mu p_1(T) - 3\kappa c_1(\mathrm{U}(1)_F)^2\Big). \tag{5.2.11}$$

On the other hand, the anomaly polynomial of the hypermultiplets is given as

$$I_8^{\text{hypers}} = \frac{c_2(D)^2}{24} + \frac{c_2(D)p_1(T)}{48} \\ - \frac{3c_1(\mathrm{U}(1)_F)^2p_1(T)}{16} + \frac{27c_1(\mathrm{U}(1)_F)^4}{8} + 2\frac{7p_1(T)^2 - 4p_2(T)}{5760}. \tag{5.2.12}$$

Comparing these two equations, we can find a unique solution given by

$$\alpha = \frac{5}{12}, \beta = -\frac{1}{24},$$
$$\gamma = \frac{7}{2880}, \delta = -\frac{1}{720},$$
$$\kappa = \frac{3}{8}, \lambda = -\frac{3}{4}, \mu = \frac{1}{16}. \tag{5.2.13}$$

To our knowledge, a 6d SCFT with these values of anomalies and a SCFT with Higgs branch $M_{\mathrm{SU}(3)}$ are either not known.

5.2.2 G is one of classical groups

Let us consider the cases when $G = \mathrm{Sp}(n), \mathrm{SO}(2n), \mathrm{SU}(n)$. For these groups, there are two independent fourth order Casimir invariants, $\mathrm{tr}_{\text{fund}} F^4$ and $(\mathrm{Tr}\, F^2)^2$ to be considered.

When $G = \mathrm{Sp}(n)$. In this case, the anomaly of the free hypers is given by

$$I_8^{\text{hypers}} = \frac{1}{48}\left(\mathrm{tr}_{\text{fund}} F^4_{\mathrm{Sp}(n-1)} + 2c_2(D)^2\right) + \frac{(2c_2(D) + \mathrm{tr}_{\text{fund}} F^2_{\mathrm{Sp}(n-1)})p_1(T)}{96} + n\frac{7p_1(T)^2 - 4p_2(T)}{5760}. \tag{5.2.14}$$

Since the purely gravitational part of the anomaly can be reproduced from that of the free hypers, we focus on the R-symmetry and the flavor symmetry part. By using some unknown coefficients, they can be written as

$$I_8^{\text{origin}} = \alpha c_2(R)^2 + \beta c_2(R)p_1(T) + x\, \mathrm{tr}_{\text{fund}} F^4_{\mathrm{Sp}(n)} + y(\mathrm{tr}_{\text{fund}} F^2_{\mathrm{Sp}(n)})^2 + \mathrm{tr}_{\text{fund}} F^2_{\mathrm{Sp}(n)}\left(\kappa c_2(R) + \lambda p_1(T)\right). \tag{5.2.15}$$

Decomposing the characteristic classes for $\mathrm{Sp}(n)$ to those for $\mathrm{Sp}(n-1)$ using (D.3) and (D.4), the anomaly at the origin becomes,

$$\begin{aligned} I_8^{\text{generic}} = (\alpha + 2x + 4y + 2\kappa)c_2(D)^2 + (\beta + 2\lambda)c_2(D)p_1(T) \\ + x\, \mathrm{tr}_{\text{fund}} F^4_{\mathrm{Sp}(n-1)} + 4yc_2(D)\, \mathrm{tr}_{\text{fund}} F^2_{\mathrm{Sp}(n-1)} \\ + y\left(\mathrm{tr}_{\text{fund}} F^2_{\mathrm{Sp}(n-1)}\right)^2 + \mathrm{tr}_{\text{fund}} F^2_{\mathrm{Sp}(n-1)}\left(\kappa c_2(D) + \lambda p_1(T)\right). \end{aligned} \tag{5.2.16}$$

Comparing (5.2.14) and (5.2.16), we find

$$\alpha = 0,\ \beta = 0,\ x = \frac{1}{48},\ y = 0,\ \kappa = 0,\ \lambda = \frac{1}{96}. \tag{5.2.17}$$

The full anomaly polynomial at the origin thus computed precisely agrees with that of $\mathrm{O}(1) \times \mathrm{Sp}(n)$ half-hyper. In fact, This half-hypermultplet is the ADHM gauge theory for $\mathrm{Sp}(n)$.

When $G = \mathrm{SO}(n)$. In this case, the anomaly of the hypermultiplet is given by

$$I_8^{\text{hypers}} = \frac{(n-4)\,\mathrm{tr}_{\text{fund}} F_F^4 + 6\,\mathrm{tr}_{\text{fund}} F_F^2\,\mathrm{tr}_{\text{fund}} F_{\mathrm{SO}(n-4)}^2 + 2\,\mathrm{tr}_{\text{fund}} F_{\mathrm{SO}(n-4)}^4 + 2c_2(D)^2}{48}$$
$$+ \frac{((n-4)\,\mathrm{tr}_{\text{fund}} F_F^2 + 2\,\mathrm{tr}_{\text{fund}} F_{\mathrm{SO}(n-4)}^2 + 2c_2(D))p_1(T)}{96} + (n-3)\frac{7p_1(T)^2 - 4p_2(T)}{5760}. \tag{5.2.18}$$

Since the purely gravitational part already reproduces the anomaly at the origin, we concentrate on the part involving the R-symmetry and the flavor symmetry. We write the anomaly at the origin using some unknown coefficients as

$$I_8^{\text{origin}} = \alpha c_2(R)^2 + \beta c_2(R)p_1(T) + x\,\mathrm{tr}_{\text{fund}} F_{\mathrm{SO}(n)}^4 + y(\mathrm{tr}_{\text{fund}} F_{\mathrm{SO}(n)}^2)^2 + \mathrm{tr}_{\text{fund}} F_{\mathrm{SO}(n)}^2(\kappa c_2(R) + \lambda p_1(T)). \tag{5.2.19}$$

We can use equations (D.5) and (D.6) to decompose the anomaly (5.2.19) to

$$\begin{aligned} I_8^{\text{generic}} &= (\alpha + 4x + 16y + 4\kappa)c_2(R)^2 + (\beta + 4\lambda)c_2(D)p_1(T) + (x+4y)(\mathrm{tr}_{\text{fund}} F_F^2)^2 \qquad (5.2.20) \\ &+ (12x + 16y + 2\kappa)c_2(D)\,\mathrm{tr}_{\text{fund}} F_F^2 + x\,\mathrm{tr}_{\text{fund}} F_{\mathrm{SO}(n-4)}^4 + (16y+\kappa)c_2(D)\,\mathrm{tr}_{\text{fund}} F_{\mathrm{SO}(n-4)}^2 \\ &+ y(\mathrm{tr}_{\text{fund}} F_{\mathrm{SO}(n-4)}^2)^2 + 4y\,\mathrm{tr}_{\text{fund}} F_F^2\,\mathrm{tr}_{\text{fund}} F_{\mathrm{SO}(n-4)}^2 + \lambda p_1(T)(\mathrm{tr}_{\text{fund}} F_{\mathrm{SO}(n-4)}^2 + 2\,\mathrm{tr}_{\text{fund}} F_F^2). \end{aligned}$$

We try to match (5.2.18) and (5.2.21) and solve for $\alpha, \beta, x, y, \kappa, \lambda$. We find that the $\mathrm{SU}(2)_F$ independent terms can be matched by setting $\alpha = -\frac{1}{8}, \beta = -\frac{1}{16}, x = \frac{1}{24}, \lambda = \frac{1}{48}, y = \kappa = 0$. These values reproduce the anomalies of $\mathrm{SU}(2)$ gauge theory with n half-hypermultiplets, though it is anomalous in 6d. However it is not possible to match other $\mathrm{SU}(2)_F$ dependent terms. Therefore, there is no solution in this case.

When $G = \mathrm{SU}(n)$. We only need to consider $n \geq 4$. Then, the anomaly of the hypermultiplets is given by

$$\begin{aligned} I_8^{\text{hypers}} &= \frac{c_2(D)^2}{24} + \frac{c_2(D)p_1(T)}{48} - \frac{n^2(n-2)c_1(\mathrm{U}(1)_F)^2 p_1(T)}{48} + \frac{\mathrm{tr}_{\text{fund}} F_{\mathrm{SU}(n-2)}^2 p_1(T)}{48} \\ &- \frac{n^2 c_1(\mathrm{U}(1)_F)^2\,\mathrm{tr}_{\text{fund}} F_{\mathrm{SU}(n-2)}^2}{4} + \frac{n^4(n-2)c_1(\mathrm{U}(1)_F)^4}{24} + \frac{\mathrm{tr}_{\text{fund}} F_{\mathrm{SU}(n-2)}^4}{24} \\ &- \frac{nc_1(\mathrm{U}(1)_F)\,\mathrm{tr}_{\text{fund}} F_{\mathrm{SU}(n-2)}^3}{6} + (n-1)\frac{7p_1(T)^2 - 4p_2(T)}{5760}. \end{aligned} \tag{5.2.21}$$

We take the flavor and R-symmetry part of the anomaly to be written as (5.2.15) with the replacement $\mathrm{Sp}(n) \to \mathrm{SU}(n)$. Decomposing the characteristic classes of $\mathrm{SU}(n)$ to those for

$\mathrm{U}(1)_F \times \mathrm{SU}(n-2)$ by using (D.7) and (D.8), we find

$$\begin{aligned} I_8^{\text{generic}} &= (\alpha + 2x + 4y + 2\kappa)c_2(D)^2 + (\beta + 2\lambda)c_2(D)p_1(T) + x\,\mathrm{tr}_{\text{fund}}\,F^4_{\mathrm{SU}(n-2)} + y(\mathrm{tr}_{\text{fund}}\,F^2_{\mathrm{SU}(n-2)})^2 \\ &- 8xc_1(\mathrm{U}(1)_F)\,\mathrm{tr}_{\text{fund}}\,F^3_{\mathrm{SU}(n-2)} - 4(6x + n(n-2)y)c_1(\mathrm{U}(1)_F)^2\,\mathrm{tr}_{\text{fund}}\,F^2_{\mathrm{SU}(n-2)} \\ &- 2(n-2)(n\kappa + 4yn + 6(n-2)x)c_1(\mathrm{U}(1)_F)^2 c_2(D) + 2n(n-2)(x(n^2 - 6n + 12) \\ &+ 2yn(n-2))c_1(\mathrm{U}(1)_F)^4 + (\kappa + 4y)c_2(D)\,\mathrm{tr}_{\text{fund}}\,F^2_{\mathrm{SU}(n-2)} \\ &+ \lambda\,\mathrm{tr}_{\text{fund}}\,F^2_{\mathrm{SU}(n-2)}p_1(T) - 4n(n-2)\lambda c_1(\mathrm{U}(1)_F)^2 p_1(T). \end{aligned} \tag{5.2.22}$$

Matching two equations (5.2.21) and (5.2.22), we find that the $\mathrm{U}(1)_F$ independent terms can be matched by setting $\alpha = -x = -\frac{1}{24}, \beta = -\lambda = -\frac{1}{48}, y = \kappa = 0$. These values coincide the anomalies of $\mathrm{U}(1)$ gauge theory with n hypermultiplets, though it is anomalous in 6d. The $\mathrm{U}(1)_F$ dependent terms can be matched only if $n = 2$. However, we have assumed that $n \geq 4$. Then we again conclude that there is no solution in this case.

5.2.3 Excluding $G = G_2$ by global anomaly

In this short section, we exclude the case of $G = G_2$ by using the anomalies under large gauge transformations. Such a global anomaly exist only when the group has a non-trivial 6th homotopy group; $\pi_6(G) \neq 0$. It is known that this can only happen when

$$\begin{aligned} &G = \mathrm{SU}(2), \quad \pi_6(\mathrm{SU}(2)) = \mathbb{Z}_{12}, \\ &G = \mathrm{SU}(3), \quad \pi_6(\mathrm{SU}(3)) = \mathbb{Z}_6, \\ &G = G_2, \qquad \pi_6(G_2) = \mathbb{Z}_3. \end{aligned} \tag{5.2.23}$$

These anomalies[1] are mapped to each another by the embedding

$$\mathrm{SU}(2) \to \mathrm{SU}(3) \to G_2. \tag{5.2.24}$$

More explicitly, the relations between global anomalies can be described in terms of hypermultiplets as follows. Let us consider a hyper in the $\mathbf{7}$ of G_2, one in the $\mathbf{3}$ of $\mathrm{SU}(3)$, and a half-hyper in the $\mathbf{2}$ of $\mathrm{SU}(2)$. In fact, they contribute to the global anomaly as the generator of $\pi_6(G)$ for their respective groups [53]. Under the mapping (5.2.24), the $\mathbf{7}$ of G_2 decomposes to the $\mathbf{3} + \overline{\mathbf{3}}$ of $\mathrm{SU}(3)$ and further decomposes to the $2 \times \mathbf{2} +$ singlets of $\mathrm{SU}(2)$. Therefore the global anomaly is consistently mapped across each groups in (5.2.23).

[1]Note that these groups have no independent quartic Casimir invariant. If we embed the groups in (5.2.23) into another group with an independent fourth Casimir, the global anomaly is matched with the standard gauge anomaly $\mathrm{tr}\,F^4$.

Global anomaly matching on Higgs branch. Then let us examine whether $\mathrm{SU}(2), \mathrm{SU}(3)$ and G_2 cases are excluded by the global anomaly matching test on the Higgs branch. For $\mathrm{SU}(2)$ and $\mathrm{SU}(3)$, the global anomaly doesn't exist on the Higgs branch. This simply means that the global anomaly also vanishes even at the origin.

The situation for G_2 is more subtle. The putative SCFT at the origin possesses the G_2 global symmetry which is potentially globally anomalous. However, we will argue below that the G_2 global anomaly at the origin cannot be matched with the $\mathrm{SU}(2)_2$ anomaly at the Higgs branch, thus excluding the case of $G = G_2$.

First, let us analyze the G_2 anomaly at the origin. We consider a $\mathbf{7}$ of G_2, the generator of the global anomaly of G_2 in (5.2.23). Under the $\mathrm{SU}(2)_1 \times \mathrm{SU}(2)_2$ subgroup of G_2, it decomposes as $\mathbf{7} \to (\mathbf{2}, \mathbf{2}) \oplus (\mathbf{1}, \mathbf{3})$. Since the anomaly must be preserved under this decomposition and the $\mathbf{3}$ of $\mathrm{SU}(2)$ contribute to the anomaly just as 8 half-hyper doublets [53], we conclude that the G_2 anomaly is fully included in the $\mathrm{SU}(2)_1$ part, while $\mathrm{SU}(2)_2$ is non-anomalous. Therefore $\mathrm{SU}(2)_2$, which is the remaining global symmetry on a generic point on the Higgs branch, must be non-anomalous.

However on a generic point on the Higgs branch we have an half-hyper in the $\mathbf{4}$ of $\mathrm{SU}(2)_2$ which does contribute to the global anomaly. This can be easily checked by decomposing the $\mathbf{14}$ of G_2 under the $\mathrm{SU}(2)_1 \times \mathrm{SU}(2)_2$ subgroup. Thus we find that we cannot match the $\mathrm{SU}(2)_2$ anomaly on the Higgs branch with the anomaly of G_2 at the origin. Therefore the G_2 theory is inconsistent.

5.3 Two-dimensional theories

In this section we analyze the 2d $\mathcal{N}=(0,4)$ theories since these appear as the worldsheet theories on strings in 6d theories. In two dimensions, the scalars always fluctuate all over the moduli space, and the continuous symmetry never breaks. Therefore, it is imprecise to consider the theory at the origin of the moduli space and compare the anomaly between the origin and the generic point. Rather, what we do is to match the anomaly polynomial calculated using a semi-classical analysis at the generic point using the unbroken symmetry at that point, with the anomaly polynomial written in terms of the full symmetry.

