学位論文

# Higher-order quantum computation for equivalence determination of unitary operations

（高階量子計算を用いたユニタリ演算の同値性判別）

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# Abstract

Quantum state discrimination is one of the most fundamental tasks in quantum information since it describes the read-out process of classical information from quantum states in quantum communications. In quantum state discrimination, a given *test state* is guaranteed to be identical to one of a set of candidate quantum states. The task is to determine the identity of the test state by performing an appropriate quantum measurement based on the classical descriptions of the candidate states. Quantum state discrimination also has several connections to the principles of quantum mechanics such as the no-cloning theorem and the no-signaling principle.

One of fundamental properties of quantum mechanics is that the exact classical description of unknown quantum states cannot be retrieved from a single copy of the physical state. Quantum state discrimination in the case that the candidate states are given as two unknown physical states (*reference states*) but their classical descriptions are not given, has been investigated. The two reference states are labeled one and two, respectively. The task is to determine which of the two reference states is equivalent to the test state, whereas the descriptions of the states are not of interest. We call such a task as *equivalence determination* of quantum states.

Discrimination of quantum operations has been also investigated and it has been shown that there is an intrinsic difference between discrimination of quantum state and discrimination of quantum operations. We analyze a unitary operation version of the equivalence determination task by using the concept of *higher-order quantum computation* to investigate properties of discrimination of other quantum objects. We consider three black-boxes (*test box* and *reference boxes*) implementing unknown unitary operations given as physical systems. The test box is guaranteed to implement one of the two unitary operations given by the reference boxes. Equivalence determination of unitary operations is a task to determine the reference box implementing the same unitary operation as the one by the test box. We assume that the unitary operations of the reference boxes are randomly sampled from the Haar measure of SU(2).

We first consider the case that each of three black-boxes can be used only once both in parallel and ordered uses of the black-boxes. We show that the optimal success probability for equivalence determination is 7/8 by obtaining analytical solutions of the corresponding semidefinite programmings (SDPs) both in the parallel and ordered strategies. The optimal success probability can be achieved even without using one of the reference boxes. We also showed that entanglement of an initial state across the systems on which the reference boxes

act and the test box acts is necessary to achieve the optimal success probability. We then consider the case that the multiple uses of the reference boxes are allowed. We consider the parallel strategies up to four uses of each of the reference boxes and obtain the optimal success probabilities by numerically solving the semidefinite programmings. We numerically show that the ordered strategies give improvement over the parallel strategies and the optimal success probability varies depending on the order of the black-boxes when one reference box is used once and the other reference box is used twice. This result indicates that an appropriate order of the black-boxes under the ordered strategies is necessary to obtain improvement over the parallel strategies.

# Contents

# Chapter 1

# Introduction

Recently it has been revealed that the predictions of quantum mechanics affect various information processing tasks. There are several information processing tasks that can be performed more efficiently in the framework of quantum mechanics than in that of classical mechanics. For example, Shor's algorithm [1] enables to find prime factors of an integer in polynomial time, which can not be possible by any known algorithms for classical computers. Another example is Simon's problem [2], which was proved to be solvable strictly more efficient in the framework of quantum mechanics. The laws of quantum mechanics also impose restrictions on information processing tasks. One example is the no-cloning theorem [3] that prevents making an identical copy of an unknown quantum state. Interdisciplinary research efforts between quantum mechanics and information science comprise what is presently known as quantum information.

Information processing tasks typically combine various primitive information processing tasks. An information processing task can be described by a function, the output of which is a result of the information processing task. Discrimination is a task to determine a label of a given input when the input is chosen from a set of labeled candidates. Discrimination is used in decoding process of information processing tasks and classification of given inputs. Therefore discrimination is regarded as a fundamental information processing task.

In quantum information, *quantum state discrimination* has been investigated as a typical example of discrimination [4] . In quantum state discrimination, one quantum *test state* is chosen from a set of candidate quantum states by an apparatus. The task for a discriminator is to determine which quantum state has been chosen by performing an appropriate quantum measurement on the state generated by the apparatus. The quantum measurement is a shingle shot measurement that gives an the measurement outcome is a label to identify a chosen state, not the expectation value of a certain observable. A schematics view of discrimination of quantum states is presented in Fig. 1.1.

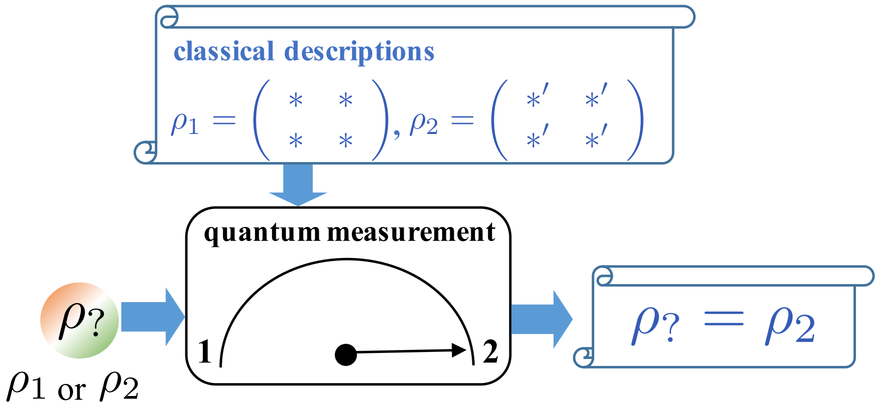



Figure 1.1: A schematic representation of quantum state discrimination

Quantum state discrimination is related to the fundamental properties of quantum mechanics. When a test state is chosen from two non-orthogonal candidate quantum states, it is impossible to perfectly determine which candidate state is identical to the test state whereas two different sequences of classical bits can always be discriminated perfectly. Perfect discrimination of non-orthogonal quantum states contradicts no-cloning theorem and no-signaling principle [5,6]. The optimality of quantum state discrimination is shown to be determined by the no-signaling principle [7].

The task of quantum state discrimination appears in various quantum information processing tasks. One example is quantum communication over a noisy transmission. In quantum communication a sender encodes a classical message on orthogonal quantum states and transmits the state to the receiver. Since the state transmitted suffers from noises over the communication, the receiver needs to discriminate non-orthogonal states to decode the message. Another example is Grover's algorithm. In Grover's algorithm, an initial state is transformed to one of candidate states that are different from each other depending on the solution of the problem. To obtain the result, it is necessary to discriminate which state is generated.

The target of quantum system to be discriminated are not limited to quantum states. Quantum operations are also regarded as a carrier of information as well as quantum states. Discrimination of quantum operations can be useful. In the example of Grover's algorithm, a solutions of the problem is considered to be encoded on an operation, not a state and discriminating the operations is necessary to obtain the result of the problem. In discrimination of quantum operations, a black-box called a *test box* that implements a quantum operation chosen from a set of quantum operations is given. The task is to determine which operation is performed by the test box. Since quantum measurements can performed only on quantum states, discrimination of quantum operations is reduced to discrimination of quantum state where the test state is obtained by applying the operation by the test box on an initial state and the candidate states are the states obtained by applying the candidate operations on the same initial states. However, there is an intrinsic difference between discrimination of quantum states and discrimination of quantum operations. It was shown that perfect discrimination is possible for discrimination of a finite number of unitary operations with the finite uses of the operation whereas perfect discrimination is not possible for non-orthogonal quantum states with the finite copies of the state [8, 9].

The goal of information theory is to construct theories that hold in every information processing device, not ones satisfied in specific information processing devices. Therefore a description of quantum information processing that does not depend on specific quantum systems is needed. The quantum circuit model [10] is a way to describe quantum information processing without depending on the specific quantum systems. Any quantum operations can be represented by a quantum circuit in the quantum circuit models. Equivalence to other models such as measurement based quantum computation [11], adiabatic quantum computation [12] and topological quantum computation [13] is shown.

Formulations of quantum information processing including quantum operations as inputs and outputs have been recently developed and such quantum information processing called *higher-order quantum computation*. One of the operations of higher-order quantum computation is *quantum supermaps* [14, 15] that describe transformations between quantum operations. Higher-order quantum computation is hierarchical since the operations of higher-order quantum computation includes quantum supermaps as well as transformations of quantum supermaps and even higher-order transformations.

Discrimination of quantum operations is also considered to be implemented by higher-order quantum computation since the goal of the task is to implement a function from a quantum operation to the label identifying one of the quantum operations in the set. Discrimination of quantum operations can be described by *quantum combs* [16–18], which are general schemes to formulate higher-order quantum computation in the quantum circuit model. Quantum combs have been shown to have various applications for discrimination and estimation of quantum operations [19], cloning of transformations [20, 21] and quantum learning [22].

In quantum state discrimination, classical descriptions of candidate states are given. Depending on the classical descriptions, an optimal quantum measurement is chosen so as to maximize a certain figure of merit such as the success probability for obtaining the correct answer. The optimal measurement depends on the candidate states and it is necessary to derive the another optimal measurement every-time the candidate states are changed. For the case of discrimination of quantum operations, derivations and adaptations of the pair of the optimal initial state and the measurement are required as well every time the candidate operations are changed since both of the optimal initial state for the operations and a measurement depend on the candidate operations for discrimination of quantum operations.

Classical descriptions of quantum states are considered as stronger resources for quantum state discrimination than the physical states when we assume the existence of a *quantum computer* that can implement any quantum operations allowed by the principles of quantum mechanics. From the classical description of a quantum state, the arbitrary number of the physical state can be prepared by using a quantum computer. In contrast, if a physical state is given but its classical description is unknown, it is impossible to exactly identify the classical description of the state by finite copies of the physical state. The no-cloning theorem prevents making identical copies of unknown states either. This difference appears in quantum mechanics, not in classical mechanics.

For quantum state discrimination, it is possible to consider the case that the candidate states for discrimination are given as physical states, but their classical descriptions are not given [23]. For the case with two candidate states, two unknown states (reference states) as candidates and a state (test state) that is guaranteed to be one of the two reference state are given as physical states. The task is to determine which of the two reference states is equivalent to the test state by performing an appropriate quantum measurement on the two reference states and the test state, whereas the classical description of the test state is not of interest. We call this task as *equivalence determination* of quantum states. The optimal measurement on the three states does not depend on the candidate states. Therefore the adaptation for the optimal measurement is not required when the candidate states are changed. This task was first introduced in [23] for a special setting and the optimal protocols in various general settings have been investigated [24–32].

We analyze a unitary operation version of the equivalence determination task to investigate properties of discrimination of other quantum objects as the next step. We consider that three black-boxes (a test box and two reference boxes) implementing unknown unitary operations are given as physical systems. The test box is guaranteed to implement one of the two unitary operations. As is the case with equivalence determination of quantum states, the two reference boxes implement the two unitary operations, respectively, instead of being given classical descriptions of the two unitary operations. Thus the equivalence determination task for unitary operations is to determine which of the two reference boxes implements the same unitary operation of the test box, whereas the classical description of the unitary operations are not of interest similarly to the case of states. For equivalence determination of unitary operations, the operations implemented by the test box and the two reference boxes are performed on an appropriate initial state to obtain an output state and the measurement outcome is obtained by performing an appropriate quantum measurement on the output state. Equivalence determination of unitary operations can be regarded as higher-order quantum computation taking three unitary operations as inputs and a binary number representing the reference box identical to the test box as an output. We analyze the optimal success probability of the protocols realizing higher-order quantum computation of equivalence determination for the case that the unitary operations are uniform-randomly sampled for a qubit system, namely the special unitary group SU(2). A schematics view of equivalence determination of unitary operations are given in Fig. 1.2.

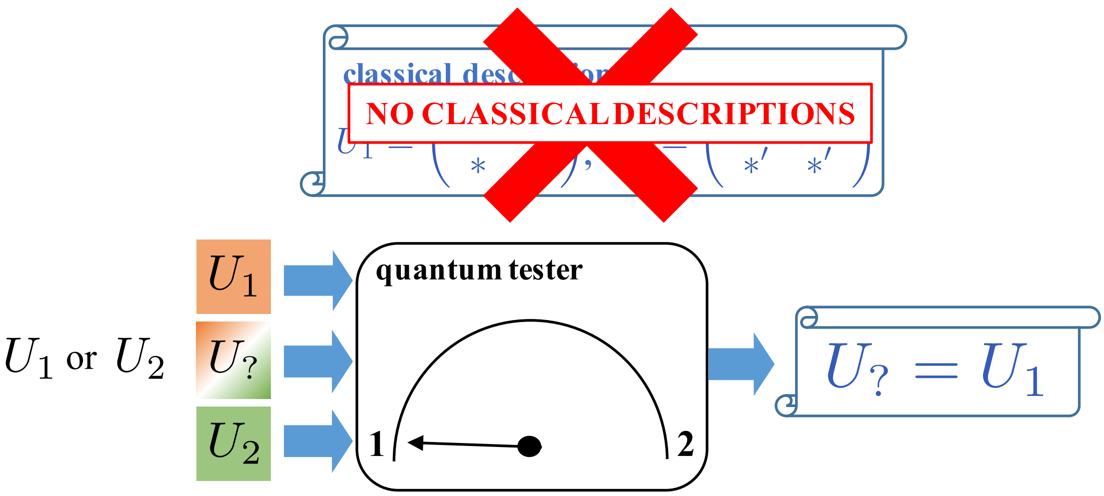



Figure 1.2: A schematics view of equivalence determination of unitary operations

One of the significant differences between equivalence determination of quantum states and unitary operations is that there is an extra degree of freedom to optimize for the case of unitary operations, an initial state for the unitary operations. Another difference is that there are two types of strategies, parallel strategies and ordered strategies for equivalence determination of unitary operations. In parallel strategies, any operations between the uses of the black-boxes is not allowed. But in general, arbitrary operations between the uses of black-boxes can be inserted, which we call the ordered strategies. There are several tasks [33–36], in which the ordered strategies outperform the parallel strategies, but conditions for improvements under the ordered strategies not yet known. Further, equivalence determination of unitary operations is a problem described by higher-order quantum computation of which inputs are more than two different operations. To the best of our knowledge, such a task considered before is only the problem of implementing quantum switch [37]. In quantum mechanics, when the order of operations is changed, the resulting operations may be different since quantum operations do not commute each other in general. Therefore the order of the use of the black-boxes is expected to affect the performance of equivalence determination of unitary operations. As an application, equivalence determination of unitary operations can be useful for checking whether two different quantum circuit representations implement the same quantum dynamics or not.

In Chap. 4, we first analyze equivalence determination unitary operations with a single use of each of the two reference boxes under both the parallel strategies and the ordered strategies, and show that the ordered use of the black-boxes does not give improvement in any order for this case. We also show that one of the two reference boxes is not necessary for achieving the optimal success probability since the optimal success probability can be also achieved by a simplified task called *comparison of two unitary operations* [38], which is a task to decide whether two black-boxes implement the same unitary operations or not. We analyze another case that the classical description of the unitary operation of one of the reference box is given and derive the optimal success probability for the cases where the other reference box is not given or allowed to use only once.

In Chap. 5, we analyze equivalence determination of unitary operations when the multiple uses of the reference boxes are allowed. We obtain the optimal success probability by formulating the optimization problem as semidefinite programmings and solving the semidefinite programmings numerically. We first consider the parallel strategies up to four uses of each of the reference boxes and see the behavior of the optimal success probability when the number of the use of the reference boxes increases. We numerically show that the ordered strategies give improvement over the parallel strategies and the optimal success probability varies depending on the deferent orders of the black-boxes when one reference box is be used once and the other reference box is be used twice.

This thesis is organized as follows. In Chap. 2 we review the basic mathematical tools used in the following chapters. In Chap. 3, we introduce quantum state discrimination, discrimination of quantum channels and related topics. In Chap. 4, we formulate and solve equivalence determination of unitary operations when each of the the reference boxes can be used only once. In Chap. 5 we investigate equivalence determination of unitary operations with the multiple uses of the reference boxes. We close this thesis with a conclusion in Chap. 6.

# Notations

- $\mathcal{H}_i$, $\mathcal{K}_j$ : Hilbert spaces labeled with $i$ and $j$.
- $\mathcal{L}(\mathcal{H})$: The set of linear operators on $\mathcal{H}$.
- $\mathcal{L}(\mathcal{H}, \mathcal{K})$: The set of linear operators from $\mathcal{H}$ to $\mathcal{K}$.
- $\mathcal{S}(\mathcal{H})$ : The set of positive semidefinite linear operation of unit trace on $\mathcal{H}$
- $\mathrm{Tr}_{\mathcal{H}}$ : The partial trace over $\mathcal{H}$
- $U^\dagger$: The conjugate transpose of $U$
- $\{\Pi_i\}$: The positive operator-valued measure
- $\{\widetilde{\Pi}_i\}$: The quantum tester
- $\delta_{ij}$: The Kronecker delta, equal to 1 if $i = j$ and 0 otherwise
- $d\mu(\cdot)$ : The Haar measure, the uniform distribution of the space of unitary operations
- $|I\rangle\rangle$ : Unnormalized maximally entangled vector defined as $|I\rangle\rangle = \sum_i |i\rangle|i\rangle$
- $\mathcal{M}$ : A quantum channel
- $M$ : Choi operator of a quantum channel $\mathcal{M}$
- $\mathcal{U}_j$ : The irreducible subspace of the total angular momentum $j$.
- $\mathcal{V}_j^{[N]}$: The multiplicity subspace of the irreducible subspace $\mathcal{U}_j$ with $N$ spin-1/2 particles
- $\lfloor N \rfloor$ : The value is equal to 0 when $N$ is even and 1/2 otherwise.
- $\|\rho\|_1$: The trace norm of $\rho$
- $\|\mathcal{M}\|_\diamond$: The diamond norm of $\mathcal{M}$
- $I_{\mathcal{H}}$ : The identity operator on $\mathcal{H}$
- $I_j$ : The identity operator on $\mathcal{U}_j$
- $\mathcal{I}_H$ : The identity channel from $\mathcal{L}(\mathcal{H})$ to $\mathcal{L}(\mathcal{H})$

# Chapter 2

# Preliminaries

In this chapter, we introduce terminologies and notations used in this thesis. In Sec. 2.1, we briefly summarize quantum states, quantum operations and quantum measurements. In Sec. 2.1.2, we introduce quantum channels that describe transformations between quantum states. In Sec. 2.1.3, two norms for quantum states and quantum channels are defined. We briefly explain the quantum circuit model in Sec. 2.2 and the Choi-Jamiołkowski isomorphism which is the duality of quantum channels and quantum states in Sec. 2.3. In Sec. 2.4, we give a detailed explanation of quantum combs, which is a main technique used in this thesis.

## 2.1 States and Operations

### 2.1.1 States

A quantum system is associated with a Hilbert space denoted as $\mathcal{H} = \mathbb{C}^d$. A linear operator $A$ on $\mathcal{H}$ is *positive semidefinite* if $\langle\psi|A|\psi\rangle \geq 0$ for an arbitrary normalized vector in $\mathcal{H}$ and that is denoted as $A \geq 0$. A quantum state of the system $\mathcal{H}$ is represented by a positive semidefinite operator $\rho \geq 0$ on $\mathcal{H}$ satisfying $\mathrm{Tr}\rho = 1$. The operator representing a state is called *density operator*. When a quantum state $\rho$ is a projective operator, *i.e.*, $\rho^2 = \rho$, there is a normalized vector $|\psi\rangle \in \mathcal{H}$ satisfying $\rho = |\psi\rangle\langle\psi|$ and the state is called a *pure state*. For the representation of a pure state, we use both a density operator $|\psi\rangle\langle\psi|$ and a ket vector $|\psi\rangle$. When a quantum state is not a pure state, the state is called a *mixed state*. In quantum information, the *computational basis* is often used for a standard basis and represented as $\{|i\rangle\}_{i=1,2,\cdots,d}$, where $\langle i|j\rangle = \delta_{ij}$ for the Kronecker delta $\delta_{ij}$. We deal with only finite dimensional Hilbert spaces and always assume that operators on the Hilbert spaces are normal in this thesis.

We also note that $A \geq B \Leftrightarrow A - B \geq 0$ for arbitrary two linear operators $A$ and $B$.

For independent two Hilbert spaces $\mathcal{H}_1$ and $\mathcal{H}_2$, the composite Hilbert space of these Hilbert spaces is represented as the tensor product of the two spaces $\mathcal{H}_1 \otimes \mathcal{H}_2$. A pure state $|\psi\rangle$ in the composite space is called a *product* state if there are two states $|\phi\rangle$ and $|\eta\rangle$ in $\mathcal{H}_1$ and $\mathcal{H}_2$, respectively, such that $|\psi\rangle = |\psi\rangle \otimes |\eta\rangle$. A pure state is called *entangled* if the state is not a product state. Similarly, a mixed state $\rho \in \mathcal{S}(\mathcal{H}_1 \otimes \mathcal{H}_2)$ is called a *product* state if the state is written as a tensor product of two mixed states in $\mathcal{S}(\mathcal{H}_1)$ and $\mathcal{S}(\mathcal{H}_1)$, respectively. A mixed state $\rho$ is called a *separable* state if $\rho$ is represented as $\rho = \sum_i p_i \sigma_i \otimes \eta_i$ with a probability distribution $\{p_i\}$ and sets of mixed states $\{\sigma_i\}$ and $\{\eta_i\}$ in $\mathcal{S}(\mathcal{H}_1)$ and $\mathcal{S}(\mathcal{H}_2)$, respectively. A mixed state is called *entangled* if the state is not separable.

We consider a bipartite Hilbert space $\mathcal{H} \otimes \mathcal{H}$ and define a unnormalized *maximally entangled* vector $|I\rangle\rangle$ as

$$|I\rangle\rangle := \sum_{i=1}^{\dim \mathcal{H}} |i\rangle|i\rangle, \tag{2.1}$$

in the computational basis $\{|i\rangle\}_{i=1}^{\dim \mathcal{H}}$. For any pure state $|\psi\rangle \in \mathcal{H} \otimes \mathcal{H}$, there exists a positive operator $X$ satisfying $\mathrm{Tr}[X] = 1$ such that $|\psi\rangle = I_{\mathcal{H}} \otimes \sqrt{X}|I\rangle\rangle$. Note that $\sqrt{X}$ is define as a positive operator satisfying $\sqrt{X}\sqrt{X} = X$. Considering the singular value decomposition of $X$, one can always find the *Schmidt decoposition* of $|\psi\rangle$ represented as

$$|\psi\rangle = \sum_{i=1}^{\dim \mathcal{H}} \sqrt{\lambda_i}|e_i\rangle|f_i\rangle, \tag{2.2}$$

where $\lambda_i$ is a non-negative real number called the *Schmidt coefficient* and $\{|e_i\rangle\}$ and $\{|f_i\rangle\}$ form orthonormal bases called the *Schmidt bases*. The number of non-zero Schmidt coefficients is called the *Schmidt rank*. When the dimensions of the two subsystems $\mathcal{H}_1$ and $\mathcal{H}_2$ are different, the Schmidt rank is at most $\min\{\dim \mathcal{H}_1, \dim \mathcal{H}_2\}$.

We call the Hilbert space $\mathcal{H}$, where $\dim \mathcal{H} = 2$ as a *qubit* system and a state in the system as a *qubit* state.

### 2.1.2 Operations

#### Completely-positive Trace-preserving Maps

Any quantum state can be represented by a density operator that is a positive semidefinite operator with unit trace. The most general deterministic transformations allowed for quantum states described by density operators are called *quantum channels* and they are given by the linear maps called *completely-positive and trace-preserving (CPTP)* maps.

For a linear map $\mathcal{M}$ from $\mathcal{L}(\mathcal{H})$ to $\mathcal{L}(\mathcal{K})$, the trace-preserving property is written as

$$\mathrm{Tr}[\mathcal{M}(A)] = \mathrm{Tr}[A] \tag{2.3}$$

for any operator $A \in \mathcal{L}(\mathcal{H})$. Thus $\mathrm{Tr}\mathcal{M}(\rho) = 1$ is guaranteed for $\mathrm{Tr}\rho = 1$ if $\mathcal{M}$ is a quantum channel. The completely-positive condition is represented as

$$(\mathcal{M} \otimes \mathcal{I}_{\mathcal{H}_A})(\rho) \geq 0, \tag{2.4}$$

for any positive semidefinite operator $\rho$ on $\mathcal{H} \otimes \mathcal{H}_A$ and any ancillary system $\mathcal{H}_A$, where $\mathcal{I}_{\mathcal{H}_A}$ is the identity channel from $\mathcal{L}(\mathcal{H}_A)$ to $\mathcal{L}(\mathcal{H}_A)$, which is defined as $\mathcal{I}_{\mathcal{H}_A}(\sigma) = \sigma$ for any $\sigma \in \mathcal{L}(\mathcal{H}_A)$. We use quantum channels to represent deterministic quantum operations that satisfy the properties of completely-positive and trace-preserving.

**Remark.** One may consider that the set of positive maps, which transforms any positive semidefinite operator to a positive semidefinite operator, is a class of most general maps describing transformations between quantum states. But it is not the case. This is because positive maps can be applied on a part of a entangled state of a composite system. In such a case, positivity of a map does not guarantee the output state to be positive. Examples of positive, but non completely-positive maps are transposition and the Choi maps [39].

There are several ways to represent a quantum channel. One useful representation is the *Kraus representation*. For any quantum channel $\mathcal{M}$ from $\mathcal{L}(\mathcal{H})$ to $\mathcal{L}(\mathcal{K})$, there is a set of linear operators $\{F_i\}$ with $F_i \in \mathcal{L}(\mathcal{H}, \mathcal{K})$ satisfying $\sum_i F_i^\dagger F_i = I_{\mathcal{H}}$ and the action of the map is represented as

$$\mathcal{M}(\rho) = \sum_i F_i \rho F_i^\dagger. \tag{2.5}$$

The right hand side of Equation (2.5) is called the Kraus representation. The sufficient number of the Kraus operators is at most $\dim \mathcal{H} \times \dim \mathcal{K}$.

A *unitary operation* is a special case of quantum channels. A unitary operation has only single Kraus operator, which is a unitary operator $U \in \mathcal{L}(\mathcal{H})$ satisfying $U^\dagger U = UU^\dagger = I_{\mathcal{H}}$, where $I_{\mathcal{H}}$ is the identity operator on $\mathcal{H}$. Unitary operators transform pure states $|\psi\rangle$ and $|\phi\rangle$ to pure states and the inner product of the two pure states is preserved under the unitary operator $\langle\psi|\phi\rangle = \langle\psi|U^\dagger U|\phi\rangle$.

One class of quantum channels used in Chap. 4 is a *random unitary* channel. A random unitary channel is associated with a probability distribution $\{p_i\}_{i=1}^N$

and a set of unitary operators $\{U_i\}_{i=1}^N$ with $U_i \in \mathcal{L}(\mathcal{H}, \mathcal{K})$. A random unitary channel $\mathcal{M}_U$ is represented as

$$\mathcal{M}_U(\rho) = \sum_{i=1}^{N} p_i U_i \rho U_i^\dagger. \tag{2.6}$$

### Quantum Measurements

A quantum measurement can be understood as a probabilistic process mapping a state to classical bits based on the Born rule. Such a quantum measurement is mathematically described by a set of operators $\{\Pi_i\}$ satisfying

$$\Pi_i \geq 0, \sum_i \Pi_i = I, \tag{2.7}$$

where $I$ is the identity operator of the system. The set $\{\Pi_i\}$ is called a *positive operator-valued measure* (POVM) and this type of measurements is called *POVM measurements*.

When a POVM measurement given by $\{\Pi_i\}$ is performed on a quantum state $\rho$, the probability for obtaining an outcome $i$ is given by the Born rule as

$$p_i = \mathrm{Tr}[\rho \Pi_i]. \tag{2.8}$$

The value $p_i$ is non-negative since the trace of the product of two positive operators is non-negative, namely, for positive operators $A$ and $B$, we have

$$\mathrm{Tr}[AB] = \mathrm{Tr}[A \sum_j b_j |\phi_i\rangle\langle\phi_j|] = \sum_j b_j \langle\phi_j|A|\phi_j\rangle \geq 0, \tag{2.9}$$

where $B = \sum_j b_j |\phi_j\rangle\langle\phi_j|$ is the eigenvalue decomposition of $B$. The sum of the $p_i$ is give by

$$\sum_i p_i = \sum_i \mathrm{Tr}[\rho \Pi_i] = \mathrm{Tr}[\rho \sum_i \Pi_i] = \mathrm{Tr}[\rho] = 1. \tag{2.10}$$

Therefore the set $\{p_i\}$ is a probability distribution.

### Quantum Instruments

A *quantum instrument* is a probabilistic process mapping a quantum state to another state labeled by an outcome. It represent a probabilistic state transformation performed by the result of a quantum measurements. A quantum instrument is mathematically described by a set of trace non-increasing and completely-positive maps $\{\mathcal{E}_i\}$, the sum of which $\mathcal{E} = \sum_i \mathcal{E}$ is a quantum channel. The set $\{\mathcal{E}_i\}$ is called a *quantum instrument*. The subscript $i$ corresponds to the measurement outcome and the post measurement state is given by $\mathcal{E}_i(\rho)$ for an input state $\rho$ of the measurement. The probability for obtaining the outcome $i$ is give by $\mathrm{Tr}[\mathcal{E}_1(\rho)]$.

### 2.1.3 Norms

In this section, we review two norms for our analysis used in the following chapters. One is for linear operators and the other is for quantum channels.

#### Trace Norms

The *trace norm* $\|\cdot\|_1$ of an linear operator $A \in \mathcal{L}(\mathcal{H})$ is defined as

$$\|A\|_1 := \mathrm{Tr}\sqrt{A^\dagger A}. \tag{2.11}$$

The trace norm of $A$ is equivalent to the sum of all singular values of $A$. For $\rho, \sigma$ in $\mathcal{S}(\mathcal{H})$, the *trace distance* is defined as

$$d_1(\rho,\sigma) = \frac{1}{2}\|\rho - \sigma\|_1. \tag{2.12}$$

The trace distance satisfies conditions for distance measures. The trace distance of two quantum states is use to represent the success probability for minimum-error discrimination of the two quantum states, which is presented in Chap. 3.

#### Diamond Norms

For a quantum channel $\mathcal{M}$ from $\mathcal{L}(\mathcal{H})$ to $\mathcal{L}(\mathcal{K})$, the *diamond norm* is define as

$$\|\mathcal{M}\|_\diamond = \sup_{\rho \in \mathcal{S}(\mathcal{H}\otimes\mathcal{H})} \|(\mathcal{M}\otimes\mathcal{I}_\mathcal{H})(\rho)\|_1. \tag{2.13}$$

The distance of two quantum channels $\mathcal{M}_1$ and $\mathcal{M}_2$ induced by the diamond norm is

$$d_\diamond(\mathcal{M}_1,\mathcal{M}_2) = \frac{1}{2}\|\mathcal{M}_1 - \mathcal{M}_2\|_\diamond. \tag{2.14}$$

We will see this distance is related to the optimal success probability in minimum-error discrimination of the two quantum channels as we will see in Chap. 3.

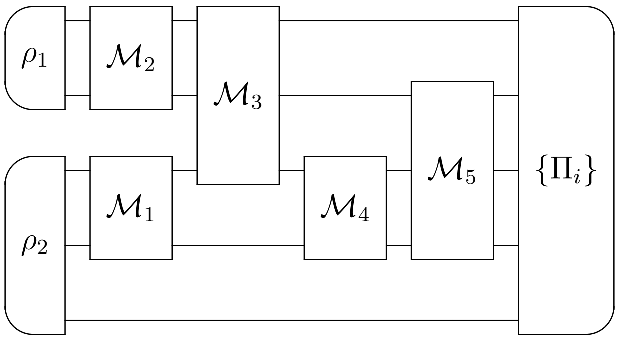



Figure 2.1: A schematic view of a quantum circuit. In this circuit, an input state is $\rho_1 \otimes \rho_2$. On the state, five quantum channels $\{\mathcal{M}_i\}_{i=1}^5$ are applied, followed by a POVM measurement $\{\Pi_i\}$. At this stage the outcome of the POVM measurement is not yet specified.

## 2.2 Quantum Circuit Model

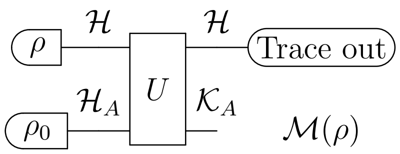
There are several models for quantum computation, for example the quantum circuit model [10], measurement based quantum computation [11], adiabatic quantum computation [12] and topological quantum computation [13]. In this section, we present a brief explanation for the quantum circuit model used for the analysis in this thesis. We follow the notation presented in [17]. The quantum circuit model consists of quantum states, quantum operations and horizontal lines (sometimes called quantum wires) connecting states and operations (see Fig. 2.1). Input states are placed on the far left side of the circuit and a sequence of operations are applied from left to right along the horizontal lines. When a state enters a box representing a quantum channel, the channel is applied on the state. When horizontal lines enter a box representing a POVM measurement, the POVM measurement is performed on the state and the measurement outcome is obtained.

Any quantum channel can be represented by adding an auxiliary system, prepared in a fixed state, applying a unitary operations on the composite system and tracing out the auxiliary system. This representation is called the *Stinespring* representation. In the Stinespring representation, an input state $\rho \in \mathcal{S}(\mathcal{H})$ and a fixed state $\rho_0 \in \mathcal{S}(\mathcal{H}_A)$ are prepared. A unitary operator $U$ from $\mathcal{H} \otimes \mathcal{H}_A$ to $\mathcal{K} \otimes \mathcal{K}_A$ is applied. Finally the auxiliary system $\mathcal{H}_A$ is discarded by partially tracing out. For any quantum channel $\mathcal{M}$ from $\mathcal{L}(\mathcal{H})$ to $\mathcal{L}(\mathcal{K})$, there always exists a unitary operator $U$ from $\mathcal{H} \otimes \mathcal{H}_A$ to $\mathcal{K} \otimes \mathcal{K}_A$ and a fixed state

$\rho_0 \in \mathcal{S}(\mathcal{H}_A)$ such that

$$\mathcal{M}(\rho) = \mathrm{Tr}_{\mathcal{K}_A}[U_{\mathcal{M}}(\rho \otimes \rho_0)U_{\mathcal{M}}^{\dagger}]. \tag{2.15}$$

Note that the fixed state $\rho_0$ can be chosen to be independent on $\mathcal{M}$. The circuit for the Stinspring representation of $\mathcal{M}$ is given in Fig. 2.2.




(2.16)

Figure 2.2: The Stinespring representation for a quantum channel $\mathcal{M}$. The state $\rho_0$ in the ancillary system is fixed.

## 2.3 Choi-Jamiołkowski Isomorphism

To analyze quantum channels the channel-state duality is useful. We consider a linear map $\mathcal{M}$ from $\mathcal{L}(\mathcal{H})$ to $\mathcal{L}(\mathcal{K})$. The *Choi operator* $M$ of the linear map $\mathcal{M}$ is defined as

$$M := (\mathcal{M} \otimes \mathcal{I})(|I\rangle\rangle\langle\langle I|) \in \mathcal{L}(\mathcal{K} \otimes \mathcal{H}), \tag{2.17}$$

where $|I\rangle\rangle$ is an unnormalized maximally entangled vector defined by Equation (2.1). Here we use the same character $\mathcal{H}$ for representing the two systems. When we apply a CPTP map on the part of a bipartite system, we always assume that the CPTP map is applied on the first system of the tensor product $\mathcal{H} \otimes \mathcal{H}$ and the second one is the ancillary system. Thus the linear map $\mathcal{M}$ is acting on the left system of $\mathcal{H} \otimes \mathcal{H}$.

The Choi operator $M$ has complete information about the map $\mathcal{M}$. With the Choi operator, the action of the original map can be described as

$$\mathcal{M}(\rho) = \mathrm{Tr}_{\mathcal{H}}[M(I_{\mathcal{K}} \otimes \rho^T)], \tag{2.18}$$

for an input state $\rho \in \mathcal{S}(\mathcal{H})$ .

The following proposition gives a relation between a channel and the corresponding Choi operator [40].

**Proposition 1.** *For a linear map $\mathcal{M}$ from $\mathcal{L}(\mathcal{H})$ to $\mathcal{L}(\mathcal{K})$, we have*

- *$\mathcal{M}$ is completely positive if and only if the corresponding Choi operator $M$ is positive semidefinite.*

- *$\mathcal{M}$ is trace preserving if and only if the corresponding Choi operator $M$ satisfies* $\mathrm{Tr}_{\mathcal{H}} M = I_{\mathcal{K}}$.

An important point is the map $\mathcal{M} \to M$ is a bijection.

## 2.4 Quantum Combs

In this section, we introduce *quantum combs*, which are the Choi operators representing a transformation between quantum operations, formulated in [16–18]. We first define the *link product* of the two Choi operators and describe the concatenation of two quantum channels in terms of the corresponding Choi operators. Then we introduce quantum 2-combs transforming a single quantum operation to another quantum operation. After that, deterministic quantum combs are introduced. Finally probabilistic quantum combs and *quantum testers* are defined.

### 2.4.1 Deterministic Quantum Combs

#### The Link Product

For two CPTP maps $\mathcal{M}_1$ from $\mathcal{L}(\mathcal{H}_1)$ to $\mathcal{L}(\mathcal{H}_2)$ and $\mathcal{M}_2$ from $\mathcal{L}(\mathcal{H}_2)$ to $\mathcal{L}(\mathcal{H}_3)$, we denote the corresponding Choi operators as $M_1$ and $M_2$, respectively. We first derive the Choi operator of the composition of the two channels $\mathcal{M}_3 := \mathcal{M}_2 \circ \mathcal{M}_1$ from $\mathcal{L}(\mathcal{H}_1)$ to $\mathcal{L}(\mathcal{H}_3)$ (see Fig. 2.3). When an input state $\rho_{in} \in \mathcal{S}(\mathcal{H}_1)$ is transformed by the channel $\mathcal{M}_3$, the output state $\rho_{out} \in \mathcal{S}(\mathcal{H}_3)$ is represented as

$$\rho_{out} = \mathrm{Tr}_{\mathcal{H}_2}[M_2(I_{\mathcal{H}_3} \otimes (\mathrm{Tr}_{\mathcal{H}_1}[M_1(I_{\mathcal{H}_2} \otimes \rho^{T_{\mathcal{H}_1}})])^{T_{\mathcal{H}_2}}]) \tag{2.19}$$

$$= \mathrm{Tr}_{\mathcal{H}_1}[\mathrm{Tr}_{\mathcal{H}_2}[(M_2 \otimes I_{\mathcal{H}_1})(I_{\mathcal{H}_3} \otimes M_1^{T_{\mathcal{H}_2}})](I_{\mathcal{H}_3} \otimes \rho^T)], \tag{2.20}$$

where we used Equation (2.18) to represent the channel in terms of the Choi operator $M_i$. Using Equation (2.18) with the Choi operator of $\mathcal{M}_3$, $\rho_{out}$ is also represented as

$$\rho_{out} = \mathrm{Tr}_{\mathcal{H}_1}[M_3(I_{\mathcal{H}_3} \otimes \rho^T)]. \tag{2.21}$$

Since the Choi isomorphism $\mathcal{M} \to M$ is a bijection, the Choi operator of $\mathcal{M}_3$ is represented as

$$M_3 = \mathrm{Tr}_{\mathcal{H}_2}[(M_2 \otimes I_{\mathcal{H}_1})(I_{\mathcal{H}_3} \otimes M_1^{T_{\mathcal{H}_2}})]. \tag{2.22}$$

To generalize the way to represent the Choi operator corresponding to the quantum channels, we define the *link product* $*$ of two operators.

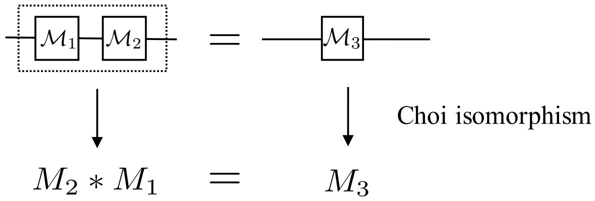



Figure 2.3: The concatenation of two quantum channels $\mathcal{M}_1$ and $\mathcal{M}_2$. The Choi operator $M_3$ of a composite map $\mathcal{M}_3 = \mathcal{M}_2 \circ \mathcal{M}_1$ is given by the link product $M_2 * M_3$.

**Definition 1.** *Let $C$ be an operator in $\mathcal{L}(\bigotimes_{i\in \mathrm{I}} \mathcal{H}_i)$ and $D$ be an operator in $\mathcal{L}(\bigotimes_{j\in \mathrm{J}} \mathcal{H}_j)$ with finite sets of indices* I *and* J. *The link product $*$ of the two operators $C$ and $D$ is defined as*

$$C * D := \mathrm{Tr}_{\mathcal{H}_{\mathrm{I}\bigcap \mathrm{J}}}[(I_{\mathcal{H}_{\mathrm{J}\backslash \mathrm{I}}} \otimes C)(I_{\mathcal{H}_{\mathrm{I}\backslash \mathrm{J}}} D^{T_{\mathcal{H}_{\mathrm{I}\cap \mathrm{J}}}})], \tag{2.23}$$

*where* $\mathrm{I}\backslash \mathrm{J} = \{i \in \mathrm{I} | i \notin \mathrm{J}\}$ *and* $\mathcal{H}_{\mathrm{K}} = \bigotimes_{k\in \mathrm{K}} \mathcal{H}_k$.

The Choi operator of a composition of two general CPTP maps is derived by the following theorem.

**Theorem 1.** *Let $\mathcal{M}$ be a quantum channel from $\mathcal{L}(\bigotimes_{i\in \mathrm{I}_\mathcal{M}} \mathcal{H}_i)$ to $\mathcal{L}(\bigotimes_{j\in \mathrm{O}_\mathcal{M}} \mathcal{H}_j)$ and $\mathcal{N}$ be a quantum channel from $\mathcal{L}(\bigotimes_{i\in \mathrm{I}_\mathcal{N}} \mathcal{H}_i)$ to $\mathcal{L}(\bigotimes_{j\in \mathrm{O}_\mathcal{N}} \mathcal{H}_j)$, where the sets of finite indices $\mathrm{I}_\mathcal{M}$ and $\mathrm{O}_\mathcal{M}$ ($\mathrm{I}_\mathcal{N}$ and $\mathrm{O}_\mathcal{N}$) specify the input and output Hilbert spaces of the channel $\mathcal{M}(\mathcal{N})$ with $\mathrm{O}_\mathcal{M} \bigcap \mathrm{I}_\mathcal{N}$. Let $M$ and $N$ be the Choi operators of $\mathcal{M}$ and $\mathcal{N}$, respectively. Then the Choi operator of the composition of $\mathcal{M}$ and $\mathcal{N}$ is given by $M * N$.*

**Proof.** Calculate the action of the composite channel

$$((M) \otimes \mathcal{I}_{\mathcal{H}_{\mathrm{O}_\mathcal{N}\backslash \mathrm{I}_\mathcal{M}}})(\mathcal{I}_{\mathcal{H}_{\mathrm{I}_\mathcal{M}\backslash \mathrm{O}_\mathcal{N}}} \otimes \mathcal{N})$$

of $\mathcal{M}$ and $\mathcal{N}$ in two ways as we showed for $\mathcal{M}_3 = \mathcal{M}_2 \circ \mathcal{M}_1$, $\mathcal{M}_1$ and $\mathcal{M}_2$, and use the bijective property between the Choi operator and a quantum channel. □

### Quantum 2-Combs

To introduce quantum combs, we consider a composite channel of two quantum channels linked subsystems of each channel. For a quantum channel $\mathcal{M}_1$ from $\mathcal{L}(\mathcal{K}_0)$ to $\mathcal{L}(\mathcal{H}_1 \otimes \mathcal{H}_{A_{12}})$ and a quantum channel $\mathcal{M}_2$ from $\mathcal{L}(\mathcal{K}_1 \otimes \mathcal{H}_{A_{12}})$ to $\mathcal{L}(\mathcal{H}_2)$, the Choi operator $M \in \mathcal{L}(\mathcal{H}_2 \otimes \mathcal{K}_1 \otimes \mathcal{H}_1 \otimes \mathcal{K}_0)$ of the composite channel of $\mathcal{M}_1$ and $\mathcal{M}_2$ from $\mathcal{L}(\mathcal{K}_0 \otimes \mathcal{K}_1)$ to $\mathcal{L}(\mathcal{H}_1 \otimes \mathcal{H}_2)$ shown in Fig. 2.4 is represented by

$$M = M_2 * M_1 = \mathrm{Tr}_{\mathcal{H}_{A_{12}}}[(M_2 \otimes I_{\mathcal{H}_1\mathcal{K}_0})(I_{\mathcal{H}_2\mathcal{K}_1} \otimes M_1^{T_{\mathcal{H}_{A_{12}}}}). \tag{2.24}$$

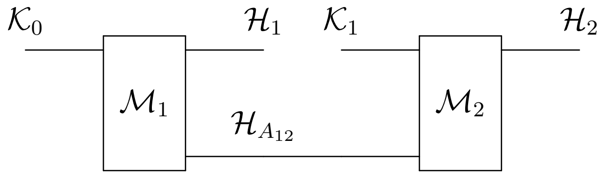



Figure 2.4: Two quantum channels $\mathcal{M}_1$ and $\mathcal{M}_2$ are connected by an ancillary system $\mathcal{H}_{A_{12}}$.

We consider conditions for the Choi operator $M$ of a composite channel to satisfy. First, $M$ is positive semidefinite since $M$ is the Choi operator of a quantum channel. Taking the partial trace on $\mathcal{H}_2$, we obtain

$$\mathrm{Tr}_{\mathcal{H}_2} M = \mathrm{Tr}_{\mathcal{H}_{A_{12}}}[(\mathrm{Tr}_{\mathcal{H}_2}[M_2] \otimes I_{\mathcal{H}_1\mathcal{K}_0})(I_{\mathcal{K}_1} \otimes M_1^{T_{\mathcal{H}_{A_{12}}}}) \tag{2.25}$$
$$= \mathrm{Tr}_{\mathcal{H}_{A_{12}}}[(I_{\mathcal{K}_1} \otimes I_{\mathcal{H}_1\mathcal{K}_0})(I_{\mathcal{K}_1} \otimes M_1^{T_{\mathcal{H}_{A_{12}}}}) \tag{2.26}$$
$$= I_{\mathcal{K}_1} \otimes \mathrm{Tr}_{\mathcal{H}_{A_{12}}}[M_1^{T_{\mathcal{H}_{A_{12}}}}] \tag{2.27}$$
$$= I_{\mathcal{K}_1} \otimes M^{(1)}, \tag{2.28}$$

where we define $M^{(1)} := \mathrm{Tr}_{\mathcal{H}_{A_{12}}}[M_1^{T_{\mathcal{H}_{A_{12}}}}] = \mathrm{Tr}_{\mathcal{H}_{A_{12}}}[M_1]$. Taking the partial trace on $\mathcal{H}_1$ for $M^{(1)}$, we obtain

$$\mathrm{Tr}_{\mathcal{H}_1} M^{(1)} = \mathrm{Tr}_{\mathcal{H}_1\mathcal{H}_{A_{12}}}[M_1] = I_{\mathcal{K}_0}. \tag{2.29}$$

Therefore $M$ satisfies

$$M \geq 0, \mathrm{Tr}_{\mathcal{H}_2} = I_{\mathcal{K}_1} \otimes M^{(1)}, \mathrm{Tr}_{\mathcal{H}_1} M^{(1)} = I_{\mathcal{K}_0}. \tag{2.30}$$

Conversely, for an operator $M \in \mathcal{L}(\mathcal{H}_2 \otimes \mathcal{K}_1 \otimes \mathcal{H}_1 \otimes \mathcal{K}_0)$ satisfying Conditions (2.30) for a certain positive semidefinite operator $M^{(1)}$, it is shown in [18] that there exist quantum channels $M_1$ from $\mathcal{L}(\mathcal{K}_0)$ to $\mathcal{L}(\mathcal{H}_1 \otimes \mathcal{H}_{A_{12}})$ and $\mathcal{M}_2$ from $\mathcal{L}(\mathcal{K}_1 \otimes \mathcal{H}_{A_{12}})$ to $\mathcal{L}(\mathcal{H}_2)$ such that the Choi operator of the composite channel of $\mathcal{M}_1$ and $\mathcal{M}_2$ is $M$ .

This type of the Choi operator $M$ is called a *quantum comb*, more specifically, a *quantum 2-comb*. The quantum 2-comb represents a transformation between quantum channels. Consider a quantum channel $\mathcal{N}$ from $\mathcal{L}(\mathcal{H}_1)$ to $\mathcal{L}(\mathcal{K}_1)$. Connecting the output of $\mathcal{M}_1$ to the input of $\mathcal{N}$ and the output of $\mathcal{N}$ to the input of $\mathcal{M}_2$, we obtain a quantum channel $\mathcal{R}$ from $\mathcal{L}(\mathcal{K}_0)$ to $\mathcal{L}(\mathcal{H}_2)$ as a sequence of

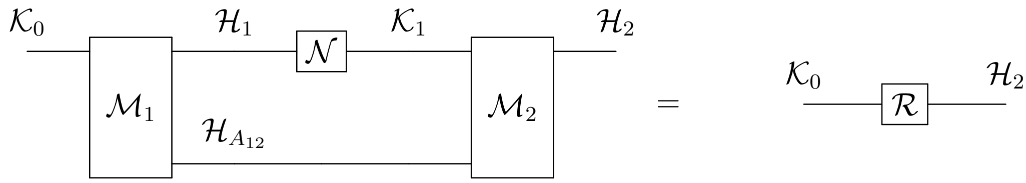



Figure 2.5: A quantum channel $\mathcal{N}$ from $\mathcal{L}(\mathcal{H}_1)$ to $\mathcal{L}(\mathcal{K}_1)$ is inserted into the open slot between $\mathcal{H}_1$ and $\mathcal{K}_1$ of $M = M_2 * M_1$ shown in Figure 2.4. The resulting channel is $\mathcal{R}$ from $\mathcal{L}(\mathcal{K}_0)$ to $\mathcal{L}(\mathcal{H}_2)$. This can be considered that $\mathcal{N}$ is transformed to $\mathcal{R}$ by applying $\mathcal{M}_1$ and $\mathcal{M}_2$ before and after $\mathcal{N}$.

the three quantum channels $\mathcal{M}_1, \mathcal{N}$ and $\mathcal{M}_2$ as shown in Fig. 2.5. In terms of the corresponding Choi operators $M$ given by Equation (2.24) and $N$, the Choi operator $R$ of the resulting channel $\mathcal{R}$ is represented as

$$R = M * N \tag{2.31}$$
$$= \mathrm{Tr}_{\mathcal{H}_1\mathcal{K}_1}[M(I_{\mathcal{H}_2\mathcal{K}_0} \otimes N^T)]. \tag{2.32}$$

That is, the quantum 2-comb $M$ can be describes the transformation from $N$ to $R$.

### Deterministic Quantum Combs

The quantum 2-comb can be generalized by considering the composition of arbitrary number of quantum channels. Let $\mathcal{M}_i$ be a quantum channel from $\mathcal{L}(\mathcal{K}_i \otimes \mathcal{H}_{A_{i-1i}})$ to $\mathcal{L}(\mathcal{H}_i \otimes \mathcal{H}_{A_{ii+1}})$ for $i = 1, 2, \cdots, N$, where we set $\mathcal{H}_{A_{01}} = \mathcal{H}_{A_{NN+1}} = \mathbb{C}$. Denoting the Choi operator of $\mathcal{M}_i$ as $M_i$, the Choi operator of the composite channel of these $N$ quantum channels is given by

$$M = M_1 * M_2 * \cdots * M_N. \tag{2.33}$$

The resulting $M$ satisfies

$$M \geq 0, \quad \mathrm{Tr}_{\mathcal{H}_i} M^{(N)} = I_{\mathcal{K}_i} \otimes M^{(i-1)}, \tag{2.34}$$

for $i = 1, 2, \cdots, N$ where $M = M^{(N)}$ and $M^{(0)} = 1$.

Conversely, for an operator $M$ satisfying Equation (2.34), it is shown in [18] that there exists a set of quantum channels $\{\mathcal{M}_i\}$ from $\mathcal{L}(\mathcal{K}_{i-1} \otimes \mathcal{H}_{A_{i-1i}})$ to $\mathcal{L}(\mathcal{H}_i \otimes \mathcal{H}_{A_{ii+1}})$ for $i = 1, 2, \cdots, N$, where $\mathcal{H}_{A_{01}} = \mathcal{H}_{A_{NN+1}} = \mathbb{C}$ such that the Choi operator of the composite channel of $\mathcal{M}_i$'s is given by $M$. This $M$ is called quantum $N$-comb [18].

### 2.4.2 Probabilistic Quantum Combs

Probabilistic transformations of quantum channels can be also formulated in a similar way of quantum combs. A probabilistic quantum comb $F$ is the Choi operator of a composition of a set of completely-positive, trace non-increasing maps $\{\mathcal{C}_i\}$. The condition for the operator $F$ to be a probabilistic quantum comb is the existence of a deterministic quantum comb $S$ such that $0 \le F \le S$.

In addition, a *generalized instrument* is defined as a set of probabilistic quantum combs such that the sum of the probabilistic quantum combs is a deterministic quantum comb. Generalized instruments can be represented as a sequence of quantum channels and a POVM measurement on the ancillary system.

#### Quantum Testers

A generalized instrument is called a *quantum tester* when $\dim(\mathcal{K}_0) = \dim(\mathcal{H}_N) = 1$. The quantum tester can be regarded as a generalized POVM measurement, which is a probabilistic transformation from a quantum quantum comb to a real number. Since a quantum tester is one of the main techniques used in this thesis, we give the rigorous definition.

**Definition 2.** *Quantum $N$-tester is a set of operators $\{\widetilde{\Pi}_i\}$ when each $\widetilde{\Pi}_i \in \mathcal{L}(\bigotimes_{i=1}^{N-1} \mathcal{H}_i \otimes \bigotimes_{j=1}^{N-1} \mathcal{K}_j)$ satisfies*

$$\widetilde{\Pi}_i \ge 0 \tag{2.35}$$

$$\sum_i \widetilde{\Pi}_i = I_{\mathcal{K}_N} \otimes X^{(N-1)} \tag{2.36}$$

$$\mathrm{Tr}_{\mathcal{H}_j} X^{(j)} = I_{\mathcal{K}_{j-1}} \otimes X^{(j-1)}, \textit{for } j = 2, \cdots, N-1 \tag{2.37}$$

$$\mathrm{Tr} X^{(1)} = 1. \tag{2.38}$$

*When the quantum $N$-tester $\{\widetilde{\Pi}_i\}$ is combined with a quantum $(N-1)$-comb $R$, the probability obtaining the outcome $i$ is given by*

$$p_i = \mathrm{Tr}[M_i R^T]. \tag{2.39}$$

For distinguishing a quantum tester and a POVM, we represent an element of a quantum tester as $\widetilde{\Pi}_i$. Quantum 2-testers describe both of the initial state on the part of which a quantum channel is applied and a quantum measurement after the application of the quantum channel.

# Chapter 3

# Discrimination of States and Operations

In this chapter, we review several discrimination tasks studied in quantum information. In Sec. 3.1, we give a formulation of quantum state discrimination, which is one of the fundamentally important tasks in quantum information as a way to decode classical information encoded in quantum states. In Sec. 3.2, we formulate discrimination of quantum channels. In particular, a detailed explanation about discrimination of unitary operations is given in order to show an intrinsic difference between discrimination of states and unitary operations in Sec. 3.2.2. In Sec. 3.3, we review a universal state discriminator, in which the description of candidate states are not given by classical information, but given as an unknown quantum states. In Sec. 3.4, we introduce tasks called comparison of quantum states and comparison of unitary operations, which are special cases of the tasks analyzed in the following chapters.

## 3.1 Quantum State Discrimination

### 3.1.1 Setting for Quantum State Discrimination

We consider a set of states $\{\rho_i\}_{i=1}^N$ in $\mathcal{S}(\mathcal{H})$ and the corresponding probability distribution $\{q_i\}_{i=1}^N$ of which descriptions are given, that is, the matrix representation of $\rho_i$ and the probability distribution $\{q_i\}$ are provided. An apparatus generates a state $\rho_i$ with the probability $q_i$. A discrimination task is to determine which quantum state is generated by the apparatus by a measurement on the generated state. Note that the goal of the task is not to identify the matrix representation of the state.

The task is analyzed by using a POVM measurement given by $\{\Pi_i\}_{i=1}^L$, $L$

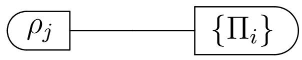



Figure 3.1: A schematic figure to represent the discrimination tasks. An apparatus generates a state $\rho_i$ chosen from $\{\rho_i\}$ and a POVM $\{\Pi_i\}$ is applied by the discriminator.

is the number of the outcomes. When a state $\rho_i$ is generated, the conditional probability $p(k|i)$ to obtain measurement outcome $k$ is given by

$$p(k|i) = \mathrm{Tr}[\rho_i \Pi_k]. \tag{3.1}$$

A quantum circuit for the representation of the task is shown in Fig. 3.1.

The POVM can be optimized according to the set of states $\{\rho_i\}$ and the probability distribution $\{q_i\}$ to maximize a certain figure of merit. For quantum state discrimination, mainly two different figures of merit have been analyzed. We review the two methods, *minimum-error* discrimination and *unambiguous* discrimination, but more focus on the minimum-error discrimination, which is used in this thesis.

### 3.1.2 Minimum-error Discrimination

In minimum-error state discrimination, the discriminator tries to minimize the error probability over all states in the set (or equivalently, maximize the success probability for obtaining correct guesses). The correct guess is achieved by obtaining the measurement outcome $j$ when the generated state is $\rho_j$. We can assume $L = N$ since $N$ is sufficient for discriminating all different states by different outcomes. The average success probability $p_{succ}$ is given by

$$p_{succ} = \sum_{i=1}^{N} p(i,i) = \sum_{i=1}^{N} p(i|i) q_i = \sum_{i=1}^{N} q_i \mathrm{Tr}[\Pi_i \rho_i]. \tag{3.2}$$

Maximizing such a success probability is a kind of convex optimization problem called a semidefinite programming (SDP). The optimization problem can be rewritten in a standard form as presented in Appx. A.

Using the formulation of the minimum-error state discrimination problem as an SDP, we can numerically calculate the optimal success probability and the optimal POVM. However the dimension of the system is $2^M$ for a $M$-qubit systems and numerical analysis is difficult for large quantum systems.

The optimal POVM is not necessarily unique and it is difficult to analytically obtain the optimal POVM and the optimal success probability in general.

However for specific cases, the optimal POVM and the optimal success probability can be derived [4, 41–43]. We explain out of such cases, the minimum-error discrimination of just two states and give the closed formulas.

### Minimum-error Discrimination of Two Quantum States

For discriminating two states $\rho_1$ and $\rho_2$ in $\mathcal{S}(\mathcal{H})$, the optimization problem is described as

$$\begin{aligned} \text{maximize} \quad & p_{succ} = \mathrm{Tr}[q_1\rho_1\Pi_1 + q_2\rho_2\Pi_2] && (3.3)\\ \text{subject to} \quad & \Pi_1, \Pi_2 \geq 0 \\ & \Pi_1 + \Pi_2 = I_{\mathcal{H}}, && (3.4)\end{aligned}$$

where $I_{\mathcal{H}}$ is the identity operator on $\mathcal{H}$. By introducing an operator $\Pi$, we define $\Pi$ as

$$\Pi_1 = \frac{1}{2}(I_{\mathcal{H}} + \Pi),\ \Pi_2 = \frac{1}{2}(I_{\mathcal{H}} - \Pi). \tag{3.5}$$

To satisfy the positivity condition for $\Pi_1$ and $\Pi_2$, the operator $\Pi$ should satisfy $-I_{\mathcal{H}} \leq \Pi \leq I_{\mathcal{H}}$. Substituting $\Pi_1$ and $\Pi_2$ given in Equation (3.5) into Equation (3.3), we have

$$\begin{aligned} p_{succ} &= \frac{1}{2}\mathrm{Tr}[q_1\rho_1(I_{\mathcal{H}} + \Pi) + q_2\rho_2(I_{\mathcal{H}} - \Pi)] && (3.6)\\ &= \frac{1}{2}\mathrm{Tr}[(q_1\rho_1 + q_2\rho_2) + (q_1\rho_1 - q_2\rho_2)\Pi] && (3.7)\\ &= \frac{1}{2} + \frac{1}{2}\mathrm{Tr}[(q_1\rho_1 - q_2\rho_2)\Pi], && (3.8)\end{aligned}$$

where we used $\mathrm{Tr}[\rho_1] = \mathrm{Tr}[\rho_2] = 1$ and $q_1 + q_2 = 1$.

The spectral decomposition of the operator $q_1\rho_1 - q_2\rho_2$ is represented by

$$q_1\rho_1 - q_2\rho_2 = \sum_{i=1}^{d_+} \lambda_i^+ |\psi_i^+\rangle\langle\psi_i^+| + \sum_{i=1}^{d_-} \lambda_i^- |\psi_i^-\rangle\langle\psi_i^-|, \tag{3.9}$$

where $\lambda_i^+ (\lambda_i^-)$ is the non-negative (non-positive) eigenvalue, $|\phi_i^\pm\rangle$ is the eigenvector and $d_+ + d_- = d$. Then the operator $\Pi$ maximizing the success probability is given by

$$\Pi = \sum_{i=1}^{d_+} |\psi_i^+\rangle\langle\psi_i^+| - \sum_{i=1}^{d_-} |\psi_i^-\rangle\langle\psi_i^-|. \tag{3.10}$$

Thus the optimal success probability is given by

$$\begin{aligned} p_{succ}^{opt} &= \frac{1}{2} + \frac{1}{2}\Big(\sum_{i=1}^{d_+} \lambda_i^+ - \sum_{i=1}^{d_-} \lambda_i^-\Big) && (3.11)\\ &= \frac{1}{2} + \frac{1}{2}\|q_1\rho_1 - q_2\rho_2\|_1, && (3.12)\end{aligned}$$

where $\| \cdot \|_1$ is the trace norm introduced in Sec 2.1.3 and the optimal POVM is given by

$$\Pi_1 = \sum_{i=1}^{d_+} |\psi_i^+\rangle\langle\psi_i^+|, \ \Pi_2 = \sum_{i=1}^{d_-} |\psi_i^-\rangle\langle\psi_i^-|, \tag{3.13}$$

which are the projections onto the positive and the negative part of $q_1\rho_1 - q_2\rho_2$.

### 3.1.3 Unambiguous Discrimination

In minimum-error discrimination, the measurement outcome $i$ does not necessary imply the generated state is certainly $\rho_i$ if the states in $\{\rho_1\}$ are not mutually orthogonal. In unambiguous state discrimination, we can set that the measurement result $i$ indicates the prepared state is $\rho_i$ with confidence by introducing more than $N$ outcomes. This condition is represented as

$$\mathrm{Tr}[\Pi_i \rho_j] = 0 \text{ for } i \neq j \text{ and } i, j = 1, 2, \cdots, N. \tag{3.14}$$

The additional POVM element $\Pi_{N+1}$ indicates the inconclusive result. Note that unambiguous state discrimination is not always possible. For pure states, it is only possible for a linearly independent set of states [44] and for mixed states, the support of each state in the set $\{\rho_i\}$ has not to be overlapped [45].

Unambiguous state discrimination is also formulated as a semidefinite programming. The optimization problem is represented as

$$\text{maximize} \quad p_{succ} = \sum_{i=1}^{N} q_i \mathrm{Tr}[\Pi_i \rho_i] \tag{3.15}$$

$$\text{subject to} \quad \Pi_i \geq 0 \tag{3.16}$$

$$\sum_{i=1}^{N+1} \Pi_i = I_{\mathcal{H}} \tag{3.17}$$

$$\mathrm{Tr}[\Pi_i \rho_j] = 0 \text{ for } i \neq j, \text{ and } i, j = 1, 2, \cdots, N. \tag{3.18}$$

Derivation of the optimal POVM and the optimal success probability for unambiguous state discrimination is also not easy in general. Recently a geometric approach for unambiguous discrimination for pure states is analyzed [46]. As applications of unambiguous discrimination, quantum state comparison [47] and state filtering [48] are introduced.

## 3.2 Discrimination For Quantum Channels

In this section, we review a task to discriminate a set of quantum channels. First, the difference between discrimination of quantum states and quantum channels is discussed. Detailed discussion on the case where quantum channels are restricted to unitary operations.

### 3.2.1 Formulation

We consider a set of quantum channels $\{\mathcal{M}_i\}_{i=1}^N$ from $\mathcal{L}(\mathcal{H})$ to $\mathcal{L}(\mathcal{K})$. The classical description of the channels, for instance matrix representations of Kraus operators, Choi operators or unitary operators for Stinespring representation of each channel, is given. Instead of apparatus generating a state as in quantum state discrimination, a black-box that implements one of the channels with probability $q_i$ is given. The task in this case is to determine which channel is implemented by the black-box.

To determine the channel, we can choose an initial state to apply the channel in question implemented by the black-box. Then we obtain an output state of the channel, on which a POVM measurement is performed. In general, one can prepare an initial state entangled with an ancillary system $\mathcal{H}_A$.

We denote an initial state as $\rho \in \mathcal{S}(\mathcal{H} \otimes \mathcal{H}_A)$ and a POVM $\{\Pi_j\}_{j=1}^L$ with $L$ outcomes satisfying $\Pi_i \geq 0$ and $\sum_{j=1}^L \Pi_j = I_{\mathcal{K}\mathcal{H}_A}$. The conditional probability of obtaining the measurement outcome $j$ when the implemented channel is $M_i$ is given by

$$p(j|i) = q_i \mathrm{Tr}[\Pi_j(\mathcal{M}_j \otimes \mathcal{I}_{\mathcal{H}_A})(\rho))]. \tag{3.19}$$

For minimum-error discrimination of quantum channels, it is enough to take $L = N$ and the optimization problem is given by

$$\begin{aligned} \text{maximize} \quad & p_{succ} = \sum_{i=1}^N q_i \mathrm{Tr}[\Pi_i(\mathcal{M}_i \otimes \mathcal{I}_{\mathcal{H}_A})(\rho))] && (3.20) \\ \text{subject to} \quad & \rho \in \mathcal{S}(\mathcal{H} \otimes \mathcal{H}_A) && (3.21) \\ & \Pi_j \geq 0, \sum_j^N \Pi_j = I_{\mathcal{K}\mathcal{H}_A} && (3.22) \end{aligned}$$

For unambiguous discrimination of quantum channels, take $L = N + 1$ and the optimization problem is give by

$$\begin{aligned} \text{maximize} \quad & p_{succ} = \sum_{i=1}^N q_i \mathrm{Tr}[\Pi_i(\mathcal{M}_i \otimes \mathcal{I}_{\mathcal{H}_A})(\rho))] && (3.23) \\ \text{subject to} \quad & \rho \in \mathcal{S}(\mathcal{H} \otimes \mathcal{H}_A) && (3.24) \\ & \Pi_j \geq 0, \sum_j^N \Pi_j = I_{\mathcal{K}\mathcal{H}_A} && (3.25) \end{aligned}$$

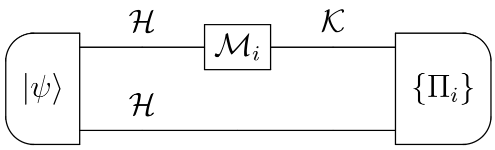



Figure 3.2: A quantum channel $\mathcal{M}_j$ is chosen from the set $\{\mathcal{M}_i\}$. To determine which channel is chosen, a part of a pure state $|\psi\rangle$ on $\mathcal{H} \otimes \mathcal{H}$ is sent to the channel followed by the POVM measurement described by $\{\Pi_i\}$.

$$\mathrm{Tr}[\Pi_i(\mathcal{M}_j \otimes \mathcal{I}_{\mathcal{H}_A})(\rho))] = 0 \text{ for } i \neq j \text{ and } i, j = 1, 2, \cdots, N. \quad (3.26)$$

Without loss of generality, the initial state can be chosen to be a pure state in both minimum-error and unambiguous discrimination. This is because if a mixed initial state can achieve the optimal success probability, any eigenstates of the mixed state also achieve the same success probability. In addition, the dimension of the ancillary system is at most the dimension of the input system of the channels since the Schmidt rank of a pure state in $\mathcal{H} \otimes \mathcal{H}_A$ is at most $\min\{\dim \mathcal{H}, \dim \mathcal{H}_A\}$. Therefore we can assume $\mathcal{H}_A = \mathcal{H}$. An quantum circuit representation of quantum channel discrimination is presented in Fig. 3.2.

We define an initial state $|\psi\rangle = I_{\mathcal{H}} \otimes \sqrt{X}|I\rangle\rangle$. Applying the channel $\mathcal{M}_i$ on $\mathcal{H}$, the state is transformed to

$$\mathcal{M}_i \otimes \mathcal{I}_{\mathcal{H}}(|\psi\rangle\langle\psi|) = (I_{\mathcal{K}} \otimes \sqrt{X})M_i(I_{\mathcal{K}} \otimes \sqrt{X}), \quad (3.27)$$

where $M_i$ is the Choi operator of $\mathcal{M}_i$. We redefine each positive operators $\widetilde{\Pi}_i$ as $\widetilde{\Pi}_i = (I_{\mathcal{K}} \otimes \sqrt{X})\Pi(I_{\mathcal{K}} \otimes \sqrt{X})$. Then the success probability is represented as

$$p_{succ} = \sum_{i=1}^{N} q_i \mathrm{Tr}[\widetilde{\Pi}_i M_i], \quad (3.28)$$

where $\widetilde{\Pi}_i \geq 0$ and $\sum_{i=1}^{N} \widetilde{\Pi}_i = I_{\mathcal{K}} \otimes X$.

This set of the positive operators $\{\widetilde{\Pi}_i\}$ is a quantum 2-tester introduced in Sec. 2.4. The correspondence between states $\rho_i$ and Choi operators $M_i$, and a POVM measurement $\{\Pi_i\}$ and quantum 2-tester $\{\widetilde{\Pi}_i\}$ is understood by a comparing Equation (3.2) and Equation (3.28).

A derivation of the optimal success probabilities in both minimum-error discrimination for quantum channels and unambiguous discrimination are represented in terms of semidefinite programmings as follows. For minimum-error discrimination, we have

$$\begin{aligned} \text{maximize} \quad & p_{succ} = \sum_{i=1}^{N} q_i \mathrm{Tr}[\widetilde{\Pi}_i M_i] && (3.29) \\ \text{subject to} \quad & \widetilde{\Pi} \geq 0, \quad \sum_{i=1}^{N} \widetilde{\Pi}_i = I_{\mathcal{K}} \otimes X && (3.30) \\ & X \geq 0, \quad \mathrm{Tr} X = 1 && (3.31) \end{aligned}$$

and for unambiguous discrimination, we have

$$\begin{aligned} \text{maximize} \quad & p_{succ} = \sum_{i} q_i \mathrm{Tr}[\widetilde{\Pi}_i M_i] && (3.32) \\ \text{subject to} \quad & \mathrm{Tr}[\widetilde{\Pi}_i M_j] = 0 \text{ for } i \neq j \text{ and } i, j = 1, 2, \cdots, N && (3.33) \\ & \Pi \geq 0, \quad \sum_{i=1}^{N+1} \widetilde{\Pi}_i = I_{\mathcal{K}} \otimes X && (3.34) \\ & X \geq 0, \quad \mathrm{Tr} X = 1. && (3.35) \end{aligned}$$

It is not easy to find the optimal initial state and the optimal POVM in general. However, for minimum-error discrimination with $N = 2$, the optimal POVM measurement is given by a measurement called the Helstrom measurement. The Helstrom measurement is obtained by $\Pi_{1/2} = \{q_1(\mathcal{M}_1 \otimes \mathcal{I}_{\mathcal{H}})(|\psi\rangle\langle\psi|) - q_2(\mathcal{M}_2 \otimes \mathcal{I}_{\mathcal{H}})(|\psi\rangle\langle\psi|)\}_{+/-}$, where $\{A\}_{+/-}$ is the projection onto the positive/negative part of $A$ and $|\psi\rangle$ is the initial state of $\mathcal{M}_i \otimes \mathcal{I}_{\mathcal{H}}$. The optimal success probability is written as

$$\begin{aligned} p_{succ} &= \frac{1}{2} + \frac{1}{2} \sup_{|\psi\rangle \in \mathcal{H} \otimes \mathcal{H}} \| q_1(\mathcal{M}_1 \otimes \mathcal{I}_{\mathcal{H}})(|\psi\rangle\langle\psi|) - q_2(\mathcal{M}_2 \otimes \mathcal{I}_{\mathcal{H}})(|\psi\rangle\langle\psi|) \|_1 \\ &= \frac{1}{2} + \frac{1}{2} \| q_1 \mathcal{M}_1 - q_2 \mathcal{M}_2 \|_{\diamond}, \qquad (3.36) \end{aligned}$$

where $\| \cdot \|_{\diamond}$ is the diamond norm introduced in Sec. 2.1.3. To derive the optimal success probability of minimum-error discrimination of the two quantum channels, one still needs to derive the optimal initial state.

One distinctive difference between quantum state discrimination and discrimination of quantum channels is the room for choosing an initial state. Entanglement of the initial state has been analyzed in terms of the performance of properties enhancing the success probability for discrimination of quantum channel [49, 50]. Even for the discrimination of *entanglement-breaking* channels, which map any entangled state to a separable state, entangled initial states achieve the better success probability, although the output states of channels are not entangled [51]. There are two known classes that do not require entangled initial states for the optimal minimum-error discrimination, unitary operations [52] and classical channels.

Another significant difference is whether ordered strategies would help to improve the success probability when the multiple use of the black box is allowed. For states cases, the ordered strategies since any operation can be absorbed into the last POVM measurement. On the other hand, there are several instances in which the ordered strategies lead an the improvement of success probability in discrimination of quantum channels [33, 35]. It is important to know when the ordered strategies improve the success probability in order to construct efficient protocols for quantum information processing.

In general, perfect discrimination of quantum channels is impossible similarly to discrimination of quantum states. Necessary and sufficient conditions for a set of quantum channels to be perfectly discriminated with a finite use of the black-boxes has been derived [53]. The authors of [53] have also constructed a protocol for perfect discrimination of two quantum channels and showed that entanglement of the initial state is not necessary for perfect discrimination between two isometry channels.

### 3.2.2 Discrimination of Unitary Operations

An important class of quantum channel is unitary operations. The optimal initial state is derived in [54] for minimum-error discrimination of two unitary operations. Consider a black-box implementing one of unitary operations $U_1$ and $U_2$ with probability $q_1$ and $q_2$, respectively, and denote an initial state as $|\psi\rangle = I_{\mathcal{H}} \otimes \sqrt{X}|I\rangle\rangle$, using a positive semidefinite operator $X$ of unit trace. Then the two candidate states $|\psi_1\rangle = U_1 \otimes I|\psi\rangle$ and $|\psi_2\rangle = U_2 \otimes I|\psi\rangle$ are obtained after applying the unitary operation implemented by the black-box. The success probability is given by

$$p_{succ} = \frac{1}{2} + \frac{1}{2}\|q_1|\psi_1\rangle\langle\psi_1| - q_2|\psi_2\rangle\langle\psi_2|\|_1 \tag{3.37}$$

$$= \frac{1}{2}\left(1 + \sqrt{1 - 4q_1q_2|\langle\psi|U_1^\dagger U_2|\psi\rangle|^2}\right), \tag{3.38}$$

where we used the eigenvalues of $q_1|\phi\rangle\langle\phi| - q_2|\eta\rangle\langle\eta|$ given by

$$(q_1 - q_2 \pm \sqrt{1 - 4q_1q_2|\langle\phi|\eta\rangle|^2})/2. \tag{3.39}$$

We are going to minimize the overlap $|\langle\psi|U_1^\dagger U_2|\psi\rangle|$.

We denote the spectral decomposition of $X$ as $X = \sum_{i=1}^{d} r_i |\lambda_i\rangle\langle\lambda_i|$. The overlap is is rewritten in terms of the eigenstates $\{|\lambda_i\rangle\}$ as

$$|\langle\psi|U_1^\dagger U_2|\psi\rangle| = \sum_{i=1}^{d} r_i |\langle\lambda_i|U_1^\dagger U_2|\lambda_i\rangle|. \tag{3.40}$$

This is a probabilistic mixture of non-negative values since $r_i \geq 0$ and $\sum_i r_1 = 1$. Minimization of $|\langle\psi|U_1^\dagger U_2|\psi\rangle|$ can be achieved by choosing $r_{i_{min}} = 1$ and $r_j = 0$ for $j \neq i_{min}$, where $i_{min} = \arg\min\{|\langle\lambda_i|U_1^\dagger U_2|\lambda_i\rangle|\}$. That means the optimal initial state can be chosen to be unentangled with other systems.

We denote the spectral decomposition $U_1^\dagger U_2 = \sum_{j=1}^{d} e^{i\theta_j}|\zeta_i\rangle\langle\zeta_i|$ and an initial state $|\phi\rangle = \sum_{i=1}^{d} \alpha_i|\zeta_i\rangle$ with $\sum_{i=1}^{d} |\alpha_i|^2 = 1$. The success probability is then represented by

$$|\langle\psi|U_1^\dagger U_2|\psi\rangle| = \left|\sum_{i=1}^{d} |\alpha_i|^2 e^{i\theta_i}\right|. \tag{3.41}$$

The right hand side of Equation (3.41) can be interpreted as a distance from the origin and a point in a polytope made from the eigenvalues $\{e^{i\theta_i}\}$ on the unit circle in the complex plane (see Fig. 3.3). Assuming $-\pi \leq \theta_i < \pi$ and $\theta_1 \leq \theta_2 \leq \cdots \leq \theta_d$, we obtain

$$\min_{|\psi\rangle\in\mathcal{H}} |\langle\psi|U_1^\dagger U_2|\psi\rangle| = \begin{cases} 0 & (\theta_d - \theta_1 \geq \pi) \\ \cos\frac{\theta_d-\theta_1}{2} & (otherwise). \end{cases} \tag{3.42}$$

Therefore the optimal success probability for minimum-error discrimination is derived as

$$p_{succ}^{opt} = \begin{cases} 1 & (\theta_d - \theta_1 \geq \pi) \\ \frac{1}{2}(1 + \sqrt{1 - 4q_1q_2\cos^2\frac{\theta_d-\theta_1}{2}}) & (otherwise). \end{cases} \tag{3.43}$$

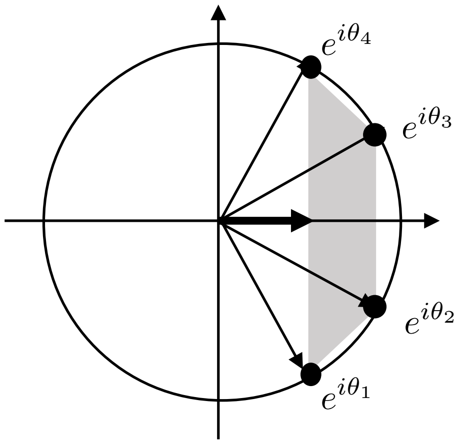



Figure 3.3: Four eigenvalues $\{e^{i\theta_i}\}_{i=1}^4$ on the unit circle in the complex plane. The shaded area is the polytope that consists of the four eigenvalues. $\sum_{i=1}^d r_i|\langle\lambda_i|U_1^\dagger U_2|\lambda_i\rangle|$ is the distance from the origin and a point in the polytope. The minimum value of that is the norm of the bold arrow in the figure.

From this result, we can see that perfect discrimination of unitary operations is possible by the finite uses of the black-box [8, 54]. We consider the case where the black-box can be use $M$ times and the black-box is used in a parallel way. In this case, the task is to discriminate $U_1^{\otimes M}$ and $U_2^{\otimes M}$. The largest difference of the angle of the eigenvalues of $(U_1^\dagger U_2)^{\otimes M}$ is $M(\theta_d - \theta_1)$. From Equation 3.2.2, perfect discrimination is possible if the largest difference of the angle of the eigenvalues is greater than $\pi$. Therefore perfect discrimination is possible with $M_{min} = \lceil \pi/(\theta_d - \theta_1) \rceil$ times use of the black-box, where $\lceil a \rceil$ is the minimum integer greater than $a$.

For discrimination of a set of $N$ unitary operations, we can extend the method for discrimination of to unitary operations by introducing tests, each of which exclude one of the unitary operations out of $N$ unitary operations, and using the tests for $N-1$ times. For instance, to exclude the possibility of $U_1$, perform measurements to discriminate $U_1$ and $U_i$ for $i = 2, 3, \cdots, N$. If every measurement result indicates that the actual unitary operation applied is $U_1$, one can conclude that the unitary operation is $U_1$, or exclude $U_1$ otherwise. Therefore perfect discrimination of $N$ unitary operations is possible with the finite use of the black-box is achieved.

This result is is in contrast with discrimination of quantum states. There is an intrinsic difference between discrimination of states and discrimination of unitary operations.

Perfect discrimination of unitary operations with finite use of the black-box can be also achieved by applying appropriate operations and the unitary operations implemented by the black-box, interchangeably on the same system [9]. This results implies that entanglement is not necessary for perfect discrimination of unitary operations. Note that the minimum number of the use of the black-box necessary to achieve perfect discrimination is same in both parallel and sequential uses of the black-box.

## 3.3 Equivalence Determination of Quantum States

In discrimination of states, unitary operations or quantum channels presented in the previous sections, the complete descriptions of the candidate states or channels are given. The cases introduced in the previous sections, the POVM for the optimal success probability depends on the classical descriptions. Therefore it is necessary to derive the optimal POVM every time the candidate are changed. On the other hand, it is possible to consider universal schemes that do not depend on the candidate states to discriminate.

To this end, candidate quantum state can be given as quantum states instead of classical descriptions of the states. Two quantum states $|\psi_1\rangle$ and $|\psi_2\rangle$ (*reference states*) are given as well as a state $|\psi_i\rangle$ (*test state*) that is generated to be one of the two states $|\psi_1\rangle$ and $|\psi_2\rangle$ with the probability $q_j$ for $j = 1, 2$. In this case, discrimination of a test state generated by the apparatus is determining that it is identical to the given physical state $|\psi_1\rangle$ and $|\psi_2\rangle$ (see Fig. 3.4). We call the task as *equivalence determination* of quantum states.

Since the complete classical description of an unknown quantum state $|\psi\rangle$ cannot be obtained from a finite copies of $|\psi\rangle$, classical descriptions of candidate states are stronger resources for discrimination than candidate states.

Consider that $N_1$ copies of $|\psi_1\rangle$ and $N_2$ copies of $|\psi_2\rangle$ are given. For the simplicity, we assume $q_1 = q_2 = 1/2$ and denote $|\psi\rangle\langle\psi|$ as $\psi$. For minimum-error discrimination, the success probability is given by

$$p_{succ} = \frac{1}{2}\mathrm{Tr}\left[(\psi_1^{\otimes N_1} \otimes \psi_2^{\otimes N_2} \otimes \psi_1)\Pi_1 + (\psi_1^{\otimes N_1} \otimes \psi_2^{\otimes N_2} \otimes \psi_2)\Pi_2\right], \tag{3.44}$$

where $\{\Pi_1, \Pi_2\}$ is a POVM. This success probability depends on the specific states $\psi_1$ and $\psi_2$. The POVM cannot depend on specific choices of $\psi_1$ and $\psi_2$ since no prior information about $\psi_1$ and $\psi_2$ is given. Therefore as a figure of merit, the *averaged* success probability over the uniform distribution of all pure states is employed.

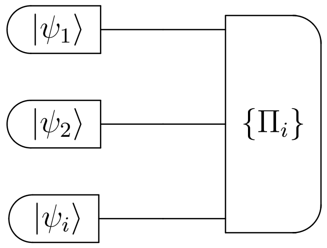



Figure 3.4: The simplest case of equivalence determination of quantum states. The test state $|\psi_j\rangle$ is guaranteed to be one of two reference states $|\psi_1\rangle$ and $|\psi_2\rangle$ and the task is to determine which state is identical to the test state $|\psi_j\rangle$. The POVM cannot depend $|\psi_1\rangle$ and $|\psi_2\rangle$.

The averaged success probability is written as

$$\begin{aligned}\bar{p}_{succ} &= \frac{1}{2}\int d\mu(\psi_1)\int d\mu(\psi_2)\mathrm{Tr}[(\psi_1^{\otimes N_1}\otimes\psi_2^{\otimes N_2}\otimes\psi_1)\Pi_1 + (\psi_1^{\otimes N_1}\otimes\psi_2^{\otimes N_2}\otimes\psi_2)\Pi_2]\\ &= \frac{1}{2}\mathrm{Tr}[\sigma_1\Pi_1+\sigma_2\Pi_2], \end{aligned} \tag{3.45}$$

where $d\mu(\phi)$ is the uniform distribution over all pure states and

$$\sigma_i = \int d\mu(\psi_1)\int d\mu(\psi_2)\psi_1^{\otimes N_1}\otimes\psi_2^{\otimes N_2}\otimes\psi_i, \tag{3.46}$$

for $i = 1, 2$. The averaged state $\sigma_i$ can be calculated for the uniform distribution. Thus equivalence determination of states is reduced to discriminate the mixed states $\sigma_1$ and $\sigma_2$ of which classical descriptions are given by Equation (3.46). The optimal POVM is guaranteed to be given by the Helstrom measurement, although it is not easy to calculate the success probability for the Helstrom measurement in general.

For unambiguous discrimination, the corresponding optimization problem is represented as

$$\text{maximize} \quad p_{succ} = \frac{1}{2}\mathrm{Tr}[\sigma_1\Pi_1+\sigma_2\Pi_2] \tag{3.47}$$

$$\text{subject to} \quad \mathrm{Tr}[\Pi_1\sigma_2] = \mathrm{Tr}[\Pi_2\sigma_1] = 0 \tag{3.48}$$

$$\Pi_1, \Pi_2 \geq 0, \Pi_1+\Pi_2+\Pi_0 = I. \tag{3.49}$$

Unambiguous equivalence determination of states is first introduced in [23] for a restricted class of pure states as the name of the programmable quantum-state discriminator. In [24], the case for $N_1 = N_2 = 1$ is considered and the optimal success probability is derived. Several extensions are analyzed in [23,25–28]. General solutions of the optimal success probability are derived in [29–31] for minimum-error equivalence determination and in [32] for unambiguous one with $N_1 = N_2$. As an application of this task, detection of change points is considered in [30].

## 3.4 Comparison of Quantum States and Operations

### 3.4.1 Comparison of Quantum States

In this section, we introduce *comparison* of quantum states. Comparison of quantum states is a task to determine whether given two quantum states are same or not. This task is a special case of equivalence determination of quantum states by taking $N_1 = 1$ and $N_2 = 0$.

For minimum-error comparison, the success probability can be written by $p_{succ} = \frac{1}{2}[\eta_1\Pi_1 + \eta_2\Pi_2]$, where $\eta_1 = \int d\mu(\psi_1)\psi_1^{\otimes 2}$ and $\eta_2 = \int d\mu(\psi_1) \int d\mu(\psi_2)\psi_1 \otimes \psi_2$. The mixed state $\eta_1$ and $\eta_2$ can be calculated without specifying $\psi_1$ and $\psi_2$ if $\psi_1$ and $\psi_2$ are completely unknown, that is, they are both uniformly distributed. Thus the optimal POVM of comparison of two quantum states is the Helstrom measurement.

For unambiguous comparison, the corresponding optimization problem is represented as

$$\begin{aligned} \text{maximize} \quad & p_{succ} = \frac{1}{2}\mathrm{Tr}[\eta_1\Pi_1 + \eta_2\Pi_2] && (3.50) \\ \text{subject to} \quad & \mathrm{Tr}[\Pi_1\eta_2] = \mathrm{Tr}[\Pi_2\eta_1] = 0 && (3.51) \\ & \Pi_1, \Pi_2 \geq 0, \Pi_1 + \Pi_2 + \Pi_0 = I. && (3.52) \end{aligned}$$

Unambiguous comparison of two quantum states is first introduced in [47] and the authors showed that probability of conclusive result corresponding to $\psi_1 = \psi_2$ is zero. In [55], the optimal solution is derived for the arbitrary prior probabilities of states and comparison of mixed states is discussed. The optimal measurement using multiple copies of each state is derived in [56]. In [57], the task to determine whether all of $n$ mixed states are the same or not with an unambiguous quantum measurement is considered. Unambiguous comparison of quantum states can be used verification tasks such as quantum signature [58, 59] and quantum fingerprinting [60, 61].

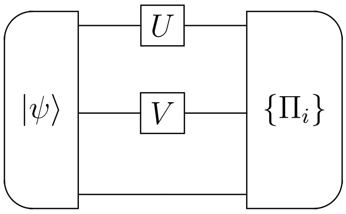



Figure 3.5: Comparison of two unitary operations to determine whether two unitary operations $U$ and $V$ are the identical or not. The initial state $|\psi\rangle$ and POVM $\{\Pi_i\}$ cannot depend on the classical descriptions of $U$ and $V$ since there are no prior information about the two unitary operations.

### 3.4.2 Comparison of Unitary Operations

Consider that two black-boxes implementing unitary operations are give. The two unitary operations are guaranteed to be identical with probability 1/2 and different with the probability 1/2. The task to determine whether the two unitary operations are same or not is called *comparison of unitary operations*. In contrast to minimum-error comparison of quantum states, the optimal protocol for minimum-error comparison of unitary operations requires to choose an appropriate initial state, and its optimization is not straightforward.

We denoted the two unitary operations as $U$ and $V$ acting on $\mathcal{H}$. We assume that the system $\mathcal{H}$ is a $d$-dimensional system. We denote an initial state as $|\psi\rangle \in \mathcal{H}^{\otimes 4}$ and a POVM as $\{\Pi_i\}$ (see Fig. 3.5). Assigning the measurement outcome 1 for $U = V$ and 2 for $U \neq V$.Then the success probability is given by

$$\begin{aligned} p_{succ} =& \frac{1}{2}\mathrm{Tr}[(U^{\otimes 2} \otimes I_{\mathcal{H}}^{\otimes 2})|\psi\rangle\langle\psi|(U^{\dagger\otimes 2} \otimes I_{\mathcal{H}^{\otimes 2}})\Pi_1 \\ &+ (U \otimes V \otimes I_{\mathcal{H}}^{\otimes 2})|\psi\rangle\langle\psi|(U^\dagger \otimes V^\dagger \otimes I_{\mathcal{H}}^{\otimes 2})\Pi_2]. \end{aligned} \tag{3.53}$$

There are no prior information about unitary operations $U$ and $V$. Therefore the POVM and the initial state cannot depend on $U$ and $V$. As a figure of merit, the averaged success probability over the Haar measure $d\mu(U)$ is employed. The averaged success probability is written as

$$\bar{p}_{succ} = \frac{1}{2}\mathrm{Tr}[\rho_1\Pi_1 + \rho_2\Pi_2], \tag{3.54}$$

where $\rho_1 = \int d\mu(U)(U^{\otimes 2} \otimes I_{\mathcal{H}}^{\otimes 2})|\psi\rangle\langle\psi|(U^{\dagger\otimes 2} \otimes I_{\mathcal{H}^{\otimes 2}})$ and $\rho_2 = \int d\mu(U)d\mu(V)(U \otimes V \otimes I_{\mathcal{H}}^{\otimes 2})|\psi\rangle\langle\psi|(U^\dagger \otimes V^\dagger \otimes I_{\mathcal{H}}^{\otimes 2})$.

For minimum-error comparison of unitary operations, the averaged success probability $\bar{p}_{succ}$ with conditions $\Pi_1, \Pi_2 \geq 0$ and $\Pi_1 + \Pi_2 = I$ should be maximized. For unambiguous comparison, an inconclusive result $\Pi_3$ is added and the conditions are $\Pi_i \geq 0$ for $i = 1, 2, 3$, $\mathrm{Tr}[\Pi_1\rho_2] = \mathrm{Tr}[\Pi_2\rho_1] = 0$ and $\sum_{i=1}^{3} \Pi_i = I$.

Comparison of unitary operations is first introduced in [38] and the authors showed that the optimal success probability is 7/8 for minimum-error comparison and 3/7 for unambiguous comparison for qubit systems ($d = 2$). In [38], optimization was made with an 2-dimensional ancillary system, which is too restricted for proving the general optimal success probability. The authors extended to the case with two copies of each unitary operation are given [62] and proposed an implementation of comparison of unitary operations using the Franson interferometry [63]. Unambiguous comparison of unitary operations is extended for general $d$-dimensional systems and the optimal success probability is derived to be $(d+1)/2d$ with the enough size of ancillary systems in [64]. An extension for comparison of POVM measurements is discussed in [65].

# Chapter 4

# Equivalence Determination: Single Use of the Reference Boxes

In this chapter, we investigate equivalence determination of unitary operations. We consider that three black-boxes (a *test box* and two *reference boxes*) implementing unknown unitary operations are given. The test box is guaranteed to implement one of the two unitary operations implemented by the reference boxes. The two reference boxes are given, but the classical descriptions of their implementing unitary operations are not. The equivalence determination of unitary operations is to determine which of the two reference boxes implements the same unitary operation of the test box. A classification of discrimination task is summarized in Table 4.1.

| targets \ candidates given | Classical descriptions | Physical states/systems |
|---|---|---|
| Quantum states | Sec. 3.1.2 - 3.1.3 | Sec. 3.3 |
| Unitary operations | Sec. 3.2.2 | Chap. 4, 5 |

Table 4.1: A classification of the discrimination tasks introduced in this thesis. The shaded part is the contribution of the thesis.

One significant difference between equivalence determination of states and unitary operations is an extra freedom for choosing an initial state for the case of unitary operations. The initial state can be entangled with ancillary systems. In addition, the initial state can be an entangled state on between the systems where the test box and the reference boxes are applied. This property does not appear in discrimination of unitary operations with the classical descriptions of candidate unitary operations.

Another difference is the strategies of how to use the black-boxes in the quantum circuit. In general, the black-boxes can be used in arbitrary order and any quantum operations can be applied between the uses of the black-boxes. We consider two types of strategies. One is parallel strategies, in which the black-boxes is used in a parallel way without introducing the order and no quantum operation is applied between the use of the black-boxes. The other is ordered strategies, in which arbitrary quantum operations can be inserted between the use of the black-boxes. To pursue efficient quantum information processing, clarifying when the ordered strategies outperform the parallel strategies is important [33–36].

In equivalence determination of unitary operations, two kinds of black-boxes are given. In the ordered strategies, each black-box can be used in different orders. The relation between the order of the black-boxes and the performance is interesting.

The organization of this chapter is as follows. In Sec. 4.1, we define and analyze equivalence determination of unitary operations with a single use of two reference boxes. We analytically show the optimal success probability for the parallel strategies in Sec. 4.1.1 and the ordered strategies in 4.1.2. In Sec. 4.2, we consider the case that two candidate unitary operations, where the classical description of one of the reference box is given.

## 4.1 Single Use of Reference Boxes

In this section, we consider the simplest case that each of a test box and reference boxes can be used only once. We denote the unitary operation implemented by the reference box $j$ as $U_j$ for $j = 1, 2$ and assume that the test box implements one of two unitary operations $U_1$ and $U_2$ with probability 1/2. The equivalence determination task is to determine which reference box implements the unitary operation implemented by the test box.

We denote input and output Hilbert spaces of the the reference box $i$ by $\mathcal{H}_i$ and $\mathcal{K}_i$, respectively for $i = 1, 2$ and input and output spaces of the test box by $\mathcal{H}_3$ and $\mathcal{K}_3$. For simplicity, we define $\mathcal{H} := \bigotimes_{j=1}^{3} \mathcal{H}_j$ and $\mathcal{K} := \bigotimes_{j=1}^{3} \mathcal{K}_j$.

### 4.1.1 Parallel Strategies

First we consider the parallel strategies for equivalence determination of unitary operations. We can represent a tensor products of three unitary operators as a

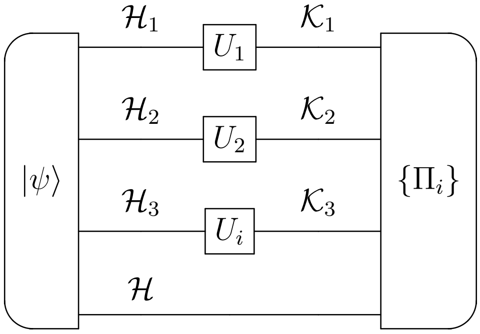



Figure 4.1: A quantum circuit representation for the equivalence determination of unitary operations of the test box $U_i$ under the parallel strategy with a single use of two reference boxes $U_1$ and $U_2$.

single unitary operator $W_i \in L(\mathcal{H}, \mathcal{K})$. When the test box implements a unitary operations $U_1$, $W_1 := U_1 \otimes U_2 \otimes U_1$ and $W_2 := U_1 \otimes U_2 \otimes U_2$ for the other case. Then equivalence determination task is to determine which unitary operation, $W_1$ or $W_2$, is implemented by the three boxes.

We denote an initial state as $|\psi\rangle \in \mathcal{H} \otimes \mathcal{H}$. Without loss of generality, an initial state can be assumed to be pure states. This is because if the optimal initial state is a mixed state, any eigenstate of the optimal mixed state can also achieve the optimal success probability. We represent the initial state $|\psi\rangle$ as

$$|\psi\rangle = I_{\mathcal{H}} \otimes \sqrt{X}|I\rangle\rangle,$$

using the $|I\rangle\rangle = \sum_{i=1}^{\dim \mathcal{H}} |i\rangle|i\rangle$ defined in terms of the computational basis $\{|i\rangle\}_{i=1}^{\dim \mathcal{H}}$ of $\mathcal{H}$ and $X$ is a positive semidefinite operator on $\mathcal{H}$ with $\mathrm{Tr}X = 1$.

Application of the unitary operation $W_i$ on the initial state $|\psi\rangle$ generates a candidate stat $|\psi_i\rangle = W_i \otimes I|\psi\rangle$ for $i = 1, 2$. A quantum measurement on $|\psi_1\rangle \in \mathcal{K} \otimes \mathcal{H}$ is chosen to detect which state is obtained, namely, $i = 1, 2$. We introduce a positive operator-valued measure (POVM) $\{\Pi_1, \Pi_2\}$. The subscript of each elements denotes the measurement outcome (see Fig. 4.1).

The success probability of discrimination is a sum of the probability that the measurement result $i$ coincides with the unitary operation implemented by the test box $U_i$. Thus the success probability $p_{succ}$ is represented as

$$p_{succ} = \frac{1}{2}\mathrm{Tr}[\psi_1\Pi_1 + \psi_2\Pi_2],$$

where we denote a pure state $|\psi_i\rangle\langle\psi_i|$ as $\psi_i$.

The success probability $p_{succ}$ depends on the specific choice of the candidate unitary operators $U_1$ and $U_2$ whereas we assumed that there is no prior information about the candidate unitary operations. Thus we average the success probability over the uniform distribution of unitary operations, the Haar measure. Then the *averaged* success probability $\bar{p}_{succ}$ is given by

$$\bar{p}_{succ} = \frac{1}{2}\int d\mu(U_1)\int d\mu(U_2)\mathrm{Tr}[\psi_1\Pi_1 + \psi_2\Pi_2] \tag{4.1}$$

$$= \frac{1}{2}\mathrm{Tr}[\int d\mu(U_1)\int d\mu(U_2)\psi_1\Pi_1 + \int d\mu(U_1)\int d\mu(U_2)\psi_2\Pi_2] \tag{4.2}$$

$$= \frac{1}{2}\mathrm{Tr}[\widetilde{M_1}\Pi_1 + \widetilde{M_2}\Pi_2], \tag{4.3}$$

where $d\mu(U)$ is the Haar measure satisfying $d\mu(U) = d\mu(VU)$ for any unitary operations $U$ and $V$ and $\widetilde{M_i}$ is defined as

$$\widetilde{M_i} = \int d\mu(U_1)\int d\mu(U_2)\psi_i \tag{4.4}$$

$$= \int d\mu(U_1)\int d\mu(U_2)(U_1\otimes U_2\otimes U_i\otimes\sqrt{X})|I\rangle\!\rangle\langle\!\langle I|(U_1^\dagger\otimes U_2^\dagger\otimes U_i^\dagger\otimes\sqrt{X}). \tag{4.5}$$

From Equation (4.3), the optimization problem is regarded as the discrimination of mixed states $\{\widetilde{M_1}, \widetilde{M_2}\}$ using the measurement described by the POVM $\{\Pi_1, \Pi_2\}$ are required. To maximize the success probability, optimization of both of $X$ and the POVM $\{\Pi_1, \Pi_2\}$. The mixed state $\widetilde{M_1}$ still depends on $X$. To make the analysis easier, the dependence of $X$ on $M_i$ is to be removed. Then let us define the operators $M_1$ and $M_2$ as

$$M_i := \int d\mu(U_1)\int d\mu(U_2)(U_1\otimes U_2\otimes U_i\otimes I_{\mathcal{H}})|I\rangle\!\rangle\langle\!\langle I|(U_1^\dagger\otimes U_2^\dagger\otimes U_i^\dagger\otimes I_{\mathcal{H}}), \tag{4.6}$$

where $I_{\mathcal{H}}$ is the identity operator on $\mathcal{H}$. Define a new positive semidefinite operator $\widetilde{\Pi}_i$ as $\widetilde{\Pi}_i = (I\otimes\sqrt{X})\Pi_i(I\otimes\sqrt{X})$, we obtain

$$\bar{p}_{succ} = \frac{1}{2}\mathrm{Tr}[(I\otimes\sqrt{X})M_1(I\otimes\sqrt{X})\Pi_1 + (I\otimes\sqrt{X})M_2(I\otimes\sqrt{X})\Pi_2] \tag{4.7}$$

$$= \frac{1}{2}\mathrm{Tr}[M_1(I\otimes\sqrt{X})\Pi_1(I\otimes\sqrt{X}) + M_2(I\otimes\sqrt{X})\Pi_2](I\otimes\sqrt{X}) \tag{4.8}$$

$$= \frac{1}{2}\mathrm{Tr}[M_1\widetilde{\Pi}_1 + M_2\widetilde{\Pi}_2]. \tag{4.9}$$

Note that $\{\widetilde{\Pi}_1, \widetilde{\Pi}_2\}$ satisfies $\widetilde{\Pi}_1 + \widetilde{\Pi}_2 = I\otimes X$. Therefore $\{\widetilde{\Pi}_1, \widetilde{\Pi}_2\}$ is a quantum 2-tester.

Although the classical description of unitary operations $U_1$ and $U_2$ are not given, the operator $M_i$ can be calculated. Thus the equivalence determination task is reduced to discriminating Choi operators $M_1$ and $M_2$ using a quantum 2-tester $\{\widetilde{\Pi}_1, \widetilde{\Pi}_2\}$.

Before giving the optimization problem, we define the equivalence determination task.

**Definition 3.** *We denote the unitary operation implemented by the reference box $j$ as $U_j$ for $j = 1, 2$ and assume that the test box implements one of two unitary operations $U_1$ and $U_2$ with probability $1/2$. The $(N_1, N_2)$-equivalence determination task is to determine which reference box implements the unitary operation implemented by the test box by using the reference box $i$ $N_j$ times and the test box only once.*

Then we obtain the optimization problem in the form of a semidefinite programming (SDP) as follows.

**Proposition 2.** *The averaged optimal success probability of $(1, 1)$-equivalence determination under the parallel strategies is given by the following semidefinite programming.*

$$\begin{aligned}
\textit{max:} \quad & p_{succ} = \frac{1}{2}\mathrm{Tr}\left[M_1\widetilde{\Pi}_1 + M_2\widetilde{\Pi}_2\right] \\
\textit{subject to:} \quad & \widetilde{\Pi}_i \geq 0, \; i = 1, 2 \\
& \widetilde{\Pi}_1 + \widetilde{\Pi}_2 = I_{\mathcal{K}} \otimes X \\
& X \geq 0 \\
& \mathrm{Tr}X = 1, \qquad (4.10)
\end{aligned}$$

*where the definition of $M_i$ is given by Equation (4.6)*

Due to the symmetry introduced by averaging over the Haar measure, the following lemma can be proven.