We will find that the anomaly polynomials on the Higgs branch can consistently be matched only for

$$\mathrm{SU}(2), \quad \mathrm{SU}(3), \quad \mathrm{SO}(8), \quad \mathrm{Sp}(n), \quad E_{6,7,8}, \quad F_4, \quad \text{and} \quad G_2. \tag{5.3.1}$$

The data is summarized in Table 5.2, where n_v, d_H, k_G are the coefficients in the anomaly polynomial written as

$$I_4^{\text{full}} = -n_v c_2(R) + d_H c_2(I) + \frac{2d_H - n_f}{24} p_1(T) + \frac{k_G}{4} \operatorname{Tr}(F_G^2) \tag{5.3.2}$$

where $\mathrm{SU}(2)_R$ and $\mathrm{SU}(2)_I$ are the R-symmetries. Note that the $\mathrm{SU}(2)_I$ and the gravitational part of the anomaly can be matched directly on the Higgs branch. We also note that there are no global gauge anomalies in 2d since $\pi_2(G) = 0$ for all Lie groups.

G	n	n_v	d_H	k_G
$\mathrm{SU}(2)$		$x-1$	1	x
$\mathrm{Sp}(n)$		0	n	1
$\mathrm{SU}(3)$	3	2	2	3
$\mathrm{SO}(8)$	4	3	5	4
F_4	5	4	8	5
E_6	6	5	11	6
E_7	8	7	17	8
E_8	12	11	29	12
G_2		$\frac{7}{3}$	3	$\frac{10}{3}$

Table 5.2: The cases without Fermi multiplets in two dimensions. We explicitly show the value of self-Dirac-Zwazinger paring as n when the theory is realized on a single string in minimal 6d $\mathcal{N}=(1,0)$ theories.

In this section, we also consider the slightly generalized situation: we can also have Fermi multiplets in addition to hypermultiplets on a generic point of the Higgs branch. This is indeed the case for many 2d theories on the strings of 6d $\mathcal{N}=(1,0)$ theories. Here Fermi multiplet is nothing but a single left-moving Weyl fermion transforming some representation R_F under G. Therefore, we should also examine how the anomaly matching is affected when we allow the Fermi multiplets as the massless spectrum.[2]

5.3.1 G is of type Sp or one of the exceptionals

In this case, the unbroken subgroup G' is simple. If we denote the representations of Fermi multiplets under G' as $\sum_m \mathbf{N_m}$, then the anomaly polynomial of multiplets on Higgs branch is given as

$$I_4^{\text{free}} = c_2(D) + \frac{2+d_{R'}}{2}c_2(I) + \frac{2+d_{R'}-\sum_m N_m}{24}p_1(T) + \frac{2T^{G'}(R') - 2\sum_m T^{G'}(\mathbf{N_m})}{4}\,\mathrm{Tr}(F_{G'}^2). \tag{5.3.3}$$

[2]In this thesis, we only consider Fermi multiplets transforming non-trivially under the unbroken subgroup on Higgs branch G'. The only effect of neutral Fermi multiplets is to change the value of the gravitational anomaly.

On the other hand, by using (D.2) and (D.3), the anomaly (5.3.2) becomes

$$I_4^{\text{generic}} = (k_G - n_v)c_2(D) + d_H c_2(I) + \frac{2d_H - n_f}{24}p_1(T) + \frac{mk_G}{4}\operatorname{Tr}(F_{G'}^2), \tag{5.3.4}$$

where m is 3 for G_2 and 1 for other cases.

Without Fermi multiplets. If we assume that there are no Fermi multiplets, the anomalies (5.3.3) and (5.3.4) can be matched by the data summarized in Table 5.2.

The cases with $G = E_8, E_7, E_6, F_4$ reproduce the anomaly on a single self-dual string[3] in minimal 6d $\mathcal{N}=(1,0)$ theories for $n = 12, 8, 7, 5$:

$$I_4^{\text{string}} = -(n-1)c_2(R) + (3n-7)c_2(I) + \frac{3n-7}{12}p_1(T) + \frac{n}{4}\operatorname{Tr} F_G^2. \tag{5.3.5}$$

The case with $G = \mathrm{Sp}(n)$ reproduces the anomaly of $O(1) \times \mathrm{Sp}(n)$ half-hypers as expected. To the best of our knowledge, we do not know an example of 2d $\mathcal{N}=(0,4)$ SCFT with Higgs branch M_{G_2} and no Fermi multiplets.

With Fermi multiplets. Next we consider the cases with Fermi multiplets on the Higgs branch.

As examples, let us consider n_f fundamental Fermi multiplets of G'. For the $G = E_7$, the anomaly is written as

$$I_4^{\text{full}} = -(7-n_f)c_2(R) + 17c_2(I) + \frac{17-3n_f}{12}p_1(T) + \frac{8-n_f}{4}\operatorname{Tr}(F_{E_7}^2) + \frac{1}{4}\operatorname{Tr}(F_{\mathrm{SO}(n_f)}^2). \tag{5.3.6}$$

where we included the $\mathrm{SO}(n_f)$ symmetry acting on Fermi multiplets. This anomaly precisely agrees with that of a single string in 6d E_7 gauge theory with $n_f/2$ hypermultplets. Similarly, $G = E_6, F_4$ cases reproduce the anomaly of a single string in 6d $G = E_6, F_4$ gauge theory with n_f fundamental hypermultiplets.

Finally, we consider $G = G_2$. The anomaly can be matched by

$$I_4^{\text{full}} = -\frac{7-n_f}{3}c_2(R) + 3c_2(I) + \frac{3-n_f}{12}p_1(T) + \frac{10-n_f}{12}\operatorname{Tr}(F_{G_2}^2) + \frac{1}{4}\operatorname{Tr}(F_{\mathrm{SU}(n_f)}^2), \tag{5.3.7}$$

where we included the $\mathrm{SU}(n_f)$ flavor symmetry acting on the Fermi multiplets. For $n_f = 1, 4, 7$, (5.3.7) reproduces the anomaly of a string in the 6d G_2 gauge theory with $n_f = 1, 4, 7$ fundamental hypermultiplets.

[3] We have subtracted the anomaly of the center-of-mass mode from the result presented in [37].

5.3.2 G is of type SO

In this case, the unbroken group is $\mathrm{SU}(2)_F \times \mathrm{SO}(n-4)$. If we denote the representation of the Fermi multiplets by $\sum_m(\mathbf{n_m}, \mathbf{N_m})$, the anomaly of the free multiplets is written as

$$
\begin{aligned}
I_4^{\text{free}} &= c_2(D) + (n-3)c_2(I) + \frac{2n-6-\sum_m n_m N_m}{24} p_1(T) \\
&+ \frac{n-4-2\sum_m N_m T^{\mathrm{SU}(2)_F}(\mathbf{n_m})}{4} \mathrm{Tr}(F_F^2) + \frac{2-\sum_m n_m T^{\mathrm{SO}(n-4)}(\mathbf{N_m})}{2} \mathrm{Tr}(F_{\mathrm{SO}(n-4)}^2).
\end{aligned} \tag{5.3.8}
$$

On the other hand, by using (D.5), the anomaly (5.3.2) becomes

$$
I_4^{\text{generic}} = (k_G - n_v)c_2(D) + d_H c_2(I) + \frac{2d_H - n_F}{24} p_1(T) + \frac{k_G}{4} \mathrm{Tr}(F_F^2) + \frac{k_G}{4} \mathrm{Tr}(F_{\mathrm{SO}(n-4)}^2), \tag{5.3.9}
$$

Without Fermi multiplets. Comparing (5.3.8) and (5.3.9) in the case without Fermi multiplets, the anomaly can be determined as

$$
n_v = 3, \qquad d_H = n-3, \qquad k_G = 4. \tag{5.3.10}
$$

when we only consider the $\mathrm{SU}(2)_F$-independent part. This precisely agrees with the values of the anomalies of the $\mathrm{SU}(2)$ gauge theory with n half-hypers, though it is anomalous in 2d. If we would like to match the remaining $\mathrm{SU}(2)_F$-dependent part, the solution exists only for $G = \mathrm{SO}(8)$. The solution precisely agrees with the anomaly of (5.3.5) for $n = 4$. Indeed, the worldsheet theory on a single string in minimal 6d $\mathcal{N}=(1,0)$ SCFT for $n = 4$ has the Higgs branch $M_{\mathrm{SO}(8)}$ without Fermi multiplets.

With Fermi multiplets. Let us consider the cases with Fermi multiplets. The matching of $\mathrm{SU}(2)_F$ requires the following equation to be satisfied

$$
n - 4 - 2\sum_m N_m T^{\mathrm{SU}(2)_F}(\mathbf{n_m}) = 4 - 2\sum_m n_m T^{\mathrm{SO}(n-4)}(\mathbf{N_m}) \tag{5.3.11}
$$

An example of solution of it is obtained by setting $1 \leq m \leq n-8$, $\mathbf{N_m} = \mathbf{1}$ and $\mathbf{n_m} = \mathbf{2}$ for all m. Then, the anomaly polynomial is given by

$$
I_4^{\text{full}} = -3c_2(R) + (n-3)c_2(I) + \frac{5}{12} p_1(T) + \mathrm{Tr}(F_{\mathrm{SO}(n)}^2) + \frac{1}{4} \mathrm{Tr}(F_{\mathrm{Sp}(n-8)}^2), \tag{5.3.12}
$$

where we have included the global symmetry acting on $(n-8)$ free Fermi multiplets. This is precisely the anomaly of a single string in 6d $\mathrm{SO}(n)$ gauge theory with $(n-8)$ fundamental hypermultiplets.

5.3.3 G is of type SU

If we denote the representation of the Fermi multiplets as $\oplus_m(\mathbf{N_m})_{\mathbf{n_m}}$ under $\mathrm{SU}(n-2)\times \mathrm{U}(1)_F$, the anomaly of the multiplets on the Higgs branch is

$$I_4^{\text{free}} = c_2(D) + (n-1)c_2(I) + \frac{2n-2-\sum_m N_m}{24}p_1(T) + \frac{2-2\sum_m T^{\mathrm{SU}(n-2)}(\mathbf{N_m})}{4}\,\mathrm{Tr}(F^2_{\mathrm{SU}(n-2)})$$
$$- \left(n^2(n-2) - \frac{1}{2}\sum_m N_m n_m^2\right)c_1(\mathrm{U}(1)_F)^2. \qquad (5.3.13)$$

On the other hand, by using the decomposition (D.7), we have the anomaly

$$I_4^{\text{generic}} = (k_{\mathrm{SU}(n)} - n_v)c_2(D) + d_H c_2(I) + \frac{2d_H - \sum_m N_m}{24}p_1(T) + \frac{k_{\mathrm{SU}(n)}}{4}\,\mathrm{Tr}(F^2_{\mathrm{SU}(n-2)})$$
$$- k_{\mathrm{SU}(n)} n(n-2)c_1(\mathrm{U}(1)_F)^2. \qquad (5.3.14)$$

Without Fermi multiplets. Let us first consider the case $n \geq 4$. If we only consider the $\mathrm{U}(1)_F$-independent part, the matching problem between (5.3.13) and (5.3.14) can be solved by

$$n_v = 1, \qquad d_H = n-1, \qquad k_{\mathrm{SU}(n)} = 2 \qquad (5.3.15)$$

which precisely reproduce the anomalies the $\mathrm{U}(1)$ gauge theory with n hypermultiplets, though it is anomalous in 2d. The matching of $\mathrm{U}(1)_F$-dependent terms requires us to set $n = 2$, which contradicts with the original assumption $n \geq 4$.

When $G = \mathrm{SU}(2)$, the matching can be solved by

$$I_4^{\text{full}} = -(k_{\mathrm{SU}(2)} - 1)c_2(R) + c_2(I) + \frac{1}{12}p_1(T) + \frac{k_{\mathrm{SU}(2)}}{4}\,\mathrm{Tr}(F^2_{\mathrm{SU}(2)}), \qquad (5.3.16)$$

where $k_{\mathrm{SU}(2)}$ is an undetermined coefficient. If we set $k_{\mathrm{SU}(2)} = 1$, we reproduce the anomaly of a $\mathrm{O}(1)\times\mathrm{SU}(2)$ half-hyper.

When $G = \mathrm{SU}(3)$, the matching can be solved by

$$I_4^{\text{full}} = -2c_2(R) + 2c_2(I) + \frac{1}{6}p_1(T) + \frac{3}{4}\,\mathrm{Tr}(F^2_{\mathrm{SU}(3)}), \qquad (5.3.17)$$

which coincides with the anomaly (5.3.5) for $n = 3$. Indeed, the worldsheet theory of a single string in minimal 6d $\mathcal{N}=(1,0)$ SCFT for $n = 3$ is the SCFT with Higgs branch $M_{\mathrm{SU}(3)}$ and no Fermi multiplets.

With Fermi multiplets. Let us consider the case with Fermi multiplets for $n \geq 4$. For simplicity, we only consider two examples.

- When $\mathbf{N_{m\geq 1}} = \mathbf{1}$, the matching can be solved by

$$n_V = 1, \qquad d_H = n-1, \qquad k_{\mathrm{SU}(n)} = 2 \tag{5.3.18}$$

as long as the $\mathrm{U}(1)_F$ charges satisfy

$$2n(n-2) = n^2(n-2) - \frac{1}{2}\sum_m n_m^2. \tag{5.3.19}$$

An example of solutions of (5.3.19) is obtained by setting $1 \leq m \leq 2n$ and $n_m = (n-2)$ for all m. The full anomaly is written as

$$I_4^{\text{full}} = -c_2(R) + (n-1)c_2(I) - \frac{1}{12}p_1(T) + \frac{2}{4}\,\mathrm{Tr}(F^2_{\mathrm{SU}(n)}) + \frac{1}{4}\,\mathrm{Tr}(F^2_{\mathrm{SU}(2n)_F}), \tag{5.3.20}$$

where we have included the contribution of the flavor symmetry $\mathrm{SU}(2n)_F$, acting on the Fermi multiplets of the same $\mathrm{U}(1)_F$ charges. This is precisely the anomaly of a single string in 6d $\mathrm{SU}(n)$ gauge theory with $2n$ fundamental hypermultiplets.

- When $\mathbf{N_1} = (\mathbf{n}-\mathbf{2}), \mathbf{N_{m\geq 2}} = \mathbf{1}$, the matching can be solved by

$$n_v = 0, \qquad d_H = n-1, \qquad k_{\mathrm{SU}(n)} = 1 \tag{5.3.21}$$

as long as the $\mathrm{U}(1)_F$ charges satisfy

$$n(n-2) = n^2(n-2) - \frac{(n-2)n_1^2}{2} - \frac{1}{2}\sum_{m\geq 2} n_m^2. \tag{5.3.22}$$

An example of solutions of (5.3.22) is obtained by setting $1 \leq m \leq n+9$, $n_1 = (n-4)$ and $n_m = (n-2)$ for all $m \geq 2$. The total anomaly is written as

$$I_4^{\text{full}} = (n-1)c_2(I) - \frac{1}{3}p_1(T) + \frac{1}{4}\,\mathrm{Tr}(F^2_{\mathrm{SU}(n)}) + \frac{1}{4}\,\mathrm{Tr}(F^2_{\mathrm{SU}(n+8)_F}), \tag{5.3.23}$$

where we have included the global symmetry $\mathrm{SU}(n+8)$ acting on the Fermi multiplets of the same $\mathrm{U}(1)_F$ charge. This precisely agrees with the anomaly of a single string in 6d $\mathrm{SU}(n)$ gauge theory with $N_f = n+8$, $N_{\Lambda^2} = 1$ hypermultiplets.