**Lemma 1.** *The optimal initial state $|\psi\rangle = I_{\mathcal{K}} \otimes \sqrt{X}|I\rangle\rangle$ for maximizing the success probability of $(1, 1)$-equivalence determination can be chosen to satisfy*

$$[A^{\otimes 3}, X] = 0, \qquad (4.11)$$

*with arbitrary unitary operator $A \in SU(2)$.*

The proof of this Lemma is given in Sec. 4.1.3.

When we consider that the reference box 1 and the reference box 2 are exchanged, the subscript of $\{\Pi_1, \Pi_2\}$ are exchanged, namely, the labels of the measurement results are exchanged. From the symmetry of changing the labels of measurement outcomes, the following lemma is obtained.

**Lemma 2.** *Let $S_{\mathcal{H}_{12}}$ be a swap operator of system $\mathcal{H}_1$ and $\mathcal{H}_2$. Then the optimal initial state $|\psi\rangle = I \otimes \sqrt{X}|I\rangle\rangle$ for maximizing the success probability* $(1,1)$-*equivalence determination can be chosen to satisfy*

$$[S_{\mathcal{H}_{12}} \otimes I_{\mathcal{H}_3}, X] = 0. \tag{4.12}$$

The proof of this lemma is given in Sec. 4.1.4.

To derive the optimal success probability, we formulate the optimization problem of discrimination of two random unitary channels. In parallel use of unitary operations, the equivalence determination task can be regarded as discrimination of two random unitary channels $\mathcal{M}_1$ and $\mathcal{M}_2$ defined as

$$\mathcal{M}_i(\rho) := \int d\mu(U_1) \int d\mu(U_2)(U_1 \otimes U_2 \otimes U_i)\rho(U_1^\dagger \otimes U_2^\dagger \otimes U_i^\dagger), \tag{4.13}$$

for $i = 1, 2$. As explained in Sec. 3.2, the optimal success probability $p_{succ}^{opt}$ of discriminating two channels is represented in terms of the diamond norm $\|\cdot\|_\diamond$ as

$$p_{succ}^{opt} = \frac{1}{2} + \frac{1}{4}\|\mathcal{U}_1 - \mathcal{U}_2\|_\diamond \tag{4.14}$$

$$= \frac{1}{2} + \frac{1}{4} \max_{X \geq 0, \mathrm{Tr}X=1} ||(I_\mathcal{K} \otimes \sqrt{X})(M_1 - M_2)(I_\mathcal{K} \otimes \sqrt{X})||_1. \tag{4.15}$$

Equivalence determination of unitary operations under the parallel strategies can be formulated as discrimination of two (known) random unitary channels since our figure of merit is given by the *averaged* success probability.

Now we are ready to derive the optimal success probability in the form of Equation (4.15) using Lemma 1 and Lemma 2. We obtain the following theorem.

**Theorem 2.** *The optimal averaged success probability of* $(1,1)$-*equivalence determination under the parallel strategies is* $7/8$ *when unitary operations are chosen from the Haar measure.*

The proof of the Theorem 2 is given in Sec. 4.1.5.

### Separable Input States

We consider an initial state for equivalence determination that is not entangled across the systems on which the reference boxes act and the test box acts (see Fig. 4.2). We show that with such an initial state the averaged optimal success probability is strictly less than 7/8. Therefore entanglement between the input systems of the reference boxes and the test box is essential for achieving the optimal value.

We denote an product initial state $|\psi\rangle \otimes |\phi\rangle$, where $|\psi\rangle \in (\mathcal{H}_1 \otimes \mathcal{H}_2)^{\otimes 2}$ and $|\phi\rangle \in \mathcal{H}_3^{\otimes 2}$. Without loss of generality, we can represent the initial state as

$$|\psi\rangle \otimes |\phi\rangle = \sqrt{X_1} \otimes \sqrt{X_2} \otimes I_{\mathcal{H}}|I\rangle\rangle, \tag{4.16}$$

where $X_1$ and $X_2$ are positive semidefinite operators on $\mathcal{H}_1 \otimes \mathcal{H}_2$ and $\mathcal{H}_3$, respectively, satisfy $\mathrm{Tr}X_1 = 1$ and $\mathrm{Tr}X_2 = 1$ and $|I\rangle\rangle$ is the unnormalized maximally entangled vector in $(\mathcal{H}_1 \otimes \mathcal{H}_2 \otimes \mathcal{H}_2)$. Therefore the assumption that $X = X_1 \otimes X_2$ should be added in the discussion presented in the previous section.

From Lemma 1, the condition $[X, A^{\otimes 3} \otimes B^{\otimes 3}] = [X_1 \otimes X_2, A^{\otimes 3} \otimes B^{\otimes 3}] = 0$ for arbitrary unitary operators $A, B \in \mathrm{SU}(2)$ should be satisfied. We can assume

$$X_1 = qI_0 \oplus (1-q)\frac{I_1}{3} \tag{4.17}$$

$$X_2 = \frac{I_{\mathcal{H}_3}}{2}, \tag{4.18}$$

where $I_j$ is the identity operator on the $\mathcal{U}_j$ the subspace on which the total angular momentum is $j$ in spin-1/2 systems (see Appx. B) and $I_{\mathcal{H}_3}$ is the identity operator on $\mathcal{H}_3$. Therefore we have

$$X = \frac{I_{\frac{1}{2}}}{2} \otimes (q|\hat{0}\rangle\langle\hat{0}| + \frac{(1-q)}{3}|\hat{1}\rangle\langle\hat{1}|) \oplus \frac{I_{\frac{3}{2}}}{4} \otimes \frac{2}{3}(1-q). \tag{4.19}$$

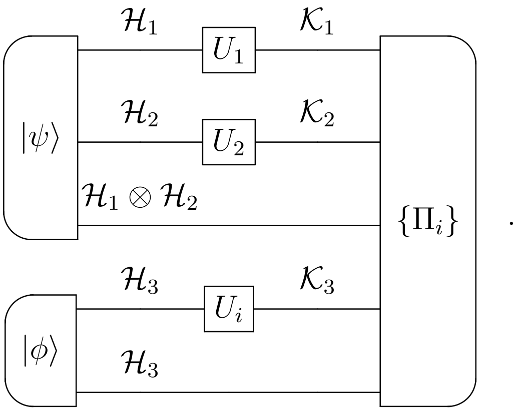



Figure 4.2: The quantum circuit for equivalence determination of unitary operations under the parallel strategy with single uses of two reference boxes when the initial state is unentangled.

Thus the optimal average success probability can be calculated by

$$p_{succ}^{opt} = \frac{1}{2} + \frac{1}{4}\max_{X \geq 0, \mathrm{Tr}X=1} \|(I_{\mathcal{K}} \otimes \sqrt{X})(M_1 - M_2)(I_{\mathcal{K}} \otimes \sqrt{X})\|_1, \tag{4.20}$$

where $M_1$ and $M_2$ are defined in Equation (4.80) and (4.84),respectively. Then, we can obtain

$$\|(I_{\mathcal{K}} \otimes \sqrt{X})(M_1 - M_2)(I_{\mathcal{K}} \otimes \sqrt{X})\|_1 \tag{4.21}$$

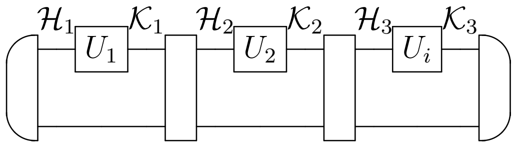
$$= \frac{1}{4}\left(\frac{1}{3}\sin 2t + \frac{2\cos^2 t}{3\sqrt{3}} + \frac{2\cos t\sqrt{2-\cos 2t}}{3\sqrt{3}}\right) \tag{4.22}$$

The above equation can be derived by summing the absolute value of all of eigenvalues of $\|(I_{\mathcal{K}} \otimes \sqrt{X})(M_1 - M_2)(I_{\mathcal{K}} \otimes \sqrt{X})\|_1$. The optimal averaged success probability is numerically derived as $p_{succ}^{opt} \simeq 0.746399 < 0.875 = 7/8$. Without entanglement between the input systems of the test box and the reference boxes are crucial for achieving the optimal averaged success probability.

### 4.1.2 Ordered Strategies

In general, the ordered use of three boxes and arbitrary quantum operations can be applied between the use of the black-boxes. In this section, we show that the ordered use of the test box and the reference boxes can be considered and arbitrary quantum operations can be inserted between the use of the black-boxes. In this section, we show that the ordered use of the black-boxes does not give improvement. That means the optimal success probability of discriminate is still 7/8.

In the ordered strategies, three different orders can be considered. We assign the Hilbert spaces denoted by $\mathcal{H}_i$ and $\mathcal{K}_i$ as the input and output system of the $i$-th black boxes, respectively. First we consider the case that the reference box 1 is used first and the reference box 2 second followed by the use of the test box (see Fig. 4.3).




(4.23)

Figure 4.3: A quantum circuit representation of the case of three unitary operations inserted in the quantum 4-tester.

To analyze the ordered strategies, quantum combs, more specifically, quantum testers are useful. Quantum 4-tester $\{\widetilde{\Pi}_1, \widetilde{\Pi}_2\}$ is considered to be a generalized POVM. When reference box 1 and 2 implement $U_1$ and $U_2$, respectively, and the test box implement one of $U_1$ and $U_2$, the success probability of obtaining the correct answer is given by

$$\begin{aligned}
\text{maximize} \quad & p_{succ} = \frac{1}{2}\mathrm{Tr}\left[|W_1\rangle\rangle\langle\langle W_1|\widetilde{\Pi}^{[1]} + |W_2\rangle\rangle\langle\langle W_2|\widetilde{\Pi}^{[2]}\right] \\
\text{subject to} \quad & \widetilde{\Pi}_i \geq 0,\ i = 1,2 \\
& \widetilde{\Pi}_1 + \widetilde{\Pi}_2 = I_{\mathcal{K}_3} \otimes Y \\
& \mathrm{Tr}_{\mathcal{H}_3} Y = I_{\mathcal{K}_2} \otimes Y^{(1)} \\
& \mathrm{Tr}_{\mathcal{H}_2} Y = I_{\mathcal{K}_1} \otimes Y^{(0)} \\
& \mathrm{Tr} Y^{(0)} = 1,
\end{aligned} \tag{4.24}$$

where $|W_i\rangle\rangle$ is defined as

$$|W_i\rangle\rangle_{\mathcal{K}\mathcal{H}} := |U_1\rangle\rangle \otimes |U_2\rangle\rangle \otimes |U_i\rangle\rangle \tag{4.25}$$

and

$$|U\rangle\rangle := U \otimes I|I\rangle. \tag{4.26}$$

This success probability depends on the choice of $U_1$ and $U_2$. By taking the average over the Haar measure, we obtain the following SPD.

$$\text{maximize} \quad p_{succ} = \frac{1}{2}\mathrm{Tr}\left[\widetilde{\Pi}_1 M_1 + \widetilde{\Pi}_2 M_2\right] \tag{4.27}$$

$$\text{subject to} \quad \widetilde{\Pi}_i \geq 0,\ i = 1,2 \tag{4.28}$$

$$\widetilde{\Pi}_1 + \widetilde{\Pi}_2 = I_{\mathcal{K}_3} \otimes Y \tag{4.29}$$

$$\mathrm{Tr}_{\mathcal{H}_3} Y = I_{\mathcal{K}_2} \otimes Y^{(1)} \tag{4.30}$$

$$\mathrm{Tr}_{\mathcal{H}_2} Y^{(1)} = I_{\mathcal{K}_1} \otimes Y^{(0)} \tag{4.31}$$

$$\mathrm{Tr} Y^{(0)} = 1, \tag{4.32}$$

where $Y$, $Y^{(1)}$ and $Y^{(0)}$ are positive semidefinite operators and $M_1$ and $M_2$ are given by Equation 4.6.

For the above SDP, the following lemma can be proven.

**Lemma 3.** *The quantum* 4*-tester* $\{\Pi_i\}$ *and positive semidefinite operators* $Y$*,* $Y^{(1)}$ *and* $Y^{(0)}$ *can be chosen to satisfy*

$$[\widetilde{\Pi}_i, (A^{\otimes 3})_{\mathcal{K}} \otimes (B^{\otimes 3})_{\mathcal{H}}] = 0 \tag{4.33}$$

$$[Y, (A^{\otimes 2})_{\mathcal{K}_1\mathcal{K}_2} \otimes (B^{\otimes 3})_{\mathcal{H}}] = 0 \tag{4.34}$$

$$[Y^{(1)}, (A^{\otimes 1})_{\mathcal{K}_1} \otimes (B^{\otimes 2})_{\mathcal{H}_1\mathcal{H}_2}] = 0 \tag{4.35}$$

$$[Y^{(0)}, B_{\mathcal{H}_1}] = 0, \tag{4.36}$$

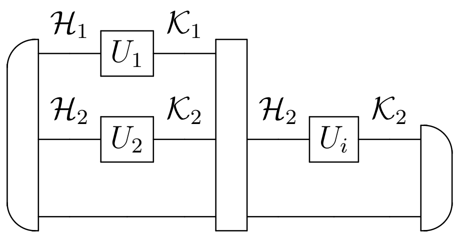
*for* $i = 1, 2$ *and arbitrary* $A, B \in SU(2)$*.*

The proof of the lemma is given in Sec. 4.1.6.

From Lemma 3, we can assume $Y^{(0)} = I_{\mathcal{H}_1}/2$, and $Y^{(1)} = I_{\mathcal{K}_1} \otimes Y'^{(1)}$. The new conditions for the quantum tester $\{\Pi_1, \Pi_2\}$ is

$$\begin{aligned}
&\widetilde{\Pi}_i \geq 0,\ \ i = 1,2 \\
&\widetilde{\Pi}_1 + \widetilde{\Pi}_2 = I_{\mathcal{K}_3} \otimes Y \\
&\mathrm{Tr}_{\mathcal{H}_3} Y = I_{\mathcal{K}_1\mathcal{K}_2} \otimes Y^{(1)} \\
&\mathrm{Tr}_{\mathcal{H}_2} Y^{(1)} = \frac{I_{\mathcal{H}_1}}{2},
\end{aligned}$$

where we rewrite $Y'^{(1)}$ as $Y^{(1)}$. This new conditions correspond to a quantum 3-tester described in Fig. 4.4 and the first two boxes can be used in a parallel ways. This parallelizability of the black-boxes property always holds for $(N_1, N_2)$-equivalence determination. There are only two cases of non-trivial orders of black boxes, the test box being used first or last.




(4.37)

Figure 4.4: A quantum circuit representation of the case of three unitary operations inserted in the quantum 3-tester.

**Lemma 4.** *For the task of* $(1,1)$*-equivalence determination under the ordered strategies, the optimization problem is represented as semidefinite programming given by*

$$\begin{aligned}
\textit{maximize}\quad & p_{succ} = \frac{1}{2}\mathrm{Tr}\left[M_1\widetilde{\Pi}_1 + M_2\widetilde{\Pi}_2\right] \\
\textit{subject to}\quad & \widetilde{\Pi}_i \geq 0,\ \ i = 1,2 \\
& \widetilde{\Pi}_1 + \widetilde{\Pi}_2 = I_{\mathcal{K}_3} \otimes Y \\
& \mathrm{Tr}_{\mathcal{H}_3} Y = I_{\mathcal{K}_1\mathcal{K}_2} \otimes Y^{(1)} \\
& \mathrm{Tr}_{\mathcal{H}_2} Y^{(1)} = \frac{I_{\mathcal{H}_1}}{2},
\end{aligned}$$

*where* $M_1$ *and* $M_2$ *are the Choi operators corresponding the order of boxes. Without loss of generality, the first two boxes can be used parallel way.*

Since we formulate the optimization problem as SDP, the dual problem of the SDP can be derived. The solution of the dual problem gives the upper bound of the original problem and in most cases the optimal values coincide. The *lower* bound of the solution of the original problem is 7/8 since the ordered strategies include the parallel strategies. In the following, we present the dual problem and give a solution of the dual problem that achieves the value 7/8, which shows that ordered strategies do not give improvements.

**Lemma 5.** *The dual problem of the primal SDP given in Equations (4.27) - (4.32) is represented as*

$$\begin{aligned}
\textit{minimize} \quad & a \\
\textit{subject to} \quad & \frac{M_1}{2} - \Omega \leq 0, && (4.38)\\
& \frac{M_2}{2} - \Omega \leq 0, && (4.39)\\
& \mathrm{Tr}_{\mathcal{K}_3}\Omega - I_{\mathcal{H}_3} \otimes \Omega^{[1]} \leq 0, && (4.40)\\
& \mathrm{Tr}_{\mathcal{K}_2}\Omega - I_{\mathcal{H}_2} \otimes \Omega^{[0]} \leq 0, && (4.41)\\
& \mathrm{Tr}_{\mathcal{K}_1}\Omega - aI_{\mathcal{H}_1} \leq 0 && (4.42)
\end{aligned}$$

The proof of this lemma is given in Sec. 4.1.7

Now we are ready to conclude the following theorem.

**Theorem 3.** *For the task of* $(1,1)$*-equivalence determination under the ordered strategies, the optimal averaged success probability is 7/8 when unitary operations are chosen from the Haar measure.*

The proof of this theorem is given in Sec. 4.1.8.

### 4.1.3 The Proof of Lemma 1

*proof* Suppose that a quantum 2-tester $\widetilde{\Pi}_i$ gives the success probability $p$, satisfying $\Pi_i \geq 0$ for $i = 1, 2$ and $\widetilde{\Pi}_1 + \widetilde{\Pi}_2 = I_{\mathcal{K}} \otimes X_{\mathcal{H}}$ with $\mathrm{Tr}X = 1$. Let us define an averaged quantity of $\widetilde{\Pi}_i$ as

$$\widetilde{\Pi}'_i := \int d\mu(A) \int d\mu(B)((A^{\otimes 3})_{\mathcal{K}} \otimes (B^{\otimes 3})_{\mathcal{H}})\Pi_i((A^{\dagger\otimes 3})_{\mathcal{K}} \otimes (B^{\dagger\otimes 3})_{\mathcal{H}}). \quad (4.43)$$

One can easily see that

$$\widetilde{\Pi}'_1 + \widetilde{\Pi}'_2 = \int d\mu(A) \int d\mu(B)((A^{\otimes 3})_{\mathcal{K}} \otimes (B^{\otimes 3})_{\mathcal{H}})(\Pi_1 + \Pi_2)((A^{\dagger\otimes 3})_{\mathcal{K}} \otimes (B^{\dagger\otimes 3})_{\mathcal{H}}) \tag{4.44}$$

$$= \int d\mu(A) \int d\mu(B)((A^{\otimes 3})_{\mathcal{K}} \otimes (B^{\otimes 3})_{\mathcal{H}})(I_{\mathcal{K}} \otimes X_{\mathcal{H}})((A^{\dagger\otimes 3})_{\mathcal{K}} \otimes (B^{\dagger\otimes 3})_{\mathcal{H}}) \tag{4.45}$$

$$= I_{\mathcal{K}} \otimes \int d\mu(B)(B^{\otimes 3})_{\mathcal{H}} X_{\mathcal{H}} (B^{\dagger\otimes 3})_{\mathcal{H}} \tag{4.46}$$

$$= I_{\mathcal{K}} \otimes X'_{\mathcal{H}}, \tag{4.47}$$

where $X'_{\mathcal{H}}$ is defined as

$$X'_{\mathcal{H}} = \int d\mu(B)(B^{\otimes 3})_{\mathcal{H}} X_{\mathcal{H}} (B^{\dagger\otimes 3})_{\mathcal{H}}. \tag{4.48}$$

Thus $\widetilde{\Pi}'_1 + \widetilde{\Pi}'_2 = I_{\mathcal{K}} \otimes X'_{\mathcal{H}}$ with $\mathrm{Tr}X'_{BD} = 1$ and $\widetilde{\Pi}'_1, \widetilde{\Pi}'_2, X'_{\mathcal{H}} \geq 0$. We show that $\widetilde{\Pi}'_1$ and $\widetilde{\Pi}'_2$ also give the success probability $p$. Due to the definition of $X'_{\mathcal{H}}$, since for arbitrary unitary operator $T_{\mathcal{H}}$, $[X'_{\mathcal{H}}, (T^{\otimes 3})_{\mathcal{H}}] = 0$is satisfied. This is because we have

$$T^{\otimes 3} X' T^{\dagger\otimes 3} = \int d\mu(B)(TB)^{\otimes 3}_{\mathcal{H}} X_{\mathcal{H}} (TB)^{\dagger\otimes 3}_{\mathcal{H}} \tag{4.49}$$

$$= \int d\mu(TB)(TB)^{\otimes 3}_{\mathcal{H}} X_{\mathcal{H}} (TB)^{\dagger\otimes 3}_{\mathcal{H}} \tag{4.50}$$

$$= \int d\mu(B')(B')^{\otimes 3}_{\mathcal{H}} X_{\mathcal{H}} (B')^{\dagger\otimes 3}_{\mathcal{H}} \tag{4.51}$$

$$= X', \tag{4.52}$$

where we used the property of the Haar measure $d\mu(AB) = d\mu(B)$ for arbitrary unitary operators $A$ and $B$ in SU(2).

Finally we show that $\{\widetilde{\Pi}'_i\}$ gives the same success probability as $\{\widetilde{\Pi}_i\}$. The success probability $p'_{succ}$ can be transformed as

$$p'_{succ} = \frac{1}{2}\mathrm{Tr}\left[M_1 \widetilde{\Pi}'_1 + M_2 \widetilde{\Pi}'_2\right] \tag{4.53}$$

$$\begin{aligned} &= \frac{1}{2}\mathrm{Tr} \int d\mu(A) \int d\mu(B) \big[(A^{\otimes 3} \otimes B^{\otimes 3})\widetilde{\Pi}_1 (A^{\dagger\otimes 3} \otimes B^{\dagger\otimes 3}) M_1 \\ &\quad + (A^{\otimes 3} \otimes B^{\otimes 3})\widetilde{\Pi}_2 (A^{\dagger\otimes 3} \otimes B^{\dagger\otimes 3}) M_2\big] \\ &= \frac{1}{2}\mathrm{Tr} \int d\mu(A) \int d\mu(B) \big[(A^{\dagger\otimes 3} \otimes B^{\dagger\otimes 3}) M_1 (A^{\otimes 3} \otimes B^{\otimes 3})\widetilde{\Pi}_1 \\ &\quad + (A^{\dagger\otimes 3} \otimes B^{\dagger\otimes 3}) M_2 (A^{\otimes 3} \otimes B^{\otimes 3})\widetilde{\Pi}_2\big] \end{aligned}$$

$$= \frac{1}{2}\mathrm{Tr}\left[M_1 \widetilde{\Pi}_1 + M_2 \widetilde{\Pi}_2\right] = p_{succ}, \tag{4.54}$$

In the last equality, the following properties,

$$\int dA \int dB (A^{\dagger\otimes 3} \otimes B^{\dagger\otimes 3}) M_i (A^{\otimes 3} \otimes B^{\otimes 3})$$

$$= \int d\mu(A) \int d\mu(B) \int d\mu(U) \int d\mu(V) \tag{4.55}$$

$$\times ((A^\dagger U_1 B^*) \otimes (A^\dagger U_2 B^*) \otimes (A^\dagger U_i B^*))|I\rangle\rangle\langle\langle I|^{\otimes 3}((A^\dagger U_1 B^*) \otimes (A^\dagger U_2 B^*) \otimes (A^\dagger U_i B^*))^\dagger$$

$$= \int d\mu(A) \int d\mu(B) \int d\mu(U') \int d\mu(V') (U'_1 \otimes U'_2 \otimes U'_i)|I\rangle\rangle\langle\langle I|^{\otimes 3}(U'_1 \otimes U'_2 \otimes U'_i)^\dagger$$

$$= \int d\mu(A) \int d\mu(B) M_i \tag{4.56}$$

$$= M_i, \tag{4.57}$$

for $i = 1, 2$ where we defined $U'_i := A^\dagger U_i B^*$. Therefore without loss of generality the positive semidefinite operator $X$ can be chosen satisfying $[T^{\otimes 3}, X] = 0$ for arbitrary unitary operator $T$. □

### 4.1.4 The Proof of Lemma 2

**Proof.** Suppose that a set of positive semidefinite operators $\{\Pi_1, \Pi_2\}$ gives the success probability $p$. By using a product of the swap operators $S_{\mathcal{K}_{12}} \otimes S_{\mathcal{H}_{12}}$, where $S_{\mathcal{K}_{12}}$ acts on $\mathcal{K}_1 \otimes \mathcal{K}_2$ as $S_{\mathcal{K}_{12}}(|\psi\rangle \otimes |\phi\rangle) = (|\phi\rangle \otimes |\psi\rangle)$ for any $|\psi\rangle \in \mathcal{K}_1$ and $|\phi\rangle \in \mathcal{K}_2$, and $S_{\mathcal{H}_{12}}$ acts similarly on $\mathcal{H}_1 \otimes \mathcal{H}_2$, we define another quantum 2-tester as

$$\widetilde{\Pi}'_i := \frac{1}{2}\{\widetilde{\Pi}_i + (S_{\mathcal{K}_{12}} \otimes S_{\mathcal{H}_{12}} \otimes I)\widetilde{\Pi}_{\bar{i}}(S_{\mathcal{K}_{12}} \otimes S_{\mathcal{H}_{12}} \otimes I)\}, \tag{4.58}$$

where $\bar{1} = 2$ and $\bar{2} = 1$. By definition, the equality $\widetilde{\Pi}'_i = (S_{\mathcal{K}_{12}} \otimes S_{\mathcal{H}_{12}} \otimes I)\widetilde{\Pi}'_{\bar{i}}(S_{\mathcal{K}_{12}} \otimes S_{\mathcal{H}_{12}} \otimes I)$ holds. Then we have

$$\widetilde{\Pi}'_1 + \widetilde{\Pi}'_2 = \frac{1}{2}\{\widetilde{\Pi}_1 + \widetilde{\Pi}_2 + (S_{\mathcal{K}_{12}} \otimes S_{\mathcal{H}_{12}} \otimes I)(\widetilde{\Pi}_1 + \widetilde{\Pi}_2)(S_{\mathcal{K}_{12}} \otimes S_{\mathcal{H}_{12}} \otimes I)\} \tag{4.59}$$

$$= \frac{1}{2}\{I_{\mathcal{K}} \otimes X_{\mathcal{H}} + (S_{\mathcal{K}_{12}} \otimes S_{\mathcal{H}_{12}} \otimes I)(I_{\mathcal{K}} \otimes X_{\mathcal{H}})(S_{\mathcal{K}_{12}} \otimes S_{\mathcal{H}_{12}} \otimes I)\} \tag{4.60}$$

$$= I_{\mathcal{K}} \otimes \frac{1}{2}(X_{\mathcal{H}} + (S_{\mathcal{H}_{12}} \otimes I)X_{\mathcal{H}}(S_{\mathcal{H}_{12}} \otimes I) \tag{4.61}$$

$$= I_{\mathcal{K}} \otimes X'_{\mathcal{H}}, \tag{4.62}$$

where $X'_{\mathcal{H}} = 1/2(X_{\mathcal{H}} + (S_{\mathcal{H}_{12}} \otimes I)X_{\mathcal{H}}(S_{\mathcal{H}_{12}} \otimes I))$ satisfying $\mathrm{Tr}X'_{\mathcal{H}} = 1$. The quantum 2-tester $\{\widetilde{\Pi}'_i\}$ also give the success probability $p$ since By easy calculation, we have

$$\frac{1}{2}\mathrm{Tr}\left[M_1\widetilde{\Pi}'_1 + M_2\widetilde{\Pi}'_2\right] \tag{4.63}$$

$$= \frac{1}{4}\mathrm{Tr}\big[M_1\{\widetilde{\Pi}_1 + (S_{\mathcal{K}_{12}} \otimes S_{\mathcal{H}_{12}} \otimes I)\widetilde{\Pi}_2(S_{\mathcal{K}_{12}} \otimes S_{\mathcal{H}_{12}} \otimes I)\} \tag{4.64}$$

$$+ M_2\{\widetilde{\Pi}_2 + (S_{\mathcal{K}_{12}} \otimes S_{\mathcal{H}_{12}} \otimes I)\widetilde{\Pi}_1(S_{\mathcal{K}_{12}} \otimes S_{\mathcal{H}_{12}} \otimes I)\}\big]$$

$$= \frac{1}{4}\mathrm{Tr}\big[M_1\widetilde{\Pi}_1 + (S_{\mathcal{K}_{12}} \otimes S_{\mathcal{H}_{12}} \otimes I)M_1(S_{\mathcal{K}_{12}} \otimes S_{\mathcal{H}_{12}} \otimes I)\widetilde{\Pi}_2 \tag{4.65}$$

$$+ M_2\widetilde{\Pi}_2 + (S_{\mathcal{K}_{12}} \otimes S_{\mathcal{H}_{12}} \otimes I)M_2(S_{\mathcal{K}_{12}} \otimes S_{\mathcal{H}_{12}} \otimes I)\widetilde{\Pi}_1\big]$$

$$= \frac{1}{2}\mathrm{Tr}\left[M_1\widetilde{\Pi}_1 + M_2\widetilde{\Pi}_2\right], \tag{4.66}$$

the last equality is derived by using

$$(S_{\mathcal{K}_{12}} \otimes S_{\mathcal{H}_{12}} \otimes I)M_i(S_{\mathcal{K}_{12}} \otimes S_{\mathcal{H}_{12}} \otimes I) = M_{\bar{i}}, \tag{4.67}$$

for $i = 1, 2$. □

### 4.1.5 The Proof of Theorem 2

*proof* Outline: first, the explicit calculations of $M_1$ and $M_2$ are given. Then the suitable form of $X$ are derived by using Lemma 1 and Lemma 2. Finally the optimal success probability is obtained.

From Equation (4.1.3), we have

$$[M_i, (A_{\mathcal{K}})^{\otimes 3} \otimes (B_{\mathcal{H}})^{\otimes 3}] = 0, \tag{4.68}$$

for any unitary operators $A, B$ in SU(2)and $i = 1, 2$. The Hilbert space of three qubit system is decomposed as

$$\mathcal{K} \cong (\mathbb{C}^2)^{\otimes 3} = \bigoplus_{J=\frac{1}{2}}^{\frac{3}{2}} \mathcal{U}_J \otimes \mathcal{V}_J^{[3]}, \tag{4.69}$$

where $\mathcal{V}_J^{[3]}$ corresponds to the multiplicity subspace of the irreducible subspace $\mathcal{U}_J$. For any unitary operator $A$ on $\mathbb{C}^2$, $A^{\otimes 3}$ can be decomposed as

$$A^{\otimes 3} = \bigoplus_{J=\frac{1}{2}}^{\frac{3}{2}} A_J \otimes I_{\mathcal{V}_J^{[3]}}, \tag{4.70}$$

where $I_{\mathcal{V}_J^{[3]}}$ denotes the identity operator on the multiplicity subspace $\mathcal{V}_J^{[3]}$.

Then $\mathcal{K} \otimes \mathcal{H}$ is decomposed as

$$\mathcal{K} \otimes \mathcal{H} = \bigoplus_{J=\frac{1}{2}}^{\frac{3}{2}} \bigoplus_{L=\frac{1}{2}}^{\frac{3}{2}} \mathcal{U}_J \otimes \mathcal{U}_L \otimes \mathcal{V}_{m_J} \otimes \mathcal{V}_{m_L}. \tag{4.71}$$

Here we changed the order of the spaces for convenience. The irreducible representation of tensor products of unitary operators is given as

$$A_{\mathcal{K}}^{\otimes 3} \otimes B_{\mathcal{H}}^{\otimes 3} = \bigoplus_{J=\frac{1}{2}}^{\frac{3}{2}} \bigoplus_{L=\frac{1}{2}}^{\frac{3}{2}} A_J \otimes B_L \otimes I_{\mathcal{V}_J^{[3]} \mathcal{V}_L^{[3]}}, \tag{4.72}$$

where $A_J$ and $B_L$ are the irreducible representations acting on $\mathcal{U}_J$ and $\mathcal{U}_L$, respectively and $I_{\mathcal{V}_J^{[3]} \mathcal{V}_L^{[3]}}$ is the identity operator on $\mathcal{V}_J^{[3]} \otimes \mathcal{V}_L^{[3]}$.

From Schur's lemma and Equation (4.68), $M_i$ is represented as

$$M_i = \bigoplus_{J=\frac{1}{2}}^{\frac{3}{2}} \bigoplus_{L=\frac{1}{2}}^{\frac{3}{2}} \frac{I_J}{d_J} \otimes I_L \otimes M_{JL}^{(i)}, \tag{4.73}$$

where $I_J$ and $I_L$ are the identity operators on $\mathcal{U}_J$ and $\mathcal{U}_L$, respectively, and $M_{JL}^{(i)}$ is an operator on $\mathcal{V}_J^{[3]} \otimes \mathcal{V}_L^{[3]}$.

The next step is to derive $M_{JL}^{(i)}$ for $i = 1, 2$. Define $\eta^{[N]}$ as

$$\eta^{[N]} = \int d\mu(U)(U^{\otimes N} \otimes I_{\mathcal{H}})|I\rangle\!\rangle\langle\!\langle I|^{\otimes N}(U^{\dagger \otimes N} \otimes I_{\mathcal{H}}). \tag{4.74}$$

$M_1$ and $M_2$ are represented as

$$M_1 = \eta^{[2]}_{\mathcal{K}_1 \mathcal{K}_3 \mathcal{H}_1 \mathcal{H}_3} \otimes \eta^{[1]}_{\mathcal{K}_2 \mathcal{H}_2}, \tag{4.75}$$

$$M_2 = \eta^{[2]}_{\mathcal{K}_1 \mathcal{H}_1} \otimes \eta^{[2]}_{\mathcal{K}_2 \mathcal{K}_3 \mathcal{H}_2 \mathcal{H}_3}. \tag{4.76}$$

By inserting Equation (4.74), we obtain

$$\eta^{[2]} = I_0 \otimes I_0 \oplus \frac{1}{3} I_1 \otimes I_1 \tag{4.77}$$

$$\eta^{[1]} = \frac{I_{\frac{1}{2}}}{2} \otimes I_{\frac{1}{2}}. \tag{4.78}$$

Since the dimension of all of the multiplicity subspaces is 1 for $N = 1, 2$. $M_1$ is decomposed as

$$M_1 = (I_0 \otimes I_0 \oplus \frac{1}{3} I_1 \otimes I_1) \otimes \frac{1}{2} I_{\frac{1}{2}} \otimes I_{\frac{1}{2}} \tag{4.79}$$

$$= (\frac{1}{2} I_{\frac{1}{2}} \otimes I_{\frac{1}{2}} \otimes |00\rangle\langle 00|_{\frac{1}{2}\frac{1}{2}} + \frac{1}{6} I_{\frac{1}{2}} \otimes I_{\frac{1}{2}} \otimes |11\rangle\langle 11|_{\frac{1}{2}\frac{1}{2}})$$

$$\oplus \frac{1}{6} I_{\frac{3}{2}} \otimes I_{\frac{1}{2}} \otimes |1\rangle\langle 1|_{\frac{1}{2}} \oplus \frac{1}{6} I_{\frac{1}{2}} \otimes I_{\frac{3}{2}} \otimes |1\rangle\langle 1|_{\frac{1}{2}} \oplus \frac{1}{6} I_{\frac{3}{2}} \otimes I_{\frac{3}{2}}, \tag{4.80}$$

where the basis $\{|0\rangle, |1\rangle\}$ of the multiplicity subspace $\mathcal{V}^{[3]}_{\frac{1}{2}}$ is defined as

$$\textstyle\sum_{m=-1/2}^{1/2} |\frac{1}{2}\frac{1}{2}(0)_{\mathcal{K}_1\mathcal{K}_3}\frac{1}{2};\frac{1}{2}m\rangle\langle\frac{1}{2}\frac{1}{2}(0)_{\mathcal{K}_1\mathcal{K}_3}\frac{1}{2};\frac{1}{2}m| =: I^{\mathcal{K}}_{\frac{1}{2}} \otimes |0\rangle\langle 0|_{\frac{1}{2}} \tag{4.81}$$

$$\textstyle\sum_{m=-1/2}^{1/2} |\frac{1}{2}\frac{1}{2}(1)_{\mathcal{K}_1\mathcal{K}_3}\frac{1}{2};\frac{1}{2}m\rangle\langle\frac{1}{2}\frac{1}{2}(1)_{\mathcal{K}_1\mathcal{K}_3}\frac{1}{2};\frac{1}{2}m| =: I^{\mathcal{K}}_{\frac{1}{2}} \otimes |1\rangle\langle 1|_{\frac{1}{2}}. \tag{4.82}$$

As explained Appx. B, one way to define the basis of the multiplicity subspace is the order of coupling of spin-1/2 systems. The basis $\{|0\rangle, |1\rangle\}$ corresponds to the order in which qubits in $\mathcal{K}_1$ and $\mathcal{K}_3$ are first coupled, followed by a qubit in $\mathcal{K}_2$.

Similarly, $M_2$ is also derived as

$$M_2 = \left(I_0 \otimes I_0 \oplus \frac{1}{3} I_1 \otimes I_1\right) \otimes \frac{1}{2} I_{\frac{1}{2}} \otimes I_{\frac{1}{2}} \tag{4.83}$$

$$\begin{aligned} &= \left(\frac{1}{2} I_{\frac{1}{2}} \otimes I_{\frac{1}{2}} \otimes |\tilde{0}\tilde{0}\rangle\langle\tilde{0}\tilde{0}|_{\frac{1}{2}\frac{1}{2}} + \frac{1}{6} I_{\frac{1}{2}} \otimes I_{\frac{1}{2}} \otimes |\tilde{1}\tilde{1}\rangle\langle\tilde{1}\tilde{1}|_{\frac{1}{2}\frac{1}{2}}\right) \\ &\quad \oplus \frac{1}{6} I_{\frac{3}{2}} \otimes I_{\frac{1}{2}} \otimes |\tilde{1}\rangle\langle\tilde{1}|_{\frac{1}{2}} \oplus \frac{1}{6} I_{\frac{1}{2}} \otimes I_{\frac{3}{2}} \otimes |\tilde{1}\rangle\langle\tilde{1}|_{\frac{1}{2}} \oplus \frac{1}{6} I_{\frac{3}{2}} \otimes I_{\frac{3}{2}}, \end{aligned} \tag{4.84}$$

where the basis $\{|\tilde{0}\rangle, |\tilde{1}\rangle\}$ is defined as of the multiplicity subspace $\mathcal{V}^{[3]}_{\frac{1}{2}}$

$$\textstyle\sum_{m=-\frac{1}{2}}^{\frac{1}{2}} |\frac{1}{2}\frac{1}{2}(0)_{\mathcal{K}_2\mathcal{K}_3}\frac{1}{2};\frac{1}{2}m\rangle\langle\frac{1}{2}\frac{1}{2}(0)_{\mathcal{K}_2\mathcal{K}_3}\frac{1}{2};\frac{1}{2}m| =: I^{\mathcal{K}}_{\frac{1}{2}} \otimes |\tilde{0}\rangle\langle\tilde{0}|_{\frac{1}{2}} \tag{4.85}$$

$$\textstyle\sum_{m=-\frac{1}{2}}^{\frac{1}{2}} |\frac{1}{2}\frac{1}{2}(1)_{\mathcal{K}_2\mathcal{K}_3}\frac{1}{2};\frac{1}{2}m\rangle\langle\frac{1}{2}\frac{1}{2}(1)_{\mathcal{K}_2\mathcal{K}_3}\frac{1}{2};\frac{1}{2}m| =: I^{\mathcal{K}}_{\frac{1}{2}} \otimes |\tilde{1}\rangle\langle\tilde{1}|_{\frac{1}{2}} \tag{4.86}$$

Thus we obtain

$$\begin{aligned} M_1 - M_2 = &\frac{I_{\frac{1}{2}}}{2} \otimes I_{\frac{1}{2}} \otimes \left(|00\rangle\langle 00| + \frac{1}{3}|11\rangle\langle 11| - |\tilde{0}\tilde{0}\rangle\langle\tilde{0}\tilde{0}| - \frac{1}{3}|\tilde{1}\tilde{1}\rangle\langle\tilde{1}\tilde{1}|\right) \\ &\oplus \frac{I_{\frac{3}{2}}}{4} \otimes I_{\frac{1}{2}} \otimes \frac{2}{3}(|1\rangle\langle 1| - |\tilde{1}\rangle\langle\tilde{1}|) \\ &\oplus \frac{I_{\frac{1}{2}}}{2} \otimes I_{\frac{3}{2}} \otimes \frac{1}{3}(|1\rangle\langle 1| - |\tilde{1}\rangle\langle\tilde{1}|) \end{aligned} \tag{4.87}$$

Next we derive the suitable form of $X$ for applying Lemma 1 and Lemma 2. From Lemma 1, $X$ can be chosen as

$$X = p\frac{I_{\frac{1}{2}}}{2} \otimes X^{(\frac{1}{2})} \oplus (1-p)\frac{I_{\frac{3}{2}}}{4}, \tag{4.88}$$

where $X^{(\frac{1}{2})}$ is a two by two positive semidefinite operator on the multiplicity subspace $\mathcal{V}^{[3]}_{\frac{1}{2}}$ with unit trace and $0 \le p \le 1$. In order to utilize Lemma 2, the basis $\{|\hat{0}\rangle, |\hat{1}\rangle\}$ is defined as

$$\sum_{m=-\frac{1}{2}}^{\frac{1}{2}} |\tfrac{1}{2}\tfrac{1}{2}(0)_{\mathcal{K}_1\mathcal{K}_2}\tfrac{1}{2};\tfrac{1}{2}m\rangle\langle\tfrac{1}{2}\tfrac{1}{2}(0)_{\mathcal{K}_1\mathcal{K}_2}\tfrac{1}{2};\tfrac{1}{2}m| =: I^{\mathcal{K}}_{\frac{1}{2}} \otimes |\hat{0}\rangle\langle\hat{0}|_{\frac{1}{2}} \tag{4.89}$$

$$\sum_{m=-\frac{1}{2}}^{\frac{1}{2}} |\tfrac{1}{2}\tfrac{1}{2}(1)_{\mathcal{K}_1\mathcal{K}_2}\tfrac{1}{2};\tfrac{1}{2}m\rangle\langle\tfrac{1}{2}\tfrac{1}{2}(1)_{\mathcal{K}_1\mathcal{K}_2}\tfrac{1}{2};\tfrac{1}{2}m| =: I^{\mathcal{K}}_{\frac{1}{2}} \otimes |\hat{1}\rangle\langle\hat{1}|_{\frac{1}{2}} \tag{4.90}$$

For $m = -1/2, 1/2$ we have

$$(S_{\mathcal{K}_{12}} \otimes I)|\frac{1}{2}\frac{1}{2}(0)_{\mathcal{K}_1\mathcal{K}_2}\frac{1}{2};\frac{1}{2}m\rangle = -|\frac{1}{2}\frac{1}{2}(0)_{\mathcal{K}_1\mathcal{K}_2}\frac{1}{2};\frac{1}{2}m\rangle \tag{4.91}$$

$$(S_{\mathcal{K}_{12}} \otimes I)|\frac{1}{2}\frac{1}{2}(1)_{\mathcal{K}_1\mathcal{K}_2}\frac{1}{2};\frac{1}{2}m\rangle = |\frac{1}{2}\frac{1}{2}(1)_{\mathcal{K}_1\mathcal{K}_2}\frac{1}{2};\frac{1}{2}m\rangle. \tag{4.92}$$

The condition of Lemma 2, *i.e.*, $[S_{\mathcal{H}_{12}}, X] = 0$ implies that $X^{(\frac{1}{2})}$ is diagonalized with a positive parameter $q$ in the basis $\{|\hat{0}\rangle, |\hat{1}\rangle\}$, namely,

$$X_q^{(\frac{1}{2})} = q|\hat{0}\rangle\langle\hat{0}| + (1-q)|\hat{1}\rangle\langle\hat{1}|, \tag{4.93}$$

where $0 \leq q \leq 1$.

Calculating Wigner's $6j$-coefficients [66], we obtain the relations of these three bases as

$$|\hat{0}\rangle = \frac{1}{2}|0\rangle + \frac{\sqrt{3}}{2}|1\rangle \tag{4.94}$$

$$|\hat{1}\rangle = \frac{\sqrt{3}}{2}|0\rangle - \frac{1}{2}|1\rangle \tag{4.95}$$

$$|\tilde{0}\rangle = -\frac{1}{2}|0\rangle + \frac{\sqrt{3}}{2}|1\rangle \tag{4.96}$$

$$|\tilde{1}\rangle = \frac{\sqrt{3}}{2}|0\rangle + \frac{1}{2}|1\rangle. \tag{4.97}$$

By substituting this $X$ in Equation (4.88), the diamond norm $\|\mathcal{U}_1 - \mathcal{U}_2\|_\diamond$ in Equation (4.14) is calculated as

$$\begin{aligned}
\|\mathcal{U}_1 - \mathcal{U}_2\|_\diamond &= \max_{0\leq p,q\leq 1} \|\frac{1}{2}I_{\frac{1}{2}} \otimes I_{\frac{1}{2}} \\
&\otimes p(I_{m_{1/2}} \otimes \sqrt{X_q^{(\frac{1}{2})}})(|00\rangle\langle 00| + \frac{1}{3}|11\rangle\langle 11| - |\tilde{0}\tilde{0}\rangle\langle\tilde{0}\tilde{0}| - \frac{1}{3}|\tilde{1}\tilde{1}\rangle\langle\tilde{1}\tilde{1}|)(I_{m_{1/2}} \otimes \sqrt{X_q^{(\frac{1}{2})}}) \\
&\oplus \frac{1}{4}I_{\frac{3}{2}} \otimes I_{\frac{1}{2}} \otimes p\frac{2}{3}\sqrt{X_q^{(\frac{1}{2})}}(|1\rangle\langle 1| - |\tilde{1}\rangle\langle\tilde{1}|)\sqrt{X_q^{(\frac{1}{2})}} \oplus \frac{1}{2}I_{\frac{1}{2}} \otimes I_{\frac{3}{2}} \otimes (1-p)\frac{1}{3}(|1\rangle\langle 1| - |\tilde{1}\rangle\langle\tilde{1}|)\|_1 \\
&= \max_{0\leq p,\alpha\leq 1} p\Big\{\Big\|(I_{m^A_{1/2}} \otimes \sqrt{X_q^{(\frac{1}{2})}})(|00\rangle\langle 00| + \frac{1}{3}|11\rangle\langle 11| - |\tilde{0}\tilde{0}\rangle\langle\tilde{0}\tilde{0}| - \frac{1}{3}|\tilde{1}\tilde{1}\rangle\langle\tilde{1}\tilde{1}|)(I_{m^A_{1/2}} \otimes \sqrt{X_q^{(\frac{1}{2})}})\Big\|_1 \\
&+ \frac{2}{3}\Big\|\sqrt{X_q^{(\frac{1}{2})}}(|1\rangle\langle 1| - |\tilde{1}\rangle\langle\tilde{1}|)\sqrt{X_q^{(\frac{1}{2})}}\Big\|_1\Big\} + (1-p)\frac{1}{3}\Big\||1\rangle\langle 1| - |\tilde{1}\rangle\langle\tilde{1}|\Big\|_1.
\end{aligned} \tag{4.98}$$

To maximize the diamond norm, $p = 0$ or $p = 1$ should be satisfied.

For the case of $p = 1$, we obtain

$$\left\|(I_{m^A_{1/2}}\otimes\sqrt{X_q^{(\frac{1}{2})}})(|00\rangle\langle 00|+\frac{1}{3}|11\rangle\langle 11|-|\tilde{0}\tilde{0}\rangle\langle\tilde{0}\tilde{0}|-\frac{1}{3}|\tilde{1}\tilde{1}\rangle\langle\tilde{1}\tilde{1}|)(I_{m^A_{1/2}}\otimes\sqrt{X_q^{(\frac{1}{2})}})\right\|_1=\frac{2}{\sqrt{3}}\cos t, \tag{4.99}$$

$$\begin{aligned}\frac{2}{3}\left\|\sqrt{X_q^{(\frac{1}{2})}}(|1\rangle\langle 1|-|\tilde{1}\rangle\langle\tilde{1}|)\sqrt{X_q^{(\frac{1}{2})}}\right\|_1 &= \frac{2}{3}\left\|-\frac{\sqrt{3}}{2}\sin t\cos t(|0\rangle\langle 1|+|1\rangle\langle 0|)\right\|_1\\ &= \frac{2}{\sqrt{3}}\sin t\cos t, \end{aligned}\tag{4.100}$$

where $t$ is defined as $q =: \sin t$. This calculation is derived as follows done by the followings. From the fact that the operator $\rho$ defined as

$$\rho := |00\rangle\langle 00| + \frac{1}{3}|11\rangle\langle 11| - |\tilde{0}\tilde{0}\rangle\langle\tilde{0}\tilde{0}| - \frac{1}{3}|\tilde{1}\tilde{1}\rangle\langle\tilde{1}\tilde{1}|,$$

is at most rank 2 since

$$\rho(|01\rangle - |10\rangle) = 0, \tag{4.101}$$

$$\rho(|00\rangle + 2\sqrt{3}|01\rangle + 3|11\rangle) = 0. \tag{4.102}$$

The rank of $\rho_x := (I_{m^A_{1/2}} \otimes \sqrt{X_q^{(\frac{1}{2})}})\rho(I_{m^A_{1/2}} \otimes \sqrt{X_q^{(\frac{1}{2})}})$ is also at most two. Then $\rho_x$ is represented as

$$\rho_x = \frac{\sin t}{4}\begin{pmatrix} 0 & \sqrt{3}\sin t & 0 & -\sin t\\ \sqrt{3}\sin t & 2\cos t & -\sin t & \frac{2\cos t}{\sqrt{3}}\\ 0 & -\sin t & 0 & \frac{\sin t}{\sqrt{3}}\\ -\sin t & \frac{2\cos t}{\sqrt{3}} & \frac{\sin t}{\sqrt{3}} & -2\cos t \end{pmatrix}$$

in the basis of $\{|0\hat{0}\rangle, |0\hat{1}\rangle, |1\hat{0}\rangle, |1\hat{1}\rangle\}$. Eigenvectors $|f_\pm\rangle$ of $\rho_x$ with non-zero eigenvalues are represented as

$$\begin{aligned}|f_\pm\rangle = &\frac{1}{2\sqrt{6}\sqrt{4-2\sqrt{3}\cos t-\sin^2 t}}\\ &\Big(3(\sqrt{3}\pm\cos t)\sin t, 6\cos t\pm\sqrt{3}(4-\sin^2 t), -(3\pm\sqrt{3}\cos t)\sin t, 2\sqrt{3}\cos t\mp 3\sin^2 t\Big)^T,\end{aligned}\tag{4.103}$$

with eigenvalues $\pm\cos t/\sqrt{3}$. The optimal success probability of this case is

$$p_{succ} = \frac{1}{2} + \frac{1}{4}\max_{0\le t\le\pi/2}\frac{2}{\sqrt{3}}\sin t(1+\cos t) = 7/8,$$

where the maximization is achieved with $t = \pi/3$.