Chapter 6

Anomaly of strings in 6d $\mathcal{N}=(1,0)$ theories

6.1 Inflow to the strings from 6d

The 6d theories on the tensor branch have a rich spectrum of 2d strings [76] For example, when the 6d theory is engineered by F-theory compactification with elliptic Calabi-Yau threefold, the defect strings come from D3 branes wrapping a curve on the base B_2. When the 6d theory is obtained by intersecting branes in type IIA superstring, they come from D2 branes suspended between NS5 branes.

At sufficiently low energy scale below the vev of the scalars in tensor multiplets, the string can be considered as a non-dynamical defect put on a definite position in the 6d spacetime. Just like D-branes in string theory, these defect strings have non-zero chiral anomalies localized on their worldsheets. In this chapter, we will determine the anomaly of the string defects in 6d theories on the tensor branch by using the inflow from 6d [37] (See also [34, 35]).

To use the anomaly inflow, we need to know the Chern-Simons like terms in the 6d effective Lagrangian. The relevant part of the effective action is given as[1]

$$2\pi \int \eta^{ij} \left(\frac{1}{2} dB_i \wedge \star dB_j + B_i \wedge I_j \right). \tag{6.1.1}$$

The strings in the 6d theory are charged under those 2-form fields and the charge matrix is given by η^{ij}. The Dirac-Zwazinger quantization implies that the matrix η^{ij} is symmetric, positive definite and integral. In the following, we sill raise/lower the indices by using η^{ij}.

The Green-Schwarz coupling $B_i \wedge I_j$ contributes to the anomaly polynomial of the 6d theory by $I^{\mathrm{GS}} = \frac{1}{2}\eta^{ij} I_i \wedge I_j$. As argued in Section 3.2, the 4-form I_i in (6.1.1) is given as

$$\eta^{ij} I_j = \frac{1}{4}\left(\eta^{ia} \operatorname{Tr} F_a^2 - (2 - \eta^{ii}) p_1(T) \right) + h^\vee_{G_i} c_2(I) \tag{6.1.2}$$

[1] Because B_i is self-dual, it is imprecise to write the kinetic term as in (6.1.1), but we will see that it is convenient to include it here for the inflow computation.

for all 6d theories known so far. Here we sum over the indices j and a, but the index in η^{ii} is not summed. The field strengths F_a include both dynamical and background gauge fields. The charge matrix η is extended to include both dynamical and background tensor multiplets.[2] $p_1(T)$ is the first Pontrjagin class of the tangent bundle of the 6d spacetime and $c_2(I)$ is the second Chern class of the background $\mathrm{SU}(2)_I$ R-symmetry bundle of the 6d $\mathcal{N}=(1,0)$ supersymmetry. When $G_i=\varnothing$, the dual Coxeter number $h^\vee_G$ is interpreted as 1, which occurs only when $\eta^{ii}=1$ or 2.

Symmetries. Let us consider the string with the charge vector Q_i put on $x_2=\cdots=x_5=0$ in 6d spacetime. The global symmetries of the 2d theory on the string is $\mathrm{SU}(2)_L\times\mathrm{SU}(2)_R\times\mathrm{SU}(2)_I\times\prod_a G_a$. Here $\mathrm{SU}(2)_L\times\mathrm{SU}(2)_R\simeq\mathrm{SO}(4)_N$ comes from the normal directions to the string. The $\mathrm{SU}(2)_I\times\prod_a G_a$ is the R, gauge and global symmetries of the bulk 6d theory.[3]

The decomposition of the 6d supercharges into the 2d supersymmetry along $x_{0,1}$ is

$$(\mathbf{2},\mathbf{1},\mathbf{2})_+ + (\mathbf{1},\mathbf{2},\mathbf{2})_-. \tag{6.1.3}$$

Here we denote the representations under $\mathrm{SU}(2)_L\times\mathrm{SU}(2)_R\times\mathrm{SU}(2)_I$, with the subscript $\pm$ for the 2d chirality. The string preserves only half of the original 6d supercharges; $(\mathbf{1},\mathbf{2},\mathbf{2})_-$. We note that the $\mathrm{SO}(4)$ R-symmetry of 2d $\mathcal{N}=(0,4)$ supersymmetry is identified with $\mathrm{SU}(2)_R\times\mathrm{SU}(2)_I$.

Main formula. Let us present the main formula in this chapter. The anomaly 4-form I_4 of the 2d $\mathcal{N}=(0,4)$ theory on the string is given as

$$I_4=\frac{\eta^{ij}Q_iQ_j}{2}\Big(c_2(L)-c_2(R)\Big)+\eta^{ij}Q_iI_j. \tag{6.1.4}$$

By using the concrete form of the 4-form I_j in (6.1.2), we can obtain a more concrete formula;

$$\begin{aligned}I_4=&\frac{\eta^{ij}Q_iQ_j}{2}\Big(c_2(L)-c_2(R)\Big)\\&+Q_i\left(\frac{1}{4}\eta^{ia}\,\mathrm{Tr}\,F_a^2-\frac{2-\eta^{ii}}{4}(p_1(T)-2c_2(L)-2c_2(R))+h^\vee_{G_i}c_2(I)\right).\end{aligned} \tag{6.1.5}$$

In (6.1.4) and (6.1.5), we decompose the 6d $p_1(T)$ as 2d $p_1(T)+p_1(N)$. We also used the relations $\chi(N)=c_2(L)-c_2(R)$ and $p_1(N)=-2c_2(L)-2c_2(R)$.

[2] The indices $a,b,\ldots$ are for both dynamical and background fields, while the indices $i,j,\ldots$ are only for dynamical ones.

[3] Here we should not confuse $\mathrm{SU}(2)_R$ with the R-symmetry of 6d supersymmetry.

Derivation by inflow. Here we will derive the formula (6.1.4) by performing the anomaly inflow computation for the self-dual string. The anomaly inflow of self-dual strings was studied in other references such as [77, 78, 79], mostly in the case of 6d $\mathcal{N}=(2,0)$ theory.

The modified Bianchi identity for the 3-form field strength under the presence of the string is written as

$$dH_i = I_i + Q_i \prod_{j=2}^{5} \delta(x_j)dx_j, \tag{6.1.6}$$

where Q_i is the charge of the string. We can solve this identity by

$$H_i = Q_i \frac{e_3^{(0)}}{2} + \text{(regular)}, \tag{6.1.7}$$

where $e_3^{(0)}$ is the global angular form of the S^3 bundle of the tubular neighborhood of the string. We note that it is related to the Euler class $\chi_4(N)$ of the normal bundle by $de_3^{(0)} = 2\chi_4(N)$.

We rewrite the relevant part of the 6d Lagrangian for the inflow computation as

$$2\pi \int_{Y_7} \eta^{ij} \big(\frac{1}{2} dH_i \wedge H_j + H_i \wedge I_j\big), \tag{6.1.8}$$

instead of (6.1.1). Here Y_7 is an auxiliary 7d manifold whose boundary is the physical 6d spacetime. We also extend the worldsheet of the string to Y_7 and denote it as M_3.

We will extract the most singular term in (6.1.8) in the presence of the string. It is given as

$$2\pi \int_{Y_7} \big(\frac{\eta^{ij}Q_iQ_j}{4} \chi_4(N) e_3^{(0)} + \eta^{ij} Q_i I_j \frac{e_3^{(0)}}{2}\big). \tag{6.1.9}$$

Anomaly inflow tells us that the anomaly on the string worldsheet is given by integrating out (6.1.9). Integration is straightforward since there is a factor of the Euler class $\chi_4(N)$ which reduces the integral over Y_7 to M_3. The result is

$$2\pi \int_{M_3} \big(\frac{\eta^{ij}Q_iQ_j}{4} e_3^{(0)} + \eta^{ij} Q_i I_{j,3}^{(0)}\big) \tag{6.1.10}$$

which is nothing but the formula (6.1.4).[4]

6.2 Comparison with quiver gauge theory description

When the 6d theory can be engineered by a brane web, it is not so hard to explicitly write the string worldsheet theory as a 2d $\mathcal{N}=(0,4)$ gauge theory. If so, we can easily compute its chiral

[4]The computation of the contribution from the kinetic term of (6.1.1) involves some hand-waving, due to the self-duality of the tensor fields. However, a more careful derivation, based on the method in [80], gives the same result.

anomalies by counting the number of multiplets. The purpose of this subsection is to provide further evidence for the formula (6.1.5) by checking with the multiplet counting.

To this end let us briefly recall the free multiplets of 2d $\mathcal{N}=(0,4)$ supersymmetry [81].

- Vector multiplet contains a gauge field and right moving fermions which is a doublet of $\mathrm{SU}(2)_R$ and $\mathrm{SU}(2)_I$.
- Hypermultiplet contains complex scalars which is a doublet of $\mathrm{SU}(2)_R$ and left moving fermions which is a doublet of $\mathrm{SU}(2)_I$.
- Twisted hypermultiplet contains complex scalars which is a doublet of $\mathrm{SU}(2)_I$ and left moving fermions which is a doublet of $\mathrm{SU}(2)_R$.
- Fermi multiplet contains a right moving fermion which is neutral under the R-symmetry.

Here we identified the R-symmetry of 2d $\mathcal{N}=(0,4)$ as $\mathrm{SU}(2)_R \times \mathrm{SU}(2)_I$. For the computation of the anomalies, we note the fact that a left moving complex Weyl fermion in 2d in the fundamental representation of $\mathrm{SU}(2)$ gives the anomaly $-\frac{1}{2}\operatorname{tr}_{\text{fund}} F^2 = -\frac{1}{4}\operatorname{Tr} F^2 = -c_2(\mathrm{SU}(2))$.

E-strings. The formula (6.1.5) states that the anomaly 4-form of the bound state of Q E-strings is given as

$$I_4^{\text{E-string}}(Q) = \frac{Q^2+Q}{2}c_2(L) - \frac{Q^2-Q}{2}c_2(R) - \frac{Q}{4}\operatorname{Tr} F_{E_8}^2 - \frac{Q}{4}p_1(T) + Qc_2(I). \tag{6.2.1}$$

Here we used the fact that η^{ia} in (6.1.2) for the E_8 global symmetry is -1.

On the other hand, when we engineer the E-string theory by an NS5 probing the 16 D8s with O8$^-$, the Q E-strings can be introduced by putting Q D2s connecting the NS5 and the D8-O8. The matter content of the gauge theory on the D2 was determined in [21]. It is summarized in Table 6.1. Since all the fermions in the Table are real, we have to multiply $1/2$ in the anomaly computation.

It is straightforward to check that the $\mathrm{SU}(2)_{L,R,I}$ anomaly matches with (6.2.1). The anomaly of the E_8 global symmetry also matches under the assumption that the $\mathrm{SO}(16)$ symmetry enhances to E_8 in the infrared of 2d gauge theory. In fact, the Fermi multiplet Ψ_+^l contributes to the anomaly by $\frac{1}{2}Q(-\frac{1}{2}\operatorname{tr}_{\text{fund}} F_{\mathrm{SO}(16)}^2) = -\frac{Q}{4}\operatorname{Tr} F_{\mathrm{SO}(16)}^2$. For the gravitational anomaly, the vector multiplet gives $-\frac{1}{24}(Q^2-Q)p_1(T)$, the hypermultiplet gives $+\frac{1}{24}(Q^2+Q)p_1(T)$ and the Fermi multiplet gives $-\frac{1}{3}Qp_1(T)$. if we sum the three contributions, we can reproduce the coefficient of $p_1(T)$ in (6.2.1).

	vector $(A_\mu, \lambda_+^{\dot\alpha A})$	hyper $(\phi_{\alpha\dot\alpha}, \lambda_-^{\alpha A})$	Fermi (Ψ_+^l)
$\mathrm{O}(Q)$	antisymmetric	symmetric	fund
$\mathrm{SU}(2)_L$	-	fund	-
$\mathrm{SU}(2)_R$	fund	-	-
$\mathrm{SU}(2)_I$	fund	fund	-
$\mathrm{SO}(16)$	-	-	fund

Table 6.1: The gauge theory on the worldsheet theory of Q E-strings. $\alpha, \dot\alpha = 1, 2$ are indices for $\mathrm{SU}(2)_{L,R}$, $A = 1, 2$ are for $\mathrm{SU}(2)_I$ and $l = 1, \ldots, 16$ are for $\mathrm{SO}(16)$ flavor symmetry. We explicitly write the representations of the fermions in the multiplets. $O(Q)$ is the gauge symmetry while the other symmetries are global. It should note that all the fermions listed above are real.

	vector $(A_\mu, \lambda_+^{\dot\alpha A})$	hyper $(\phi_{\alpha\dot\alpha}, \lambda_-^{\alpha A})$	hyper $(q_{\dot\alpha}, \psi_-^A)$	Fermi (ψ_+^F)
$\mathrm{U}(Q)$	adjoint	adjoint	fund	fund
$\mathrm{SU}(2)_L$	-	fund	-	-
$\mathrm{SU}(2)_R$	fund	-	-	-
$\mathrm{SU}(2)_I$	fund	fund	fund	-
$\mathrm{SU}(2)_F$	-	-	-	fund

Table 6.2: The gauge theory on the worldsheet of Q M-strings. Here the indices $\alpha, \dot\alpha, A$ are the same as in the E-string. $F = 1, 2$ is the fundamental indices of $\mathrm{SU}(2)_F$ symmetry. $\mathrm{U}(Q)$ is the gauge symmetry and the others are global. Again, the fermions λ are real.

M-strings. The anomaly of the bound states of Q M-string (i.e., the Q M2s suspended between two M5s) can be computed using our formula (6.1.5). It is given as

$$I_4^{\text{M-string}}(Q) = Q^2(c_2(L) - c_2(R)) + Q(c_2(I) - c_2(F)). \tag{6.2.2}$$

Here we decompose the $\mathrm{SO}(5)$ R-symmetry as $\mathrm{SU}(2)_F \times \mathrm{SU}(2)_I \subset \mathrm{SO}(5)$ and regard $\mathrm{SU}(2)_I$ as the R-symmetry of 6d $\mathcal{N}{=}(1,0)$ supersymmetry and $\mathrm{SU}(2)_F$ as the flavor symmetry. Note that the worldsheet theory on M-string possess both $\mathrm{SU}(2)$ symmetries, as can be seen in (6.2.2).

The matter contents of the gauge theory description of the 2d theory [18] is listed in Table 6.2. It is straightforward reproduces (6.2.2) from the counting of multiplets.

Instanton strings of $\mathrm{SO}(8)$**.** Let us consider the charge-Q string in $n = 4$ minimal 6d theory in Section 2.3.1. This is the instanton-string of $\mathrm{SO}(8)$ with instanton number Q. Then, the gauge theory description follows from the ADHM construction of $\mathrm{SO}(8)$ instantons, as explained in [22].

	vector $(A_\mu, \lambda_+^{\dot\alpha A})$	hyper $(\phi_{\alpha\dot\alpha}, \lambda_-^{\alpha A})$	hyper $(q_{\dot\alpha}, \psi_-^A)$
$\mathrm{Sp}(Q)$	symmetric	antisymmetric	fund
$\mathrm{SU}(2)_L$	-	fund	-
$\mathrm{SU}(2)_R$	fund	-	-
$\mathrm{SU}(2)_I$	fund	fund	fund
$\mathrm{SO}(8)$	-	-	fund

Table 6.3: The gauge theory on the charge Q string in $n=4$ minimal 6d $\mathcal{N}=(1,0)$ theory. $\mathrm{Sp}(Q)$ is the gauge symmetry and $\mathrm{SO}(8), \mathrm{SU}(2)_{L,R,I}$ are global symmetries. The indices $\alpha, \dot\alpha, A$ are the same as in the E-string case. Again the fermions λ are all real. The bifundamental hyper of $\mathrm{Sp}(Q)\times\mathrm{SO}(8)$ is in fact a half-hypermultiplet.

The matter content are summarized in Table 6.3.

On the other hand, our formula (6.1.5) state that the anomaly 4-form of the charge Q string in the $n=4$ minimal 6d $\mathcal{N}=(1,0)$ theory is given as

$$I_4^{n=4}(Q) = (2Q^2 - Q)c_2(L) - (2Q^2+Q)c_2(R) + Q\,\mathrm{Tr}\,F_{\mathrm{SO}(8)}^2 + \frac{Q}{2}p_1(T) + 6Qc_2(I). \quad (6.2.3)$$

We can check that the matter contents in Table 6.3 indeed match with the anomaly (6.2.3).

String chains. We can also consider more complicated bound states of the strings in 6d theories with several tensor multiplets. The reference [23] studied the strings in 6d $\mathcal{N}=(1,0)$ theories on the multiple M5-branes probing the ALE singularity or the E_8 wall. The 2d $\mathcal{N}=(0,4)$ gauge theory on the string can be read off from the IIA realization of the 6d theory. The basic feature of the 2d theories is that they are linear quiver gauge theories whose ranks of gauge groups are determined by the string charge vector $\{Q_i\}_{i=1}^N$. Since the precise matter content and gauge groups depend on which 6d theory we consider, we do not try to write them down here.

For these 2d theories, we can also check the agreement of the formula (6.1.5) with the multiplet counting. As a simple example, we compute the $\mathrm{SU}(2)_I$ anomaly of the charge-$\{Q_i\}_{i=1}^N$ string in $\mathcal{T}_{\mathrm{SU}(k)}(N-1)$ [3] in two ways. From the formula (6.1.5), the coefficient of $c_2(I)$ is equal to $k\sum_{i=1}^N Q_i$. On the other hand, the multiplets in the gauge theory description which have a non-trivial $\mathrm{SU}(2)_I$ anomaly are $\mathrm{U}(Q_i)$ vectors, $\mathrm{U}(Q_i)$-adjoint hypers and $\mathrm{U}(Q_i)\times\mathrm{SU}(k)_i$-bifundamental hypers [23]. Since the anomalies from vectors and adjoint hypers cancel each other, the total anomaly comes from only bifundamental hypers and is given by $(k\sum_{i=1}^N Q_i)c_2(I)$, as expected.

6.3 Some applications of anomalies of strings

In this section, we will present some applications of the anomaly formula (6.1.5).

ADE classification of 6d $\mathcal{N}=(2,0)$ theories. We will derive the ADE classification of 6d $\mathcal{N}=(2,0)$ theories from the viewpoint of the string, slightly extending the argument in [77]. Let us consider an $\mathcal{N}=(2,0)$ theory with r tensor multiplets. On a generic point on the tensor branch, there are strings charged under the tensor multiplets. Given two strings with charges $\vec{Q}$ and $\vec{Q}'$ respectively, we denote the Dirac pairing as

$$\langle \vec{Q}, \vec{Q}' \rangle = \eta^{ij} Q_i Q'_j. \tag{6.3.1}$$

The Dirac-Zwazinger quantization law implies $\langle \vec{Q}, \vec{Q}' \rangle \in \mathbb{Z}$.

Let us consider a single string with charge $\vec{Q}$. Since the string breaks translational invariance and its fermionic counterpart, there are four bosonic zero-modes and eight chiral Majorana fermionic zero-modes on the string. They comprise a hypermultiplet of the 2d $\mathcal{N}=(4,4)$ supersymmetry., whose fermionic components transform as $(\mathbf{2},\mathbf{1},\mathbf{2})_- + (\mathbf{1},\mathbf{2},\mathbf{2})_+$ under $\mathrm{SU}(2)_L \times \mathrm{SU}(2)_R \times \mathrm{SU}(2)_I$, with the subscript $\pm$ denoting the 2d chirality. The anomaly polynomial of the hypermultiplet is

$$I_4^{\text{zero modes}} = (c_2(L) + \frac{1}{12}p_1(T)) - (c_2(R) + \frac{1}{12}p_1(T)) = c_2(L) - c_2(R). \tag{6.3.2}$$

The crucial assumption in [77] was that the worldsheet theory is given purely by these zero modes. Then the anomaly (6.3.2) needs to be reproduced from the anomaly inflow. Since we do not know the 6d Green-Schwarz coupling at this point, we just assume a generic one

$$dH_i = c_i p_1(T). \tag{6.3.3}$$

Here we neglected contributions from the 6d R-symmetry $c_2(I)$ since they do not affect the 2d anomaly terms which will be important in the following.

Using the inflow formula (6.1.4), we obtain

$$I_4^{\text{worldsheet}} = \frac{\langle \vec{Q}, \vec{Q} \rangle}{2}(c_2(L) - c_2(R)) + \langle \vec{Q}, \vec{c} \rangle (p_1(T) - 2c_2(L) - 2c_2(R)). \tag{6.3.4}$$

Comparing (6.3.2) and (6.3.4), we obtain

$$\langle \vec{Q}, \vec{Q} \rangle = 2, \tag{6.3.5}$$

and $c_i = 0$ in (6.3.3).

We have found that the charge lattice of strings of a 6d $\mathcal{N}=(2,0)$ theory is an integral lattice generated by vectors whose length squared is two. Since this condition is equivalent to the fact that the charge lattice is a simply-laced root lattice, we have an ADE classification of the 6d $\mathcal{N}=(2,0)$ theory. We also obtained $c_i = 0$ in (6.3.3), which agrees with a different computation in [28].

Existence of E_8 flavor symmetry of the smallest 6d $\mathcal{N}=(1,0)$ theory. Let us consider an $\mathcal{N}=(1,0)$ theory with one tensor multiplet whose Dirac pairing is given by

$$\eta = 1, \tag{6.3.6}$$

and whose Green-Schwarz term[5] is given as

$$dH = I = \frac{\eta - 2}{4} p_1(T). \tag{6.3.7}$$

We further assume that there is no dynamical gauge field on the tensor branch. We conjecture that the 6d theory introduced above is in fact the E-string theory. If so, this smallest theory somehow has the E_8 flavor symmetry.

To see how the symmetry arises automatically, let us consider a string of charge $Q = 1$. By substituting (6.3.6) and (6.3.7) to the inflow formula (6.1.4), we find the anomaly on the string

$$\begin{aligned} I_4^{\text{inflow}} &= \frac{1}{2}(c_2(L) - c_2(R)) - \frac{1}{4}(p_1(T) - 2c_2(L) - 2c_2(R)) + c_2(I) \\ &= c_2(L) + c_2(I) - \frac{1}{4}p_1(T). \end{aligned} \tag{6.3.8}$$

On the string worldsheet, there are bosonic and fermionic zero modes coming from the breaking of the bulk translational symmetry and its fermionic partner. They form a hypermultiplet of 2d $\mathcal{N}=(0,4)$ supersymmetry, and have the anomaly polynomial

$$I_4^{\text{zero modes}} = c_2(L) + c_2(I) + \frac{1}{12}p_1(T). \tag{6.3.9}$$

Comparing (6.3.8) and (6.3.9), we find that there must be some additional degrees of freedom on the worldsheet since there is a mismatch in the gravitational anomaly by $-\frac{1}{3}p_1(T)$. The simplest possibility to account for the difference is to add a chiral CFT on the left-moving side of the string worldsheet, with $c = 8$.

Assuming that the partition function of the 2d theory on the string is well-defined up to a phase, this additional chiral CFT with $c = 8$ must be the E_8 current algebra of level one. Although this argument is not a complete derivation of the E_8 flavor symmetry, it does at least indicate that the E_8 symmetry needs to arise automatically.

World-sheet structure of strings of minimal 6d $\mathcal{N}=(1,0)$ theories. Let us consider the charge-Q string of minimal 6d $\mathcal{N}=(1,0)$ theories reviewed in Section 2.3.1. We can find the anomaly on

[5] For 6d $\mathcal{N}=(1,0)$ theories constructed from F-theory, this relation follows from the geometric consideration [50]. It would be interesting to find a purely field theoretical derivation of (6.3.7)

the string as

$$I_4(n,Q) = \frac{nQ^2-(n-2)Q}{2}c_2(L) - \frac{nQ^2+(n-2)Q}{2}c_2(R) + \frac{nQ}{4}\operatorname{Tr}F_G^2 + \frac{(n-2)Q}{4}p_1(T) + Qh_G^\vee c_2(I) \quad (6.3.10)$$

by applying the inflow formula (6.1.5).

An instanton configuration of $\mathfrak{g}$ vector multiplet is charged under the tensor field and its charge Q is identified with the instanton number. Therefore, the 2d world-sheet theory with $\mathcal{N}=(0,4)$ supersymmetry on the strings of the minimal 6d $\mathcal{N}=(1,0)$ theory at least have a Higgs branch which is the instanton moduli space of gauge group $\mathfrak{g}$ of instanton number Q, whose quaternionic dimension is $h^\vee Q$. Indeed, for $n=4$ and $\mathfrak{g}=\mathfrak{so}(8)$, the worldsheet gauge theory is known to be the ADHM construction of charge-Q $\mathrm{so}(8)$ instanton.

Now, let us move onto the Higgs branch. Since it does not break the $\mathrm{SU}(2)_I$ and the diffeomorphism symmetry, the $p_1(T)$ and $c_2(I)$ terms of the anomaly can be computed straightforwardly at the generic point on the Higgs branch. There, the string is a finite-sized instanton-string on which there are $4h^\vee Q$ bosonic and chiral fermionic zero-modes. Therefore, the anomaly should contain the terms

$$Qh_G^\vee(\frac{1}{12}p_1(T) + c_2(I)). \quad (6.3.11)$$

Comparing with (6.3.10), we need to have

$$h_G^\vee = 3(n-2). \quad (6.3.12)$$

This explains the numerology (2.3.4) pointed out in Section 2.3.1.

Chapter 7

Anomaly polynomials of frozen conformal matters

7.1 Anomaly polynomials of unfrozen conformal matters

Let us start by reviewing the computation of anomalies of the 6d theory $\mathcal{T}_G(Q-1)$; the theory on Q M5-branes probing the ALE singularities $\mathbb{C}^2/\Gamma_G$. We sill use the method presented in Section 3.1.2. We note that we will ignore the $\mathrm{U}(1)$ part of the anomaly for simplicity. This $\mathrm{U}(1)$ exists as acting on $\mathbb{C}^2/\Gamma$ when Γ is of type A_k.

The matter content of the theory $\mathcal{T}_G(Q-1)$ at the generic point of the tensor branch is already presented in Section 2.4.1 and it is straightforward to use the method in Section 3.1.2. However, it is easier to divide the computation into two steps; the first is to compute the anomalies for $Q=1$.The second is to compute the anomalies for general Q theory.

Minimal conformal matter $\mathcal{T}_G(0)$. Even when $Q=1$, to compute the anomalies it is more convenient to make some fractional M5-branes coincident, rather than staying on the generic point on the tensor branch. Specifically, if there is no nontrivial gauge groups on a segment between two fractions, we make them coalesce. Then the theory consists of an equal number of tensor and vector multiplets, coupled to hypermultiplets and/or E-string theories of rank 1 and 2. Then we apply the method in Section 3.1.2.

As an illustration, we consider the single M5-brane on the E_6 singularity and compute the anomaly. In order to make the numbers of gauge groups and tensor multiplets equal, we make pairs of fractional M5-branes to coalesce. The anomalies of two rank-1 E-string theories and a

SU(3) vector multiplet plus one $\mathcal{N}=(1,0)$ tensor multiplet is given as

$$\begin{aligned}
&I^{\text{one-loop}}\\
=&I^{\text{rank 1}}_{\text{E-string}}(\mathrm{Tr}\,F_L^2+\mathrm{Tr}\,F_{\mathrm{SU}(3)}^2)+I^{\text{vec}}_{\mathrm{SU}(3)}(\mathrm{Tr}\,F^2_{\mathrm{SU}(3)})+I^{\text{tensor}}+I^{\text{rank 1}}_{\text{E-string}}(\mathrm{Tr}\,F^2_{\mathrm{SU}(3)}+\mathrm{Tr}\,F_R^2)\\
=&\frac{1}{32}(\mathrm{Tr}\,F_L^2)^2+\frac{1}{32}(\mathrm{Tr}\,F_R^2)^2+\left(\mathrm{Tr}\,F_L^2+\mathrm{Tr}\,F_R^2\right)\left(\frac{1}{16}p_1(T)-\frac{1}{4}c_2(R)\right)\\
&+\frac{19}{24}c_2^2(R)-\frac{29}{48}c_2(R)p_1(T)+\frac{373}{5760}p_1^2(T)-\frac{79}{1440}p_2(T)\\
&-\frac{1}{32}(\mathrm{Tr}\,F^2_{\mathrm{SU}(3)})^2+\mathrm{Tr}\,F^2_{\mathrm{SU}(3)}\left(-\frac{5}{4}c_2(R)+\frac{1}{16}p_1(T)+\frac{1}{16}\mathrm{Tr}\,F_L^2+\frac{1}{16}\mathrm{Tr}\,F_R^2\right),
\end{aligned}$$

where F_L, $F_{\mathrm{SU}(3)}$ and F_R are background field strength of E_6^L, SU(3) and E_6^R, respectively. The anomaly of the rank-1 E-string $I^{\text{rank 1}}_{\text{E-string}}$ is taken from (3.1.13), subtracting the contribution from a free hypermultiplet. Although there is no concept of loop computations in the E-string theory, we will call these contributions as the 'one-loop' contribution by the abuse of terminology.

The Green-Schwarz term which cancels the SU(3) part of the anomalies is given as

$$I^{\text{GS}}=\frac{1}{2}\left(\frac{1}{4}\mathrm{Tr}\,F^2_{\mathrm{SU}(3)}+5c_2(R)-\frac{1}{4}p_1(T)-\frac{1}{4}\mathrm{Tr}\,F_L^2-\frac{1}{4}\mathrm{Tr}\,F_R^2\right)^2. \qquad (7.1.1)$$

Therefore, the total anomalies is

$$\begin{aligned}
I^{\text{bif}}_{E_6,E_6}(F_L,F_R)=I^{\text{one-loop}}+I^{\text{GS}}=&\frac{1}{16}(\mathrm{Tr}\,F_L^2)^2+\frac{1}{16}\mathrm{Tr}\,F_L^2\,\mathrm{Tr}\,F_R^2\\
&+\frac{1}{16}(\mathrm{Tr}\,F_R^2)^2+\left(\mathrm{Tr}\,F_L^2+\mathrm{Tr}\,F_R^2\right)\left(\frac{1}{8}p_1(T)-\frac{3}{2}c_2(R)\right)\\
&+\frac{319}{24}c_2^2(R)-\frac{89}{48}c_2(R)p_1(T)+\frac{553}{5760}p_1^2(T)-\frac{79}{1440}p_2(T). \qquad (7.1.2)
\end{aligned}$$

By applying the same procedure for the E_6 case to all G, where G is a simply-laced group, we obtain the following anomaly polynomial

$$\begin{aligned}
I^{\text{bif}}_{G,G}(F_L,F_R)=&\frac{\alpha}{24}c_2(R)^2-\frac{\beta}{48}c_2(R)p_1(T)+\gamma\frac{7p_1(T)^2-4p_2(T)}{5760}\\
&+\left(-\frac{x}{8}c_2(R)+\frac{y}{96}p_1(T)\right)(\mathrm{Tr}\,F_L^2+\mathrm{Tr}\,F_R^2)\\
&+\frac{1}{48}\left(\mathrm{tr}_G\,F_L^4+\mathrm{tr}_G\,F_R^4\right)-\frac{1}{2}\left(\frac{1}{4}\mathrm{Tr}\,F_L^2-\frac{1}{4}\mathrm{Tr}\,F_R^2\right)^2, \qquad (7.1.3)
\end{aligned}$$

where coefficients are listed in Table 7.1. From this table, we can easily find that $\gamma=\dim_G+1$, $x=|\Gamma_G|-h^\vee_G$ and $y=h^\vee_G$. α and β are more complicated combinations of group theoretical data, which we will display as a part of the formula for a general number Q of M5-branes.