For the case of $p = 0$, the success probability is

$$p^{opt}_{succ} = \frac{1}{2} + \frac{1}{12}\left\||1\rangle\langle 1| - |\tilde{1}\rangle\langle\tilde{1}|\right\|_1 = \frac{1}{2} + \frac{\sqrt{3}}{12} < \frac{7}{8}. \tag{4.104}$$

Thus the optimal averaged success probability is given by $p^{opt}_{succ} = 7/8$. □

### 4.1.6 The Proof of Lemma 3

*proof* Suppose that a quantum 4-tester $\widetilde{\Pi}_i$ gives the success probability $p_{succ}$, satisfying Equations (4.28) - (4.32). Let us define an averaged quantity of $\widetilde{\Pi}_i$ as

$$\widetilde{\Pi}'_i := \int d\mu(U) \int d\mu(V)((U^{\otimes 3})_{\mathcal{K}} \otimes (V^{\otimes 3})_{\mathcal{H}})\widetilde{\Pi}_i((U^{\dagger\otimes 3})_{\mathcal{K}} \otimes (V^{\dagger\otimes 3})_{\mathcal{H}}). \quad (4.105)$$

Similarly to the proof of Lemma 1, it can be shown that the success probability $p'_{succ}$ achieved the averaged 4-tester $\{\widetilde{\Pi}'_i\}$ is same as $p_{succ}$. For arbitrary unitary operators $A, B \in \mathrm{SU}(2)$, we have

$$((A^{\otimes 3})_{\mathcal{K}} \otimes (B^{\otimes})_{\mathcal{K}})\widetilde{\Pi}'_i((A^{\otimes 3})_{\mathcal{K}} \otimes (B^{\otimes})_{\mathcal{K}})^{\dagger} \quad (4.106)$$

$$= \int d\mu(U') \int d\mu(V')((U'^{\otimes 3})_{\mathcal{K}} \otimes (V'^{\otimes 3})_{\mathcal{H}})\Pi_i((U'^{\dagger\otimes 3})_{\mathcal{K}} \otimes (V'^{\dagger\otimes 3})_{\mathcal{H}}) \quad (4.107)$$

$$= \widetilde{\Pi}_i, \quad (4.108)$$

where we define $U' = AU$ and $V' = BV$. We obtain

$$[\widetilde{\Pi}'_i, (A^{\otimes 3})_{\mathcal{K}} \otimes (B^{\otimes 3})_{\mathcal{H}}] = 0, \quad (4.109)$$

for $i = 1, 2$ and arbitrary unitary operators $A, B \in \mathrm{SU}(2)$.

The sum of $\widetilde{\Pi}'_1$ and $\widetilde{\Pi}'_2$ is given by

$$\widetilde{\Pi}'_1 + \widetilde{\Pi}'_2 \quad (4.110)$$

$$= \int d\mu(U) \int d\mu(V)((U^{\otimes 3})_{\mathcal{K}} \otimes (V^{\otimes 3})_{\mathcal{H}})(\widetilde{\Pi}_1 + \widetilde{\Pi}_2)((U^{\dagger\otimes 3})_{\mathcal{K}} \otimes (V^{\dagger\otimes 3})_{\mathcal{H}}) \quad (4.111)$$

$$= \int d\mu(U) \int d\mu(V)((U^{\otimes 3})_{\mathcal{K}} \otimes (V^{\otimes 3})_{\mathcal{H}})(I_{\mathcal{K}_3} \otimes Y)((U^{\dagger\otimes 3})_{\mathcal{K}} \otimes (V^{\dagger\otimes 3})_{\mathcal{H}}) \quad (4.112)$$

$$= \int d\mu(U) \int d\mu(V)((U^{\otimes 2})_{\mathcal{K}_1\mathcal{K}_2} \otimes (V^{\otimes 3})_{\mathcal{H}})Y((U^{\dagger\otimes 3})_{\mathcal{K}_1\mathcal{K}_2} \otimes (V^{\dagger\otimes 3})_{\mathcal{H}}) \quad (4.113)$$

$$=: Y'. \quad (4.114)$$

By definition, we obtain

$$[Y', (A^{\otimes 2})_{\mathcal{K}_1\mathcal{K}_2} \otimes (B^{\otimes 3})_{\mathcal{H}}] = 0, \quad (4.115)$$

for arbitrary unitary operators $A, B \in \mathrm{SU}(2)$. Next, we have

$$\begin{aligned}
&\mathrm{Tr}_{\mathcal{H}_3} Y' && (4.116)\\
&= \mathrm{Tr}_{\mathcal{H}_3} \int d\mu(U) \int d\mu(V)((U^{\otimes 2})_{\mathcal{K}_1\mathcal{K}_2} \otimes (V^{\otimes 3})_{\mathcal{H}}) Y ((U^{\dagger\otimes 3})_{\mathcal{K}_1\mathcal{K}_2} \otimes (V^{\dagger\otimes 3})_{\mathcal{H}}) && (4.117)\\
&= \int d\mu(U) \int d\mu(V)((U^{\otimes 2})_{\mathcal{K}_1\mathcal{K}_2} \otimes (V^{\otimes 3})_{\mathcal{H}})(I_{\mathcal{K}_2} \otimes Y^{(1)})((U^{\dagger\otimes 3})_{\mathcal{K}_1\mathcal{K}_2} \otimes (V^{\dagger\otimes 3})_{\mathcal{H}}) && (4.118)\\
&= \int d\mu(U) \int d\mu(V)(U_{\mathcal{K}_1} \otimes (V^{\otimes 2})_{\mathcal{H}_1\mathcal{H}_2}) Y^{(1)} (U^{\dagger}_{\mathcal{K}_1} \otimes (V^{\dagger\otimes 2})_{\mathcal{H}_1\mathcal{H}_2}) && (4.119)\\
&=: Y'^{(1)}. && (4.120)
\end{aligned}$$

By definition, we obtain

$$[Y'^{(1)}, A_{\mathcal{K}_1} \otimes (B^{\otimes 2})_{\mathcal{H}_1\mathcal{H}_2}] = 0, \tag{4.121}$$

for arbitrary unitary operators $A, B \in \mathrm{SU}(2)$. Finally we have

$$\begin{aligned}
&\mathrm{Tr}_{\mathcal{H}_2} Y'^{(1)} && (4.122)\\
&= \mathrm{Tr}_{\mathcal{H}_2} \int d\mu(U) \int d\mu(V)(U_{\mathcal{K}_1} \otimes (V^{\otimes 2})_{\mathcal{H}_1\mathcal{H}_2}) Y^{(1)} (U^{\dagger}_{\mathcal{K}_1} \otimes (V^{\dagger\otimes 2})_{\mathcal{H}_1\mathcal{H}_2}) && (4.123)\\
&= \int d\mu(U) \int d\mu(V)(U_{\mathcal{K}_1} \otimes (V^{\otimes 2})_{\mathcal{H}_1\mathcal{H}_2})(I_{\mathcal{K}_1} \otimes Y^{(0)})(U^{\dagger}_{\mathcal{K}_1} \otimes (V^{\dagger\otimes 2})_{\mathcal{H}_1\mathcal{H}_2}) && (4.124)\\
&= \int d\mu(V) V_{\mathcal{H}_1}) Y^{(0)} V^{\dagger}_{\mathcal{H}_1} && (4.125)\\
&=: Y'^{(0)}. && (4.126)
\end{aligned}$$

By definition, we obtain

$$[Y'^{(0)}, B_{\mathcal{H}_1}] = 0, \tag{4.127}$$

for arbitrary unitary operators $B \in \mathrm{SU}(2)$. Therefore the quantum 4-tester $\{\Pi_i\}$ and positive semidefinite operators $Y$, $Y^{(1)}$ and $Y^{(0)}$ can be chosen to satisfy

$$\begin{aligned}
&[\Pi'_i, (A^{\otimes 3})_{\mathcal{K}} \otimes (B^{\otimes 3})_{\mathcal{H}}] = 0 && (4.128)\\
&[Y', (A^{\otimes 2})_{\mathcal{K}_1\mathcal{K}_2} \otimes (B^{\otimes 3})_{\mathcal{H}}] = 0 && (4.129)\\
&[Y'^{(1)}, (A^{\otimes 1})_{\mathcal{K}_1} \otimes (B^{\otimes 2})_{\mathcal{H}_1\mathcal{H}_2}] = 0 && (4.130)\\
&[Y'^{(0)}, B_{\mathcal{H}_1}] = 0. && (4.131)
\end{aligned}$$

□

### 4.1.7 The Proof of Lemma 5

We derive the dual problem using Lagrange multipliers. Lagrangian $L$ is defined as

$$\begin{aligned} L = &\frac{1}{2}\mathrm{Tr}\left[\Pi_1 M_1 + \Pi_2 M_2\right] - \mathrm{Tr}\left[\Omega(\Pi_1 + \Pi_2 - I_{\mathcal{K}} \otimes Y)\right] \\ &- \mathrm{Tr}\left[\Omega^{[1]}(\mathrm{Tr}_{\mathcal{H}} Y - I_{\mathcal{K}_3} \otimes Y^{(1)})\right] - \mathrm{Tr}\left[\Omega^{[0]}(\mathrm{Tr}_{\mathcal{H}} Y^{(1)} - I_1 \otimes Y^{(0)})\right] \\ &- a(\mathrm{Tr} Y^{(0)} - 1) \end{aligned} \tag{4.132}$$

where $\Omega, \Omega^{[1]}, \Omega^{[0]}$ and $a$ are Lagrange multipliers. If the conditions in Equations (4.27) - (4.32) are satisfied, additional terms in Lagrangian are 0 for any Lagrange multipliers $\Omega, \Omega^{[1]}, \Omega^{[0]}$ and $a$. By rewriting Lagrangian, we have

$$\begin{aligned} L = \mathrm{Tr}\left[\Pi_1(\frac{M_1}{2} - \Omega)\right] + \mathrm{Tr}\left[\Pi_2(\frac{M_2}{2} - \Omega)\right] + \mathrm{Tr}\left[Y(\mathrm{Tr}_{\mathcal{K}_3}\Omega - I_{\mathcal{H}_3} \otimes \Omega^{[1]})\right] \\ \mathrm{Tr}\left[Y^{(1)}(\mathrm{Tr}_{\mathcal{K}_2}\Omega - I_{\mathcal{H}_2} \otimes \Omega^{[0]})\right] + \mathrm{Tr}\left[Y^{(0)}(\mathrm{Tr}_{\mathcal{K}_1}\Omega - aI_{\mathcal{H}_1} \otimes \Omega^{0})\right] + a. \end{aligned}$$

Note that the trace of the product of two positive semidefinite operators is non-negative. Therefor, we obtain

$$L \leq a, \tag{4.133}$$

if the following inequalities

$$\frac{M^{[1]}}{2} - \Omega \leq 0, \tag{4.134}$$

$$\frac{M^{[2]}}{2} - \Omega \leq 0, \tag{4.135}$$

$$\mathrm{Tr}_{\mathcal{K}_3}\Omega - I_{\mathcal{H}_3} \otimes \Omega^{[1]} \leq 0, \tag{4.136}$$

$$\mathrm{Tr}_{\mathcal{K}_2}\Omega - I_{\mathcal{H}_2} \otimes \Omega^{[0]} \leq 0, \tag{4.137}$$

$$\mathrm{Tr}_{\mathcal{K}_1}\Omega - aI_{\mathcal{H}_1} \otimes \Omega^{0} \leq 0 \tag{4.138}$$

are satisfied. If the above Conditions (4.134) - (4.138) are satisfied, Inequality (4.133) is always satisfied for arbitrary positive semidefinite operators $\Pi_1, \Pi_2, Y, Y^{(1)}$ and $Y^{(0)}$. Therefore minimizing $a$ satisfying Conditions (4.134) - (4.134) gives the upper bound of the solution of the primal problem. □

### 4.1.8 The Proof of Theorem 3

The outline of the proof is the following. First we show that the positive semidefinite operator $\Omega$ can be chosen so that the non-trivial elements are only on the multiplicity subspaces. Then we rewrite the dual problem in terms of the operators in the multiplicity subspaces. Finally we give the solution for the dual problem in the multiplicity subspaces.

First we assume that a positive semidefinite operator $\Omega$ and $a$ are satisfying Equations (4.38) - (4.42). Then a new positive semidefinite operator $\Omega'$ defined as

$$\Omega' = \int d\mu(U) \int d\mu(V)((U^{\otimes 3})_{\mathcal{K}} \otimes V_{\mathcal{H}}^{\otimes 3})\Omega((U^{\dagger \otimes 3})_{\mathcal{K}} \otimes (V^{\dagger \otimes 3})_{\mathcal{H}}). \tag{4.139}$$

$\Omega'$ gives the same solution $a$ as $\Omega$. By definition $\Omega'$ satisfies

$$[\Omega', A^{\otimes 3} \otimes B^{\otimes 3}] = 0 \tag{4.140}$$

for arbitrary unitary operators $A$ and $B$ in SU(2). Therefore to minimize $a$, without loss of generality, $\Omega$ can be chosen satisfying Equation (4.140).

Taking the partial trace $\mathcal{K}_3 \otimes \mathcal{H}_3$ and $\mathcal{K}_3 \otimes \mathcal{H}_3 \otimes \mathcal{K}_2 \otimes \mathcal{H}_2$ of Equation (4.140) commutation relations

$$[\Omega'^{[1]}, A^{\otimes 2} \otimes B^{\otimes 2}] = 0 \tag{4.141}$$

$$[\Omega'^{[0]}, A \otimes B] = 0 \tag{4.142}$$

are obtained for $\Omega'^{[1]}$ and $\Omega'^{[0]}$ defined as

$$\Omega'^{[1]} = \int d\mu(U) \int d\mu(V)((U^{\otimes 2})_{\mathcal{K}_1\mathcal{K}_2} \otimes V_{\mathcal{H}_1\mathcal{H}_2}^{\otimes 2})\Omega((U^{\dagger \otimes 2})_{\mathcal{K}_1\mathcal{K}_2} \otimes (V^{\dagger \otimes 2})_{\mathcal{H}_1\mathcal{H}_2}), \tag{4.143}$$

and

$$\Omega'^{[0]} = \int d\mu(U) \int d\mu(V)(U_{\mathcal{K}_1} \otimes V_{\mathcal{H}_1})\Omega(U_{\mathcal{K}}^{\dagger} \otimes V_{\mathcal{H}_1}^{\dagger}), \tag{4.144}$$

respectively.

We can assume that $\Omega, \Omega^{[1]}$ and $\Omega^{[0]}$ are represented as

$$\Omega = \bigoplus_{J=\frac{1}{2}}^{\frac{3}{2}} \bigoplus_{L=\frac{1}{2}}^{\frac{3}{2}} \frac{I_J}{d_J} \otimes I_L \otimes \Omega_{(JL)} \tag{4.145}$$

$$\Omega^{[1]} = \bigoplus_{J=0}^{1} \bigoplus_{L=0}^{1} \frac{I_J}{d_J} \otimes I_L \otimes \Omega_{JL}^{[1]} \tag{4.146}$$

$$\Omega^{[0]} = \Omega_{\frac{1}{2}\frac{1}{2}}^{[0]} \frac{I_{\frac{1}{2}}}{d_{\frac{1}{2}}} \otimes I_{\frac{1}{2}}, \tag{4.147}$$

where $I_J$ is the identity operator on the irreducible subspace $\mathcal{U}_J$ and $\Omega^{(JL)}$ is an operator on $\mathcal{V}_J^{[3]} \otimes \mathcal{V}_L^{[3]}$ for $J, L = 1/2, 2/3$ and $\Omega_{JL}^{[1]}$ and $\Omega_{\frac{1}{2}\frac{1}{2}}^{[0]}$ are some positive numbers for $J, L = 0, 1$.

We rewrite Equations (4.38) - (4.39) in terms of the operator on the multiplicity subspaces. The operator $M_i$ is also represented as

$$M_i = \bigoplus_{J=\frac{1}{2}}^{\frac{3}{2}} \bigoplus_{L=\frac{1}{2}}^{\frac{3}{2}} \frac{I_J}{d_J} \otimes I_L \otimes M_i^{(JL)}, \tag{4.148}$$

for $i = 1, 2$. Thus Equations (4.38) and (4.39) are rewritten as

$$\Omega_{JL} - \frac{M_{JL}^{[i]}}{2} \geq 0, \tag{4.149}$$

for $J, L = 1/2, 3/2$ and $i = 1, 2$.

Next we rewrite Equation (4.40). The multiplicity in $\mathcal{V}_{\frac{1}{2}}^{[3]}$ is generated by the following two ways. The first one corresponds to the coupling of total spin 0 and 1/2, that is, $\mathcal{U}_0 \otimes \mathcal{U}_{\frac{1}{2}} = \mathcal{U}_{\frac{1}{2}}$. The other one is the coupling of total spin 1 and 1/2, that is $\mathcal{U}_1 \otimes \mathcal{U}_{\frac{1}{2}} = \mathcal{U}_{\frac{1}{2}} \oplus \mathcal{U}_{\frac{3}{2}}$. One can easily calculate that

$$\mathrm{Tr}_{\mathcal{K}_3}\Omega = \bigoplus_{L=\frac{1}{2}}^{\frac{3}{2}} \Big[ \frac{I_0}{d_0} \otimes I_L \otimes (P_{\frac{1}{2},0}^{[3]} \otimes I_{\mathcal{V}_L^{[3]}}) \Omega_{\frac{1}{2}L} (P_{\frac{1}{2},0}^{[3]} \otimes I_{\mathcal{V}_L^{[3]}}) \tag{4.150}$$

$$\oplus \frac{I_1}{d_1} \otimes I_L \otimes ((P_{\frac{1}{2},1}^{[3]} \otimes I_{\mathcal{V}_L^{[3]}}) \Omega_{\frac{1}{2}L} (P_{\frac{1}{2},1}^{[3]} \otimes I_{\mathcal{V}_L^{[3]}}) + \Omega_{\frac{3}{2}L}) \Big], \tag{4.151}$$

where we define isometries $P_{j,j\pm\frac{1}{2}}^{[N]}$ from the subspace of $\mathcal{V}_j^{[N]}$ that is generated from the multiplicity subspace $\mathcal{V}_{j\pm\frac{1}{2}}^{[N-1]}$ to the multiplicity subspace $\mathcal{V}_{j\pm\frac{1}{2}}^{[N-1]}$.

Using the Clepsch-Gordon decomposition, we have

$$I_{\mathcal{H}_3} \otimes \Omega^{[1]} = \frac{I_0}{d_0} \otimes [I_{\frac{1}{2}} \otimes (\Omega_{00}^{[1]} \oplus \Omega_{01}^{[1]}) \oplus I_{\frac{3}{2}} \otimes \Omega_{01}^{[1]}] \oplus \frac{I_1}{d_1} \otimes [I_{\frac{1}{2}} \otimes (\Omega_{10}^{[1]} \oplus \Omega_{11}^{[1]}) \oplus I_{\frac{3}{2}} \otimes \Omega_{11}^{[1]}].$$

Condition (4.40) is rewritten as

$$\Omega_{00}^{[1]} \oplus \Omega_{01}^{[1]} - \Omega_{\frac{1}{2}\frac{1}{2}}^{0\to 1/2} \geq 0, \tag{4.152}$$

$$\Omega_{01}^{[1]} - \Omega_{\frac{1}{2}\frac{3}{2}}^{0\to 1/2} \geq 0, \tag{4.153}$$

$$\Omega_{00}^{[1]} \oplus \Omega_{01}^{[1]} - \Omega_{\frac{1}{2}\frac{1}{2}}^{1\to 1/2} - \Omega_{\frac{3}{2}\frac{1}{2}} \geq 0, \tag{4.154}$$

$$\Omega_{11}^{[1]} - \Omega_{\frac{1}{2}\frac{3}{2}}^{1\to 1/2} - \Omega_{\frac{3}{2}\frac{3}{2}} \geq 0, \tag{4.155}$$

where we define $\Omega_{\frac{1}{2}\frac{1}{2}}^{j\to 1/2} = (P_{\frac{1}{2},j}^{[3]} \otimes I_{\mathcal{V}_L^{[3]}}) \Omega_{\frac{1}{2}L} (P_{\frac{1}{2},j}^{[3]} \otimes I_{\mathcal{V}_L^{[3]}})$. Similarly, we obtain

$$\mathrm{Tr}_{\mathcal{K}_2}\Omega^{[1]} = \frac{I_{\frac{1}{2}}}{d_{\frac{1}{2}}} \otimes [(\Omega_{00} + \Omega_{10}) I_0 \oplus (\Omega_{01} + \Omega_{11}) I_1] \tag{4.156}$$

$$I_{\mathcal{H}_2} \otimes \Omega^{[0]} = \frac{I_{\frac{1}{2}}}{d_{\frac{1}{2}}} \otimes (\Omega_{\frac{1}{2}\frac{1}{2}}^{[0]} I_0 + \Omega_{\frac{1}{2}\frac{1}{2}}^{[0]} I_1) \tag{4.157}$$

and

$$\mathrm{Tr}_{\mathcal{K}_1}\Omega^{[0]} = \Omega^{[0]}_{\frac{1}{2}\frac{1}{2}} I^{[0]}. \tag{4.158}$$

Equations (4.41) and (4.41) are rewritten as

$$\Omega^{[0]}_{\frac{1}{2}\frac{1}{2}} - \Omega^{[1]}_{00} - \Omega^{[1]}_{10} \geq 0, \tag{4.159}$$

$$\Omega^{[0]}_{\frac{1}{2}\frac{1}{2}} - \Omega^{[1]}_{01} - \Omega^{[1]}_{11} \geq 0, \tag{4.160}$$

$$a - \Omega^{[0]}_{\frac{1}{2}\frac{1}{2}} \geq 0. \tag{4.161}$$

Obviously we can assume $\Omega^{[0]}_{\frac{1}{2}\frac{1}{2}} = a$ without loss of generality. We finally obtain a dual problem on the multiplicity subspace as follows.

$$\text{minimize:} \quad a \tag{4.162}$$

$$\text{subject to:} \quad \Omega_{KL} - \frac{M^{[i]}_{KL}}{2} \geq 0 \text{ for } K, L = 1/2, 3/2 \text{ and } i = 1, 2, \tag{4.163}$$

$$\Omega^{[1]}_{00} \oplus \Omega^{[1]}_{01} - \Omega^{0\to 1/2}_{\frac{1}{2}\frac{1}{2}} \geq 0, \tag{4.164}$$

$$\Omega^{[1]}_{01} - \Omega^{0\to 1/2}_{\frac{1}{2}\frac{3}{2}} \geq 0, \tag{4.165}$$

$$\Omega^{[1]}_{00} \oplus \Omega^{[1]}_{01} - \Omega^{1\to 1/2}_{\frac{1}{2}\frac{1}{2}} - \Omega_{\frac{3}{2}\frac{1}{2}} \geq 0, \tag{4.166}$$

$$\Omega^{[1]}_{11} - \Omega^{1\to 1/2}_{\frac{1}{2}\frac{3}{2}} - \Omega_{\frac{3}{2}\frac{3}{2}} \geq 0. \tag{4.167}$$

$$a - \Omega^{[1]}_{00} - \Omega^{[1]}_{10} \geq 0 \tag{4.168}$$

$$a - \Omega^{[1]}_{01} - \Omega^{[1]}_{11} \geq 0. \tag{4.169}$$

As we have shown in Lemma 4, two different kind of orders can be considered. The first case is the order in which the reference box 1 and the reference box 2 are first used in the parallel way and the test box is used later. Note that we use the bases $\{|0\rangle, |1\rangle\}$ defined in Equation (4.81) and (4.82) to represent the following matrices in the multiplicity subspaces.

In this case, $M_1$ is derived as

$$\begin{aligned} M_1 = &\frac{1}{2} I^{\mathcal{K}}_{\frac{1}{2}} \otimes I^{\mathcal{H}}_{\frac{1}{2}} \otimes (|00\rangle\langle 00|_{\frac{1}{2}\frac{1}{2}} + \frac{1}{3}|11\rangle\langle 11|_{\frac{1}{2}\frac{1}{2}}) \\ &\oplus \frac{1}{4} I^{\mathcal{K}}_{\frac{3}{2}} \otimes I^{\mathcal{H}}_{\frac{1}{2}} \otimes \frac{2}{3}|1\rangle\langle 1|_{\frac{1}{2}} \oplus \frac{1}{2} I^{\mathcal{K}}_{\frac{1}{2}} \otimes I^{\mathcal{H}}_{\frac{3}{2}} \otimes \frac{1}{3}|1\rangle\langle 1|_{\frac{1}{2}} \oplus \frac{1}{6} I^{\mathcal{K}}_{\frac{3}{2}} \otimes I^{\mathcal{H}}_{\frac{3}{2}}, \end{aligned} \tag{4.170}$$

which is same as Equation (4.80). In matrix representations, we have

$$M^{[1]}_{\frac{1}{2}\frac{1}{2}} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{3} \end{pmatrix}, M^{[1]}_{\frac{1}{2}\frac{3}{2}} = \begin{pmatrix} 0 & 0 \\ 0 & \frac{1}{3} \end{pmatrix}, \tag{4.171}$$

$$M^{[1]}_{\frac{3}{2}\frac{1}{2}} = \begin{pmatrix} 0 & 0 \\ 0 & \frac{2}{3} \end{pmatrix}, M^{[1]}_{\frac{3}{2}\frac{3}{2}} = \frac{2}{3}. \tag{4.172}$$

$M_2$ is derived as

$$\begin{aligned} M_2 = (&\frac{1}{2} I^{\mathcal{K}}_{\frac{1}{2}} \otimes I^{\mathcal{H}}_{\frac{1}{2}} \otimes |\tilde{0}\tilde{0}\rangle\langle\tilde{0}\tilde{0}|_{\frac{1}{2}\frac{1}{2}} + \frac{1}{6} I^{\mathcal{K}}_{\frac{1}{2}} \otimes I^{\mathcal{H}}_{\frac{1}{2}} \otimes |\tilde{1}\tilde{1}\rangle\langle\tilde{1}\tilde{1}|_{\frac{1}{2}\frac{1}{2}}) \\ &\oplus \frac{1}{6} I^{\mathcal{K}}_{\frac{3}{2}} \otimes I^{\mathcal{H}}_{\frac{1}{2}} \otimes |\tilde{1}\rangle\langle\tilde{1}|_{\frac{1}{2}} \oplus \frac{1}{6} I^{\mathcal{K}}_{\frac{1}{2}} \otimes I^{\mathcal{H}}_{\frac{3}{2}} \otimes |\tilde{1}\rangle\langle\tilde{1}|_{\frac{1}{2}} \oplus \frac{1}{6} I^{\mathcal{K}}_{\frac{3}{2}} \otimes I^{\mathcal{H}}_{\frac{3}{2}}. \end{aligned} \tag{4.173}$$

The matrix representations are

$$M^{[2]}_{\frac{1}{2}\frac{1}{2}} = \frac{1}{4} \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & -\frac{2}{\sqrt{3}} \\ 0 & 1 & 1 & -\frac{2}{\sqrt{3}} \\ 1 & -\frac{2}{\sqrt{3}} & -\frac{2}{\sqrt{3}} & \frac{7}{3} \end{pmatrix}, M^{[2]}_{\frac{1}{2}\frac{3}{2}} = \frac{1}{4} \begin{pmatrix} 1 & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} & \frac{1}{3} \end{pmatrix}, \tag{4.174}$$

$$M^{[2]}_{\frac{3}{2}\frac{1}{2}} = \frac{1}{2} \begin{pmatrix} 1 & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} & \frac{1}{3} \end{pmatrix}, M^{[2]}_{\frac{3}{2}\frac{3}{2}} = \frac{2}{3}. \tag{4.175}$$

Therefore a solution of the dual problem represented by Equations (4.162) and (4.169) is given by

$$a = \frac{7}{8} \tag{4.176}$$

$$\Omega_{\frac{1}{2}\frac{1}{2}} = \frac{1}{2} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{1}{4} & \frac{1}{4} & -\frac{1}{2\sqrt{3}} \\ 0 & \frac{1}{4} & \frac{1}{4} & -\frac{1}{2\sqrt{3}} \\ 0 & -\frac{1}{2\sqrt{3}} & -\frac{1}{2\sqrt{3}} & \frac{2}{3}, \end{pmatrix}, \Omega_{\frac{1}{2}\frac{3}{2}} = \begin{pmatrix} \frac{1}{8} & \frac{1}{8\sqrt{3}} \\ \frac{1}{8\sqrt{3}} & \frac{1}{3} \end{pmatrix}, \tag{4.177}$$

$$\Omega_{\frac{3}{2}\frac{1}{2}} = \frac{1}{4} \begin{pmatrix} 1 & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} & \frac{5}{3} \end{pmatrix}, \Omega_{\frac{3}{2}\frac{3}{2}} = \frac{1}{3} \tag{4.178}$$

$$\Omega^{[1]}_{00} = \frac{1}{2}, \Omega^{[1]}_{01} = \frac{1}{8}, \Omega^{[1]}_{10} = \frac{3}{8}, \Omega^{[1]}_{11} = \frac{3}{4}, \tag{4.179}$$

It is easy to check these solutions satisfy Conditions (4.163) to (4.169).

The second case is the order in which the reference box 1 and the test box are used first and the reference box 2 is next. In this case, $M_2$ is same in the first case. $M_1$ can be derived by exchanging the spaces $\mathcal{H}_1$ and $\mathcal{H}_3$ and changing $U_1$ and $U_2$. $U_1$ and $U_2$ can be exchanged freely since $U_1$ and $U_2$ are just labels in the integration over the Haar measure. In the multiplicity subspace of the basis $\{|0\rangle, |1\rangle\}$, exchanging the spaces $\mathcal{H}_1$ and $\mathcal{H}_3$ corresponds to applying $\sigma_z$ defined as

$$\sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{4.180}$$

Then we obtain the matrix representations

$$M^{[1]}_{\frac{1}{2}\frac{1}{2}} = \frac{1}{4}\begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & \frac{2}{\sqrt{3}} \\ 0 & 1 & 1 & \frac{2}{\sqrt{3}} \\ 1 & \frac{2}{\sqrt{3}} & \frac{2}{\sqrt{3}} & \frac{7}{3} \end{pmatrix}, M^{[1]}_{\frac{1}{2}\frac{3}{2}} = \frac{1}{4}\begin{pmatrix} 1 & -\frac{1}{\sqrt{3}} \\ -\frac{1}{\sqrt{3}} & \frac{1}{3} \end{pmatrix}, \tag{4.181}$$

$$M^{[1]}_{\frac{3}{2}\frac{1}{2}} = \frac{1}{2}\begin{pmatrix} 1 & -\frac{1}{\sqrt{3}} \\ -\frac{1}{\sqrt{3}} & \frac{1}{3} \end{pmatrix}, M^{[1]}_{\frac{3}{2}\frac{3}{2}} = \frac{2}{3} \tag{4.182}$$

A solution of the dual problem given in Equation (**??**) in this case is given by

$$a = \frac{7}{8} \tag{4.183}$$

$$\Omega_{\frac{1}{2}\frac{1}{2}} = \frac{1}{2}\begin{pmatrix} \frac{1}{4} & 0 & 0 & \frac{1}{4} \\ 0 & \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & \frac{1}{2} & \frac{1}{2} & 0 \\ \frac{1}{4} & 0 & 0 & \frac{11}{12} \end{pmatrix}, \Omega_{\frac{1}{2}\frac{3}{2}} = \frac{1}{4}\begin{pmatrix} 1 & 0 \\ 0 & \frac{1}{3} \end{pmatrix}, \tag{4.184}$$

$$\Omega_{\frac{3}{2}\frac{1}{2}} = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & \frac{1}{3} \end{pmatrix}, \Omega_{\frac{3}{2}\frac{3}{2}} = \frac{1}{3} \tag{4.185}$$

$$\Omega^{[1]}_{00} = \frac{1}{8},\ \Omega^{[1]}_{01} = \frac{1}{4},\ \Omega^{[1]}_{10} = \frac{3}{4},\ \Omega^{[1]}_{11} = \frac{5}{8}. \tag{4.186}$$

It is easy to check that these solutions also satisfy Conditions (4.163) to (4.169). Thus the optimal averaged success probability of obtaining correct answer is 7/8 under the ordered strategies. □

## 4.2 One of Unitary Operations is Known

In this section, we assume that classical descriptions of one of the reference boxes, $U_1$, is given so that any quantum operations depending on the description of the known reference box $U_1$ are allowed to discriminate which unitary operation is performed by the test box, whereas the classical description of another reference box is not give. This situation help to understand how the success probability changes when classical descriptions of the candidate unitary operations are given.

### 4.2.1 No Reference Box

First we consider the case in which only the test box is given and there is no reference box but a description of a unitary operator $U_1$ is given and we need to find out that the test implements $U_1$ or not. In this case, there is no clue to specify what kind operation $U_2$ is. Here, instead of being given the reference box 1, we have complete information about the unitary operation $U_1$. That means, we can apply the the reference box 1 arbitrary times since any quantum operations depending on $U_1$ is allowed. Contrary to the difference of the given resource, we show that the optimal success probability is still 7/8.

We denote the input and output spaces of the test box as $\mathcal{H}$ and $\mathcal{K}$. An initial state $|\psi\rangle \in \mathcal{H} \otimes \mathcal{H}$ is represented as $|\psi\rangle = I \otimes \sqrt{X}|I\rangle\rangle$, with a positive semidefinite operator $X$ on $\mathcal{H}$ and maximally entangled vector $|I\rangle\rangle$ in $\mathcal{H} \otimes \mathcal{H}$. The POVM is denoted as $\{\Pi_1, \Pi_2\}$.

Consider a new initial state $|\phi\rangle = U_1^\dagger \otimes I|\psi\rangle$, two candidate states for discrimination are $U_2U_1^\dagger \otimes I|\phi\rangle$ and $|\phi\rangle$, where $U_2$ is the unitary operator implemented by the reference box 2. Then the average success probability $p_{succ}$ over $U_1$

$$p_{succ} = \frac{1}{2}\int d\mu(U_1)\mathrm{Tr}[(U_2U_1^\dagger \otimes I)|\phi\rangle\langle\phi|(U_1U_2^\dagger \otimes I) + |\phi\rangle\langle\phi|] \tag{4.187}$$

$$= \frac{1}{2}\mathrm{Tr}[E\Pi_1 + |\phi\rangle\langle\phi|\Pi_2], \tag{4.188}$$

where

$$E = \int d\mu(U_1)(U_1 \otimes I)|\phi\rangle\langle\phi|(U_1^\dagger \otimes I) = \frac{I}{2} \otimes X. \tag{4.189}$$

Thus the averaged success probability over $U_1$ does not depend on $U_2$ when we consider the initial state $|\phi\rangle$. Without loss of generality it is possible to assume that $U_2 = I$ in the following discussion. In this case, the test box can be considered to implement one of unitary operations $U_1$ and $I$, and $U_1$ is probabilistically chosen from the Haar measure. To maximize the averaged success probability, we define a 2 tester $\widetilde{\Pi}_i = (I \otimes \sqrt{X})\Pi_i(I \otimes \sqrt{X})$ and obtain

$$p_{succ} = \frac{1}{2}\mathrm{Tr}[E\Pi_1 + |\phi\rangle\langle\phi|\Pi_2] \tag{4.190}$$

$$= \frac{1}{2}\mathrm{Tr}\left[\left(\frac{I}{2} \otimes I\right)\widetilde{\Pi}_1 + |I\rangle\rangle\langle\langle I|\widetilde{\Pi}_2\right], \tag{4.191}$$

where $\widetilde{\Pi_1} + \widetilde{\Pi_2} = I \otimes X$.

We assume that $\{\widetilde{\Pi}_1, \widetilde{\Pi}_2\}$ realize the averaged success probability $p_{succ}$. Then a positive semidefinite operator $\Pi_i'$ defined as

$$\widetilde{\Pi}_i' = \int d\mu(A)(A \otimes A^*)\widetilde{\Pi}_i(A \otimes A^*)^\dagger \tag{4.192}$$

also achieves the same averaged success probability $p_{succ}$. Since

$$\begin{aligned}
p_{succ}' &= \frac{1}{2}\mathrm{Tr}[(\frac{I}{2} \otimes I)\widetilde{\Pi}_1' + |I\rangle\rangle\langle\langle I|\widetilde{\Pi}_2'] \\
&= \frac{1}{2}\int d\mu(A)\mathrm{Tr}[(\frac{I}{2} \otimes I)(A \otimes A^*)\widetilde{\Pi}_1(A \otimes A^*)^\dagger + |I\rangle\rangle\langle\langle I|(A \otimes A^*)\widetilde{\Pi}_2(A \otimes A^*)^\dagger] \\
&= \frac{1}{2}\int d\mu(A)\mathrm{Tr}[(A \otimes A^*)^\dagger(\frac{I}{2} \otimes I)(A \otimes A^*)\widetilde{\Pi}_1 + (A \otimes A^*)^\dagger|I\rangle\rangle\langle\langle I|(A \otimes A^*)\widetilde{\Pi}_2] \\
&= \frac{1}{2}\int d\mu(A)\mathrm{Tr}[(\frac{I}{2} \otimes I)\widetilde{\Pi}_1 + |I\rangle\rangle\langle\langle I|\widetilde{\Pi}_2] \\
&= \frac{1}{2}\mathrm{Tr}[(\frac{I}{2} \otimes I)\widetilde{\Pi}_1 + |I\rangle\rangle\langle\langle I|\widetilde{\Pi}_2] \\
&= p_{succ},
\end{aligned} \tag{4.193}$$

where we used the property $(A \otimes A^*)|I\rangle\rangle = |I\rangle\rangle$ for an arbitrary unitary operator $A \in \mathrm{SU}(2)$. By definition, $\Pi_i'$ satisfies $[\Pi_i', A \otimes A^*] = 0$ for an arbitrary unitary operator $A \in \mathrm{SU}(2)$. To maximize the averaged success probability, we assume this commutation relation.

Then from the relation $\widetilde{\Pi}_1 + \widetilde{\Pi}_2 = I \otimes X$ we can assume that

$$[X, A] = 0, \tag{4.194}$$

can be assumed for arbitrary unitary operator $A$. This implies that without loss of generality $X = I/2$.

From $[\widetilde{\Pi}_i, A \otimes A^*] = 0$ for any unitary $A \in \mathrm{SU}(2)$ for $i = 1, 2$, we have

$$\widetilde{\Pi}_i = \alpha_i \frac{|I\rangle\rangle\langle\langle I|}{2} + \beta_i Q, \tag{4.195}$$

where $Q$ is the projector onto the subspace orthogonal to $|I\rangle\rangle\langle\langle I|$ defined as $Q := I - |I\rangle\rangle\langle\langle I|/2$. From the condition $\widetilde{\Pi}_1 + \widetilde{\Pi}_2 = I \otimes I/2$ we obtain

$$\alpha_1 + \alpha_2 = \beta_1 + \beta_2 = \frac{1}{2}. \tag{4.196}$$

Now the averaged success probability $p_{succ}$ is represented as

$$p_{succ} = \frac{1}{2}\mathrm{Tr}\left[(\frac{I}{2}\otimes I)\widetilde{\Pi}_1 + |I\rangle\rangle\langle\langle I|\widetilde{\Pi}_2\right]$$

$$= \frac{1}{2}\left[\frac{1}{2}(\alpha_1 + 3\beta_1) + 2\alpha_2\right] \quad (4.197)$$

$$= \frac{1}{4}(\alpha_1 + 3\beta_1 + 4\alpha_2) \quad (4.198)$$

$$\leq \frac{7}{8}, \quad (4.199)$$

where the inequality saturates when $\alpha_2 = \beta_1 = 1/2$ and $\alpha_1 = \beta_2 = 0$.

Providing a the complete classical description of $U_1$ implies ability to use the black-box for any number of times. But the result shown in this subsection indicates that the multiple uses of reference box 1 alone without use of the reference box 2 does not help to improve the averaged success probability when only are use of the test box is allowed. As we show in the next section, the classical descriptions of $U_1$ give advantages when the reference box 2, can be used even if the classical description of the reference box 2 is not provided.

### 4.2.2 Single Use of The Reference Box

In this section, we consider the case that a single use of the reference box 2 implementing $U_2$ is allowed in addition to the classical descriptions of $U_1$ under the parallel strategies.

Similarly to the analysis made in the previous sections, the task is to discriminate unitary operations $V \otimes V$ and $V \otimes I$, where $V = U_2 U_1^\dagger$ and $U_1$ and $U_2$ are randomly chosen from the Haar measure. The averaged success probability $p_{succ}$ for this case is represented in terms of the 2-tester $\{\widetilde{\Pi}_i\}$ as

$$p_{succ} = \frac{1}{2}\mathrm{Tr}[E_1\widetilde{\Pi}_1 + E_2\widetilde{\Pi}_2], \quad (4.200)$$

where $E_1$ and $E_2$ are defined as

$$E_1 = \int d\mu(U_2)(U_2 \otimes U_2 \otimes I_{\mathcal{H}_1\mathcal{H}_1})|I\rangle\rangle\langle\langle I|(U_2 \otimes U_2 \otimes I_{\mathcal{H}_1\mathcal{H}_1})^\dagger \quad (4.201)$$

$$= I_0 \otimes I_0 \oplus \frac{I_1}{3} \otimes I_1, \quad (4.202)$$

and

$$E_2 = \int d\mu(U_2)(U_2 \otimes I_{\mathcal{H}_2} \otimes I_{\mathcal{H}_1\mathcal{H}_1})|I\rangle\rangle\langle\langle I|(U_2 \otimes I_{\mathcal{H}_2} \otimes I_{\mathcal{H}_1\mathcal{H}_1})^\dagger \quad (4.203)$$

$$= \frac{I_{\mathcal{K}_1}}{2} \otimes I_{\mathcal{H}_1} \otimes |I\rangle\rangle\langle\langle I|_{\mathcal{K}_2\mathcal{H}_2}, \quad (4.204)$$

and $\widetilde{\Pi}_1, \widetilde{\Pi}_2 \geq 0$ and $\widetilde{\Pi}_1 + \widetilde{\Pi}_2 = I_{\mathcal{K}} \otimes X_{\mathcal{H}}$.

From the condition $[E_i, A_{\mathcal{K}}^{\otimes 2} \otimes A_{\mathcal{H}}^{*\otimes 2}] = 0$ for arbitrary unitary operator $A \in \mathrm{SU}(2)$, without loss of generality one can assume that

$$[\widetilde{\Pi}_i, A_{\mathcal{K}}^{\otimes 2} \otimes A_{\mathcal{H}}^{*\otimes 2}] = 0. \tag{4.205}$$

This condition and the relation $\widetilde{\Pi}_1 + \widetilde{\Pi}_2 = I_{\mathcal{K}} \otimes X_{\mathcal{H}}$ lead

$$[X_{\mathcal{H}}, A^{\otimes 2}] = 0, \tag{4.206}$$

for arbitrary unitary operator $A \in \mathrm{SU}(2)$. Without loss of generality, we have

$$X = \sin^2[t] I_0 \oplus \cos^2[t] \frac{I_1}{3}, \tag{4.207}$$

with $0 \leq t \leq \pi$.

The task is to discriminate two random unitary channels where $E_1$ and $E_2$ are the Choi operators of the random unitary channels. The optimal average success probability is calculated as

$$p_{succ}^{opt} = \frac{1}{2} + \frac{1}{4} \max_{X \geq 0, \mathrm{Tr} X = 1} \|(I_{\mathcal{K}} \otimes \sqrt{X})(E_1 - E_2)(I_{\mathcal{K}} \otimes \sqrt{X})\|_1, \tag{4.208}$$

which was already derived in Sec. 3.2. The maximization in the second term of the right hand side of Equation (4.208) can be calculated as

$$\|(I_{\mathcal{K}} \otimes \sqrt{X})(E_1 - E_2)(I_{\mathcal{K}} \otimes \sqrt{X})\|_1 \tag{4.209}$$

$$= \frac{5\cos^2 t}{36} + \frac{3}{144}\sqrt{87 - 4\cos 2t - 10\cos 4t} \tag{4.210}$$

$$+ \frac{1}{36}\sqrt{357 - 352\cos 2t + 20\cos 4t}. \tag{4.211}$$

The above equation can be calculated by summing the absolute value of the all eigenvalues of $(I_{\mathcal{K}} \otimes \sqrt{X})(E_1 - E_2)(I_{\mathcal{K}} \otimes \sqrt{X})$. The optimal averaged success probability is numerically derived as $p_{succ}^{opt} \simeq 0.902127 > 0.875 = 7/8$.

# Chapter 5

# Equivalence Determination: Multiple Uses of the Reference Boxes

In the previous chapter, we analyzed equivalence determination of unitary operations when each of the reference boxes and the test box can be used only once and showed that the ordered strategies do not improve the optimal success probability comparing with the parallel strategies. One question is whether this no improvement property holds for the case of multiple uses of the reference boxes. Another interesting question is how the averaged optimal success probability behaves when the number of the uses of the reference boxes increases. To answer these questions, we deal with equivalence determination with the multiple uses of the two reference boxes in this chapter.

In Sec. 5.1, we analyze the case that all black-boxes are used in parallel ways with up to four uses of each of the reference boxes and derive the averaged success probability by numerically solving the corresponding semidefinite programmings. In Sec. 5.2, we consider ordered strategies for the case that reference box 1 can be used twice and reference box 2 can be used only once and investigate all orders of the black-boxes and all configurations of quantum testers.

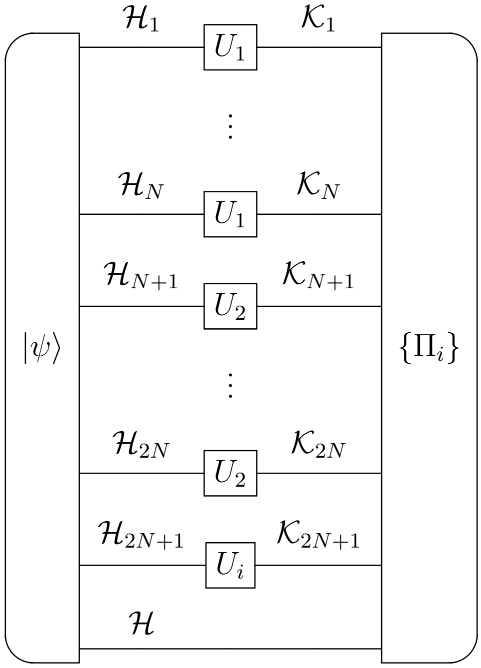



Figure 5.1: A quantum circuit representation for $(N, N)$-equivalence determination under the parallel strategies.

## 5.1 $(N, N)$-Equivalence Determination under Parallel Strategies

### 5.1.1 Formulation with Semidefinite Programmings

We first formulate the optimization problems for $(N, N)$-equivalence determination in terms of semidefinite programmings (SDP). We consider two unitary operators $U_1$ and $U_2$ in SU(2) and assign Hilbert spaces $\mathcal{H}_i$ and $\mathcal{K}_i$ as $U_1^{\otimes N} \in \mathcal{L}(\bigotimes_{i=1}^{N} \mathcal{H}_i, \bigotimes_{i=1}^{N} \mathcal{K}_i)$ and $U_2^{\otimes N} \in \mathcal{L}(\bigotimes_{j=N+1}^{2N} \mathcal{H}_j, \bigotimes_{j=N+1}^{2N} \mathcal{K}_j)$. The test-box implements a unitary operation $U_i$ that is guaranteed to be one of two unitary operations $U_1$ and $U_2$. Let $U_i$ be in $\mathcal{L}(\mathcal{H}_{2N+1}, \mathcal{K}_{2N+1})$. We define $\mathcal{H} = \bigotimes_{i=1}^{2N+1} \mathcal{H}_i$ and $\mathcal{K} = \bigotimes_{i=1}^{2N+1} \mathcal{K}_i$ and (see Fig. 5.1).

We define an initial state $|\psi\rangle = I_{\mathcal{H}} \otimes \sqrt{X}|I\rangle\rangle$ Let $\{\Pi_1, \Pi_2\}$ be a POVM. We adopt the averaged success probability over the Haar measure as the figure of merit similarly to the analysis in the previous chapter. The success probability is given by

$$p_{succ} = \frac{1}{2} \int d\mu(U_1) d\mu(U_2) \mathrm{Tr}[|\psi_1\rangle\langle\psi_1|\Pi_1 + |\psi_2\rangle\langle\psi_2|\Pi_2] \tag{5.1}$$

$$= \frac{1}{2}\mathrm{Tr}[M_1\widetilde{\Pi_1} + M_2\widetilde{\Pi_2}], \tag{5.2}$$

where we defined $|\psi_i\rangle = U_1^{\otimes N} \otimes U_2^{\otimes N} \otimes U_i \otimes I_{\mathcal{H}}|\psi\rangle$ and

$$M_i = \int d\mu(U_1) \int d\mu(U_2)(U_1^{\otimes N} \otimes U_2^{\otimes N} \otimes U_i \otimes I_{\mathcal{H}})|I\rangle\rangle\langle\langle I|(U_1^{\dagger\otimes N} \otimes U_2^{\dagger\otimes N} \otimes U_i^{\dagger} \otimes I_{\mathcal{H}}) \tag{5.3}$$

for $i = 1, 2$ and a quantum two tester $\widetilde{\Pi_i} = (I_{\mathcal{K}} \otimes \sqrt{X})\Pi_i(I_{\mathcal{K}} \otimes \sqrt{X})$.

The corresponding optimization problem is given as follows.

**Proposition 3.** *The averaged optimal success probability for $(N, N)$-equivalence determination under the parallel strategies obtaining the correct answer with $N$ use of reference boxes under parallel strategies is given by the following semidefinite programming.*

$$\begin{aligned}
\textit{maximize} \quad & p_{succ} = \frac{1}{2}\mathrm{Tr}\left[M_1\widetilde{\Pi}_1 + M_2\widetilde{\Pi}_2\right] && (5.4)\\
\textit{maximize} \quad & \widetilde{\Pi}_i \geq 0, \; i = 1, 2 && (5.5)\\
& \widetilde{\Pi}_1 + \widetilde{\Pi}_2 = I_{\mathcal{K}} \otimes X && (5.6)\\
& X \geq 0 && (5.7)\\
& \mathrm{Tr}X = 1, && (5.8)
\end{aligned}$$

*where $M_i$ is defined in Equation (5.3).*

We rewrite the SDP in Prop. 3 in terms of operators on the multiplicity subspaces. Following the similar logic to the one presented in Sec. 4.1.3, it is possible to choose $X$ and $\{\widetilde{\Pi}_i\}$ satisfying $[X, A^{\otimes 2N+1}] = 0$ and $[\widetilde{\Pi}_i, (A^{\otimes 2N+1})_{\mathcal{H}} \otimes (B^{\otimes 2N+1})_{\mathcal{K}}] = 0$ for any $A, B \in \mathrm{SU}(2)$ for $i = 1, 2$. Thus without loss of generality, we can assume

$$X = \bigoplus_{L=\lfloor N \rfloor}^{N/2} \frac{I_L}{d_L} \otimes X_L \tag{5.9}$$

$$\widetilde{\Pi}_i = \bigoplus_{J=\lfloor N \rfloor}^{N/2} \bigoplus_{L=\lfloor N \rfloor}^{N/2} I_J \otimes \frac{I_L}{d_L} \otimes \widetilde{\Pi}_{JL}^{[i]} \tag{5.10}$$

$$M_i = \bigoplus_{J=\lfloor N \rfloor}^{N/2} \bigoplus_{L=\lfloor N \rfloor}^{N/2} \frac{I_J}{d_J} \otimes I_L \otimes M_{JL}^{[i]}, \tag{5.11}$$

where $\lfloor N \rfloor$ is 0 when $N$ is odd and 1 otherwise.

Then the SDP on multiplicity subspaces is represented as

$$\text{maximize} \quad p_{succ} = \frac{1}{2}\mathrm{Tr} \sum_{J=\lfloor N \rfloor}^{N/2} \sum_{L=\lfloor N \rfloor}^{N/2} \left[ M_{JL}^{[1]} \Pi_{JL}^{[1]} + M_{JL}^{[2]} \Pi_{JL}^{[2]} \right] \tag{5.12}$$

$$\text{subject to} \quad \Pi_{JL}^{[i]} \geq 0, \quad i = 1, 2 \text{ and } J, L = \lfloor N \rfloor \cdots N/2 \tag{5.13}$$

$$\Pi_{JL}^{[1]} + \Pi_{JL}^{[2]} = I_{\mathcal{V}_J^{[N]}} \otimes X_L, \quad J, L = \lfloor N \rfloor \cdots N/2 \tag{5.14}$$

$$X_L \geq 0, \quad L = \lfloor N \rfloor \cdots N/2 \tag{5.15}$$

$$\sum_{L=\lfloor N \rfloor}^{N/2} \mathrm{Tr} X_L = 1, \tag{5.16}$$

where $M_{JL}^{[i]}$ is derived in Sec. 5.1.3.

### 5.1.2 Numerical Results

We solve the SDP presented in the previous subsectionby numerical calculations for $N = 2, 3, 4$. The numerical results are shown in Fig. 5.2.

If the classical descriptions of two unitary operations are given, this situation corresponds to $N \to \infty$, since unitary operation can be applied on many times when the classical description are given. Therefore averaging the optimal success probability of discrimination of two unitary operations given by Equation (3.2.2) where the classical descriptions of the candidate unitary operations is given provides the upper bound of the averaged success probability of $(N, N)$-equivalence determination. The upper bound is $1/2+4/3\pi \simeq 0.9244$ and the optimal success probability for equivalence determination with $N = 4$ is $p_{succ} \approx 0.9183$. The success probabilities are already close for $N = 4$.

For numerical calculation, we did not directly solve the SPD given by the Conditions (5.4) - (5.8) but the Conditions (5.12) - (5.16) utilizing the group theoretical properties. The concrete calculation of $M_{JL}^{[i]}$ for $N = 2$ is given in the next section and for $N = 3, 4$ is given Appx. C.

### 5.1.3 Calculations of Choi operators in Prop. 3 for $N = 2$

In this section, we give the explicit calculations for the Choi operators defined in Equation (5.3) for $N = 2$. First we denote $\mathcal{H}_{R_1} = \bigotimes_{i=1}^{N} \mathcal{H}_i$, $\mathcal{H}_{R_2} = \bigotimes_{j=N+1}^{2N} \mathcal{H}_j$ and $\mathcal{H}_T = \mathcal{H}_{2N+1}$. We use the same labels for the corresponding output spaces $\mathcal{K}$'s. We assume that spin-1/2 systems $\mathcal{H}_{R_i}$ are first coupled for $i = 1, 2$. Then for $M_1$, $\mathcal{H}_{R_1}$ and $\mathcal{H}_T$ are coupled followed by the coupling with $\mathcal{H}_{R_2}$. For $M_2$, $\mathcal{H}_{R_2}$ and $\mathcal{H}_T$ are coupled followed by the coupling with $\mathcal{H}_{R_1}$. Then we represent

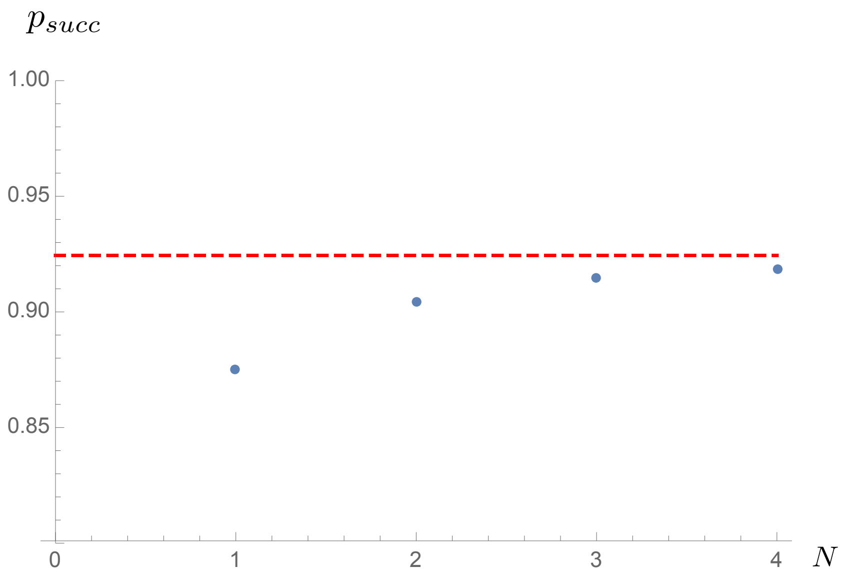



Figure 5.2: The averaged optimal success probability for $(N, N)$-equivalence determination under the parallel strategies for $N = 1, 2, 3, 4$. The red dashed line indicates the optimal averaged success probability when classical descriptions of two unitary operations are given, namely, for the case of discrimination of unitary operations. Since the value represented by red dashed line corresponds to the asymptotic limit for $N \to \infty$ in equivalence determination, the value with $N = 4$ reaches close to this asymptotic limit.

the relation between the two basis in the multiplicity subspaces due to the order of the coupling. The calculations for $N = 3, 4$ are given in Appx. C.

### $N = 2$ case

The Choi operator $M_i$ is represented as

$$M_1 = \eta^{[3]}_{\mathcal{H}_{R_1}\mathcal{K}_{R_1}\mathcal{H}_T\mathcal{K}_T} \otimes \eta^{[2]}_{\mathcal{H}_{R_2}\mathcal{K}_{R_2}} \tag{5.17}$$

$$M_2 = \eta^{[2]}_{\mathcal{H}_{R_1}\mathcal{K}_{R_1}} \otimes \eta^{[2]}_{\mathcal{H}_{R_2}\mathcal{K}_{R_2}\mathcal{H}_T\mathcal{K}_T}, \tag{5.18}$$

where the definition of $\eta^{[N]}$ is given in Equation (4.74). Then we have

$$\eta^{[3]}_{\mathcal{H}_{R_1}\mathcal{K}_{R_1}\mathcal{H}_T\mathcal{K}_T} = \frac{I_{\frac{1}{2}}}{2} \otimes I_{\frac{1}{2}} \otimes \left\{ \left[ (|\alpha_0^{[2]}\rangle\rangle + |\alpha_1^{[2]}\rangle\rangle) \otimes |\alpha_1^{[1]}\rangle\rangle \right] \right\} \tag{5.19}$$

$$\oplus \frac{I_{\frac{1}{2}}}{2} \otimes I_{\frac{1}{2}} \otimes \left[ |\alpha_1^{[2]}\rangle\rangle \otimes |\alpha_1^1\rangle\rangle \right] \tag{5.20}$$

and

$$\eta^{[2]}_{\mathcal{H}_{R_2}\mathcal{K}_{R_2}} = I_0 \otimes I_0 \otimes [|\alpha_0^{[2]}\rangle\rangle] \oplus \frac{I_1}{3} \otimes I_1 \otimes [|\alpha_1^{[2]}\rangle\rangle], \tag{5.21}$$

where we define $|\alpha_J^N\rangle\rangle$ is the unnormalized maximally entangled vector on $\mathcal{V}_J^{[N]\otimes 2}$ and $[|\psi\rangle] = |\psi\rangle\langle\psi|$.

We denote

$$M_i = \bigoplus_{J=\frac{1}{2}}^{\frac{5}{2}} \bigoplus_{L=\frac{1}{2}}^{\frac{5}{2}} \frac{I_J}{d_J} \otimes I_L \otimes M_{JL}^{[i]}, \tag{5.22}$$

where $M_{JL}^{[i]}$ is an operator on the multiplicity subspace. Then we have

$$M_{\frac{1}{2}\frac{1}{2}}^{[1]} = \left[|00\rangle_{\frac{1}{2}\frac{1}{2}} + |22\rangle_{\frac{1}{2}\frac{1}{2}}\right] + \frac{1}{3}\left[|11\rangle_{\frac{1}{2}\frac{1}{2}} + |33\rangle_{\frac{1}{2}\frac{1}{2}}\right] + \frac{1}{6}\left[|44\rangle_{\frac{1}{2}\frac{1}{2}}\right], \tag{5.23}$$

$$M_{\frac{1}{2}\frac{3}{2}}^{[1]} = \frac{1}{3}\left[|00\rangle_{\frac{1}{2}\frac{3}{2}} + |32\rangle_{\frac{1}{2}\frac{3}{2}}\right] + \frac{1}{6}\left[|43\rangle_{\frac{1}{2}\frac{3}{2}}\right], \tag{5.24}$$

$$M_{\frac{1}{2}\frac{5}{2}}^{[1]} = \frac{1}{6}\left[|40\rangle_{\frac{1}{2}\frac{5}{2}}\right], \tag{5.25}$$

$$M_{\frac{3}{2}\frac{1}{2}}^{[1]} = \frac{2}{3}\left[|00\rangle_{\frac{3}{2}\frac{1}{2}} + |23\rangle_{\frac{3}{2}\frac{1}{2}}\right] + \frac{1}{3}\left[|34\rangle_{\frac{3}{2}\frac{1}{2}}\right], \tag{5.26}$$

$$M_{\frac{3}{2}\frac{3}{2}}^{[1]} = \left[|00\rangle_{\frac{3}{2}\frac{3}{2}}|22\rangle_{\frac{3}{2}\frac{3}{2}}\right] + \left[|11\rangle_{\frac{3}{2}\frac{3}{2}}\right] + \frac{1}{3}\left[|33\rangle_{\frac{3}{2}\frac{3}{2}}\right], \tag{5.27}$$

$$M_{\frac{3}{2}\frac{5}{2}}^{[1]} = \frac{1}{3}\left[|30\rangle_{\frac{3}{2}\frac{5}{2}}\right], \tag{5.28}$$

$$M_{\frac{5}{2}\frac{1}{2}}^{[1]} = \frac{1}{2}\left[|04\rangle_{\frac{5}{2}\frac{5}{2}}\right], \tag{5.29}$$

$$M_{\frac{5}{2}\frac{3}{2}}^{[1]} = \frac{1}{2}\left[|03\rangle_{\frac{5}{2}\frac{5}{2}}\right], \tag{5.30}$$

$$M_{\frac{5}{2}\frac{5}{2}}^{[1]} = \frac{1}{2}\left[|00\rangle_{\frac{5}{2}\frac{5}{2}}\right], \tag{5.31}$$

where we defined the basis states for the multiplicity subspaces $\mathcal{V}_{\frac{1}{2}}^{[5]}$ as

$$\textstyle\sum_{m=-1/2}^{1/2} |0\tfrac{1}{2}(\tfrac{1}{2})\mathcal{K}_1\mathcal{K}_T 0; \tfrac{1}{2}m\rangle\langle 0\tfrac{1}{2}(\tfrac{1}{2})\mathcal{K}_1\mathcal{K}_T 0; \tfrac{1}{2}m| =: I_{\frac{1}{2}} \otimes |0\rangle\langle 0|_{\frac{1}{2}}, \tag{5.32}$$

$$\textstyle\sum_{m=-1/2}^{1/2} |0\tfrac{1}{2}(\tfrac{1}{2})\mathcal{K}_1\mathcal{K}_T 1; \tfrac{1}{2}m\rangle\langle 0\tfrac{1}{2}(\tfrac{1}{2})\mathcal{K}_1\mathcal{K}_T 1; \tfrac{1}{2}m| =: I_{\frac{1}{2}} \otimes |1\rangle\langle 1|_{\frac{1}{2}}, \tag{5.33}$$

$$\textstyle\sum_{m=-1/2}^{1/2} |1\tfrac{1}{2}(\tfrac{1}{2})\mathcal{K}_1\mathcal{K}_T 0; \tfrac{1}{2}m\rangle\langle 1\tfrac{1}{2}(\tfrac{1}{2})\mathcal{K}_1\mathcal{K}_T 0; \tfrac{1}{2}m| =: I_{\frac{1}{2}} \otimes |2\rangle\langle 2|_{\frac{1}{2}}, \tag{5.34}$$

$$\textstyle\sum_{m=-1/2}^{1/2} |1\tfrac{1}{2}(\tfrac{1}{2})\mathcal{K}_1\mathcal{K}_T 1; \tfrac{1}{2}m\rangle\langle 1\tfrac{1}{2}(\tfrac{1}{2})\mathcal{K}_1\mathcal{K}_T 1; \tfrac{1}{2}m| =: I_{\frac{1}{2}} \otimes |3\rangle\langle 3|_{\frac{1}{2}}, \tag{5.35}$$

$$\textstyle\sum_{m=-1/2}^{1/2} |1\tfrac{1}{2}(\tfrac{3}{2})\mathcal{K}_1\mathcal{K}_T 1; \tfrac{1}{2}m\rangle\langle 1\tfrac{1}{2}(\tfrac{3}{2})\mathcal{K}_1\mathcal{K}_T 1; \tfrac{1}{2}m| =: I_{\frac{1}{2}} \otimes |4\rangle\langle 4|_{\frac{1}{2}}, \tag{5.36}$$

and for $\mathcal{V}_{\frac{3}{2}}^{[5]}$ as

$$\textstyle\sum_{m=-3/2}^{3/2} |0\tfrac{1}{2}(\tfrac{1}{2})\mathcal{K}_1\mathcal{K}_T 1; \tfrac{3}{2}m\rangle\langle 0\tfrac{1}{2}(\tfrac{1}{2})\mathcal{K}_1\mathcal{K}_T 1; \tfrac{3}{2}m| =: I_{\frac{3}{2}} \otimes |0\rangle\langle 0|_{\frac{3}{2}}, \tag{5.37}$$

$$\textstyle\sum_{m=-3/2}^{3/2} |1\tfrac{3}{2}(\tfrac{1}{2})\mathcal{K}_1\mathcal{K}_T 0; \tfrac{3}{2}m\rangle\langle 1\tfrac{1}{2}(\tfrac{3}{2})\mathcal{K}_1\mathcal{K}_T 0; \tfrac{3}{2}m| =: I_{\frac{3}{2}} \otimes |0\rangle\langle 0|_{\frac{3}{2}}, \tag{5.38}$$

$$\textstyle\sum_{m=-3/2}^{3/2} |1\tfrac{1}{2}(\tfrac{1}{2})\mathcal{K}_1\mathcal{K}_T 1; \tfrac{3}{2}m\rangle\langle 1\tfrac{1}{2}(\tfrac{1}{2})\mathcal{K}_1\mathcal{K}_T 1; \tfrac{3}{2}m| =: I_{\frac{3}{2}} \otimes |0\rangle\langle 0|_{\frac{3}{2}}, \tag{5.39}$$

$$\textstyle\sum_{m=-3/2}^{3/2} |1\tfrac{1}{2}(\tfrac{3}{2})\mathcal{K}_1\mathcal{K}_T 1; \tfrac{3}{2}m\rangle\langle 1\tfrac{1}{2}(\tfrac{3}{2})\mathcal{K}_1\mathcal{K}_T 1; \tfrac{3}{2}m| =: I_{\frac{3}{2}} \otimes |0\rangle\langle 0|_{\frac{3}{2}} \tag{5.40}$$

and for $\mathcal{V}^{[5]}_{\frac{5}{2}}$ as

$$\textstyle\sum_{m=-5/2}^{5/2} |1\frac{1}{2}(\frac{3}{2})_{\mathcal{K}_1\mathcal{K}_T}1;\frac{5}{2}m\rangle\langle 1\frac{1}{2}(\frac{3}{2})_{\mathcal{K}_1\mathcal{K}_T}1;\frac{5}{2}m| =: I_{\frac{5}{2}} \otimes |0\rangle\langle 0|_{\frac{3}{2}}, \tag{5.41}$$

where $|j_1\frac{1}{2}(j_1 \pm \frac{1}{2})_{\mathcal{K}_1\mathcal{K}_T}j_2;Jm\rangle$ represents a state with the total angular momentum $J$ and the angular momentum along the $z$-axis $m$ obtained by coupling a spin $j_1$ on $\mathcal{K}_1$ and a spin 1/2 on $\mathcal{K}_T$ followed by coupling spin $j_2$ in $\mathcal{K}_2$.

For $M_2$ we have

$$M^{[2]}_{\frac{1}{2}\frac{1}{2}} = \left[|\hat{0}\hat{0}\rangle_{\frac{1}{2}\frac{1}{2}}|\hat{2}\hat{2}\rangle_{\frac{1}{2}\frac{1}{2}}\right] + \frac{1}{3}\left[|\hat{1}\hat{1}\rangle_{\frac{1}{2}\frac{1}{2}}|\hat{3}\hat{3}\rangle_{\frac{1}{2}\frac{1}{2}}\right] + \frac{1}{6}\left[|\hat{4}\hat{4}\rangle_{\frac{1}{2}\frac{1}{2}}\right], \tag{5.42}$$

$$M^{[2]}_{\frac{1}{2}\frac{3}{2}} = \frac{1}{3}\left[|\hat{0}\hat{0}\rangle_{\frac{1}{2}\frac{3}{2}} + |\hat{3}\hat{2}\rangle_{\frac{1}{2}\frac{3}{2}}\right] + \frac{1}{6}\left[|\hat{4}\hat{3}\rangle_{\frac{1}{2}\frac{3}{2}}\right], \tag{5.43}$$

$$M^{[2]}_{\frac{1}{2}\frac{5}{2}} = \frac{1}{6}\left[|\hat{4}\hat{0}\rangle_{\frac{1}{2}\frac{5}{2}}\right], \tag{5.44}$$

$$M^{[2]}_{\frac{3}{2}\frac{1}{2}} = \frac{2}{3}\left[|\hat{0}\hat{0}\rangle_{\frac{3}{2}\frac{1}{2}} + |\hat{2}\hat{3}\rangle_{\frac{3}{2}\frac{1}{2}}\right] + \frac{1}{3}\left[|\hat{3}\hat{4}\rangle_{\frac{3}{2}\frac{1}{2}}\right], \tag{5.45}$$

$$M^{[2]}_{\frac{3}{2}\frac{3}{2}} = \left[|\hat{0}\hat{0}\rangle_{\frac{3}{2}\frac{3}{2}}|\hat{2}\hat{2}\rangle_{\frac{3}{2}\frac{3}{2}}\right] + \left[|\hat{1}\hat{1}\rangle_{\frac{3}{2}\frac{3}{2}}\right] + \frac{1}{3}\left[|\hat{3}\hat{3}\rangle_{\frac{3}{2}\frac{3}{2}}\right], \tag{5.46}$$

$$M^{[2]}_{\frac{3}{2}\frac{5}{2}} = \frac{1}{3}\left[|\hat{3}\hat{0}\rangle_{\frac{3}{2}\frac{5}{2}}\right], \tag{5.47}$$

$$M^{[2]}_{\frac{5}{2}\frac{1}{2}} = \frac{1}{2}\left[|\hat{0}\hat{4}\rangle_{\frac{5}{2}\frac{5}{2}}\right], \tag{5.48}$$

$$M^{[2]}_{\frac{5}{2}\frac{3}{2}} = \frac{1}{2}\left[|\hat{0}\hat{3}\rangle_{\frac{5}{2}\frac{5}{2}}\right], \tag{5.49}$$

$$M^{[2]}_{\frac{5}{2}\frac{5}{2}} = \frac{1}{2}\left[|\hat{0}\hat{0}\rangle_{\frac{5}{2}\frac{5}{2}}\right], \tag{5.50}$$

where we defined the basis states for the multiplicity subspaces $\mathcal{V}^{[5]}_{\frac{1}{2}}$ as

$$\textstyle\sum_{m=-1/2}^{1/2} |0\frac{1}{2}0(\frac{1}{2})_{\mathcal{K}_1\mathcal{K}_T};\frac{1}{2}m\rangle\langle 0\frac{1}{2}0(\frac{1}{2})_{\mathcal{K}_1\mathcal{K}_T};\frac{1}{2}m| =: I_{\frac{1}{2}} \otimes |\hat{0}\rangle\langle\hat{0}|_{\frac{1}{2}}, \tag{5.51}$$

$$\textstyle\sum_{m=-1/2}^{1/2} |1\frac{1}{2}1(\frac{1}{2})_{\mathcal{K}_1\mathcal{K}_T};\frac{1}{2}m\rangle\langle 1\frac{1}{2}1(\frac{1}{2})_{\mathcal{K}_1\mathcal{K}_T};\frac{1}{2}m| =: I_{\frac{1}{2}} \otimes |\hat{1}\rangle\langle\hat{1}|_{\frac{1}{2}}, \tag{5.52}$$

$$\textstyle\sum_{m=-1/2}^{1/2} |0\frac{1}{2}1(\frac{1}{2})_{\mathcal{K}_1\mathcal{K}_T};\frac{1}{2}m\rangle\langle 0\frac{1}{2}1(\frac{1}{2})_{\mathcal{K}_1\mathcal{K}_T};\frac{1}{2}m| =: I_{\frac{1}{2}} \otimes |\hat{2}\rangle\langle\hat{2}|_{\frac{1}{2}}, \tag{5.53}$$

$$\textstyle\sum_{m=-1/2}^{1/2} |1\frac{1}{2}1(\frac{1}{2})_{\mathcal{K}_1\mathcal{K}_T};\frac{1}{2}m\rangle\langle 1\frac{1}{2}1(\frac{1}{2})_{\mathcal{K}_1\mathcal{K}_T};\frac{1}{2}m| =: I_{\frac{1}{2}} \otimes |\hat{3}\rangle\langle\hat{3}|_{\frac{1}{2}}, \tag{5.54}$$

$$\textstyle\sum_{m=-1/2}^{1/2} |1\frac{1}{2}1(\frac{3}{2})_{\mathcal{K}_1\mathcal{K}_T};\frac{1}{2}m\rangle\langle 1\frac{1}{2}1(\frac{3}{2})_{\mathcal{K}_1\mathcal{K}_T};\frac{1}{2}m| =: I_{\frac{1}{2}} \otimes |\hat{4}\rangle\langle\hat{4}|_{\frac{1}{2}}, \tag{5.55}$$

and for $\mathcal{V}^{[5]}_{\frac{3}{2}}$ as

$$\textstyle\sum_{m=-3/2}^{3/2} |1\frac{1}{2}0(\frac{1}{2})_{\mathcal{K}_1\mathcal{K}_T};\frac{3}{2}m\rangle\langle 1\frac{1}{2}0(\frac{1}{2})_{\mathcal{K}_1\mathcal{K}_T};\frac{3}{2}m| =: I_{\frac{3}{2}} \otimes |\hat{0}\rangle\langle\hat{0}|_{\frac{3}{2}}, \tag{5.56}$$

$$\textstyle\sum_{m=-3/2}^{3/2} |0\frac{1}{2}1(\frac{3}{2})_{\mathcal{K}_1\mathcal{K}_T};\frac{3}{2}m\rangle\langle 0\frac{1}{2}1(\frac{3}{2})_{\mathcal{K}_1\mathcal{K}_T};\frac{3}{2}m| =: I_{\frac{3}{2}} \otimes |\hat{1}\rangle\langle\hat{1}|_{\frac{3}{2}}, \tag{5.57}$$

$$\textstyle\sum_{m=-3/2}^{3/2} |1\frac{1}{2}1(\frac{1}{2})_{\mathcal{K}_1\mathcal{K}_T};\frac{3}{2}m\rangle\langle 1\frac{1}{2}1(\frac{1}{2})_{\mathcal{K}_1\mathcal{K}_T};\frac{3}{2}m| =: I_{\frac{3}{2}} \otimes |\hat{2}\rangle\langle\hat{2}|_{\frac{3}{2}}, \tag{5.58}$$

$$\textstyle\sum_{m=-3/2}^{3/2} |1\frac{1}{2}1(\frac{3}{2})_{\mathcal{K}_1\mathcal{K}_T};\frac{3}{2}m\rangle\langle 1\frac{1}{2}1(\frac{3}{2})_{\mathcal{K}_1\mathcal{K}_T};\frac{3}{2}m| =: I_{\frac{3}{2}} \otimes |\hat{3}\rangle\langle\hat{3}|_{\frac{3}{2}}, \tag{5.59}$$

and for $J = 5/2$,

$$\sum_{m=-5/2}^{5/2} |1\tfrac{1}{2}1(\tfrac{3}{2})_{\mathcal{K}_1\mathcal{K}_T}; \tfrac{5}{2}m\rangle\langle 1\tfrac{1}{2}1(\tfrac{3}{2})_{\mathcal{K}_1\mathcal{K}_T}; \tfrac{5}{2}m| =: I_{\frac{3}{2}} \otimes |\hat{0}\rangle\langle\hat{0}|_{\frac{5}{2}}, \tag{5.60}$$

where $|j_1\frac{1}{2}j_2(j_2 \pm \frac{1}{2})_{\mathcal{K}_T\mathcal{K}_2}; Jm\rangle$ represents a state with the total angular momentum $J$ and the angular momentum along the $z$-axis $m$ obtained by coupling a spin $j_2$ on $\mathcal{K}_2$ and a spin 1/2 on $\mathcal{K}_T$ followed by coupling spin $j_1$ in $\mathcal{K}_1$.

Using Lemma 6, the relation between the two bases can be calculated. We define a two-by-two unitary matrix $U(j_1, j_3, j)$ as

$$U(j_1,j_3,j)_{11} = \frac{(-1)^{2(j_1+j_3+j)}}{\sqrt{(2j_3+1)(2j_1+1)}}\sqrt{(j+\frac{1}{2})^2-(j_3-j_1)^2}, \tag{5.61}$$

$$U(j_1,j_3,j)_{12} = -\frac{(-1)^{2(j_1+j_3+j)}}{\sqrt{(2j_3+1)(2j_1+1)}}\sqrt{(j_3+j_1+1)^2-(j+\frac{1}{2})^2}, \tag{5.62}$$

$$U(j_1,j_3,j)_{21} = \frac{(-1)^{1+2(j_1+j_3+j)}}{\sqrt{(2j_3+1)(2j_1+1)}}\sqrt{(j_3+j_1+1)^2-(j+\frac{1}{2})^2}, \tag{5.63}$$

$$U(j_1,j_3,j)_{22} = \frac{(-1)^{1+2(j_1+j_3+j)}}{\sqrt{(2j_3+1)(2j_1+1)}}\sqrt{(j+\frac{1}{2})^2-(j_3-j_1)^2}. \tag{5.64}$$

Then we obtain

$$|\hat{0}\rangle_{\frac{1}{2}} = U(0,0,1/2)_{22}|0\rangle_{\frac{1}{2}}, \tag{5.65}$$

$$|\hat{1}\rangle_{\frac{1}{2}} = U(1,0,1/2)_{21}|2\rangle_{\frac{1}{2}}, \tag{5.66}$$

$$|\hat{2}\rangle_{\frac{1}{2}} = U(0,1,1/2)_{12}|1\rangle_{\frac{1}{2}}, \tag{5.67}$$

$$|\hat{3}\rangle_{\frac{1}{2}} = U(1,1,1/2)_{11}|3\rangle_{\frac{1}{2}} + U(1,1,1/2)_{12}|4\rangle_{\frac{1}{2}}, \tag{5.68}$$

$$|\hat{4}\rangle_{\frac{1}{2}} = U(1,1,1/2)_{21}|3\rangle_{\frac{1}{2}} + U(1,1,1/2)_{22}|4\rangle_{\frac{1}{2}}, \tag{5.69}$$

$$|\hat{0}\rangle_{\frac{3}{2}} = U(1,0,3/2)_{22}|1\rangle_{\frac{3}{2}}, \tag{5.70}$$

$$|\hat{1}\rangle_{\frac{3}{2}} = U(0,1,3/2)_{22}|2\rangle_{\frac{3}{2}}, \tag{5.71}$$

$$|\hat{2}\rangle_{\frac{3}{2}} = U(1,1,3/2)_{11}|3\rangle_{\frac{3}{2}} + U(1,1,3/2)_{12}|3\rangle_{\frac{3}{2}}, \tag{5.72}$$

$$|\hat{3}\rangle_{\frac{3}{2}} = U(1,1,3/2)_{21}|3\rangle_{\frac{3}{2}} + U(1,1,3/2)_{22}|3\rangle_{\frac{3}{2}}, \tag{5.73}$$

$$|\hat{0}\rangle_{\frac{5}{2}} = U(1,1,5/2)_{22}|0\rangle_{\frac{5}{2}}. \tag{5.74}$$

## 5.2 (1, 2)-Equivalence Determination

In this section, we deal with equivalence determination with multiple uses of the reference boxes both under the parallel and the ordered strategies. For the

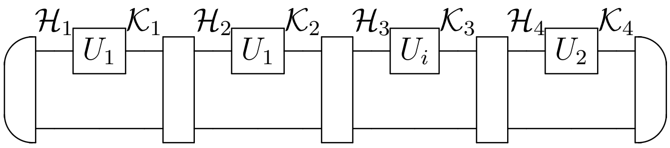



Figure 5.3: A configuration of the quantum 5-tester in case 1.

case of single uses of each of the reference boxes, the ordered strategies do not help as we have shown in the previous chapter. The question is whether this property holds for the case of the multiple uses of the reference boxes. The answer turn out to be no. In the next two sections, we show that the ordered use of the black-boxes gives improvements. Another interesting question is if the order of the test box and the reference boxes affects the averaged optimal success probability, and if so, which order gives the best one. We give a solution for $(2,1)$-equivalence determination under the ordered strategies by numerically solving the corresponding SDPs.

### 5.2.1 Formulation with Semidefinite Programmings

In this section, we investigate $(2,1)$-equivalence determination both under the ordered strategies and the parallel strategies. There are four different configurations of quantum testers. We first give the semidefinite programmings for every configuration of the quantum testers. Then give the numerical results for all of orders of the black-boxes in each configurations.

#### Case 1

First we consider the Choi operators corresponding to the case that quantum operations forming the quantum tester are applied after every use of the black-boxes. A configuration of this type of the quantum testers in this case is shown in Fig. 5.4.

We denote the quantum tester as $\{\widetilde{\Pi}_1, \widetilde{\Pi}_2\}$. The corresponding semidefinite programming is given by

$$\begin{aligned}
\max\ p_{succ} &= \frac{1}{2}\mathrm{Tr}\left[M_1\Pi_1 + M_2\Pi_2\right] \\
\text{subject to } & \Pi_i \geq 0,\ i = 1,2 \\
& \Pi_1 + \Pi_2 = I_{\mathcal{K}_4} \otimes X^{(3)} \\
& \mathrm{Tr}_{\mathcal{H}_4} X^{(3)} = I_{\mathcal{K}_3} \otimes X^{(2)} \\
& \mathrm{Tr}_{\mathcal{H}_3} X^{(2)} = I_{\mathcal{K}_2} \otimes X^{(1)}
\end{aligned}$$

$$\mathrm{Tr}_{\mathcal{H}_2} X^{(1)} = I_{\mathcal{K}_1} \otimes X^{(0)}$$
$$\mathrm{Tr} X^{(0)} = 1. \tag{5.75}$$

From the symmetry of $M_i$, for any unitary operations $A, B \in \mathrm{SU}(2)$, the following conditions

$$[\widetilde{\Pi}_i, A^{\otimes 4} \otimes B^{\otimes 4}] = 0 \tag{5.76}$$
$$[X^{(3)}, A^{\otimes 3} \otimes B^{\otimes 4}] = 0 \tag{5.77}$$
$$[X^{(2)}, A^{\otimes 2} \otimes B^{\otimes 3}] = 0 \tag{5.78}$$
$$[X^{(1)}, A^{\otimes 1} \otimes B^{\otimes 2}] = 0 \tag{5.79}$$
$$[X^{(0)}, B] = 0 \tag{5.80}$$

hold for $i = 1, 2$. The Equation (5.79) implies

$$X^{(1)} = I_{\mathcal{K}_1} \otimes X'^{(1)}, \tag{5.81}$$

for an operator $X'^{(1)}$ on $\mathcal{H}_1 \otimes \mathcal{H}_2$. This relation corresponds to the condition that the first two uses of the black-boxes can be applied in the parallel ways, namely,

$$\text{subject to} \quad \widetilde{\Pi}_i \geq 0, \ i = 1, 2$$
$$\widetilde{\Pi}_1 + \widetilde{\Pi}_2 = I_{\mathcal{K}_4} \otimes X^{(3)} \tag{5.82}$$
$$\mathrm{Tr}_{\mathcal{H}_4} X^{(3)} = I_{\mathcal{K}_3} \otimes X^{(2)} \tag{5.83}$$
$$\mathrm{Tr}_{\mathcal{H}_3} X^{(2)} = I_{\mathcal{K}_2 \mathcal{K}_1} \otimes X^{(1)} \tag{5.84}$$
$$\mathrm{Tr} X^{(1)} = 1. \tag{5.85}$$

The configuration of the quantum tester for this case is shown in Fig. 5.4.

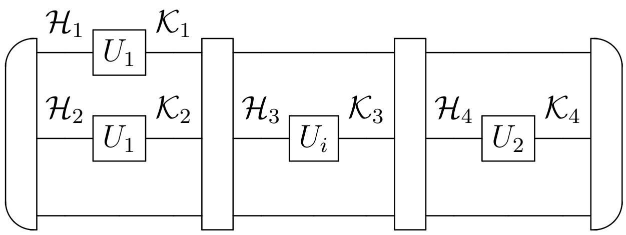



Figure 5.4: Quantum circuit representation of the quantum 4-tester when quantum operations is applied after every use of the black-box.

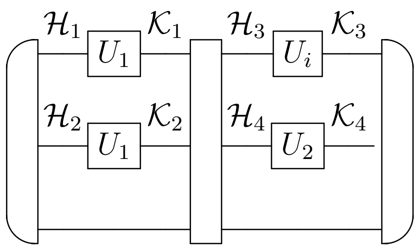



Figure 5.5: A configuration of the quantum 3-tester in case 2.

### Case 2

The next configuration of a quantum tester is that the first uses of the black-boxes and the other two black-boxes are applied in parallel ways. An example of this type of the configuration is represented in Fig. 5.5.

The corresponding SDP is give by

$$\begin{aligned} \text{maximize} \quad & p_{succ} = \frac{1}{2}\text{Tr}\left[M_1\widetilde{\Pi}_1 + M_2\widetilde{\Pi}_2\right] \\ \text{subject to} \quad & \widetilde{\Pi}_i \geq 0, \ i = 1, 2 \\ & \widetilde{\Pi}_1 + \widetilde{\Pi}_2 = I_{\mathcal{K}_3\mathcal{K}_4} \otimes X^{(2)} && (5.86) \\ & \text{Tr}_{\mathcal{H}_3\mathcal{H}_2} X^{(2)} = I_{\mathcal{K}_2\mathcal{K}_1} \otimes X^{(1)} && (5.87) \\ & \text{Tr} X^{(1)} = 1, && (5.88) \end{aligned}$$

with the quantum tester $\{\widetilde{\Pi}_1, \widetilde{\Pi}_2\}$.

### Case 3

The third configuration is that the first three uses of the black-boxes are in the parallel way followed by the use of a black-box. An example of this type of the configuration is shown in Fig. 5.6.

The corresponding SDP is give by

$$\begin{aligned} \text{maximize} \quad & p_{succ} = \frac{1}{2}\text{Tr}\left[M_1\widetilde{\Pi}_1 + M_2\widetilde{\Pi}_2\right] \\ \text{subject to} \quad & \widetilde{\Pi}_i \geq 0, \ i = 1, 2 \\ & \widetilde{\Pi}_i \geq 0, \ i = 1, 2 \\ & \widetilde{\Pi}_1 + \widetilde{\Pi}_2 = I_{\mathcal{K}_4} \otimes X^{(2)} && (5.89) \\ & \text{Tr}_{\mathcal{H}_4} X^{(2)} = I_{\mathcal{K}_1\mathcal{K}_2\mathcal{K}_2} \otimes X^{(1)} && (5.90) \\ & \text{Tr} X^{(1)} = 1, && (5.91) \end{aligned}$$

with the quantum tester $\{\widetilde{\Pi}_1, \widetilde{\Pi}_2\}$.

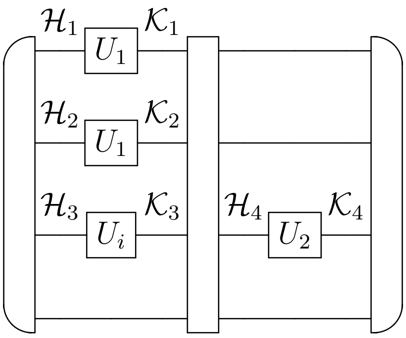



Figure 5.6: Quantum circuit representation of the quantum 3-tester in case 3.

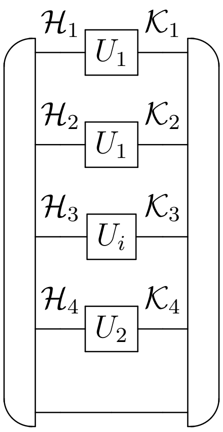



Figure 5.7: The configuration of the quantum 2-tester in case 4.

### Case 4

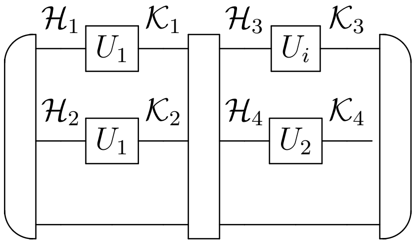
The last configuration is the parallel strategy. The circuit representation of the quantum 2-tester in this case is given in Fig. 5.7.

The constraint of the quantum tester $\{\widetilde{\Pi}_1, \widetilde{\Pi}_2\}$ is give by

$$\begin{aligned} \text{maximize} \quad & p_{succ} = \frac{1}{2}\mathrm{Tr}\left[M_1\widetilde{\Pi}_1 + M_2\widetilde{\Pi}_2\right], \\ \text{subject to} \quad & \widetilde{\Pi}_i \geq 0, \ i = 1, 2, \\ & \widetilde{\Pi}_i \geq 0, \ i = 1, 2, \\ & \widetilde{\Pi}_1 + \widetilde{\Pi}_2 = I_{\mathcal{K}} \otimes X, && (5.92) \\ & \mathrm{Tr}X = 1. && (5.93) \end{aligned}$$

### 5.2.2 Numerical Results

In numerical calculations, we investigate all of the four types of the configurations of the quantum testers and all of the orders of the uses of black-boxes. The result is summarized in Table. 5.2. In the column of the configurations in Table 5.2, only the order and configurations of the uses of the black-boxes inserted in the quantum testers are shown. For instance, the following configuration of case 2 is represented in the table as




$$\rightarrow \quad \begin{array}{|c|c|} \hline U_1 & U_i \\ \hline U_1 & U_2 \\ \hline \end{array} \; .$$

The configurations and the orders are divided into two classes depending on the optimal success probability. It is interesting that the ordered uses of the black-boxes do not necessarily give improvement comparing with the parallel uses of the black-boxes. The order of the uses of the black-boxes should be appropriately chosen to improve the optimal success probability comparing to the parallel strategies.

Based on classical information, the success probability becomes higher when the test box is used after the uses of the reference boxes since the test box can be used after obtaining information necessary to a measurement for equivalence determination. However, we find the counter example to that intuition. The examples are the following two orders of case 2, given as

$$\begin{array}{|c|ccc} \hline U_i & & & \\ \hline U_1 & U_1 & U_2 & \\ \hline \end{array} \quad \rightarrow \quad p_{succ}^{opt} \simeq 0.910516 \tag{5.94}$$

$$\begin{array}{|c|ccc} \hline U_1 & & & \\ \hline U_1 & U_2 & U_i & \\ \hline \end{array} \quad \rightarrow \quad p_{succ}^{opt} \simeq 0.902127. \tag{5.95}$$

The order given in Equation (5.94) in which the test box is used first achieves higher success probability than the one by the order given in Equation (5.95).

Similarly to the former section, we did not directly solve the SPDs given in this section, but the ones on each multiplicity subspaces obtained by utilizing the group theoretical properties. The derivations of the SDPs on the multiplicity subspaces are given in the next section.

Note that the success probability for $(2, 1)$-equivalence determination under the parallel strategies is same as the one for the case that the classical descriptions of $U_1$ is given and the reference box 2 is used once. A summary on the optimal success probabilities obtained is shown in Table 5.1.

By comparing the results obtained in Sec. 4.1.1 and Sec. 4.2.1, the optimal success probability for $(N_1, 0)$-equivalence determination under the parallel strategies can be achieved with $N_1 = 1$. That means, the additional $N_1 - 1$ uses of the reference box is not actually necessary for achieving the optimal success probability. Similarly, the results obtained in Sec. 4.1.1 and Sec. 4.2.1 indicate that $(N_1, 1)$-equivalence determination under the parallel strategies can be achieved with $N_1 = 2$. The two results are examples of the cases that the optimal success probability using the classical description of the unitary operation can be achieved by a finite uses of the black-box implementing the unitary operation, whereas the classical description of a unitary operator cannot be exactly determined by finite uses of the black-box.

| Section | Sec. 4.1.1 | Sec. 4.2.1 | Sec. 4.2.2 | Sec. 5.2.2 |
|---|---|---|---|---|
| $N$ | 1 | 1 | 1 | 1 |
| $N_1$ | known ($N_1 \to \infty$) | 1 | known ($N_1 \to \infty$) | 2 |
| $N_2$ | 0 | 0 | 1 | 1 |
| $p_{succ}^{opt}$ | $7/8 = 0.875$ | | $\approx 0.902127$ | |

Table 5.1: A comparison of the optimal success probabilities of $(N_1, N_2)$-equivalence determination. Note that $N$ and $N_i$ are the numbers of use of the test box and the reference box $i$, respectively.

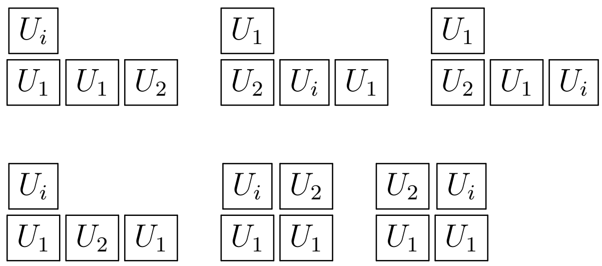
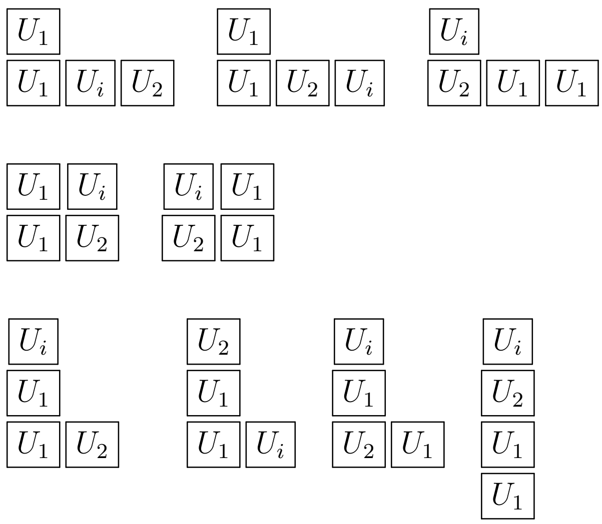
| $p_{succ}^{opt}$ | Configurations of the black-boxes |
|---|---|
| 0.910516 |  |
| 0.902127 |  |

Table 5.2: Numerical results of the optimal success probabilities for $(2, 1)$-equivalence determination. All of configurations of the quantum testers and all of (non-trivial) orders of the black-boxes are investigated. The configurations and the orders are divided into two classes depending on the optimal success probability.

### 5.2.3 Calculations for Semidefinite Programmings in the Multiplicity Subspaces

In this section, we rewrite the SDPs represented in Sec 5.2.1 in terms of operators on the multiplicity subspaces.

The multiplicity subspace $\mathcal{V}_j^{[N]}$ consists of the direct sum of the multiplicity subspaces of $\mathcal{V}_{j-1/2}^{[N-1]}$ and $\mathcal{V}_{j+1/2}^{N-1}$. We define isometries $P_{j,j\pm\frac{1}{2}}^{[N]}$ from the subspace of $\mathcal{V}_j^{[N]}$ that is generated from the multiplicity subspace $\mathcal{V}_{j\pm\frac{1}{2}}^{[N-1]}$ to the multiplicity subspace $\mathcal{V}_{j\pm\frac{1}{2}}^{[N-1]}$.

#### Case 1

We rewrite conditions (5.82) - (5.85). To this end, without loss of generality, we can represent

$$\widetilde{\Pi}_i = \bigoplus_{J,L=0}^{2} I_J \otimes \frac{I_L}{d_L} \otimes \widetilde{\Pi}_{JL}^{[i]} \tag{5.96}$$

$$X^{(3)} = \bigoplus_{J=\frac{1}{2}}^{\frac{3}{2}} \bigoplus_{L=0}^{2} I_J \otimes \frac{I_L}{d_L} \otimes X_{JL}^{(3)} \tag{5.97}$$

$$X^{(2)} = \bigoplus_{J=0}^{1} \bigoplus_{L=\frac{1}{2}}^{\frac{3}{2}} I_J \otimes \frac{I_L}{d_L} \otimes X_{JL}^{(2)} \tag{5.98}$$

$$X^{(1)} = \bigoplus_{L=0}^{1} \frac{I_L}{d_L} \otimes X_{JL}^{(1)}, \tag{5.99}$$

for $i = 1, 2$.

First we rewrite Equation (5.82). We have

$$\widetilde{\Pi}_1 + \widetilde{\Pi}_2 = \bigoplus_{J,L=0}^{2} I_J \otimes \frac{I_L}{d_L} \otimes (\widetilde{\Pi}_{JL}^{[1]} + \widetilde{\Pi}_{JL}^{[2]}) \tag{5.100}$$

$$I_{\mathcal{K}_4} \otimes X^{(3)} = \bigoplus_{L=0}^{2} I_0 \otimes \frac{I_L}{d_L} \otimes X_{\frac{1}{2}L}^{(3)} \tag{5.101}$$

$$= \bigoplus_{L=0}^{2} I_1 \otimes \frac{I_L}{d_L} \otimes (X_{\frac{1}{2}L}^{(3)} \oplus X_{\frac{3}{2}L}^{(3)}) \tag{5.102}$$

$$= \bigoplus_{L=0}^{2} I_2 \otimes \frac{I_L}{d_L} \otimes X_{\frac{3}{2}L}^{(3)}. \tag{5.103}$$

Thus Equation (5.82) is equivalent to conditions given as

$$(P^{[4]}_{0,\frac{1}{2}} \otimes I_{\mathcal{V}^{[4]}_L})(\widetilde{\Pi}^{[1]}_{0L} + \widetilde{\Pi}^{[2]}_{0L})(P^{[4]}_{0,\frac{1}{2}} \otimes I_{\mathcal{V}^{[4]}_L}) = X^{(3)}_{\frac{1}{2}L} \tag{5.104}$$

$$(P^{[4]}_{1,l+\frac{1}{2}} \otimes I_{\mathcal{V}^{[4]}_L})(\widetilde{\Pi}^{[1]}_{1L} + \widetilde{\Pi}^{[2]}_{1L})(P^{[4]}_{1,l'+\frac{1}{2}} \otimes I_{\mathcal{V}^{[4]}_L}) = \delta_{ll'} X^{(3)}_{l+\frac{1}{2}L} \tag{5.105}$$

$$(P^{[4]}_{2,\frac{3}{2}} \otimes I_{\mathcal{V}^{[4]}_L})(\widetilde{\Pi}^{[1]}_{2L} + \widetilde{\Pi}^{[2]}_{2L})(P^{[4]}_{2,\frac{3}{2}} \otimes I_{\mathcal{V}^{[4]}_L}) = X^{(3)}_{\frac{3}{2}L}, \tag{5.106}$$

for $L = 0, 1, 2$ and $l, l' = 0, 1$, where $I_{\mathcal{V}^{[4]}_L}$ is the identity operator on $\mathcal{V}^{[4]}_L$.

Then we rewrite Equation (5.83). We have

$$\begin{aligned}\mathrm{Tr}_{\mathcal{H}_3} X^{(3)} = &\bigoplus_{J=\frac{1}{2}}^{\frac{3}{2}} \bigoplus_{L=\frac{1}{2}}^{\frac{3}{2}} I_J \otimes \frac{I_L}{d_L} \\ &\otimes \Big[(I_{\mathcal{V}^{[3]}_J} \otimes P^{[4]}_{L-\frac{1}{2},\frac{1}{2}}) X_{JL-\frac{1}{2}} (I_{\mathcal{V}^{[3]}_J} \otimes P^{[4]}_{L-\frac{1}{2},\frac{1}{2}}) + (I_{\mathcal{V}^{[3]}_J} \otimes P^{[4]}_{L+\frac{1}{2},\frac{1}{2}}) X_{JL+\frac{1}{2}} (I_{\mathcal{V}^{[3]}_J} \otimes P^{[4]}_{L+\frac{1}{2},\frac{1}{2}})\Big],\end{aligned} \tag{5.107}$$

and

$$\begin{aligned} I_{\mathcal{K}_3} \otimes X^{(2)} = &\bigoplus_{L=\frac{1}{2}}^{\frac{3}{2}} I_{\frac{1}{2}} \otimes \frac{I_L}{d_L} \otimes (X^{(2)}_{0L} \oplus X^{(2)}_{1L}) \\ &\bigoplus_{L=\frac{1}{2}}^{\frac{3}{2}} I_{\frac{3}{2}} \otimes \frac{I_L}{d_L} \otimes X^{(2)}_{1L}. \end{aligned} \tag{5.108}$$

Therefore Equation (5.83) is equivalent to conditions given as

$$\begin{aligned} \delta_{JJ'} X^{(2)}_{JL'+\frac{1}{2}} = &(P^{[3]}_{\frac{1}{2},J} \otimes P^{[4]}_{0,L'+\frac{1}{2}}) X^{(3)}_{\frac{1}{2}L'} (P^{[3]}_{\frac{1}{2},J'} \otimes P^{[4]}_{0,L'+\frac{1}{2}}) \\ &+ (P^{[3]}_{\frac{1}{2},J} \otimes P^{[4]}_{1,L'+\frac{1}{2}}) X^{(3)}_{\frac{1}{2}L'+1} (P^{[3]}_{\frac{1}{2},J'} \otimes P^{[4]}_{1,L'+\frac{1}{2}}) \end{aligned} \tag{5.109}$$

$$\begin{aligned} X^{(2)}_{1L'+\frac{1}{2}} = &(P^{[3]}_{\frac{3}{2},1} \otimes P^{[4]}_{0,L'+\frac{1}{2}}) X^{(3)}_{\frac{3}{2}L'} (P^{[3]}_{\frac{3}{2},1} \otimes P^{[4]}_{0,L'+\frac{1}{2}}) \\ &+ (P^{[3]}_{\frac{3}{2},1} \otimes P^{[4]}_{1,L'+\frac{1}{2}}) X^{(3)}_{\frac{3}{2}1} (P^{[3]}_{\frac{3}{2},1} \otimes P^{[4]}_{1,L'+\frac{1}{2}}), \end{aligned} \tag{5.110}$$

for $L' = 0, 1$.

For Equation (5.83), we have

$$\mathrm{Tr}_{\mathcal{H}_3} X^{(2)} = \bigoplus_{J=0}^{1} I_J \otimes \Big[I_0 \otimes P^{[3]}_{0,\frac{1}{2}} X_{J\frac{1}{2}} P^{[3]}_{0,\frac{1}{2}} \tag{5.111}$$

$$\oplus \frac{I_0}{d_0} \otimes (P^{[3]}_{1,\frac{1}{2}} X_{J\frac{1}{2}} P^{[3]}_{1,\frac{1}{2}} + P^{[3]}_{1,\frac{3}{2}} X_{J\frac{1}{2}} P^{[3]}_{1,\frac{3}{2}})\Big], \tag{5.112}$$

$$I_{\mathcal{K}_1\mathcal{K}_2} \otimes X^{(1)} = \bigoplus_{J=0}^{1} I_J \otimes \left[I_0 \otimes X^{(1)}_0 \oplus \frac{I_1}{d_1} \otimes X^{(1)}_1\right]. \tag{5.113}$$

Therefore Equation (5.82) is equivalent to conditions given as

$$P^{[3]}_{0,\frac{1}{2}} X_{J\frac{1}{2}} P^{[3]}_{0,\frac{1}{2}} = X^{(1)}_0 \tag{5.114}$$

$$P^{[3]}_{1,\frac{1}{2}} X_{J\frac{1}{2}} P^{[3]}_{1,\frac{1}{2}} + P^{[3]}_{1,\frac{3}{2}} X_{J\frac{1}{2}} P^{[3]}_{1,\frac{3}{2}} = X^{(1)}_1, \tag{5.115}$$

for $J = 0, 1$.

For Equation (5.83), we have

$$X^{(1)}_0 + X^{(1)}_1 = 1. \tag{5.116}$$

### Case 2

We rewrite Equations (5.86) - (5.88). To this end, without loss of generality, we can represent

$$\widetilde{\Pi}_i = \bigoplus_{J,L=0}^{2} I_J \otimes \frac{I_L}{d_L} \otimes \widetilde{\Pi}^{[i]}_{JL}, \tag{5.117}$$

$$X^{(2)} = \bigoplus_{J=1}^{1} \bigoplus_{L=0}^{2} I_J \otimes \frac{I_L}{d_L} \otimes X^{(3)}_{JL}, \tag{5.118}$$

$$X^{(1)} = \bigoplus_{L=0}^{1} \frac{I_L}{d_L} \otimes X^{(1)}_{JL}, \tag{5.119}$$

for $i = 1, 2$.