G	$\mathrm{SU}(k)$	$\mathrm{SO}(2k)$	E_6	E_7	E_8
α	0	$10k^2-57k+81$	319	1670	12489
β	0	$2k^2-3k-9$	89	250	831
γ	k^2	$k(2k-1)+1$	79	134	249
x	0	$2k-6$	12	30	90
y	k	$2k-2$	12	18	30

Table 7.1: Table of anomaly coefficients for (G,G) conformal matters.

Non-minimal conformal matter $\mathcal{T}_G(Q-1)$. We would like to determine the anomaly polynomial of the theory $\mathcal{T}_G(Q-1)$. We go to a point on the tensor branch where the theory becomes $[G]$—$(G)\cdots(G)$—$[G]$ where — represents $\mathcal{T}_G(0)$. The effective theory consists of $Q-1$ free tensor multiplets, describing the relative positions of the full M5s, vector multiplets for each G, and the 6d theories $\mathcal{T}_G(0)$ as generalized bifudamental matters.

We choose to include the center-of-mass motion of Q M5-branes for convenience of computation and to compare the result with the inflow argument where the center-of-mass contribution is automatically included. At the end of this section, we will comment on how to subtract the contribution of the center-of-mass mode (both one-loop and Green-Schwarz).

The one-loop anomaly is given by

$$I^{\text{one-loop}} = \sum_{i=0}^{Q-1} I^{\text{bif}}_{G,G}(F_i, F_{i+1}) + \sum_{i=1}^{Q-1} I^{\text{vec}}_G(F_i) + Q I^{\text{tensor}}. \tag{7.1.4}$$

We find that the gauge anomalies can be canceled by the Green-Schwarz term

$$I^{\text{GS}} = \frac{1}{2}\sum_{i=0}^{Q-1} \mathcal{I}_i \mathcal{I}_i \tag{7.1.5}$$

for the self-dual tensor fields with the Bianchi identity

$$d\mathcal{H}_i = \mathcal{I}_i = \frac{1}{4}\operatorname{Tr} F_i^2 - \frac{1}{4}\operatorname{Tr} F_{i+1}^2 + \frac{1}{2}(2i-Q+1)|\Gamma| c_2(R), \tag{7.1.6}$$

where $\mathcal{H}_i$ $(i=0,1,\cdots,Q-1)$ are the three-form fields in the tensor multiplets whose scalars represent the absolute positions of Q full M5s on $\mathbb{R}^1$, not the relative positions of them. Combining all of them, we obtain the total anomaly polynomial:

$$\begin{aligned} I^{\text{tot}}_G = I^{\text{GS}} + I^{\text{one-loop}} &= |\Gamma|^2 Q^3 \frac{c_2^2(R)}{24} - \frac{Q}{48} c_2(R)\Big(|\Gamma|(r_G+1)-1\Big)\Big(4c_2(R)+p_1(T)\Big) \\ &\quad - \frac{Q}{8}|\Gamma| c_2(R)(\operatorname{Tr} F_0^2 + \operatorname{Tr} F_Q^2) + \frac{Q}{8}\left(\frac{1}{6}c_2(R)p_1(T) - \frac{1}{6}p_2(T) + \frac{1}{24}p_1^2(T)\right) \\ &\quad - \frac{1}{2}I^{\text{vec}}(F_0) - \frac{1}{2}I^{\text{vec}}(F_Q). \end{aligned} \tag{7.1.7}$$

Remark. As already mentioned, we give a comment about the center of mass tensor multiplet. In fact, the anomaly polynomial of the UV SCFT is given by subtracting the contributions of the center of mass tensor multiplet from (7.1.7),

$$I^{\text{tot}} = I^{\text{SCFT}} + I^{\text{ten}} + \frac{1}{2Q}\left(\frac{1}{4}\operatorname{Tr}F_0^2 - \frac{1}{4}\operatorname{Tr}F_Q^2\right)^2. \tag{7.1.8}$$

Here the third term is a Green-Schwarz term for the center of mass tensor multiplet; it has the Bianchi identity

$$d(\frac{1}{Q}\sum_i \mathcal{H}_i) = \frac{1}{Q}\sum_i \mathcal{I}_i = \frac{1}{Q}\left(\frac{1}{4}\operatorname{Tr}F_0^2 - \frac{1}{4}\operatorname{Tr}F_Q^2\right). \tag{7.1.9}$$

We note that the additional factor Q comes from the factor Q in front of the kinetic term of the center-of-mass tensor multiplet, i.e. $\eta^{\text{center-of-mass}} = Q$.

7.2 Comparison with the inflow

We will compute the anomaly polynomial the theory $\mathcal{T}_G(Q-1)$ by using the anomaly inflow and check that the result agrees with (7.1.7). We will denote the 11d spacetime as $X_{11} = X_6 \times (\mathbb{R}\times\mathbb{C}^2/\Gamma)$ and put Q M5-branes at the origin of $\mathbb{R}\times\mathbb{C}^2/\Gamma$. Let y^a $(a = 1,2,3,4,5)$ be the coordinates of the covering space $\mathbb{R}^5 = \mathbb{R}\times\mathbb{C}^2$.

7.2.1 7d Chern-Simons terms on ALE singularities

First of all, we would like to determine the additional two types of Chern-Simons terms localized on the singularity in the M-theory spacetime $X_{11} = X_7 \times \mathbb{C}^2/\Gamma$.

Gravitational Chern-Simons terms. The Chern-Simons terms which only contain gravitational terms can be determined as follows. For $X_{11} = X_7 \times \mathbb{C}^2/\Gamma$, the structure group of the tangent bundle is decomposed as $\mathrm{SO}(11) \to \mathrm{SO}(7)\times\mathrm{SU}(2)_L\times\mathrm{SU}(2)_R$. Since the discrete group Γ is contained in $\mathrm{SU}(2)_L$, the $\mathrm{SU}(2)_R$ symmetry still acts on $\mathbb{C}^2/\Gamma$.[1] Let $c_2(L)$ and $c_2(R)$ be the Chern classes of $\mathrm{SU}(2)_L$ and $\mathrm{SU}(2)_R$ respectively. Then I_8 becomes

$$I_8 = -\frac{1}{48}c_2(L)(4c_2(R) + p_1(TX_7)) + \frac{1}{48}\left[p_2(T) - p_1(T)c_2(R) - \frac{1}{4}p_1(T)^2\right]. \tag{7.2.1}$$

[1] As we already mentioned, when G is of type A_k, there is an additional $\mathrm{U}(1)$ symmetry acting on $\mathbb{C}^2/\Gamma$ which we ignore for simplicity.

The singularity can be regarded as a gravitational instanton with nontrivial localized curvature $c_2(L)$ given as

$$\int_{\mathbb{C}^2/\Gamma} c_2(L) =: \chi_\Gamma. \tag{7.2.2}$$

where χ_Γ can be thought of as a version of the "Euler number" of the singularity. Therefore, the Chern-Simons term S_{CI_8} in M-theory can be decomposed as

$$S_{CI_8} = S_{CI_8}^{\text{bulk}} + 2\pi \int_{X_7\times\{0\}} \frac{\chi_\Gamma}{48} C\wedge (4c_2(R) + p_1(T)). \tag{7.2.3}$$

Here $S_{CI_8}^{\text{bulk}}$ is the Chern-Simons term (3.3.2) while the second terms is localized on the singularity. The value of χ_Γ is given by [82]

$$\chi_\Gamma = r_\Gamma + 1 - \frac{1}{|\Gamma|}, \tag{7.2.4}$$

where r_Γ is the rank of the A_r, D_r, E_r group corresponding to Γ, and $|\Gamma|$ is the order of Γ.

This formula can be derived as follows. Let $M = \{z \in \mathbb{C}^2/\Gamma; |z|^2 \le 1\}$. whose boundary is $\partial M = S^3/\Gamma$. The topological Euler number of this space is $\chi(M) = r+1$ because $\dim H_2(M) = r$, $\dim H_0(M) = 1$ and others are zero. On the other hand, the topological Euler number is also written as an integral of local quantities; $\chi(M) = \int_M E_4 + \int_{\partial M}(\text{local term})$, where we denote the Euler density as E_4. When Γ is trivial so that $\partial M = S^3$, the contribution from the boundary integral is 1. This is becuase $r = 0$ and $E_4 = 0$ in that case. Then, the boundary contribution to $\chi(M)$ is $1/|\Gamma|$ when $\partial M = S^3/\Gamma$. Therefore we obtain $\int_M E_4 = r + 1 - 1/|\Gamma|$.

Gauge Chern-Simons terms. Let us next consider the Chern-Simons terms involving gauge fields localized on the singularity. The gauge fields A_i $(i = 1, \cdots, r_\Gamma)$ in the Cartan of the G gauge algebra localized on the singularity comes from the M-theory 3-form C as follows,

$$C = C^{\text{bulk}} + i\sum_{i=1}^{r} \omega^i \wedge A_i, \tag{7.2.5}$$

where ω^i are Poincare duals to the 2-cycles which are collapsed at the singularity. The factor $i = \sqrt{-1}$ was introduced to make A_i anti-Hermitian. Then we obtain

$$\begin{aligned} S_{CGG} &= S_{CGG}^{\text{bulk}} + \frac{2\pi}{2}\eta^{ij} \int_{X_7\times\{0\}} C^{\text{bulk}} \wedge F_i \wedge F_j \\ &= S_{CGG}^{\text{bulk}} + \frac{2\pi}{4} \int_{X_7\times\{0\}} C^{\text{bulk}} \wedge \operatorname{Tr} F^2, \end{aligned} \tag{7.2.6}$$

where $\eta^{ij} = -\int \omega^i \wedge \omega^j$ is -1 times the intersection matrix of the two-cycles given by the Cartan matrix of G. Although we derived the formula only for gauge fields in the Cartan sugalgebra, the last expression is valid for the whole G gauge field.

Summary. Combining all the results, we obtain the Chern-Simons terms localized on the singularity as follows;

$$S_\Gamma = 2\pi \int_{X_7\times\{0\}} C^{\text{bulk}} \wedge J_4, \tag{7.2.7}$$

$$J_4 \equiv \frac{\chi_\Gamma}{48}(4c_2(R) + p_1(T)) + \frac{1}{4}\operatorname{Tr} F^2. \tag{7.2.8}$$

7.2.2 Inflow computation

Let us put Q coincident M5s and determine the anomalies by the inflow from the Chern-Simons terms. We start from the modification of the Bianchi identity for G. If Γ is trivial, the Bianchi equation for G is the same as (3.3.9). In the case of $\mathbb{C}^2/\Gamma$, we restrict the $\mathrm{SO}(5)_R$ bundle to the subbundle $\mathrm{SU}(2)_R \subset \mathrm{SO}(4)_R \subset \mathrm{SO}(5)_R$, which will be preserved even when the spacetime is orbifolded by Γ. Then, the only change in G is the additional factor of $|\Gamma|$:

$$G = |\Gamma| Q \frac{e_4}{2} + (\text{regular}) \tag{7.2.9}$$

where y^as are understood as the coordinates of the covering space.

Because G is singular at the position of the M5s, we remove a small tubular neighborhood of them in the integral of Chern-Simons terms. By an abuse of notation, we denote the worldvolume of M5s as X_6 (or Y_7 depending on whether we consider X_{11} or Y_{12}). The integral formulas for global angular forms, when we divide $\mathbb{C}^2$ by Γ is given by including additional factors of $|\Gamma|$;

$$|\Gamma| \int_{S^4} e_4 = 2, \ |\Gamma| \int_{S^4} (e_4)^3 = 2c_2(R)^2 \tag{7.2.10}$$

where S^4 is a sphere around the origin of $\mathbb{R}\times\mathbb{C}^2$.

Then, the most singular part of the Chern-Simons term S^{bulk}_{CGG} is given by

$$\begin{aligned} S^{\text{bulk}}_{CGG} =& \frac{2\pi}{6} \lim_{\epsilon\to 0} \int_{Y_{12}\backslash D_\epsilon(Y_7)} G^{\text{bulk}} \wedge G^{\text{bulk}} \wedge G^{\text{bulk}} \\ \sim& 2\pi \cdot \frac{Q^3|\Gamma|^3}{48} \lim_{\epsilon\to 0} \int_{Y_{12}\backslash D_\epsilon(Y_7)} e_4^3 = -2\pi \cdot \frac{Q^3|\Gamma|^3}{48} \lim_{\epsilon\to 0} \int_{\partial D_\epsilon(Y_7)} e_3^{(0)} e_4^2 \\ =& -2\pi \cdot \frac{Q^3|\Gamma|^2}{24} \int_{Y_7} c_2(R)^{(0)} c_2(R), \end{aligned} \tag{7.2.11}$$

where $dc_2(R)^{(0)} = c_2(R)$ and we have used the fact that because e_4 is closed, it is locally written as $e_4 = de_3^{(0)}$. Therefore, the contribution to the anomaly polynomial is $-(Q^3|\Gamma|^2/24)c_2(R)^2$.

Similarly, we obtain

$$\begin{aligned} S^{\text{bulk}}_{CI_8} \sim& 2\pi \cdot Q \int_{Y_7} I_7^{(0)}, \\ S_\Gamma \sim& 2\pi \cdot \frac{Q|\Gamma|}{2} \int_{Y_7} c_2(R)^{(0)} \left(J_{4,L} + J_{4,R}\right), \end{aligned} \tag{7.2.12}$$

where $dI_7^{(0)} = I_8$. We note that the two sides of the M5-branes become the independent gauge fields for G_L and G_R and that the $J_{4,L}$ and $J_{4,L}$ are the J_4 defined in (7.2.8) on the left and right of the M5-branes respectively.

Combining the three contributions, the inflow terms are given as

$$-\frac{Q^3|\Gamma|^2}{24}c_2(R)^2 + QI_8 + \frac{Q|\Gamma|}{2}c_2(R)(J_{4,L} + J_{4,R}). \tag{7.2.13}$$

which must be cancelled by the anomaly of the theory $\mathcal{T}_G(Q-1)$.

Moreover, there is another source of the anomaly; the boundary condition of gauge fields $G_{L,R}$. In fact, the M5-branes set the boundary condition of these gauge fields, such that a gauge theory on the singularity is described by a 6d $\mathcal{N}=(1,0)$ vector multiplet instead of a 6d $\mathcal{N}=(1,1)$ vector multiplet. In other words, among the 7d $\mathcal{N}=1$ vector fields on the singularity, three scalars and a component of vector field normal to M5-branes have Dirichlet boundary condition, while vector fields tangent to M5-branes have Neumann boundary condition.