The LHS of Equations (5.86) is represented as

$$\widetilde{\Pi}_1 + \widetilde{\Pi}_2 = \bigoplus_{J,L=0}^{2} I_J \otimes \frac{I_L}{d_L} \otimes (\widetilde{\Pi}^{[1]}_{JL} + \widetilde{\Pi}^{[2]}_{JL}), \tag{5.120}$$

and the RHS is give by

$$I_{\mathcal{K}_3 4} \otimes X^{(2)} = \bigoplus_{j=0}^{2} I_0 \otimes \frac{I_L}{d_L} \otimes (X^{(2)}_{0L} \oplus X^{(2)}_{1L}) \tag{5.121}$$

$$\bigoplus_{j=0}^{2} I_1 \otimes \frac{I_L}{d_L} \otimes (X^{(2)}_{0L} \oplus X^{(2)}_{1L} \oplus X^{(2)}_{1L}) \tag{5.122}$$

$$\bigoplus_{j=0}^{2} I_2 \otimes \frac{I_L}{d_L} \otimes (X^{(2)}_{1L}). \tag{5.123}$$

Therefore Equations (5.86) is equivalent to the equations given as

$$(P^{[3]}_{\frac{1}{2},l}P^{[4]}_{j\frac{1}{2}} \otimes I_{\mathcal{V}^{[4]}_L})(\widetilde{\Pi}^{[1]}_{jL} + \widetilde{\Pi}^{[2]}_{jL})(P^{[4]}_{j\frac{1}{2}}P^{[3]}_{\frac{1}{2},l'} \otimes I_{\mathcal{V}^{[4]}_L}) = \delta_{ll'}X_{jL} \tag{5.124}$$

$$(P^{[3]}_{\frac{3}{2},1}P^{[4]}_{1,m+\frac{1}{2}} \otimes I_{\mathcal{V}^{[4]}_L})(\widetilde{\Pi}^{[1]}_{1L} + \widetilde{\Pi}^{[2]}_{1L})(P^{[4]}_{1,m'+\frac{1}{2}}P^{[3]}_{\frac{1}{2},1} \otimes I_{\mathcal{V}^{[4]}_L}) = \delta_{mm'}X_{jL} \tag{5.125}$$

$$P^{[3]}_{\frac{3}{2},1}P^{[4]}_{2,\frac{3}{2}}(\widetilde{\Pi}^{[1]}_{jL} + \widetilde{\Pi}^{[2]}_{jL})P^{[4]}_{2,\frac{3}{2}}P^{[3]}_{\frac{3}{2},1} = X^{[2]}_{1L}. \tag{5.126}$$

The LHS of Equation (5.87) is given as

$$\mathrm{Tr}_{\mathcal{H}_3\mathcal{H}_4}X^{(2)} = \tag{5.127}$$

$$\bigoplus_{J=0}^{1} I_J \otimes I_0 \otimes \left[\sum_{l=0}^{1}(I_{\mathcal{V}^{[2]}_J} \otimes P^{[3]}_{\frac{1}{2},0}P^{[4]}_{l,\frac{1}{2}})X^{[2]}_{Jl}(I_{\mathcal{V}^{[2]}_J} \otimes P^{[4]}_{l,\frac{1}{2}}P^{[3]}_{\frac{1}{2},0})\right] \tag{5.128}$$

$$\bigoplus_{J=0}^{1} I_J \otimes \frac{I_1}{d_1} \otimes \Big[(I_{\mathcal{V}^{[2]}_J} \otimes P^{[3]}_{\frac{1}{2},1}P^{[4]}_{1,\frac{1}{2}})X^{[2]}_{Jl}(I_{\mathcal{V}^{[2]}_J} \otimes P^{[4]}_{1,\frac{1}{2}}P^{[3]}_{\frac{1}{2},1}) \tag{5.129}$$

$$+\sum_{l=1}^{2}(I_{\mathcal{V}^{[2]}_J} \otimes P^{[3]}_{\frac{1}{2},1}P^{[4]}_{l,\frac{1}{2}})X^{[2]}_{Jl}(I_{\mathcal{V}^{[2]}_J} \otimes P^{[4]}_{l,\frac{1}{2}}P^{[3]}_{\frac{1}{2},1})\Big], \tag{5.130}$$

and the RHS of Equation (5.87) is given by

$$I_{\mathcal{K}_1\mathcal{K}_2} \otimes X^{(1)} = \bigoplus_{J=0}^{1} I_J \otimes \left[I_0 \otimes X^{(1)}_0 \oplus \frac{I_1}{d_1} \otimes X^{(1)}_1\right]. \tag{5.131}$$

Therefore Equation (5.87) is equivalent to conditions given as

$$X^{[1]}_0 = \sum_{l=0}^{1}(I_{\mathcal{V}^{[2]}_J} \otimes P^{[3]}_{\frac{1}{2},0}P^{[4]}_{l,\frac{1}{2}})X^{[2]}_{Jl}(I_{\mathcal{V}^{[2]}_J} \otimes P^{[4]}_{l,\frac{1}{2}}P^{[3]}_{\frac{1}{2},0}) \tag{5.132}$$

$$X^{[1]}_1 = (I_{\mathcal{V}^{[2]}_J} \otimes P^{[3]}_{\frac{1}{2},1}P^{[4]}_{1,\frac{1}{2}})X^{[2]}_{Jl}(I_{\mathcal{V}^{[2]}_J} \otimes P^{[4]}_{1,\frac{1}{2}}P^{[3]}_{\frac{1}{2},1}) \tag{5.133}$$

$$+\sum_{l=1}^{2}(I_{\mathcal{V}^{[2]}_J} \otimes P^{[3]}_{\frac{1}{2},1}P^{[4]}_{l,\frac{1}{2}})X^{[2]}_{Jl}(I_{\mathcal{V}^{[2]}_J} \otimes P^{[4]}_{l,\frac{1}{2}}P^{[3]}_{\frac{1}{2},1}). \tag{5.134}$$

Equation (5.87) is rewritten as

$$X^{[1]}_0 + X^{[1]}_1 = 1. \tag{5.135}$$

**Case 3**

We rewrite Equations (5.89) - (5.91). To this end, without loss of generality, we can represent

$$\widetilde{\Pi}_i = \bigoplus_{J,L=0}^{2} I_J \otimes \frac{I_L}{d_L} \otimes \widetilde{\Pi}_{JL}^{[i]} \tag{5.136}$$

$$X^{(2)} = \bigoplus_{J=\frac{1}{2}}^{\frac{3}{2}} \bigoplus_{L=0}^{2} I_J \otimes \frac{I_L}{d_L} \otimes X_{JL}^{(2)} \tag{5.137}$$

$$X^{(1)} = \bigoplus_{L=\frac{1}{2}}^{\frac{3}{2}} \frac{I_L}{d_L} \otimes X_{JL}^{(1)}, \tag{5.138}$$

for $i = 1, 2$. The Equation (5.89) is same as Equations (5.82) in the case 1. For Equation (5.90), the LHS is calculated as

$$\begin{aligned}\mathrm{Tr}_{\mathcal{K}_3} X^{(3)} = &\bigoplus_{J=\frac{1}{2}}^{\frac{3}{2}} \bigoplus_{L=\frac{1}{2}}^{\frac{3}{2}} I_J \otimes \frac{I_L}{d_L} \\ &\otimes \Big[ (I_{\mathcal{V}_J^{[3]}} \otimes P^{[4]}_{L-\frac{1}{2},\frac{1}{2}}) X_{JL-\frac{1}{2}} (I_{\mathcal{V}_J^{[3]}} \otimes P^{[4]}_{L-\frac{1}{2},\frac{1}{2}}) + (I_{\mathcal{V}_J^{[3]}} \otimes P^{[4]}_{L+\frac{1}{2},\frac{1}{2}}) X_{JL+\frac{1}{2}} (I_{\mathcal{V}_J^{[3]}} \otimes P^{[4]}_{L+\frac{1}{2},\frac{1}{2}}) \Big]\end{aligned} \tag{5.139}$$

and the RHS is represented as

$$I_{\mathcal{K}_1\mathcal{K}_2\mathcal{K}_3} \otimes X^{(1)} = \bigoplus_{J=\frac{1}{2}}^{\frac{3}{2}} \bigoplus_{L=\frac{1}{2}}^{\frac{3}{2}} I_J \otimes \frac{I_L}{d_L} \otimes X_L^{(1)}. \tag{5.140}$$

Therefore Equation (5.90) is equivalent to the conditions represented as

$$\begin{aligned} I_{\mathcal{V}_J^{[3]}} \otimes X_L^{(1)} = &\Big[ (I_{\mathcal{V}_J^{[3]}} \otimes P^{[4]}_{L-\frac{1}{2},\frac{1}{2}}) X_{JL-\frac{1}{2}} (I_{\mathcal{V}_J^{[3]}} \otimes P^{[4]}_{L-\frac{1}{2},\frac{1}{2}}) \\ &+ (I_{\mathcal{V}_J^{[3]}} \otimes P^{[4]}_{L+\frac{1}{2},\frac{1}{2}}) X_{JL+\frac{1}{2}} (I_{\mathcal{V}_J^{[3]}} \otimes P^{[4]}_{L+\frac{1}{2},\frac{1}{2}}) \Big], \end{aligned} \tag{5.141}$$

for $J, L = \frac{1}{2}, \frac{3}{2}$.

**Case 4**

We rewrite Equations (5.92) - (5.93). To this end, without loss of generality, we can represent

$$\widetilde{\Pi}_i = \bigoplus_{J,L=0}^{2} I_J \otimes \frac{I_L}{d_L} \otimes \widetilde{\Pi}_{JL}^{[i]} \tag{5.142}$$

$$X = \bigoplus_{L=0}^{2} \frac{I_L}{d_L} \otimes X_L. \tag{5.143}$$

Equation (5.93) is equivalent to conditions given in

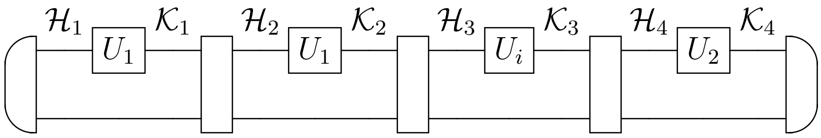
$$\widetilde{\Pi}_{JL}^{[1]} + \Pi_{JL}^{[2]} = I_{V_J^{[N]}} \otimes X_L \tag{5.144}$$

$$\sum_{L=0}^{2} \mathrm{Tr} X_L = 1, \tag{5.145}$$

for $J, L = 0, 1, 2$.

### 5.2.4 Calculations of the Choi Operators

First we consider the Choi operator corresponding to the order of $(U_1, U_1, U_i, U_2)$, namely,




. (5.146)

For $M_i$, we obtain

$$M_1 = \eta^{[3]}_{\mathcal{H}_1\mathcal{H}_2\mathcal{H}_3\mathcal{K}_1\mathcal{K}_2\mathcal{K}_3} \otimes \eta^{[1]}_{\mathcal{H}_4\mathcal{K}_4}, \tag{5.147}$$

$$M_2 = \eta^{[2]}_{\mathcal{H}_1\mathcal{H}_2\mathcal{K}_1\mathcal{K}_2} \otimes \eta^{[2]}_{\mathcal{H}_3\mathcal{H}_4\mathcal{K}_3\mathcal{K}_4}. \tag{5.148}$$

For $M_1$, we can calculate as

$$\eta^{[3]}_{\mathcal{H}_1\mathcal{H}_2\mathcal{H}_3\mathcal{K}_1\mathcal{K}_2\mathcal{K}_3} = \frac{I_{\frac{1}{2}}}{2} \otimes I_{\frac{1}{2}} \otimes \left[ (|\alpha_0^{[2]}\rangle\rangle + |\alpha_1^{[2]}\rangle\rangle) \otimes |\alpha_{\frac{1}{2}}^{[1]}\rangle\rangle \right] \oplus \frac{I_{\frac{3}{2}}}{4} \otimes I_{\frac{3}{2}} \otimes \left[ |\alpha_1^{[2]}\rangle\rangle \otimes |\alpha_{\frac{1}{2}}^{[1]}\rangle\rangle \right], \tag{5.149}$$

$$\eta^{[1]}_{\mathcal{H}_4\mathcal{K}_4} = \frac{I_{\frac{1}{2}}}{2} \otimes I_{\frac{1}{2}} \otimes \left[ |\alpha_{\frac{1}{2}}^{[1]}\rangle\rangle \right]. \tag{5.150}$$

Therefore we have

$$M_{00}^{[1]} = \frac{1}{4}\left[|00\rangle_{00} + |11\rangle_{00}\right], \tag{5.151}$$

$$M_{01}^{[1]} = \frac{1}{4}\left[|00\rangle_{01} + |11\rangle_{01}\right], \tag{5.152}$$

$$M_{02}^{[1]} = 0, \tag{5.153}$$

$$M_{10}^{[1]} = \frac{3}{4}\left[|00\rangle_{10} + |11\rangle_{10}\right], \tag{5.154}$$

$$M_{11}^{[2]} = \frac{3}{4}\left[|00\rangle_{11} + |11\rangle_{11}\right] + \frac{3}{8}\left[|22\rangle_{11}\right], \tag{5.155}$$

$$M_{12}^{[2]} = \frac{3}{8}\left[|20\rangle_{12}\right], \tag{5.156}$$

$$M_{20}^{[2]} = 0, \tag{5.157}$$

$$M_{21}^{[2]} = \frac{5}{8}\left[|02\rangle_{21}\right], \tag{5.158}$$

$$M_{22}^{[2]} = \frac{5}{8}\left[|00\rangle_{22}\right], \tag{5.159}$$

where we defined the basis states on the multiplicity subspaces $\mathcal{V}_0^{[4]}$ as

$$|0\tfrac{1}{2}(\tfrac{1}{2})\tfrac{1}{2};0\rangle\langle 0\tfrac{1}{2}(\tfrac{1}{2})\tfrac{1}{2};0| =: I_0 \otimes |\hat{0}\rangle\langle\hat{0}|_0, \tag{5.160}$$

$$|1\tfrac{1}{2}(\tfrac{1}{2})\tfrac{1}{2};0\rangle\langle 1\tfrac{1}{2}(\tfrac{1}{2})\tfrac{1}{2};0| =: I_0 \otimes |\hat{1}\rangle\langle\hat{1}|_0, \tag{5.161}$$

$$\tag{5.162}$$

for the multiplicity subspaces $\mathcal{V}_1^{[4]}$ as

$$\textstyle\sum_{m=-1}^{1} |0\tfrac{1}{2}(\tfrac{1}{2})\tfrac{1}{2};1\rangle\langle 0\tfrac{1}{2}(\tfrac{1}{2})\tfrac{1}{2};1| =: I_1 \otimes |\hat{0}\rangle\langle\hat{0}|_0, \tag{5.163}$$

$$\textstyle\sum_{m=-1}^{1} |1\tfrac{1}{2}(\tfrac{1}{2})\tfrac{1}{2};1\rangle\langle 1\tfrac{1}{2}(\tfrac{1}{2})\tfrac{1}{2};1| =: I_1 \otimes |\hat{2}\rangle\langle\hat{2}|_0, \tag{5.164}$$

$$\textstyle\sum_{m=-1}^{1} |1\tfrac{1}{2}(\tfrac{3}{2})\tfrac{1}{2};1\rangle\langle 1\tfrac{1}{2}(\tfrac{3}{2})\tfrac{1}{2};1| =: I_1 \otimes |\hat{3}\rangle\langle\hat{3}|_0, \tag{5.165}$$

for the multiplicity subspaces $\mathcal{V}_2^{[4]}$ as

$$\textstyle\sum_{m=-1}^{1} |1\tfrac{1}{2}(\tfrac{3}{2})\tfrac{1}{2};2\rangle\langle 1\tfrac{1}{2}(\tfrac{3}{2})\tfrac{1}{2};2| =: I_1 \otimes |\hat{0}\rangle\langle\hat{0}|_0, \tag{5.166}$$

For $M_2$, we have

$$\eta^{[2]}_{\mathcal{H}_1\mathcal{H}_2\mathcal{K}_1\mathcal{K}_2} = I_0 \otimes I_0 \otimes \left[|\alpha_0^{[2]}\rangle\!\rangle\right] \oplus \tfrac{I_1}{3} \otimes I_1 \otimes \left[|\alpha_1^{[2]}\rangle\!\rangle\right], \tag{5.167}$$

$$\eta^{[2]}_{\mathcal{H}_3\mathcal{H}_4\mathcal{K}_3\mathcal{K}_4} = I_0 \otimes I_0 \otimes \left[|\alpha_{\frac{1}{2}^{[1]}}\rangle\!\rangle \otimes |\alpha_{\frac{1}{2}^{[1]}}\rangle\!\rangle\right] \oplus \tfrac{I_1}{3} \otimes I_1 \otimes \left[|\alpha_{\frac{1}{2}^{[1]}}\rangle\!\rangle \otimes |\alpha_{\frac{1}{2}^{[1]}}\rangle\!\rangle\right] \tag{5.168}$$

and therefore, we obtain

$$M_{00}^{[2]} = \left[|\hat{0}\hat{0}\rangle_{00}\right] + \frac{1}{9}\left[|\hat{1}\hat{1}\rangle_{00}\right], \tag{5.169}$$

$$M_{01}^{[2]} = \frac{1}{9}\left[|\hat{1}\hat{2}\rangle_{01}\right], \tag{5.170}$$

$$M_{02}^{[2]} = \frac{1}{9}\left[|\hat{1}\hat{0}\rangle_{02}\right], \tag{5.171}$$

$$M_{10}^{[2]} = \frac{1}{3}\left[|\hat{2}\hat{1}\rangle_{10}\right], \tag{5.172}$$

$$M_{11}^{[2]} = \left[|\hat{1}\hat{1}\rangle_{11}\right] + \left[|\hat{0}\hat{0}\rangle_{11}\right] + \frac{1}{3}\left[|\hat{2}\hat{2}\rangle_{11}\right], \tag{5.173}$$

$$M_{12}^{[2]} = \frac{1}{3}\left[|\hat{2}\hat{0}\rangle_{10}\right], \tag{5.174}$$

$$M_{02}^{[2]} = \frac{5}{9}\left[|\hat{0}\hat{1}\rangle_{02}\right], \tag{5.175}$$

$$M_{12}^{[2]} = \frac{5}{9}\left[|\hat{0}\hat{2}\rangle_{12}\right], \tag{5.176}$$

$$M_{22}^{[2]} = \frac{5}{9}\left[|\hat{0}\hat{0}\rangle_{22}\right], \tag{5.177}$$

where we defined the basis states on the multiplicity subspaces $\mathcal{V}_0^{[4]}$ as

$$|0\tfrac{1}{2}\tfrac{1}{2}(0);0\rangle\langle 0\tfrac{1}{2}\tfrac{1}{2}(0);0| =: I_0 \otimes |\hat{0}\rangle\langle\hat{0}|_0, \tag{5.178}$$

$$|1\tfrac{1}{2}\tfrac{1}{2}(1);0\rangle\langle 1\tfrac{1}{2}\tfrac{1}{2}(1);0| =: I_0 \otimes |\hat{1}\rangle\langle\hat{1}|_0, \tag{5.179}$$

and for the multiplicity subspaces $\mathcal{V}_1^{[4]}$ as

$$\textstyle\sum_{m=-1}^{1} |0\tfrac{1}{2}\tfrac{1}{2}(1);1\rangle\langle 0\tfrac{1}{2}\tfrac{1}{2}(1);1| =: I_1 \otimes |\hat{0}\rangle\langle\hat{0}|_0, \tag{5.180}$$

$$\textstyle\sum_{m=-1}^{1} |1\tfrac{1}{2}\tfrac{1}{2}(0);1\rangle\langle 1\tfrac{1}{2}\tfrac{1}{2}(0);1| =: I_1 \otimes |\hat{0}\rangle\langle\hat{1}|_1, \tag{5.181}$$

$$\textstyle\sum_{m=-1}^{1} |1\tfrac{1}{2}\tfrac{1}{2}(1);1\rangle\langle 1\tfrac{1}{2}\tfrac{1}{2}(1);1| =: I_1 \otimes |\hat{0}\rangle\langle\hat{2}|_2, \tag{5.182}$$

$$\tag{5.183}$$

and for the multiplicity subspaces $\mathcal{V}_1^{[4]}$ as

$$\textstyle\sum_{m=-2}^{2} |1\tfrac{1}{2}\tfrac{1}{2}(1);2\rangle\langle 1\tfrac{1}{2}\tfrac{1}{2}(1);2| =: I_0 \otimes |\hat{0}\rangle\langle\hat{0}|_0. \tag{5.184}$$

Using Lemma 6, one can calculate the relation between the two basis states. We obtain

$$|\hat{0}\rangle_0 = U(0,1/2,0)_{12}|0\rangle_0, \quad (5.185)$$
$$|\hat{1}\rangle_0 = U(0,1/2,0)_{21}|1\rangle_0, \quad (5.186)$$
$$|\hat{0}\rangle_1 = U(0,1/2,1)_{22}|0\rangle_1, \quad (5.187)$$
$$|\hat{1}\rangle_1 = U(1,1/2,1)_{11}|1\rangle_1 + U(1,1/2,1)_{12}|2\rangle_1, \quad (5.188)$$
$$|\hat{2}\rangle_1 = U(1,1/2,1)_{21}|1\rangle_1 + U(1,1/2,1)_{22}|2\rangle_1, \quad (5.189)$$
$$|\hat{0}\rangle_2 = U(1,1/2,2)_{22}|0\rangle_2. \quad (5.190)$$

Swap operations on $\mathcal{H}_i \otimes \mathcal{H}_j$ and $\mathcal{K}_i \otimes \mathcal{K}_j$ are applied to change the order of the use of the black-boxes. In the multiplicity subspaces, these swap operations correspond to unitary operations within each multiplicity subspaces. Such a unitary operations $U_{ij}^{[J]}$ on the multiplicity subspace $\mathcal{V}_J^{[4]}$ when the $i$-th and $j$-th systems are exchanged.

We define a two-by-two matrix $V(j_1, j_3, j_{13})$ as

$$V(j_1,j_3,j_{13})_{11} = \frac{(-1)^{2(j_1+j_3+j_{13})}}{\sqrt{(2j_{13}+2)(2j_1+1)}}\sqrt{(j_3+\frac{1}{2})^2-(j_{13}-j_1+\frac{1}{2})^2}, \quad (5.191)$$
$$V(j_1,j_3,j_{13})_{12} = \frac{(-1)^{2(j_1+j_3+j_{13})}}{\sqrt{(2j_{13}+2)(2j_1+1)}}\sqrt{(j_{13}+j_1+3/2)^2-(j_3+\frac{1}{2})^2}, \quad (5.192)$$
$$V(j_1,j_3,j_{13})_{21} = \frac{(-1)^{2(j_1+j_3+j_{13})}}{\sqrt{(2j_{13}+2)(2j_1+1)}}\sqrt{(j_{13}+j_1+\frac{3}{2})^2-(j_3+\frac{1}{2})^2}, \quad (5.193)$$
$$V(j_1,j_3,j_{13})_{22} = -\frac{(-1)^{2(j_1+j_3+j_{13})}}{\sqrt{(2j_{13}+2)(2j_1+1)}}\sqrt{(j_3+\frac{1}{2})^2-(j_{13}-j_1+\frac{1}{2})^2}. \quad (5.194)$$

From Lemma 6, we can derive

$$U_{12}^{[0]} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \quad U_{12}^{[1]} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad U_{12}^{[2]} = 1, \quad (5.195)$$

$$U_{23}^{[0]} = V(1/2,1/2,0)^T, \quad U_{23}^{[1]} = \begin{pmatrix} \multicolumn{2}{c}{V(1/2,1/2,0)^T} & \begin{matrix} 0 \\ 0 \end{matrix} \\ 0 & 0 & V(1,1/2,1/2)_{22} \end{pmatrix},$$
$$U_{23}^{[2]} = V(1/2,1/2,1)_{12}, \quad (5.196)$$
$$U_{34}^{[0]} = \begin{pmatrix} V(0,1/2,-1/2)_{22} & 0 \\ 0 & V(1,1/2,-1/2) \end{pmatrix},$$

$$U_{34}^{[1]} = \left( \begin{array}{ccc} V(0,1/2,1/2) & 0 & 0 \\ 0 & \multicolumn{2}{c}{\multirow{2}{*}{$V(1,1/2,1/2)^T$}} \\ 0 & & \end{array} \right), \quad U_{34}^{[2]} = V(1,1/2,3/2)_{12}. \tag{5.197}$$

# Chapter 6

# Conclusion

In this thesis, we introduced equivalence determination of unitary operations, which is a discrimination task of candidate unitary operations whose classical descriptions are not available. Three black-boxes (one test box and two reference boxes) each implementing a unitary operation are given as a physical system. The unitary operation implemented by the test box is guaranteed to be one of the unitary operations implemented by the two reference boxes but the classical description of the unitary operations are unknown. If $i$-th reference box is allowed to be used $N_i$ times and the test box once, then is is called $(N_1, N_2)$-equivalence determination of unitary operations. We considered two types of strategies called parallel strategies and ordered strategies and investigated when one outperforms the other. In the parallel strategies, the black-boxes are used simultaneously during the computation and no quantum operation is applied between the uses of the black-boxes. In contrast, the ordered strategies permit arbitrary quantum operations to be inserted between the uses of the black-boxes. To perform equivalence determination of unitary operations, we used quantum testers that generalize quantum measurements to higher-order quantum computation. A tester takes quantum operations as inputs of computation. We formulated optimization problems to maximize the success probability for obtaining the correct guess in terms of semidefinite programmings.

In Chap. 4, we showed that the optimal success probability of $(1,1)$-equivalence determination of unitary operations is $7/8$ in any configuration of the three black-boxes. We found that the optimal success probability for $(1,1)$-equivalence determination can be achieved by a simplified task called comparison of unitary operations. We proved that to achieve the optimal success probability requires an entangled state as input state, where entanglement exists between the systems on which the reference boxes act and the test box acts. We investigated the case that the classical description of one of the reference boxes is given, whereas the classical description of another reference box is not given. We derived the optimal success probability analytically for the case of no use of the reference box and numerically obtained for the case of a single use of the reference box.

In Chap. 5, we numerically analyzed equivalence determination of unitary operations with multiple uses of the black-boxes. We first numerically calculated the optimal success probability for $(N, N)$-equivalence determination under the parallel strategies for the cases of $N = 2, 3, 4$, by formulating the optimization problem in terms of the semidefinite programmings. We then investigated $(2, 1)$-equivalence determination of unitary operations both under the parallel strategies and ordered strategies. We numerically analyzed the optimal success probabilities for all configurations of the quantum testers and all orders of the black boxes. We found that the configurations and the orders can be divided into two classes according to the optimal success probability, and that the ordered strategies outperform the parallel strategies.

We summarize implications and observations of note.

- **Parallelizable case:** We showed that the task of $(1, 1)$-equivalence determination of unitary operations can be parallelized, thus that the ordered strategies do not give improvements over the parallel. This contrasts with the general expectation that the adaptive operations allowed under the ordered strategies provide advantages over the parallel strategies in optimization. Ordered strategies are at least as strong as parallel strategies in success probability and strictly stronger in some cases [33–36], but less efficient in execution time. General conditions for improvements under the ordered strategies, however, are not known. The difficulty of the analysis originates from the complexity of deriving the optimum under the ordered strategies. The result obtained in this thesis is an example of the cases that the ordered and parallel strategies exhibit equivalent performance.

- **Order-dependent case:** The task of $(2, 1)$-equivalence determination of unitary operations is the first example in which the optimal performance of the task varies depending on the order of the operation in higher-order quantum computation with more than two different quantum operations as inputs. To the best of our knowledge, such a task considered before is only the quantum switch [37], where the order dependence is trivial as its exact implementation is impossible with the single use of each operation in the quantum circuit model.

  The general method to find the order to obtain the higher success probability is still an open problem. As shown in Table 5.2, the ordered strategies outperform the parallel strategies in $(2, 1)$-equivalence determination of unitary operations whereas the task of $(1, 1)$-equivalence determination of unitary operations can be parallelized by keeping the same success probability. Further investigation of these tasks may provide a clue to understand the condition for parallelizability of the black-boxes.

- **Finite use as powerful as full classical description:** We found the examples in which the performance obtained with the classical description of a unitary operation can be achieved with the finite uses of the black-boxes implementing the unitary operations as shown in Table 5.1. The classical description of a unitary operation enables to implement arbitrary number of the use of the black-box implementing the unitary operation. The results presented in Table 5.1 imply that the additional use of the black-box offers no improvement to the optimal success probability. The difference of the classical description of a unitary operation and the power of implementing the operation is also mentioned in the context of quantum learning of unitary operations [22]. To the best our knowledge, however, such a phenomenon indicating no difference between the two resources has not been found before.

We hope that our results contribute to the development of the complete theory of resources in higher-order quantum computation and, especially, to revealing fundamental consequences of time-ordering in quantum mechanics.

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# Appendix A

# Semidefinite Programmings

A semidefinite programming (SDP) is a special case of convex optimizations [67]. It is a useful technique in both analytic and numerical calculations. Several optimization problems are represented in terms of SDPs. For example, optimization problems for deriving the fidelity of two quantum states [68], the optimal success probability for discriminating quantum channels [69] and the zero-error capacity of quantum channels [70] are formulated in terms of SDPs.

There are several formalizations for representing an SDP. Here we follow the formalization in Watrous lecture note [68], which is adopted in a suitable form for problems appearing in quantum information theory. First we give the definition of an SDP and introduce both primal problems and dual problems. The weak duality is introduced, which indicates the optimal value of a primal problem is bounded by the one by the corresponding dual problem. Then Slator's condition, under which the two optimal values of the primal and the dual problems coincides, is introduced. Finally we briefly summarize the numerical calculation method for SPDs.

For determining semidefinite programming, several terms should be fixed. A map $\Phi$ from $L(\mathcal{H})$ to $L(\mathcal{K})$ is *Hermitian preserving* if it maps Hermitian operators on $\mathcal{H}$ to Hermitian operators on $\mathcal{K}$.

**Definition 4.** *A semidefinite programming is a set of triplet* $(\Phi, A, B)$*, where*

- $\Phi$ *is an Hermitian preserving channel from* $L(\mathcal{H})$ *to* $L(\mathcal{K})$ *and*
- $A$ *and* $B$ *are Hermitian operators on* $\mathcal{H}$ *and* $\mathcal{K}$*, respectively.*

From the triplet, two optimization problems called the *primal* and *dual* problems are defined. The primal problem is represented as

$$\begin{aligned} \text{maximize} \quad & \mathrm{Tr}[AX] && \text{(A.1)} \\ \text{subject to} \quad & \Phi(X) = B && \text{(A.2)} \\ & X \geq 0, && \text{(A.3)} \end{aligned}$$

and the dual problem is represented as

$$\begin{aligned} \text{minimize} \quad & \mathrm{Tr}[BY] && \text{(A.4)} \\ \text{subject to} \quad & \Phi^*(Y) \geq A && \text{(A.5)} \\ & Y \text{ is on a Hermitian operator}, && \text{(A.6)} \end{aligned}$$

where $\Phi^*(\cdot)$ is a dual map defined as $\mathrm{Tr}[\Phi(A)B] = \mathrm{Tr}[A\Phi^*(B)]$ for any operator $A$ and $B$.

The primal problem is said to be *feasible* if there exists at least one positive operator $X$ satisfying $\Phi(X) = B$ and *infeasible* otherwise. The set of feasible solutions is called a *feasible set* and denoted as

$$\mathcal{A} = \{X | \Phi(X) = B, X \geq 0\}. \tag{A.7}$$

Feasibility can be also defined for the dual problem with similar terminology. The feasible set for the dual problem is denoted as

$$\mathcal{B} = \{X | \Phi^*(Y) = A, Y \text{ is Hermitian}\}. \tag{A.8}$$

We denote the optimal values for the primal and dual problems as $\alpha$ and $\beta$ given by $\alpha = \sup_{X\in\mathcal{A}} \mathrm{Tr}[AX]$ and $\beta = \inf_{Y\in\mathcal{B}} \mathrm{Tr}[BY]$, respectively. The *weak duality* is stated as follows:

**Proposition 4.** *For any semidefinite programming* $(\Phi, A, B)$*, the inequality* $\alpha \leq \beta$ *holds.*

**Proof.** We only consider the case that both of feasible sets for the primal and dual problems are not empty. Therefore any feasible solutions $X \in \mathcal{A}$ and $Y \in \mathcal{B}$, we have

$$\mathrm{Tr}[AX] \leq \mathrm{Tr}[\Phi^*(Y)X] = \mathrm{Tr}[Y\Phi(X)] = \mathrm{Tr}[YB]. \tag{A.9}$$

Since the optimum value $\alpha$ is the supremum over all feasible solutions $X \in \mathcal{A}$ and the optimum value $\beta$ is the infimum over all feasible solutions $Y \in \mathcal{B}$, the inequality $\alpha \leq \beta$ holds. □

The weak duality indicates that the optimal value of one problem is just the bound for the one of the other problem. The equality condition $\alpha = \beta$ is called *strong duality*, which does not hold necessarily for all semidefinite programmings. Several conditions are known for the strong duality. Here we give one of the conditions called Slator's condition without the proof.

**Proposition 5** (Slator's condition)**.** *If $\mathcal{A} \neq \emptyset$ and there exists a feasible operator $Y \in \mathcal{B}$ for the dual problem satisfying $\Phi^*(Y) > A$, then $\alpha = \beta$. If $\mathcal{B} \neq \emptyset$ and there exists a feasible operator $X \in \mathcal{A}$ for the primal problem satisfying $X > 0$, then $\alpha = \beta$.*

Slator's condition is useful especially for discrimination of states and operations considered in this thesis. In semidefinite programming for the discrimination, it is usually easy to find the feasible solution satisfying Slator's condition.

We give several comments on semidefinite programming. First comment is about the strong duality. The strong duality is considered to hold in most cases unless one tries to fail it [68]. Thus the strategy to find the optimum value is divided into two cases. One strategy is to show a condition indicating the strong duality and solve one of the primal and dual problems. The other is find the feasible solutions in both primal and dual problems that give $\alpha = \beta$.

The second comment is about the conditions in the primal problem. The standard form given by conditions (A.1) and (A.2) contains only an equality condition. In general, multiple conditions including both equalities and inequalities can appear in the primal problem. Rewriting the problem as the standard form to obtain the dual problem is often not straightforward. In that case, Lagrange dual function [67] is used to derive the dual problem from a primal problem with complicated conditions. This technique is used in this thesis.

The final comment is about numerical calculations for the primal and dual problems. Numerical calculations are efficient with the size of the operator $X$. The numerical values of $\alpha$ and $\beta$ checked if the two value are close. The optimum solution is also close to the two values due to the weak duality.

There are several solvers and interpreters for SDPs many programming languages such as Matlab and Python. In Chap. 5, we deal with numerical calculations of several SDPs. We use PICOS [71] as a interpreter and MOSEK [72] as a solver in Python.

# Appendix B

# Irreducible Representation of Unitary Operators

We summarize the irreducible representation of SU(2) and the multiplicity subspaces. We only discuss SU(2) since we only analyze qubit systems in this thesis. In this appendix, we use the correspondence between the computational basis state of qubits and standard angular momentum states of a spin-1/2 particles given $|0(1)\rangle \leftrightarrow |\frac{1}{2}, \frac{1}{2}(-\frac{1}{2})\rangle$. First we give the definition of the irreducible subspace of $U^{\otimes N}$ in SU(2), then give some remarks about the basis in the multiplicity subspace and the order of coupling of multiple spins.

We consider a $N$-qubits system. The Hilbert space of the system is denoted as $(\mathcal{H}^2)^{\otimes N}$ with $\dim \mathcal{H} = 2$, which can be decomposed as

$$(\mathcal{H}^2)^{\otimes N} = \bigoplus_{j=\lfloor N \rfloor}^{N/2} \mathcal{U}_j \otimes \mathcal{V}_j^{[N]}, \tag{B.1}$$

where $\mathcal{V}_j^{[N]}$ corresponds to the multiplicity subspace of the irreducible subspace $\mathcal{U}_j$ and $\lfloor N \rfloor$ is 0 when $N$ is even and $\lfloor N \rfloor$ is 1/2 when $N$ is odd. The dimension of $\mathcal{U}_j$ is $2j+1$, independent of $N$ and the basis is denoted as $\{|j,m\rangle\}$ with $m = -j, \cdots, j$. Note that for spin-1/2 systems $j$ is the total angular momentum and $m$ is the angular momentum along the $z$ axis.

For the tensor product of arbitrary unitary operator $U \in \mathrm{SU}(2)$, the irreducible representation of $U^{\otimes N}$ is given by

$$U^{\otimes N} = \bigoplus_{j=\lfloor N \rfloor}^{N/2} U_j \otimes I_{\mathcal{V}_j^{[N]}}, \tag{B.2}$$

where $U_j$ is the irreducible representation on $\mathcal{U}_j$ and $I_{\mathcal{V}_j^{[N]}}$ is the identity operator on $\mathcal{V}_j^{[N]}$. The dimension of the multiplicity subspaces is given by

$$\dim \mathcal{V}_j^{[N]} = \frac{N!(2j+1)}{(N/2-j)!(N/2+1+j)!}. \tag{B.3}$$

### Haar measure

We define the *Haar measure* $d\mu(U)$ for $U \in \mathrm{SU}(2)$. The Haar measure is the invariant measure, which means $d\mu(U) = d\mu(VU)$ for an arbitrary unitary operator $V \in \mathrm{SU}(2)$. In this thesis, we use integration over the Haar measure. We show a brief explanation for the calculation of the integral using Schur's lemma.

We consider the most simple case and introduce an operator $X$ on a qubit system $\mathcal{H}$. The operator averaged over the Haar measure $\widetilde{X}$ is defined as

$$\widetilde{X} = \int_U d\mu(U) U X U^\dagger. \tag{B.4}$$

For another unitary operator $V$, we have

$$V\widetilde{X}V^\dagger = \int_U d\mu(U) VUXU^\dagger V^\dagger \tag{B.5}$$

$$= \int_{U'} d\mu(V^\dagger U') U' X U'^\dagger \tag{B.6}$$

$$= \int_{U'} d\mu(U') U' X U'^\dagger \tag{B.7}$$

$$= \widetilde{X}, \tag{B.8}$$

where we defined $U' = VU$ and used $d\mu(U) = d\mu(WU)$ for an arbitrary unitary operator $V \in \mathrm{SU}(2)$ operation. Thus we obtain

$$[\widetilde{X}, V] = 0, \tag{B.9}$$

for an arbitrary unitary operator $V$. This commutation relation and Schur's lemma implies that $\widetilde{X}$ is proportional to the identity operator. The trace of an operator is preserved under the application of unitary operations and integration over the Haar measure. We now obtain

$$\widetilde{X} = \mathrm{Tr}[X]\frac{I}{2}, \tag{B.10}$$

where $I$ is the identity operator on $\mathcal{H}$.

An extension for integration with an $N$ tensor product of unitary operations is not difficult whereas particular attentions are paid for the multiplicity subspace for $N \geq 3$. Let $Y$ be an operator on an $N$-qubits system and $\widetilde{Y}$ be defined as

$$\widetilde{Y} = \int_U U^{\otimes N} Y U^{\otimes N}. \tag{B.11}$$

Similarly to the case of $N = 1$ case, $[\widetilde{Y}, V^{\otimes N}] = 0$ is obtained for an arbitrary unitary operator $V$. From this commutation relation and Schur's lemma, we obtain

$$\widetilde{Y} = \sum_{j=\lfloor N \rfloor}^{N/2} \mathrm{Tr}[P_{\mathcal{U}_j} Y] \frac{I_j}{d_j} \otimes P_{\mathcal{V}_j} Y P_{\mathcal{V}_j}, \tag{B.12}$$

where $P_{\mathcal{U}_j}$ and $P_{\mathcal{V}_j}$ are the projection onto the irreducible subspace $\mathcal{U}_j$ and the multiplicity subspace $\mathcal{V}_j$. Therefore nontrivial elements of $\widetilde{Y}$ are only on the multiplicity subspaces.

### Multiplicity subspace

To explain the dimension and basis in the multiplicity subspaces, let us consider the coupling of three spin-1/2 particles. First, by coupling two of the three spin-1/2 particles, we have $\mathcal{U}_{1/2} \otimes \mathcal{U}_{1/2} = \mathcal{U}_0 \oplus \mathcal{U}_1$. Then coupling with the other spin-1/2 particles, we obtain

$$\mathcal{U}_{1/2}^{\otimes 3} = (\mathcal{U}_0 \oplus \mathcal{U}_1) \otimes \mathcal{U}_{1/2} = \mathcal{U}_{1/2} \oplus \mathcal{U}_{1/2} \oplus \mathcal{U}_{3/2} = \mathcal{U}_{1/2} \otimes \mathcal{V}_{1/2} \oplus \mathcal{U}_{3/2}, \tag{B.13}$$

where $\mathcal{V}_{1/2}$ is the multiplicity subspace corresponding to the irreducible subspace with $j = 1/2$ and $N = 3$. Therefore the dimension of the multiplicity subspace with the total angular momentum $j$ is the number of the paths to obtain the total angular momentum $j$ in coupling of $N$ spin-1/2 particles. To define a basis in the multiplicity subspace, one can assign each path to the number $0, 1, \cdots$. For example of $N = 3$ and $j = 1/2$, the path $(1/2 \to 0 \to 1/2)$ is assigned to 0 and the path $(1/2 \to 1 \to 1/2)$ is 1 as illustrated in Fig. B.1.

For coupling three spin-1/2 particles it is necessary to choose which two spins-1/2 particles are among three spin-1/2 particles. A different basis is obtained when the order of coupling of multiple spin-1/2 particles is changed. For $N = 3$, there are three different orders of coupling, which are used in the proofs in Chap. 4. When three angular momenta $j_1, j_2$ and $j_3$ are coupled to give an angular momentum $j$, three ways of coupling would be represented as $|j_1 j_2 (j_{12}) j_3; jM\rangle, |j_1 j_2 j_3 (j_{23}); jM\rangle$ and $|j_1 j_3 (j_{13}) j_2; jM\rangle$. These three vectors are transformed to each other by unitary operations represented by using Wigner's $6j$ coefficients [66]. For our case, set $j_2 = 1/2$ and we obtain the following lemma.

**Lemma 6.** *Suppose that three angular momenta $j_1, 1/2$ and $j_3$ are coupled to give an angular momentum $j$ and the three ways of coupling are represented as $|j_1 1/2 (j_{12}) j_3; jM\rangle, |j_1 1/2 j_3 (j_{23}); jM\rangle$ and $|j_1 j_3 (j_{13}) 1/2; jm\rangle$. These three vectors*

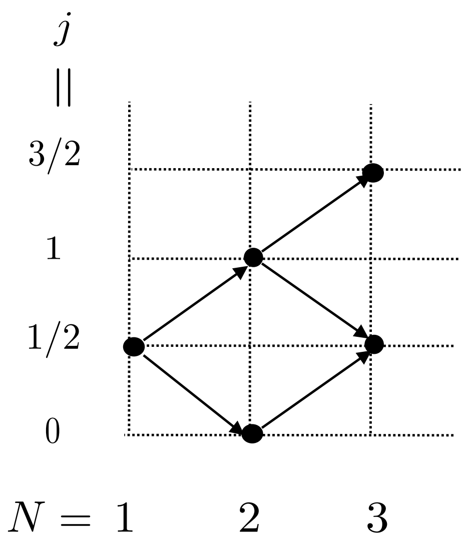


Figure B.1: The paths to couple three 1/2-spins. For $N = 3$ and $j = 1/2$ there are two paths, which corresponds to the multiplicity subspace.

*are related as*

$$|j_1\tfrac{1}{2}j_3(j_3-\tfrac{1}{2});jm\rangle = \tfrac{(-1)^{2(j_1+j_3+j)}}{\sqrt{(2j_3+1)(2j_1+1)}}\Big\{\sqrt{(j+\tfrac{1}{2})^2-(j_3-j_1)^2}|j_1\tfrac{1}{2}(j_1-\tfrac{1}{2})j_3;jm\rangle - \sqrt{(j_3+j_1+1)^2-(j+\tfrac{1}{2})^2}|j_1\tfrac{1}{2}(j_1+\tfrac{1}{2})j_3;jm\rangle\Big\}$$

$$|j_1\tfrac{1}{2}j_3(j_3+\tfrac{1}{2});jm\rangle = \tfrac{(-1)^{1+2(j_1+j_3+j)}}{\sqrt{(2j_3+1)(2j_1+1)}}\Big\{\sqrt{(j_3+j_1+1)^2-(j+\tfrac{1}{2})^2}|j_1\tfrac{1}{2}(j_1-\tfrac{1}{2})j_3;jm\rangle + \sqrt{(j+\tfrac{1}{2})^2-(j_3-j_1)^2}|j_1\tfrac{1}{2}(j_1+\tfrac{1}{2})j_3;jm\rangle\Big\} \tag{B.14}$$

$$|j_1j_3(j_{13})\tfrac{1}{2};j_{13}+\tfrac{1}{2}\ m\rangle = \tfrac{(-1)^{2(j_1+j_3+j_{13})}}{\sqrt{(2j_{13}+2)(2j_1+1)}}\Big\{\sqrt{(j_3+\tfrac{1}{2})^2-(j_{13}-j_1+\tfrac{1}{2})^2}|j_1\tfrac{1}{2}(j_1-\tfrac{1}{2})j_3;j_{13}+\tfrac{1}{2}m\rangle + \sqrt{(j_{13}+j_1+3/2)^2-(j_3+\tfrac{1}{2})^2}|j_1\tfrac{1}{2}(j_1+\tfrac{1}{2})j_3;j_{13}+\tfrac{1}{2}m\rangle\Big\} \tag{B.15}$$

$$|j_1j_3(j_{13}+1)\tfrac{1}{2};j_{13}+\tfrac{1}{2}\ m\rangle = \tfrac{(-1)^{2(j_1+j_3+j_{13})}}{\sqrt{(2j_{13}+2)(2j_1+1)}}\Big\{\sqrt{(j_{13}+j_1+\tfrac{3}{2})^2-(j_3+\tfrac{1}{2})^2}|j_1\tfrac{1}{2}(j_1-\tfrac{1}{2})j_3;j_{13}+\tfrac{1}{2}m\rangle - \sqrt{(j_3+\tfrac{1}{2})^2-(j_{13}-j_1+\tfrac{1}{2})^2}|j_1\tfrac{1}{2}(j_1+\tfrac{1}{2})j_3;j_{13}+\tfrac{1}{2}M\rangle\Big\}. \tag{B.16}$$

# Appendix C

# Calculations of Choi operators in Prop. 3 for $N = 3, 4$

In this section, we give the explicit calculations for the Choi operators defined in Equation (5.3) for $N = 3, 4$. We use the same notations for the Hilbelt spaces in Sec. 5.1.3.

**N = 3**

The Choi operator $M_i$ is represented as

$$M_1 = \eta^{[4]}_{\mathcal{H}_{R_1}\mathcal{K}_{R_1}\mathcal{H}_T\mathcal{K}_T} \otimes \eta^{[3]}_{\mathcal{H}_{R_2}\mathcal{K}_{R_2}}, \tag{C.1}$$
$$M_2 = \eta^{[3]}_{\mathcal{H}_{R_1}\mathcal{K}_{R_1}} \otimes \eta^{[4]}_{\mathcal{H}_{R_2}\mathcal{K}_{R_2}\mathcal{H}_T\mathcal{K}_T}, \tag{C.2}$$

where the definition of $\eta^{[N]}$ is given in Equation (4.74). We have

$$\eta^{[4]}_{\mathcal{H}_{R_1}\mathcal{K}_{R_1}\mathcal{H}_T\mathcal{K}_T} = I_0 \otimes I_0 \otimes \left[|\alpha^{[3]}_{\frac{1}{2}}\rangle\rangle \otimes |\alpha^{[1]}_{\frac{1}{2}}\rangle\rangle\right], \tag{C.3}$$
$$\oplus \tfrac{I_1}{3} \otimes I_1 \otimes \left[(|\alpha^{[3]}_{\frac{1}{2}}\rangle\rangle + |\alpha^{[3]}_{\frac{1}{2}}\rangle\rangle) \otimes |\alpha^{[1]}_{\frac{1}{2}}\rangle\rangle\right], \tag{C.4}$$
$$\oplus \tfrac{I_2}{5} \otimes I_2 \otimes \left[|\alpha^{[3]}_{\frac{3}{2}}\rangle\rangle \otimes |\alpha^{[1]}_{\frac{1}{2}}\rangle\rangle\right], \tag{C.5}$$

and

$$\eta^{[3]}_{\mathcal{H}_{R_2}\mathcal{K}_{R_2}} = \tfrac{I_{1/2}}{2} \otimes I_{1/2} \otimes \left[|\alpha^{[3]}_{\frac{1}{2}}\rangle\rangle\right] \oplus \tfrac{I_{3/2}}{4} \otimes I_{3/2} \otimes \left[|\alpha^{[3]}_{\frac{3}{2}}\rangle\rangle\right] \tag{C.6}$$

Then we obtain

$$M^{[1]}_{\frac{1}{2}\frac{1}{2}} = \left[|00\rangle_{\frac{1}{2}\frac{1}{2}}\right] + \frac{1}{3}\left[|11\rangle_{\frac{1}{2}\frac{1}{2}} + |33\rangle_{\frac{1}{2}\frac{1}{2}}\right] + \frac{1}{6}\left[|22\rangle_{\frac{1}{2}\frac{1}{2}} + |44\rangle_{\frac{1}{2}\frac{1}{2}}\right] + \frac{1}{10}\left[|55\rangle_{\frac{1}{2}\frac{1}{2}}\right], \tag{C.7}$$

$$M^{[1]}_{\frac{1}{2}\frac{3}{2}} = \frac{1}{2}\left[|10\rangle_{\frac{1}{2}\frac{3}{2}} + |33\rangle_{\frac{1}{2}\frac{3}{2}}\right] + \frac{1}{6}\left[|22\rangle_{\frac{1}{2}\frac{3}{2}} + |45\rangle_{\frac{1}{2}\frac{3}{2}}\right] + \frac{1}{10}\left[|56\rangle_{\frac{1}{2}\frac{3}{2}}\right], \tag{C.8}$$

$$M^{[1]}_{\frac{1}{2}\frac{5}{2}} = \frac{1}{6}\left[|20\rangle_{\frac{1}{2}\frac{5}{2}} + |42\rangle_{\frac{1}{2}\frac{5}{2}}\right] + \frac{1}{10}\left[|53\rangle_{\frac{1}{2}\frac{5}{2}}\right], \tag{C.9}$$

$$M^{[1]}_{\frac{1}{2}\frac{7}{2}} = \frac{1}{10}\left[|50\rangle\right], \tag{C.10}$$

$$M^{[1]}_{\frac{3}{2}\frac{1}{2}} = \frac{2}{3}\left[|01\rangle_{\frac{3}{2}\frac{1}{2}} + |33\rangle_{\frac{3}{2}\frac{1}{2}}\right] + \frac{1}{3}\left[|22\rangle_{\frac{3}{2}\frac{1}{2}} + |54\rangle_{\frac{3}{2}\frac{1}{2}}\right] + \frac{1}{10}\left[|65\rangle_{\frac{3}{2}\frac{1}{2}}\right], \tag{C.11}$$

$$\begin{aligned} M^{[1]}_{\frac{3}{2}\frac{3}{2}} = {} & \left[|11\rangle_{\frac{3}{2}\frac{3}{2}}\right] + \frac{2}{3}\left[|00\rangle_{\frac{3}{2}\frac{3}{2}} + |33\rangle_{\frac{3}{2}\frac{3}{2}}\right] + \frac{1}{3}\left[|22\rangle_{\frac{3}{2}\frac{3}{2}} + |55\rangle_{\frac{3}{2}\frac{3}{2}}\right], \\ & + \frac{2}{5}\left[|44\rangle_{\frac{3}{2}\frac{3}{2}}\right] + \frac{1}{5}\left[|66\rangle_{\frac{3}{2}\frac{3}{2}}\right], \end{aligned} \tag{C.12}$$

$$M^{[1]}_{\frac{3}{2}\frac{5}{2}} = \frac{1}{3}\left[|20\rangle_{\frac{3}{2}\frac{5}{2}} + |52\rangle_{\frac{3}{2}\frac{5}{2}}\right] + \frac{2}{5}\left[|41\rangle_{\frac{3}{2}\frac{5}{2}}\right] + \frac{1}{5}\left[|63\rangle_{\frac{3}{2}\frac{5}{2}}\right], \tag{C.13}$$

$$M^{[1]}_{\frac{3}{2}\frac{7}{2}} = \frac{1}{5}\left[|60\rangle_{\frac{3}{2}\frac{7}{2}}\right], \tag{C.14}$$

$$M^{[1]}_{\frac{5}{2}\frac{1}{2}} = \frac{1}{2}\left[|02\rangle_{\frac{5}{2}\frac{1}{2}} + |42\rangle\right] + \frac{3}{10}\left[|35\rangle_{\frac{5}{2}\frac{1}{2}}\right], \tag{C.15}$$

$$M^{[1]}_{\frac{5}{2}\frac{3}{2}} = \frac{1}{2}\left[|02\rangle_{\frac{5}{2}\frac{3}{2}} + |25\rangle_{\frac{5}{2}\frac{3}{2}}\right] + \frac{3}{5}\left[|14\rangle_{\frac{5}{2}\frac{3}{2}}\right] + \frac{3}{10}\left[|36\rangle\right], \tag{C.16}$$

$$M^{[1]}_{\frac{5}{2}\frac{5}{2}} = \frac{1}{2}\left[|00\rangle_{\frac{5}{2}\frac{5}{2}} + |22\rangle_{\frac{5}{2}\frac{5}{2}}\right] + \frac{3}{5}\left[|11\rangle_{\frac{5}{2}\frac{5}{2}}\right] + \frac{3}{10}\left[|33\rangle_{\frac{5}{2}\frac{5}{2}}\right], \tag{C.17}$$

$$M^{[1]}_{\frac{5}{2}\frac{7}{2}} = \frac{3}{10}\left[|30\rangle_{\frac{5}{2}\frac{7}{2}}\right], \tag{C.18}$$

$$M^{[1]}_{\frac{7}{2}\frac{1}{2}} = \frac{2}{5}\left[|05\rangle_{\frac{7}{2}\frac{1}{2}}\right], \tag{C.19}$$

$$M^{[1]}_{\frac{7}{2}\frac{3}{2}} = \frac{2}{5}\left[|06\rangle_{\frac{7}{2}\frac{3}{2}}\right], \tag{C.20}$$

$$M^{[1]}_{\frac{7}{2}\frac{5}{2}} = \frac{2}{5}\left[|03\rangle_{\frac{7}{2}\frac{5}{2}}\right], \tag{C.21}$$

$$M^{[1]}_{\frac{7}{2}\frac{7}{2}} = \frac{2}{5}\left[|00\rangle_{\frac{7}{2}\frac{7}{2}}\right], \tag{C.22}$$

where we defined the basis states for the multiplicity subspaces $\mathcal{V}^{[7]}_{\frac{1}{2}}$ as

$$\textstyle\sum_{m=-1/2}^{1/2} |\frac{1}{2}\frac{1}{2}(0)\mathcal{K}_1\mathcal{K}_T\,\frac{1}{2};\frac{1}{2}\rangle\langle\frac{1}{2}\frac{1}{2}(0)\mathcal{K}_1\mathcal{K}_T\,\frac{1}{2};\frac{1}{2}| =: I_{\frac{1}{2}} \otimes |0\rangle\langle 0|_{\frac{1}{2}}, \tag{C.23}$$

$$\textstyle\sum_{m=-1/2}^{1/2} |\frac{1}{2}\frac{1}{2}(1)\mathcal{K}_1\mathcal{K}_T\,\frac{1}{2};\frac{1}{2}\rangle\langle\frac{1}{2}\frac{1}{2}(1)\mathcal{K}_1\mathcal{K}_T\,\frac{1}{2};\frac{1}{2}| =: I_{\frac{1}{2}} \otimes |1\rangle\langle 1|_{\frac{1}{2}}, \tag{C.24}$$

$$\textstyle\sum_{m=-1/2}^{1/2} |\frac{1}{2}\frac{1}{2}(1)\mathcal{K}_1\mathcal{K}_T\,\frac{3}{2};\frac{1}{2}\rangle\langle\frac{1}{2}\frac{1}{2}(1)\mathcal{K}_1\mathcal{K}_T\,\frac{3}{2};\frac{1}{2}| =: I_{\frac{1}{2}} \otimes |2\rangle\langle 2|_{\frac{1}{2}}, \tag{C.25}$$

$$\textstyle\sum_{m=-1/2}^{1/2} |\frac{3}{2}\frac{1}{2}(1)_{\mathcal{K}_1\mathcal{K}_T}\frac{1}{2};\frac{1}{2}\rangle\langle\frac{3}{2}\frac{1}{2}(1)_{\mathcal{K}_1\mathcal{K}_T}\frac{1}{2};\frac{1}{2}| =: I_{\frac{1}{2}} \otimes |3\rangle\langle 3|_{\frac{1}{2}}, \tag{C.26}$$

$$\textstyle\sum_{m=-1/2}^{1/2} |\frac{3}{2}\frac{1}{2}(1)_{\mathcal{K}_1\mathcal{K}_T}\frac{3}{2};\frac{1}{2}\rangle\langle\frac{3}{2}\frac{1}{2}(1)_{\mathcal{K}_1\mathcal{K}_T}\frac{3}{2};\frac{1}{2}| =: I_{\frac{1}{2}} \otimes |4\rangle\langle 4|_{\frac{1}{2}}, \tag{C.27}$$

$$\textstyle\sum_{m=-1/2}^{1/2} |\frac{3}{2}\frac{1}{2}(2)_{\mathcal{K}_1\mathcal{K}_T}\frac{3}{2};\frac{1}{2}\rangle\langle\frac{3}{2}\frac{1}{2}(2)_{\mathcal{K}_1\mathcal{K}_T}\frac{3}{2};\frac{1}{2}| =: I_{\frac{1}{2}} \otimes |5\rangle\langle 5|_{\frac{1}{2}}, \tag{C.28}$$

and for $\mathcal{V}_{\frac{3}{2}}^{[7]}$ as

$$\textstyle\sum_{m=-3/2}^{3/2} |\frac{1}{2}\frac{1}{2}(1)_{\mathcal{K}_1\mathcal{K}_T}\frac{1}{2};\frac{3}{2}\rangle\langle\frac{1}{2}\frac{1}{2}(1)_{\mathcal{K}_1\mathcal{K}_T}\frac{1}{2};\frac{3}{2}| =: I_{\frac{3}{2}} \otimes |0\rangle\langle 0|_{\frac{3}{2}}, \tag{C.29}$$

$$\textstyle\sum_{m=-3/2}^{3/2} |\frac{1}{2}\frac{1}{2}(0)_{\mathcal{K}_1\mathcal{K}_T}\frac{3}{2};\frac{3}{2}\rangle\langle\frac{1}{2}\frac{1}{2}(0)_{\mathcal{K}_1\mathcal{K}_T}\frac{3}{2};\frac{3}{2}| =: I_{\frac{3}{2}} \otimes |1\rangle\langle 1|_{\frac{3}{2}}, \tag{C.30}$$

$$\textstyle\sum_{m=-3/2}^{3/2} |\frac{1}{2}\frac{1}{2}(1)_{\mathcal{K}_1\mathcal{K}_T}\frac{3}{2};\frac{3}{2}\rangle\langle\frac{1}{2}\frac{1}{2}(1)_{\mathcal{K}_1\mathcal{K}_T}\frac{3}{2};\frac{3}{2}| =: I_{\frac{3}{2}} \otimes |2\rangle\langle 2|_{\frac{3}{2}}, \tag{C.31}$$

$$\textstyle\sum_{m=-3/2}^{3/2} |\frac{3}{2}\frac{1}{2}(1)_{\mathcal{K}_1\mathcal{K}_T}\frac{1}{2};\frac{3}{2}\rangle\langle\frac{3}{2}\frac{1}{2}(1)_{\mathcal{K}_1\mathcal{K}_T}\frac{1}{2};\frac{3}{2}| =: I_{\frac{3}{2}} \otimes |3\rangle\langle 3|_{\frac{3}{2}}, \tag{C.32}$$

$$\textstyle\sum_{m=-3/2}^{3/2} |\frac{3}{2}\frac{1}{2}(2)_{\mathcal{K}_1\mathcal{K}_T}\frac{1}{2};\frac{3}{2}\rangle\langle\frac{3}{2}\frac{1}{2}(2)_{\mathcal{K}_1\mathcal{K}_T}\frac{1}{2};\frac{3}{2}| =: I_{\frac{3}{2}} \otimes |4\rangle\langle 4|_{\frac{3}{2}}, \tag{C.33}$$

$$\textstyle\sum_{m=-3/2}^{3/2} |\frac{3}{2}\frac{1}{2}(1)_{\mathcal{K}_1\mathcal{K}_T}\frac{3}{2};\frac{3}{2}\rangle\langle\frac{3}{2}\frac{1}{2}(1)_{\mathcal{K}_1\mathcal{K}_T}\frac{3}{2};\frac{3}{2}| =: I_{\frac{3}{2}} \otimes |5\rangle\langle 5|_{\frac{3}{2}}, \tag{C.34}$$

$$\textstyle\sum_{m=-3/2}^{3/2} |\frac{3}{2}\frac{1}{2}(2)_{\mathcal{K}_1\mathcal{K}_T}\frac{3}{2};\frac{3}{2}\rangle\langle\frac{3}{2}\frac{1}{2}(2)_{\mathcal{K}_1\mathcal{K}_T}\frac{3}{2};\frac{3}{2}| =: I_{\frac{3}{2}} \otimes |6\rangle\langle 6|_{\frac{3}{2}}, \tag{C.35}$$

and for $\mathcal{V}_{\frac{5}{2}}^{[7]}$ as

$$\textstyle\sum_{m=-5/2}^{5/2} |\frac{1}{2}\frac{1}{2}(1)_{\mathcal{K}_1\mathcal{K}_T}\frac{3}{2};\frac{5}{2}\rangle\langle\frac{1}{2}\frac{1}{2}(1)_{\mathcal{K}_1\mathcal{K}_T}\frac{3}{2};\frac{5}{2}| =: I_{\frac{5}{2}} \otimes |0\rangle\langle 0|_{\frac{5}{2}}, \tag{C.36}$$

$$\textstyle\sum_{m=-5/2}^{5/2} |\frac{3}{2}\frac{1}{2}(2)_{\mathcal{K}_1\mathcal{K}_T}\frac{1}{2};\frac{5}{2}\rangle\langle\frac{3}{2}\frac{1}{2}(2)_{\mathcal{K}_1\mathcal{K}_T}\frac{1}{2};\frac{5}{2}| =: I_{\frac{5}{2}} \otimes |1\rangle\langle 1|_{\frac{5}{2}}, \tag{C.37}$$

$$\textstyle\sum_{m=-5/2}^{5/2} |\frac{3}{2}\frac{1}{2}(1)_{\mathcal{K}_1\mathcal{K}_T}\frac{3}{2};\frac{5}{2}\rangle\langle\frac{3}{2}\frac{1}{2}(1)_{\mathcal{K}_1\mathcal{K}_T}\frac{3}{2};\frac{5}{2}| =: I_{\frac{5}{2}} \otimes |2\rangle\langle 2|_{\frac{5}{2}}, \tag{C.38}$$

$$\textstyle\sum_{m=-5/2}^{5/2} |\frac{3}{2}\frac{1}{2}(2)_{\mathcal{K}_1\mathcal{K}_T}\frac{3}{2};\frac{5}{2}\rangle\langle\frac{3}{2}\frac{1}{2}(2)_{\mathcal{K}_1\mathcal{K}_T}\frac{3}{2};\frac{5}{2}| =: I_{\frac{5}{2}} \otimes |3\rangle\langle 3|_{\frac{5}{2}}, \tag{C.39}$$

and for $\mathcal{V}_{\frac{7}{2}}^{[7]}$ as

$$\textstyle\sum_{m=-7/2}^{7/2} |\frac{3}{2}\frac{1}{2}(2)_{\mathcal{K}_1\mathcal{K}_T}\frac{3}{2};\frac{7}{2}\rangle\langle\frac{3}{2}\frac{1}{2}(2)_{\mathcal{K}_1\mathcal{K}_T}\frac{3}{2};\frac{7}{2}| =: I_{\frac{7}{2}} \otimes |0\rangle\langle 0|_{\frac{7}{2}}. \tag{C.40}$$

For $M_2$, we have

$$M_{\frac{1}{2}\frac{1}{2}}^{[1]} = \left[|\hat{0}\hat{0}\rangle_{\frac{1}{2}\frac{1}{2}}\right] + \frac{1}{3}\left[|\hat{1}\hat{1}\rangle_{\frac{1}{2}\frac{1}{2}} + |\hat{3}\hat{3}\rangle_{\frac{1}{2}\frac{1}{2}}\right] + \frac{1}{6}\left[|\hat{2}\hat{2}\rangle_{\frac{1}{2}\frac{1}{2}} + |\hat{4}\hat{4}\rangle_{\frac{1}{2}\frac{1}{2}}\right] + \frac{1}{10}\left[|\hat{5}\hat{5}\rangle_{\frac{1}{2}\frac{1}{2}}\right], \tag{C.41}$$

$$M_{\frac{1}{2}\frac{3}{2}}^{[1]} = \frac{1}{2}\left[|\hat{1}\hat{0}\rangle_{\frac{1}{2}\frac{3}{2}} + |\hat{3}\hat{3}\rangle_{\frac{1}{2}\frac{3}{2}}\right] + \frac{1}{6}\left[|\hat{2}\hat{2}\rangle_{\frac{1}{2}\frac{3}{2}} + |\hat{4}\hat{5}\rangle_{\frac{1}{2}\frac{3}{2}}\right] + \frac{1}{10}\left[|\hat{5}\hat{6}\rangle_{\frac{1}{2}\frac{3}{2}}\right], \tag{C.42}$$

$$M_{\frac{1}{2}\frac{5}{2}}^{[1]} = \frac{1}{6}\left[|\hat{2}\hat{0}\rangle_{\frac{1}{2}\frac{5}{2}} + |\hat{4}\hat{2}\rangle_{\frac{1}{2}\frac{5}{2}}\right] + \frac{1}{10}\left[|\hat{5}\hat{3}\rangle_{\frac{1}{2}\frac{5}{2}}\right], \tag{C.43}$$

$$M_{\frac{1}{2}\frac{7}{2}}^{[1]} = \frac{1}{10}\left[|\hat{5}\hat{0}\rangle\right], \tag{C.44}$$

$$M_{\frac{3}{2}\frac{1}{2}}^{[1]} = \frac{2}{3}\left[|\hat{0}\hat{1}\rangle_{\frac{3}{2}\frac{1}{2}} + |\hat{3}\hat{3}\rangle_{\frac{3}{2}\frac{1}{2}}\right] + \frac{1}{3}\left[|\hat{2}\hat{2}\rangle_{\frac{3}{2}\frac{1}{2}} + |\hat{5}\hat{4}\rangle_{\frac{3}{2}\frac{1}{2}}\right] + \frac{1}{10}\left[|\hat{6}\hat{5}\rangle_{\frac{3}{2}\frac{1}{2}}\right], \tag{C.45}$$

$$M^{[1]}_{\frac{3}{2}\frac{3}{2}} = \left[|\hat{1}\hat{1}\rangle_{\frac{3}{2}\frac{3}{2}}\right] + \frac{2}{3}\left[|\hat{0}\hat{0}\rangle_{\frac{3}{2}\frac{3}{2}} + |\hat{3}\hat{3}\rangle_{\frac{3}{2}\frac{3}{2}}\right] + \frac{1}{3}\left[|\hat{2}\hat{2}\rangle_{\frac{3}{2}\frac{3}{2}} + |\hat{5}\hat{5}\rangle_{\frac{3}{2}\frac{3}{2}}\right] + \frac{2}{5}\left[|\hat{4}\hat{4}\rangle_{\frac{3}{2}\frac{3}{2}}\right] + \frac{1}{5}\left[|\hat{6}\hat{6}\rangle_{\frac{3}{2}\frac{3}{2}}\right], \quad (C.46)$$

$$M^{[1]}_{\frac{3}{2}\frac{5}{2}} = \frac{1}{3}\left[|\hat{2}\hat{0}\rangle_{\frac{3}{2}\frac{5}{2}} + |\hat{5}\hat{2}\rangle_{\frac{3}{2}\frac{5}{2}}\right] + \frac{2}{5}\left[|\hat{4}\hat{1}\rangle_{\frac{3}{2}\frac{5}{2}}\right] + \frac{1}{5}\left[|\hat{6}\hat{3}\rangle_{\frac{3}{2}\frac{5}{2}}\right], \quad (C.47)$$

$$M^{[1]}_{\frac{3}{2}\frac{7}{2}} = \frac{1}{5}\left[|\hat{6}\hat{0}\rangle_{\frac{3}{2}\frac{7}{2}}\right], \quad (C.48)$$

$$M^{[1]}_{\frac{5}{2}\frac{1}{2}} = \frac{1}{2}\left[|\hat{0}\hat{2}\rangle_{\frac{5}{2}\frac{1}{2}} + |\hat{4}\hat{2}\rangle\right] + \frac{3}{10}\left[|\hat{3}\hat{5}\rangle_{\frac{5}{2}\frac{1}{2}}\right], \quad (C.49)$$

$$M^{[1]}_{\frac{5}{2}\frac{3}{2}} = \frac{1}{2}\left[|\hat{0}\hat{2}\rangle_{\frac{5}{2}\frac{3}{2}} + |\hat{2}\hat{5}\rangle_{\frac{5}{2}\frac{3}{2}}\right] + \frac{3}{5}\left[|\hat{1}\hat{4}\rangle_{\frac{5}{2}\frac{3}{2}}\right] + \frac{3}{10}\left[|\hat{3}\hat{6}\rangle\right], \quad (C.50)$$

$$M^{[1]}_{\frac{5}{2}\frac{5}{2}} = \frac{1}{2}\left[|\hat{0}\hat{0}\rangle_{\frac{5}{2}\frac{5}{2}} + |\hat{2}\hat{2}\rangle_{\frac{5}{2}\frac{5}{2}}\right] + \frac{3}{5}\left[|\hat{1}\hat{1}\rangle_{\frac{5}{2}\frac{5}{2}}\right] + \frac{3}{10}\left[|\hat{3}\hat{3}\rangle_{\frac{5}{2}\frac{5}{2}}\right], \quad (C.51)$$

$$M^{[1]}_{\frac{5}{2}\frac{7}{2}} = \frac{3}{10}\left[|\hat{3}\hat{0}\rangle_{\frac{5}{2}\frac{7}{2}}\right], \quad (C.52)$$

$$M^{[1]}_{\frac{7}{2}\frac{1}{2}} = \frac{2}{5}\left[|\hat{0}\hat{5}\rangle_{\frac{7}{2}\frac{1}{2}}\right], \quad (C.53)$$

$$M^{[1]}_{\frac{7}{2}\frac{3}{2}} = \frac{2}{5}\left[|\hat{0}\hat{6}\rangle_{\frac{7}{2}\frac{3}{2}}\right], \quad (C.54)$$

$$M^{[1]}_{\frac{7}{2}\frac{5}{2}} = \frac{2}{5}\left[|\hat{0}\hat{3}\rangle_{\frac{7}{2}\frac{5}{2}}\right], \quad (C.55)$$

$$M^{[1]}_{\frac{7}{2}\frac{7}{2}} = \frac{2}{5}\left[|\hat{0}\hat{0}\rangle_{\frac{7}{2}\frac{7}{2}}\right], \quad (C.56)$$

where we defined the basis states for the multiplicity subspaces $\mathcal{V}^{[7]}_{\frac{1}{2}}$ as

$$\textstyle\sum_{m=-1/2}^{1/2} |\frac{1}{2}\frac{1}{2}\frac{1}{2}(0)_{\mathcal{K}_T\mathcal{K}_2}; \frac{1}{2}\rangle\langle\frac{1}{2}\frac{1}{2}\frac{1}{2}(0)_{\mathcal{K}_T\mathcal{K}_2}; \frac{1}{2}| =: I_{\frac{1}{2}} \otimes |\hat{0}\rangle\langle\hat{0}|_{\frac{1}{2}}, \quad (C.57)$$

$$\textstyle\sum_{m=-1/2}^{1/2} |\frac{1}{2}\frac{1}{2}\frac{1}{2}(1)_{\mathcal{K}_T\mathcal{K}_2}; \frac{1}{2}\rangle\langle\frac{1}{2}\frac{1}{2}\frac{1}{2}(1)_{\mathcal{K}_T\mathcal{K}_2}; \frac{1}{2}| =: I_{\frac{1}{2}} \otimes |\hat{1}\rangle\langle\hat{1}|_{\frac{1}{2}}, \quad (C.58)$$

$$\textstyle\sum_{m=-1/2}^{1/2} |\frac{3}{2}\frac{1}{2}\frac{1}{2}(1)_{\mathcal{K}_T\mathcal{K}_2}; \frac{1}{2}\rangle\langle\frac{3}{2}\frac{1}{2}\frac{1}{2}(1)_{\mathcal{K}_T\mathcal{K}_2}; \frac{1}{2}| =: I_{\frac{1}{2}} \otimes |\hat{2}\rangle\langle\hat{2}|_{\frac{1}{2}}, \quad (C.59)$$

$$\textstyle\sum_{m=-1/2}^{1/2} |\frac{1}{2}\frac{1}{2}\frac{3}{2}(1)_{\mathcal{K}_T\mathcal{K}_2}; \frac{1}{2}\rangle\langle\frac{1}{2}\frac{1}{2}\frac{3}{2}(1)_{\mathcal{K}_T\mathcal{K}_2}; \frac{1}{2}| =: I_{\frac{1}{2}} \otimes |\hat{3}\rangle\langle\hat{3}|_{\frac{1}{2}}, \quad (C.60)$$

$$\textstyle\sum_{m=-1/2}^{1/2} |\frac{3}{2}\frac{1}{2}\frac{3}{2}(1)_{\mathcal{K}_T\mathcal{K}_2}; \frac{1}{2}\rangle\langle\frac{3}{2}\frac{1}{2}\frac{3}{2}(1)_{\mathcal{K}_T\mathcal{K}_2}; \frac{1}{2}| =: I_{\frac{1}{2}} \otimes |\hat{4}\rangle\langle\hat{4}|_{\frac{1}{2}}, \quad (C.61)$$

$$\textstyle\sum_{m=-1/2}^{1/2} |\frac{3}{2}\frac{1}{2}\frac{3}{2}(2)_{\mathcal{K}_T\mathcal{K}_2}; \frac{1}{2}\rangle\langle\frac{3}{2}\frac{1}{2}\frac{3}{2}(2)_{\mathcal{K}_T\mathcal{K}_2}; \frac{1}{2}| =: I_{\frac{1}{2}} \otimes |\hat{4}\rangle\langle\hat{4}|_{\frac{1}{2}}, \quad (C.62)$$

(C.63)

and for $\mathcal{V}^{[7]}_{\frac{3}{2}}$ as

$$\textstyle\sum_{m=-3/2}^{3/2} |\frac{1}{2}\frac{1}{2}\frac{1}{2}(1)_{\mathcal{K}_T\mathcal{K}_2}; \frac{3}{2}\rangle\langle\frac{1}{2}\frac{1}{2}\frac{1}{2}(1)_{\mathcal{K}_T\mathcal{K}_2}; \frac{3}{2}| =: I_{\frac{3}{2}} \otimes |\hat{0}\rangle\langle\hat{0}|_{\frac{3}{2}}, \quad (C.64)$$

$$\textstyle\sum_{m=-3/2}^{3/2} |\frac{3}{2}\frac{1}{2}\frac{1}{2}(0)_{\mathcal{K}_T\mathcal{K}_2}; \frac{3}{2}\rangle\langle\frac{3}{2}\frac{1}{2}\frac{1}{2}(0)_{\mathcal{K}_T\mathcal{K}_2}; \frac{3}{2}| =: I_{\frac{3}{2}} \otimes |\hat{1}\rangle\langle\hat{1}|_{\frac{3}{2}}, \quad (C.65)$$

$$\textstyle\sum_{m=-3/2}^{3/2} |\frac{3}{2}\frac{1}{2}\frac{1}{2}(1)_{\mathcal{K}_T\mathcal{K}_2}; \frac{3}{2}\rangle\langle\frac{3}{2}\frac{1}{2}\frac{1}{2}(1)_{\mathcal{K}_T\mathcal{K}_2}; \frac{3}{2}| =: I_{\frac{3}{2}} \otimes |\hat{2}\rangle\langle\hat{2}|_{\frac{3}{2}}, \quad (C.66)$$

$$\sum_{m=-3/2}^{3/2} |\tfrac{1}{2}\tfrac{1}{2}\tfrac{3}{2}(1)_{\mathcal{K}_T\mathcal{K}_2}; \tfrac{3}{2}\rangle\langle\tfrac{1}{2}\tfrac{1}{2}\tfrac{3}{2}(1)_{\mathcal{K}_T\mathcal{K}_2}; \tfrac{3}{2}| =: I_{\frac{3}{2}} \otimes |\hat{3}\rangle\langle\hat{3}|_{\frac{3}{2}}, \tag{C.67}$$

$$\sum_{m=-3/2}^{3/2} |\tfrac{1}{2}\tfrac{1}{2}\tfrac{3}{2}(2)_{\mathcal{K}_T\mathcal{K}_2}; \tfrac{3}{2}\rangle\langle\tfrac{1}{2}\tfrac{1}{2}\tfrac{3}{2}(2)_{\mathcal{K}_T\mathcal{K}_2}; \tfrac{3}{2}| =: I_{\frac{3}{2}} \otimes |\hat{4}\rangle\langle\hat{4}|_{\frac{3}{2}}, \tag{C.68}$$

$$\sum_{m=-3/2}^{3/2} |\tfrac{3}{2}\tfrac{1}{2}\tfrac{3}{2}(1)_{\mathcal{K}_T\mathcal{K}_2}; \tfrac{3}{2}\rangle\langle\tfrac{3}{2}\tfrac{1}{2}\tfrac{3}{2}(1)_{\mathcal{K}_T\mathcal{K}_2}; \tfrac{3}{2}| =: I_{\frac{3}{2}} \otimes |\hat{5}\rangle\langle\hat{5}|_{\frac{3}{2}}, \tag{C.69}$$

$$\sum_{m=-3/2}^{3/2} |\tfrac{3}{2}\tfrac{1}{2}\tfrac{3}{2}(2)_{\mathcal{K}_T\mathcal{K}_2}; \tfrac{3}{2}\rangle\langle\tfrac{3}{2}\tfrac{1}{2}\tfrac{3}{2}(2)_{\mathcal{K}_T\mathcal{K}_2}; \tfrac{3}{2}| =: I_{\frac{3}{2}} \otimes |\hat{6}\rangle\langle\hat{6}|_{\frac{3}{2}}, \tag{C.70}$$

and for $\mathcal{V}_{\frac{5}{2}}^{[7]}$ as

$$\sum_{m=-5/2}^{5/2} |\tfrac{3}{2}\tfrac{1}{2}\tfrac{1}{2}(1)_{\mathcal{K}_T\mathcal{K}_2}; \tfrac{5}{2}\rangle\langle\tfrac{3}{2}\tfrac{1}{2}\tfrac{1}{2}(1)_{\mathcal{K}_T\mathcal{K}_2}; \tfrac{5}{2}| =: I_{\frac{5}{2}} \otimes |\hat{0}\rangle\langle\hat{0}|_{\frac{5}{2}}, \tag{C.71}$$

$$\sum_{m=-5/2}^{5/2} |\tfrac{1}{2}\tfrac{1}{2}\tfrac{3}{2}(2)_{\mathcal{K}_T\mathcal{K}_2}; \tfrac{5}{2}\rangle\langle\tfrac{1}{2}\tfrac{1}{2}\tfrac{3}{2}(2)_{\mathcal{K}_T\mathcal{K}_2}; \tfrac{5}{2}| =: I_{\frac{5}{2}} \otimes |\hat{1}\rangle\langle\hat{1}|_{\frac{5}{2}}, \tag{C.72}$$

$$\sum_{m=-5/2}^{5/2} |\tfrac{3}{2}\tfrac{1}{2}\tfrac{3}{2}(1)_{\mathcal{K}_T\mathcal{K}_2}; \tfrac{5}{2}\rangle\langle\tfrac{3}{2}\tfrac{1}{2}\tfrac{3}{2}(1)_{\mathcal{K}_T\mathcal{K}_2}; \tfrac{5}{2}| =: I_{\frac{5}{2}} \otimes |\hat{2}\rangle\langle\hat{2}|_{\frac{5}{2}}, \tag{C.73}$$

$$\sum_{m=-5/2}^{5/2} |\tfrac{3}{2}\tfrac{1}{2}\tfrac{3}{2}(2)_{\mathcal{K}_T\mathcal{K}_2}; \tfrac{5}{2}\rangle\langle\tfrac{3}{2}\tfrac{1}{2}\tfrac{3}{2}(2)_{\mathcal{K}_T\mathcal{K}_2}; \tfrac{5}{2}| =: I_{\frac{5}{2}} \otimes |\hat{3}\rangle\langle\hat{3}|_{\frac{5}{2}}, \tag{C.74}$$

and for $\mathcal{V}_{\frac{7}{2}}^{[7]}$ as

$$\sum_{m=-7/2}^{7/2} |\tfrac{3}{2}\tfrac{1}{2}\tfrac{3}{2}(2)_{\mathcal{K}_T\mathcal{K}_2}; \tfrac{7}{2}\rangle\langle\tfrac{3}{2}\tfrac{1}{2}\tfrac{3}{2}(2)_{\mathcal{K}_T\mathcal{K}_2}; \tfrac{7}{2}| =: I_{\frac{5}{2}} \otimes |\hat{0}\rangle\langle\hat{0}|_{\frac{7}{2}}. \tag{C.75}$$

Using Lemma 6, one can calculate the relation between the two bases. We obtain

$$|\hat{0}\rangle_{\frac{1}{2}} = U(1/2, 1/2, 1/2)_{11}|0\rangle_{\frac{1}{2}} + U(1/2, 1/2, 1/2)_{12}|0\rangle_{\frac{1}{2}}, \tag{C.76}$$

$$|\hat{1}\rangle_{\frac{1}{2}} = U(1/2, 1/2, 1/2)_{21}|0\rangle_{\frac{1}{2}} + U(1/2, 1/2, 1/2)_{32}|0\rangle_{\frac{1}{2}}, \tag{C.77}$$

$$|\hat{2}\rangle_{\frac{1}{2}} = U(3/2, 1/2, 1/2)_{21}|3\rangle_{\frac{1}{2}}, \tag{C.78}$$

$$|\hat{3}\rangle_{\frac{1}{2}} = U(1/2, 3/2, 1/2)_{12}|2\rangle_{\frac{1}{2}}, \tag{C.79}$$

$$|\hat{4}\rangle_{\frac{1}{2}} = U(3/2, 3/2, 1/2)_{11}|4\rangle_{\frac{1}{2}} + U(3/2, 3/2, 1/2)_{12}|5\rangle_{\frac{1}{2}}, \tag{C.80}$$

$$|\hat{5}\rangle_{\frac{1}{2}} = U(3/2, 3/2, 1/2)_{21}|4\rangle_{\frac{1}{2}} + U(3/2, 3/2, 1/2)_{22}|5\rangle_{\frac{1}{2}}, \tag{C.81}$$

$$|\hat{0}\rangle_{\frac{3}{2}} = U(1/2, 1/2, 3/2)_{22}|0\rangle_{\frac{3}{2}}, \tag{C.82}$$

$$|\hat{1}\rangle_{\frac{3}{2}} = U(3/2, 1/2, 3/2)_{11}|3\rangle_{\frac{3}{2}} + U(3/2, 1/2, 3/2)_{12}|4\rangle_{\frac{3}{2}}, \tag{C.83}$$

$$|\hat{2}\rangle_{\frac{3}{2}} = U(3/2, 1/2, 3/2)_{21}|3\rangle_{\frac{3}{2}} + U(3/2, 1/2, 3/2)_{22}|4\rangle_{\frac{3}{2}}, \tag{C.84}$$

$$|\hat{3}\rangle_{\frac{3}{2}} = U(1/2, 3/2, 3/2)_{11}|1\rangle_{\frac{3}{2}} + U(1/2, 3/2, 3/2)_{12}|2\rangle_{\frac{3}{2}}, \tag{C.85}$$

$$|\hat{4}\rangle_{\frac{3}{2}} = U(1/2, 3/2, 3/2)_{21}|1\rangle_{\frac{3}{2}} + U(1/2, 3/2, 3/2)_{22}|2\rangle_{\frac{3}{2}}, \tag{C.86}$$

$$|\hat{5}\rangle_{\frac{3}{2}} = U(3/2, 3/2, 3/2)_{11}|5\rangle_{\frac{3}{2}} + U(3/2, 3/2, 3/2)_{12}|6\rangle_{\frac{3}{2}}, \tag{C.87}$$

$$|\hat{6}\rangle_{\frac{3}{2}} = U(3/2, 3/2, 3/2)_{21}|5\rangle_{\frac{3}{2}} + U(3/2, 3/2, 3/2)_{22}|6\rangle_{\frac{3}{2}}, \tag{C.88}$$

$$|\hat{0}\rangle_{\frac{5}{2}} = U(3/2, 1/2, 5/2)_{22}|1\rangle_{\frac{5}{2}}, \tag{C.89}$$

$$|\hat{1}\rangle_{\frac{5}{2}} = U(1/2,3/2,5/2)_{22}|0\rangle_{\frac{5}{2}}, \tag{C.90}$$
$$|\hat{2}\rangle_{\frac{5}{2}} = U(3/2,3/2,5/2)_{11}|2\rangle_{\frac{5}{2}} + U(3/2,3/2,5/2)_{12}|3\rangle_{\frac{5}{2}}, \tag{C.91}$$
$$|\hat{3}\rangle_{\frac{5}{2}} = U(3/2,3/2,5/2)_{21}|2\rangle_{\frac{5}{2}} + U(3/2,3/2,5/2)_{22}|3\rangle_{\frac{5}{2}}, \tag{C.92}$$
$$|\hat{0}\rangle_{\frac{7}{2}} = U(3/2,3/2,7/2)_{22}|0\rangle_{\frac{7}{2}}. \tag{C.93}$$

$N = 4$ **case**

The Choi operator $M_i$ is represented as

$$M_1 = \eta^{[5]}_{\mathcal{H}_{R_1}\mathcal{K}_{R_1}\mathcal{H}_T\mathcal{K}_T} \otimes \eta^{[4]}_{\mathcal{H}_{R_2}\mathcal{K}_{R_2}}, \tag{C.94}$$
$$M_2 = \eta^{[4]}_{\mathcal{H}_{R_1}\mathcal{K}_{R_1}} \otimes \eta^{[5]}_{\mathcal{H}_{R_2}\mathcal{K}_{R_2}\mathcal{H}_T\mathcal{K}_T}, \tag{C.95}$$

where the definition of $\eta^{[N]}$ is given in Equation (4.74). We have

$$\eta^{[5]}_{\mathcal{H}_{R_1}\mathcal{K}_{R_1}\mathcal{H}_T\mathcal{K}_T} = \frac{I_{\frac{1}{2}}}{2} \otimes I_{\frac{1}{2}} \otimes \left[(|\alpha^{[4]}_0\rangle\rangle + |\alpha^{[4]}_1\rangle\rangle) \otimes |\alpha^{[1]}_{\frac{1}{2}}\rangle\rangle\right], \tag{C.96}$$
$$\oplus \frac{I_{\frac{3}{2}}}{4} \otimes I_{\frac{3}{2}} \otimes \left[(|\alpha^{[4]}_1\rangle\rangle + |\alpha^{[4]}_2\rangle\rangle) \otimes |\alpha^{[1]}_{\frac{1}{2}}\rangle\rangle\right], \tag{C.97}$$
$$\oplus \frac{I_{\frac{5}{2}}}{6} \otimes I_{\frac{5}{2}} \otimes \left[|\alpha^{[4]}_2\rangle\rangle \otimes |\alpha^{[1]}_{\frac{1}{2}}\rangle\rangle\right] \tag{C.98}$$

and

$$\eta^{[4]}_{\mathcal{H}_{R_1}\mathcal{K}_{R_1}\mathcal{H}_T\mathcal{K}_T} \oplus I_0 \otimes I_0 \otimes \left[|\alpha^{[4]}_0\rangle\rangle\right] \oplus \tfrac{I_1}{3} \otimes I_1 \otimes \left[|\alpha^{[4]}_1\rangle\rangle\right], \tag{C.99}$$
$$\oplus \tfrac{I_2}{5} \otimes I_2 \otimes \left[|\alpha^{[4]}_2\rangle\rangle\right]. \tag{C.100}$$

Then we have

$$M^{[1]}_{\frac{1}{2}\frac{1}{2}} = \left[|00\rangle_{\frac{1}{2}\frac{1}{2}} + |22\rangle_{\frac{1}{2}\frac{1}{2}}\right] + \frac{1}{3}\left[|11\rangle_{\frac{1}{2}\frac{1}{2}} + |33\rangle_{\frac{1}{2}\frac{1}{2}}\right] + \frac{1}{6}\left[|44\rangle_{\frac{1}{2}\frac{1}{2}} + |66\rangle_{\frac{1}{2}\frac{1}{2}}\right], \tag{C.101}$$
$$+ \frac{1}{10}\left[|55\rangle_{\frac{1}{2}\frac{1}{2}} + |77\rangle_{\frac{1}{2}\frac{1}{2}}\right] + \frac{1}{15}\left[|88\rangle_{\frac{1}{2}\frac{1}{2}}\right], \tag{C.102}$$
$$M^{[1]}_{\frac{1}{2}\frac{3}{2}} = \frac{1}{3}\left[|10\rangle_{\frac{1}{2}\frac{3}{2}} + |34\rangle_{\frac{1}{2}\frac{3}{2}}\right] + \frac{1}{6}\left[|45\rangle_{\frac{1}{2}\frac{3}{2}} + |68\rangle_{\frac{1}{2}\frac{3}{2}}\right],$$
$$+ \frac{1}{10}\left[|57\rangle_{\frac{1}{2}\frac{3}{2}} + |7,10\rangle_{\frac{1}{2}\frac{3}{2}}\right] + \frac{1}{15}\left[|8,11\rangle_{\frac{1}{2}\frac{3}{2}}\right], \tag{C.103}$$
$$M^{[1]}_{\frac{1}{2}\frac{5}{2}} = \frac{1}{6}\left[|42\rangle_{\frac{1}{2}\frac{5}{2}} + |65\rangle_{\frac{1}{2}\frac{5}{2}}\right] + \frac{1}{10}\left[|54\rangle_{\frac{1}{2}\frac{5}{2}} + |77\rangle_{\frac{1}{2}\frac{5}{2}}\right] + \frac{1}{15}\left[|88\rangle_{\frac{1}{2}\frac{5}{2}}\right], \tag{C.104}$$
$$M^{[1]}_{\frac{1}{2}\frac{7}{2}} = \frac{1}{10}\left[|50\rangle_{\frac{1}{2}\frac{7}{2}} + |72\rangle_{\frac{1}{2}\frac{7}{2}}\right] + \frac{1}{15}\left[|83\rangle_{\frac{1}{2}\frac{7}{2}}\right], \tag{C.105}$$
$$M^{[1]}_{\frac{1}{2}\frac{9}{2}} = \frac{1}{15}\left[|80\rangle_{\frac{1}{2}\frac{9}{2}}\right], \tag{C.106}$$

$$
\begin{aligned}
M^{[1]}_{\frac{3}{2}\frac{1}{2}} &= \frac{2}{3}\left[|01\rangle_{\frac{3}{2}\frac{1}{2}} + |43\rangle_{\frac{3}{2}\frac{1}{2}}\right] + \frac{1}{3}\left[|54\rangle_{\frac{3}{2}\frac{1}{2}} + |86\rangle_{\frac{3}{2}\frac{1}{2}}\right], \\
&\quad + \frac{1}{5}\left[|75\rangle_{\frac{3}{2}\frac{1}{2}} + |10,7\rangle_{\frac{3}{2}\frac{1}{2}}\right] + \frac{2}{15}\left[|11,8\rangle_{\frac{3}{2}\frac{1}{2}}\right], \qquad \text{(C.107)} \\
M^{[1]}_{\frac{3}{2}\frac{3}{2}} &= \left[|11\rangle_{\frac{3}{2}\frac{3}{2}} + |33\rangle_{\frac{3}{2}\frac{3}{2}}\right] + \frac{2}{3}\left[|00\rangle_{\frac{3}{2}\frac{3}{2}} + |44\rangle_{\frac{3}{2}\frac{3}{2}}\right] + \frac{1}{3}\left[|55\rangle_{\frac{3}{2}\frac{3}{2}} + |88\rangle_{\frac{3}{2}\frac{3}{2}}\right] \\
&\quad + \frac{2}{9}\left[|99\rangle_{\frac{3}{2}\frac{3}{2}}\right] + \frac{2}{5}\left[|22\rangle_{\frac{3}{2}\frac{3}{2}} + |66\rangle_{\frac{3}{2}\frac{3}{2}}\right] + \frac{1}{5}\left[|77\rangle_{\frac{3}{2}\frac{3}{2}} + |10,10\rangle_{\frac{3}{2}\frac{3}{2}}\right], \\
&\quad + \frac{2}{15}\left[|11,11\rangle_{\frac{3}{2}\frac{3}{2}}\right], \qquad \text{(C.108)} \\
M^{[1]}_{\frac{3}{2}\frac{5}{2}} &= \frac{1}{3}\left[|52\rangle_{\frac{3}{2}\frac{5}{2}} + |85\rangle_{\frac{3}{2}\frac{5}{2}}\right] + \frac{2}{9}\left[|96\rangle_{\frac{3}{2}\frac{5}{2}}\right] + \frac{2}{5}\left[|20\rangle_{\frac{3}{2}\frac{5}{2}} + |63\rangle_{\frac{3}{2}\frac{5}{2}}\right] \\
&\quad + \frac{1}{5}\left[|74\rangle_{\frac{3}{2}\frac{5}{2}} + |10,7\rangle_{\frac{3}{2}\frac{5}{2}}\right] + \frac{2}{15}\left[|11,8\rangle_{\frac{3}{2}\frac{5}{2}}\right], \qquad \text{(C.109)} \\
M^{[1]}_{\frac{3}{2}\frac{7}{2}} &= \frac{2}{9}\left[|9,1\rangle_{\frac{3}{2}\frac{7}{2}}\right] + \frac{1}{5}\left[|70\rangle_{\frac{3}{2}\frac{7}{2}} + |10,2\rangle_{\frac{3}{2}\frac{7}{2}}\right] + \frac{2}{15}\left[|11,3\rangle_{\frac{3}{2}\frac{7}{2}}\right], \qquad \text{(C.110)} \\
M^{[1]}_{\frac{3}{2}\frac{9}{2}} &= \frac{2}{15}\left[|11,0\rangle_{\frac{3}{2}\frac{9}{2}}\right], \qquad \text{(C.111)} \\
M^{[1]}_{\frac{5}{2}\frac{1}{2}} &= \frac{1}{2}\left[|24\rangle_{\frac{5}{2}\frac{1}{2}} + |56\rangle_{\frac{5}{2}\frac{1}{2}}\right] + \frac{3}{10}\left[|45\rangle_{\frac{5}{2}\frac{1}{2}} + |77\rangle_{\frac{5}{2}\frac{1}{2}}\right] + \frac{1}{5}\left[|88\rangle_{\frac{5}{2}\frac{1}{2}}\right], \qquad \text{(C.112)} \\
M^{[1]}_{\frac{5}{2}\frac{3}{2}} &= \frac{1}{2}\left[|25\rangle_{\frac{5}{2}\frac{3}{2}} + |58\rangle_{\frac{5}{2}\frac{3}{2}}\right] + \frac{1}{3}\left[|69\rangle_{\frac{5}{2}\frac{3}{2}}\right] + \frac{3}{5}\left[|02\rangle_{\frac{5}{2}\frac{3}{2}} + |36\rangle_{\frac{5}{2}\frac{3}{2}}\right] \\
&\quad + \frac{3}{10}\left[|47\rangle_{\frac{5}{2}\frac{3}{2}} + |7,10\rangle_{\frac{5}{2}\frac{3}{2}}\right] + \frac{1}{5}\left[|8,11\rangle_{\frac{5}{2}\frac{3}{2}}\right], \qquad \text{(C.113)} \\
M^{[1]}_{\frac{5}{2}\frac{5}{2}} &= \left[|11\rangle_{\frac{5}{2}\frac{5}{2}}\right] + \frac{1}{2}\left[|22\rangle_{\frac{5}{2}\frac{5}{2}} + |55\rangle_{\frac{5}{2}\frac{5}{2}}\right] + \frac{3}{5}\left[|00\rangle_{\frac{5}{2}\frac{5}{2}} + |33\rangle_{\frac{5}{2}\frac{5}{2}}\right] \\
&\quad + \frac{1}{3}\left[|66\rangle_{\frac{5}{2}\frac{5}{2}}\right] + \frac{3}{10}\left[|44\rangle_{\frac{5}{2}\frac{5}{2}} + |77\rangle_{\frac{5}{2}\frac{5}{2}}\right] + \frac{1}{5}\left[|88\rangle_{\frac{5}{2}\frac{5}{2}}\right], \qquad \text{(C.114)} \\
M^{[1]}_{\frac{5}{2}\frac{7}{2}} &= \frac{1}{3}\left[|01\rangle_{\frac{5}{2}\frac{7}{2}}\right] + \frac{3}{10}\left[|40\rangle_{\frac{5}{2}\frac{7}{2}} + |72\rangle_{\frac{5}{2}\frac{7}{2}}\right] + \frac{1}{5}\left[|83\rangle_{\frac{5}{2}\frac{7}{2}}\right], \qquad \text{(C.115)} \\
M^{[1]}_{\frac{5}{2}\frac{9}{2}} &= \frac{1}{5}\left[|80\rangle_{\frac{5}{2}\frac{9}{2}}\right], \qquad \text{(C.116)} \\
M^{[1]}_{\frac{7}{2}\frac{1}{2}} &= \frac{2}{5}\left[|05\rangle_{\frac{7}{2}\frac{1}{2}} + |27\rangle_{\frac{7}{2}\frac{1}{2}}\right] + \frac{4}{15}\left[|38\rangle_{\frac{7}{2}\frac{1}{2}}\right], \qquad \text{(C.117)} \\
M^{[1]}_{\frac{7}{2}\frac{3}{2}} &= \frac{2}{9}\left[|19\rangle_{\frac{7}{2}\frac{3}{2}}\right] + \frac{2}{5}\left[|07\rangle_{\frac{7}{2}\frac{3}{2}} + |2,10\rangle_{\frac{7}{2}\frac{3}{2}}\right] + \frac{4}{15}\left[|3,11\rangle_{\frac{7}{2}\frac{3}{2}}\right], \qquad \text{(C.118)} \\
M^{[1]}_{\frac{7}{2}\frac{5}{2}} &= \frac{4}{9}\left[|10\rangle_{\frac{7}{2}\frac{5}{2}}\right] + \frac{2}{5}\left[|04\rangle_{\frac{7}{2}\frac{5}{2}} + |27\rangle_{\frac{7}{2}\frac{5}{2}}\right] + \frac{4}{15}\left[|38\rangle_{\frac{7}{2}\frac{5}{2}}\right], \qquad \text{(C.119)} \\
M^{[1]}_{\frac{7}{2}\frac{7}{2}} &= \frac{2}{5}\left[|00\rangle_{\frac{7}{2}\frac{7}{2}} + |22\rangle_{\frac{7}{2}\frac{7}{2}}\right] + \frac{4}{9}\left[|11\rangle_{\frac{7}{2}\frac{7}{2}}\right] + \frac{4}{15}\left[|33\rangle_{\frac{7}{2}\frac{7}{2}}\right], \qquad \text{(C.120)} \\
M^{[1]}_{\frac{7}{2}\frac{9}{2}} &= \frac{4}{15}\left[|30\rangle_{\frac{7}{2}\frac{9}{2}}\right], \qquad \text{(C.121)} \\
M^{[1]}_{\frac{9}{2}\frac{1}{2}} &= \frac{1}{3}\left[|08\rangle_{\frac{9}{2}\frac{1}{2}}\right], \qquad \text{(C.122)}
\end{aligned}
$$

$$M^{[1]}_{\frac{9}{2}\frac{3}{2}} = \frac{1}{3}\left[|0,11\rangle_{\frac{9}{2}\frac{3}{2}}\right], \quad (C.123)$$
$$M^{[1]}_{\frac{9}{2}\frac{5}{2}} = \frac{1}{3}\left[|08\rangle_{\frac{9}{2}\frac{5}{2}}\right], \quad (C.124)$$
$$M^{[1]}_{\frac{9}{2}\frac{7}{2}} = \frac{1}{3}\left[|03\rangle_{\frac{9}{2}\frac{7}{2}}\right], \quad (C.125)$$
$$M^{[1]}_{\frac{9}{2}\frac{9}{2}} = \frac{1}{3}\left[|00\rangle_{\frac{9}{2}\frac{9}{2}}\right], \quad (C.126)$$

where we defined the basis states for the multiplicity subspaces $\mathcal{V}^{[9]}_{\frac{1}{2}}$ as

$$\textstyle\sum_{m=-1/2}^{1/2} |0\frac{1}{2}(\frac{1}{2})\mathcal{K}_{R_1}\mathcal{K}_T 0; \frac{1}{2}\rangle\langle 0\frac{1}{2}(\frac{1}{2})\mathcal{K}_{R_1}\mathcal{K}_T 0; \frac{1}{2}| =: I_{\frac{1}{2}} \otimes |0\rangle\langle 0|_{\frac{1}{2}}, \quad (C.127)$$
$$\textstyle\sum_{m=-1/2}^{1/2} |0\frac{1}{2}(\frac{1}{2})\mathcal{K}_{R_1}\mathcal{K}_T 1; \frac{1}{2}\rangle\langle 0\frac{1}{2}(\frac{1}{2})\mathcal{K}_{R_1}\mathcal{K}_T 1; \frac{1}{2}| =: I_{\frac{1}{2}} \otimes |1\rangle\langle 1|_{\frac{1}{2}}, \quad (C.128)$$
$$\textstyle\sum_{m=-1/2}^{1/2} |1\frac{1}{2}(\frac{1}{2})\mathcal{K}_{R_1}\mathcal{K}_T 0; \frac{1}{2}\rangle\langle 1\frac{1}{2}(\frac{1}{2})\mathcal{K}_{R_1}\mathcal{K}_T 0; \frac{1}{2}| =: I_{\frac{1}{2}} \otimes |2\rangle\langle 2|_{\frac{1}{2}}, \quad (C.129)$$
$$\textstyle\sum_{m=-1/2}^{1/2} |1\frac{1}{2}(\frac{1}{2})\mathcal{K}_{R_1}\mathcal{K}_T 1; \frac{1}{2}\rangle\langle 1\frac{1}{2}(\frac{1}{2})\mathcal{K}_{R_1}\mathcal{K}_T 1; \frac{1}{2}| =: I_{\frac{1}{2}} \otimes |3\rangle\langle 3|_{\frac{1}{2}}, \quad (C.130)$$
$$\textstyle\sum_{m=-1/2}^{1/2} |1\frac{1}{2}(\frac{3}{2})\mathcal{K}_{R_1}\mathcal{K}_T 1; \frac{1}{2}\rangle\langle 1\frac{1}{2}(\frac{3}{2})\mathcal{K}_{R_1}\mathcal{K}_T 1; \frac{1}{2}| =: I_{\frac{1}{2}} \otimes |4\rangle\langle 4|_{\frac{1}{2}}, \quad (C.131)$$
$$\textstyle\sum_{m=-1/2}^{1/2} |1\frac{1}{2}(\frac{3}{2})\mathcal{K}_{R_1}\mathcal{K}_T 2; \frac{1}{2}\rangle\langle 1\frac{1}{2}(\frac{3}{2})\mathcal{K}_{R_1}\mathcal{K}_T 2; \frac{1}{2}| =: I_{\frac{1}{2}} \otimes |5\rangle\langle 5|_{\frac{1}{2}}, \quad (C.132)$$
$$\textstyle\sum_{m=-1/2}^{1/2} |2\frac{1}{2}(\frac{1}{2})\mathcal{K}_{R_1}\mathcal{K}_T 1; \frac{1}{2}\rangle\langle 2\frac{1}{2}(\frac{1}{2})\mathcal{K}_{R_1}\mathcal{K}_T 1; \frac{1}{2}| =: I_{\frac{1}{2}} \otimes |6\rangle\langle 6|_{\frac{1}{2}}, \quad (C.133)$$
$$\textstyle\sum_{m=-1/2}^{1/2} |2\frac{1}{2}(\frac{3}{2})\mathcal{K}_{R_1}\mathcal{K}_T 2; \frac{1}{2}\rangle\langle 2\frac{1}{2}(\frac{3}{2})\mathcal{K}_{R_1}\mathcal{K}_T 2; \frac{1}{2}| =: I_{\frac{1}{2}} \otimes |7\rangle\langle 7|_{\frac{1}{2}}, \quad (C.134)$$
$$\textstyle\sum_{m=-1/2}^{1/2} |2\frac{1}{2}(\frac{5}{2})\mathcal{K}_{R_1}\mathcal{K}_T 2; \frac{1}{2}\rangle\langle 2\frac{1}{2}(\frac{5}{2})\mathcal{K}_{R_1}\mathcal{K}_T 2; \frac{1}{2}| =: I_{\frac{1}{2}} \otimes |8\rangle\langle 8|_{\frac{1}{2}}, \quad (C.135)$$