This boundary condition gives contribution to the anomaly. This is the same in the case of the end-of-the-world brane where the change of the gravitino boundary condition gave contributions to the anomaly by $\frac{1}{4}p_1(TX_{11})$. In our case, the contribution is written as

$$-\frac{1}{2}I_L^{\text{vec}} - \frac{1}{2}I_R^{\text{vec}}, \tag{7.2.14}$$

where I_L^{vec} and I_R^{vec} are the anomalies of 6d $\mathcal{N}=(1,0)$ vector multiplets with gauge groups G_L and G_R, respectively.

Taking into account the contribution (7.2.14), we finally obtain the anomaly polynomial $I^{\text{tot}}(Q,G)$ of $\mathcal{T}_G(Q-1)$ as

$$\begin{aligned} &I^{\text{tot}}(Q,G) \\ &= \frac{Q^3|\Gamma|^2}{24}c_2(R)^2 - QI_8 - \frac{Q|\Gamma|}{2}c_2(R)(J_{4,L} + J_{4,R}) - \frac{1}{2}I_L^{\text{vec}} - \frac{1}{2}I_R^{\text{vec}}, \end{aligned} \tag{7.2.15}$$

which coincides with the result (7.1.7) in the previous section.

7.3 Anomaly polynomials of frozen conformal matters

We will now compute the anomaly polynomial for the 6d theories $\mathcal{T}^{\text{fr}}_{G\to G_{\text{fr}}}(Q-1)$ in a similar way to Section 7.1. The theory is a chain of Q copies of $(G_{\text{fr}}, G_{\text{fr}})$ conformal matter theories:

$$[G_{\text{fr}}] \text{---} (G_{\text{fr}}) \text{---} \cdots \text{---} (G_{\text{fr}}) \text{---} [G_{\text{fr}}]\ , \tag{7.3.1}$$

where G_{fr} can also be non-simply-laced or trivial.

By repeating the same computation in Section 7.1, we find that the anomaly polynomial, including the center of mass tensor multiplet, can be written in an elegant expression which suggest the anomaly inflow interpretation;

$$I^{\text{tot}} = \frac{1}{24}Q^3|\Gamma_G|^2 c_2(R)^2 - QI_8 - \frac{1}{2}Q|\Gamma_G|(J_{4,L} + J_{4,R}) - \frac{1}{2}I_L^{\text{vec}} - \frac{1}{2}I_R^{\text{vec}} \ , \tag{7.3.2}$$

where G is the simply-laced group from which G_{fr} is obtained,. Other definitions are summarized as follows;

- The expression for I_8 is

$$I_8 = \frac{1}{48}\left[p_2(T) - p_1(T)c_2(R) - \frac{1}{4}p_1(T)^2\right] \ . \tag{7.3.3}$$

- The expression for $J_{4,L/R}$ is

$$J_{4,L/R} = \frac{1}{48}(4c_2(R) + p_1(T))\chi_{G\to G_{\text{fr}}} + \frac{1}{4d_{G\to G_{\text{fr}}}}\operatorname{tr} F_{L/R}^2 \ , \tag{7.3.4}$$

  where

$$\chi_{G\to G_{\text{fr}}} = r_G - 11 + \frac{12}{d_{G\to G_{\text{fr}}}} - \frac{1}{|\Gamma_G|} \ . \tag{7.3.5}$$

  r_G is the rank of G, $|\Gamma_G|$ is the number of elements of Γ_G, and $d_{G\to G_{\text{fr}}}$ is the parameter given in (2.4.10). When $G_{\text{fr}} = G$, $d_{G\to G_{\text{fr}}}$ is taken to be 1. The last term in (7.3.4) is present only when G_{fr} is non-empty. The quantity $\chi_{G\to G_{\text{fr}}}$ can alternatively be expressed using $\operatorname{rank}(G_{\text{fr}})$ as

$$\chi_{G\to G_{\text{fr}}} = \begin{cases} r_G + 1 - \frac{1}{|\Gamma_G|} & \text{unfrozen, } G_{\text{fr}} = G \ ; \\ -\frac{3}{5} - \frac{1}{|\Gamma_G|} & \text{“exotically frozen”: (4.4.6) ;} \\ r_{G_{\text{fr}}} - 1 - \frac{1}{|\Gamma_G|} & \text{otherwise .} \end{cases} \tag{7.3.6}$$

- The expression for $I_{L/R}^{\text{vec}}$ is

$$\begin{aligned} I_{L/R}^{\text{vec}} = &-\frac{1}{24}\left[\operatorname{tr}_{\text{adj}} F_{L/R}^4 + 6c_2(R)\operatorname{tr}_{\text{adj}} F_{L/R}^2 + \dim(G_{\text{fr}})c_2(R)^2\right] \\ &- \frac{1}{48}p_1(T)\left[\operatorname{tr}_{\text{adj}} F_{L/R}^2 + \dim(G_{\text{fr}})c_2(R)\right] - \frac{1}{5760}\left[7p_1(T)^2 - 4p_2(T)\right] \ , \end{aligned} \tag{7.3.7}$$

  where

$$\operatorname{tr}_{\text{adj}} F_{L/R}^2 = h^\vee(G_{\text{fr}})\operatorname{tr} F_{L/R}^2 \ . \tag{7.3.8}$$

  The terns involving $F_{L,R}$ are only present when $G_{\text{fr}} \neq \varnothing$.

The center of mass contribution to the above anomaly polynomial is similar to (7.1.8);

$$
\begin{aligned}
I^{\mathrm{CM}} = &\left[\frac{1}{24}c_2(R)^2 + \frac{1}{48}c_2(R)p_1(T) + \frac{23}{5760}p_1(T)^2 - \frac{116}{5760}p_2(T)\right] \\
&+ \frac{1}{2Q}\frac{1}{16d^2_{G\to G_{\mathrm{fr}}}}\left(\mathrm{tr}\,F_L^2 - \mathrm{tr}\,F_R^2\right)^2 \ .
\end{aligned}
\tag{7.3.9}
$$

In particular, the second line is a Green–Schwarz contribution.

Therefore, the anomaly polynomial of the UV SCFT associated with (7.3.1) is given as

$$
\begin{aligned}
I^{\mathrm{tot}} - I^{\mathrm{CM}} = &\alpha c_2(R)^2 + \beta c_2(R)p_1(T) + \gamma p_1(T)^2 + \delta p_2(T) \\
&+ \left(-\frac{x}{8}c_2(R) + \frac{h^\vee_{G_{\mathrm{fr}}}}{96}p_1(T)\right)(\mathrm{tr}\,F_L^2 + \mathrm{tr}\,F_R^2) \\
&+ \frac{1}{48}(\mathrm{tr}_{\mathrm{adj}}\,F_L^4 + \mathrm{tr}_{\mathrm{adj}}\,F_R^4) - \frac{1}{2Q}\frac{1}{16d^2_{G\to G_{\mathrm{fr}}}}\left(\mathrm{tr}\,F_L^2 - \mathrm{tr}\,F_R^2\right)^2
\end{aligned}
\tag{7.3.10}
$$

where

$$
\begin{aligned}
\alpha &= \frac{1}{24}|\Gamma_G|^2Q^3 - \frac{1}{12}Q|\Gamma_G|\chi_{G\to G_{\mathrm{fr}}} + \frac{1}{24}(\dim(G_{\mathrm{fr}}) - 1) \\
\beta &= \frac{1}{48}Q\left(1 - |\Gamma_G|\chi_{G\to G_{\mathrm{fr}}}\right) + \frac{1}{48}(\dim(G_{\mathrm{fr}}) - 1) \\
\gamma &= \frac{1}{5760}\left[30(Q-1) + 7(\dim(G_{\mathrm{fr}}) + 1)\right] \\
\delta &= -\frac{1}{1440}\left[30(Q-1) + \dim(G_{\mathrm{fr}}) + 1\right] \\
x &= \frac{|\Gamma_G|}{d_{G\to G_{\mathrm{fr}}}}Q - h^\vee_{G_{\mathrm{fr}}} \ .
\end{aligned}
\tag{7.3.11}
$$

Here we comment on a special case where $G_{\mathrm{fr}} = \varnothing$ and G_{fr} is not exotically frozen. Using (7.3.5) or (7.3.6), we take $\chi_{G\to G_{\mathrm{fr}}} = -1 - \frac{1}{|\Gamma_G|}$ and we obtain

$$
\begin{aligned}
\alpha &= \frac{1}{24}|\Gamma_G|^2Q^3 - \frac{1}{12}(-|\Gamma_G| - 1)Q - \frac{1}{24} \\
\beta &= \frac{Q}{48}(2 + |\Gamma_G|) - \frac{1}{48} \\
\gamma &= \frac{1}{5760}[30(Q-1) + 7] \\
\delta &= -\frac{1}{1440}[30(Q-1) + 1] \ .
\end{aligned}
\tag{7.3.12}
$$

This can be obtained formally from (7.1.7) and (7.1.8) by setting $\dim G = \mathrm{rank}\, G = 0$ and with $|\Gamma_G|$ replaced by $-|\Gamma_G|$.

The fact that the our formulas for the anomaly polynomial are a minimal modification of those in Section 7.2 give interesting indications on the physics of frozen singularities. Especially, we find that there are new parameter $d_{G\to G_{\rm fr}}$ of (2.4.10), and of $\chi_{G\to G_{\rm fr}}$ in (7.3.6) associated with the frozen singularity. For example, $\chi_{G\to G_{\rm fr}}$ seems to modify the Euler number of the singularity χ_Γ in (7.2.4). It would be very interesting to see how these parameters should be interpreted in the framework of M-theory.

Appendix A

Group theoretical data

Let us summerize group theoretic notations. In this thesis we do not concern about subtleties arising from global structures of gauge groups and we are careless about whether we are talking about groups or algebras. It is convenient to define the symbol Tr_G to be the trace in the adjoint representation divided by the dual Coxeter number $h^\vee_G$ of the gauge group G, listed in Table A.1.

One of the properties of Tr is that $\frac{1}{4}\int \mathrm{Tr}\, F^2$ is one when there is one instanton on a four-manifold. Moreover, if we have subgroup G' in a group G with Dynkin index of embedding 1, for an element f of universal enveloping algebra of Lie algebra of G' , the following equation holds:

$$\mathrm{Tr}_{G'}\, f = \mathrm{Tr}_G\, f. \tag{A.1}$$

All of the embeddings we consider in this paper have index 1, so we often omit the subscript G in Tr_G.

To convert the above anomaly polynomials to a convenient form, we define some constants and write those values in Table A.1. We define the constant s_G which relates the trace of F^2 in the fundamental representation and $\mathrm{Tr}\, F^2$ as $\mathrm{tr}_{\text{fund}}\, F^2 = s_G\, \mathrm{Tr}\, F^2$. Then we have

$$\mathrm{tr}_{\text{adj}} F^2 = h^\vee_G\, \mathrm{Tr}\, F^2, \qquad \mathrm{tr}_{\text{fund}} F^2 = s_G\, \mathrm{Tr}\, F^2, \tag{A.2}$$

where the first equation is just the definition of Tr. For trace of F^4, we define t_G and u_G by

$$\mathrm{tr}_{\text{adj}} F^4 = t_G \mathrm{tr}_{\text{fnd}} F^4 + \frac{3}{4} u_G (\mathrm{Tr}\, F^2)^2. \tag{A.3}$$

For gauge groups $G = \mathrm{SU}(2), \mathrm{SU}(3)$ and all exceptional groups, there are no independent quadratic Casimir operator, so we can relate $\mathrm{tr}_\rho F^4$ and $(\mathrm{Tr}\, F^2)^2$ by

$$\mathrm{tr}_{\text{adj}} F^4 = \frac{3}{4} w_G (\mathrm{Tr}\, F^2)^2, \qquad \mathrm{tr}_{\text{fund}} F^4 = \frac{3}{4} x_G (\mathrm{Tr}\, F^2)^2 \tag{A.4}$$

These constants are tabulated in Table A.2. Note that because $t_{\mathrm{SO}(8)} = 0$, we can also relate $\mathrm{tr}_{\text{adj}}\, F^4$ to $(\mathrm{Tr}\, F^2)^2$ for $G = \mathrm{SO}(8)$.

G	$\mathrm{SU}(k)$	$\mathrm{SO}(k)$	$\mathrm{USp}(2k)$	G_2	F_4	E_6	E_7	E_8
r_G	$k-1$	$\lfloor k/2 \rfloor$	k	2	4	6	7	8
$h^\vee_G$	k	$k-2$	$k+1$	4	9	12	18	30
d_G	k^2-1	$k(k-1)/2$	$k(2k+1)$	14	52	78	133	248
d_{fnd}	k	k	$2k$	7	26	27	56	248
s_G	$\frac{1}{2}$	1	$\frac{1}{2}$	1	3	3	6	30
t_G	$2k$	$k-8$	$2k+8$	0	0	0	0	0
u_G	2	4	1	$\frac{10}{3}$	5	6	8	12

Table A.1: Group theoretical constants defined for all G. Those constants are also listed in Appendix of [83].

G	$\mathrm{SU}(2)$	$\mathrm{SU}(3)$	G_2	F_4	E_6	E_7	E_8
w_G	$\frac{8}{3}$	3	$\frac{10}{3}$	5	6	8	12
x_G	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{3}$	1	1	2	12

Table A.2: Group theoretical constants defined only G without independent quartic Casimir.

All representations we use in this paper are fundamental or adjoint, except for the spin representation $\mathbf{8}$ of $\mathrm{SO}(7)$. The conversion constant for this representation is

$$\begin{aligned} \mathrm{tr}_{\mathbf{8}} F^2 &= \mathrm{Tr}\, F^2, \\ \mathrm{tr}_{\mathbf{8}} F^4 &= -\frac{1}{2}\mathrm{tr}_{\text{fund}} F^4 + \frac{3}{8}(\mathrm{Tr}\, F^2)^2. \end{aligned} \tag{A.5}$$

Finally, let us note that the finite subgroup Γ_G of $\mathrm{SU}(2)$ of type $G = A_n$, D_n and E_n has the following order:

$$|\Gamma_{\mathrm{SU}(k)}| = k, \quad |\Gamma_{\mathrm{SO}(2k)}| = 4k-8, \quad |\Gamma_{E_6}| = 24, \quad |\Gamma_{E_7}| = 48, \quad |\Gamma_{E_8}| = 120. \tag{A.6}$$

Appendix B

Gauge triples and fractional branes

In this appendix, we will explain the relation between the Chern-Simons invariants on T^3, called as gauge triples and fractional M5-brane in Chapter 2.

Let us consider the M-theory on $\mathbb{R}^{1,2} \times S^1 \times T^2 \times \mathbb{R} \times \mathbb{C}^2/\Gamma_G$ with a single M5 along $\mathbb{R}^{1,2} \times S^1 \times T^2$. After reducing along one S^1 of T^2 and taking the T-dual along the other S^1 of T^2, we obtain the IIB on $\mathbb{R}^{1,2} \times S^1 \times S^1 \times \mathbb{R} \times \mathbb{C}^2/\Gamma_G$ with a single D3-brane filling $\mathbb{R}^{1,2} \times S^1$.

Taking another T-dual along S^1 and lifting the whole system back to M-theory, we have M-theory on $\mathbb{R}^{1,2} \times T^3 \times \mathbb{R} \times \mathbb{C}^2/\Gamma_G$ and a single M2-brane filling $\mathbb{R}^{1,2}$. The singularity has the G gauge multiplet along $\mathbb{R}^{1,2} \times T^3 \times \mathbb{R}$, and the M2-brane can be absorbed into an instanton configuration of G on $T^3 \times \mathbb{R}$.

We examine more carefully about an instanton configuration on $T^3 \times \mathbb{R}$. By restricting the gauge field to T^3 at a constant "time" $t \in \mathbb{R}$, we define the Chern-Simons invariant $CS(t)$. Since we have a single M2 that becomes one instanton, we simply set $CS(-\infty) = 0$ and $CS(+\infty) = 1$.