and for $\mathcal{V}^{[9]}_{\frac{3}{2}}$ as

$$\textstyle\sum_{m=-3/2}^{3/2} |0\frac{1}{2}(\frac{1}{2})\mathcal{K}_{R_1}\mathcal{K}_T 1; \frac{1}{2}\rangle\langle 0\frac{1}{2}(\frac{1}{2})\mathcal{K}_{R_1}\mathcal{K}_T 1; \frac{1}{2}| =: I_{\frac{3}{2}} \otimes |0\rangle\langle 0|_{\frac{3}{2}}, \quad (C.136)$$
$$\textstyle\sum_{m=-3/2}^{3/2} |1\frac{1}{2}(\frac{3}{2})\mathcal{K}_{R_1}\mathcal{K}_T 0; \frac{1}{2}\rangle\langle 1\frac{1}{2}(\frac{3}{2})\mathcal{K}_{R_1}\mathcal{K}_T 0; \frac{1}{2}| =: I_{\frac{3}{2}} \otimes |1\rangle\langle 1|_{\frac{3}{2}}, \quad (C.137)$$
$$\textstyle\sum_{m=-3/2}^{3/2} |0\frac{1}{2}(\frac{1}{2})\mathcal{K}_{R_1}\mathcal{K}_T 2; \frac{1}{2}\rangle\langle 0\frac{1}{2}(\frac{1}{2})\mathcal{K}_{R_1}\mathcal{K}_T 2; \frac{1}{2}| =: I_{\frac{3}{2}} \otimes |2\rangle\langle 2|_{\frac{3}{2}}, \quad (C.138)$$
$$\textstyle\sum_{m=-3/2}^{3/2} |2\frac{1}{2}(\frac{3}{2})\mathcal{K}_{R_1}\mathcal{K}_T 0; \frac{1}{2}\rangle\langle 2\frac{1}{2}(\frac{3}{2})\mathcal{K}_{R_1}\mathcal{K}_T 0; \frac{1}{2}| =: I_{\frac{3}{2}} \otimes |3\rangle\langle 3|_{\frac{3}{2}}, \quad (C.139)$$
$$\textstyle\sum_{m=-3/2}^{3/2} |1\frac{1}{2}(\frac{1}{2})\mathcal{K}_{R_1}\mathcal{K}_T 1; \frac{1}{2}\rangle\langle 1\frac{1}{2}(\frac{1}{2})\mathcal{K}_{R_1}\mathcal{K}_T 1; \frac{1}{2}| =: I_{\frac{3}{2}} \otimes |4\rangle\langle 4|_{\frac{3}{2}}, \quad (C.140)$$
$$\textstyle\sum_{m=-3/2}^{3/2} |1\frac{1}{2}(\frac{3}{2})\mathcal{K}_{R_1}\mathcal{K}_T 1; \frac{1}{2}\rangle\langle 1\frac{1}{2}(\frac{3}{2})\mathcal{K}_{R_1}\mathcal{K}_T 1; \frac{1}{2}| =: I_{\frac{3}{2}} \otimes |5\rangle\langle 5|_{\frac{3}{2}}, \quad (C.141)$$
$$\textstyle\sum_{m=-3/2}^{3/2} |1\frac{1}{2}(\frac{1}{2})\mathcal{K}_{R_1}\mathcal{K}_T 2; \frac{1}{2}\rangle\langle 1\frac{1}{2}(\frac{1}{2})\mathcal{K}_{R_1}\mathcal{K}_T 2; \frac{1}{2}| =: I_{\frac{3}{2}} \otimes |6\rangle\langle 6|_{\frac{3}{2}}, \quad (C.142)$$
$$\textstyle\sum_{m=-3/2}^{3/2} |1\frac{1}{2}(\frac{3}{2})\mathcal{K}_{R_1}\mathcal{K}_T 2; \frac{1}{2}\rangle\langle 1\frac{1}{2}(\frac{3}{2})\mathcal{K}_{R_1}\mathcal{K}_T 2; \frac{1}{2}| =: I_{\frac{3}{2}} \otimes |7\rangle\langle 7|_{\frac{3}{2}}, \quad (C.143)$$
$$\textstyle\sum_{m=-3/2}^{3/2} |2\frac{1}{2}(\frac{3}{2})\mathcal{K}_{R_1}\mathcal{K}_T 1; \frac{1}{2}\rangle\langle 2\frac{1}{2}(\frac{3}{2})\mathcal{K}_{R_1}\mathcal{K}_T 1; \frac{1}{2}| =: I_{\frac{3}{2}} \otimes |8\rangle\langle 8|_{\frac{3}{2}}, \quad (C.144)$$
$$\textstyle\sum_{m=-3/2}^{3/2} |2\frac{1}{2}(\frac{3}{2})\mathcal{K}_{R_1}\mathcal{K}_T 1; \frac{1}{2}\rangle\langle 2\frac{1}{2}(\frac{3}{2})\mathcal{K}_{R_1}\mathcal{K}_T 1; \frac{1}{2}| =: I_{\frac{3}{2}} \otimes |9\rangle\langle 9|_{\frac{3}{2}}, \quad (C.145)$$
$$\textstyle\sum_{m=-3/2}^{3/2} |2\frac{1}{2}(\frac{3}{2})\mathcal{K}_{R_1}\mathcal{K}_T 2; \frac{1}{2}\rangle\langle 2\frac{1}{2}(\frac{3}{2})\mathcal{K}_{R_1}\mathcal{K}_T 2; \frac{1}{2}| =: I_{\frac{3}{2}} \otimes |10\rangle\langle 10|_{\frac{3}{2}}, \quad (C.146)$$

$$\textstyle\sum_{m=-3/2}^{3/2} |2\frac{1}{2}(\frac{5}{2})\mathcal{K}_{R_1}\mathcal{K}_T 2; \frac{1}{2}\rangle\langle 2\frac{1}{2}(\frac{5}{2})\mathcal{K}_{R_1}\mathcal{K}_T 2; \frac{1}{2}| =: I_{\frac{3}{2}} \otimes |11\rangle\langle 11|_{\frac{3}{2}}, \tag{C.147}$$

and for $\mathcal{V}_{\frac{5}{2}}^{[9]}$ as

$$\textstyle\sum_{m=-5/2}^{5/2} |0\frac{1}{2}(\frac{1}{2})\mathcal{K}_{R_1}\mathcal{K}_T 2; \frac{5}{2}\rangle\langle 0\frac{1}{2}(\frac{1}{2})\mathcal{K}_{R_1}\mathcal{K}_T 2; \frac{5}{2}| =: I_{\frac{5}{2}} \otimes |0\rangle\langle 0|_{\frac{5}{2}}, \tag{C.148}$$
$$\textstyle\sum_{m=-5/2}^{5/2} |2\frac{1}{2}(\frac{5}{2})\mathcal{K}_{R_1}\mathcal{K}_T 0; \frac{5}{2}\rangle\langle 2\frac{1}{2}(\frac{5}{2})\mathcal{K}_{R_1}\mathcal{K}_T 0; \frac{5}{2}| =: I_{\frac{5}{2}} \otimes |1\rangle\langle 1|_{\frac{5}{2}}, \tag{C.149}$$
$$\textstyle\sum_{m=-5/2}^{5/2} |1\frac{1}{2}(\frac{3}{2})\mathcal{K}_{R_1}\mathcal{K}_T 1; \frac{5}{2}\rangle\langle 1\frac{1}{2}(\frac{3}{2})\mathcal{K}_{R_1}\mathcal{K}_T 1; \frac{5}{2}| =: I_{\frac{5}{2}} \otimes |2\rangle\langle 2|_{\frac{5}{2}}, \tag{C.150}$$
$$\textstyle\sum_{m=-5/2}^{5/2} |1\frac{1}{2}(\frac{1}{2})\mathcal{K}_{R_1}\mathcal{K}_T 2; \frac{5}{2}\rangle\langle 1\frac{1}{2}(\frac{1}{2})\mathcal{K}_{R_1}\mathcal{K}_T 2; \frac{5}{2}| =: I_{\frac{5}{2}} \otimes |3\rangle\langle 3|_{\frac{5}{2}}, \tag{C.151}$$
$$\textstyle\sum_{m=-5/2}^{5/2} |1\frac{1}{2}(\frac{3}{2})\mathcal{K}_{R_1}\mathcal{K}_T 2; \frac{5}{2}\rangle\langle 1\frac{1}{2}(\frac{3}{2})\mathcal{K}_{R_1}\mathcal{K}_T 2; \frac{5}{2}| =: I_{\frac{5}{2}} \otimes |4\rangle\langle 4|_{\frac{5}{2}}, \tag{C.152}$$
$$\textstyle\sum_{m=-5/2}^{5/2} |2\frac{1}{2}(\frac{3}{2})\mathcal{K}_{R_1}\mathcal{K}_T 1; \frac{5}{2}\rangle\langle 2\frac{1}{2}(\frac{3}{2})\mathcal{K}_{R_1}\mathcal{K}_T 1; \frac{5}{2}| =: I_{\frac{5}{2}} \otimes |5\rangle\langle 5|_{\frac{5}{2}}, \tag{C.153}$$
$$\textstyle\sum_{m=-5/2}^{5/2} |2\frac{1}{2}(\frac{5}{2})\mathcal{K}_{R_1}\mathcal{K}_T 1; \frac{5}{2}\rangle\langle 2\frac{1}{2}(\frac{5}{2})\mathcal{K}_{R_1}\mathcal{K}_T 1; \frac{5}{2}| =: I_{\frac{5}{2}} \otimes |6\rangle\langle 6|_{\frac{5}{2}}, \tag{C.154}$$
$$\textstyle\sum_{m=-5/2}^{5/2} |2\frac{1}{2}(\frac{3}{2})\mathcal{K}_{R_1}\mathcal{K}_T 2; \frac{5}{2}\rangle\langle 2\frac{1}{2}(\frac{3}{2})\mathcal{K}_{R_1}\mathcal{K}_T 2; \frac{5}{2}| =: I_{\frac{5}{2}} \otimes |7\rangle\langle 7|_{\frac{5}{2}}, \tag{C.155}$$
$$\textstyle\sum_{m=-5/2}^{5/2} |2\frac{1}{2}(\frac{5}{2})\mathcal{K}_{R_1}\mathcal{K}_T 2; \frac{5}{2}\rangle\langle 2\frac{1}{2}(\frac{5}{2})\mathcal{K}_{R_1}\mathcal{K}_T 2; \frac{5}{2}| =: I_{\frac{5}{2}} \otimes |8\rangle\langle 8|_{\frac{5}{2}}, \tag{C.156}$$

and for $\mathcal{V}_{\frac{7}{2}}^{[9]}$ as

$$\textstyle\sum_{m=-7/2}^{7/2} |1\frac{1}{2}(\frac{3}{2})\mathcal{K}_{R_1}\mathcal{K}_T 2; \frac{7}{2}\rangle\langle 1\frac{1}{2}(\frac{3}{2})\mathcal{K}_{R_1}\mathcal{K}_T 2; \frac{7}{2}| =: I_{\frac{7}{2}} \otimes |0\rangle\langle 0|_{\frac{7}{2}}, \tag{C.157}$$
$$\textstyle\sum_{m=-7/2}^{7/2} |2\frac{1}{2}(\frac{5}{2})\mathcal{K}_{R_1}\mathcal{K}_T 1; \frac{7}{2}\rangle\langle 2\frac{1}{2}(\frac{5}{2})\mathcal{K}_{R_1}\mathcal{K}_T 1; \frac{7}{2}| =: I_{\frac{7}{2}} \otimes |1\rangle\langle 1|_{\frac{7}{2}}, \tag{C.158}$$
$$\textstyle\sum_{m=-7/2}^{7/2} |2\frac{1}{2}(\frac{3}{2})\mathcal{K}_{R_1}\mathcal{K}_T 2; \frac{7}{2}\rangle\langle 2\frac{1}{2}(\frac{3}{2})\mathcal{K}_{R_1}\mathcal{K}_T 2; \frac{7}{2}| =: I_{\frac{7}{2}} \otimes |2\rangle\langle 2|_{\frac{7}{2}}, \tag{C.159}$$
$$\textstyle\sum_{m=-7/2}^{7/2} |2\frac{1}{2}(\frac{5}{2})\mathcal{K}_{R_1}\mathcal{K}_T 2; \frac{7}{2}\rangle\langle 2\frac{1}{2}(\frac{5}{2})\mathcal{K}_{R_1}\mathcal{K}_T 2; \frac{7}{2}| =: I_{\frac{7}{2}} \otimes |3\rangle\langle 3|_{\frac{7}{2}}, \tag{C.160}$$

and for $\mathcal{V}_{\frac{7}{2}}^{[9]}$ as

$$\textstyle\sum_{m=-9/2}^{9/2} |2\frac{1}{2}(\frac{5}{2})\mathcal{K}_{R_1}\mathcal{K}_T 2; \frac{9}{2}\rangle\langle 2\frac{1}{2}(\frac{5}{2})\mathcal{K}_{R_1}\mathcal{K}_T 2; \frac{9}{2}| =: I_{\frac{9}{2}} \otimes |0\rangle\langle 0|_{\frac{9}{2}}. \tag{C.161}$$

For $M_2$, we have

$$\begin{aligned} M_{\frac{1}{2}\frac{1}{2}}^{[2]} &= \left[|\hat{0}\hat{0}\rangle_{\frac{1}{2}\frac{1}{2}} + |\hat{2}\hat{2}\rangle_{\frac{1}{2}\frac{1}{2}}\right] + \frac{1}{3}\left[|\hat{1}\hat{1}\rangle_{\frac{1}{2}\frac{1}{2}} + |\hat{3}\hat{3}\rangle_{\frac{1}{2}\frac{1}{2}}\right] + \frac{1}{6}\left[|\hat{4}\hat{4}\rangle_{\frac{1}{2}\frac{1}{2}} + |\hat{6}\hat{6}\rangle_{\frac{1}{2}\frac{1}{2}}\right] \\ &\quad + \frac{1}{10}\left[|\hat{5}\hat{5}\rangle_{\frac{1}{2}\frac{1}{2}} + |\hat{7}\hat{7}\rangle_{\frac{1}{2}\frac{1}{2}}\right] + \frac{1}{15}\left[|\hat{8}\hat{8}\rangle_{\frac{1}{2}\frac{1}{2}}\right], \end{aligned} \tag{C.162}$$

$$\begin{aligned} M_{\frac{1}{2}\frac{3}{2}}^{[2]} &= \frac{1}{3}\left[|\hat{1}\hat{0}\rangle_{\frac{1}{2}\frac{3}{2}} + |\hat{3}\hat{4}\rangle_{\frac{1}{2}\frac{3}{2}}\right] + \frac{1}{6}\left[|\hat{4}\hat{5}\rangle_{\frac{1}{2}\frac{3}{2}} + |\hat{6}\hat{8}\rangle_{\frac{1}{2}\frac{3}{2}}\right] \\ &\quad + \frac{1}{10}\left[|\hat{5}\hat{7}\rangle_{\frac{1}{2}\frac{3}{2}} + |7, 10\rangle_{\frac{1}{2}\frac{3}{2}}\right] + \frac{1}{15}\left[|8, 11\rangle_{\frac{1}{2}\frac{3}{2}}\right], \end{aligned} \tag{C.163}$$

$$M_{\frac{1}{2}\frac{5}{2}}^{[2]} = \frac{1}{6}\left[|\hat{4}\hat{2}\rangle_{\frac{1}{2}\frac{5}{2}} + |\hat{6}\hat{5}\rangle_{\frac{1}{2}\frac{5}{2}}\right] + \frac{1}{10}\left[|\hat{5}\hat{4}\rangle_{\frac{1}{2}\frac{5}{2}} + |\hat{7}\hat{7}\rangle_{\frac{1}{2}\frac{5}{2}}\right] + \frac{1}{15}\left[|\hat{8}\hat{8}\rangle_{\frac{1}{2}\frac{5}{2}}\right], \tag{C.164}$$

$$M^{[2]}_{\frac{1}{2}\frac{7}{2}} = \frac{1}{10}\left[|\hat{5}\hat{0}\rangle_{\frac{1}{2}\frac{7}{2}} + |\hat{7}\hat{2}\rangle_{\frac{1}{2}\frac{7}{2}}\right] + \frac{1}{15}\left[|\hat{8}\hat{3}\rangle_{\frac{1}{2}\frac{7}{2}}\right], \tag{C.165}$$

$$M^{[2]}_{\frac{1}{2}\frac{9}{2}} = \frac{1}{15}\left[|\hat{8}\hat{0}\rangle_{\frac{1}{2}\frac{9}{2}}\right], \tag{C.166}$$

$$\begin{aligned} M^{[2]}_{\frac{3}{2}\frac{1}{2}} = &\frac{2}{3}\left[|\hat{0}\hat{1}\rangle_{\frac{3}{2}\frac{1}{2}} + |\hat{4}\hat{3}\rangle_{\frac{3}{2}\frac{1}{2}}\right] + \frac{1}{3}\left[|\hat{5}\hat{4}\rangle_{\frac{3}{2}\frac{1}{2}} + |\hat{8}\hat{6}\rangle_{\frac{3}{2}\frac{1}{2}}\right] \\ &+ \frac{1}{5}\left[|\hat{7}\hat{5}\rangle_{\frac{3}{2}\frac{1}{2}} + |10,7\rangle_{\frac{3}{2}\frac{1}{2}}\right] + \frac{2}{15}\left[|11,8\rangle_{\frac{3}{2}\frac{1}{2}}\right], \end{aligned} \tag{C.167}$$

$$\begin{aligned} M^{[2]}_{\frac{3}{2}\frac{3}{2}} = &\left[|\hat{1}\hat{1}\rangle_{\frac{3}{2}\frac{3}{2}} + |\hat{3}\hat{3}\rangle_{\frac{3}{2}\frac{3}{2}}\right] + \frac{2}{3}\left[|\hat{0}\hat{0}\rangle_{\frac{3}{2}\frac{3}{2}} + |\hat{4}\hat{4}\rangle_{\frac{3}{2}\frac{3}{2}}\right] + \frac{1}{3}\left[|\hat{5}\hat{5}\rangle_{\frac{3}{2}\frac{3}{2}} + |\hat{8}\hat{8}\rangle_{\frac{3}{2}\frac{3}{2}}\right] \\ &+ \frac{2}{9}\left[|\hat{9}\hat{9}\rangle_{\frac{3}{2}\frac{3}{2}}\right] + \frac{2}{5}\left[|\hat{2}\hat{2}\rangle_{\frac{3}{2}\frac{3}{2}} + |\hat{6}\hat{6}\rangle_{\frac{3}{2}\frac{3}{2}}\right] + \frac{1}{5}\left[|\hat{7}\hat{7}\rangle_{\frac{3}{2}\frac{3}{2}} + |10,10\rangle_{\frac{3}{2}\frac{3}{2}}\right] \\ &+ \frac{2}{15}\left[|11,11\rangle_{\frac{3}{2}\frac{3}{2}}\right], \end{aligned} \tag{C.168}$$

$$\begin{aligned} M^{[2]}_{\frac{3}{2}\frac{5}{2}} = &\frac{1}{3}\left[|\hat{5}\hat{2}\rangle_{\frac{3}{2}\frac{5}{2}} + |\hat{8}\hat{5}\rangle_{\frac{3}{2}\frac{5}{2}}\right] + \frac{2}{9}\left[|\hat{9}\hat{6}\rangle_{\frac{3}{2}\frac{5}{2}}\right] + \frac{2}{5}\left[|\hat{2}\hat{0}\rangle_{\frac{3}{2}\frac{5}{2}} + |\hat{6}\hat{3}\rangle_{\frac{3}{2}\frac{5}{2}}\right] \\ &+ \frac{1}{5}\left[|\hat{7}\hat{4}\rangle_{\frac{3}{2}\frac{5}{2}} + |10,7\rangle_{\frac{3}{2}\frac{5}{2}}\right] + \frac{2}{15}\left[|11,8\rangle_{\frac{3}{2}\frac{5}{2}}\right], \end{aligned} \tag{C.169}$$

$$M^{[2]}_{\frac{3}{2}\frac{7}{2}} = \frac{2}{9}\left[|9,1\rangle_{\frac{3}{2}\frac{7}{2}}\right] + \frac{1}{5}\left[|\hat{7}\hat{0}\rangle_{\frac{3}{2}\frac{7}{2}} + |10,2\rangle_{\frac{3}{2}\frac{7}{2}}\right] + \frac{2}{15}\left[|11,3\rangle_{\frac{3}{2}\frac{7}{2}}\right], \tag{C.170}$$

$$M^{[2]}_{\frac{3}{2}\frac{9}{2}} = \frac{2}{15}\left[|11,0\rangle_{\frac{3}{2}\frac{9}{2}}\right], \tag{C.171}$$

$$M^{[2]}_{\frac{5}{2}\frac{1}{2}} = \frac{1}{2}\left[|\hat{2}\hat{4}\rangle_{\frac{5}{2}\frac{1}{2}} + |\hat{5}\hat{6}\rangle_{\frac{5}{2}\frac{1}{2}}\right] + \frac{3}{10}\left[|\hat{4}\hat{5}\rangle_{\frac{5}{2}\frac{1}{2}} + |\hat{7}\hat{7}\rangle_{\frac{5}{2}\frac{1}{2}}\right] + \frac{1}{5}\left[|\hat{8}\hat{8}\rangle_{\frac{5}{2}\frac{1}{2}}\right], \tag{C.172}$$

$$\begin{aligned} M^{[2]}_{\frac{5}{2}\frac{3}{2}} = &\frac{1}{2}\left[|\hat{2}\hat{5}\rangle_{\frac{5}{2}\frac{3}{2}} + |\hat{5}\hat{8}\rangle_{\frac{5}{2}\frac{3}{2}}\right] + \frac{1}{3}\left[|\hat{6}\hat{9}\rangle_{\frac{5}{2}\frac{3}{2}}\right] + \frac{3}{5}\left[|\hat{0}\hat{2}\rangle_{\frac{5}{2}\frac{3}{2}} + |\hat{3}\hat{6}\rangle_{\frac{5}{2}\frac{3}{2}}\right] \\ &+ \frac{3}{10}\left[|\hat{4}\hat{7}\rangle_{\frac{5}{2}\frac{3}{2}} + |7,10\rangle_{\frac{5}{2}\frac{3}{2}}\right] + \frac{1}{5}\left[|8,11\rangle_{\frac{5}{2}\frac{3}{2}}\right], \end{aligned} \tag{C.173}$$

$$\begin{aligned} M^{[2]}_{\frac{5}{2}\frac{5}{2}} = &\left[|\hat{1}\hat{1}\rangle_{\frac{5}{2}\frac{5}{2}}\right] + \frac{1}{2}\left[|\hat{2}\hat{2}\rangle_{\frac{5}{2}\frac{5}{2}} + |\hat{5}\hat{5}\rangle_{\frac{5}{2}\frac{5}{2}}\right] + \frac{3}{5}\left[|\hat{0}\hat{0}\rangle_{\frac{5}{2}\frac{5}{2}} + |\hat{3}\hat{3}\rangle_{\frac{5}{2}\frac{5}{2}}\right] \\ &+ \frac{1}{3}\left[|\hat{6}\hat{6}\rangle_{\frac{5}{2}\frac{5}{2}}\right] + \frac{3}{10}\left[|\hat{4}\hat{4}\rangle_{\frac{5}{2}\frac{5}{2}} + |\hat{7}\hat{7}\rangle_{\frac{5}{2}\frac{5}{2}}\right] + \frac{1}{5}\left[|\hat{8}\hat{8}\rangle_{\frac{5}{2}\frac{5}{2}}\right], \end{aligned} \tag{C.174}$$

$$M^{[2]}_{\frac{5}{2}\frac{7}{2}} = \frac{1}{3}\left[|\hat{0}\hat{1}\rangle_{\frac{5}{2}\frac{7}{2}}\right] + \frac{3}{10}\left[|\hat{4}\hat{0}\rangle_{\frac{5}{2}\frac{7}{2}} + |\hat{7}\hat{2}\rangle_{\frac{5}{2}\frac{7}{2}}\right] + \frac{1}{5}\left[|\hat{8}\hat{3}\rangle_{\frac{5}{2}\frac{7}{2}}\right], \tag{C.175}$$

$$M^{[2]}_{\frac{5}{2}\frac{9}{2}} = \frac{1}{5}\left[|\hat{8}\hat{0}\rangle_{\frac{5}{2}\frac{9}{2}}\right], \tag{C.176}$$

$$M^{[2]}_{\frac{7}{2}\frac{1}{2}} = \frac{2}{5}\left[|\hat{0}\hat{5}\rangle_{\frac{7}{2}\frac{1}{2}} + |\hat{2}\hat{7}\rangle_{\frac{7}{2}\frac{1}{2}}\right] + \frac{4}{15}\left[|\hat{3}\hat{8}\rangle_{\frac{7}{2}\frac{1}{2}}\right], \tag{C.177}$$

$$M^{[2]}_{\frac{7}{2}\frac{3}{2}} = \frac{2}{9}\left[|\hat{1}\hat{9}\rangle_{\frac{7}{2}\frac{3}{2}}\right] + \frac{2}{5}\left[|\hat{0}\hat{7}\rangle_{\frac{7}{2}\frac{3}{2}} + |2,10\rangle_{\frac{7}{2}\frac{3}{2}}\right] + \frac{4}{15}\left[|3,11\rangle_{\frac{7}{2}\frac{3}{2}}\right], \tag{C.178}$$

$$M^{[2]}_{\frac{7}{2}\frac{5}{2}} = \frac{4}{9}\left[|\hat{1}\hat{0}\rangle_{\frac{7}{2}\frac{5}{2}}\right] + \frac{2}{5}\left[|\hat{0}\hat{4}\rangle_{\frac{7}{2}\frac{5}{2}} + |\hat{2}\hat{7}\rangle_{\frac{7}{2}\frac{5}{2}}\right] + \frac{4}{15}\left[|\hat{3}\hat{8}\rangle_{\frac{7}{2}\frac{5}{2}}\right], \tag{C.179}$$

$$M^{[2]}_{\frac{7}{2}\frac{7}{2}} = \frac{2}{5}\left[|\hat{0}\hat{0}\rangle_{\frac{7}{2}\frac{7}{2}} + |\hat{2}\hat{2}\rangle_{\frac{7}{2}\frac{7}{2}}\right] + \frac{4}{9}\left[|\hat{1}\hat{1}\rangle_{\frac{7}{2}\frac{7}{2}}\right] + \frac{4}{15}\left[|\hat{3}\hat{3}\rangle_{\frac{7}{2}\frac{7}{2}}\right], \tag{C.180}$$

$$M^{[2]}_{\frac{7}{2}\frac{9}{2}} = \frac{4}{15}\left[|\hat{3}\hat{0}\rangle_{\frac{7}{2}\frac{9}{2}}\right], \tag{C.181}$$
$$M^{[2]}_{\frac{9}{2}\frac{1}{2}} = \frac{1}{3}\left[|\hat{0}\hat{8}\rangle_{\frac{9}{2}\frac{1}{2}}\right], \tag{C.182}$$
$$M^{[2]}_{\frac{9}{2}\frac{3}{2}} = \frac{1}{3}\left[|0,11\rangle_{\frac{9}{2}\frac{3}{2}}\right], \tag{C.183}$$
$$M^{[2]}_{\frac{9}{2}\frac{5}{2}} = \frac{1}{3}\left[|\hat{0}\hat{8}\rangle_{\frac{9}{2}\frac{5}{2}}\right], \tag{C.184}$$
$$M^{[2]}_{\frac{9}{2}\frac{7}{2}} = \frac{1}{3}\left[|\hat{0}\hat{3}\rangle_{\frac{9}{2}\frac{7}{2}}\right], \tag{C.185}$$
$$M^{[2]}_{\frac{9}{2}\frac{9}{2}} = \frac{1}{3}\left[|\hat{0}\hat{0}\rangle_{\frac{9}{2}\frac{9}{2}}\right], \tag{C.186}$$

where we defined the basis states for the multiplicity subspaces $\mathcal{V}^{[9]}_{\frac{1}{2}}$ as

$$\textstyle\sum_{m=-1/2}^{1/2} |0\tfrac{1}{2}0(\tfrac{1}{2});\tfrac{1}{2}\rangle\langle 0\tfrac{1}{2}0(\tfrac{1}{2});\tfrac{1}{2}| =: I_{\frac{1}{2}} \otimes |\hat{0}\rangle\langle\hat{0}|_{\frac{1}{2}}, \tag{C.187}$$
$$\textstyle\sum_{m=-1/2}^{1/2} |1\tfrac{1}{2}0(\tfrac{1}{2});\tfrac{1}{2}\rangle\langle 1\tfrac{1}{2}0(\tfrac{1}{2});\tfrac{1}{2}| =: I_{\frac{1}{2}} \otimes |\hat{1}\rangle\langle\hat{1}|_{\frac{1}{2}}, \tag{C.188}$$
$$\textstyle\sum_{m=-1/2}^{1/2} |0\tfrac{1}{2}1(\tfrac{1}{2});\tfrac{1}{2}\rangle\langle 0\tfrac{1}{2}1(\tfrac{1}{2});\tfrac{1}{2}| =: I_{\frac{1}{2}} \otimes |\hat{2}\rangle\langle\hat{2}|_{\frac{1}{2}}, \tag{C.189}$$
$$\textstyle\sum_{m=-1/2}^{1/2} |1\tfrac{1}{2}1(\tfrac{1}{2});\tfrac{1}{2}\rangle\langle 1\tfrac{1}{2}1(\tfrac{1}{2});\tfrac{1}{2}| =: I_{\frac{1}{2}} \otimes |\hat{3}\rangle\langle\hat{3}|_{\frac{1}{2}}, \tag{C.190}$$
$$\textstyle\sum_{m=-1/2}^{1/2} |1\tfrac{1}{2}1(\tfrac{3}{2});\tfrac{1}{2}\rangle\langle 1\tfrac{1}{2}1(\tfrac{3}{2});\tfrac{1}{2}| =: I_{\frac{1}{2}} \otimes |\hat{4}\rangle\langle\hat{4}|_{\frac{1}{2}}, \tag{C.191}$$
$$\textstyle\sum_{m=-1/2}^{1/2} |2\tfrac{1}{2}1(\tfrac{3}{2});\tfrac{1}{2}\rangle\langle 2\tfrac{1}{2}1(\tfrac{3}{2});\tfrac{1}{2}| =: I_{\frac{1}{2}} \otimes |\hat{5}\rangle\langle\hat{5}|_{\frac{1}{2}}, \tag{C.192}$$
$$\textstyle\sum_{m=-1/2}^{1/2} |1\tfrac{1}{2}2(\tfrac{1}{2});\tfrac{1}{2}\rangle\langle 1\tfrac{1}{2}2(\tfrac{1}{2});\tfrac{1}{2}| =: I_{\frac{1}{2}} \otimes |\hat{6}\rangle\langle\hat{6}|_{\frac{1}{2}}, \tag{C.193}$$
$$\textstyle\sum_{m=-1/2}^{1/2} |2\tfrac{1}{2}2(\tfrac{3}{2});\tfrac{1}{2}\rangle\langle 2\tfrac{1}{2}2(\tfrac{3}{2});\tfrac{1}{2}| =: I_{\frac{1}{2}} \otimes |\hat{7}\rangle\langle\hat{7}|_{\frac{1}{2}}, \tag{C.194}$$
$$\textstyle\sum_{m=-1/2}^{1/2} |2\tfrac{1}{2}2(\tfrac{5}{2});\tfrac{1}{2}\rangle\langle 2\tfrac{1}{2}2(\tfrac{5}{2});\tfrac{1}{2}| =: I_{\frac{1}{2}} \otimes |\hat{8}\rangle\langle\hat{8}|_{\frac{1}{2}}, \tag{C.195}$$

and for the multiplicity subspaces $\mathcal{V}^{[9]}_{\frac{3}{2}}$ as

$$\textstyle\sum_{m=-3/2}^{3/2} |1\tfrac{1}{2}0(\tfrac{1}{2});\tfrac{3}{2}\rangle\langle 1\tfrac{1}{2}0(\tfrac{1}{2});\tfrac{3}{2}| =: I_{\frac{3}{2}} \otimes |\hat{0}\rangle\langle\hat{0}|_{\frac{3}{2}}, \tag{C.196}$$
$$\textstyle\sum_{m=-3/2}^{3/2} |0\tfrac{1}{2}1(\tfrac{3}{2});\tfrac{3}{2}\rangle\langle 0\tfrac{1}{2}1(\tfrac{3}{2});\tfrac{3}{2}| =: I_{\frac{3}{2}} \otimes |\hat{1}\rangle\langle\hat{1}|_{\frac{3}{2}}, \tag{C.197}$$
$$\textstyle\sum_{m=-3/2}^{3/2} |2\tfrac{1}{2}0(\tfrac{1}{2});\tfrac{3}{2}\rangle\langle 2\tfrac{1}{2}0(\tfrac{1}{2});\tfrac{3}{2}| =: I_{\frac{3}{2}} \otimes |\hat{2}\rangle\langle\hat{2}|_{\frac{3}{2}}, \tag{C.198}$$
$$\textstyle\sum_{m=-3/2}^{3/2} |0\tfrac{1}{2}2(\tfrac{3}{2});\tfrac{3}{2}\rangle\langle 0\tfrac{1}{2}2(\tfrac{3}{2});\tfrac{3}{2}| =: I_{\frac{3}{2}} \otimes |\hat{3}\rangle\langle\hat{3}|_{\frac{3}{2}}, \tag{C.199}$$
$$\textstyle\sum_{m=-3/2}^{3/2} |1\tfrac{1}{2}1(\tfrac{1}{2});\tfrac{3}{2}\rangle\langle 1\tfrac{1}{2}1(\tfrac{1}{2});\tfrac{3}{2}| =: I_{\frac{3}{2}} \otimes |\hat{4}\rangle\langle\hat{4}|_{\frac{3}{2}}, \tag{C.200}$$
$$\textstyle\sum_{m=-3/2}^{3/2} |1\tfrac{1}{2}1(\tfrac{3}{2});\tfrac{3}{2}\rangle\langle 1\tfrac{1}{2}1(\tfrac{3}{2});\tfrac{3}{2}| =: I_{\frac{3}{2}} \otimes |\hat{5}\rangle\langle\hat{5}|_{\frac{3}{2}}, \tag{C.201}$$
$$\textstyle\sum_{m=-3/2}^{3/2} |2\tfrac{1}{2}1(\tfrac{1}{2});\tfrac{3}{2}\rangle\langle 2\tfrac{1}{2}1(\tfrac{1}{2});\tfrac{3}{2}| =: I_{\frac{3}{2}} \otimes |\hat{6}\rangle\langle\hat{6}|_{\frac{3}{2}}, \tag{C.202}$$
$$\textstyle\sum_{m=-3/2}^{3/2} |2\tfrac{1}{2}1(\tfrac{3}{2});\tfrac{3}{2}\rangle\langle 2\tfrac{1}{2}1(\tfrac{3}{2});\tfrac{3}{2}| =: I_{\frac{3}{2}} \otimes |\hat{7}\rangle\langle\hat{7}|_{\frac{3}{2}}, \tag{C.203}$$

$$\textstyle\sum_{m=-3/2}^{3/2} |1\frac{1}{2}2(\frac{1}{2});\frac{3}{2}\rangle\langle 1\frac{1}{2}2(\frac{1}{2});\frac{3}{2}| =: I_{\frac{3}{2}} \otimes |\hat{8}\rangle\langle\hat{8}|_{\frac{3}{2}}, \tag{C.204}$$
$$\textstyle\sum_{m=-3/2}^{3/2} |1\frac{1}{2}2(\frac{3}{2});\frac{3}{2}\rangle\langle 1\frac{1}{2}2(\frac{3}{2});\frac{3}{2}| =: I_{\frac{3}{2}} \otimes |\hat{9}\rangle\langle\hat{9}|_{\frac{3}{2}}, \tag{C.205}$$
$$\textstyle\sum_{m=-3/2}^{3/2} |2\frac{1}{2}2(\frac{3}{2});\frac{3}{2}\rangle\langle 2\frac{1}{2}2(\frac{3}{2});\frac{3}{2}| =: I_{\frac{3}{2}} \otimes |\hat{10}\rangle\langle\hat{10}|_{\frac{3}{2}}, \tag{C.206}$$
$$\textstyle\sum_{m=-3/2}^{3/2} |2\frac{1}{2}2(\frac{5}{2});\frac{3}{2}\rangle\langle 2\frac{1}{2}2(\frac{5}{2});\frac{3}{2}| =: I_{\frac{3}{2}} \otimes |\hat{11}\rangle\langle\hat{11}|_{\frac{3}{2}}, \tag{C.207}$$

and for the multiplicity subspaces $\mathcal{V}_{\frac{5}{2}}^{[9]}$ as

$$\textstyle\sum_{m=-5/2}^{5/2} |2\frac{1}{2}0(\frac{1}{2});\frac{5}{2}\rangle\langle 2\frac{1}{2}0(\frac{1}{2});\frac{5}{2}| =: I_{\frac{5}{2}} \otimes |\hat{0}\rangle\langle\hat{0}|_{\frac{5}{2}}, \tag{C.208}$$
$$\textstyle\sum_{m=-5/2}^{5/2} |0\frac{1}{2}2(\frac{5}{2});\frac{5}{2}\rangle\langle 0\frac{1}{2}2(\frac{5}{2});\frac{5}{2}| =: I_{\frac{5}{2}} \otimes |\hat{1}\rangle\langle\hat{1}|_{\frac{5}{2}}, \tag{C.209}$$
$$\textstyle\sum_{m=-5/2}^{5/2} |1\frac{1}{2}1(\frac{3}{2});\frac{5}{2}\rangle\langle 1\frac{1}{2}1(\frac{3}{2});\frac{5}{2}| =: I_{\frac{5}{2}} \otimes |\hat{2}\rangle\langle\hat{2}|_{\frac{5}{2}}, \tag{C.210}$$
$$\textstyle\sum_{m=-5/2}^{5/2} |2\frac{1}{2}1(\frac{1}{2});\frac{5}{2}\rangle\langle 2\frac{1}{2}1(\frac{1}{2});\frac{5}{2}| =: I_{\frac{5}{2}} \otimes |\hat{3}\rangle\langle\hat{3}|_{\frac{5}{2}}, \tag{C.211}$$
$$\textstyle\sum_{m=-5/2}^{5/2} |2\frac{1}{2}1(\frac{3}{2});\frac{5}{2}\rangle\langle 2\frac{1}{2}1(\frac{3}{2});\frac{5}{2}| =: I_{\frac{5}{2}} \otimes |\hat{4}\rangle\langle\hat{4}|_{\frac{5}{2}}, \tag{C.212}$$
$$\textstyle\sum_{m=-5/2}^{5/2} |1\frac{1}{2}2(\frac{3}{2});\frac{5}{2}\rangle\langle 1\frac{1}{2}2(\frac{3}{2});\frac{5}{2}| =: I_{\frac{5}{2}} \otimes |\hat{5}\rangle\langle\hat{5}|_{\frac{5}{2}}, \tag{C.213}$$
$$\textstyle\sum_{m=-5/2}^{5/2} |1\frac{1}{2}2(\frac{5}{2});\frac{5}{2}\rangle\langle 1\frac{1}{2}2(\frac{5}{2});\frac{5}{2}| =: I_{\frac{5}{2}} \otimes |\hat{6}\rangle\langle\hat{6}|_{\frac{5}{2}}, \tag{C.214}$$
$$\textstyle\sum_{m=-5/2}^{5/2} |2\frac{1}{2}2(\frac{3}{2});\frac{5}{2}\rangle\langle 2\frac{1}{2}2(\frac{3}{2});\frac{5}{2}| =: I_{\frac{5}{2}} \otimes |\hat{7}\rangle\langle\hat{7}|_{\frac{5}{2}}, \tag{C.215}$$
$$\textstyle\sum_{m=-5/2}^{5/2} |2\frac{1}{2}2(\frac{5}{2});\frac{5}{2}\rangle\langle 2\frac{1}{2}2(\frac{5}{2});\frac{5}{2}| =: I_{\frac{5}{2}} \otimes |\hat{8}\rangle\langle\hat{8}|_{\frac{5}{2}}, \tag{C.216}$$

and $\mathcal{V}_{\frac{7}{2}}^{[9]}$ as

$$\textstyle\sum_{m=-7/2}^{7/2} |2\frac{1}{2}1(\frac{3}{2});\frac{7}{2}\rangle\langle 2\frac{1}{2}1(\frac{3}{2});\frac{7}{2}| =: I_{\frac{5}{2}} \otimes |\hat{0}\rangle\langle\hat{0}|_{\frac{7}{2}}, \tag{C.217}$$
$$\textstyle\sum_{m=-7/2}^{7/2} |1\frac{1}{2}2(\frac{5}{2});\frac{7}{2}\rangle\langle 1\frac{1}{2}2(\frac{5}{2});\frac{7}{2}| =: I_{\frac{5}{2}} \otimes |\hat{1}\rangle\langle\hat{1}|_{\frac{7}{2}}, \tag{C.218}$$
$$\textstyle\sum_{m=-7/2}^{7/2} |2\frac{1}{2}2(\frac{3}{2});\frac{7}{2}\rangle\langle 2\frac{1}{2}2(\frac{3}{2});\frac{7}{2}| =: I_{\frac{5}{2}} \otimes |\hat{2}\rangle\langle\hat{2}|_{\frac{7}{2}}, \tag{C.219}$$
$$\textstyle\sum_{m=-7/2}^{7/2} |2\frac{1}{2}2(\frac{5}{2});\frac{7}{2}\rangle\langle 2\frac{1}{2}2(\frac{5}{2});\frac{7}{2}| =: I_{\frac{5}{2}} \otimes |\hat{3}\rangle\langle\hat{3}|_{\frac{7}{2}}, \tag{C.220}$$

for $J = 7/2$,

$$\textstyle\sum_{m=-9/2}^{9/2} |2\frac{1}{2}(\frac{5}{2})2;\frac{9}{2}\rangle\langle 2\frac{1}{2}(\frac{5}{2})2;\frac{9}{2}| =: I_{\frac{7}{2}} \otimes |\hat{0}\rangle\langle\hat{0}|_{\frac{9}{2}}. \tag{C.221}$$

Using Lemma 6, one can calculate the relation between the two bases. We obtain

$$|\hat{0}\rangle_{\frac{1}{2}} = U(0,0,1/2)_{22}|0\rangle_{\frac{1}{2}}, \tag{C.222}$$
$$|\hat{1}\rangle_{\frac{1}{2}} = U(1,0,1/2)_{21}|2\rangle_{\frac{1}{2}}, \tag{C.223}$$
$$|\hat{2}\rangle_{\frac{1}{2}} = U(0,1,1/2)_{12}|1\rangle_{\frac{1}{2}}, \tag{C.224}$$
$$|\hat{3}\rangle_{\frac{1}{2}} = U(1,1,1/2)_{11}|3\rangle_{\frac{1}{2}} + U(1,1,1/2)_{12}|4\rangle_{\frac{1}{2}}, \tag{C.225}$$
$$|\hat{4}\rangle_{\frac{1}{2}} = U(1,1,1/2)_{21}|3\rangle_{\frac{1}{2}} + U(1,1,1/2)_{22}|4\rangle_{\frac{1}{2}}, \tag{C.226}$$
$$|\hat{5}\rangle_{\frac{1}{2}} = U(2,1,1/2)_{21}|6\rangle_{\frac{1}{2}}, \tag{C.227}$$
$$|\hat{6}\rangle_{\frac{1}{2}} = U(1,2,1/2)_{12}|5\rangle_{\frac{1}{2}}, \tag{C.228}$$
$$|\hat{7}\rangle_{\frac{1}{2}} = U(2,2,1/2)_{11}|7\rangle_{\frac{1}{2}} + U(2,2,12)_{12}|8\rangle, \tag{C.229}$$
$$|\hat{8}\rangle_{\frac{1}{2}} = U(2,2,1/2)_{21}|7\rangle_{\frac{1}{2}} + U(2,2,12)_{22}|8\rangle, \tag{C.230}$$
$$|\hat{0}\rangle_{\frac{3}{2}} = U(1,0,3/2)_{22}|1\rangle_{\frac{3}{2}}, \tag{C.231}$$
$$|\hat{1}\rangle_{\frac{3}{2}} = U(0,1,3/2)_{22}|0\rangle_{\frac{3}{2}}, \tag{C.232}$$
$$|\hat{2}\rangle_{\frac{3}{2}} = U(2,0,3/2)_{21}|3\rangle_{\frac{3}{2}}, \tag{C.233}$$
$$|\hat{3}\rangle_{\frac{3}{2}} = U(0,2,3/2)_{12}|2\rangle_{\frac{3}{2}}, \tag{C.234}$$
$$|\hat{4}\rangle_{\frac{3}{2}} = U(1,1,3/2)_{11}|4\rangle_{\frac{3}{2}} + U(1,1,3/2)_{12}|5\rangle_{\frac{3}{2}}, \tag{C.235}$$
$$|\hat{5}\rangle_{\frac{3}{2}} = U(1,1,3/2)_{21}|4\rangle_{\frac{3}{2}} + U(1,1,3/2)_{22}|5\rangle_{\frac{3}{2}}, \tag{C.236}$$
$$|\hat{6}\rangle_{\frac{3}{2}} = U(2,1,3/2)_{11}|8\rangle_{\frac{3}{2}} + U(2,1,3/2)_{12}|9\rangle_{\frac{3}{2}}, \tag{C.237}$$
$$|\hat{7}\rangle_{\frac{3}{2}} = U(2,1,3/2)_{21}|8\rangle_{\frac{3}{2}} + U(2,1,3/2)_{22}|9\rangle_{\frac{3}{2}}, \tag{C.238}$$
$$|\hat{8}\rangle_{\frac{3}{2}} = U(1,2,3/2)_{11}|6\rangle_{\frac{3}{2}} + U(1,2,3/2)_{12}|7\rangle_{\frac{3}{2}}, \tag{C.239}$$
$$|\hat{9}\rangle_{\frac{3}{2}} = U(1,2,3/2)_{21}|6\rangle_{\frac{3}{2}} + U(1,2,3/2)_{22}|7\rangle_{\frac{3}{2}}, \tag{C.240}$$
$$|\hat{10}\rangle_{\frac{3}{2}} = U(2,2,3/2)_{11}|10\rangle_{\frac{3}{2}} + U(2,2,3/2)_{12}|11\rangle_{\frac{3}{2}}, \tag{C.241}$$
$$|\hat{11}\rangle_{\frac{3}{2}} = U(2,2,3/2)_{21}|10\rangle_{\frac{3}{2}} + U(2,2,3/2)_{22}|11\rangle_{\frac{3}{2}}, \tag{C.242}$$
$$|\hat{0}\rangle_{\frac{5}{2}} = U(2,0,5/2)_{22}|1\rangle_{\frac{5}{2}}, \tag{C.243}$$
$$|\hat{1}\rangle_{\frac{5}{2}} = U(0,2,5/2)_{22}|0\rangle_{\frac{5}{2}}, \tag{C.244}$$
$$|\hat{2}\rangle_{\frac{5}{2}} = U(1,1,5/2)_{22}|2\rangle_{\frac{5}{2}}, \tag{C.245}$$
$$|\hat{3}\rangle_{\frac{5}{2}} = U(2,1,5/2)_{11}|5\rangle_{\frac{5}{2}} + U(2,1,5/2)_{12}|6\rangle_{\frac{5}{2}}, \tag{C.246}$$
$$|\hat{4}\rangle_{\frac{5}{2}} = U(2,1,5/2)_{21}|5\rangle_{\frac{5}{2}} + U(2,1,5/2)_{22}|6\rangle_{\frac{5}{2}}, \tag{C.247}$$
$$|\hat{5}\rangle_{\frac{5}{2}} = U(1,2,5/2)_{11}|3\rangle_{\frac{5}{2}} + U(1,2,5/2)_{12}|4\rangle_{\frac{5}{2}}, \tag{C.248}$$
$$|\hat{6}\rangle_{\frac{5}{2}} = U(1,2,5/2)_{21}|3\rangle_{\frac{5}{2}} + U(1,2,5/2)_{22}|4\rangle_{\frac{5}{2}}, \tag{C.249}$$

$$|\hat{7}\rangle_{\frac{5}{2}} = U(2,2,5/2)_{11}|7\rangle_{\frac{5}{2}} + U(2,2,5/2)_{12}|8\rangle_{\frac{5}{2}}, \tag{C.250}$$

$$|\hat{8}\rangle_{\frac{5}{2}} = U(2,2,5/2)_{21}|7\rangle_{\frac{5}{2}} + U(2,2,5/2)_{22}|8\rangle_{\frac{5}{2}}, \tag{C.251}$$

$$|\hat{0}\rangle_{\frac{7}{2}} = U(2,1,7/2)_{22}|1\rangle_{\frac{7}{2}}, \tag{C.252}$$

$$|\hat{1}\rangle_{\frac{7}{2}} = U(1,2,7/2)_{22}|0\rangle_{\frac{7}{2}}, \tag{C.253}$$

$$|\hat{2}\rangle_{\frac{7}{2}} = U(2,2,7/2)_{11}|2\rangle_{\frac{7}{2}} + U(2,2,7/2)_{12}|3\rangle_{\frac{7}{2}}, \tag{C.254}$$

$$|\hat{3}\rangle_{\frac{7}{2}} = U(2,2,7/2)_{21}|2\rangle_{\frac{7}{2}} + U(2,2,7/2)_{22}|3\rangle_{\frac{7}{2}}, \tag{C.255}$$

$$|\hat{0}\rangle_{\frac{9}{2}} = U(2,2,9/2)_{22}|0\rangle. \tag{C.256}$$

**Remark.** Note that the number of the vector in the bases of the multiplicity subspaces defined in the case of $N = 3, 4$ is less than the dimension of the multiplicity subspaces given by

$$\dim \mathcal{V}_j^{[M]} = \frac{M!(2j+1)}{(M/2-j)!(M/2+1+j)!}. \tag{C.257}$$

For example, $\dim \mathcal{V}_{\frac{1}{2}}^{[9]} = 42$, but we only defined nine vectors. This is because the multiplicities of spin $j_1$ and $j_2$ in $\mathcal{K}_{R_1}$ and $\mathcal{K}_{R_1}$ do not appear in the calculation. More precisely, in the calculation of $\eta^{[N]}$, a maximally entangled vector in $\mathcal{V}_{j_i}^{[N]} \otimes \mathcal{V}_{j_i}^{[N]}$ is associated to spin $j_i$. But the vector rank of the vector is 1 and the vector is invariant under the integration over the Haar measure. That is the reason why the number of vector in the basis defined in this section is smaller than the dimension of the multiplicity subspaces.

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