At $t = \pm\infty$, we need a zero-energy configuration, so the three holonomies $g_{1,2,3}$ around three edges of T^3 should commute. $g_{1,2,3}$ are called as commuting triple. For simplicity, let us set $g_{1,2,3} = 1$ at $t = \pm\infty$. If we take them to be in the Cartan of G, the Chern-Simons invariant of the flat gauge field on T^3 is 0 mod 1. Something interesting happens when we have a commuting triple which cannot be simultaneously conjugated into the Cartan. We examine each G separately.

$G = A_N$. The commuting triple can be simultaneously conjugated into the Cartan

$G = D_N$. There is a unique commuting triple (g_1^*, g_2^*, g_3^*) that cannot be simultaneously conjugated into the Cartan; they can be chosen to be in a common $\mathrm{Spin}(7)$ subgroup, see Appendix I of [84]. The Chern-Simons invariant is $1/2$ mod 1 [85], and the unbroken subgroup is $\mathfrak{so}(2N-7)$.

Using this fact, we can have the following one-instanton configuration on $T^3 \times \mathbb{R}$:

- For $-\infty < t < t_0$, the configuration on T^3 is basically flat and given by $(g_1, g_2, g_3) = (1, 1, 1)$. $CS(t)$ stays almost constant close to 0.
- At around $t = t_0$, the gauge configuration suddenly changes to $(g_1, g_2, g_3) = (g_1^*, g_2^*, g_3^*)$ dressed with holonomies in the Cartan of the commutant, $\mathfrak{so}(2N-7)$. $CS(t)$ jumps to 1/2.
- Again, for $t_0 < t < t_1$, the configuration remains almost constant.
- And then at around $t = t_1$, it suddenly changes back to $(g_1, g_2, g_3) = (1, 1, 1)$, making $CS(t)$ to jump to 1.

In these configurations, there are two parameters $t_{0,1}$ corresponding to the positions of the two fractional M5-branes. Moreover, the $\mathrm{USp}(2N-8)$ gauge group between the two fractions is interpreted as the S-dual of $\mathfrak{so}(2N-7)$ found above.

$G = E_n$. The analysis is similar to the D_N cases, using the data in [85]. For $G = E_6$, we have the following commuting triples:

$$\begin{array}{r|cccc} \text{value } v \text{ of } CS & 0 & \frac{1}{3} & \frac{1}{2} & \frac{2}{3} \\ \text{commutant } G_v & \mathfrak{e}_6 & \varnothing & \mathfrak{su}(3) & \varnothing \end{array} . \tag{B.1}$$

Then the one-instanton configuration can go through these commuting triples. The "time" of the jump from one commuting triple characterized by $CS = v_i$ to the next $CS = v_{i+1}$ corresponds to the position of the M5 fraction. The dual of G_v is the unbroken gauge algebra in between the two M5 fractions.

For $G = E_7$, the list of the commuting triples are

$$\begin{array}{r|cccccc} \text{value } v \text{ of } CS & 0 & \frac{1}{4} & \frac{1}{3} & \frac{1}{2} & \frac{2}{3} & \frac{3}{4} \\ \text{commutant } G_v & \mathfrak{e}_7 & \varnothing & \mathfrak{su}(2) & \mathfrak{usp}(6) & \mathfrak{su}(2) & \varnothing \end{array} . \tag{B.2}$$

and for $G = E_8$, these are

$$\begin{array}{r|cccccccccccc} \text{value } v \text{ of } CS & 0 & \frac{1}{6} & \frac{1}{5} & \frac{1}{4} & \frac{1}{3} & \frac{2}{5} & \frac{1}{2} & \frac{3}{5} & \frac{2}{3} & \frac{3}{4} & \frac{4}{5} & \frac{5}{6} \\ \text{commutant } G_v & \mathfrak{e}_8 & \varnothing & \varnothing & \mathfrak{su}(2) & \mathfrak{g}_2 & \varnothing & \mathfrak{f}_4 & \varnothing & \mathfrak{g}_2 & \mathfrak{su}(2) & \varnothing & \varnothing \end{array} . \tag{B.3}$$

In both cases, one can check that the sequence of the groups are the S-dual of the ones that appear in between two consecutive M5 fractions. The possible values of fractional 3-form flux of C in (2.4.9) can also be read off from the table above.

Appendix C

Global angular forms

In this appendix, we recall the properties of a smoothed-out version of differential forms with delta-function support. They are constructed using the help of the so-called global angular forms.

Let M denote an oriented manifold and E be an oriented rank $2k+1$ real vector bundle over M. Assume that E admits a metric and a connection Θ compatible with its metric. Denote its zero section by s_0. We can construct an S^{2k} bundle $\pi : S(E) \to M$ which is homeomorphic to $E_0 = E \setminus s_0(M)$ by assigning each point p of M a unit sphere in the fibre of E around $s_0(p)$.

Then, we can construct a form e_{2k} on $S(E)$ which has the following properties:

- e_{2k} is a globally well-defined $2k$-form on $S(E)$.
- $de_{2k} = 0$.
- $\int_{\pi^{-1}(p)} e_{2k}|_{\pi^{-1}(p)} = 2$ for any point p of M. In other words, $\pi_* e_{2k} = 2$.

Let ρ be a compactly supported function on E which satisfies $\rho(s(p)) = -1$. We can explicitly write a Thom class $\Phi(E)$ of the bundle E, the smooth analogue of $\delta(M \hookrightarrow E)$, as

$$\Phi(E) = d\rho\, e_{2k}/2. \tag{C.1}$$

Here we have identified the form e_{2k} on $S(E)$ and its pullback in terms of the homeomorphism $S(E) \simeq E_0$.

We can apply the usual decent notation:

$$de^{(0)}_{2k-1} = e_{2k}, \quad \delta e^{(0)}_{2k-1} = e^{(1)}_{2k-2}. \tag{C.2}$$

Here δ denote a $SO(2k+1)$ gauge transformation associated with the connection Θ.

Let us concentrate on the cases $k = 0$ and $k = 2$, which are relevant for our calculation. e_0 is just a step function whose value is $+1$ or -1. The explicit form of e_4 is given by

$$e_4 = \frac{1}{32\pi^2}\epsilon_{a_1\cdots a_5}\Big[(D\hat{y})^{a_1}(D\hat{y})^{a_2}(D\hat{y})^{a_3}(D\hat{y})^{a_4}\hat{y}^{a_5} - 2F^{a_1a_2}(D\hat{y}^{a_3})(D\hat{y})^{a_4}\hat{y}^{a_5} + F^{a_1a_2}F^{a_3a_4}\hat{y}^{a_5}\Big] \quad \text{(C.3)}$$

Here, $a_i = 1\cdots 5$ labels the fiber coordinates and $\hat{y}^{a_i}$ are coordinates of the unit sphere S^4. A covariant derivative and 2-form is defined using the connection Θ by

$$D\hat{y}^a = d\hat{y}^a - \Theta^{ab}\hat{y}^b, \qquad F^{ab} = d\Theta^{ab} - \Theta^{ac}\wedge\Theta^{cb}. \quad \text{(C.4)}$$

Using this explicit form, we can prove the formulae

$$\pi_*(e_4) = 2, \quad \pi_*(e_4^3) = 2p_2(E). \quad \text{(C.5)}$$

The formula $\pi_*(e_4^3) = 2p_2(N)$ is first proved by Bott and Cattaneo [86].

When the $\mathrm{SO}(5)$ connection reduces to $\mathrm{SO}(4)$, we can consider e_0 and e_4 at the same time; e_0 is a step function taking $+1$ and -1 on the northern and the southern hemispheres of S^4, respectively. Then we have

$$\pi_*(e_4e_0^2) = 2, \quad \pi_*(e_4^2e_0) = 2\chi_4(F), \quad \text{(C.6)}$$

where $\chi_4(F)$ is the Euler class of the rank 4 bundle F which is associated with the $\mathrm{SO}(4)$ connection.

Appendix D

Decomposition of characteristic classes

We would like to collect the formulas relating the characteristic classes for G and G' used in Chapter 5. We recall the definition of the symbol Tr; the trace in the adjoint representation divided by $h^\vee_G$. Then, the Dynkin index $T(R)$ for a representation R relates the Tr to tr_R by the formula

$$\mathrm{tr}_R\, F_G^2 = T^G(R)\,\mathrm{Tr}\, F_G^2. \tag{D.1}$$

. We list the values of $T^G(R)$ in Table D.1.

G	E_7	SO(12)	SU(6)	Sp(3)	SU(2)	Sp(n)	SO(n)	SU(n)
R	**56**	**32**	**20**	$\mathbf{14}'$	**4**	**2n**	**n**	**n**
$T^G(R)$	6	4	3	$\frac{5}{2}$	5	$\frac{1}{2}$	1	$\frac{1}{2}$

Table D.1: The values of $T^G(R)$ for various representations.

G: exceptional. Since there are no independent quartic Casimir invariants in this case, we only have to consider $\mathrm{Tr}\, F_G^2$. As already remarked in the main body of the thesis, the unbroken subgroup G' is simple for these cases. The formula is given as

$$\mathrm{Tr}(F_G^2) = 4c_2(D) + m\,\mathrm{Tr}\, F_{G'}^2, \tag{D.2}$$

where $m = 3$ for G_2 and 1 for any other group.

$G = \mathrm{Sp}$**.** The unbroken subgroup is $\mathrm{Sp}(n-1)$ in this case. Then $\mathrm{Tr}\, F_{\mathrm{Sp}(n)}^2$ is related that of $\mathrm{Sp}(n-1)$ by the formula

$$\mathrm{Tr}(F_{\mathrm{Sp}(n)}^2) = 4c_2(D) + \mathrm{Tr}\, F_{\mathrm{Sp}(n-1)}^2. \tag{D.3}$$

The $\mathrm{tr}_{\text{fund}}\, F_{\mathrm{Sp}(n)}^4$ can be decomposed as

$$\mathrm{tr}_{\text{fund}}\, F_{\mathrm{Sp}(n)}^4 = 2c_2(D)^2 + \mathrm{tr}_{\text{fund}}\, F_{\mathrm{Sp}(n-1)}^4. \tag{D.4}$$

$G = \mathrm{SO}$**.** The unbroken subgroup in this case is not simple; $\mathrm{SU}(2)_F \times \mathrm{SO}(n-4)$. The $\mathrm{Tr}\, F^2_{\mathrm{SO}(n)}$ can be decomposed as follows;

$$\mathrm{Tr}\, F^2_{\mathrm{SO}(n)} = 4c_2(D) + \mathrm{Tr}\, F_F^2 + \mathrm{Tr}\, F^2_{\mathrm{SO}(n-4)}. \tag{D.5}$$

The $\mathrm{tr}_{\text{fund}}\, F^4_{\mathrm{SO}(n)}$ is also decomposed;

$$\mathrm{tr}_{\text{fund}}\, F^4_{\mathrm{SO}(n)} = 4c_2(D)^2 + 2\,\mathrm{tr}_{\text{fund}}\, F_F^4 + 12c_2(D)\,\mathrm{tr}_{\text{fund}}\, F^2_{\mathrm{SO}(n-4)} + \mathrm{tr}_{\text{fund}}\, F^4_{\mathrm{SO}(n-4)}. \tag{D.6}$$

$G = \mathrm{SU}$**.** The situation is different whether $n \geq 4$ or not. First, we consider the case of $n \geq 4$. Then, $\mathrm{Tr}\, F^2_{\mathrm{SU}(n)}$ is decomposed as

$$\mathrm{Tr}\, F^2_{\mathrm{SU}(n)} = 4c_2(D) - 4n(n-2)c_1(\mathrm{U}(1)_F)^2 + \mathrm{Tr}(F^2_{\mathrm{SU}(n-2)}), \tag{D.7}$$

and $\mathrm{tr}_{\text{fund}}\, F^4_{\mathrm{SU}(n)}$ is related by

$$\begin{aligned}\mathrm{tr}_{\text{fund}}\, F^4_{\mathrm{SU}(n)} = {} & \mathrm{tr}_{\text{fund}}\, F^4_{\mathrm{SU}(n-2)} - 8c_1(\mathrm{U}(1)_F)\,\mathrm{tr}_{\text{fund}}\, F^3_{\mathrm{SU}(n-2)} - 24c_1(\mathrm{U}(1)_F)^2\,\mathrm{tr}_{\text{fund}}\, F^2_{\mathrm{SU}(n-2)} + \\ & 2c_2(D)^2 - 12(n-2)^2 c_1(\mathrm{U}(1)_F)^2 c_2(D) + 2n(n-2)(n^2-6n+12)c_1(\mathrm{U}(1)_F)^4.\end{aligned} \tag{D.8}$$

When $G = \mathrm{SU}(2)$ or $G = \mathrm{SU}(3)$, the situation is similar to the exceptional groups. In fact, these groups have no independent quartic Casimir invariants and we only have to consider $\mathrm{Tr}\, F^2$. When $G = \mathrm{SU}(2)$, $\mathrm{SU}(2)_R$ is identified with the original G and the formula is simply given as

$$\mathrm{Tr}\, F^2_{\mathrm{SU}(2)} = 4c_2(D) = 4c_2(R). \tag{D.9}$$

When $G = \mathrm{SU}(3)$, we use the formula;

$$\mathrm{Tr}\, F^2_{\mathrm{SU}(3)} = 4c_2(D) - 12c_1(\mathrm{U}(1)_F)^2. \tag{D.10}$$

Bibliography

[1] O. Aharony, A. Hanany, and B. Kol, "Webs of (p,q) five-branes, five-dimensional field theories and grid diagrams," *JHEP* **01** (1998) 002, arXiv:hep-th/9710116 [hep-th].

[2] K. A. Intriligator, D. R. Morrison, and N. Seiberg, "Five-dimensional supersymmetric gauge theories and degenerations of Calabi-Yau spaces," *Nucl. Phys.* **B497** (1997) 56–100, arXiv:hep-th/9702198 [hep-th].

[3] M. Del Zotto, J. J. Heckman, A. Tomasiello, and C. Vafa, "6d Conformal Matter," *JHEP* **02** (2015) 054, arXiv:1407.6359 [hep-th].

[4] D. R. Morrison and C. Vafa, "Compactifications of F theory on Calabi-Yau threefolds. 1," *Nucl. Phys.* **B473** (1996) 74–92, arXiv:hep-th/9602114 [hep-th].

[5] D. R. Morrison and C. Vafa, "Compactifications of F theory on Calabi-Yau threefolds. 2.," *Nucl. Phys.* **B476** (1996) 437–469, arXiv:hep-th/9603161 [hep-th].

[6] J. J. Heckman, D. R. Morrison, and C. Vafa, "On the Classification of 6D SCFTs and Generalized ADE Orbifolds," *JHEP* **05** (2014) 028, arXiv:1312.5746 [hep-th]. [Erratum: JHEP06,017(2015)].

[7] J. J. Heckman, D. R. Morrison, T. Rudelius, and C. Vafa, "Atomic Classification of 6D SCFTs," *Fortsch. Phys.* **63** (2015) 468–530, arXiv:1502.05405 [hep-th].

[8] L. Bhardwaj, "Classification of 6d $\mathcal{N} = (1, 0)$ gauge theories," *JHEP* **11** (2015) 002, arXiv:1502.06594 [hep-th].

[9] P. Jefferson, H.-C. Kim, C. Vafa, and G. Zafrir, "Towards Classification of 5d SCFTs: Single Gauge Node," arXiv:1705.05836 [hep-th].

[10] P. Jefferson, S. Katz, H.-C. Kim, and C. Vafa, "On Geometric Classification of 5d SCFTs," arXiv:1801.04036 [hep-th].

[11] D. Gaiotto, "N=2 dualities," *JHEP* **08** (2012) 034, `arXiv:0904.2715 [hep-th]`.

[12] G. 't Hooft, "Naturalness, chiral symmetry, and spontaneous chiral symmetry breaking," *NATO Sci. Ser. B* **59** (1980) 135–157.

[13] S. Weinberg, *The quantum theory of fields. Vol. 2: Modern applications*. Cambridge University Press, 2013.

[14] L. Alvarez-Gaume and P. H. Ginsparg, "The Structure of Gauge and Gravitational Anomalies," *Annals Phys.* **161** (1985) 423. [Erratum: Annals Phys.171,233(1986)].

[15] J. A. Harvey, "TASI 2003 lectures on anomalies," 2005. `arXiv:hep-th/0509097 [hep-th]`.

[16] C. Cordova, T. T. Dumitrescu, and X. Yin, "Higher Derivative Terms, Toroidal Compactification, and Weyl Anomalies in Six-Dimensional (2,0) Theories," `arXiv:1505.03850 [hep-th]`.

[17] C. Cordova, T. T. Dumitrescu, and K. Intriligator, "Anomalies, renormalization group flows, and the a-theorem in six-dimensional (1, 0) theories," *JHEP* **10** (2016) 080, `arXiv:1506.03807 [hep-th]`.

[18] B. Haghighat, A. Iqbal, C. Kozaz, G. Lockhart, and C. Vafa, "M-Strings," *Commun. Math. Phys.* **334** no. 2, (2015) 779–842, `arXiv:1305.6322 [hep-th]`.

[19] B. Haghighat, C. Kozcaz, G. Lockhart, and C. Vafa, "Orbifolds of M-strings," *Phys. Rev.* **D89** no. 4, (2014) 046003, `arXiv:1310.1185 [hep-th]`.

[20] B. Haghighat, G. Lockhart, and C. Vafa, "Fusing E-strings to heterotic strings: E+E $\rightarrow$ H," *Phys. Rev.* **D90** no. 12, (2014) 126012, `arXiv:1406.0850 [hep-th]`.

[21] J. Kim, S. Kim, K. Lee, J. Park, and C. Vafa, "Elliptic Genus of E-strings," *JHEP* **09** (2017) 098, `arXiv:1411.2324 [hep-th]`.

[22] B. Haghighat, A. Klemm, G. Lockhart, and C. Vafa, "Strings of Minimal 6d SCFTs," *Fortsch. Phys.* **63** (2015) 294–322, `arXiv:1412.3152 [hep-th]`.

[23] A. Gadde, B. Haghighat, J. Kim, S. Kim, G. Lockhart, and C. Vafa, "6d String Chains," `arXiv:1504.04614 [hep-th]`.

[24] C. Cordova, T. T. Dumitrescu, and K. Intriligator, "Multiplets of Superconformal Symmetry in Diverse Dimensions," `arXiv:1612.00809 [hep-th]`.

[25] C. Cordova, T. T. Dumitrescu, and K. Intriligator, "Deformations of Superconformal Theories," *JHEP* **11** (2016) 135, arXiv:1602.01217 [hep-th].

[26] C. Beem, M. Lemos, L. Rastelli, and B. C. van Rees, "The (2, 0) superconformal bootstrap," *Phys. Rev.* **D93** no. 2, (2016) 025016, arXiv:1507.05637 [hep-th].

[27] C.-M. Chang and Y.-H. Lin, "Carving Out the End of the World or (Superconformal Bootstrap in Six Dimensions)," *JHEP* **08** (2017) 128, arXiv:1705.05392 [hep-th].

[28] K. Ohmori, H. Shimizu, Y. Tachikawa, and K. Yonekura, "Anomaly polynomial of general 6d SCFTs," *PTEP* **2014** no. 10, (2014) 103B07, arXiv:1408.5572 [hep-th].

[29] K. Intriligator, "6d, $\mathcal{N} = (1,\ 0)$ Coulomb branch anomaly matching," *JHEP* **10** (2014) 162, arXiv:1408.6745 [hep-th].

[30] H. Shimizu, Y. Tachikawa, and G. Zafrir, "Anomaly matching on the Higgs branch," arXiv:1703.01013 [hep-th].

[31] N. Mekareeya, K. Ohmori, H. Shimizu, and A. Tomasiello, "Small instanton transitions for M5 fractions," *JHEP* **10** (2017) 055, arXiv:1707.05785 [hep-th].

[32] K. Ohmori, H. Shimizu, Y. Tachikawa, and K. Yonekura, "6d $\mathcal{N} = (1, 0)$ theories on T^2 and class S theories: Part I," *JHEP* **07** (2015) 014, arXiv:1503.06217 [hep-th].

[33] K. Ohmori, H. Shimizu, Y. Tachikawa, and K. Yonekura, "6d $\mathcal{N}$=$(1, 0)$ theories on S^1/T^2 and class S theories: part II," arXiv:1508.00915 [hep-th].

[34] H.-C. Kim, S. Kim, and J. Park, "6d strings from new chiral gauge theories," arXiv:1608.03919 [hep-th].

[35] M. Del Zotto and G. Lockhart, "On Exceptional Instanton Strings," *JHEP* **09** (2017) 081, arXiv:1609.00310 [hep-th].

[36] M. Del Zotto, J. Gu, M.-x. Huang, A.-K. Kashani-Poor, A. Klemm, and G. Lockhart, "Topological Strings on Singular Elliptic Calabi-Yau 3-folds and Minimal 6d SCFTs," arXiv:1712.07017 [hep-th].

[37] H. Shimizu and Y. Tachikawa, "Anomaly of strings of 6d $\mathcal{N} = (1, 0)$ theories," *JHEP* **11** (2016) 165, arXiv:1608.05894 [hep-th].

[38] C. G. Callan, Jr. and J. A. Harvey, "Anomalies and Fermion Zero Modes on Strings and Domain Walls," *Nucl. Phys.* **B250** (1985) 427–436.

[39] D. Freed, J. A. Harvey, R. Minasian, and G. W. Moore, "Gravitational anomaly cancellation for M theory five-branes," *Adv. Theor. Math. Phys.* **2** (1998) 601–618, arXiv:hep-th/9803205 [hep-th].

[40] J. A. Harvey, R. Minasian, and G. W. Moore, "NonAbelian tensor multiplet anomalies," *JHEP* **09** (1998) 004, arXiv:hep-th/9808060 [hep-th].

[41] K. Ohmori, H. Shimizu, and Y. Tachikawa, "Anomaly polynomial of E-string theories," *JHEP* **08** (2014) 002, arXiv:1404.3887 [hep-th].

[42] F. Benini, Y. Tachikawa, and B. Wecht, "Sicilian gauge theories and N=1 dualities," *JHEP* **01** (2010) 088, arXiv:0909.1327 [hep-th].

[43] S. S. Razamat, C. Vafa, and G. Zafrir, "4d $\mathcal{N} = 1$ from 6d (1, 0)," *JHEP* **04** (2017) 064, arXiv:1610.09178 [hep-th].

[44] H.-C. Kim, S. S. Razamat, C. Vafa, and G. Zafrir, "E - String Theory on Riemann Surfaces," *Fortsch. Phys.* **66** no. 1, (2018) 1700074, arXiv:1709.02496 [hep-th].

[45] H.-C. Kim, S. S. Razamat, C. Vafa, and G. Zafrir, "D-type Conformal Matter and SU/USp Quivers," arXiv:1802.00620 [hep-th].

[46] M. Henningson, "Self-Dual Strings in Six Dimensions: Anomalies, the ADE-Classification, and the World-Sheet WZW-Model," *Commun. Math. Phys.* **257** (2005) 291–302, arXiv:hep-th/0405056.

[47] L. Alvarez-Gaume and E. Witten, "Gravitational Anomalies," *Nucl. Phys.* **B234** (1984) 269.

[48] N. Seiberg and W. Taylor, "Charge Lattices and Consistency of 6D Supergravity," *JHEP* **06** (2011) 001, arXiv:1103.0019 [hep-th].

[49] A. Sagnotti, "A Note on the Green-Schwarz mechanism in open string theories," *Phys. Lett.* **B294** (1992) 196–203, arXiv:hep-th/9210127 [hep-th].

[50] J. H. Schwarz, "Anomaly-Free Supersymmetric Models in Six-Dimensions," *Phys.Lett.* **B371** (1996) 223–230, arXiv:hep-th/9512053 [hep-th].

[51] V. Sadov, "Generalized Green-Schwarz mechanism in F theory," *Phys. Lett.* **B388** (1996) 45–50, arXiv:hep-th/9606008 [hep-th].

[52] N. Seiberg, "Non-Trivial Fixed Points of the Renormalization Group in Six Dimensions," *Phys. Lett.* **B390** (1997) 169–171, arXiv:hep-th/9609161.

[53] U. H. Danielsson, G. Ferretti, J. Kalkkinen, and P. Stjernberg, "Notes on supersymmetric gauge theories in five-dimensions and six-dimensions," *Phys. Lett.* **B405** (1997) 265–270, arXiv:hep-th/9703098 [hep-th].

[54] M. Bershadsky and C. Vafa, "Global Anomalies and Geometric Engineering of Critical Theories in Six Dimensions," arXiv:hep-th/9703167.

[55] E. Witten, "Some comments on string dynamics," in *Future perspectives in string theory. Proceedings, Conference, Strings'95, Los Angeles, USA, March 13-18, 1995*, pp. 501–523. 1995. arXiv:hep-th/9507121 [hep-th].

[56] A. Strominger, "Open p-branes," *Phys. Lett.* **B383** (1996) 44–47, arXiv:hep-th/9512059 [hep-th].

[57] E. Witten, "Five-Branes and M-theory on an Orbifold," *Nucl. Phys.* **B463** (1996) 383–397, arXiv:hep-th/9512219.

[58] I. Garca-Etxebarria and D. Regalado, "Exceptional $\mathcal{N} = 3$ theories," *JHEP* **12** (2017) 042, arXiv:1611.05769 [hep-th].

[59] P. Horava and E. Witten, "Heterotic and type I string dynamics from eleven-dimensions," *Nucl.Phys.* **B460** (1996) 506–524, arXiv:hep-th/9510209 [hep-th].

[60] P. Horava and E. Witten, "Eleven-Dimensional Supergravity on a Manifold with Boundary," *Nucl.Phys.* **B475** (1996) 94–114, arXiv:hep-th/9603142 [hep-th].

[61] O. J. Ganor and A. Hanany, "Small E(8) instantons and tensionless noncritical strings," *Nucl. Phys.* **B474** (1996) 122–140, arXiv:hep-th/9602120 [hep-th].

[62] N. J. Evans, C. V. Johnson, and A. D. Shapere, "Orientifolds, branes, and duality of 4-D gauge theories," *Nucl. Phys.* **B505** (1997) 251–271, arXiv:hep-th/9703210 [hep-th].

[63] J. de Boer, R. Dijkgraaf, K. Hori, A. Keurentjes, J. Morgan, *et al.*, "Triples, fluxes, and strings," *Adv.Theor.Math.Phys.* **4** (2002) 995–1186, arXiv:hep-th/0103170 [hep-th].

[64] Y. Tachikawa, "Frozen singularities in M and F theory," *JHEP* **06** (2016) 128, arXiv:1508.06679 [hep-th].

[65] S. Cecotti, C. Cordova, J. J. Heckman, and C. Vafa, "T-Branes and Monodromy," *JHEP* **07** (2011) 030, arXiv:1010.5780 [hep-th].

[66] I. Brunner and A. Karch, "Branes at Orbifolds Versus Hanany Witten in Six-Dimensions," *JHEP* **9803** (1998) 003, arXiv:hep-th/9712143 [hep-th].

[67] A. Hanany and A. Zaffaroni, "Branes and six-dimensional supersymmetric theories," *Nucl. Phys.* **B529** (1998) 180–206, arXiv:hep-th/9712145 [hep-th].

[68] J. J. Heckman, T. Rudelius, and A. Tomasiello, "6D RG Flows and Nilpotent Hierarchies," *JHEP* **07** (2016) 082, arXiv:1601.04078 [hep-th].

[69] N. Mekareeya, T. Rudelius, and A. Tomasiello, "T-branes, Anomalies and Moduli Spaces in 6D SCFTs," arXiv:1612.06399 [hep-th].

[70] F. Bonetti, T. W. Grimm, and S. Hohenegger, "One-Loop Chern-Simons Terms in Five Dimensions," *JHEP* **1307** (2013) 043, arXiv:1302.2918 [hep-th].

[71] I. R. Klebanov and A. A. Tseytlin, "Entropy of near extremal black p-branes," *Nucl. Phys.* **B475** (1996) 164–178, arXiv:hep-th/9604089 [hep-th].

[72] C. Vafa and E. Witten, "A One Loop Test of String Duality," *Nucl. Phys.* **B447** (1995) 261–270, arXiv:hep-th/9505053.

[73] E. Witten, "Five-Brane Effective Action in M-theory," *J. Geom. Phys.* **22** (1997) 103–133, arXiv:hep-th/9610234.

[74] E. Witten, "On Flux Quantization in M Theory and the Effective Action," *J.Geom.Phys.* **22** (1997) 1–13, arXiv:hep-th/9609122 [hep-th].

[75] P. Yi, "Anomaly of (2,0) Theories," *Phys. Rev.* **D64** (2001) 106006, arXiv:hep-th/0106165.

[76] D. Gaiotto, A. Neitzke, and Y. Tachikawa, "Argyres-Seiberg Duality and the Higgs Branch," *Commun. Math. Phys.* **294** (2010) 389–410, arXiv:0810.4541 [hep-th].

[77] M. Del Zotto, J. J. Heckman, D. S. Park, and T. Rudelius, "On the Defect Group of a 6D SCFT," *Lett. Math. Phys.* **106** no. 6, (2016) 765–786, arXiv:1503.04806 [hep-th].

[78] D. S. Berman and J. A. Harvey, "The Self-Dual String and Anomalies in the M5-Brane," *JHEP* **0411** (2004) 015, arXiv:hep-th/0408198 [hep-th].

[79] M. Henningson and E. P. G. Johansson, "Dyonic Anomalies," *Phys. Lett.* **B627** (2005) 203–207, arXiv:hep-th/0508103.

[80] H. Kim and P. Yi, "D-Brane Anomaly Inflow Revisited," *JHEP* **02** (2012) 012, `arXiv:1201.0762 [hep-th]`.

[81] D. Tong, "The Holographic Dual of $AdS_3 \times S^3 \times S^3 \times S^1$," *JHEP* **04** (2014) 193, `arXiv:1402.5135 [hep-th]`.

[82] G. Gibbons, C. Pope, and H. Romer, "Index Theorem Boundary Terms for Gravitational Instantons," *Nucl.Phys.* **B157** (1979) 377.

[83] J. Erler, "Anomaly cancellation in six-dimensions," *J.Math.Phys.* **35** (1994) 1819–1833, `arXiv:hep-th/9304104 [hep-th]`.

[84] E. Witten, "Toroidal Compactification Without Vector Structure," *JHEP* **02** (1998) 006, `arXiv:hep-th/9712028`.

[85] A. Borel, R. Friedman, and J. W. Morgan, "Almost commuting elements in compact Lie groups," `arXiv:math/9907007 [math]`.

[86] R. Bott and A. S. Cattaneo, "Integral invariants of 3-manifolds," *J. Differential Geom.* **48** no. 1, (1998) 91–133, `arXiv:dg-ga/9710001`.